

The static force from generalized Wilson loops on the lattice using gradient flow

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The static QCD force from the lattice can be used to extract $\Lambda_{\overline{\text{MS}}}$, which determines the running of the strong coupling. Usually, this is done with a numerical derivative of the static potential. However, this introduces additional systematic uncertainties; thus, we use another observable to measure the static force directly. This observable consists of a Wilson loop with a chromoelectric field insertion. We work in the pure SU(3) gauge theory. We use gradient flow to improve the signal-to-noise ratio and to address the field insertion. We extract $\Lambda_{\overline{\text{MS}}}^{n_f=0}$ from the data by exploring different methods to perform the zero flow time limit. We obtain the value $\sqrt{8t_0}\Lambda_{\overline{\text{MS}}}^{n_f=0} = 0.629_{-34}^{+23}$, where t_0 is a flow time reference scale. We also obtain precise determinations of several scales: r_0/r_1 , $\sqrt{8t_0}/r_0$, $\sqrt{8t_0}/r_1$ and we compare to the literature. The gradient flow appears to be a promising method for calculations of Wilson loops with chromoelectric and chromomagnetic insertions in quenched and unquenched configurations.

I. INTRODUCTION

The Standard Model of particle physics is one of the most precisely tested theories. A precise knowledge of the Standard Model parameters is a necessary condition to work out accurate perturbative predictions, and to compare them with high-precision experimental measurements. Quantum Chromodynamics (QCD) is the sector of the Standard Model that describes the strong interaction. It is a field theory based on the gauge group SU(3) that depends on just one coupling, g or equivalently $\alpha_s = g^2/(4\pi)$. The coupling may be traded at any time with an intrinsic scale, in the $\overline{\text{MS}}$ scheme this is $\Lambda_{\overline{\text{MS}}}$. Once renormalized, α_s is small at energy scales much larger than $\Lambda_{\overline{\text{MS}}}$, a property known as asymptotic freedom, but becomes of order one at energy scales close to $\Lambda_{\overline{\text{MS}}}$. At high energies, we can rely on weak coupling perturbation theory to compute QCD observables.

The value of $\Lambda_{\overline{\text{MS}}}$, or equivalently α_s at a large energy scale, can be determined by comparing some high-energy observable computed in weak coupling perturbation theory with data. A viable alternative is to replace data with lattice QCD computations, i.e. the exact evaluation of the observable in QCD via Monte Carlo computations. For the current status of $\Lambda_{\overline{\text{MS}}}$ extractions from lattice QCD, see for example the recent reviews [1, 2]. While in the last years lattice extractions of $\Lambda_{\overline{\text{MS}}}$ have been mostly done in QCD with dynamical quarks, the interest towards the running of the coupling in the pure gauge version of QCD, i.e. without dynamical quarks, also called quenched QCD, has been recently reignited [3]. With modern lattice methods, the extraction of the coupling from the pure gauge theory can be done much more precisely nowadays than in the past when quenched calculations were the only viable option.

The static energy is a well-understood quantity in lattice QCD [4, 5] that can be used to set the lattice scale [6]. Furthermore, the static energy is an observable that can be used to extract $\Lambda_{\overline{\text{MS}}}$ by comparing its perturbative expression with lattice data at short distances. The QCD scale $\Lambda_{\overline{\text{MS}}}$ has been extracted from the static energy both in pure gauge [7–11] and with dynamical fermions [12–18].

In lattice QCD, the static energy suffers from a linear divergence and needs to be renormalized. In dimensionally regularized perturbation theory, this divergence becomes a renormalon of mass dimension one. For these reasons, it presents some advantages to look at the derivative of the static energy, which is the static force. The static force is not affected by linear divergences in lattice QCD and by renormalons of mass dimension one in dimensional regularization,

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yet it contains all the relevant information on the running of the strong coupling, as this is entirely encoded in the slope of the static energy. The force is known up to next-to-next-to-next-to-leading logarithmic accuracy in the coupling [19–24].

The derivative of the static energy performed numerically on lattice data introduces additional systematic uncertainties. Therefore, it may be advantageous to compute the force directly from a suitable observable [25–27]. This observable consists of a Wilson loop with a chromoelectric field insertion. A difficulty related to this observable is that field insertions on Wilson loops when evaluated on the lattice have a bad signal-to-noise ratio and a slow convergence to the continuum limit originating from the discretization of the field components. This was studied in a previous work [28]. In this work, to overcome the difficulty, we rely on the gradient flow method [29–31] to improve the signal-to-noise ratio and to remove the discretization effects of the field components. Wilson loops smeared with gradient flow have been studied before in terms of Creutz ratios [32–34]. In Ref. [35], the Wilson line correlator in Coulomb gauge was measured at finite T with gradient flow. To our knowledge, this is the first time the gradient flow is applied to the force directly. Furthermore, this study also serves as a preparation for the study of similar Wilson loops with field insertions appearing in the computation of several observables in the context of nonrelativistic effective field theories. New methods for integrating the gradient flow equations, based on Runge–Kutta methods, have been developed and implemented during the last few years [36]. The force in gradient flow, besides with lattice QCD, can also be computed analytically in perturbation theory. In perturbation theory, the static force at finite flow time is known at 1-loop order [37].

The paper is structured as follows. In section II, we discuss the theoretical background: we introduce the gradient flow and the static force at zero and finite flow time, in continuum and on the lattice. The lattice setup is described in section III, and in section IV we show our numerical results and perform the continuum limits. Finally, in section V, we discuss our results and extract $\Lambda_{\overline{\text{MS}}}^{n_f=0}$. Preliminary results based on these data have appeared before in conference proceedings [38, 39].

II. THEORETICAL BACKGROUND

A. The static force

The static energy $V(r)$ in Euclidean QCD is related to a rectangular Wilson loop $W_{r \times T}$ with temporal extent from 0 to T and spatial extent r [40] by

$$V(r) = - \lim_{T \rightarrow \infty} \frac{\ln \langle \text{Tr}(P W_{r \times T}) \rangle}{T} = - \frac{1}{a} \lim_{T \rightarrow \infty} \frac{\langle \text{Tr}(P W_{r \times (T+a)}) \rangle}{\langle \text{Tr}(P W_{r \times T}) \rangle}, \quad (1)$$

$$W_{r \times T} = \left\{ \exp \left(i \oint_{r \times T} dz_\mu g A_\mu \right) \right\}, \quad (2)$$

where a is the lattice spacing, g the strong coupling, Tr the trace over the color matrices, and P is the path ordering operator for the color matrices. In dimensional regularization, the static energy has a renormalon ambiguity of order Λ_{QCD} , while on the lattice there is a linear divergence of order $1/a$ coming from the self energy of the Wilson line. Both, the perturbative and lattice problems, manifest as a constant shift to the potential. This may be renormalized by fixing the potential to a given value at a given point r^* :

$$V^r(r) = V(r) - V(r^*). \quad (3)$$

Alternatively, taking the derivative removes the divergent constant, and we obtain a renormalized quantity, the static force.

The static force $F(r)$ is defined as the derivative of the static energy:

$$F_{\partial V}(r) = \partial_r V(r). \quad (4)$$

In perturbation theory, the static force is known up to next-to-next-to-next-to-leading logarithmic order (N³LL) [19–24]. On the lattice, this derivative is evaluated from the static energy data either with interpolations or with finite differences, which leads to increased systematic errors. It is possible, however, to carry out the derivative of the Wilson lines at the level of eq. (2), and rewrite the force as [25–27]

$$F_E(r) = - \lim_{T \rightarrow \infty} \frac{i}{\langle \text{Tr}(W_{r \times T}) \rangle} \left\langle \text{Tr} \left(P \left\{ \exp \left(i \oint_{r \times T} dz_\mu g A_\mu \right) \hat{\mathbf{r}} \cdot g \mathbf{E}(\mathbf{r}, t^*) \right\} \right) \right\rangle \quad (5)$$

$$= - \lim_{T \rightarrow \infty} i \frac{\langle \text{Tr} \{ P W_{r \times T} \hat{\mathbf{r}} \cdot g \mathbf{E}(\mathbf{r}, t^*) \} \rangle}{\langle \text{Tr}(W_{r \times T}) \rangle}, \quad (6)$$

where the expression in the numerator consists of a static Wilson loop with a chromoelectric field insertion on the temporal Wilson line at position t^* , and $\hat{\mathbf{r}}$ is the spatial direction of the quark-antiquark pair separation; t^* can be chosen arbitrarily. Since both expressions for the force represent the same renormalized quantity, it holds in the continuum that $F_E = F_{\partial V} = F$.

B. Gradient flow

We rely on the gradient flow method [29–31] for measuring Wilson loops with and without chromoelectric field insertions. The gradient flow is a continuous transformation of the gauge fields towards the minimum of the Yang–Mills gauge action along a fictitious flow time τ_F :

$$\dot{B}_\mu(\tau_F, x) = D_\nu G_{\nu\mu} = -g_0^2 \frac{\delta S_{\text{YM}}[B]}{\delta B_\mu(\tau_F, x)}, \quad (7)$$

$$B_\mu|_{\tau_F=0} = A_\mu, \quad (8)$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu], D_\mu = \partial_\mu + [B_\mu, \cdot], \quad (9)$$

where $B_\mu(\tau_F)$ are the flowed gauge fields at flow time τ_F with the SU(3) QCD gauge fields A_μ as initial condition at zero flow time, S_{YM} is the Yang–Mills action evaluated with the flowed gauge fields, $G_{\mu\nu}$ is the field strength tensor evaluated with the flowed gauge fields, and D_μ is the gauge covariant derivative. The flow depends on the local neighboring gauge field values through the derivative of the action with respect to the gauge field at position x , and its characteristic range is given by the flow radius $\sqrt{8\tau_F}$. This results in a smearing that cools off systematically the ultraviolet physics and automatically renormalizes gauge invariant observables [41, 42]. Furthermore, we introduce the reference scale t_0 [31] defined implicitly through the expectation value of the action density

$$E = \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a, \quad (10)$$

as

$$\tau_F^2 \langle E \rangle|_{\tau_F=t_0} = 0.3. \quad (11)$$

The gradient flow equation is adapted for flowed link variables $V_{\tau_F}(\mu, x)$ on the lattice as

$$\begin{aligned} \dot{V}_{\tau_F}(x, \mu) &= -g_0^2 (\partial_{x,\mu} S_{\text{Gauge}}(V_{\tau_F})) V_{\tau_F}(x, \mu), \\ V_{\tau_F}(x, \mu)|_{\tau_F=0} &= U_\mu(x), \end{aligned} \quad (12)$$

where $S_{\text{Gauge}}(V_{\tau_F})$ is some lattice gauge action evaluated with the flowed link variables, $\partial_{x,\mu}$ is the derivative with respect to $V_{\tau_F}(x, \mu)$, and $U_\mu(x)$ is the original SU(3) link variable. The flowed link variables of a gauge field configuration depend uniquely on the initial gauge field configuration U , i.e., $V_{\tau_F} = V_{\tau_F}(U)$ and flowed observables are obtained by replacing the original link variables with the flowed link variables: $O(\tau_F) = O|_{U=V_{\tau_F}}$. The flowed expectation value of O is evaluated on the flowed gauge ensemble and can be written as

$$\langle O(\tau_F) \rangle = \frac{1}{Z} \int \mathcal{D}[U] e^{-S_E[U]} O[V_{\tau_F}(U)], \quad (13)$$

which is still a path integral with the Euclidean action S_E of the original zero flow time theory. Therefore, on the lattice, the expectation value is given by

$$\langle O(\tau_F) \rangle \approx \frac{1}{N} \sum_{U, p(U) \propto e^{-S_E}} O(V_{\tau_F}(U)), \quad (14)$$

where N is the number of gauge fields. We solve the gradient flow on the lattice by an iterative Runge–Kutta implementation for the SU(3) matrices. We use either a fixed step size algorithm [31], or an adaptive step size algorithm [36, 43].

C. The perturbative static force at finite flow time

The 1-loop formula for the static force in gradient flow is [37]:

$$F(r, \tau_F) = \frac{\alpha_S(\mu) C_F}{r^2} \left[\left(1 + \frac{\alpha_S}{4\pi} a_1 \right) \mathcal{F}_0(r, \tau_F) + \frac{\alpha_S}{4\pi} \beta_0 \mathcal{F}_{\text{NLO}}^L(r, \tau_F, \mu) + \frac{\alpha_S C_A}{4\pi} \mathcal{F}_{\text{NLO}}^F(r, \tau_F) \right] + \mathcal{O}(\alpha_S^3), \quad (15)$$

c_1	-0.0501648
c_2	0.526758
c_3	-5.55177
c_4	45.8753
c_5	-147.8
c_6	463.906
c_7	-851.741
c_8	884.315
c_9	-499.105
c_{10}	121.773

TABLE I: Numerical values of the coefficients c_n appearing in Eq. (20).

with $C_F = (N_C^2 - 1)/(2N_C)$, $C_A = N_C$, $N_C = 3$ the number of colors, $\beta_0 = 11C_A/3 - 2n_f/3$, n_f the number of flavors, and $a_1 = 31C_A/9 - 10n_f/9$. The functions $\mathcal{F}_0(r, \tau_F)$ and $\mathcal{F}_{\text{NLO}}^L(r, \tau_F, \mu)$ are given analytically with

$$\mathcal{F}_0(r, \tau_F) = \text{erf}\left(\frac{r}{\sqrt{8\tau_F}}\right) - \frac{r}{\sqrt{2\pi\tau_F}} \exp\left(-\frac{r^2}{8\tau_F}\right), \quad (16)$$

$$\begin{aligned} \mathcal{F}_{\text{NLO}}^L(r, \tau_F, \mu) = & \log(\mu^2 r^2) \mathcal{F}_0(r, \tau_F) + \log\left(\frac{8\tau_F}{r^2} e^{\gamma_E}\right) \mathcal{F}_0(r, \tau_F) \\ & - \frac{r}{\sqrt{2\pi\tau_F}} \left[e^{-\frac{r^2}{8\tau_F}} M^{(1,0,0)}\left(0, \frac{1}{2}, \frac{r^2}{8\tau_F}\right) + M^{(1,0,0)}\left(\frac{1}{2}, \frac{3}{2}, -\frac{r^2}{8\tau_F}\right) \right], \end{aligned} \quad (17)$$

where γ_E is the Euler–Mascheroni constant and $M(a, b, z)$ is the confluent hypergeometric function defined by

$$M(a, b, z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}, \quad (18)$$

with $(x)_k = \Gamma(x+k)/\Gamma(x)$, and

$$M^{(1,0,0)}(a, b, z) = \frac{\partial}{\partial a} M(a, b, z). \quad (19)$$

For $r > \sqrt{\tau_F}$, we approximate $\mathcal{F}_{\text{NLO}}^F$ with the polynomial

$$\mathcal{F}_{\text{NLO}}^F(\xi = r/\sqrt{\tau_F}) = \sum_{n=1}^{10} \frac{c_n}{n!} \left(\frac{\xi}{1 + \xi/C_a} \right)^n e^{-\xi} + \frac{44 - C_b}{C_a + \xi^2} + \frac{C_b \xi^2}{\xi^4 + C_c}, \quad (20)$$

where $C_a = 109.358$, $C_b = 43.8438$, $C_c = 404.790$, and c_n are listed in table I. In practice, the computation of $M^{(1,0,0)}$ takes a considerable amount of time, which is a problem for fitting this function. Those terms depend only on the flow time ratio τ_F/r^2 , hence, we precompute them on a fine flow time ratio grid, and use spline interpolations for the further calls of the perturbative formula.

The 1-loop formula has an explicit dependence on the renormalization scale μ in the form of $\log(\mu^2 r^2)$ and an implicit dependence through the perturbative strong coupling $\alpha_s(\mu)$. At zero flow time, the only scale is the distance r , therefore, setting $\mu = 1/r$ is a natural choice. At large flow time, r is negligible with respect to $\sqrt{8\tau_F}$, and, therefore, the natural choice is $\mu = 1/\sqrt{8\tau_F}$. A parameterization of μ that interpolates between these two limiting cases is $\mu = 1/\sqrt{r^2 + 8\tau_F}$ [37]. Because in our lattice calculations we are not collecting data at large flow time, we adopt in this work the more general parameterization

$$\mu(r, \tau_F) = \frac{1}{\sqrt{sr^2 + 8b\tau_F}}. \quad (21)$$

At zero flow time $\mu(r, 0) = 1/(\sqrt{s}r)$ and we can interpret \sqrt{s} as a scale variation parameter with central value 1. At intermediate flow time values, i.e. $\sqrt{8\tau_F}$ of the order of r , the parameter b defines an effective flowed distance. Starting from three loops, an ultrasoft (us) scale of order α_s/r also enters the static force equations at zero flow time [19]. For the scope of this paper, we set the ultrasoft scale to be $\mu_{\text{us}} = C_A \alpha_s(1/r)/(2r)$.

We renormalize the coupling in the $\overline{\text{MS}}$ scheme, hence, both α_s and the scale $\Lambda_{\overline{\text{MS}}}$ are defined in that scheme. The scale $\Lambda_{\overline{\text{MS}}}$ can be obtained by comparing the perturbative expression of the force with lattice data. Since we work in the pure SU(3) theory, the comparison provides $\Lambda_0 \equiv \Lambda_{\overline{\text{MS}}}^{n_f=0}$.

The small flow time expansion of $r^2 F$ reads

$$r^2 F(r, \tau_F) \approx r^2 F(r, \tau_F = 0) + \frac{\alpha_s^2 C_F}{4\pi} [-12\beta_0 - 6C_A c_L] \frac{\tau_F}{r^2}, \quad (22)$$

with $c_L = -22/3$. We remark that $[-12\beta_0 - 6C_A c_L] = 8n_f$, which is 0 in this study ($n_f = 0$). This means that at small flow time, the static force approaches a constant behavior and that corrections to it are smaller than $\alpha_s^2 \tau_F / r^2$; $r^2 F(r, \tau_F = 0)$ is r^2 times the 1-loop perturbative force at zero flow time.

The static force at zero flow time is known up to N³LL accuracy [19–24] and the higher loop contributions are crucial for the extraction of the Λ_0 parameter. To benefit from this knowledge, we model the flowed force with the 1-loop expression at finite flow time, and we demand it to converge to the expression at arbitrary order at zero flow time. In terms of equations, this is given by

$$r^2 F(r, \tau_F) = r^2 F(r, \tau_F = 0) + f^{1\text{-loop}}(r, \tau_F), \quad (23)$$

$$f^{1\text{-loop}}(r, \tau_F) = r^2 F^{1\text{-loop}}(r, \tau_F) - r^2 F^{1\text{-loop}}(r, \tau_F = 0), \quad (24)$$

where $r^2 F(r, \tau_F = 0)$ is the static force at a given order at zero flow time, and $r^2 F^{1\text{-loop}}$ is the full 1-loop expression in Eq. (15). In this way, we correct for the change of the force due to the flow time up to 1-loop order. The accuracy of the flow time correction is consistent with the 3-loop accuracy of the zero flow time part, as long as the flow time correction subleading to $\alpha_s^2 \tau_F / r^2$ is small compared to α_s^4 . This appears to be the case in our study, where we restrict to $\tau_F / r^2 \lesssim 0.05$.

For the rest of the paper, we refer to Eq. (23) when dealing with the flowed static force at higher orders. We label the 1-loop order force (next-to leading order (NLO)) as F1l, the 2-loop force (next-to-next-to leading order (N²LO)) as F2l, the 2-loop force with leading ultrasoft logarithms resummed (next-to-next-to leading logarithmic order (N²LL)) as F2lLus, the 3-loop force (next-to-next-to-next-to leading order (N³LO)) with F3l, and the 3-loop force with leading ultrasoft logarithms resummed as F3lLus. For the reasons discussed in [15], we also restrict the present study to the F3lLus force, although the force at 3-loop order with next-to-leading ultrasoft logarithms resummed (next-to-next-to-next-to leading logarithmic order (N³LL)) would be available.

D. The force on the lattice

The Wilson loops $W_{r \times T}$ are constructed as closed, path ordered product of link variables, consisting of two straight spatial Wilson lines in the spatial plane separated by T in temporal direction. The spatial Wilson lines have the length r into the direction $\hat{\mathbf{r}}$. The ends of both spatial Wilson lines are connected by two straight temporal Wilson lines. The static force can be obtained as the numerical derivative of Eq. (1) from the symmetric finite difference

$$F_{\partial V} = \frac{V(r+a) - V(r-a)}{2a}. \quad (25)$$

Other methods of defining the derivative of Eq. (1) consist, for example, in using the derivative of interpolating functions; these methods, however, add additional systematic uncertainties.

The main purpose of this work is to obtain the static force directly by computing $P W_{r \times T} g E_j(\mathbf{r} = r\hat{\mathbf{j}}, t^*)$, which consists in inserting a discretized j -th chromoelectric field component into the path ordered product at the temporal position t^* in one of the temporal Wilson lines. In general, t^* is arbitrary. Nevertheless, we choose $t^* = T/2$ for even-spaced separations, and an average over $t^* = T/2 \pm a/2$ for odd-spaced separations. This reduces the interactions between the chromoelectric field and the corners of the Wilson loop. We use the clover discretization for the field strength tensor

$$a^2 F_{\mu\nu} = \frac{-i}{8} (Q_{\mu\nu} - Q_{\nu\mu}), \quad (26)$$

$$Q_{\mu\nu} = U_{\mu,\nu} + U_{\nu,-\mu} + U_{-\mu,-\nu} + U_{-\nu,\mu} = Q_{\nu\mu}^\dagger, \quad (27)$$

where $U_{\mu\nu}$ is a plaquette in the μ - ν -plane. This symmetric definition of the chromoelectric field corresponds to the symmetric center difference according to Eq. (25) at tree level. Finally, we replace $a^2 F_{\mu\nu}$ by $a^2 F_{\mu\nu} - \mathbb{1} \text{Tr}(a^2 F_{\mu\nu})/3$, which makes the components of the field strength tensor traceless and corresponds to an a^2 improvement [44]. The chromoelectric field components are accessible through the components $a^2 E_i = -a^2 F_{i,4}$.

The direct determination of the force, F_E , on the lattice follows from Eq. (6) and the discretized version of $P W_{r \times T} g E_j(\mathbf{r} = r\hat{\mathbf{j}}, t^*)$. The finite extent of the chromoelectric field through its discretization introduces additional

N_S	N_T	β	a [fm]	t_0/a^2	N_{conf}	Label
20^3	40	6.284	0.060	7.868(8)	6000	L20
26^3	52	6.481	0.046	13.62(3)	6000	L26
30^3	60	6.594	0.040	18.10(5)	6000	L30
40^3	80	6.816	0.030	32.45(7)	3300	L40

TABLE II: The parameters for the lattice ensembles. The scale t_0 was set on a smaller subset of lattice configurations.

self energy contributions with a non-trivial lattice spacing dependence. These self-energy contributions slow down the convergence to the continuum limit; they have been studied in lattice perturbation theory [45]. They are absent in the force obtained through the derivative, $F_{\partial V}$. Since both calculations provide the same physical quantity, we may set

$$Z_E F_E(r) = F_{\partial V}, \quad (28)$$

where the constant Z_E reabsorbs the additional self energy contributions at finite lattice spacing. If $Z_E = 1$, no self-energy contributions are present, and we can assume that the quantity behaves in a trivial way in the continuum limit. Z_E from the static force was investigated non-perturbatively in a former study [28], and it was found that Z_E has only a weak r -dependence. For a different discretization of the chromoelectric field insertion needed for determining transport coefficients, the renormalization constant Z_E was computed up to NLO in lattice perturbation theory [46]. More recent studies [47, 48] rely on the gradient flow method to renormalize the field insertions. In this study, we also use gradient flow to show the renormalization property explicitly, and to improve the signal-to-noise ratio. In the rest of this work, the default force measurement is given in terms of the chromoelectric field, i.e. $F \equiv F_E$, while we still call the force obtained from the derivative of the static energy $\partial_r V \equiv F_{\partial V}$.

III. LATTICE SETUP AND TECHNICAL DETAILS

On the lattice, we measure $a^2 F(t, r, T)$, which multiplied by r^2/a^2 yields the dimensionless quantity $r^2 F(t, r, T)$. We have data on a $N_s^3 \times N_t$ (N_s : spatial lattice extent, N_t : temporal lattice extent) grid, in our case the available grids are $20^3 \times 40$, $26^3 \times 52$, $30^3 \times 60$, and $40^3 \times 80$. Table II shows our lattice parameters. We use the scaling from [4], which is based on the scale r_0 , to fine-tune the simulation parameters. We produce the lattice configurations using overrelaxation and heatbath algorithms with Wilson action.

We solve the gradient flow equation (12) with a fixed step size integrator for the coarsest lattice ($20^3 \times 40$) and an adaptive step size integrator for the finer lattices. In all gradient flow integrations, we use the Symanzik action. We measure pure Wilson loops and loops with chromoelectric field insertions. That way, we can determine both, the static potential and its numerical derivative for obtaining the static force, and the force directly. In appendix B, we show briefly the impact of gradient flow on the bare Wilson loops with and without chromoelectric field insertions. To find the reference scale t_0 , we use the clover discretization in Eq. (26) to measure $\langle E \rangle$ and solve Eq. (11) for t_0 . We use this reference scale to express our quantities r and τ_F in units of $\sqrt{t_0}$ and t_0 , respectively, and to perform the continuum limit as $a^2/t_0 \rightarrow 0$.

The adaptive step size integrator changes the gradient flow step sizes after every integration step, dependent on the lattice configuration. This means that the lattice measurement is done at different flow time grids for each individual lattice configuration. Therefore, we need to interpolate the data to a common flow time grid among the different lattice configurations. We use simple spline interpolations, since the gradient flow is a continuous transformation of the fields that produces a continuous function of the flow time. The interpolation can be done to a fixed flow time grid in physical units, or to a fixed flow time ratio τ_F/r^2 grid. Data along a flow time grid in physical units at a given fixed r can be easily presented on a flow time ratio axis by setting the x -axis to τ_F/r^2 .

We check the fluctuation of the topological charge and observe no full freezing of the topology. At our largest lattice (L40), the fluctuation of topology slows down, which increases the autocorrelation times of the topological charge. The static energy (and consequently, its derivative) is known to be less affected by topological slowing down [49] than a bare gradient flow coupling measurement would be [50]. However, to be safe, we block our data such that the block size is larger than the topological autocorrelation times we see on any of the ensembles and settle to 30 jackknife blocks per ensemble. While the static force measurement is only weakly correlated with the topological charge, the scale t_0 can have a stronger dependence on it. To counter this, we measure t_0 only on a subset of configurations, having longer Monte Carlo time in between the measurements.

IV. PREPARATORY ANALYSES

In this section, we analyze and prepare the raw lattice data, and perform the continuum limits needed for the Λ_0 -extraction. The preparation contains the plateau extraction for the static energy and the $T \rightarrow \infty$ limit for the direct force measurement, which is covered in Sec. IV A. Renormalization properties of gradient flow are discussed in Sec. IV B, followed by the continuum extraction, worked out in Sec. IV C. To finalize this section, we investigate the behavior of the various scales r_0 , r_1 , and t_0 .

A. Plateau extraction

We extract the plateaus in the $T \rightarrow \infty$ limit based on a procedure from [51] that relies on an Akaike information criterion. We summarize it here briefly. This procedure is applied to data at every fixed r and τ_F combination for each lattice ensemble.

In the first step, we perform constant fits for all possible continuous ranges within $T/a = 1$ and a certain T_{\max} with a minimum of at least three support points, each fit minimizing χ^2 as

$$\mathbf{a}_{i_1, i_2}^* = \underset{\mathbf{a}_{i_1, i_2}}{\operatorname{argmin}} \chi^2(\mathbf{a}_{i_1, i_2}) = \underset{\mathbf{a}_{i_1, i_2}}{\operatorname{argmin}} \sum_{i, j=i_1}^{i_2} (f(x_i, \mathbf{a}_{i_1, i_2}) - y_i) C_{i,j}^{-1} (f(x_j, \mathbf{a}_{i_1, i_2}) - y_j), \quad (29)$$

where i_1 and i_2 are the indices defining the specific lower and upper limit of the range, f represents the model function, which, in our case, is the constant function $f(x, \mathbf{a}) = a_0$, and \mathbf{a} is the parameter vector, which consists of only one component for the constant fit. The data set D is specified by y_i , the force measurement at $T = x_i$, and C_{ij} the covariance matrix along the T -axis.

In the next step, we define for a given fit and for a specific range i_1 to i_2 the Akaike information criterion (AIC) as

$$\text{AIC}_{i_1, i_2} = \chi^2(\mathbf{a}_{i_1, i_2}^*) + 2k + 2(i_1 - i_2), \quad (30)$$

where k is the number of parameters of the fit function, $k = 1$ for the constant fit, and $i_1 - i_2$ is the relative number of discarded data points within the total range. In theory, we are required to have the absolute number of discarded data points, which is given by $N_{\text{tot}} - (i_1 - i_2)$, however, this corresponds to a global shift for all AIC values, which can be eliminated in the next step.

In the third step, we find the model probability of a specific fit range as

$$p(i_1, i_2|D) = \frac{1}{Z} \exp^{-\frac{1}{2} \text{AIC}_{i_1, i_2}}, \quad (31)$$

where Z is a normalization constant such that the sum of the probabilities over all considered fit ranges reduces to 1:

$$\sum_{i_1 < i_2} p(i_1, i_2|D) = \frac{1}{Z} \sum_{i_1 < i_2} \exp^{-\frac{1}{2} \text{AIC}_{i_1, i_2}} \stackrel{!}{=} 1. \quad (32)$$

At this stage, we observe that a global shift of all AIC values has no effect on the model probability, since it is absorbed into a redefinition of Z . Thus, we perform the global shift $\text{AIC}_{i_1, i_2} \rightarrow \text{AIC}_{i_1, i_2} - \min_{i_1, i_2} \text{AIC}_{i_1, i_2}$ because it makes the computations of the model probability on the most important ranges numerically more stable.

In the last step, we compute model expectation values and deviations with the given model probabilities. The final results and their deviations from the plateau fit are, thus, given by

$$\bar{a}_n = \langle a_n \rangle = \sum_{i_1, i_2} a_{n, i_1, i_2}^* p(i_1, i_2|D), \quad (33)$$

$$\sigma_n^2 = \langle (a_n - \bar{a}_n)^2 \rangle = \langle a_n^2 \rangle - \langle a_n \rangle^2. \quad (34)$$

This procedure can be generalized to more complex model functions which we use in this work for the continuum limit and the Λ_0 extraction. Furthermore, to achieve a better understanding of the process, we compute the average fit ranges

$$\langle x_{i_1} \rangle \equiv \sum_{i_1, i_2} x_{i_1} p(i_1, i_2|D), \quad (35)$$

$$\langle x_{i_2} \rangle \equiv \sum_{i_1, i_2} x_{i_2} p(i_1, i_2|D), \quad (36)$$

and their deviations, defined equivalently to Eq. (34).

A crucial part of this procedure is the selection of a certain T_{\max} . In principle, T_{\max} can be chosen to cover the whole temporal range of the Wilson loops, since the information criterion should eventually select the important ranges. However, especially at larger T , the statistical errors for the covariance matrix are underestimated. These give inaccurate model probabilities for insignificant fit ranges. Therefore, we have to select a proper T_{\max} to prevent this behavior. To find a suitable T_{\max} , we perform the whole procedure for several T_{\max} and select the one where for a small variation of T_{\max} the final result stays invariant. This is justified by the assumption that the important fit ranges, selected by the information criterion, should be fully captured by T_{\max} . Hence, a small variation of T_{\max} should not modify the important fit ranges. The T_{\max} selection can be automatized by taking data up to a T_{\max} where the relative statistical error is less than 10%, and starting decreasing T_{\max} iteratively until an invariant range of T_{\max} is found. However, the automatized procedure does not work in every case and sometimes a selection has to be made manually. To do so, it is enough to identify a representative selection of a few flow times and distances r , and use piecewise linear interpolations to cover the whole data set.

The Akaike information procedure provides us also an error estimate of the plateau extraction. Nevertheless, to propagate the statistical error, we use jackknife resampling with 30 blocks and perform the plateau extraction for every jackknife block. If the resulting jackknife error is comparable with the fit error, we use only the jackknife samples to propagate the error. In several cases, however, the choice of the fit range is a significant systematic error source, and we need to take it into account. In these cases, we label the systematic error by the Akaike information criterion of the corresponding observable as σ_{AIC}^2 , and the statistical error as σ_{stat}^2 .

B. Implications of the gradient flow

Gradient flow has not only an impact on the signal-to-noise ratio improvement, but also on reducing discretization effects that occur through self interaction contributions. Therefore, we are also interested in the renormalization factor Z_E of the chromoelectric field insertion from Eq. (28). We find Z_E non-perturbatively by solving Eq. (28) for Z_E , which gives the ratio of the numerical derivative of the static potential and the direct force measurement according to Eq. (6):

$$Z_E(r) = \frac{\partial_r V(r)}{F(r)}. \quad (37)$$

In [28], it was shown that $Z_E(r)$ has a weak r -dependence. Hence, we extract Z_E as a plateau fit, similar to the $T \rightarrow \infty$ limit discussed in Sec. IV A, over r at fixed τ_F ; we keep r approximately between $0.3 r_0$ and $0.65 r_0$. Figure 1 shows the result for Z_E for all lattice sizes against the flow time. At minimal flow times, we obtain that $Z_E > 1$, meaning that at minimal flow times the direct force measurement is affected by self energy contributions originating from the chromoelectric field discretization. Furthermore, we obtain that $Z_E = 1$ within 1% deviation for flow radii larger than one lattice spacing, which is required for reliable continuum limits. We recognize a small bump within the 1% range for flow radii within $1.5 < \sqrt{8\tau_F}/a < 5$ for all lattice sizes. This is expected, as the $\partial_r V(r)$ -part is a simple finite difference and hence only approximates the static force. We observe that this systematic difference between the definitions of the force seems to vanish at larger flow radii ($\sqrt{8\tau_F}/a > 6$). In conclusion, we find that a minimum amount of flow time ($\sqrt{8\tau_F} > a$) has to be applied to be in the regime where the gradient flow has practically eliminated the non-trivial discretization effects.

C. Continuum extrapolation

1. Continuum interpolation

Preparing for the continuum limit, we interpolate $r^2 F$ on all ensembles at a fixed flow time to a common r -range in $\sqrt{t_0}$ -units. We use polynomial interpolations at different orders:

$$P_n(r) = \sum_{k=0}^n a_{k,n} r^k, \quad (38)$$

with $a_{k,n}$ being the coefficients. We have different coefficients for different fixed order polynomials. An interpolation at fixed order corresponds to a single fit, where we minimize the χ^2 .

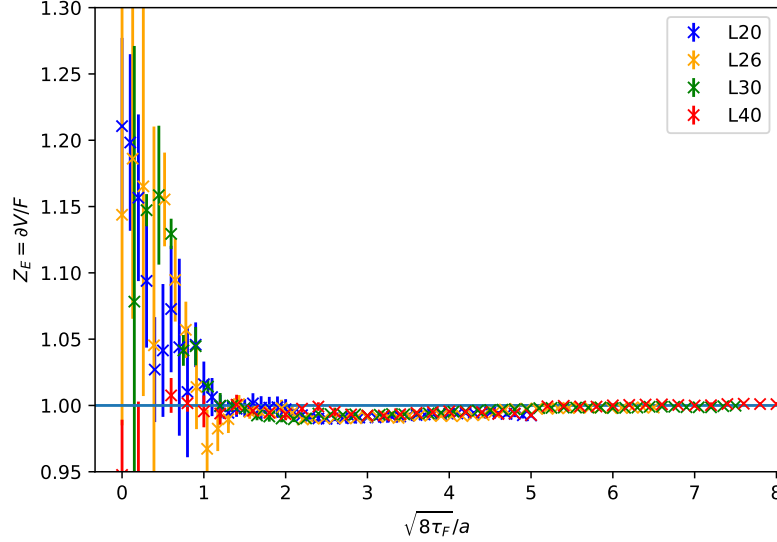


FIG. 1: Z_E for all lattice sizes with increasing flow time. We see that for flow radii larger than one lattice spacing, the factor Z_E becomes practically one.

In addition, interpolations have the advantage that small fluctuations within the data get smoothed. Polynomial interpolations can have fluctuations, especially higher order polynomial fits, and hence, we average over different polynomial orders to reduce those fluctuations:

$$P_{\text{inter}}(r) = \sum_{n=n_{\min}}^{n_{\max}} w_n P_n(r), \quad (39)$$

$$\sum_{n=n_{\min}}^{n_{\max}} w_n = 1, \quad (40)$$

where w_n are normalized, adjustable weights.

For the L20, L26, and L30 lattices, we use equal weights for all polynomial orders. We choose polynomials of orders $n = 4, 5, 6, 7$ for L20, orders $n = 7, 8, 9, 10$ for L26, and orders $n = 8, 9, 10, 11$ for L30. For L40, we use an Akaike average [51] over the orders from 4 to 12 because at some flow times the plateau extraction at larger r gives fluctuating results. This results from underestimated systematic effects and causes a change in the effective polynomial orders, which is considered through the Akaike average. The weights are found analogously to Eqs. (30) and (31) through

$$\text{AIC}_n = \chi_n^2 + 2 \text{dof}, \quad (41)$$

$$w_n \propto e^{-\frac{1}{2} \text{AIC}_n}, \quad (42)$$

where the final values of w_n are fixed by the normalization condition. To propagate the statistical error, the continuum interpolation is done for every jackknife block.

This procedure works for most of the data. An exception is the data at small r (up to $r/a = 3$) for the L26 lattice. We obtain a miscarried interpolation in this case due to large r effects in the interpolation. For the L26 lattice, it turns out that spline interpolations up to $r/a = 3$ and changing to the polynomial interpolation for larger r works properly.

2. Tree level improvement

To reduce the effects of finite lattice spacings, we apply a tree level improvement procedure to the data at finite flow time by dividing out the leading lattice perturbation theory result. Following [52], the static energy in lattice perturbation theory at finite flow time can be written as:

$$V_{\text{lat}}(r, \tau_F) = -C_F g^2 \int \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{r}} e^{-2\tau_F D_{\text{GF}}^{-1}} D_{\text{MC}}, \quad (43)$$

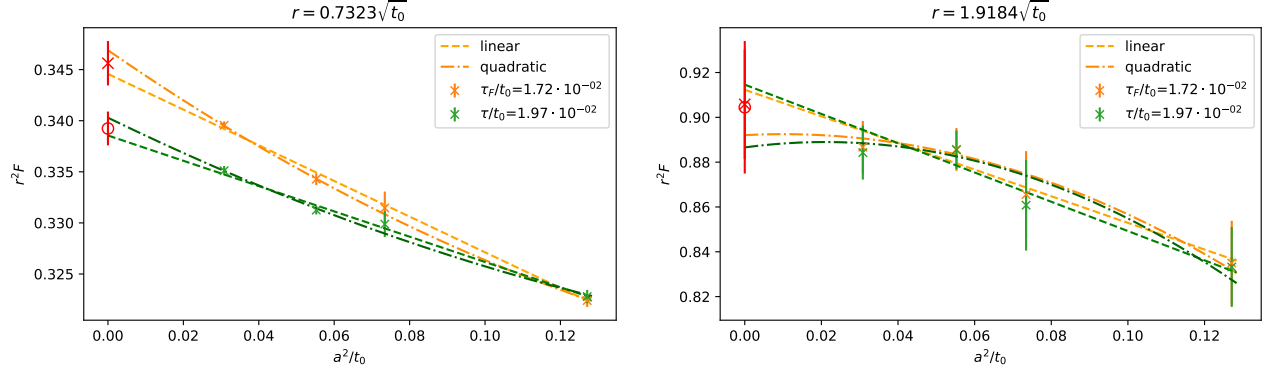


FIG. 2: Examples of continuum extrapolation for two different separations r . The figure shows the linear and quadratic continuum limits: the line corresponds to the linear limit, the curve to the quadratic one.

where we have assumed $a = 1$ for simplicity and D is the lattice propagator:

$$D^{-1} = 4 \sum_{i=1}^4 \left(\sin^2 \frac{k_i}{2} + c_w \sin^4 \frac{k_i}{2} \right); \quad (44)$$

$c_w = 0$ for the Wilson action, which we use for the simulation part, D_{MC} , and $c_w = 1/3$ for the Symanzik action, which we use in the gradient flow equations, D_{GF} . Similarly to the zero flow time case that was derived in [28], the static force coming from the chromoelectric field insertion reduces to a symmetric finite difference:

$$F_{\text{lat}}(r, \tau_F) = \frac{V_{\text{lat}}(r + a, \tau_F) - V_{\text{lat}}(r - a, \tau_F)}{2a}. \quad (45)$$

Now, we can tree level improve the measured force F_{meas} by dividing out the leading lattice expression and multiplying by the continuum 1-loop gradient flow expression

$$F_{\text{improved}}(r, \tau_F) = \frac{\mathcal{F}_0(r, \tau_F)}{F_{\text{lat}}(r, \tau_F)} F_{\text{meas}}(r, \tau_F), \quad (46)$$

where \mathcal{F}_0 is given by Eq. (16).

3. Continuum extrapolation

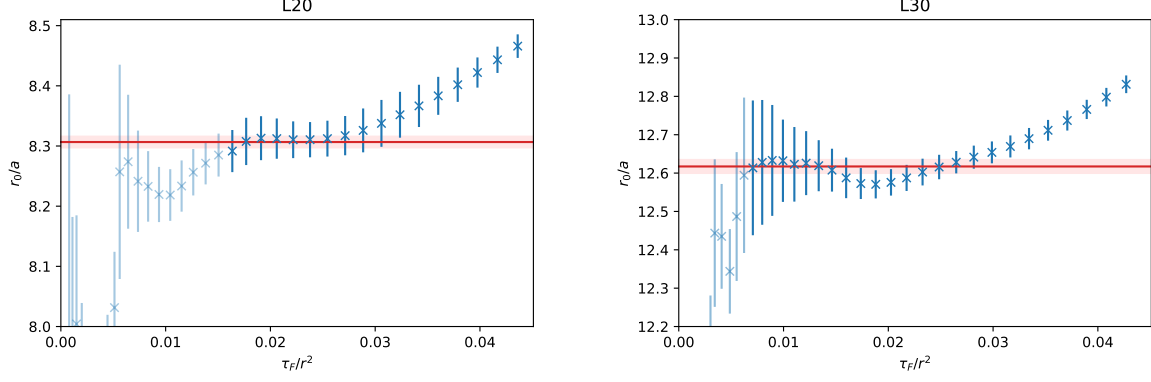
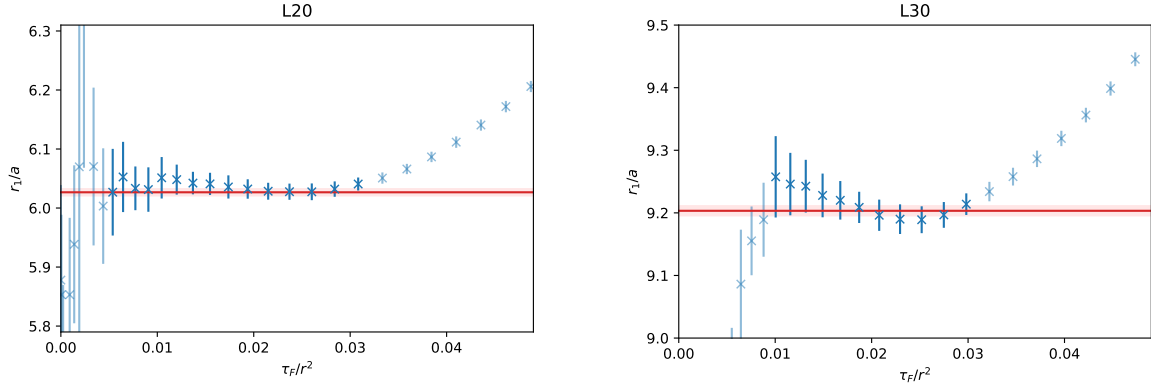
We use the interpolated and tree level improved data for the continuum limit, which is obtained from extrapolations linear and quadratic in a^2/t_0 at fixed physical distances r and fixed physical flow times τ_F :

$$f^{\text{linear}}(a^2/t_0) = m \frac{a^2}{t_0} + c_l, \quad (47)$$

$$f^{\text{quadratic}}(a^2) = A \frac{a^4}{t_0^2} + B \frac{a^2}{t_0} + C_l, \quad (48)$$

where m, c_l, A, B , and C_l are the fit parameters, and c_l and C_l are the continuum limits of the linear and quadratic extrapolation, respectively. We take an Akaike average, defined for polynomials as in Eqs. (41) and (42). Figure 2 shows a working example for the continuum limit at two fixed distances r and for each distance at two different flow times.

Although we use tree level improved data and Akaike average over linear and quadratic continuum limits, often both χ^2/dof are too large. Hence, we restrict to data which accomplish that at least one of the extrapolations gives $\chi^2/\text{dof} < 3.0$ and that the flow radius fulfills $\sqrt{8\tau_F} > a$. The remaining, filtered data represents reliable continuum limit results, with which we continue the further analyses.

FIG. 3: The r_0 scale along the flow time axis.FIG. 4: The r_1 scale along the flow time axis.

D. r_0 and r_1 scales

In terms of the force $F(r)$, we define a reference scale as

$$r^2 F(r, \tau_F)|_{r(c, \tau_F)} = c, \quad (49)$$

with a dimensionless number c . Common choices are $r_0(\tau_F) \equiv r(c = 1.65, \tau_F)$, $r_0 \equiv r_0(\tau_F = 0)$ [6], and $r_1(\tau_F) \equiv r(c = 1, \tau_F)$, $r_1 \equiv r_1(\tau_F = 0)$ [53]. In the original proposals, the scales $r(c)$ are defined with the force at zero flow time, however, in our case, the force is computed at different flow times. Thus, we obtain flow time dependent r_0 and r_1 as shown in Figs. 3 and 4. To find r_0 and r_1 , we perform multiple polynomial interpolations of $r^2 F(r, \tau_F)$ for larger r along the r -axis and at fixed flow time ratio τ_F/r^2 , and find the roots of the individual interpolations as $(r^2 F)^{\text{inter}} - c = 0$. The final scales are given in terms of an Akaike average over the roots of the polynomial interpolators.

Both scales, r_0 and r_1 , approach a constant plateau within a recognizable flow time regime and start deviating from this plateau at larger flow times. We perform plateau fits within this range for a zero flow time extraction of both

	r_0/a	σ_{stat}	σ_{AIC}	$\sigma_{\text{stat+AIC}}$	r_1/a	σ_{stat}	σ_{AIC}	$\sigma_{\text{stat+AIC}}$
L20	8.306	0.017	0.010	0.020	6.026	0.0090	0.006	0.011
L26	10.833	0.031	0.025	0.040	7.849	0.014	0.010	0.018
L30	12.617	0.032	0.018	0.040	9.203	0.0130	0.008	0.015
L40	16.933	0.042	0.015	0.044	12.316	0.0280	0.011	0.030

TABLE III: Results for r_0 and r_1 in the zero flow time limit at finite lattice spacing. The table includes the statistical error (stat subscript), the systematic error by choosing different fit ranges (AIC subscript), and the errors added in quadrature.

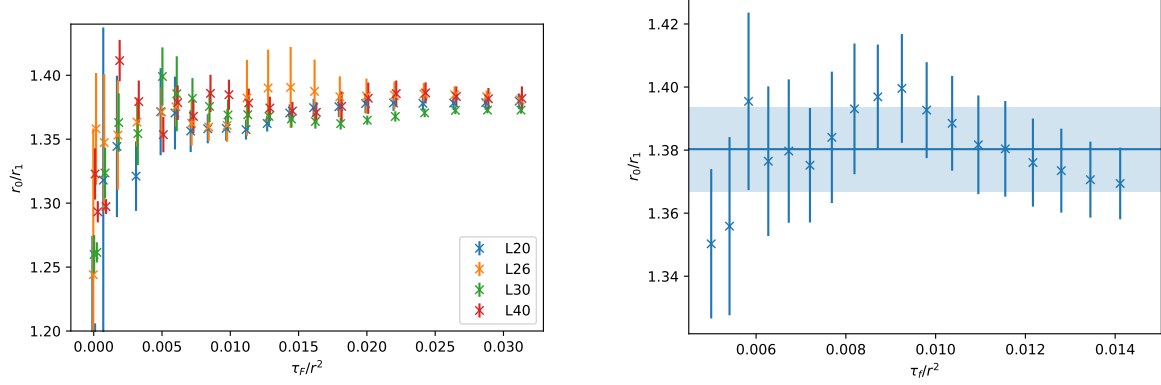


FIG. 5: The ratio of the scales r_0 and r_1 . The left side shows the ratio at finite lattice spacing, the right side the continuum limit with the constant zero flow time extraction.

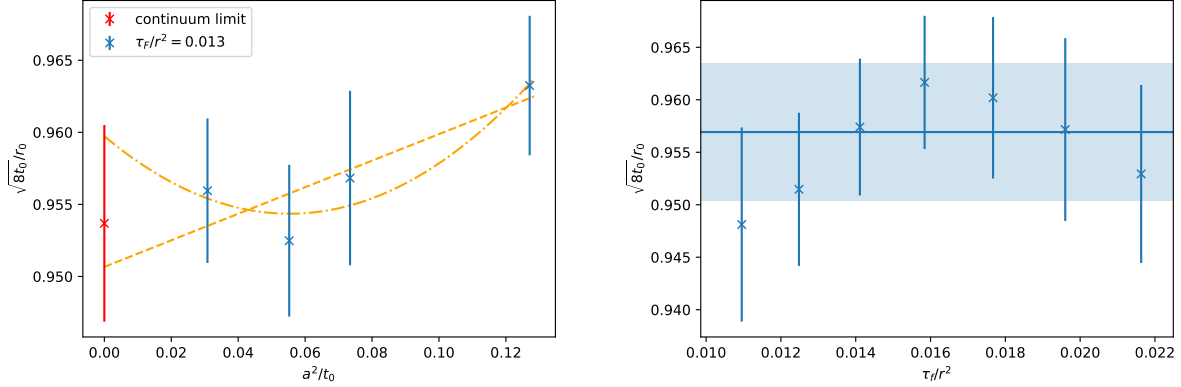


FIG. 6: Left: An example for the continuum limit of the ratio $\sqrt{8t_0}/r_0$ at a fixed flow time ratio. The straight line corresponds to a continuum limit linear in a^2/t_0 , the curved line to a polynomial up to quadratic order in a^2/t_0 . The red cross and bar is the final continuum limit of a weighted average of both extrapolations. Right: The flowed continuum ratio with the constant zero flow time limit.

scales, and we find them comparable to the scales fixed by [4]. The results are shown in table III. We see that the error of the scale setting is dominated by the statistical fluctuation rather than by the Akaike error of the polynomial interpolators.

The ratio of both scales, $r_0(\tau_F)/r_1(\tau_F)$, is shown in Fig. 5. We take the a^2/t_0 continuum limit followed by a constant zero flow time limit. Finally, we obtain

$$\frac{r_0}{r_1} = 1.380(14). \quad (50)$$

To our knowledge, there are no prior direct determinations of the scale ratio r_0/r_1 in pure gauge. However, in Ref. [54] this ratio was roughly estimated to be ~ 1.37 based on the curves shown in [4], which agrees well with our extraction within errors. The extracted value is about 9% larger than the ratio in unquenched theories with 2+1 or 2+1+1 fermion flavors [1, 47]. Such a shift is to be expected, due to the effect of unquenching of the quark flavors. A similar effect between quenched and unquenched scales has been seen for the gradient flow scale ratios in Ref. [55].

We repeat the same procedure for the ratios $\sqrt{8t_0}/r_0$ and $\sqrt{8t_0}/r_1$. Fig. 6 shows an example for the ratio $\sqrt{8t_0}/r_0$.

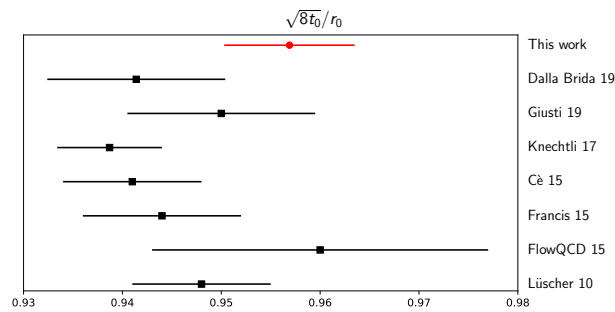


FIG. 7: Our extracted ratio (red circle) of the scale ratio $\sqrt{8t_0}/r_0$ compared to the existing results (black squares) in the literature.

The final results for the ratios of the scales are

$$\frac{\sqrt{8t_0}}{r_0} = 0.9569(66), \quad (51)$$

$$\frac{\sqrt{8t_0}}{r_1} = 1.325(13). \quad (52)$$

Our extracted ratio $\sqrt{8t_0}/r_0$ agrees within errors with the previous determinations [31, 56–62],¹ albeit the mean value is slightly above most of the existing results. We show the comparison to the existing literature in Fig. 7. The previous results are somewhat correlated, since most of them focus on t_0 calculation and use the data from Refs. [4, 5] at least for part of their dataset for r_0 . Again, the quenched ratios are larger than the unquenched ones [1] as has been previously seen in Ref. [55]. To our knowledge, this is the first direct measurement of the scale ratio $\sqrt{8t_0}/r_1$ in pure gauge.

V. ANALYSES OF THE CONTINUUM RESULTS

After having worked out the continuum limit in Sec. IV C, we compare the lattice results with the perturbative expressions to extract Λ_0 . Since gradient flow introduces another scale next to $1/r$, we have several possibilities to compare the lattice results with the perturbative expressions. In the first approach, we use the simple expression of the flowed force in Eq. (22), which turns out to be also applicable to large r results. In the second approach, we compare the lattice results either by keeping the scale $1/r$ fixed and inspecting the behavior along the flow time, or vice versa by keeping τ_F fixed and inspecting the behavior along the distance r . We show plots only for the perturbative 1-loop (F1l) and 3-loop with leading ultrasoft resummation (F3lLus) expressions, however, in the result tables we also provide extractions based on the 2-loop with and without leading ultrasoft resummation (F2l and F2lLus), and 3-loop without ultrasoft resummation (F3l) expressions.

A. Constant zero flow time limit at fixed r

We know from Eq. (22) that the static force has a constant behavior at small flow times. Physical quantities are defined at zero flow time; hence, we need to perform the zero flow time limit, $\tau_F \rightarrow 0$, while we keep r fixed. In the constant regime, we perform this by a constant fit at fixed distance r . Figure 8 shows data where we obtain a constant behavior of the flowed force. The left side shows the data for the smallest r before the smaller flow time comes into conflict with the $\sqrt{8\tau_F} > a$ condition. The right side shows data at larger r where the condition $\sqrt{8\tau_F} > a$ is fulfilled at even minimal flow time ratios. The small flow time expansion is performed in the ratio τ_F/r^2 , thus, small flow time is defined in the sense of small flow time ratio, which is a dimensionless quantity. The condition $\sqrt{8\tau_F} > a$ is given in terms of flow times in physical units; hence, considering this condition in terms of ratio moves it to smaller flow time ratios for larger r , since the r in the denominator decreases the ratio.

¹ The FlowQCD result from Ref. [56] is inferred from their results of $\sqrt{t_0}/w_0$ and r_0/w_0 neglecting error correlation. Therefore, the error shown in Fig. 7 is certainly overestimated.

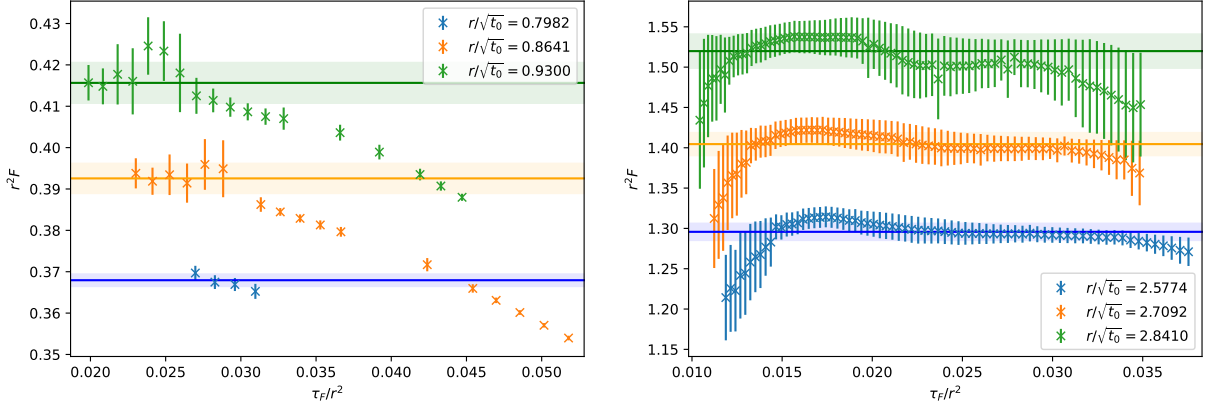


FIG. 8: Examples of the flowed force in a regime where a constant behavior can be obtained. The left side shows the data for the smallest r where a constant regime can be obtained. The right side shows the flowed force at larger r where we can apply a constant fit in a proper range of τ_F/r^2 .

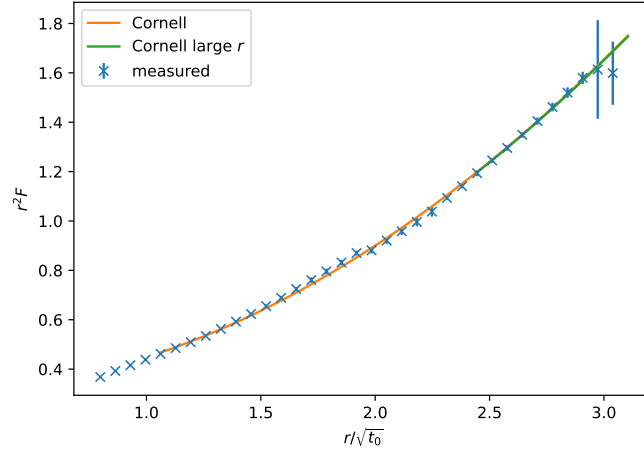


FIG. 9: The extracted force from the constant zero flow time limits with two Cornell fits. In a window from $r/\sqrt{t_0} \approx 2.1$ to $r/\sqrt{t_0} \approx 2.3$, the continuum limit turns out to be difficult to reach. We lose the signal at larger r around $r/\sqrt{t_0} \approx 3.0$.

In Fig. 9, we see the final result of the constant zero flow time limits. We identify an increasing behavior with small errors up to $r/\sqrt{t_0} \approx 3.0$. Around the distance of $r/\sqrt{t_0} \approx 2.25$, we are faced with difficulties to extrapolate to the continuum, and obtain χ^2/dof of order 4 and larger.

We perform a Cornell-fit to the data from $r/\sqrt{t_0} \approx 1.1$ to $r/\sqrt{t_0} \approx 3.0$ and we obtain

$$r^2 F(r) = A + \sigma r^2, \quad (53)$$

$$A = 0.297(6), \quad (54)$$

$$\sigma t_0 = 0.151(3). \quad (55)$$

The string tension σ is a quantity that is dominated by the large r regime; hence, in addition, we perform the fit only for data beyond the region where the continuum limits are problematic up to $r/\sqrt{t_0} \approx 3.0$. In this case, we obtain

$$A = 0.268(33), \quad (56)$$

$$\sigma t_0 = 0.154(6). \quad (57)$$

The uncertainty for A is five times larger now, which is to be expected, since it is a small r quantity. The results for σ from both ranges agree within their uncertainties. With the result in Eq. (51), we can express the string tension in

r_0 units:

$$\sigma r_0^2 = 1.345(54). \quad (58)$$

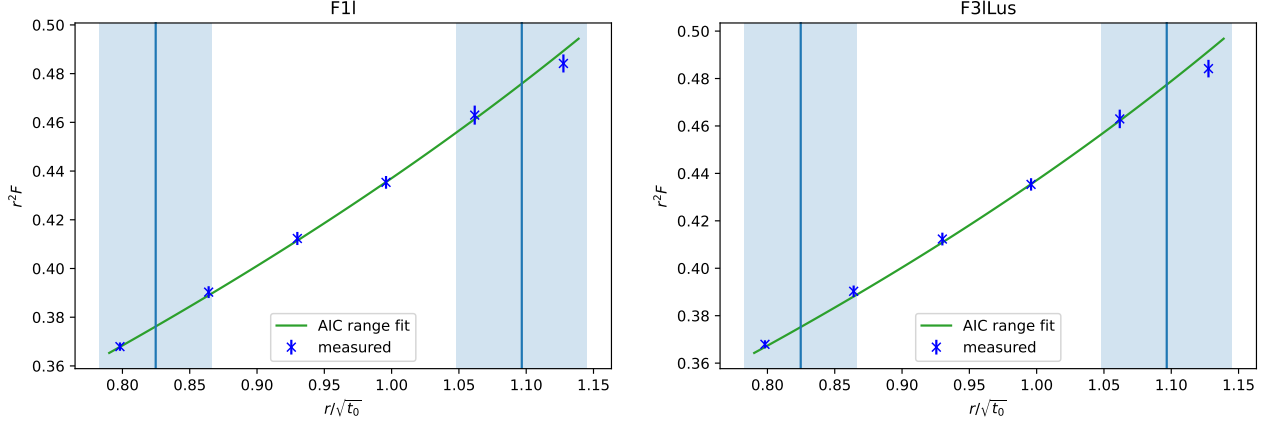


FIG. 10: The $\sqrt{8t_0}\Lambda_0$ extraction with constant zero flow times extrapolation at smallest possible distances r . The vertical lines with the dimmer bar represent the lower and upper averaged fit range according to Eqs. (35) and (36).

In the past, the string tension was found to be $\sigma r_0^2 = 1.353(3)$ [63] at finite lattice spacing with $\beta = 6.0$, which is in good agreement with our result. Continuum results were obtained in [34, 64] in another reference scale \bar{r} . With the ratio $\sqrt{8t_0}/\bar{r}$ from [34], these results become $\sigma t_0 = 0.133(2)$ and $\sigma t_0 = 0.143(2)$ respectively. Nevertheless, the ratio $\sqrt{8t_0}/\bar{r}$ in [34] is only an approximation over several lattices sizes, and is not extrapolated to the continuum limit. Reliable continuum results for the string tension can be found in [65, 66] in units of the critical temperature T_c . With the conversion factor $T_c\sqrt{t_0}$ from [57], these results become $\sigma t_0 = 0.1484(22)$ and $\sigma t_0 = 0.156(3)$, which agree well with our result.

Order	$\sqrt{8t_0}\Lambda_0$	σ_{stat}	σ_{AIC}	$\sigma_{\text{stat+AIC}}$
F1l	0.8214	0.0044	0.0018	0.0047
F2l	0.6635	0.0048	0.0029	0.0056
F2lLus	0.6961	0.0057	0.0039	0.0069
F3l	0.6197	0.0036	0.0024	0.0043
F3lLus	0.6353	0.0032	0.0013	0.0035

TABLE IV: The fit result for $\sqrt{t_0}\Lambda_0$ from the constant zero flow time limit extracted data at different orders. We see that the error is dominated by statistical fluctuations rather than the choice of the proper fit window.

At small r , we extract Λ_0 from the data by fitting the perturbative force at the available orders. Figure 10 shows examples of the fit for two different orders. Table IV shows the fit results. We observe that the error is dominated by the statistical fluctuations rather than by the systematic uncertainty due to the choice of the fit ranges.

B. Flow time behavior of the force at fixed r or fixed τ_F

In the very small r regime, the requirement $\sqrt{8\tau_F} > a$ moves the data points along the τ_F/r^2 axis outside the regime where they are flow time independent even for the smallest possible flow time. Therefore, we compare the lattice data with the full expression of the force, Eq. (23), i.e. without expanding for small τ_F .

We inspect the small r flow time behavior, first, at fixed distances r , which corresponds to the classical zero flow time limit. In a second approach, we fit along the r axis at fixed flow times to extract $\sqrt{8t_0}\Lambda_0$. Since the dependence on τ_F of the numerical extraction turns out to be negligible within the distance and flow time regions used for the fits to the lattice data, getting $\sqrt{8t_0}\Lambda_0$ provides in practice its zero flow time limit.

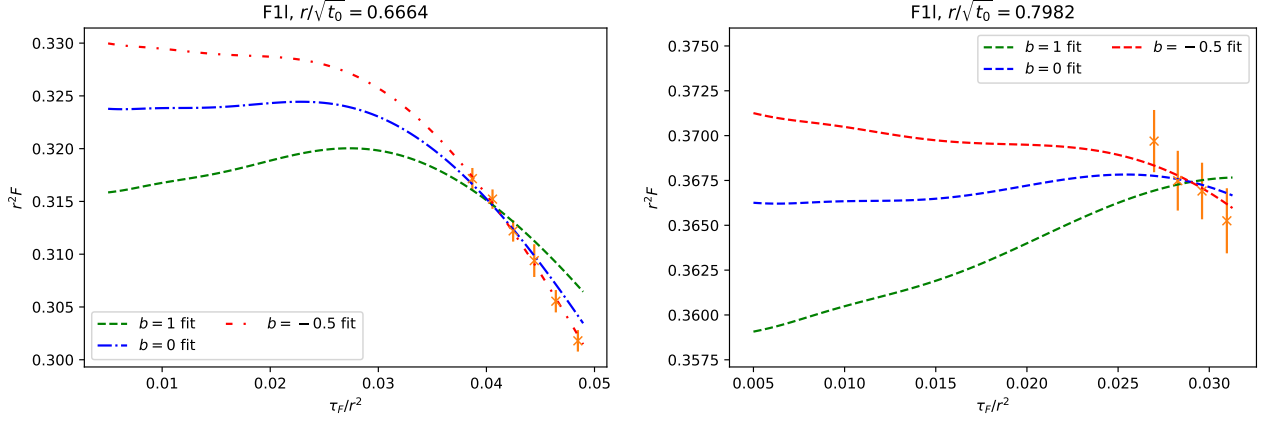


FIG. 11: Comparison of different zero flow time limits at 1-loop order at fixed r . We compare fits with different scale choices obtained by varying b according to Eq. (21).

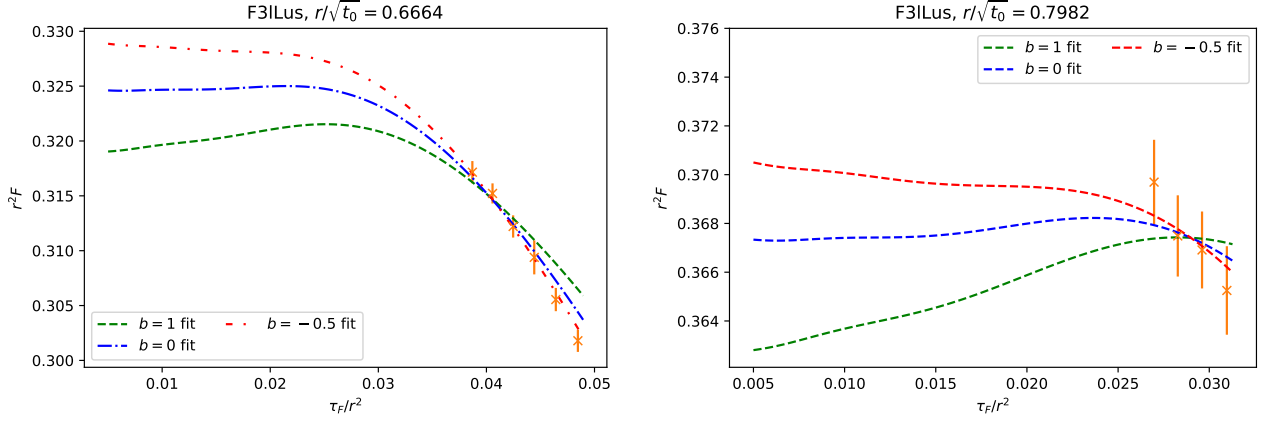


FIG. 12: As in Fig. 11, but with the fit function being now the force at 3 loops with leading ultrasoft resummation.

1. Fixed r

Figures 11 and 12 show the flow time behavior of the force at two different fixed r along the flow time axis. We compare our lattice data with the perturbative expressions of the force at different orders and fit them to the data. Λ_0 serves as the fit parameter. In the figures, we show the fit results for the case of different, but fixed choices of b in Eq. (21). The fit range starts at the smallest possible flow time. For the upper bound, we take an Akaike average [51] over different fit ranges to reduce the systematics due to the fit range choice.

From the figures, we get that the choice of the scale parameter b has a definitive effect on how well the perturbative curve describes the data. A value of $b = 1$ guarantees the correct asymptotic behavior at large flow time [37]. However, we see that $b = 1$ is the worst of our b choices at describing the actual lattice data in the range of flow times we are interested in. Hence, we use negative values of b , which still lead to valid scaling in our range of flow times, as discussed in Appendix A. At small r (left side plots in Figs. 11 and 12), the slope along the flow time is strong, while at the largest r to which we can reasonably apply fixed r fits (right side plots in Figs. 11 and 12), the data points seem already sensitive to the constant behavior of the force expected at small flow times. At small flow times within our considered flow time range, the fits with $b = 0$ and $b = -0.5$ agree reasonably well with the data. Table V shows an incomplete part of the fit results.

We conclude that for smaller r , which requires going to larger flow time ratios τ_F/r^2 , a negative value of b describes the data better in the large flow time regime than $b = 0$ or 1, whereas for larger r all choices might describe the data within the given uncertainties. That means that fixing b introduces another source of uncertainty which has to be considered. In the zero flow time limit, all choices of b give $\mu = 1/r$, which is the natural scale choice for the static

Order	$r/\sqrt{t_0}$	b	$\sqrt{8t_0}\Lambda_0$	σ_{stat}	σ_{AIC}	χ^2/dof	$\overline{\tau_F/r^2}_{\text{Max}}$
F3lLus	0.6664	1	0.6020	0.0030	0.0010	0.57	0.0449(9)
		0	0.6229	0.0031	0.0010	0.52	0.0463(16)
		-0.5	0.6388	0.0029	0.0005	0.22	0.0473(14)
	0.7323	1	0.6128	0.0037	0.0011	0.93	0.0381(14)
		0	0.6300	0.0037	0.0009	0.37	0.0395(20)
		-0.5	0.6427	0.0037	0.0004	0.21	0.0403(19)
	0.7982	1	0.6210	0.0036		0.72	
		0	0.6348	0.0037		0.39	
		-0.5	0.6440	0.0038		0.23	
F2l	0.6664	1	0.6132	0.0032	0.0010	0.59	0.0449(9)
		0	0.6361	0.0033	0.0011	0.51	0.0463(16)
		-0.5	0.6538	0.0031	0.0005	0.20	0.0474(14)
	0.7323	1	0.6274	0.0040	0.0012	0.97	0.0381(13)
		0	0.6468	0.0041	0.0010	0.38	0.0395(20)
		-0.5	0.6612	0.0040	0.0005	0.20	0.0404(19)
	0.7982	1	0.6394	0.0040		0.77	
		0	0.6555	0.0041		0.41	
		-0.5	0.6663	0.0042		0.23	
F1l	0.6664	1	0.7550	0.0036	0.0010	0.86	0.0446(6)
		0	0.7952	0.0038	0.0012	0.43	0.0466(16)
		-0.5	0.8276	0.0037	0.0001	0.04	0.0476(13)
	0.7323	1	0.7696	0.0045	0.0013	1.33	0.0376(9)
		0	0.8042	0.0048	0.0013	0.49	0.0393(20)
		-0.5	0.8304	0.0048	0.0003	0.12	0.0405(19)
	0.7982	1	0.7829	0.0045		1.29	
		0	0.8125	0.0048		0.59	
		-0.5	0.8328	0.0050		0.26	

TABLE V: Fit results at fixed r along the flow time axis for three different choices of the perturbative order of the static force. We start at the smallest possible flow time and use an Akaike range fit for different upper limits. No value for σ_{AIC} indicates that there are not enough points along the flow time to perform an Akaike range fit. The given χ^2/dof corresponds to the most likely fit range. $\overline{\tau_F/r^2}_{\text{Max}}$ gives the Akaike averaged upper flow time limit.

b -scale	F1l	F2l	F2lLus	F3l	F3lLus
1	0.7972(56)	0.6591(49)	0.6911(52)	0.6062(52)	0.6218(50)
0	0.8134(57)	0.6649(47)	0.6982(54)	0.6154(44)	0.6287(44)
-0.5	0.8334(42)	0.6709(47)	0.7017(53)	0.6285(35)	0.6415(36)

TABLE VI: Results of the $\sqrt{8t_0}\Lambda_0$ extraction from the constant zero flow time limit of Λ_0 at fixed flow time. The error includes the statistical and the AIC error from the fit.

force and energy at zero flow time.

2. Fixed τ_F

In the next step, we use data at fixed flow times τ_F and fit the perturbative force along the r -axis. We use Akaike average [51] over different fit windows for reducing systematics by choosing the right fit window. We perform one-parametric fits at fixed b for $\sqrt{8t_0}\Lambda_0$. Figure 13 shows an example fit for $b = 0, 1, -0.5$ for F1l and F3lLus at the same flow time. The left vertical line with the dimmer band corresponds to the average lower fit limit for the $b = 0$ fit, and the right vertical line to its average upper limit.

We observe that the $b = 0$ fit describes the data over a wide r range from small to larger r . The $b = 1$ fit describes the data around $r/\sqrt{t_0} = 1$ and up to larger r in the same way as the $b = 0$ fit, but deviates from the data at smaller r in contrast to the $b = 0$ fit. This matches with its lower fit range limit being at larger r , but the upper fit range limit being the same as in the $b = 0$ fit, indicating that the effective fit range for $b = 1$ is more on the larger r side. The $b = -0.5$ fit describes the data around $r/\sqrt{t_0} = 1$ likewise the $b = 0$ and $b = 1$ fits, but in contrast to the $b = 1$ case, it fits better to the data at small r and deviates from the data at larger r . This matches with its upper fit range limit being at smaller r , but the lower fit range limit being the same as in the $b = 0$ fit, indicating that the effective fit range for $b = -0.5$ is more on the smaller r side. We conclude that the range for b from -0.5 to 1 fits well to

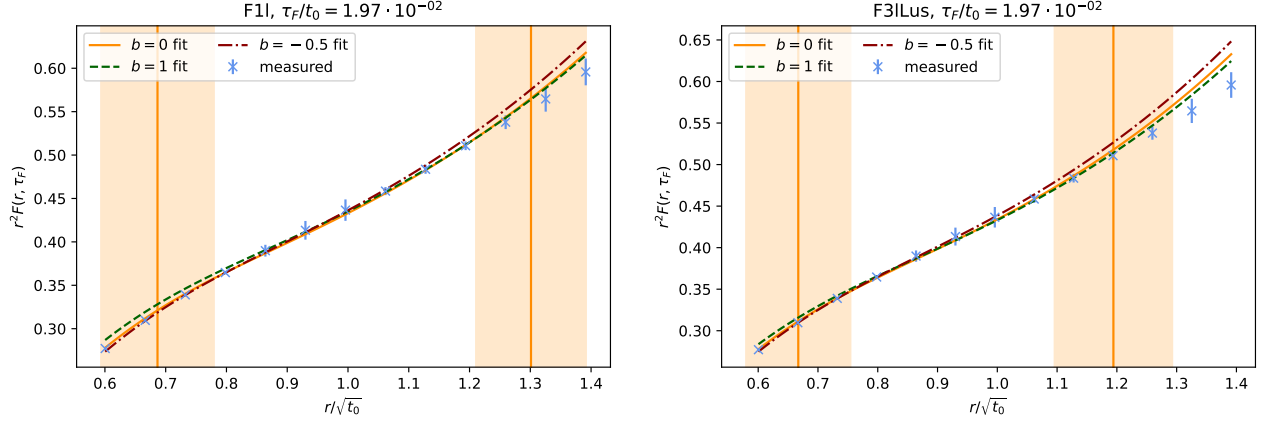


FIG. 13: $\sqrt{8t_0}\Lambda_0$ fit at fixed flow time for different orders. The left vertical line corresponds to the average lower fit limit for $b = 0$, and the right line to its average upper fit limit. The bands around the lines represent the errors of the fit window.

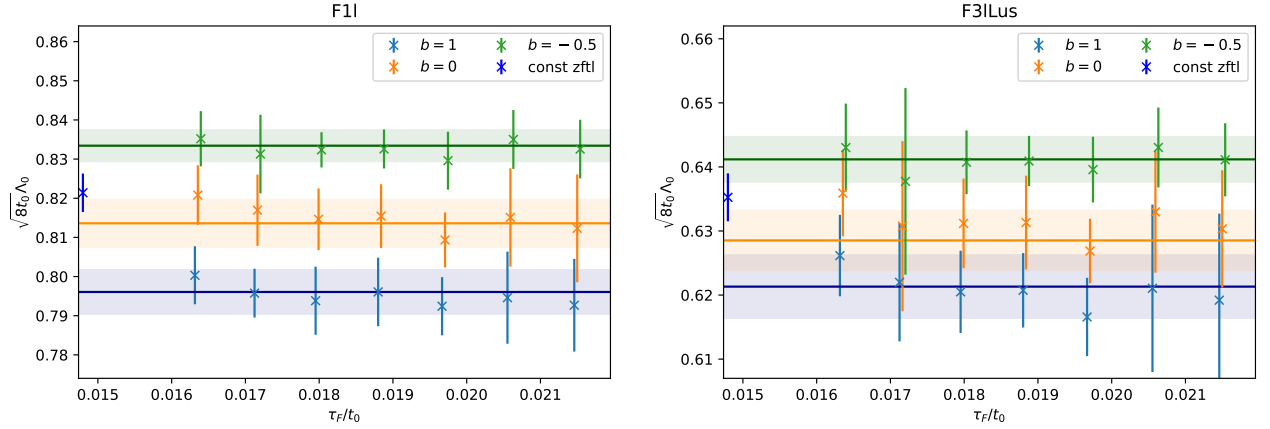


FIG. 14: $\sqrt{8t_0}\Lambda_0$ in the zero flow time limit from a constant fit including the statistical and fit errors. We compare different orders and different choices of b . The blue points with the label “const zftl” are the results from table IV.

the data, but with different effective fit ranges. The $b = 0$ fit has the widest effective fit range and will serve as our default choice.

Figure 14 shows the fit results for $\sqrt{8t_0}\Lambda_0$ at the valid flow time positions for $b = 1, 0, -0.5$ at F1l and F3lLus. We observe that for a fixed choice of b , the values of Λ_0 are constant along the flow time axis. This indicates that the flow time dependence of the static force at finite flow time in the distance and flow time ranges explored in this fixed τ_F analysis is well captured by a constant 1-loop gradient flow correction to the static force. We then extrapolate Λ_0 to the zero flow time limit with a constant function. The final results of the constant zero flow time limits are shown in table VI.

C. Estimate of the perturbative systematic uncertainties and final results

Up to this point, we have presented results for Λ_0 with error estimates that include only the statistical errors and the systematic errors from choosing different fit ranges. We still need to include the perturbative uncertainty from the unknown higher-order terms in the perturbative expansion. We can do this by varying the scale μ (21). In previous studies of the static energy [14, 15, 17], the zero flow time scale $\mu = 1/r$ was varied by a factor of $\sqrt{2}$. We make here the same choice and vary the parameter s in Eq. (21) from $s = 0.5$ to $s = 2$. We vary the s -parameter only in the zero flow time part of Eq. (23) and keep it fixed at $s = 1$ in the finite flow time part. In principle, we could vary the

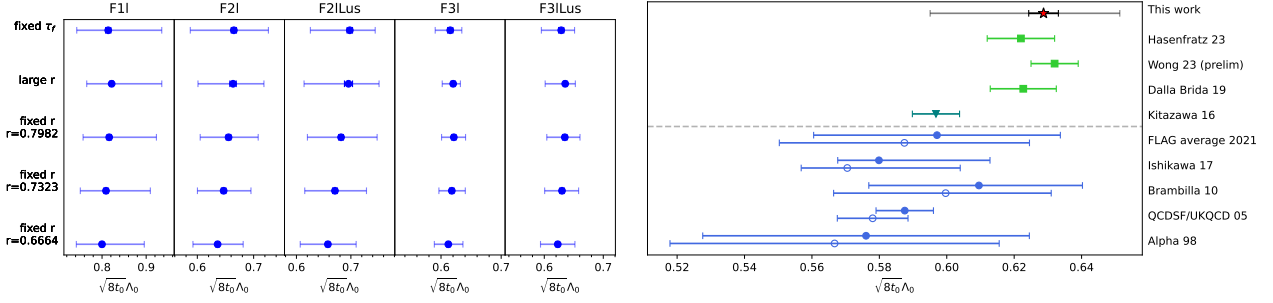


FIG. 15: The final results for $\sqrt{8t_0}\Lambda_0$. Left: Comparison of all fit results for all methods and for all orders. The inner error bar corresponds to the combined statistical and AIC errors, when applicable. The outer error bar represents the total error, including the systematic uncertainties from the scale variation. The row “large r ” shows the results from the first method, where we perform the constant zero flow time limit of the force first. Right: Our final result compared to the literature. The points above the dashed line were determined with gradient flow based scale settings, t_0 (squares) or w_0 (triangle), while the lower points (circles) are converted from r_0 units with either our ratio Eq. (51) (filled points), or with the ratio from [62] (hollow points).

scale by a factor of 2 instead of $\sqrt{2}$, but it was noted in Ref. [17] that this requires access to quite small distances r . Our current data does not contain small enough distances to allow for this wider variation.

As already stated in the previous sections, the finite flow time part of the static force has a considerable dependence on the choice of the parameter b in Eq. (21). To match the conventions of zero flow time studies, we choose $b = 0$ as our main result. To estimate the systematic error due to the choice of b and the missing higher order finite flow time perturbative terms, we vary the b parameter between $b = -0.5$ and $b = 1$; for this choice, we refer to the discussion in appendix A. We vary b only in the finite flow time part of the Eq. (23).

To arrive at a final result for Λ_0 , we have explored several possibilities in sections VA, VB1, and VB2:

1. We have performed the constant zero flow time limit of the force first, followed by fitting the perturbative expression to the data. This method has the advantage that we do not need to combine the zero flow time expression of the force with the 1-loop correction coming from the gradient flow. The method does not work, however, at the smallest r , but only for $r/\sqrt{t_0} \geq 0.8$.
2. We have performed the fit of the combined equation at fixed r along τ_F . This method corresponds to the classical zero flow time limit. With this method, the $1/r$ scale has a minor effect and the method can be applied for only few r , while the gradient flow scale has a dominant role, which can be seen by the large dependence on the choice of b .
3. We have performed the fit of the combined equation at fixed τ_F along r . In this method, the $1/r$ scale has a major impact, while the flow time scale has a minor role. This can be seen by the fact that various choices of b fit well to the data.

On the left side of Fig. 15, we compare the results of all three methods at the available perturbative orders. All results agree very well within the errors.

Based on the advantages and disadvantages of all three methods, we chose method 3 with $b = 0$ at F3lLus as the main result. Including the uncertainties due to variations of the parameters s and b , we obtain our final result, which reads

$$\sqrt{8t_0}\Lambda_0 = 0.629^{+23}_{-34}, \quad (59)$$

$$\delta(\sqrt{8t_0}\Lambda_0) = (4)^{\text{lattice}} \binom{+18}{-25} s\text{-scale} \binom{+13}{-22} b\text{-scale}, \quad (60)$$

where $(\dots)^{\text{lattice}}$ stands for the error coming from the statistics and choosing different fit windows. As we allow the parameters b and s in Eq. (21) to be varied only in the finite and zero flow time part respectively, it follows that the systematic uncertainties from these variations are nearly independent one from the other. Hereby, we quote the uncertainty from the b scale variation measured at $s = 1$ and the uncertainty from the s scale variation measured at $b = 0$, and add these in quadrature. This accounts for a conservative estimate of the perturbative error.

On the right side of Fig. 15, we compare our final result with results from previous measurements of Λ_0 [9, 62, 67–72] and the FLAG average [1]. We only show previous studies that contribute to the FLAG average and a couple of newer

studies that have come out since the latest FLAG average. In Fig. 15, the points above the dashed line have been obtained using the gradient flow based scales t_0 and w_0 ,² while the points below the dashed line have been obtained from the scale r_0 . For the measurements done in the r_0 scale, we convert to the t_0 scale with our own ratio (51) (filled points) or with one of the ratios from Ref. [62] (empty points). We find it remarkable that all the newer studies done in t_0 units are on the higher end of the measurements. Furthermore, we note that our error is larger than other recent studies. Our error is dominated by the perturbative error from the scale variation. Since the scale variation is more prominent at larger distances, it is to be expected that future access to finer lattices could bring this error down. Lastly, with the ratio in Eq. (51), we can convert our final result into r_0 units and obtain

$$r_0\Lambda_0 = 0.657^{+61}_{-66}. \quad (61)$$

VI. SUMMARY AND CONCLUSION

We have shown that the gradient flow renormalizes an operator made of a Wilson loop with a chromoelectric field insertion by reducing discretization effects and in this way improving the convergence towards the continuum limit. This result can be of use for further studies on operators with different field insertions, which typically show up in nonrelativistic effective field theories.

Thanks to the above property, we are able to perform the continuum limit of the static force at finite flow time, and extrapolate to zero flow time in three different ways. In the first method, we extrapolate the static force from a constant zero flow time limit. This works for large and intermediate r , but not at very short distances in the regime $r/\sqrt{t_0} < 0.8$. At large distances, we extract the scales r_0 and r_1 , and are able to perform a Cornell fit. For the scales, we find the ratios

$$\frac{r_0}{r_1} = 1.380(14), \quad (62)$$

$$\frac{\sqrt{8t_0}}{r_0} = 0.9569(66), \quad (63)$$

$$\frac{\sqrt{8t_0}}{r_1} = 1.325(13), \quad (64)$$

and for the string tension parameter in the Cornell fit, we find

$$\sigma t_0 = 0.154(6), \quad (65)$$

$$\sigma r_0^2 = 1.345(54), \quad (66)$$

where we have used our result for the ratio $\sqrt{8t_0}/r_0$ in Eq. (63) to convert into r_0 units. At short distances, we fit the perturbative force to the data, and obtain at F3lLus order

$$\sqrt{8t_0}\Lambda_0 = 0.635(4). \quad (67)$$

In the second and third method, we fit with a function that combines the force at zero flow time up to 3 loops with the 1-loop flow time correction. In this way, the fit function depends on two scales, r and τ_F . In the second method, we keep r fixed and perform the fit along τ_F . In the third method, we keep τ_F fixed and perform the fit along r , and extrapolate the resulting $\Lambda_0(\tau_F)$ to the zero flow time limit. The third method has, in comparison to the second method, a strong dependence on the scale $1/r$, which is the dominant physical scale. Furthermore, the third method reaches out to small r in contrast to the first method. Therefore, we take the third method as our reference method and obtain

$$\sqrt{8t_0}\Lambda_0 = 0.629^{+23}_{-34}, \quad (68)$$

$$\delta(\sqrt{8t_0}\Lambda_0) = (4)^{\text{lattice}} \binom{+18}{-25}^{\text{s-scale}} \binom{+13}{-22}^{\text{b-scale}}. \quad (69)$$

Using the ratio in Eq. (63), we can give our final result in r_0 units as

$$r_0\Lambda_0 = 0.657^{+61}_{-66}. \quad (70)$$

Nevertheless, all methods agree well within their errors, with an overlap of almost 70 %.

² The measurement of Ref. [70] in w_0 units was transformed to t_0 units using the ratios from [56].

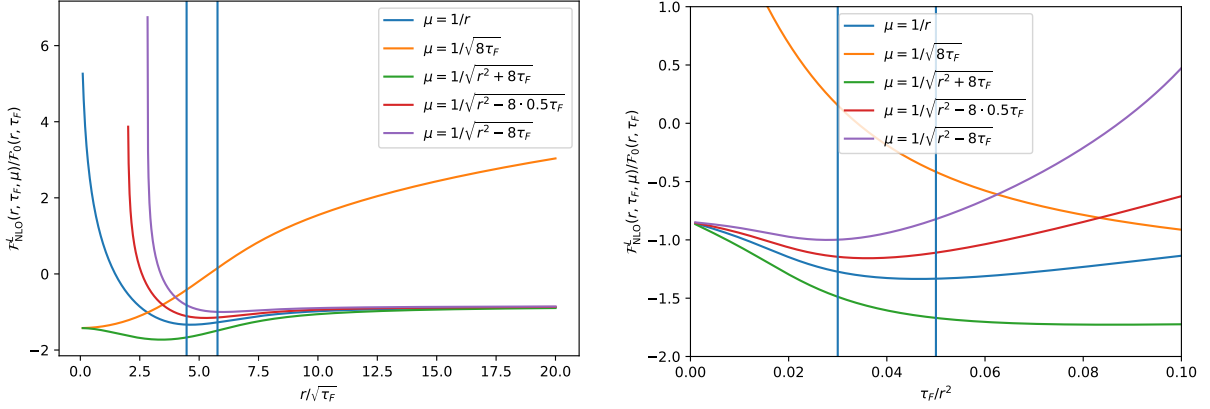


FIG. 16: Left: The ratio $\mathcal{F}_{\text{NLO}}^L(r, \tau_F, \mu)/\mathcal{F}_0(r, \tau_F)$ (see Eq. (15)) as a function of $r/\sqrt{\tau_F}$ plotted for various values of s and b . The vertical lines represent the characteristic flow time window of this study. The left limit on the x -axis corresponds to the infinite flow time limit, while the right limit to the zero flow time limit. Right: The same ratio, but as a function of τ_F/r^2 zoomed around the region of interest.

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Appendix A: About negative values of b in μ

In the analysis done in section V, we chose the scale (21) with different values for b , including negative values. The ratio $\mathcal{F}_{\text{NLO}}^L(r, \tau_F, \mu)/\mathcal{F}_0(r, \tau_F)$ for different values of the parameters s and b is shown in Fig. 16. In [37], the choices $b = 1, 0$ with $s = 1$, and $b = 1$ with $s = 0$ were also analyzed. The choice $s = 1, b = 0$ is the natural choice at zero flow time, since for it $\log(\mu r)$ terms vanish. However, this choice does not capture $\log(\mu\sqrt{\tau_F})$ terms that become important at large flow time. This is shown by the ratio $\mathcal{F}_{\text{NLO}}^L(r, \tau_F, \mu)/\mathcal{F}_0(r, \tau_F)$ becoming large at large flowtimes for this choice of parameters. The choice $s = 0, b = 1$ is the natural choice at large flow time, since for it $\log(\mu\sqrt{\tau_F})$ terms vanish. However, this choice does not capture $\log(\mu r)$ terms that become important at small flow time. This is shown by the ratio $\mathcal{F}_{\text{NLO}}^L(r, \tau_F, \mu)/\mathcal{F}_0(r, \tau_F)$ becoming large at small flow times for this choice of parameters. The choice $s = 1$ and $b = 1$ interpolates between these two extreme and provides a small $\mathcal{F}_{\text{NLO}}^L(r, \tau_F, \mu)$ correction with respect to the leading gradient flow term $\mathcal{F}_0(r, \tau_F)$ over the whole range of flow times. Also the overall scale dependence turns out to be weak with this scale choice.

Nevertheless, our lattice data explore a very specific and limited region of flow time values, the one in between the vertical lines in Fig. 16. This region is zoomed in in the right plot that makes manifest that different choices of b , keeping $s = 1$, provide, indeed, even smaller and more stable corrections $\mathcal{F}_{\text{NLO}}^L(r, \tau_F, \mu)$ in the region of interest than the choice $b = 1$. In particular, this is the case for $b = 0$ and $b = -0.5$, which indeed best fit our lattice data, as we have discussed in the main body of the paper. More negative values of b further reduce the relative size of $\mathcal{F}_{\text{NLO}}^L(r, \tau_F, \mu)$, but make it more scale dependent. Hence, the 1-loop expression of the gradient flow expression of the force suggests that for $0.03 \lesssim \tau_F/r^2 \lesssim 0.05$ the ideal choice of the parameter b is in between 0 and a negative number larger than -1 . This is confirmed by the lattice data. Clearly also a parameterization with negative b must

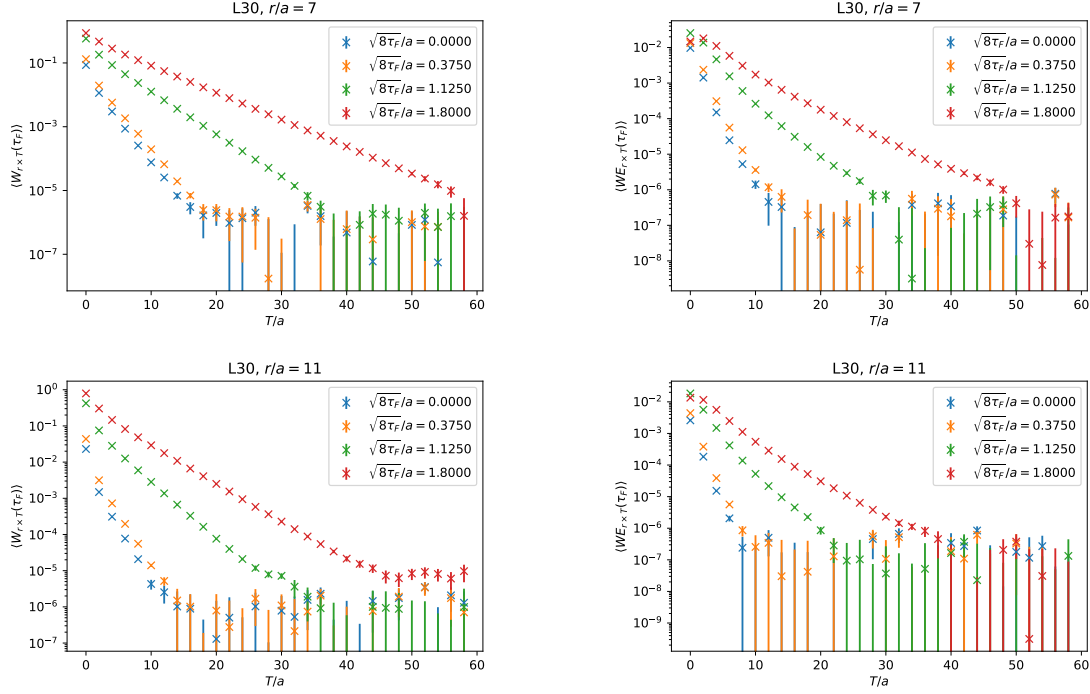


FIG. 17: The left sides show the T -dependence of the Wilson loops at four different fixed flow times and for two fixed r . The right sides show the same for the Wilson loops with a chromoelectric field insertion.

smoothly go over $\mu = 1/\sqrt{\tau_F}$ at large flow time. However, the specific form of the parametrization at large flow times, $\tau_F/r^2 > 0.05$, cannot be explored with the present data.

Appendix B: Flow time dependence of the Wilson loops with and without chromoelectric field insertions

The Wilson loops with and without chromoelectric field insertion are the main objects of this work, therefore, it is worth to have a closer look on the flow time dependence of them. Figure 17 shows the T dependence of the Wilson loops at different flow times. In a logarithmic y -scale, we see linear, decreasing curves for large T , which correspond to the exponential fall off controlled by the static energy. For zero and small flow times, the slope is the same, for larger flow times the slope becomes more flat. This reflects the flow time dependency of the static energy. This observation holds for Wilson loops with and without chromoelectric field insertions.

Figure 18 shows the flow time dependence of the Wilson loops with and without chromoelectric field insertion at fixed r and T . We see a strong flow time dependence for both cases caused by the divergence of the static quark propagator. The strong flow time dependence cancels in the ratio of the Wilson loops.

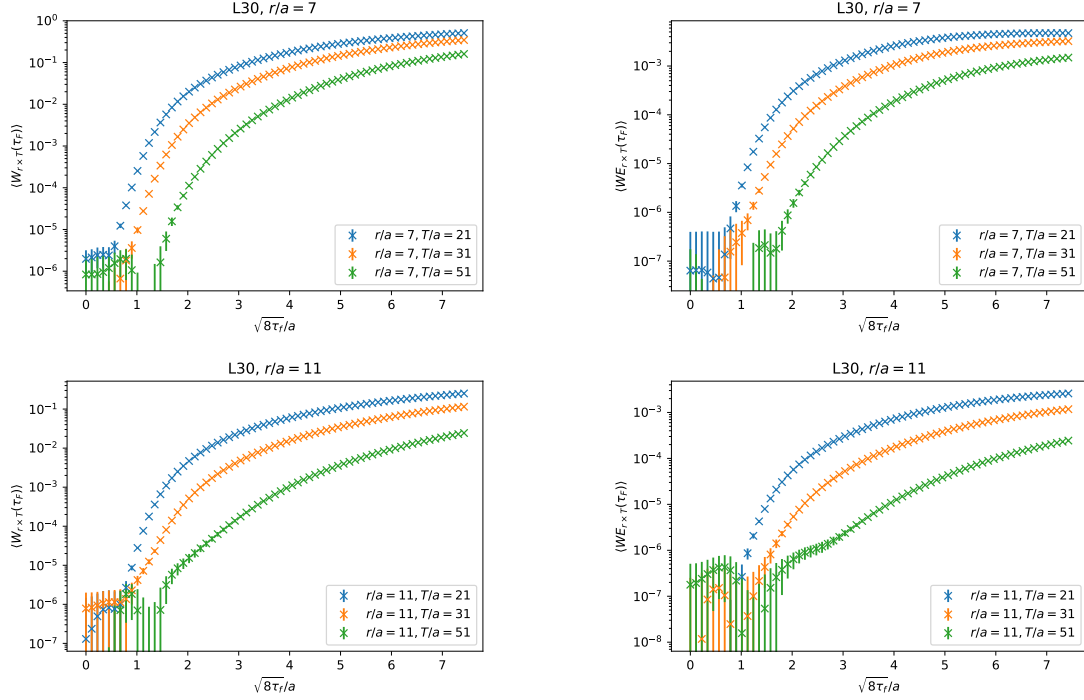


FIG. 18: The left sides show the flow time dependence of the Wilson loop at several fixed T and at two different r . The right sides show the same for the Wilson loops with a chromoelectric field insertion.

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