

CHEVALLEY FORMULAE FOR THE MOTIVIC CHERN CLASSES OF SCHUBERT CELLS AND FOR THE STABLE ENVELOPES

LEONARDO C. MIHALCEA, HIROSHI NARUSE, AND CHANGJIAN SU

ABSTRACT. We prove a Chevalley formula to multiply the motivic Chern classes of Schubert cells in a generalized flag manifold G/P by the class of any line bundle \mathcal{L}_λ . Our formula is given in terms of the λ -chains of Lenart and Postnikov. Its proof relies on a change of basis formula in the affine Hecke algebra due to Ram, and on the Hecke algebra action on torus-equivariant K-theory of the complete flag manifold G/B via left Demazure–Lusztig operators. We revisit some wall-crossing formulae for the stable envelopes in $T^*(G/B)$. We use our Chevalley formula, and the equivalence between motivic Chern classes of Schubert cells and K-theoretic stable envelopes in $T^*(G/B)$, to give formulae for the change of polarization, and for the change of slope for stable envelopes. We prove several additional applications, including Serre, star, and Dynkin, dualities of the Chevalley coefficients, new formulae for the Whittaker functions, and for the Hall–Littlewood polynomials. We also discuss (mostly conjectural) positivity and log concavity properties of special cases of the Chevalley coefficients.

CONTENTS

1. Introduction	2
Notation	5
2. Affine Hecke algebra via alcove walk algebra	6
2.1. Affine Hecke algebra	6
2.2. Alcove walk algebra	6
3. A λ -chain formula for the transition coefficients $c_{u,\mu}^{w,\lambda}$	11
3.1. Alcove paths and λ -chains	11
3.2. λ -chain formulae	13
4. Motivic Chern classes of Schubert cells	15
4.1. Definition of the Motivic Chern classes	15
4.2. Complete flag variety case	17
4.3. Partial flag variety case	18
5. Chevalley formulae for the motivic Chern classes	19
5.1. Chevalley coefficients	19
5.2. Chevalley formulae via alcove walks and the λ -chains	20
5.3. Positivity conjectures	21
5.4. Operator formula	24
5.5. Parabolic case	25
6. Dualities of Chevalley coefficients	26

Date: December 29, 2023.

2020 Mathematics Subject Classification. Primary 14M15, 14C17; Secondary 14N15, 17B10, 33D80.

Key words and phrases. motivic Chern classes of Schubert cells, stable envelopes, λ -chains, Hecke algebra, Demazure–Lusztig operators.

L. C. Mihalcea was supported in part by the NSF grant DMS-2152294 and a Simons Collaboration Grant. H. Naruse was supported in part by JSPS KAKENHI Grant Number 16H03921.

7.	K-theoretic stable envelopes for $T^*(G/B)$	28
7.1.	Definition of the stable envelopes	28
7.2.	Changing the polarizations	30
7.3.	Changing the alcoves	31
8.	Whittaker functions and Hall–Littlewood polynomials	34
8.1.	Whittaker functions	34
8.2.	Hall–Littlewood polynomials	36
Appendix A.	Chevalley formulae for the Chern–Schwartz–MacPherson classes of Schubert cells	38
A.1.	Degenerate affine Hecke algebra	38
A.2.	Definition of the CSM/SM classes	39
A.3.	The Chevalley formula in cohomology	40
Appendix B.	An example of the λ -chain formula	41
References		43

1. INTRODUCTION

Let G be a complex, semisimple, Lie group and $T \subset B \subset P \subset G$ be a parabolic subgroup containing a Borel subgroup and the (standard) maximal torus. Let W be the Weyl group determined by (G, T) . In the study of cohomology and K theory rings of (generalized) flag manifolds G/P , the Chevalley formula expresses the multiplication of a Schubert class by the class of line bundle, or, equivalently, a Schubert divisor in G/P . If one works equivariantly, this formula determines completely the multiplication in the equivariant K ring. In this paper we prove a Chevalley formula for the coefficients $C_{u,\lambda}^w(y) \in K_T(pt)[y]$ arising in the multiplication

$$(1) \quad MC_y(X(w)^\circ) \cdot \mathcal{L}_\lambda = \sum C_{u,\lambda}^w(y) MC_y(X(u)^\circ)$$

in the equivariant K theory ring $K_T(G/B)[y]$; see (27) and Theorem 5.2 below. Here $MC_y(X(w)^\circ) \in K_T(G/B)[y]$ is the *motivic Chern class* of a Schubert cell $X(w)^\circ \subset G/B$, and $\mathcal{L}_\lambda = G \times^B \mathbb{C}_\lambda$ is the line bundle on G/B associated to the one dimensional B -module of weight λ .

The motivic Chern classes $MC_y(X(w)^\circ) \in K_T(G/B)[y]$ have been defined by Brasselet, Schürmann, and Yokura [BSY10] more generally for elements $[Y \rightarrow X]$ in the Grothendieck group $G_0(\text{var}/X)$ of varieties over X . They are the unique classes which are functorial with respect to proper morphisms $f : X_1 \rightarrow X_2$, and which satisfy the normalization condition $MC[id_X : X \rightarrow X] = \lambda_y(T_X^*)$ for X smooth, where $\lambda_y(T_X^*) = \sum y^i [\wedge^i T_X^*]$ is the Hirzebruch λ_y class; see §4 below. They may be thought of as the K-theoretic generalizations of Chern–Schwartz–MacPherson classes defined by MacPherson [Mac74].

The motivic Chern classes of Schubert cells generalize well studied classes from Schubert calculus. If $y = 0$, the motivic class $MC_y(X(w)^\circ)$ is equal to the class of the ideal sheaf $[\mathcal{O}_{X(w)}(-\partial X(w))]$ of the boundary $\partial X(w) = X(w) \setminus X(w)^\circ$, where $X(w) = \overline{X(w)^\circ}$ is the Schubert variety. If $y = -1$, then $MC_y(X(w)^\circ)$ is equal to the class of the unique T -fixed point in $X(w)^\circ$; see [AMSS22]. The Poincaré duals of the classes $MC_y(X(w)^\circ)$, the *Segre motivic classes* [AMSS19, MSA22], specialize when $y = 0$ to the Grothendieck classes of the structure sheaves of the opposite Schubert varieties. Our Chevalley formula (1), and its analogous formula for the motivic Segre classes, specialize to known Chevalley formulae for K-theoretic Schubert classes and ideal sheaves from [GR04, LP07].

The formula for the coefficients $C_{u,\lambda}^w(y) \in K_T(pt)[y]$ from (1) follows from a formula of Ram [Ram06] in the affine Hecke algebra \mathbb{H} , calculating transition coefficients between two bases $\{T_w X^\lambda\}$ and $\{X^\lambda T_w\}$ of the affine Hecke algebra:

$$(2) \quad T_w X^{-\lambda} = \sum_{\mu \in X^*(T), u \in W} (-q)^{\ell(w) - \ell(u)} c_{u,\mu}^{w,\lambda} X^{-\mu} T_u.$$

Here T_w is an element in the standard basis of \mathbb{H} , $X^{-\lambda}$ is an affine element in \mathbb{H} , and $X^*(T)$ denotes the weight lattice of T . Ram's formula is stated in terms of a combinatorial model utilizing alcove walks, and it is convenient for our purposes to rewrite it utilizing in terms of λ -chains, a model introduced and studied by Lenart and Postnikov [LP07, LP08] in relation to equivariant K theory of flag manifolds. We refer to Theorem 3.9 and Theorem 3.10 for the precise statements in the Hecke algebra in terms of λ -chains, and to §5.1 for the formulae involving motivic Chern classes. We also note that (affine) Hecke algebras have long been used to obtain Chevalley formulas in various contexts, for example in [PR99, LP07]. We state next our main result.

Assume λ is an integral weight and fix a reduced λ -chain $(\beta_1, \beta_2, \dots, \beta_l)$. The chain corresponds to an alcove walk from the fundamental alcove A_\circ to $A_\circ - \lambda$, with separating hyperplanes $H_{-\beta_j, d_j}$. Denote by s_β the reflection determined by the root β . We refer the reader to §3 below for full definitions. The following is our main result, cf. Theorem 5.5.

Theorem 1.1. *The following Chevalley formula holds in $K_T(G/B)[y]$:*

$$\mathcal{L}_{-\lambda} \otimes MC_y(X(w)^\circ) = \sum_{\mu \in X^*(T), u \in W} C_{u,-\lambda}^w MC_y(X(u)^\circ),$$

where the Chevalley coefficients are given by

$$(3) \quad C_{u,-\lambda}^w = \sum_{J \subset \{1,2,\dots,l\}} (-1)^{n(J)} (1+y)^{|J|} (-y)^{\frac{\ell(w) - \ell(u) - |J|}{2}} e^{-w\tilde{r}_{J>}(\lambda)},$$

and the sum is over subsets $J = \{j_1 < \dots < j_t\} \subset \{1, 2, \dots, l\}$ such that $u < us_{\beta_{j_1}} < us_{\beta_{j_1}} s_{\beta_{j_2}} < \dots < us_{\beta_{j_1}} s_{\beta_{j_2}} \dots s_{\beta_{j_t}} = w$; the Weyl group element $\tilde{r}_{J>}$ is defined in (19). For the multiplication $\mathcal{L}_\lambda \otimes MC_y(X(w)^\circ)$, the Chevalley coefficients are given by

$$(4) \quad C_{u,\lambda}^w = \sum_{J \subset \{1,2,\dots,l\}} (-1)^{n(J)} (-1-y)^{|J|} (-y)^{\frac{\ell(w) - \ell(u) - |J|}{2}} e^{-w\hat{r}_{J<}(-\lambda)},$$

where the sum is over subsets $J = \{j_1 < \dots < j_t\} \subset \{1, 2, \dots, l\}$ such that $u < us_{\beta_{j_t}} < us_{\beta_{j_t}} s_{\beta_{j_{t-1}}} < \dots < us_{\beta_{j_t}} \dots s_{\beta_{j_1}} = w$, and with $\hat{r}_{J<}$ defined in (16).

The connection between the Hecke algebra coefficients from (2) and the Chevalley coefficients above is given by

$$(5) \quad C_{u,-\lambda}^w = \sum_{\mu \in X^*(T)} y^{\ell(w) - \ell(u)} e^{-\mu} c_{u,\mu}^{w,\lambda}|_{q=-y}.$$

The coefficients $c_{u,\mu}^{w,\lambda}$ are in general Laurent polynomials in y , while $C_{u,-\lambda}^w$ are polynomials in $K_T(pt)[y]$. In fact, the power $y^{\ell(w) - \ell(u)}$ from the formula (5) is absorbed into $c_{u,\mu}^{w,\lambda}$ so it becomes polynomial in y .

As mentioned above, our Chevalley formula generalizes to the motivic situation the classical Chevalley multiplication in $K_T(G/B)$. It also generalizes the Chevalley multiplication by (equivariant) Chern–Schwartz–MacPherson classes of Schubert cells from [AMSS17]; a short, self-contained, proof of this is added in an Appendix. All these specializations are

appropriately positive, in the sense of [Buc02, Bri02, AGM11]. In §5.3 below we state several positivity results and conjectures for the general formula. Notably, our formula for the multiplication by \mathcal{L}_λ with λ dominant (i.e., when $\mathcal{L}_{-\lambda}$ is globally generated) may be written as a positive combination of products $q^a(q-1)^b$, with $q = -y$; see Proposition 5.7. This positivity is similar to the one satisfied by R -polynomials in Kazhdan–Lusztig theory.

We conjecture that a different positivity holds in the special case when $\lambda = \varpi_i$ is a minuscule fundamental weight. Then the Chevalley coefficients satisfy

$$C_{u, \varpi_i}^w = e^{u(\varpi_i)} P_{u, \varpi_i}^w(y),$$

with $P_{u, \varpi_i}^w(y) \in \mathbb{Z}[y]$. These polynomials are palindromic (cf. Proposition 6.5), and we conjecture that they have alternating coefficients, i.e.,

$$(-1)^{\deg P_{u, \varpi_i}^w} P_{u, \varpi_i}^w(y) \in \mathbb{Z}_{\geq 0}[y];$$

see Conjecture 2 below. By the ‘Star duality’ proved in Proposition 6.2, a similar positivity must hold for the coefficients $C_{u, -\varpi_i}^w$. We note that our formulae from Theorem 1.1 do involve cancellations in terms of y ; see Example 5.11 below. Another, most intriguing, positivity is the one from Conjecture 3. It states that, for simply laced G , or $G = G_2$, the coefficients $A_{u, \lambda}^w$ in the (non-equivariant) multiplication

$$\mathcal{L}_\lambda \otimes MC_y(X(w)) = \sum A_{u, \lambda}^w MC_y(X(u)^\circ) \in K(G/B)[y]$$

of an ample line bundle by the motivic class of the Schubert *variety*, are positive and log concave.

We now give a rough idea on the proof of Theorem 1.1. The key connection between the Chevalley formula in the Hecke algebra to motivic Chern classes, proved in [MNS22b], and ultimately based on results from [AMSS19], is that the motivic Chern classes are recursively obtained by certain *left* Demazure–Lusztig operators \mathcal{T}_w^L acting on $K_T(G/B)[y]$:

$$MC_y(X(w)^\circ) = \mathcal{T}_w^L[\mathcal{O}_{1.B}].$$

These operators commute with elements in $K_G(G/B)[y]$ (i.e., the Weyl-group invariants of $K_T(G/B)$), and an argument based on equivariant localization shows that

$$MC_y(X(w)^\circ) \cdot \mathcal{L}_\lambda = \mathcal{T}_w^L[\mathcal{O}_{1.B}] \cdot \mathcal{L}_\lambda = \mathcal{T}_w^L(\mathcal{L}_\lambda \cdot [\mathcal{O}_{1.B}]) = \mathcal{T}_w^L(e^\lambda \cdot [\mathcal{O}_{1.B}]).$$

Therefore, the knowledge of the expansion from (2) implies the Chevalley formula in the geometric case. This argument may be generalized to any homogeneous bundle, see Remark 5.3 below. In cohomology (i.e., for the Chern–Schwartz–MacPherson classes), and for $G = \mathrm{SL}_n$, this argument is implicitly utilized in the paper [FGX22] to obtain a Murnaghan–Nakayama formula.

Having established formula to calculate the Chevalley coefficients, in §6 we utilize several dualities with geometric origin (the Serre duality, the star duality, and the Dynkin automorphism duality) to obtain several symmetries of the coefficients $C_{v, \lambda}^w(y)$. See e.g. Proposition 6.5. Combining these dualities shows that the polynomials $C_{v, \lambda}^w(y)$ are palindromic.

A remarkable property of the motivic Chern classes of Schubert cells, proved in [AMSS19, FRW21], is that they are equivalent to the K-theoretic version of Maulik and Okounkov’s stable envelopes, see [MO19, AO21]. The stable envelopes are elements in the $T \times \mathbb{C}^*$ -equivariant K-theory of the cotangent bundle, $K_{T \times \mathbb{C}^*}(T^*(G/B))$. In this context, the formal variable y may be identified to the (inverse) of the character given by the \mathbb{C}^* fibre dilation on the cotangent bundle. If $\iota : G/B \hookrightarrow T^*(G/B)$ is the inclusion of the zero section, then

$\iota^*(\text{stab}(w))$ is a multiple of the motivic Chern class of the (opposite) Schubert cell for w , where stab is a stable envelope, appropriately normalized.

The stable envelopes depend on three parameters: a chamber, a polarization, and a slope, and the precise normalizations are essential for this paper. A variation in the chamber results in conjugating by the Borel subgroup [AMSS19], and it is encoded in the left Weyl group action [MNS22b] and certain R -matrix operators [RTV15, RTV19]. Varying the polarization, or the slope, results in the multiplication of $\text{stab}(w)$ by a line bundle \mathcal{L}_λ pulled back from G/B ; cf. [AMSS19, Oko17], see also §4.1 below. In particular, the coefficients $C_{v,\lambda}^w(y)$ from (1) give ‘wall-crossing’ formulae, recording the change of stable envelopes when its defining parameters are varied. While these wall crossing formulae have been worked out in [Oko17, SZZ20, SZZ21] (see also [KW23]), in §4.1 we revisit some of these from the point of view of Theorem 1.1. In particular, we utilize the Chevalley formula to give an explicit combinatorial rule relating the stable envelope for the fundamental alcove A_\circ to the one corresponding to any translation $A_\circ + \lambda$; see Theorem 7.8.

In addition to our application mentioned above to wall crossing formulae for stable envelopes, in §8 we utilize known relations between motivic Chern classes of Schubert cells, Whittaker functions, and Hall–Littlewood polynomials, to obtain new formulae for the latter.

Finally, in an Appendix we obtain an analogue of the Chevalley formula (1) for the homological analogue of the motivic Chern classes, the Chern–Schwartz–MacPherson classes. While this formula may be obtained by a specialization argument as in [AMSS22], the degenerate affine Hecke algebra is much simpler in this case, and the proof of the Chevalley formula may be obtained rather quickly.

Acknowledgments. HN and CS thank Arun Ram and Andrei Okounkov for answering their questions and providing references; LM and CS thank Bogdan Ion for a related collaboration on Hall–Littlewood polynomials, and Paolo Aluffi and Jörg Schürmann for previous collaborations on motivic Chern classes and CSM classes; LM thanks Andrzej Weber for stimulating discussions. This project started in Spring 2021 while LM was in residence at the Institute for Computational and Experimental Research in Mathematics in Providence, RI, during the Combinatorial Algebraic Geometry program; LM wishes to thank the institute for excellent working conditions.

Notation. We fix the notation utilized throughout the paper. Let G be a simply connected complex Lie group with Borel subgroup B and maximal torus $T \subset B$. Denote by $\mathfrak{g} = \text{Lie}(G)$ and by $\mathfrak{h} = \text{Lie}(T)$ be corresponding Lie algebras. Let $R^+ \subset \mathfrak{h}^* := \mathfrak{h}_{\mathbb{Q}}^*$ denote the positive roots, which by convention are the roots in B , and by $\Sigma = \{\alpha_i : i \in I\}$ the set of simple roots. The set of all roots is $R := R^+ \sqcup -R^+$. We use $\alpha > 0$ (resp. $\alpha < 0$) to denote $\alpha \in R^+$ (resp. $\alpha \in -R^+$). For any root $\alpha \in R$, let $\alpha^\vee \in \mathfrak{h}$ denote the corresponding coroot. Denote by $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \rightarrow \mathbb{Q}$ the evaluation pairing, and let $X^*(T) \subset \mathfrak{h}^*$ be the weight lattice. For any weight $\lambda \in X^*(T)$, let $\mathcal{L}_\lambda := G \times^B \mathbb{C}_\lambda$ be the line bundle on G/B associated to λ . The Weyl group is $W = N_G(T)/T$ and it is generated by simple reflections $s_i = s_{\alpha_i}$ ($i \in I$). It is equipped with a length function $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$ defined as the length of a minimal expression of w in terms of the simple reflections; we denote by w_0 the longest element. The Bruhat order on W is a partial order determined by the covering relations $u \leq us_\alpha$ where $\alpha \in R$ and $\ell(us_\alpha) = \ell(u) + 1$.

For any $w \in W$, let $X(w)^\circ := BwB/B \subset G/B$ and $Y(w)^\circ := B^-wB/B \subset G/B$ be Schubert cells, where B^- is the opposite Borel subgroup. Let $X(w) := \overline{X(w)^\circ}$ and $Y(w) := \overline{Y(w)^\circ}$ be the Schubert varieties, respectively. Let $P(\supseteq B)$ be a parabolic subgroup with simple roots $\Sigma_P \subset \Sigma$. Let R_P^+ denote the positive roots spanned by Σ_P . Let W_P be the

Weyl group generated by the simple reflections s_α , $\alpha \in \Sigma_P$. Let $W^P \simeq W/W_P$ denote the set of minimal length representatives. For any $w \in W^P$, let $X(wW_P)^\circ := BwP/P \subset G/P$ (resp. $Y(wW_P)^\circ := B^-wP/P \subset G/P$) denote the Schubert cell with closure $X(wW_P)$ (resp. $Y(wW_P)$). Let $X^*(T)_P := \{\lambda \in X^*(T) \mid \langle \lambda, \gamma^\vee \rangle = 0 \text{ for all } \gamma \in R_P^+\}$ be the set of integral weights which vanish on $(R_P^+)^\vee$. For any $\lambda \in X^*(T)_P$, we still use \mathcal{L}_λ to denote the line bundle $G \times^P \mathbb{C}_\lambda \in \text{Pic}(G/P)$, which has fiber over $1.P$ the T -module of weight λ .

2. AFFINE HECKE ALGEBRA VIA ALCOVE WALK ALGEBRA

In this section, we introduce the alcove walk algebra, and a formula of Ram [Ram06] describing a change of bases matrix for the affine Hecke algebra.

2.1. Affine Hecke algebra. The affine Hecke algebra \mathbb{H} is a free $\mathbb{Z}[q, q^{-1}]$ module with basis $\{T_w X^\lambda \mid w \in W, \lambda \in X^*(T)\}$, such that

- For any $\lambda, \mu \in X^*(T)$, $X^\lambda X^\mu = X^{\lambda+\mu}$.
- For any simple root α , $(T_{s_\alpha} + 1)(T_{s_\alpha} - q) = 0$.
- For any $w, y \in W$, such that $\ell(wy) = \ell(w) + \ell(y)$, $T_w T_y = T_{wy}$.
- For any simple root α and $\lambda \in X^*(T)$,

$$T_{s_\alpha} X^\lambda - X^{s_\alpha \lambda} T_{s_\alpha} = (1 - q) \frac{X^{s_\alpha \lambda} - X^\lambda}{1 - X^{-\alpha}}.$$

For our geometric application we will need two other bases of the affine Hecke algebra \mathbb{H} : $\{T_{w^{-1}}^{-1} X^\lambda \mid w \in W, \lambda \in X^*(T)\}$ and $\{X^\lambda T_{w^{-1}}^{-1} \mid w \in W, \lambda \in X^*(T)\}$. Define the transition matrix coefficients $c_{u,\mu}^{w,\lambda} \in \mathbb{Z}[q, q^{-1}]$ by

$$(6) \quad T_{w^{-1}}^{-1} X^\lambda = \sum_{\mu \in X^*(T), u \in W} c_{u,\mu}^{w,\lambda} X^\mu T_u^{-1}.$$

The main result of this section is a formula for $c_{u,\mu}^{w,\lambda}$ obtained by Ram [Ram06], see Theorem 2.4 below.

For the later application to the motivic Chern classes, we also introduce the Iwahori–Matsumoto $\mathbb{Z}[q, q^{-1}]$ -algebra involution Θ on \mathbb{H} defined by

$$\Theta(T_{s_\alpha}) = -q T_{s_\alpha}^{-1}, \quad \text{and} \quad \Theta(X^\lambda) = X^{-\lambda},$$

where s_α is a simple reflection; see [EM97, Section 5.1]. Hence, $\Theta(T_{w^{-1}}^{-1}) = (-q)^{-\ell(w)} T_w$. Applying Θ to Equation (6), we obtain:

$$(7) \quad T_w X^{-\lambda} = \sum_{\mu \in X^*(T), u \in W} (-q)^{\ell(w) - \ell(u)} c_{u,\mu}^{w,\lambda} X^{-\mu} T_u$$

2.2. Alcove walk algebra. In this section, we review Ram's definition of the alcove walk algebra, and state his formula for the matrix coefficients $c_{u,\mu}^{w,\lambda}$. We refer the reader to [Ram06] for a more detailed account of the alcove walk algebras.

2.2.1. Alcoves. Let $\mathfrak{t}_\mathbb{R}^*$ be the dual of the Lie algebra of the maximal torus T . For any root α and $j \in \mathbb{Z}$, define

$$H_{\alpha,j} := \{\lambda \in \mathfrak{t}_\mathbb{R}^* \mid \langle \lambda, \alpha^\vee \rangle = j\}.$$

Notice that $H_{\alpha,j} = H_{-\alpha,-j}$. The connected components of $\mathfrak{t}_\mathbb{R}^* \setminus \bigcup_{\alpha > 0, j \in \mathbb{Z}} H_{\alpha,j}$ are called **alcoves**. The codimension 1 faces of any alcove are called the walls of that alcove. The **fundamental alcove** A_\circ is defined by:

$$A_\circ = \{\lambda \in \mathfrak{t}_\mathbb{R}^* \mid 0 < \langle \lambda, \alpha^\vee \rangle < 1, \text{ for any positive root } \alpha\}.$$

If $\alpha_1, \alpha_2, \dots, \alpha_r$ denote the simple roots, and θ^\vee denotes the highest coroot, then the walls of the fundamental alcove A_\circ are $H_{\theta,1}$ and $H_{\alpha_i,0}$ ($1 \leq i \leq r$). We label these walls of A_\circ by $0, 1, \dots, r$ respectively.

The affine Weyl group for the dual root system is defined by $W_{\text{aff}} := Q \rtimes W$, where Q is the root lattice. Then W_{aff} acts simply transitively on the set of alcoves, and this action is determined by the reflections across the hyperplanes $h = H_{\alpha,j}$, given by

$$(8) \quad s_{\alpha,j}(\mu) = \hat{r}_h(\mu) := s_\alpha(\mu) + j\alpha \text{ for } \mu \in X^*(T),$$

The affine Weyl group is a Coxeter group generated by the reflections $s_0 := s_{\theta,1}$ and s_i ($1 \leq i \leq r$) along the walls of A_\circ . In fact, A_\circ is a fundamental domain for the action of W_{aff} on the set of alcoves, in the sense that any element in $\mathfrak{t}_{\mathbb{R}}^* \setminus \bigcup_{\alpha > 0, j \in \mathbb{Z}} H_{\alpha,j}$ is sent to exactly one element in A_\circ . See [Ram06, Hum90] for more details.

The extended affine Weyl group for the dual root system is $W_{\text{aff}}^{\text{ext}} = X^*(T) \rtimes W$, where $X^*(T)$ is the weight lattice. For any $\lambda \in X^*(T)$, let t_λ denote the corresponding element in $W_{\text{aff}}^{\text{ext}}$. There is a length function ℓ on $W_{\text{aff}}^{\text{ext}}$ defined by the following formula (see [Mac96, Equation (2.8)]):

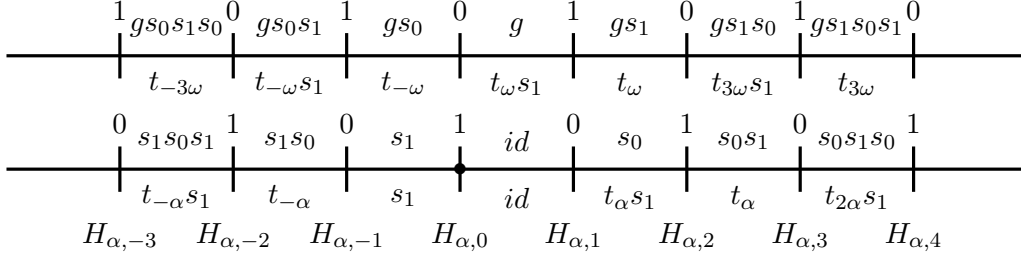
$$\ell(t_\mu w) = \sum_{\alpha \in R^+} |\langle \mu, w(\alpha^\vee) \rangle + \chi(w(\alpha))| \quad \text{where} \quad \chi(\alpha) = \begin{cases} 0 & \alpha \in R^+ \\ 1 & \alpha \in R^- \end{cases}.$$

Let $\Omega \subset W_{\text{aff}}^{\text{ext}}$ be the subgroup of length zero elements in $W_{\text{aff}}^{\text{ext}}$. Then $W_{\text{aff}}^{\text{ext}} \simeq W_{\text{aff}} \rtimes \Omega$, see [Mac96, Equation (2.10)], and $\ell(wg) = \ell(w)$ for any $w \in W_{\text{aff}}$ and $g \in \Omega$. The elements in Ω preserve the fundamental alcove A_\circ and act as automorphisms.

Utilizing a bijection between W_{aff} and the alcoves in $\mathfrak{t}_{\mathbb{R}}^*$, one can define a bijection between $W_{\text{aff}}^{\text{ext}} \simeq W_{\text{aff}} \rtimes \Omega$ and the alcoves in $\Omega \times \mathfrak{t}_{\mathbb{R}}^*$ ($|\Omega|$ copies of $\mathfrak{t}_{\mathbb{R}}^*$, each tiled by alcoves). We label the walls of every alcove in $\Omega \times \mathfrak{t}_{\mathbb{R}}^*$ in an $W_{\text{aff}}^{\text{ext}}$ -equivariant way. This means that for each $w \in W_{\text{aff}}^{\text{ext}}$ the walls of wA_\circ are $wH_{\alpha_i,0}$ ($1 \leq i \leq n$) and $wH_{\theta,1}$, and they are labeled by i and 0 , respectively. In particular, if two adjacent alcoves A_1 and A_2 are separated by a wall labeled by i (in both A_1 and A_2), and $A_1 = wA_\circ$ for some $w \in W_{\text{aff}}^{\text{ext}}$, then $A_2 = ws_iA_\circ$. Equivalently, in terms of the wall crossings, if $w = gs_{i_1}s_{i_2} \cdots s_{i_\ell} \in W_{\text{aff}}^{\text{ext}}$, with $g \in \Omega$ and $0 \leq i_j \leq r$, then the alcove wA_\circ in $\Omega \times \mathfrak{t}_{\mathbb{R}}^*$ is the alcove obtained from rotating the fundamental alcove A_\circ according to the automorphism g , then reflect along the walls labelled (in order) by i_1, \dots, i_ℓ ; see also Lemma 3.3 below.

Example 2.1. We consider the example of the root system of type A_1 . Let α be the positive root, and $\omega = \alpha/2$ be the fundamental weight. The weight lattice $X^*(T) = \mathbb{Z}\omega$, the root lattice $Q = \mathbb{Z}\alpha$, and the finite Weyl group $W = \{id, s_1 = s_\alpha\}$. The affine Weyl group $W_{\text{aff}} = Q \rtimes W$ has Coxeter generators s_1 and $s_0 = t_\alpha s_1$. The subgroup of length zero elements in $W_{\text{aff}}^{\text{ext}}$ is $\Omega = \{id, g = t_\omega s_1\} \simeq \mathbb{Z}/2\mathbb{Z}$.

In the following picture for $\Omega \times \mathfrak{t}_{\mathbb{R}}^*$, the lower sheet is the identity sheet, while the upper sheet is the sheet $g \times \mathfrak{t}_{\mathbb{R}}^*$. Each alcove wA_\circ is labeled by the corresponding $w \in W_{\text{aff}}^{\text{ext}}$, both in the Coxeter presentation and the translation presentation. In the lower sheet, the walls $H_{\alpha,n}$ are labeled by 1 if n is even, and 0 if n is odd. On the upper sheet, the labelings are in the opposite way.

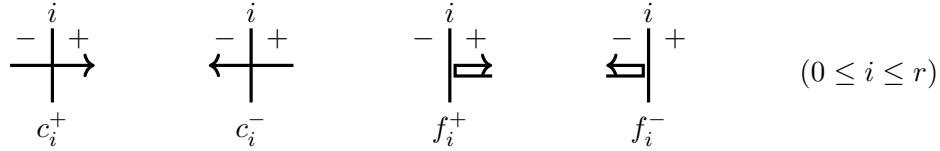


2.2.2. *Alcove walk algebra.* In this section we recall a realization of the Hecke algebra in terms of alcove walks; see [Ram06]. For each positive root α and hyperplane $H_{\alpha, j}$, set the positive side of it to be $\{\lambda \in \mathfrak{t}_{\mathbb{R}}^* \mid \langle \lambda, \alpha^\vee \rangle > j\}$.

Definition 2.2. The **alcove walk algebra** is generated over $\mathbb{Z}[q, q^{-1}]$ by elements $g \in \Omega$, and for $0 \leq i \leq r$, the elements c_i^+ (positive i -crossing), c_i^- (negative i -crossing), f_i^+ (positive i -fold) and f_i^- (negative i -fold), subject to the following relations, sometimes called straightening laws:

$$c_i^+ = c_i^- + f_i^+, \quad c_i^- = c_i^+ + f_i^-, \quad \text{and} \quad gc_i^\pm = c_{g(i)}^\pm g, \quad gf_i^\pm = f_{g(i)}^\pm g.$$

In terms of pictures, these generators can be drawn as follows:



Here, c_i^+ represents a crossing of a wall labeled by i from its negative to its positive side, and similarly for the other generators. The product is given as concatenation. An **alcove walk** is a word in the generators such that,

- the tail of the first step is in the fundamental alcove A_\circ ;
- at every step, either we change the sheet according to an element in Ω (thus rotating the alcove according to this elements), or the head of each arrow is in the same alcove as the tail of the next arrow.

An alcove walk p is called **nonfolded** if there is no f_i^\pm in its word. The **length** of an alcove walk is the number of letters c_i^\pm, f_i^\pm in an alcove walk. (In particular, rotation with respect to an element of Ω does not contribute to the length.) For a *minimal* alcove walk between two alcoves, one can show that the walk is non-folded, thus its length is the number of c_i^\pm in the walk [Ram06]. From this it follows that if $w \in W_{\text{aff}}^{\text{ext}}$, then $\ell(w)$ =length of a minimal length walk from A_\circ to wA_\circ .

Pick a square root $q^{\frac{1}{2}}$ of q . The following is proved by A. Ram.

Proposition 2.3. [Ram06, §3.2]

- (a) The affine Hecke algebra \mathbb{H} is isomorphic as a $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ -algebra to the quotient of the alcove walk algebra by the relations

$$(9) \quad c_i^+ = (c_i^-)^{-1}, \quad f_i^+ = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}), \quad f_i^- = -(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$$

and

$$(10) \quad p = p' \text{ if } p \text{ and } p' \text{ are nonfolded alcove walks with } \text{end}(p) = \text{end}(p'),$$

where $\text{end}(p)$ means the final alcove of p .

- (b) Under the previous isomorphism, for any $w \in W$ and $\lambda \in X^*(T)$ ¹:
- a minimal length alcove walk from A_\circ to $w A_\circ$ is sent to $q^{\frac{\ell(w)}{2}} T_{w^{-1}}^{-1}$, and,
 - a minimal length alcove walk from A_\circ to $t_\lambda A_\circ$ is sent to X^λ .

For an alcove walk p , define the functions **weight** $wt(p) \in X^*(T)$, and **final direction** $\varphi(p) \in W$ of p by the condition that p ends in the alcove $t_{wt(p)} \varphi(p) A_\circ$. Let

$$\begin{aligned} f^-(p) &= \text{number of negative folds of } p, \\ f^+(p) &= \text{number of positive folds of } p, \text{ and} \\ f(p) &= f^+(p) + f^-(p). \end{aligned}$$

Now we can state the formula for the matrix coefficients $c_{u,\mu}^{w,\lambda}$ defined in Equation (6), see also Equation (7).

Theorem 2.4. [Ram06, Theorem 3.3] *Let $\lambda \in X^*(T)$ and $w \in W$. Fix a minimal length walk $p_w = c_{i_1}^- c_{i_2}^- \cdots c_{i_r}^-$ from A_\circ to $w A_\circ$ and a minimal length walk $p_\lambda = c_{j_1}^{\epsilon_1} c_{j_2}^{\epsilon_2} \cdots c_{j_s}^{\epsilon_s} g$ from A_\circ to $t_\lambda A_\circ$, where $g \in W_{\text{aff}}^{\text{ext}}$ is defined by the condition $gW_{\text{aff}} = t_\lambda W_{\text{aff}}$ ², and $\epsilon_i = \pm$ for each i . Then*

$$(11) \quad T_{w^{-1}}^{-1} X^\lambda = \sum_p (-1)^{f^-(p)} (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{f(p)} q^{\frac{\ell(\varphi(p)) - \ell(w)}{2}} X^{wt(p)} T_{\varphi(p)^{-1}}^{-1},$$

where the sum is over all alcove walks p of the form

$$(12) \quad p = c_{i_1}^- c_{i_2}^- \cdots c_{i_r}^- p_{j_1} p_{j_2} \cdots p_{j_s} g \text{ such that } p_{j_k} \in \{c_{j_k}^\pm, f_{j_k}^{\epsilon_k}\}.$$

Therefore the matrix coefficients $c_{u,\mu}^{w,\lambda}$ in Equation (6) are given by:

$$(13) \quad c_{u,\mu}^{w,\lambda} = \sum_{\substack{p \text{ of the form (12)} \\ \varphi(p)=u, wt(p)=\mu}} (-1)^{f^+(p)} (1-q)^{f(p)} q^{\frac{\ell(u) - \ell(w) - f(p)}{2}}.$$

Example 2.5. Let $G = \text{SL}(2, \mathbb{C})$. We use the same notation as in Example 2.1. We check the above theorem for $w = s_1$ and $\lambda = \omega$. From the alcove picture in Example 2.1, $T_{s_1}^{-1}$ is represented by the minimal length walk $q^{-1/2} c_1^{-1}$, while X^ω is represented by the walk $g c_1^+ = c_0^+ g$. Thus, the sum in the right hand side of the Theorem is over the alcove walks $c_1^- c_0^- g$ and $c_1^- f_0^+ g$, which end at the alcoves $t_{-\omega} s_1 A_\circ$ and $t_{-\omega} A_\circ$, respectively. (Note that $c_1^- c_0^+ g$, which represents $T_{s_1}^{-1} X^\omega$, is not an alcove walk.) Therefore, the identity in the theorem is

$$T_{s_1}^{-1} X^\omega = X^{-\omega} T_{s_1}^{-1} - q^{-1} (1-q) X^{-\omega}.$$

On the other hand, it is easy to check the above equation using the definition of the affine Hecke algebra in Section 2.1.

Remark 2.6. In Theorem 2.4, one may relax the hypotheses about the alcove walks p_w and p_λ to be of non-minimal length. This follows from analyzing the proof of Ram's result in [Ram06]. We do not use this, but it is consistent with our use of non-reduced λ -chains in §3 below.

¹The q^2 and T_{s_i} in [Ram06, Proposition 3.2(b)] are our q and $q^{-\frac{1}{2}} T_i$, respectively.

²We need to add this extra $g \in \Omega$ since $t_\lambda A_\circ$ may not on the same sheet as A_\circ , see Example 2.5. The stated conditions determine g uniquely.

Consider an ordered collection of hyperplanes $\mathcal{H} = \{H_{\beta_1, k_1}, \dots, H_{\beta_s, k_s}\}$ and set $h_i := H_{\beta_i, k_i}$. Associated to this sequence we define the elements

$$\hat{r}_{\mathcal{H}} = s_{\beta_1, k_1} \cdots s_{\beta_s, k_s} \in W_{\text{aff}}; \quad r_{\mathcal{H}} = s_{\beta_1} \cdots s_{\beta_s} \in W.$$

(Note that these depend on the order of \mathcal{H} .) We also define an \mathcal{H} -restricted version of the Bruhat order on W by

$$(14) \quad w \xrightarrow{\mathcal{H}} u \iff w > ws_{\beta_1} > ws_{\beta_1} s_{\beta_2} > \cdots > ws_{\beta_1} s_{\beta_2} \cdots s_{\beta_s} = u.$$

The following is a mild generalization of [Len11, Proposition 2.5].

Lemma 2.7. *Let $w \in W$ and $\lambda \in X^*(T)$. Fix:*

- an alcove walk p_w from A_{\circ} to wA_{\circ} ;
- an alcove walk $p_{\lambda} = c_{j_1}^{\epsilon_1} c_{j_2}^{\epsilon_2} \cdots c_{j_s}^{\epsilon_s} g$ from A_{\circ} to $t_{\lambda} A_{\circ}$.

Let $h_i = H_{\beta_i, k_i}$, for $1 \leq i \leq s$ be the sequence of hyperplanes defined by the walls of alcoves crossed by p_{λ} , with $\beta_i \in R^{\epsilon_i}$ and $k_i \in \mathbb{Z}$.

Then there is a bijection between the following two sets:

- (1) The set of alcove walks of the form $\bar{p} = p_w p_{j_1} \cdots p_{j_s}$ such that $p_{j_k} \in \{c_{j_k}^{\pm}, f_{j_k}^{\epsilon_k}\}$;
- (2) The set of subsets $\mathcal{M} = \{h_{m_1}, \dots, h_{m_t}\} \subset \mathcal{H} = \{h_1, \dots, h_s\}$ with $m_1 < m_2 < \cdots < m_t$, s.t. $w \xrightarrow{\mathcal{M}} wr_{\mathcal{M}}$.

Under this bijection, the indices m_i correspond to the positions of foldings $f_{m_i}^{\epsilon_{m_i}}$. Furthermore,

$$\varphi(\bar{p}g) = wr_{\mathcal{M}} \text{ and } \text{wt}(\bar{p}g) = w\hat{r}_{\mathcal{M}}(\lambda).$$

Remark 2.8. This statement is a slight generalization of Lenart's result. We do not require λ to be dominant, and therefore we need to allow $\beta_i \in R^{\epsilon_i}$ to be a negative root.

Proof. The proof follows the same outline as that of [Len11, Proposition 2.5], so we will be brief. To start, consider the *unique* unfolded alcove walk $p_0 = p_w p_{j_1} \cdots p_{j_s}$ such that $p_{j_k} \in \{c_{j_k}^{\pm}\}$. Any other alcove walk \bar{p} as in the statement is of the form

$$\bar{p} = \phi_{m_t} \cdots \phi_{m_1}(p_0),$$

where ϕ_{m_j} is the folding operation at position m_j (cf. [Len11]) and the m_i 's with $m_{i-1} < m_i$ are the folding positions of \bar{p} . This alcove walk has the property that if $k \notin \{m_1, \dots, m_t\}$ then $p_{j_k} \in \{c_{j_k}^{\pm}\}$ and $p_{j_{m_i}} = f_{j_{m_i}}^{\epsilon_{m_i}}$ ($1 \leq i \leq t$). In addition $\varphi(\bar{p}g) = wr_{\mathcal{M}}$ and $\text{wt}(\bar{p}g) = w\hat{r}_{\mathcal{M}}(\lambda)$. Let $\mathcal{M}_i = \{h_{m_1}, \dots, h_{m_i}\} \subset \mathcal{H} = \{h_1, \dots, h_s\}$ for $1 \leq i \leq t$ with the convention that $\mathcal{M}_0 = \emptyset$ and $r_{\emptyset} = \text{id}$. A key point is that $wr_{\mathcal{M}_{i-1}} > wr_{\mathcal{M}_i}$ if and only if the folding orientations satisfy $\epsilon_{m_i} = \epsilon'_{m_i}$. (The proof uses the condition that if $\beta > 0$, then $ws_{\beta} > w$ if and only if $w(\beta) > 0$.) All this put together implies that $\{h_{m_1}, \dots, h_{m_t}\} \leftrightarrow \phi_{m_t} \cdots \phi_{m_1}(p_0)$ gives the bijection in the statement. \square

For a subset $\mathcal{M} \subset \mathcal{H}$ as in Lemma 2.7, set $p(\mathcal{M})$ to be the alcove walk $\bar{p}g$ associated to \mathcal{M} , and set $f^+(\mathcal{M}) = f^+(p(\mathcal{M}))$.

The following is a reformulation of Ram's result from Theorem 2.4 in terms of paths in the Bruhat order.

Corollary 2.9. *Let $u, w \in W$ and $\lambda, \mu \in X^*(T)$. Assume the same hypotheses and notation as in Lemma 2.7. Then*

$$c_{u, \mu}^{w, \lambda} = \sum_{\mathcal{M}} (-1)^{f^+(\mathcal{M})} (1 - q)^{|\mathcal{M}|} q^{\frac{\ell(u) - \ell(w) - |\mathcal{M}|}{2}},$$

where the sum is over subsets $\mathcal{M} \subset \mathcal{H}$ which satisfy $w \xrightarrow{\mathcal{M}} u = wr_{\mathcal{M}}$ and $\mu = w\hat{r}_{\mathcal{M}}(\lambda)$.

Remark 2.10. A priori, the coefficients $c_{u,\mu}^{w,\lambda}$ appearing in the walk algebra are in $\mathbb{Z}[q^{\pm\frac{1}{2}}]$. However, after matching these with the initial definition of Hecke algebras, it turns out that $c_{u,\mu}^{w,\lambda} \in \mathbb{Z}[q^{\pm 1}]$. This can also be seen directly in the formula above, by observing that $\ell(u) - \ell(w) - |\mathcal{M}|$ is even. (This uses that a reflection has odd length, and the cancellation property of non-reduced expressions.)

3. A λ -CHAIN FORMULA FOR THE TRANSITION COEFFICIENTS $c_{u,\mu}^{w,\lambda}$

In this section, we reformulate the alcove walk formula Corollary 2.9 in terms of the notion of λ -chains introduced in [LP07]. This notion was utilized to obtain a K -theory Chevalley formula for the structure sheaves of Schubert varieties. The main results of this section are Theorem 3.9 and Theorem 3.10. Specializing $y \mapsto 0$, these recover [LP07, Theorem 6.1]. Throughout this section, we only consider alcoves on the identity sheet $\mathfrak{t}_{\mathbb{R}}^*$. Recall that $A_{\circ} + \lambda := \{x + \lambda \mid x \in A_{\circ}\}$ is the alcove on $\mathfrak{t}_{\mathbb{R}}^*$. (If λ is not a root, the alcove $t_{\lambda}(A_{\circ})$ is not the alcove $A_{\circ} + \lambda$.)

3.1. Alcove paths and λ -chains. For any two alcoves A and B , which are separated by a common wall lying on a hyperplane $H_{\beta,k}$, write $A \xrightarrow{\beta} B$ if the root β points from A to B .

Definition 3.1. [LP07, Definition 5.2] An **alcove path** is a sequence of alcoves (A_0, A_1, \dots, A_l) such that A_{j-1} and A_j are adjacent, $1 \leq j \leq l$. We denote this by

$$A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \dots \xrightarrow{-\beta_l} A_l.$$

When the length l is minimal among all alcove paths from A_0 to A_l , it is called a **reduced alcove path**.

Remark 3.2. By definition, there is a one-to-one correspondence between alcove paths

$$(15) \quad A_{\circ} = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \dots \xrightarrow{-\beta_l} A_l = A_{\circ} - \lambda$$

from A_{\circ} to $A_l = v_{-\lambda}(A_{\circ})$ and alcove walks from A_0 to A_l of the form $c_{i_1}^{\epsilon_1} c_{i_2}^{\epsilon_2} \dots c_{i_l}^{\epsilon_l}$, where $-\beta_j \in R^{\epsilon_j}$ for $1 \leq j \leq l$.

Recall that $s_0 = s_{\theta,1}$ is the affine reflection along the hyperplane $H_{\theta,1}$ with θ^{\vee} the highest coroot. Define $\alpha_0 = -\theta$, $\bar{s}_0 = s_{\theta}$ and $\bar{s}_i = s_i$ for $1 \leq i \leq r$.

Lemma 3.3. [LP07, Lemma 5.3] *For any $v \in W_{\text{aff}}$, there is a one-to-one correspondence between decompositions of v in the simple reflections $s_i \in W_{\text{aff}}$ ($0 \leq i \leq r$) and alcove paths from A_{\circ} to $v(A_{\circ})$ as follows.*

For any decomposition $v = s_{i_1} s_{i_2} \dots s_{i_l}$, define

$$\beta_j := \bar{s}_{i_1} \bar{s}_{i_2} \dots \bar{s}_{i_{j-1}}(\alpha_{i_j}), \quad 1 \leq j \leq l.$$

Then

$$A_{\circ} = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \dots \xrightarrow{-\beta_l} A_l = v(A_{\circ})$$

is an alcove path from A_{\circ} to $v(A_{\circ})$.

Moreover, the affine reflection along the j -th hyperplane separating A_{j-1} and A_j is $s_{i_1} s_{i_2} \dots s_{i_{j-1}} s_{i_j} s_{i_{j-1}} \dots s_{i_2} s_{i_1}$; in particular, the separating wall is labeled by i_j . Under this correspondence, a reduced alcove path corresponds to a reduced decomposition for v .

Let λ be an integral weight and let $v_{-\lambda} \in W_{\text{aff}}$ be the affine Weyl group element which satisfies $v_{-\lambda}(A_o) = A_o - \lambda$, i.e., $t_{-\lambda} = v_{-\lambda}g$ with $g \in \Omega$. Choose a (possibly non-reduced) decomposition $v_{-\lambda} = s_{i_1} \cdots s_{i_l}$ and let β_j be defined by

$$\beta_j := \bar{s}_{i_1} \bar{s}_{i_2} \cdots \bar{s}_{i_{j-1}}(\alpha_{i_j}), \quad 1 \leq j \leq l,$$

with the convention that $\alpha_0 = -\theta$ (see Lemma 3.3). Then

$$A_o = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_l} A_o - \lambda$$

is an alcove path from A_o to $A_o - \lambda$.

Definition 3.4. [LP07, Definition 5.4] The sequence of roots $(\beta_1, \dots, \beta_l)$ is a λ -**chain** of roots associated to the decomposition of $v_{-\lambda}$. A λ -chain is called reduced if the decomposition $v_{-\lambda} = s_{i_1} \cdots s_{i_l}$ is reduced.

Let $H_{-\beta_j, d_j}$ be the hyperplane separating the alcoves A_j and A_{j+1} . The sequence of integers d_j are determined by the sequence of roots β_j , but we occasionally keep the information of d_j 's in the notation, and we refer to the sequence of pairs $(\beta_1, d_1), (\beta_2, d_2), \dots, (\beta_l, d_l)$ as a λ -chain³. Following [LP07, Prop. 6.8] we recall a combinatorial construction of a λ -chain for an integral weight λ .

Fix a linear order on the index of Dynkin nodes (for example $1 < 2 < \cdots < r$ in $I = \{1, 2, \dots, r\}$). The set $\mathcal{R}_\lambda \subset W_{\text{aff}}$ of affine reflections $s_{\alpha, k}$ for the hyperplanes $H_{\alpha, k}$ separating the fundamental alcove A_o and $A_o - \lambda$ is given by

$$\mathcal{R}_\lambda = \bigcup_{\alpha \in R^+} \begin{cases} \{s_{\alpha, k} : 0 \geq k > -\langle \lambda, \alpha^\vee \rangle\} & \text{if } \langle \lambda, \alpha^\vee \rangle > 0 \\ \{s_{\alpha, k} : 0 < k \leq -\langle \lambda, \alpha^\vee \rangle\} & \text{if } \langle \lambda, \alpha^\vee \rangle < 0 \\ \emptyset & \text{otherwise} \end{cases}$$

One defines a ‘height function’

$$h : \mathcal{R}_\lambda \rightarrow \mathbb{R}^{r+1}; \quad h(s_{\alpha, k}) = \frac{1}{\langle \lambda, \alpha^\vee \rangle} (-k, \langle \varpi_1, \alpha^\vee \rangle, \dots, \langle \varpi_r, \alpha^\vee \rangle).$$

It turns out that this function is injective. Now order the images of h in lexicographic order, so we obtain $h(s_{\alpha_1, k_1}) < \dots < h(s_{\alpha_l, k_l})$. Define another function $b : \mathcal{R}_\lambda \rightarrow R^+ \cup R^-$ by

$$b(s_{\alpha, k}) = \begin{cases} \alpha & \text{if } k \leq 0, \alpha \in R^+; \\ -\alpha & \text{if } k > 0, \alpha \in R^+. \end{cases}$$

Then $b(s_{\alpha_1, k_1}), \dots, b(s_{\alpha_l, k_l})$ is a (reduced) λ -chain of roots.

Remark 3.5. A particularly nice situation occurs for a minuscule fundamental weight ϖ_i , i.e. when $\langle \varpi_i, \alpha^\vee \rangle \in \{0, 1\}$ for any positive root α . In this case all $k_i = 0$ and $v_{-\varpi_i} = (w^{P_i})^{-1} \in W$, with w^{P_i} being the longest minimal length representative for cosets of W/W_{P_i} , where $W_{P_i} = \text{Stab}_W(\varpi_i)$; equivalently, the Schubert variety $X(w^{P_i}W_{P_i}) = G/P_i$. A reduced decomposition and the associated ϖ_i -chain may be read from the associated Young poset of G/P_i - see e.g. [Pro99, Ste01] and also [BCMP18, §3.1] and Example 3.6 below. It may also be obtained as a reverse linear extension of the heap $H(w^{P_i})$, and this gives a one-to-one correspondence between reduced ϖ_i -chains and reverse linear extensions of the heap $H(w^{P_i})$. We refer the readers to [MNS22a, NO19] for the heap perspective.

³If λ is dominant, this definition was extended to the Kac–Moody situation in [LP08].

Example 3.6. Consider $G = \mathrm{SL}_5$ and the fundamental weight ϖ_2 . The stabilizer is the maximal parabolic P_2 so that G/P_2 is the Grassmannian $\mathrm{Gr}(2, 5)$. The inversion set and a reduced decomposition of $v_{-\varpi_2}$ may be read from the Young diagrams below, with the notation $(i - j) = \alpha_i + \dots + \alpha_{j-1}$.

2 - 3	2 - 4	2 - 5
1 - 3	1 - 4	1 - 5

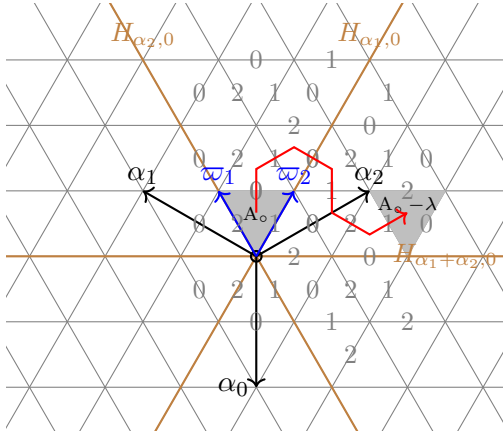
α_2	α_3	α_4
α_1	α_2	α_3

Then $v_{-\varpi_2} = s_2 s_3 s_4 s_1 s_2 s_3$ and a ϖ_2 -chain of roots is given by $\{\alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$.

Example 3.7. Let $G = \mathrm{SL}_3$, with the Weyl group $W = S_3$, and consider $\lambda = 2\varpi_1 - 2\varpi_2$. An example of alcove path from A_0 to $A_0 - \lambda$ is

$$A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} A_2 \xrightarrow{-\beta_3} A_3 \xrightarrow{-\beta_4} A_4 \xrightarrow{-\beta_5} A_5 \xrightarrow{-\beta_6} A_6 = A_0 - \lambda \quad (A_i = r_i A_{i-1}, 1 \leq i \leq 6),$$

which is the red path below, and it corresponds to the decomposition $v_{-\lambda} = s_0 s_1 s_0 s_1 s_2 s_1$. The corresponding alcove walk is $\bar{p} = c_0^+ c_1^+ c_0^- c_1^- c_2^- c_1^+$, and the corresponding λ -chain of roots is $(\beta_1, 1), (\beta_2, 1), (\beta_3, 0), (\beta_4, -1), (\beta_5, 1), (\beta_6, 2)$, as calculated below. Here we included the d 's in the notation.



$$\begin{aligned} i_1 &= 0, & \beta_1 &= \alpha_0 = -(\alpha_1 + \alpha_2) \\ i_2 &= 1, & \beta_2 &= \bar{s}_0(\alpha_1) = -\alpha_2 \\ i_3 &= 0, & \beta_3 &= \bar{s}_0 \bar{s}_1(\alpha_0) = \alpha_1 \\ i_4 &= 1, & \beta_4 &= \bar{s}_0 \bar{s}_1 \bar{s}_0(\alpha_1) = \alpha_1 + \alpha_2 \\ i_5 &= 2, & \beta_5 &= \bar{s}_0 \bar{s}_1 \bar{s}_0 \bar{s}_1(\alpha_2) = \alpha_1 \\ i_6 &= 1, & \beta_6 &= \bar{s}_0 \bar{s}_1 \bar{s}_0 \bar{s}_1 \bar{s}_2(\alpha_1) = -\alpha_2 \end{aligned}$$

$$\begin{aligned} s_0 &= \hat{r}_{H_{-\beta_1,1}} \\ s_0 s_1 s_0 &= \hat{r}_{H_{-\beta_2,1}} \\ (s_0 s_1) s_0 (s_1 s_0) &= s_1 = \hat{r}_{H_{-\beta_3,0}} \\ (s_0 s_1 s_0) s_1 (s_0 s_1 s_0) &= s_0 = \hat{r}_{H_{-\beta_4,-1}} \\ (s_0 s_1 s_0 s_1) s_2 (s_1 s_0 s_1 s_0) &= \hat{r}_{H_{-\beta_5,1}} \\ (s_0 s_1 s_0 s_1 s_2) s_1 (s_2 s_1 s_0 s_1 s_0) &= \hat{r}_{H_{-\beta_6,2}} \end{aligned}$$

We can choose another λ -chain for $\lambda = 2\varpi_1 - 2\varpi_2$ by using the reduced decomposition $v_{-\lambda} = s_1 s_0 s_2 s_1$. The corresponding reduced λ -chain is $(\alpha_1, 0), (-\alpha_2, 1), (\alpha_1, 1), (-\alpha_2, 2)$.

Lemma 3.8. [LP07, Remark 5.5] *Let $L = (\beta_1, \dots, \beta_l)$ be a λ -chain. Then $\bar{L} := (-\beta_l, \dots, -\beta_1)$ is a $(-\lambda)$ -chain. If $H_{-\beta_j, d_j}$ is the j -th hyperplane of the alcove path from A_0 to $A_0 - \lambda$ determined by L , then the j -th hyperplane of alcove path from A_0 to $A_0 + \lambda$ determined by \bar{L} is $H_{\beta_{l+1-j}, \langle \lambda, \beta_{l+1-j}^\vee \rangle - d_{l+1-j}}$. If L is reduced, then \bar{L} is also reduced.*

3.2. λ -chain formulae. Next we state the main theorem of this section. For any root hyperplane $h = H_{\beta, k}$, let r_h denote the reflection along the hyperplane $H_{\beta, 0}$, and \hat{r}_h be the reflection along h .

Assume λ is an integral weight and fix a reduced λ -chain $(\beta_1, \beta_2, \dots, \beta_l)$, which corresponds to an alcove walk from A_0 to $A_0 - \lambda$, with separating hyperplanes $H_{-\beta_j, d_j} =: h_j$.

For a subset $J = \{j_1 < j_2 < \dots < j_t\} \subset \{1, 2, \dots, l\}$, define the following relation

$$u \xrightarrow{J} w \stackrel{\text{def.}}{\iff} u < ur_{h_{j_t}} < ur_{h_{j_t}} r_{h_{j_{t-1}}} < \dots < ur_{h_{j_t}} \dots r_{h_{j_1}} = w,$$

and let

$$(16) \quad \hat{r}_{J_{<}} := \hat{r}_{h_{j_1}} \cdots \hat{r}_{h_{j_t}}.$$

Hence, $w \xrightarrow{J} u$ from Equation (14) is equivalent to $u \xrightarrow{J_{>}} w$. Therefore, Corollary 2.9 can be restated as follows.

Theorem 3.9. *In the above setting,*

$$(17) \quad c_{u,\mu}^{w,-\lambda} = \sum (-1)^{n(J)} (1-q)^{|J|} q^{\frac{\ell(u)-\ell(w)-|J|}{2}},$$

where the sum is over subsets $J \subset \{1, 2, \dots, l\}$ such that $u \xrightarrow{J_{>}} w$ and $w\hat{r}_{J_{<}}(-\lambda) = \mu$, and where $n(J) := \#\{j \in J \mid \beta_j < 0\}$.

Proof. In the summation in Corollary 2.9, we let $J \subset \{1, 2, \dots, l\}$ be the indices of the hyperplanes in \mathcal{M} . Then by Remark 3.2, $f^+(\mathcal{M}) = n(J)$. This finishes the proof. \square

Using the transformation between a λ -chain and a $(-\lambda)$ -chain, we can get rid of the negative sign in front of λ in Theorem 3.9 as follows.

First of all, $(-\beta_l, -\beta_{l-1}, \dots, -\beta_1)$ is a $(-\lambda)$ -chain, which corresponds to an alcove walk from A_\circ to $A_\circ + \lambda$, with the j -th separating hyperplane being

$$(18) \quad h'_j := H_{\beta_{l+1-j}, \langle \lambda, \beta_{l+1-j}^\vee \rangle - d_{l+1-j}}.$$

Then $r_{h'_j} = r_{h_{l+1-j}}$. Let $\tilde{r}_{h_{l+1-j}} := \hat{r}_{h'_j}$ be the reflection along h'_j . Define

$$u \xrightarrow{J_{<}} w \xLeftrightarrow{def} u < ur_{h_{j_1}} < ur_{h_{j_1}} r_{h_{j_2}} < \cdots < ur_{h_{j_1}} \cdots r_{h_{j_t}} = w,$$

and

$$(19) \quad \tilde{r}_{J_{>}} := \tilde{r}_{h_{j_t}} \cdots \tilde{r}_{h_{j_1}}.$$

Theorem 3.10. *The following holds,*

$$(20) \quad c_{u,\mu}^{w,\lambda} = \sum (-1)^{n(J)} (q-1)^{|J|} q^{\frac{\ell(u)-\ell(w)-|J|}{2}},$$

where the sum is over subsets $J \subset \{1, 2, \dots, l\}$ such that $u \xrightarrow{J_{<}} w$ and $w\tilde{r}_{J_{>}}(\lambda) = \mu$, where $n(J) := \#\{j \in J \mid \beta_j < 0\}$.

Proof. Applying Theorem 3.9 to the $(-\lambda)$ -chain $(-\beta_l, -\beta_{l-1}, \dots, -\beta_1)$, we get

$$c_{u,\mu}^{w,\lambda} = \sum (-1)^{\#\{j_i \mid -\beta_{l+1-j_i} < 0\}} (1-q)^t q^{\frac{\ell(u)-\ell(w)-t}{2}},$$

where the summation is over subsets $J' := \{1 \leq j_1 < j_2 < \cdots < j_t \leq l\}$, such that

$$(21) \quad u < ur_{h'_{j_t}} < \cdots < ur_{h'_{j_t}} \cdots r_{h'_{j_1}} = w$$

and

$$(22) \quad w\hat{r}_{h'_{j_1}} \cdots \hat{r}_{h'_{j_t}}(\lambda) = \mu.$$

Let

$$J := \{1 \leq l+1-j_t < l+1-j_{t-1} < \cdots < l+1-j_1 \leq l\}.$$

Then

$$(-1)^{\#\{j_i \mid -\beta_{l+1-j_i} < 0\}} (1-q)^t = (-1)^{n(J)} (q-1)^t,$$

where $n(J)$ is defined in Theorem 3.9. Since $r_{h'_j} = r_{h_{l+1-j}}$, the condition Equation (21) is equivalent to $u \xrightarrow{J_{<}} w$. On the other hand, the condition Equation (22) is equivalent to $w\tilde{r}_{J_{>}}(\lambda) = \mu$ as $\tilde{r}_{h_{l+1-j}} = \hat{r}_{h'_j}$. This finishes the proof. \square

For further use, we also record the following technical result, which will allow to rewrite the elements $w\hat{r}_{J_{<}}(-\lambda)$ from Theorem 3.9 and $w\tilde{r}_{J_{<}}$ from Theorem 3.10.

Proposition 3.11. *Consider a λ -chain β_1, \dots, β_l . For any subsequence $J = \{j_1 < j_2 < \dots < j_t\} \subset \{1, 2, \dots, l\}$, we have*

$$-\hat{r}_{J_{<}}(-\lambda) = r_J \tilde{r}_{J_{>}}(\lambda), \text{ where } r_J = r_{h_{j_1}} r_{h_{j_2}} \cdots r_{h_{j_t}}.$$

Proof. Let $m_j = \langle \lambda, \beta_j^\vee \rangle$ ($1 \leq j \leq l$). By induction on $|J|$, it is easy to show the following three equalities:

$$\begin{aligned} \hat{r}_{h_{j_1}} \hat{r}_{h_{j_2}} \cdots \hat{r}_{h_{j_t}}(-\lambda) &= -\lambda + (m_{j_1} - d_{j_1})\beta_{j_1} + (m_{j_2} - d_{j_2})r_{h_{j_1}}\beta_{j_2} + \cdots \\ &\quad + (m_{j_t} - d_{j_t})r_{h_{j_1}}r_{h_{j_2}} \cdots r_{h_{j_{t-1}}}\beta_{j_t}, \\ \tilde{r}_{h_{j_t}} \tilde{r}_{h_{j_{t-1}}} \cdots \tilde{r}_{h_{j_1}}(\lambda) &= \lambda - d_{j_t}\beta_{j_t} - d_{j_{t-1}}r_{h_{j_t}}\beta_{j_{t-1}} - \cdots - d_{j_1}r_{h_{j_t}}r_{h_{j_{t-1}}} \cdots r_{h_{j_2}}\beta_{j_1}, \\ r_{h_{j_1}}r_{h_{j_2}} \cdots r_{h_{j_t}}(\lambda) &= \lambda - m_{j_1}\beta_{j_1} - m_{j_2}r_{h_{j_1}}\beta_{j_2} - \cdots - m_{j_t}r_{h_{j_1}}r_{h_{j_2}} \cdots r_{h_{j_{t-1}}}\beta_{j_t}. \end{aligned}$$

As $r_J = r_{h_{j_1}} r_{h_{j_2}} \cdots r_{h_{j_t}}$ is a linear transformation, we get

$$r_J \tilde{r}_{J_{>}}(\lambda) = r_J(\lambda) + d_{j_t}r_{h_{j_1}} \cdots r_{h_{j_{t-1}}}\beta_{j_t} + d_{j_{t-1}}r_{h_{j_1}} \cdots r_{h_{j_{t-2}}}\beta_{j_{t-1}} + \cdots + d_{j_1}\beta_{j_1} = -\hat{r}_{J_{<}}(-\lambda).$$

□

Remark 3.12. With the notation as above, the condition that $u \xrightarrow{J_{>}} w$ in Theorem 3.9 implies that $ur_J^{-1} = w$, thus by Proposition 3.11,

$$w\hat{r}_{J_{<}}(-\lambda) = -u\tilde{r}_{J_{>}}(\lambda).$$

Similarly, in Theorem 3.10, we have that $w = ur_J$, thus

$$w\tilde{r}_{J_{>}}(\lambda) = -u\hat{r}_{J_{<}}(-\lambda).$$

This leads to alternative formulae in the aforementioned theorems 3.9 and 3.10.

In the appendix §B below we included a fully worked out example illustrating calculations of some coefficients $c_{u,\mu}^{w,\lambda}$ utilizing Theorem 3.9 and Theorem 3.10.

4. MOTIVIC CHERN CLASSES OF SCHUBERT CELLS

In this section, we recall basic properties of the motivic Chern classes of the Schubert cells in the (partial) flag varieties.

4.1. Definition of the Motivic Chern classes. Let X be a quasi-projective complex algebraic variety, and let $G_0(\text{var}/X)$ be the (relative) Grothendieck group of varieties over X . It consists of isomorphism classes of morphisms $[f : Z \rightarrow X]$ modulo the scissor relations. Brasselet, Schürmann and Yokura [BSY10] defined the *motivic Chern transformation* $MC_y : G_0(\text{var}/X) \rightarrow K(X)[y]$ with values in the K-theory group of coherent sheaves in X to which one adjoins a formal variable y . The transformation MC_y is a group homomorphism, it is functorial with respect to proper push-forwards, and if X is smooth it satisfies the normalization condition

$$MC_y[id_X : X \rightarrow X] = \sum [\wedge^j T_X^*] y^j.$$

Here $[\wedge^j T_X^*]$ is the K-theory class of the bundle of degree j differential forms on X . As explained in [BSY10], the motivic Chern class is related by a Hirzebruch-Riemann-Roch type statement to the Chern-Schwartz-MacPherson (CSM) class in the homology of X ; see Appendix A.2.

There is also an equivariant version of the motivic Chern class transformation, which uses equivariant varieties and morphisms, and has values in the suitable equivariant K-theory

group. Its definition was given in [AMSS19, FRW21], following closely the approach of [BSY10].

Assume X is smooth and there is a torus T acting on X . Let $K_T(X)$ denote the equivariant K theory of X , see [CG09]. If X is a point, $K_T(pt) = K^0(\text{Rep}(T)) = \mathbb{Z}[T]$. For any $\mathcal{F} \in K_T(X)$, let

$$\chi_T(X, \mathcal{F}) := \sum_i (-1)^i H^i(X, \mathcal{F}) \in K_T(pt).$$

Let $\langle -, - \rangle$ be the non-degenerate pairing on $K_T(X)$ defined by

$$\langle \mathcal{F}, \mathcal{G} \rangle = \chi_T(X, \mathcal{F} \otimes \mathcal{G}) \in K_T(pt),$$

where $\mathcal{F}, \mathcal{G} \in K_T(X)$. For a vector bundle E , the λ_y -class of E is the class

$$\lambda_y(E) := \sum_k [\wedge^k E] y^k \in K_T(X)[y].$$

The λ_y -class is multiplicative, i.e. for any short exact sequence $0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$ of equivariant vector bundles there is an equality $\lambda_y(E) = \lambda_y(E_1) \lambda_y(E_2)$ as elements in $K_T(X)[y]$.

Recall the (relative) motivic Grothendieck group $G_0^T(\text{var}/X)$ of varieties over X is the free abelian group generated by the isomorphism classes $[f : Z \rightarrow X]$ where Z is a quasi-projective T -variety and $f : Z \rightarrow X$ is a T -equivariant morphism modulo the usual additivity relations

$$[f : Z \rightarrow X] = [f : U \rightarrow X] + [f : Z \setminus U \rightarrow X]$$

for $U \subset Z$ an open T -invariant subvariety. For any equivariant morphism $f : X \rightarrow Y$ of quasi-projective T -varieties there are well defined push-forwards $f_! : G_0^T(\text{var}/X) \rightarrow G_0^T(\text{var}/Y)$ given by composition. The equivariant motivic Chern class is defined by the following theorem.

Theorem 4.1. [AMSS19, FRW21] *Let X be a quasi-projective, non-singular, complex algebraic variety with an action of the torus T . There exists a unique natural transformation $MC_y : G_0^T(\text{var}/X) \rightarrow K_T(X)[y]$ satisfying the following properties:*

- (1) *It is functorial with respect to T -equivariant proper morphisms of non-singular, quasi-projective varieties.*
- (2) *It satisfies the normalization condition*

$$MC_y[id_X : X \rightarrow X] = \lambda_y(T_X^*) = \sum y^i [\wedge^i T_X^*]_T \in K_T(X)[y].$$

The transformation MC_y satisfies a Verdier-Riemann-Roch (VRR) formula: for any smooth, T -equivariant morphism $\pi : X \rightarrow Y$ of quasi-projective and non-singular algebraic varieties, and any $[f : Z \rightarrow Y] \in G_0^T(\text{var}/Y)$, the following holds:

$$\lambda_y(T_\pi^*) \cap \pi^* MC_y[f : Z \rightarrow Y] = MC_y[\pi^* f : Z \times_Y X \rightarrow X].$$

Define the Grothendieck-Serre dual operator $\mathcal{D} : K_T(X) \rightarrow K_T(X)$ as follows. For any $[\mathcal{F}] \in K_T(X)$, define

$$(23) \quad \mathcal{D}[\mathcal{F}] := [R\text{Hom}(\mathcal{F}, \omega_X^\bullet)] := [\omega_X^\bullet \otimes \mathcal{F}^\vee] \in K_T(X),$$

where $\omega_X^\bullet \simeq \omega_X[\dim X]$ is the (equivariant) dualizing complex of X (the canonical bundle ω_X shifted by dimension). The class $[\mathcal{F}^\vee]$ is obtained by taking an equivariant resolution of \mathcal{F} by vector bundles, and then taking duals. Extend the operators \mathcal{D} and $(-)^\vee$ to $K_T(X)[y^{\pm 1}]$ by sending $y \mapsto y^{-1}$. We also let $(-)^\vee$ denote the operator on $K_T(pt)[y^{\pm 1}]$, which sends e^λ to $e^{-\lambda}$ and y to y^{-1} .

Definition 4.2. Assume $\Omega \hookrightarrow X$ is a T -stable subvariety.

- (1) The *motivic Chern* (MC) class of Ω is

$$MC_y(\Omega) := MC_y[\Omega \hookrightarrow X] \in K_T(X)[y].$$

- (2) If Ω is pure dimensional, the *Segre motivic class* (SMC) $SMC_y(\Omega) \in K_T(X)[[y]]$ is the class

$$SMC_y(\Omega) := (-y)^{\dim \Omega} \frac{\mathcal{D}(MC_y(\Omega))}{\lambda_y(T_X^*)} \in K_T(X)[[y]].$$

4.2. Complete flag variety case. Both $\{MC_y(X(w)^\circ) \mid w \in W\}$ and $\{SMC_y(Y(w)^\circ) \mid w \in W\}$ are bases for the localized equivariant K-theory

$$K_T(G/B)[[y]]_{\text{loc}} := K_T(G/B)[[y]] \otimes_{K_T(\text{pt})} \text{Frac } K_T(\text{pt}).$$

These classes can be calculated recursively using the Demazure–Lusztig operators as follows.

The left multiplication action of G on G/B induces a left Weyl group action on $K_T(G/B)$. For any $w \in W$, let w^L denote this action.

Definition 4.3. [MNS22b, Section 5.3] For any simple reflection s_i , we define the following left Demazure–Lusztig operator on $K_T(G/B)_{\text{loc}}$:

$$(24) \quad \mathcal{T}_i^L := \frac{1 + ye^{-\alpha_i}}{1 - e^{-\alpha_i}} s_i^L - \frac{1 + y}{1 - e^{-\alpha_i}}.$$

These operators enjoy the following properties.

Lemma 4.4. [MNS22b] *The operators \mathcal{T}_i^L satisfy the braid relation and the quadratic relation*

$$(\mathcal{T}_i^L + 1)(\mathcal{T}_i^L + y) = 0.$$

Moreover, they commute with the operators of tensoring by elements in $K_G(G/B)$.

Moreover, it is easy to verify the following lemma.

Lemma 4.5. *The following map Ψ defines an action of the affine Hecke algebra \mathbb{H} (with $q = -y$) on $K_T(G/B)_{\text{loc}}$,*

$$T_i \mapsto \mathcal{T}_i^L, \quad \text{and} \quad X^\lambda \mapsto e^\lambda,$$

where $e^\lambda \in K_T(\text{pt})$ acts on $K_T(G/B)_{\text{loc}}$ by multiplication.

We have the following theorem.

Theorem 4.6. (1) [MNS22b, Theorem 7.6] *For $w \in W$ and a simple root α_i ,*

$$\mathcal{T}_i^L(MC_y(X(w)^\circ)) = \begin{cases} MC_y(X(s_i w)^\circ) & \text{if } s_i w > w; \\ -(y+1)MC_y(X(w)^\circ) - yMC_y(X(s_i w)^\circ) & \text{if } s_i w < w. \end{cases}$$

In particular,

$$MC_y(X(w)^\circ) = \mathcal{T}_w^L([\mathcal{O}_{X(\text{id})}]).$$

- (2) [MNS22b, Theorem 7.1] *For any $w, u \in W$:*

$$\langle MC_y(X(w)^\circ), SMC_y(Y(u)^\circ) \rangle = \delta_{u,w}.$$

Remark 4.7. (1) By [AMSS22, Thm. 5.1 and Cor. 5.3],

$$MC_0(X(w)^\circ) = [\mathcal{O}_{X(w)}(-\partial X(w))], \quad \text{and} \quad SMC_0(Y(w)^\circ) = [\mathcal{O}_{Y(w)}],$$

where $\partial X(w) := X(w) \setminus X(w)^\circ$. Thus, the duality in the second part of the theorem reduces to the classical fact

$$\langle [\mathcal{O}_{X(w)}(-\partial X(w))], [\mathcal{O}_{Y(u)}] \rangle = \delta_{u,w}.$$

(2) By definition, for any $w \in W$,

$$MC_y(Y(w)) = \sum_{u \geq w} MC_y(Y(u)^\circ).$$

Besides, by the linearity of the Grothendieck-Serre dual operator \mathcal{D} ,

$$SMC_y(Y(w)) = \sum_{u \geq w} (-y)^{\ell(u) - \ell(w)} SMC_y(Y(u)^\circ).$$

Therefore,

$$SMC_0(Y(w)) = [\mathcal{O}_{Y(w)}].$$

4.3. Partial flag variety case. For any $w \in W$, let $\ell(wW_P)$ denote the length of the minimal length representative in wW_P . Let $\pi : G/B \rightarrow G/P$ be the natural projection. It is proved in [AMSS19, Remark 4.8] that

$$(25) \quad \pi_* MC_y(X(w)^\circ) = (-y)^{\ell(w) - \ell(wW_P)} MC_y(X(wW_P)^\circ).$$

In particular, if w is a minimal length representative, then $\pi_* MC_y(X(w)^\circ) = MC_y(X(wW_P)^\circ)$; this also follows directly from the functoriality property of the motivic classes.

Proposition 4.8. (1) [MNS22b, Proposition 6.3] *Let $\Omega \subset G/P$ be a T -stable subvariety of G/P and $\pi : G/B \rightarrow G/P$ be the projection. Then*

$$\pi^* SMC_y(\Omega) = SMC_y(\pi^{-1}\Omega) \in K_T(G/B)[[y]].$$

(2) [MNS22b, Theorem 7.2] *Let $u, w \in W^P$. The Segre motivic Chern classes are dual to the motivic Chern classes for any G/P , i.e.*

$$\langle MC_y(X(wW_P)^\circ), SMC_y(Y(uW_P)^\circ) \rangle = \delta_{u,w}.$$

Remark 4.9. By the first property in the proposition, for any $w \in W^P$,

$$\pi^*(SMC_y(Y(wW_P))) = SMC_y(Y(w)).$$

Letting $y = 0$, we get

$$\pi^*(SMC_0(Y(wW_P))) = SMC_0(Y(w)) = [\mathcal{O}_{Y(w)}],$$

where the second equality follows from Remark 4.7. By the definition of the SMC class,

$$(26) \quad SMC_y(Y(wW_P)) = \sum_{u \in W^P, u \geq w} (-y)^{\ell(u) - \ell(w)} SMC_y(Y(uW_P)^\circ).$$

Hence,

$$\pi^*(SMC_0(Y(wW_P)^\circ)) = \pi^*(SMC_0(Y(wW_P))) = [\mathcal{O}_{Y(w)}].$$

Since $\pi^*([\mathcal{O}_{Y(wW_P)}]) = [\mathcal{O}_{Y(w)}]$ and π^* is injective, we get

$$SMC_0(Y(wW_P)^\circ) = SMC_0(Y(wW_P)) = [\mathcal{O}_{Y(wW_P)}].$$

Combining with the second part of the proposition, we get

$$MC_0(X(wW_P)^\circ) = [\mathcal{O}_{X(wW_P)}(-\partial X(wW_P))].$$

This can also be proved using Remark 4.7(1), Equation (25), and the fact that the push-forward of an ideal sheaf is an ideal sheaf, see [Bri02].

5. CHEVALLEY FORMULAE FOR THE MOTIVIC CHERN CLASSES

In this section, we obtain several Chevalley formulae for the motivic Chern classes, in terms of alcove walks, λ -chains, and certain operators. The main technique is to reinterpret formulae from Hecke algebras such as Theorem 3.9 in terms of multiplications of motivic Chern classes by line bundles. Our main results are Theorem 5.5 and Theorem 5.15. In §5.3 we discuss several positivity properties and conjectures of the Chevalley coefficients. Finally, in §5.5 we discuss parabolic Chevalley formulae.

5.1. Chevalley coefficients. Consider a torus weight $\lambda \in X^*(T)$ and $u, w \in W$. The Chevalley coefficient $C_{u,\lambda}^w$ is defined by the following formula:

$$(27) \quad \mathcal{L}_\lambda \otimes MC_y(X(w)^\circ) = \sum_{u \leq w} C_{u,\lambda}^w MC_y(X(u)^\circ).$$

Note that for any simple reflection s_i there is a short exact sequence of equivariant sheaves

$$0 \rightarrow \mathcal{L}_{\varpi_i} \otimes \mathbb{C}_{-w_0(\varpi_i)} \rightarrow \mathcal{O}_{G/B} \rightarrow \mathcal{O}_{X(w_0 s_i)} \rightarrow 0$$

with ϖ_i the fundamental weight, see e.g., [BM15, §8]. Therefore,

$$[\mathcal{O}_{X(w_0 s_i)}] = 1 - e^{-w_0(\varpi_i)} \mathcal{L}_{\varpi_i} \in K_T(G/B),$$

and the coefficients from (27) for $\lambda = \varpi_i$ also recover the multiplication of $[\mathcal{O}_{X(w_0 s_i)}]$ with the MC classes of the Schubert cells.

The coefficients $C_{u,\lambda}^w$ also arise from Chevalley formulae involving Segre motivic classes:

Lemma 5.1. *The following equation holds:*

$$(28) \quad \mathcal{L}_\lambda \otimes SMC_y(Y(u)^\circ) = \sum_{w \geq u} C_{u,\lambda}^w SMC_y(Y(w)^\circ).$$

Proof. This follows from the (Poincaré) duality in Theorem 4.6(2), as the Chevalley coefficients in Equation (27) and Equation (28) are given by

$$C_{u,\lambda}^w = \langle \mathcal{L}_\lambda \otimes MC_y(X(w)^\circ), SMC_y(Y(u)^\circ) \rangle.$$

□

We now relate the Chevalley coefficients above to the coefficients $c_{u,\mu}^{w,\lambda}$ from Equation (7) in the Hecke algebra.

Theorem 5.2. *Let λ be any weight in $X^*(T)$. The following Chevalley formula holds in $K_T(G/B)[y]$:*

$$\mathcal{L}_{-\lambda} \otimes MC_y(X(w)^\circ) = \sum_{\mu \in X^*(T), u \in W} y^{\ell(w)-\ell(u)} e^{-\mu} c_{u,\mu}^{w,\lambda}|_{q=-y} MC_y(X(u)^\circ).$$

In particular, the following equation holds for the Chevalley coefficients in (27):

$$(29) \quad C_{u,-\lambda}^w = \sum_{\mu \in X^*(T)} y^{\ell(w)-\ell(u)} e^{-\mu} c_{u,\mu}^{w,\lambda}|_{q=-y}.$$

From Equation (27), the Chevalley coefficients $C_{u,\lambda}^v$ belong to a localization $K_T(pt)[y]$ which allows division by $1 + e^\alpha y$ for any root α . However, the expansion from Equation (7) implies that the coefficients $(-q)^{\ell(w)-\ell(v)} c_{u,\mu}^{w,\lambda}$ are *polynomials* in $\mathbb{Z}[q]$. Then it follows that $C_{u,\lambda}^v$ are in fact polynomials in $K_T(pt)[y]$. This will be seen explicitly in Theorem 5.4 below.

Proof of Theorem 5.2. Applying the map Ψ in Lemma 4.5 to Equation (7), we get

$$(30) \quad \mathcal{T}_w^L e^{-\lambda} = \sum_{\mu \in X^*(T), u \in W} y^{\ell(w) - \ell(u)} e^{-\mu} c_{u, \mu}^{w, \lambda}|_{q=-y} \mathcal{T}_u^L \in \text{End}_{\mathbb{C}} K_T(G/B)[y].$$

Then the theorem follows by applying both sides to $[\mathcal{O}_{X(id)}]$, and utilizing that

$$\begin{aligned} \mathcal{T}_w^L e^{-\lambda}([\mathcal{O}_{X(id)}]) &= \mathcal{T}_w^L(\mathcal{L}_{-\lambda} \otimes [\mathcal{O}_{X(id)}]) \\ &= \mathcal{L}_{-\lambda} \otimes MC_y(X(w)^\circ). \end{aligned}$$

Here the first equality follows from $\mathcal{L}_{-\lambda} \otimes [\mathcal{O}_{X(id)}] = e^{-\lambda}[\mathcal{O}_{X(id)}]$, while the second one follows from Lemma 4.4 and Theorem 4.6. \square

Remark 5.3. The above argument can be generalized to the case when the line bundle \mathcal{L}_λ is replaced by any homogeneous bundle $\mathcal{V} = G \times^B V \rightarrow G/B$ associated to a B -representation of V . If the character of V is $ch(V) = \sum_\lambda a_\lambda e^\lambda$, then a localization argument shows that the class of \mathcal{V} in $K_T(G/B)$ is equal to

$$[\mathcal{V}] = \sum a_\lambda \mathcal{L}_\lambda.$$

It follows that for any $w \in W$,

$$\mathcal{V} \otimes MC(X(w)^\circ) = \sum_\lambda a_\lambda \mathcal{L}_\lambda \otimes MC(X(w)^\circ) = \sum_u \sum_\lambda a_\lambda C_{u, \lambda}^w MC(X(u)^\circ).$$

We illustrate this for $G/B = \text{Fl}(n)$, the complete flag manifold. This is equipped with the tautological sequence $\mathcal{F}_0 = 0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \subset \mathcal{F}_n = \mathbb{C}^n$. For $1 \leq i \leq n-1$ define $X_i = \mathcal{F}_i/\mathcal{F}_{i-1}$ regarded in $K_T(G/B)$. Then

$$\wedge^j \mathcal{F}_i = e_j(X_1, \dots, X_i), \quad \text{Sym}^j \mathcal{F}_i = h_j(X_1, \dots, X_i)$$

where e_j and h_j denote the elementary symmetric function, respectively the complete homogeneous symmetric function. Note that if ϖ_i denotes the i th fundamental weight, then $X_i = \mathcal{L}_{\varpi_i - \varpi_{i-1}}$ for $1 \leq i \leq n-1$, with the convention that $\varpi_0 = 0$. Theorem 5.2 gives a formula for the multiplication by monomials $X_1^{a_1} \dots X_{n-1}^{a_{n-1}}$, which in turn gives formulae to multiply by $e_j(X_1, \dots, X_i)$ and $h_j(X_1, \dots, X_i)$.

In the next section, we give explicit formulae for the Chevalley coefficients $C_{\lambda, w}^u$ based on the formulae for the Hecke algebra coefficients $c_{u, \mu}^{w, \lambda}$.

5.2. Chevalley formulae via alcove walks and the λ -chains. Let us recall the setting of Corollary 2.9. For $\lambda \in X^*(T)$, choose a minimal length alcove walk $p_{v_\lambda} = c_{j_1}^{\epsilon_1} c_{j_2}^{\epsilon_2} \dots c_{j_s}^{\epsilon_s}$ from A_\circ to $A_\circ + \lambda$, and let $\mathcal{H} = \{h_1, h_2, \dots, h_s\}$ be the ordered sequence of hyperplanes defined by the walls of alcoves crossed by p_λ . Define $\beta_i \in R^{\epsilon_i}$ and $k_i \in \mathbb{Z}$ ($1 \leq i \leq s$) by the condition $h_i = H_{\beta_i, k_i}$. Combining Corollary 2.9 and Theorem 5.2, we get the following formula.

Theorem 5.4 (Alcove walk formula for the Chevalley coefficients).

$$C_{u, -\lambda}^w = \sum_{\mathcal{M}} (-1)^{f^+(\mathcal{M})} (-1-y)^{|\mathcal{M}|} (-y)^{\frac{\ell(w) - \ell(u) - |\mathcal{M}|}{2}} e^{-w\hat{r}_{\mathcal{M}}(\lambda)}$$

where the sum is over ordered subsets $\mathcal{M} \subset \mathcal{H}$ such that $w \xrightarrow{\mathcal{M}} u = wr_{\mathcal{M}}(\lambda)$.

We now recall the setup of Theorem 3.9. Assume λ is an integral weight and fix a reduced λ -chain $(\beta_1, \beta_2, \dots, \beta_l)$, which corresponds to an alcove walk from A_\circ to $A_\circ - \lambda$, with separating hyperplanes $h_j := H_{-\beta_j, d_j}$.

Combining Theorem 3.9, Theorem 3.10, and Theorem 5.2, we get the following formulae.

Theorem 5.5 (λ -chain formula for the Chevalley coefficients). *The following hold:*

$$(31) \quad C_{u, -\lambda}^w = \sum_{J \subset \{1, 2, \dots, l\}} (-1)^{n(J)} (1+y)^{|J|} (-y)^{\frac{\ell(w) - \ell(u) - |J|}{2}} e^{-w\tilde{r}_{J>}(\lambda)},$$

where the sum is over subsets $J = \{j_1 < \dots < j_t\} \subset \{1, 2, \dots, l\}$ such that $u < ur_{h_{j_1}} < ur_{h_{j_1}} r_{h_{j_2}} < \dots < ur_{h_{j_1}} r_{h_{j_2}} \dots r_{h_{j_t}} = w$ and $\tilde{r}_{J>}$ is defined in (19). Furthermore,

$$(32) \quad C_{u, \lambda}^w = \sum_{J \subset \{1, 2, \dots, l\}} (-1)^{n(J)} (-1-y)^{|J|} (-y)^{\frac{\ell(w) - \ell(u) - |J|}{2}} e^{-w\hat{r}_{J<}(-\lambda)},$$

where the sum is over subsets $J = \{j_1 < \dots < j_t\} \subset \{1, 2, \dots, l\}$ such that $u < ur_{h_{j_t}} < ur_{h_{j_t}} r_{h_{j_{t-1}}} < \dots < ur_{h_{j_t}} \dots r_{h_{j_1}} = w$ and $\hat{r}_{J<}$ is defined in (16).

Remark 5.6. (1) It follows from Remark 3.12 that $-w\tilde{r}_{J>}(\lambda) = u\hat{r}_{J<}(-\lambda)$ in Equation (31), and that $-w\hat{r}_{J<}(-\lambda) = u\tilde{r}_{J>}(\lambda)$ in Equation (32), giving alternative ways to calculate these.

(2) By Remark 4.7 (1), the MC and SMC classes specialize at $y = 0$ to the ideal sheaves and the structure sheaves, respectively. Under this specialization, and using that $-w\tilde{r}_{J>}(\lambda) = u\hat{r}_{J<}(-\lambda)$, Equation (31) reduces to the equivariant K -theory Chevalley formula of Lenart-Postnikov [LP07, Theorem 6.1].

One can also consider the more general situation of Kac-Moody flag varieties defined e.g. in [Kum02]. These are ind-varieties, and one can define the motivic Chern classes of the finite-dimensional Schubert cells using the ind-structure. There are analogues of the (left and right) Demazure operators, and the notion of λ -chains extends to this setting, by results of Lenart and Postnikov [LP08]. Note that for an infinite Weyl group W , a λ -chain may be an infinite sequence, i.e. $l = \infty$. But, for given $u \leq w$, $t = |J|$ is finite, and the number of J which satisfies the condition in Theorem 5.5 is also finite. Based on these similarities to the finite case, we expect the following conjecture to hold.

Conjecture 1. *For a Kac-Moody Weyl group W and a dominant integral weight λ , the analogues of the equations Equation (31) and Equation (32) hold.*

5.3. Positivity conjectures. In this section we discuss several positivity properties and conjectures of the coefficients from the Chevalley formula. We start with the following consequence of Theorem 5.5.

Proposition 5.7. *Let λ be a dominant weight and set $q = -y$. Then:*

(a) $C_{u, \lambda}^w$ may be written as a combination of terms of the form $e^\mu q^a (q-1)^b$ with non-negative integer coefficients.

(b) $C_{u, -\lambda}^w$ may be written as a combination of terms of the form $(-1)^b e^\mu q^a (q-1)^b$ with non-negative integer coefficients.

Furthermore, in both situations b has the same parity as $\ell(w) - \ell(u)$.

Proof. Both statements follow from Theorem 5.5, using that for a reduced λ -chain with λ dominant we have $n(J) = 0$, and that $\ell(w) - \ell(u) - |J|$ is an even integer. \square

Example 5.8. Consider type A_2 , with $\lambda = 2\varpi_1 + \varpi_2$, and with the λ -chain from §B. Take $w = s_2 s_1$. Then from Theorem 5.5 we get (with $q = -y$):

$$\begin{aligned} \mathcal{L}_\lambda \otimes MC_{-q}(X(s_2 s_1)^\circ) &= e^{\varpi_1 - 3\varpi_2} MC_{-q}(X(s_2 s_1)^\circ) \\ &\quad + (q-1)(e^{-\varpi_2} + e^{-\varpi_1 + \varpi_2} + e^{-2\varpi_1 + 3\varpi_2}) MC_{-q}(X(s_1)^\circ) \\ &\quad + (q-1)(e^{2\varpi_1 - 2\varpi_2} + e^{3\varpi_1 - \varpi_2}) MC_{-q}(X(s_2)^\circ) \\ &\quad + (q-1)^2(e^{2\varpi_1 + \varpi_2} + e^{2\varpi_2} + e^{\varpi_1}) MC_{-q}(X(id)^\circ). \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{-\lambda} \otimes MC_{-q}(X(s_2 s_1)^\circ) &= e^{-\varpi_1 + 3\varpi_2} MC_{-q}(X(s_2 s_1)^\circ) \\ &\quad - (q-1)(e^{\varpi_1 - \varpi_2} + e^{\varpi_2} + e^{-\varpi_1 + 3\varpi_2}) MC_{-q}(X(s_1)^\circ) \\ &\quad - (q-1)(e^{-2\varpi_1 + 2\varpi_2} + e^{-\varpi_1 + 3\varpi_2}) MC_{-q}(X(s_2)^\circ) \\ &\quad + (q-1)^2(e^{-\varpi_1} + e^{\varpi_1 - \varpi_2} + e^{-2\varpi_1 + 2\varpi_2} + e^{\varpi_2} + e^{-\varpi_1 + 3\varpi_2}) MC_{-q}(X(id)^\circ). \end{aligned}$$

We now investigate the special multiplication by $\mathcal{L}_{\pm\varpi_i}$ in the case of minuscule fundamental weights. In this case the coefficients have a particularly pleasing factorization.

Lemma 5.9. *If $\lambda = \varpi_i$ is a minuscule weight, then for any $u, w \in W$,*

$$C_{u,\lambda}^w = e^{u(\lambda)} P_{u,\lambda}^w(y) \quad \text{and} \quad C_{u,-\lambda}^w = (-1)^{\ell(w) - \ell(u)} e^{-w(\lambda)} P_{w_0 w, \lambda}^{w_0 u}(y),$$

where $P_{u,\lambda}^w(y) \in \mathbb{Z}[y]$. Furthermore, these polynomials satisfy

$$P_{u,\lambda}^w(y^{-1}) \cdot y^{\ell(w) - \ell(u)} = P_{u,\lambda}^w(y),$$

i.e., they are palindromic.

Proof. Since $\lambda = \varpi_i$ is minuscule, for any reduced λ -chain $(\beta_1, \beta_2, \dots, \beta_l)$, the separating hyperplanes must be of the form $h_j := H_{-\beta_j, 0}$, thus $\hat{r}_{h_j} = r_{h_{j_1}}$. Therefore the first equality follows from Equation (32), since $-w\hat{r}_{J_{<}}(-\lambda) = -wr_{J_{<}}(-\lambda) = u(\lambda)$. The second equality follows this and from the ‘Star duality’ in Proposition 6.2 below. The palindromic property follows from Proposition 6.5(a) below. \square

Based on computer experiments in all Lie types we conjecture the following.

Conjecture 2. *Consider the multiplication:*

$$MC(X(w)^\circ) \cdot \mathcal{L}_\lambda = \sum C_{u,\lambda}^w MC(X(u)^\circ).$$

Then the following hold:

- (1) *If $\lambda = \varpi_i$ is a minuscule weight, then $C_{u,\lambda}^w = e^{u(\lambda)} P_{u,\lambda}^w(y)$ where*

$$(-1)^{\deg P_{u,\lambda}^w} P_{u,\lambda}^w \in \mathbb{Z}_{\geq 0}[y]$$

is a polynomial in y with non-negative integer coefficients.

- (2) *If $\lambda = \varpi_i$ is a minuscule weight, then*

$$C_{u,-\lambda}^w = (-1)^{\ell(w) - \ell(u)} e^{-w(\lambda)} P_{w_0 w, \lambda}^{w_0 u}(y)$$

and

$$(-1)^{\deg P_{w_0 w, \lambda}^{w_0 u}} P_{w_0 w, \lambda}^{w_0 u} \in \mathbb{Z}_{\geq 0}[y]$$

is a polynomial in y with non-negative coefficients.

Of course, parts (1) and (2) of this conjecture are equivalent, by Lemma 5.9.

Example 5.10. Consider type A_2 , $\lambda = \varpi_1$. An ϖ_1 -chain is $(\beta_1 = \alpha_1, \beta_2 = \alpha_1 + \alpha_2)$. Take $w = s_1 s_2 s_1$. Then Theorem 5.5 gives

$$\begin{aligned} \mathcal{L}_{-\varpi_1} \otimes MC_y(X(s_1 s_2 s_1)^\circ) &= e^{\varpi_2} MC_y(X(s_1 s_2 s_1)^\circ) + (1+y)e^{\varpi_2} MC_y(X(s_1 s_2)^\circ) \\ &\quad - (1+y)ye^{\varpi_2} MC_y(X(id)^\circ) \end{aligned}$$

Here each term corresponds to the choice of $J = \{\}, \{1\}, \{2\}$. Furthermore,

$$\begin{aligned} \mathcal{L}_{\varpi_1} \otimes MC_y(X(s_1 s_2 s_1)^\circ) &= e^{-\varpi_2} MC_y(X(s_1 s_2 s_1)^\circ) - (1+y)e^{-\varpi_1 + \varpi_2} MC_y(X(s_1 s_2)^\circ) \\ &\quad + (1+y)^2 e^{\varpi_1} MC_y(X(s_2)^\circ) + (1+y)ye^{\varpi_1} MC_y(X(id)^\circ) \end{aligned}$$

with the terms corresponding to the choice of $J = \{\}, \{1\}, \{1, 2\}, \{2\}$.

The next example shows that cancellations may occur in the formula for the coefficients $C_{u,\lambda}^w$.

Example 5.11. Consider type A_3 , $\lambda = \varpi_2$, $w = s_1 s_2 s_3 s_1 s_2 s_1$ and $u = s_3 s_1$. An ϖ_2 -chain is $\beta_1 = \alpha_2, \beta_2 = \alpha_2 + \alpha_3, \beta_3 = \alpha_1 + \alpha_2, \beta_4 = \alpha_1 + \alpha_2 + \alpha_3$; see Example 3.6. We have two paths from u to w in (32):

- $J_1 = \{2, 3\}$ which gives $u < us_{\beta_3} < us_{\beta_3} s_{\beta_2} = w$, where $\ell(us_{\beta_3}) + 3 = \ell(w)$;
- $J_2 = \{1, 2, 3, 4\}$ which gives $u < us_{\beta_4} < us_{\beta_4} s_{\beta_3} < us_{\beta_4} s_{\beta_3} s_{\beta_2} < us_{\beta_4} s_{\beta_3} s_{\beta_2} s_{\beta_1} = w$.

The path J_1 gives coefficient $(-1-y) \times (-1-y)(-y)e^{u(\lambda)} = -y(y+1)^2 e^{u(\lambda)}$, and the path J_2 gives coefficient $(-1-y)^4 e^{u(\lambda)} = (y+1)^4 e^{u(\lambda)}$. Therefore

$$C_{u,\lambda}^w = e^{u(\lambda)} ((y+1)^4 - y(y+1)^2) = e^{u(\lambda)} (y^2 + y + 1)(y+1)^2.$$

Conjecture 3. Consider the non-equivariant multiplication by the motivic Chern class of Schubert varieties:

$$(33) \quad MC_y(X(w)) \cdot \mathcal{L}_\lambda = \sum A_{u,\lambda}^w(y) MC_y(X(u)^\circ).$$

Assume that G is simply laced or $G = G_2$, and that \mathcal{L}_λ is ample, i.e., $\lambda = \sum a_i \varpi_i$ and $a_i < 0$ for each i . Then $A_{u,\lambda}^w(y)$ is a polynomial of degree $\ell(w) - \ell(u)$ in y , with no internal zeros, and with positive and log concave coefficients.

We note that if $y = 0$, the multiplication (33) specializes to the multiplication $\mathcal{O}_w \cdot \mathcal{L}_\lambda = \sum A_{u,\lambda}^w(0) \mathcal{I}_u$ where \mathcal{I}_u is the class of the ideal sheaf of the boundary of the Schubert variety. The coefficients are indeed positive by results of Brion, see e.g., [Bri05, Thm. 4.2.1(b)].

Example 5.12. We record the multiplications $\mathcal{L}_{-(\varpi_1 + \varpi_2)} \cdot MC_y(X(w))$ in $K(\text{Fl}(3))$:

$$\begin{aligned} w = id &: MC_y(X(id)^\circ) \\ s_1 &: MC_y(X(s_1)^\circ) + (y+2)MC_y(X(id)^\circ) \\ s_2 &: MC_y(X(s_2)^\circ) + (y+2)MC_y(X(id)^\circ) \\ s_2 s_1 &: MC(X(s_2 s_1)^\circ) + (2y+3)MC_y(X(s_1)^\circ) + (y+2)MC_y(X(s_2)^\circ) \\ &\quad + (2y^2 + 6y + 5)MC_y(X(id)^\circ) \\ s_1 s_2 &: MC(X(s_1 s_2)^\circ) + (y+2)MC_y(X(s_1)^\circ) + (2y+3)MC_y(X(s_2)^\circ) \\ &\quad + (2y^2 + 6y + 5)MC_y(X(id)^\circ) \\ s_1 s_2 s_1 &: MC(X(s_1 s_2 s_1)^\circ) + (y+2)(MC(X(s_1 s_2)^\circ) + MC(X(s_2 s_1)^\circ)) \\ &\quad + (y^2 + 5y + 5)(MC_y(X(s_1)^\circ) + MC_y(X(s_2)^\circ)) \\ &\quad + (y^3 + 5y^2 + 11y + 8)MC(X(id)^\circ). \end{aligned}$$

Example 5.13. Let $G = G_2$ and consider $MC_y(X(w_0)) = \lambda_y(T_{G_2/B}^*)$. Then

$$\begin{aligned} \mathcal{L}_{-(\varpi_1+\varpi_2)} \cdot MC_y(X(w_0)) &= MC_y(X(w_0)^\circ) \\ &+ (y+2)MC_y(X(s_2s_1s_2s_1s_2)^\circ) + (y+2)MC_y(X(s_1s_2s_1s_2s_1)^\circ) \\ &+ (y^2+5y+5)MC_y(X(s_1s_2s_1s_2)^\circ) + (y^2+7y+7)MC_y(X(s_2s_1s_2s_1)^\circ) \\ &+ (y^3+9y^2+23y+16)MC_y(X(s_2s_1s_2)^\circ) + (y^3+8y^2+21y+15)MC_y(X(s_1s_2s_1)^\circ) \\ &+ (y^4+8y^3+35y^2+63y+36)MC_y(X(s_2s_1)^\circ) \\ &+ (y^4+9y^3+33y^2+54y+30)MC_y(X(s_1s_2)^\circ) \\ &+ (y^5+8y^4+29y^3+72y^2+99y+50)MC_y(X(s_1)^\circ) \\ &+ (y^5+8y^4+35y^3+91y^2+119y+57)MC_y(X(s_2)^\circ) \\ &+ (y^6+8y^5+29y^4+69y^3+125y^2+141y+64)MC(X(id)^\circ) \end{aligned}$$

Remark 5.14. Examples also suggest that there is an equivariant version of Conjecture 3. In this case, the expansion $MC_y(X(w)) \cdot \mathcal{L}_\lambda = \sum A_{u,\lambda}^w(y) MC_y(X(u)^\circ)$ has coefficients $A_{u,\lambda}^w(y) \in K_T(pt)[y]$, i.e.,

$$A_{u,\lambda}^w(y) = \sum a_{u,\lambda}^{w,\mu,p} e^\mu y^p$$

with $a_{u,\lambda}^{w,\mu,p} \in \mathbb{Z}$. Then we observed that $a_{u,\lambda}^{w,\mu,p} \geq 0$. We checked this for $\text{Fl}(n)$, $n \leq 5$ and G_2 .

5.4. Operator formula. In this section, we reformulate the λ -chain Chevalley formula from Theorem 5.5 via operators generalizing to motivic Chern classes similar ones from [LP07].

Let $h := (\rho, \theta^\vee) + 1$ be the Coxeter number, where $\rho := \sum_{i=1}^r \varpi_i$ and θ^\vee is the highest coroot. Let $\tilde{R}(T) := \mathbb{Z}[e^{\pm\varpi_1/h}, \dots, e^{\pm\varpi_r/h}]$, and let

$$\tilde{K}_T(G/B) := K_T(G/B)[y] \otimes_{K_T(pt)[y]} \text{Frac}(\tilde{R}(T)[y]).$$

Then $\tilde{K}_T(G/B)$ has a basis over $\text{Frac}(\tilde{R}(T)[y])$ given by the motivic Chern classes of the Schubert cells. Define $\text{Frac}(\tilde{R}(T)[y])$ -linear operators B_β ($\beta \in R^+$) and E^μ ($\mu \in X^*(T)$) on $\tilde{K}_T(G/B)$ by

$$B_\beta(MC_y(X(w)^\circ)) := \begin{cases} (-1-y)(-y)^{\frac{\ell(w)-\ell(ws_\beta)-1}{2}} MC_y(X(ws_\beta)^\circ) & \text{if } ws_\beta < w \\ 0 & \text{otherwise} \end{cases},$$

$$E^\mu(MC_y(X(w)^\circ)) := e^{\frac{w(\mu)}{h}} MC_y(X(w)^\circ).$$

If $\beta \in R^-$, define $B_\beta := -B_{-\beta}$. Then

$$E^\mu E^{\mu'} = E^{\mu+\mu'}, \quad \text{and} \quad B_\beta E^{s_\beta \mu} = E^\mu B_\beta.$$

Given a λ -chain $(\beta_1, \dots, \beta_l)$, define

$$R^{[\lambda]} := R_{\beta_l} R_{\beta_{l-1}} \cdots R_{\beta_1}, \quad \text{where} \quad R_\beta := E^\rho (E^\beta + B_\beta) E^{-\rho} = E^\beta + E^{(\rho, \beta^\vee)\beta} B_\beta.$$

Then we have the following operator formula.

Theorem 5.15 (Operator Chevalley formula). *For any integral weight $\lambda \in X^*(T)$,*

$$(34) \quad \mathcal{L}_\lambda \otimes MC_y(X(w)^\circ) = R^{[\lambda]} (MC_y(X(w)^\circ)).$$

- Remark 5.16.* (1) The theorem implies that the definition of $R^{[\lambda]}$ does not depend on the choice of the λ -chain, which is equivalent to the Yang–Baxter equations ([LP07, Definition 9.1]) satisfied by the operators R_β .
- (2) The formula analogous to (34) involving SMC classes is obtained by replacing the operator $R^{[\lambda]}$ by an operator defined via the adjoint operators of B_β and E^μ .
- (3) Recall from Remark 4.7(1) that $MC_0(X(w)^\circ) = [\mathcal{O}_{X(w)}(-\partial X(w))]$. Specializing $y = 0$ in the Theorem, we get a dual version of Lenart–Postnikov’s formula [LP07, Theorem 13.1].

For the proof, we need to recall the following result from [LP07]. Recall $(\beta_1, \dots, \beta_l)$ is a λ -chain. The hyperplane $h_j := H_{-\beta_j, d_j}$ separates the alcoves A_{j-1} and A_j in the corresponding alcove path, and \hat{r}_{h_j} is the reflection along h_j .

Lemma 5.17. [LP07, Proof of Prop. 14.5] *For any $1 \leq j_1 < j_2 < \dots < j_t \leq l$,*

$$\begin{aligned} & -\rho + \beta_1 + \dots + \beta_{j_1-1} + s_{\beta_{j_1}}(\beta_{j_1+1} + \dots + \beta_{j_2-1}) + \dots + s_{\beta_{j_1}} \dots s_{\beta_{j_t}}(\beta_{j_t+1} + \dots + \beta_l + \rho) \\ & = -h\hat{r}_{j_1} \dots \hat{r}_{j_t}(-\lambda). \end{aligned}$$

Proof of Theorem 5.15. By definition,

$$\begin{aligned} R^{[\lambda]} &= R_{\beta_l} R_{\beta_{l-1}} \dots R_{\beta_1} \\ &= E^\rho (E^{\beta_l} + B_{\beta_l}) \dots (E^{\beta_2} + B_{\beta_2}) (E^{\beta_1} + B_{\beta_1}) E^{-\rho} \\ &= \sum_J R_J^{[\lambda]}. \end{aligned}$$

Here $J = \{j_1 < j_2 < \dots < j_t\}$ is a subset of $\{1, 2, \dots, l\}$, and $R_J^{[\lambda]}$ is the term that contains B_{β_j} if $j \in J$, and E^{β_j} , otherwise. Moving all the B -operators to the left, we get

$$R_J^{[\lambda]} = B_{\beta_{j_t}} \dots B_{\beta_{j_1}} E^{-h\hat{r}_{j_1} \dots \hat{r}_{j_t}(-\lambda)} = B_{\beta_{j_t}} \dots B_{\beta_{j_1}} E^{-h\hat{r}_J(-\lambda)}.$$

Here we have used the relation $B_\beta E^{s_\beta \mu} = E^\mu B_\beta$ and Lemma 5.17. Then the Theorem follows from this and Theorem 5.5. \square

5.5. Parabolic case. In this section, we extend the above Chevalley formula to the partial flag variety case G/P .

Theorem 5.18 (Chevalley formula for the G/P case). *Let $w \in W^P$ and $\lambda \in X^*(T)_P$. Then we have*

$$\mathcal{L}_\lambda \otimes MC_y(X(wW_P)^\circ) = \sum_{u \in W^P} \left(\sum_{v \in uW_P} (-y)^{\ell(v)-\ell(u)} C_{v,\lambda}^w \right) MC_y(X(uW_P)^\circ),$$

and

$$\mathcal{L}_\lambda \otimes SMC_y(Y(uW_P)^\circ) = \sum_{w \in W^P} \left(\sum_{v \in uW_P} (-y)^{\ell(v)-\ell(u)} C_{v,\lambda}^w \right) SMC_y(Y(wW_P)^\circ),$$

where $C_{u,\lambda}^v$ are the Chevalley coefficients for full flag G/B given explicitly Theorem 5.5.

Proof. The first equality follows from equations (25) and (27), while the second equality follows from the first one and the duality in Proposition 4.8(2). \square

Remark 5.19. In a paper in preparation we use this theorem to give a combinatorial Chevalley formula for minuscule flag varieties and a K -theoretic generalization of Nakada's colored hook formula. We also recently learned about work by Neil Fan, Peter Guo, Changjian Su, and Rui Xiong who proved Pieri formulae for the motivic Chern classes on Grassmannians.

6. DUALITIES OF CHEVALLEY COEFFICIENTS

Recall the expansion from Equation (27):

$$\mathcal{L}_\lambda \otimes MC_y(X(w)^\circ) = \sum_{u \leq w} C_{u,\lambda}^w MC_y(X(u)^\circ).$$

In this section we state and prove several duality properties of the Chevalley coefficients $C_{u,\lambda}^w$. All these dualities have a geometric origin (Serre duality, star duality, Dynkin automorphisms) and we name the Chevalley dualities correspondingly.

Recall the Grothendieck-Serre operator \mathcal{D} from (23), and the duality functor $(-)^\vee$ on $K_T(G/B)[y^{\pm 1}]$ and $K_T(\text{pt})[y^{\pm 1}]$, which sends a vector bundle to its dual and y^i to y^{-i} . This induces the following property of the Chevalley coefficients in Equation (27).

Proposition 6.1 (Serre duality).

$$C_{u,\lambda}^w = (-y)^{\ell(w)-\ell(u)} w_0(C_{w_0w, -\lambda}^{w_0u})^\vee.$$

Proof. Since

$$SMC_y(Y(w)^\circ) = (-y)^{\dim Y(w)} \frac{\mathcal{D}(MC_y(Y(w)^\circ))}{\lambda_y(T^*(G/B))},$$

Equation (28) can be written as

$$\mathcal{L}_\lambda \otimes \mathcal{D}(MC_y(Y(u)^\circ)) = \sum_{w \in W} C_{u,\lambda}^w (-y)^{\ell(u)-\ell(w)} \mathcal{D}(MC_y(Y(w)^\circ)).$$

Taking the dual \mathcal{D} on both sides, we have

$$\mathcal{L}_{-\lambda} \otimes MC_y(Y(u)^\circ) = \sum_{w \in W} (C_{u,\lambda}^w)^\vee (-y)^{\ell(w)-\ell(u)} MC_y(Y(w)^\circ).$$

Finally, applying the left w_0 action to the above identity, we get

$$\mathcal{L}_{-\lambda} \otimes MC_y(X(w_0u)^\circ) = \sum_{w \in W} w_0(C_{u,\lambda}^w)^\vee (-y)^{\ell(w)-\ell(u)} MC_y(X(w_0w)^\circ).$$

This finishes the proof of the theorem by comparing the above equation with Equation (27). \square

The star duality $*$ acts on $K_T(G/B)[y^{\pm 1}]$ and $K_T(\text{pt})[y^{\pm 1}]$ by sending a vector bundle to its dual, and leaving y^i unchanged. Consider the composition

$$\iota := w_0* : K_T(\text{pt})[y^{\pm 1}] \rightarrow K_T(\text{pt})[y^{\pm 1}].$$

Then we have the following result by combining Theorem 9.1(a) and Remark 4.7 of [AMSS22].

$$(35) \quad \mathbb{C}_{-\rho} \otimes \mathcal{L}_{-\rho} \otimes MC_y(X(w)^\circ) = (-1)^{\dim G/B - \ell(w)} \prod_{\alpha > 0} (1 + ye^{-\alpha}) * \left(SMC_y(X(w)^\circ) \right).$$

Proposition 6.2 (Star duality).

$$C_{u,\lambda}^w = (-1)^{\ell(w)-\ell(u)} \iota(C_{w_0w, -\lambda}^{w_0u}).$$

Proof. Applying the star duality functor $*$ to Equation (27), then use Equation (35) to get

$$\mathcal{L}_{-\lambda} \otimes SMC_y(X(w)^\circ) = \sum_u (-1)^{\ell(w)-\ell(u)} * (C_{u,\lambda}^w) SMC_y(X(u)^\circ).$$

Applying the left w_0 action to the above equation and comparing with Equation (28), we get the result. \square

The map sending $\alpha_i \mapsto -w_0(\alpha_i)$ for every simple root α_i induces an automorphism on the Dynkin diagram, hence also on the flag variety G/B . This automorphism maps $X(w)^\circ$ to $X(w_0ww_0)^\circ$, and it induces a ring automorphism ϕ on $K_T(G/B)$, which sends \mathcal{L}_λ to $\mathcal{L}_{-w_0\lambda}$, and twists the base ring $K_T(\text{pt})$ by the map ι above.

Proposition 6.3 (Dynkin duality).

$$C_{u,\lambda}^w = \iota(C_{w_0uw_0,-w_0\lambda}^{w_0ww_0}).$$

Proof. Applying the Dynkin automorphism ϕ to Equation (27), we obtain

$$\mathcal{L}_{-w_0\lambda} \otimes MC_y(X(w_0ww_0)^\circ) = \sum_{u \in W} \iota(C_{u,\lambda}^w) MC_y(X(w_0uw_0)^\circ).$$

Then the claim follows from the definition of the coefficients $C_{u,\lambda}^w$. \square

Combining the above dualities we obtain the following.

Proposition 6.4. (1) (*Serre duality + star duality*)

$$(C_{u,\lambda}^w)|_{y \rightarrow y^{-1}} \times y^{\ell(w)-\ell(u)} = C_{u,\lambda}^w.$$

(2) (*Serre duality + star duality + Dynkin duality*)

$$C_{u,\lambda}^w = (-1)^{\ell(w)-\ell(u)} C_{w_0w_0,\lambda}^{w_0ww_0}.$$

Proof. The first part follows directly from Proposition 6.1 and Proposition 6.2. By Proposition 6.1 and Proposition 6.3, we get

$$\iota(C_{w_0uw_0,-w_0\lambda}^{w_0ww_0}) = (-y)^{\ell(w)-\ell(u)} w_0(C_{w_0w,-\lambda}^{w_0u})^\vee.$$

Notice that $\iota = w_0*$. The above equation is equivalent to

$$C_{w_0uw_0,-w_0\lambda}^{w_0ww_0} = C_{w_0w,-\lambda}^{w_0u}|_{y \rightarrow y^{-1}} (-y)^{\ell(w)-\ell(u)} = (-1)^{\ell(w)-\ell(u)} C_{w_0w,-\lambda}^{w_0u},$$

where the second equality follows from part (1). This finishes the proof. \square

Using the λ -chain formula in Theorem 5.5, we can also give a direct combinatorial proof for the various duality identities in this section. The proofs are similar to [LP07, Theorem 8.6] and [LP07, Theorem 8.7].

Proposition 6.5. For any integral weight λ and $w, u \in W$,

- (a) $C_{u,\lambda}^w \in K_T(\text{pt})[y]$ and $(C_{u,\lambda}^w)|_{y \rightarrow y^{-1}} \times y^{\ell(w)-\ell(u)} = C_{u,\lambda}^w$ i.e., $C_{u,\lambda}^w$ is palindromic regarded as a polynomial in y ; (cf. Proposition 6.4 (1));
- (b) $C_{u,\lambda}^w = (-1)^{\ell(w)-\ell(u)} C_{w_0w_0,\lambda}^{w_0ww_0}$; (cf. Proposition 6.4 (2));
- (c) $C_{u,\lambda}^w = (-1)^{\ell(w)-\ell(u)} \iota \left(C_{w_0w,-\lambda}^{w_0u} \right)$; (cf. Proposition 6.2).
- (d) $C_{u,\lambda}^w = \iota \left(C_{w_0uw_0,-w_0\lambda}^{w_0ww_0} \right)$; (cf. Proposition 6.3).

Proof. Part (a) is straightforward from Equation (32); part (d) follows from (b) and (c). Parts (b) and (c) follow from known properties of λ -chain, i.e., if $(\beta_1, \dots, \beta_l)$ is a λ -chain, then $(w_0\beta_l, \dots, w_0\beta_1)$ is a $(w_0\lambda)$ -chain and $(-\beta_l, \dots, -\beta_1)$ is a $(-\lambda)$ -chain. \square

Remark 6.6. The Serre duality extends to any G/P . For $w, u \in W^P$ and $\lambda \in X^*(T)_P$ recall the expansion:

$$\mathcal{L}_\lambda \otimes MC_y(X(wW_P)^\circ) = \sum_{u \in W^P} C_{u,\lambda}^{w,P} MC_y(X(uW_P)^\circ).$$

Then Theorem 5.18 can be rewritten as $C_{u,\lambda}^{w,P} = \sum_{v \in uW_P} (-y)^{\ell(v)-\ell(u)} C_{v,\lambda}^w$. By the same argument as in the proof of Proposition 6.1, we have the Serre duality on parabolic Chevalley coefficients;

$$(36) \quad C_{u,\lambda}^{w,P} = (-y)^{\ell(w)-\ell(u)} w_0 (C_{\overline{w_0 w}, -\lambda}^{\overline{w_0 u}, P})^\vee,$$

where $\overline{w_0 w}$ and $\overline{w_0 u} \in W^P$ are minimal coset representatives of $w_0 w W_P$ and $w_0 u W_P$.

7. K-THEORETIC STABLE ENVELOPES FOR $T^*(G/B)$

In this section we apply the Chevalley formula for motivic classes to calculate the transformation of stable envelopes in $T^*(G/B)$ under the change of arbitrary alcoves. For alcoves adjacent to the fundamental alcove, a formula for this transformation was obtained in [SZZ20, Theorem 5.4], see also [KW23].

7.1. Definition of the stable envelopes. The stable envelopes were defined by Maulik and Okounkov in their seminal work on quantum cohomology of Nakajima quiver varieties [MO19]. Later, this was generalized by Okounkov and his collaborators to K -theory and elliptic cohomology [Ok17, AO21]. We recall next the definition of the stable envelopes for $T^*(G/B)$.

The torus T acts by left multiplication on G/B . Hence, it induces a natural action on $T^*(G/B)$. There is also a natural dilation \mathbb{C}^* action on the cotangent fibers by a character of q^{-1} . Throughout this section, we use $q^{1/2}$ to denote the standard representation of \mathbb{C}^* , so that $K_{T \times \mathbb{C}^*}(\text{pt}) = K_T(\text{pt})[q^{\pm 1/2}]$.

The definition of the stable envelopes depends on three parameters:

- a *chamber* \mathcal{C} in the Lie algebra of the maximal torus T .
- a *polarization* $T^{\frac{1}{2}} \in K_{T \times \mathbb{C}^*}(T^*(G/B))$ of the tangent bundle $T(T^*(G/B))$, i.e., a solution of the equation

$$T^{\frac{1}{2}} + q^{-1}(T^{\frac{1}{2}})^\vee = T(T^*(G/B))$$

in the ring $K_{T \times \mathbb{C}^*}(T^*(G/B))_{\text{loc}}$.

- an *alcove* A in $\mathfrak{t}_{\mathbb{R}}^*$, which is also called a *slope* in [Ok17].

Given a polarization $T^{\frac{1}{2}}$, there is an opposite polarization defined by $T_{\text{opp}}^{\frac{1}{2}} := q^{-1}(T^{\frac{1}{2}})^\vee$. A typical example of a polarization is $T(G/B)$ (pulled back from G/B to $T^*(G/B)$). Its opposite is equal to $T^*(G/B)$. Because we will work with fibers of polarizations over fixed points, in what follows we will assume that the polarization is given by a subbundle of $T(T^*(G/B))$, or possibly a virtual vector subbundle, i.e., a formal difference of such subbundles.

The torus fixed point set $(T^*(G/B))^T = (G/B)^T$ is in one-to-one correspondence with the Weyl group W . For every $w \in W$, recall that e_w denotes the corresponding fixed point. For a chosen Weyl chamber \mathfrak{C} in $\text{Lie } T$, pick any cocharacter $\sigma \in \mathfrak{C}$. The attracting set of the fixed point e_w , also called the Białynicki-Birula cell in the literature, is defined by

$$\text{Attr}_{\mathfrak{C}}(w) = \left\{ x \in T^*(G/B) \mid \lim_{z \rightarrow 0} \sigma(z) \cdot x = e_w \right\}.$$

By analyzing the (signs of the roots in the) weight space decomposition of $T_w(T^*(G/B))$, one may show that $\text{Attr}_{\mathfrak{C}}(w)$ is the conormal bundle of the attracting variety of w in G/B , i.e., the conormal bundle of the Schubert cell stable under the Borel subgroup associated to the chamber \mathfrak{C} .⁴ Define a partial order on the fixed point set W to be the (transitive closure of the) following relation:

$$e_w \preceq_{\mathfrak{C}} e_v \quad \text{if} \quad \overline{\text{Attr}_{\mathfrak{C}}(v)} \cap e_w \neq \emptyset.$$

Then the order determined by the positive (resp., negative) chamber is the same as the Bruhat order (resp., the opposite Bruhat order).

Any chamber \mathfrak{C} determines a decomposition of the tangent space $N_w := T_w(T^*(G/B))$ as $N_w = N_{w,+} \oplus N_{w,-}$ into T -weight spaces which are positive and negative with respect to \mathfrak{C} respectively. For every polarization $T^{\frac{1}{2}}$, denote $N_w \cap T^{1/2}|_w$ by $N_w^{\frac{1}{2}}$. Similarly, we have $N_{w,+}^{\frac{1}{2}}$ and $N_{w,-}^{\frac{1}{2}}$. In particular, $N_{w,-} = N_{w,-}^{\frac{1}{2}} \oplus q^{-1}(N_{w,+}^{\frac{1}{2}})^{\vee}$. Consequently, we have

$$N_{w,-} - N_w^{\frac{1}{2}} = q^{-1}(N_{w,+}^{\frac{1}{2}})^{\vee} - N_{w,+}^{\frac{1}{2}}$$

as virtual vector bundles. The determinant bundle of the virtual bundle $N_{w,-} - N_w^{\frac{1}{2}}$ is a complete square and its square root will be denoted by $\left(\frac{\det N_{w,-}}{\det N_w^{\frac{1}{2}}}\right)^{\frac{1}{2}}$; cf. [Oko17, §9.1.5]. For instance, if we choose the polarization $T^{1/2} = T(G/B)$, the positive chamber, and $w = \text{id}$ then both $N_{\text{id}}^{\frac{1}{2}}$ and $N_{\text{id},-}$ have weights $-\alpha$, where α varies in the set of positive roots; in this case the virtual bundle $N_{\text{id},-} - N_{\text{id}}^{\frac{1}{2}} = 0$.

Let $f := \sum_{\mu} f_{\mu} e^{\mu} \in K_{T \times \mathbb{C}^*}(\text{pt})$ be a Laurent polynomial, where $e^{\mu} \in K_T(\text{pt})$ and $f_{\mu} \in \mathbb{Q}[q^{1/2}, q^{-1/2}]$. The *Newton polytope* of f , denoted by $\deg_T f$, is

$$\deg_T f = \text{Convex hull}(\{\mu | f_{\mu} \neq 0\}) \subseteq X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $X^*(T)$ denotes the character lattice of T . The following theorem defines the K -theoretic stable envelopes.

Theorem 7.1. [Oko17, §9.1], [OS22, Thm. 1]. *For every chamber \mathfrak{C} , a polarization $T^{1/2}$, and an alcove A , there exists a unique map of $K_{T \times \mathbb{C}^*}(\text{pt})$ -modules*

$$\text{stab}_{\mathfrak{C}, T^{\frac{1}{2}}, A} : K_{T \times \mathbb{C}^*}((T^*(G/B))^T) \rightarrow K_{T \times \mathbb{C}^*}(T^*(G/B))$$

such that for every $w \in W$, the class $\Gamma := \text{stab}_{\mathfrak{C}, T^{\frac{1}{2}}, A}(w)$ satisfies:

- (1) (*Support*) $\text{Supp } \Gamma \subseteq \cup_{z \preceq_{\mathfrak{C}} w} \overline{\text{Attr}_{\mathfrak{C}}(z)}$;
- (2) (*Normalization*) $\Gamma|_w = (-1)^{\text{rk } N_{w,+}^{\frac{1}{2}}} \left(\frac{\det N_{w,-}}{\det N_w^{\frac{1}{2}}}\right)^{\frac{1}{2}} \mathcal{O}_{\text{Attr}_{\mathfrak{C}}(w)}|_w$;
- (3) (*Degree*) For every $e_v \prec_{\mathfrak{C}} e_w$,

$$\deg_T \Gamma|_v \subseteq \deg_T \text{stab}_{\mathfrak{C}, T^{\frac{1}{2}}, A}(v)|_v + v\lambda - w\lambda,$$

where $\lambda \in (X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}) \cap A$ is any rational weight in the alcove A .

⁴We note that $T^*(G/B)$ is not compact, so not all points have well defined limits at 0. For example, if \mathfrak{C} is the positive chamber, then the points in the open set $T^*(X(w_0)^{\circ}) \setminus X(w_0)^{\circ} \subset T^*(G/B)$ do not have limits at 0.

Strictly speaking, $\text{stab}_{\mathfrak{C}, T^{\frac{1}{2}}, A}(w) \in K_{T \times \mathbb{C}^*}(G/B)$ denotes the image of $1 \in K_{T \times \mathbb{C}^*}(e_w)$ under the map $\text{stab}_{\mathfrak{C}, T^{\frac{1}{2}}, A}$. From the definition, it is immediate to see that $\{\text{stab}_{\mathfrak{C}, T^{\frac{1}{2}}, A}(w) \mid w \in W\}$ forms a basis for the localized equivariant K-theory $K_{T \times \mathbb{C}^*}(T^*(G/B))_{\text{loc}}$, called the *stable basis*. Explicit combinatorial formulae and recursions for the localizations of stable envelopes in $T^*(G/B)$ may be found in [AMSS19, §8.3].

7.2. Changing the polarizations. A natural question is to study the change of the stable envelopes when we vary the above three parameters. To start, the change of chambers is encoded in the left Weyl group action. More precisely, the group G acts on $T^*(G/B)$ by left multiplication, which induces a left Weyl group action on $K_{T \times \mathbb{C}^*}(T^*(G/B))$, see [MNS22b]. If we change the chamber \mathfrak{C} to another chamber $w(\mathfrak{C})$ ($w \in W$), we have the following formula (see [AMSS19, Lemma 8.2(a)])

$$w \cdot (\text{stab}_{\mathfrak{C}, T^{\frac{1}{2}}, A}(u)) = \text{stab}_{w(\mathfrak{C}), w(T^{\frac{1}{2}}), A}(wu).$$

Next we consider the change of polarizations. In this case the results are stated e.g. in [Oko17] in the more general setting of symplectic resolutions; for the convenience of the reader we include proofs for $T^*(G/B)$. We have the following lemma relating two arbitrary polarizations see [Oko17, Section 7.5.8].

Lemma 7.2. *For any two polarizations $T_1^{\frac{1}{2}}$ and $T_2^{\frac{1}{2}}$, there exists a class $\mathcal{F} \in K_{T \times \mathbb{C}^*}(T^*(G/B))$ such that $T_1^{\frac{1}{2}} - T_2^{\frac{1}{2}} = \mathcal{F} - q^{-1}\mathcal{F}^\vee$.*

Remark 7.3. Classes of the form $\mathcal{F} - q^{-1}\mathcal{F}^\vee$ are called balanced classes in *loc. cit.*

Proof. It suffices to prove the following fact: any solution of the equation $\mathcal{G} + q^{-1}\mathcal{G}^\vee = 0 \in K_{T \times \mathbb{C}^*}(T^*(G/B))$ must be of the form $\mathcal{G} = \mathcal{F} - q^{-1}\mathcal{F}^\vee$ for some $\mathcal{F} \in K_{T \times \mathbb{C}^*}(T^*(G/B))$. Since $\mathcal{G} \in K_{T \times \mathbb{C}^*}(T^*(G/B)) \simeq K_T(G/B)[q, q^{-1}]$, we can write $\mathcal{G} = \mathcal{F} + q^{-1}\mathcal{F}'$ for some $\mathcal{F} \in K_T(G/B)[q]$ and $\mathcal{F}' \in q^{-1}K_T(G/B)[q^{-1}]$. Thus,

$$\mathcal{F} + q^{-1}\mathcal{F}' + q^{-1}\mathcal{F}^\vee + (\mathcal{F}')^\vee = 0.$$

Since $\mathcal{F}, (\mathcal{F}')^\vee \in K_T(G/B)[q]$ and $q^{-1}\mathcal{F}', q^{-1}\mathcal{F}^\vee \in q^{-1}K_T(G/B)[q^{-1}]$, we get

$$\mathcal{F} + (\mathcal{F}')^\vee = 0, \quad \text{and} \quad q^{-1}\mathcal{F}' + q^{-1}\mathcal{F}^\vee = 0.$$

Therefore, $\mathcal{F}' = -\mathcal{F}^\vee$, and $\mathcal{G} = \mathcal{F} - q^{-1}\mathcal{F}^\vee$. \square

For any two polarizations $T_1^{\frac{1}{2}}$ and $T_2^{\frac{1}{2}}$ let \mathcal{F} be defined by $T_1^{\frac{1}{2}} - T_2^{\frac{1}{2}} = \mathcal{F} - q^{-1}\mathcal{F}^\vee$ as in the above lemma. Define a $T \times \mathbb{C}^*$ -equivariant line bundle \mathcal{L} on $T^*(G/B)$ by the following formula

$$\mathcal{L} := \det \mathcal{F}.$$

Here for a virtual bundle $\mathcal{F} = \mathcal{V}_1 - \mathcal{V}_2$, $\det \mathcal{F} := \det \mathcal{V}_1 / \det \mathcal{V}_2$.

Example 7.4. Consider the polarization $T_1^{\frac{1}{2}} := T(G/B)$ and the opposite polarization $T_2^{\frac{1}{2}} = q^{-1}(T_1^{\frac{1}{2}})^\vee = T^*(G/B)$. Then the element \mathcal{F} in Lemma 7.2 can be taken to be $T(G/B)$, therefore $\mathcal{L} = \det \mathcal{F} = \mathcal{L}_{-2\rho}$.

The next proposition shows that the change of polarizations results in a multiplication by a line bundle, therefore it is encoded in the Chevalley formula for the stable envelopes.

Proposition 7.5. [Oko17, Exercise 9.1.12] *The following holds:*

$$\text{stab}_{\mathfrak{c}, T_2^{\frac{1}{2}}, A}(w) = (-1)^{\text{rk } N_{w,+}^{\frac{1}{2}} - \text{rk } N_{w,+}^{\frac{1}{2}}} q^{\frac{\text{rk } \mathcal{F}|_w}{2}} \mathcal{L} \otimes \text{stab}_{\mathfrak{c}, T_1^{\frac{1}{2}}, A}(w),$$

where $N_{w,+}^{\frac{1}{2}} := N_{w,+} \cap T_i^{\frac{1}{2}}$ for $i = 1, 2$, and if $\mathcal{F} = T_1^{1/2} - T_2^{1/2} = \mathcal{V}_1 - \mathcal{V}_2$ is the virtual bundle from Lemma 7.2, then $\text{rk } \mathcal{F}|_w := \text{rk } \mathcal{V}_1|_w - \text{rk } \mathcal{V}_2|_w$.

Proof. From the characterization Theorem 7.1, it suffices to show that the right hand side satisfies the defining properties of the stable envelope on the left hand side. The support condition is immediate. For the degree condition, we need to check that for every $e_v \prec_{\mathfrak{c}} e_w$,

$$\deg_T(\mathcal{L} \otimes \text{stab}_{\mathfrak{c}, T_1^{\frac{1}{2}}, A}(w))|_v \subseteq \deg_T(\mathcal{L} \otimes \text{stab}_{\mathfrak{c}, T_2^{\frac{1}{2}}, A}(w))|_v + v\lambda - w\lambda,$$

for some $\lambda \in A$. The terms $\mathcal{L}|_v$ on both sides cancel, and the above inclusion reduces to the degree condition for $\text{stab}_{\mathfrak{c}, T_1^{\frac{1}{2}}, A}(w)$. The normalization condition follows from the next calculation. We utilize our assumption that \mathcal{F} is represented by a subbundle of $T(T^*(G/B))$, and we note also that the weights of $N_w = T_w(T^*(G/B))$ are distinct.

$$\begin{aligned} & (-1)^{\text{rk } N_{w,+}^{\frac{1}{2}} - \text{rk } N_{w,+}^{\frac{1}{2}}} \frac{\text{stab}_{\mathfrak{c}, T_2^{\frac{1}{2}}, A}(w)|_w}{\text{stab}_{\mathfrak{c}, T_1^{\frac{1}{2}}, A}(w)|_w} \\ &= \left(\frac{\det N_{w,-}}{\det N_{w,2}^{\frac{1}{2}}} \right)^{\frac{1}{2}} \left(\frac{\det N_{w,-}}{\det N_{w,1}^{\frac{1}{2}}} \right)^{-\frac{1}{2}} \\ &= \left(\frac{\det N_{w,1}^{\frac{1}{2}}}{\det N_{w,2}^{\frac{1}{2}}} \right)^{\frac{1}{2}} \\ &= (\det N_w \cap (\mathcal{F}|_w - q^{-1} \mathcal{F}^\vee|_w))^{\frac{1}{2}} \\ &= \left(\frac{\det(N_w \cap \mathcal{F}|_w)}{\det(N_w \cap q^{-1}(\mathcal{F}|_w)^\vee)} \right)^{\frac{1}{2}} \\ &= q^{\frac{\text{rk } \mathcal{F}|_w}{2}} \mathcal{L}|_w. \end{aligned}$$

The reason for the last equality is as follows. We have that $N_w \cap \mathcal{F}|_w = \mathcal{F}|_w$, and suppose e^λ is a torus weight of it. Since N_w is a symplectic vector space, $q^{-1}e^{-\lambda}$ is a weight of N_w . Hence $q^{-1}e^{-\lambda}$ is also a weight of the intersection $N_w \cap q^{-1}(\mathcal{F}|_w)^\vee$ and in fact $N_w \cap q^{-1}(\mathcal{F}|_w)^\vee = q^{-1}(\mathcal{F}|_w)^\vee$, giving the equality. The case of $\mathcal{F} = \mathcal{V}_1 - \mathcal{V}_2$ being a virtual bundle follows from linearity of the constructions. \square

7.3. Changing the alcoves. We now turn to what happens under the change of alcoves. This can be answered utilizing the recursive formulae from [SZZ20, SZZ21], see Theorem 7.6 below. Our Chevalley formula provides a non-recursive answer, in terms of λ -chains, relating the stable envelope for the fundamental alcove A_\circ to the stable envelope for a translate $A_\circ + \lambda$; see Theorem 7.8 below. We recall some of the formulae below, and state our Chevalley based formula in Theorem 7.8.

The alcoves in $\mathfrak{t}_{\mathbb{R}}^*$ are of the form $x(A_{\circ}) + \lambda$ for some $x \in W$ and some λ in the root lattice. It was proved in [AMSS19, Lemma 8.2], [SZZ20, Rmk. 2.3] that

$$(37) \quad \text{stab}_{\mathcal{C}, T^{\frac{1}{2}}, A+\lambda}(w) = e^{-w\lambda} \mathcal{L}_{\lambda} \otimes \text{stab}_{\mathcal{C}, T^{\frac{1}{2}}, A}(w),$$

where \mathcal{L}_{λ} is the pullback of $G \times^B \mathbb{C}_{\lambda}$ from G/B to $T^*(G/B)$. Fix the chamber \mathcal{C} to be the anti-dominant Weyl chamber

$$\mathcal{C} := \{\lambda \in \mathfrak{t}_{\mathbb{R}}^* \mid \langle \lambda, \alpha^{\vee} \rangle < 0, \text{ for any positive root } \alpha\}$$

and the polarization $T^{\frac{1}{2}} = T^*(G/B)$. To simplify the notations, we denote $\text{stab}_{\mathcal{C}, T^*(G/B), A}(w)$ by $\text{stab}_A(w)$.

Let $Z = T^*(G/B) \times_{\mathcal{N}} T^*(G/B)$ be the Steinberg variety where $\mathcal{N} \subset \mathfrak{g}$ denotes the nilpotent cone. There is an algebra isomorphism due to Kazhdan–Lusztig and Ginzburg [KL87, CG09]

$$\mathbb{H} \simeq K_{G \times \mathbb{C}^*}(Z),$$

where $K_{G \times \mathbb{C}^*}(Z)$ has an algebra structure given by convolution. The convolution induces an action of the Hecke algebra \mathbb{H} on $K_{T \times \mathbb{C}^*}(T^*(G/B))$ which we recall next. For a simple root α_i , and the corresponding minimal parabolic subgroup $P_i \supset B$, define the operator T_i on $K_{T \times \mathbb{C}^*}(T^*(G/B))$ by the following formula:

$$T_i(\mathcal{F}) := -\mathcal{F} - \pi_{1*}(\pi_2^* \mathcal{F} \otimes \pi_2^* \mathcal{L}_{\alpha_i}).$$

Here $\mathcal{F} \in K_{A \times \mathbb{C}^*}(T^*(G/B))$, $Y_i := G/B \times_{G/P_i} G/B \subset G/B \times G/B$, $T_{Y_i}^*$ is the conormal bundle of Y_i inside $G/B \times G/B$, and $\pi_j : T_{Y_i}^* \rightarrow T^*(G/B)$ ($j = 1, 2$) are the two projections. These operators satisfy the quadratic relations and the braid relations in \mathbb{H} . In particular, T_w is well defined for any $w \in W$. Recall that A_{\circ} denotes the fundamental alcove.

The following have been proved in [SZZ20, SZZ21].

Theorem 7.6. (a) [SZZ20, Theorem 4.5]

$$T_i(\text{stab}_{A_{\circ}}(w)) = \begin{cases} (q-1) \text{stab}_{A_{\circ}}(w) + q^{1/2} \text{stab}_{A_{\circ}}(ws_i), & \text{if } ws_i < w, \\ q^{1/2} \text{stab}_{A_{\circ}}(ws_i), & \text{if } ws_i > w. \end{cases}$$

(b) [SZZ21, Theorem 5.4] *Let $x \in W$. Then*

$$\text{stab}_{x(A_{\circ})}(w) = q^{-\ell(x)/2} T_x(\text{stab}_{A_{\circ}}(wx)).$$

(c) [SZZ21, Lemma 3.5 and Corollary 5.3] *Assume A_1 and A_2 are two adjacent alcoves separated by a wall of the form $H_{\alpha, n}$, where $\alpha > 0$. Assume A_2 is on the positive side of $H_{\alpha, n}$, i.e., for any $\mu \in A_2$, $(\mu, \alpha^{\vee}) > n$. Then*

$$\text{stab}_{A_1}(w) = \begin{cases} \text{stab}_{A_2}(w) + e^{-nw\alpha}(q^{1/2} - q^{-1/2}) \text{stab}_{A_2}(ws_{\alpha}) & \text{if } ws_{\alpha} > w, \\ \text{stab}_{A_2}(w) & \text{if } ws_{\alpha} < w. \end{cases}$$

Part (a) of the theorem implies that $T_x(\text{stab}_{A_{\circ}}(wx))$ may be recursively calculated in terms of $\{\text{stab}_{A_{\circ}}(w) \mid w \in W\}$. Part (b) implies that the same is true for $\text{stab}_{x(A_{\circ})}(w)$. Finally, part (c) may be used to relate directly the stable bases for any two adjacent alcoves, and therefore recursively relate the stable bases for two arbitrary alcoves.

We focus next on relating (37) to our Chevalley formulae obtained earlier in this paper. Together with (a) and (b) from Theorem 7.6 above, this gives an alternative recursion to (c), calculating the stable envelope for an arbitrary alcove $x A_{\circ} + \lambda$ starting from the stable envelope for A_{\circ} .

Fix λ an integral weight. By Theorem 7.6(a), $\text{stab}_{x(A_o)}(w)$ may be written as a linear combination of $\{\text{stab}_{A_o}(w) \mid w \in W\}$. By (37), to determine $\text{stab}_{x(A_o)+\lambda}(w)$, it suffices to find a formula for $\mathcal{L}_\lambda \otimes \text{stab}_{A_o}(w)$. This can be achieved by the Chevalley formula for the motivic Chern classes. The key is the following result.

Lemma 7.7. [AMSS19, Theorem 8.6] *Let $\iota : G/B \hookrightarrow T^*(G/B)$ denote the inclusion of the zero section. For any $w \in W$,*

$$\iota^*(\text{stab}_{A_o}(w)) = (-1)^{\dim G/B} q^{\dim G/B - \frac{\ell(w)}{2}} MC_{-q^{-1}}(Y(w)^\circ) \otimes \mathcal{L}_{-2\rho}.$$

Recall the operator $(-)^\vee$ on $K_T(pt)[y^{\pm 1}]$, which sends e^μ to $e^{-\mu}$ and y to y^{-1} . We have the following Chevalley formula for the stable bases.

Theorem 7.8. *Let $\lambda \in X^*(T)$ be a weight and fix β_1, \dots, β_l a reduced λ -chain corresponding to an alcove walk from A_o to $A_o - \lambda$. Then*

$$\mathcal{L}_\lambda \otimes \text{stab}_{A_o}(u) = \sum_w q^{\frac{\ell(u) - \ell(w)}{2}} (C_{u, -\lambda}^w)^\vee|_{y=-q^{-1}} \text{stab}_{A_o}(w),$$

where $C_{u, \lambda}^w$ are the coefficients defined in Equation (27).

One can write this in terms of λ -chains as follows (with the notation from Theorem 5.5):

$$\mathcal{L}_\lambda \otimes \text{stab}_{A_o}(u) = \sum_{J \subset \{1, 2, \dots, l\}} (-1)^{n(J)} (q^{-1/2} - q^{1/2})^{|J|} e^{w_{\tilde{r}_J}(\lambda)} \text{stab}_{A_o}(ur_{h_{j_1}} r_{h_{j_2}} \cdots r_{h_{j_t}}),$$

where the sum is over subsets $J = \{j_1 < \dots < j_t\} \subset \{1, 2, \dots, l\}$ such that $u < ur_{h_{j_1}} < ur_{h_{j_1}} r_{h_{j_2}} < \dots < ur_{h_{j_1}} r_{h_{j_2}} \cdots r_{h_{j_t}}$.

Proof. By the definition of the SMC classes from definition 4.2(2), (28) becomes

$$\mathcal{L}_\lambda \otimes \mathcal{D}(MC_y(Y(u)^\circ)) = \sum_{w \geq u} (-y)^{\ell(u) - \ell(w)} C_{u, \lambda}^w \mathcal{D}(MC_y(Y(w)^\circ)).$$

Taking $(-)^\vee$ on both sides of the equation, we get

$$\mathcal{L}_{-\lambda} \otimes MC_y(Y(u)^\circ) = \sum_{w \geq u} (-y)^{\ell(w) - \ell(u)} (C_{u, \lambda}^w)^\vee MC_y(Y(w)^\circ).$$

Then the first equation of the theorem follows from this and Lemma 7.7. The second equation is a consequence of the first and of Equation (31) in Theorem 5.5. \square

Example 7.9. In type A_2 , set $u = s_2 s_1$ and $\lambda = 2\varpi_1 + \varpi_2$. A λ -chain of roots is given by $\beta_1 = \alpha_2$, $\beta_2 = \alpha_1 + \alpha_2$, $\beta_3 = \alpha_1$, $\beta_4 = \alpha_1 + \alpha_2$, $\beta_5 = \alpha_1$, $\beta_6 = \alpha_1 + \alpha_2$; see Appendix B. From Theorem 7.8, we have

$$\mathcal{L}_\lambda \otimes \text{stab}_{A_o}(s_2 s_1) = e^{\varpi_1 - 3\varpi_2} \text{stab}_{A_o}(s_2 s_1) + (q^{-\frac{1}{2}} - q^{\frac{1}{2}}) e^{-\varpi_1 - 2\varpi_2} \text{stab}_{A_o}(s_1 s_2 s_1).$$

Therefore by (37), using that $-u(\lambda) = -\varpi_1 + 3\varpi_2$, we have

$$\text{stab}_{A_o + \lambda}(s_2 s_1) = \text{stab}_{A_o}(s_2 s_1) + (q^{-\frac{1}{2}} - q^{\frac{1}{2}}) e^{-\alpha_1} \text{stab}_{A_o}(s_1 s_2 s_1).$$

Example 7.10. Consider $u = s_2$, $\lambda = 2\varpi_1 + \varpi_2$, $w_0 u = s_2 s_1$.

We can use Serre duality (Proposition 6.1) and Example 5.8 to get

$$\begin{aligned} \mathcal{L}_\lambda \otimes \text{stab}_{A_o}(s_2) &= e^{3\varpi_1 - \varpi_2} \text{stab}_{A_o}(s_2) \\ &\quad + (q^{-\frac{1}{2}} - q^{\frac{1}{2}}) (e^{\varpi_1} + e^{-\varpi_1 + \varpi_2} + e^{-3\varpi_1 + 2\varpi_2}) \text{stab}_{A_o}(s_1 s_2) \\ &\quad + (q^{-\frac{1}{2}} - q^{\frac{1}{2}}) (e^{2\varpi_1 - 2\varpi_2} + e^{\varpi_1 - 3\varpi_2}) \text{stab}_{A_o}(s_2 s_1) \\ &\quad + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 (e^{-\varpi_1 - 2\varpi_2} + e^{-2\varpi_1} + e^{-\varpi_2}) \text{stab}_{A_o}(s_1 s_2 s_1). \end{aligned}$$

As $-s_2(\lambda) = -3\varpi_1 + \varpi_2$, we have

$$\begin{aligned}
\text{stab}_{A_o + \lambda}(s_2) &= \text{stab}_{A_o}(s_2) \\
&\quad + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})(e^{-\alpha_1} + e^{-2\alpha_1} + e^{-3\alpha_1}) \text{stab}_{A_o}(s_1 s_2) \\
&\quad + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})(e^{-\alpha_1 - \alpha_2} + e^{-2\alpha_1 - 2\alpha_2}) \text{stab}_{A_o}(s_2 s_1) \\
&\quad + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 (e^{-3\alpha_1 - 2\alpha_2} + e^{-3\alpha_1 - \alpha_2} + e^{-2\alpha_1 - \alpha_2}) \text{stab}_{A_o}(s_1 s_2 s_1).
\end{aligned}$$

8. WHITTAKER FUNCTIONS AND HALL–LITTLEWOOD POLYNOMIALS

In this section we apply the Chevalley formula to obtain combinatorial expressions for Whittaker functions and Hall–Littlewood polynomials. Variants of the formulae we obtain were already available in the literature, and our approach based on the cohomological calculations adds a geometric perspective to this.

8.1. Whittaker functions. In this section we study Whittaker functions. These appear in p -adic representation theory, and in this note we utilize a cohomological construction of these functions from [AMSS19], see also [MSA22]. Recall the definition of the Demazure–Lusztig operators on $K_T(pt)[y]$:

$$\widetilde{T}_i(e^\lambda) = -e^\lambda \frac{1+y}{1-e^{-\alpha_i}} + e^{s_i \lambda} \frac{1+ye^{\alpha_i}}{1-e^{-\alpha_i}},$$

and

$$\widetilde{T}_i^\vee(e^\lambda) = -e^\lambda \frac{1+y}{1-e^{-\alpha_i}} + e^{s_i \lambda} \frac{1+ye^{-\alpha_i}}{1-e^{-\alpha_i}}.$$

The operators satisfy the usual quadratic and braid relations in the Hecke algebra, therefore for any $w \in W$ there are operators \widetilde{T}_w and \widetilde{T}_w^\vee acting on $K_T(pt)[y]$, and defined using any reduced decomposition of w . The following has been proved in [MSA22, Thm. 1.1]:

Proposition 8.1. *Let $\lambda_y(id) := \prod_{\alpha > 0} (1 + ye^\alpha)$, and denote by*

$$MC'_y(X(w)^\circ) := \lambda_y(id) \frac{MC_y(X(w)^\circ)}{\lambda_y(T_{G/B}^*)}.$$

Then for any $\lambda \in X^(T)$,*

- (1) $\chi_T(G/B, \mathcal{L}_\lambda \otimes MC_y(X(w)^\circ)) = \widetilde{T}_w^\vee(e^\lambda);$
- (2) $\chi_T(G/B, \mathcal{L}_\lambda \otimes MC'_y(X(w)^\circ)) = \widetilde{T}_w(e^\lambda).$

We note that for any anti-dominant weight λ and $w \in W$,

$$(38) \quad \chi_T(G/B, \mathcal{L}_\lambda \otimes MC'_y(X(w)^\circ)) = \widetilde{T}_w(e^\lambda) = \mathcal{W}_{\lambda, w},$$

where $\mathcal{W}_{\lambda, w}$ is the Iwahori–Whittaker function for the Langlands dual group over a non-archimedean local field; we refer to [MSA22] for more details, including the number theoretic definition of $\mathcal{W}_{\lambda, w}$. Here we identified y with $-q^{-1}$, where q is the number of elements in the residue field. As explained in [MSA22], from the fact that

$$\sum_{w \in W} MC'_y(X(w)^\circ) = 1$$

by the additivity motivic Chern classes, one recovers the Casselman–Shalika formula for the spherical Whittaker function [CS80]:

$$(39) \quad \sum_{w \in W} \mathcal{W}_{\lambda, w} = \prod_{\alpha > 0} (1 + ye^\alpha) \chi_T(G/B, \mathcal{L}_\lambda) = \prod_{\alpha > 0} (1 + ye^\alpha) \chi_{w_0 \lambda}.$$

Here $\chi_{w_0\lambda}$ denotes the character for the irreducible representation of G of highest weight $w_0(\lambda)$. We also note that, in type A, an interpretation of the Iwahori–Whittaker function in terms of the partition function of the Iwahori lattice model has been obtained in [BBBG19].

Using the Chevalley coefficients in Equation (27), we obtain the following formula for the Iwahori–Whittaker function $\mathcal{W}_{\lambda,w}$. Let ρ denote the half sum of the positive roots.

Theorem 8.2. *For any anti-dominant weight λ and $w \in W$,*

$$\mathcal{W}_{\lambda,w} = e^\rho \sum_u (-1)^{\ell(u)} C_{\lambda-\rho,u}^w|_{y \mapsto y^{-1}} y^{\ell(w)-\ell(u)}.$$

Proof. For any $u \in W$,

$$(40) \quad \chi_T(G/B, MC_y(X(u)^\circ)) = MC_y[X(u)^\circ \rightarrow \text{pt}] = MC_y[\mathbb{A}^1 \rightarrow \text{pt}]^{\ell(u)} = (-y)^{\ell(u)},$$

where the second equality follows from [AMSS19, Theorem 4.2(3)]. Therefore, taking the equivariant Euler characteristics of both sides of Equation (27), we get

$$\widetilde{T}_w^\vee(e^\lambda) = \chi_T(G/B, \mathcal{L}_\lambda \otimes MC_y(X(w)^\circ)) = \sum_u C_{\lambda,u}^w (-y)^{\ell(u)}.$$

On the other hand, it is immediate to check the following relation between the two Demazure–Lusztig operators:

$$(41) \quad \widetilde{T}_w = e^\rho \widetilde{T}_w^\vee|_{y \mapsto y^{-1}} e^{-\rho} y^{\ell(w)}.$$

Hence,

$$\chi_T(G/B, \mathcal{L}_\lambda \otimes MC'_y(X(w)^\circ)) = \widetilde{T}_w(e^\lambda) = e^\rho \sum_u (-1)^{\ell(u)} C_{\lambda-\rho,u}^w|_{y \mapsto y^{-1}} y^{\ell(w)-\ell(u)}.$$

The proof ends by applying Proposition 8.1. \square

Remark 8.3. Notice that $C_{0,u}^w = \delta_{u,w}$. The above proof shows

$$\chi_T(G/B, \mathcal{L}_\rho \otimes MC'_y(X(w)^\circ)) = (-1)^{\ell(w)} e^\rho.$$

As a corollary we prove a variant of the Casselman–Shalika formula (39), obtained by Li [Li92], see also [BBBG19, Proposition 9.4]. First define

$$(42) \quad R_\lambda(y) := \chi_T(G/B, \lambda_y(T_{G/B}^*) \otimes \mathcal{L}_\lambda) \in K_T(pt)[y].$$

Corollary 8.4. *Let λ be an anti-dominant integral weight. Then*

$$\sum_w y^{-\ell(w)} \mathcal{W}_{\lambda,w} = e^\rho R_{\lambda-\rho}(y^{-1}).$$

Proof. By the additivity of the motivic Chern classes and Proposition 8.1(1),

$$(43) \quad R_\lambda(y) = \sum \chi_T(G/B, \mathcal{L}_\lambda \otimes MC_y(X(w)^\circ)) = \sum_w \widetilde{T}_w^\vee(e^\lambda).$$

On the other hand, for an anti-dominant weight λ , we have

$$\begin{aligned} \sum_w y^{-\ell(w)} \mathcal{W}_{\lambda,w} &= \sum_w y^{-\ell(w)} \widetilde{T}_w(e^\lambda) \\ &= e^\rho \sum_w y^{-\ell(w)} e^{-\rho} \widetilde{T}_w^\vee(e^{\lambda-\rho}) \\ &= e^\rho \sum_w \widetilde{T}_w^\vee|_{y \mapsto y^{-1}}(e^{\lambda-\rho}) \\ &= e^\rho R_{\lambda-\rho}(y^{-1}). \end{aligned}$$

Here, the first equality follows from Proposition 8.1(2), the third one follows from Equation (41), and the last one follows from Equation (43). \square

8.2. Hall–Littlewood polynomials. In this section, we assume either λ or $-\lambda$ to be a dominant integral weight. Set $\Sigma_\lambda := \{\alpha \in \Sigma \mid \langle \lambda, \alpha^\vee \rangle = 0\}$, and let R_λ^+ be the set of positive roots which are linear combinations of the simple roots in Σ_λ . Denote by $W_\lambda \subset W$ the subgroup generated by the simple reflections s_α , where $\alpha \in \Sigma_\lambda$. Let W^λ be the set of minimal length representatives for the cosets W/W_λ . Finally, let P_λ be the parabolic subgroup containing the Borel subgroup B defined by the condition that $W_{P_\lambda} = W_\lambda$.

Definition 8.5. (1) Define $H_\lambda(y) := \chi_T(G/P_\lambda, \lambda_y(T_{G/P_\lambda}^*) \otimes \mathcal{L}_\lambda) \in K_T(pt)[y]$.
 (2) (Hall–Littlewood polynomial cf. [Mac98, P.208 (2.2)]) For a dominant weight λ , define

$$HL_\lambda(\mathbf{x}; t) := \sum_{w \in W^\lambda} w \left(\mathbf{x}^\lambda \prod_{\alpha \in R^+ \setminus R_\lambda^+} \frac{1 - t\mathbf{x}^{-\alpha}}{1 - \mathbf{x}^{-\alpha}} \right)$$

where \mathbf{x}^λ denotes e^λ .

Let $\pi_\lambda : G/B \rightarrow G/P_\lambda$ be the natural projection. Then

$$\lambda_y(G/B) = \pi_\lambda^*(\lambda_y(G/P_\lambda)) \cdot \lambda_y(P_\lambda/B).$$

By the projection formula, and using that $\pi_\lambda^*(\mathcal{L}_\lambda) = \mathcal{L}_\lambda$, we have that

$$H_\lambda(y) = \chi_T(G/P_\lambda, \lambda_y(T_{G/P_\lambda}^*) \otimes \mathcal{L}_\lambda) = \frac{\chi_T(G/B, \lambda_y(T_{G/B}^*) \otimes \mathcal{L}_\lambda)}{\chi_T(P_\lambda/B, \lambda_y(T_{P_\lambda/B}^*))} = \frac{R_\lambda(y)}{\sum_{w \in W_\lambda} (-y)^{\ell(w)}}$$

Here the last equality follows from Equation (42), and the fact that

$$\chi_T(P_\lambda/B, \lambda_y(T_{P_\lambda/B}^*)) = \sum_{w \in W_\lambda} \chi(MC_y(X(w)^\circ)) = \sum_{w \in W_\lambda} (-y)^{\ell(w)}.$$

The relation between H_λ and the Hall–Littlewood polynomial, summarized next, was obtained in a related (upcoming) collaboration with B. Ion.

Lemma 8.6. *We have the following formulae for $H_\lambda(y)$:*

$$(44) \quad H_\lambda(y) = \sum_{w \in W^P} \sum_{u \in W} C_{u, \lambda}^w (-y)^{\ell(u)},$$

and

$$(45) \quad H_\lambda(y) = \sum_{w \in W^\lambda} w \left(e^\lambda \prod_{\alpha \in R^+ \setminus R_\lambda^+} \frac{1 + ye^\alpha}{1 - e^\alpha} \right).$$

Proof. Equation (44) follows because $\lambda_y(T_{G/P}^*) = MC_y(G/P) = \sum_{w \in W^P} MC_y(X(wW_P)^\circ)$,

Theorem 5.18, and $\chi_T(MC_y(X(uW_P)^\circ)) = (-y)^{\ell(u)}$ for any $u \in W^P$. Equation (45) follows from the localization formula (cf. [Nie74], [MSA22, Theorem 2.1 (c)]). To be more specific, the torus fixed points in G/P_λ are $\{wP_\lambda \mid w \in W^P\}$, and the torus weights of the tangent space at the fixed point wP_λ are $\{-w\alpha \mid \alpha \in R^+ \setminus R_\lambda^+\}$. \square

Corollary 8.7. *For a dominant integral λ , the Hall–Littlewood polynomial $HL_\lambda(x; t)$ can be expressed using $H_{-\lambda}(y)$ or $H_\lambda(y)$ as follows:*

$$(46) \quad HL_\lambda(\mathbf{x}; t) = H_{-\lambda}(y)|_{e^\alpha \mapsto \mathbf{x}^{-\alpha}, y \mapsto -t},$$

and

$$(47) \quad HL_\lambda(\mathbf{x}; t) = \left(\frac{1}{(-y)^{\dim G/P_\lambda}} H_\lambda(y) \right) |_{e^\alpha \mapsto \mathbf{x}^\alpha, y \mapsto -t^{-1}}.$$

We assume next that λ is a dominant integral weight. Fix a reduced $(-\lambda)$ -chain $\Gamma = (-\beta_1, -\beta_2, \dots, -\beta_l)$ and the sequence of hyperplanes $H_{\beta_1, d_1}, H_{\beta_2, d_2}, \dots, H_{\beta_l, d_l}$. This corresponds to an alcove path from A_\circ to $A_\circ + \lambda$. Since λ is dominant, $\beta_j > 0, d_j > 0$.

Then we can recover the following known formula for the Hall–Littlewood polynomial, see [Len11, Theorem 2.7].

Proposition 8.8. [Sch06, Ram06, Len11]

$$(48) \quad HL_\lambda(\mathbf{x}; t) = \sum_{(w, J, u) \in \mathcal{A}(\Gamma)} t^{\frac{\ell(w) + \ell(u) - |J|}{2}} (1-t)^{|J|} \mathbf{x}^{w\hat{r}_{J <}(\lambda)},$$

where (with notation as in §3.2),

$$\mathcal{A}(\Gamma) = \{(w, J, u) \mid w \in W^P, u \in W, J \subset \{1, 2, \dots, l\}, u \xrightarrow{J >} w\}.$$

Proof. This is a direct consequence of Equation (46), Equation (44), and Equation (32) from Theorem 5.5, applied to the $(-\lambda)$ -chain $(-\beta_1, -\beta_2, \dots, -\beta_l)$. \square

We also get a new formula for $HL_\lambda(\mathbf{x}; t)$ as follows.

Proposition 8.9.

$$(49) \quad HL_\lambda(\mathbf{x}; t) = \sum_{(u, J, w) \in \mathcal{A}^{op}(\Gamma)} t^{\frac{2 \dim G/P_\lambda - \ell(w) - \ell(u) - |J|}{2}} (1-t)^{|J|} \mathbf{x}^{u\hat{r}_{J <}(\lambda)},$$

where $\mathcal{A}^{op}(\Gamma) = \{(u, J, w) \mid w \in W^P, u \in W, J \subset \{1, 2, \dots, l\}, u \xrightarrow{J <} w\}$.

Proof. This is a direct consequence of Equation (47), Equation (44), and Equation (31) from Theorem 5.5, applied to the $(-\lambda)$ -chain together with Remark 5.6 (1). \square

Remark 8.10. When $P_\lambda = B$, the equations (48) and (49) give the same formula. The correspondence may be seen using the Serre duality in Equation (36). However, the formulae are in general different, as shown by the examples below.

Example 8.11. (Type A_2) Let $G = GL_3(\mathbb{C})$, $T = (\mathbb{C}^*)^3$, and $x_i = e^{\varepsilon_i}$, for $i = 1, 2, 3$. Let $\lambda = \varpi_1 = \varepsilon_1$, then $W_\lambda = \langle s_2 \rangle \subset W = \langle s_1, s_2 \rangle$, and $W^\lambda = \{id, s_1, s_2 s_1\}$. Fix a reduced $(-\lambda)$ -chain is $(-\beta_1 = -\alpha_1 - \alpha_2, -\beta_2 = -\alpha_1)$.

Proposition 8.8 sums over the following seven terms:

w	J	u	
id	\emptyset	id	x_1
s_1	\emptyset	s_1	tx_2
	$\{2\}$	id	$(1-t)x_2$
$s_2 s_1$	\emptyset	$s_2 s_1$	$t^2 x_3$
	$\{1\}$	s_1	$t(1-t)x_3$
	$\{2\}$	s_2	$t(1-t)x_3$
	$\{1, 2\}$	id	$(1-t)^2 x_3$

Hence, with $\mathbf{x} = (x_1, x_2, x_3)$,

$$HL_\lambda(\mathbf{x}; t) = (x_1) + (tx_2 + (1-t)x_2) + (t^2 x_3 + t(1-t)x_3 + t(1-t)x_3 + (1-t)^2 x_3) = x_1 + x_2 + x_3.$$

On the other hand, Proposition 8.9 sums over the following six terms:

w	J	u	
id	\emptyset	id	$t^2 x_1$
s_1	\emptyset	s_1	tx_2
	$\{2\}$	id	$t(1-t)x_1$

$s_2 s_1$	\emptyset	$s_2 s_1$	x_3
	$\{1\}$	s_1	$(1-t)x_2$
	$\{2\}$	s_2	$(1-t)x_1$

Hence,

$$HL_\lambda(\mathbf{x}; t) = (t^2 x_1) + (tx_2 + t(1-t)x_1) + (x_3 + (1-t)x_2 + (1-t)x_1) = x_1 + x_2 + x_3.$$

Example 8.12. Same setup as in the above example, but with $\lambda = 2\varpi_2$. Then $W_\lambda = \langle s_1 \rangle$, $W^\lambda = \{id, s_2, s_1 s_2\}$, and $d = \dim G/P_\lambda = 2$. A $(-\lambda)$ -chain is $(-\beta_1 = -(\alpha_1 + \alpha_2), -\beta_2 = -\alpha_2, -\beta_3 = -(\alpha_1 + \alpha_2), -\beta_4 = -\alpha_2)$, and $d_1 = 1, d_2 = 1, d_3 = 2, d_4 = 2$.

Proposition 8.8 sums over the following twelve terms:

w	J	u		$w = s_1 s_2$	\emptyset	$s_1 s_2$	$t^2 x_2^2 x_3^2$
id	\emptyset	id	$x_1^2 x_2^2$		$\{1\}$	s_2	$t(1-t)x_1 x_2 x_3^2$
					$\{2\}$	s_1	$t(1-t)x_1 x_2^2 x_3$
s_2	\emptyset	s_1	$tx_1^2 x_3^2$		$\{3\}$	s_2	$t(1-t)x_2^2 x_3^2$
	$\{2\}$	id	$(1-t)x_1^2 x_2 x_3$		$\{4\}$	s_1	$t(1-t)x_2^2 x_3^2$
	$\{4\}$	id	$(1-t)x_1^2 x_3^2$		$\{1, 2\}$	id	$(1-t)^2 x_1 x_2^2 x_3$
					$\{1, 4\}$	id	$(1-t)^2 x_1 x_2 x_3^2$
					$\{3, 4\}$	id	$(1-t)^2 x_2^2 x_3^2$

We get $HL_\lambda(\mathbf{x}; t) = s_{22} - ts_{211}$, where

$$s_{22} = x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2,$$

and

$$s_{211} = x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2.$$

The Proposition 8.9 sums over the following ten terms:

w	J	u		$s_1 s_2$	\emptyset	$s_1 s_2$	$x_2^2 x_3^2$
id	\emptyset	id	$t^2 x_1^2 x_2^2$		$\{1\}$	s_2	$(1-t)x_1 x_2 x_3^2$
					$\{2\}$	s_1	$(1-t)x_1 x_2^2 x_3$
s_2	\emptyset	s_2	$tx_1^2 x_3^2$		$\{3\}$	s_2	$(1-t)x_2^2 x_3^2$
	$\{2\}$	id	$t(1-t)x_1^2 x_2 x_3$		$\{4\}$	s_1	$(1-t)x_2^2 x_3^2$
	$\{4\}$	id	$t(1-t)x_1^2 x_2^2$		$\{2, 3\}$	id	$(1-t)^2 x_1^2 x_2 x_3$

The summation also gives $HL_\lambda(\mathbf{x}; t) = (x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2) + (1-t)(x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2) = s_{22} - ts_{211}$.

APPENDIX A. CHEVALLEY FORMULAE FOR THE CHERN–SCHWARTZ–MACPHERSON CLASSES OF SCHUBERT CELLS

In this appendix, we give a short proof of the Chevalley formulae for the equivariant Chern–Schwartz–MacPherson (CSM) and Segre–MacPherson (SM) classes of the Schubert cells in the (partial) flag varieties, see [AMSS17, Thm 9.10] and Theorem A.4 below. Our proof again relies on the action of the appropriate Hecke algebra, this time on the equivariant cohomology of G/P .

A.1. Degenerate affine Hecke algebra.

A.1.1. A change of bases formula. Recall that the degenerate affine Hecke algebra \mathcal{H} is generated by $T_w, w \in W$ and $x_\lambda, \lambda \in X^*(T)$, such that

- $T_w T_u = T_{wu}$ for any $w, u \in W$;
- $x_\lambda x_\mu = x_\mu x_\lambda$ for any $\lambda, \mu \in X^*(T)$;

- $x_{\lambda+\mu} = x_\lambda + x_\mu$ for any $\lambda, \mu \in X^*(T)$;
- for any simple root α_i , $T_{s_i}x_\lambda - x_{s_i\lambda}T_{s_i} = -\langle \lambda, \alpha_i^\vee \rangle$.

We have the following commutation relation.

Lemma A.1. *For any $w \in W$ and $\lambda \in X^*(T)$, the following holds in \mathcal{H}*

$$T_w x_\lambda = x_{w\lambda} T_w - \sum_{\alpha > 0, ws_\alpha < w} \langle \lambda, \alpha^\vee \rangle T_{ws_\alpha}.$$

Proof. We utilize induction on $\ell(w)$. The claim is clear when $\ell(w) = 0$ or 1 . Now assume $\ell(w) > 1$, and that the claim holds for any Weyl group element with length smaller than $\ell(w)$. Pick a simple root α_i , such that $w > ws_i$. By induction,

$$T_{ws_i} x_{s_i\lambda} = x_{ws_i\lambda} T_{ws_i} - \sum_{\alpha > 0, ws_i s_\alpha < ws_i} \langle s_i\lambda, \alpha^\vee \rangle T_{ws_i s_\alpha}.$$

Using the commutation of T_{s_i} and x_λ and the induction hypothesis we obtain:

$$\begin{aligned} T_w x_\lambda - x_{w\lambda} T_w &= T_{ws_i} x_{s_i\lambda} T_{s_i} - \langle \lambda, \alpha_i^\vee \rangle T_{ws_i} - x_{w\lambda} T_w \\ &= - \sum_{\alpha > 0, ws_i s_\alpha < ws_i} \langle \lambda, s_i \alpha^\vee \rangle T_{ws_i s_\alpha} T_{s_i} - \langle \lambda, \alpha_i^\vee \rangle T_{ws_i} \\ &= - \sum_{\alpha > 0, ws_\alpha < w} \langle \lambda, \alpha^\vee \rangle T_{ws_\alpha}. \end{aligned}$$

Here in the last equality, we have used the facts that if $ws_i < w$ then

$$\begin{aligned} \{\alpha > 0 \mid ws_\alpha < w\} &= \{\alpha > 0 \mid w\alpha < 0\} \\ &= \{\alpha_i\} \sqcup \{s_i\alpha \mid \alpha > 0, ws_i s_\alpha < ws_i\}, \end{aligned}$$

and $T_{ws_i s_\alpha} T_{s_i} = T_{ws_i s_\alpha s_i} = T_{ws_{s_i(\alpha)}}$. □

A.1.2. The Hecke action on the equivariant cohomology. Since G acts on G/P by left multiplication, there is a natural left Weyl group action on $H_T^*(G/P)$ for any partial flag variety G/P and which acts on the base ring $H_T^*(\text{pt})$ by the usual Weyl group action; see e.g. [MNS22b]. For any $w \in W$, we use w^L to denote this action.

For any simple root α_i , define the left Demazure–Lusztig operator on $H_T^*(G/P)$ by the following formula (see [MNS22b, Section 3.2]):

$$\mathcal{T}_i^L := \frac{\alpha_i + 1}{\alpha_i} s_i^L - \frac{1}{\alpha_i}.$$

Then it is proved in *loc. cit.* that these operators satisfy the braid relations and $(\mathcal{T}_i^L)^2 = \text{id}$. Moreover, it is immediate to check the following lemma.

Lemma A.2. *There is an action Ψ of the degenerate affine Hecke algebra \mathcal{H} on $H_T^*(G/P)$, sending T_i to \mathcal{T}_i^L and x_λ to $\lambda \in H_T^*(\text{pt})$.*

A.2. Definition of the CSM/SM classes. Next we recall the basic definitions and properties of CSM classes for the Schubert cells in G/P , we will be brief. We refer the reader e.g. to [Ohm06, AMSS17] for details, including a construction of these classes in the equivariant setting and for general varieties.

The (additive) group of constructible functions $\mathcal{F}(X)$ consists of functions $\varphi = \sum_Z c_Z \mathbb{1}_Z$, where the sum is over a finite set of constructible subsets $W \subset X$, $c_Z \in \mathbb{Z}$ are integers, and $\mathbb{1}_Z$ is the characteristic function of Z . For a proper morphism $f : Y \rightarrow X$, there is a linear map $f_* : \mathcal{F}(Y) \rightarrow \mathcal{F}(X)$, such that for any constructible subset $Z \subset Y$, $f_*(\mathbb{1}_Z)(x) =$

$\chi_{top}(f^{-1}(x) \cap Z)$, where $x \in X$ and χ_{top} denotes the topological Euler characteristic. A conjecture attributed to Deligne and Grothendieck states that there is a unique natural transformation $c_* : \mathcal{F} \rightarrow H_*$ from the functor of constructible functions on a complex algebraic variety X to the homology functor, where all morphisms are proper, such that if X is smooth then $c_*(\mathbb{1}_X) = c(T_X) \cap [X]$. This conjecture was proved by MacPherson [Mac74]; the class $c_*(\mathbb{1}_X)$ for possibly singular X was shown to coincide with a class defined earlier by M.-H. Schwartz [Sch65a, Sch65b, BS81].

There is an equivariant version of MacPherson's transformation defined by Ohmoto [Ohm06]. In this case one starts with a variety X with a T -action, and the equivariant version $\mathcal{F}^T(X)$ of the group of constructible functions $\mathcal{F}(X)$ contains the characteristic functions $\mathbb{1}_Z$ for Z stable under the T -action. If $f : X \rightarrow Y$ is a proper T -equivariant morphism of algebraic varieties the induced homomorphism and $Z \subset X$ is constructible and T -stable then one defines $f_*^T : \mathcal{F}^T(X) \rightarrow \mathcal{F}^T(Y)$ with the property that $f_*^T(\mathbb{1}_Z) = f_*(\mathbb{1}_Z)$. Ohmoto proves [Ohm06, Theorem 1.1] that there is an equivariant version of MacPherson transformation $c_*^T : \mathcal{F}^T(X) \rightarrow H_*^T(X)$ that satisfies $c_*^T(\mathbb{1}_X) = c^T(T_X) \cap [X]_T$ if X is a non-singular T -variety, which and is functorial with respect to proper push-forwards. The last statement means that for all proper T -equivariant morphisms $Y \rightarrow X$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}^T(Y) & \xrightarrow{c_*^T} & H_*^T(Y) \\ f_*^T \downarrow & & \downarrow f_*^T \\ \mathcal{F}^T(X) & \xrightarrow{c_*^T} & H_*^T(X). \end{array}$$

Definition A.3. Let Z be a T -invariant constructible subvariety of X .

- (1) We denote by $c_{SM}(Z) := c_*^T(\mathbb{1}_Z) \in H_*^T(X)$ the *equivariant Chern–Schwartz–MacPherson (CSM) class* of Z .
- (2) If X is smooth, we denote by $s_M(Z \subset X) := \frac{c_*^T(\mathbb{1}_Z)}{c(T_X)} \in H_*^T(X)_{loc}$ the *equivariant Segre–MacPherson (SM) class* of Z , where $H_*^T(X)_{loc} := H_*^T(X) \otimes_{H_*^T(pt)} \text{Frac } H_*^T(pt)$ denotes the localization of $H_*^T(X)$, and $\text{Frac } H_*^T(pt)$ is the fraction field of $H_*^T(pt)$.

A.3. The Chevalley formula in cohomology. We now specialize to $X = G/P$ with the usual T -action. For simplicity we will denote by $s_M(Z \subset G/P)$ simply by $s_M(Z)$. We will identify the equivariant (Borel–Moore) homology group $H_*^T(G/P)$ with the equivariant cohomology $H_T^*(G/P)$, using the Poincaré duality. The sets of CSM classes of Schubert cells $\{c_{SM}(X(wW_P)^\circ) \mid w \in W^P\}$ and of the SM classes $\{s_M(X(wW_P)^\circ) \mid w \in W^P\}$ form bases for $H_T^*(G/P)_{loc} := H_T^*(G/P) \otimes_{H_T^*(pt)} \text{Frac } H_T^*(pt)$. Moreover, if one takes the opposite Schubert cells in any of these sets, then the two bases are dual under the usual intersection pairing, see [AMSS17, Theorem 9.4]:

$$(50) \quad \langle c_{SM}(X(wW_P)^\circ), s_M(Y(uW_P)^\circ) \rangle_{G/P} = \delta_{w,u} \text{ for any } w, u \in W^P.$$

The left Demazure–Lusztig operator acts on the CSM classes by the following formula (see [MNS22b, Theorem 4.3])

$$\mathcal{T}_i^L(c_{SM}(X(wW_P)^\circ)) = c_{SM}(X(s_i w W_P)^\circ).$$

Hence, for any $w \in W$,

$$(51) \quad c_{SM}(X(wW_P)^\circ) = \mathcal{T}_w^L([X(\text{id})]).$$

Recall for any $\lambda \in X^*(T)_P$, $\mathcal{L}_\lambda := G \times^P \mathbb{C}_\lambda \in \text{Pic}_T(G/P)$. The following is our main result in this Appendix, and it has also been proved in [AMSS17, Thm. 9.10] using the Chevalley formula for the cohomological stable envelopes from [Su16, Thm. 3.7]. Here we give a direct proof based on the action of the degenerate affine Hecke algebra.

Theorem A.4. *For any $w \in W^P$ and $\lambda \in X^*(T)_P$, the following holds in $H_T^*(G/P)$:*

$$c_1^T(\mathcal{L}_\lambda) \cup c_{\text{SM}}(X(wW_P)^\circ) = w(\lambda)c_{\text{SM}}(X(wW_P)^\circ) - \sum_{\alpha > 0, ws_\alpha < w} \langle \lambda, \alpha^\vee \rangle c_{\text{SM}}(X(ws_\alpha W_P)^\circ),$$

and

$$c_1^T(\mathcal{L}_\lambda) \cup s_{\text{M}}(Y(wW_P)^\circ) = w(\lambda)s_{\text{M}}(Y(wW_P)^\circ) - \sum_{\alpha > 0, ws_\alpha > w} \langle \lambda, \alpha^\vee \rangle s_{\text{M}}(Y(ws_\alpha W_P)^\circ).$$

Proof. Applying the Hecke action Ψ in Lemma A.2 to the equation in Lemma A.1, and acting on the point class $[X(\text{id})]$, we get

$$\begin{aligned} c_1^T(\mathcal{L}_\lambda) \cup c_{\text{SM}}(X(wW_P)^\circ) &= c_1^T(\mathcal{L}_\lambda) \cup \mathcal{T}_w^L([X(\text{id})]) \\ &= \mathcal{T}_w^L(c_1^T(\mathcal{L}_\lambda) \cup [X(\text{id})]) \\ &= \mathcal{T}_w^L(\lambda \cdot [X(\text{id})]) \\ &= \Psi(T_w x_\lambda)([X(\text{id})]) \\ &= \Psi(x_{w\lambda} T_w - \sum_{\alpha > 0, ws_\alpha < w} \langle \lambda, \alpha^\vee \rangle T_{ws_\alpha})([X(\text{id})]) \\ &= w(\lambda)c_{\text{SM}}(X(wW_P)^\circ) - \sum_{\alpha > 0, ws_\alpha < w} \langle \lambda, \alpha^\vee \rangle c_{\text{SM}}(X(ws_\alpha W_P)^\circ). \end{aligned}$$

The second equality follows from the fact that the left operator \mathcal{T}_w^L commutes with $c_1^T(\mathcal{L}_\lambda)$ because the latter is Weyl-group invariant, as \mathcal{L}_λ is a G -equivariant line bundle; see [MNS22b]. Finally, the Chevalley formula for the SM classes follows from the one on CSM via the duality in Equation (50), similar to the proof of Lemma 5.1 above. \square

APPENDIX B. AN EXAMPLE OF THE λ -CHAIN FORMULA

We consider Lie type A_2 , with the Weyl group $W = S_3$, with $\lambda = 2\varpi_1 + \varpi_2$, and $w = s_2 s_1$. We can find an alcove walk $p_{-\lambda}$ from A_\circ to $A_\circ - \lambda$ as indicated by the red path in diagram (b). This gives the corresponding reduced expression $v_{-\lambda} = s_2 s_1 s_2 s_0 s_1 s_2$ and the corresponding alcove path

$$A_\circ = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} A_2 \xrightarrow{-\beta_3} A_3 \xrightarrow{-\beta_4} A_4 \xrightarrow{-\beta_5} A_5 \xrightarrow{-\beta_6} A_6 = A_\circ - \lambda \quad (A_i = r_i A_{i-1}, 1 \leq i \leq 6),$$

with λ -chain $\beta_1 = \alpha_2$, $\beta_2 = \alpha_1 + \alpha_2$, $\beta_3 = \alpha_1$, $\beta_4 = \alpha_1 + \alpha_2$, $\beta_5 = \alpha_1$, $\beta_6 = \alpha_1 + \alpha_2$.

The associated sequence of hyperplanes is

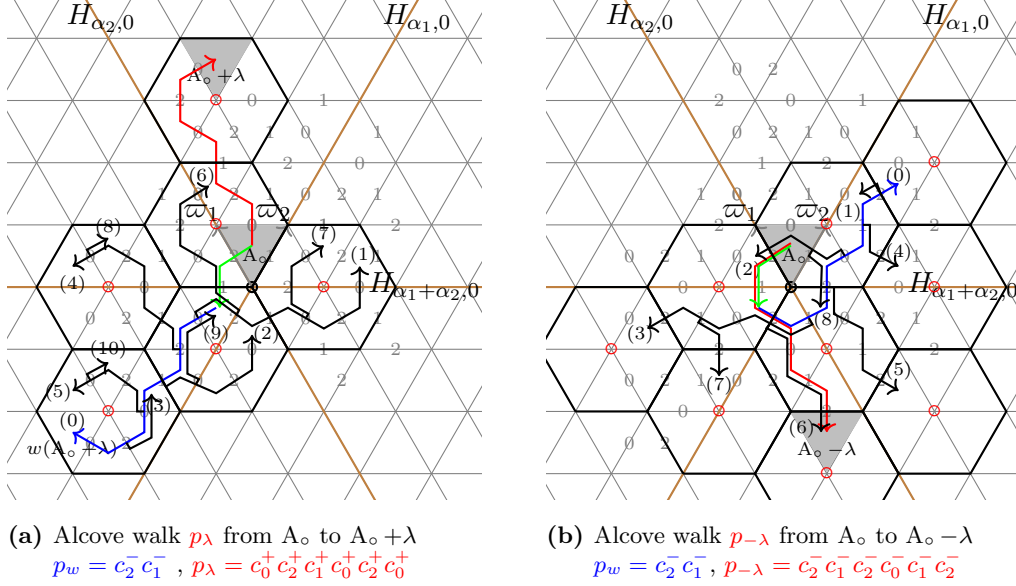
$$h_1 = H_{\alpha_2, 0}, h_2 = H_{\alpha_1 + \alpha_2, 0}, h_3 = H_{\alpha_1, 0}, h_4 = H_{\alpha_1 + \alpha_2, -1}, h_5 = H_{\alpha_1, -1}, h_6 = H_{\alpha_1 + \alpha_2, -2}.$$

We first calculate $c_{u, \mu}^{w, \lambda}$ according to the formula Theorem 3.10. In the proof we introduced λ -shifted (reversed) hyperplane sequence which can be seen in diagram (a).

$$h'_1 = H_{\alpha_1 + \alpha_2, 1}, h'_2 = H_{\alpha_1, 1}, h'_3 = H_{\alpha_1 + \alpha_2, 2}, h'_4 = H_{\alpha_1, 2}, h'_5 = H_{\alpha_1 + \alpha_2, 3}, h'_6 = H_{\alpha_2, 1}.$$

According to the formula, we need to choose $J \subset \{1, 2, \dots, l\}$ such that $u \xrightarrow{J} w$.

If $u = s_1$, $J = \{6\}, \{4\}, \{2\}$ as $s_{\alpha_1 + \alpha_2} = s_1 s_2 s_1$. For the weight μ , when $J = \{6\}$, we need to calculate $\mu = w\tilde{r}_{h_6}(\lambda)$. But as is explained in the proof, $\tilde{r}_{h_6} = \hat{r}_{h'_1}$, so, $\mu =$



$w\hat{r}_{h'_1}(\lambda) = w(-\varpi_2) = -\varpi_1 + \varpi_2$. Likewise when $J = \{4\}$, $\mu = w\tilde{r}_{h_4}(\lambda) = w\hat{r}_{h'_3}(\lambda) = -\varpi_2$, and when $J = \{2\}$, $\mu = w\tilde{r}_{h_2}(\lambda) = w\hat{r}_{h'_5}(\lambda) = \varpi_1 - 3\varpi_2$. If $u = id$, there are two possible Bruhat chains $u = id < s_1 < s_2 s_1 = w$ and $u = id < s_2 < s_2 s_1 = w$. For the first case, $J = \{5, 6\}, \{3, 6\}, \{3, 4\}$, and for the second case, $J = \{1, 5\}, \{1, 3\}$. When $J = \{5, 6\}$, $\mu = w\tilde{r}_{h_6}\tilde{r}_{h_5}(\lambda) = w\hat{r}_{h'_1}\hat{r}_{h'_2}(\lambda) = w\hat{r}_{h'_1}(2\varpi_2) = w(-\varpi_1 + \varpi_2) = \varpi_1$. All the possibilities and the corresponding (folded) alcove walks \tilde{p}_λ are listed in the table below. (We can also see the bijection of Lemma 2.7, cf. diagram (a).)

$p_w \tilde{p}_\lambda$	\tilde{p}_λ	\mathcal{M}	J	$\varphi(p) = u$	$\text{wt}(p) = w\tilde{r}_{J>}^\lambda(\lambda)$
(0)	$c_0^- c_2^- c_1^- c_0^- c_2^- c_0^+$	$\{\}$	$\{\}$	$s_2 s_1$	$\varpi_1 - 3\varpi_2$
(1)	$f_0^+ c_2^- c_1^+ c_0^- c_2^+ c_0^+$	$\{h_1\}$	$\{6\}$	s_1	$-\varpi_1 + \varpi_2$
(2)	$c_0^- c_2^- f_1^+ c_0^- c_2^+ c_0^+$	$\{h_3\}$	$\{4\}$	s_1	$-\varpi_2$
(3)	$c_0^- c_2^- c_1^- c_0^- f_2^+ c_0^+$	$\{h_5\}$	$\{2\}$	s_1	$\varpi_1 - 3\varpi_2$
(4)	$c_0^- f_2^+ c_1^+ c_0^+ c_2^+ c_0^-$	$\{h_2\}$	$\{5\}$	s_2	$2\varpi_1 - 2\varpi_2$
(5)	$c_0^- c_2^- c_1^- f_0^+ c_2^+ c_0^-$	$\{h_4\}$	$\{3\}$	s_2	$\varpi_1 - 3\varpi_2$
(6)	$f_0^+ f_2^+ c_1^+ c_0^+ c_2^+ c_0^+$	$\{h_1, h_2\}$	$\{5, 6\}$	id	ϖ_1
(7)	$f_0^+ c_2^- c_1^+ f_0^+ c_2^+ c_0^+$	$\{h_1, h_4\}$	$\{3, 6\}$	id	$-\varpi_1 + \varpi_2$
(8)	$c_0^- f_2^+ c_1^+ c_0^+ c_2^+ f_0^+$	$\{h_2, h_6\}$	$\{1, 5\}$	id	$2\varpi_1 - 2\varpi_2$
(9)	$c_0^- c_2^- f_1^+ f_0^+ c_2^+ c_0^+$	$\{h_3, h_4\}$	$\{3, 4\}$	id	$-\varpi_2$
(10)	$c_0^- c_2^- c_1^- f_0^+ c_2^+ f_0^+$	$\{h_4, h_6\}$	$\{1, 3\}$	id	$\varpi_1 - 3\varpi_2$

By this table, we get all the coefficients $c_{u,\mu}^{w,\lambda}$ as follows.

$$\begin{aligned}
c_{s_2 s_1, \mu}^{w, \lambda} &= 1 \text{ for } \mu = \varpi_1 - 3\varpi_2, \\
c_{s_1, \mu}^{w, \lambda} &= (q-1)q^{-1} \text{ for } \mu = -\varpi_1 + \varpi_2, -\varpi_2, \varpi_1 - 3\varpi_2, \\
c_{s_2, \mu}^{w, \lambda} &= (q-1)q^{-1} \text{ for } \mu = 2\varpi_1 - 2\varpi_2, \varpi_1 - 3\varpi_2, \\
c_{id, \mu}^{w, \lambda} &= (q-1)^2 q^{-2} \text{ for } \mu = \varpi_1, -\varpi_1 + \varpi_2, 2\varpi_1 - 2\varpi_2, -\varpi_2, \varpi_1 - 3\varpi_2.
\end{aligned}$$

Next we calculate $c_{u,\mu}^{w,-\lambda}$ according to Theorem 3.9. For this case we need to chose $J \subset \{1, 2, \dots, l\}$ such that $u \xrightarrow{J>} w$.

For example $u = id$, then there are three possible J corresponding to the Bruhat chains $u < us_{\beta_3} < us_{\beta_3}s_{\beta_2} = w$, $u < us_{\beta_5} < us_{\beta_5}s_{\beta_2} = w$, $u < us_{\beta_5} < us_{\beta_5}s_{\beta_4} = w$. For the first case the weight μ can be calculated as (using diagram (b)),

$$\mu = w\hat{r}_{J_{<}}(-\lambda) = s_2s_1\hat{r}_{h_2}\hat{r}_{h_3}(-\lambda) = s_2s_1(3\varpi_1 - 2\varpi_2) = -2\varpi_1 - \varpi_2.$$

All the possible J and corresponding alcove walks $\tilde{p}_{-\lambda}$ are listed in the table below.

$p_w\tilde{p}_{-\lambda}$	$\tilde{p}_{-\lambda}$	\mathcal{M}	J	$\varphi(p) = u$	$wt(p) = w\hat{r}_{J_{<}}(-\lambda)$
(0)	$c_2^-c_1^+c_2^+c_0^+c_1^+c_2^+$	$\{\}$	$\{\}$	s_2s_1	$-\varpi_1 + 3\varpi_2$
(1)	$c_2^-c_1^+c_2^+c_0^+c_1^+f_2^-$	$\{h_6\}$	$\{6\}$	s_1	ϖ_2
(2)	$c_2^-c_1^+c_2^+f_0^-c_1^+c_2^-$	$\{h_4\}$	$\{4\}$	s_1	$\varpi_1 - \varpi_2$
(3)	$c_2^-f_1^-c_2^+c_0^-c_1^+c_2^-$	$\{h_2\}$	$\{2\}$	s_1	$2\varpi_1 - 3\varpi_2$
(4)	$c_2^-c_1^+c_2^+c_0^+f_1^-c_2^-$	$\{h_5\}$	$\{5\}$	s_2	$-2\varpi_1 + 2\varpi_2$
(5)	$c_2^-c_1^+f_2^-c_0^-c_1^-c_2^-$	$\{h_3\}$	$\{3\}$	s_2	$-3\varpi_1 + \varpi_2$
(6)	$c_2^-f_1^-f_2^-c_0^-c_1^-c_2^-$	$\{h_2, h_3\}$	$\{2, 3\}$	id	$-2\varpi_1 - \varpi_2$
(7)	$c_2^-f_1^-c_2^+c_0^-f_1^-c_2^-$	$\{h_2, h_5\}$	$\{2, 5\}$	id	$-2\varpi_2$
(8)	$c_2^-c_1^+c_2^+f_0^-f_1^-c_2^-$	$\{h_4, h_5\}$	$\{4, 5\}$	id	$-\varpi_1$

By this table, we get all the coefficients $c_{u,\mu}^{w,-\lambda}$ as follows.

$$\begin{aligned} c_{s_2s_1,\mu}^{w,-\lambda} &= 1 \text{ for } \mu = -\varpi_1 + 3\varpi_2, \\ c_{s_1,\mu}^{w,-\lambda} &= (1-q)q^{-1} \text{ for } \mu = \varpi_2, \varpi_1 - \varpi_2, 2\varpi_1 - 3\varpi_2, \\ c_{s_2,\mu}^{w,-\lambda} &= (1-q)q^{-1} \text{ for } \mu = -2\varpi_1 + 2\varpi_2, -3\varpi_1 + \varpi_2, \\ c_{id,\mu}^{w,-\lambda} &= (1-q)^2q^{-2} \text{ for } \mu = -2\varpi_1 - \varpi_2, -2\varpi_2, -\varpi_1. \end{aligned}$$

REFERENCES

- [AGM11] Dave Anderson, Stephen Griffeth, and Ezra Miller. Positivity and Kleiman transversality in equivariant K -theory of homogeneous spaces. *J. Eur. Math. Soc. (JEMS)*, 13(1):57–84, 2011.
- [AMSS17] Paolo Aluffi, Leonardo C. Mihalcea, Jörg Schürmann, and Changjian Su. Shadows of characteristic cycles, Verma modules, and positivity of Chern–Schwartz–MacPherson classes of Schubert cells. *to appear in Duke Mathematical Journal, arXiv preprint arXiv:1709.08697*, 2017.
- [AMSS19] Paolo Aluffi, Leonardo C. Mihalcea, Jörg Schürmann, and Changjian Su. Motivic Chern classes of Schubert cells, Hecke algebras, and applications to Casselman’s problem. *to appear in Ann. Sci. Éc. Norm. Supér.*, *arXiv preprint arXiv:1902.10101*, 2019.
- [AMSS22] Paolo Aluffi, Leonardo C. Mihalcea, Jörg Schürmann, and Changjian Su. From motivic Chern classes of Schubert cells to their Hirzebruch and CSM classes. *arXiv preprint arXiv:2212.12509, to appear in Comtemp. Math.*, 2022.
- [AO21] Mina Aganagic and Andrei Okounkov. Elliptic stable envelopes. *Journal of the American Mathematical Society*, 34(1):79–133, 2021.
- [BBBG19] Ben Brubaker, Valentin Buciumas, Daniel Bump, and Henrik Gustafsson. Colored vertex models and Iwahori Whittaker functions. *arXiv preprint arXiv:1906.04140*, 2019.
- [BCMP18] Anders S. Buch, Pierre-Emmanuel Chaput, Leonardo C. Mihalcea, and Nicolas Perrin. A Chevalley formula for the equivariant quantum K -theory of cominuscule varieties. *Algebr. Geom.*, 5(5):568–595, 2018.
- [BM15] Anders S. Buch and Leonardo C. Mihalcea. Curve neighborhoods of Schubert varieties. *J. Differential Geom.*, 99(2):255–283, 2015.
- [Bri02] Michel Brion. Positivity in the Grothendieck group of complex flag varieties. *J. Algebra*, 258(1):137–159, 2002. Special issue in celebration of Claudio Procesi’s 60th birthday.
- [Bri05] Michel Brion. Lectures on the geometry of flag varieties. In *Topics in cohomological studies of algebraic varieties*, Trends Math., pages 33–85. Birkhäuser, Basel, 2005.
- [BS81] Jean-Paul Brasselet and Marie-Hélène Schwartz. Sur les classes de Chern d’un ensemble analytique complexe. In *The Euler–Poincaré characteristic (French)*, volume 83 of *Astérisque*, pages 93–147. Soc. Math. France, Paris, 1981.

- [BSY10] Jean-Paul Brasselet, Jörg Schürmann, and Shoji Yokura. Hirzebruch classes and motivic Chern classes for singular spaces. *J. Topol. Anal.*, 2(1):1–55, 2010.
- [Buc02] Anders Skovsted Buch. A Littlewood–Richardson rule for the K -theory of Grassmannians. *Acta Math.*, 189(1):37–78, 2002.
- [CG09] Neil Chriss and Victor Ginzburg. *Representation theory and complex geometry*. Springer Science & Business Media, 2009.
- [CS80] William Casselman and Joseph Shalika. The unramified principal series of p -adic groups. ii. the Whittaker function. *Compositio Mathematica*, 41(2):207–231, 1980.
- [EM97] Sam Evens and Ivan Mirković. Fourier transform and the Iwahori–Matsumoto involution. *Duke Math. J.*, 86(3):435–464, 1997.
- [FGX22] Neil JY Fan, Peter L Guo, and Rui Xiong. Pieri and Murnaghan–Nakayama type rules for Chern classes of Schubert cells. *arXiv preprint arXiv:2211.06802*, 2022.
- [FRW21] Laszlo M Feher, Richard Rimanyi, and Andrzej Weber. Motivic Chern classes and K-theoretic stable envelopes. *Proceedings of the London Mathematical Society*, 122(1):153–189, 2021.
- [GR04] Stephen Griffeth and Arun Ram. Affine Hecke algebras and the Schubert calculus. *European J. Combin.*, 25(8):1263–1283, 2004.
- [Hum90] James E Humphreys. *Reflection groups and Coxeter groups*. Number 29. Cambridge university press, 1990.
- [KL87] David Kazhdan and George Lusztig. Proof of the Deligne–Langlands conjecture for Hecke algebras. *Inventiones mathematicae*, 87(1):153–215, 1987.
- [Kum02] Shrawan Kumar. *Kac-Moody groups, their flag varieties and representation theory*, volume 204 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2002.
- [KW23] Jakub Koncki and Andrzej Weber. Hecke algebra action on twisted motivic Chern classes and K-theoretic stable envelopes. *arXiv preprint arXiv:2301.12746*, 2023.
- [Len11] Cristian Lenart. Hall–Littlewood polynomials, alcove walks, and fillings of Young diagrams. *Discrete mathematics*, 311(4):258–275, 2011.
- [Li92] Jian-Shu Li. Some results on the unramified principal series of p -adic groups. *Mathematische Annalen*, 292(1):747–761, 1992.
- [LP07] Cristian Lenart and Alexander Postnikov. Affine Weyl groups in K -theory and representation theory. *Int. Math. Res. Not. IMRN*, (12):Art. ID rnm038, 65, 2007.
- [LP08] Cristian Lenart and Alexander Postnikov. A combinatorial model for crystals of Kac–Moody algebras. *Transactions of the American Mathematical Society*, 360(8):4349–4381, 2008.
- [Mac74] Robert D. MacPherson. Chern classes for singular algebraic varieties. *Ann. of Math. (2)*, 100:423–432, 1974.
- [Mac96] Ian G Macdonald. Affine Hecke algebras and orthogonal polynomials. *Astérisque, Société mathématique de France*, 237:189–208, 1996.
- [Mac98] Ian Grant Macdonald. *Symmetric functions and Hall polynomials*. Oxford university press, 1998.
- [MNS22a] Leonardo C Mihalcea, Hiroshi Naruse, and Changjian Su. Hook formulae from Segre–MacPherson classes. *arXiv preprint arXiv:2203.16461*, 2022.
- [MNS22b] Leonardo C. Mihalcea, Hiroshi Naruse, and Changjian Su. Left Demazure–Lusztig operators on equivariant (quantum) cohomology and K-theory. *Int. Math. Res. Not. IMRN*, (16):12096–12147, 2022.
- [MO19] Daves Maulik and Andrei Okounkov. Quantum groups and quantum cohomology. *Astérisque*, 408, 2019.
- [MSA22] Leonardo C Mihalcea, Changjian Su, and Dave Anderson. Whittaker functions from motivic Chern classes. *Transformation Groups*, pages 1–23, 2022.
- [Nie74] H. Andreas Nielsen. Diagonalizably linearized coherent sheaves. *Bull. Soc. Math. France*, 102:85–97, 1974.
- [NO19] Hiroshi Naruse and Soichi Okada. Skew hook formula for d -complete posets via equivariant K-theory. *Algebraic Combinatorics*, 2(4):541–571, 2019.
- [Ohm06] Toru Ohmoto. Equivariant Chern classes of singular algebraic varieties with group actions. *Math. Proc. Cambridge Philos. Soc.*, 140(1):115–134, 2006.
- [Oko17] Andrei Okounkov. Lectures on K-theoretic computations in enumerative geometry. In *Geometry of moduli spaces and representation theory*, volume 24 of *IAS/Park City Math. Ser.*, pages 251–380. Amer. Math. Soc., Providence, RI, 2017.
- [OS22] Andrei Okounkov and Andrey Smirnov. Quantum difference equation for Nakajima varieties. *Inventiones mathematicae*, 229(3):1203–1299, 2022.

- [PR99] Harsh Pittie and Arun Ram. A Pieri–Chevalley formula in the K -theory of a G/B -bundle. *Electron. Res. Announc. Amer. Math. Soc.*, 5:102–107, 1999.
- [Pro99] Robert A. Proctor. Dynkin diagram classification of λ -minuscule Bruhat lattices and of d -complete posets. *J. Algebraic Combin.*, 9(1):61–94, 1999.
- [Ram06] Arun Ram. Alcove walks, Hecke algebras, spherical functions, crystals and column strict tableaux. *Pure and Applied Mathematics Quarterly*, 2(4):963–1013, 2006.
- [RTV15] Richárd Rimányi, Vitaly Tarasov, and Alexander Varchenko. Trigonometric weight functions as K -theoretic stable envelope maps for the cotangent bundle of a flag variety. *J. Geom. Phys.*, 94:81–119, 2015.
- [RTV19] R Rimányi, V Tarasov, and Alexander Varchenko. Elliptic and K -theoretic stable envelopes and Newton polytopes. *Selecta Mathematica*, 25(1):16, 2019.
- [Sch65a] Marie-Hélène Schwartz. Classes caractéristiques définies par une stratification d’une variété analytique complexe. I. *C. R. Math. Acad. Sci. Paris*, 260:3262–3264, 1965.
- [Sch65b] Marie-Hélène Schwartz. Classes caractéristiques définies par une stratification d’une variété analytique complexe. II. *C. R. Math. Acad. Sci. Paris*, 260:3535–3537, 1965.
- [Sch06] Christoph Schwer. Galleries, Hall–Littlewood polynomials, and structure constants of the spherical Hecke algebra. *International Mathematics Research Notices*, 2006(9):75395–75395, 2006.
- [Ste01] John R Stembridge. Minuscule elements of Weyl groups. *Journal of Algebra*, 235(2):722–743, 2001.
- [Su16] Changjian Su. Equivariant quantum cohomology of cotangent bundle of G/P . *Adv. Math.*, 289:362–383, 2016.
- [SZZ20] Changjian Su, Gufang Zhao, and Changlong Zhong. On the K -theory stable bases of the Springer resolution. *Ann. Sci. Éc. Norm. Supér. (4)*, 53(3):663–711, 2020.
- [SZZ21] Changjian Su, Gufang Zhao, and Changlong Zhong. Wall-crossings and a categorification of K -theory stable bases of the Springer resolution. *Compositio Mathematica*, 157(11):2341–2376, 2021.

225 STANGER ST., 460 MCBRYDE HALL, DEPARTMENT OF MATHEMATICS, VIRGINIA TECH UNIVERSITY, BLACKSBURG, VA 24061 USA

Email address: `lmihalce@vt.edu`

GRADUATE SCHOOL OF EDUCATION, UNIVERSITY OF YAMANASHI, KOFU, 400-8510, JAPAN

Email address: `hmaruse@yamanashi.ac.jp`

YAU MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, BEIJING, CHINA

Email address: `changjiansu@mail.tsinghua.edu.cn`