Classical Mechanics Special Relativity for Starters

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0.1 Introducing Classical Mechanics & Special Relativity

0.1.1 About this book

Classical mechanics is the starting point of physics. Over the centuries, via Newton's three fundamental laws formulated around 1687, we have built a solid framework describing the material world around us. On these pages, you will find a text book with animations, demos and exercises for studying introductory classical mechanics. Moreover, we will consider the first steps of Einstein's Special Theory of Relativity published 1905.

This material is made to support first year students from the BSc Applied Physics at Delft University of Technology during their course *Classical Mechanics and Relativity Theory*, MechaRela for short. But, of course, anybody interested in Classical Mechanics and Special Relativity is more than welcome to use this book.

With this e-book our aim is to provide learning material that is:

- self-contained
- easy to modify and thus improve over the years
- interactive, provide additional demos and exercises next to the lectures

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Exercises

- exercises in the chapter, die moet je maken als check of je het hebt begrepen
- exercises los om zelf te doen
- spicy: niveau 1 2 en 3

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0.2 Mechanics

0.2.1 The language of Physics

Physics is the science that seeks to understand the fundamental workings of the universe: from the motion of everyday objects to the structure of atoms and galaxies. To do this, physicists have developed a precise and powerful language: one that combines mathematics, both colloquial and technical language, and visual representations. This language allows us not only to describe how the physical world behaves, but also to predict how it will behave under new conditions.

In this chapter, we introduce the foundational elements of this language, covering how to express physical ideas using equations, graphs, diagrams, and words. You'll also get a first taste of how physics uses numerical simulations as an essential complement to analytical problem solving.

This language is more than just a set of tools—it is how physicists *think*. Mastering it is the first step in becoming fluent in physics.

Representations

Physics problems and concepts can be represented in multiple ways, each offering a different perspective and set of insights. The ability to translate between these representations is one of the most important skills you will develop as a physics student. In this section, we examine three key forms of representation: equations, graphs and drawings, and verbal descriptions.

We use the context of a base jumper, see Figure 1.

Figure 1: A base jumper is used as context to get familiair with representation, picture from https://commons.wikimedia.org/wiki/File:04SHANG4963.jpg

Verbal descriptions Words are indispensable in physics. Language is used to describe a phenomenon, explain concepts, pose problems and interpret results. A good verbal description makes clear:

- What is happening in a physical scenario
- What assumptions are being made (e.g., frictionless surface, constant mass)
- What is known and what needs to be found

Visual representations Visual representations help us interpret physical behavior at a glance. Graphs, motion diagrams, free-body diagrams, and vector sketches are all ways to make abstract ideas more concrete.

- **Graphs** (e.g., position vs. time, velocity vs. time) reveal trends and allow for estimation of slopes and areas, which have physical meanings like velocity and displacement.

- **Drawings** help illustrate the situation: what objects are involved, how they are moving, and what forces act on them.

Equations Equations are the compact, symbolic expressions of physical relationships. They tell us how quantities like velocity, acceleration, force, and energy are connected.

A physicist is able to switch between these representations, carefully considering with representations suits best for the given situation. We will practice these when solving problems.

> **Danger**
> Note that in the example above we neglected directions. The problem should be using vector notation, which we will cover in one of the next sections in this chapter.

> **On quantities and units**
> Each quantity has a unit. As there are only so many letters in the alphabet (even when including the Greek alphabet), letters are used for multiple quantities. How can we distinguish then meters from mass, both denoted with the letter m? Quantities are expressed in italics (m) and units are not (m).
>
> We make extensively use of the International System of Units (SI) to ensure consistency and precision in measurements across all scientific disciplines. The seven base SI units are:
>
> - Meter (m) – length
> - Kilogram (kg) – mass
> - Second (s) – time
> - Ampere (A) – electric current
> - Kelvin (K) – temperature
> - Mole (mol) – amount of substance
> - Candela (cd) – luminous intensity
>
> All other quantities can be derived from these using dimension analysis:
>
> $$\begin{aligned} W &= F \cdot s = ma \cdot s = m\frac{\Delta v}{\Delta t} \cdot s \\ [\mathrm{J}] &= [.] \cdot [\mathrm{m}] = [\mathrm{kg}] \cdot [.] \cdot [\mathrm{m}] = [\mathrm{kg}] \cdot \frac{[\mathrm{m}]}{[\mathrm{s}] \cdot [\mathrm{s}]} \cdot [\mathrm{m}] = [\frac{\mathrm{kg\ m}^2}{\mathrm{s}^2}] \end{aligned} \tag{1}$$

How to solve a physics problem?

One of the most common mistakes made by 'novices' when studying problems in physics is trying to jump as quickly as possible to the solution of a given problem or exercise. For simple questions, this may work. But when stuff gets more complicated, it is almost a certain route to frustration.

There is, however, a structured way of problem solving, that is used by virtually all scientists and engineers. Later this will be second nature to you, and you apply this way of working automatically. It is called IDEA, an acronym that stands for:

- **Interpret** - First think about the problem. What does it mean? Usually, making a sketch helps. Actually *always start with a sketch*;
- **Develop** - Build a model, from coarse to fine, that is, first think in the governing phenomena and then subsequently put in more details. Work towards writing down the equation of motion and boundary conditions;
- **Evaluate** - Solve your model, i.e. the equation of motion;
- **Assess** - Check whether your answer makes any sense (e.g. units OK?).

We will practice this and we will see that it actually is a very relaxed way of working and thinking. We strongly recommend to apply this strategy for your homework and exams (even though it seems

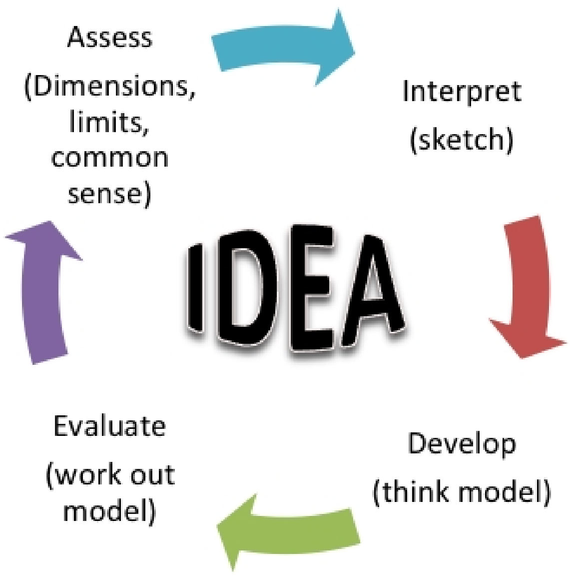


Figure 2: IDEA

strange in the beginning). The first two steps (Interpret and Develop) typically take up most of the time spend on a problem.

Good Practice

It is a good habit to make your mathematical steps small: one-by-one. Don't make big jumps or multiple steps in one step. If you make a mistake, it will be very hard to trace it back. Next: check constantly the dimensional correctness of your equations: that is easy to do and you will find the majorities of your mistakes.

Finally, use letters to denote quantities, including π. The reason for this is:

- letters have meaning and you can easily put dimensions to them;
- letters are more compact;
- your expressions usually become easier to read and characteristic features of the problem at hand can be recognized.

powers of ten

In physics, powers of ten are used to express very large or very small quantities compactly and clearly, from the size of atoms (~10^{-10} m) to the distance between stars (~10^{16} m). This notation helps compare scales, estimate orders of magnitude, and maintain clarity in calculations involving extreme values.

Prefix	Symbol	Math	Prefix	Symbol	Math
Yocto	y	10^{-24}	Base	-	10^{0}
Zepto	z	10^{-21}	Deca	da	10^{1}
Atto	a	10^{-18}	Hecto	h	10^{2}
Femto	f	10^{-15}	Kilo	k	10^{3}
Pico	p	10^{-12}	Mega	M	10^{6}
Nano	n	10^{-9}	Giga	G	10^{9}
Micro	ţ	10^{-6}	Tera	T	10^{12}
Milli	m	10^{-3}	Peta	P	10^{15}
Centi	c	10^{-2}	Exa	E	10^{18}
Deci	d	10^{-1}	Zetta	Z	10^{21}
Base	-	10^{0}	Yotta	Y	10^{24}

Tip
For a more elaborate description of quantities, units and dimension analysis, see the manual of the first year physics lab course.

Example

Calculus

Most of the undergraduate theory in physics is presented in the language of Calculus. We do a lot of differentiating and integrating. And for good reasons. The basic concepts and laws of physics can be cast in mathematical expressions, providing us the rigor and precision that is needed in our field. Moreover, once we have solved a certain problem using calculus, we obtain a very rich solution, usually in terms of functions. We can quickly recognize and classify the core features, that help us understanding the problem and its solution much deeper.

Given the example of the base jumper, we would like to know how the jumper's position as a function of time. We can answer this question by applying Newton's second law (though it is covered in secondary school, the next chapter explains in detail Newton's laws of motion):

$$\sum F = F_g - F_f = ma = m\frac{dv}{dt} \tag{2}$$

$$m\frac{dv}{dt} = mg - \frac{1}{2}\rho_{air} C_D A v^2 \tag{3}$$

Clearly this is some kind of differential equation: the change in the velocity depends on the velocity itself. Before we even try to solve this problem ($v(t) = ...$), we have to dig deeper in the precise notation, otherwise we will get lost in directions and sign conventions.

Differentiation Many physical phenomena are described by differential equations. That may be because a system evolves in time or because it changes from location to location. In our treatment of the principles of classical mechanics, we will use differentiation with respect to time a lot. The reason is obviously found in Newton's 2^{nd} law: $F = ma$.

The acceleration a is the derivative of the velocity with respect to time; velocity in itself is the derivative of position with respect to time. Or when we use the equivalent formulation with momentum: $\frac{dp}{dt} = F$. So the change of momentum in time is due to forces. Again, we use differentiation, but now of momentum.

There are three common ways to denote differentiation. The first one is by 'spelling it out':

$$v = \frac{dx}{dt} \text{ and } a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \tag{4}$$

- Advantage: it is crystal clear what we are doing.
- Disadvantage: it is a rather lengthy way of writing.

Newton introduced a different flavor: he used a dot above the quantity to indicate differentiation with respect to time. So,

$$v = \dot{x}, \text{ or } a = \dot{v} = \ddot{x} \tag{5}$$

- Advantage: compact notation, keeping equations compact.
- Disadvantage: a dot is easily overlooked or disappears in the writing.

Finally, in math often the prime is used: $\frac{df}{dx} = f'(x)$ or $\frac{d^2f}{dx^2} = f''(x)$. Similar advantage and disadvantage as with the dot notation.

Important

$$v = \frac{dx}{dt} = \dot{x} = x' \tag{6}$$

$$a = \frac{dv}{dt} = \dot{v} = \frac{d^2x}{dt^2} = \ddot{x} \tag{7}$$

It is just a matter of notation.

Vectors Position, velocity, momentum, force: they are all vectors. In physics we will use vectors a lot. It is important to use a proper notation to indicate that you are working with a vector. That can be done in various ways, all of which you will probably use at some point in time. We will use the position of a particle located at point P as an example.

Tip

See the linear algebra book on vectors.

Position vector We write the position**vector** of the particle as $\vec{r}$. This vector is a 'thing', it exists in space independent of the coordinate system we use. All we need is an origin that defines the starting point of the vector and the point P, where the vector ends.

A coordinate system allows us to write a representation of the vector in terms of its coordinates. For instance, we could use the familiar Cartesian Coordinate system {x,y,x} and represent $\vec{r}$ as a column.

$$\vec{r} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{8}$$

Alternatively, we could use unit vectors in the x, y and z-direction. These vectors have unit length and point in the x, y or z-direction, respectively. They are denoted in varies ways, depending on taste. Here are 3 examples:

$$\begin{aligned} &\hat{x},\ \hat{\imath},\ \vec{e}_x \\ &\hat{y},\ \hat{\jmath},\ \vec{e}_y \\ &\hat{z},\ \hat{k},\ \vec{e}_z \end{aligned} \tag{9}$$

With this notation, we can write the position vector $\vec{r}$ as follows:

$$\begin{aligned} \vec{r} &= x\hat{x} + y\hat{y} + z\hat{z} \\ \vec{r} &= x\hat{\imath} + y\hat{\jmath} + z\hat{k} \\ \vec{r} &= x\vec{e}_x + y\vec{e}_y + z\vec{e}_z \end{aligned} \tag{10}$$

Note that these are completely equivalent: the difference is in how the unit vectors are named. Also note, that these three representations are all given in terms of vectors. That is important to realize: in contrast to the column notation, now all is written at a single line. But keep in mind: $\hat{x}$ and $\hat{y}$ are perpendicular **vectors**.

Differentiating a vector We often have to differentiate physical quantities: velocity is the derivative of position with respect to time; acceleration is the derivative of velocity with respect to time. But you will also come across differentiation with respect to position.
As an example, let's take velocity:

$$\vec{v} \equiv \frac{d\vec{r}}{dt} \tag{11}$$

What does it mean? Differentiating is looking at the change of your 'function' when you go from x to $x + dx$:

$$\frac{df}{dx} \equiv \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{12}$$

In 3 dimensions we will have that we go from point P, represented by $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ to 'the next point' $\vec{r} + d\vec{r}$. The small vector $d\vec{r}$ is a small step forward in all three directions, that is a bit dx in the x-direction, a bit dy in the y-direction and a bit dz in the z-direction.
Consequently, we can write $\vec{r} + d\vec{r}$ as

$$\vec{r} + d\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} + dx\hat{dx} + dy\hat{y} + dz\hat{z} \tag{13}$$

$$= (x + dx)\hat{x} + (y + dy)\hat{y} + (z + dz)\hat{z} \tag{14}$$

Now, we can take a look at each component of the position and define the velocity component as, e.g., in the x-direction

$$v_x = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{dx}{dt} \tag{15}$$

Applying this to the 3-dimensional vector, we get

$$\begin{aligned} \vec{v} \equiv \frac{d\vec{r}}{dt} &= \frac{d}{dt}\left(x\hat{x} + y\hat{y} + z\hat{z}\right) \\ &= \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z} \\ &= v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \end{aligned} \tag{16}$$

Note that in the above, we have used that according to the product rule:

$$\frac{d}{dt}(x\hat{x}) = \frac{dx}{dt}\hat{x} + x\frac{d\hat{x}}{dt} = \frac{dx}{dt}\hat{x} \tag{17}$$

since $\frac{d\hat{x}}{dt} = 0$ (the unit vectors in a Cartesian system are constant). This may sound trivial: how could they change; they have always length 1 and they always point in the same direction. Trivial as this may be, we will come across unit vectors that are not constant as their direction may change. But we will worry about those examples later.

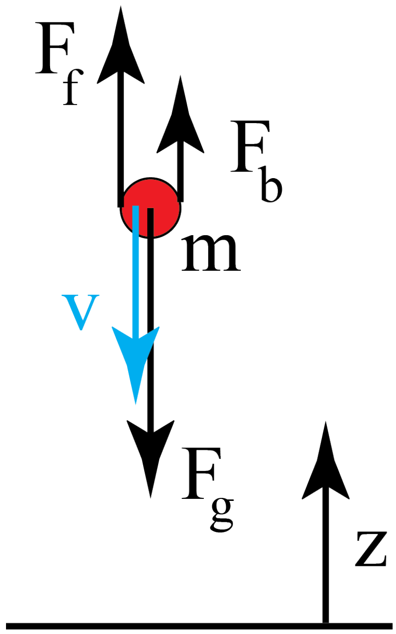
The base jumper Given the above explanation, we can now reconsider our description of the base jumper.

We see a z-coordinate pointing upward, where the velocity. As gravitational force is in the direction of the ground, we can state

$$\vec{F}_g = -mg\hat{z} \tag{18}$$

Buoyancy is clearly along the z-direction, hence

$$\vec{F}_b = \rho_{air} V g\hat{z} \tag{19}$$



The drag force is a little more complicated as the direction of the drag force is always against the direction of the velocity $-\vec{v}$. However, in the formula for drag we have v^2. To solve this, we can write

$$\vec{F_f} = -\frac{1}{2}\rho_{air}C_D A|v|\vec{v} \tag{20}$$

Note that $\hat{z}$ is missing in (20) on purpose. That would be a simplification that is valid in the given situation, but not in general.

Numerical computation and simulation

In simple cases we can come to an analytical solution. In the case of the base jumper, an analytical solution exists, though it is not trivial and requires some advanced operations as separation of variables and partial fractions:

$$v(t) = \sqrt{\frac{mg}{k}}tanh(\sqrt{\frac{kg}{m}}t) \tag{21}$$

with

$$k = \frac{1}{2}\rho_{air}C_D A \tag{22}$$

In that case there is nothing to add or gain from a numerical analysis. Nevertheless, it is instructive to see how we could have dealt with this problem using numerical techniques. One way of solving the problem is, to write a computer code (e.g. in python) that computes from time instant to time instant the force on the jumper, and from that updates the velocity and subsequently the position.

```
some initial conditions
t = 0
x = x_0
v = 0
dt = 0.1

for i is 1 to N:
    compute F: formula
    compute new v: v[i+1] = v[i] - F[i]/m*dt
    compute new x: x[i+1] = x[i] + v[i]*dt
    compute new t: t[i+1] = t[i] + dt
```

You might already have some experience with numerical simulations. (Figure 3) presents a script for the software Coach, which you might have encountered in secondary school.

'Stop condition is set 'Computations are based on Euler x := x + flow_1*Δt1 v := v + flow_2*Δt1 t1 := t1 + Δt1 flow_1 := v Fz := m*g Fw := 6*d*d*v*v f := Fz - Fw a := f/m flow_2 := a	t1 := 0 's Δt1 := 0.01 's x := 0 'm v := 0 'm/s m := 75 'kg g := 9.81 'm/s^2 d := 2.5 'm flow_1 := v 'm/s Fz := m*g 'N Fw := 6*d*d*v*v 'N f := Fz - Fw 'N a := f/m 'm/s^2 flow_2 := a 'm/s^2

Figure 3: An example of a numerical simulation made in Coach. At the left the iterative calculation proces, at the right the initial conditions.

The base jumper Let us go back to the context of the base jumper and write some code.

First we take: $k = \frac{1}{2}\rho_{air}C_D A$ which eases writing. The force balance then becomes:

$$m\vec{a} = -m\vec{g} - k|v|\vec{v} \tag{23}$$

We rewrite this to a proper differential equation for v into a finite difference equation. That is, we go back to how we came to the differential equation:

$$m \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} = \vec{F}_{net} \tag{24}$$

with $\vec{F}_{net} = -m\vec{g} - k|v|\vec{v}$

On a computer, we can not literally take the limit of Δt to zero, but we can make Δt very small. If we do that, we can rewrite the difference equation (thus not taken the limit):

$$\vec{v}(t+\Delta t) = \vec{v}(t) + \frac{\vec{F}}{m}\Delta t \tag{25}$$

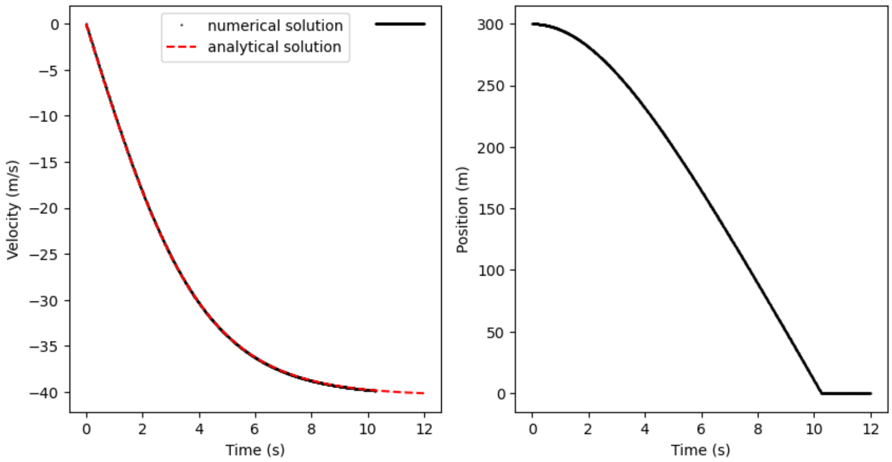
This expression forms the heart of our numerical approach. We will compute v at discrete moments in time: $t_i = 0, \Delta t, 2\Delta t, 3\Delta t, ...$ We will call these values v_i. Note that the force can be calculated at time t_i once we have v_i.

$$\begin{aligned} F_i &= mg - k|v_i|v_i \\ v_{i+1} &= v_i + \frac{F_i}{m}\Delta t \end{aligned} \tag{26}$$

Similarly, we can keep track of the position:

$$\frac{dx}{dt} = v \Rightarrow x_{i+1} = x_i + v_i\Delta t \tag{27}$$

With the above rules, we can write an iterative code:



Important to note is the sign-convention which we adhere to. Rather than using v^2 we make use of $|v|v$ which takes into account that drag is always against the direction of movement. Note as well the similarity between the analytical solution and the numerical solution.

To come back to our initial problem:
It roughly takes 10 seconds to get close to terminal velocity (note that without friction the velocity would be 98 m/s). The building is not high enough to reach this velocity (safely).

Exercises

Unit analysis
Given the formula $F = kv^2$. Derive the unit of k, expressed only in SI-units .

```
# Gravitatie ding

import numpy as np

# Zon
m1 = 1.989e30 # kg

# Aarde
m2 = 5.972e24 # kg
R = 149e9 # m

# Gravitatieconstante
G = 6.674e -11 # m^3/(kg*s^2)

# Parameters
dt = 24*3600 # s
t = np.arange(0, 3*365*dt, dt) # s

x_aarde = np.zeros((len(t),3)) # m
x_aarde[0] = np.array([0, R, 0]) # m
```

```
import numpy as np
import matplotlib.pyplot as plt

m2 = 1
m1 = 1
dt = 0.001
t = np.arange(0, 100, dt) # s

x1 = np.zeros(len(t)) # m
x1[0] = 3
y1 = np.zeros(len(t)) # m
vx = 0
vy = 7

def R(x1, x2, y1, y2):
    return np.sqrt((x1 -x2)**2 + (y1 -y2)**2)


for i in range(0, len(t) -1):

    F = G*m1*m2/(R**2)
    ax = -F*(x1[i] -0)/np.sqrt(x1[i]**2 + y1[i]**2)
    ay = -F*(y1[i] -0)/np.sqrt(x1[i]**2 + y1[i]**2)
```

```
    vx = vx + ax*dt
    vy = vy + ay*dt
    x1[i+1] = x1[i] + vx*dt
    y1[i+1] = y1[i] + vy*dt


plt.figure(figsize=(10,10))
plt.plot(x1, y1, 'k.', markersize=1)
plt.xlabel('x (m)')
plt.ylabel('y (m)')
plt.show()
```

0.2.2 Newton's Laws

Now we turn to one of the most profound breakthroughs in the history of science: the laws of motion formulated by Isaac Newton. These laws provide a systematic framework for understanding how and why objects move and form the backbone of classical mechanics. With them, we can predict the motion of a falling apple, a car accelerating down the road, or a satellite orbiting Earth (though some adjustments are required in this context to make use of e.g. GPS!). More than just equations, they express deep principles about the nature of force, mass, and interaction.

In this chapter, you will begin to develop the core physicist's skill: building a simplified model of the real world, applying physical principles, and using mathematical tools to reach meaningful conclusions.

Newton's Three Laws

Much of physics, in particular Classical Mechanics, rests on three laws that carry Newton's name:

Newton's first law (N1)
If no force acts on an object, the object moves with constant velocity.

Newton's second law (N2)
If a (net) force acts on an object, the momentum of the object will change according to:

$$\frac{d\vec{p}}{dt} = \vec{F} \tag{28}$$

Newton's third law (N3)
If object 1 exerts a force $\vec{F}_{12}$ on object 2, then object 2 exerts a force $\vec{F}_{21}$ equal in magnitude and opposite in direction on object 1:

$$\vec{F}_{21} = -\vec{F}_{12} \tag{29}$$

N1 has, in fact, been formulated by Galileo Galilei. Newton has, in his N2, build upon it: N1 is included in N2, after all:
if $\vec{F} = 0$, then $\frac{d\vec{p}}{dt} = 0 \rightarrow \vec{p} = const \rightarrow \vec{v} = cst$, provided m is a constant.

Most people know N2 as

$$\vec{F} = m\vec{a} \tag{30}$$

For particles of constant mass, the two are equivalent:
if $m = const$, then $\frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}$.
Nevertheless, in many cases using the momentum representation is beneficial. The reason is that momentum is one of the key quantities in physics. This is due to the underlying conservation law, that we will derive in a minute. Moreover, using momentum allows for a new interpretation of force: force is that quantity that - provided it is allowed to act for some time interval on an object - changes the momentum of that object. This can be formally written as:

$$d\vec{p} = \vec{F}dt \leftrightarrow \Delta\vec{p} = \int \vec{F}dt \tag{31}$$

The latter quantity $\vec{I} \equiv \int \vec{F}dt$ is called the impulse.

Note
Momentum is in Dutch **impuls** in Dutch; the English **impulse** is in Dutch **stoot**.

Example 2.1
Consider a point particle of mass m, moving at a velocity v_0 along the x-axis. At $t = 0$ a constant force acts on the particle in the positive x-direction. The force lasts for a small time interval Δt.
What is the velocity of the particle for $t > \Delta t$?

Solution 2.1

Important
In Newtons mechanics time does not have a preferential direction. That means, in the equations derived based on the three laws of Newton, we can replace t by $-t$ and the motion will have different sign, but that's it. The path/orbit will be the same, but traversed in opposite direction. Also in special relativity this stays the same.

However, in daily life we experience a clear distinction between past, present and future. This difference is not present in this lecture at all. Only by the second of law thermodynamics the time axis obtains a direction, more about this in classes on Statistical Mechanics.

Acceleration due to gravity In most cases the forces acting on an object are not constant. However, there is a classical case that is treated in physics (already at secondary school level) where only one force acts and other forces are neglected. Hence, according to Newton's second law, the acceleration is constant.

When we first consider only the motion in the z-direction, we can derive:

$$a = \frac{F}{m} = const. \tag{32}$$

Hence, for the velocity:

$$v(t) = \int_{t_0}^{t_e} a dt = a(t_e - t_0) + v_0 \tag{33}$$

assuming $t_0 = 0$ and $t_e = t \Rightarrow v(t) = v_0 + at$ the position is described by

$$s(t) = \int_0^t v(t) dt = \int_0^t at + v_0 dt = \frac{1}{2}at^2 + v_0 t + s_0 \tag{34}$$

Rearranging:

$$s(t) = \frac{1}{2}at^2 + v_0 t + s_0 \tag{35}$$

2D-motion We only considered motion in the vertical direction, however, objects tend to move in three dimension. We consider now the two-dimensional situation, starting with an object which is horizontally thrown from a height.

In the situation given in Figure 4 the object is thrown with a horizontal velocity of v_x. As no forces in the horizontal direction act on the object (N1), its horizontal motion can be described by

$$s_x(t) = v_x t \tag{36}$$

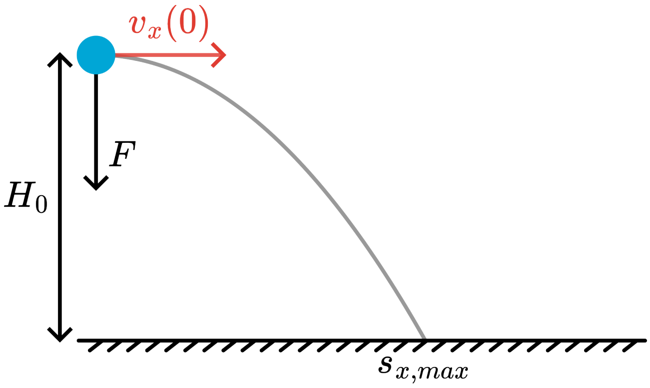


Figure 4: A sketch of the situation where an object is thrown horizontally and the horizontal distance should be calculated.

In the vertical direction only the gravitational force acts (N2), hence the motion can be described by (35). Taking the z-direction upward, a starting height H_0 and $v_y(0) = 0$ it becomes:

$$s_y(t) = H_0 - \frac{1}{2}gt^2 \tag{37}$$

The total horizontal traveled distance of the object before hitting the ground then becomes:

$$s_{x,max} = v_x\sqrt{\frac{2H_0}{g}} \tag{38}$$

This motion is visualized in Figure 5. The trajectory is shown with s_x on the horizontal axis and s_y on the vertical axis. At regular time intervals Δt, velocity vectors are drawn to illustrate the motion. Note that the horizontal and vertical components of velocity, v_x and v_y, vary independently throughout the trajectory. Moreover, $v(t)$ is the tangent of $s(t)$.

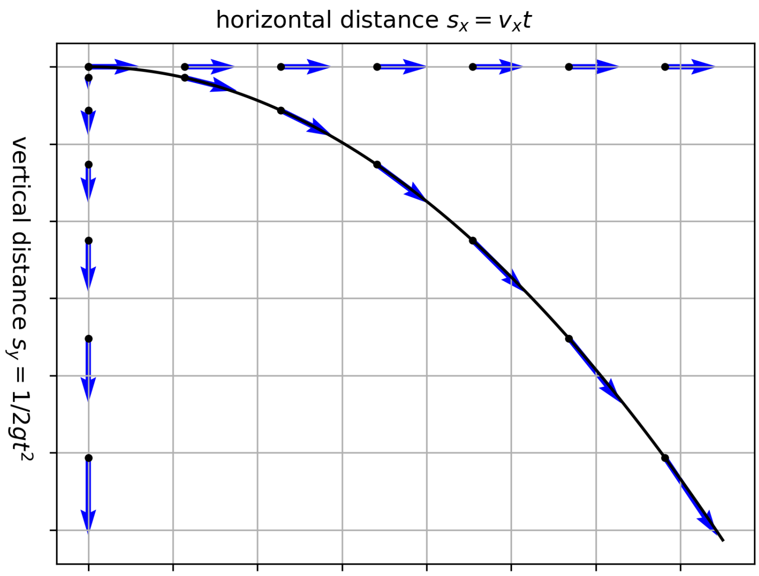


Figure 5: The parabolic motion is visualized with blue velocity vectors v, v_x and v_y shown at various points along the trajectory.

Danger
Understand that the case above is specific in physics: in most realistic contexts multiple forces are acting upon the object. Hence the equation of motion does not become $s(t) = s_0 + v_0 t + 1/2gt^2$

Conservation of Momentum

From Newton's 2^{nd} and 3^{rd} law we can easily derive the law of conservation of momentum.
Assume there are only two point-particle (i.e. particles with no size but with mass), that exert a force on each other. No other forces are present. From N2 we have:

$$\frac{d\vec{p}_1}{dt} = \vec{F}_{21} \frac{d\vec{p}_2}{dt} = \vec{F}_{12} \tag{39}$$

From N3 we know:

$$\vec{F}_{21} = -\vec{F}_{12} \tag{40}$$

And, thus by adding the two momentum equations we get:

$$\left.\begin{aligned} \tfrac{d\vec{p}_1}{dt} &= \vec{F}_{21} \\ \tfrac{d\vec{p}_2}{dt} &= \vec{F}_{12} = -\vec{F}_{21} \end{aligned}\right\} \Rightarrow \tag{41}$$

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0 \rightarrow \frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2\right) = 0 \tag{42}$$

$$\Rightarrow \vec{p}_1 + \vec{p}_2 = const \text{ i.e. does } \textit{not} \text{ depend on time} \tag{43}$$

Note the importance of the last conclusion: **if objects interact via a mutual force then the total momentum of the objects can not change.** No matter what the interaction is. It is easily extended to more interacting particles. The crux is that particles interact with one another via forces that obey N3. Thus for three interacting point particles we would have (with $\vec{F}_{ij}$ the force from particle i felt by particle j):

$$\left.\begin{aligned} \tfrac{d\vec{p}_1}{dt} &= \vec{F}_{21} + \vec{F}_{31} \\ \tfrac{d\vec{p}_2}{dt} &= \vec{F}_{12} + \vec{F}_{32} = -\vec{F}_{21} + \vec{F}_{32} \\ \tfrac{d\vec{p}_3}{dt} &= \vec{F}_{13} + \vec{F}_{23} = -\vec{F}_{31} - \vec{F}_{32} \end{aligned}\right\} \tag{44}$$

Sum these three equations:

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} + \frac{d\vec{p}_3}{dt} = 0 \rightarrow \frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2 + \vec{p}_3\right) = 0 \Rightarrow \vec{p}_1 + \vec{p}_2 + \vec{p}_3 = const \text{ i.e. does } \textit{not} \text{ depend on time} \tag{45}$$

For a system of N particles, extension is straight forward.

Momentum example The above theoretical concept is simple in its ideas:

- a particle changes its momentum whenever a force acts on it;
- momentum is conserved;
- action = - reaction.

But it is incredible powerful and so generic, that finding when and how to use it is much less straight forward. The beauty of physics is its relatively small set of fundamental laws. The difficulty of physics is these laws can be applied to almost anything. The trick is how to do that, how to start and get the machinery running. That can be very hard. Luckily there is a recipe to master it: it is called practice.

Example 2.2
A point particle (mass m) is dropped from rest at a height h above the ground. Only gravity acts on the particle with a constant acceleration g (=9.813 m/s^2).

- Find the momentum when the particle hits the ground.
- What would be the earth' velocity upon impact?

Solution 2.2
Let's do this one together. We follow the standard approach of IDEA: Interpret (and make your sketch!), develop (think 'model'), evaluate (solve your model) and assess (does it make any sense?).

Forces & Inertia

Newton's laws introduce the concept of force. Forces have distinct features:

- forces are vectors, that is, they have magnitude and direction;
- forces change the motion of an object:
  - they change the velocity, i.e. they accelerate the object

$$\vec{a} = \frac{\vec{F}}{m} \leftrightarrow d\vec{v} = \vec{a}dt = \frac{\vec{F}dt}{m} \tag{46}$$

- or, equally true, they change the momentum of an object

$$\frac{d\vec{p}}{dt} = \vec{F} \leftrightarrow d\vec{p} = \vec{F}dt \tag{47}$$

Many physicists like the second bullet: forces change the momentum of an object, but for that they need time to act.

Momentum is a more fundamental concept in physics than acceleration. That is another reason why physicists prefer the second way of looking at forces.

Connecting physics and calculus
Let's look at a particle of mass m, that has initially (say at $t = 0$) a velocity v_0. For $t > 0$ the particle is subject to a force that is of the form $F = -bv$. This is a kind of frictional force: the faster the particle goes, the larger the opposing force will be.

We would like to know how the position of the particle is as a function of time.

We can answer this question by applying Newton 2:

$$m\frac{dv}{dt} = F \Rightarrow m\frac{dv}{dt} + bv = 0 \tag{48}$$

Clearly, we have to solve a differential equation which states that if you take the derivative of v you should get something like $-v$ back. From calculus we know, that exponential function have the feature that when we differentiate them, we get them back. So, we will try $v(t) = Ae^{-\mu t}$ with A and μ to be determined constants.

We substitute our trial v:

$$m \cdot A \cdot -\mu e^{-\mu t} + b \cdot Ae^{-\mu t} = 0 \tag{49}$$

This should hold for all t. Luckily, we can scratch out the term $e^{-\mu t}$, leaving us with:

$$-mA\mu + Ab = 0 \tag{50}$$

We see, that also our unknown constant A drops out. And, thus, we find

$$\mu = \frac{b}{m} \tag{51}$$

Next we need to find A: for that we need an initial condition, which we have: at $t = 0$ is $v = v_)$. So, we know:

$$v(t) = Ae^{-\frac{b}{m}t} \text{ and } v(0) = v_0 \tag{52}$$

From the above we see: $A = v_0$ and our final solution is:

$$v(t) = v_0 e^{-\frac{b}{m}t} \tag{53}$$

From the solution for v, we easily find the position of m as a function of time. Let's assume that the particle was in the origin at $t = 0$, thus $x(0) = 0$. So, we find for the position

$$\frac{dx}{dt} \equiv v = v_0 e^{-\frac{b}{m}t} \Rightarrow x = v_0 \cdot \left(-\frac{m}{b} e^{-\frac{b}{m}t}\right) + B \tag{54}$$

We find B with the initial condition and get as final solution:

$$x(t) = \frac{mv_0}{b}\left(1 - e^{-\frac{b}{m}t}\right) \tag{55}$$

If we inspect and assess our solution, we see: the particle slows down (as is to be expected with a frictional force acting on it) and eventually comes to a stand still. At that moment, the force has also decreased to zero, so the particle will stay put.

Inertia Inertia is denoted by the letter m for mass. And mass is that property of an object that characterizes its resistance to changing its velocity. Actually, we should have written something like m_i, with subscript i denoting inertia.

Why? There is another property of objects, also called mass, that is part of Newton's Gravitational Law.

Two bodies of mass m_1 and m_2 attract each other via the so-called gravitational force:

$$\vec{F}_{12} = -G\frac{m_1 m_2}{r_{12}^2}\hat{r}_{12} \tag{56}$$

Here, we should have used a different symbol, rather than m. Something like m_g, as it is by no means obvious that the two 'masses' m_i and m_g refer to the same property. If you find that confusing, think about inertia and electric forces: you denote the property associated with electric forces by q and call it charge. We have no problem writing

$$\vec{F} = m\vec{a}\vec{F}_C = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\hat{r} \tag{57}$$

We do not confuse q by m or vice versa. They are really different quantities: q tells us that the particle has a property we call 'charge' and that it will respond to other charges, either being attracted to, or repelled from. How fast it will respond to this force of another charged particle depends on m. If m is big, the particle will only get a small acceleration; the strength of the force does not depend on m at all. So far, so good. But what about m_g? That property of a particle that makes it being attracted to another particle with this same property, that we could have called 'gravitational charge'. It is clearly different from 'electrical charge'. But would it have been logical that it was also different from the property inertial mass, m_i?

$$\begin{aligned} \vec{F} &= m_i\vec{a} \\ \vec{F}_g &= -G\frac{m_g M_g}{r^2}\hat{r} \end{aligned} \tag{58}$$

As far as we can tell (via experiments) m_i and m_g are the same. Actually, it was Einstein who postulated that the two are referring to the same property of an object: there is no difference.

Force field As far as we've seen, forces like gravity and electrostatics act between objects. When you push a car, the force is applied locally, through direct contact. In contrast, gravitational and electrostatic forces act over a distance—they are present throughout space, though they still depend on the positions of the objects involved.

One powerful way to describe how a force acts at different locations in space is through the concept of a **force field**. A force field assigns a force vector (indicating both direction and magnitude) to every point in space, telling you what force an object would experience if placed there.

For example, Figure 6 shows a gravitational field, described by $\vec{F}_g = G\frac{mM}{r^2}\hat{r}$. Any object entering this field is attracted toward the central mass with a force that depends on its distance from that mass's center.

Figure 6: Two force fields, left a gravitational field, right an electric field

Measuring mass or force So far we did not address how to measure force. Neither did we discuss how to measure mass. This is less trivial than it looks at first side. Obviously, force and mass are coupled via N2: $F = ma$.

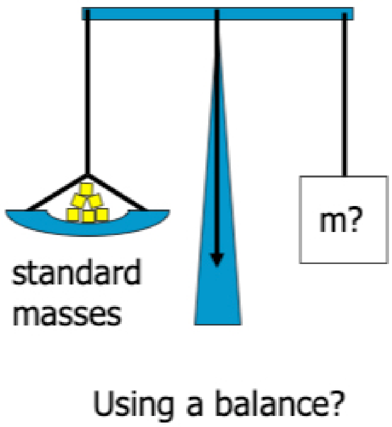


Figure 7: Can force be measured using a balance?

The acceleration can be measured when we have a ruler and a clock, i.e. once we have established how to measure distance and how to measure time intervals, we can measure position as a function of time and from that velocity and acceleration.

But how to find mass? We could agree upon a unit mass, an object that represents by definition 1kg. In fact we did. But that is only step one. The next question is: how do we compare an unknown mass to our standard. A first reaction might be: put them on a balance and see how many standard kilograms you need (including fractions of it) to balance the unknown mass. Sounds like a good idea, but is it? Unfortunately, the answer is not a 'yes'.

As on second thought: the balance compares the pull of gravity. Hence, it 'measures' gravitational mass, rather than inertia. Luckily, Newton's laws help. Suppose we let two objects, our standard mass and the unknown one, interact under their mutual interaction force. Every other force is excluded. Then, on account on N2 we have

$$\begin{cases} m_1 a_1 = F_{21} \\ m_2 a_2 = F_{12} = -F_{21} \end{cases} \tag{59}$$

where we used N3 for the last equality. Clearly, if we take the ratio of these two equations we get:

$$\frac{m_1}{m_2} = \left|\frac{a_2}{a_1}\right| \tag{60}$$

irrespective of the strength or nature of the forces involved. We can measure acceleration and thus with this rule express the unknown mass in terms of our standard.

> **Note**
> We will not use this method to measure mass. We came to the conclusion that we can't find any difference in the gravitational mass and the inertial mass. Hence, we can use scales and balances for all practical purposes. But the above shows, that we can safely work with inertial mass: we have the means to measure it and compare it to our standard kilogram.

Now that we know how to determine mass, we also have solved the problem of measuring force. We just measure the mass and the acceleration of an object and from N2 we can find the force. This allows us to develop 'force measuring equipment' that we can calibrate using the method discussed above.

> **Intermezzo: kilogram, unit of mass**
> In 1795 it was decided that 1 gram is the mass of 1 cm^3 of water at its melting point. Later on, the kilogram became the unit for mass. In 1799, the *kilogramme des Archives* was made, being from then on the prototype of the unit of mass. It has a mass equal to that of 1 liter of water at 4řC (when water has its maximum density).

In recent years, it became clear that using such a standard kilogram does not allow for high precision: the mass of the standard kilogram was, measured over a long time, changing. Not by much (on the order of 50 micrograms), but sufficient to hamper high precision measurements and setting of other standards. In modern physics, the kilogram is now defined in terms of Planck's constant. As Planck's constant has been set (in 2019) at exactly $h = 6.62607015 \cdot 10^{-34}\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-1}$, the kilogram is now defined via h, the meter and second.

```
### Eötvös experiment on mass ###
The question whether inertial mass and gravitational mass are the same has put experimentalist

A couple of years later, Galilei used both fall experiments and pendula to improve this to: le

An important step forward was set by the Hungarian physicist, Loránd Eötvös (1848 -1918). We w

#### The experiment
The essence of the Eötvös experiment is finding a set up in which both gravity (sensitive to t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Torsion balance used by Eötvös.
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This is a sensitive device: any mismatch in forces or torques will have the setup either tilt or rotate a bit. Eötvös attached a tiny mirror to one of the arms of the rod. If you shine a light beam on the mirror and let it reflect and be projected on a wall, then the smallest deviation in position will be amplified to create a large motion of the light spot on the wall.

In Eötvös experiment two forces are acting on each of the masses: gravity, proportional to m_g, but also the centrifugal force $F_c = m_i R\omega^2$, the centrifugal force stemming from the fact that the experiment is done in a frame of reference rotating with the earth. This force is proportional to the inertial mass. The experiment is designed such that if the rod does not show any rotation around the vertical axis, then the gravitational mass and inertial mass must be equal. It can be done with great precision and Eötvös observed no measurable rotation of the rod. From this he could conclude that the ratio of the gravitational over inertial mass differed less from 1 than $5 \cdot 10^{-8}$. Currently, experimentalist have brought this down to $1 \cdot 10^{-15}$.

Note

The question is not if m_i/m_g is different from 1. If that was the case but the ratio would always be the same, then we would just rescale m_g, that is redefine the value of the gravitational const G to make m_g equal to m_i. No, the question is whether these two properties are separate things, like mass and charge. We can have two objects with the same inertial mass but give them very different charges. In analogy: if m_i and m_g are fundamentally different quantities then we could do the same but now with inertial and gravitational mass.

Tip

Want to know more about this experiment? Watch this videoclip.

Examples & Exercises

Here are some examples and exercises that deals with forces. Make sure you practice IDEA.

Force on a particle
Consider a point particle of mass m, moving at a velocity v_0 along the x-axis. At $t = 0$ a constant force acts on the particle in the negative x-direction. The force lasts for a small time interval Δt.

What is the strength of the force, if it brings the particle exactly to a zero-velocity? Start by making a drawing.

A ball is shot from a 10m high hill with a velocity of 10m/s under an angle of 30°, see Figure **??**.

1. How long is the ball in the air?
2. How far does it travel in the horizontal direction?
3. With what velocity does the ball hit the ground?

A particle of mass m moves along the x-axis. At time $t = 0$ it is at the origin with velocity v_0. For $t > 0$, a constant force acts on the particle. This is a 1-dimensional problem.

- Derive the acceleration of the particle as a function of time.
- Derive the velocity of the particle as a function of time.
- Derive the position of the particle as a function of time.

A particle of mass m moves along the x-axis. At time $t = 0$ it is at the origin with velocity v_0. For $t > 0$ the particle is subject to a force $F_0 \sin(2\pi f_0 t)$. This is a 1-dimensional problem.

- Calculate the acceleration of the particle as a function of time.
- Calculate the velocity of the particle as a function of time.
- Calculate the position of the particle as a function of time.

A particle follows a straight path with a constant velocity. At $t = 0$ the particle is at point A with coordinate $(0, y_A)$, while at $t = t_1$ it is at B with coordinate $(x_B, 0)$. The coordinates are given in a Cartesian system. The problem is 2-dimensional.

1. Make a sketch.
2. Find the position of the particle at arbitrary time $0 < t < t_1$.
3. Derive the velocity of the particle from position as function of time.

Represent vectors in a Cartesian coordinate system using the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

In Classical Mechanics we often use a coordinate system to describe motion of object. In this exercise, you will look at two Cartesian coordinate systems. System S has coordinates (x, y) and

corresponding unit vectors $\hat{x}$ and $\hat{y}$.
The second system, S', uses (x', y') and corresponding unit vectors. The x'-axis makes an angle of 30° with the x-axis (measured counter-clockwise).

1. Make a sketch.
2. Determine the relations between $\hat{x}'$ and $\hat{x}, \hat{y}$ as well as between $\hat{y}'$ and $\hat{x}, \hat{y}$
   An object has, according to S, a velocity of $\vec{v} = 3\hat{x} + 5\hat{y}$.
3. Determine the velocity according to S'.

According to your observations, a particle is located at position (1,0) (you use a Cartesian coordinate system). The particle has no velocity and no forces are acting on it.
Another observer, S', uses a Cartesian coordinate system described by (x', y'). You notice that her unit vectors rotate at a constant speed compared to your unit vectors:

$$\hat{x}' = \cos(2\pi ft)\hat{x} + \sin(2\pi ft)\hat{y} \tag{61}$$

$$\hat{y}' = -\sin(2\pi ft)\hat{x} + \cos(2\pi ft)\hat{y} \tag{62}$$

1. Find the position of the particle according to the other observer, S'.
2. Calculate the velocity of the particle according to S'.

A particle of mass m moves at a constant velocity v_0 over a friction less table. The direction it is moving in, is at 45° with the positive x-axis. At some point in time, the particle experiences a force $\vec{F} = -b\vec{v}$ with $b > 0$.
We call this time $t = 0$ and take the position of the particle at that time as our origin.

1. Make a sketch.
2. Determine whether this problem needs to be analyzed as a 1D or a 2D problem.
3. Set up N2 in the form $m\frac{d\vec{v}}{dt} = \vec{F}$
4. Solve N2 and find the velocity of the particle as a function of time.
5. What happens to the particle for large t?

Exercises set 1

Answers

Who is strongest? Two strong boys try to keep a rope straight by each pulling hard at one end. A not so strong third person is pulling in the middle of the rope, but at an angle of 90ř to the rope. The two strong boys have the task to keep the deviation of the rope to a small value, set by you.
Or open the widget full screen

Interact with the widget about forces being vectors.

You drop a stone from a height of 50m the tower of the church. Calculate the velocity of the stone when it hits the ground (ignore friction). In the video (click on the image below) you will see on the left a quick and dirty solution, NOT using IDEA. The right hand side uses IDEA and Newton's 2^{nd} law.

Or open the widget full screen
Inertia game. Mass has inertia; can you catch the red mass?

Or open the widget full screen
Two point particles slide down a slope: one feels friction the other doesn't. Can you analyse the situation and understand the graphs?

Or open the widget full screen
What kind of forces are acting?

Or open the widget full screen
A particle pulled by another.

A point particle (mass m) is from position $z = 0$ shot with a velocity v_0 straight upwards into the air. On this particle only gravity acts, i.e. friction with the air can be ignored. The acceleration of gravity, g, may be taken as a constant.

The following questions should be answered.

- What is the maximum height that the particle reaches?
- How long doe it take to reach that highest point?

Solve this exercise using IDEA.

- Sketch the situation and draw the relevant quantities.
- Reason that this exercise can be solved using F=ma (or dp/dt = F).
- Formulate the equation of motion (N2) for m.
- Classify what kind of mathematical equation this is and provide initial or boundary conditions that are needed to solve the equation.
- Solve the equation of motion and answer the two questions.
- Check your math and the result for dimensional correctness. Inspect the limit: $F_{zw} \to 0$.

First try it yourself (and try seriously) before peeking at the solution. Click solution to open a pdf.

Acceleration of Gravity

- Find an object that you can safely drop from some height.
- Drop the object from any (or several heights) and measure using a stop watch or you mobile the time from dropping to hitting the ground.
- Measure the dropping height.

Find from these data the value of gravity's acceleration constant.

Don't forget to first make an analysis of this experiment in terms of a physical model and make clear what your assumptions are.

Hint: think about the effect of air resistance: is dropping from a small, a medium or a high height best? Any arguments?

Use numerical analysis to assess influence of air friction
If you want to learn also how to use numerical methods ...

Try using an air drag force: $F_{drag} = -A_{\perp} C_D \frac{1}{2} \rho_{air} v^2$. With $A_{\perp}$ the cross-sectional area of your object perpendicular to the velocity vector and $C_D \approx 1$ the drag coefficient (in real life it is actually a function of the velocity). ρ_{air} is the density of air which is about 1.2 kg/m^3.

Write a computer program (e.g. in python) that calculates the motion of your object. See Solution with Python how you could do that.

Forces on your bike

On a bicycle you will have to apply a force to the peddles to move forward, right? What force actually moves you forward, where is it located and who/what is providing that force?

- Make sketch and draw the relevant force. Give the force that actually propels you a different color.
- Think for a minute about the nature of this force: are you surprised?
  N.B. Consider while thinking about this problem: what would happen if you were biking on an extremely slippery floor?

To check your thoughts: click Riding a Bike .

Getting of the boat

You are stepping from a boat onto the shore. Use Newton's laws to describe why you will end up in the water.

N.B. A calculation is not required, but focus on the physics and describe in words why you didn't make it to the jetty.

To check your thoughts: click Stepping off the boat .

Newton's Laws

Close this book (or don't peak at it ;-)) and write down Newton's laws. Explain in words the meaning of each of the laws. Try to come up with several, different ways of describing what is in these equations.

Click Newtons Laws for a solution.

Exercises set 2

0.2.3 Work & Energy

Work

Work and energy are two important concepts. Work is calculated as 'force times path', but we need a formal definition:

if a point particle moves from $\vec{r}$ to $\vec{r} + d\vec{r}$ and during this interval a force $\vec{F}$ acts on the particle, then this force has performed an amount of work equal to:

$$dW = \vec{F} \cdot d\vec{r} \tag{63}$$

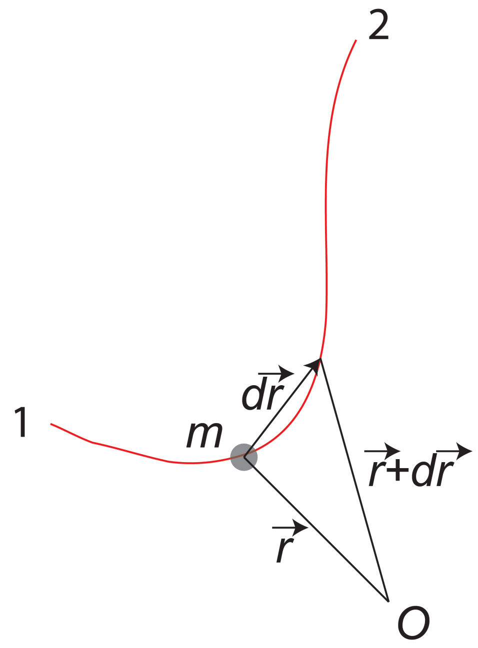


Figure 11: Path of a particle.

Note that this an inner product between two vectors, resulting in a scalar (See also the chapter on dot product here). In other words, work is a number, not a vector. It has no direction. That is one of the advantages over force.

$$dW = \vec{F} \cdot d\vec{r} = F_x dx + F_y dy + F_z dz \tag{64}$$

Work done on m by F during motion from 1 to 2 over a prescribed trajectory:

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} \tag{65}$$

Keep in mind: in general the work depends on the starting point 1, the end point 2 and on the trajectory. Different trajectories from 1 to 2 may lead to different amounts of work.

Kinetic Energy

Kinetic energy is defined and derived using the definition of work and Newton's 2^{nd} Law.

The following holds: if work is done on a particle, then its kinetic energy must change. And vice versa: if the kinetic energy of an object changes, then work must have been done on that particle. The following derivation shows this.

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} = \int_1^2 \vec{F} \cdot \frac{d\vec{r}}{dt} dt = \int_1^2 \vec{F} \cdot \vec{v} dt = \int_1^2 m \frac{d\vec{v}}{dt} \cdot \vec{v} dt = m \int_1^2 \vec{v} \cdot d\vec{v} = m[\frac{1}{2}\vec{v}^2]_1^2 = \frac{1}{2} m\vec{v}_2^2 - \frac{1}{2} m\vec{v}_1^2 \tag{66}$$

It is from the above that we indicate $\frac{1}{2}m\vec{v}^2$ as kinetic energy. It is important to realize that the concept of kinetic energy does not bring anything that is not contained in N2 to the table. But it does give a new perspective: kinetic energy can only be gained or lost if a force performs work on the particle. And vice versa: if a force performs work on a particle, the particle will change its kinetic energy.

Obviously, if more than one force acts, the net work done on the particle determines the change in kinetic energy. It is perfectly possible that force 1 adds an amount W to the particle, whereas at the same time force 2 will take out an amount $-W$. This is the case for a particle that moves under the influence of two forces that cancel each other: $\vec{F}_1 = -\vec{F}_2$. From Newton 2, we immediately infer that if the two forces cancel each other, then the particle will move with a constant velocity. Hence, its kinetic energy stays constant. This is completely in line with the fact that in this case the net work done on the particle is zero:

$$W_1 + W_2 = \int_1^2 \vec{F}_1 \cdot d\vec{r} + \int_1^2 \vec{F}_2 \cdot d\vec{r} = \int_1^2 \vec{F}_1 \cdot d\vec{r} - \int_1^2 \vec{F}_1 \cdot d\vec{r} = 0 \tag{67}$$

Worked Examples

Carrying a weight

You carry a heavy backpack m =20 kg from Delft to Rotterdam (20 km). What is the work that you have done against the gravitational force?

Work in a force field

Now we consider a force field $\vec{F}(x,y) = (-y, x^2) = -y\hat{x} + x^2\hat{y}$. We compute the work done over a path from the origin $(0,0)$ to $(1,0)$ and then to $(1,1)$ first along the x-axis and then parallel to the y-axis.

Reminder of path/line integral from Analysis

As long as the path can be split along coordinate axis the separation above is a good recipe. If that is not the case, then we need to turn back to the way how things have been introduced in the Analysis class. We need to make a 1D parameterization of the path.

Line integral of a vector valued function $\vec{F}(x,y,z) : \mathbb{R}^3 \to \mathbb{R}^3$ over a curve $\mathcal{C}$ is given as

$$\int_{\mathcal{C}} \vec{F}(x,y,z) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(\tau)) \cdot \frac{d\vec{r}(\tau)}{d\tau} \, d\tau \tag{68}$$

We integrate in the definition of the work from point 1 to 2 over an implicitly given path. To compute this actually, you need to parameterize the path by $\vec{r}(\tau) = (x(\tau), y(\tau), z(\tau))$. The integration variable τ tells you where you are on the path, $\tau \in [a,b] \in \mathbb{R}$. The derivative of $\vec{r}$ with respect to τ gives the tangent vector to the curve, the "speed" of walking along the curve. In the analysis class you used $\vec{v}(\tau) \equiv \frac{d\vec{r}(\tau)}{d\tau}$ for the speed. The value of the line integral is independent of the chosen parameterization. However, it changes sign when reversing the integration boundaries.

Example 4.3

Now we integrate from $(0,0) \to (1,1)$ but along the diagonal. A parameterization of this path is

$\vec{r}(\tau) = (0,0) + (1,1)\tau = (\tau,\tau), \tau \in [0,1]$. The derivative is $\frac{d\vec{r}(\tau)}{d\tau} = (1,1)$. Therefore we can write the work of $\vec{F}(x,y) = -y\hat{x} + x^2\hat{y}$ along the diagonal as

$$\int_0^1 \vec{F}(\tau,\tau) \cdot (1,1)\, d\tau = \int_0^1 (-\tau,\tau^2) \cdot (1,1)\, d\tau = \int_0^1 -\tau + \tau^2\, d\tau = -\frac{1}{6} \tag{69}$$

Integration of the same force $\vec{F}(x,y) = -y\hat{x} + x^2\hat{y}$ from $(0,0) \to (1,1)$ but along a normal parabola. A parameterization of the path is $\vec{r}(\tau) = (0,0) + (\tau,\tau^2), \tau \in [0,1]$ and the derivative is $\frac{d\vec{r}}{d\tau} = (1,2\tau)$. The work then is

$$\int_0^1 \vec{F}(\tau,\tau^2) \cdot (1,2\tau)\, d\tau = \int_0^1 (-\tau^2,\tau^2) \cdot (1,2\tau)\, d\tau = \int_0^1 -\tau^2 + 2\tau^3\, d\tau = \frac{1}{6} \tag{70}$$

Exercise

Simulations Below is a physlet showing the relation between work done by a constant force and kinetic energy of a particle. Can you understand the curves plotted?

Or open the widget full screen

HIERONDER IN PYTHON

```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.animation as animation

# Parameters
m = 1.0       # kg
F = 1.0       # N (instelbaar)
v0 = 0.0      # m/s (instelbaar)
t_stop = 2.0  # s
dt = 0.05     # s

# Tijdstappen
t_array = np.arange(0, t_stop, dt)

# Arrays voor opslag
x_array = []
W_array = []
Ekin_array = []

# Beginwaarden
x0 = 0
x_ref = 0     # referentiepunt (zoals 185 px in JS)
x = x0
velo = v0

for t in t_array:
    # Beweging onder constante kracht
    x = x0 + v0 * t + 0.5 * F / m * t**2
    v = v0 + F / m * t
    W = F * (x - x0)              # Arbeid: W = F * \Deltax
    Ekin = 0.5 * m * v**2         # Kinetische energie

    x_array.append(x)
    W_array.append(W)
    Ekin_array.append(Ekin)

# Plotten
fig, ax = plt.subplots(3, 1, figsize=(8, 10), sharex=True)

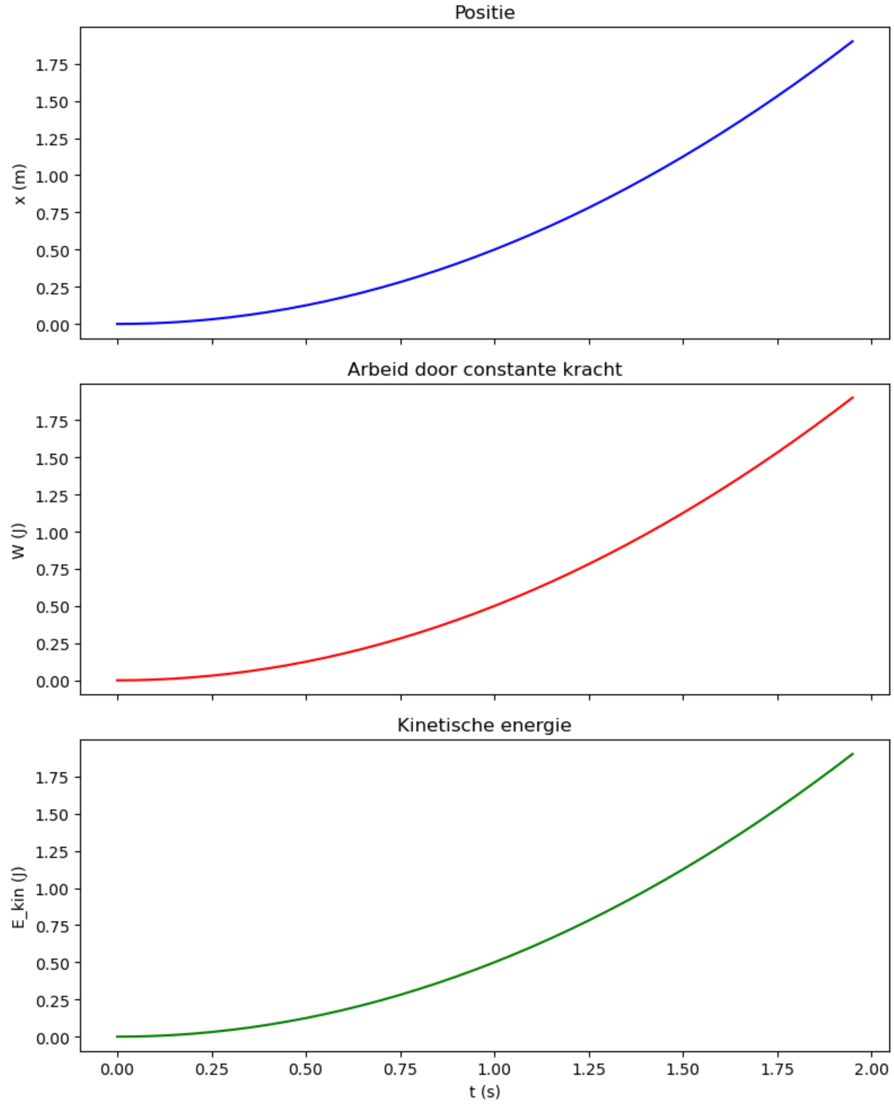
# Positie
ax[0].plot(t_array, x_array, color='blue')
ax[0].set_ylabel('x (m)')
ax[0].set_title('Positie')

# Arbeid
ax[1].plot(t_array, W_array, color='red')
ax[1].set_ylabel('W (J)')
```

```
ax[1].set_title('Arbeid door constante kracht')

# Kinetische energie
ax[2].plot(t_array, Ekin_array, color='green')
ax[2].set_ylabel('E_kin (J)')
ax[2].set_xlabel('t (s)')
ax[2].set_title('Kinetische energie')

plt.tight_layout()
plt.show()
```



Of course, forces are not always constant. This physlet show the relation between work done by a force that increases linearly with position. Can you understand the curves plotted?

Or open the widget full screen

Exercises Here are some more exercises that deals with forces and work. Make sure you practice IDEA.

A hockey puck (m = 160 gram) is hit and slides over the ice-floor. It starts at an initial velocity of 10 m/s. The hockey puck experiences a frictional force from the ice that can be modeled as $-\mu F_g$ with F_g the gravitational force on the puck. The friction force eventually stops the motion of the puck. Then the friction is zero (otherwise it would accelerate the puck from rest to some velocity :smiley:). Constant $\mu = 0.01$.

- Set up the equation of motion (i.e. N2) for m.

- Solve the equation of motion and find the trajectory of the puck.
- Calculate the amount of work done by gravity by solving the integral $W_{12} = \int_1^2 \vec{F} \cdot d\vec{r}$.
- Show that the amount of work calculated is indeed equal to the change in kinetic energy.

Solve this exercise using IDEA.

```
‘‘‘{exercise}
:label: 43


A hockey puck (m = 160 gram) is hit and slides over the ice -floor. It starts at an initial ve
<ol type="a">
 <li>Set up the equation of motion (i.e. N2) for m.</li>
 <li>Solve the equation of motion and find the trajectory of the puck.</li>
    <li>Calculate the amount of work done by gravity by solving the integral $ W_{12} = \int_1
 <li>Show that the amount of work calculated is indeed equal to the change in kinetic energy.<
</ol>
Solve this exercise using IDEA.

First try it yourself (and try seriously) before peeking at the [solution](Solutions/ExerciseH
```

0.2.4 Potential Energy

Conservative force

Work done on m by F during motion from 1 to 2 over a prescribed trajectory, is defined as:

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} \tag{71}$$

In general, the amount of work depends on the path followed. That is, the work done when going from $\vec{r}_1$ to $vecr_2$ over the red path in the figure below, will be different when going from $\vec{r}_1$ to $\vec{r}_2$ over the blue path. Work depends on the specific trajectory followed.

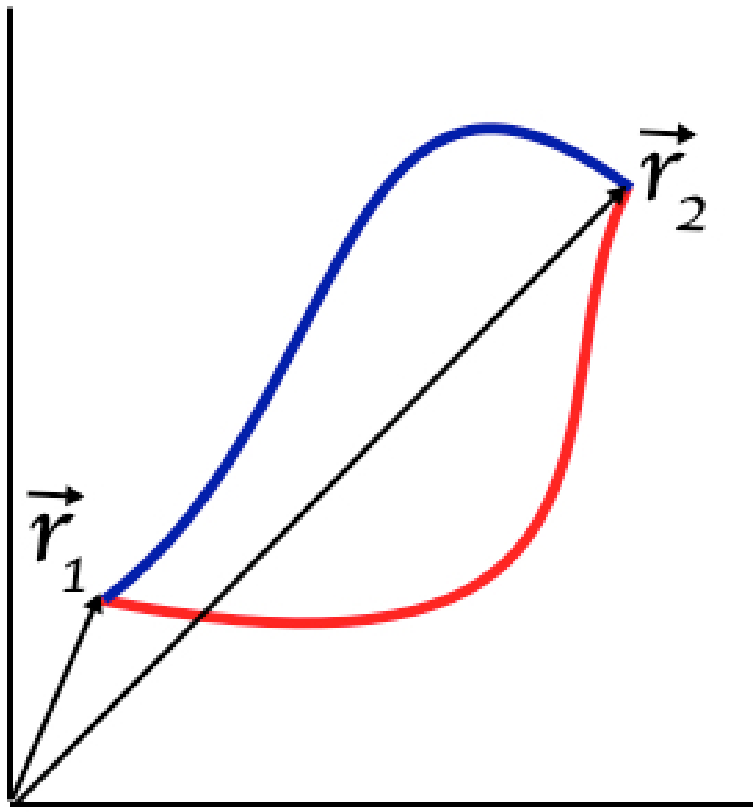


Figure 12: Two different paths.

However, there is a certain class of forces for which the path does not matter, only the start and end point do. These forces are called conservative forces. As a consequence, the work done by a conservative force over a closed path, i.e start and end are the same, is always zero. No matter which closed path is taken.

$$\text{conservative force } \Leftrightarrow \oint \vec{F} \cdot d\vec{r} = 0 \text{ for } \textbf{ALL} \text{ closed paths} \tag{72}$$

Stokes' Theorem

It was George Stokes who proved an important theorem, that we will use to turn the concept of conservative forces into a new and important concept.

His theorem reads as:

Figure 13: Sir George Stokes (1819-1903). From Wikimedia Commons, public domain.

$$\oint \vec{F} \cdot d\vec{r} = \iint \vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} \tag{73}$$

In words: the integral of the force over a closed path equals the surface integral of the curl of that force. The surface being 'cut out' by the close path. The term $\vec{\nabla} \times \vec{F}$ is called the curl of F: it is a vector. The meaning of it and some words on the theorem are given below.

Intermezzo: intuitive proof of Stokes' Theorem

Consider a closed curve in the xy-plane. We would like to calculate the work done when going around this curve. In other words: what is $\oint \vec{F} \cdot d\vec{r}$ if we move along this curve?

We can visualize what we need to do: we cut the curve in small part; compute $\vec{F} \cdot d\vec{r}$ for each part (i.e. the red, green, blue, etc. in fig.(\ref{Stokes2.jpg}) and sum these to get the total along the curve. If we make the parts infinitesimally small, we go from a (Riemann) sum to an integral.

We are going to compute much more: take a look at Figure **??**. We have put a grid in the xy-plane over a closed curve Γ. Hence, the interior of our curve is fool op squares. We are not only computing the parts along the curve, but also along the sides of alle curves. This will sound like way too much work, but we will see that it actually is a very good idea.

See Figure **??**: we calculate $\oint \vec{F} \cdot d\vec{r}$ counter clockwise for the green square. Than we have at least the green part of our $\oint \vec{F} \cdot d\vec{r}$ done in the right direction. Hence, we compute $\int \vec{F} \cdot d\vec{r}$ along the right side of the green square. We do that from bottom to top as we go counter clockwise along the green square. Let's call that $\int_g \vec{F} \cdot d\vec{r}$.

Then we move to the blue square and repeat in counter clockwise direction our calculation. But this means that we compute along the left side of blue the square from top to bottom. We will call this $\int_b \vec{F} \cdot d\vec{r}$.

Note that we will add all contributions. Thus we get $\int_g \vec{F} \cdot d\vec{r} + \int_b \vec{F} \cdot d\vec{r}$. But these two cancel each other as they are exactly the same but done in opposite directions. Thus if we use that $\int_1^2 f dx = -\int_2^1 f dx$ for any integration, it becomes obvious that $\int_g \vec{F} \cdot d\vec{r} + \int_b \vec{F} \cdot d\vec{r} = 0$.

Note that this will happen for all side of the squares that are in the interior of our curve. Thus, the integral over all squares is exactly the integral along the curve Γ.

It seems, we do a lot of work for nothing. But there is another way of looking at the path-integrals along the squares. If we make the square small enough, the calculation along one square can be approximated:

$$\begin{aligned} \oint_{square} \vec{F} \cdot d\vec{r} &\approx F_x(x,y)dx + F_y(x+dx,y)dy - F_x(x,y+dy)dx - F_y(x,y)dy \\ &\approx \frac{F_x(x,y) - F_x(x,y+dy)}{dy} dxdy + \frac{F_y(x+dx,y) - F_y(x,y)}{dx} dxdy \\ &\approx \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dxdy \end{aligned} \tag{74}$$

The results get more accurate the smaller we make the square.

If we now sum up all squares and make these squares infinitesimally small, the sum becomes an integral, but now an integral over the surface enclosed by the curve:

$$\oint_\Gamma \vec{F} \cdot d\vec{r} = \iint \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dxdy \tag{75}$$

The right hand side of the above equation is an surface integral of the 'vector' $\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$. Obviously, we did not provide a rigorous proof, but only an intuitive one. For a mathematical proof, see your calculus classes.

Moreover, we only worked in the xy-plane. If we would extend our reasoning to a closed curve in 3 dimensions, we would get Stokes theorem, which reads as:

$$\oint_\Gamma \vec{F} \cdot d\vec{r} = \iint \vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} \tag{76}$$

Here, $d\vec{\sigma}$ is a small element out of the surface. Note that it is a vector: it has size and directions. The latter is perpendicular to the surface element itself. Furthermore, we have the vector $\vec{\nabla} \times \vec{F}$. This is the cross-product of the nabla-operator and our vector field $\vec{F}$. The nabla operator is a strange kind of vector. Its components are: partial differentiation. In a Cartesian coordinate system this can be written as:

$$\vec{\nabla} \equiv \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} \tag{77}$$

or if you prefer a column notation:

$$\vec{\nabla} \equiv \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \tag{78}$$

The curl of F can be found from e.g.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right)\hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right)\hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)\hat{z} \tag{79}$$

Note of warning: do be careful with the nabla-operator. It is not a standard vector. For instance, ordinary vectors have the property $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$. This does not hold for the nabla-operator.

Second note of warning: the representation of the nabla-operator does change quite a bit when using other coordinate systems like cylindrical or spherical. For instance, in cylindrical coordinates it is ***not*** equal to $\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial z} \end{pmatrix}$. This can be easily seen as both r, z have units length, i.e. meters, but ϕ has no units.

Example 5.1
Suppose we need to calculate the integral of the vectorfield $\vec{F}(x, y) = y\hat{x} - x\hat{y}$ over the closed curve formed by a square from $(0, 0)$ to $(1, 0)$, $(1, 1)$, $(0, 1)$ and back to $(0, 0)$.
We go counter clockwise.

$$\begin{aligned} \oint \vec{F} \cdot d\vec{r} &= \int_{x=0}^{1} F_x(x, y = 0)dx + \int_{y=0}^{1} F_y(x = 1, y)dy + \\ &+ \int_{x=1}^{0} F_x(x, y = 1)dx + \int_{y=1}^{0} F_y(x = 0, y)dy \\ &= \int_0^1 0dx + \int_0^1 -1dy + \int_1^0 1dx + \int_1^0 -0dx \\ &= 0 - [y]_0^1 + [x]_1^0 - 0 \\ &= -2 \end{aligned} \tag{80}$$

Now we try this using Stokes' Theorem:

$$\oint \vec{F} \cdot d\vec{r} = \iint \vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} \tag{81}$$

We first calculate $\vec{\nabla} \times \vec{F}$:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = \left(\frac{\partial(-x)}{\partial x} - \frac{\partial(y)}{\partial y} \right) \hat{z} = -2\hat{z} \tag{82}$$

Thus, in this example $\vec{\nabla} \times \vec{F}$ has only a z-component.

An elementary surface element of the square is: $d\vec{\sigma} = dxdy\hat{z}$. This also has only a z-component. Note that it points in the positive z-direction. This is a consequence of the counter clockwise direction that we use to go along the square.

According to Stokes Theorem, we this find:

$$\oint \vec{F} \cdot d\vec{r} = \iint \vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} = \int_{x=0}^{1} \int_{y=0}^{1} (-2)dxdy = -2 \tag{83}$$

Indeed, we find the same outcome.

Conservative force and $\vec{\nabla} \times \vec{F}$

For a conservative force the integral over the closed path is zero for any closed path. Consequently, $\vec{\nabla} \times \vec{F} = 0$ everywhere. How do we know this? Suppose $\vec{\nabla} \times \vec{F} \neq 0$ at some point in space. Then, since we deal with continuous differentiable vector fields, in the close vicinity of this point, it must also be non-zero. Without loss of generality, we can assume that in that region $\vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} > 0$. Next, we draw a closed curve around this point, in this region. We now calculate the $\oint \vec{F} \cdot d\vec{r}$ along this curve. That is, we invoke Stokes Theorem. But we know that $\vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} > 0$ on the surface formed by the closed curve. Consequently, the outcome of the surface integral is non-zero. But that is a contradiction as we started with a conservative force and thus the integral should have been zero.

The only way out, is that $\vec{\nabla} \times \vec{F} = 0$ everywhere.

Thus we have:

$$\text{conservative force } \Leftrightarrow \vec{\nabla} \times \vec{F} = 0 \text{ everywhere} \tag{84}$$

Potential Energy

A direct consequence of the above is:

if $\vec{\nabla} \times \vec{F} = 0$ everywhere, a function $V(\vec{r}))$ exists such that $\vec{F} = -\vec{\nabla} V$

$$\text{conservative force } \Leftrightarrow \vec{\nabla} \times \vec{F} = 0 \text{ everywhere } \Updownarrow \vec{F} = -\vec{\nabla} V \Leftrightarrow V(\vec{r}) = -\int_{ref} \vec{F} \cdot d\vec{r} \tag{85}$$

where in the last integral, the lower limit is taken from some, self picked, reference point. The upper limit is the position $\vec{r}$.

This function V is called the potential energy or the potential for short. It has a direct connection to work and kinetic energy.

$$E_{kin,2} - E_{kin,1} = W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} = V(\vec{r}_2) - V(\vec{r}_1) \tag{86}$$

or rewritten:

$$E_{kin,1} + V(\vec{r}_1) = E_{kin,2} + V(\vec{r}_2) \tag{87}$$

In words: **for a conservative force, the sum of kinetic and potential energy stays constant**.

Energy versus Newton's Second Law We, starting from Newton's Laws, arrived at an energy formulation for physical problems.

Question: can we also go back? That is: suppose we would start with formulating the energy rule for a physical problem, can we then back out the equation of motion?

Answer: yes, we can!

It goes as follows. Take a system that can be completely described by its kinetic plus potential energy. Then: take the time-derivative and simplify, we will do it for a 1-dimensional case first.

$$\begin{aligned}
&\frac{1}{2}mv^2 + V(x) = E_0 \Rightarrow \\
&\frac{d}{dt}\left[\frac{1}{2}mv^2 + V(x)\right] = \frac{dE_0}{dt} = 0 \Rightarrow \\
&m v \dot{v} + \frac{dV}{dx}\underbrace{\frac{dx}{dt}}_{=v} = 0 \Rightarrow \\
&v\left(m\dot{v} + \frac{dV}{dx}\right) = 0
\end{aligned} \tag{88}$$

The last equation must hold for all times and all circumstances. Thus, the term in brackets must be zero.

$$m\dot{v} + \frac{dV}{dx} = 0 \Rightarrow m\ddot{x} = -\frac{dV}{dx} = F \tag{89}$$

And we have recovered Newton's second law.

In 3 dimensions it is the same procedure. What is a bit more complicated, is using the chain rule. In the above 1-d case we used $\frac{dV}{dt} = \frac{dV(x(t))}{dt} = \frac{dV}{dx}\frac{dx(t)}{dt}$. In 3-d this becomes:

$$\frac{dV}{dt} = \frac{dV(\vec{r}(t))}{dt} = \frac{dV}{d\vec{r}} \cdot \frac{d\vec{r}(t)}{dt} = \vec{\nabla}V \cdot \vec{v} \tag{90}$$

Thus, if we repeat the derivation, we find:

$$\begin{aligned}
&\frac{1}{2}mv^2 + V(\vec{r}) = E_0 \Rightarrow \\
&\frac{d}{dt}\left[\frac{1}{2}mv^2 + V(\vec{r})\right] = 0 \Rightarrow \\
&m\vec{v} \cdot \dot{\vec{v}} + \vec{\nabla}V \cdot \vec{v} = 0 \Rightarrow \\
&v\left(m\vec{a} + \vec{\nabla}V\right) = 0 \Rightarrow \\
&m\vec{a} = -\vec{\nabla}V = \vec{F}
\end{aligned} \tag{91}$$

And we have recovered the 3-dimensional form of Newton's second Law. This is a great result. It allows us to pick what we like: formulate a problem in terms of forces and momentum, i.e. Newton's second law, or reason from energy considerations. It doesn't matter: they are equivalent. It is a matter of taste, a matter of what do you see first, understand best, find easiest to start with. Up to you!

Stable/Unstable Equilibrium

A particle (or system) is in equilibrium when the sum of forces acting on it is zero. Then, it will keep the same velocity, and we can easily find an inertial system in which the particle is at rest, at an equilibrium position.

The equilibrium position (or more general state) can also be found directly from the potential energy.

Potential energy and (conservative) forces are coupled via:

$$\vec{F} = -\vec{\nabla} V \tag{92}$$

The equilibrium positions $\left(\sum_i \vec{F}_i = 0\right)$ can be found by finding the extremes of the potential energy:

$$\text{equilibrium position } \Leftrightarrow \vec{\nabla} V = 0 \tag{93}$$

Once we find the equilibrium points, we can also quickly address their nature: is it a stable or unstable solution? That follows directly from inspecting the characteristics of the potential energy around the equilibrium points.

For a stable equilibrium, we require that a small push or a slight displacement will result in a force pushing back such that the equilibrium position is restored (apart from the inertia of the object that might cause an overshoot or oscillation).

However, an unstable equilibrium is one for which the slightest push or displacement will result in motion away from the equilibrium position.

The second derivative of the potential can be investigated to find the type of extremum. For 1D functions that is easy, for scalar valued functions of more variables that is a bit more complicated. Here we only look at the 1D case $V(x) : \mathbb{R} \rightarrow \mathbb{R}$

$$\text{equilibrium: } \vec{\nabla} V = 0 \begin{cases} \text{stable:} & \frac{d^2V}{dx^2} > 0 \\ \text{unstable:} & \frac{d^2V}{dx^2} < 0 \end{cases} \tag{94}$$

Luckily, the definition of potential energy is such that these rules are easy to visualize in 1D and remember, see fig.(?.?).

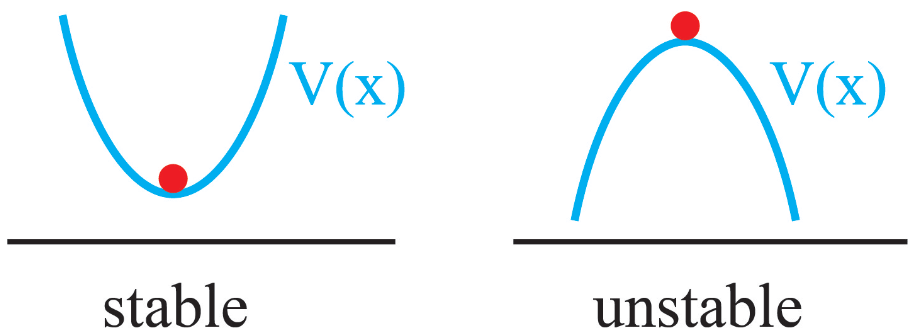


Figure 16: Stable and unstable position of a particle in a potential.

A valley is stable; a hill top is unstable.

NB: Now the choice of the minus sign in the definition of the potential is clear . Otherwise a hill would be stable, but that does not feel natural at all.

It is also easy to visualize what will happen if we distort that particle from the equilibrium state:

- The valley, i.e., the stable system, will make the particle move back to the lowest point. Due to inertia, it will not stop but will continue to move. As the lowest position is one of zero force, the particle will 'climb' toward the other end of the valley and start an oscillatory motion.

- The top, i.e., the unstable point, will make the particle move away from the stable point. The force acting on the particle is now pushing it outwards, down the slope of the hill.

Taylor Series Expansion of the Potential The Taylor expansion or Taylor series is a mathematical approximation of a function in the vicinity of a specific point. It uses an infinite series of polynomial terms with coefficients given by value of the derivative of the function at that specific point: the more terms you use, the better the approximation. If you use all terms, then it is exact. Mathematically, it reads for a 1D scalar function $f : \mathbb{R} \to \mathbb{R}$:

$$f(x) \approx f(x_0) + \frac{1}{1!}f'(x_0)(x-x_0) + \frac{1}{2!}f''(x_0)(x-x_0)^2 + \frac{1}{3!}f'''(x_0)(x-x_0)^3 + ... \tag{95}$$

For our purpose here, it suffices to stop after the second derivative term:

$$f(x) \approx f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \mathcal{O}(x^3) \tag{96}$$

A way of understanding why the Taylor series actually works is the following.

Imagine you have to explain to someone how a function looks around some point x_0, but you are not allowed to draw it. One way of passing on information about $f(x)$ is to start by giving the value of $f(x)$ at the point x_0:

$$f(x) \approx f(x_0) \tag{97}$$

Next, you give how the tangent at x_0 is; you pass on the first derivative at x_0. The other person can now see a bit better how the function changes when moving away from x_0:

$$f(x) \approx f(x_0) + f'(x_0)(x-x_0) \tag{98}$$

Then, you tell that the function is not a straight line but curved, and you give the second derivative. So now the other one can see how it deviates from a straight line:

$$f(x) \approx f(x_0) + \frac{1}{1!}f'(x_0)(x-x_0) + \frac{1}{2!}f''(x_0)(x-x_0)^2 \tag{99}$$

Note that the prefactor is placed back. But the function is not necessarily a parabola; it will start deviating more and more as we move away from x_0. Hence we need to correct that by invoking the third derivative that tells us how fast this deviation is. And this process can continue on and on.

Important to note: if we stay close enough to x_0 the terms with the lowest order terms will always prevail as higher powers of $(x-x_0)$ tend to zero faster than a lower powers (for instance: $0.5^4 << 0.5^2$).

Here is a youtube movie that explains the 1D Taylor series nicely (for physicists).

For scalar valued functions as our potentials $V(\vec{r}) : \mathbb{R}^3 \to \mathbb{R}$ the extension of the Taylor series is not too difficult. If we expand the function around a point

$$\begin{aligned} V(\vec{r}) &\approx V(\vec{r}_0) + \vec{\nabla}V(\vec{r}_0)\cdot(\vec{r}-\vec{r}_0) \\ &\quad + \frac{1}{2}(\vec{r}-\vec{r}_0)\cdot(\partial^2 V)(\vec{r}_0)\cdot(\vec{r}-\vec{r}_0) + \mathcal{O}(r^3) \end{aligned} \tag{100}$$

The second derivative of the potential indicated by $\partial^2 V$ is the Hessian matrix.

Conceptually the extrema of the function are again the hills and valleys. The classification of the extrema has next to hills and valleys also saddle points etc. In this course we will not bother about these more dimensional cases, but only stick to simple ones.

Exercises

Example 5.2
Is gravity $\vec{F}_g = m\vec{g}$ a conservative force? If yes, what is the corresponding potential energy?
To find the answer we can do two things:

a. Show $\vec{\nabla} \times m\vec{g} = 0$

b. Find a V that satisfies $-m\vec{g} = -\vec{\nabla}V$

Solution Example 5.2
a. $\vec{\nabla} \times m\vec{g} = 0$? How to compute it? For **Cartesian** coordinates there is an easy to remember rule:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \tag{101}$$

If we chose our coordinates such that $\vec{g} = -g\hat{z}$ we get:

$$\vec{\nabla} \times \vec{F}_g = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & -mg \end{vmatrix} = 0 \tag{102}$$

Thus $\vec{F}_g$ is conservative.

b. Secondly: does $-m\vec{g} = -\vec{\nabla}V$ have a solution for V? Let's try, using the same coordinates as above.

$$\begin{aligned} -\vec{\nabla}V &= -m\vec{g} \Rightarrow \\ \frac{\partial V}{\partial x} &= 0 \rightarrow V(x,y,z) = f(y,z) \\ \frac{\partial V}{\partial y} &= 0 \rightarrow V(x,y,z) = g(x,z) \\ \frac{\partial V}{\partial z} &= mg \rightarrow V(x,y,z) = mgz + h(x,y) \end{aligned} \tag{103}$$

f,g,h are unknow functions. But all we need to do, is find one V that satisfies $-m\vec{g} = -\vec{\nabla}V$.

So, if we take $V(x,y,z) = mgz$ we have shown, that gravity in this form is conservative and that we can take $V(x,y,z) = mgz$ for its corresponding potential energy.

By the way: from the first part (curl F = 0), we know that the force is conservative and we know that we could try to find V from

$$\begin{aligned} V(x,y,z) &= -\int_{ref} m\vec{g} \cdot d\vec{r} = \int_{ref} mg\hat{z} \cdot d\vec{r} \\ &= \int_{ref} mgdz = mgz + const \end{aligned} \tag{104}$$

A simple model for the frictional force experienced by a body sliding over a horizontal, smooth surface is $F_f = -\mu F_g$ with F_g the gravitational force on the object.The friction force is opposite the direction of motion of the object.

- Show that this frictional force is not conservative (and, consequently, a potential energy associated does not exist!).

Hint: think of two different trajectories to go from point 1 to point 2 and show that the amount of work along these trajectories is not the same.

A force is given by: $\vec{F} = x\hat{x} + y\hat{y} + z\hat{z}$

1. Show that this force is conservative.
2. Find the corresponding potential energy.

A second force is given by: $\vec{F} = y\hat{x} + x\hat{y} + z\hat{z}$

1. Show that this force is also conservative.
2. Find the corresponding potential energy.

Another force is given by: $\vec{F} = y\hat{x} - x\hat{y}$

1. Show that this frictional force is not conservative.
2. Compute the work done when moving an object over the unit circle in the xy-plane in an anti-clockwise direction.

   Hint: use Stokes theorem.
3. Discuss the meaning of your answer: is it positive or negative? And what does that mean in terms of physics?

Given a potential energy $E_{pot} = xy$.

a. Find the corresponding force (field).

b. Make a plot of $\vec{F}$ as a function of (x,y,z).

c. Describe the force and comment on what the potential itself already reveals about the force.

Given a force field $\vec{F} = -xy\hat{x} + xy\hat{y}$. A particle moves from $(x, y) = (0, 0)$ over the x-axis to $(x, y) = (1, 0)$ and then parallel to the y-axis to $(x, y) = (1, 1)$. In a second motion, the same particle goes from $(x, y) = (0, 0)$ over the y-axis to $(x, y) = (0, 1)$ and then parallel to the x-axis to end also in $(x, y) = (1, 1)$.

1. Show that not necessarily the work done over the two paths is equal.
2. Compute the amount of work done over each of the paths.

A particle of mass m is initially at position x=0. It has zero velocity.
On the particle a force is acting. The force can be described by F = F_0 \sin \frac{x}{L} with F_0 and L positive constants.

a. Show that this force is conservative and find the corresponding potential. Take as reference point for the potential energy x = \frac{\pi}{2} L .
b. The particle gets a tiny push, such that it starts moving in the positive x-direction. Its initial velocity is so small that, for all practical calculations, it can be set to zero. Under the influence of the force, the particle will move. Why did the particle get its tiny push?
c. Find the maximum velocity that the particle can get. At which location(s) will this take place?.

Note: this is a 1-dimensional problem.

Answers

0.3 Special Relativity

0.3.1 Special Relativity - Lorentz Transformation

As we discusses in the second half of the nineteenth century it became clear that there was something wrong in classical mechanics. However, people would not easily give up the ideas of classical mechanics. We saw that the luminiferous aether was introduced as a cure and as a medium in which Electromagnetic waves could travel and oscillate. Moreover, Lorentz and Fitzgerald managed managed to find a coordinate transformation that made the wave equation of Maxwell invariant. Fitzgerald came even up with length contraction: if the arm moving parallel to the aether of the interferometer of Michelson and Morley would contract according to $L_n = L\sqrt{1-\frac{V^2}{c^2}}$ then, the M&M experiment should result in no time difference for the two paths, in agreement with the experimental findings. However, there was no fundamental reasoning, no physics under pinning the transformation and the length contraction. The proposals worked, but they had an ad hoc character. very unsatisfying for physicists!

And as we have mentioned, it took the work of a single man to change this and underpin the Lorentz Transformation, making Classical Mechanics a valid limit of Relativity Theory, only applicable at velocities small compared to the speed of light and to small distances compared to those of interest in cosmology.

Lorentz Transformation

$$\begin{split}
ct' &= \gamma \left(ct - \frac{V}{c} x \right) \\
x' &= \gamma \left(x - \frac{V}{c} ct \right) \\
y' &= y \\
z' &= z
\end{split} \tag{105}$$

with

$$\gamma = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \tag{106}$$

But there is more! Einstein also changed our view on the universe and on time itself. In the world of Newton and Galilei, people could not even think about relativity of time. Of course time was the same for everyone. There was only one time, one master clock - the same for all of us. It is hard coded in the Galilei Transformation:

```
\begin{split}
ct’ &= \gamma \left ( ct - \frac{V}{c} x \right ) \\
x’ &= \gamma \left ( x - \frac{V}{c} ct \right ) \\
y’ &= y \\
z’ &= z
\end{split}
```

	= t
x'	= x - Vt
y'	= y
z'	= z

Galilei Transformation