Optimization: Linear programming

Hongxin Zhang

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Content

- Linear programming
- Non-linear optimization

- Text books
  - 《线性规划》
    - 张建中，许绍吉，科学出版社
  - 《最优化理论与方法》
    - 袁亚湘，孙文瑜，科学出版社
  - 《数学规划》
    - 黄红选，韩继业，清华大学出版社

Linear programming

- In the production practice, we often need to consider how to make full use of available resources for maximum economic efficiency.

$Profit-driven$

Operational Research →

Mathematical Programming →

Linear Programming, LP

- In 1947, G. B. Dantzig proposed the simplex method to solve linear programming problem.
- Linear programming are more widely used since current computer is able to deal with complex linear programming problem, which has thousands of constraints and decision variables.

Basic Theory

Problem definition (linear programming, LP)

- A mobile game company is going to develop and operate large mobile games. Now they have three technical teams A,B,C.
  - RPG game system
    - Gain 40 million profit per game
    - Need team A cooperate with team B
    - Team A spends 2 month and team B spends 1 month;
  - Social mobile game
    - Gain 30 million profit per game
    - Need cooperation among team A,B, C
    - Every team spends 1 month
  - In 2013~2014, team A can work 10 month, team B 8 month, team C 7 month.
- Question: What is the best stratege to gain maximum profit?

Problem definition (linear programming, LP)

- Suppose that develop x_1 RPG games, x_2 social mobile games
- Objective function

$$\max \quad z = 4x_1 + 3x_2$$

- Constraint conditions $\begin{cases} 2x_1 + x_2 \le 10 \\ x_1 + x_2 \le 8 \\ x_2 \le 7 \\ x_1, x_2 \ge 0 \end{cases}$

Additional conditions :

X_1 and x_2 are integer.

Discussion ? ! ?

Problem definition (linear programming, LP)

- Minimize or maximize a linear cost function subject to a set of linear equality and inequality constraints.
- Formal definition：

$$
\begin{aligned}
&\min c_1x_1 + \ldots + c_nx_n \\
&\text{s. t.} \\
&\qquad a_{11}x_1 + \ldots + a_{1n}x_n \le b_1 \\
&\qquad \ldots \\
&\qquad a_{m1}x_1 + \ldots + a_{mn}x_n \le b_m \\
&\qquad x_1, \ldots, x_n \ge 0
\end{aligned}
$$

Problem definition (linear programming, LP)

- Minimize or maximize a linear cost function subject to a set of linear equality and inequality constraints.
- Formal definition：

objective function

$$
\begin{aligned}
&\min \mathbf{c}^T\mathbf{x} \\
&\text{s.t. } A\mathbf{x} \le \mathbf{b} \\
&\qquad \mathbf{x} \ge \mathbf{0}
\end{aligned}
$$

$\mathbf{c}$: cost vector
A: constraint matrix
$\mathbf{b}$: right-hand-side vector

Problem definition (linear programming, LP)

- **X**: A vector satisfies all of the constraints, called a feasible solution or feasible point

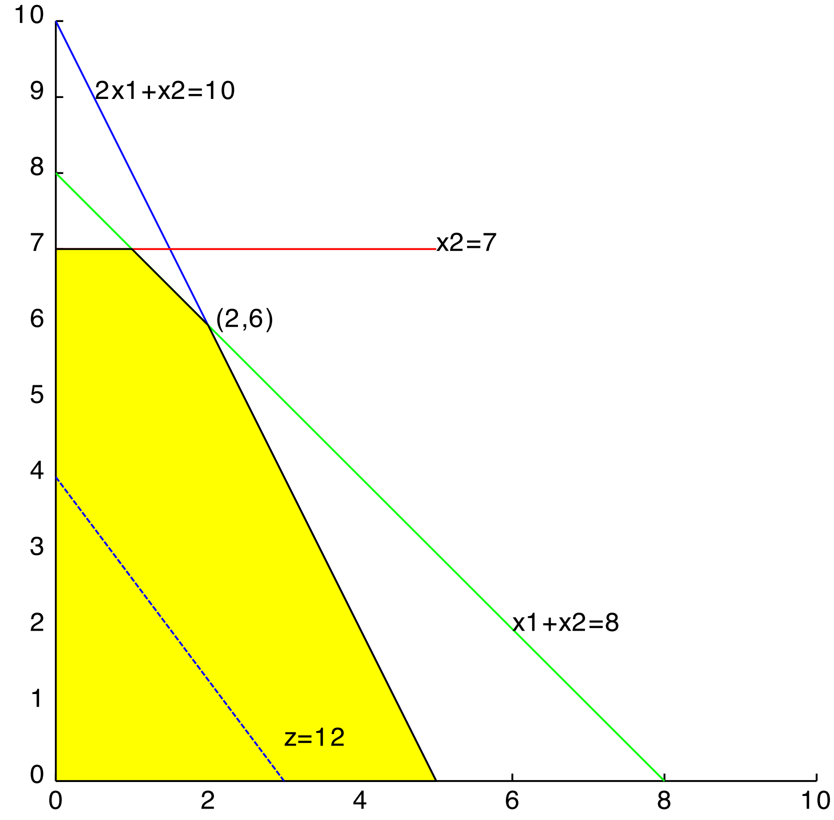
- D: A set of all feasible points, called feasible region

- LP problem：
  - D = $\emptyset$, no solution or infeasible
  - D ≠ $\emptyset$, but the objective finction is boundless in D: boundless
  - D ≠ $\emptyset$, and the objective function has finite optimal solution: has optimal solution

Graphical solution of a linear programming problem

$$\max \quad z = 4x_1 + 3x_2$$

$$\begin{cases} 2x_1 + x_2 \le 10 \\ x_1 + x_2 \le 8 \\ x_2 \le 7 \\ x_1, x_2 \ge 0 \end{cases}$$



Implementation of linear programming in Matlab

- Matlab specify the standard form of linear programming as:

$$\min_{\mathbf{x}} \mathbf{c}^T \mathbf{x} \quad \text{s. t.} \quad \mathbf{A}\mathbf{x} \le \mathbf{b}$$

- Including:
  - $\mathbf{C}$, $\mathbf{x}$ are *n-dimesional* column vectors,
  - $\mathbf{B}$ is a *m-dimensional* column vector,
  - $\mathbf{A}$ is a $m \times n$ matrix.
- http://www.mathworks.cn/cn/help/optim/ug/linprog.html

Implementation of linear programming in matlab

- Baisic functional form :

  linprog(c, A, b)

  - Return the value of vector x
- Example:

  [x,fval]=linprog(c, A, b, Aeq, beq, LB, UB, X0, OPTIONS)

  fval returns the value of the objective function ,

  Aeq, beq correspond to the equality constraints, Aeq X = beq

  LB, UB are lower and upper bounds of the variable x, respectively. X0 is the starting point of X

  OPTIONS is a structure of control parameters.

Implementation of linear programming in Matlab

- Example:

$$\max z = 2x_1 + 3x_2 - 5x_3$$

$$\begin{cases} x_1 + x_2 + x_3 = 7 \\ 2x_1 - 5x_2 + x_3 \geq 10 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

- Matlab code:

```
c=[2,3,-5];
a=[-2,5,-1];
b=-10;
aeq=[1,1,1];
beq=7;
x=linprog(
  -c,a,b,aeq,beq,
  zeros(3,1))
value=c'*x
```

Implementation of linear programming in Python

- CVXOPT：
- http://abel.ee.ucla.edu/cvxopt/index.html

Standard LP problem

- In practice, the LP form we get is quite different:
  - Objective function is a maximization problem or minimization problem
  - Constraints are a set of linear equality and inequality constraints
  - Variables have upper bound or lower bound or no bound

$$
\boxed{\begin{aligned} &\min z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ &\text{s.t. } \mathrm{A}\mathbf{x} = \mathbf{b} \\ &\qquad \mathbf{x} \geq \mathbf{0} \end{aligned}}
$$

Standard form LP problem

$$\begin{aligned} &\min z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ &\text{s.t. } \mathrm{A}\mathbf{x} = \mathbf{b} \\ &\qquad \mathbf{x} \geq \mathbf{0} \end{aligned}$$

- Standardization
  - Objective function transformation $\max z \rightarrow \min(-z)$
  - Constraints transformation (introduce slack variable)

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \Leftrightarrow \begin{cases} \sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i \\ x_{n+i} \geq 0 \end{cases}$$

  - Variables nonnegtive constraints

$$x_j \geq l_j \Leftrightarrow y_j \geq 0, y_j = x_j - l_j$$

$$x_j \text{ free variable} \Leftrightarrow u_j \geq 0, v_j \geq 0, x_j = u_j - v_j$$

Standard form LP problem

$$\begin{aligned} &\min z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ &\text{s.t. } \mathrm{A}\mathbf{x} = \mathbf{b} \\ &\qquad \mathbf{x} \geq \mathbf{0} \end{aligned}$$

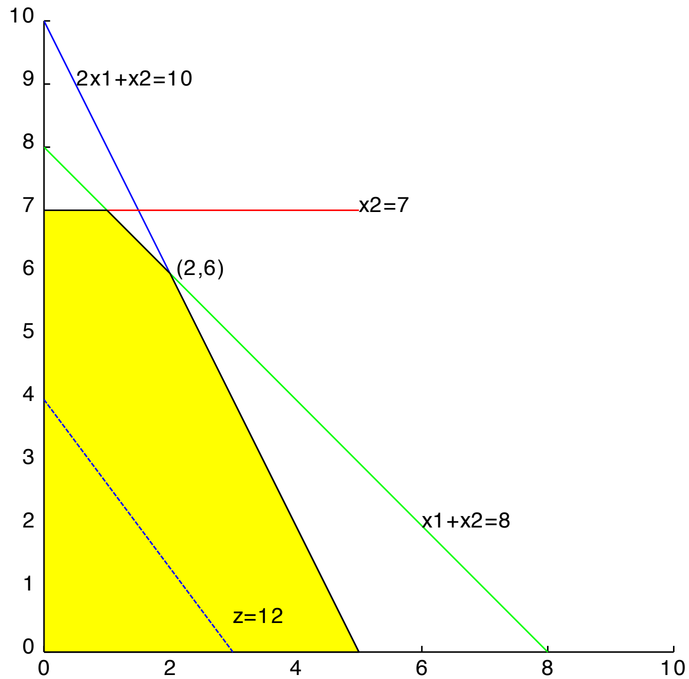
- Standardization
  - Objective function transformation $\min |x_1| + |x_2|$?
  - Variable substitution

$$x_j \text{ free variable} \Leftrightarrow x_j^- \geq 0, x_j^+ \geq 0$$

$$z = \sum_j |x_j| = \sum_j \left(x_j^+ - x_j^-\right)$$

Feasible region D

- Discuss the structure of feasible region $D=\{\mathbf{x} \mid \mathrm{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$
  - First, discuss the set $K=\{\ \mathbf{x} \mid \mathrm{A}\mathbf{x} = \mathbf{b}\ \}$



Affine set

- Affine set: A set $S \subseteq E^n$, for any $\mathbf{x}, \mathbf{y} \in S$, $\lambda \in E^1$, we have $\lambda\mathbf{x}+(1-\lambda)\mathbf{y} \in S$. Then we call S an affine set.
- Example:
  - Both line and plane of E^3 are affine sets, and empty set Ø is also an affine set
  - The sufficient and necessary condition that S is a subset of E^n is that S is an affine set containing the origin
  - Every non-empty affine set S is parallel to an unique subspace L
    - Set shift: $S+\mathbf{p}=\{\ \mathbf{x}+\mathbf{p} \mid \mathbf{x} \in S\ \}$
  - The dimensions of a non-empty affine set S is defined as those of its parallel subspace, dim(S)

Affine set

- Affine sets with dimensions 0,1,2 are point, line, plane, respectively.
- An affine set in E^n with n-1 dimensions is called hyperplane
- Given $b \in E^1$ and nonzero vector $\mathbf{a} \in E^n$, then the set

$$H=\{ \mathbf{x} \mid \mathbf{a}^T\mathbf{x}=b \}$$

is hyperplane in E^n, and any hyperplane in E^n can be expressed as this form.

  - $\mathbf{a}$ is a normal vector of H
  - Fix $\mathbf{a}$, b varies, $\mathbf{a}^T\mathbf{x}=b$ represent a set of hyperplane in E^n

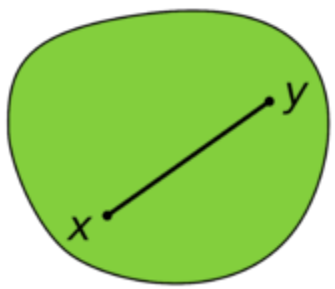
Affine set

- Properties of general affine sets:
  - Given mxn real matrix A and $\mathbf{b} \in E^m$, then $K=\{\mathbf{x} \in E^n \mid A\mathbf{x}=\mathbf{b}\}$ is an affine set in E^n , and any affine set in E^n can be expressed as this form.
- The set $K=\{\ \mathbf{x} \mid A\mathbf{x} = \mathbf{b}\ \}$ is an affine set in E^n
  - If $\mathrm{rank}(A) = m$,
  - then $\dim(K) = n - m$

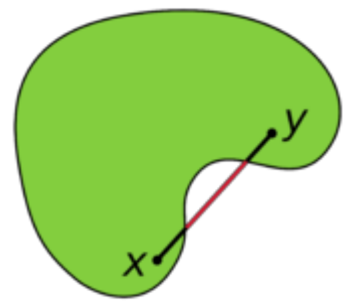
Convex set

- A set C $\subseteq E^n$ is convex if for any $x, y \in C$, and any $\lambda \in (0,1)$, we have

$$\lambda x + (1-\lambda) y \in C,$$



Convex



Not convex

- A polyhedron is equal to the intersection of a finite number of closed halfspaces of E^n, indicated as $\{\mathbf{x} \mid A\mathbf{x} \leq \mathrm{b}\}$
- Feasible region $D=\{\mathbf{x} \mid A\mathbf{x}=\mathbf{b},\ \mathbf{x}\geq 0\}$ is a convex set

Convex set and face

- If non-empty subset C' of convex set C satisfies the following conditions:
  - If **u** $\in C'$, and **u** is a point on a line in C, there exists **x**,**y** $\in C$ to make u $\in$(**x**,**y**),
  - We can infer that **x**,**y** $\in C'$, C' is called a face of C.
  - Apparently, C is also a face.

Face, extreme point and extreme direction

- Theorem: If and only if there exists an index set $Q \subseteq \{1,\ldots,n\}$ to make

$$D' = \{\mathbf{x} \mid A\mathbf{x} = b, \mathbf{x} \geq 0, \text{ when } x_j = 0 \text{ and } j \in Q\}$$

Then D' is a subset and face of feasible region

  - If $\dim(D) = n-m$, D' is a face of D, and $\dim(D') = n-m-k$, then $|Q| \geq k$.

- 1-dimensional face of convex C is called edge, 0-dimensional face is called extreme point. If two extreme points are on the same line, they are said to be adjacent.
- Equivalent definition of extreme point: A vector $\mathbf{x} \in C$ is an extreme point, if $\mathbf{x} = \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$, wherein $\mathbf{x}_1, \mathbf{x}_2 \in C$ and $\lambda \in (0, 1)$, there must be $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}$.
- Intuitively, extreme point is not an interior point on any hyperplane of C.

Extreme direction

- Let set $C\subseteq E^n$ be convex set and $C\neq 0$, $\mathbf{d}\in E^n$, $\mathrm{d}\neq 0$. If for any $\mathbf{x}\in C$, $\lambda>0$, $\mathbf{x}+\lambda\mathbf{d}\in C$, $\mathbf{d}$ is a extreme direction of C.

- If a direction $\mathbf{d}$ can't be expressed as linear combination of two directions, $\mathbf{d}$ is called extreme direction of C.
- The proposition that $\mathbf{d}$ is the direction of $D=\{\mathbf{x} \mid A\mathbf{x}=\mathbf{b}, \mathbf{x}\geq\mathbf{0}\}\neq\varnothing$ is equivalent to:
  - $\mathbf{d}$ satisfies conditions $A\mathbf{d}=\mathbf{0}$, $\mathbf{d}\geq 0$, $\mathbf{d}\neq 0$

Basic Feasible Solution

$$\min z = \mathbf{c}^T\mathbf{x}$$
$$\text{s.t. } A\mathbf{x} = \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

Assume that rank$(A)=m$, then there must be m linearly independent column vectors constituting the full rank matrix B, the remaining columns of A constitute sub-matrix N, so $A=(B, N)$. Accordingly, $\mathbf{x}=(\mathbf{x}_B, \mathbf{x}_N)$, then $A\mathbf{x}=\mathbf{b}$ can be rewritten as

$$A\mathbf{x} = B\mathbf{x}_B+N\mathbf{x}_N=\mathbf{b}.$$

Because B is a full rank matrix, then there exists B^{-1}, so

$$\mathbf{x}_B = B^{-1}\mathbf{b}-B^{-1}N\mathbf{x}_N$$

Choose $\mathbf{x}_N$ arbitrarily, then obtain the corresponding $\mathbf{x}_B$,

$$(\mathbf{x}_B, \mathbf{x}_N)$$

is a solution to $A\mathbf{x}=\mathbf{b}$.

Let $\mathbf{x}_N=0$, then $\mathbf{x}_B=B^{-1}\mathbf{b}$, $\mathbf{x}=(B^{-1}\mathbf{b}, 0)$ is called a basic solution to the constraint equations.

Basic feasible solution

- Suppose that rank(A)=m
  - m linearly independent columns of A constitute full rank matrix B
  - The remaining columns of A constitute sub-matrix N
  - A=(B, N)
- $\boldsymbol{x}=(\boldsymbol{x}_B, \boldsymbol{x}_N)$
  - $A\boldsymbol{x}=\mathbf{b}$ can be rewritten as $B\boldsymbol{x}_B + N\boldsymbol{x}_N = \mathbf{b}$

B is named basis, m linearly independent columns of B are named basic vectors, m components of $\mathbf{x}_B$ are named basic variables, the remaining variables are called nonbasic variable.

A basic solution maybe not be nonnegtive, so it may not feasible. If a basic solution is nonnegtive, then it is a basic feasible solution, and name B feasible basis.

Basic feasible solution

- Theorem:
  - If the feasible solution **x** is a basic feasible solution, the columns corresponding to positive components of **x** are linearly independent.
  - If the feasible solution **x** is a basic feasible solution, **x** is an extreme point of D.

Proof: (sufficiency)

Let x be an extreme point of D, top k components be positive, we are aim to prove that $\mathrm{a}_1,\ldots,\mathrm{a}_k$ are linearly independent.

Reduction to absurdity. If they are linearly dependent, there exists nonzero $\mathrm{y}=(y_1,\ldots,y_k,0,\ldots0)^T$, to make

$$\sum_{j=1}^{k} y_j \mathrm{a}_j = 0.$$

For any positive real number δ, we have

$$\sum_{j=1}^{k}\left(x_j \pm \delta y_j\right)\mathrm{a}_j = \mathrm{b}$$

Then take $\mathrm{x}^1=\mathrm{x}+\delta\mathrm{y}, \mathrm{x}^2=\mathrm{x}-\delta\mathrm{y}$,

It's easy to verify that when δ is small enough, $\mathrm{x}^1,\mathrm{x}^2\in D$.

Because $y\neq 0$, then $\mathrm{x}^1\neq\mathrm{x}^2$, while

$$\mathrm{x}=\frac{1}{2}\mathrm{x}^1+\frac{1}{2}\mathrm{x}^2,$$ this conflicts with that x is an extreme point.

Therefore, $\mathrm{a}_1,\ldots,\mathrm{a}_k$ are linearly independent, x is a basic feasible solution.

- Proof: (necessity)

Let x be a basic feasible solution, top k components be positive. Suppose that there exists $\mathrm{x}^1, \mathrm{x}^2 \in D$ and $\lambda \in (0,1)$, to make

$$\mathrm{x} = \lambda \mathrm{x}^1 + (1-\lambda)\mathrm{x}^2$$

Because when $j \ge k+1$, $x_j = 0$, there is $x_j^1 = x_j^2 = 0$. According to $A\mathrm{x}^1 = A\mathrm{x}^2 = \mathrm{b}$, we obtain that,

$$\sum_{j=1}^{k} \left(x_j^1 - x_j^2\right)\mathrm{a}_j = 0$$

Because x is a basic feasible solution, and $\mathrm{a}_1, \ldots, \mathrm{a}_k$ are linearly independent. Therefore, $\mathrm{x}_j^1 = \mathrm{x}_j^2 (j = 1, \ldots, k)$, so as to $\mathrm{x}^1 = \mathrm{x}^2 = \mathrm{x}$. x is an extreme point.

Degeneracy

- If the basic feasible solution **x**, all its basic variables are positive, then x is non-degenerated; Otherwise, it is degenerated.
- A feasible basis uniquely determines a basic feasible solution; Otherwise, if a basic feasible solution is non-degenerated, it uniquely corresponds to a feasible basis.
- If a basic feasible solution is degenerated, the corres-ponding basis is not unique.
- If all the basic feasible solutions to a LP problem are non-degenerated, then the problem is non-degenerated, otherwise it is degenerated.

Extreme direction of feasible region

- Algebraic properties of extreme direction: The direction **d** of $D=\{\mathbf{x} \mid A\mathbf{x}=\mathbf{b}, \mathbf{x}\geq\mathbf{0}\}$ has k nonzero components, then the sufficient and necessary condition that **d** is the extreme direction of D is that the rank of the columns corresponding to nonzero components in **d** is k-1
- Geometric properties of extreme direction: The sufficient and necessary condition that **d** is an extreme direction of D is that **d** is a direction of a certain half-line face in D.
- If D has extreme directions, D is a boundless set; Conversely, if D is a boundless set, D has directions and they are extreme directions.

Linear programming basic theorem

Basic theorem 1 (Representation theorem)

Theorem: Let all extreme points of $D=\left\{\mathrm{x} \mid A\mathrm{x}=\mathrm{b}, \mathrm{x} \geq 0\right\}$ be $\mathrm{x}^1,\ldots,\mathrm{x}^k$, all extreme directions be $\mathrm{d}^1,\ldots,\mathrm{d}^l$, the sufficient and necessary condition to $x \in D$ is that there exists a set of $\lambda_i (i=1,\ldots,k)$ and $\mu_j (j=1,\ldots,l)$ to make

Extreme point

Extreme direction

$$\mathbf{x}=\sum_{i=1}^{k}\lambda_i \mathbf{x}^i+\sum_{j=1}^{l}\mu_j \mathbf{d}^j,$$

$$\lambda_i \geq 0, i=1,\ldots,k; \mu_j \geq 0, j=1,\ldots,l$$

$$\sum_{i=1}^{k}\lambda_i=1$$

Basic Theorem 2

Given a LP problem

$$\min z = \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

$$\text{s.t. } A\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

(1)If there exists a finite optimal solution, the optimal solution must be an extreme point in the feasible region D.

(2)The sufficient and necessary condition that objective function has optimal solution is that for any extreme direction $\mathbf{d}^j$ of D we have $\mathbf{c}\mathbf{d}^j \geq 0$.

Proof: Let all extreme points in $D = \{ x \mid Ax = b, x \geq 0 \}$ be $x^1, \ldots, x^k$, all extreme directions be $d^1, \ldots, d^l$, according to representation theorem, we know that any $x \in D$ can be expressed as

$$x = \sum_{i=1}^{k} \lambda_i x^i + \sum_{j=1}^{l} \mu_j d^j,$$

$$\lambda_i \geq 0, i = 1, \ldots, k; \quad \mu_j \geq 0, j = 1, \ldots, l$$

$$\sum_{i=1}^{k} \lambda_i = 1$$

So transform the LP problem about x into the one about λ and μ.

$$\min \sum_{i=1}^{k} \left(c^T x^i \right) \lambda_i + \sum_{j=1}^{l} \left(c^T d^j \right) \mu_j$$

$$\text{s.t.} \quad \sum_{i=1}^{k} \lambda_i = 1; \ \lambda_i \geq 0, i = 1, \ldots, k; \quad \mu_j \geq 0, j = 1, \ldots, l$$

Because μ_j can be arbitrarily large, to realize the objective function has lower boundary, there must be

$$c^T d^j \geq 0 \qquad \left(j = 1, \ldots, l \right)$$

or there are no extreme direction.

Let $c^T x^p = \min\{ c^T x^i \mid 1 \leq i \leq k \}$, then we know that the objective function is not less than $c^T x^p$. So let $\lambda_p = 1$ the remaining $\lambda_i = 0$, that is the objective function reaches the minimum. That is to say we have verified that the function reaches the optimal at the extreme point x_p.

Annotation and Interpretation of The Theorem

- Explanation of the theorem:
  - The optimal to the objective function of stand form LP problem must be at a basic feasible solution.
  - To solve stand form LP problem, only need to search in the set of basic feasible solutions.
- The way that work out and compare all the basic feasible solutions is not feasible. And when *n* is large, the number of basic feasible solution is quite large.
- A general method is called the somplex method, in which we only need to search in a subset of basic feasible solutions.

Simplex Method

Basic simplex method

- Main idea: First, find a basic feasible solution, and discriminate whether it is optimal. If not, find a better basic feasible solution and discriminate it again. The method continues until finding the optimal solution or discriminating it is boundless.

- Two major problems:
  - Find initial solution
  - The way to discrimination and iteration (consider first)

Discrimination and iteration

- Suppose that rank$(A) = m<n$, and we have found a non-degenerated feasible basic solution, i.e. found a basis B. Then rewrite $A\mathbf{x}=\mathbf{b}$ as

$$\mathbf{x}_B+B^{-1}N\mathbf{x}_N=B^{-1}\mathbf{b} \qquad (1)$$

Denote $B=(a_1,\ldots,a_m)$

$$\bar{\mathrm{a}}_j = B^{-1}\mathrm{a}_j = (\bar{a}_{1j},\ldots,\bar{a}_{mj})^T, j=1,\ldots,n$$

$$\bar{\mathrm{b}} = B^{-1}\mathrm{b} = (\bar{b}_1,\ldots,\bar{b}_m)^T$$

$$\bar{N} = B^{-1}N$$

Then rewrite formula (1) as $\mathrm{x}_B+\bar{N}\mathrm{x}_N=\bar{\mathrm{b}} \qquad (2)$

Obviously, different bases correspond to different equations.

Formula (2) is called canonical system of equations,

canonical form for short.

Discrimination and iteration

$$\mathbf{x}_B + \bar{N}\mathbf{x}_N = \bar{\mathbf{b}} \qquad (2-2)$$

Obviously, different bases correspond to different equations.

Formula (2) is called canonical system of equations of basis B, canonical form for short.

If $\bar{\mathbf{b}} \geq 0$, then formula (2) corresponds to the feasible basic solution $\bar{\mathbf{x}} = \begin{pmatrix} \bar{\mathbf{b}} \\ 0 \end{pmatrix}$. Transform the objective function,

$$z = \mathbf{c}\mathbf{x} = \mathbf{c}_B\bar{\mathbf{b}} - \left(\mathbf{c}_B\bar{N} - \mathbf{c}_N\right)\mathbf{x}_N = \mathbf{c}_B\bar{\mathbf{b}} - \sum_{j=m+1}^{n}\left(\mathbf{c}_B\bar{\mathbf{a}}_j - c_j\right)\mathbf{x}_j \qquad (3)$$

Subsitute z_0 for the objective function value at $\bar{\mathbf{x}}$,

then $z_0 = \mathbf{c}\bar{\mathbf{x}} = \mathbf{c}_B\bar{\mathbf{b}}$ （i.e. the constant term of the equation above）

Discrimination and iteration

$\bar{\mathbf{a}}_1, \ldots \bar{\mathbf{a}}_m$ are linearly independent,

so when $j = 1, \ldots, m$

$\mathbf{c}_B \bar{\mathbf{a}}_j - c_j = 0.$

Introduce the notation

$\zeta_j = \mathbf{c}_B \bar{\mathbf{a}}_j - c_j, \ j = 1, \ldots, n$

or in the form of vectors

$\zeta = \mathbf{c}_B B^{-1} A - \mathbf{c} = (\boldsymbol{\zeta}_B, \boldsymbol{\zeta}_N) = (\mathbf{0}, \mathbf{c}_B B^{-1} N - \mathbf{c}_N)$

Then rewrite equation (3) as $z = \mathbf{c}_B \bar{\mathbf{b}} - \boldsymbol{\zeta}\mathbf{x}.$

Discrimination and iteration

$$z = \mathbf{c}_B \bar{\mathbf{b}} - \zeta \mathbf{x}, \quad \zeta = \mathbf{c}_B B^{-1} A - \mathbf{c} = (\zeta_B, \zeta_N) = (\mathbf{0}, \mathbf{c}_B B^{-1} N - \mathbf{c}_N)$$

After variable transformation, describe LP problem as

$$\min z = z_0 - \zeta \mathbf{x} \qquad (4)$$

$$s.t. \quad \mathbf{x}_B + B^{-1} N \mathbf{x}_N = \bar{\mathbf{b}}, \qquad (5)$$

$$\mathbf{x} \geq \mathbf{0}.$$

Theorem 2.1: If ζ in formula(4) satisfies $\zeta \leq 0$, $\bar{\mathbf{x}}$ is the optimal.

Theorem 2.2: If there exists a component in ζ $\zeta_k > 0$, and $\bar{\mathbf{a}}_k \leq \mathbf{0}$, then there is no solution to the primal problem.

Theorem 2.3: If there is $\zeta_k > 0$ in formula(4), and there is at least one positive compnent in $\bar{\mathbf{a}}_k$,

then we can find the basic feasible solution $\hat{\mathbf{x}}$ to make $\hat{\mathbf{x}} < \mathbf{c}\bar{\mathbf{x}}$。

Discrimination and iteration

Construct(a better basic feasible solution)$\hat{\mathbf{x}}$. Let

$$\theta = \min\left\{ \frac{\bar{b}_i}{\bar{a}_{ik}} \middle| \bar{a}_{ik} > 0, i = 1, \dots, m \right\} = \frac{\bar{b}_r}{\bar{a}_{rk}} > 0$$

$$\hat{\mathbf{x}} = \bar{\mathbf{x}} + \theta\left(\begin{pmatrix} -\bar{\mathbf{a}}_k \\ 0 \end{pmatrix} + \mathbf{e}_k \right) = \begin{pmatrix} \bar{\mathbf{b}} - \theta\bar{\mathbf{a}}_k \\ 0 \end{pmatrix} + \theta\mathbf{e}_k .$$

The following will prove $\hat{\mathbf{x}}$ is a feasible solution. Plug in the formula(5),

$$\hat{\mathbf{x}}_B + \bar{N}\hat{\mathbf{x}}_N = \bar{\mathbf{b}} - \theta\bar{\mathbf{a}}_k + \sum_{j=m+1}^{n} \hat{x}_j \bar{\mathbf{a}}_j = \bar{\mathbf{b}} - \theta\bar{\mathbf{a}}_k + \theta\bar{\mathbf{a}}_k = \bar{\mathbf{b}}.(\text{Notice } \hat{x}_N = (0, \dots, \theta, 0..0))$$

Due to θ construction, it's clear that $\hat{x} \geq 0$. So $\hat{x}$ is a feasible solution.

Discrimination and iteration

The following will prove $\hat{\mathrm{x}}$ is a basic solution. Each component of $\hat{\mathbf{x}}$ is

$$\hat{x}_i = \bar{b}_i - \left(\bar{b}_r / \bar{a}_{rk}\right)\bar{a}_{ik}, i = 1, \ldots, m$$

$$\hat{x}_k = \bar{b}_k / \bar{a}_{rk}$$

$$\hat{x}_i = 0, j = m+1, \ldots, n, j \neq k.$$

Because of $\hat{x}_r = 0$, while $\hat{x}_k > 0$,

So only need to prove $a_1, \ldots, a_{r-1}, a_k, a_{r+1}, \ldots, a_m$ are linearly independent.

Proof: $a_1, \ldots, a_{r-1}, a_k, a_{r+1}, \ldots, a_m$ are linearly independent

Proof by contradiction. If they are linearly depenent, because of $\{\mathbf{a}_1, \ldots \mathbf{a}_m\}$ linearly dependent, $\mathbf{a}_k$ must be a linear combination of the other $m-1$ vectors.

i.e. $\mathbf{a}_k = \sum_{i=1, i \neq r}^{m} y_i \mathbf{a}_i$

Additionally, $\bar{\mathbf{a}}_k = B^{-1}\mathbf{a}_k$, so as to $\mathbf{a}_k = B\bar{\mathbf{a}}_k = \sum_{i=1}^{m} \bar{a}_{ik}\mathbf{a}_i$

Two equations above subtracting to yeild

$$\bar{a}_{rk}\mathbf{a}_r + \sum_{i=1, i \neq r}^{m} (\bar{a}_{ik} - y_i)\mathbf{a}_i = \mathbf{0}$$

Linearly dependent

Because of $\bar{a}_{rk} \neq 0$, $\mathbf{a}_1, \ldots \mathbf{a}_m$ are linearly independent, they're contradictory.

Finally,

$$z(\hat{\mathbf{x}}) = z_0 - \boldsymbol{\zeta}\hat{\mathbf{x}} = z_0 - \zeta_k \theta < z_0.$$

Simplex method procedure

ζ is called test vector. In iteration process, if ζ has more than one positive component, then choose colum $\mathbf{a}_k$ correspongding to the maximum component ζ_k to enter the basis , making objective function descend more quickly .

1. Find an initial feasible basis;
2. Compute the corresponding canonical form;
3. Compute $\zeta_k = \max\{\zeta_j \mid j = 1, \ldots, n\}$
4. If $\zeta_k \le 0$, terminate. The optimum solution $\mathbf{x} = \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{b}} \\ \mathbf{0} \end{pmatrix}$

   and optimun value $z = \mathbf{c}_B \bar{\mathbf{b}}$ have been found.
5. If $\bar{\mathbf{a}}_k \le \mathbf{0}$, terminate, primal problem is boundless.
6. Compute the minimum ratio $\dfrac{\bar{b}_r}{\bar{a}_{rk}} = \min\left\{ \dfrac{\bar{b}_i}{\bar{a}_{ik}} \middle| \bar{a}_{ik} > 0 \right\}$
7. Replace $\mathbf{a}_{Br}$ with $\mathbf{a}_k$ to obtain a new basis, go to step 2.

Simplex tableau

According to the way to construct θ and $\hat{x}$, we obtain the modified basic feasible solution, i.e. transform the nonbasic variable x_k into positive basic varialble.

Meanwhile, make the basic variable x_r zero(turn into nonbasic variable).

Equivalently, basis $B = \left[a_1, \ldots, a_m\right]$ turns into another basis $\hat{B} = \left[a_1, \ldots, a_r, a_k, a_{r+1}, \ldots a_m\right]$.

*Basis transform

*x_k is called entering basis variable, x_r is exiting basis variable.

With basis B, the coefficient of canonical form augmented matrix is,

$$
\begin{array}{cccccccccc}
x_1 & \cdots & x_r & \cdots & x_m & x_{m+1} & \cdots & x_k & \cdots & x_n & \\
1 & \cdots & 0 & \cdots & 0 & \bar{a}_{1,m+1} & \cdots & \bar{a}_{1,k} & \cdots & \bar{a}_{1,n} & \bar{b}_1 \\
\vdots & \ddots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & \cdots & 1 & \cdots & 0 & \bar{a}_{r,m+1} & \cdots & \bar{a}_{r,k} & \cdots & \bar{a}_{r,n} & \bar{b}_r \\
\vdots & & \vdots & \ddots & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & 1 & \bar{a}_{m,m+1} & \cdots & \bar{a}_{m,k} & \cdots & \bar{a}_{m,n} & \bar{b}_m
\end{array}
$$

Exchange x_r and x_k.

Transform the kth column into unit vector $\mathbf{e}_r$ by elementary transformation.

1) Divide the rth row $\bar{a}_r^T$ by $\bar{a}_{rk}$, $\hat{a}_r^T = \bar{a}_r^T / \bar{a}_{rk}$

2) Subtract the new rth row $\hat{a}_r^T$ mutiplying with $\bar{a}_{ik}$ from the ith row$(i \neq r)$ $\bar{a}_i^T$, $\hat{a}_r^T = \bar{a}_i^T - \hat{a}_r^T \bar{a}_{ik}$

The last column is the basic feasible solution corresponding to basis $\hat{B}$

$$\hat{b}_r = \bar{b}_r / \bar{a}_{rk}, \hat{b}_i = \bar{b}_i - (\bar{b}_r / \bar{a}_{rk})\bar{a}_{ik}, i \neq r$$

When changing basis, change B into $\hat{B}$ in the objective function $z = \mathrm{c}_B\bar{\mathrm{b}} - \left(\mathrm{c}_B\bar{N} - \mathrm{c}_N\right)\mathrm{x}_N$.

Regard the following equations(equivalent form of the above equation)

$$z + \left(\mathrm{c}_B B^{-1}N - \mathrm{c}_N\right)\mathrm{x}_N = \mathrm{c}_B\bar{\mathrm{b}}, \text{ or } z + \zeta_N \mathrm{x}_N = z_0$$

as an equation, make elementary transformation both to this equation and canonical form.

$$\begin{array}{c|cccccccccccc}
 & z & x_1 & \cdots & x_r & \cdots & x_m & x_{m+1} & \cdots & x_k & \cdots & x_n & RHS \\
z & 1 & 0 & \cdots & 0 & \cdots & 0 & \zeta_{m+1} & \cdots & \zeta_k & \cdots & \zeta_n & z_0 \\
x_1 & 0 & 1 & \cdots & 0 & \cdots & 0 & \bar{a}_{1,m+1} & \cdots & \bar{a}_{1,k} & \cdots & \bar{a}_{1,n} & \bar{b}_1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
x_r & 0 & 0 & \cdots & 1 & \cdots & 0 & \bar{a}_{r,m+1} & \cdots & \bar{a}_{r,k} & \cdots & \bar{a}_{r,n} & \bar{b}_r \\
\vdots & \vdots & \vdots & & \vdots & \ddots & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
x_m & 0 & 0 & \cdots & 0 & \cdots & 1 & \bar{a}_{m,m+1} & \cdots & \bar{a}_{m,k} & \cdots & \bar{a}_{m,n} & \bar{b}_m
\end{array}$$

Simplex tableau

Pivot column

Pivot element

Pivot row

Denote simplex tableau as

$$\begin{array}{c|cccc}
 & z & \mathrm{x}_B & \mathrm{x}_N & RHS \\
z & 1 & 0 & \zeta_N & z_0 \\
\mathrm{x}_B & 0 & I & \bar{N} & \bar{\mathrm{b}}
\end{array}$$

The above transformation is called pivot, similar to principal component elemination method to solve linear eqations .

Initial solution: Two-phase simplex method

Let the original problem be

$$\begin{aligned} &\min \mathbf{cx} \\ &s.t.\ \ A\mathbf{x} = \mathbf{b}\ (\mathbf{b} \geq \mathbf{0}) \\ &\qquad \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Introduce m artificial variables $\mathbf{x}_a = \left(x_{n+1}, \ldots, x_{n+m}\right)^T$,

consider the following auxiliary problem

$$\begin{aligned} &\min g = \mathbf{1x}_a \\ &s.t.\ \ A\mathbf{x} + \mathbf{x}_a = \mathbf{b} \\ &\qquad \mathbf{x}, \mathbf{x}_a \geq 0 \end{aligned}$$

wherein $\mathbf{1} = (1, \ldots, 1)$. Let D and D' respectively be the feasible region of the primal problem and the auxiliary problem,

Obviously, $\mathbf{x} \in D \Leftrightarrow \begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix} \in D'$. And $\begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix} \in D' \Leftrightarrow \min g = 0.$

Initial solution: Two-phase simplex method

m artificial variables are $\mathbf{x}_a = \left(x_{n+1}, \dots, x_{n+m}\right)^T$, consider the following auxiliary problem:

$$\begin{aligned} &\min g = \mathbf{1}\mathbf{x}_a \\ &s.t. \quad A\mathbf{x} + \mathbf{x}_a = \mathbf{b} \\ &\qquad\quad \mathbf{x}, \mathbf{x}_a \geq 0 \end{aligned}$$

$\begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix} \in D' \Leftrightarrow \min g = 0.$

$\begin{pmatrix} \mathbf{x} \\ \mathbf{x}_a \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{b} \end{pmatrix}$ is a basic feasible solution to the auxiliary problem,

Then we can solve the problem iteratively by simplex method, there are two possible results :

1) $\min g > 0$. Note that $D = \varnothing$.

2) $\min g = 0$. It's quite natural that $\mathbf{x}_a = 0$, we get a feasible solution to the primal problem after removing it.

Initial Solution: Two-phase simplex method

m artificial vatiables $\mathbf{x}_a = \left(x_{n+1}, \ldots, x_{n+m}\right)^T$, we obtain the auxiliary problem

$$\begin{aligned} &\min g = \mathbf{1x}_a \\ &s.t. \quad A\mathbf{x} + \mathbf{x}_a = \mathbf{b} \\ &\qquad \mathbf{x}, \mathbf{x}_a \geq 0 \end{aligned}$$

$$\begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix} \in D' \Leftrightarrow \min g = 0.$$

* If all the artificial variables are nonbasic variables, the solution is a basic feasible solution.
* Otherwise, eliminate artificial variables from basic variables by pivot.

Let basic varable x_r be an artficial variable,

▹ Transform with top n nonzero elements in the rth row as pivot element , then non-artificial variable x_s enters the basis as well as eliminate x_r from basic variables.

▹ If top n elements in the rth row are zero, eliminate this row and its correspongding atificial variables directly.

Initial Solution: The Big-M method

Let the primal problem be

$$\begin{aligned} &\min \mathrm{cx} \\ &s.t.\ \ A\mathrm{x} = \mathrm{b}\ (\mathrm{b} \geq 0) \\ &\qquad \mathrm{x} \geq 0 \end{aligned}$$

Introduce $\mathrm{x}_a \in E^m$ and a large positive constant M, we obtain the new problem as follows:

$$\begin{aligned} &\min \mathrm{cx} + M\mathbf{1}\mathrm{x}_a \\ &s.t.\ \ A\mathrm{x} + \mathrm{x}_a = \mathrm{b} \\ &\qquad \mathrm{x}, \mathrm{x}_a \geq 0 \end{aligned}$$

Wherein $\mathbf{1} = (1,\ldots,1)$. When M is large enough,

x is the optimal solution to the primal problem $\Leftrightarrow \begin{pmatrix} \mathrm{x} \\ 0 \end{pmatrix}$ is the optimal solution to the new peoblem.

And the new problem has initial solution $\begin{pmatrix} 0 \\ \mathrm{b} \end{pmatrix}$, so it can be solved by the simplex method.

Degeneracy and anticycling rule

- If there exists zero component among the vector right to simplex tableau, i.e. the solution is degenerate, the algorithm may loop indefinitely. We add some pivoting rules to avoid cycling.
  - Dictionary ordered method
  - Bland criteria

Dictionary ordered method

Nonzero vector x, and its first nonzero component is nonnegtive, called dictionary sequence nonnegative, denoted as $\mathrm{x} \succ= 0$.

Vectors x, y, if $\mathrm{x} - \mathrm{y} \succ= 0$, x is greater than or equal to y according to dictionary sequence.

If a set of x^{i}, there exists a x^{t}, for all x^{i} satisfy $\mathrm{x}^{i} \succ= \mathrm{x}^{t}$, then say x^{t} is the minimum one among these vectors according to dictionary sequence.

Denoted as $\mathrm{x}^{t} = lex \min \mathrm{x}^{i}$.

The so-called dictionary ordered method is as follows:

After the choice of entering variable x_k, let

$$\frac{\mathrm{p}_r}{\bar{a}_{rk}} = lex \min \left\{ \frac{\mathrm{p}_i}{\bar{a}_{ik}} \middle| \bar{a}_{ik} > 0, i = 1, \ldots, m \right\},$$

wherein p_i is the ith row of $\left(\bar{\mathrm{b}}, B^{-1}\right)$.

Bland criteria

1) Choose nonbasic variable x_k, which corresponds to the positive test number ζ_k with the minmum subscript, as entering basis variable.

2) Determine exiting variable x_l: If there are several $\dfrac{b_r}{\bar{a}_{rk}}$ reaching minimum, choose the basic variable with the minimum subscript as exiting basis variable. i.e.

$$l = \min\left\{ r \,\middle|\, \frac{b_r}{\bar{a}_{rk}} = \min\left\{ \frac{b_i}{\bar{a}_{ik}} \,\middle|\, \bar{a}_{ik} > 0 \right\} \right\}$$

- The advantage of Bland rules is simple and practicable, but it only cares about the minimal subscript, without considering the decline speed of objective function. So its efficiency is lower than dictionary ordered method and simplex method.
- In practice, degeneration is usual, but degeneration doesn’t bring about loop necessarily. In fact, loop phenomenon is unusual.

Revised simplex method

- When a LP problem is quite large in dimension, we must consider how to reduce the storage and computational burden. In practice, we usually use revised simplex method.

Inverse matrix method

General simplex tableau is as follows :

$$\begin{bmatrix} & z & \mathrm{x}_B & \mathrm{x}_N & RHS \\ z & 1 & 0 & \mathrm{c}_B B^{-1}N-\mathrm{c}_N & \mathrm{c}_B B^{-1}\mathrm{b} \\ \mathrm{x}_B & 0 & I & B^{-1}N & B^{-1}\mathrm{b} \end{bmatrix}$$

Inverse matrix method only store these data as follows in each simplex tableau:

$$\begin{bmatrix} \mathrm{w} & z_o \\ B^{-1} & \overline{\mathrm{b}} \end{bmatrix} = \begin{bmatrix} \mathrm{c}_B B^{-1} & \mathrm{c}_B B^{-1}\mathrm{b} \\ B^{-1} & B^{-1}\mathrm{b} \end{bmatrix}$$, which is called revised simplex tableau.

We compute the original data iteratively according to this tableau. Determine Take x_k as entering variable according to $\zeta_k = \max\left\{\mathrm{wa}_j - c_j \mid x_j \text{ nonbasic variable}\right\}$

Compute $\overline{\mathrm{a}}_k = B^{-1}\mathrm{a}_k$, determine the exiting variable $\mathrm{x_r}$ by the smallest ratio,then obtain new basis $\hat{B}$, finally construct the revised simplex tableau corresponding to $\hat{\mathrm{B}}$.

If we must construct B^{-1} at each iteration, the calculation is very large.

Theorem: add a column $\begin{pmatrix} \zeta_k \\ \overline{\mathrm{a}}_k \end{pmatrix}$ right to the revised simplex tableau corresponding to B, pivot with $\overline{\mathrm{a}}_{rk}$ as pivot element, then we obtain the revised simplex tableau corresponding to basis $\hat{B}$.

Optimality Conditions and Duality Theory

Duality

- Given a linear programming problem P, we can associate it with another(dual) linear programming D. People can solve the primal problem P by solving the dual problem D.

Karush-Kuhn-Tucker condition

- Theorem

Let A a $m \times n$ matrix, $\mathrm{b} \in E^m, \mathrm{c} \in E^n, \mathrm{x} \in E^n$, LP problem:

$$\begin{aligned} &\min \mathrm{c}^{\mathrm{T}}\mathrm{x} \\ &s.t. \quad A\mathrm{x} \geq \mathrm{b} \\ &\qquad\quad \mathrm{x} \geq 0 \end{aligned}$$

x^* is the optimal solution, which its sufficient and necessary condition is that there exists $\mathrm{w} \in E^m, \mathrm{v} \in E^n$ making the following KKT conditions be satisfied

$$\begin{aligned} &A\mathrm{x}^* \geq \mathrm{b}, \mathrm{x}^* \geq 0 \\ &\mathrm{c} - \mathrm{w}^{\mathrm{T}}A - \mathrm{v} = 0, \mathrm{w} \geq 0, \mathrm{v} \geq 0 \\ &\mathrm{w}^{\mathrm{T}}(A\mathrm{x}^* - \mathrm{b}) = 0, \mathrm{v}^{\mathrm{T}}\mathrm{x}^* = 0 \end{aligned}$$

If v is omitted(by the equation c-w$^{\mathrm{T}}$A-v=0）, we obtian another form of KTT condition

$$\begin{aligned} &A\mathrm{x}^* \geq \mathrm{b}, \quad \mathrm{x}^* \geq 0 \\ &\mathrm{w}A \leq \mathrm{c}, \quad \mathrm{w} \geq 0 \\ &\mathrm{w}(A\mathrm{x}^* - \mathrm{b}) = 0, \quad (\mathrm{w}A - \mathrm{c})\mathrm{x}^* = 0 \end{aligned}$$

Kuhn Tucker condition

The $K-T$ condition for the standard form LP problem

As long as rewrite $A\mathrm{x} = \mathrm{b}$ as $\begin{pmatrix} A \\ -A \end{pmatrix} \mathrm{x} \geq \begin{pmatrix} \mathrm{b} \\ -\mathrm{b} \end{pmatrix}$, then we can infer the corresponding $K-T$ condition

$$A\mathrm{x} = \mathrm{b}, \mathrm{x} \geq 0$$

$$\mathrm{c} - \mathrm{w}A - \mathrm{v} = 0, \mathrm{v} \geq 0$$

$$\mathrm{vx} = 0$$

or the equivalent form

$$A\mathrm{x} = \mathrm{b}, \mathrm{x} \geq 0$$

$$\mathrm{w}A \leq \mathrm{c}$$

$$(\mathrm{w}A - \mathrm{c})\mathrm{x} = 0.$$

Kuhn Tucker condition

The sufficient and necessary condition that for a feasible basic solution $\begin{pmatrix} \mathrm{x}_B \\ \mathrm{x}_N \end{pmatrix} = \begin{pmatrix} \bar{\mathrm{b}} \\ 0 \end{pmatrix} (\bar{\mathrm{b}} > 0)$

is optimal is that the test vector $\zeta \le 0$, and what's the relation with $K-T$ condition?

Because of $\mathrm{vx} = \mathrm{v}_B \mathrm{x}_B + \mathrm{v}_N \mathrm{x}_N = 0$, and $\mathrm{x}_B > 0$, we infer that $\mathrm{v}_B = 0$.

Condition $\mathrm{c} - \mathrm{w}A - \mathrm{v} = 0$ can be rewritten as $(\mathrm{c}_B, \mathrm{c}_N) - \mathrm{w}(B, N) - (\mathrm{v}_B, \mathrm{v}_N) = (0, 0)$, so as to

$$\begin{cases} \mathrm{c}_B - \mathrm{w}B = 0 \\ \mathrm{c}_N - \mathrm{w}N - \mathrm{v}_N = 0 \end{cases} \Rightarrow \begin{cases} \mathrm{w} = \mathrm{c}_B B^{-1} \\ \mathrm{v}_N = \mathrm{c}_N - \mathrm{w}N = \mathrm{c}_N - \mathrm{c}_B B^{-1} N \end{cases}$$

i.e. given a basic feasible solution x, if we let

$\mathrm{v}_B = 0, \mathrm{w} = \mathrm{c}_B B^{-1}, \mathrm{v}_N = \mathrm{c}_N - \mathrm{w}N = \mathrm{c}_N - \mathrm{c}_B B^{-1} N,$

except $\mathrm{v} \ge 0$, the rest $K-T$ conditions are satisfied.

And according to the choise of v, it's easy to know $v \ge 0 \Leftrightarrow \zeta \le 0$.

Duality theorem

- Regard the dual problem as transposition of the primal problem

$$\boxed{\min \quad \mathbf{c}^T\mathbf{x} \qquad \text{s.t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0}$$

$$\min \quad c^T x \qquad \text{s.t.} \quad \begin{bmatrix} A \\ -A \end{bmatrix} x \geq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0$$

- The dual problem is:

$$\max \quad \begin{bmatrix} b^T & -b^T \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \text{s.t.} \quad \begin{bmatrix} A^T & -A^T \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \leq c$$

- Wherein y_1 and y_2 are dual variables corresponding to constraints Ax>=b and -Ax<=B, respectively. Let $y=y_1-y_2$

$$\boxed{\max \quad \mathbf{b}^T\mathbf{y} \qquad \text{s.t.} \qquad \mathbf{A}^T\mathbf{y} \leq \mathbf{c}}$$

sufficient and necessary condition

Duality theorem

- Let **x** and **w** be the feasible solution to (P) and (D), respectively. So as to **cx**≥**wb**.
- Let **x*** and **w*** be the feasible solution to (P) and (D), respectively. So as to the sufficient and necessary condition is **cx***=**w*****b**, for which **x*** and **w*** are optimal to (P) and (D), respectively.
- Corollary: If (P) has an optimal solution **x***, so does its dual (D), optimal solution **w***, and **cx***=**w*****b**; If (P) is boundless, (D) has no solution and vice versa.
- Complementary slackness:Let **x*** and **w*** be optimal solution to the primal and the dual problem, respectively. So as to **w***(A**x***-**b**)=**0**, (**c**-**w***A)**x***=**0**。

$K-T$ conditions:

$$A\mathrm{x}^* \geq \mathrm{b},\ \ \mathrm{x}^* \geq 0 \qquad \text{(primal feasibility)}$$

$$\mathrm{w}A \leq \mathrm{c}, \mathrm{w} \geq 0 \qquad \text{(dual feasibility)}$$

$$\mathrm{w}(A\mathrm{x}^* - \mathrm{b}) = 0,\ \ (\mathrm{w}A - \mathrm{c})\mathrm{x}^* = 0 \qquad \text{(complementary slackness)}$$

Duality theorem

- The relation between K-T conditions and the optimal criterion of simplex condition: Simplex method satisfies all K-T condition except for condition $\mathbf{v}\geq\mathbf{0}$. i.e. simplex method keeps primal feasibility and complementary slackness, and modifies the feasible basic solution under the condition $\mathbf{c}-\mathbf{w}A-\mathbf{v}=\mathbf{0}$, to make $\mathbf{v}=\mathbf{c}-\mathbf{w}A\geq\mathbf{0}$ in dual feasibility be satisfied.
- Inspiration: When dual feasibility and complementary slackness are satisfied, we can modify the solution to satisfy primal feasibility.

Dual simplex method

$$
\begin{array}{ll}
(P)\ \min \mathrm{cx} & (D)\ \max \mathrm{wb} \\
s.t. A\mathrm{x} = b & s.t.\ \ \mathrm{w}A \le \mathrm{c} \\
\mathrm{x} \ge 0 &
\end{array}
$$

The basic feasible solution to D: Let $A=(B, N)$, wherein B is a full rank matrix, then the solution $\bar{\mathrm{w}} = \mathrm{c}_B B^{-1}$ to $\mathrm{w}B = \mathrm{c}_B$ is a basic solution to (D),

If $\bar{\mathrm{w}}N \le \mathrm{c}_N$, $\bar{\mathrm{w}}$ is the basic feasible solution to (D).

The regular solution to P: If the test vector ζ corresponding to the basic solution $\mathrm{x} = \begin{pmatrix} B^{-1}\mathrm{b} \\ 0 \end{pmatrix}$ to the primal(P) satisfies $\zeta = (0, \mathrm{c}_B B^{-1}N - \mathrm{c}_N) \le 0$, x is called regular solution to problem (P).

Basis B is called ragular basis. It's easy to verify that the basic feasible solutions to D and the regular solutions to P are one-to-one relationship. The same as simplex method, to solve the dual programming(D), the method moves from one basic feasible solution to another to increase the objective function value. Equivalently, to solve the primal programming(P), the method moves one regular solution to another to increase the objective function value $z = \mathrm{wb} = \mathrm{c}_B B^{-1}\mathrm{b} = \mathrm{cx}$, until $B^{-1}\mathrm{b} \ge 0$. i.e. when regular solution satisfies primal feasibility, the optimal solution is found. We call this method dual simplex method.

Dual simplex method

$$(P)\ \min \mathbf{cx} \qquad\qquad (D)\ \max \mathbf{wb}$$
$$s.t. A\mathbf{x} = b \qquad\qquad s.t.\ \ \mathbf{w}A \le \mathbf{c}$$
$$\mathbf{x} \ge 0$$

1) Find a normal basis B, construct its corresponding simplex tableau.

2) If $\overline{\mathrm{b}} = B^{-1}\mathrm{b} \ge 0$, terminate, the primal optimal solution has been found;

Otherwise, let $\overline{b}_r = \min\{\overline{b}_i \mid i = 1, \ldots, m\}$.

3) If $\overline{\mathrm{a}}^r \ge 0$, terminate, the primal has no optimal solution;

Otherwise, let $\dfrac{\zeta_k}{\overline{a}_{rk}} = \min\{\dfrac{\zeta_j}{\overline{a}_{rj}} \mid a_{rj} < 0\}$

4) Pivot with pivot element $\overline{a}_{rk}$, go to 2.

Primal-Dual simplex method

- Solve both the dual problem and the auxiliary problem at the first phase. Start with any feasible solution to the dual. In iterations, dual feasibility, complementary slackness and $x \geq 0$ are maintained. In this case, elemiate artificial variables to satisfy $Ax=b$.

$(P)\ \min \mathrm{cx}$ $(D)\ \max \mathrm{wb}$

$s.t.\ \ A\mathrm{x} = b$ $s.t.\ \ \mathrm{w}A \le \mathrm{c}$

$\mathrm{x} \ge 0$

After introducing artificial vatiables in (P), we obtain the auxiliary problem

$$\min g = 1\mathrm{x}_a$$

$$s.t\ \ A\mathrm{x} + \mathrm{x}_a = \mathrm{b} \qquad (\mathrm{b} \ge 0)$$

$$\mathrm{x}, \mathrm{x}_a \ge 0$$

If a feasible solution $\bar{\mathrm{w}}$ to (D) is known, to maintain complementary slackness $(\mathrm{x}(\bar{\mathrm{w}}A - \mathrm{c}) = 0)$, let $x_j = 0$（when $\bar{\mathrm{w}}\mathrm{a}_j \ne c_j$), so that the auxiliary problem is

$$(P')\ \min g = 1\mathrm{x}_a$$

$$s.t\ \ A\mathrm{x} + \mathrm{x}_a = \mathrm{b}$$

$$x_j = 0,\ j \notin Q$$

$$x_j \ge 0,\ j \in Q$$

$$\mathrm{x}_a \ge 0$$

Wherein $Q = \{j \mid \bar{\mathrm{w}}\mathrm{a}_j = c_j, j = 1,\ldots,n\}$. Problem (P') is called restrict problem corresponding to $\bar{\mathrm{w}}$.

Solve (P'), obtain the optimal solution $\begin{pmatrix} \mathrm{x}^* \\ \mathrm{x}_a^* \end{pmatrix}$, it's a basic feasible solution to auxiliary problem of (P).

If $\mathrm{x}_a^* = 0$, x^* is the optimal solution to (P)(Because x^* and $\bar{\mathrm{w}}$ are feasible solutions to (P) and (D), respectively. And complementary slackness is satisfied).

If $\mathrm{x}_a^* \ne 0$, find another basic feasible solution $\hat{\mathrm{w}}$ to (D) to increase the objetive function value of (D). Meanwhile, the optimal value corresponding to the limit problem of $\hat{\mathrm{w}}$ has relative reduction to that of $\bar{\mathrm{w}}$(equivalent to eliminate artificial varialbles).

Consider the dual problem (D') to (P')

(D') $\max \mathrm{vb}$

$s.t.$ $\mathrm{va}_j \le 0, j \in Q$

$\mathrm{v} \le 1$

Let v^* be its optimal solution. If $\mathrm{v}^*\mathrm{a}_j \le 0$ for any $j = 1,\dots,n$, v^* is still the optimal solution to the dual of auxiliary problem.

So $\begin{pmatrix} \mathrm{x}^* \\ \mathrm{x}_a^* \end{pmatrix}$ is the optimal solution to the auxiliary problem. Because of $\mathrm{x}_a^* \ne 0$, the primal (P) has no optimal solution.

Construct $\hat{\mathrm{w}} = \bar{\mathrm{w}} + \theta \mathrm{v}^*$, wherein $\theta = \min\left\{ -\dfrac{\bar{\mathrm{w}}\mathrm{a}_j - c_j}{\mathrm{v}^*\mathrm{a}_j} \middle| \mathrm{v}^*\mathrm{a}_j > 0, j = 1,\dots,n \right\}$.

We can prove that $\hat{\mathrm{w}}$ saticfies conditions.Meanwahile, $\begin{pmatrix} \mathrm{x}^* \\ \mathrm{x}_a^* \end{pmatrix}$ is a basic feasible solution to the limit problem corresponding to $\hat{\mathrm{w}}$, so take it as an initial solution to sovle the limit problem corresponding to $\hat{\mathrm{w}}$. Loop repeats.

1) Convert the programming problem into the form

$$\begin{aligned} &\min \mathrm{cx} \\ &s.t.\ A\mathrm{x} = b \\ &\quad\ \mathrm{x} \geq 0 \end{aligned}$$

Find its dual initial solution w, which saticfies w$A \leq$ c.(When c ≥ 0, choose w$=$0 as the initial solution)

2)Let $Q = \{j \mid \bar{\mathrm{w}}\mathrm{a}_j = c_j, j = 1,\ldots,n\}$. Solve the limit problem corresponding to w

$$\begin{aligned} &\min g = 1\mathrm{x}_a \\ &s.t\ \ A\mathrm{x} + \mathrm{x}_a = \mathrm{b} \\ &\quad x_j = 0, j \notin Q \\ &\quad x_j \geq 0, j \in Q \\ &\quad \mathrm{x}_a \geq 0 \end{aligned}$$

If its optimal value $g = 0$, terminates, the primal optimal solution is found; Otherwise, solve its dual and let the optimal solution be v.

3)If v$A \leq 0$, terminates, the primal has no optimal solution; Otherwise, let

$$\theta = \min\left\{ -\frac{\bar{\mathrm{w}}\mathrm{a}_j - c_j}{\mathrm{v}^*\mathrm{a}_j} \middle| \mathrm{v}^*\mathrm{a}_j > 0, j = 1,\ldots,n \right\}$$, and construct $\mathrm{w} = \mathrm{w} + \theta \mathrm{v}^*$, go to step 2.

Dual initial solution

- Both dual simplex method and primal-dual simplex method need a dual feasible solution. Furthermore, dual simplex method requires a basic feasible solution.

For any LP problem, we can find a basis B by Gausse elimination method, and transform it into canonical form.If $\zeta_N \le 0$, we has found a basic feasible solution $w = c_B B^{-1}$ to the dual.Otherwise, add a constraint $1\mathrm{x}_N \le M$, M is large enough constant.(So this constraint has no influence on the primal) It's called extended problem after adding constraint.(Provable: The primal has feasible problem $\Leftrightarrow$ The extended problem has feasible solution, when M is large enough)

	z	x_B	x_N	x_{n+1}	RHS
z	1	0	ζ_N	0	z_0
x_B	0	I	$\bar{N}$	0	$\bar{\mathrm{b}}$
x_{n+1}	0	0	1	1	M

Let $\zeta_k = \max\left\{\zeta_i\right\}$, pivot with pivot element $\bar{a}_{m+1,k}$ to get a new simplex tableau. Because of $\zeta \le 0$, the new simplex tableau corresponds to a regular solution. Then we can solve the problem by dual simplex method or primal-dual simplex method.

There are two possible results:

1) The extended problem has no optimal solution $\Rightarrow$ The primal has no optimal solution.

2) The extended problem has optimal solution $\bar{\mathrm{x}} = \bar{\mathrm{x}}_1 + \bar{\mathrm{x}}_2 M$, the optimal value is $z_0 = z_1 + z_2 M$. Then the primal must have a feasible solution $\hat{\mathrm{x}}$, and $z_1 + z_2 M \le \mathrm{c}\hat{\mathrm{x}}$. Because M is large enough, there must be $z_2 \le 0$.(Otherwise, $\mathrm{c}\hat{\mathrm{x}}$ can be arbitrary large.)

(i) $z_2 < 0$. When $M \to \infty, z_0 \to -\infty$. The problem is boundless.

(ii) z_2=0. $z_0 = z_1$ is the optimal value of the primal, $\bar{\mathrm{x}} \ge 0$ is the optimal solution to the primal. If $\bar{\mathrm{x}}_2 = 0$, $\bar{\mathrm{x}}$ is still a basic feasible solution.Otherwise, $(\bar{\mathrm{x}}_2 \ne 0)$, let $M_o = \min\{M \mid \bar{\mathrm{x}}_1 + \bar{\mathrm{x}}_2 M \ge 0\}$, then $\bar{\mathrm{x}}_0 = \bar{\mathrm{x}}_1 + \bar{\mathrm{x}}_2 M_0$ is also a basic feasible solution.

And $\{\mathrm{x} \mid \mathrm{x} = \bar{\mathrm{x}}_1 + \bar{\mathrm{x}}_2 M, M \ge M_0\}$ reprensents a half-line face of the feasible region, every point in it is a optimal solution.