线性系统的根轨迹法

根轨迹绘制基本法则

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根轨迹分支数、对称性和连续性

- 根轨迹分支数等于开环零点数与开环极点数中的较大者，即 $\max(m, n)$
- 根轨迹关于实轴对称
- 根轨迹是连续的

开环零极点数相等时

$$K^* \frac{\prod_{j=1}^{m}(s-z_j)}{\prod_{i=1}^{n}(s-p_i)} = -1 \quad (1)$$

$$\prod_{i=1}^{n}(s-p_i) + K^* \prod_{j=1}^{m}(s-z_j) = 0 \quad (2)$$

$$\frac{1}{K^*}\prod_{i=1}^{n}(s-p_i) + \prod_{j=1}^{m}(s-z_j) = 0 \quad (3)$$

- 起点：开环极点
- 终点：开环零点

开环极点数大于零点数

$$K^* = \frac{\prod_{i=1}^{n}|s-p_i|}{\prod_{j=1}^{m}|s-z_j|} \tag{4}$$

$$K^* = \frac{|s|^{n-m}\prod_{i=1}^{n}|1-\frac{p_i}{s}|}{\prod_{j=1}^{m}|1-\frac{z_j}{s}|} \tag{5}$$

根轨迹的渐近线

$$K^* \frac{1}{(s-c)^{n-m}} \approx -1 \tag{6}$$

$$K^* \frac{\prod_{j=1}^{m}(1-\frac{z_j}{s})}{s^{n-m}\prod_{i=1}^{n}(1-\frac{p_i}{s})} \approx \frac{K^*}{s^{n-m}(1-\frac{c}{s})^{n-m}} \tag{7}$$

$$(1-\frac{c}{s})^{n-m}\prod_{j=1}^{m}(1-\frac{z_j}{s}) \approx \prod_{i=1}^{n}(1-\frac{p_i}{s}) \tag{8}$$

根轨迹的渐近线

$$
\begin{aligned}
(1-\frac{c}{s})^{n-m} &= & (9)\\
\prod_{j=1}^{m}(1-\frac{z_j}{s}) &= & (10)\\
\prod_{i=1}^{n}(1-\frac{p_i}{s}) &= & (11)
\end{aligned}
$$

$$(1-\frac{c}{s})^{n-m} = 1-\frac{(n-m)c}{s}+\cdots \tag{9}$$

$$\prod_{j=1}^{m}(1-\frac{z_j}{s}) = \tag{10}$$

$$\prod_{i=1}^{n}(1-\frac{p_i}{s}) = \tag{11}$$

根轨迹的渐近线

$$(1-\frac{c}{s})^{n-m} = \tag{9}$$

$$\prod_{j=1}^{m}(1-\frac{z_j}{s}) = 1-\frac{\sum_{j=1}^{m} z_j}{s}+\cdots \tag{10}$$

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根轨迹的渐近线

$$1-\frac{(n-m)c}{s}-\frac{\sum_{j=1}^{m} z_j}{s}+\cdots \approx 1-\frac{\sum_{i=1}^{n}}{s}+\cdots$$

$$-(n-m)c-\sum_{j=1}^{m} z_j = \sum_{i=1}^{n} p_i \tag{12}$$

$$c = \frac{\sum_{i=1}^{n} p_i-\sum_{j=1}^{m} z_j}{n-m} \tag{13}$$

根轨迹在实轴上的分布 $s \in R$(开环零极点全为实数)

$$K^* \frac{\prod_{j=1}^{m}(s - z_j)}{\prod_{i=1}^{n}(s - p_i)} = -1 \qquad (14)$$

$$K^* \frac{\prod_{j=1}^{m}\rho_j}{\prod_{i=1}^{n}A_i} = -1 \qquad (15)$$

$$K^* \frac{\prod_{j=1}^{m}|\rho_j|}{\prod_{i=1}^{n}|A_i|} \frac{j = 1(-1)^q}{(-1)^r} = -1 \qquad (16)$$

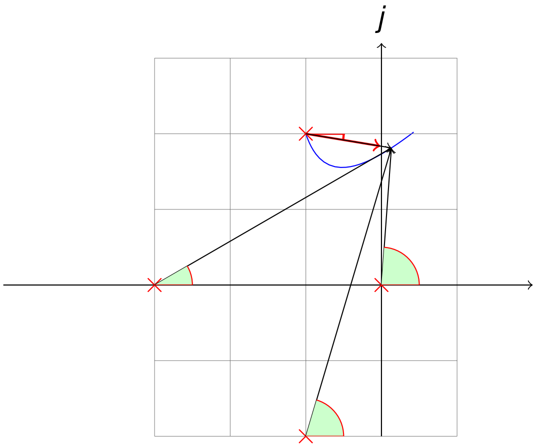
实轴上某区域，若其右边开环实数零、极点个数之和为奇数，则该区域必是根轨迹。

根轨迹在实轴上的分布 $s \in R$(开环零极点存在共轭复数)

$$
\begin{aligned}
K^* G_q(s)(s - z_q)(s - \bar{z}_q) &= -1 \quad (17) \\
K^* G_i(s)(s - a - jb_q)(s - a + jb_q) &= -1 \quad (18) \\
K^* G_i(s)((s - a)^2 + b_q^2) &= -1 \quad (19)
\end{aligned}
$$

实轴上某区域，若其右边开环实数零、极点个数之和为奇数，则该区域必是根轨迹。

根轨迹的起始角与终止角



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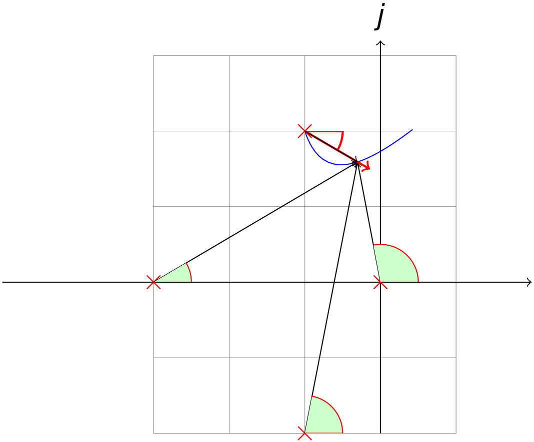

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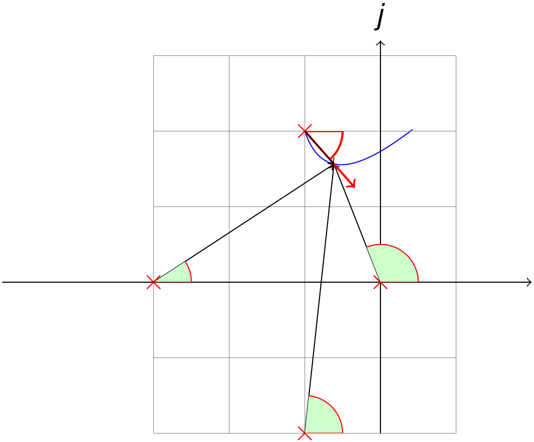

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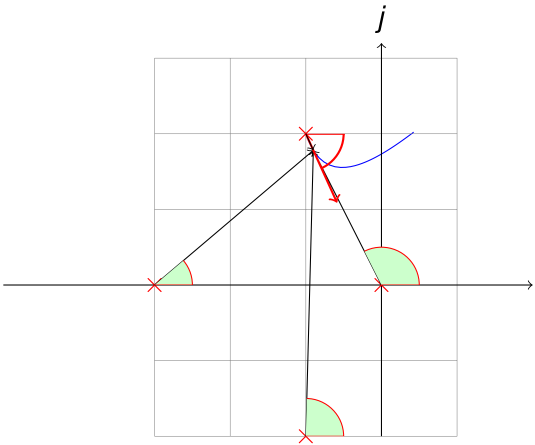

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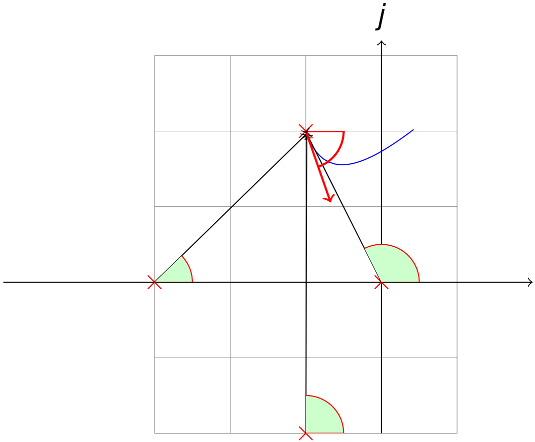

起始角

$$
\begin{aligned}
\sum_{l=1}^{m} \angle(s - z_l) - \sum_{i=1}^{n} \angle(s - p_i) &= (2k+1)\pi \\
s &= p_q + \delta r e^{j\theta} \\
\sum_{l=1}^{m} \angle(p_q + \delta r e^{j\theta} - z_l) - \sum_{i=1}^{n} \angle(p_q + \delta r e^{j\theta} - p_i) &= (2k+1)\pi \\
\lim_{\delta r \to 0} \sum_{l=1}^{m} \angle(p_q + \delta r e^{j\theta} - z_l) - \sum_{i=1}^{n} \angle(p_q + \delta r e^{j\theta} - p_i) &= (2k+1)\pi \\
\sum_{l=1}^{m} \angle(p_q - z_l) - \sum_{p_q = p_i} \theta - \sum_{p_q \neq p_i} \angle(p_q - p_i) &= (2k+1)\pi
\end{aligned}
$$

终止角

$$
\begin{aligned}
\sum_{l=1}^{m} \angle(s - z_l) - \sum_{i=1}^{n} \angle(s - p_i) &= (2k+1)\pi \\
s &= z_q + \delta r e^{j\theta} \\
\sum_{l=1}^{m} \angle(z_q + \delta r e^{j\theta} - z_l) - \sum_{i=1}^{n} \angle(z_q + \delta r e^{j\theta} - p_i) &= (2k+1)\pi \\
\lim_{\delta r \to 0} \sum_{l=1}^{m} \angle(z_q + \delta r e^{j\theta} - z_l) - \sum_{i=1}^{n} \angle(z_q + \delta r e^{j\theta} - p_i) &= (2k+1)\pi \\
\sum_{z_q \neq z_l} \angle(z_q - z_l) + \sum_{z_q = z_l} \theta - \sum_{i=1}^{n} \angle(z_q - p_i) &= (2k+1)\pi
\end{aligned}
$$

分离点与分离角

分离点与分离角

- 分离点：两条或两条以上根轨迹分支在 S 平面上相交后又分开的点称为根轨迹的分离点。

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- 分离点：两条或两条以上根轨迹分支在 S 平面上相交后又分开的点称为根轨迹的分离点。
- 分离角：根轨迹进入分离点的切线方向与离开分离点的切线方向之间的夹角

分离点计算

$$G(s) = \frac{K^* M(s)}{N(s)}$$

$$D(s) = N(s) + K^* M(s)$$

$$N(s) = -K^* M(s) \qquad D(s) = 0$$

$$N'(s) = -K^* M'(s) \qquad D'(s) = 0$$

$$\frac{N'(s)}{N(s)} = \frac{M'(s)}{M(s)}$$

$$\frac{\frac{d}{ds}\prod_{i=1}^{n}(s-p_i)}{\prod_{i=1}^{n}(s-p_i)} = \frac{\frac{d}{ds}\prod_{j=1}^{m}(s-z_j)}{\prod_{j=1}^{m}(s-z_j)}$$

$$\sum_{i=1}^{n}\frac{1}{s-p_i} = \sum_{j=1}^{m}\frac{1}{s-z_j}$$

分离角计算:

设 $K^* = K_0$ 时有重极点。可将原特征方程变换

$$
\begin{aligned}
D(s) &= N(s) + K^* M(S) \\
D(s) &= N(s) + (K' + K_0) M(S) \\
D(s) &= N(s) + K_0 M(S) + K' M(s) \\
D'(s) &= N'(s) + K' M(s)
\end{aligned}
$$

- 原系统的分离点等于新系统的起始点
- 分离角等于 180° 根轨迹与 0 度根轨迹起始角之差
  - 180° 根轨迹起始角：θ_1
  - 0 度根轨迹起始角：θ_2
  - 其它零（极）点到分离点的辐角之和（差）：θ_2

$$
\begin{aligned}
L\theta_1 &= \theta_0 + (2k_1 + 1)\pi \\
L\theta_2 &= \theta_0 + 2k_2\pi \\
L\Delta\theta &= (2(k_1 - k_2) + 1)\pi \\
\Delta\theta &= \frac{(2k+1)\pi}{L}
\end{aligned}
$$

根轨迹与虚轴的交点

$$\begin{aligned} 1 + G(s)H(s) &= 0 \\ s &= j\omega \\ 1 + G(j\omega)H(j\omega) &= 0 \end{aligned}$$

根之和 $(n-1>m)$

$$\prod_{i=1}^{n}(s-p_i)+K^*\prod_{j=1}^{m}(s-z_j) = 0$$

$$\prod_{i=1}^{n}(s-p_i) = s^n+\sum_{i=1}^{n}(-p_i)s^{n-1}+\cdots$$

$$\prod_{j=1}^{m}(s-z_j) = s^m+\sum_{j=1}^{m}(-z_j)s^{m-1}+\cdots$$

$$\prod_{i=1}^{n}(s-p_i)+K^*\prod_{j=1}^{m}(s-z_j) = s^n+\sum_{i=1}^{n}(-p_i)s^{n-1}+\cdots$$

根之和 $(n-1>m)$

$$\prod_{i=1}^{n}(s-p_i)+K^*\prod_{j=1}^{m}(s-z_j) = s^n+\sum_{i=1}^{n}(-p_i)s^{n-1}+\cdots$$

$$\prod_{i=1}^{n}(s-p_i)+K^*\prod_{j=1}^{m}(s-z_j) = \prod_{i=1}^{n}(s-s_i)$$

$$\prod_{i=1}^{n}(s-s_i) = s^n+\sum_{i=1}^{n}(-s_i)s^{n-1}+\cdots$$

$$\sum_{i=1}^{n}(-p_i)s^{n-1} = \sum_{i=1}^{n}(-s_i)s^{n-1}$$

$$\sum_{i=1}^{n}(-p_i) = \sum_{i=1}^{n}(-s_i)$$

$$\sum_{i=1}^{n}p_i = \sum_{i=1}^{n}s_i$$