

QUIVER SEMI-INVARIANTS AND SAGBI BASES

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ABSTRACT. We introduce a new combinatorial structure of *linked tableaux*, which generalize the semi-standard tableaux that index a SAGBI basis of the Plücker coordinate ring of a flag variety. We show that linked tableaux index Domokos-Zubkov semi-invariants, which generate the semi-invariant ring of a quiver, which in many cases coincides with the Cox ring of an associated quiver moduli space. We show that these semi-invariants satisfy straightening laws. In the case of the generalized Kronecker quiver, we prove that the semi-invariants associated to semi-standard linked tableaux are a (possibly infinite) SAGBI basis. For the generalized Kronecker quiver with dimension vector $(2, 2)$, we show that the SAGBI basis is finite and describe it explicitly.

1. INTRODUCTION

Many interesting varieties can be constructed as GIT quotients of vector spaces $V//G$: important examples include toric varieties, Grassmannians, and flag varieties. When $G = T$ is a torus, the quotient is a toric variety, for which many questions are highly computable. For a general group G , and GIT quotient $V//G$, one can replace G with a maximal torus T . The associated toric variety $V//T$ is sometimes called the Abelianization of $V//G$.

Understanding the connection between $V//G$ and $V//T$ can allow one to extend the combinatorial control of toric varieties to other, more general, varieties. This strategy, which can be called the *Abelian/non-Abelian correspondence* has been applied successfully in many different contexts: cohomology [10, 18], I and J functions [3, 15, 22], and quantum cohomology [14]. The inspiration for this paper was to ask whether this strategy could give insight in understanding the Cox ring of $V//G$. The Cox ring of a variety X , informally defined, is the algebra generated by all global sections of all line bundles of X .

Examples of Cox rings include such well-studied objects as the Plücker coordinate ring of the Grassmannian and the homogeneous coordinate ring of a toric variety. Finding generators of the Cox ring – or even just generators of sub-algebras of the Cox ring – gives information about the variety; for example, through understanding maps into projective space. One important application is that a SAGBI basis of the Cox ring induces a toric degeneration of the variety. Toric degenerations extend the combinatorial control of toric varieties to other varieties and also have important applications in mirror symmetry.

In this paper, we study the Cox ring of quiver moduli spaces via quiver semi-invariants and the Abelian/non-Abelian correspondence. Our main results are:

- We introduce pairs of linked tableaux as a combinatorial structure indexing Domokos–Zubkov [9] generators of the Cox ring. We show that these generators satisfy straightening laws.
- In the case of a generalized Kronecker quiver, we show that semi-standard and primitive linked tableaux index a (possibly infinite) SAGBI basis of the Cox ring.

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- We prove that this SAGBI basis is finite in the case of the generalized Kronecker quiver with dimension vector $(2, 2)$.

We describe these results in more detail in Theorems 1.1-1.3.

Let G be a reductive group acting on a vector space V . A stability condition θ , which is a character of G , gives a GIT quotient $V//_{\theta}G = V^{ss}/G$ where the semi-stable locus V^{ss} is determined by θ . Given another character $\alpha \in \chi(G)$, we can consider the projection

$$V^{ss} \times \mathbb{C}/G \rightarrow V^{ss}/G = V//_{\theta}G.$$

The action of G on the product is the diagonal action defined by the representation V of G on the first factor and the character α on the second. When the GIT quotient is smooth, by Kempf's descent lemma $V^{ss} \times \mathbb{C}/G$ is a line bundle over $V//_{\theta}G$, which we denote L_{α} . Any element $f \in \mathbb{C}[V]$ such that, for all $g \in G$,

$$g \cdot f = \chi_{\alpha}(g)f,$$

defines a section of L_{α} . Such an f is called a G -semi-invariant of weight α .

In many cases of interest – such as quiver flag varieties – the Cox ring of $V//_{\theta}G$ is the algebra of semi-invariants. This holds [1] whenever the unstable locus $V^{us} = V \setminus V^{ss}$ has codimension at least two and

$$\chi(G) \cong \text{Pic}(V//_{\theta}G).$$

The results in this paper concern the semi-invariant ring, and hence (when they coincide) the Cox ring.

The Weyl group W acts on $\chi(T)$ and $\chi(G) \cong \chi(T)^W$. If a polynomial $f \in \mathbb{C}[V]$ is G -semi-invariant, then it is T -semi-invariant of the same weight. Moreover, since T scales the monomials in $\mathbb{C}[V]$, it follows that if f is G -semi-invariant, it is a sum of T -semi-invariant monomials.

This suggests three immediate questions:

- (1) Given a T -semi-invariant monomial with weight in $\chi(T)^W$, can we construct a G -semi-invariant monomial using the Weyl group?
- (2) Which T -semi-invariant monomials appear as summands of a G -semi-invariant monomial?
- (3) Given a term order, which T -semi-invariant monomials appear as the leading term in a G -semi-invariant monomial?

This paper is motivated by these questions in the setting of quiver moduli spaces. A quiver moduli space is a GIT quotient $V//_{\theta}G$ constructed from a quiver Q . The Abelianization $V//_{\theta}T$ is the quiver moduli space associated to the Abelianized quiver Q^{ab} of Q .

We introduce some notation to state our results. Let $Q = (Q_0, Q_1)$ be a quiver, so that $Q_0 = \{0, 1, \dots, \rho\}$ is the set of vertices, and Q_1 the set of arrows. All quivers in this paper are assumed to be acyclic. There are two maps $s, t : Q_1 \rightarrow Q_0$ taking arrows to their source and targets respectively. Given a choice of dimension for each vertex $\mathbf{r} = (r_0, \dots, r_{\rho}) \in \mathbb{N}^{\rho+1}$, set

$$\text{Rep}(Q, \mathbf{r}) = \bigoplus_{a \in Q_1} \text{Hom}(\mathbb{C}^{r_{s(a)}}, \mathbb{C}^{r_{t(a)}}).$$

There is a natural $\text{GL}(\mathbf{r}) = \prod_{i=0}^{\rho} \text{GL}(r_i)$ action on $\text{Rep}(Q, \mathbf{r})$. The GIT quotient

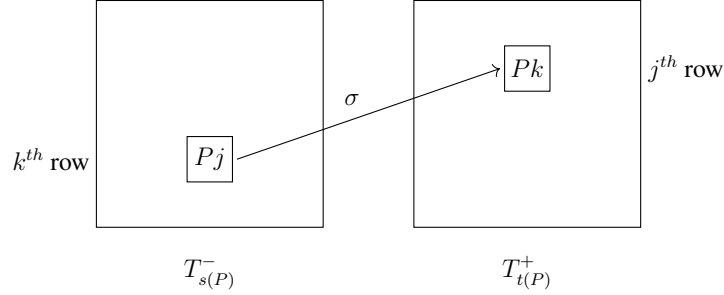
$$\text{Rep}(Q, \mathbf{r})//_{\theta} \text{GL}(\mathbf{r})$$

is the quiver moduli space associated to (Q, \mathbf{r}) and was first considered in [16]. Semi-invariants of the $\text{GL}(\mathbf{r})$ -action on $\text{Rep}(Q, \mathbf{r})$ are called *quiver semi-invariants*, and have been amply studied [8, 9, 20, 21].

In the first part of the paper, we explain how to produce the semi-invariants of Domokos–Zubkov [9] from this perspective, answering the first question above. We show how a new combinatorial

structure, *linked tableaux pairs*, can be used to index these semi-invariants, which we call DZ semi-invariants. Tableaux are key in understanding the structure of the Cox ring of flag varieties, and linked tableaux pairs play an analogous role for quiver moduli spaces.

For the definition of a linked tableaux pair, see Definition 3.6, but roughly speaking these are pairs $(\underline{T}^-, \underline{T}^+)$, where \underline{T}^\pm is a tuple of rectangular tableaux, one for each vertex in the quiver, together with a *link*. The entries of the tableaux are of the form Pj , where P is a path in the quiver and j is an integer. The link is a bijection σ between the labels of \underline{T}^- and \underline{T}^+ , that takes a label Pj in the k^{th} row of $T_{s(P)}^-$ to a label Pk in the j^{th} row of $T_{t(P)}^+$, as illustrated below:



By ordering the paths of the quiver, and then using the lex ordering, we can extend the notion of semi-standard to linked tableaux pairs.

We exploit the generalization of tableaux to linked tableaux to extend results for Grassmannians to quiver moduli spaces. In particular, we show that the analogue of the straightening laws for Grassmannians hold in this context:

Theorem 1.1. *Domokos–Zubkov semi-invariants satisfy straightening laws coming from linked tableaux. Linked tableaux with weakly increasing columns and rows span the semi-invariant ring.*

In the special case of the generalized Kronecker quiver, we can use linked tableaux to answer the third question as well. The main motivation in considering this question is finding SAGBI bases of quiver semi-invariants. A SAGBI basis of an algebra $R \subset \mathbb{C}[V]$ with a term order is a set S of elements in R such that the leading term of any element in R can be written as a product of the leading terms of elements from S .

One powerful application of SAGBI bases is in producing toric degenerations. Toric degenerations extend the combinatorial control we have over toric varieties to other varieties; for example, they are used in various mirror symmetry constructions. From this perspective, the central example of a toric degeneration is the Gelfand–Cetlin toric degeneration of type A flag varieties, due to Gonciulea–Lakshmibai [13]. This is a SAGBI basis degeneration where the basis is indexed by semi-standard Young tableaux.

We show that semi-standard linked tableaux index a (possibly infinite) SAGBI basis for the generalized Kronecker quiver. The generalized Kronecker quiver (henceforth, just Kronecker quiver) is the quiver with two vertices 0 and 1, with K arrows from 0 to 1:

$$0 \xrightarrow{K} 1$$

Fix a dimension vector $\mathbf{r} = (r_0, r_1)$.

Linked tableaux pairs are simpler in this case, as the tuples of tableaux are of length one and all paths are in fact arrows, which we label from $1, \dots, K$. Let α^- be the partition of shape $r_0 \times ar_1$ and α^+ the partition of shape $r_1 \times ar_0$, for $a \in \frac{1}{\gcd(r_0, r_1)}\mathbb{Z}$. Let T^- be a tableau of shape α^- , filled

with entries of the form ij , where

$$i \in \{1, \dots, K\}, \quad j \in \{1, \dots, r_1\}.$$

Let T^+ be a tableau of shape α^+ , filled with entries of the form ij , where

$$i \in \{1, \dots, K\}, \quad j \in \{1, \dots, r_0\}.$$

A link σ is a bijection between the labels of T^- and the labels of T^+ taking a label iq in the p^{th} row of T^- to a label ip in the q^{th} row of T^+ . Define an ordering on the labels by setting $ip < jq$ if $i < j$ or $i = j$ and $p < q$. In this way, we extend the notion of semi-standard to these tableaux with double entries: a linked pair is *semi-standard* if both T^+ and T^- are semi-standard.

Theorem 1.2. *Let Q be the Kronecker quiver. Then pairs of primitive semi-standard linked tableaux index a SAGBI basis for the algebra of quiver semi-invariants.*

For the definition of *primitive* semi-standard linked tableaux pairs see Definition 6.4. It is not clear from the combinatorics that the set of primitive semi-standard linked tableaux is finite. We do not know of any examples where it is not finite, and we show that it holds for the following family:

Theorem 1.3. *Let Q be the Kronecker quiver with dimensions $r_0 = r_1 = 2$. Then this SAGBI basis is finite.*

Plan of the paper. In §2, we give the necessary background on quiver moduli spaces and quiver semi-invariants. In §3, we describe the DZ semi-invariants and show in §4 that they satisfy straightening laws. In §5 and §6, we discuss the SAGBI basis of the Kronecker quiver, and describe it explicitly in some small examples. In §7 we prove that this basis is finite in the $r_0 = r_1 = 2$ case. In the last section of the paper, §8, we use the SAGBI basis to produce a toric degeneration of Fano quiver moduli space, and give a conjectural Laurent polynomial mirror for this variety.

2. BACKGROUND

As in the introduction, let $Q = (Q_0, Q_1)$ be a quiver. Label the vertices $Q_0 = \{0, 1, \dots, \rho\}$. All quivers in this paper are assumed to be acyclic, so we can assume that if $a \in Q_1$, then $s(a) < t(a)$. We fix a dimension vector $\mathbf{r} = (r_0, \dots, r_\rho) \in \mathbb{N}^{\rho+1}$. Recall that the set of representations of this quiver of dimension \mathbf{r} is

$$\text{Rep}(Q, \mathbf{r}) = \bigoplus_{a \in Q_1} \text{Hom}(\mathbb{C}^{r_{s(a)}}, \mathbb{C}^{r_{t(a)}}).$$

The change of basis action is given by $\text{GL}(\mathbf{r}) = \prod_{i=0}^{\rho} \text{GL}(r_i)$.

Given $\theta \in \chi(\text{GL}(\mathbf{r}))$, there is a GIT quotient

$$M_\theta(Q, \mathbf{r}) := \text{Rep}(Q, \mathbf{r}) //_\theta G.$$

The character group $\chi(\text{GL}(\mathbf{r}))$ is naturally identified with $\mathbb{Z}^{\rho+1}$. When $r_0 = 1$ and

$$\theta = \left(-\sum_{i=1}^{\rho} r_i, 1, \dots, 1\right)$$

under this identification, the resulting GIT quotient is a smooth projective fine moduli space, called a *quiver flag variety* [7]. Examples of quiver flag varieties include type A flag varieties and Grassmannians.

2.1. Quiver semi-invariants. The group $\mathrm{GL}(\mathbf{r})$ acts naturally on the elements of $\mathbb{C}[\mathrm{Rep}(Q, \mathbf{r})]$. If Q is acyclic, there are no non-trivial $\mathrm{GL}(\mathbf{r})$ -invariant polynomials in the ring $\mathbb{C}[\mathrm{Rep}(Q, \mathbf{r})]$. Instead, we can consider semi-invariants of this group action. We choose coordinates on $\mathbb{C}[\mathrm{Rep}(Q, \mathbf{r})]$:

$$x_{ij}^a, \quad a \in Q_1, i \in \{1, \dots, r_{s(a)}\}, j \in \{1, \dots, r_{t(a)}\}.$$

Definition 2.1. Let f be a polynomial in $\mathbb{C}[\mathrm{Rep}(Q, \mathbf{r})]$. Then f is a semi-invariant of weight $\alpha \in \chi(\mathrm{GL}(\mathbf{r}))$ if, for all $g \in \mathrm{GL}(\mathbf{r})$,

$$g \cdot f = \chi_\alpha(g)f.$$

The ring of semi-invariants is denoted

$$\mathrm{SI}(Q, \mathbf{r}),$$

and is graded by $\chi(\mathrm{GL}(\mathbf{r}))$. For $\alpha \in \chi(\mathrm{GL}(\mathbf{r}))$, we denote the α -graded piece as

$$\mathrm{SI}(Q, \mathbf{r})_\alpha.$$

Apart from semi-invariants being a natural construction to consider in invariant theory, they also play an important role in the study of the related GIT quotients. The GIT quotient $X = \mathrm{Rep}(Q, \mathbf{r}) //_\theta G$ is constructed as

$$\mathrm{Proj}(\oplus_{k \in \mathbb{Z}_{\geq 0}} \mathrm{SI}(Q, \mathbf{r})_{k\theta}).$$

As described in the introduction, given a character $\alpha \in \chi(\mathrm{GL}(\mathbf{r}))$, we can consider

$$\mathrm{Rep}(Q, \mathbf{r})^{ss} \times \mathbb{C} / \mathrm{GL}(\mathbf{r}) \rightarrow \mathrm{Rep}(Q, \mathbf{r})^{ss} / \mathrm{GL}(\mathbf{r}) = \mathrm{Rep}(Q, \mathbf{r}) // \mathrm{GL}(\mathbf{r}).$$

When the GIT quotient is smooth, by Kempf's descent lemma this is a line bundle, which we denote L_α . For GIT quotients X with sufficiently nice properties, this induces isomorphisms

$$\chi(\mathrm{GL}(\mathbf{r})) \cong \mathrm{Pic}(X),$$

$$\Gamma(X, L_\alpha) \cong \mathrm{SI}(Q, \mathbf{r})_\alpha,$$

and hence

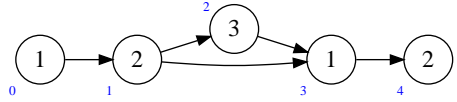
$$\mathrm{Cox}(X) \cong \mathrm{SI}(Q, \mathbf{r}).$$

These isomorphisms hold when the unstable locus has codimension at least 2 and there are no strictly semi-stable points [1]. In particular, they hold for quiver flag varieties [7].

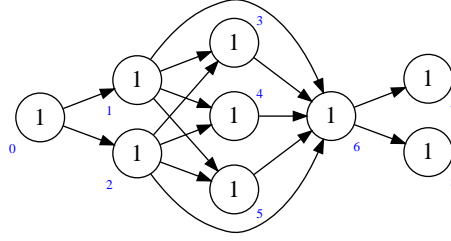
Studying the semi-invariant ring can give information on embeddings of $M_\theta(Q, \mathbf{r})$ coming from ample line bundles. Another application, as will be discussed, is in finding toric degenerations.

2.2. Abelianization. Since a quiver GIT quotient is of the form $V //_\theta G$, we can consider its Abelianization $V //_\theta T$, where $T \subset G$ is the diagonal maximal torus. In [15], this shown to be another quiver GIT quotient: the one associated to the *abelianized* quiver Q^{ab} of Q , with dimension vector $(1, \dots, 1)$.

The Abelianization Q^{ab} of a quiver Q is formed by replacing a vertex p with dimension r_i with i separate vertices p_1, \dots, p_{r_i} all of dimension one. We call p_i a *lift* of p to Q^{ab} . If there is an arrow $a : p \rightarrow q$ between vertex p and q in Q , there is an arrow $a_{ij} : p_i \rightarrow q_j$ in Q^{ab} . For example, consider the quiver



Here the number inside the vertices indicates the dimension associated to that vertex. The abelianization of this quiver is



The two vector spaces

$$\text{Rep}(Q, \mathbf{r}) = \text{Rep}(Q^{ab}, (1, \dots, 1))$$

are naturally identified. A lift of an arrow $a : p \rightarrow q$ in Q to an arrow $p_i \rightarrow q_j$ in Q^{ab} corresponds to the coordinate x_{ij}^a , in $\mathbb{C}[\text{Rep}(Q, \mathbf{r})]$.

Denote the T -weight of x_{ij}^a as

$$D_{a_{ij}}^T \in \chi(T).$$

The monomial $\prod_{a_{ij} \in S} x_{ij}^{a_{ij}}$ for some collection of arrows S has weight

$$\sum_{a_{ij} \in S} D_{a_{ij}}^T.$$

Definition 2.2. Let S be a collection of arrows in Q^{ab} . We say that S is Weyl-invariant if the difference between the number of arrows in and out of each vertex is constant over the lifts of p for all $p \in Q_0$. That is, for every vertex $p \in Q_0$, $n_{p_i} = n_{p_j}$ for all lifts p_i and p_j of p to Q^{ab} , where

$$n_{p_i} := \#\{a \in S : t(a) = p_i\} - \#\{a \in S : s(a) = p_i\}.$$

The set of monomial semi-invariants of T with Weyl-invariant weight are in one-to-one correspondence with the Weyl-invariant arrow sets. Given an arrow set, we can partition it into paths.

Definition 2.3. Let S be a collection of paths in Q^{ab} . We say that S is Weyl-invariant if for every vertex $p \in Q_0$, the difference between the number of paths in and out of each vertex is constant over the lifts of p for all $p \in Q_0$. That is, $n_{p_i} = n_{p_j}$ for all lifts p_i and p_j of p to Q^{ab} , where

$$n_{p_i} := \#\{P \in S : t(P) = p_i\} - \#\{P \in S : s(P) = p_i\}.$$

An arrow $a \in Q_1$ is associated to the matrix of coordinates on $\text{Rep}(Q, \mathbf{r})$:

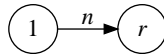
$$M_a := (x_{ij}^a) \in \text{Mat}(r_{t(a)} \times r_{s(a)}).$$

For a path P in Q , let

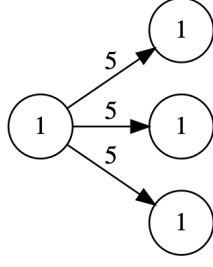
$$(1) \quad M_P := \prod_{a \in P} M_a.$$

3. DETERMINANTAL SEMI-INVARIANTS

3.1. The Grassmannian. Before describing the general case, we consider the simplest example: the Grassmannian. The Grassmannian $\text{Gr}(n, r)$ is the quiver flag variety associated to the quiver



The Abelianization of this quiver has one source vertex and r target vertices, each with n arrows from the source. For example, the Abelianization of the quiver of $\text{Gr}(5, 3)$ is



The Weyl group is Sym_r and permutes the r target vertices. Let x_j^i be the coordinate associated to the arrow a_j^i , where a_j^i is the i^{th} arrow into the j^{th} vertex. A collection S is W -invariant if there is the same number of arrows to each target vertex.

Consider collections with one arrow into each target vertex, i.e. sets $\{a_1^{i_1}, \dots, a_r^{i_r}\}$ for any choice i_1, \dots, i_r . This is associated to the monomial

$$\prod_{j=1}^r x_j^{i_j}.$$

The Weyl group acts on this monomial by permuting the lower indices. To make this a semi-invariant, we certainly need it to be Weyl-anti-invariant under the Weyl group action, so we can consider the sum

$$\sum_{\sigma \in \text{Sym}_r} \text{sign}(\sigma) \sigma \cdot \prod_{j=1}^r x_j^{i_j} = \sum_{\sigma \in \text{Sym}_r} \text{sign}(\sigma) \prod_{j=1}^r x_{\sigma(j)}^{i_j} = \det([x_k^{i_j}]).$$

This sum is now not just semi-invariant for the Weyl group, but actually for all of G , as these are of course the Plücker coordinates of the Grassmannian. One can visualize this on Q^{ab} by drawing the arrows, and then letting W act on them in the natural way.

Now consider collections with two arrows into each target vertex. Label such a set

$$\{a_1^{i_1}, \dots, a_r^{i_r}, a_1^{k_1}, \dots, a_r^{k_r}\}.$$

We get the product of monomials

$$\prod_{j=1}^r x_j^{i_j} x_j^{k_j}.$$

We could symmetrize using the Weyl group action:

$$\sum_{\sigma \in \text{Sym}_r} \text{sign}(\sigma)^2 \sigma \cdot \prod_{j=1}^r x_j^{i_j},$$

but this is not a G -semi-invariant. Instead, we need to remember that since this is actually of weight $2 \in \mathbb{Z} \cong \chi(G)$, that we should have two copies of the Weyl group acting. Notice that we have labeled the arrows in such a way that we can act by one copy of W on the $a_j^{i_j}$ and one copy on the $a_j^{k_j}$. Doing this gives a semi-invariant. However, we now have more data than just the set of arrows: we have partitioned the arrows into two separate sets, each of which are Weyl-invariant. To record this data, we can use a tableau. Each column in the tableau is of length r , and corresponds to the

set that one copy of the Weyl group acts on:

i_1	j_1
i_2	j_2
\vdots	\vdots
i_r	j_r

For weight l , we write down a $r \times l$ tableau. Each tableau T gives a monomial m_T : a box labeled i in the j^{th} row contributes a variable a_j^i . There is a $W^{\times l}$ action on tableaux of shape $r \times l$, where the first copy of W permutes the labels in the first column, the second in the second column, and so forth. The semi-invariant is

$$\sum_{\underline{w} \in W^{\times l}} \text{sign}(\underline{w}) m_{\underline{w} \cdot T}.$$

3.2. The general case. Let Q be an acyclic quiver with dimension vector \mathbf{r} . Let Q^{ab} denote the Abelianization of the quiver. Recall that T -semi-invariant monomials of Weyl-invariant weights are in one-to-one correspondence with Weyl-invariant collections of arrows. The goal of this section is to associate to Weyl-invariant collections of arrows in Q^{ab} semi-invariants of Q . As in the Grassmannian case, we will need tableaux and actions of copies of the Weyl group on tableaux to do this.

We'll do this in two steps:

- (1) Partition the arrows into a set of paths S_P .
- (2) Assign to the set of paths two tuples of partitions that are linked.

3.2.1. Partitioning the arrows.

Definition 3.1. Let S_P be a collection of paths in Q^{ab} . We say that S is bipartite if for every vertex $p_i \in Q^{ab}$, at least one of

$$\#\{P \in S : t(P) = p_i\} \text{ and } \#\{P \in S : s(P) = p_i\}$$

is zero.

Lemma 3.2. Let S be a Weyl-invariant collection of arrows. Then there is at least one partition of S into a collection of paths S_P such that S_P is Weyl-invariant and bipartite.

Proof. Consider a vertex $p \in Q_0$ such that both

$$\#\{P \in S : t(P) = p_i\} \text{ and } \#\{P \in S : s(P) = p_i\}$$

are non-zero for some lift p_i . By connecting

$$\min\{\#\{P \in S : t(P) = p_i\}, \#\{P \in S : s(P) = p_i\}\}$$

arrows into p_i with with arrows out of p_i , we get a new set \tilde{S}_P which is Weyl-invariant and satisfies the bi-partite condition at p_i . We can do this at every vertex, and in such a way produce S_P . \square

Remark 3.3. Instead of considering Weyl-invariant arrow sets, we consider Weyl-invariant, bi-partite path sets. This simplifies the combinatorics of the tableaux that we will introduce.

Fix a bipartite, Weyl-invariant path set S . The monomial associated to the underlying arrow set has a Weyl-invariant weight, and let

$$(w_p)_{p \in Q_0} = (w_p^+) - (w_p^-), \quad w_p^\pm \geq 0$$

be this weight in $\chi(G)$ under the identification $\chi(G) \cong \chi(T)^W$. Then

- $w_p^- = \#\{P \in S : s(P) = p_i \text{ for some lift } p_i \text{ of } p\},$
- $w_p^+ = \#\{P \in S : t(P) = p_i \text{ for some lift } p_i \text{ of } p\}.$

Lemma 3.4. $\sum_p r_p w_p = 0.$

Proof. We observe that

$$\sum_p r_p w_p = \sum_{p_i \in Q^{ab}} n_{p_i},$$

where n_{p_i} is, as before,

$$n_{p_i} := \#\{P \in S : t(P) = p_i\} - \#\{P \in S : s(P) = p_i\}.$$

Since each path has exactly one source and one target, it follows that this sum is 0. \square

3.2.2. Linked pairs of tableaux. We now define linked pairs of tableaux, which will play an analogous role to the tableaux the Grassmannian case. Here, we need to have pairs of tableaux because there are non-trivial Weyl group actions on both the source and target of every arrow.

Let \mathcal{P} be the set of all paths in the quiver Q . Choose a labeling of this set $\{P_1, \dots, P_N\}$. Given a path P in Q^{ab} , we can project it onto a path in Q .

Choose any weight

$$(w_p)_{p \in Q_0} = (w_p^+) - (w_p^-), \quad w_p^\pm \geq 0$$

satisfying $\sum_p r_p w_p = 0$. For this weight, we define two tuples of partitions: the *target partitions*

$$(r_p \times w_p^+)_{p \in Q_0}$$

and the *source partitions*,

$$(r_p \times w_p^-)_{p \in Q_0}.$$

Lemma 3.5. *The total number of boxes in the source partitions equals the total number of boxes in the target partitions.*

Next, we consider tableaux with shapes given by the target and source partitions. That is, let $(\underline{T}^+, \underline{T}^-)$ be two tuples of tableaux satisfying

$$\underline{T}^+ = (T_p^+)_{p \in Q_0}, \quad sh(T_p^+) = r_p \times w_p^+,$$

and

$$\underline{T}^- = (T_p^-)_{p \in Q_0}, \quad sh(T_p^-) = r_p \times w_p^-.$$

The labels of \underline{T}^\pm are given by pairs of integers:

- (1) T_p^+ is filled by entries of the form ij , where i is the index of a path P_i such that $t(P_i) = p$ and $j \in \{1, \dots, r_{s(P_i)}\}.$
- (2) T_p^- is filled by entries of the form ij , where $s(P_i) = p$ and $j \in \{1, \dots, r_{t(P_i)}\}.$

Definition 3.6. *Let \underline{T}^+ and \underline{T}^- be a pair of tableaux tuples filled as above. We say that \underline{T}^+ and \underline{T}^- form a linked pair if there is a bijection σ from the set of boxes in \underline{T}^+ to the set of boxes in \underline{T}^- satisfying the following. If σ takes box B_1 in T_p^+ with label ij to box B_2 in T_q^- with label kl , then*

- $i = k$, so that P_i is a path from $q \rightarrow p$, and $j \in \{1, \dots, r_q\}, l \in \{1, \dots, r_p\}.$

- B_1 appears in the l^{th} row of T_p^+ , a tableau of shape $r_p \times w_p^+$.
- B_2 appears in the j^{th} row of T_q^+ , a tableau of shape $r_q \times w_q^+$.

3.2.3. *From path sets to linked pairs of tableaux.* Next, we explain how to go from path sets to linked tableaux pairs. Let S be a bipartite, Weyl-invariant path set of Q^{ab} . As above, let w be the Weyl-invariant weight of the associated monomial, and write

$$w = (w_p)_{p \in Q_0} = (w_p^+) - (w_p^-), \quad w_p^\pm \geq 0$$

for this weight in $\chi(G)$ under the identification $\chi(G) \cong \chi(T)^W$.

The number of paths in S is the same as the total number boxes in the target partitions for this weight, which is the same as the total number boxes in the source partitions. We use the paths to label the source and target partitions corresponding to the weight w as follows. A path $P \in S$ projects to a path P_i in Q . Then the source of P is a vertex $q_j, j \in \{1, \dots, r_q\}$ and the target of P is a vertex $p_l, l \in \{1, \dots, r_p\}$. The path P then gives a label ij which can be placed in l^{th} row of the q^{th} source tableau, and a label il which can be placed in the j^{th} row of the p^{th} target tableau.

That is, for the path set S , we consider tuples of tableaux $\underline{T}^+ = (T_p^+)$ and (T_q^-) that satisfy the following two conditions:

- The labels in the l^{th} row of T_p^+ are the set

$$\{ij : P \in S \text{ with projection } P_i \text{ and } s(P) = q_j, t(P) = p_l.\}$$

- The labels of the j^{th} row of T_q^- are the set

$$\{il : P \in S \text{ with projection } P_i \text{ and } s(P) = q_j, t(P) = p_l.\}$$

Then the two tuples are naturally a linked pair, using the path set P : the label in the target tableaux \underline{T}^+ corresponding to P is linked to the label in the source tableau \underline{T}^- corresponding to P .

3.2.4. *Weyl-type group actions on linked pairs of tableaux.* As in the case of the Grassmannian, choose such a tuple of tableaux corresponding to a path set S allows us to construct a G -semi-invariant out of the monomial with W -invariant weight that is given by S . The structure of linked pairs of tableaux allows us to define a more refined action of (copies of the) Weyl group on the monomial.

Let \underline{T}^\pm be a linked tableaux pair, linked by σ . Let

$$G^+ = \prod_{p \in Q_0} \text{Sym}_{r_p}^{w_p^+}.$$

Let

$$G^- = \prod_{p \in Q_0} \text{Sym}_{r_p}^{w_p^-}.$$

Set $\text{sign}(\alpha^\pm)$ to be the product of the signs of the permutations in $\alpha^\pm \in G^\pm$.

Then both groups act on each of \underline{T}^- and \underline{T}^+ . One action is given by permuting the rows, and one by permuting the labels.

Permuting the rows of \underline{T}^\pm . This is an action of G^\pm on \underline{T}^\pm .

There are w_p^\pm columns of T_p^\pm , and w_p^\pm copies of Sym_{r_p} in G^\pm . The action is defined by using the l^{th} copy of Sym_{r_p} to permute the labels of the l^{th} column of T_p^\pm .

Permuting the labels of \underline{T}^\pm . This is an action of G^\pm on \underline{T}^\mp . Because $(\underline{T}^+, \underline{T}^-)$ are a linked pair, there are $r_p w_p^-$ labels ij in \underline{T}^+ satisfying $s(P_i) = p$. Each of the w_p^- columns in T_p^- corresponds to a set of labels in \underline{T}^+ of the form $\{i_1 1, \dots, i_{r_p} r_p\}$. The action of G^- on \underline{T}^+ is defined by having the l^{th} copy of Sym_{r_p} in G^- permutes the second entry of the l^{th} set of labels $\{i_1 1, \dots, i_{r_p} r_p\}$.

Note that for all $\alpha^+ \in G^+$,

$$(\alpha^+ \cdot \underline{T}^+, \alpha^+ \cdot \underline{T}^-)$$

is naturally a linked pair. Similarly,

$$(\alpha^- \cdot \underline{T}^+, \alpha^- \cdot \underline{T}^-)$$

is a linked pair for all $\alpha^- \in G^-$.

3.2.5. From linked pairs of tableaux to semi-invariants. T -semi-invariant monomials of Weyl-invariant weights are in one-to-one correspondence with Weyl-invariant arrow sets. Given such an arrow set, we choose a Weyl-invariant, bipartite path set, and for this path set, we choose a linked tableaux pair. The final stage is to explain how to produce a semi-invariant from a linked pair.

Recall that for a path P in Q , there is a matrix M_P , see (1). This matrix is of size $r_{t(P)} \times r_{s(P)}$. Let $(M_P)_{ij}$ be the ij entry of M_P . To a tuple of tableaux \underline{T}^+ , define

$$\text{Mon}(\underline{T}^+) = \prod (M_{P_i})_{lj},$$

where the product is over all labels ij appearing in some T_p^+ , and l is the row in which this label appears.

Analogously, to a tuple of tableaux \underline{T}^- define

$$\text{Mon}(\underline{T}^-) = \prod (M_{P_i})_{lj},$$

where the product is over all labels il appearing in some T_p^- , and j is the row in which this label appears.

The following lemma is clear from the definition:

Lemma 3.7. *If \underline{T}^\pm is a linked pair, then*

$$\text{Mon}(\underline{T}^+) = \text{Mon}(\underline{T}^-).$$

Definition 3.8. *Define*

$$f_{\underline{T}^\pm} = \sum_{\alpha^- \in G^-, \alpha^+ \in G^+} \text{sign}(\alpha^-) \text{sign}(\alpha^+) \text{Mon}(\alpha^- \cdot \alpha^+ \cdot \underline{T}^\pm).$$

Remark 3.9. Since $\alpha^- \cdot \alpha^+ \cdot \underline{T}^+$ and $\alpha^- \cdot \alpha^+ \cdot \underline{T}^-$ are canonically linked, this can be equivalently written as

$$f_{\underline{T}^\pm} = \sum_{\alpha^- \in G^-, \alpha^+ \in G^+} \text{sign}(\alpha^-) \text{sign}(\alpha^+) \text{Mon}(\alpha^- \cdot \alpha^+ \cdot \underline{T}^-).$$

As written, it is not clear that this is a semi-invariant of Q . This will follow from two easy lemmas, but we first state the theorem.

Theorem 3.10. *For any linked pair, $f_{\underline{T}^\pm}$ is a semi-invariant of Q .*

Let $C_i(M_P)$ be the i^{th} column of M_P and $R_i(M_P)$ the i^{th} row.

We first describe this in words. Given a tableau T_p^- , $D(T_p^-)$ is given by taking a product of determinants, where there is one determinant for each column of T_p^- . Each determinant is the determinant of matrix with rows taken from the path matrixes showing up in the first index of the

labels. The row contributed by a particular path matrix is determined by the second index in that label. More precisely, if the labels of the l^{th} column of T_p^- are

$$i_m^l j_m^l, m = 1, \dots, r_p,$$

set

$$D(T_p^-) = \prod_{l=1}^{r_p} \det([R_{j_m^l}(M_{P_{i_m^l}})]_{m=1, \dots, r_p}).$$

Set $D(\underline{T}^-) = \prod_{p \in Q_0} D(T_p^-)$.

Lemma 3.11. *If \underline{T}^\pm is a linked pair,*

$$f_{\underline{T}^\pm} = \sum_{\alpha^+ \in G^+} \text{sign}(\alpha^+) D(\alpha^+ \underline{T}^-).$$

Proof. Follows from the Leibnitz formula for the determinant. \square

Similarly, given a tableau T_p^+ , $D(T_p^+)$ is given by taking a product of determinants, where there is one determinant for each column of T_p^+ . Each determinant is the determinant of matrix with one column taken from each of the path matrixes showing up in the first index of the labels in this tableau column. The column contributed by a particular path matrix is determined by the second index in that label. More precisely, if the labels of the l^{th} column of T_p^+ are

$$i_m^l j_m^l, m = 1, \dots, r_p,$$

set

$$D(T_p^+) = \prod_{l=1}^{r_p} \det([C_{j_m^l}(M_{P_{i_m^l}})]_{m=1, \dots, r_p}).$$

As above, set $D(\underline{T}^+) = \prod_{p \in Q_0} D(T_p^+)$. Then we have, analogously,

Lemma 3.12. *If \underline{T}^\pm is a linked pair,*

$$f_{\underline{T}^\pm} = \sum_{\alpha^- \in G^-} \text{sign}(\alpha^-) D(\alpha^- \underline{T}^+).$$

Proof of Theorem 3.10. Write $G = H^+ \times H^-$, where

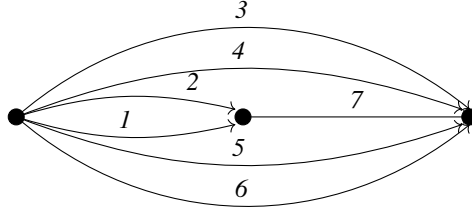
$$H^+ = \prod_{p: w_p^+ > 0} \text{GL}(r_p),$$

and

$$H^- = \prod_{p: w_p^- > 0} \text{GL}(r_p).$$

Then the two expressions of $f_{\underline{T}^\pm}$ from the lemmas are, respectively, $H^+ \times \text{id}$ and $\text{id} \times H^-$ semi-invariant of weights w^+ and $-w^-$. It follows that $f_{\underline{T}^\pm}$ is semi-invariant for the entire group G with weight $w^+ - w^-$. \square

Example 3.13. Consider the quiver



with dimension vector $(2, 2, 3)$. Label the arrows from $1, \dots, 7$, ordered by their source vertex and then by their target, as drawn above. There are two paths given by concatenating arrows from vertices 0 to 1 with the arrow from 1 to 2, which we label 8, 9.

Consider the weight $[-2, -1, 2]$. Let $\underline{T}^+ = (\emptyset, \emptyset, T_2^+)$, where

$$T_2^+ = \begin{array}{|c|c|} \hline 31 & 52 \\ \hline 41 & 71 \\ \hline 72 & 62 \\ \hline \end{array}.$$

Let $\underline{T}^- = (T_0^-, T_1^-, \emptyset)$, where

$$T_1^- = \begin{array}{|c|} \hline 72 \\ \hline 73 \\ \hline \end{array}, \quad T_0^- = \begin{array}{|c|c|} \hline 31 & 42 \\ \hline 63 & 51 \\ \hline \end{array}.$$

This is a linked pair, with a unique link. If M_i denotes the matrix corresponding to the i^{th} path, then we can write $f_{\underline{T}^\pm}$ as

$$\sum_{\sigma=(\sigma_i) \in S_2^3} \text{sign}(\sigma) \det[C_{\sigma_2(1)}(M_3)|C_{\sigma_3(1)}(M_4)|C_{\sigma_1(2)}(M_7)] \det[C_{\sigma_3(2)}(M_5)|C_{\sigma_1(1)}(M_7)|C_{\sigma_2(2)}(M_6)],$$

or equivalently

$$f_{\underline{T}^\pm} = \sum_{(\tau_1, \tau_2) \in S_3^2} \text{sign}(\tau_1) \text{sign}(\tau_2) \det \begin{bmatrix} R_{\tau_2(2)}(M_7) \\ R_{\tau_1(3)}(M_7) \end{bmatrix} \det \begin{bmatrix} R_{\tau_1(1)}(M_3) \\ R_{\tau_2(3)}(M_6) \end{bmatrix} \det \begin{bmatrix} R_{\tau_1(2)}(M_4) \\ R_{\tau_2(1)}(M_5) \end{bmatrix}.$$

Using Proposition 5.4 of [9], it is straightforward to check that the semi-invariants $f_{\underline{T}^\pm}$ are exactly the semi-invariants defined there (up to multiplication by a scalar). These semi-invariants therefore span the algebra of all semi-invariants.

4. STRAIGHTENING LAWS

A tableau is called semi-standard if it is weakly increasing along rows and strictly increasing along columns.

A foundational fact for Grassmannians and flag varieties is that the semi-invariants indexed by semi-standard tableaux span all semi-invariants. Moreover, every leading term of *any* semi-invariant is the leading term of a semi-invariant coming from a semi-standard tableau. Moreover, the leading term of a semi-standard tableau is simply the monomial read off of that tableau. This fact is proven by showing that the semi-invariants of these varieties satisfy straightening laws. For

example, the semi-invariant corresponding to the non-semi-standard tableau

1	2
4	3

can be written as a linear combination of the semi-invariants of the semi-standard tableaux

1	3	1	2
2	4	3	4

We now investigate to what extent this holds for linked tableaux pairs. We'll see that they satisfy straightening laws, but that the condition on leading terms is weaker.

4.1. Straightening laws for products of minors. Suppose A is an $r \times N$ matrix. For $I = \{i_1, \dots, i_r\}$, $i_j \in \{1, \dots, N\}$, let p_I denote the $r \times r$ minor corresponding to the columns indexed by I . The p_I are Plücker coordinates, and it is well-known that they satisfy straightening laws. That is, if a set of k Plücker coordinates I_1, \dots, I_k does not correspond to a semi-standard tableau, then

$$p_{I_1} \cdots p_{I_k} = \sum a_{J_1, \dots, J_k} p_{J_1} \cdots p_{J_k},$$

where each set J_1, \dots, J_k satisfies:

- The associated tableau is semi-standard.
- The J_1, \dots, J_k are a re-partitioning of the elements in the union of I_1, \dots, I_k .
- Under a term ordering on the Plücker coordinates, the monomial on the left hand side has lower term order than all monomials on the right hand side.

Viewing the data of the tuples of subsets as a tableau, we can write such relations as

$$p_T = \sum_{\alpha} a_{\alpha} p_{\alpha \cdot T},$$

where α ranges over permutations of the labels in T (only some coefficients are non-zero).

These relations are true for any matrix. Fix a vertex p in the quiver, and consider the matrix A_p^+ formed by concatenating from left to right the matrices M_P for all paths P such that $t(P) = p$. We can analogously form A_p^- , by concatenating the matrices from top to bottom.

Now consider a linked pair $(\underline{T}^+, \underline{T}^-)$. Fix a vertex p such that T_p^+ is non-empty. Recall that

$$D(T_p^+) = \prod_{l=1}^{r_p} \det([C_{j_m^l}(M_{P_{i_m^l}})]_{m=1, \dots, r_p}),$$

where the labels of the l^{th} column of T_p^+ are

$$i_m^l j_m^l, m = 1, \dots, r_p.$$

Thus, $D(T_p^+)$ is a product of minors of A_p^+ . Any straightening law of the form above can then be applied to $D(T_p^+)$. Of course, when we act by a permutation α , blocks can change rows, which affects the link between \underline{T}^+ and \underline{T}^- . We write $\alpha \cdot \underline{T}^-$ for the induced action on the tableaux in \underline{T}^- given by changing the second entry of the labels. That is, we have a linked pair,

$$\alpha \cdot (\underline{T}^{\pm}) := (\alpha \cdot \underline{T}^+, \alpha \cdot \underline{T}^-).$$

Lemma 4.1. *Suppose there is a straightening relation giving*

$$D(T_p^+) = \sum_{\alpha} a_{\alpha} D(\alpha \cdot T_p^+).$$

Then for any $\beta \in G^-$, there is a relation

$$D(\beta \cdot T_p^+) = \sum_{\alpha} a_{\alpha} D(\beta \cdot \alpha \cdot T_p^+).$$

Proof. Using the straightening law, there is a relation:

$$D(\beta \cdot T_p^+) = \sum_{\alpha} a_{\alpha} D(\alpha \cdot \beta \cdot T_p^+).$$

It suffices to show that

$$D(\alpha \cdot \beta \cdot T_p^+) = D(\beta \cdot \alpha \cdot T_p^+),$$

for all α . We first observe that the right hand side makes sense: the action of α on \underline{T}^- does not change the columns of \underline{T}^- , and thus does not affect the action of elements in G^- on $\alpha \cdot \underline{T}^+$. The actions of α and β therefore commute. \square

Theorem 4.2 (Straightening Laws for semi-invariants of quivers). *Suppose there is a straightening relation giving*

$$D(T_p^+) = \sum_{\alpha} a_{\alpha} D(\alpha \cdot T_p^+)$$

for some target p . Then

$$f_{\underline{T}^{\pm}} = \sum_{\alpha} a_{\alpha} f_{\alpha \cdot \underline{T}^{\pm}}.$$

In the same way, straightening laws applied to source tableaux give rise to straightening laws for semi-invariants.

Proof. From the lemma, we see that for any $\beta \in G^-$, there is a relation

$$D(\beta \cdot T_p^+) = \sum_{\alpha} a_{\alpha} D(\beta \cdot \alpha \cdot T_p^+).$$

By summing over $\beta \in G^-$, we obtain the statement of the theorem. \square

Example 4.3. *We illustrate the theorem in an example, continued from Example 3.13. Consider the semi-invariant coming from the linked pair $\underline{T}^+ = (\emptyset, \emptyset, T_2^+)$ and $\underline{T}^- = (T_1^-, T_2^-, \emptyset)$, where*

$$T_2^+ = \begin{array}{|c|c|} \hline 31 & 52 \\ \hline 41 & 71 \\ \hline 72 & 62 \\ \hline \end{array}.$$

Let $\underline{T}^- = (T_0^-, T_1^-, \emptyset)$, where

$$T_1^- = \begin{array}{|c|} \hline 72 \\ \hline 73 \\ \hline \end{array}, \quad T_0^- = \begin{array}{|c|c|} \hline 31 & 42 \\ \hline 63 & 51 \\ \hline \end{array}.$$

There is a Plücker relation that gives

$$\begin{array}{|c|c|} \hline 31 & 42 \\ \hline 63 & 51 \\ \hline \end{array} = - \begin{array}{|c|c|} \hline 31 & 51 \\ \hline 42 & 63 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 31 & 42 \\ \hline 51 & 63 \\ \hline \end{array},$$

where we have omitted the $D(-)$ notation. Each of the tableaux on the right hand side give rise to a new \underline{T}^- , which corresponds to some \underline{T}^+ with T_2^+ respectively

$$\begin{array}{|c|c|} \hline 31 & 51 \\ \hline 42 & 71 \\ \hline 72 & 62 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 31 & 52 \\ \hline 41 & 71 \\ \hline 72 & 62 \\ \hline \end{array}.$$

Call the two new linked pairs \underline{U}^\pm and \underline{V}^\pm respectively.

For an element $(\tau_1, \tau_2) \in S_3^2$, there is a relation

$$\begin{array}{|c|c|} \hline 3\tau_1(1) & 4\tau_1(2) \\ \hline 6\tau_2(3) & 5\tau_2(1) \\ \hline \end{array} = - \begin{array}{|c|c|} \hline 3\tau_1(1) & 5\tau_2(1) \\ \hline 4\tau_1(2) & 6\tau_2(3) \\ \hline \end{array} + \begin{array}{|c|c|} \hline 3\tau_1(1) & 4\tau_1(2) \\ \hline 5\tau_2(1) & 6\tau_2(3) \\ \hline \end{array}.$$

Summing over all of the elements of S_3^2 , we obtain that

$$f_{\underline{T}^\pm} = -f_{\underline{U}^\pm} + f_{\underline{V}^\pm}.$$

Fix a quiver Q and choose an order on the path set of Q . We define an order on the labels of the tableaux:

$$ij \leq kl \text{ if } i \leq k \text{ or } i = k \text{ and } j \leq l.$$

Definition 4.4. A pair of linked tableaux is semi-standard if each tableau in \underline{T}^\pm is semi-standard; i.e. it is strictly increasing along the columns and weakly increasing along the rows.

A pair of linked tableaux is weakly semi-standard if, for each tableau in \underline{T}^\pm , the first digits of the labels are weakly increasing along columns and weakly increasing along the rows.

Example 4.5. Both the first and second tableau below are weakly semi-standard, while only the second tableau is semi-standard.

$$\begin{array}{|c|c|} \hline 12 & 11 \\ \hline 13 & 42 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 11 & 12 \\ \hline 13 & 42 \\ \hline \end{array}.$$

Theorem 4.6. The weakly semi-standard quiver semi-invariants generate the semi-invariant ring of a quiver moduli space.

Proof. Suppose we have a quiver semi-invariant corresponding to a pair of linked tableaux \underline{T}^\pm that is not semi-standard. We can use the straightening laws on the target vertices to write it as a linear combination of linked tableaux whose source vertices are all semi-standard. So we can assume that the target tableaux are all semi-standard. If one of the source tableau is not semi-standard, we can again use the straightening laws to re-write it as a sum of semi-standard tableaux – however, in doing so we may have changed the labels of \underline{T}^- , so that they are no longer semi-standard. However, since the changes are all in the second digit of the labels, the tableaux remain weakly semi-standard. \square

Example 4.7. Continuing Example 4.3, we can now write $f_{\underline{U}^\pm}$ and $f_{\underline{V}^\pm}$ as a sum of weakly semi-standard tableaux. For example, for U^\pm , note that by noting that

$$U_2^+ = \begin{array}{|c|c|} \hline 31 & 51 \\ \hline 42 & 71 \\ \hline 72 & 62 \\ \hline \end{array} = - \begin{array}{|c|c|} \hline 31 & 51 \\ \hline 42 & 62 \\ \hline 72 & 71 \\ \hline \end{array}.$$

If we call the tableau on the right hand side W_2^+ , then $\underline{W}^+ = (\emptyset, \emptyset, W_2^+)$ forms a weakly semi-standard linked pair together with

$$\underline{W}^- = \left(\begin{array}{|c|c|} \hline 31 & 51 \\ \hline 42 & 62 \\ \hline \end{array}, \begin{array}{|c|} \hline 73 \\ \hline 73 \\ \hline \end{array}, \emptyset \right),$$

and $f_{\underline{W}^\pm} = -f_{\underline{U}^\pm}$.

In the case of the Kronecker quiver, we now use these results to study the leading terms of semi-invariants.

5. THE KRONECKER QUIVER

In this section, we fix a Kronecker quiver,

$$0 \xrightarrow{K} 1$$

and a dimension vector $\mathbf{r} = (r_0, r_1)$. The semi-invariant ring of the Abelianization is

$$\mathbb{C}[\text{Mat}(r_1 \times r_0)^{\oplus K}].$$

We label generators x_{jk}^i , where x_{jk}^i represents the (j, k) entry of the i^{th} matrix, so that the matrix corresponding to the i^{th} arrow is $M_i = [x_{j,k}^i]$.

We can concatenate the M_i two ways:

$$A_R := [M_1 \mid \cdots \mid M_K],$$

$$A_C := \begin{bmatrix} M_1 \\ \vdots \\ M_K \end{bmatrix}.$$

We fix a term order given by ordering the variables by going along the rows of M_1 , then M_2 and so forth.

Lemma 5.1. *The leading term of a full-size minor of A_R or A_C is the monomial corresponding to the diagonal.*

Proof. The result follows because both A_R and A_C have the property that the entry in a submatrix of any size of highest weight is the entry in the upper left corner. \square

Proposition 5.2. *Let f be a semi-invariant of the Kronecker quiver. Then the leading monomial of f , which we denote $\text{LM}(f)$, is*

$$\text{Mon}(\underline{T}^+) = \text{Mon}(\underline{T}^-)$$

for some semi-standard pair of linked tableaux T^\pm .

Proof. Since f is a semi-invariant of the Kronecker quiver, it is a semi-invariant with respect to both the $\mathrm{GL}(r_0)$ and $\mathrm{GL}(r_1)$ natural actions. It therefore can be written both as a polynomial in the $r_0 \times r_0$ minors of A_C and as a polynomial in the $r_1 \times r_1$ minors of A_R . Using standard results for these rings, we know that the leading term of f can be written as

$$\mathrm{Mon}(T^+) = \mathrm{Mon}(T^-),$$

for two semi-standard tableaux, T^+ and T^- . It suffices to show that this pair is linked. We construct a link using the leading term of f . Consider x_{jk}^i appearing in the leading term of f . Then x_{jk}^i corresponds to a label ik appearing in the j^{th} row of T^+ , and also a box label in ij appearing the k^{th} row of T^- . \square

Remark 5.3. Note that there are potentially multiple links supporting the same pair of semi-standard tableaux T^\pm ; these lead to different semi-invariants. The ambiguity is up to multiple labels ij appearing in a fixed row k of T^+ , or equivalently labels ik appearing in row j of T^- . We say a link is *semi-standard* if it maps the boxes labeled ij in row k of T^+ to the boxes labeled ik in T^- in an order preserving way. From now on, we include this in the definition of a semi-standard linked pair: both the tableaux and the link are required to be semi-standard.

Corollary 5.4. *Let (T^-, T^+) be a semi-standard linked tableaux pair. Then there is no other semi-standard linked tableaux pair (T_1^-, T_1^+) such that $T^- = T_1^-$ or $T^+ = T_1^+$.*

Proposition 5.5. *Let (T^+, T^-) be a linked, semi-standard pair, and let f_{T^\pm} be the associated semi-invariant. Then*

$$\mathrm{Mon}(T^+) = \mathrm{Mon}(T^-)$$

is the leading monomial of f_{T^\pm} .

Proof. The monomials that appear (with possibly non-zero coefficient – there could be cancellation) in f_{T^\pm} are

$$\mathrm{Mon}(\sigma^+ \cdot \sigma^- T^+)$$

for $(\sigma^+, \sigma^-) \in G^+ \times G^-$. The element σ^+ acts by permuting the rows of each column of T^+ , and the element σ^- acts by permuting the second digit in the label of the boxes of T^+ . Since T^+ and T^- are semi-standard, if the action is non-trivial the resulting monomial is of strictly lower order than $\mathrm{Mon}(T^+)$. If the action is trivial, then $\mathrm{sign}(\sigma^+) = \mathrm{sign}(\sigma^-)$, and the coefficient is 1. Therefore there is no cancellation of the leading term. \square

Theorem 5.6. *The semi-invariants corresponding to semi-standard linked tableaux form a vector space basis of the semi-invariant ring.*

Proof. We first show that these semi-invariants span. Let f be any semi-invariant. By Proposition 5.2, its leading term corresponds to the monomial of some semi-standard linked pair T^\pm with semi-standard link. Then the leading term of $f - f_{T^\pm}$ is of strictly lower order. By induction, we can therefore write f as a sum of semi-standard semi-invariants. Since there is a unique semi-standard linked pair with semi-standard link, these semi-invariants are linearly independent. \square

6. SAGBI BASES FOR THE KRONECKER QUIVER

Recall the definition of a SAGBI basis:

Definition 6.1. *A collection of elements $S \subset R$ of a graded ring R is a SAGBI basis if, for every $f \in R$, the leading term of f is a product of leading terms of elements from S .*

So far, the results for quiver semi-invariants of a Kronecker quiver follow what is known about the semi-invariants of the Grassmannian. However, to move from a vector space basis of the semi-invariant ring to a SAGBI basis, we need to consider factorization. Let f be a semi-invariant. There are two questions one can ask about factorization:

- (1) Does f factor into a product of two semi-invariants?
- (2) Does the leading monomial, $\text{LM}(f)$, of f factor into the product of the leading term of two semi-invariants?

For the Grassmannian, the answer to both questions always coincide. If T is a semi-standard tableau, $D(T)$ is a product of the determinants corresponding to each of its columns. This implies that the semi-invariant ring of the Grassmannian has a SAGBI basis corresponding to the semi-standard tableaux with a single column.

The following example illustrates how the leading term of a semi-invariant can factor, while the semi-invariant itself does not.

Example 6.2. Consider the Kronecker quiver with dimension vector $(2, 2)$ and 6 arrows. The following tableaux are a linked pair:

11	31	52
21	42	62

,

11	22	31
42	51	62

Then the leading term of the associated semi-invariant is the product of the leading terms of the semi-invariants of the associated linked pairs:

11	52
21	62

,

11	22
51	62

,

31
42

,

31
42

However, the semi-invariant of the original pair is not a product of the lower degree semi-invariants.

In contrast, the linked pair

11	32	52
21	41	62

,

11	22	42
31	51	62

has a leading term that cannot be written as a product of leading terms of any semi-invariants.

In general, the link prevents such a factorization property for quiver semi-invariants. We consider the requirements on the link for such a factorization to exist.

Lemma 6.3. If (T^\pm, σ) is a pair of linked tableaux such that there are subtableaux T_i^+ and T_i^- of T^\pm respectively such that the link σ pairs labels in T_i^+ with labels in T_j^- , then

$$f_{T^\pm, \sigma} = f_{T_1^\pm, \sigma_1} f_{T_2^\pm, \sigma_2}.$$

Here σ_i is the link for T_i^\pm induced by σ .

Proof. Follows from the definition of f_{T^\pm} . □

Definition 6.4. Let T^\pm be a semi-standard linked tableaux pair. If there are two semi-standard linked tableaux pairs T_1^\pm, T_2^\pm such that

$$\text{LM}(f_{T^\pm}) = \text{LM}(f_{T_1^\pm})\text{LM}(f_{T_2^\pm}),$$

then we say that T_\pm^1 and T_\pm^2 induce a splitting of T^\pm . If there are no splittings of T^\pm , then it is primitive. If

The motivation for this definition is the following immediate corollary of Theorem 5.6.

Corollary 6.5. The semi-invariants corresponding to primitive, semi-standard linked tableaux are a (possibly infinite) SAGBI basis of the semi-invariant ring of the Kronecker quiver.

Giving a combinatorial description of primitive semi-standard tableaux pairs is thus the key step to explicitly writing down SAGBI bases of Kronecker quivers. This seems to be a non-trivial problem. A related question is whether the set of primitive, semi-standard tableaux is finite for a fixed r_1, r_2 , and K . We have no examples where it is infinite. In the following section, we show that the set is finite when $r_1 = r_2 = 2$ for any K and give an explicit, combinatorial description. Before proving this, we give some small examples where we have used the computer algebra program MAGMA to determine the set of primitive, semi-standard linked tableaux.

Example 6.6. Consider the Kronecker quiver with $r_0 = 2$ and $r_1 = 3$, and 2 arrows between the vertices. Using MAGMA, we determine that there is exactly one primitive semi-standard tableaux pair of degree $(-3, 2)$ and no primitive semi-standard tableaux pairs of degree $(-6, 4)$. The primitive tableaux pair is

$$T^+ = \begin{array}{|c|c|} \hline 11 & 11 \\ \hline 12 & 21 \\ \hline 22 & 22 \\ \hline \end{array}, \quad T^- = \begin{array}{|c|c|c|} \hline 11 & 11 & 12 \\ \hline 12 & 23 & 23 \\ \hline \end{array}.$$

This is clearly a SAGBI basis, as it is the only generator of the semi-invariant ring.

Example 6.7. Consider the Kronecker quiver with $r_0 = 2$ and $r_1 = 3$, and 3 arrows between the vertices. Using MAGMA, we determine that there are 20 primitive semi-standard tableaux pairs of degree $(-3, 2)$ and no primitive semi-standard tableaux pairs of degree $[-6, 4]$. The T^+ of the 20 primitive tableaux pairs are:

$$\begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|} \\ \hline 11 & 22 \\ \hline 31 & 31 \\ \hline 32 & 32 \\ \hline \end{array}, & \begin{array}{|c|c|} \\ \hline 11 & 21 \\ \hline 12 & 31 \\ \hline 32 & 32 \\ \hline \end{array}, & \begin{array}{|c|c|} \\ \hline 11 & 21 \\ \hline 12 & 31 \\ \hline 22 & 32 \\ \hline \end{array}, & \begin{array}{|c|c|} \\ \hline 11 & 21 \\ \hline 22 & 31 \\ \hline 32 & 32 \\ \hline \end{array}, & \begin{array}{|c|c|} \\ \hline 11 & 22 \\ \hline 21 & 31 \\ \hline 32 & 32 \\ \hline \end{array}, \\ \\ \hline \begin{array}{|c|c|} \\ \hline 11 & 22 \\ \hline 21 & 31 \\ \hline 22 & 32 \\ \hline \end{array}, & \begin{array}{|c|c|} \\ \hline 11 & 11 \\ \hline 12 & 21 \\ \hline 32 & 32 \\ \hline \end{array}, & \begin{array}{|c|c|} \\ \hline 11 & 11 \\ \hline 12 & 21 \\ \hline 22 & 32 \\ \hline \end{array}, & \begin{array}{|c|c|} \\ \hline 11 & 11 \\ \hline 12 & 22 \\ \hline 21 & 32 \\ \hline \end{array}, & \begin{array}{|c|c|} \\ \hline 11 & 11 \\ \hline 12 & 12 \\ \hline 21 & 32 \\ \hline \end{array}, \\ \\ \hline \begin{array}{|c|c|} \\ \hline 11 & 11 \\ \hline 21 & 22 \\ \hline 32 & 32 \\ \hline \end{array}, & \begin{array}{|c|c|} \\ \hline 11 & 11 \\ \hline 12 & 21 \\ \hline 22 & 22 \\ \hline \end{array}, & \begin{array}{|c|c|} \\ \hline 11 & 11 \\ \hline 21 & 22 \\ \hline 22 & 32 \\ \hline \end{array}, & \begin{array}{|c|c|} \\ \hline 11 & 11 \\ \hline 12 & 31 \\ \hline 32 & 32 \\ \hline \end{array}, & \begin{array}{|c|c|} \\ \hline 11 & 11 \\ \hline 12 & 31 \\ \hline 22 & 32 \\ \hline \end{array}, \end{array}$$

11	11	11	21	11	21	11	21	21	21
22	31	12	22	21	22	21	22	22	31
32	32	21	32	32	32	22	32	32	32

Using MAGMA, we computed the semi-invariants associated to each of the tableaux pairs, and checked that they are a SAGBI basis. If we increase the number of arrows to 4, then there are 232 primitive semi-standard tableaux pairs of degree $d(-3, 2)$, for $1 \leq d \leq 3$, but there may be more in higher degree.

Example 6.8. Consider the Kronecker quiver with $r_0 = 3$ and $r_1 = 3$, and 3 arrows between the vertices. Using MAGMA, we determine that there are 18 primitive semi-standard tableaux pairs of degree $(-d, d)$ for $1 \leq d \leq 3$ and no primitive semi-standard tableaux pairs of degree $[-4, 4]$. The T^+ of the 18 primitive tableaux pairs are:

$i1$
$j2$
$k3$

, where $1 \leq i \leq j \leq k \leq 3$,

and

11	11	11	23	11	22	11	22	11	21	11	22
12	23	12	32	21	32	21	32	12	23	21	23
22	33	21	33	23	33	33	33	22	33	22	33

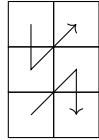
11	11	23	11	11	22
12	21	32	12	23	32
22	33	33	21	33	33

The following reformulation of the link between a pair of linked tableaux will be helpful in the next section. Let (T^-, T^+) be a linked pair. Each column T^- corresponds, using the link, to some set of r_0 boxes in T^+ with labels $i_1 1, \dots, i_{r_0} r_0$. We can connect these boxes in T^+ using an arrow (or path). After doing this for each column of T^- , all boxes of T^+ will be contained in exactly one of these paths. The roles of T^- and T^+ can be reversed.

To illustrate this, consider the following linked pair:

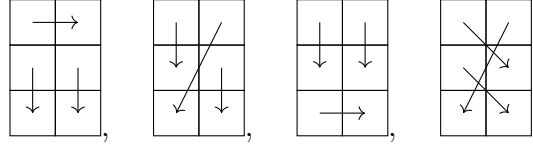
$$T^+ = \begin{array}{|c|c|} \hline 11 & 23 \\ \hline 12 & 32 \\ \hline 21 & 33 \\ \hline \end{array}, \quad T^- = \begin{array}{|c|c|} \hline 11 & 23 \\ \hline 12 & 32 \\ \hline 21 & 33 \\ \hline \end{array}$$

The arrow diagram for T^+ is



The arrow diagram for T^- the same since T^+ is self-dual.

Example 6.9. Consider the Kronecker quiver with $r_0 = 2$ and $r_1 = 3$, and 3 arrows between the vertices. The primitive semi-standard tableaux pairs of Example 6.7 all have one of the following arrow diagrams:



7. THE $r_0 = r_1 = 2$ CASE AND FINITENESS

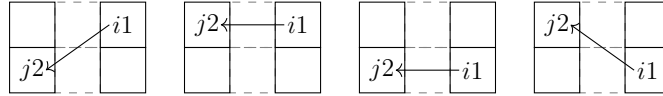
In this section, we give an explicit description of the primitive semi-standard tableaux pairs of the Kronecker quiver, when $r_0 = r_1 = 2$, for any number of arrows K between the vertices. Our strategy will be as follows: in the lemmas and propositions, we rule out some basic configurations of arrows in arrow diagrams. This gives strong control over the possible shapes of primitive semi-standard tableaux. We then use an inductive argument to show finiteness; in fact, we give an explicit description of the primitive semi-standard tableaux.

As $r_0 = r_1 = 2$, any semi-standard linked tableaux T^- and T^+ will consist of two rows, and each of the $a \cdot r_2$ arrows in the arrow configuration of T^\pm will be of length one.

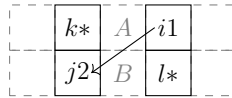
Lemma 7.1. Let T^\pm be a semi-standard, primitive, linked pair. Then the arrow configurations satisfy the following conditions.

- (1) Arrows in T^\pm cannot point backwards, i.e for an arrow $i1 \rightarrow j2$, $j2$ is not positioned to the left of $i1$ in T^\pm .
- (2) Arrows in T^\pm cannot point downwards, i.e there is no arrow $i1 \rightarrow j2$ where $i1$ is positioned on the first row of T^\pm and $j2$ on the second row of T^\pm .

Proof. (1) We wish to exclude the following four configurations:



Without loss of generality, we can assume these configurations are in T^+ . Since T^- is semi-standard, we have that $i1 < j2$. Since T^+ is also semi-standard, every configuration except the first one violates this inequality. We show that the first configuration also does not occur, as it cannot be contained in any primitive tableau. More precisely, assume we have the following:



where we denote the intermediate blocks by $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$ and $k* \in \{k1, k2\}$, $l* \in \{l1, l2\}$. We claim that this tableau splits into a 2×1 block formed by the $i1$ and $j2$ cells, and a second tableau containing the rest of T^+ as follows:

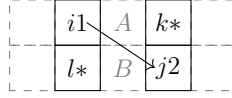


Indeed, both factors are semi-standard. The $i1$ - $j2$ block is semi-standard by hypothesis. The semi-standardness of T^+ implies the following inequalities

$$k* \leq A_1 \leq \dots \leq A_n \leq i1 < j2 \leq B_1 \leq \dots \leq B_n \leq l*.$$

One can now see that the right-hand tableau is semi-standard, as both non-strict and strict inequalities required in the definition are immediately deduced from the sequence above: in particular, we have that $A_i < B_{i+1}$, which addresses the “shift” that appears when removing the $i1$ - $j2$ block from T^+ . The splitting exists, therefore any tableau with a backwards-pointing arrow is not primitive.

(2) Here we wish to exclude only one configuration, namely



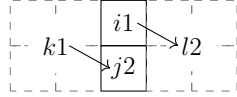
where the notation is as before. We now compare the relative positions of $k*$ and $l*$ to exclude all possible configurations.

If $k < l$ then removing the $i1 - j2$ block splits T^+ into two SSYT. Indeed, there is nothing to check about T^- , while in T^+ we have

$$i1 \leq A_1 \leq \dots \leq A_n \leq k* < l* \leq B_1 \leq \dots \leq B_n \leq j2.$$

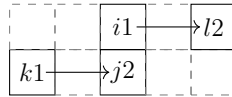
After splitting the $i1 - j2$ block, what remains is an SSYT due to the strict inequality $A_s < B_t$ for all s, t (in particular if $s = t + 1$), in addition to $A_1 < l*$ and $k* < B_n$.

If $k \geq l$, since there are no backwards arrows in either T^\pm , the configuration in T^- is:



There are four cases:

- We have $k* = k1$ and $l* = l1$ in T^+ . In other words, both $k1$ and $l1$ are on the first row of T^- . This implies that the condition $k1 \leq i1 \leq l2$ becomes an equality, which is a contradiction since $i1 < l1$ in T^+ .
- A similar argument works for $k* = k2$ and $l* = l2$ in T^+ .
- If $k* = k1$ and $l* = l2$, then $k1 \leq i1 < j2 \leq l2$ from T^- , so that $i = j = k = l$. Even further, in T^+ we obtain $(A_1, \dots, A_n) = (i_1, \dots, i_1)$ and $(B_1, \dots, B_n) = (i_2, \dots, i_2)$. It is then immediate that we can split off the initial $i1 - j2$ (now $i1 - i2$) block without violating the strict inequalities in the remaining tableau.
- $k* = k2, l* = l1$ in T^+ , in which case T^- becomes:



We claim that we can split the 2×2 block containing i, j, k and l . In fact, we only need to show that the following sub-block of T^-



is semistandard: its complement in T^- is semi-standard by construction, and we automatically obtain that T^+ splits into two rectangular SSYT. The above 2×2 block is not semistandard only if $i = k$ or $l = j$. But from T^+ and our hypothesis we know that $j > k \geq l > i$, so neither of these cases occur.

We are thus able to conclude that there is no primitive linked SSYT-pair in which one tableau contains a *downwards* arrow. □

Lemma 7.2. *Let (T^\pm) be a pair of primitive, semi-standard linked tableaux. Consider labels $a1$ and $d2$ that appear in the first and second row respectively of T^+ (and hence also, respectively, in the first and second row of T^-). Then either $a1$ is strictly the left of $d2$ in both T^+ and T^- , or $a1$ is strictly to the right of $d2$ in both T^+ and T^- .*

Proof. We first rule out the case where $a1$ is strictly above $d2$ in T^+ . Then there must be a downwards arrow in T^- , which is impossible by Lemma 7.1.

Suppose that $a1$ is strictly to the left of $d2$ in T^- , and strictly to the right of $d2$ in T^+ . That is, the configuration in T^- is of the following form, where the second digits of the labels above and below $a1$ and $d2$ are determined by using Lemma 7.1.

$a1$	$c2$
$b1$	$d2$

In T^+ , again by using Lemma 7.1, the configuration must therefore be:

$c1$	$d2$	$a1$	$b2$
------	------	------	------

If $c < b$, then split T^\pm into the tableaux pair

$a1$	$a1$
$d2$	$d2$

and the tableaux T_1^\pm obtained by removing $a1$ and $d2$ from T^\pm . Since $a < d$, the tableaux above are a semi-standard linked tableaux pair. T_1^- is clearly semi-standard and T_1^\pm are a linked pair. It remains to check that T_1^+ is semi-standard. Since $c < d$, every label in the first row of T^+ , to the right of $a1$ and the left of $d2$ is strictly smaller than every label in the second row of T^+ , to the right of $a1$ and the left of $d2$. So even after removing $a1$ and $d2$, the tableau remains semi-standard. Therefore, T^\pm is not primitive.

If $c \geq b$, split T^\pm into the tableaux pair

$a1$	$c2$	$a1$	$b2$
$b1$	$d2$	$c1$	$d2$

and the tableaux T_1^\pm obtained by removing these four labels from T^\pm . The tableaux above are semi-standard as

$$\begin{aligned} a &< b \leq c, \\ b &\leq c < d. \end{aligned}$$

It's clear by the shape of what we've removed that T_1^\pm are also semi-standard. So T^\pm is not primitive.

By symmetry we can swap the roles of T^+ and T^- and use the same argument in the last remaining case. □

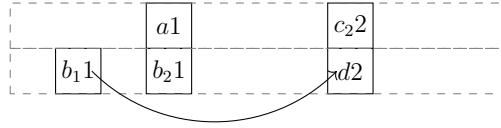
The proof the next proposition unfortunately involves many cases. It is, however, essential to the main theorem of this section.

Proposition 7.3. *Let T^\pm be a primitive, semi-standard linked tableaux pair. Then neither T^+ nor T^- contains a configuration of the following form:*



We assume that $b_1 1$ is to the left of, and $d2$ to the right of, the vertical double line.

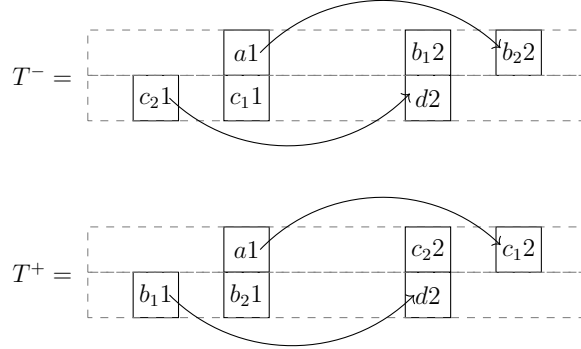
Proof. To prove the statement, we need a bit more notation. Without loss of generality, we assume that the offending configuration appears in T^+ . Let $b_2 1$ be the label under $a1$ in T^+ . Let $c_2 2$ be label above $d2$ in T^+ . Then the configuration in T^+ is



In T^- , by Lemma 7.2, $a1$ is again to the left of $d2$, i.e. $b_1 \geq a$. Let $c_1 1$ be the label below $a1$ in T^- . There are two cases, which we consider separately:

- (1) $c_2 \leq c_1$,
- (2) $c_2 > c_1$.

Suppose first that $c_2 \leq c_1$. Then, by changing the link trivially if necessary, we have the configurations:



We have four inequalities:

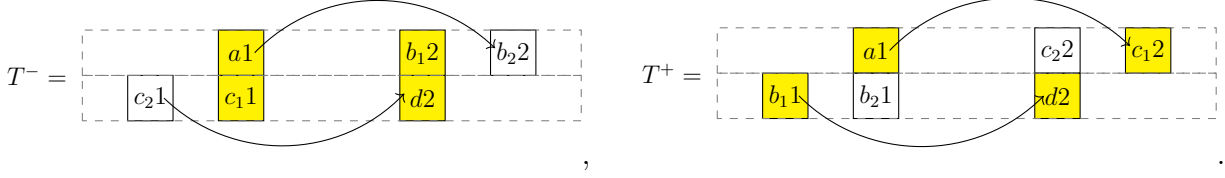
$a \leq b_1$	$a \leq c_2$
$d \geq c_1$	$d \geq b_2$

We consider three subcases, where some of these inequalities are strict or equalities, that together cover all possibilities:

- Suppose the inequalities are of one of the forms:

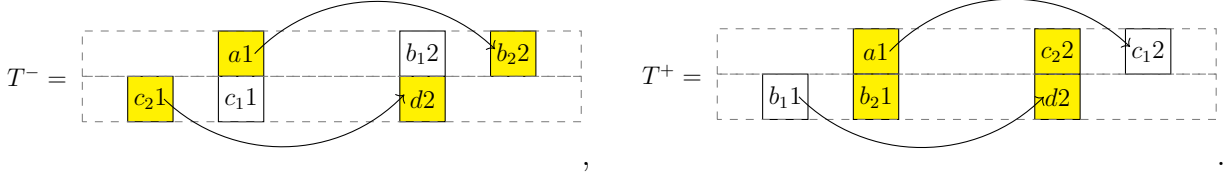
$<$	\leq	\leq	$<$
$>$	\geq	\geq	$>$

In the first case, we remove the highlighted boxes below:



Using the inequalities, this gives a splitting of T^\pm , which therefore is not primitive. We leave it to the reader to check that the new tableaux are semi-standard.

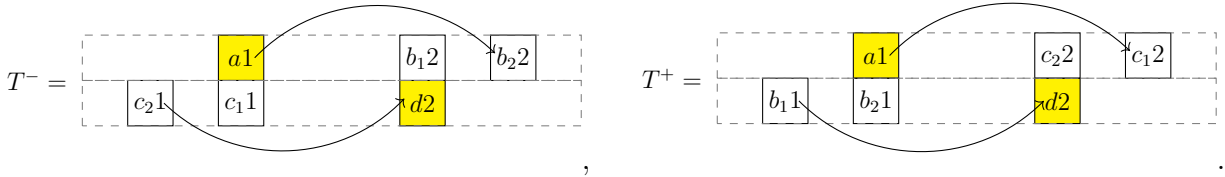
In the second case, we remove the highlighted boxes below:



- Suppose the inequalities are of one of the forms:

$$\begin{array}{|c|c|} \hline \leq & = \\ \hline = & \geq \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline \leq & = \\ \hline = & \geq \\ \hline \end{array}.$$

In either case, we remove the highlighted boxes below:



It is clear that the tableaux removed from T^\pm is semi-standard. The equalities can be used to show that what remains is semi-standard. For example, in the first case, this follows from the inequalities

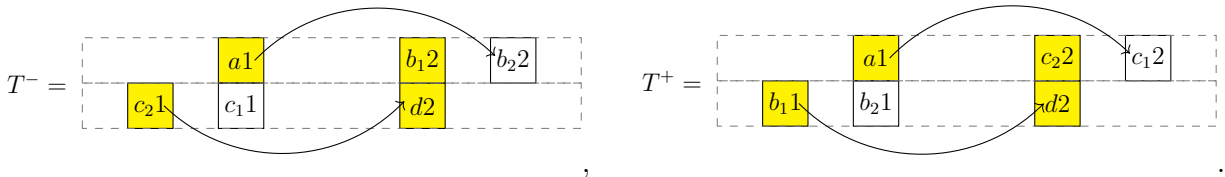
$$c_2 = a < b_2, c_1 = d > b_1.$$

The second case is similar.

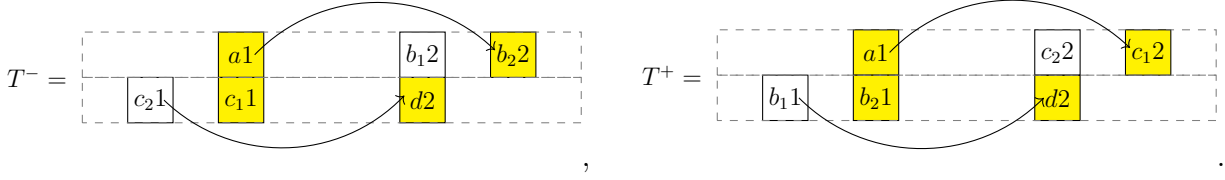
- Suppose the inequalities are of one of the forms:

$$\begin{array}{|c|c|} \hline < & < \\ \hline \geq & \geq \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline \leq & \leq \\ \hline > & > \\ \hline \end{array}.$$

In the first case, we remove the highlighted boxes below:

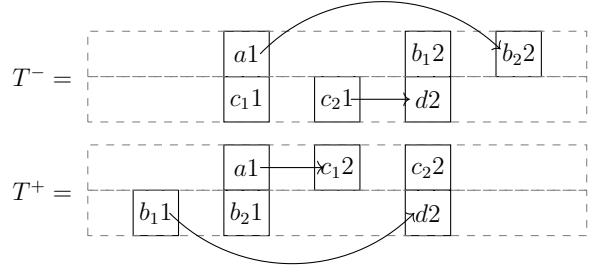


In the second case, we remove the highlighted boxes below:

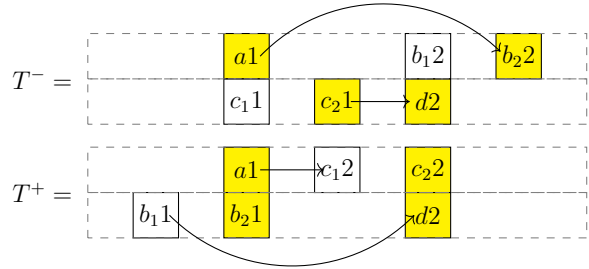


We leave it to the reader to check that the new tableaux are semi-standard.

This shows that the lemma holds when $c_1 \geq c_2$. We now need to consider what happens when $c_2 > c_1$. Then the configuration is



We consider the possible ordering of the pairs (c_2, b_2) and (c_1, b_1) . If $b_1 < c_1$ and $c_2 < b_2$, then it is clear that we can remove $a1$ and $d2$, so the tableaux pair is not primitive. If only one of these inequalities hold, it is also easy to see that the tableaux pair splits: for example, if $b_1 < c_1$ and $c_2 \geq b_2$, then we can remove the highlighted labels:



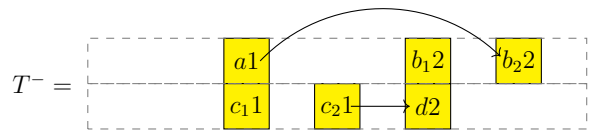
The strict inequality $b_1 < c_1$ ensures that what remains in T^- is semi-standard, and as $d > c_2 \geq b_2$, what is removed from T^- is also semi-standard. The situation when $b_1 \geq c_1$ and $c_2 < b_2$ is analogous.

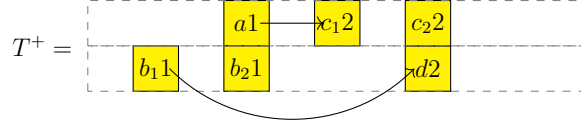
We now consider the situation when $b_1 \geq c_1$ and $c_2 \geq b_2$. Since $c_1 < c_2$ and $b_2 \geq b_1$, we have the chain of inequalities:

$$d > c_2 \geq b_2 \geq b_1 \geq c_1 > a,$$

and at least one of the middle three inequalities must be strict.

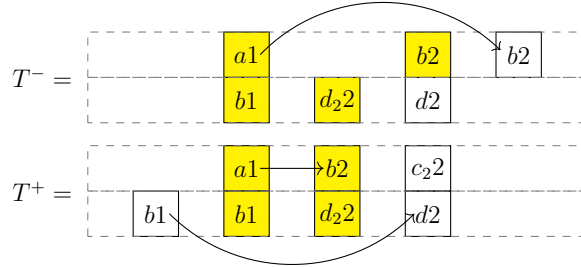
If $b_2 > b_1$ or both $c_2 > b_2$ and $b_1 > c_1$ then we can remove the six highlighted boxes below to obtain a splitting of T^\pm .



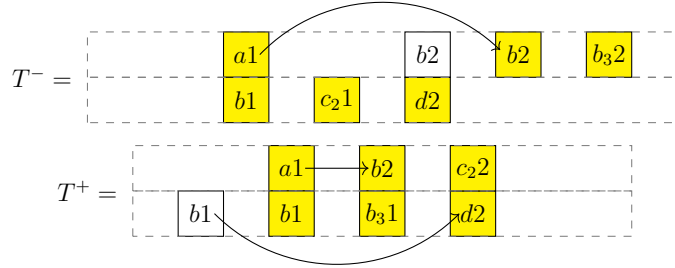


It remains to check what happens with $b_2 = b_1$ and either $c_2 = b_2$ or $b_1 = c_1$. We check the case where $b_2 = b_1 = c_1 < c_2$, and leave the other case to the reader (it is symmetric to the one we check). In this case, we need to consider the label below b_2 in T^+ . It is either of the form d_22 or of the form b_31 . We consider these separately:

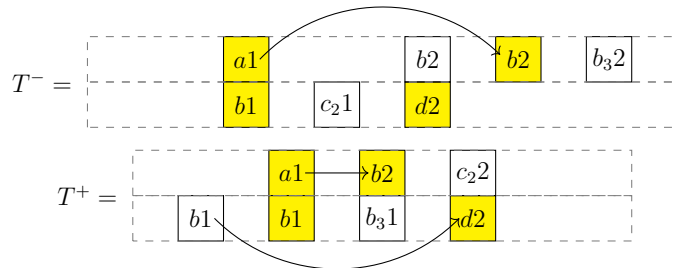
- Suppose the label is of the form d_22 . Then the configuration is below and the highlighted boxes induce a splitting.



- Suppose the label is of the form b_31 . If $b_3 < d$, then the configuration is below and the highlighted boxes induce a splitting.



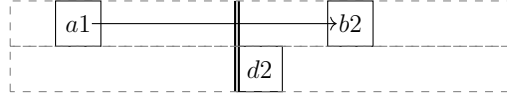
If $b_3 \geq d$, then in fact $b_3 = d > c_2$ and the highlighted boxes below induce a splitting.



□

There is a version for d of Proposition 7.3:

Proposition 7.4. *Let T^\pm be a primitive, semi-standard linked tableaux pair. Then neither T^\pm contains a configuration of the following form:*



We assume that $a1$ is to the left of, and $b2$ to the right of, the vertical double line.

Proof. Since the tableaux set up is symmetric across the first and second rows, this follows from Proposition 7.3. \square

In preparation of the proof of finiteness, we introduce some notation. Let T^\pm be a primitive, semi-standard linked pair. We label the entries of the first row of T^- with second digit 1,

$$a_11 \leq \cdots \leq a_k1,$$

and the entries of the first row with second digit 2

$$b_12 \leq \cdots \leq b_l2.$$

For the second row, we label the entries

$$c_11 \leq \cdots \leq c_l1,$$

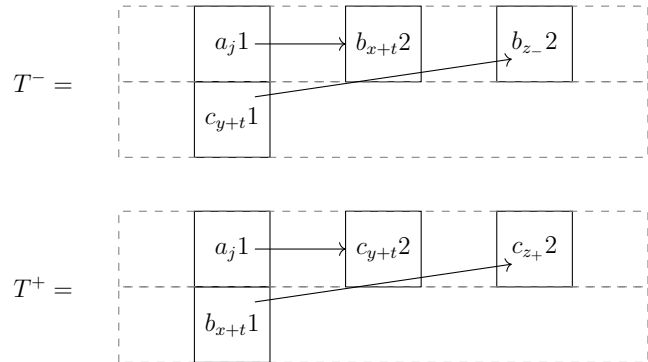
and

$$d_12 \leq \cdots \leq d_k2.$$

It is not hard to see that the number of a s equal the number of d s, and the number of b s equal the number of c s, so the indexing above is justified. Because there are no backwards arrows by Lemma 7.1, as we move through the columns of T^\pm from left to right, we have always passed strictly more a labels than d labels, until we reach the last column. Now consider the configuration near some a_j . Define

- s to be the number of d s to the left of a_j in T^+ and hence also in T^- , by Lemma 7.2;
- $t = j - s > 0$, by the above observation;
- y to be the number of b s to the left of a_j in T^- ;
- x to be the number of c s to the left of a_j in T^+ .

Then near a_j the tableaux look like



The labels have been filled in using the following observations:

- The label a_j1 is the $(j + y)^{th}$ box in the first row of T^- . Since there are s d s to the left of a_j1 , the label b_p1 below a_j1 in T^- must have $p = j + y - s = y + t$. Similar reasoning gives that the label below a_j1 in T^+ is $c_{x+t}1$.

- It follows from Proposition 7.3 that the target of the arrow in the configuration above in T^+ with source $b_{x+t}1$ is on the first row. The analogous statement holds for $c_{y+t}2$ in T^- .

Lemma 7.5. *Using the above notation,*

$$|x - y| < t.$$

Proof. By definition b_y2 is strictly the left of a_j1 in T^- . Since there are no backwards arrows, and a_j1 maps to $b_{x+t}2$, it follows that

$$x + t > y.$$

Looking at T^+ , we similarly obtain that $c_{y+t}2$ is strictly to the right of c_x2 , and so

$$y + t > x.$$

This gives the claim. \square

We now define an *initial set* for each a_j . Assume $x \leq y$. If $y < x$, then the definition is completely analogous using symmetry.

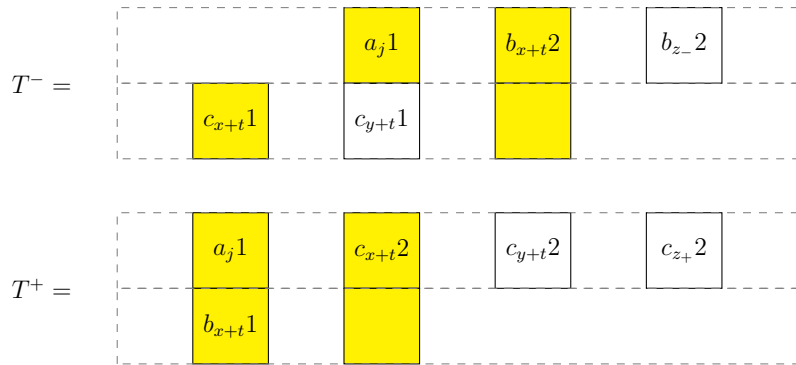
Definition 7.6. *Using the notation above, we define the initial set of a_j to be*

- $I_j^a := \{a_j, c_{x+t}, b_{x+t}\}$ if $a_j < c_{x+t}$,
- $I_j^a := \{a_j, c_{y+t}, b_{y+t}\}$ if $a_j \geq c_{x+t}$.

The initial set is defined so that it begins the description of a sub-tableaux pair of T^\pm that induces a splitting.

Lemma 7.7. *Near a_j , the initial set defines the part of sub-tableaux pair that induces a splitting of T^\pm .*

Proof. Consider the first case, when $a_j < c_{x+t}$. Note by Lemma 7.5, $x + t > y$, and therefore $b_{x+t}2$ is to the right of a_j1 in T^- . Then if we highlight the initial set, plus extra tableaux to indicate the pattern, both what remains and what is removed is semi-standard:



Consider the second case, when $a_j \geq c_{x+t}$. Since by assumption, $c_{x+t} \leq a_j$, in fact $c_{x+t} = a_j$. Applying Proposition 7.3 twice, we obtain that $c_{x+t}2$ is above a label of the form b_p1 and that there are no d type labels between $b_{x+t}1$ and b_p1 in T^+ . Therefore

$$p \geq x + 2t = (x + t) + t > y + t,$$

applying Lemma 7.5. We conclude that in T^+ , c_{x+t} is strictly to the right of b_{y+t} . Again, if we highlight the initial set, plus extra tableaux to indicate the pattern, both what remains and what is removed is semi-standard. What remains is semi-standard because of the inequality:

$$c_{x+t} = a_j < b_{x+t}.$$

We illustrate this below:

$$\begin{array}{l}
 T^- = \begin{array}{|c|c|c|c|} \hline & & a_j 1 & b_{y+t} 2 & b_{z-} 2 \\ \hline c_{x+t} 1 & c_{y+t} 1 & & & \\ \hline \end{array}, \\
 \\
 T^+ = \begin{array}{|c|c|c|c|} \hline a_j 1 & & c_{x+t} 2 & c_{y+t} 2 & c_{z+} 2 \\ \hline b_{x+t} 1 & b_{y+t} 2 & & & \\ \hline \end{array}.
 \end{array}$$

□

We define the notion of an initial set for each d_j – it is the analogue of the definition for the a_j . Set $\hat{j} = k - j + 1$, so that d_j is the \hat{j}^{th} d from the right. Define

- s to be the number of a s to the right of d_j in T^+ and hence also in T^- , by Lemma 7.2;
- $t = \hat{j} - s > 0$,
- y to be the number of b s to the right of d_j in T^+ , and $\hat{y} =: l - y + 1$ (recall that l is the total number of b s, which is equal to the total number of c s),
- x to be the number of c s to the right of d_l in T^- , and $\hat{x} := l - x + 1$.

Then near d_l the tableaux look like

$$\begin{array}{l}
 T^- = \begin{array}{|c|c|c|} \hline & & b_{\hat{y}-t} 2 \\ \hline c_{z-} 2 & c_{\hat{x}-t} 1 & d_l 2 \\ \hline \end{array}, \\
 \\
 T^+ = \begin{array}{|c|c|c|} \hline & & c_{\hat{x}-t} 2 \\ \hline b_{z+} 1 & b_{\hat{y}-t} 1 & d_l 2 \\ \hline \end{array}.
 \end{array}$$

Again, it holds that $|x - y| = |\hat{x} - \hat{y}| < t$.

We now define an *initial set* for each d_j . Assume $x \leq y$. If $y < x$, then the definition is completely analogous using symmetry between T^+ and T^- .

Definition 7.8. Using the notation above, we define the initial set of d_j to be

- $I_j^d := \{d_j, c_{x-t}, b_{x-t}\}$ if $d_j \leq c_{y-t}$,
- $I_j^d := \{d_j, c_{y-t}, b_{y-t}\}$ if $d_j > c_{y-t}$.

As in the a_j case, one can check that the initial sets define the first few boxes of sub-tableaux that give a semi-standard splitting of T^\pm : that is, Lemma 7.7 holds for I_j^d as well.

We are ready to state and prove the main theorem of this section. Before we do that, however, we illustrate the main idea of the theorem as well as the role of initial sets in the example below.

Example 7.9. Consider the semi-standard linked pair, where labels a_i, b_i, c_i, d_i are any choice such that the tableaux are semi-standard. Suppose that $a_4 < b_4$, $c_7 < d_1$, and $c_{12} \geq d_2$ – this determines the initial sets. As described in Theorem 7.10, one can find a splitting of this tableaux pair by starting with the initial set of the last a . In this case, this initial set is $\{a_4, c_4, b_4\}$. Add to this set

labels $c_{4+4\alpha}, b_{4+4\alpha}$, as long as no d s are passed; in case we add only b_8 and c_8 . When d_1 is passed, we start adding multiples of 3 as long as we have not passed d_2 : our set is now

$$\{a_4, c_4, b_4, c_8, b_8, c_{11}, b_{11}\}.$$

As we pass d_2 , we start adding multiples of 2, until we reach d_3 and d_4 (simultaneously in this case). We add d_4 to complete the last column, so the labels are:

$$\{a_4, c_4, b_4, c_8, b_8, c_{11}, b_{11}, c_{13}, b_{13}, c_{15}, b_{15}, d_4\}.$$

We highlight these labels in yellow below.

$T^- =$	a_11	a_21	a_31	a_41	b_{12}	b_{22}	b_{32}	b_{42}	b_{52}	b_{62}	b_{72}	b_{82}	b_{92}	b_{102}	b_{112}	b_{122}	b_{132}	b_{142}	b_{152}
	c_{11}	c_{21}	c_{31}	c_{41}	c_{51}	c_{61}	c_{71}	c_{81}	c_{91}	c_{101}	d_{12}	c_{111}	c_{121}	c_{131}	c_{141}	d_{22}	c_{151}	d_{32}	d_{42}

$T^+ =$	a_11	c_{12}	a_21	a_31	c_{22}	a_{41}	c_{32}	c_{42}	c_{52}	c_{62}	c_{72}	c_{82}	c_{92}	c_{102}	c_{112}	c_{122}	c_{132}	c_{142}	c_{152}
	b_{11}	b_{21}	b_{31}	b_{41}	b_{51}	b_{61}	b_{71}	b_{81}	d_{12}	b_{91}	b_{101}	b_{111}	b_{121}	d_{22}	b_{131}	b_{141}	b_{151}	d_{32}	d_{42}

Whenever we have highlighted entire columns, there is nothing to check as far as semi-standardness goes. In T^+ , there is one location where we do not have an entire column: at the beginning, where Lemma 7.7 ensures that semi-standardness is preserved. In T^- , there is the pair $b_{11}2$ and $c_{13}1$ to consider. Note that these labels fall between the elements of the initial set of d_2 , highlighted in orange. In fact, Lemma 7.7 (or rather its analogue for d s) once again ensures that the tableau that is removed is semi-standard. We check this explicitly: by assumption, $c_{12} \geq d_2$ and $c_{12} \leq d_2$, so $c_{12} = d_2$, hence

$$b_{11} \leq b_{12} < d_2 = c_{12} \leq c_{13}.$$

Theorem 7.10. *The semi-invariant ring of the Kronecker quiver with dimension vector $(2, 2)$ and K arrows has a finite SAGBI basis indexed by linked pairs of semi-standard tableaux of the form:*

$T^- =$

a_1	$\rightarrow b_12$	b_22	b_32	\rightarrow
c_11	c_21	c_31	c_41	

...

$b_{l-3}2$	$b_{l-2}2$	$b_{l-1}2$	b_l2	
$c_{l-2}1$	$c_{l-1}1$	c_l1	$\rightarrow d2$	

$T^+ =$

a_1	$\rightarrow c_12$	c_22	c_32	\rightarrow
b_11	b_21	b_31	b_41	

...

$c_{l-3}2$	$c_{l-2}2$	$c_{l-1}2$	c_l2	
$b_{l-2}1$	$b_{l-1}1$	b_l1	$\rightarrow d2$	

Proof. Note that every primitive semi-standard tableaux pair T^\pm with $k = 1$ is of the form in the theorem (recall that k is the number a s, i.e. the number of labels with second digit 1 in the first row of T^+). We first show that the set as described in the theorem is finite. Note that as the tableaux are semi-standard, the labels satisfy two chains of inequalities:

$$a < c_1 < b_2 < c_3 < \cdots < b_l < d, \quad a < b_1 < c_2 < b_3 < \cdots < c_l < d.$$

Since every label lies between 1 and K , as there are K arrows in the quiver, this is only possible if $l + 2 \leq K$. Therefore, only tableaux pairs of dimensions $2 \times j$, $j \leq K - 1$ can possibly be primitive. This implies finiteness of the SAGBI basis.

It remains to show that if T^\pm is a primitive semi-standard tableaux pair, then $k = 1$. This uses the construction of the initial set set above. Suppose that $k > 1$. Begin with furthest right a label, a_k , and take its initial set. Near a_k , the tableaux pair looks like:

$$T^- = \begin{array}{|c|c|c|} \hline a_k 1 & \rightarrow b_{x+t} 2 & \rightarrow b_{z-} 2 \\ \hline c_{y+t} 1 & & \\ \hline \end{array},$$

$$T^+ = \begin{array}{|c|c|c|} \hline a_k 1 & \rightarrow c_{y+t} 2 & \rightarrow c_{z+} 2 \\ \hline b_{x+t} 1 & & \\ \hline \end{array}.$$

In T^- , if $p1$ is a label weakly to left of c_{y+t} , then $p \leq c$. There are $y + t + k - 1$ such labels, and $y + 2t - 1$ of them have an arrow to the first row. If $p1$ is a label strictly to the right of c_{y+t} , then $p1$ must appear in the second row of T^- , and hence $p \geq c$. Therefore $z_- = y + 2t$. Similar reasoning demonstrates that $z_+ = x + 2t$.

As long as only bs and cs appear, this pattern continues: the target of the arrow from $b_p 1$ in T^+ is $c_{p+t} 2$, and the target of the arrow from $c_p 1$ in T^- is $b_{p+t} 2$.

Until a d type box is passed in either T^\pm , we add boxes to our initial set by adding t to the b and c labels appearing in I_k^a . This determines a sub-tableau that, other than for the initial set, contains columns t apart. That is, we obtain either the set

$$\{a_k, c_{x+t}, b_{x+t}\} \cup \{c_{x+\alpha t}, b_{x+\alpha t} : 2 \leq \alpha \leq A, \}$$

or

$$\{a_k, c_{y+t}, b_{y+t}\} \cup \{c_{y+\alpha t}, b_{y+\alpha t} : 2 \leq \alpha \leq A, \}$$

where A is an integer that ensures that all elements are to the left of d_{s+1} . Recall that d_{s+1} is the first d on the right of a_k . We illustrate this in the case where $I_k^a := \{a_k, c_{x+t}, b_{x+t}\}$ and $x \leq y$:

$$T^- = \begin{array}{|c|c|c|c|c|} \hline & a_k 1 & b_{x+t} 2 & b_{x+2t} 2 & b_{x+3t} 2 \\ \hline c_{x+t} 1 & c_{y+t} 1 & c_{x+2t} 2 & c_{x+3t} 2 & c_{x+4t} 2 \\ \hline \end{array},$$

$$T^+ = \begin{array}{|c|c|c|c|} \hline a_k 1 & c_{x+t} 2 & c_{x+2t} 2 & c_{x+3t} 2 \\ \hline b_{x+t} 1 & b_{x+2t} 2 & b_{x+3t} 2 & b_{x+4t} 2 \\ \hline \end{array}.$$

There is similar start to sub-tableaux beginning with initial set of d_{s+1} , and moving to the left. We set $\underline{x}, \underline{y}, \underline{t}, \underline{s}$ to be indices associated with d_{s+1} . As $\underline{s} = 0$, note that

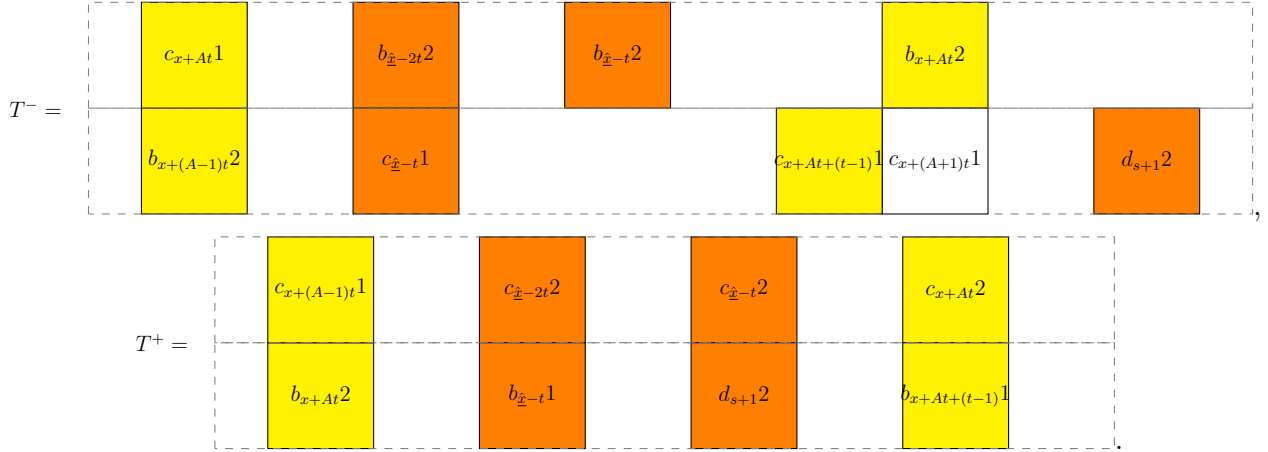
$$\underline{t} = k - (s + 1) + 1 - \underline{s} = k - s = t.$$

Therefore the columns in the sub-tableaux starting at d_{s+1} are also t apart. They therefore either completely coincide with the columns from a_k or are disjoint. If they coincide, we have a sub-tableaux pair of T^\pm , which, by Lemma 7.7 is semi-standard and, once removed, what remains is also semi-standard. This contradicts primitivity.

Otherwise, the columns from the initial set of a_k have no common labels with the initial set of d_{s+1} . In this case, we do the following. While adding boxes to our initial set starting at a_k , when we reach a box that lies between d_{s+1} and one of the other two labels in I_{s+1}^d , we add the other box not in the column of T^\pm but in the column that would appear when I_{s+1}^d is removed. We illustrate this in the case where $I_k^a := \{a_k, c_{x+t}, b_{x+t}\}$, $I_{s+1}^d = \{d_{s+1}, c_{\hat{x}-t}, b_{\hat{x}-t}\}$, and $\hat{x} \leq \underline{y}$. We can define A now more precisely as the smallest integer satisfying

$$x + At \geq \hat{x} - t,$$

(this could be expressed using the floor function). Note that the inequality is necessarily strict as we have assumed that $x \not\equiv \hat{x}$ modulo t .



Here we have highlighted in yellow the subtableaux starting at a_k , and in orange the subtableaux starting at d_{s+1} . Until we pass d_{s+2} , we can keep adding boxes to the sub-tableaux starting at a_k so that columns are now $t - 1$ apart. That is, in the case illustrated above, the sub-tableaux contain labels

$$\{a_k, c_{x+t}, b_{x+t}\} \cup \{c_{x+\alpha t}, b_{x+\alpha t} : 2 \leq \alpha \leq A, \} \cup \{c_{x+At+\alpha(t-1)}, b_{x+At+\alpha(t-1)} : 2 \leq \alpha \leq B\},$$

where B is an integer that ensures that all elements are to the left of d_{s+2} . If $I_k^a = \{a_k, c_{y+t}, b_{y+t}\}$, then the sub-tableaux contains the labels

$$\{a_k, c_{y+t}, b_{y+t}\} \cup \{c_{y+\alpha t}, b_{y+\alpha t} : 2 \leq \alpha \leq A, \} \cup \{c_{y+At+\alpha(t-1)}, b_{y+At+\alpha(t-1)} : 2 \leq \alpha \leq B\}.$$

The situation when $I_{s+1}^d = \{d_{s+1}, c_{\hat{y}-t}, b_{\hat{y}-t}\}$ and $\hat{x} \leq \underline{y}$, is very similar: the only change is in the definition of A as the smallest integer satisfying

$$x + At \geq \hat{y} - t.$$

This process continues for each d_{s+p} : either we end the sub-tableaux using the initial set of d_{s+p} , or we by-pass it using the same method as for d_{s+1} , and end up with a sub-tableaux where the columns at this stage are now $t - p$ apart, and the \underline{t} for d_{s+p+1} is $t - p$. If we pass d_{k-1} , then the columns are 1 apart, and so we necessarily end our sub-tableaux using the initial set coming from d_k .

The final result is a sub-tableaux pair that induces a semi-standard splitting of T^\pm . The fact that all tableaux in the splitting are semi-standard follows from Lemma 7.7 and its d analogue. This concludes the proof of the theorem. \square

8. AN EXAMPLE OF A TORIC DEGENERATION

In this section, we describe a toric degeneration of a Fano quiver moduli space using the SAGBI basis described above. Let Q be the Kronecker quiver with three arrows and dimension vector $\mathbf{r} = (2, 3)$.

Let $\theta := (-9, 6)$ be the anti-canonical stability condition. Then the unstable locus with respect to θ has co-dimension at least 2 by [19, Proposition 5.2]. By [11], the stable and semi-stable locus coincide. The GIT quotient $M_\theta(Q, \mathbf{r})$ is a six-dimensional smooth Fano variety with Fano index 3. Its Cox ring coincides with the semi-invariant ring, and therefore a SAGBI basis is given by the linked tableaux in Example 6.7. Therefore, there exists a toric degeneration from $M_\theta(Q, \mathbf{r})$ to the toric variety given by the leading terms of the semi-invariants. This toric variety corresponds to the Fano polytope P with vertices:

$$(2) \quad \begin{aligned} &\{(2, -1, 2, 0, -1, 0), (2, -1, 2, 1, -1, -1), (1, 0, 1, 0, -1, 0), (1, 0, 1, 1, -1, -1), \\ &(0, 0, 0, 0, 0, 1), (0, 0, 0, 0, 1, 0), (0, 0, 0, 1, 0, 0), (0, 0, 1, 0, 0, 0), (1, -1, 0, -1, 0, 0), \\ &(2, -1, 1, 0, 0, 0), (1, -1, 1, -1, -1, 0), (-1, 1, -1, 0, 1, 0), (-1, 0, -1, 0, 0, 0)\}. \end{aligned}$$

We interpret this degeneration in the context of mirror symmetry for Fano varieties. For some background on mirror symmetry for Fano varieties, see [4, 5]. The mirror to a deformation class of n -dimensional smooth Fano varieties, also known as its (weak) *Landau–Ginzburg model*, is a mutation class of certain Laurent polynomials in n variables. The Laurent polynomials that are mirror to smooth Fano varieties are, conjecturally, *rigid maximally mutable* Laurent polynomials [6].

To check that representatives X and f belong to mirror classes, one shows that two power series coincide. The power series associated to a Fano variety X is the *quantum period*, which is built out of genus zero Gromov–Witten invariants, and hence deformation invariant. The power series associated to the Laurent polynomial f is called the *classical period*; it is easy to compute and is mutation invariant. See [5] for a definition of both periods. Checking that the two periods coincide can be very difficult, as it is often hard or impossible to find a closed formula for the quantum period of a Fano variety. For Fano toric complete intersections, this is a consequence of the celebrated mirror theorem [12, 17].

The Newton polytope P of a Laurent polynomial f which is a *rigid maximally-mutable* Laurent polynomial [6] is a Fano polytope, i.e. P spans the fan of a (singular) toric Fano variety X_P . Part of the mirror symmetry conjectures is that X_P can smooth to a Fano variety X which is mirror to f . Thus, if we want to find a conjectural mirror to a Fano variety X , we find a toric degeneration of X to some X_P . This determines the monomials of f , and coefficients are then chosen to ensure that f is rigid maximally mutable.

The toric variety associated to the face-fan of the polytope P with vertices (2) is Gorenstein Fano, with terminal singularities. Since P contains no other lattice points except for the origin and its vertices, there exists a unique rigid maximally-mutable Laurent polynomial supported on P . In other words,

$$\begin{aligned} f = & x_1^2 x_3^2 x_4 / (x_2 x_5 x_6) + x_1^2 x_3^2 / (x_2 x_5) + x_1^2 x_3 / x_2 + x_1 x_3 x_4 / (x_5 x_6) \\ & + x_1 x_3 / x_5 + x_1 x_3 / (x_2 x_4 x_5) + x_1 / (x_2 x_4) + x_3 + x_4 + x_5 + x_6 + x_2 x_5 / (x_1 x_3) + 1 / (x_1 x_3), \end{aligned}$$

where each monomial corresponds to one of the 13 vertices of P and f has coefficient 1 for each of these. We can now consider the mirror condition for $M_\theta(Q, \mathbf{r})$ and f .

The first 20 terms of the period sequence of this Laurent polynomial are

$[1, 0, 0, 18, 0, 0, 4590, 0, 0, 1728720, 0, 0, 876610350, 0, 0, 520461209268, 0, 0, 343838539188144, 0]$.

As a sanity check, the periodicity of pairs of zeroes in this sequence confirms the Fano index 3 of $M_\theta(Q, \mathbf{r})$. Conjecturally, the sequence should coincide with the quantum period of $M_\theta(Q, \mathbf{r})$. The Fano variety $M_\theta(Q, \mathbf{r})$ can also be described [2] as the zero locus of a generic section of $E^* \otimes \det(E) = \wedge^5 E$ on $\mathrm{Gr}(8, 6)$, where E is the rank six tautological quotient bundle. Then, in theory, terms of the quantum period of $M_\theta(Q, \mathbf{r})$ can be computed [15]. In practice this is very expensive computationally; however, we have verified that the first 7 terms agree with the sequence above.

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