Back Propogation Explanation

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1 Scalar f , w and x

1.1 Single layer

$$
\begin{aligned}
o &= f(wx-b)\\
\frac{do}{dw} &= f'\frac{d}{dw}(wx)\\
&= f'x\\
e &= \frac{1}{2}(t-o)^2\\
\frac{de}{dw} &= (o-t)\frac{\mathrm{do}}{dw}\\
&= (o-t)f'x\\
\Delta w &= -\eta\frac{de}{dw}\\
&= -\eta(o-t)f'x
\end{aligned}
$$

1.2 Double layers

$$
\begin{aligned}
o^{(2)} &= f_2\big(w^{(2)}o^{(1)}-b^{(2)}\big)\\
o^{(1)} &= f_1(w^{(1)}x-b^{(1)})\\
e &= \frac{1}{2}(t-o^{(2)})^2\\
\frac{\partial e}{\partial w^{(1)}} &= \frac{\partial e}{\partial o^{(2)}}\cdot\frac{\partial o^{(2)}}{\partial o^{(1)}}\cdot\frac{\partial o^{(1)}}{\partial w^{(1)}}\\
&= (o^{(2)}-t)f_2'w^{(2)}f_1'x\\
\Delta w^{(1)} &= -\eta\frac{\partial e}{\partial w^{(1)}}\\
&= -\eta(o^{(2)}-t)f_2'w^{(2)}f_1'x
\end{aligned}
$$

let

$$
\begin{aligned}
\delta^{(2)} &= (o^{(2)}-t)f_2'\\
\delta^{(1)} &= \delta^{(2)}w^{(2)}f_1'
\end{aligned}
$$

then

$$
\Delta w^{(1)} = -\eta\delta^{(1)}x
$$

1.3 Multi layers

$$
\begin{aligned}
o^{(i)} &= f_i\big(w^{(i)}o^{(i-1)} - b^{(i)}\big) \\
e &= \frac{1}{2}(t - o^{(n)})^2 \\
\frac{\partial e}{\partial w^{(i)}} &= \frac{\partial e}{\partial o^{(n)}} \cdot \frac{\partial o^{(n)}}{\partial o^{(n-1)}} \cdots \frac{\partial o^{(i)}}{\partial w^{(i)}} \\
&= (o^{(n)} - t) f_n' \left[\prod_{k=i+1}^{n} w^{(k)} f_{k-1}'\right] o^{(i-1)} \\
\Delta w^{(i)} &= -\eta \frac{\partial e}{\partial w^{(i)}} \\
&= -\eta (o^{(n)} - t) f_n' \left[\prod_{k=i+1}^{n} w^{(k)} \cdot f_{k-1}'\right] \cdot o^{(i-1)}
\end{aligned}
$$

let

$$
\begin{aligned}
\delta^{(n)} &= 2(o^{(n)} - t) f_n' \\
\delta^{(k)} &= \delta^{(k+1)} w^{(k+1)} f_k'
\end{aligned}
$$

then

$$
\Delta w^{(i)} = -\eta \delta^{(k)} o^{(i-1)}
$$

2 Multi nodes

2.1 Matrix $\boldsymbol{W}$, vector $\boldsymbol{X}, \boldsymbol{O}$ and $\boldsymbol{F}$

$$
\begin{aligned}
O^{(i)} &= F_i(W^{(i)} O^{(i-1)}) \\
e &= \frac{1}{2}(T - O^{(n)})^T (T - O^{(n)}) \\
\Delta e &= \frac{\partial e}{\partial O^{(n)}} \cdot \frac{\partial O^{(n)}}{\partial O^{(n-1)}} \cdots \frac{\partial O^{(i+1)}}{\partial O^{(i)}} \Delta O^{(i)} \\
\Delta O^{(i)} &= \operatorname{diag}(F_i') \Delta W^{(i)} O^{(i-1)} \\
\Delta e &= (O^{(n)} - T)^T \operatorname{diag}(F_n') \left[\prod_{k=i+1}^{n} W^{(k)} \cdot \operatorname{diag}(F_{k-1}')\right] \Delta W^{(i)} O^{(i-1)} \\
\frac{\partial e}{\partial W^{(i)}} &= \left[\prod_{k=i+1}^{n} \operatorname{diag}(F_{k-1}') [W^{(k)}]^T\right] \operatorname{diag}(F_n') (O^{(n)} - T) [O^{(i-1)}]^T
\end{aligned}
$$

where

$$
\begin{aligned}
F_i(X) &= F_i\big(\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T\big) \\
&= \begin{pmatrix} f_i(x_1) & f_i(x_2) & \cdots & f_i(x_n) \end{pmatrix}^T \\
\frac{\partial}{\partial X} F_i(X) &= \begin{pmatrix} f_i'(x_1) & & & \\ & f_i'(x_2) & & \\ & & \ddots & \\ & & & f_i'(x_2) \end{pmatrix} \\
&= \operatorname{diag}(F_i')
\end{aligned}
$$

let

$$\begin{aligned}\delta^{(n)} &= \operatorname{diag}(F_n')\cdot 2(O^{(n)} - T)\\ \delta^{(k)} &= \operatorname{diag}(F_k')[W^{(k+1)}]^T\cdot\delta^{(k+1)}\end{aligned}$$

then

$$\Delta W^{(i)} = -\eta\delta^{(i)}[O^{(i-1)}]^T$$

2.2 Summation notation

Output of the j'th unit in the i'th layer:

$$\begin{aligned}o_j^{(i)} &= f_{i,j}\big(y_j^{(i)}\big)\\ y_j^{(i)} &= \sum_{k=1}^{K} w_{j,k}^{(i)} o_k^{(i-1)} - b_j^{(i)}\\ \frac{\partial y_p^{(i+1)}}{\partial y_j^{(i)}} &= \frac{\partial y_p^{(i+1)}}{\partial o_j^{(i)}}\cdot\frac{\partial o_j^{(i)}}{\partial y_j^{(i)}}\\ &= w_{p,j}^{(i+1)}\cdot f_{i,j}'\\ \frac{\partial y_j^{(i)}}{\partial w_{j,k}^{(i)}} &= o_k^{(i-1)}\end{aligned}$$

$$\begin{aligned}e &= \frac{1}{2}\sum_{j=1}^{J}\big(t_j - o_j^{(n)}\big)^2\\ \frac{\partial e}{\partial y_j^{(n)}} &= \frac{\partial e}{\partial o_j^{(n)}}\cdot\frac{\partial o_j^{(n)}}{\partial y_j^{(n)}}\\ &= \big(o_j^{(n)} - t_j\big)\cdot f_{n,j}'\\ \frac{\partial e}{\partial y_j^{(i)}} &= \frac{\partial e}{\partial o_j^{(i)}}\cdot\frac{\partial o_j^{(i)}}{\partial y_j^{(i)}}\\ &= \sum_{P=1}^{P}\frac{\partial e}{\partial o_p^{(i+1)}}\cdot\frac{\partial o_p^{(i+1)}}{\partial o_j^{(i)}}\cdot\frac{\partial o_j^{(i)}}{\partial y_j^{(i)}}\\ &= \sum_{P=1}^{P}\frac{\partial e}{\partial o_p^{(i+1)}}\cdot\frac{\partial o_p^{(i+1)}}{\partial y_p^{(i+1)}}\cdot\frac{\partial y_p^{(i+1)}}{\partial o_j^{(i)}}\cdot\frac{\partial o_j^{(i)}}{\partial y_j^{(i)}}\\ &= \sum_{P=1}^{P}\frac{\partial e}{\partial y_p^{(i+1)}}\cdot\frac{\partial y_p^{(i+1)}}{\partial y_j^{(i)}}\\ &= \sum_{p=1}^{P}\frac{\partial e}{\partial y_p^{(i+1)}}\cdot w_{p,j}^{(i+1)} f_{i,j}'\\ \frac{\partial e}{\partial w_{j,k}^{(i)}} &= \frac{\partial e}{\partial y_j^{(i)}}\cdot\frac{\partial y_j^{(i)}}{\partial w_{j,k}^{(i)}}\\ &= \frac{\partial e}{\partial y_j^{(i)}}\cdot o_k^{(i-1)}\end{aligned}$$

let

$$\begin{aligned}
\delta_j^{(n)} &= \frac{\partial e}{\partial y_j^{(n)}} \\
&= \left(o_j^{(n)} - t_j\right) \cdot f'_{n,j} \\
\delta_j^{(i)} &= \frac{\partial e}{\partial y_j^{(i)}} \\
&= \sum_{p=1}^{P} \delta_p^{(i+1)} \cdot w_{p,j}^{(i+1)} f'_{i,j}
\end{aligned}$$

then

$$\begin{aligned}
\Delta w_{j,k}^{(i)} &= -\eta \frac{\partial e}{\partial w_{j,k}^{(i)}} \\
&= -\eta \delta_j^{(i)} o_k^{(i-1)}
\end{aligned}$$