线性系统的频域分析法

频率特性分析

邢超

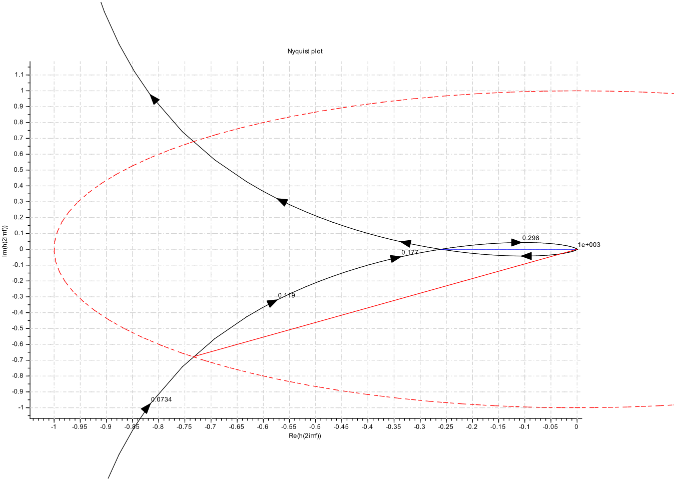
Outline

Topic

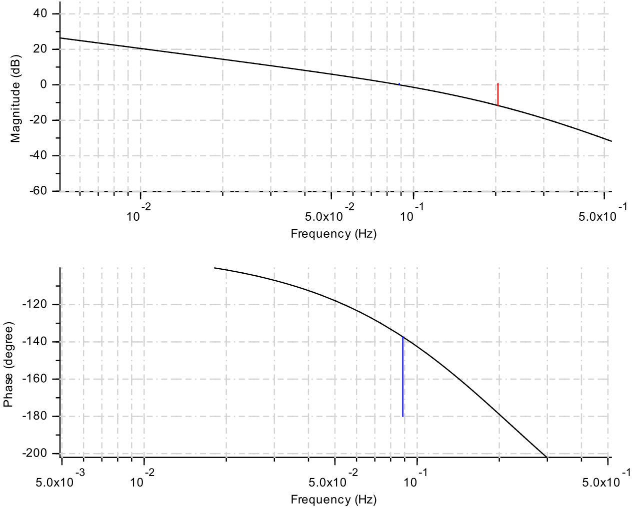
稳定裕度

- 相角裕度 γ：$\gamma = 180^\circ + \phi(\omega_c)$
- 幅值裕度 h：$h = -20\lg A(\omega_x)$

Nyquist 图与稳定裕度



Bode 图与稳定裕度


Magnitude (dB)
40
20
0
-20
-40
-60
10^{-2}
$5.0\text{x}10^{-2}$
10^{-1}
$5.0\text{x}10^{-1}$
Frequency (Hz)
Phase (degree)
-120
-140
-160
-180
-200
$5.0\text{x}10^{-3}$
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Frequency (Hz)

ω_c 近似计算

例: $G_o(s)=\frac{100(s+4)}{s(s+1)(s+2)(s+3)}$ 近似计算求解 ω_c

- $\omega_c<1$ 时, $A(\omega)=\frac{200}{3}\cdot\frac{1}{\omega_c}$, $\omega_c=\frac{200}{3}>1$ 矛盾.
- $1<\omega_c<2$ 时, $A(\omega)=\frac{200}{3}\cdot\frac{1}{\omega_c\cdot\omega_c}$, $\omega_c=\sqrt{\frac{200}{3}}>2$ 矛盾.
- $2<\omega_c<3$ 时, $A(\omega)=\frac{200}{3}\cdot\frac{1}{\omega_c\cdot\omega_c\cdots\frac{\omega_c}{2}}$, $\omega_c=\sqrt[3]{\frac{400}{3}}>3$ 矛盾.
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ω_x 计算

例: $G_o(s) = \frac{100(s+4)}{s(s+1)(s+2)(s+3)}$ 求解 ω_x，即求
$G_o(s) = \frac{K(s+4)}{s(s+1)(s+2)(s+3)}$
的根轨迹与虚轴交点.
$D(s) = s^4 + 6s^3 + 11s^2 + (K+6)s + 4K$

根轨迹与虚轴交点

Routh 表:

s^4	1	11	$4K$
s^3	6	$K+6$	
s^2	$\frac{60-K}{6}$	$4K$	
s^1	0	0	

$$\frac{60-K}{6}(K+6) = 4K \times 6$$

$$\frac{K^2}{6} + 15K - 60 = 0$$

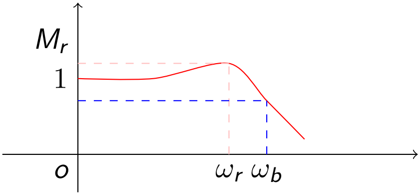
$$K = -45 \pm 3\sqrt{265}$$

$$\frac{60-K}{6}s^2 = 4K$$

$$\omega_x \approx 1.28$$

频带宽度

- 设闭环系统频率特性为 $\Phi(j\omega)$，若 $\omega > \omega_b$ 时，有 $20\lg|\Phi(j\omega)| < 20\lg|\Phi(j0)| - 3$，则称 ω_b 为带宽频率.



Topic

等 α 曲线

$$
\begin{aligned}
G(j\omega) &= Ae^{j\phi} \\
\Phi(j\omega) &= Me^{j\alpha} \\
&= \frac{Ae^{j\phi}}{1+Ae^{j\phi}} \\
\frac{Ae^{j\phi}}{Me^{j\alpha}} &= 1 + Ae^{j\phi} \\
\frac{A}{M} &= e^{-j(\phi-\alpha)} + Ae^{j\alpha} \\
0 &= \sin(\alpha-\phi) + A\sin\alpha \\
20\lg A &= 20\lg\frac{\sin(\phi-\alpha)}{\sin\alpha}
\end{aligned}
$$

等 M 曲线

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\begin{aligned}
\frac{Ae^{j\phi}}{Me^{j\alpha}} &= 1 + Ae^{j\phi} \\
\frac{A}{M} &= |1 + Ae^{j\phi}| \\
\frac{A^2}{M^2} &= (1 + A\cos\phi)^2 + A^2\sin^2\phi \\
0 &= (1 - M^{-2})A^2 + 2\cos\phi A + 1 \\
A &= \frac{\cos\phi \pm \sqrt{\cos^2\phi + M^{-2} - 1}}{M^{-2} - 1} \\
20\lg A &= 20\lg\frac{\cos\phi \pm \sqrt{\cos^2\phi + M^{-2} - 1}}{M^{-2} - 1}
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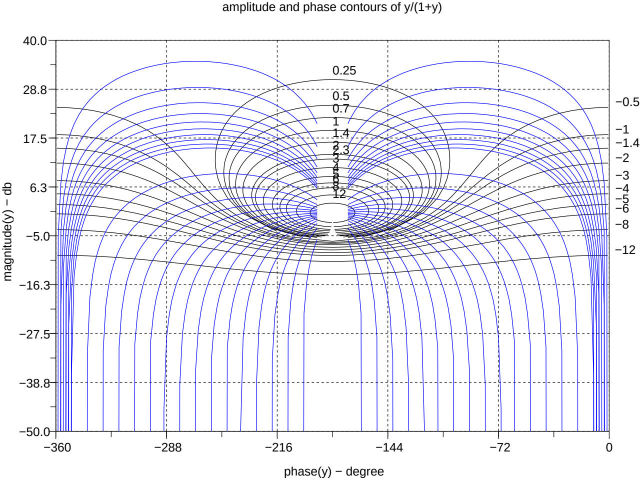
Nichols Chart



Topic

系统闭环和开环和频域指标的关系

$$
\begin{aligned}
G(j\omega) &= Ae^{-j(180^\circ - \gamma(\omega))} \\
&= A\left(-\cos\gamma(\omega) - j\sin\gamma(\omega)\right) \\
M &= \left|\frac{G(j\omega)}{1+G(j\omega)}\right| \\
&= \frac{A}{\sqrt{1+A^2-2A\cos\gamma(\omega)}} \\
&= \frac{1}{\sqrt{\left[\frac{1}{A}-\cos\gamma(\omega)\right]^2+\sin^2\gamma(\omega)}} \\
M_r &= \frac{1}{\sin\gamma(\omega_r)} \approx \frac{1}{sin\gamma} \qquad (\omega_r \approx \omega_c)
\end{aligned}
$$

2 阶系统频域指标

$$
\begin{aligned}
G(j\omega) &= \frac{\omega_n^2}{j\omega(j\omega + 2\xi\omega_n)} \\
&= \frac{\omega_n^2}{\omega\sqrt{\omega^2 + 4\xi^2\omega_n^2}} \angle(-\arctan\frac{\omega}{2\xi\omega_n} - 90^\circ) \\
\omega_c &= \omega_n(\sqrt{4\xi^4 + 1} - 2\xi^2)^{\frac{1}{2}} \\
\gamma &= 180^\circ + \angle G(j\omega_c) \\
&= \arctan\frac{2\xi\omega_n}{\omega_c}
\end{aligned}
$$

2 阶系统频域指标 (M_r, ω_r)

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\begin{aligned}
M_r &= \frac{1}{2\xi\sqrt{1-\xi^2}} \\
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- M_r 与 $\sigma\%$ 一一对应，且成正比

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高阶系统频域指标

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\begin{aligned}
M_r &= \frac{1}{\sin\gamma} \\
\sigma\% &= 16\% + 0.4(M_r - 1), (1 \le M_r \le 1.8) \\
t_s &= \frac{K\pi}{\omega_c} \\
K &= 2 + 1.5(M_r - 1) + 2.5(M_r - 1)^2
\end{aligned}
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- $\gamma \uparrow \rightarrow \sigma\% \downarrow \rightarrow \xi \uparrow$
- $\omega_c \uparrow \rightarrow t_s \downarrow$

频域要求:

- 低频段: 稳态性能
- 中频段: 瞬态性能
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