Theory Problems:

Problem 1.1:

Norm Requirements:

1) Positivity: $\|x\|_p \geq 0$, $\|x\|_p = 0$ iff $x = \emptyset$
2) Homogenity: $\|\alpha x\|_p = |\alpha| \|x\|_p$
3) Sub-additivity: $\|x+y\|_p \leq \|x\|_p + \|y\|_p$

l_1:

✓1) $\|x\|_1 = 0 \iff \sum_{i=1}^{n} |x_i| = 0 \iff x_i = 0, \forall i$

✓2) $\|\alpha x\|_1 = \sum_{i=1}^{n} |\alpha x_i| = \sum |\alpha| \cdot |x_i| = |\alpha| \cdot \underbrace{\sum |x_i|}_{\|x\|_1}$

$\Rightarrow |\alpha| \|x\|_1$

✓3) Triangle Inequality: $|a+b| \leq |a| + |b|$

✓$\|x+y\|_1 = \sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} (|x_i| + |y_i|) = \|x\|_1 + \|y\|_1$

l_∞:

✓1) holds based on definition of "l_∞"

✓2) $\|\alpha x\|_\infty = \max_i |\alpha x_i| = \max_i |\alpha| \cdot |x_i|$

$$= |\alpha| \cdot \underbrace{\max_i |x_i|}_{\|x\|_\infty} = |\alpha| \cdot \|x\|_\infty$$

3) $\|x+y\|_\infty = \max |x_i + y_i| = x_j + y_j$

↳ jth entry is the max entry. of the Vector "$x+y$"

$$\Rightarrow x_j + y_j \leq \max_i |x_i| + \max_i |y_i|$$

$$\Rightarrow x_j + y_j \leq \|x\|_\infty + \|y\|_\infty$$

Problem 1.2:

Exercise 2.6 (Norm inequalities)

1. Show that the following inequalities hold for any vector x:

$$\frac{1}{\sqrt{n}}\|x\|_2 \overset{1}{\leq} \|x\|_\infty \overset{2}{\leq} \|x\|_2 \overset{3}{\leq} \|x\|_1 \overset{4}{\leq} \sqrt{n}\|x\|_2 \overset{5}{\leq} n\|x\|_\infty.$$

   Hint: use the Cauchy–Schwartz inequality.

2. Show that for any nonzero vector x,

$$\operatorname{card}(x) \geq \frac{\|x\|_1^2}{\|x\|_2^2},$$

   where $\operatorname{card}(x)$ is the *cardinality* of the vector x, defined as the number of nonzero elements in x. Find vectors x for which the lower bound is attained.

1) $\frac{\|x\|_2}{\sqrt{n}} \leq \|x\|_\infty \Rightarrow \|x\|_2^2 \leq n \cdot \|x\|_\infty^2$

$\Rightarrow \sum_{i=1}^{n} x_i^2 \leq n \cdot \max |x_i|^2$ ✓

✓2) $\|x\|_\infty \leq \|x\|_2 \Rightarrow \|x\|_\infty \leq \sqrt{x_1^2 + x_2^2 + \cdots}$ Set $|x_j| = \max_i |x_i|$

$\Rightarrow |x_j| \leq \sqrt{|x_j|^2 + \sum \cdots}$

✓3) $\|x\|_2 \leq \|x\|_1 \Rightarrow \|x\|_2^2 \leq \|x\|_1^2 \Rightarrow \sum_{i}^{n} x_i^2 \leq \left(\sum x_i\right)^2$

4) $\|x\|_1 \leq \sqrt{n}\|x\|_2 \Rightarrow$ did not Complet.

4) Continued: Based on Cauchy-Schwartz inequality ($|z^T y| \leq \|y\|_2 \cdot \|z\|_2$)

we can assume $y = (\underbrace{1, \dots, 1}_{n \text{ items}})$ & $z = |x| = (|x_1|, \dots, |x_n|)$

$\Rightarrow z^T y = \sum z_i y_i = \sum |x_i| = \|x\|_1$

$\Rightarrow \|y\|_2 = \sqrt{n}$ & $\|z\|_2 = \sqrt{\sum |x_i|^2} = \|x\|_2$

$$\left.\begin{array}{l} \\ \end{array}\right\} \begin{array}{l} |z^T y| \leq \|y\|_2 \cdot \|z\|_2 \\ \therefore \|x\|_1 \leq \sqrt{n} \cdot \|x\|_2 \end{array}$$

5) Not covered by Solutions. However, the Proof follows 1)

6) Again based on Cauchy-Schwartz we can assume:

$$z = |x| \quad \& \quad y = \begin{cases} y_i = 0, & x_i = 0 \\ y_i = 1, & x_i \neq 0 \end{cases}$$

$\Rightarrow \|y\|_2 = \sqrt{k}$ with "k" being equal to card(x).

$\Rightarrow |z^T y| \leq \|y\|_2 \cdot \|z\|_2$

$\|x\|_1 \leq \sqrt{k}\, \|x\|_2 \quad \therefore \quad \dfrac{\|x\|_1^2}{\|x\|_2^2} \leq k = \text{Card}(x)$ This True for $\forall x$

with "k" non zero elements of equal magnitude.

Did not Complete Problems 1.3-1.6

Problem 1.3:

a) Based on the linear Independance of $a_1, \ldots a_k$ we know that no linear Combination of "c_i" vectors could be equal to zero without the multiples being zero.

b) No Conclusion can be made regarding the linear dependents of c_i as no Information is provided for b_i and therefore the state of the c_i dependants is unknown.

Problem 1.4:

Since x, y are unit norm. We can derive:

$$(x-y)^T(x+y) = x^T x - y^T y - y^T x + x^T y = x^T x - y^T y = 0$$

For any $z = \lambda x + \mu y$, $\lambda, \mu \in \mathbb{R}$ it can be expressed as:

$$z = \alpha u + \beta v \text{ where } \alpha = \frac{\lambda+\mu}{2}, \beta = \frac{\lambda-\mu}{2}$$

and $u = x+y$, $v = x-y$

Since $(x-y)^T(x+y)=0$ & orthogonal then u & v are orthogonal.

& Since $\text{span}(\{u, v\}) = \text{span}(\{x, y\})$

Finally normalize u, v $\Rightarrow \left\{\frac{u}{\|u\|_2}, \frac{v}{\|v\|_2}\right\} = \left\{\frac{(x-y)}{\|(x-y)\|_2}, \frac{(x+y)}{\|x+y\|_2}\right\}$

Problem 1.5:

By definition $f(x, y)$ meets all requirements for an inner product except $f(x,x) \geq 0$ & $f(x,x)=0$ for $x=\phi$ since if any α_k is equal or smaller than zero any vector x that is a unit vector in $\mathbb{R}^n$ would not pass the condition.

$\therefore$ for any α s.t $card(\alpha)=0$ the function is an inner product.

Problem 1.6:

a) using $n=1$, $z_1 = 1$, $z_2 = 10$, $x = -1$, we have:

$$\|x - z_1\| = 2 \qquad \|x - z_2\| = 9$$

$$\angle(x, z_1) = \frac{\pi}{2}, \ \angle(x, z_2) = 0$$

b) $\|x - z_j\| \leq \|x - z_i\| \iff \|x - z_j\|^2 \leq \|x - z_i\|^2$

$$\iff \|x\|^2 - 2x^T z_j + \|z_j\|^2 \leq \|x\|^2 - 2x^T z_i + \|z_i\|^2$$

$$\iff -2x^T z_j \leq -2x^T z_i$$

$$\iff x^T z_j \geq x^T z_i$$

$$\iff \frac{x^T z_j}{\|x\|\|z_j\|} \geq \frac{x^T z_i}{\|x\|\|z_i\|}$$

$$\iff \arccos\left(\frac{x^T z_j}{\|x\|\|z_j\|}\right) \leq \arccos\left(\frac{x^T z_i}{\|x\|\|z_i\|}\right)$$

$$\iff \angle(x, z_j) \leq \angle(x, z_i)$$

```
% PROBLEM 1.7 PART a
[d,i]=pdist2(V',V',"euclidean","Smallest",2);
[min_distance,index] = mink(d(2,:),1);
pair = i(2,index);
display([min_distance,index,pair])
```

```
   24.7184    7.0000    8.0000
```

```
[angles,i]=pdist2(V',V',"cosine","Smallest",2);
[min_angle,index] = mink(angles(2,:),1);
pair = i(2,index);
display([min_angle,index,pair])
```

```
    0.1386    9.0000   10.0000
```

```
% PROBLEM 1.7 PART b
S = sum(V); % find the sum of all word frequencies for each document
Vnorm = V./S; % normalize the dataset
```

```
[d,i]=pdist2(Vnorm',Vnorm',"euclidean","Smallest",2);
[min_distance,index] = mink(d(2,:),1);
pair = i(2,index);
display([min_distance,index,pair])
```

```
    0.0643    9.0000   10.0000
```

```
[angles,i]=pdist2(Vnorm',Vnorm',"cosine","Smallest",2);
[min_angle,index] = mink(angles(2,:),1);
pair = i(2,index);
display([min_angle,index,pair])
```

```
    0.1386    9.0000   10.0000
```

```
% PROBLEM 1.7 PART c
VTF = zeros(size(V));
for i = 1:10
    for j = 1:1651
        VTF(j,i) = tf_idf(j,i,V);
    end
end
[d,i]=pdist2(VTF',VTF',"euclidean","Smallest",2);
[min_distance,index] = mink(d(2,:),1);
pair = i(2,index);
display([min_distance,index,pair])
```

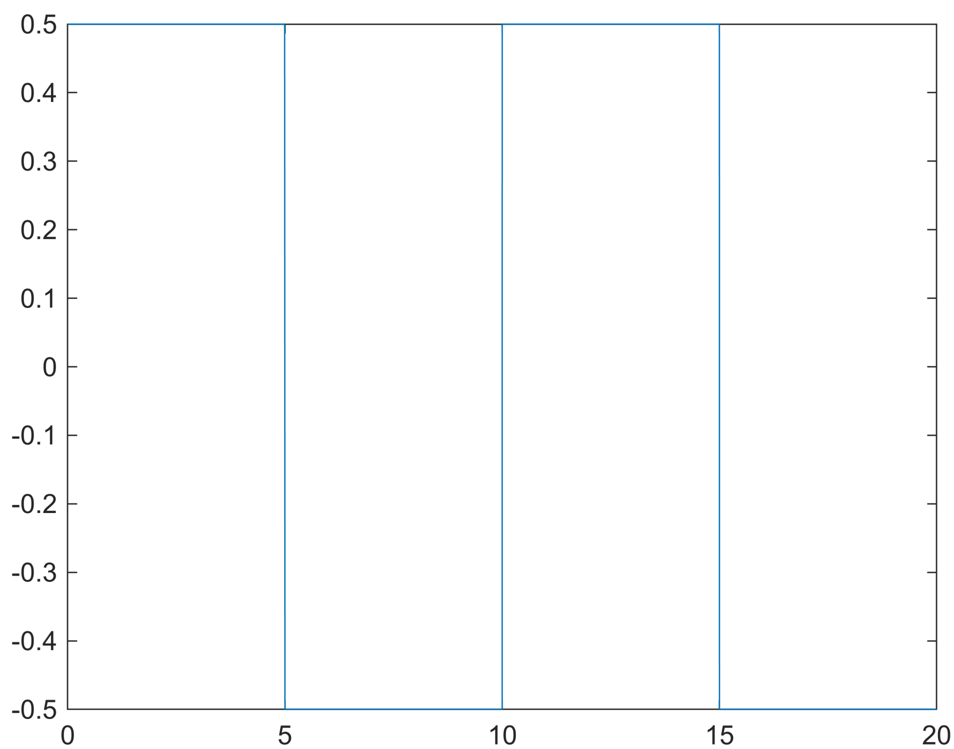
```
    0.0821    9.0000   10.0000
```

```
% PROBLEM 1.7 PART d
% the adjustment is multiplying the rarety of the a word in a document by
% the rarety of the word in all the document. This will make all words
% weighting equal so that words that are very frequent like "the" arent
% skewing results due to how often they apear.
```

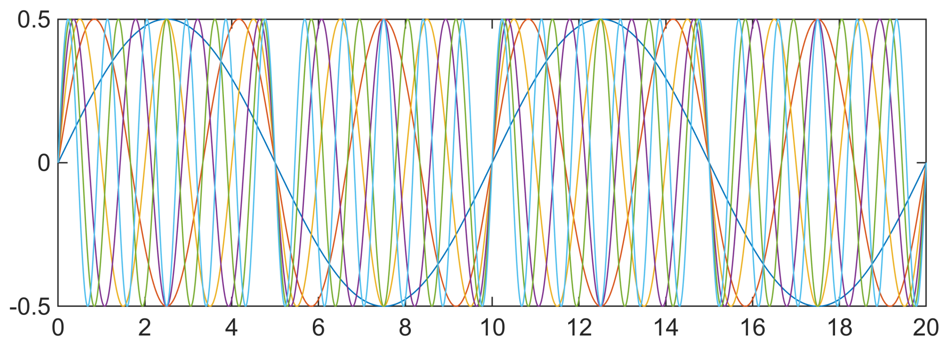
```
% PROBLEM 1.8 PART a
[ time_pos , sq_wave , B_unnorm ] = generate_data
```

```
time_pos = 1×200001
         0    0.0001    0.0002    0.0003    0.0004    0.0005    0.0006    0.0007 ···
sq_wave = 1×200001
    0.5000    0.5000    0.5000    0.5000    0.5000    0.5000    0.5000    0.5000 ···
B_unnorm = 30×200001
         0    0.0000    0.0001    0.0001    0.0001    0.0002    0.0002    0.0002 ···
         0    0.0001    0.0002    0.0003    0.0004    0.0005    0.0006    0.0007
         0    0.0002    0.0003    0.0005    0.0006    0.0008    0.0009    0.0011
         0    0.0002    0.0004    0.0007    0.0009    0.0011    0.0013    0.0015
         0    0.0003    0.0006    0.0008    0.0011    0.0014    0.0017    0.0020
         0    0.0003    0.0007    0.0010    0.0014    0.0017    0.0021    0.0024
         0    0.0004    0.0008    0.0012    0.0016    0.0020    0.0025    0.0029
         0    0.0005    0.0009    0.0014    0.0019    0.0024    0.0028    0.0033
         0    0.0005    0.0011    0.0016    0.0021    0.0027    0.0032    0.0037
         0    0.0006    0.0012    0.0018    0.0024    0.0030    0.0036    0.0042
      ⋮
```

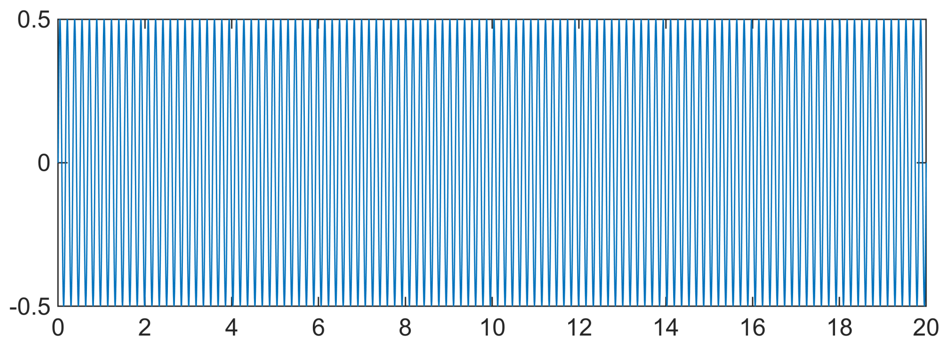
```
subplot(1,1,1);
plot(time_pos,sq_wave)
```



```
% PROBLEM 1.8 PART b
% I would use the gram schmidt procedure to turn the basis into an
% orthonormal set of basis vectors
subplot(2,1,1);
plot(time_pos, B_unnorm(1:6,:))
subplot(2,1,2);
plot(time_pos,B_unnorm(30,:))
```



X wrong display




```
% PROBLEM 1.8 PART c
[q,r]=qr(B_unnorm);
```

X Projection not done.

```
function wd = tf_idf(t,d,V)
    wd = V(t,d) / sum( V(:,d) ) * sqrt( log( 10 /  length( nonzeros( V(t,:) ) ) ) );
end
function [ time_pos , sq_wave , B_unnorm ] = generate_data
    n_comps = 30; period = 10; fundFreq = 1/ period;
    time_pos = 0:0.0001:2* period ; harmonics = 2*(1: n_comps ) -1;
    sq_wave = floor (0.9* sin (2* pi * fundFreq * time_pos ) ) +.5; % % generate the signal
    B_unnorm = sin (2* pi * fundFreq *( harmonics' * time_pos ) ) /2; % % generate the basis
end
```