# Back Propogation Explanation

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## 1 Scalar $f$ , $w$ and $x$

### 1.1 Single layer

$$\begin{aligned}
o &= f\,(w\,x - b)\\
\frac{d\,o}{d\,w} &= f'\,\frac{d}{d\,w}\,(w\,x)\\
&= f'\,x\\
e &= (t - o)^2\\
\frac{d\,e}{d\,w} &= 2\,(o - t)\,\frac{\mathrm{do}}{d\,w}\\
&= 2\,(o - t)\,f'\,x\\
\Delta\,w &= -\eta\,\frac{d\,e}{d\,w}\\
&= -2\,\eta\,(o - t)\,f'\,x
\end{aligned}$$

### 1.2 Double layers

$$\begin{aligned}
o^{(2)} &= f_2\,(w^{(2)}\,o^{(1)} - b^{(2)})\\
o^{(1)} &= f_1\,(w^{(1)}\,x - b^{(1)})\\
e &= (t - o^{(2)})^2\\
\frac{\partial\,e}{\partial\,w^{(1)}} &= \frac{\partial\,e}{\partial\,o^{(2)}}\cdot\frac{\partial\,o^{(2)}}{\partial\,o^{(1)}}\cdot\frac{\partial\,o^{(1)}}{\partial\,w^{(1)}}\\
&= 2\,(o^{(2)} - t)\,f_2'\,w^{(2)}\,f_1'\,x\\
\Delta\,w^{(1)} &= -\eta\,\frac{\partial\,e}{\partial\,w^{(1)}}\\
&= -2\,\eta\,(o^{(2)} - t)\,f_2'\,w^{(2)}\,f_1'\,x
\end{aligned}$$

### 1.3 Multi layers

$$\begin{aligned}
o^{(i)} &= f_i\,(w^{(i)}\,o^{(i-1)} - b^{(i)})\\
e &= (t - o^{(n)})^2\\
\frac{\partial\,e}{\partial\,w^{(i)}} &= \frac{\partial\,e}{\partial\,o^{(n)}}\cdot\frac{\partial\,o^{(n)}}{\partial\,o^{(n-1)}}\cdots\frac{\partial\,o^{(i)}}{\partial\,w^{(i)}}\\
&= 2\,(o^{(n)} - t)\,\prod_{k=i+1}^{n}\,[f_k'\,w^{(k)}]\cdot f_i'\,o^{(i-1)}\\
\Delta\,w^{(i)} &= -\eta\,\frac{\partial\,e}{\partial\,w^{(i)}}\\
&= -2\,\eta\,(o^{(n)} - t)\,\prod_{k=i+1}^{n}\,[f_k'\,w^{(k)}]\cdot f_i'\,o^{(i-1)}
\end{aligned}$$

# 2 Multi nodes

## 2.1 Matrix $\boldsymbol{W}$ , vector $\boldsymbol{X}, \boldsymbol{O}$ and $\boldsymbol{F}$

$$\begin{aligned}
O^{(i)} &= F_i\,(W^{(i)}\,O^{(i-1)}) \\
e &= (T - O^{(n)})^T\,(T - O^{(n)}) \\
\frac{\partial e}{\partial W^{(i)}} &= \frac{\partial e}{\partial O^{(n)}} \cdot \frac{\partial O^{(n)}}{\partial O^{(n-1)}} \cdots \frac{\partial O^{(i)}}{\partial W^{(i)}} \\
&= 2\,(O^{(n)} - T)^T \prod_{k=i+1}^{n} [\mathrm{diag}(F_k)'\,W^{(k)}] \cdot \mathrm{diag}(F_i')\,O^{(i-1)}
\end{aligned}$$

where

$$\begin{aligned}
F_i(X) &= F_i((\ x_1\ \ x_2\ \ \cdots\ \ x_n\ )^T) \\
&= (\ f_i(x_1)\ \ f_i(x_2)\ \ \cdots\ \ f_i(x_n)\ )^T \\
\frac{\partial}{\partial X}\,F_i(X) &= \begin{pmatrix} f_i'(x_1) & & & \\ & f_i'(x_2) & & \\ & & \ddots & \\ & & & f_i'(x_2) \end{pmatrix} \\
&= \mathrm{diag}(F_i')
\end{aligned}$$

## 2.2 Summation notation

$$\begin{aligned}
o_j^{(i)} &= f_{i,j}\left(\sum_{k=1}^{N_{i-1}} w_{j,k}^{(i)}\,o_k^{(i-1)} - b_j^{(i)}\right) \\
e &= \sum_{j=1}^{N_n} (t_j - o_j^{(n)})^2 \\
\frac{\partial e}{\partial w_{j,k}^{(i)}} &= \sum_{k_n=1}^{N_n} \frac{\partial e}{\partial o_{k_n}^{(n)}} \cdot \sum_{k_{n-1}=1}^{N_{n-1}} \frac{\partial o_{k_n}^{(n)}}{\partial o_{k_{n-1}}^{(n-1)}} \cdots \frac{\partial o_j^{(i)}}{\partial w_{j,k}^{(i)}} \\
&= 2 \sum_{k_n=1}^{N_n} (o_{k_n}^{(n)} - t_{k_n}) \sum_{k_{n-1}=1}^{N_{n-1}} f_{n,k_n}'\,w_{k_n,k_{n-1}}^{(n)} \cdots f_i'\,o_k^{(i-1)} \\
&= 2 \sum_{k_n=1}^{N_n} \sum_{k_{n-1}=1}^{N_{n-1}} \sum_{k_{n-2}=1}^{N_{n-2}} \cdots \\
&\quad (o_{k_n}^{(n)} - t_{k_n})\,f_{n,k_n}'\,w_{k_n,k_{n-1}}^{(n)}\,f_{n-1,k_{n-1}}'\,w_{k_{n-1},k_{n-2}}^{(n-1)} \cdots f_i'\,o_k^{(i-1)} \\
&= 2 \sum_{k_n=1}^{N_n} (o_{k_n}^{(n)} - t_{k_n}) \prod_{m=n}^{i+1} \left[\sum_{k_{m-1}=1}^{N_{m-1}} f_{m,k_m}'\,w_{k_m,k_{m-1}}^{(m)}\right] f_i'\,o_k^{(i-1)} \\
\Delta\,w^{(i)} &= -\eta\,\frac{\partial e}{\partial w_{j,k}^{(i)}} \\
&= -2\,\eta \sum_{k_n=1}^{N_n} (o_{k_n}^{(n)} - t_{k_n}) \prod_{m=n}^{i+1} \left[\sum_{k_{m-1}=1}^{N_{m-1}} f_{m,k_m}'\,w_{k_m,k_{m-1}}^{(m)}\right] f_i'\,o_k^{(i-1)}
\end{aligned}$$