

Next-to-next-to-leading-order QCD corrections to pion electromagnetic form factors

Long-Bin Chen ^{*,1} Wen Chen ^{†,2} Feng Feng ^{‡,3,4} and Yu Jia ^{§,4,5}

¹ *School of Physics and Materials Science, Guangzhou University, Guangzhou 510006, China*

² *Zhejiang Institute of Modern Physics, School of Physics, Zhejiang University, Hangzhou, Zhejiang 310027, China*

³ *China University of Mining and Technology, Beijing 100083, China*

⁴ *Institute of High Energy Physics and Theoretical Physics Center for Science Facilities, Chinese Academy of Sciences, Beijing 100049, China*

⁵ *School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China*

(Dated: December 29, 2023)

We investigate the next-to-next-to-leading order (NNLO) QCD radiative corrections to the pion electromagnetic form factor with large momentum transfer. We explicitly verify the validity of the collinear factorization to two-loop order for this observable, and obtain the respective IR-finite two-loop hard-scattering kernel in the closed form. The NNLO QCD correction turns to be positive and significant. Incorporating this new ingredient of correction, we then make a comprehensive comparison between the finest theoretical predictions and various pion form factor data in both space-like and time-like regions. Our phenomenological analysis provides strong constraints on the second Gegenbauer moments of the pion light-cone distribution amplitude (LCDA) obtained from recent lattice QCD studies.

Introduction. Originally proposed by Yukawa as the strong nuclear force carrier in 1935 [1], the π mesons have always occupied the central stage throughout the historic advancement of the strong interaction. As the lightest particles in the hadronic world (hence the highly-relativistic bound systems composed of light quark and gluons), π mesons entail extremely rich QCD dynamics, exemplified by the color confinement and chiral symmetry breaking. Notwithstanding extensive explorations during the past decades, there still remain some great myths about internal structure of π mesons.

A classic example of probing the internal structure of charged pions is the pion electromagnetic (EM) form factor:

$$\langle \pi^+(P') | J_{\text{em}}^\mu | \pi^+(P) \rangle = F_\pi(Q^2)(P^\mu + P'^\mu), \quad (1)$$

with the electromagnetic current defined by $J_{\text{em}}^\mu = \sum_f e_f \bar{\psi}_f \gamma^\mu \psi_f$, and $Q^2 \equiv -(P - P')^2$.

During the past half century, the pion EM form factor has been extensively studied experimentally [2–28]. At small Q^2 , dominated by the soft strong interaction, the pion EM form factor is investigated by chiral perturbation theory [29] and lattice QCD [30–34], from which one can infer the pion charge radius. On the other hand, at large momentum transfer, the $F_\pi(Q^2)$ is expected to be adequately described by perturbative QCD. Within the collinear factorization framework tailored for hard exclusive reactions [35–41] (for a review, see [42]), at the lowest

order in $1/Q$, the pion EM form factor can be expressed in the following form:

$$F_\pi(Q^2) = \iint dx dy \Phi_\pi^*(x, \mu_F) T(x, y, \frac{\mu_R^2}{Q^2}, \frac{\mu_F^2}{Q^2}) \Phi_\pi(y, \mu_F), \quad (2)$$

where $T(x, y)$ signifies the perturbatively calculable hard-scattering kernel, and $\Phi_\pi(x, \mu_F)$ represents the nonperturbative yet universal leading-twist pion light-cone distribution amplitude (LCDA), *i.e.*, the probability amplitude of finding the valence u and \bar{d} quarks inside π^+ carrying the momentum fractions x and $\bar{x} \equiv 1 - x$, respectively. The pion LCDA can be parameterized as

$$\Phi_\pi(x, \mu_F) = \frac{f_\pi}{2\sqrt{2}N_c} \phi_\pi(x, \mu_F), \quad \int_0^1 \phi_\pi(x, \mu_F) dx = 1, \quad (3)$$

with $N_c = 3$ is the number of colors, and the pion decay constant $f_\pi = 0.131$ GeV. The dimensionless leading-twist pion LCDA assumes the following operator definition:

$$\begin{aligned} \phi_\pi(x, \mu_F) &= \frac{i}{f_\pi} \int \frac{dz^-}{2\pi} e^{-ixz^- P^+} \langle \pi(P) | \bar{\psi}(0) \gamma^+ \gamma_5 \\ &\times \mathcal{W}(0, z^-) \psi(z^-) | 0 \rangle, \end{aligned} \quad (4)$$

with \mathcal{W} signifies the light-like gauge link to ensure the gauge invariance. Conducting the UV renormalization for (4), one is led to the celebrated Efremov-Radyushkin-Brodsky-Lepage (ERBL) evolution equation [37, 39]:

$$\frac{d\phi_\pi(x, \mu_F)}{d \ln \mu_F^2} = \int_0^1 dy V(x, y) \phi_\pi(y, \mu_F). \quad (5)$$

Eq. (2) is expected to hold to all orders in perturbative expansion. The hard-scattering kernel

*chenlb@gzhu.edu.cn

†chenwenphy@gmail.com

‡f.feng@outlook.com

§jiay@ihep.ac.cn

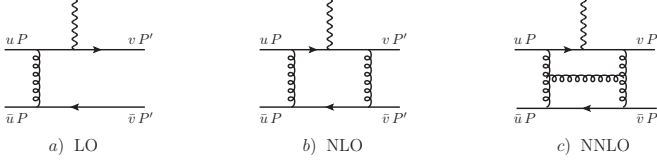


FIG. 1: Sample parton-level Feynman diagrams for the reaction $\gamma^* \pi(P) \rightarrow \pi(P')$ at various perturbative orders.

$T(x, y, \mu_R^2/Q^2, \mu_F^2/Q^2)$ can thus be expanded in the power series:

$$T = \frac{\alpha_s}{\pi Q^2} \left\{ T^{(0)} + \frac{\alpha_s}{\pi} T^{(1)} + \left(\frac{\alpha_s}{\pi} \right)^2 T^{(2)} + \dots \right\}, \quad (6)$$

The leading order (LO) result was known shortly after the advent of QCD [36, 37, 39, 41]. The next-to-leading order (NLO) correction was originally computed by three groups in early 80s [43–45]. Unfortunately these results are not compatible with each other. In 1987, scrutinizing the previous calculations, Braaten and Tse traced the origin of the discrepancies among the earlier work and presented the correct expression of the NLO hard-scattering kernel [46]. In 1998, Melić, Nižić, and Passek conducted a comprehensive phenomenological study by incorporating the NLO correction as well as the evolution effect of pion LCDA [47]. The central goal of this work is to compute the next-to-next-to-leading order (NNLO) perturbative correction to pion EM form factor, and critically examine its phenomenological impact.

Setup of perturbative matching. The strategy of deducing the short-distance coefficients is through the standard matching procedure. Since the hard-scattering kernel is insensitive to the long-distance physics, it is legitimate to replace the physical $|\pi^+\rangle$ state by a fictitious pion state, *i.e.*, a free massless quark-antiquark pair $|u\bar{d}\rangle$, and compute both sides of (2) in perturbation theory, order by order in α_s . To make things simpler, we neglect the transverse motion, assign the momenta of the u and \bar{d} in the incoming “pion” to be uP and $\bar{u}P$, and assign the momenta of the u and \bar{d} in the outgoing “pion” to be vP and $\bar{v}P$, with u, v range from 0 to 1.

In the left-hand side of (2), we extract the scalar form factor $F(u, v)$ through the partonic reaction $\gamma^* + u(uP)\bar{d}(\bar{u}P) \rightarrow u(vP')\bar{d}(\bar{v}P')$. Some typical Feynman diagrams through two-loop are depicted in Fig. 1. It is subject to a perturbative expansion:

$$F(u, v) = F^{(0)}(u, v) + \frac{\alpha_s}{\pi} F^{(1)}(u, v) + \left(\frac{\alpha_s}{\pi} \right)^2 F^{(2)}(u, v) + \dots \quad (7)$$

In the right-hand side of (2), one can expand the renormalized “pion” LCDA as

$$\phi(x|u) = \phi^{(0)}(x|u) + \frac{\alpha_s}{\pi} \phi^{(1)}(x|u) + \left(\frac{\alpha_s}{\pi} \right)^2 \phi^{(2)}(x|u) + \dots \quad (8)$$

At tree level, the fictitious pion DA in (4) simply reduces to $\phi^{(0)}(x|u) = \delta(x - u)$. By equating both sides of

(2), one reproduces the well-known tree-level expression $T^{(0)}(x, y)$ [35–41]

$$T^{(0)}(x, y) = e_u C_F \frac{16\pi^2}{\bar{x}\bar{y}} (1 - \epsilon) - \left[x \rightarrow \bar{x}, y \rightarrow \bar{y} \right], \quad (9)$$

with $e_u = 2/3$ and $e_d = -1/3$ are electric charges of the u and \bar{d} quarks. This equation is exact in $d = 4 - 2\epsilon$ spacetime dimension.

Once beyond the tree level, the UV and IR divergences inevitably arise and we use the dimensional regularization to regularize both types of divergences. Nevertheless, the bare “pion” LCDA remains intact due to the scaleless integrals vanish in the dimensional regularization scheme. The renormalized “pion” LCDA is related to the bare one via

$$\phi(x|u) = \int dy Z(x, y) \phi_{\text{bare}}(y|u) = Z(x, u), \quad (10)$$

which is solely comprised of various IR poles.

$Z(x, y)$ in (10) signifies the renormalization function in the $\overline{\text{MS}}$ scheme, which can be cast into the following Laurent-expanded form in ϵ :

$$Z(x, y) = \delta(x - y) + \sum_{k=1}^{\infty} \frac{1}{\epsilon^k} Z_k(x, y), \quad (11)$$

Note that the prefactor of single pole in (11) is related to the ERBL kernel $V(x, y)$ in (5) via [48]

$$V(x, y) = -\alpha_s \frac{\partial Z_1}{\partial \alpha_s}. \quad (12)$$

Note that the two-loop [45, 49–52] and three-loop corrections [53] to the ERBL kernel $V(x, y)$ have been known.

The two-loop renormalized LCDA $\phi^{(2)}$ also contains double IR pole. The Z_2 can be obtained through the recursive relation [54]

$$\alpha_s \frac{\partial Z_2}{\partial \alpha_s} = \alpha_s \frac{\partial Z_1}{\partial \alpha_s} \otimes Z_1 + \beta(\alpha_s) \frac{\partial Z_1}{\partial \alpha_s}, \quad (13)$$

where $d\alpha_s/d\ln\mu^2 = \beta(\alpha_s) - \epsilon\alpha_s$.

With the aid of (10) and (11), we then determine the $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s^2)$ corrections to the renormalized “pion” LCDA in (8).

At one-loop order, the matching condition for fictitious pion states becomes

$$\begin{aligned} Q^2 F^{(1)}(u, v) = & T^{(1)}(u, v) + \phi^{(1)}(x|u) \otimes_x T^{(0)}(x, v) \\ & + \phi^{(1)}(y|v) \otimes_y T^{(0)}(u, y), \end{aligned} \quad (14)$$

where \otimes_x is the shorthand for the convolution integration over x . Note that the renormalized scalar form factor $F^{(1)}(u, v)$ still contains single collinear pole. However, the renormalized $\phi^{(1)}(x|u)$ and $\phi^{(1)}(y|v)$ also contains the same IR poles. Upon solving this matching equation,

one ends with both UV and IR-finite $T^{(1)}(x, y)$. Our expressions agree with the known NLO result [47].

To the desired two-loop order, the following matching equation descends from (2):

$$\begin{aligned}
Q^2 F^{(2)}(u, v) = & T^{(2)}(u, v) + \phi^{(2)}(x|u) \otimes_x T^{(0)}(x, v) \\
& + \phi^{(2)}(y|v) \otimes_y T^{(0)}(u, y) \\
& + \phi^{(1)}(x|u) \otimes_x T^{(1)}(x, v) \\
& + \phi^{(1)}(y|v) \otimes_y T^{(1)}(u, y) \\
& + \phi^{(1)}(x|u) \otimes_x T^{(0)}(x, y) \otimes_y \phi^{(1)}(y|v),
\end{aligned} \tag{15}$$

Clearly one needs compute $T^{(1)}(x, y)$ to $\mathcal{O}(\epsilon)$. There are more severe IR divergences in both $F^{(2)}(u, v)$ and $\phi^{(1,2)}(x|u)$ compared with the NLO case. However, for the collinear factorization to hold, these IR poles must exactly cancel so that one ends up with an IR-finite $T^{(2)}(u, v)$.

Description of the calculation. We use **HepLib** [55] and **FeynArts** [56] to generate Feynman diagrams and the corresponding amplitudes for the partonic reaction $\gamma^* + u(P)\bar{d}(\bar{u}P) \rightarrow u(vP')\bar{d}(\bar{v}P')$. We then employ the covariant projector technique to enforce each $u\bar{d}$ pair to bear zero helicity. For our purpose it suffices to adopting the naive anticommutation relation to handle γ_5 . There are about 1600 two-loop diagrams, one of which is depicted in Fig. 1c). We employ the package **Apart** [57] to conduct partial fraction, and **FIRE** [58] for integration-by-part reduction. We end up with 116 independent master integrals (MIs). The MIs are calculated by using the differential equations method [59–61]. Note that these MIs are considerably more involved than what are encountered in the two-loop corrections for the $\pi - \gamma$ transition form factor [62, 63]. We have attempted two independent ways to construct and solve the differential-equation systems, one of which is based on the method developed in [64–67]. The analytic results are expressed in terms of Goncharov Polylogarithms (GPLs) [68]. Two independent calculations yield the identical answer. We also numerically check our results against the package **AMFLOW** [69] and found perfect agreement. More technical details will be included in the future long write-up.

Upon renormalizing the QCD coupling in $\overline{\text{MS}}$ scheme, we end up with a lengthy expression of $F^{(2)}(u, v)$. Being UV finite, it still contains severe IR divergences which start at order- $1/\epsilon_{\text{IR}}^2$. Inspecting the matching equation (15), piecing all the known ingredients together, we are able to solve for the intended two-loop hard-scattering kernel. Hearteningly, $T^{(2)}(x, y)$ is indeed IR finite. Therefore, our explicit calculation verifies that the collinear factorization does hold at two-loop level for the pion EM form factor. The analytical expression of $T^{(2)}(x, y)$ is too lengthy to be reproduced here. For the sake of clarity, in Appendix we provide the asymptotic expressions for $T^{(1,2)}(x, y)$ near the end point regions.

Master formula for pion EM form factor at NNLO. Given a certain parametrized form of pion LCDA, the two-fold

convolution integration in (2) turns out to be difficult to conduct, mainly due to numerical instability caused by the fake singularity as $x \rightarrow y$ and $x \rightarrow \bar{y}$ in $T^{(2)}(x, y)$. Here we provide an efficient recipe to predict the pion EM form factor to two-loop order with high accuracy.

The leading-twist pion LCDA can be conveniently expanded in the Gegenbauer polynomial basis:

$$\phi_\pi(x, \mu_F) = \sum_{n=0}' a_n(\mu_F) \psi_n(x), \tag{16a}$$

$$\psi_n(x) = 6x\bar{x}C_n^{3/2}(2x-1) \tag{16b}$$

where \sum' signifies the sum over even integers, and all the nonperturbative dynamics is encoded in the Gegenbauer moments $a_n(\mu_F)$.

Substituting (16) into (2), conducting two-fold integration, we can reexpress the pion EM form factor as

$$Q^2 F_\pi(Q^2) = \frac{f_\pi^2}{24} \sum_{k=0} \left(\frac{\alpha_s}{\pi} \right)^{k+1} \sum_{m,n}' a_n(\mu_F) a_m(\mu_F) \mathcal{T}_{mn}^{(k)}, \tag{17}$$

with $\mathcal{T}_{mn}^{(k)}$ defined by

$$\mathcal{T}_{mn}^{(k)} = \psi_m(x) \otimes_x T^{(k)}(x, y, \frac{\mu_R^2}{Q^2}, \frac{\mu_F^2}{Q^2}) \otimes_y \psi_n(y). \tag{18}$$

For simplicity, we will set $\mu_R = \mu_F = \mu$ and $n_L = 3$ from now on. The two-fold convolution integrals in (18) can be readily worked out at tree and one-loop levels. For instance, we have

$$\mathcal{T}_{mn}^{(0)} = 192\pi^2(e_u - e_d), \tag{19a}$$

$$\mathcal{T}_{00}^{(1)} = 16\pi^2(e_u - e_d)(27L_\mu + 79), \tag{19b}$$

with $L_\mu \equiv \ln(\mu^2/Q^2)$.

(m,n)	c_1	c_2	d_1	d_2	d_3
(0,0)	4263.669	12475.18	19186.51	100735.4	146588.2
(0,2)	6895.564	23681.40	35818.62	230425.7	426544.1
(0,4)	8095.708	31085.44	44616.34	324507.5	675234.9
(0,6)	8897.307	36711.64	50915.90	400305.4	898014.2
(2,2)	9527.458	40609.34	56106.14	458633.6	1042937
(2,4)	10727.60	50570.49	66570.73	605456.2	1523871
(2,6)	11529.20	57896.28	73983.62	719123.7	1931327
(4,4)	11927.75	61685.28	77795.41	779945.0	2152367
(4,6)	12729.35	69779.17	85715.98	913298.2	2673762
(6,6)	13530.95	78384.61	93975.64	1060605	3282163

TABLE I: The numerical values for $\mathcal{T}_{mn}^{(1)} = (e_u - e_d)(c_1 L_\mu + c_2)$ and $\mathcal{T}_{mn}^{(2)} = (e_u - e_d)(d_1 L_\mu^2 + d_2 L_\mu + d_3)$, with $0 \leq m, n \leq 6$.

As a bonus, the coefficients $\mathcal{T}_{mn}^{(2)}$ can also be computed analytically, since $T^{(2)}$ are expressed in terms of GPLs. Although the integrand involving $T^{(2)}$ in (18) contain about $\mathcal{O}(10^5)$ terms, the final result after two-fold integration becomes exceedingly compact, which can be expressed in terms of the rational numbers and Riemann zeta function. For instance, the expression of $\mathcal{T}_{00}^{(2)}$ reads

$$\mathcal{T}_{00}^{(2)} = \pi^2(e_u - e_d) \left\{ 1944L_\mu^2 - L_\mu \left[\frac{512\zeta_3}{3} + \frac{1120\pi^2}{9} - 11640 \right] + \frac{13120\zeta_5}{3} - 8096\zeta_3 - \frac{16\pi^4}{5} - \frac{3658\pi^2}{9} + \frac{73118}{3} \right\}. \quad (20)$$

Due to the length restriction, we refrain from providing the analytical expressions for other $\mathcal{T}_{mn}^{(1,2)}$. For reader's convenience, in Table I we tabulate the numerical values of $\mathcal{T}_{mn}^{(1,2)}$ for $0 \leq m, n \leq 6$, which is sufficient for most phenomenological purpose.

With the input from Table I, Eq. (17) is our master formula for yielding phenomenological predictions through the two-loop accuracy. Compared with (2), we have simplified an integration task into an algebraic one.

It is straightforward to adapt our master formula for the space-like pion EM form factor to the time-like one, provided that one makes the replacement $L_\mu \rightarrow L_\mu + i\pi$ in Table I, with Q^2 now indicating the squared invariant mass of the $\pi^+\pi^-$ pair.

Input parameters. As the key nonperturbative input, our knowledge on the pion LCDA is still not confirmative enough. In the early days, it is popular to assume asymptotic form, CZ parametrization [42] and the BSM parameterizations [70, 71]. In recent years there have emerged extensive investigations on the profile of the pion DA from different methodology, including QCD light-cone sum rule [72] with nonlocal condensate [70, 73] or fitted from dispersion relation [74] or Platykurtic [75], Dyson-Schwinger equation [76, 77], basis light-front quantization [78], light-front quark model [79], holographic QCD [80], and very recently, from the lattice simulation [81, 82]. Various Gegenbauer moments predicted from these approaches are scattered in a wide range.

Since lattice QCD provides the first-principle approach, we will take the recent lattice results as inputs. In 2019 RQCD Collaboration has presented a precise prediction for the second Gegenbauer moment of pion LCDA in $\overline{\text{MS}}$ scheme [81]:

$$a_2(2 \text{ GeV}) = 0.116_{-0.020}^{+0.019}, \quad (21)$$

An important progress in lattice QCD is brought by the advent of the Large-momentum effective theory (LaMET) a decade ago [83, 84], which allows one to access the light-cone distributions in Euclidean lattice directly in the x space. Very recently, the LPC Collaboration has presented the whole profile of the pion LCDA [82], from which various Gegenbauer moments can be inferred:

$$\begin{aligned} a_2(2 \text{ GeV}) &= 0.258 \pm 0.087, \\ a_4(2 \text{ GeV}) &= 0.122 \pm 0.056, \\ a_6(2 \text{ GeV}) &= 0.068 \pm 0.038. \end{aligned} \quad (22)$$

It is curious that the value of a_2 reported by LPC Collaboration is about twice greater than that reported by the RQCD Collaboration. This discrepancy might be

attributed to the fact that the LaMET approach receives large power correction in the endpoint region. On the other hand, it is very challenging for the OPE approach [81] to compute the higher Gegenbauer moments, thus difficult to reconstruct the whole profile of the LCDA.

Phenomenological exploration. We use the three-loop evolution equation [53, 85] to evolve each a_n evaluated at 2 GeV by lattice simulation to any intended scale μ_F . In the phenomenological study, we only retain those a_n with n up to 6. We also use the package FAPT [86] to evaluate the running QCD coupling constant to three-loop accuracy.

For the sake of comparison, we take the pion EM form factor data in the spacetime region from NA7 collaboration [10], Corne11 data compiled by Bebek *et al.* [4], and reanalyzed JLab data [15], and take the pion EM form factor data in the timelike region entirely from BaBar [26]. We discard many irrelevant data at small momentum transfer.

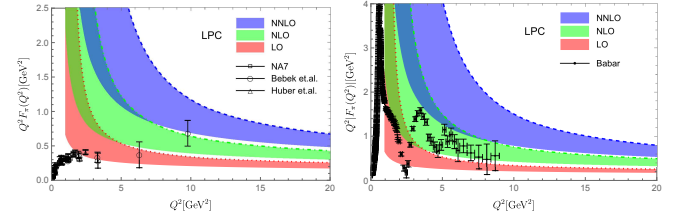


FIG. 2: Theoretical predictions vs. data for $Q^2 F_\pi(Q^2)$ in the space-like (left panel) and time-like (right panel) regimes. We take the various Gegenbauer moments of pion LCDA from the central values of (22) given by LPC collaboration. The red, green and blue curves correspond to the LO, NLO and NNLO results, and the respective bands are obtained by sliding μ from $Q/2$ to Q . Experimental data points are taken from NA7 [10], Bebek *et al.* [4], Huber *et al.* [15] and BaBar [26].

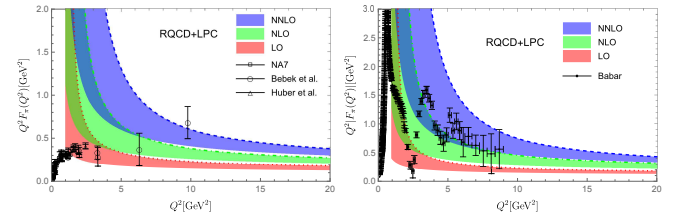


FIG. 3: Same as Fig. 2, except the predictions are made by taking a_2 from (21) by RQCD, and taking the values of a_4 and a_6 as the lower bounds in (22) by LPC.

In Fig. 2 and Fig. 3, we confront our predictions at various perturbative accuracy with the available data, in-

cluding both space-like and time-like regime. One clearly visualizes that NNLO correction is important and positive. In Fig. 2, we set the various Gegenbauer moments of pion LCDA to the central values of (22) given by LPC Collaboration [82]. It appears the NNLO predictions are significantly overshooting the experimental data at large Q^2 ($> 5 \text{ GeV}^2$), especially for the time-like regime with high statistics data. This symptom is mainly due to the large value of a_2 given in (22).

In Fig. 3 we represent our predictions with a_2 taken from (21) given by RQCD, yet still quote the LPC values for a_4 and a_6 (as the lower bounds in (22)). We find satisfactory agreement between the NNLO predictions and the data, both in space-like and time-like regimes. This might indicate that the value of a_2 given by RQCD might be more trustworthy. It is of great important for RQCD and LPC collaborations to settle the discrepancy in the value of a_2 .

The prospective Electron-Ion Collider (EIC) program plans to measure the space-like pion EM form factor with Q^2 as large as 30 GeV^2 [87], where perturbative QCD should be very reliable. We are eagerly awaiting to confronting our NNLO predictions with the future EIC data.

Summary. In this work we report the first calculation of the NNLO QCD corrections to the pion electromagnetic form factor. We have explicitly verified the validity of the collinear factorization to two-loop order for this case, and

obtain the UV, IR-finite two-loop hard-scattering kernel in closed form. The NNLO QCD correction turns to be positive and substantial. We then confront our finest theoretical predictions with various space-like and time-like pion form factor data. Our phenomenological study reveals that adopting the second Gegenbauer moment computed by RQCD can yield a decent agreement with large- Q^2 data. Nevertheless, to make definite conclusion, it seems to be imperative to resolve the discrepancy between LPC and RQCD for the value of a_2 . Furthermore, we look forward to the future high-statistics larger- Q^2 data for critically testing our NNLO predictions.

Acknowledgments

We are indebted to Jun Hua for providing the value of the Gegenbauer moment a_6 based on the recent LPC study [82]. The work of L.-B. C. is supported by the NNSFC Grant No. 12175048. The work of W.C. is supported by National Natural Science Foundation of China under contract No. 11975200. The work of F. F. is supported by the NNSFC Grant No. 12275353. The work of Y. J. is supported in part by the NNSFC Grants No. 11925506, No. 12070131001 (CRC110 by DFG and NSFC).

In this appendix, we present the asymptotic expressions of the one-loop hard-scattering kernels near the endpoint regimes. For the one-loop hard-scattering kernel, we have

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} T^{(1)}(x, y) = -\frac{16\pi^2 e_d}{27xy} \left[12 \ln^2(xy) - 18 \ln(xy) - \pi^2 + 30 - 3(8 \ln(xy) - 3)L_\mu \right], \quad (23a)$$

$$\begin{aligned} \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 1}} T^{(1)}(x, y) &= -\frac{16\pi^2 e_d}{9x} \left[4 \ln^2 x - (1 + \ln x) \ln \bar{y} - 7 \ln x + 15 - (8 \ln x - 3)L_\mu \right] \\ &+ [x \rightarrow \bar{y}, \bar{y} \rightarrow x, e_d \leftrightarrow -e_u]. \end{aligned} \quad (23b)$$

The limiting behavior of $T^{(1)}(x, y)$ in the corners $x \rightarrow 1, y \rightarrow 1$ and $x \rightarrow 1, y \rightarrow 0$ can be obtained from the above formulas by making the substitution $x \rightarrow \bar{x}, y \rightarrow \bar{y}, e_u \leftrightarrow -e_d$.

For the two-loop hard-scattering kernel, we have

$$\begin{aligned} \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} T^{(2)}(x, y) = & -\frac{32e_d\pi^2}{27xy} \left[\ln^4(xy) - \frac{33}{2} \ln^3(xy) - \left(\frac{5}{3}\pi^2 - \frac{313}{8}\right) \ln^2(xy) - 81 \ln x \ln y \right. \\ & + (73\zeta_3 + \frac{461}{48}\pi^2 - \frac{1925}{8}) \ln(xy) + \frac{83}{60}\pi^4 - 123\zeta_3 - \frac{619}{48}\pi^2 + \frac{3681}{16} \\ & + \frac{1}{4}(8 \ln(xy) - 3)(2 \ln(xy) - 21)L_\mu^2 \\ & \left. - (4 \ln^3(xy) - 48 \ln^2(xy) - \frac{1}{3}(10\pi^2 - 228) \ln(xy) + 4\zeta_3 + \frac{17}{2}\pi^2 - \frac{513}{4})L_\mu \right]. \end{aligned} \quad (24a)$$

$$\begin{aligned} \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 1}} T^{(2)}(x, y) = & -\frac{32e_d\pi^2}{27x} \left[\ln^4 x - \frac{1}{2} \ln^3 x \ln \bar{y} - \frac{5}{32} \ln^2 x \ln^2 \bar{y} - \frac{1}{6} \ln x \ln^3 \bar{y} \right. \\ & - 17 \ln^3 x + \frac{51}{8} \ln^2 x \ln \bar{y} + \frac{23}{4} \ln x \ln^2 \bar{y} - \frac{1}{6} \ln^3 \bar{y} \\ & - \frac{1}{48} ((89\pi^2 - 2019) \ln^2 x + 2(225 - 14\pi^2) \ln x \ln \bar{y} - 327 \ln^2 \bar{y}) \\ & - \frac{1}{8} ((-395\zeta_3 - 83\pi^2 + 1700) \ln x + (29\zeta_3 + \pi^2 + 186) \ln \bar{y}) \\ & + \frac{1}{80} (32\pi^4 - 7670\zeta_3 - 745\pi^2 + 15105) + \\ & + (4 \ln^2 x - \frac{87}{2} \ln x - \frac{2}{3}\pi^2 + \frac{47}{4})L_\mu^2 - (4 \ln^3 x - \ln^2 x \ln \bar{y} \\ & - \frac{1}{2} \ln^2 \bar{y} \ln x - 49 \ln^2 x + \frac{21}{2} \ln x \ln \bar{y} - \frac{1}{2} \ln^2 \bar{y} + (\frac{165}{2} - 4\pi^2) \ln x \\ & \left. + \frac{23}{2} \ln \bar{y} - 7\zeta_3 + \frac{14}{3}\pi^2 - \frac{471}{4})L_\mu \right] \\ & + [x \rightarrow \bar{y}, \bar{y} \rightarrow x, e_d \rightarrow -e_u]. \end{aligned} \quad (24b)$$

Similar to the one-loop case, the limiting behavior of $T^{(2)}(x, y)$ in the corners $x \rightarrow 1, y \rightarrow 1$ and $x \rightarrow 1, y \rightarrow 0$ can be obtained by making the substitution $x \rightarrow \bar{x}, y \rightarrow \bar{y}, e_u \leftrightarrow -e_d$ in (24).

Inspecting (23) and (24), one observes that $T^{(1)}(x, y)$ contains double endpoint logarithms, while $T^{(2)}(x, y)$ contains quartic endpoint logarithms. It is curious whether such endpoint logarithms can be resummed to all orders in α_s .

-
- | | |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p>[1] H. Yukawa, Proc. Phys. Math. Soc. Jap. 17, 48-57 (1935) doi:10.1143/PTPS.1.1</p> <p>[2] C. J. Bebek, C. N. Brown, M. Herzlinger, S. D. Holmes, C. A. Lichtenstein, F. M. Pipkin, S. Raither and L. K. Sisterson, Phys. Rev. D 13 (1976), 25 doi:10.1103/PhysRevD.13.25</p> <p>[3] C. J. Bebek, A. Browman, C. N. Brown, K. M. Hanson, S. D. Holmes, R. V. Kline, D. Larson, F. M. Pipkin, S. W. Raither and A. Silverman, <i>et al.</i> Phys. Rev. Lett. 37 (1976), 1326 doi:10.1103/PhysRevLett.37.1326</p> <p>[4] C. J. Bebek, C. N. Brown, S. D. Holmes, R. V. Kline, F. M. Pipkin, S. Raither, L. K. Sisterson, A. Browman, K. M. Hanson and D. Larson, <i>et al.</i> Phys. Rev. D 17 (1978), 1693 doi:10.1103/PhysRevD.17.1693</p> <p>[5] H. Ackermann, T. Azemoon, W. Gabriel, H. D. Mer-tiens, H. D. Reich, G. Specht, F. Janata and D. Schmidt, Nucl. Phys. B 137 (1978), 294-300 doi:10.1016/0550-3213(78)90523-0</p> | <p>[6] P. Brauel, T. Canzler, D. Cords, R. Felst, G. Grind-hammer, M. Helm, W. D. Kollmann, H. Kre-hbiel and M. Schadlich, Z. Phys. C 3 (1979), 101 doi:10.1007/BF01443698</p> <p>[7] E. B. Dally, D. J. Drickey, J. M. Hauptman, C. F. May, D. H. Stork, J. A. Poirier, C. A. Rey, R. J. Wojslaw, P. F. Shepard and A. J. Lennox, <i>et al.</i> Phys. Rev. D 24 (1981), 1718-1735 doi:10.1103/PhysRevD.24.1718</p> <p>[8] E. B. Dally, J. M. Hauptman, J. Kubic, D. H. Stork, A. B. Watson, Z. Guzik, T. S. Nigmanov, V. D. Ryabtsov, E. N. Tsyganov and A. S. Vodopi-anov, <i>et al.</i> Phys. Rev. Lett. 48 (1982), 375-378 doi:10.1103/PhysRevLett.48.375</p> <p>[9] S. R. Amendolia, B. Badelek, G. Batignani, G. A. Beck, F. Bedeschi, E. H. Bellamy, E. Bertolucci, D. Bettoni, H. Bilokon and G. Bologna, <i>et al.</i> Phys. Lett. B 146 (1984), 116-120 doi:10.1016/0370-2693(84)90655-5</p> <p>[10] S. R. Amendolia <i>et al.</i> [NA7], Nucl. Phys. B 277 (1986),</p> |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

- 168 doi:10.1016/0550-3213(86)90437-2
- [11] J. Volmer *et al.* [Jefferson Lab F(pi)], Phys. Rev. Lett. **86** (2001), 1713-1716 doi:10.1103/PhysRevLett.86.1713 [arXiv:nucl-ex/0010009 [nucl-ex]].
- [12] V. Tadevosyan *et al.* [Jefferson Lab F(pi)], Phys. Rev. C **75** (2007), 055205 doi:10.1103/PhysRevC.75.055205 [arXiv:nucl-ex/0607007 [nucl-ex]].
- [13] T. Horn *et al.* [Jefferson Lab F(pi)-2], Phys. Rev. Lett. **97** (2006), 192001 doi:10.1103/PhysRevLett.97.192001 [arXiv:nucl-ex/0607005 [nucl-ex]].
- [14] H. P. Blok *et al.* [Jefferson Lab], Phys. Rev. C **78** (2008), 045202 doi:10.1103/PhysRevC.78.045202 [arXiv:0809.3161 [nucl-ex]].
- [15] G. M. Huber *et al.* [Jefferson Lab], Phys. Rev. C **78** (2008), 045203 doi:10.1103/PhysRevC.78.045203 [arXiv:0809.3052 [nucl-ex]].
- [16] R. R. Akhmetshin *et al.* [CMD-2], Phys. Lett. B **527** (2002), 161-172 doi:10.1016/S0370-2693(02)01168-1 [arXiv:hep-ex/0112031 [hep-ex]].
- [17] R. R. Akhmetshin *et al.* [CMD-2], Phys. Lett. B **578** (2004), 285-289 doi:10.1016/j.physletb.2003.10.108 [arXiv:hep-ex/0308008 [hep-ex]].
- [18] M. N. Achasov, K. I. Beloborodov, A. V. Berdyugin, A. G. Bogdanchikov, A. V. Bozhenok, A. D. Bukin, D. A. Bukin, T. V. Dimova, V. P. Druzhinin and V. B. Golubev, *et al.* J. Exp. Theor. Phys. **101** (2005) no.6, 1053-1070 doi:10.1134/1.2163921 [arXiv:hep-ex/0506076 [hep-ex]].
- [19] M. N. Achasov, K. I. Beloborodov, A. V. Berdyugin, A. G. Bogdanchikov, A. V. Bozhenok, A. D. Bukin, D. A. Bukin, T. V. Dimova, V. P. Druzhinin and V. B. Golubev, *et al.* J. Exp. Theor. Phys. **103** (2006), 380-384 doi:10.1134/S106377610609007X [arXiv:hep-ex/0605013 [hep-ex]].
- [20] V. M. Aul'chenko, R. R. Akhmetshin, V. S. Banzarov, L. M. Barkov, N. S. Bashtovoi, D. V. Bondarev, A. E. Bondar', A. V. Bragin, N. I. Gabyshev and D. A. Gorbachev, *et al.* JETP Lett. **84** (2006), 413-417 doi:10.1134/S0021364006200021 [arXiv:hep-ex/0610016 [hep-ex]].
- [21] R. R. Akhmetshin *et al.* [CMD-2], Phys. Lett. B **648** (2007), 28-38 doi:10.1016/j.physletb.2007.01.073 [arXiv:hep-ex/0610021 [hep-ex]].
- [22] F. Ambrosino *et al.* [KLOE], Phys. Lett. B **670** (2009), 285-291 doi:10.1016/j.physletb.2008.10.060 [arXiv:0809.3950 [hep-ex]].
- [23] B. Aubert *et al.* [BaBar], Phys. Rev. Lett. **103** (2009), 231801 doi:10.1103/PhysRevLett.103.231801 [arXiv:0908.3589 [hep-ex]].
- [24] F. Ambrosino *et al.* [KLOE], Phys. Lett. B **700** (2011), 102-110 doi:10.1016/j.physletb.2011.04.055 [arXiv:1006.5313 [hep-ex]].
- [25] D. Babusci *et al.* [KLOE], Phys. Lett. B **720** (2013), 336-343 doi:10.1016/j.physletb.2013.02.029 [arXiv:1212.4524 [hep-ex]].
- [26] J. P. Lees *et al.* [BaBar], Phys. Rev. D **86** (2012), 032013 doi:10.1103/PhysRevD.86.032013 [arXiv:1205.2228 [hep-ex]].
- [27] A. Anastasi *et al.* [KLOE-2], JHEP **03** (2018), 173 doi:10.1007/JHEP03(2018)173 [arXiv:1711.03085 [hep-ex]].
- [28] M. Ablikim *et al.* [BESIII], Phys. Lett. B **753** (2016), 629-638 [erratum: Phys. Lett. B **812** (2021), 135982] doi:10.1016/j.physletb.2015.11.043 [arXiv:1507.08188 [hep-ex]].
- [29] J. Gasser and H. Leutwyler, Annals Phys. **158**, 142 (1984) doi:10.1016/0003-4916(84)90242-2
- [30] G. Martinelli and C. T. Sachrajda, Nucl. Phys. B **306** (1988), 865-889 doi:10.1016/0550-3213(88)90445-2
- [31] T. Draper, R. M. Woloshyn, W. Wilcox and K. F. Liu, Nucl. Phys. B **318** (1989), 319-336 doi:10.1016/0550-3213(89)90609-3
- [32] X. Feng, Y. Fu and L. C. Jin, Phys. Rev. D **101** (2020) no.5, 051502 doi:10.1103/PhysRevD.101.051502 [arXiv:1911.04064 [hep-lat]].
- [33] G. Wang *et al.* [chiQCD], Phys. Rev. D **104** (2021), 074502 doi:10.1103/PhysRevD.104.074502 [arXiv:2006.05431 [hep-ph]].
- [34] X. Gao, N. Karthik, S. Mukherjee, P. Petreczky, S. Syritsyn and Y. Zhao, Phys. Rev. D **104** (2021) no.11, 114515 doi:10.1103/PhysRevD.104.114515 [arXiv:2102.06047 [hep-lat]].
- [35] G. P. Lepage and S. J. Brodsky, Phys. Lett. B **87**, 359-365 (1979) doi:10.1016/0370-2693(79)90554-9
- [36] G. P. Lepage and S. J. Brodsky, Phys. Rev. Lett. **43**, no.21, 545-549 (1979) [erratum: Phys. Rev. Lett. **43**, 1625-1626 (1979)] doi:10.1103/PhysRevLett.43.1625.2
- [37] G. P. Lepage and S. J. Brodsky, Phys. Rev. D **22**, 2157 (1980) doi:10.1103/PhysRevD.22.2157
- [38] A. V. Efremov and A. V. Radyushkin, Theor. Math. Phys. **42**, 97-110 (1980) doi:10.1007/BF01032111
- [39] A. V. Efremov and A. V. Radyushkin, Phys. Lett. B **94**, 245-250 (1980) doi:10.1016/0370-2693(80)90869-2
- [40] A. Duncan and A. H. Mueller, Phys. Lett. B **90**, 159-163 (1980) doi:10.1016/0370-2693(80)90074-X
- [41] A. Duncan and A. H. Mueller, Phys. Rev. D **21**, 1636 (1980) doi:10.1103/PhysRevD.21.1636
- [42] V. L. Chernyak and A. R. Zhitnitsky, Phys. Rept. **112** (1984), 173 doi:10.1016/0370-1573(84)90126-1
- [43] R. D. Field, R. Gupta, S. Otto and L. Chang, Nucl. Phys. B **186** (1981), 429-474 doi:10.1016/0550-3213(81)90022-5
- [44] F. M. Dittes and A. V. Radyushkin, Sov. J. Nucl. Phys. **34** (1981), 293
- [45] M. H. Sarmadi, Phys. Lett. B **143** (1984), 471 doi:10.1016/0370-2693(84)91504-1
- [46] E. Braaten and S. M. Tse, Phys. Rev. D **35** (1987), 2255 doi:10.1103/PhysRevD.35.2255
- [47] B. Melic, B. Nizic and K. Passek, Phys. Rev. D **60** (1999), 074004 doi:10.1103/PhysRevD.60.074004 [arXiv:hep-ph/9802204 [hep-ph]].
- [48] E. G. Floratos, D. A. Ross and C. T. Sachrajda, Nucl. Phys. B **129** (1977), 66-88 [erratum: Nucl. Phys. B **139** (1978), 545-546] doi:10.1016/0550-3213(77)90020-7
- [49] F. M. Dittes and A. V. Radyushkin, Phys. Lett. B **134** (1984), 359-362 doi:10.1016/0370-2693(84)90016-9
- [50] G. R. Katz, Phys. Rev. D **31** (1985), 652 doi:10.1103/PhysRevD.31.652
- [51] S. V. Mikhailov and A. V. Radyushkin, Nucl. Phys. B **254** (1985), 89-126 doi:10.1016/0550-3213(85)90213-5
- [52] A. V. Belitsky, D. Mueller and A. Freund, Phys. Lett. B **461** (1999), 270-279 doi:10.1016/S0370-2693(99)00837-0 [arXiv:hep-ph/9904477 [hep-ph]].
- [53] V. M. Braun, A. N. Manashov, S. Moch and M. Strohmaier, JHEP **06**, 037 (2017) doi:10.1007/JHEP06(2017)037 [arXiv:1703.09532 [hep-ph]].
- [54] T. Becher and M. Neubert, Phys. Lett. B **633**,

- 739-747 (2006) doi:10.1016/j.physletb.2006.01.006 [arXiv:hep-ph/0512208 [hep-ph]].
- [55] F. Feng, Y. F. Xie, Q. C. Zhou and S. R. Tang, Comput. Phys. Commun. **265** (2021), 107982 doi:10.1016/j.cpc.2021.107982 [arXiv:2103.08507 [hep-ph]].
- [56] T. Hahn, Comput. Phys. Commun. **140** (2001), 418-431 doi:10.1016/S0010-4655(01)00290-9 [arXiv:hep-ph/0012260 [hep-ph]].
- [57] F. Feng, Comput. Phys. Commun. **183** (2012), 2158-2164 doi:10.1016/j.cpc.2012.03.025 [arXiv:1204.2314 [hep-ph]].
- [58] A. V. Smirnov and F. S. Chuharev, Comput. Phys. Commun. **247** (2020), 106877 doi:10.1016/j.cpc.2019.106877 [arXiv:1901.07808 [hep-ph]].
- [59] A. V. Kotikov, Phys. Lett. B **254** (1991), 158-164 doi:10.1016/0370-2693(91)90413-K
- [60] E. Remiddi, Nuovo Cim. A **110** (1997), 1435-1452 doi:10.1007/BF03185566 [arXiv:hep-th/9711188 [hep-th]].
- [61] J. M. Henn, Phys. Rev. Lett. **110** (2013), 251601 doi:10.1103/PhysRevLett.110.251601 [arXiv:1304.1806 [hep-th]].
- [62] J. Gao, T. Huber, Y. Ji and Y. M. Wang, Phys. Rev. Lett. **128** (2022) no.6, 6 doi:10.1103/PhysRevLett.128.062003 [arXiv:2106.01390 [hep-ph]].
- [63] V. M. Braun, A. N. Manashov, S. Moch and J. Schoenleber, Phys. Rev. D **104** (2021) no.9, 094007 doi:10.1103/PhysRevD.104.094007 [arXiv:2106.01437 [hep-ph]].
- [64] W. Chen, JHEP **02** (2020), 115 doi:10.1007/JHEP02(2020)115 [arXiv:1902.10387 [hep-ph]].
- [65] W. Chen, Eur. Phys. J. C **81** (2021) no.3, 244 doi:10.1140/epjc/s10052-021-09036-5 [arXiv:1912.08606 [hep-ph]].
- [66] W. Chen, Eur. Phys. J. C **80** (2020) no.12, 1173 doi:10.1140/epjc/s10052-020-08757-3 [arXiv:2007.00507 [hep-ph]].
- [67] W. Chen, M. x. Luo, T. Z. Yang and H. X. Zhu, [arXiv:2309.03832 [hep-ph]].
- [68] A. B. Goncharov, Math. Res. Lett. **5** (1998), 497-516 doi:10.4310/MRL.1998.v5.n4.a7 [arXiv:1105.2076 [math.AG]].
- [69] X. Liu and Y. Q. Ma, Comput. Phys. Commun. **283** (2023), 108565 doi:10.1016/j.cpc.2022.108565 [arXiv:2201.11669 [hep-ph]].
- [70] S. V. Mikhailov, A. V. Pimikov and N. G. Stefanis, Phys. Rev. D **93** (2016) no.11, 114018 doi:10.1103/PhysRevD.93.114018 [arXiv:1604.06391 [hep-ph]].
- [71] A. P. Bakulev, S. V. Mikhailov and N. G. Stefanis, Phys. Lett. B **508** (2001), 279-289 [erratum: Phys. Lett. B **590** (2004), 309-310] doi:10.1016/S0370-2693(01)00517-2 [arXiv:hep-ph/0103119 [hep-ph]].
- [72] P. Ball, V. M. Braun and A. Lenz, JHEP **05** (2006), 004 doi:10.1088/1126-6708/2006/05/004 [arXiv:hep-ph/0603063 [hep-ph]].
- [73] N. G. Stefanis, Phys. Rev. D **102** (2020) no.3, 034022 doi:10.1103/PhysRevD.102.034022 [arXiv:2006.10576 [hep-ph]].
- [74] S. Cheng, A. Khodjamirian and A. V. Rusov, Phys. Rev. D **102** (2020) no.7, 074022 doi:10.1103/PhysRevD.102.074022 [arXiv:2007.05550 [hep-ph]].
- [75] N. G. Stefanis, Phys. Lett. B **738** (2014), 483-487 doi:10.1016/j.physletb.2014.10.018 [arXiv:1405.0959 [hep-ph]].
- [76] L. Chang, I. C. Cloet, J. J. Cobos-Martinez, C. D. Roberts, S. M. Schmidt and P. C. Tandy, Phys. Rev. Lett. **110** (2013) no.13, 132001 doi:10.1103/PhysRevLett.110.132001 [arXiv:1301.0324 [nucl-th]].
- [77] K. Raya, L. Chang, A. Bashir, J. J. Cobos-Martinez, L. X. Gutiérrez-Guerrero, C. D. Roberts and P. C. Tandy, Phys. Rev. D **93** (2016) no.7, 074017 doi:10.1103/PhysRevD.93.074017 [arXiv:1510.02799 [nucl-th]].
- [78] J. Lan, C. Mondal, S. Jia, X. Zhao and J. P. Vary, Phys. Rev. D **101** (2020) no.3, 034024 doi:10.1103/PhysRevD.101.034024 [arXiv:1907.01509 [nucl-th]].
- [79] H. M. Choi and C. R. Ji, Phys. Rev. D **91** (2015) no.1, 014018 doi:10.1103/PhysRevD.91.014018 [arXiv:1412.2507 [hep-ph]].
- [80] Q. Chang, S. J. Brodsky and X. Q. Li, Phys. Rev. D **95** (2017) no.9, 094025 doi:10.1103/PhysRevD.95.094025 [arXiv:1612.05298 [hep-ph]].
- [81] G. S. Bali *et al.* [RQCD], JHEP **08**, 065 (2019) doi:10.1007/JHEP08(2019)065 [arXiv:1903.08038 [hep-lat]].
- [82] J. Hua *et al.* [Lattice Parton], Phys. Rev. Lett. **129** (2022) no.13, 132001 doi:10.1103/PhysRevLett.129.132001 [arXiv:2201.09173 [hep-lat]].
- [83] X. Ji, Phys. Rev. Lett. **110** (2013), 262002 doi:10.1103/PhysRevLett.110.262002 [arXiv:1305.1539 [hep-ph]].
- [84] X. Ji, Sci. China Phys. Mech. Astron. **57**, 1407-1412 (2014) doi:10.1007/s11433-014-5492-3 [arXiv:1404.6680 [hep-ph]].
- [85] M. Strohmaier, *Conformal symmetry breaking and evolution equations in Quantum Chromodynamics*, Ph.D. thesis, Regensburg U. (2018).
- [86] A. P. Bakulev and V. L. Khandramai, Comput. Phys. Commun. **184** (2013) no.1, 183-193 doi:10.1016/j.cpc.2012.08.014 [arXiv:1204.2679 [hep-ph]].
- [87] A. Bylinkin, C. T. Dean, S. Fegan, D. Gangadharan, K. Gates, S. J. D. Kay, I. Korover, W. B. Li, X. Li and R. Montgomery, *et al.* Nucl. Instrum. Meth. A **1052** (2023), 168238 doi:10.1016/j.nima.2023.168238 [arXiv:2208.14575 [physics.ins-det]].