线性系统的频域分析法

Contents

1 频率法介绍

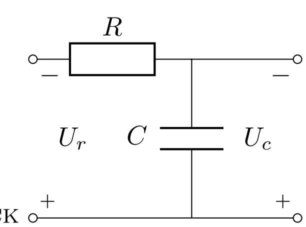
1.1 频率法基本概念

1.1.1 频域法特点:

1. 工程使用广泛, 有自己一套指标体系
2. 时域与频域指标可由经验公式相互转化
3. 根据 Nyquist 判据可由系统的开环频率特性判断闭环系统的稳定性, 频率特性可由实验测定
4. 频率特性还可适用于典型非线性环节系统
5. 可以方便地设计出各种滤波器

1.1.2 频率特性基本概念

1. RC 网络: BMCOL:B_BLOCK



$$
\begin{aligned}
U_r &= U_c + RC\dot{U}_c \\
U_r(s) &= U_c(s) + RsU_c(s)
\end{aligned}
$$

2. 传递函数: B_BLOCK:BMCOL

$$
\begin{aligned}
G(s) &= \frac{U_c(s)}{U_r(s)} \\
&= \frac{1}{1+RCs} \\
&= \frac{1}{1+Ts}
\end{aligned}
$$

其中, $T = RC$,

1.1.3 频率特性基本概念 (续)

- 当 $U_r = A\sin\omega t$ 时,

$$
\begin{aligned}
U_c(s) &= G(s)U_r(s) \\
U_c(t) &= \frac{A\omega t}{1+\omega^2T^2}e^{-\frac{t}{T}} + \frac{A}{\sqrt{1+\omega^2T^2}}\sin(\omega t-\beta)
\end{aligned}
$$

- 稳态分量为

$$\frac{A}{\sqrt{1+\omega^2T^2}}\sin(\omega t-\beta)$$

其中, $\tan\beta = \omega T$.

- 结论:
  - · 对线性系统而言, 输入为正弦信号, 输出也为相同频率的正弦信号, 但幅值与相位发生变化.
  - · 幅值变化: 输出是输入的 $\frac{1}{\sqrt{1+\omega^2T^2}}$ 倍
  - · 相角变化: 输出比输入滞后 $\arctan\omega T$.

1.1.4 频率特性定义

- 幅频特性: 系统稳态正弦输出量与输入量的幅值比 $A(\omega)$
- 相频特性: 系统稳态正弦输出量与输入量的相角差 $\phi(\omega)$
- 令 $s = j\omega$, 则

$$
\begin{aligned}
A(j\omega) &= |G(j\omega)| \\
\phi(j\omega) &= \angle G(j\omega)
\end{aligned}
$$

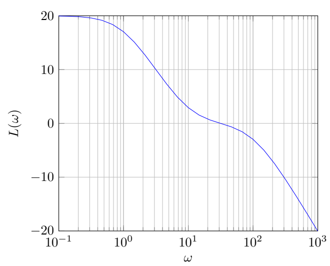
1.2 频率特性的图示表示法

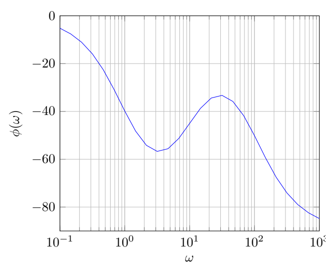
1.2.1 Bode 图

- 横坐标: $\log_{10}\omega$
- 纵坐标:$L(\omega) = 20\log_{10} A(\omega), \phi(\omega)$

1. 例: B_BLOCK

$$G(s) = \frac{10(0.1s+1)}{(s+1)(0.01s+1)}$$



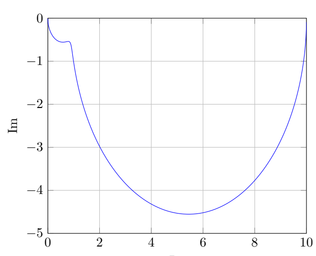


1.2.2 Nyquist 图

- 横坐标:$\Re[G(j\omega)]$
- 纵坐标:$\Im[G(j\omega)]$

1. 例: B_BLOCK

$$G(s) = \frac{10(0.1s+1)}{(s+1)(0.01s+1)}$$

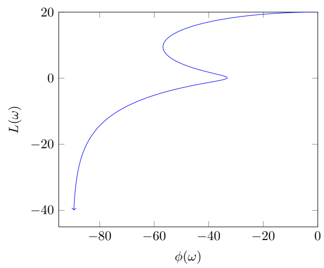


1.2.3 Nichols 图

- 横坐标:$\phi(j\omega)$
- 纵坐标:$20\log_{10} A(\omega)$

1. 例: B__BLOCK

$$G(s) = \frac{10(0.1s+1)}{(s+1)(0.01s+1)}$$



2 典型环节频率特性

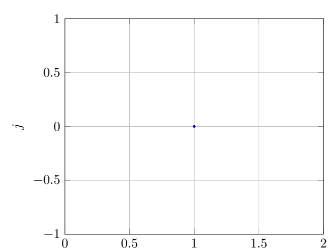
2.1 比例, 积分微分环节

2.1.1 比例环节

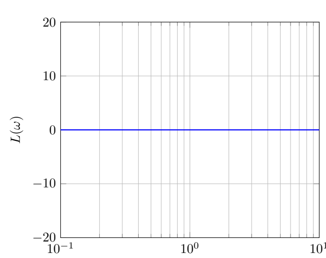
$$\begin{aligned} G(s) &= K \\ G(j\omega) &= K \\ A(\omega) &= K \\ \phi(\omega) &= 0 \\ L(\omega) &= 20\lg K \end{aligned}$$

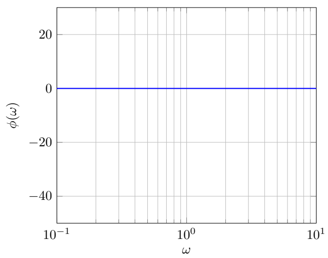
2.1.2 比例环节 (续) $G(j\omega) = K, K = 1$

1. Nyquist 图 BMCOL:B__BLOCK



2. Bode 图 BMCOL:B__BLOCK

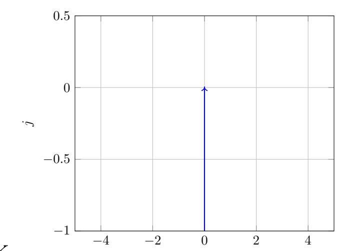




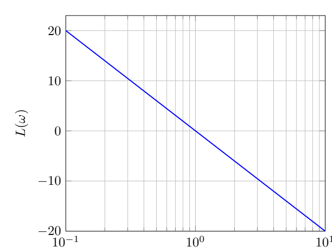
2.1.3 积分环节

$$
\begin{aligned}
G(s) &= \frac{1}{s} \\
G(j\omega) &= \frac{1}{j\omega} \\
A(\omega) &= \frac{1}{\omega} \\
\phi(\omega) &= -90^\circ \\
L(\omega) &= -20\lg\omega
\end{aligned}
$$

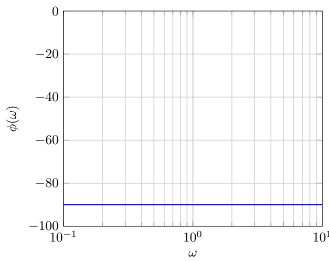
2.1.4 积分环节 (续)$G(j\omega) = \frac{1}{j\omega}$



1. Nyquist 图 BMCOL:B__BLOCK



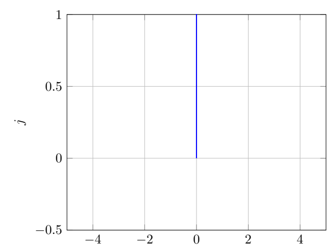
2. Bode 图 BMCOL:B__BLOCK



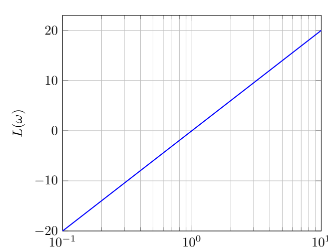
2.1.5 微分环节

$$
\begin{aligned}
G(s) &= s \\
G(j\omega) &= j\omega \\
A(\omega) &= \omega \\
\phi(\omega) &= 90^\circ \\
L(\omega) &= 20\lg\omega
\end{aligned}
$$

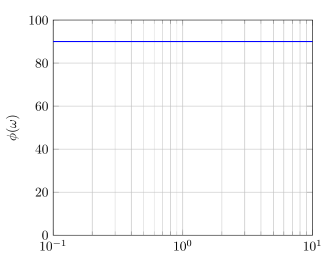
2.1.6 微分环节 (续) $G(j\omega) = j\omega$



1. Nyquist 图 BMCOL:B_BLOCK



2. Bode 图 BMCOL:B_BLOCK

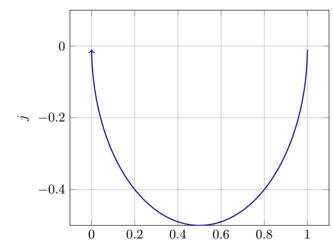


2.2 惯性, 一阶微分环节

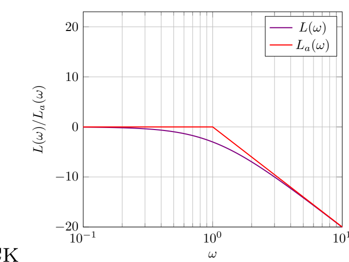
2.2.1 惯性环节

$$
\begin{aligned}
G(s) &= \frac{1}{Ts+1} \\
G(j\omega) &= \frac{1}{j\omega T+1} \\
A(\omega) &= \sqrt{\frac{1}{1+\omega^2T^2}} \\
\phi(\omega) &= -\arctan \omega T \\
L(\omega) &= -20\lg\sqrt{1+\omega^2T^2} \\
L_a(\omega) &= \begin{cases} 0 & \omega < \frac{1}{T} \\ -20\lg \omega T & \omega > \frac{1}{T} \end{cases}
\end{aligned}
$$

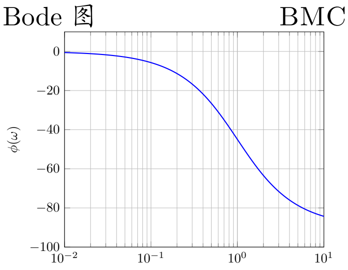
2.2.2 惯性环节 (续) $G(j\omega) = \frac{1}{j\omega T+1}, T = 1$



1. Nyquist 图 B__BLOCK:BMCOL



2. Bode 图 BMCOL:B__BLOCK

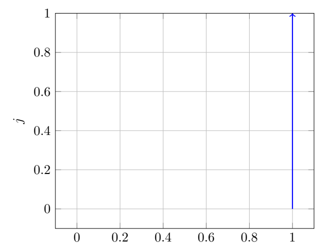


2.2.3 一阶微分环节

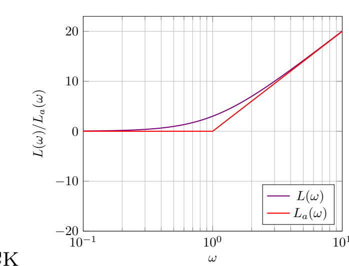
$$
\begin{aligned}
G(s) &= Ts+1 \\
G(j\omega) &= j\omega T+1 \\
A(\omega) &= \sqrt{1+\omega^2T^2} \\
\phi(\omega) &= \arctan \omega T \\
L(\omega) &= 20\lg\sqrt{1+\omega^2T^2} \\
L_a(\omega) &= \begin{cases} 0 & \omega < \frac{1}{T} \\ 20\lg \omega T & \omega > \frac{1}{T} \end{cases}
\end{aligned}
$$

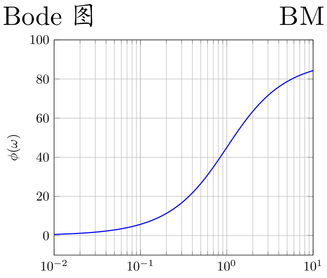
2.2.4 一阶微分环节 (续) $G(j\omega) = j\omega T + 1, T = 1$

1. Nyquist 图 BMCOL:B_BLOCK



2. Bode 图 BMCOL:B_BLOCK





2.3 二阶环节

2.3.1 二阶振荡环节

$$
\begin{aligned}
G(s) &= \frac{\omega_n^2}{\omega_n^2 + 2\xi\omega_n s + s^2} = \frac{1}{(Ts)^2 + 2\xi Ts + 1} \\
G(j\omega) &= \frac{1}{1 + 2j\xi\omega T - \omega^2 T^2} \\
A(\omega) &= \sqrt{\frac{1}{(1-\omega^2T^2)^2 + (2\xi\omega T)^2}} \\
\phi(\omega) &= \begin{cases} -\arctan\frac{2\xi\omega T}{1-\omega^2T^2} & \omega T < 1 \\ -90^\circ & \omega T = 1 \\ -180 - \arctan\frac{2\xi\omega T}{1-\omega^2T^2} & \omega T > 1 \end{cases} \\
L(\omega) &= -20\lg\sqrt{(1-\omega^2T^2)^2 + (2\xi\omega T)^2} \\
L_a(\omega) &= \begin{cases} 0 & \omega T < 1 \\ -40\lg\omega T & \omega T > 1 \end{cases}
\end{aligned}
$$

2.3.2 二阶振荡环节 (续)$G(j\omega) = \frac{1}{1+2j\xi\omega T - \omega^2T^2}$

- Nyquist 曲线与虚轴交点:

$$
\begin{aligned}
\Re[G(j\omega)] &= 0 \\
1 - \omega^2T^2 &= 0 \\
\omega T &= 1 \\
G(j\frac{1}{T}) &= -\frac{1}{2\xi}j
\end{aligned}
$$

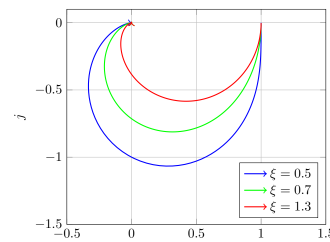
2.3.3 二阶振荡环节 (续)$G(j\omega) = \frac{1}{1+2j\xi\omega T - \omega^2T^2}$

- 谐振频率与谐振峰值

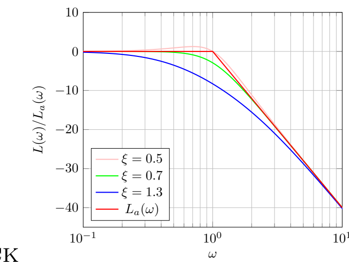
$$
\begin{aligned}
A(\omega) &= \sqrt{\frac{1}{(1-\omega^2T^2)^2 + (2\xi\omega T)^2}} \\
\frac{dA(\omega)}{d\omega} &= -\frac{-2(1-\omega^2T^2)\omega T^2 + 4\xi^2\omega T^2}{\sqrt{(1-\omega^2T^2)^2 + (2\xi\omega T)^2}}
\end{aligned}
$$

  - 令 $\frac{dA(\omega)}{d\omega} = 0$，得
  - 谐振频率:$\omega_r = \omega_n\sqrt{1-2\xi^2}$，其中 $0 < \xi \le \frac{\sqrt{2}}{2}$
  - 谐振峰值:$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$

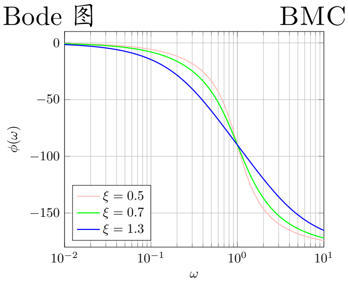
2.3.4 二阶振荡环节 (续):$G(j\omega) = \frac{1}{1+2j\xi\omega T-\omega^2T^2}, T = 1$



1. Nyquist 图 BMCOL:B_BLOCK



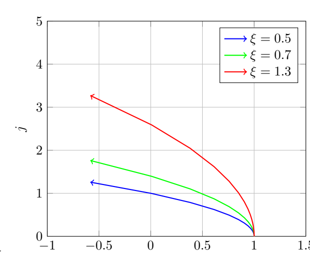
2. Bode 图 BMCOL:B_BLOCK



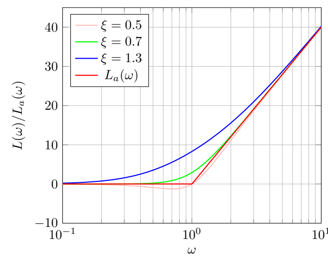
2.3.5 二阶微分环节

$$
\begin{aligned}
G(s) &= (Ts)^2 + 2\xi Ts + 1 \\
G(j\omega) &= 1 + 2j\xi\omega T - \omega^2 T^2
\end{aligned}
$$

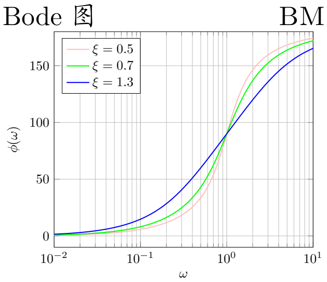
2.3.6 二阶微分环节 (续)$G(j\omega) = 1 + 2j\xi\omega T - \omega^2T^2, T = 1$



1. Nyquist 图 BMCOL:B_BLOCK



2. Bode 图 BMCOL:B__BLOCK



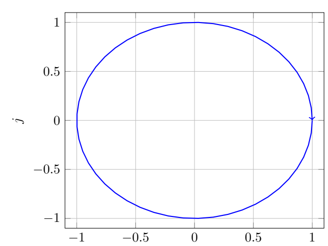
2.4 非最小相位环节

2.4.1 延迟环节

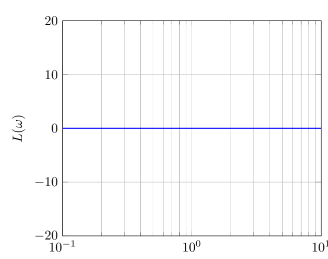
$$\begin{aligned} G(s) &= e^{-\tau s} \\ G(j\omega) &= e^{-j\omega\tau} \\ A(\omega) &= 1 \\ \phi(\omega) &= -\omega\tau \end{aligned}$$

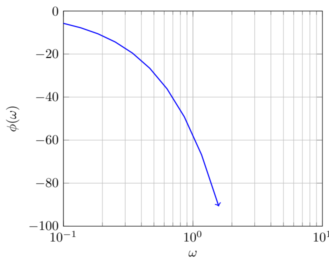
2.4.2 延迟环节 (续)$G(j\omega) = e^{-j\omega\tau}, \tau = 1$

1. Nyquist 图 BMCOL:B__BLOCK



2. Bode 图 BMCOL:B__BLOCK





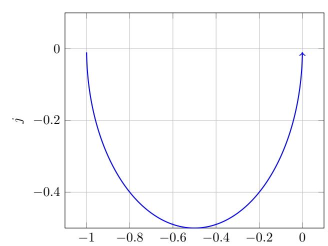
2.4.3 非最小相位惯性环节

最小相位系统: 在右半平面无零极点

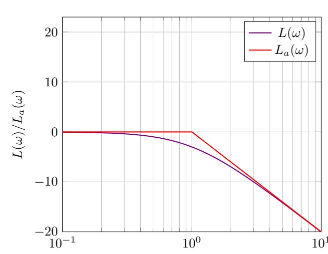
$$
\begin{aligned}
G(s) &= \frac{1}{Ts-1} \\
G(j\omega) &= \frac{1}{j\omega T-1} \\
A(\omega) &= \sqrt{\frac{1}{1+\omega^2T^2}} \\
\phi(\omega) &= -180^\circ + \arctan \omega T \\
L(\omega) &= -20\lg\sqrt{1+\omega^2T^2} \\
L_a(\omega) &= \begin{cases} 0 & \omega < \frac{1}{T} \\ -20\lg \omega T & \omega > \frac{1}{T} \end{cases}
\end{aligned}
$$

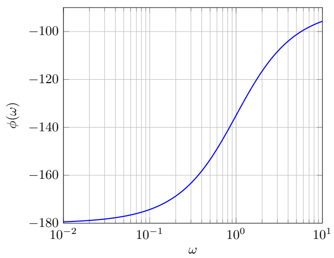
2.4.4 非最小相位惯性环节 (续)$G(j\omega) = \frac{1}{j\omega T-1}, T = 1$

1. Nyquist 图 BMCOL:B_BLOCK



2. Bode 图 BMCOL:B_BLOCK





3 系统开环频率特性

3.1 开环系统 Nyquist 图

3.1.1 开环系统 Nyquist 图 $G_o(s) = \frac{K\prod_{j=1}^{m}(\tau_j s+1)}{s^\nu \prod_{i=1}^{n-\nu}(T_i s+1)}$

- 当 $\nu = 0$ 时, 为零型系统:

$$\begin{aligned} A(\omega)|_{\omega=0} &= K \\ \phi(\omega)|_{\omega=0} &= 0 \\ \lim_{\omega\to\infty} A(\omega) &= 0 \\ \lim_{\omega\to\infty} \phi(\omega) &= -(n-m)\times\frac{\pi}{2} \end{aligned}$$

3.1.2 开环系统 Nyquist 图 (续) $G_o(s) = \frac{K\prod_{j=1}^{m}(\tau_j s+1)}{s^\nu \prod_{i=1}^{n-\nu}(T_i s+1)}$

- 当 $\nu = 1$ 时, 为 I 型系统:

$$\begin{aligned} \lim_{\omega\to 0} A(\omega) &= \infty \\ \lim_{\omega\to 0} \phi(\omega) &= -\frac{\pi}{2} \\ \lim_{\omega\to\infty} A(\omega) &= 0 \\ \lim_{\omega\to\infty} \phi(\omega) &= -(n-m)\times\frac{\pi}{2} \end{aligned}$$

3.1.3 开环系统 Nyquist 图 (续)$G_o(s) = \frac{K\prod_{j=1}^{m}(\tau_j s+1)}{s^\nu \prod_{i=1}^{n-\nu}(T_i s+1)}$

- 当 $\nu = 2$ 时, 为 II 型系统:

$$\begin{aligned} \lim_{\omega\to 0} A(\omega) &= \infty \\ \lim_{\omega\to 0} \phi(\omega) &= -\pi \\ \lim_{\omega\to\infty} A(\omega) &= 0 \\ \lim_{\omega\to\infty} \phi(\omega) &= -(n-m)\times\frac{\pi}{2} \end{aligned}$$

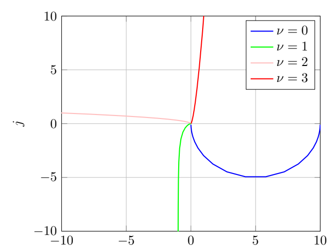
3.1.4 开环系统 Nyquist 图 (续)$G_o(s)=\frac{K\prod_{j=1}^{m}(\tau_j s+1)}{s^{\nu}\prod_{i=1}^{n-\nu}(T_i s+1)}$

- 当 $\nu=3$ 时, 为 III 型系统:

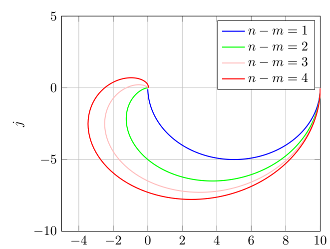
$$
\begin{aligned}
\lim_{\omega\to 0} A(\omega) &= \infty \\
\lim_{\omega\to 0} \phi(\omega) &= -\frac{3}{2}\pi \\
\lim_{\omega\to\infty} A(\omega) &= 0 \\
\lim_{\omega\to\infty} \phi(\omega) &= -(n-m)\times\frac{\pi}{2}
\end{aligned}
$$

3.1.5 开环系统 Nyquist 图, 例 1

1. $G_o(s)=\frac{10}{s^{\nu}(0.1s+1)}$ BMCOL:B_BLOCK



2. $G_o(s)=\frac{10}{(0.1s+1)^n}$ BMCOL:B_BLOCK



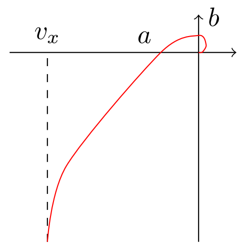
3.1.6 开环系统 Nyquist 图, 例 2:$G(s)=\frac{10}{s(s+1)(2s+1)(4s+1)}$

- 绘制 Nyquist 图, 求出各特征点坐标:
- 由于 $\nu=1$

$$
\begin{aligned}
\lim_{\omega\to 0} A(\omega) &= \infty \\
\lim_{\omega\to 0} \phi(\omega) &= -\frac{\pi}{2} \\
\lim_{\omega\to\infty} A(\omega) &= 0 \\
\lim_{\omega\to\infty} \phi(\omega) &= -2\pi
\end{aligned}
$$

3.1.7 开环系统 Nyquist 图, 例 2(续),$G(s)=\frac{10}{s(s+1)(2s+1)(4s+1)}$

概略 Nyquist 图:



3.1.8 开环系统 Nyquist 图, 例 2(续),$G(s)=\frac{10}{s(s+1)(2s+1)(4s+1)}$

- 起始点实部 v_x:

$$\begin{aligned}
G(j\omega) &= \frac{10}{j\omega(j\omega+1)(2j\omega+1)(4j\omega+1)} \\
&= \frac{10\omega(8\omega^2-7)+10(14\omega^2-1)j}{\omega(1+\omega^2)(1+4\omega^2)(1+16\omega^2)} \\
\lim_{\omega\to 0}\Re[G(j\omega)] &= -70
\end{aligned}$$

3.1.9 开环系统 Nyquist 图, 例 2(续),$G(s)=\frac{10}{s(s+1)(2s+1)(4s+1)}$

- 与实轴交点 a:

$$\begin{aligned}
\Im[G(j\omega)] &= 0 \\
\frac{10(14\omega^2-1)}{(1+\omega^2)(1+4\omega^2)(1+16\omega^2)} &= 0 \\
\omega &= \sqrt{\frac{1}{14}} \\
G(j\sqrt{\frac{1}{14}}) &\approx -21.78
\end{aligned}$$

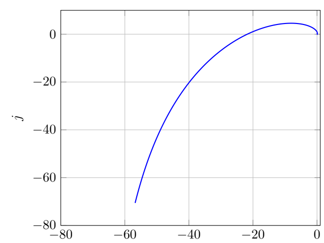
3.1.10 开环系统 Nyquist 图, 例 2(续),$G(s) = \frac{10}{s(s+1)(2s+1)(4s+1)}$

- 与虚轴交点 b:

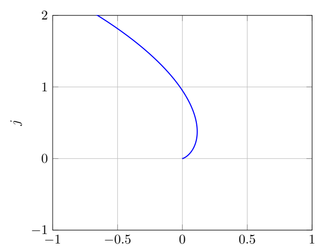
$$
\begin{aligned}
\Re[G(j\omega)] &= 0 \\
\frac{10\omega(8\omega^2-7)}{(1+\omega^2)(1+4\omega^2)(1+16\omega^2)} &= 0 \\
8\omega^2 - 7 &= 0 \\
\omega &= \sqrt{\frac{7}{8}} \\
G(j\sqrt{\frac{7}{8}}) &\approx 0.95j
\end{aligned}
$$

3.1.11 开环系统 Nyquist 图, 例 2(续),$G(s) = \frac{10}{s(s+1)(2s+1)(4s+1)}$

1. Nyquist 图 BMCOL:B__BLOCK



2. 局部放大: BMCOL:B__BLOCK



3.2 开环系统 Bode 图

3.2.1 开环系统 Bode 图

$$
\begin{aligned}
G_o(s) &= G_1(s)G_2(s)G_3(s)\cdots Gn(s) \\
A(\omega) &= A_1(\omega)A_2(\omega)A_3(\omega)\cdots A_n(\omega) \\
L(\omega) &= 20\lg A_1(\omega) + \cdots + 20\lg A_n(\omega) \\
\phi(\omega) &= \phi_1(\omega) + \cdots + \phi_n(\omega)
\end{aligned}
$$

- 结论:
  - 系统的低频段由系统的类型和开环增益 K 决定, 代表稳态性能. 由初始斜率可得系统类型.

· 系统的高频段反映系统的抗噪能力, 下降速度要快.

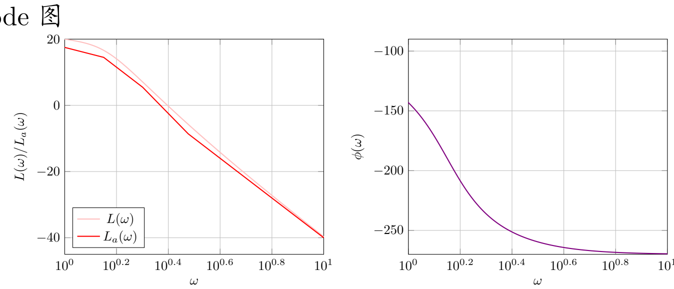
3.2.2 开环系统 Bode 图, 例 1: $G_o(s) = \frac{10(s+3)}{s(s+2)(s^2+s+2)}$

绘制 Bode 图:

1. 改写为标准形式: $G_o(s) = \frac{7.5(\frac{s}{3}+1)}{s(0.5s+1)(0.5s^2+0.5s+1)}$
2. 写出转折频率: $\omega = \sqrt{2}, 2, 3$
3. 找到点 $(1, 20\lg K)$, 其中 $K = 7.5$
4. 过点 $(1, 20\lg K)$ 作斜率为 $-20dB/dec$ 的直线
5. 找转折点依次做直线即可

3.2.3 开环系统 Bode 图, 例 1(续): $G_o(s) = \frac{10(s+3)}{s(s+2)(s^2+s+2)}$

Bode 图



{ $(1, 20\lg K)$ 在 $L(\omega)$ 上或在其延长线上}

4 频域稳定性判据

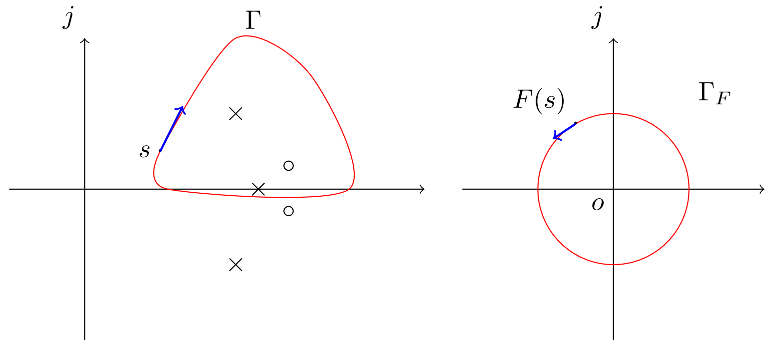
4.1 Nyquist 稳定性判据

4.1.1 辐角原理

- 设 s 为复变量, $F(s)$ 为 s 的有理分式函数. 对于 s 平面上任意一点 s , 通过复变函数 $F(s)$ 的映射关系, 可以确定 s 的象.
- 在 s 平面上任选一条闭合曲线 Γ , 且不通过 $F(s)$ 任一零点和极点, s 沿闭合曲线 Γ 运动一周, 则相应地 $F(s)$ 形成一条闭合曲线 Γ_F .

4.1.2 辐角原理 (续):

设 s 平面闭合曲线 Γ 包围 $F(s)$ 的 Z 个零点和 P 个极点, 则 s 沿 Γ 顺时针运动一周时, 在 $F(s)$ 平面上, $F(s)$ 沿闭合曲线 Γ_F 逆时针包围原点的圈数为 $R = P - Z$.

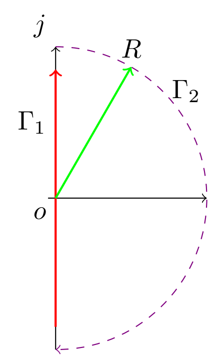


4.1.3 辐角原理的应用

$$
\begin{aligned}
\Phi(s) &= \frac{G(s)}{1+G(s)H(s)} \\
&= \frac{G(s)}{1+G_o(s)} \\
&= \frac{G(s)}{F(s)} \\
F(s) &= 1+G_o(s)
\end{aligned}
$$

- $F(s)$ 的极点是系统开环极点,
- $F(s)$ 的零点是系统的闭环极点.

4.1.4 辐角原理的应用 (续)

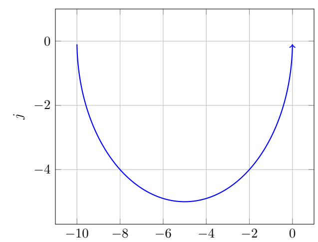


1. 示意图 B_IGNOREHEADING:BMCOL
2. 将 Γ 分为两段: BMCOL:B_BLOCK
   - $\Gamma_1: s = j\omega, \omega \in [-\infty, \infty]$
   - $\Gamma_2: s = \lim_{R\to\infty} Re^{j\theta}$, θ 从 $\frac{\pi}{2}$ 到 $-\frac{\pi}{2}$
   - 可得对应的 $G_o(s)$ 曲线.

- · s 在 Γ_1 上时, 与 Nyquist 图对应.($\omega \in [0, \infty]$)
  - · s 在 Γ_1 上时, $F(s) = 1 + G_o(s) = 1 + \lim_{R\to\infty} Re^{j\theta} G_o(s) = 1$
- Nyquist 判据
  - · 对于开环稳定系统 ($P = 0$), 若 Nyquist 曲线不包含 $(-1, 0)$ 点, 则系统稳定.
  - · 对于开环不稳定系统 ($P > 0$), 若 Nyquist 曲线逆时针包围 $(-1, 0)$ 点的次数为 $\frac{P}{2}$, 则系统稳定.

4.1.5 Nyquist 判据, 例 1:

某负反馈开环传递函数为 $G_o(s) = \frac{10}{s-1}$, 用 Nyquist 判据判断系统稳定性.



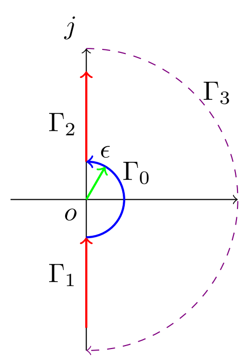
1. Nyquist 图 BMCOL:B_BLOCK
2. 稳定性判断 BMCOL:B_BLOCK

$$
\begin{aligned}
P &= 1 \\
N &= \frac{1}{2} \\
P - Z &= 2N \\
Z &= P - 2N \\
&= 0
\end{aligned}
$$

系统稳定.

4.1.6 虚轴上有极点时

- 零型系统 $F(s)$ 沿 Γ 解析且不为 0.
- I 型及以上系统 $F(s)$ 在 $s = 0$ 处不解析, 不满足辐角原理条件.

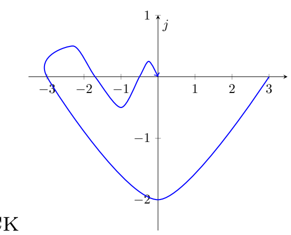


1. 示意图 B_IGNOREHEADING:BMCOL

2. 将 Γ 分为四段: BMCOL:B__BLOCK

- Γ_1 : $s = j\omega, \omega \in [-\infty, 0^-]$
- Γ_2 : $s = j\omega, \omega \in [0^+, \infty]$
- Γ_3 : $s = \lim_{R\to\infty} Re^{j\theta}, \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
- Γ_0 : $s = \lim_{\epsilon\to 0} \epsilon e^{j\theta}, \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
- 对增补后的 Nyquist 图可使用 Nyquist 判据.

4.1.7 穿越次数



1. Nyquist 图 BMCOL:B__BLOCK

2. 穿越次数 BMCOL:B__BLOCK

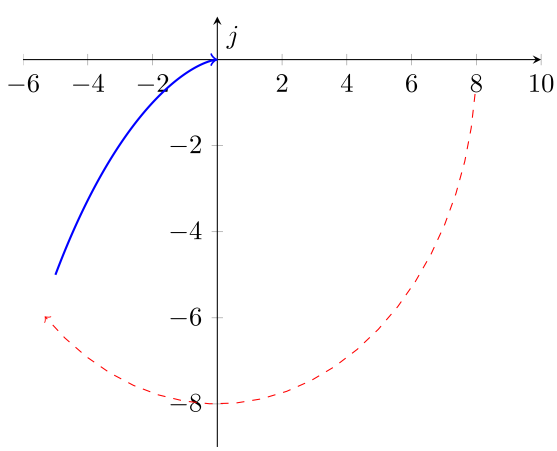
- 根据增补后的 Nyquist 曲线穿越 $(-1, 0)$ 点左侧的次数可得 Γ_F 包围原点的圈数

$$\begin{aligned} R &= 2N \\ &= 2(N_+ - N_-) \end{aligned}$$

其中,

· N_+ 为正穿越 (自上向下) 次数
· N_- 为负穿越 (自下向上) 次数

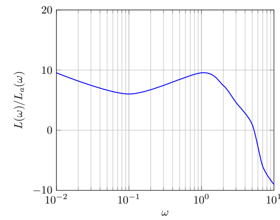
4.1.8 例: $G_o(s) = \frac{10}{s(s+1)}$

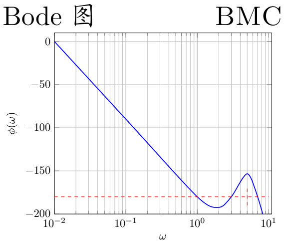


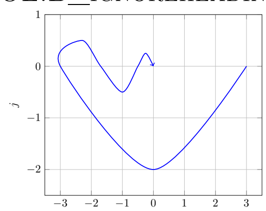
4.2 Bode 稳定性判据

4.2.1 Bode 稳定性判据

1. Bode 图 BMCOL:B_IGNOREHEADING







2. 稳定性判断 BMCOL:B_BLOCK
   - 截止频率 ω_c : $A(\omega_c) = 0$
   - 穿越频率 ω_x : $\phi(\omega_x) = (2k+1)\pi$
   - Bode 判据:
     · 最小相位系统, 若在 $\omega < \omega_c$ 前 $N_+ - N_- = 0$, 则系统稳定
     · 非最小相位系统, 若在 $\omega < \omega_c$ 前 $N_+ - N_- = \frac{P}{2}$, 则系统稳定

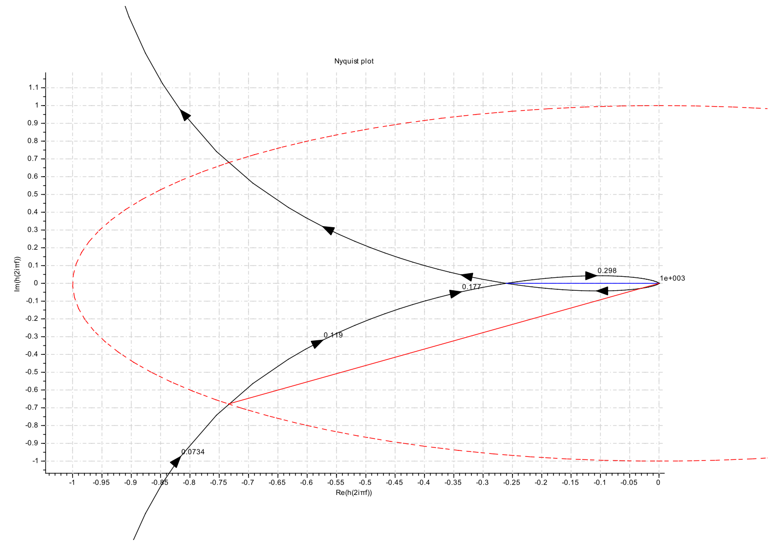
5 频率特性分析

5.1 频域性能分析

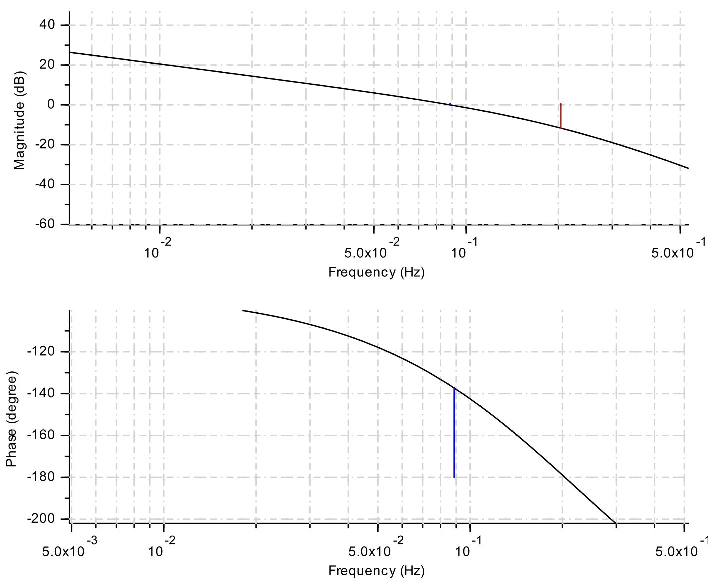
5.1.1 稳定裕度

- 相角裕度 γ : $\gamma = 180° + \phi(\omega_c)$
- 幅值裕度 h : $h = -20\lg A(\omega_x)$

5.1.2 Nyquist 图与稳定裕度



5.1.3 Bode 图与稳定裕度



5.1.4 ω_c 近似计算

例：$G_o(s) = \frac{100(s+4)}{s(s+1)(s+2)(s+3)}$ 近似计算求解 ω_c

- $\omega_c < 1$ 时, $A(\omega) = \frac{200}{3} \cdot \frac{1}{\omega_c}$, $\omega_c = \frac{200}{3} > 1$ 矛盾.
- $1 < \omega_c < 2$ 时, $A(\omega) = \frac{200}{3} \cdot \frac{1}{\omega_c \cdot \omega_c}$, $\omega_c = \sqrt{\frac{200}{3}} > 2$ 矛盾.
- $2 < \omega_c < 3$ 时, $A(\omega) = \frac{200}{3} \cdot \frac{1}{\omega_c \cdot \omega_c \cdots \frac{\omega_c}{2}}$, $\omega_c = \sqrt[3]{\frac{400}{3}} > 3$ 矛盾.
- $3 < \omega_c < 4$ 时, $A(\omega) = \frac{200}{3} \cdot \frac{1}{\omega_c \cdot \omega_c \cdots \frac{\omega_c}{2} \cdots \frac{\omega_c}{3}}$, $\omega_c = \sqrt[4]{400} > 4$ 矛盾.
- $4 < \omega_c$ 时, $A(\omega) = \frac{200}{3} \cdot \frac{\frac{\omega_c}{4}}{\omega_c \cdot \omega_c \cdots \frac{\omega_c}{2} \cdots \frac{\omega_c}{3}}$, $\omega_c = \sqrt[3]{100} > 4$ 成立.

5.1.5 ω_x 计算

例: $G_o(s) = \frac{100(s+4)}{s(s+1)(s+2)(s+3)}$ 求解 ω_x , 即求

$G_o(s) = \frac{K(s+4)}{s(s+1)(s+2)(s+3)}$

的根轨迹与虚轴交点.

$D(s) = s^4 + 6s^3 + 11s^2 + (K+6)s + 4K$

1. Routh 表: B__BLOCK:BMCOL

$$\begin{array}{cccc} s^4 & 1 & 11 & 4K \\ s^3 & 6 & K+6 & \\ s^2 & \frac{60-K}{6} & 4K & \\ s^1 & 0 & 0 & \end{array}$$

2. 根轨迹与虚轴交点 B__BLOCK:BMCOL

$$\frac{60-K}{6}(K+6) = 4K \times 6$$

$$\frac{K^2}{6} + 15K - 60 = 0$$

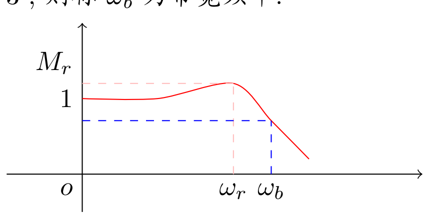
$$K = -45 \pm 3\sqrt{265}$$

$$\frac{60-K}{6}s^2 = 4K$$

$$\omega_x \approx 1.28$$

5.1.6 频带宽度

- 设闭环系统频率特性为 $\Phi(j\omega)$, 若 $\omega > \omega_b$ 时, 有 $20\lg|\Phi(j\omega)| < 20\lg|\Phi(j0)| - 3$, 则称 ω_b 为带宽频率.



5.2 闭环频率特性的确定

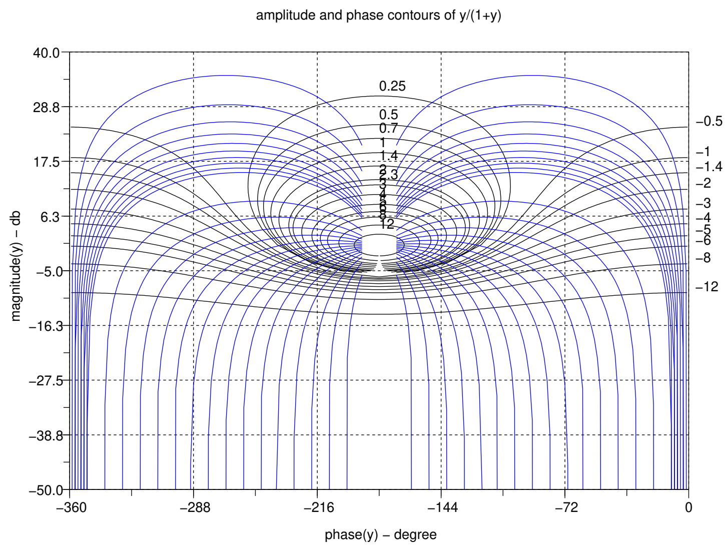
5.2.1 等 α 曲线

$$
\begin{aligned}
G(j\omega) &= Ae^{j\phi} \\
\Phi(j\omega) &= Me^{j\alpha} \\
&= \frac{Ae^{j\phi}}{1+Ae^{j\phi}} \\
\frac{Ae^{j\phi}}{Me^{j\alpha}} &= 1+Ae^{j\phi} \\
\frac{A}{M} &= e^{-j(\phi-\alpha)}+Ae^{j\alpha} \\
0 &= \sin(\alpha-\phi)+A\sin\alpha \\
20\lg A &= 20\lg\frac{\sin(\phi-\alpha)}{\sin\alpha}
\end{aligned}
$$

5.2.2 等 M 曲线

$$
\begin{aligned}
\frac{Ae^{j\phi}}{Me^{j\alpha}} &= 1+Ae^{j\phi} \\
\frac{A}{M} &= |1+Ae^{j\phi}| \\
\frac{A^2}{M^2} &= (1+A\cos\phi)^2+A^2\sin^2\phi \\
0 &= (1-M^{-2})A^2+2\cos\phi A+1 \\
A &= \frac{\cos\phi\pm\sqrt{\cos^2\phi+M^{-2}-1}}{M^{-2}-1} \\
20\lg A &= 20\lg\frac{\cos\phi\pm\sqrt{\cos^2\phi+M^{-2}-1}}{M^{-2}-1}
\end{aligned}
$$

5.2.3 Nichols Chart



5.3 指标转换

5.3.1 系统闭环和开环和频域指标的关系

$$
\begin{aligned}
G(j\omega) &= Ae^{-j(180^\circ-\gamma(\omega))} \\
&= A\left(-\cos\gamma(\omega) - j\sin\gamma(\omega)\right) \\
M &= \left|\frac{G(j\omega)}{1+G(j\omega)}\right| \\
&= \frac{A}{\sqrt{1+A^2-2A\cos\gamma(\omega)}} \\
&= \frac{1}{\sqrt{\left[\frac{1}{A}-\cos\gamma(\omega)\right]^2+\sin^2\gamma(\omega)}} \\
M_r &= \frac{1}{\sin\gamma(\omega_r)} \approx \frac{1}{sin\gamma} \qquad (\omega_r \approx \omega_c)
\end{aligned}
$$

5.3.2 2 阶系统频域指标

$$
\begin{aligned}
G(j\omega) &= \frac{\omega_n^2}{j\omega(j\omega+2\xi\omega_n)} \\
&= \frac{\omega_n^2}{\omega\sqrt{\omega^2+4\xi^2\omega_n^2}}\angle(-\arctan\frac{\omega}{2\xi\omega_n}-90^\circ) \\
\omega_c &= \omega_n(\sqrt{4\xi^4+1}-2\xi^2)^{\frac{1}{2}} \\
\gamma &= 180^\circ+\angle G(j\omega_c) \\
&= \arctan\frac{2\xi\omega_n}{\omega_c}
\end{aligned}
$$

5.3.3 2 阶系统频域指标（ M_r, ω_r ）

$$
\begin{aligned}
M_r &= \frac{1}{2\xi\sqrt{1-\xi^2}} \\
\omega_r &= \omega_n\sqrt{1-2\xi^2}
\end{aligned}
$$

- M_r 与 $\sigma\%$ 一一对应, 且成正比

5.3.4 高阶系统频域指标

1. 经验公式 BMCOL:B_IGNOREHEADING

$$
\begin{aligned}
M_r &= \frac{1}{\sin\gamma} \\
\sigma\% &= 16\%+0.4(M_r-1),(1\le M_r\le 1.8) \\
t_s &= \frac{K\pi}{\omega_c} \\
K &= 2+1.5(M_r-1)+2.5(M_r-1)^2
\end{aligned}
$$

   - $\gamma\uparrow\rightarrow\sigma\%\downarrow\rightarrow\xi\uparrow$
   - $\omega_c\uparrow\rightarrow t_s\downarrow$

2. 频域要求: BMCOL:B_BLOCK

   - 低频段: 稳态性能
   - 中频段: 瞬态性能
   - 高频段: 抗干扰能力