

# Properties of Immersions for Systems with Multiple Limit Sets with Implications to Learning Koopman Embeddings<sup>\*</sup>

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**Abstract:** Linear immersions (or Koopman eigenmappings) of a nonlinear system have wide applications in prediction and control. In this work, we study the non-existence of one-to-one linear immersions for nonlinear systems with multiple omega-limit sets. While previous research has indicated the possibility of discontinuous one-to-one linear immersions for such systems, it remained uncertain whether continuous one-to-one linear immersions are attainable. Under mild conditions, we prove that any continuous one-to-one immersion to a class of systems including linear systems cannot distinguish different omega-limit sets, and thus cannot be one-to-one. Furthermore, we show that this property is also shared by approximate linear immersions learned from data as sample size increases and sampling interval decreases. Multiple examples are studied to illustrate our results.

*Keywords:* Nonlinear Systems, Immersion, Koopman Operator, Equivalent Systems

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## 1. INTRODUCTION

Applied Koopman operator theory has drawn much attention in recent years due to its potential in the analysis, prediction, and control of nonlinear systems. The main idea behind this is fairly straightforward: As initially shown by Koopman (1931), a nonlinear system can be equivalently represented by an infinite-dimensional linear system whose states consist of single-valued observation functions of the nonlinear system. If one can further find an invariant subspace of this infinite-dimensional linear system, a finite linear representation of the nonlinear system, called the *Koopman representation*, can be extracted from a basis of this invariant subspace. This makes the prediction and control for the nonlinear system much easier since existing theoretical and algorithmic tools established for linear systems can now be applied to the nonlinear systems via their finite linear representations. Compared with local linearization by Taylor expansion, the Koopman representation can capture the global behaviors of the system (Mauroy et al. (2020); Brunton et al. (2021)) and thus opens up exciting possibilities in various applications, such as model reduction and control of PDEs (Kutz et al. (2018); Peitz and Klus (2020)), prediction of chaotic systems (Brunton et al. (2017)), modeling and control of soft robots (Bruder et al. (2020)), and model predictive control of nonlinear systems (Korda and Mezić (2018)).

The idea behind Koopman representations and embeddings of nonlinear systems in linear (or bilinear, when

there are controls) systems has been a recurring theme in the control literature, albeit under different names. Finite-dimensional embeddings correspond to finite-dimensional spaces of observables (Wang and Sontag (1989)). The Koopman representation can be interpreted as the “dual system” used in linear theory (Kalman duality) and more generally as the foundation of the duality between observability of a nonlinear system and controllability of a (generally infinite dimensional) system of observables, the *adjoint system*. See for example the work in Sontag and Rouchaleau (1976); Sontag (1979, 1995) on algebraic observability (strong reachability of the adjoint system, and surjective comorphisms into cosystems in the first reference) and a brief mention in Exercise 6.2.10 in the textbook Sontag (1998). A very closely related concept, but for infinite-dimensional linear systems, is “topological observability”, which amounts to the exact reachability of a dual system (Yamamoto (1981)).

The primary challenge in applying Koopman operator theory to prediction and control lies in identifying a suitable Koopman representation. This involves finding a nonlinear transformation of the system states such that the transformed states evolve like a linear system. We call such a transformation a *linear immersion*. As a trivial example, any constant function is a linear immersion for an arbitrary system, but this linear immersion is useless in practice since it does not include any information about the original nonlinear system. Ideally, we want to find invertible linear immersions, ensuring that the trajectories of the original nonlinear system are fully characterized by its linear representation. In instances where invertible linear immer-

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sions cannot be manually derived, especially for higher-dimensional systems, numerical approximation becomes necessary. Various numerical methods have been developed to approximate linear immersions from data (Tu et al. (2014); Williams et al. (2015); Brunton et al. (2016b)). A crucial guideline for achieving low approximation error in these methods is to carefully select a domain of interest where the linear immersions are intended to be learned. In practice, for systems with multiple equilibria, a commonly mentioned insight is that a continuous one-to-one linear immersion across multiple isolated equilibria does not exist, supported by multiple analytical and numerical examples in the literature (Mauroy and Mezić (2013); Williams et al. (2015); Brunton et al. (2016a); Page and Kerswell (2019); Bakker et al. (2019, 2020)). Consequently, numerical methods are recommended to focus on learning local linear immersions within the domain of attraction of each equilibrium point. However, a recent work from Arathoon and Kvalheim (2023) challenges this insight. In particular, Arathoon and Kvalheim (2023) construct a smooth system with multiple isolated equilibria that admits a smooth one-to-one linear immersion. These positive and negative examples suggest that the non-existence of continuous one-to-one linear immersions is not solely determined by the presence of multiple isolated equilibria. To provide more accurate guidance on approximating linear immersions, it is imperative to reassess the aforementioned insight and identify the actual factors that determine the existence or non-existence of continuous one-to-one linear immersions.

To address these inquiries, in this work we study the properties of continuous linear immersions for systems with multiple limit sets, and their implications on learning algorithms that approximate linear immersions from data. In particular, our contributions include:

- For systems with multiple  $\omega$ -limit sets, we prove that, under mild conditions, any continuous linear immersion collapses all the  $\omega$ -limit sets into one and thus can not be one-to-one. We then demonstrate the applicability of our results with multiple examples from the literature (Section 3).
- For the same class of systems, we show that approximate linear immersions learned with data converge to functions that are not one-to-one, as sampling time decreases and sample size increases (Section 4).
- We show several extensions of the main theorem that can work with a broader class of systems (Section 5).

A preliminary version of this work was presented at the IFAC World Congress (Liu et al. (2023)), focusing exclusively on one-to-one immersions. In this work, we extend the results in Liu et al. (2023) to encompass immersions that are not necessarily one-to-one in Section 3. Additionally, we introduce entirely new results in Sections 4 and 5.

**Related work:** Since the presence of multiple isolated equilibria is one of the key features that distinguish nonlinear systems from linear ones, there are many discussions in the literature on the possibility of immersing a system with more than one isolated equilibria in a linear system. It is initially observed in a numerical example from Mauroy and Mezić (2013) that the approximate linear immersion over a domain that contains two equilibria

becomes singular at one of the equilibria. Motivated by this observation, Williams et al. (2015) suggest that linear immersions should be approximated within the domain of attraction of each equilibrium to avoid singularities. This suggestion is supported by the work from Page and Kerswell (2019), who study a one-dimensional system with three equilibria where all the linear immersions can be derived manually. For this specific system, all the derived linear immersions become singular at one of the equilibria. Potentially motivated by these negative examples, Brunton et al. (2016a) claim that it is impossible to find one-to-one linear immersions for systems with multiple isolated equilibria. However, this claim is disproved by Bakker et al. (2019), who present a one-dimensional system with three equilibria that admits a discontinuous one-to-one linear immersion. Bakker et al. (2019) further conjecture that a linear immersion may exist but become discontinuous at the boundaries of the basins of attraction. This conjecture is again disproved by Arathoon and Kvalheim (2023), who construct a smooth system with two isolated equilibria that admits a smooth one-to-one linear immersion. Contrary to these varying claims, our work rigorously proves that continuous linear immersions cannot be one-to-one when the system has multiple isolated equilibria and satisfies specific conditions. Our results confirm that the negative examples from the literature do not admit continuous one-to-one linear immersions, while the counter example in Arathoon and Kvalheim (2023) is the only one not meeting our extra conditions and thus allowing a continuous one-to-one linear immersion. Notably, Kvalheim and Arathoon (2023) provide necessary and sufficient conditions for the existence of one-to-one linear immersions for a certain class of nonlinear systems. However, their results, when applied to systems with multiple isolated limit sets, are more restrictive than ours. This is because they assume that the system is defined over a compact invariant subset of a smooth manifold (e.g., a simple linear system like  $\dot{x} = -x$  does not satisfy this assumption). Interestingly, under this assumption, our results directly imply that any systems with multiple isolated equilibria do not admit a continuous one-to-one linear immersion. This implication can not be easily discerned from the necessary and sufficient conditions presented in Kvalheim and Arathoon (2023).

**Notation:** We denote the closure of a set  $X$  by  $\text{cl}(X)$ . The symbols  $\mathbb{R}$ ,  $\mathbb{R}_{\geq 0}$ , and  $\mathbb{R}_{> 0}$  denote the real line, the set of non-negative real numbers and the set of positive real numbers. The symbols  $\mathbb{Z}$  and  $\mathbb{N}$  denote the set of integers and the set of non-negative integers.

## 2. PRELIMINARIES

### 2.1 Problem Statement

We consider a continuous-time autonomous system defined on a (second countable) manifold  $\mathcal{M}$  :

$$\dot{x} = f(x), \quad x \in \mathcal{M}. \quad (1)$$

Given an initial state  $\xi \in \mathcal{M}$ , we denote the solution of the system in (1) by  $\varphi : \mathbb{R}_{\geq 0} \times \mathcal{M} \rightarrow \mathcal{M}$  satisfying  $\varphi(0, \xi) = \xi$  and for all  $t \geq 0$ ,

$$\frac{d\varphi(t, \xi)}{dt} = f(\varphi(t, \xi)). \quad (2)$$

Let  $\mathcal{X}$  be a path-connected subset of the manifold  $\mathcal{M}$  that represents the region in which we want to analyze the system behavior. We endow  $\mathcal{X}$  with the subspace topology induced from  $\mathcal{M}$ . Throughout the paper, we will assume that  $\varphi(t, \xi)$  is defined and contained in  $\mathcal{X}$  for all  $\xi \in \mathcal{X}$  and  $t \geq 0$ . Furthermore,  $f$  is smooth enough to guarantee the uniqueness and continuous dependence on the initial states of the solution  $\varphi(t, \xi)$  for all  $\xi \in \mathcal{X}$  and all  $t \geq 0$ .

*Remark 1.* Every subspace of a second countable space, such as  $\mathcal{X}$ , is also second countable, which in turn implies that a subset of  $\mathcal{X}$  is compact if and only if it is sequentially compact. We will use this last property.

Given an initial state  $\xi$  of the system in (1), we denote the  $\omega$ -limit set of  $\xi$  in  $\mathcal{X}$  by  $\omega^+(\xi)$ , that is the set of all  $x \in \mathcal{X}$  satisfying that there exists a sequence  $t_n \rightarrow \infty$  such that  $\lim_{n \rightarrow \infty} \varphi(t_n, \xi) = x$  (Hirsch et al. (2012)).

Next, we introduce the definition of immersion, which generalizes the notion of Koopman eigenfunctions (Mauroy et al. (2020)).

*Definition 1.* A system  $\dot{x} = f(x)$  on  $\mathcal{X} \subseteq \mathcal{M}$  is *immersed* in a system  $\dot{z} = g(z)$  on a manifold  $\mathcal{Z}$  if there is a mapping  $F : \mathcal{X} \rightarrow \mathcal{Z}$  (an immersion) such that, for all initial states  $\xi \in \mathcal{X}$  and all time  $t \geq 0$ ,

$$F(\varphi(t, \xi)) = \psi(t, F(\xi)), \quad (3)$$

where  $\psi(t, F(\xi))$  is the solution of  $\dot{z} = g(z)$ .

If the system  $\dot{z} = g(z)$  above is linear, the mapping  $F$  is called a *linear immersion*.

*Remark 2.* This paper investigates the properties of continuous immersions for systems with multiple  $\omega$ -limit sets. Throughout the remainder of this paper, unless otherwise specified, any immersion under consideration is assumed to be continuous.

*Remark 3.* Linear immersions are tightly related to Koopman operator theory (Brunton et al. (2016a)): A *Koopman eigenfunction*  $F$  is a (not necessarily continuous) linear immersion that immerses  $\dot{x} = f(x)$  in a one-dimensional system  $\dot{z} = \lambda z$  for some  $\lambda \in \mathbb{R}$ . The span of the entries of any linear immersion  $F$  is a *Koopman invariant subspace*.

*Remark 4.* If an immersion  $F$  is one-to-one, the inverse  $F^{-1} : F(\mathcal{X}) \rightarrow \mathcal{X}$  exists. Thus we can retrieve the solution  $\varphi(t, \xi)$  of  $\dot{x} = f(x)$  for any  $\xi \in \mathcal{X}$  from the solution  $\psi(t, F(\xi))$  of  $\dot{z} = g(z)$  via the formula  $\varphi(t, \xi) = F^{-1}(\psi(t, F(\xi)))$ .

*Remark 5.* The term “immersion” is also widely used in the study of differentiable manifolds (see for example Lee (2012)), which is unrelated to the immersion of dynamical systems considered in this work.

Given a nonlinear system  $\dot{x} = f(x)$ , we are most interested in finding a one-to-one linear immersion  $F$ , which fully encapsulates the behaviors of the nonlinear system into a linear system. However, in practice finding a one-to-one linear immersion can be very challenging, and sometimes such a linear immersion may not even exist. In particular, one might think that a one-to-one linear immersion may not exist when the  $\omega$ -limit sets of the nonlinear system are “topologically” different from those of linear systems. For instance, nonlinear systems may have limit cycles but linear systems cannot. However, the following example

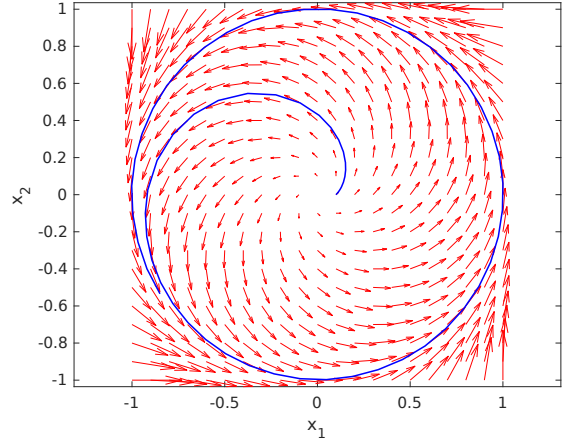


Fig. 1. The vector field (red) of the system in (4). The blue curve shows a trajectory of the system that starts from  $(0.1, 0)$  and converges to the unit circle.

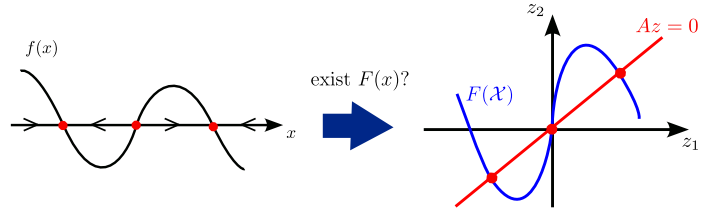


Fig. 2. For a system  $\dot{x} = f(x)$  with three equilibria, the graph of a one-to-one linear immersion (blue) must intersect with the equilibrium set (red) of the immersing linear system at exactly three points.

shows that it is possible to immerse a system with limit cycles into a linear system.

*Example 1.* Consider a two-dimensional system

$$\begin{aligned} \dot{x}_1 &= x_1 - x_2 - x_1(x_1^2 + x_2^2), \\ \dot{x}_2 &= x_1 + x_2 - x_2(x_1^2 + x_2^2). \end{aligned} \quad (4)$$

with state  $x = (x_1, x_2)$ . The system has an unstable equilibrium at the origin and one stable limit cycle on the unit circle, as shown by the phase portrait in Fig. 1. Let  $\mathcal{X} = \mathbb{R}^2 \setminus \{0\}$ . Intuitively, one may think a linear immersion does not exist for this system over  $\mathcal{X}$  since linear systems cannot have a limit cycle. However, this system does admit a one-to-one linear immersion over  $\mathcal{X}$ . Let  $F : \mathcal{X} \rightarrow \mathbb{R}^3$  be

$$F(x) = (x_1/\|x\|_2, x_2/\|x\|_2, \|x\|_2^{-2} - 1). \quad (5)$$

For a solution  $x(t)$  of the system in (4), it can be checked that  $F(x(t))$  is a solution of the following linear system

$$\begin{aligned} \dot{u} &= -v, \\ \dot{v} &= u, \\ \dot{w} &= -2w, \end{aligned} \quad (6)$$

Thus, the one-to-one function  $F$  in (5) is a linear immersion of the two-dimensional system.

This paper focuses on another well-known topological difference between linear and nonlinear systems, namely that a nonlinear system can have multiple isolated  $\omega$ -limit sets, but a linear system cannot. We wonder how the properties of linear immersions, such as injectivity, are influenced by the presence of multiple isolated  $\omega$ -limit sets. Many existing works (Mauroy and Mezić (2013); Brunton et al. (2016a); Page and Kerswell (2019)) claim

or observe that a continuous one-to-one linear immersion does not exist when the system possesses multiple isolated equilibria, a specific type of  $\omega$ -limit sets. However, a formal analysis of this phenomenon is missing in the literature. The following discussion explains why it is nontrivial to prove this claim: Suppose that an immersion  $F$  that maps a system  $\dot{x} = f(x)$  with multiple equilibria to a linear system  $\dot{z} = Az$  exists. According to (3), a one-to-one  $F$  must map equilibria of  $\dot{x} = f(x)$  to equilibria of the linear system  $\dot{z} = Az$ , and map non-equilibrium points to non-equilibrium points. However, recall that a linear system can only have one isolated equilibrium or a subspace of equilibria. If  $\dot{z} = Az$  is the former, then  $F$  maps all equilibria of  $\dot{x} = f(x)$  to the unique equilibrium of  $\dot{z} = Az$  and thus can not be one-to-one. Thus, the main challenge of proving or disproving this claim is to show if it is possible to have a one-to-one  $F$  that maps  $\dot{x} = f(x)$  to an immersing system  $\dot{z} = Az$  with a subspace of equilibria. In this case, the only possibility is that the graph of  $F$  intersects with the null space of  $A$  at exactly  $M$  points, with  $M$  the number of equilibria of  $\dot{x} = f(x)$ , as demonstrated by Fig. 2. The following example shows that this is indeed possible if we allow the immersion to be discontinuous.

*Example 2.* Consider a one-dimensional system  $\dot{x} = f(x)$  with  $M$  isolated equilibria  $\{x_{e,i}\}_{i=1}^M$ , where  $x_{e,1} < x_{e,2} < \dots < x_{e,M}$ . Assume that the solution  $\varphi(t, \xi)$  is defined for any  $\xi$  and  $t \in \mathbb{R}$  (and thus has no finite-time blow-up both forward and backward in time). Let  $\mathcal{X} = \mathbb{R}$ . This system is immersed to the following two-dimensional linear system

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \quad (7)$$

by a one-to-one discontinuous function  $F$

$$F(x) = \begin{cases} (x, 0) & f(x) = 0, \\ (\varphi^{-1}(x, x_{e,1} - 1), 1) & x < x_{e,1}, \\ \left( (i+1)\varphi^{-1}\left(x, \frac{x_{e,i} + x_{e,i+1}}{2}\right), i+1 \right) & x \in (x_{e,i}, x_{e,i+1}), \\ ((M+1)\varphi^{-1}(x, x_{e,M} + 1), M+1) & x > x_{e,M}, \end{cases} \quad (8)$$

where for any  $x$  and  $\xi \in (x_{e,i}, x_{e,i+1})$ , the inverse function  $\varphi^{-1}(x, \xi)$  is the time instance  $t$  such that  $\varphi(t, \xi) = x$ . Intuitively, the equilibria of  $\dot{x} = f(x)$  cut the real line into  $M+1$  intervals, and the function  $F$  in (8) maps all these intervals to horizontal lines in the lifted space.

To see how this immersion  $F$  works in practice, we take a concrete example of  $\dot{x} = f(x)$  with  $M = 3$

$$\dot{x} = \frac{x(1-x^2)}{1+x^2}. \quad (9)$$

The equilibria of this system include  $\pm 1$  and  $0$ . According to (8), a one-to-one linear immersion  $F$  for this system is

$$F(x) = \begin{cases} (x, 0) & f(x) = 0, \\ \left( \ln \left| \frac{3x}{-2+2x^2} \right|, 1 \right) & x < -1, \\ \left( 2 \ln \left| \frac{3x}{-2+2x^2} \right|, 2 \right) & -1 < x < 0, \\ \left( 3 \ln \left| \frac{3x}{-2+2x^2} \right|, 3 \right) & 0 < x < 1, \\ \left( 4 \ln \left| \frac{3x}{-2+2x^2} \right|, 4 \right) & x > 1. \end{cases} \quad (10)$$

Let us examine the correctness of  $F$  for  $x_0 \in (0, 1)$ . For any  $x_0 \in (0, 1)$ , the solution  $\varphi(t, x_0)$  for  $\dot{x} = f(x)$  in (9) is

$$\varphi(t, x_0) = \frac{x_0^2 - 1 + \sqrt{(x_0^2 - 1)^2 + 4x_0^2 e^{2t}}}{2x_0 e^t}. \quad (11)$$

Plugging the RHS of (11) into (10), we have

$$F(\varphi(t, x_0)) = \left( 3 \ln \left( \frac{3x_0}{2x_0^2 - 2} \right) + 3t, 3 \right). \quad (12)$$

According to (12),  $F(\varphi(t, x_0))$  is equal to the solution of the linear system in (7) with respect to the initial state  $F(x_0)$ , showing the mapping  $F$  in (10) is a one-to-one linear immersion of (9) over  $(0, 1)$ .

Note that the graph of  $F$  in (8) intersects with the subspace  $\mathbb{R} \times \{0\}$  of equilibria of (7) at precisely  $M$  points, thanks to the discontinuity. This phenomenon is also observed in continuous one-to-one linear immersions, exemplified in Arathoon and Kvalheim (2023). This work aims to elucidate the relation between the properties of linear immersions, such as injectivity and continuity, and the occurrence of multiple isolated  $\omega$ -limit sets, as indicated by these examples and the literature.

**Problem:** Identify the properties of linear immersions for systems with multiple isolated  $\omega$ -limit sets.

## 2.2 Technical Definitions

*Definition 2.* Given an initial state  $\xi \in \mathcal{X}$ , the trajectory  $\varphi(t, \xi)$  is called *precompact* in  $\mathcal{X}$  if the closure of the set  $\{\varphi(t, \xi) \mid t \geq 0\}$  is compact with respect to the subspace topology on  $\mathcal{X}$ .

The following lemma states sufficient (and necessary) conditions for the nonemptiness of  $\omega^+(\xi)$ .

*Lemma 1.* For any  $\xi \in \mathcal{X}$ , the  $\omega$ -limit set  $\omega^+(\xi)$  is nonempty if the trajectory  $\varphi(t, \xi)$  is precompact in  $\mathcal{X}$ . If the system is linear with  $\mathcal{M} = \mathbb{R}^n$  and  $\mathcal{X}$  closed, then the converse is also true.

**Proof.** The forward implication is well known. We recall the standard proof here. Suppose  $\varphi(t, \xi)$  is precompact in  $\mathcal{X}$ . Let  $t_n$  be a sequence such that  $t_n \rightarrow \infty$ . By Definition 2, there exists a subsequence  $t_{n_k}$  such that  $\varphi(t_{n_k}, \xi)$  converges to a point  $x$  in the closure of  $\{\varphi(t, \xi) \mid t \in \mathbb{R}_{\geq 0}\}$ . Thus,  $\omega^+(\xi)$  contains  $x$  and is nonempty.

Now suppose that the system in (1) is linear, that is,  $\dot{x} = Ax$  for some  $A \in \mathbb{R}^{n \times n}$ . If a solution  $\varphi(t, \xi)$  of the linear system is unbounded, it can be shown that  $\lim_{t \rightarrow \infty} \|\varphi(t, \xi)\|_2 = \infty$ . Then,  $\omega^+(\xi)$  is empty since for any sequence  $t_n \rightarrow \infty$ ,  $\|\varphi(t_n, \xi)\|_2 \rightarrow \infty$ . Thus, if  $\omega^+(\xi)$  is nonempty,  $\varphi(t, \xi)$  is bounded and thus is precompact in  $\mathbb{R}^n$ . Since  $\mathcal{X}$  is closed, the closure of  $\varphi(t, \xi)$  in  $\mathbb{R}^n$  is contained in  $\mathcal{X}$ , which implies that  $\varphi(t, \xi)$  is precompact in  $\mathcal{X}$ .  $\square$

*Definition 3.* Let  $\mathcal{W}^+$  be the set of all  $\omega$ -limit sets of  $\dot{x} = f(x)$  in  $\mathcal{X}$ . For each  $\Omega \in \mathcal{W}^+$ , we define its *domain of attraction* by

$$D^+(\Omega) = \{\xi \in \mathcal{X} \mid \omega^+(\xi) = \Omega\}. \quad (13)$$

By definition, the set  $\mathcal{W}^+$  contains all the equilibria or closed orbits in  $\mathcal{X}$ .

Finally, we introduce a class of systems with a special property of the domain of attraction  $D^+(\Omega)$ . This class



includes all linear systems. Later we show that this special property is the main reason why a one-to-one linear immersion may not exist for a system with more than one  $\omega$ -limit set.

Recall that  $\mathcal{W}^+$  is the set of all  $\omega$ -limit sets in  $\mathcal{X}$ .

*Definition 4.* A system of the form (1) has *closed basins* if the domain of attraction  $D^+(\Omega)$  is closed for all  $\omega$ -limit sets  $\Omega \in \mathcal{W}^+$ .

The following lemma provide sufficient conditions for systems to have closed basins, which are satisfied by all linear systems.

*Lemma 2.* Any system  $\dot{x} = f(x)$  defined over a subset  $\mathcal{X}$  of a normed space has closed basins if for any  $\omega$ -limit set  $\Omega$  in  $\mathcal{W}^+$ , the following two conditions are satisfied

- (C1) For any  $\xi \in \mathcal{D}^+(\Omega)$ ,  $\varphi(t, \xi)$  is precompact in  $\mathcal{X}$ .
- (C2) The system is incrementally stable in the closure of  $\mathcal{D}^+(\Omega)$ . That is, there exists a function  $\alpha$  of class  $\mathcal{K}$  such that, for any two initial states  $\xi_1$  and  $\xi_2$  in  $\text{cl}(\mathcal{D}^+(\Omega))$  and for all  $t \geq 0$ ,  $\|\varphi(t, \xi_1) - \varphi(t, \xi_2)\| \leq \alpha(\|\xi_1 - \xi_2\|)$ .

**Proof.** Let  $\Omega$  be an arbitrary  $\omega$ -limit set in  $\mathcal{W}^+$ . Let  $x$  be a limit point of  $\mathcal{D}^+(\Omega)$ . That is, there exists a sequence  $x_k \in \mathcal{D}^+(\Omega)$  such that  $x_k \rightarrow x$  as  $k \rightarrow +\infty$ .

We first show that  $\omega^+(x)$  is nonempty and includes  $\Omega$ . Pick any point  $p \in \Omega$ . For each  $x_k$ , since  $\omega^+(x_k) = \Omega$ , there exists a sequence  $t_k \rightarrow +\infty$  such that  $\|\varphi(t_k, x_k) - p\| \leq 1/k$ . According to (C2), since  $x \in \text{cl}(\mathcal{D}^+(\Omega))$ ,

$$\begin{aligned} \|\varphi(t_k, x) - p\| &\leq \|\varphi(t_k, x_k) - p\| + \|\varphi(t_k, x) - \varphi(t_k, x_k)\| \\ &\leq 1/k + \alpha(\|x - x_k\|) \rightarrow 0 \text{ as } k \rightarrow +\infty. \end{aligned}$$

Therefore,  $\varphi(t_k, x)$  converges to  $p$  and thus  $p \in \omega^+(x)$ . Since  $p$  is picked arbitrarily,  $\Omega \subseteq \omega^+(x)$ .

Next, we want to show that  $\Omega$  includes  $\omega^+(x)$ . We first prove the following claim: Given  $\xi_1$  and  $\xi_2$  in  $\text{cl}(\mathcal{D}^+(\Omega))$  and any  $p \in \omega^+(\xi_1)$ , there is a  $q \in \omega^+(\xi_2)$  such that  $\|p - q\| \leq \alpha(\|\xi_1 - \xi_2\|)$ .

Let  $t_k \rightarrow +\infty$  be a sequence such that  $\varphi(t_k, \xi_1) \rightarrow p$ . By (C1), the sequence  $\varphi(t_k, \xi_2)$  is contained in a compact subset of  $X$ . Therefore, there exists a subsequence  $t'_k$  of  $t_k$  such that  $\varphi(t'_k, \xi_2) \rightarrow q$  for some  $q$  in  $\omega^+(\xi_2)$ . By (C2),

$$\|p - q\| = \lim_{k \rightarrow +\infty} \|\varphi(t'_k, \xi_1) - \varphi(t'_k, \xi_2)\| \leq \alpha(\|\xi_1 - \xi_2\|).$$

Now we pick any point  $p \in \omega^+(x)$ . By the claim, there exists a sequence  $q_k$  in  $\omega^+(x_k)$  such that  $\|q_k - p\| \rightarrow 0$ . Since  $\Omega$  is closed, it follows that  $p \in \Omega$ . Since  $p$  is arbitrary,  $\omega^+(x)$  is a subset of  $\Omega$ .

Since  $\Omega \subseteq \omega^+(x)$  and  $\omega^+(x) \subseteq \Omega$ , we have  $\omega^+(x) = \Omega$  and thus  $x \in \mathcal{D}^+(\Omega)$ . Since  $x$  is an arbitrary limit point of  $\mathcal{D}^+(\Omega)$ ,  $\mathcal{D}^+(\Omega)$  is closed.  $\square$

*Remark 6.* If  $\mathcal{X}$  is a closed subset of a finite-dimensional normed space, the condition (C1) in Lemma 2 can be replaced with the condition that for every  $\Omega \in \mathcal{W}^+$ , there exists one trajectory in  $\mathcal{D}^+(\Omega)$  that is precompact in  $\mathcal{X}$ .

*Corollary 3.* Every linear system  $\dot{x} = Ax$  (with  $\mathcal{X} = \mathbb{R}^n$ ) has closed basins.

**Proof.** We want to prove the corollary by showing that any linear system satisfies the conditions (C1) and (C2) in

Lemma 2. The condition (C1) holds for any linear systems according to Lemma 1.

To show (C2), let  $\Omega$  be an arbitrary  $\omega$ -limit set in  $\mathcal{W}^+$ . Denote the span of  $\mathcal{D}^+(\Omega)$  by  $S$ . By the superposition property of linear systems, since  $\mathcal{D}^+(\Omega)$  is forward invariant,  $S$  is also forward invariant. Thus, without loss of generality, we can restrict the state space of the system to  $S$ . According to (C1) and the superposition property of linear systems, all trajectories with initial states in the span  $S$  of  $\mathcal{D}^+(\Omega)$  are precompact. That implies the system restricted to  $S$  is stable in the sense of Lyapunov. Thus, there exists  $M > 0$  such that  $\|\exp(At)x\|_2 \leq M\|x\|_2$  for all  $x \in S$  and  $t \geq 0$ .

Now we pick any two states  $\xi_1$  and  $\xi_2$  in  $\text{cl}(\mathcal{D}^+(\Omega))$ . Since  $S$  is the span of  $\mathcal{D}^+(\Omega)$ ,  $S$  contains  $\xi_1$ ,  $\xi_2$ , and  $\xi_1 - \xi_2$ . Therefore, for any  $t \geq 0$ ,

$$\begin{aligned} \|\exp(At)\xi_1 - \exp(At)\xi_2\|_2 &= \|\exp(At)(\xi_1 - \xi_2)\|_2 \\ &\leq M\|\xi_1 - \xi_2\|_2. \end{aligned} \quad (14)$$

Hence, any linear system satisfies the condition (C2).  $\square$

### 3. MAIN THEOREM

The following theorem states our main results:

*Theorem 1.* Suppose that:

- (T1)  $\dot{x} = f(x)$  on  $\mathcal{X}$  can be immersed in a system with closed basins by a continuous mapping  $F$ ;
- (T2) trajectories of  $\dot{x} = f(x)$  on  $\mathcal{X}$  are precompact in  $\mathcal{X}$ ;
- (T3) the set  $\mathcal{W}^+$  is finite or countable.

Then the set  $\{F(\Omega) \mid \Omega \in \mathcal{W}^+\}$  has exactly one element.

The proof of this theorem can be found in Appendix A. This theorem essentially states that if there are a countable number of  $\omega$ -limit sets and the trajectories of the system are precompact, any continuous function  $F$  that immerses the system into one with closed basins (in particular, any linear immersion) collapses all  $\omega$ -limit sets. A direct consequence of this result is as follows.

*Corollary 4.* Suppose that (T1), (T2), and (T3) hold and  $F$  is one-to-one, then  $\mathcal{W}^+$  has exactly one element.

Combining Lemma 3 and Corollary 4, the following corollary states a necessary condition for the existence of one-to-one linear immersions (or in general one-to-one immersions to systems with closed basins).

*Corollary 5.* If  $\mathcal{X}$  contains more than one, but at most countably many,  $\omega$ -limit sets and all trajectories in  $\mathcal{X}$  are precompact, then a one-to-one linear immersion does not exist for  $\dot{x} = f(x)$  on  $\mathcal{X}$ .

For non-existence of linear immersions as in Corollary 5 both precompactness of trajectories and the existence of countable but more than one  $\omega$ -limit sets are not only sufficient conditions, they are indeed necessary in the following sense. Arathoon and Kvalheim (2023) provide an example of a two dimensional system with two isolated equilibria with some trajectories that are neither precompact nor backward precompact (cf. Section 5.2) that admits a linear immersion. Similarly, there are systems with uncountably many  $\omega$ -limit sets that admit linear immersions, simplest examples being diffeomorphisms of linear systems with a nontrivial subspace as their equilibria.

Next, we demonstrate the application of our results through examples. We first show several examples where a one-to-one (linear) immersion is constructed manually when  $\mathcal{X}$  does not satisfy the conditions in Corollary 5, but these immersions become discontinuous or ill-defined when we slightly modify  $\mathcal{X}$  to violate one of these conditions. Sequentially, we offer examples from the literature where the existence of a linear immersion is uncertain, but our results establish that continuous one-to-one linear immersions do not exist.

*Example 3.* Consider the one-dimensional system

$$\dot{x} = x^2 - 1. \quad (15)$$

The  $\omega$ -limit sets of the system are  $\{-1\}$  and  $\{1\}$ . Let  $\mathcal{X} = (-\infty, 1)$ , which only contains one  $\omega$ -limit set  $\{-1\}$ . It can be shown that  $\dot{x} = x^2 - 1$  on  $\mathcal{X}$  is immersed in  $\dot{z} = -2z$  by the one-to-one mapping

$$F(x) = \frac{x+1}{x-1}. \quad (16)$$

However, if we extend  $\mathcal{X}$  by a point to  $\mathcal{X}' = (-\infty, 1]$ , the function  $F$  in (16) is not an immersion anymore, since  $F(1)$  is not defined. This observation is explained by Corollary 5: Since  $\mathcal{X}'$  contains two limit sets  $\{-1\}$  and  $\{+1\}$ , and all trajectories in  $(-\infty, 1]$  are precompact, there does not exist a one-to-one linear immersion for the system on  $\mathcal{X}'$ .

*Example 4.* Consider the one-dimensional system:

$$\dot{x} = \sin(x). \quad (17)$$

Let  $\mathcal{X} = [0, \pi]$ . The  $\omega$ -limit sets of the system are  $\{0\}$  and  $\{\pi\}$ . Define  $y = \cos(x)$ . Then, the derivative of  $y$  satisfies

$$\dot{y} = -\sin(x)^2 = \cos(x)^2 - 1 = y^2 - 1, \quad (18)$$

with  $|y| \leq 1$ . That is, the system in (17) on  $\mathcal{X}$  is immersed in the system in (15) on  $\mathcal{Z} = [-1, 1]$ . In this example,  $\mathcal{W}^+$  has two elements, and all trajectories of  $x$  in  $\mathbb{R}$  are precompact, but a one-to-one immersion exists. By Theorem 1, this is possible only if the system  $\dot{y} = y^2 - 1$  does not have closed basins. Indeed, the domain of attraction  $D^+(\{-1\})$  of the system of  $y$  on  $\mathcal{Z}$  is  $[-1, 1)$ , not a closed set.

Furthermore, by using Example 3, the system of  $y$  on  $[-1, 1)$  can be immersed in  $\dot{z} = -2z$  with the immersion in (16). Thus,  $\dot{x} = \sin(x)$  on  $\mathcal{X}' = (0, \pi]$  is immersed in  $\dot{z} = -2z$  with the one-to-one mapping

$$F(x) = \frac{\cos(x) + 1}{\cos(x) - 1}. \quad (19)$$

If we extend  $\mathcal{X}'$  to  $\mathcal{X} = [0, \pi]$ , the function  $F(x)$  in (19) is not defined at 0 and thus is not an immersion on the closed interval. This can be again explained by Corollary 5 since all the trajectories of  $x$  are precompact, and the interval  $[0, \pi]$  contains two limit sets.

*Example 5.* Consider the one-dimensional system

$$\dot{x} = x - x^3. \quad (20)$$

The  $\omega$ -limit sets of the system are  $\{-1\}$ ,  $\{0\}$  and  $\{1\}$ . Let  $\mathcal{X} = \mathbb{R} \setminus \{0\}$ . Define  $y = x^{-2} - 1$ . Then,  $\dot{y}$  satisfies

$$\dot{y} = -2x^{-3}(x - x^3) = -2y. \quad (21)$$

Thus, the system in (20) on  $\mathcal{X}$  is immersed in  $\dot{y} = -2y$  with the immersion  $F(x) = x^{-2} - 1$ . Similar to the previous example,  $\mathcal{X}$  contains two  $\omega$ -limit sets, but each of its path-connected components contains only one  $\omega$ -limit set and thus the result is consistent with Corollary 5.

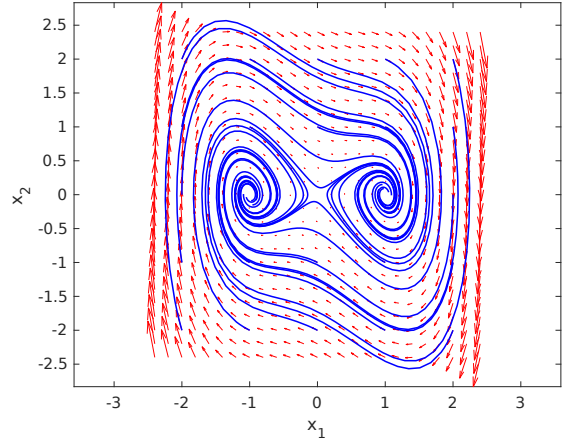


Fig. 3. The vector field (red) and phase portrait (blue) of the unforced Duffing system in (22).

*Example 6.* Consider the two-dimensional system in Example 1. The  $\omega$ -limit sets of the system are the origin  $\{0\}$  and the unit circle  $\{x \mid \|x\|_2 = 1\}$ . The linear immersion  $F(x)$  in (5) is continuous in  $\mathcal{X}$ , but extending it to  $\mathbb{R}^2$  makes  $F(x)$  singular at the origin and thus no longer an immersion. This can be explained by Corollary 5: Since all trajectories of  $x$  are precompact and the set  $\mathbb{R}^2$  contains two  $\omega$ -limit sets, there does not exist a one-to-one linear immersion for the system of  $x$  on  $\mathbb{R}^2$ .

*Example 7.* Example 2 shows that a discontinuous one-to-one linear immersion exists for the one-dimensional system in (9) with three isolated equilibria. We question if this system possesses a continuous one-to-one linear immersion over  $\mathbb{R}$ . This can be answered by Corollary 5: Since  $\mathcal{X}$  contains three equilibria and all trajectories are precompact in  $\mathcal{X}$ , a continuous one-to-one linear immersion does not exist for this system.

*Example 8.* Consider the unforced Duffing system in Williams et al. (2015)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -0.5x_2 - x_1(x_1^2 - 1). \end{aligned} \quad (22)$$

This system has two asymptotically stable equilibria  $(\pm 1, 0)$  and one unstable equilibrium  $(0, 0)$ , as shown by Fig. 3. Let  $\mathcal{X}$  be the entire plane. Williams et al. (2015) suggests that this system does not admit a one-to-one linear immersion over  $\mathcal{X}$ , which is confirmed by our results. According to Corollary 5, since the system has three  $\omega$ -limit sets and all of its trajectories are precompact in  $\mathcal{X}$ , there does not exist a one-to-one linear immersion over  $\mathcal{X}$ .

*Example 9.* Consider the Van der Pol equation

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1^3 + x_1 \\ \dot{x}_2 &= -x_1. \end{aligned} \quad (23)$$

This system has two  $\omega$ -limit sets, namely an unstable equilibrium at the origin and a stable limit cycle, as shown in Fig. 4. Let  $\mathcal{X}$  be the entire plane. Since all trajectories of the system are precompact in  $\mathcal{X}$ , by Corollary 5, there is no one-to-one linear immersion over  $\mathcal{X}$ .

*Example 10.* Consider the Lorenz system

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz, \end{aligned} \quad (24)$$

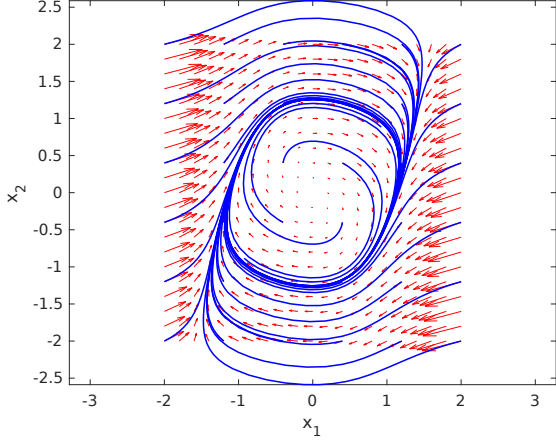


Fig. 4. The vector field (red) and phase portrait (blue) of the Van der Pol equation in (23).

with  $\sigma = 10$ ,  $b = 8/3$ , and  $r = 28$ . According to Hirsch et al. (2012), there exists an invariant ellipsoid  $\mathcal{E}$  centered at  $(0, 0, 2r)$  that contains all the  $\omega$ -limit sets of the system, which include three equilibria and the Lorenz attractor. Let  $\mathcal{X} = \mathcal{E}$ . Since  $\mathcal{X}$  is invariant and compact, all trajectories are precompact in  $\mathcal{X}$ . Thus, by Corollary 5, there does not exist a one-to-one linear immersion for the Lorenz system on  $\mathcal{X}$ .

#### 4. IMPLICATION FOR LEARNING LINEAR IMMERSIONS FROM DATA

In this section, we discuss the implications of our result for learning linear immersions of  $\dot{x} = f(x)$  from data. Throughout this section, we make the following assumption.

*Assumption 1.* The system  $\dot{x} = f(x)$  satisfies conditions (T2) and (T3) of Theorem 1 on a path-connected forward-invariant subset  $\mathcal{X}$  of  $\mathcal{M}$ .

Generally, given a fixed sampling time  $\tau$  and a set of  $N$  pairs  $\{(x_l, x_l^+)\}_{l=1}^N$  where  $x_l^+ = \varphi(\tau, x_l)$  for all  $l$ , the task of learning linear immersions  $F : \mathcal{X} \rightarrow \mathcal{Z}$  involves finding the following set

$$\mathcal{F}^*(\tau, N) = \{F^* \in \mathcal{C}(\mathcal{X}, \mathcal{Z}) \mid \exists A^* \in \mathbb{R}^{m \times m}, F^*(x_l^+) = e^{A^* \tau} F^*(x_l), \forall l = 1, \dots, N\}, \quad (25)$$

where  $\mathcal{C}(\mathcal{X}, \mathcal{Z})$  is the space of continuous functions from  $\mathcal{X}$  to  $\mathcal{Z}$ . The state pairs  $\{(x_l, x_l^+)\}_{l=1}^N$  can be extracted from a single trajectory or multiple trajectories of the system. Essentially,  $\mathcal{F}^*(\tau, N)$  is the set of continuous functions that satisfy the condition in (3) for linear immersions at finitely many points  $\{x_l\}_{l=1}^N$  and a fixed time step  $\tau$ .

Alternatively, the learning problem can be stated as:

$$(F^*, A^*) \in \arg \min_{\substack{F \in \mathcal{C}(\mathcal{X}, \mathcal{Z}), \\ A \in \mathbb{R}^{m \times m}}} \sum_{l=1}^N \|F(x_l^+) - e^{A \tau} F(x_l)\| \quad \text{s.t. } x_l^+ = \varphi(\tau, x_l), \forall l \in \{1, \dots, N\}. \quad (26)$$

The set  $\mathcal{F}^*(\tau, N)$  in (25) corresponds to the solutions  $F^*$  of this problem that give zero objective value, i.e., those interpolating the data perfectly.

Theorem 1 shows that under Assumption 1, any linear immersion  $F$  satisfies that for all pairs  $\Omega_i, \Omega_j$  in  $\mathcal{W}^+$ ,

$$F(\Omega_i) = F(\Omega_j). \quad (27)$$

A critical question here is if any learned linear immersion in  $\mathcal{F}^*(\tau, N)$  would also share this property in (27). As a sanity check, note that any constant function  $F^*(\cdot) = C$  for some  $C \in \mathcal{Z}$  belongs to  $\mathcal{F}^*(\tau, N)$  (with the corresponding  $A^* = 0$ ) and does map every  $\omega$ -limit set to the same set. However, this may not hold for every learned linear immersion in  $\mathcal{F}^*(\tau, N)$  since the learned linear immersion only satisfies (3) at finitely many points (finitely many constraints), while a true linear immersion needs to satisfy (3) everywhere in  $\mathcal{X}$  at all times (uncountably many constraints). However, in what follows we show that any function that does not collapse all  $\omega$ -limit sets would be excluded from  $\mathcal{F}^*(\tau, N)$  for small enough  $\tau$  and large enough  $N$ . The following theorem provides a crucial step for this result.

*Theorem 2.* Let  $\{x_l\}_{l=1}^\infty$  be a dense subset of  $\mathcal{X}$ . Let  $F$  be a continuous mapping from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ . If for all  $t > 0$ , there exists a sampling time  $\tau \in (0, t]$  such that  $F \in \mathcal{F}^*(\tau, N)$  for all  $N > 1$ , then  $F$  is a linear immersion of the system.

**Proof.** Suppose that a continuous function  $F$  satisfies the conditions in Theorem 2. Then, for all  $j \in \mathbb{N}$ , there exists a sampling time  $\tau_j \in (0, 2^{-j}]$  such that  $F \in \mathcal{F}^*(\tau_j, N)$  for all  $N \geq 1$ . Clearly, the positive sequence  $\tau_j$  converges to zero as  $j$  goes to infinity. Also, for each  $j$ ,  $F \in \lim_{N \rightarrow \infty} \mathcal{F}^*(\tau_j, N) =: \mathcal{F}^*(\tau_j)$ . Note that the limit  $\mathcal{F}^*(\tau_j)$  is well defined since the set sequence  $\mathcal{F}^*(\tau_j, N)$  monotonically shrinks as  $N$  increases.

Next, for a fixed  $j$ , we want to prove that there exists a matrix  $A$  such that for all  $l \geq 1$ ,

$$F(\varphi(\tau_j, x_l)) = e^{A \tau_j} F(x_l). \quad (28)$$

We denote the set of matrices  $A_d$  such that  $F(x_l^+) = A_d F(x_l)$  for all  $l$  from 1 to  $N$  by  $\mathcal{A}_d^*(N)$ . By definition,  $\mathcal{A}_d^*(N)$  is an affine subspace in  $\mathbb{R}^{m \times m}$ , and monotonically shrinks with  $N$ . Since  $\mathcal{A}_d^*(N)$  is an affine subspace for all  $N \geq 1$ , each time  $\mathcal{A}_d^*(N)$  shrinks, its dimension decreases. Thus, the sequence of sets  $\mathcal{A}_d^*(N)$  must converge at a finite  $N^*$  since the dimension of  $\mathcal{A}_d^*(N)$  can decrease at most finitely many times. Since  $F \in \mathcal{F}^*(\tau_j, N^*)$ , there exists at least one  $A \in \mathbb{R}^{m \times m}$  such that  $e^{A \tau_j} \in \mathcal{A}_d^*(N^*)$ . This matrix  $A$  satisfies (28) for all  $l \geq 1$ .

Then, we pick an arbitrary  $x \in \mathcal{X}$ . Since  $\{x_l\}_{l=1}^\infty$  is dense in  $\mathcal{X}$ , there exists a subsequence of  $x_l$ , denoted by  $\bar{x}_l$ , that converges to  $x$ . Since  $F(\cdot)$  and  $\varphi(\tau_j, \cdot)$  are continuous,

$$\begin{aligned} F(\varphi(\tau_j, x)) &= \lim_{l \rightarrow \infty} F(\varphi(\tau_j, \bar{x}_l)) \\ &= \lim_{l \rightarrow \infty} e^{A \tau_j} F(\bar{x}_l) \\ &= e^{A \tau_j} F(x). \end{aligned} \quad (29)$$

We pick an arbitrary  $t > 0$ . Since  $\tau_j$  is positive and converging to zero, there exists  $k_j \in \mathbb{N}$  such that the sequence  $\bar{t}_j := \sum_{i=1}^j k_j \cdot \tau_j$  converges to  $t$  as  $j$  goes to infinity. Note that for all  $j$ , (29) implies that

$$F(\varphi(\bar{t}_j, x)) = e^{A \bar{t}_j} F(x). \quad (30)$$

By the continuity of  $F(\cdot)$  and  $\varphi(\cdot, x)$ , we have

$$\begin{aligned}
F(\varphi(t, x)) &= F(\varphi(\lim_{j \rightarrow \infty} \bar{t}_j, x)) \\
&= \lim_{j \rightarrow \infty} F(\varphi(\bar{t}_j, x)) \\
&= \lim_{j \rightarrow \infty} e^{A\bar{t}_j} F(x) = e^{At} F(x). \quad (31)
\end{aligned}$$

Since  $x$  and  $t$  are arbitrary, (31) implies that  $F$  is a linear immersion.  $\square$

**Corollary 6.** Let  $\{x_l\}_{l=1}^\infty$  be a dense subset of  $\mathcal{X}$ . Let  $F$  be a continuous function such that  $F(\Omega_1) \neq F(\Omega_2)$  for some  $\Omega_1$  and  $\Omega_2 \in \mathcal{W}^+$ . Then, there exists  $t^* > 0$  such that for all sampling times  $\tau \in (0, t^*]$ ,  $F \notin \mathcal{F}^*(\tau, N)$  for some  $N > 0$ .

**Proof.** Let  $F$  be a continuous function that does not map every  $\Omega \in \mathcal{W}^+$  to the same subset of  $\mathbb{R}^m$ . According to Theorem 1, since (T2) and (T3) hold,  $F$  is not a linear immersion of the system. By the contrapositive of Theorem 2, for any  $F$  not a linear immersion, there exists  $t > 0$  such that for all  $\tau \in (0, t]$  and for some large enough  $N$ ,  $F$  is not in  $\mathcal{F}^*(\tau, N)$ .  $\square$

Corollary 6 reveals that any immersion candidate  $F$  that can distinguish at least two  $\omega$ -limit sets in  $\mathcal{X}$  would always be ruled out from  $\mathcal{F}(\tau, N)$  by a small enough sampling time  $\tau$  and a large enough sample size  $N$ .

**Remark 7.** Using similar arguments in the proof of Theorem 2, one can also show that for any positive time sequence  $\tau_j$  that converges to zero, we have

$$\limsup_{j \rightarrow \infty} \lim_{N \rightarrow \infty} \mathcal{F}^*(\tau_j, N) \subseteq \mathcal{C}(\mathcal{W}^+). \quad (32)$$

**Remark 8.** The condition of sampled states  $\{x_l\}_{l=1}^\infty$  being dense in  $\mathcal{X}$  in Theorem 2 indicates that the data collection process is conducted in a way such that the domain  $\mathcal{X}$  of interest is thoroughly covered by the sampled states in the limit. This condition is relatively straightforward to meet. For instance, consider a Borel probability measure  $\mu$  over  $\mathcal{X}$  such that any open subset of  $\mathcal{X}$  is not measure zero. By repeatedly drawing random initial states according to  $\mu$ , simulating trajectories for a finite time horizon, and extracting state pairs  $(x_l, x_l^+)$  from these trajectories, the resulting samples  $\{x_l\}_{l=1}^\infty$  is dense in  $\mathcal{X}$  almost surely.

## 5. EXTENSIONS OF THE MAIN THEOREM

Due to the condition (T2), Theorem 1 cannot be directly applied to show the non-existence of one-to-one linear immersions for systems with diverging trajectories in  $\mathcal{X}$ . In this section, we show several extensions to Theorem 1 that work for systems with diverging trajectories.

### 5.1 Indirect Extensions of Theorem 1

The following two propositions, in conjunction with Theorem 1, show the non-existence of one-to-one linear immersions even when a diverging trajectory is present. We refer to these propositions as “indirect extensions” to Theorem 1 because they must be utilized in conjunction with it.

**Proposition 7.** If  $F : \mathcal{X} \rightarrow \mathcal{Z}$  is an immersion of the system  $\dot{x} = f(x)$  over  $\mathcal{X}$ , then  $F$  is an immersion over any forward invariant subset of  $\mathcal{X}$ .

The proof of Proposition 7 is straightforward and omitted for brevity. By Proposition 7, if we show there is no one-to-one linear immersion over a forward invariant subdomain

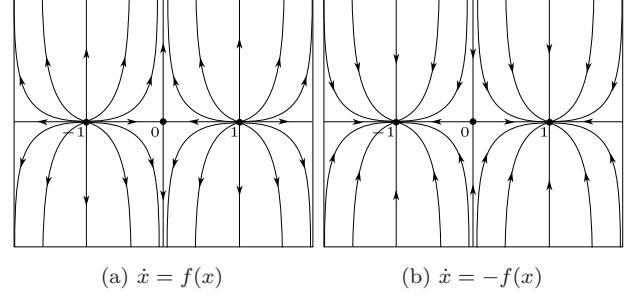


Fig. 5. Phase portraits of the system in Example 11 and its time-reversed counterpart.

of  $\mathcal{X}$ , that implies a one-to-one linear immersion does not exist over  $\mathcal{X}$ .

**Proposition 8.** Suppose that  $\mathcal{X}' \subseteq \mathcal{X}$  is forward invariant for the time-reversed system  $\dot{x} = -f(x)$ . If  $F$  immerses the system  $\dot{x} = f(x)$  over  $\mathcal{X}$  into  $\dot{z} = g(z)$  over  $\mathcal{Z}$  by  $F$ , then the same  $F$  also immerses the time-reversed system  $\dot{x} = -f(x)$  over  $\mathcal{X}'$  into  $\dot{z} = -g(z)$  over  $F(\mathcal{X}')$ .

**Proof.** We want to show that the time-reversed system  $\dot{x} = -f(x)$  over  $\mathcal{X}'$  is immersed into  $\dot{z} = -g(z)$  by  $F$ , which is equivalent to show that  $F(\varphi(-t, \xi)) = \psi(-t, F(\xi))$  for all  $\xi \in \mathcal{X}'$  and for all  $t \geq 0$ .

Pick an arbitrary  $\xi \in \mathcal{X}'$ . Since  $\mathcal{X}'$  is forward invariant for  $\dot{x} = -f(x)$ ,  $\varphi(-t, \xi)$  is well-defined for all  $t \geq 0$ . We denote  $\mathcal{X}_\xi := \{\varphi(-t, \xi) \mid t \in \mathbb{R}_{\geq 0}\} \subseteq \mathcal{X}'$ . We pick any  $x \in \mathcal{X}_\xi$ . There exists  $\tau(x) \leq 0$  such that  $x = \varphi(\tau(x), \xi)$ . Since  $F$  is an immersion over  $\mathcal{X}$  and  $x \in \mathcal{X}$ , we have for all  $t \geq 0$ ,

$$F(\varphi(t, x)) = \psi(t, F(x)). \quad (33)$$

Thus,  $F(\xi) = F(\varphi(-\tau(x), x)) = \psi(-\tau(x), F(x))$ . Since  $\varphi(t, x) = \varphi(t + \tau(x), \xi)$ , by (33),

$$\begin{aligned}
F(\varphi(t + \tau(x), \xi)) &= \psi(t, F(x)) \\
&= \psi(t + \tau(x), \psi(-\tau(x), F(x))) \\
&= \psi(t + \tau(x), F(\xi)).
\end{aligned} \quad (34)$$

Let  $t = 0$ . Then (34) implies  $F(\varphi(\tau(x), \xi)) = \psi(\tau(x), F(\xi))$ . By definition of  $\mathcal{X}_\xi$  and the uniqueness of the solution of  $\dot{x} = f(x)$ , for all  $t \leq 0$ , there exists  $x \in \mathcal{X}_\xi$  such that  $\tau(x) = t$ . Thus, for all  $t \geq 0$ , we have

$$F(\varphi(-t, \xi)) = \psi(-t, F(\xi)). \quad (35)$$

Thus, the solution  $\psi(-t, F(\xi))$  of  $\dot{z} = -g(z)$  is well defined and equal to  $F(\varphi(-t, \xi))$  for all  $t \geq 0$ .  $\square$

By Proposition 8, if we can show that a one-to-one linear immersion does not exist for the time-reversed system  $\dot{x} = -f(x)$  over a forward-invariant subdomain of  $\mathcal{X}$ , then there is no one-to-one linear immersion for the original system  $\dot{x} = f(x)$  over  $\mathcal{X}$ . The following example demonstrates how to extend our results to systems with diverging trajectories by combining the above two propositions with Theorem 1.

**Example 11.** Consider a system in  $\mathbb{R}^2$  with the phase portrait shown in Fig. 5a. The system has three limit sets  $\{-1, 0, 1\}$ , where  $-1$  and  $1$  are unstable equilibria and  $0$  is a saddle point. We want to show that there is no one-to-one linear immersion  $F$  of this system over  $\mathcal{X} = \mathbb{R}^2$ . To achieve this goal, we cannot directly apply Corollary 4 since any trajectory starting outside  $[-1, 1] \times 0$  is not precompact.



However, by applying Theorem 1 to the forward-invariant subdomain  $\mathcal{X}' = [-1, 0] \times \{0\}$  (or  $[0, 1] \times \{0\}$ ) of  $\mathcal{X}$ , we know that there is no one-to-one linear immersion over  $\mathcal{X}'$ . Then, by Proposition 7, no one-to-one linear immersion exists over  $\mathcal{X}$ .

To make this example more challenging, consider  $\mathcal{X} = \mathbb{R}^2 \setminus ((-1, 0.5) \cup (0.5, 1)) \times \{0\}$ . Note that our previous argument does not work anymore since any forward invariant  $\mathcal{X}' \subseteq \mathcal{X}$  contains at least one trajectory that is not precompact. Now consider the time-reversed system, shown by the phase portrait in Fig. 5b. Since  $\mathcal{X}$  is not forward invariant for the time-reversed system, we take the forward invariant subset  $\mathcal{X}' = \mathbb{R}^2 \setminus ((-1, 0) \cup (0, 1)) \times \{0\}$  of  $\mathcal{X}$ . Since all trajectories of the time-reversed systems in  $\mathcal{X}'$  are precompact, Corollary 4 implies that there is no one-to-one linear immersion for the time-reversed system over  $\mathcal{X}'$ . Then, by Proposition 8, we know there is no one-to-one linear immersion for the original system over  $\mathcal{X}$ .

## 5.2 Direct Extension of Theorem 1

While Propositions 7 and 8 allow us to apply Theorem 1 to systems with diverging trajectories, identifying the appropriate subdomain  $\mathcal{X}'$  for more complex examples can be challenging. In this section, we present a direct extension to Theorem 1, where we replace the original condition (T2) with a weaker condition. Before delving into this extension, we introduce several key definitions.

For a system  $\dot{x} = f(x)$  defined over  $\mathcal{X}$ , let  $\mathcal{X}'$  be the maximal subset of  $\mathcal{X}$  such that  $\varphi(-t, \xi)$  is defined and contained in  $\mathcal{X}$  for all  $t \geq 0$  and  $\xi \in \mathcal{X}'$  (Namely  $\mathcal{X}'$  is the maximal forward invariant subset of  $\mathcal{X}$  with respect to the time-reversed system). Given an initial state  $\xi \in \mathcal{X}$ , the trajectory  $\varphi(t, \xi)$  from  $\xi$  is called *backward precompact* in  $\mathcal{X}$  if  $\xi \in \mathcal{X}'$  and the trajectory from  $\xi$  is precompact in  $\mathcal{X}$  with respect to the time-reversed system  $\dot{x} = -f(x)$ .

For each initial state  $\xi \in \mathcal{X}$ , the  $\omega$ -limit set of  $\xi$  in  $\mathcal{X}$  with respect to the time-reversed system  $\dot{x} = -f(x)$  is denoted by  $\omega^-(\xi)$ . If  $\xi \notin \mathcal{X}'$ ,  $\omega^-(\xi) = \emptyset$  by default. The set  $\omega^-(\xi)$  is known as an  $\alpha$ -limit set of the original system  $\dot{x} = f(x)$  (Hirsch et al. (2012)). We denote the set of all  $\alpha$ -limit sets in  $\mathcal{X}$  by  $\mathcal{W}^-$ . For each  $\Gamma \in \mathcal{W}^-$ , its *domain of attraction* with respect to the time-reversed system is denoted by  $D^-(\Gamma)$ . Similar to  $\mathcal{W}^+$ , the set  $\mathcal{W}^-$  contains all the equilibria and closed orbits in  $\mathcal{X}$ , and thus typically the intersection of  $\mathcal{W}^+$  and  $\mathcal{W}^-$  is not empty. Now we are ready to present the extension of Theorem 1:

*Theorem 3.* Suppose that:

- (T1')  $\dot{x} = f(x)$  on  $\mathcal{X}$  is immersed in a system  $\dot{z} = g(z)$  on  $\mathcal{Z}$  by a continuous mapping  $F$ , where both  $\dot{z} = g(z)$  and its time-reversed counterpart  $\dot{z} = -g(z)$  have closed basins;
- (T2') every trajectory of  $\dot{x} = f(x)$  on  $\mathcal{X}$  is either precompact or backward precompact in  $\mathcal{X}$ ;
- (T3') the set  $\mathcal{W}^+ \cup \mathcal{W}^-$  is finite or countable.

Then, the set  $\{F(\Omega) \mid \Omega \in \mathcal{W}^+ \cup \mathcal{W}^-\}$  has exactly one maximal element, that is, there exists  $\Omega \in \mathcal{W}^+ \cup \mathcal{W}^-$  such that  $F(\Omega) \supseteq F(\Omega')$  for all  $\Omega'$  in  $\mathcal{W}^+ \cup \mathcal{W}^-$ .

If in addition to (T1')-(T3'), we also have:

- (T4') For every  $\Omega^+ \in \mathcal{W}^+$  and  $\Omega^- \in \mathcal{W}^-$ , there exist at least one precompact trajectory in  $D^+(\Omega^+)$  and one backward precompact trajectory in  $D^-(\Omega^-)$ ,

then the set  $\{F(\Omega) \mid \Omega \in \mathcal{W}^+ \cup \mathcal{W}^-\}$  has exactly one element, that is  $F(\Omega_1) = F(\Omega_2)$  for any  $\Omega_1$  and  $\Omega_2$  in  $\mathcal{W}^+ \cup \mathcal{W}^-$ .

The proof of Theorem 3 is similar to that of Theorem 1, and can be found in Appendix B. Under conditions (T1')-(T3'), Theorem 3 says that any continuous immersion  $F$  cannot fully distinguish different limit sets in  $\mathcal{X}$ . Compared with Theorem 1, (T2') is relaxed to allow diverging trajectories, as long as these trajectories converge to some limit sets (such as an unstable equilibrium) backward in time. At the same time, Theorem 3 requires that the time-reversed system  $\dot{z} = -g(z)$  also has closed basins, which is trivially satisfied by any linear system.

*Example 12.* Consider the same system in Example 11 and  $\mathcal{X} = \mathbb{R}^2 \setminus ((-1, 0.5) \cup (0.5, 1)) \times \{0\}$ . Since (T2')-(T4') in Theorem 3 are satisfied, any linear immersion over  $\mathcal{X}$  collapses the three equilibria into one. Therefore, no one-to-one linear immersions exist over this specific  $\mathcal{X}$ . Compared with the indirect extensions of Theorem 1, Theorem 3 is directly applied to show the non-existence of linear immersions, without constructing the subdomain  $\mathcal{X}'$  as in Example 11.

## 6. CONCLUSION

In this work, we first show that linear immersions collapse different  $\omega$ -limit sets into one under the condition that (i) all trajectories in  $\mathcal{X}$  are precompact and (ii) there are at most countably many  $\omega$ -limit sets. Then we bridge our theoretical findings on exact linear immersions with approximate linear immersions learned from data. We show that as the size of the data set increases and the sampling interval decreases, the learned linear immersion converges to functions incapable of distinguishing different  $\omega$ -limit sets. To extend the applicability of our results beyond the constraints of precompact trajectories, we have also present several extensions to the main theorem. These extensions broaden the scope of our results, allowing us to address systems with diverging trajectories.

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## Appendix A. PROOF OF THE MAIN THEOREM

To prove Theorem 1, we first need to introduce two lemmas. The first lemma reveals a relation between  $\omega$ -limit sets of the original system and the immersed system.

**Lemma 9.** Let  $F$  be an immersion that maps  $\dot{x} = f(x)$  on  $\mathcal{X}$  to  $\dot{z} = g(z)$  on  $\mathcal{Z}$ . For any  $\xi \in \mathcal{X}$ , if  $\omega^+(\xi)$  is nonempty, then  $\omega^+(F(\xi))$  exists and contains  $F(\omega^+(\xi))$ . Furthermore, if the trajectory starting from  $\xi$  is precompact in  $\mathcal{X}$ ,  $F(\omega^+(\xi)) = \omega^+(F(\xi))$ .

**Proof.** We first prove that  $F(\omega^+(\xi)) \subseteq \omega^+(F(\xi))$ . Indeed, suppose that  $p \in \omega^+(\xi)$ , and pick a sequence of times  $t_i \rightarrow \infty$  so that  $\varphi(t_i, \xi) \rightarrow p$  as  $t_i \rightarrow \infty$ . Therefore  $\psi(t_i, F(\xi)) = F(\varphi(t_i, \xi)) \rightarrow q := F(p)$ , showing that  $q \in \omega^+(F(\xi))$ .

Conversely, suppose that  $q' \in \omega^+(F(\xi))$ , and pick a sequence of times  $t_i \rightarrow \infty$  so that  $F(\varphi(t_i, \xi)) = \psi(t_i, F(\xi)) \rightarrow q' \in \mathcal{Z}$  as  $t_i \rightarrow \infty$ . Since the trajectory  $\varphi(t, \xi)$  is precompact in  $\mathcal{X}$ , there is a subsequence of the  $t_i$ 's, which is again denoted by  $t_i$  without loss of generality, so that  $\varphi(t_i, \xi) \rightarrow p \in \mathcal{X}$  and thus  $\psi(t_i, F(\xi)) = F(\varphi(t_i, \xi)) \rightarrow q := F(p)$ . Since we picked a subsequence, also  $\psi(t_i, F(\xi)) = F(\varphi(t_i, \xi)) \rightarrow q'$ . We conclude that  $q' = q \in F(\omega^+(\xi))$ , showing that  $\omega^+(F(\xi)) \subseteq F(\omega^+(\xi))$ . We conclude that  $F(\omega^+(\xi)) = \omega^+(F(\xi))$ .  $\square$

**Remark 9.** Under condition (T2) in Theorem 1, for any  $\Omega \in \mathcal{W}^+$ , the image  $\hat{\Omega} := F(\Omega)$  is an  $\omega$ -limit set for the system  $\dot{z} = g(z)$ . Indeed, by definition there is some  $\xi \in \mathcal{X}$  such that  $\omega^+(\xi) = \Omega$ . Thus, from Lemma 9 we have that  $\hat{\Omega} = F(\Omega) = F(\omega^+(\xi)) = \omega^+(F(\xi))$ .

Next, observe that, in general,  $F(D^+(\Omega)) \neq D^+(F(\Omega))$ , since the latter set could be larger. Examples are easy to construct by taking  $\mathcal{X}$  to be a forward-invariant subset of  $\mathcal{Z}$  and  $F$  the identity. For example, consider  $\dot{x} = -x$  on  $\mathcal{X} = (-1, 1)$  and the same system  $\dot{z} = -z$  on  $\mathcal{Z} = \mathbb{R}$ . Here  $\Omega = \{0\}$  is the only  $\omega$ -limit set, and  $F(D^+(\Omega)) = D^+(\Omega) = (-1, 1)$  but  $D^+(F(\Omega)) = \mathbb{R}$ . However, the following weaker statement is true.

**Lemma 10.** Suppose that  $F$  is an immersion. For any  $\xi \in \mathcal{X}$ , if  $\varphi(t, \xi)$  is precompact in  $\mathcal{X}$ , then  $F(D^+(\omega^+(\xi))) \subseteq D^+(F(\omega^+(\xi)))$ .

**Proof.** Since  $\varphi(t, \xi)$  is precompact in  $\mathcal{X}$ , by Lemmas 1 and 9,  $\Omega := \omega^+(\xi)$  is nonempty and  $F(\Omega) = \omega^+(F(\xi))$ . Let  $x$  be a point in  $D^+(\Omega)$ . Then, there exists a sequence  $t_n \geq 0$  such that  $t_n \rightarrow +\infty$  and  $\varphi(t_n, x) \rightarrow$

$x'$  for some  $x' \in \Omega$ . By (3) and the continuity of  $F$ ,  $\lim_{n \rightarrow +\infty} \psi(t_n, F(x)) = \lim_{n \rightarrow +\infty} F(\varphi(t_n, x)) = F(x') \in F(\Omega) = \omega^+(F(\xi))$ . Hence,  $F(x) \in D^+(\omega^+(F(\xi)))$  and thereby  $F(D^+(\omega^+(\xi))) \subseteq D^+(F(\omega^+(\xi)))$ .  $\square$

**Proof of Theorem 1:** Since by (T2) every trajectory is precompact in  $\mathcal{X}$  and by (T3) there are at most countably many  $\omega$ -limit sets in  $\mathcal{X}$ , we have that  $\mathcal{X} = \bigcup_{i \in I} D^+(\Omega_i)$ , for a finite or countable set  $I$ . Thus,  $F(\mathcal{X}) = \bigcup_{i \in I} F(D^+(\Omega_i))$ . By (T2) and Lemmas 9 and 10,  $F(\Omega_i)$  is an  $\omega$ -limit set for all  $i \in I$  and

$$F(\mathcal{X}) = \bigcup_{i \in I} (D^+(F(\Omega_i)) \cap F(\mathcal{X})). \quad (\text{A.1})$$

According (T1),  $D^+(F(\Omega_i))$  are closed in  $\mathcal{Z}$  and thus  $D^+(F(\Omega_i)) \cap F(\mathcal{X})$  are closed in the subspace topology induced on  $F(\mathcal{X})$  and disjoint for all  $i \in I$ . That is,  $F(\mathcal{X})$  is a disjoint union of a countable collection of closed sets. Since  $\mathcal{X}$  is path-connected and  $F$  is continuous,  $F(\mathcal{X})$  is path-connected as well. Thus, by a theorem of Sierpiński (1918), only one of the sets in the collection  $\{D^+(F(\Omega_i)) \cap F(\mathcal{X})\}_{i \in I}$  can be nonempty. However, since points in  $F(\Omega_i)$  are limit points of  $D^+(F(\Omega_i))$  and  $D^+(F(\Omega_i))$  is closed, the set  $D^+(F(\Omega_i)) \cap F(\mathcal{X})$  contains the nonempty set  $F(\Omega_i)$  for all  $i \in I$ . Therefore,  $D^+(F(\Omega_i)) \cap F(\mathcal{X})$  must be equal to  $F(\mathcal{X})$  for all  $i \in I$ .

Furthermore,  $F(\Omega_i)$  must be the same for all  $i \in I$  since their domains of attraction intersect each other.  $\square$

*Remark 10.* Sierpiński's Theorem states that if a continuum  $\mathcal{X}$  has a countable cover  $\{X_i\}_{i=1}^\infty$  by pairwise disjoint closed subsets, then at most one of the sets  $X_i$  is non-empty. A continuum is a compact connected Hausdorff space, but we do not assume that  $\mathcal{X}$  is compact. However, the theorem is still true if  $\mathcal{X}$  is not compact. Indeed, suppose that two of the sets  $X_i$  would be nonempty, and pick two points  $p, q$ , one in each set. Consider a (continuous) path  $\gamma : [0, 1] \rightarrow \mathcal{X}$  that joins these two points, and let  $\Gamma := \gamma([0, 1])$ . Now the sets  $\{X_i \cap \Gamma\}_{i=1}^\infty$  form a disjoint cover of the continuum  $\Gamma$ , but two of these sets are nonempty, a contradiction.

## Appendix B. PROOF OF THEOREM 3

We first show a property of systems with closed basins.

**Lemma 11.** Suppose that both the original system  $\dot{x} = f(x)$  and the time-reversed system  $\dot{x} = -f(x)$  have closed basins. Then, for any  $\omega$ -limit set  $\Omega \subseteq \mathcal{X}$  and  $\alpha$ -limit set  $\Gamma \subseteq \mathcal{X}$  of  $\dot{x} = f(x)$ ,  $D^+(\Omega) \cap D^-(\Gamma) \neq \emptyset$  implies  $\Omega = \Gamma$ .

**Proof.** Let  $x_0 \in D^+(\Omega) \cap D^-(\Gamma)$ . That is,  $\omega^+(x_0) = \Omega$  and  $\omega^-(x_0) = \Gamma$ . Denote the trajectory through  $x_0$  by  $X_0 = \{\psi(t, x_0) \mid t \in \mathbb{R}\}$ . Then, by the definition of limit sets,  $\omega^+(x) = \Omega$  and  $\omega^-(x) = \Gamma$  imply that (i)  $X_0$  is contained by  $D^+(\Omega) \cap D^-(\Gamma)$  and (ii) the closure  $\text{cl}(X_0)$  contains  $\Omega \cup \Gamma$ . Since  $D^+(\Omega)$  and  $D^-(\Gamma)$  are closed, we have

$$\Omega \cup \Gamma \subseteq \text{cl}(X_0) \subseteq D^+(\Omega) \cap D^-(\Gamma).$$

Thus, we have  $\Omega \subseteq D^-(\Gamma)$  and  $\Gamma \subseteq D^+(\Omega)$ . Next, it can be shown that any limit set is closed and invariant in  $\mathcal{X}$  (Alligood et al. (1996)). Since  $\Omega$  is invariant and  $\Omega \subseteq D^-(\Gamma)$ , we have  $\Gamma \subseteq \text{cl}(\Omega) = \Omega$ . Similarly, we have  $\Omega \subseteq \text{cl}(\Gamma) = \Gamma$ . Thus,  $\Omega = \Gamma$ .  $\square$

Next, we extend Lemmas 9 and 10 for  $\alpha$ -limit sets.

**Lemma 12.** Let  $F$  be an immersion that maps  $\dot{x} = f(x)$  on  $\mathcal{X}$  to  $\dot{z} = g(z)$  on  $\mathcal{Z}$ . For any  $\xi \in \mathcal{X}$ , if  $\omega^-(\xi)$  exists, then  $\omega^-(F(\xi))$  exists and contains  $F(\omega^-(\xi))$ . Furthermore, if the trajectory starting at  $\xi$  is backward precompact in  $\mathcal{X}$ ,  $F(\omega^-(\xi)) = \omega^-(F(\xi))$ .

**Proof.** Let  $\mathcal{X}'$  be the maximal forward invariant subset of  $\mathcal{X}$  with respect to the time-reversed system. By Proposition 8,  $F$  is an immersion that maps  $\dot{x} = -f(x)$  on  $\mathcal{X}'$  to  $\dot{z} = -g(z)$  on  $F(\mathcal{X}')$ . Then, the proof is completed by applying Lemma 9 to the time-reversed system  $\dot{x} = -f(x)$  on  $\mathcal{X}'$  and the immersion  $F$ .  $\square$

**Lemma 13.** Let  $F$  be an immersion that maps  $\dot{x} = f(x)$  on  $\mathcal{X}$  to a system  $\dot{z} = g(z)$  on  $\mathcal{Z}$ . For any  $\xi \in \mathcal{X}$ , if  $\varphi(t, \xi)$  is precompact in  $\mathcal{X}$ , then

$$F(D^+(\omega^+(\xi))) \subseteq D^+(F(\omega^+(\xi))),$$

and if  $\varphi(t, \xi)$  is backward precompact in  $\mathcal{X}$ , then

$$F(D^-(\omega^-(\xi))) \subseteq D^-(F(\omega^-(\xi))).$$

**Proof.** The inclusion relation when  $\varphi(t, \xi)$  is precompact in  $\mathcal{X}$  is directly implied by Lemma 10. The inclusion relation when  $\varphi(t, \xi)$  is backward precompact in  $\mathcal{X}$  can be shown via similar arguments in the proof of Lemma 10 with Lemma 1, Lemma 12, and (35).  $\square$

**Proof of Theorem 3:** Let  $\widehat{\mathcal{W}}^+$  be the set of  $\omega$ -limit sets  $\Omega$  in  $\mathcal{W}^+$  whose domain of attraction  $D^+(\Omega)$  contains at least one trajectory precompact in  $\mathcal{X}$ . Similarly, we define  $\widehat{\mathcal{W}}^-$  be the set of  $\alpha$ -limit sets  $\Gamma$  in  $\mathcal{W}^-$  whose domain of attraction  $D^-(\Gamma)$  contains at least one trajectory backward precompact in  $\mathcal{X}$ . Then, for any limit set  $\Omega \in \widehat{\mathcal{W}}^+ \cup \widehat{\mathcal{W}}^-$ , we define

$$D(\Omega) = \begin{cases} D^+(\Omega) & \Omega \in \widehat{\mathcal{W}}^+ \setminus \widehat{\mathcal{W}}^- \\ D^-(\Omega) & \Omega \in \widehat{\mathcal{W}}^- \setminus \widehat{\mathcal{W}}^+ \\ D^+(\Omega) \cup D^-(\Omega) & \Omega \in \widehat{\mathcal{W}}^- \cap \widehat{\mathcal{W}}^+ \end{cases}, \quad (\text{B.1})$$

$$D(F(\Omega)) = \begin{cases} D^+(F(\Omega)) & \Omega \in \widehat{\mathcal{W}}^+ \setminus \widehat{\mathcal{W}}^- \\ D^-(F(\Omega)) & \Omega \in \widehat{\mathcal{W}}^- \setminus \widehat{\mathcal{W}}^+ \\ D^+(F(\Omega)) \cup D^-(F(\Omega)) & \Omega \in \widehat{\mathcal{W}}^- \cap \widehat{\mathcal{W}}^+ \end{cases}.$$

Note that the function  $D(F(\cdot))$  is well-defined for any  $\Omega \in \widehat{\mathcal{W}}^+ \cup \widehat{\mathcal{W}}^-$ , since by Lemmas 9 and 12, for any  $\Omega \in \widehat{\mathcal{W}}^+$  and  $\Gamma \in \widehat{\mathcal{W}}^-$ ,  $F(\Omega)$  and  $F(\Gamma)$  are respectively an  $\omega$ -limit set and an  $\alpha$ -limit set of  $\dot{z} = g(z)$  on  $\mathcal{Z}$ .

Since by (T2'), for any  $\xi \in \mathcal{X}$ , the corresponding trajectory is either precompact or backward precompact in  $\mathcal{X}$ , by Lemma 1, any  $\xi \in \mathcal{X}$  must belong to  $D(\Omega)$  for some  $\Omega \in \widehat{\mathcal{W}}^+ \cup \widehat{\mathcal{W}}^-$ . Thus, the set  $\mathcal{X} = \bigcup_{i \in I} D(\Omega_i)$  for a countable set  $I$ , where  $\{\Omega_i\}_{i \in I} = \widehat{\mathcal{W}}^+ \cup \widehat{\mathcal{W}}^-$ . As a result,  $F(\mathcal{X}) = \bigcup_{i \in I} F(D(\Omega_i))$ . By Lemma 13,  $F(D(\Omega_i))$  is contained by  $D(F(\Omega_i))$ . Thus,

$$F(\mathcal{X}) = \bigcup_{i \in I} (D(F(\Omega_i)) \cap F(\mathcal{X})). \quad (\text{B.2})$$

By (T1') and Lemma 11,  $D(F(\Omega_i)) \cap F(\mathcal{X})$  is closed in the subspace topology induced by  $F(\mathcal{X})$  and is disjoint for all  $i \in I$ . So  $F(\mathcal{X})$  is a disjoint union of a countable collection of closed sets. Since  $\mathcal{X}$  is path-connected and  $F$  is continuous,  $F(\mathcal{X})$  is path-connected. Thus, by a theorem of Sierpiński (1918), only one of the sets in the collection  $\{D(F(\Omega_i)) \cap F(\mathcal{X})\}_{i \in I}$  can be nonempty. However, since

points in  $F(\Omega_i)$  are limit points of  $D(F(\Omega_i))$  and  $D(F(\Omega_i))$  is closed (as both  $\dot{z} = g(z)$  and  $\dot{z} = -g(z)$  have closed basin), the set  $D(F(\Omega_i)) \cap F(\mathcal{X})$  contains the nonempty set  $F(\Omega_i)$ , for all  $i \in I$ . Therefore, all the  $D(F(\Omega_i)) \cap F(\mathcal{X})$  must be the same, and thus equal to  $F(\mathcal{X})$ . By (T1') and Lemma 11, the limit sets  $F(\Omega_i)$  must be the same for all  $i \in I$  since their domains of attraction all contain  $F(\mathcal{X})$ .

Denote the only element in  $\{F(\Omega_i)\}_{i \in I}$  by  $\tilde{\Omega}$ . Now for any  $\Omega \in \mathcal{W}^+ \setminus \widehat{\mathcal{W}^+}$ , we pick an arbitrary  $\xi \in \Omega$ . By Lemma 9,  $F(\Omega) \subseteq \omega^+(F(\xi))$ . By the construction of  $\widehat{\mathcal{W}^+}$  and (T2'),  $\xi$  is not precompact in  $\mathcal{X}$ , and thus has to be backward precompact in  $\mathcal{X}$ . Thus, by Lemma 1,  $\omega^-(\xi)$  exists and is contained by  $\widehat{\mathcal{W}^-}$ . Furthermore, by Lemma 12,  $F(\omega^-(\xi)) = \omega^-(F(\xi))$ . By (T1') and Lemma 11, since  $F(\xi) \in D^+(\omega^+(F(\xi))) \cap D^-(\omega^-(F(\xi)))$ , we have

$$\omega^-(F(\xi)) = \omega^+(F(\xi)) \supseteq F(\Omega).$$

Since  $\omega^-(\xi)$  is contained in  $\widehat{\mathcal{W}^-}$ ,  $F(\omega^-(\xi)) = \tilde{\Omega}$ . Therefore, we have  $\tilde{\Omega} \supseteq F(\Omega)$ . Similarly, one can show that for any  $\Gamma \in \mathcal{W}^- \setminus \widehat{\mathcal{W}^-}$ ,  $\tilde{\Omega} \supseteq F(\Gamma)$ . Thus,  $\tilde{\Omega}$  is the unique maximal element in the set  $\{F(\Omega)\}_{\Omega \in \mathcal{W}^+ \cup \mathcal{W}^-}$ . When (T4') is satisfied,  $\widehat{\mathcal{W}^+} \cup \widehat{\mathcal{W}^-} = \mathcal{W}^+ \cup \mathcal{W}^-$ . Thus,  $\tilde{\Omega}$  is the only element in the set  $\{F(\Omega)\}_{\Omega \in \mathcal{W}^+ \cup \mathcal{W}^-}$ .  $\square$