Classical identification for linear time invariant systems

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classical identification method definition

A method for obtaining a mathematical model of a system from three classical input signals.

- sinusoidal input — frequency response
- step input — step response
- impulse input — impulse response

The focus of this course is on the method of obtaining the mathematical model of the system by using the impulse input signal.

Classic identification of content, purpose and method

- Classic identification content and purpose:
  - How to get the impulse response of the system?
  - How to determine the transfer function and impulse transfer function of the system from the impulse response of the system?
- Solution:
  - To get the impulse response of the system, use the correlation method;
  - To find the parameter model of the system from the impulse response, use pure analytical method.

Correlation method to obtain the impulse response of the system: system model

Refer to the SISO system impulse response function as $g(\tau)$. According to the convolution theorem of linear systems:

$$y(t) = \int_{-\infty}^{\infty} g(\sigma)x(t-\sigma)d\sigma$$

Let $x(t)$ be a stationary stochastic process with a mean of 0, then $y(t)$ is also a stationary stochastic process with a mean of 0. At any time, t_2, when $t = t_2$, the above formula is

$$y(t_2) = \int_{-\infty}^{\infty} g(\sigma)x(t_2-\sigma)d\sigma$$

Multiply the above formula with the input $x(t_1)$ at another time to get:

$$x(t_1)y(t_2) = \int_{-\infty}^{\infty} g(\sigma)x(t_1)x(t_2-\sigma)d\sigma$$

Correlation method to obtain the impulse response of the system: Wiener-Hoff equation

Take mathematics expectations on both sides and get:

$$\mathrm{E}[\mathrm{x}(\mathrm{t}_1)\mathrm{y}(\mathrm{t}_2)] = \int_{-\infty}^{\infty} \mathrm{g}(\sigma)\mathrm{E}[\mathrm{x}(\mathrm{t}_1)\mathrm{x}(\mathrm{t}_2 - \sigma)]\mathrm{d}\sigma$$

The Wiener-Hoff equation can be obtained:

$$\mathrm{R}_{\mathrm{xy}}(\tau) = \int_{-\infty}^{\infty} \mathrm{g}(\sigma)\mathrm{R}_{\mathrm{x}}(\tau)]\mathrm{d}\sigma$$

Where: $\tau = \mathrm{t}_2 - \mathrm{t}_1$ If R_{xy} and R_{x} are known in the equation, then the above equation can be solved to get $\mathrm{g}(\tau)$

Correlation method to obtain the impulse response of the system: Wiener-Hoff equation solving

When $x(t)$ is a white noise signal, there is $R_x(\tau) = K\delta(\tau)$, and $R_x(\tau - \sigma) = K\delta(\tau - \sigma)$

After substituting the Wiener Hof equation, it is available

$$\begin{aligned} R_{xy}(\tau) &= \int_{-\infty}^{\infty} g(\sigma) K\delta(\tau - \sigma) d\sigma \\ &= Kg(\tau) \\ g(\tau) &= \frac{R_{xy}(\tau)}{K} \end{aligned}$$

For the solution of $g(\tau)$, just calculate R_{xy}. If the observation time T_m is sufficiently large, then

$$\begin{aligned} R_{xy}(\tau) &= \frac{1}{T_m} \int_0^{T_m} x(t) y(t + \tau) dt \\ R_{xy}(k) &= \frac{1}{N} \sum_{i=0}^{N-1} x_i y_{i+k} \end{aligned}$$

Where x_i, y_{i+k} is the sequence of data recorded.

white noise process

If the mean of the random process w(t) is 0, the autocorrelation function is:

$$R_w(t) = \sigma^2 \delta(t)$$

The process is called a white noise process. among them:

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

- To use impulse input to get impulse response, is not possible in engineering
- white noise is artificially unproducible in engineering;

Therefore, the system's impulse response sequence must be identified by an input signal that can be repeatedly generated in the engineering practice.

- pseudo-random noise;
- discrete two-bit white noise sequence;
- pseudo-random discrete two-bit sequence; (M-sequence)
- two-level M sequence;

Pseudorandom noise is truncated by white noise and is a periodic signal.

$$\begin{aligned} R_x(\tau) &= R_x(\tau + T) \\ &= \delta(nT + \tau) \end{aligned}$$

Where $n = 0, \pm 1, \pm 2, \cdots$

Pseudo-random noise identification impulse response: Calculate R_{xy}

Pseudo-random noise signals as input signals , then:

$$
\begin{aligned}
R_{xy} &= \int_{-\infty}^{\infty} g(\sigma) R_x(\tau-\sigma) d\sigma \\
&= \int_{0}^{T} g(\sigma) R_x(\tau-\sigma) d\sigma + \int_{T}^{2T} g(\sigma) R_x(\tau-\sigma) d\sigma + \cdots \\
&= \int_{0}^{T} g(\sigma) K\delta(\tau-\sigma) d\sigma + \int_{T}^{2T} g(\sigma) K\delta(T+\tau-\sigma) d\sigma \\
&\quad + \cdots \\
&= Kg(\tau) + Kg(\tau+T) + Kg(\tau+2T) + \cdots
\end{aligned}
$$

Pseudorandom noise identification impulse response: Calculate $g(\tau)$

Select the appropriate truncation period so that $g(\tau)$ has decayed to zero at $\tau < T$. then:

$$\begin{aligned} R_{xy}(\tau) &= Kg(\tau) + 0 \\ &= Kg(\tau) \\ g(\tau) &= R_{xy}(\tau)/K \end{aligned}$$

The same identification result as white input is obtained.

Calculate $R_x(\tau), R_{xy}(\tau)$

$$
\begin{aligned}
R_x(\tau) &= \lim_{n\to\infty} \frac{1}{nT}\int_0^{nT} x(t)x(t+\tau)dt \\
&= \lim_{n\to\infty} \frac{n}{nT}\int_0^{T} x(t)x(t+\tau)dt \\
&= \frac{1}{T}\int_0^{T} x(t)x(t+\tau)dt \\
R_{xy}(\tau) &= \int_{-\infty}^{\infty} g(\sigma)R_x(\tau-\sigma)d\sigma \\
&= \int_{-\infty}^{\infty} g(\sigma)\left[\frac{1}{T}\int_0^{T} x(t)x(t+\tau-\sigma)dt\right]d\sigma \\
&= \frac{1}{T}\int_0^{T} x(t)\left[\int_{-\infty}^{\infty} g(\sigma)x(t+\tau-\sigma)d\sigma\right]dt \\
&= \frac{1}{T}\int_0^{T} x(t)y(t+\tau)dt
\end{aligned}
$$

$R_{xy}(\tau)$ only needs one cycle calculation.

A random sequence of consecutive white noise samples at equal intervals. Has the same statistical properties of continuous white noise, ie

$$\mathrm{E}(\mathrm{x_i x_j}) = \begin{cases} \sigma^2 & \mathrm{i} = \mathrm{j} \\ 0 & \mathrm{i} \neq \mathrm{j} \end{cases}$$

Where $\mathrm{i}, \mathrm{j} = 1, 2, 3, \cdots$

discrete two-bit white noise

Discrete random variables take only two values. The elements in the sequence are generally taken as 1 and -1
Example: a discrete two-bit noise
1111-1-1-11-1-111-11-1 $\cdots$
Main properties:

- -1 and 1 appear equal times;
- The total number of total runs (the segments in which the states "1" and "-1" appear consecutively are called runs) are (N+1)/2, and the runs of -1 and 1 are equal, up to one difference. (N is the length of the sequence)
- its autocorrelation function is

$$\mathrm{R_{xx}}(\tau) = \begin{cases} 1 & \tau = 0 \\ 0 & \tau \neq 0 \end{cases}$$

Features of the M sequence

In engineering practice, the M sequence is often used instead of the white noise input signal to identify the impulse response sequence of the system. Features of the M sequence:

- pseudo-random two-position sequence;
- The digital features of the M sequence are similar to white noise;
- deterministic sequence;
- can be easily regenerated in engineering practice.

M sequence generation method and its properties

M sequence: A pseudo-random sequence constructed by truncating a discrete two-bit noise sequence. Notable features:

- M sequence is a deterministic sequence that can be repeated;
- M sequence has similar properties to discrete two-bit white noise.

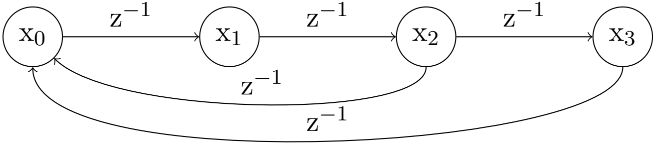
Producing method: The M sequence is generated by engineering using the shift register method.

$$
\begin{aligned}
X_0(k+1) &= a_0x_0(k) \oplus a_1x_1(k) \oplus \cdots \oplus a_nx_n(k) \\
X_1(k+1) &= x_0(k) \\
&\cdots \\
X_n(k+1) &= x_{n-1}(k)
\end{aligned}
$$

Pseudo-random sequence generating conditions : the initial state of each register is not all zero.

M sequence generation method and its properties

example:



$$
\begin{aligned}
x_0(k+1) &= x_2(k) \oplus x_3(k) \\
x_1(k+1) &= x_0(k) \\
x_2(k+1) &= x_1(k) \\
x_3(k+1) &= x_2(k)
\end{aligned}
$$

When the initial state is all 1, the status of each register is

X0: 100010011010111
X1: 110001001101011
X2: 111000100110101
X3: 111100010011010

The output sequence is: 111100010011010 (length N=15)

If the number of registers is n, then there is

- period length $\mathrm{N} = 2^{\mathrm{n}} - 1$;
- total run $= 2^{\mathrm{n}-1}$;
- The number of occurrences of "0" is (N-1)/2, and the number of occurrences of "1" is (N+1)/2. The difference is 1 time.

- turns the M sequence into a level signal,
  - "0" is taken as a, and "1" is taken as -a. The
  - shift pulse period is Δ, and the period of the two-level M sequence is $N\Delta$.
- numeric features: In a period of $N\Delta$, its mean m_x is

$$
\begin{aligned}
M_x &= \frac{1}{N\Delta}\left(\frac{N-1}{2}a\Delta - \frac{N+1}{2}a\Delta\right) = -\frac{a}{N} \\
\lim_{N\to\infty} m_x &= 0
\end{aligned}
$$

autocorrelation function $R_x(\tau)$

$$R_x\tau = \begin{cases} \dfrac{-a^2}{N} & (kN+1)\Delta<\tau<((k+1)N-1)\Delta \\ a^2\left[1-\dfrac{(N+1)|\tau|}{N\Delta}\right] & (kN-1)\Delta<\tau<(kN+1)\Delta \end{cases}$$

Triangular impulse component and DC component

$$R_x(\tau) = R_x^1(\tau) + R_x^2(\tau)$$

where:

$R_x^2(\tau) = \frac{-a^2}{N}$ is DC component

$R_x^1(\tau) = R_x(\tau) - R_x^2(\tau)$ is triangular pulse component

When Δ is small, $R_x^1(\tau)$ can be considered as a impulse function, then there is

$$\begin{aligned} R_x^1(\tau) &= \frac{N+1}{N} a^2 \Delta \delta(\tau) \\ R_x(\tau) &= \frac{N+1}{N} a^2 \Delta \delta(\tau) - \frac{a^2}{N} \end{aligned}$$

Therefore, the M sequence has a digital characteristic of a white noise sequence.

two-level M-sequence recognize system impulse response sequence $g(\tau)$: graphical method

The two-level M-sequence recognizes $g(\tau)$ in two ways: the graphical method and the formula method. First introduce the graphical method:

$$
\begin{aligned}
R_{xy}(\tau) &= \int_{-\infty}^{\infty} g(\sigma) R_x(\tau-\sigma) d\sigma \\
&= \int_{0^+}^{N\Delta^-} g(\sigma) R_x(\tau-\sigma) d\sigma \\
&= \int_{0^+}^{N\Delta^-} \left[\frac{N+1}{N} a^2 \Delta \delta(\tau-\sigma) - \frac{a^2}{N}\right] g(\sigma) d\sigma \\
&= \frac{N+1}{N} a^2 \Delta g(\tau) - \int_{0^+}^{N\Delta^-} g(\sigma) d\sigma \\
&= \frac{N+1}{N} a^2 \Delta g(\tau) - A
\end{aligned}
$$

where: $A = \int_{0^+}^{N\Delta^-} g(\sigma) d\sigma$

$R_{xy}(\tau)$ can be calculated from the input and output data sequence:

$$R_{xy}(\tau) = \frac{1}{N} \sum_{i=1}^{N-1} x(i)y(i+\tau)$$

Simply pan the $R_{xy}(\tau)$ curve up by A to get $g(\tau)$.

Analytical method to get $g(\tau)$

$$
\begin{aligned}
R_{xy}(\tau) &= \frac{N+1}{N}a^2\Delta g(\tau) - \frac{a^2}{N}\int_0^{N\Delta} g(\sigma)d\sigma \\
\int_0^{N\Delta} R_{xy}(\tau)d\tau &= \frac{N+1}{N}a^2\Delta \int_0^{N\Delta} g(\tau)d\tau \\
&\quad -\frac{a^2}{N}N\Delta \int_0^{N\Delta} g(\sigma)d\sigma \\
&= \frac{\Delta a^2}{N}\int_0^{N\Delta} g(\tau)d\tau \\
R_{xy}(\tau) &= \frac{N+1}{N}a^2\Delta g(\tau) - \frac{1}{\Delta}\int_0^{N\Delta} R_{xy}(\sigma)d\sigma \\
g(\tau) &= \frac{N}{(N+1)\Delta a^2}\left[R_{xy}(\tau) + \frac{1}{\Delta}\int_0^{N\Delta} R_{xy}(\sigma)d\sigma\right]
\end{aligned}
$$

Analytical method to get $g(\tau)$

$$
\begin{aligned}
g(\tau) &= \frac{N}{(N+1)\Delta a^2} R_{xy}(\tau) + g_0 \\
g_0 &= \frac{N}{(N+1)\Delta^2 a^2} \int_0^{N\Delta} R_{xy}(\tau) d\tau \\
\int_0^{N\Delta} R_{xy}(\tau) d\tau &\approx \Delta \sum_{i=0}^{N-1} R_{xy}(i) \\
R_{xy}(\tau) &= \frac{1}{N} \sum_{i=0}^{N-1} x(i) y(i+\tau)
\end{aligned}
$$

Matrix representation for $g(\tau)$

Discrete Wiener-Hoff equation:

$$
\begin{aligned}
R_{xy}(i\Delta) &= \sum_{k=0}^{N-1} \Delta g(k\Delta) R(i\Delta - k\Delta) \\
R_{xy} &= Rg\Delta \\
g &= \frac{R^{-1}R_{xy}}{\Delta}
\end{aligned}
$$

where:

$$
\begin{aligned}
g &= [g(0), g(1), \cdots, g(N-1)]^T \\
R_{xy} &= [R_{xy}(0), R_{xy}(1), \cdots, R_{xy}(N-1)]^T \\
R &= \begin{bmatrix} R_x(0) & R_x(-1) & \cdots & R_x(-N+1) \\ R_x(1) & R_x(0) & \cdots & R_x(-N+2) \\ \vdots & \vdots & & \vdots \\ R_x(N-1) & R_x(N-2) & \cdots & R_x(0) \end{bmatrix}
\end{aligned}
$$

Matrix representation for $g(\tau)$:calculate R^{-1}

$$R_x(k) = \begin{cases} a^2 & k = 0 \\ -\frac{a^2}{N} & 1 \leq k \leq N-1 \end{cases}$$

$$R = a^2 \begin{bmatrix} 1 & -\frac{1}{N} & \cdots & -\frac{1}{N} \\ -\frac{1}{N} & 1 & \cdots & -\frac{1}{N} \\ \vdots & \ddots & \ddots & \vdots \\ -\frac{1}{N} & -\frac{1}{N} & \cdots & 1 \end{bmatrix}$$

$$R^{-1} = \frac{N}{a^2(N+1)} \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 2 \end{bmatrix}$$

Matrix representation for $g(\tau)$:calculate R_{xy}

$$
\begin{aligned}
R_{xy} &= [R_{xy}(0), R_{xy}(1), \cdots, R_{xy}(N-1)]^T \\
&= \frac{1}{rN} XY \\
X &= \begin{bmatrix} x(0) & x(1) & \cdots & x(rN-1) \\ x(-1) & x(0) & \cdots & x(rN-2) \\ \vdots & \vdots & & \vdots \\ x(-N+1) & x(-N+2) & \cdots & x(rN-N) \end{bmatrix} \\
Y &= \begin{bmatrix} y(0) & y(1) & \cdots & y(rN-1) \end{bmatrix}^T
\end{aligned}
$$

Recursive algorithm for $g(\tau)$ (online identification)

Recursive algorithm: Suppose we get the identification result g_{m-1} for the $(m-1)$ observations, and now we have a new set of observations (x_m, y_m). Now let's discuss how to get the new $g(\tau)$ estimate g_m for g_{m-1} and (x_m, y_m) data.

The formula for the general recursive algorithm is as follows:

$$G_m = K g_{m-1} + \tilde{g}_m$$

Among them, $\tilde{g}_m$ is the information added from the newly obtained data.

Recursive formula for R_{xy}

$$
\begin{aligned}
R_{xy}(i,m) &= \frac{1}{m+1}\sum_{k=0}^{m} y(k)x(k-i) \\
&= \frac{1}{m+1}\left[\sum_{k=0}^{m-1} y(k)x(k-i) + y(m)x(m-i)\right] \\
&= \frac{1}{m+1}\left[mR_{xy}(i,m-1) + y(m)x(m-i)\right] \\
R_{xy}(m) &= \frac{1}{m+1}\left[mR_{xy}(m-1) + y(m)X(m)\right]
\end{aligned}
$$

where:

$$
\begin{aligned}
R_{xy}(m) &= [R_{xy}(0), R_{xy}(1), \cdots, R_{xy}(N-1)]^T \\
X(m) &= [x(m), x(m-1), \cdots, x(m-N+1)]^T
\end{aligned}
$$

Recursive formula for $g(\tau)$

$$
\begin{aligned}
g_m &= \frac{R^{-1}R_{xy}(m)}{\Delta} \\
&= \frac{R^{-1}}{\Delta}\frac{1}{m+1}\left[mR_{xy}(m-1)+y(m)X(m)\right] \\
&= \frac{mR^{-1}R_{xy}(m-1)}{(m+1)\Delta}+\frac{R^{-1}}{(m+1)\Delta}y(m)X(m) \\
&= \frac{m}{m+1}g_{m-1}+\frac{R^{-1}}{(m+1)\Delta}y(m)X(m)
\end{aligned}
$$

Impulse response sequence for G(z)

G(z) is called the pulse transfer function of the system and is a discrete mathematical model of the system.

$$\begin{aligned} G(z) &= \frac{C(z)}{R(z)} \\ &= \frac{b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}} \end{aligned}$$

get:

$$c_t + a_1 c_{t-1} + \cdots + a_n c_{t-n} = b_0 r_t + \cdots + b_n r_{t-n}$$

$$g(t) + a_1 g(t-1) + \cdots + a_n g(t-n) = b_0 \delta(t) + \cdots + b_n \delta(t-n)$$

$$\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ a_1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ a_n & a_{n-1} & \cdots & 1 \end{bmatrix} \begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ g(n) \end{bmatrix}$$

$$\begin{bmatrix} g(1) & g(2) & \cdots & g(n) \\ g(2) & g(3) & \cdots & g(n+1) \\ \vdots & \vdots & & \vdots \\ g(n) & g(n+1) & \cdots & g(2n-1) \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_1 \end{bmatrix} = \begin{bmatrix} -g(n+1) \\ -g(n+2) \\ \vdots \\ -g(2n) \end{bmatrix}$$

$G(s)$ is called the transfer function of the system and is a continuous mathematical model of the system.

$$G(s) = \frac{C(s)}{R(s)} = \frac{M(s)}{N(s)}$$

If the system has n closed-loop poles $s_1, s_2, \cdots, s_n$. Then the above formula can be divided into:

$$G(s) = \frac{c_1}{s - s_1} + \frac{c_2}{s - s_2} + \cdots + \frac{c_n}{s - s_n}$$

Task: $\{g(i)\}$ and n are known, find $G(s)$ in the coefficients c_i and s_i.

calculate a_i

System impulse transfer function is

$$G(z) = \frac{b_0 + b_1 z + \cdots b_n z^n}{1 + a_1 z + \cdots + a_n z^n}$$

let $r(t) = \delta(t)$, then $c(t) = g(t)$。 Substituting the above formula, write the difference equation as

$$g(k) + a_1 g(k+1) + \cdots + a_n g(k+n) = 0$$

get:

$$\begin{aligned} a_1 g(k+1) + \cdots + a_n g(k+n) &= -g(k) \\ &\cdots \\ a_1 g(k+n) + \cdots + a_n g(k+2n-1) &= -g(k+n-1) \end{aligned}$$

solving above linear equation of n unknowns， get a_i.

calculate s_i

The inverse Laplace transform from G(s) gives:

$$g(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + \ldots + c_n e^{s_n t}$$

so:

$$\begin{aligned}
g(t) &= c_1 e^{s_1 (t)} + c_2 e^{s_2 (t)} + \ldots + c_n e^{s_n (t)} \\
g(t+\Delta) &= c_1 e^{s_1 (t+\Delta)} + c_2 e^{s_2 (t+\Delta)} + \ldots + c_n e^{s_n (t+\Delta)} \\
\ldots & \quad \ldots \\
g(t+n\Delta) &= c_1 e^{s_1 (t+n\Delta)} + c_2 e^{s_2 (t+n\Delta)} + \ldots + c_n e^{s_n (t+n\Delta)} \\
0 &= c_1 e^{s_1 t}[1 + a_1 e^{s_1 \Delta} + \cdots + a_n e^{s_1 n\Delta}] \\
& \quad + c_2 e^{s_2 t}[1 + a_1 e^{s_2 \Delta} + \cdots + a_n e^{s_2 n\Delta}] + \cdots \\
& \quad + c_n e^{s_n t}[1 + a_1 e^{s_n \Delta} + \cdots + a_n e^{s_n n\Delta}]
\end{aligned}$$

the linear equation of n unkowns is required for $e^{s_i \Delta}$:

$$1 + a_1 e^{s_i \Delta} + a_2 [e^{s_i \Delta}]^2 + \cdots + a_n [e^{s_i \Delta}]^n = 0$$

where $i = 1, 2, \cdots, n$

calculate c_i

$$g(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + ... + c_n e^{s_n t}$$

get :

$$\begin{aligned} g(0) &= c_1 + c_2 + \cdots c_n \\ g(1) &= c_1 e^{s_1 \Delta} + c_2 e^{s_2 \Delta} + \cdots + c_n e^{s_n \Delta} \\ &\ldots \\ g(n-1) &= c_1 e^{s_1 (n-1)\Delta} + c_2 e^{s_2 (n-1)\Delta} + \cdots + c_n e^{s_n (n-1)\Delta} \end{aligned}$$

solution formula

$$\begin{bmatrix} g(k+1) & \cdots & g(k+n) \\ g(k+2) & \cdots & g(k+n+1) \\ \vdots & \vdots & \vdots \\ g(k+n) & \cdots & g(k+2n-1) \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} -g(k) \\ -g(k+1) \\ \vdots \\ -g(k+n-1) \end{bmatrix}$$

$$1 + a_1 x + \cdots a_n x^n = 0$$

$$s_i = \frac{\ln x_i}{\Delta}$$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \cdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ g(n-1) \end{bmatrix}$$