Notes on Monty Hall puzzle

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Cars and goats: the Monty Hall dilemma

On Sunday September 9, 1990, the following question appeared in the "Ask Marilyn" column in Parade, a Sunday supplement to many newspapers across the United States:

- Suppose you're on a game show, and you're given the choice of three doors;
- behind one door is a car; behind the others, goats.
- You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat.
- He then says to you, "Do you want to pick door No. 2?"
- Is it to your advantage to switch your choice?—Craig F. Whitaker, Columbia, Md.

https://www.boards.ie/b/thread/2056147144

Inital guess, Suppose picking No.1 door:

$$\begin{aligned}
P(\mathrm{car}_1|\mathrm{pick}_1) &= \frac{1}{3}\\
P(\mathrm{car}_2|\mathrm{pick}_1) &= \frac{1}{3}\\
P(\mathrm{car}_3|\mathrm{pick}_1) &= \frac{1}{3}
\end{aligned}$$

Open a door with a goat:

$$\begin{aligned}
P(\mathrm{open}_2|\mathrm{car}_1,\mathrm{pick}_1) &= \frac{1}{2}\\
P(\mathrm{open}_3|\mathrm{car}_1,\mathrm{pick}_1) &= \frac{1}{2}\\
P(\mathrm{open}_2|\mathrm{car}_2,\mathrm{pick}_1) &= 0\\
P(\mathrm{open}_3|\mathrm{car}_2,\mathrm{pick}_1) &= 1\\
P(\mathrm{open}_2|\mathrm{car}_3,\mathrm{pick}_1) &= 1\\
P(\mathrm{open}_3|\mathrm{car}_3,\mathrm{pick}_1) &= 0
\end{aligned}$$

Posterior probability:

$$\begin{aligned}
P(\mathrm{car}_1|\mathrm{pick}_1,\mathrm{open}_2) &= \frac{P(\mathrm{open}_2|\mathrm{car}_1,\mathrm{pick}_1)P(\mathrm{car}_1|\mathrm{pick}_1)}{\sum_{i=1}^{3} P(\mathrm{open}_2|\mathrm{car}_i,\mathrm{pick}_1)P(\mathrm{car}_i|\mathrm{pick}_1)}\\
&= \frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3}+0\cdot\frac{1}{3}+1\cdot\frac{1}{3}}\\
&= \frac{1}{3}\\
P(\mathrm{car}_1|\mathrm{pick}_1,\mathrm{open}_3) &= \frac{P(\mathrm{open}_3|\mathrm{car}_1,\mathrm{pick}_1)P(\mathrm{car}_1|\mathrm{pick}_1)}{\sum_{i=1}^{3} P(\mathrm{open}_3|\mathrm{car}_i,\mathrm{pick}_1)P(\mathrm{car}_i|\mathrm{pick}_1)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3}+1\cdot\frac{1}{3}+0\cdot\frac{1}{3}} \\
&= \frac{1}{3} \\
P(\mathrm{car}_2|\mathrm{pick}_1,\mathrm{open}_2) &= \frac{P(\mathrm{open}_2|\mathrm{car}_2,\mathrm{pick}_1)P(\mathrm{car}_2|\mathrm{pick}_1)}{\sum_{i=1}^{3} P(\mathrm{open}_2|\mathrm{car}_i,\mathrm{pick}_1)P(\mathrm{car}_i|\mathrm{pick}_1)} \\
&= \frac{0\cdot\frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3}+0\cdot\frac{1}{3}+1\cdot\frac{1}{3}} \\
&= 0 \\
P(\mathrm{car}_3|\mathrm{pick}_1,\mathrm{open}_2) &= \frac{P(\mathrm{open}_2|\mathrm{car}_3,\mathrm{pick}_1)P(\mathrm{car}_3|\mathrm{pick}_1)}{\sum_{i=1}^{3} P(\mathrm{open}_2|\mathrm{car}_i,\mathrm{pick}_1)P(\mathrm{car}_i|\mathrm{pick}_1)} \\
&= \frac{1\cdot\frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3}+0\cdot\frac{1}{3}+1\cdot\frac{1}{3}} \\
&= \frac{2}{3} \\
P(\mathrm{car}_3|\mathrm{pick}_1,\mathrm{open}_3) &= \frac{P(\mathrm{open}_3|\mathrm{car}_3,\mathrm{pick}_1)P(\mathrm{car}_3|\mathrm{pick}_1)}{\sum_{i=1}^{3} P(\mathrm{open}_3|\mathrm{car}_i,\mathrm{pick}_1)P(\mathrm{car}_i|\mathrm{pick}_1)} \\
&= \frac{0\cdot\frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3}+1\cdot\frac{1}{3}+0\cdot\frac{1}{3}} \\
&= 0 \\
P(\mathrm{car}_2|\mathrm{pick}_1,\mathrm{open}_3) &= \frac{P(\mathrm{open}_3|\mathrm{car}_2,\mathrm{pick}_1)P(\mathrm{car}_2|\mathrm{pick}_1)}{\sum_{i=1}^{3} P(\mathrm{open}_3|\mathrm{car}_i,\mathrm{pick}_1)P(\mathrm{car}_i|\mathrm{pick}_1)} \\
&= \frac{1\cdot\frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3}+1\cdot\frac{1}{3}+0\cdot\frac{1}{3}} \\
&= \frac{2}{3}
\end{aligned}
$$

Then it shows that:

Switch to another closed door can increase the probability of winning from $\frac{1}{3}$ to $\frac{2}{3}$.