

# On Carrollian and Galilean contractions of BMS algebra in 3 and 4 dimensions

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**ABSTRACT:** In this paper, we find a class of Carrollian and Galilean contractions of (extended) BMS algebra in 3+1 and 2+1 dimensions. To this end, we investigate possible embeddings of 3D/4D Poincaré into the  $BMS_3$  and  $BMS_4$  algebras, respectively. The contraction limits in the 2+1-dimensional case are then enforced by appropriate contractions of their Poincaré subalgebra. In 3+1 dimensions, we have to apply instead the analogy between the structures of Poincaré and BMS algebra. In the case of non-vanishing cosmological constant in 2+1 dimensions, we consider the contractions of  $\Lambda$ - $BMS_3$  algebras in an analogous manner.

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## 1 Introduction

The symmetries of space and time are concepts at the heart of modern physics. Poincaré symmetry is the cornerstone of classical relativistic physics and quantum field theory. Diffeomorphism invariance, expressed by the equivalence principle is the fundamental concept underlying general relativity. As shown by Arnowitt, Deser and Misner [1] and Regge and Teitelboim [2] in the case of asymptotically flat spacetime, at spatial infinity, quite expectantly the diffeomorphisms reduce to Poincaré symmetry. It came as a great surprise when shortly afterwards it was shown that the relevant symmetry algebra at null infinity is not Poincaré but an infinite-dimensional BMS (Bondi-van der Burg-Metzner-Sachs) algebra [3, 4]. After decades when asymptotic symmetries have been largely forgotten, a few years ago they reemerged as one of the most actively studied topics in high energy physics and gravity. The breakthrough here was the research carried out by Strominger and his collaborators that resulted in revealing a close relation between BMS symmetry, Weinberg’s soft theorems, and memory effect (see [5] for a review).

The original BMS algebra is a semidirect product of the standard Lorentz algebra and the infinite-dimensional commuting algebra of supertranslations. This algebra contains the Poincaré subalgebra consisting of the Lorentz algebra and four supertranslations. It should be noted however that the form of asymptotic symmetry algebra depends crucially on adopted asymptotic conditions for the metric. It was shown in [6, 7] that by relaxing boundary conditions one can extend the BMS symmetry algebra so as to make it composed of infinite supertranslational and superrotation sectors. Similarly, it turned out that by relaxing boundary conditions at spatial infinity, the asymptotic symmetry algebra there can be extended from Poincaré to BMS [8], [9].

The 2+1 dimensional counterpart of the the BMS algebra in 3+1 dimensions also attracted a lot of attention. It was originally analyzed in [10] and these investigations are recognized as a predecessor of the AdS/CFT research program. Recently, the BMS<sub>3</sub> was investigated in detail in [11, 12]. From our perspective, the case of BMS<sub>3</sub> is particularly interesting because this algebra can be extended to the case of non-vanishing cosmological constant, which is impossible in the case of BMS<sub>4</sub> [13].

In all the cases discussed above the Poincaré algebra played a crucial role being the largest subalgebra of the BMS algebra. It is well known however, that Poincaré algebra is not a unique algebra to

describe the kinematics of particles and fields. In fact, the classification of such ‘kinematical algebras’ (see [14, 15] for reviews) was provided in the seminal work by Bacry and Lévy-Leblond [16], later extended in [17] and generalized to 2+1 dimensions only recently in [18]. The idea was to find a set of 10-parameter algebras that contains space and time translations, rotations and boosts, satisfying some physically appealing conditions: the isotropy of space (the standard action of rotations on energy, momenta, and boosts), the invariance under discrete symmetries of time reversal and parity (this condition can be lifted, as done in [17]), and of noncompactness of boosts – the transformations between inertial observers. Among the kinematical algebras, apart from the Lorentzian algebras (i.e., Poincaré or (Anti)-de Sitter), two families are particularly physically relevant: the Galilean and Carrollian one [19, 20]. Both can be obtained as contraction limits of Lorentzian algebras, with speed of light being the contraction parameter; the limits are opposite, however:  $c \rightarrow \infty$  in the Galilean case and  $c \rightarrow 0$  in the Carrollian one.

Since Poincaré algebra is contained in BMS algebra, it should be possible to extend the Carrollian and Galilean contractions to the latter. This has recently been achieved [21, 22] for the original BMS, while the aim of the present paper is to define such contractions for the extended BMS algebra, both in 3+1 and 2+1 spacetime dimensions.

The Galilean contraction of BMS algebra corresponds physically to the non-relativistic limit of the theory. This contraction does not seem to be of any physical relevance in the case of the symmetries of asymptotically flat gravitational systems in the neighborhood of null infinity, in four spacetime dimensions. It is however of physical interest in the case of 3+1-dimensional asymptotically flat Newtonian systems at spatial infinity. Contrary to that, the Carrollian contraction is of physical interest not only in the neighbourhood of null infinity but also in describing gravity on manifolds with null boundaries (see e.g., [21], [22], [23], [24] and references therein). This should not be confused with an interesting fact that the original BMS algebra itself turns out to be a conformal extension of the lower-dimensional Carroll algebra [25, 26].

The plan of this paper is as follows. In the following section we shortly recall the Carrollian and Galilean contractions in 2+1 spacetime dimensions, for zero, positive, and negative cosmological constant. Section 3 is devoted to a discussion of  $\text{BMS}_3$  algebra and the embeddings of the 2+1d-Poincaré algebra into it. Then in Section 4 we discuss contractions of 2+1-dimensional BMS algebras without and with the cosmological constant. Finally, in Section 5 we consider the 3+1 dimensional extended BMS algebra and its contraction limits.

## 2 Carrollian and Galilean kinematical Lie algebras in (2+1)d

The brackets of 2+1-dimensional Poincaré and (Anti-)de Sitter algebras,  $\mathfrak{iso}(2,1) = \mathfrak{so}(2,1) \ltimes \mathcal{T}^{2,1}$ ,  $\mathfrak{so}(3,1)$  and  $\mathfrak{so}(2,2)$  (for the cosmological constant  $\Lambda = 0$ ,  $\Lambda > 0$  or  $\Lambda < 0$ , respectively), can be expressed in a unified fashion:

$$[\mathcal{J}_\mu, \mathcal{J}_\nu] = \epsilon_{\mu\nu}{}^\sigma \mathcal{J}_\sigma, \quad [\mathcal{J}_\mu, \mathcal{P}_\nu] = \epsilon_{\mu\nu}{}^\sigma \mathcal{P}_\sigma, \quad [\mathcal{P}_\mu, \mathcal{P}_\nu] = -\Lambda \epsilon_{\mu\nu}{}^\sigma \mathcal{J}_\sigma, \quad (2.1)$$

where  $\mu = 0, 1, 2$ ,  $\epsilon_{012} = 1$  and indices are raised with Minkowski metric  $(1, -1, -1)$ . Let us perform a change of basis

$$J_0 := -\mathcal{J}_0, \quad K_a := -\mathcal{J}_a, \quad P_0 := \mathcal{P}_0, \quad P_{1/2} := \mp \mathcal{P}_{2/1}, \quad (2.2)$$

so that the brackets (2.1) become

$$\begin{aligned} [J_0, K_a] &= \epsilon_a^b K_b, & [K_1, K_2] &= -J_0, & [J_0, P_a] &= \epsilon_a^b P_b, & [J_0, P_0] &= 0, \\ [K_a, P_b] &= \delta_{ab} P_0, & [K_a, P_0] &= P_a, & [P_1, P_2] &= 0, & [P_0, P_a] &= 0, \end{aligned} \quad (2.3)$$

where  $a = 1, 2$  and  $\epsilon_1^2 = 1$ . In order to perform the Carrollian contraction of either of the considered Lorentzian algebras, one needs to denote  $J := J_0$ ,  $T_a := P_a$ , define the rescaled generators  $Q_a := c K_a$ ,  $T_0 := c P_0$  and take the limit  $c \rightarrow 0$ . As a result, we obtain the brackets of 2+1-dimensional Carroll (for  $\Lambda = 0$ ) or (Anti-)de Sitter-Carroll algebra (also called the para-Poincaré if  $\Lambda < 0$  and para-Euclidean if  $\Lambda > 0$ , due to their respective isomorphisms with 2+1-dimensional Poincaré and Euclidean algebras):

$$\begin{aligned} [J, Q_a] &= \epsilon_a^b Q_b, & [Q_1, Q_2] &= 0, & [J, T_a] &= \epsilon_a^b T_b, & [J, T_0] &= 0, \\ [Q_a, T_b] &= \delta_{ab} T_0, & [Q_a, T_0] &= 0, & [T_1, T_2] &= \Lambda J_0, & [T_0, T_a] &= -\Lambda Q_a. \end{aligned} \quad (2.4)$$

On the other hand, the Galilean contraction of a Lorentzian algebra consists in denoting  $J := J_0$ ,  $T_0 := P_0$ , introducing the rescaled generators  $Q_a := c^{-1} K_a$ ,  $T_a := c^{-1} P_a$  and taking the limit  $c \rightarrow \infty$ . It allows to obtain the brackets of 2+1-dimensional Galilei (for  $\Lambda = 0$ ) or (Anti-)de Sitter-Galilei algebra (also called the oscillating Newton-Hooke if  $\Lambda < 0$  and the expanding Newton-Hooke if  $\Lambda > 0$ ):

$$\begin{aligned} [J, Q_a] &= \epsilon_a^b Q_b, & [Q_1, Q_2] &= 0, & [J, T_a] &= \epsilon_a^b T_b, & [J, T_0] &= 0, \\ [Q_a, T_b] &= 0, & [Q_a, T_0] &= T_a, & [T_1, T_2] &= 0, & [T_0, T_a] &= -\Lambda Q_a. \end{aligned} \quad (2.5)$$

### 3 BMS<sub>3</sub> algebra $\mathfrak{B}_3$ , its real forms and embeddings of $\text{iso}(2, 1)$

The non-vanishing brackets of BMS algebra in 2+1 dimensions (which we will denote by  $\mathfrak{B}_3$ ) in terms of the generators of superrotations  $l_n$  and supertranslations  $T_n$  have the form

$$[l_n, l_m] = (n - m) l_{n+m}, \quad [l_n, T_m] = (n - m) T_{n+m}, \quad (3.1)$$

where  $n, m \in \mathbb{Z}$ .

From the mathematical perspective,  $\mathfrak{B}_3$  is also known as the inhomogeneous centreless two-sided Witt algebra,  $\mathfrak{W}(2, 2)$ . Moreover, it is actually a complex algebra and in order to impose the reality conditions on its generators, one needs to introduce a real structure, i.e. a  $*$ -conjugation (an involutive, antilinear antiautomorphism). The  $\mathfrak{B}_3$  algebra has two such possible real forms, inherited from real forms of the  $\mathfrak{sl}(2, \mathbb{C})$  algebra<sup>1</sup>:

i) type  $\mathfrak{sl}(2, \mathbb{R})$ , with the reality conditions  $l_n^* = -l_n$ ,  $T_n^* = -T_n$ ,  $n \in \mathbb{Z}$ ,

ii) type  $\mathfrak{su}(1, 1)$ , with the reality conditions  $l_n^* = l_{-n}$ ,  $T_n^* = T_{-n}$ ,  $n \in \mathbb{Z}$ .

Meanwhile, the reality conditions of the  $\mathfrak{su}(2)$ -type ( $\mathfrak{su}(2)$  is the remaining compact real form of  $\mathfrak{sl}(2, \mathbb{C})$ ):  $l_0^* = l_0$ ,  $T_0^* = T_0$ ,  $l_n^* = -l_{-n}$ ,  $T_n^* = -T_{-n}$ ,  $n \neq 0$ , are not compatible with the brackets (3.1).

Let us recall that the brackets of Poincaré algebra in arbitrary dimension  $d+1$  (with the flat metric  $\eta$  of any signature and generators satisfying the anti-Hermitian reality conditions  $X^* = -X$ ) can be written as

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}, \\ [M_{\mu\nu}, P_\rho] &= \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu, \end{aligned} \quad (3.2)$$

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<sup>1</sup>Both non-compact real forms of  $\mathfrak{sl}(2, \mathbb{C})$ ,  $\mathfrak{sl}(2, \mathbb{R})$  and  $\mathfrak{su}(1, 1)$ , are known to be isomorphic, while it is no longer true for the real forms of  $\mathfrak{B}_3$ .

where indices run from 0 to  $d$ . In particular, in 2+1 dimensions we may change the basis to  $J = M_{12}$ ,  $K_i = M_{i0}$ ,  $P_0, P_i$ , with the diagonal metric  $(\eta_{00}, \eta_{11}, \eta_{22})$ , hence (3.2) becomes

$$\begin{aligned} [J, K_i] &= -\eta_{ii}\epsilon_{ij}K_j, & [K_i, K_j] &= -\eta_{00}\epsilon_{ij}J, \\ [J, P_i] &= -\eta_{ii}\epsilon_{ij}P_j, & [K_i, P_j] &= -\eta_{ij}P_0, & [K_i, P_0] &= \eta_{00}P_i, \end{aligned} \quad (3.3)$$

which is a generalization of (2.3).

There exists a one-parameter family  $\mathcal{P}_n(1, 2) = \text{span}\{l_0, l_{\pm n}, T_0, T_{\pm n}\}$ ,  $n \in \mathbb{N}$  (in this paper we adopt the convention  $0 \neq \mathbb{N}$ ) of maximal finite-dimensional subalgebras of  $\mathfrak{B}_3$ , each of them isomorphic to the Poincaré algebra  $\mathfrak{iso}(2, 1)$ . Equivalently, we can define a family of embeddings  $P^{(n)}(1, 2) = \{J, K_{i(n)}, P_0, P_{i(n)}; i = 1, 2\}$  of  $\mathfrak{iso}(2, 1)$  into  $\mathfrak{B}_3$ , given by:

$$\begin{aligned} J &= i l_0, & K_{1(n)} &= \frac{1}{2}(l_{-n} - l_n), & K_{2(n)} &= \frac{i}{2}(l_n + l_{-n}), \\ P_0 &= -i T_0, & P_{1(n)} &= \frac{i}{2}(T_n + T_{-n}), & P_{2(n)} &= \frac{1}{2}(T_n - T_{-n}), \end{aligned} \quad (3.4)$$

which satisfy the brackets (3.3) with the metric  $\eta = n(1, -1, -1)$  (if needed, the factor  $n$  can be eliminated with the help of rescaling of  $J, K_{i(n)}$  by  $1/n$  and  $P_0, P_{i(n)}$  by  $n$ ). Such embeddings are consistent with equipping the  $\mathfrak{B}_3$  algebra with the reality conditions of the  $\mathfrak{su}(1, 1)$ -type, i.e. the latter lead to the anti-Hermitian reality conditions ( $X^* = -X$ ) for each set of Poincaré generators (3.4).

On the other hand, if we choose the reality conditions of the  $\mathfrak{sl}(2, \mathbb{R})$ -type, Poincaré generators with the anti-Hermitian reality conditions are obtained in another family of embeddings, defined as:

$$\begin{aligned} J_{(n)} &= \frac{1}{2}(l_n + l_{-n}), & K_{1(n)} &= \frac{1}{2}(l_{-n} - l_n), & K_2 &= l_0, \\ P_{0(n)} &= \frac{1}{2}(T_n + T_{-n}), & P_{2(n)} &= \frac{1}{2}(T_{-n} - T_n), & P_1 &= -T_0. \end{aligned} \quad (3.5)$$

Moreover, the full family corresponds to a particular basis  $\bigcup_{n \in \mathbb{N}} \{J_{(n)}, K_{1(n)}, P_{0(n)}, P_{2(n)}\} \cup \{K_2, P_1\}$  of the  $\mathfrak{B}_3$  algebra, in which the brackets (3.1) become

$$\begin{aligned} [J_{(n)}, J_{(m)}] &= \frac{1}{2}(-(n+m)K_{1(|n-m|)} + (n-m)K_{1(n+m)}), \\ [J_{(n)}, K_{1(m)}] &= \frac{1}{2}((n+m)J_{(|n-m|)} - (n-m)J_{(n+m)}), \\ [K_{1(n)}, K_{1(m)}] &= \frac{1}{2}((n+m)K_{1(|n-m|)} - (n-m)K_{1(n+m)}), \\ [J_{(n)}, P_{0/2(m)}] &= \frac{1}{2}(\mp(n+m)P_{2/0(|n-m|)} - (n-m)P_{2/0(n+m)}), \\ [K_{1(n)}, P_{0/2(m)}] &= \frac{1}{2}(\mp(n+m)P_{0/2(|n-m|)} - (n-m)P_{0/2(n+m)}) \end{aligned} \quad (3.6)$$

for  $n \neq -m$ , in addition to the brackets (3.3) for each  $n$ , with  $\eta = (n, -n, -n)$ .

Summarizing, we see that for each  $n \in \mathbb{N}$  there exist two inequivalent real embeddings of Poincaré algebra  $\mathfrak{iso}(2, 1)$  into  $\mathfrak{B}_3$ . Both families of embeddings allow us to decompose  $\mathfrak{B}_3$  into the direct sum of its maximal finite-dimensional subalgebras (with the non-empty intersection spanned by  $l_0$  and  $T_0$ ), which are isomorphic to  $\mathfrak{iso}(2, 1)$ . In contrast,  $\mathfrak{iso}(3)$  cannot be embedded into  $\mathfrak{B}_3$ , since (as we already mentioned) the latter has no real form of the  $\mathfrak{su}(2)$ -type.

Let us note in passing that one can construct vacuum spacetimes associated with different choices of the embedding of  $\mathfrak{iso}(2, 1)$  into  $\mathfrak{B}_3$ . To this end, we consider the general solution of the vacuum Einstein equations in the Bondi gauge [7]

$$ds^2 = \Theta(\phi) du^2 - 2dudr + (u \Theta'(\phi) + \Xi(\phi)) dud\phi + r^2 d\phi^2, \quad (3.7)$$

where  $\Theta, \Xi$  are arbitrary periodic functions. It is then convenient to introduce a new variable  $z = e^{i\phi}$ , bringing the metric to the form

$$ds^2 = \Theta(z) du^2 - 2dudr + (u \Theta'(z) - i \Xi(z) z^{-1}) dud\phi - \frac{r^2}{z^2} d\phi^2. \quad (3.8)$$

Now we want to find Poincaré vacua, i.e. spacetimes whose metric, of the general form (3.8), has the maximal set of six Killing vector fields corresponding to one of the embeddings (3.4) or (3.5). This problem was analyzed in [30], where it was shown that the conditions for the vector fields corresponding to the generators  $l_n, T_n$  to be Killing vectors of the metric (3.8) are

$$\Xi = 0, \quad \Theta = -n^2. \quad (3.9)$$

We see that since the embeddings (3.4) or (3.5) differ only by arrangement of the  $l_n$  and  $T_n$  generators, they both lead to the same family of vacuum spacetimes. In other words, there is a degeneracy in the mapping of vacuum spacetimes to embeddings of the (local) Poincaré symmetry into the asymptotic BMS symmetry.

It is worthy to note that the vacuum spacetimes satisfying (3.9) generically (apart from  $n = \pm 1$ ) possess a conical singularity [30] (see also [12]), i.e. their metric can be expressed as

$$ds^2 = -dt^2 + dr^2 + r^2 n^2 d\phi^2. \quad (3.10)$$

Thus, if  $n \neq \pm 1$ , we have here to do with (2+1)d spacetimes containing a massive particle (cf. [27]) with the quantized, negative mass  $M = -(|n| - 1)/(4G)$ , where  $G$  is the three-dimensional Newton's constant. The physical meaning of such vacua is not completely clear and deserves further studies.

## 4 Contractions of the $\text{BMS}_3$ and $\Lambda\text{-BMS}_3$ algebras

Based on a family of embeddings (3.4) or (3.5), we want to extend the Carrollian and Galilean contractions of Poincaré algebra  $\mathfrak{iso}(2, 1)$  discussed in Sec. 2 to the  $\text{BMS}_3$  algebra  $\mathfrak{B}_3$  (3.1). Our guiding principle is that each subalgebra (corresponding to a member of the chosen family of embeddings) of the contracted  $\mathfrak{B}_3$  that is isomorphic to  $\mathfrak{iso}(2, 1)$  before a contraction should be isomorphic to Carroll (2.4) or Galilei (2.5) algebra after the respective contraction. An equivalent point of view is to consider this a consistent extension of Carroll/Galilei algebra by superrotations and supertranslations, which aligns with the approach of [22] to the Carrollian and Galilean contractions of the non-extended  $\text{BMS}_4$  algebra. In practice, it means that, in the contraction procedure, we need to appropriately rescale all generators of  $\mathfrak{B}_3$  that play the role of generators of boosts and time translation (in the Carrollian case) or generators of boosts and spatial translations (in the Galilean case) in any subalgebra spanned by (3.4) / (3.5).

For the family (3.4), we may express such rescalings in the standard basis of  $\mathfrak{B}_3$ , i.e.  $\{l_n, T_n\}$  satisfying the commutation relations (3.1) and it follows that:

- in the case of Carrollian contraction, we should perform the rescaling  $l_n \mapsto c l_n = \tilde{l}_n$ ,  $T_0 \mapsto c T_0 = \tilde{T}_0$  for all  $n \neq 0$  and take the limit  $c \rightarrow 0$  to obtain

$$\begin{aligned} [\tilde{l}_n, \tilde{l}_m] &= 0, & [l_0, \tilde{l}_n] &= -n \tilde{l}_n, & [l_0, T_n] &= -n T_n, \\ [l_0, \tilde{T}_0] &= 0, & [\tilde{l}_n, \tilde{T}_0] &= 0, & [\tilde{l}_n, T_m] &= 2\delta_{m,-n} n \tilde{T}_0, \end{aligned} \quad (4.1)$$

which we will call the 3D BMS-Carroll algebra, BMSC3;

- in the case of Galilean contraction, we perform the rescaling  $l_n \mapsto c^{-1} l_n = \hat{l}_n$ ,  $T_n \mapsto c^{-1} T_n = \hat{T}_n$  for all  $n \neq 0$  and take the limit  $c \rightarrow \infty$  to obtain

$$\begin{aligned} [\hat{l}_n, \hat{l}_m] &= 0, & [l_0, \hat{l}_n] &= -n \hat{l}_n, & [l_0, \hat{T}_n] &= -n \hat{T}_n, \\ [l_0, T_0] &= 0, & [\hat{l}_n, T_0] &= n \hat{T}_n, & [\hat{l}_n, \hat{T}_m] &= 0, \end{aligned} \quad (4.2)$$

which we will call the 3D BMS-Galilei algebra, BMSG3.

It is easy to check that embeddings of Carroll or Galilei algebra into (4.1) or (4.2), respectively, can be defined analogously to (3.4).

If we choose the family (3.5) instead, contractions can no longer be performed in the basis  $\{l_n, T_n\}$  (since e.g. both rotation and boost generators now depend on all  $l_n$ ,  $n \in \mathbb{Z}^*$  generators). Therefore, we keep the basis  $\{K_2, P_1, \{K_{1(n)}, J_{(n)}, P_{2(n)}, P_{0(n)}\}\}$ ,  $n \in \mathbb{N}$  and derive the contraction limit of the brackets (3.6) (the remaining ones (3.3) behave in the same way as for Poincaré algebra) but it turns out that:

- in the case of Carrollian contraction, the rescaling  $K_2 \mapsto c \tilde{K}_2$ ,  $K_{1(n)} \mapsto c \tilde{K}_{1(n)}$ ,  $P_{0(n)} \mapsto c \tilde{P}_{0(n)}$  (for all  $n \neq 0$ ) and the limit  $c \rightarrow 0$  lead to the divergence of the first and fourth bracket in (3.6);
- in the case of Galilean contraction, the rescaling  $P_1 \mapsto c^{-1} \hat{P}_1$ ,  $P_{2(n)} \mapsto c^{-1} \hat{P}_{2(n)}$ ,  $K_2 \mapsto c^{-1} \hat{K}_2$ ,  $K_{1(n)} \mapsto c^{-1} \hat{K}_{1(n)}$  (for all  $n \neq 0$ ) and the limit  $c \rightarrow \infty$  lead to the divergence of the first and fourth bracket in (3.6).

Therefore, both contractions are ill-defined. On the other hand, it is still possible to arrive at the algebra (4.1) or (4.2) starting from (3.5) if we perform a different kind of contractions. To this end, let us first change the basis of  $\mathfrak{B}_3$  from (3.5) to

$$M_{\pm 1(n)} = \pm \frac{1}{\sqrt{2}} (J_{(n)} \mp K_{1(n)}), \quad M_{+-} = -K_2, \quad P_{\pm(n)} = \frac{1}{\sqrt{2}} (P_{0(n)} \mp P_{2(n)}) \quad (4.3)$$

(keeping  $P_1$ ).  $M_{\pm 1(n)}$  are actually two generators of null rotations (parabolic Lorentz transformations) and  $P_{\pm(n)}$  are two generators of translations along the null directions, in a given embedding  $P^{(n)}(1, 2)$  of Poincaré algebra  $\mathfrak{iso}(2, 1)$ . The brackets of  $P^{(n)}(1, 2)$  become

$$\begin{aligned} [M_{+-}, M_{\pm 1(n)}] &= \pm \eta_{+-} M_{\pm 1(n)}, & [M_{+1(n)}, M_{-1(n)}] &= -\eta_{11} M_{+-}, \\ [M_{+-}, P_{\pm(n)}] &= \pm \eta_{+-} P_{\pm(n)}, & [M_{\pm 1(n)}, P_{\mp(n)}] &= -\eta_{+-} P_1, & [M_{\pm 1(n)}, P_1] &= \eta_{11} P_{\pm(n)}, \end{aligned} \quad (4.4)$$

where  $\eta$  is the Lorentzian metric with the non-zero components  $\eta_{+-} = \eta_{-+} = -\eta_{11} = n$ . If we now perform the rescalings  $M_{\pm 1(n)} \mapsto c \tilde{M}_{\pm 1(n)}$ ,  $P_1 \mapsto c \tilde{P}_1$  and take the limit  $c \rightarrow 0$ , it allows to recover the BMSC3 algebra (4.1). Similarly, performing the rescalings  $M_{\pm 1(n)} \mapsto c^{-1} \hat{M}_{\pm 1(n)}$ ,  $P_{\pm(n)} \mapsto c^{-1} \hat{P}_{\pm(n)}$  and taking the limit  $c \rightarrow \infty$  leads to the BMSG3 algebra (4.2). However, let us

stress that these contractions (i.e., the way they are performed, not their results) can not be seen as a BMS-generalization of the Carrollian or Galilean contraction – the rescaled generators do not describe boosts and time translation, or boosts and spatial translations, respectively.

A special feature of BMS algebra in 2+1 dimensions is that it can be generalized to the symmetry algebra of an asymptotically (anti-)de Sitter spacetime, with the brackets

$$\begin{aligned} [l_n, l_m] &= (n-m) l_{n+m}, & [l_n, T_m] &= (n-m) T_{n+m}, \\ [T_n, T_m] &= -\Lambda (n-m) l_{n+m}. \end{aligned} \quad (4.5)$$

Positive/negative  $\Lambda$  corresponds to one of the two real forms of a complex algebra known as  $\Lambda$ -BMS<sub>3</sub>, into which one can embed the (2+1)d de Sitter or anti-de Sitter algebra, respectively (see below). The  $\Lambda \rightarrow 0$  contraction limit of  $\Lambda$ -BMS<sub>3</sub> is the usual  $\mathfrak{B}_3$  algebra, i.e. (3.1).

Actually, if we act with an isomorphism (which is complex for  $\Lambda > 0$ )

$$L_n := \frac{1}{2} \left( l_n + \frac{1}{\sqrt{-\Lambda}} T_n \right), \quad \bar{L}_n := \frac{1}{2} \left( l_n - \frac{1}{\sqrt{-\Lambda}} T_n \right) \quad (4.6)$$

on the brackets (4.5), it shows that the complex algebra  $\Lambda$ -BMS<sub>3</sub> is equivalent to the direct sum of two copies of the Witt algebra,  $\mathfrak{W} \oplus \mathfrak{W}$ :

$$[L_n, L_m] = (n-m) L_{n+m}, \quad [\bar{L}_n, \bar{L}_m] = (n-m) \bar{L}_{n+m}, \quad [L_n, \bar{L}_m] = 0. \quad (4.7)$$

It is then easy to see that there exist infinitely many embeddings of (anti-)de Sitter algebra into  $\Lambda$ -BMS<sub>3</sub>, i.e. to check that  $\mathfrak{o}(4, \mathbb{C}) \cong \mathfrak{o}(3, \mathbb{C}) \oplus \mathfrak{o}(3, \mathbb{C})$  (of which (anti-)de Sitter algebra is a real form) can be embedded into (4.7) as any subalgebra of the form

$$\mathfrak{o}(4, \mathbb{C}) \cong \text{span}\{L_0, L_{\pm n}, \bar{L}_0, \bar{L}_{\pm n}\} \subset \mathfrak{W} \oplus \mathfrak{W}, \quad n \in \mathbb{N}, \quad (4.8)$$

up to a rescaling of  $L_n, \bar{L}_n$  by  $1/n$ . In terms of the generators  $l_n, T_n$ , these embeddings are given by

$$\mathfrak{o}(4, \mathbb{C}) \cong \text{span}\{l_0, l_{\pm n}, T_0, T_{\pm n}\} \subset \mathfrak{W} \oplus \mathfrak{W}, \quad n \in \mathbb{N}, \quad (4.9)$$

where (as one could expect due to the existence of  $\Lambda \rightarrow 0$  contraction limit) the generators are the exact counterparts of those that span the embeddings of  $\mathfrak{iso}(2, 1)$  into  $\mathfrak{B}_3$ , cf. (3.4), (3.5).

Let us now apply to (4.5) the same procedure of the Carrollian contraction as for the  $\mathfrak{B}_3$  algebra. It leads to the brackets identical to (4.1) plus additional non-trivial ones in the supertranslation sector:

$$[T_n, \tilde{T}_0] = -\Lambda n \tilde{l}_n, \quad [T_n, T_{-n}] = -2\Lambda n l_0, \quad [T_n, T_m] \rightarrow \infty, \quad m \neq -n \quad (4.10)$$

but the divergence means that, surprisingly, the contraction limit does not exist. On the other hand, applying to (4.5) the same procedure of the Galilean contraction as for the  $\mathfrak{B}_3$  algebra, we obtain the brackets identical to (4.2) plus additional non-trivial ones in the supertranslation sector:

$$[\hat{T}_n, T_0] = -\Lambda n \hat{l}_n, \quad [\hat{T}_n, \hat{T}_{-n}] = 0, \quad [\hat{T}_n, \hat{T}_m] = 0, \quad m \neq -n. \quad (4.11)$$

The corresponding algebra can be called Galilei-(A)dS-BMS<sub>3</sub>, depending on the sign of  $\Lambda$ .



## 5 Contractions of (extended) BMS<sub>4</sub> algebra $\mathfrak{B}_4$

The extended BMS algebra in 3+1 dimensions (introduced by Barnich and Troessaert [6]), which we will denote  $\mathfrak{B}_4$ , has the brackets

$$\begin{aligned} [l_n, l_m] &= (n-m) l_{n+m}, & [\bar{l}_n, \bar{l}_m] &= (n-m) \bar{l}_{n+m}, & [l_n, \bar{l}_m] &= 0, \\ [l_k, T_{nm}] &= \left(\frac{k+1}{2} - n\right) T_{n+k, m}, & [\bar{l}_k, T_{nm}] &= \left(\frac{k+1}{2} - m\right) T_{n, m+k}, & [T_{nm}, T_{n'm'}] &= 0 \end{aligned} \quad (5.1)$$

in terms of the generators of superrotations  $l_n, \bar{l}_n$  and supertranslations  $T_{nm}$ , where  $n, m \in \mathbb{Z}$ . Similarly to its 2+1-dimensional counterpart  $\mathfrak{B}_3$ ,  $\mathfrak{B}_4$  is in general a complex algebra. There are three real structures that one can introduce on it, which correspond to real forms of the  $\mathfrak{o}(4, \mathbb{C})$  algebra:

- i) type  $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ , for which  $l_n^* = -l_n$ ,  $\bar{l}_n^* = -\bar{l}_n$ ,  $T_{nm}^* = -T_{nm}$ ,  $n, m \in \mathbb{Z}$ ,
- ii) type  $\mathfrak{su}(1, 1) \oplus \mathfrak{su}(1, 1)$ , for which  $l_n^* = l_{-n}$ ,  $\bar{l}_n^* = \bar{l}_{-n}$ ,  $T_{nm}^* = -T_{1-n, 1-m}$ ,  $n, m \in \mathbb{Z}$ ,
- iii) type  $\mathfrak{so}(3, 1)$  (the ‘‘Lorentzian’’ type), for which

$$l_n^* = -\bar{l}_n, \quad T_{nm}^* = T_{nm}. \quad (5.2)$$

The real structure of the  $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{su}(1, 1)$ -type, for which  $l_n^* = -l_n$ ,  $\bar{l}_n^* = \bar{l}_{-n}$ , does not extend to the supertranslation sector.

The reality conditions (5.2) are physically the most interesting ones. Let us choose them and express the  $\mathfrak{B}_4$  algebra in the basis consisting of anti-Hermitian generators

$$\begin{aligned} k_n &:= l_n + \bar{l}_n, & \bar{k}_n &:= -i(l_n - \bar{l}_n), \\ S_{nm} &:= \frac{i}{2}(T_{nm} + T_{mn}), & A_{nm} &:= \frac{1}{2}(T_{nm} - T_{mn}) \end{aligned} \quad (5.3)$$

(let us note that  $S_{nm}$ ’s are symmetric under the exchange of indices  $n \leftrightarrow m$ , while  $A_{nm}$ ’s are anti-symmetric and hence there are no such generators with  $n = m$ ; however, for brevity, the symbol  $A_{nn}$  is implicitly used below in the sense  $A_{nn} = 0$ ), so that the brackets (5.1) become

$$\begin{aligned} [k_n, k_m] &= (n-m) k_{n+m}, & [\bar{k}_n, \bar{k}_m] &= -(n-m) k_{n+m}, \\ [k_n, \bar{k}_m] &= (n-m) \bar{k}_{n+m}, \\ [k_n, S_{pq}] &= \left(\frac{n+1}{2} - p\right) S_{p+n, q} + \left(\frac{n+1}{2} - q\right) S_{p, q+n}, \\ [\bar{k}_n, S_{pq}] &= \left(\frac{n+1}{2} - p\right) A_{p+n, q} - \left(\frac{n+1}{2} - q\right) A_{p, q+n}, \\ [k_n, A_{pq}] &= \left(\frac{n+1}{2} - p\right) A_{p+n, q} + \left(\frac{n+1}{2} - q\right) A_{p, q+n}, \\ [\bar{k}_n, A_{pq}] &= -\left(\frac{n+1}{2} - p\right) S_{p+n, q} + \left(\frac{n+1}{2} - q\right) S_{p, q+n}. \end{aligned} \quad (5.4)$$

The new basis helps us to make two interesting observations. Firstly, the subalgebra spanned by  $k_n, \bar{k}_n$  is isomorphic to (A)dS-BMS<sub>3</sub>, cf. (4.5) (up to a rescaling of the supertranslation generators of (A)dS-BMS<sub>3</sub> by  $\sqrt{\Lambda}$ ). Since (A)dS-BMS<sub>3</sub> with  $\Lambda > 0$  is the BMS generalization of  $\mathfrak{so}(3, 1)$ , which not only plays the role of 3D de Sitter algebra but also 4D Lorentz algebra, the above-mentioned

subalgebra of  $\mathfrak{B}_4$  can be seen as the BMS generalization of 4D Lorentz algebra. Secondly, it is now evident that also the algebraic structure of the full  $\mathfrak{B}_4$  (albeit infinite-dimensional) is analogous to that of a kinematical (Lie) algebra. It follows from the definition [15] that a kinematical algebra contains the subalgebra of rotations, that the generators of boosts and spatial translations transform under rotations as vectors, and that the generator of time translations transforms under rotations as a scalar. Accordingly, (5.4) contains the rotation-like subalgebra spanned by the generators  $k_n$ , which act on  $\bar{k}_n$ ,  $S_{pq}$  and  $A_{pq}$  in a vector-like way; the only missing piece seems to be an analogue of the time-translation generator. If we narrow down our analogy to Poincaré algebra, we can identify  $\bar{k}_n$ 's as corresponding to the boost generators and  $S_{pq}$ 's – to the spatial translation generators, while  $A_{pq}$ 's fill the spot of the time translation generator, with a caveat that they do not commute with  $k_n$ 's (in particular, let us note that if  $p, q \in \{-n, n\}$ , there are three generators  $S_{pq}$  and only one  $A_{pq}$ ; however, this does not allow to construct an embedding of  $\mathfrak{iso}(3,1)$  into  $\mathfrak{B}_4$ ). Consequently, it turns out that one can perform two contractions of  $\mathfrak{B}_4$  similar to the Carrollian and Galilean contraction of Poincaré algebra. As we will see, however, these contractions are not generalizations of the corresponding ones for Poincaré algebra and hence we call them quasi- Carrollian/Galilean.

The quasi-Carrollian contraction consists in the rescalings  $\bar{k}_n \mapsto c \bar{k}_n$  and  $A_{pq} \mapsto c A_{pq}$ ,  $\forall n, p, q \in \mathbb{Z}$ , and then taking the limit  $c \rightarrow 0$ , which leads to the algebra:

$$\begin{aligned} [k_n, k_m] &= (n-m) k_{n+m}, & [k_n, \bar{k}_m] &= (n-m) \bar{k}_{n+m}, & [\bar{k}_n, \bar{k}_m] &= 0, \\ [k_n, S_{pq}] &= \left(\frac{n+1}{2} - p\right) S_{p+n, q} + \left(\frac{n+1}{2} - q\right) S_{p, q+n}, \\ [\bar{k}_n, S_{pq}] &= \left(\frac{n+1}{2} - p\right) A_{p+n, q} - \left(\frac{n+1}{2} - q\right) A_{p, q+n}, \\ [k_n, A_{pq}] &= \left(\frac{n+1}{2} - p\right) A_{p+n, q} + \left(\frac{n+1}{2} - q\right) A_{p, q+n}, & [\bar{k}_n, A_{pq}] &= 0. \end{aligned} \quad (5.5)$$

Meanwhile, the quasi-Galilean contraction of (5.4) is performed by rescaling  $\bar{k}_n \mapsto c^{-1} \bar{k}_n$  and  $S_{pq} \mapsto c^{-1} S_{pq}$ ,  $\forall n, p, q \in \mathbb{Z}$ , and then taking the limit  $c \rightarrow \infty$ , which gives us the algebra:

$$\begin{aligned} [k_n, k_m] &= (n-m) k_{n+m}, & [k_n, \bar{k}_m] &= (n-m) \bar{k}_{n+m}, & [\bar{k}_n, \bar{k}_m] &= 0, \\ [k_n, S_{pq}] &= \left(\frac{n+1}{2} - p\right) S_{p+n, q} + \left(\frac{n+1}{2} - q\right) S_{p, q+n}, & [\bar{k}_n, S_{pq}] &= 0, \\ [k_n, A_{pq}] &= \left(\frac{n+1}{2} - p\right) A_{p+n, q} + \left(\frac{n+1}{2} - q\right) A_{p, q+n}, \\ [\bar{k}_n, A_{pq}] &= -\left(\frac{n+1}{2} - p\right) S_{p+n, q} + \left(\frac{n+1}{2} - q\right) S_{p, q+n}. \end{aligned} \quad (5.6)$$

The use of terms “quasi-Carrollian” and “quasi-Galilean” can be justified by referring to the analogy between the  $\mathfrak{B}_4$  and Poincaré algebra discussed below (5.4). We observe that both algebras (5.5) and (5.6) have commuting boost-like generators  $\bar{k}_n$ , as well as  $\bar{k}_n$  commuting with  $A_{pq}$ 's in the first case and  $\bar{k}_n$  commuting with  $S_{pq}$ 's in the second case, while the remaining brackets are not changed with respect to (5.4). If  $A_{pq}$ ,  $S_{pq}$  are interpreted as time and spatial “translation-like” generators (as we already postulated for  $\mathfrak{B}_4$ ), respectively, the above-mentioned commutation relations have exactly the same structure as for Carroll and Galilei algebras, respectively. Therefore, we will call the algebra defined by (5.5) quasi-Carroll-BMS<sub>4</sub> and the algebra defined by (5.6) – quasi-Galilei-BMS<sub>4</sub>.

Let us now recall that one can identify the Poincaré algebra  $\mathfrak{iso}(3,1)$  with a maximal finite-dimensional subalgebra of  $\mathfrak{B}_4$ . Similarly as in the case of 2+1 dimensions, there exist infinitely many

such subalgebras, spanned by 10 generators that satisfy the commutation relations generalizing (3.3) to 4 dimensions:

$$\begin{aligned} [J_a, J_b] &= \eta_{00} \epsilon_{abc} J_c, & [K_a, K_b] &= -\eta_{00} \epsilon_{abc} J_c, & [J_a, K_b] &= \eta_{00} \epsilon_{abc} K_c, & [J_a, P_0] &= 0, \\ [J_a, P_b] &= \eta_{00} \epsilon_{abc} P_c, & [K_a, P_0] &= \eta_{00} P_a, & [K_a, P_b] &= -\eta_{ab} P_0, & [P_\mu, P_\nu] &= 0. \end{aligned} \quad (5.7)$$

(This form of the brackets can be recovered from (3.2) by taking  $M_{ab} = \epsilon_{abc} J_c$ ,  $M_{a0} = K_a$ .) The above-mentioned subalgebras correspond to a family of embeddings of  $\mathfrak{iso}(3, 1)$  into  $\mathfrak{B}_4$ , parametrized by  $n \in 2\mathbb{N} + 1$  (cf. [30]):

$$\begin{aligned} J_{1(n)} &= \frac{1}{2}(\bar{k}_{-n} - \bar{k}_n), & J_{2(n)} &= \frac{1}{2}(k_n + k_{-n}), & J_3 &= -\bar{k}_0, \\ K_{1(n)} &= \frac{1}{2}(k_n - k_{-n}), & K_{2(n)} &= \frac{1}{2}(\bar{k}_{-n} + \bar{k}_n), & K_3 &= k_0, \\ P_{0(n)} &= \frac{1}{2}(S_{qq} + S_{pp}), & P_{3(n)} &= \frac{1}{2}(S_{qq} - S_{pp}), & P_{1(n)} &= S_{pq}, & P_{2(n)} &= A_{pq}, \end{aligned} \quad (5.8)$$

where the indices  $p = (1 + n)/2$ ,  $q = (1 - n)/2$ , so that the new generators satisfy the brackets (5.7) with the metric  $\eta_{\mu\nu} = n(1, -1, -1, -1)$ . The Poincaré generators are anti-Hermitian ( $X^* = -X$ ) if we impose the reality conditions (5.2) on the  $\mathfrak{B}_4$  algebra.

Such an embedding of the Lorentz subalgebra is well-defined also for even  $n \neq 0$ , while the same is not true for the translation generators. Consequently, in contrast to what happens in 2+1 dimensions, the union of all embeddings from the family does not correspond to a basis of the full  $\mathfrak{B}_4$  algebra. Furthermore, the brackets between  $J_{a(n)}$  and  $K_{b(m)}$  with odd parameters  $n, m$  are given by generators with even parameters  $n + m$  and  $n - m$  and hence the union of these embeddings is not even a subalgebra. Taking all this into account, we conclude that in the 3+1-dimensional case the embeddings do not allow us to straightforwardly generalize the contractions of Poincaré to BMS algebra; conversely, performing the contractions of BMS is independent from embedding Poincaré algebra into it. The form of translation generators in (5.8) does not preserve our interpretation of  $A_{pq}$ 's and  $S_{pq}$ 's as the generalizations of time and spatial translation generators, respectively, while the rotation and boost generators mix  $k_n$ 's and  $\bar{k}_n$ 's. However, this does not lead to any inconsistencies due to what we said above.

Finally, let us consider a different picture. Choosing two light-like vectors in a flat spacetime, an outgoing  $(\tau_+^\mu) = 1/\sqrt{2}(1, 0, 0, 1)$  and incoming  $(\tau_-^\mu) = 1/\sqrt{2}(1, 0, 0, -1)$ , one can perform the 2+2 spacetime decomposition and introduce light-cone (a.k.a. light-front) generators of Poincaré algebra (3.2),

$$\begin{aligned} M_{\pm a} &= \tau_\pm^\mu M_{\mu a} = \frac{1}{\sqrt{2}}(M_{0a} \pm M_{3a}), & M_{+-} &= \tau_+^\mu \tau_-^\nu M_{\mu\nu} = M_{30}, \\ P_\pm &= \frac{1}{\sqrt{2}}(P_0 \pm P_3), \end{aligned} \quad (5.9)$$

so that, for a general metric  $\eta$ , their commutation relations have the form

$$\begin{aligned} [M_{+a}, M_{-b}] &= -\eta_{+-} M_{ab} - \eta_{ab} M_{+-}, & [M_{\pm a}, M_{\pm b}] &= 0, \\ [M_{\pm a}, M_{bc}] &= \eta_{ab} M_{\pm c} - \eta_{ac} M_{\pm b}, & [M_{+-}, M_{\pm a}] &= \pm \eta_{+-} M_{\pm a}, \\ [M_{\pm a}, P_\mp] &= -\eta_{+-} P_a, & [M_{\pm a}, P_\pm] &= [M_{+-}, P_a] = 0, \\ [M_{\pm a}, P_b] &= \eta_{ab} P_\pm, & [M_{+-}, P_\pm] &= \pm \eta_{+-} P_\pm \end{aligned} \quad (5.10)$$

( $a, b, c = 1, 2$ ). Similarly to the 2+1-dimensional case (4.4),  $M_{\pm 1}$ ,  $M_{\pm 2}$  are actually generators of null rotations (transformations generated by  $M_{\pm a}$  leave invariant the null directions  $\tau_{\pm}$ , as well as the spatial directions  $x_2, x_1$  if  $a = 1, 2$ , respectively) and  $P_{\pm}$  are generators of translations along the null directions  $\tau_{\pm}$ , respectively.

Transforming the image of a given embedding (5.8) to such a light-cone basis, we obtain:

$$\begin{aligned} M_{\pm 1(n)} &= \frac{1}{\sqrt{2}} (-K_{1(n)} \pm J_{2(n)}) = \pm \frac{1}{\sqrt{2}} k_{\mp n}, & M_{+-} &= K_3 = k_0, \\ M_{\pm 2(n)} &= \frac{1}{\sqrt{2}} (-K_{2(n)} \mp J_{1(n)}) = -\frac{1}{\sqrt{2}} \bar{k}_{\mp n}, & M_{12} &= J_3 = -\bar{k}_0, \\ P_{+(n)} &= \frac{1}{\sqrt{2}} S_{qq}, & P_{-(n)} &= \frac{1}{\sqrt{2}} S_{pp}, & P_{1(n)} &= S_{pq}, & P_{2(n)} &= A_{pq} \end{aligned} \quad (5.11)$$

and then the brackets (5.10) are satisfied with the metric components  $\eta_{+-} = \frac{1}{2}(\eta_{00} - \eta_{33}) = -\eta_{aa} = n$  and  $\eta_{++} = \eta_{--} = \eta_{ab} = 0$ ,  $a \neq b$ .

Using the terminology from field theory [28], we can say that the quasi-Carroll-BMS<sub>4</sub> algebra (5.5) contains the kinematical subalgebra (not to be confused with the concept of kinematical Lie algebras, to which we refer elsewhere) spanned by the generators  $\{k_n, S_{pq}\}$  and the dynamical subalgebra spanned by  $\{\bar{k}_n, A_{pq}\}$ , while the kinematical subalgebra of the quasi-Galilei-BMS<sub>4</sub> algebra (5.6) is spanned by the generators  $\{k_n, A_{pq}\}$  and the dynamical subalgebra by  $\{\bar{k}_n, S_{pq}\}$ . In other words, from this point of view, the quasi-Carrollian contraction involves rescaling of the generators generalizing translations along the spatial direction  $x_2$ , while for the quasi-Galilean contraction one needs to rescale the generators generalizing translations along the null directions  $\tau_{\pm}$  and the spatial direction  $x_1$ ; and, in both cases, rescale the generators generalizing null rotations that leave invariant the directions  $\tau_{\pm}$  and  $x_1$ . This can now be compared with contractions of the  $\mathfrak{B}_3$  algebra based on the family of embeddings (3.5), which we discussed in Section 4. Indeed, in that case, the Carrollian contraction also involved rescaling of a spatial translation generator, while the Galilean contraction – rescaling of the null translation generators, and for both contractions we also rescaled the null rotation generators (and there is no 2+1-dimensional counterpart of  $M_{12}$ ), cf. below (4.4).

## 6 Conclusions

The aim of this paper was to try to define the notions of Carrollian and Galilean contractions of BMS algebra in 3+1 and 2+1 dimensions, as well as  $\Lambda$ -BMS in 2+1 dimensions.

We showed that both types of contractions can be extended to the BMS<sub>3</sub> algebra, leading to the BMS counterparts of Carroll and Galilei algebras, called BMSC<sub>3</sub> and BMSG<sub>3</sub>. This is achieved by using a family of embeddings of Poincaré algebra  $\mathfrak{iso}(2, 1)$  into  $\mathfrak{B}_3$ , which allows us to decompose the latter into (overlapping) subalgebras isomorphic to  $\mathfrak{iso}(2, 1)$ . We adopt the condition that the contraction limit of each such subalgebra should be isomorphic to Carroll/Galilei algebra and find that it can be satisfied if the considered embeddings are consistent with a particular real form (type  $\mathfrak{su}(1, 1)$ ) of  $\mathfrak{B}_3$ . On the other hand, BMSC<sub>3</sub> and BMSG<sub>3</sub> can also be recovered by starting with a family of embeddings consistent with the other real form (type  $\mathfrak{sl}(2, \mathbb{R})$ ). However, the contractions of BMS<sub>3</sub> that we define in such a case are not equivalent to the Carrollian/Galilean contractions when restricted to the corresponding Poincaré subalgebras. Finally, we find that only the Galilean contraction is well defined for the  $\Lambda$ -BMS<sub>3</sub> algebra.

The BMS algebra in 3+1 dimensions,  $\mathfrak{B}_4$ , turns out to be more problematic because the embeddings of Poincaré algebra  $\mathfrak{iso}(3, 1)$  do not cover the whole algebra and hence do not provide a

framework for extending the contractions of  $\mathfrak{iso}(3,1)$  to  $\mathfrak{B}_4$ . As an alternative method, we observe the analogy between the structures of these two algebras and find that it is only possible to perform the quasi-Carrollian and quasi-Galilean contractions of  $\mathfrak{B}_4$ , which lead to the quasi-Carroll-BMS<sub>4</sub> and quasi-Galilei-BMS<sub>4</sub> algebras. The structure of these algebras is superficially reminiscent of Carroll and Galilei algebras but does not agree with the embeddings of  $\mathfrak{iso}(3,1)$  into  $\mathfrak{B}_4$ .

The contractions of BMS algebras are of interest on its own, but their construction is the first step in investigations of their quantum deformations. This can be done similarly to the construction presented in [29], [30], [31] and such generalization of the results obtained for quantum deformations of finite-dimensional symmetry algebras [32] will be the subject of our follow-up paper.

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