Clemson Analysis Prelim Solutions

The Monster Group

October 16, 2024

Contents

Summer 2020

1. Let X be a Banach space. Suppose $\{x_n\} \subseteq X$ is a sequence such that for every $\varepsilon > 0$ there is a convergent sequence $\{y_n\}$ in X with $||y_n - x_n|| < \varepsilon$, for all $n \in \mathbb{N}$.

   (a) Prove that $\{x_n\}$ is convergent.

   Proof. As $\{y_n\}$ is convergent, $||y_n - y_m|| < \eta$ for m, n large enough. Therefore

   $$||x_n - x_{n+t}|| \leq ||x_n - y_n|| + ||y_n - y_{n+1}|| + ... + ||y_{n+t} - x_{n+t}|| < 2\varepsilon + t\eta.$$

   □

   (b) Prove, by providing a counterexample, that (1) is not necessarily true if the space X is assumed only to be a normed linear space.

   Proof. □

2. Let $C[0,1]$ be the space of all continuous functions on $[0,1]$ equipped with the usual supremum norm $||f||_\infty = \sup_{x\in[0,1]} |f(x)|$. Consider the functional $l : C[0,1] \to \mathbb{C}$ defined by

   $$l(f) = \int_0^1 f(x)\, dx - f(\frac{1}{2}).$$

   Prove that l is a bounded linear functional and find its norm.

   Proof.

   $$|l(f)| \leq \int_0^1 |f(x)|\, dx + |f(1/2)| \leq 2||f||_\infty.$$

   To see the opposite inequality, construct f_n's as follows: constant 1 on 0 to $1/2 - 1/n$, linear on $1/2 - 1/n$ to $1/2$, $f_n(1/2) = -1$, linear $1/2$ to $1/2 + 1/n$, then constant 1 on $1/2 + 1/n$ to 1. *This looks like a V-shape. Draw it!*

   Each $||f_n||_\infty = 1$, and they tend towards the constant 1 function with a point discontinuity at $f(1/2) = -1$. The integral is 1, then subtracting -1 gives $l(f) = 2$. Hence $||l|| = 2$. □

3. Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator. Suppose that there exists $c > 0$ such that

   $$\langle Tx, x\rangle \geq c||x||^2, \quad \forall x \in \mathcal{H}.$$

   (a) Prove that T is injective

   Proof. Let $x, y \in \text{Kern}(T)$. Then

   $$c||x - y||^2 \leq \langle T(x - y), x - y\rangle = \langle Tx, x - y\rangle - \langle Ty, x - y\rangle = 0.$$

   Thus $x = y$, so the kernel is trivial: $\text{Kern}(T) = \{0\}$. □

(b) Prove that the range of T is a closed subspace of $\mathcal{H}$.

Proof. Suppose $\{x_n\} \subseteq \mathcal{H}$ such that $Tx_n \to y$. Using $c||x||^2 \leq |\langle Tx, x\rangle| \leq ||Tx||\,||x|| \Rightarrow c||x|| \leq ||Tx||$, we see $c||x_n - x_m|| \leq ||Tx_n - Tx_m||$.
Thus $\{Tx_n\}$ Cauchy implies $\{x_n\}$ Cauchy, so $x_n \to x$. Therefore $Tx_n \to Tx$, which is equivalent to $Tx = y$, so the range is closed. □

(c) Prove that T is invertible.

Proof. For $u \in \mathrm{Ran}(T)^\perp$, note that $c||u||^2 \leq \langle Tu, u\rangle$ implies $u = 0$. That is, $\mathrm{Ran}(T)^\perp = \{0\}$. Then

$$\mathrm{Ran}(T) = \overline{\mathrm{Ran}(T)} = (\mathrm{Ran}(T)^\perp)^\perp = \{0\}^\perp = \mathcal{H}.$$

As the range is all of $\mathcal{H}$, for all y there exists a unique x such that $Tx = y$; i.e. T is invertible. □

4. Let X be a normed linear space and $S, T : X \to X$ be two linear operators.

(a) Show that if $ST - TS$ commutes with S, then for all $n \in \mathbb{N}$

$$S^nT - TS^n = nS^{n-1}(ST - TS).$$

Proof. For starters, we'll prove the $n = 2$ case.

$$\begin{aligned} S^2T - TS^2 &= S^2T - STS + STS - TS^2 \\ &= S(ST - TS) + (ST - TS)S \\ &= S(ST - TS) + S(ST - TS) \\ &= 2S(ST - TS). \end{aligned}$$

With this, we induct. Suppose that it's true for $n - 1$.

$$\begin{aligned} S^nT - TS^n &= S^nT - S^{n-1}TS + S^{n-1}TS - STS^{n-1} + STS^{n-1} - TS^n \\ &= S^{n-1}(ST - TS) + S(S^{n-2}T - TS^{n-2})S + (ST - TS)S^{n-1} \\ &= 2S^{n-1}(ST - TS) + S((n-2)S^{n-3}(ST - TS))S \\ &= 2S^{n-1}(ST - TS) + (n-2)S^{n-1}(ST - TS) \\ &= nS^{n-1}(ST - TS) \end{aligned}$$

□

(b) Show that there does not exist bounded linear operators $S, T : X \to X$ such that $ST - TS = I$, where $I : X \to X$ denotes the identity operator.
In finite dimensions, over $\mathrm{Mat}_n(\mathbb{R})$*, this is a simple consequence of the trace function. As an exercise, show that it* ***is*** *possible over* $\mathrm{Mat}_n(\mathbb{Z}_p)$ *for prime characteristic. Over infinite dimensions, this is important for quantum mechanical reasons. See Stone-von Neumann theorem.*

Proof. By above, $S^nT - TS^n = nS^{n-1}$. Then, by taking norms,

$$\begin{aligned} n||S^{n-1}|| &= ||S^nT - TS^n|| \\ &\leq ||S^nT|| + ||TS^n|| \\ &\leq ||S^n||\ ||T|| + ||T||\ ||S^n|| \\ &\leq 2||S^{n-1}||\ ||S||\ ||T||. \end{aligned}$$

Hence $n \leq 2||T||\ ||S||$ for all $n \in \mathbb{N}$. But this is in opposition to the boundedness of T and S. □

5. Consider $(\mathbb{R}, \mathcal{B}, \lambda)$, that is the real line with Lebesgue measure λ defined on the Borel sets $\mathcal{B}$. For $n \in \mathbb{N}$ define $f_n(x) := n\chi_{[1/n,2/n]}(x)$, where $\chi_{[1/n,2/n]}$ denotes the usual characteristic (indicator) function of $[1/n, 2/n]$. Decide which of the following convergence statements are true, and justify your answers.

   (a) The sequence $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to the constant 0 function.

   Proof. True. This function tends towards a tall, thin rectangle right above the origin. □

   (b) The sequence $\{f_n\}_{n\in\mathbb{N}}$ converges in $L^p(\mathbb{R}, \mathcal{B}, \lambda)$ to the constant 0 function, for every $1 \leq p < \infty$.

   Proof. Not true.

$$||f_n(x)||_p^p = \int |n\chi_{[1/n,2/n]}|^p d\lambda = n^p \int \chi_{[1/n,2/n]} d\lambda = n^{p-1}.$$

□

   (c) The sequence $\{f_n\}_{n\in\mathbb{N}}$ converges in measure to the constant 0 function.

   Proof. True. By Chebyshev's,

$$\lambda(\{|f_n(x) \geq n|\}) \leq \frac{1}{n}\int |f_n(x)| d\lambda = \frac{1}{n}.$$

□

6. Prove that for every $\varepsilon > 0$ there exists an open set $E \subseteq \mathbb{R}$ with Lebesgue measure $\lambda(E) < \varepsilon$ which is *dense* in $\mathbb{R}$.

Proof. Enumerate $\mathbb{Q}$ via the rationals. For any $\varepsilon > 0$, construct

$$A = \cup_{q_n\in\mathbb{Q}}(q_n - \frac{\varepsilon}{2^{n+1}},\ q_n + \frac{\varepsilon}{2^{n+1}}).$$

This set is open, it covers every rational so is dense, and one can calculate $\lambda(A) \leq \varepsilon$. □

7. Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose $f_n : X \to \mathbb{R}$ is a sequence of integrable functions such that $\int_X |f_n(x)|\, d\mu(x) \leq 1$ for all $n \in \mathbb{N}$.

   (a) Prove that $\lim_{n\to\infty} \frac{f_n(x)}{n^2} = 0$ for a.e. $x \in X$.

   Proof. Note
   $$\sum_{n=1}^{\infty} \int_X \frac{f_n(x)}{n^2}\, d\mu \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty.$$
   Thus
   $$\sum_{n=N}^{\infty} \int_X \frac{f_n(x)}{n^2}\, d\mu \to 0$$
   as $N \to \infty$. By applying Fubini, we see
   $$\sum_{n=N}^{\infty} \frac{f_n(x)}{n^2} \to 0 \text{ a.e.},$$
   thus $\frac{f_n(x)}{n^2} \to 0$ a.e.. $\square$

   (b) Prove, by providing an example, that $\lim_{n\to\infty} \frac{f_n(x)}{n} = 0$ for a.e. $x \in X$ doesn't have to be true.

   Proof. Let $f_n = (n - \frac{1}{n})\chi_{(\frac{1}{n}, n)}$. Each f_n integrates to 1, but
   $$\frac{(n - \frac{1}{n})\chi_{(\frac{1}{n}, n)}}{n} = \left(1 - \frac{1}{n^2}\right)\chi_{(\frac{1}{n}, n)} \to \chi_{(0,\infty)}.$$
   $\square$

8. *This problem looks annoying, so it has been excluded.*

Winter 2020

1. Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space.

   (a) Prove that every convergent sequence in $(X, \langle \cdot, \cdot \rangle)$ is bounded.

   Proof. By contrapositive, not bounded implies not convergent. □

   (b) Prove that if $x_n \to x$, $y_n \to y$, then $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

   Proof. By boundedness, let $||x_n|| < M$ for all n. Then

$$\begin{aligned}
|\langle x_n, y_n \rangle - \langle x, y \rangle| &= |\langle x_n, y_n \rangle - \langle x_n, y \rangle + \langle x_n, y \rangle - \langle x, y \rangle| \\
&= |\langle x_n, y_n - y \rangle - \langle x_n - x, y \rangle| \\
&< M||y_n - y|| + ||x_n - x|| \, ||y||.
\end{aligned}$$

   For n large enough, this becomes sufficiently small. □

2. Let $(X, ||\cdot||)$ be a normed linear space and $Y \subset X$ be a proper subspace. Prove that the interior of Y is empty, i.e., $\mathrm{Int}(Y) = \emptyset$.

   Proof. Suppose not. Then there exists some open ball in Y. Taking successive differences will fill some open ball around the origin, then taking the span will give the whole space. □

3. Let $l^2(\mathbb{N})$ be the space of square summable sequences and define $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ by

$$T(\{x_n\}) = \{x_2 - x_1, x_3 - x_2, ..., x_{n+1} - x_n\}.$$

   Prove that T is bounded and find $||T||$.

   Proof. Recall the AM-GM inequality: $2ab \le a^2 + b^2$.

$$\begin{aligned}
||Tx||^2 &= \sum_{i=1}^{\infty} |x_{i+1} - x_i|^2 \\
&= \sum_{i=1}^{\infty} |x_{i+1}^2 - 2x_{i+1}x_i + x_i^2| \\
&\le \sum_{i=1}^{\infty} |x_{i+1}|^2 + 2|x_{i+1}||x_i| + |x_i|^2 \\
&\le \sum_{i=1}^{\infty} |x_{i+1}|^2 + (|x_{i+1}|^2 + |x_i|^2) + |x_i|^2 \\
&= \sum_{i=1}^{\infty} 2|x_{i+1}|^2 + \sum_{i=1}^{\infty} 2|x_i|^2 \\
&\le \sum_{i=1}^{\infty} 4|x_i|^2 \\
&= 4||x||^2.
\end{aligned}$$

Thus $||Tx|| \leq 2||x||$, so $||T|| \leq 2$.

For equality, note that the standard basis of $\{e_n\}$ satisfies $||Te_n||^2 = 2$. Thus $||T|| = 2$. $\square$

4. Let (X, d) be a compact metric space and let $f : X \to X$ be a continuous function. Suppose that for every $\varepsilon > 0$ there exists $x \in X$ such that $d(x_\varepsilon, f(x_\varepsilon)) < \varepsilon$. Prove that there exists $x \in X$ such that $f(x) = x$.

   Proof. All bounded sequences have a convergent subsequence, so there exists $x_{n_k} \to x$.

   $$d(x, f(x)) \leq d(x, x_{n_k}) + d(x_{n_k}, f(x_{n_k})) + d(f(x_{n_k}), f(x)).$$

   The sequence of numbers $d(x_\varepsilon, f(x_\varepsilon))$ is bounded, so has convergent subsequence. With this and by continuity, these terms can all be made arbitrarily small. $\square$

5. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$. Prove that if
   $$\sup\{|\langle x, y\rangle| : ||x|| = ||y|| = 1, x \in \mathcal{M}, y \in \mathcal{N}\} < 1,$$
   then $\mathcal{M} + \mathcal{N} = \{x + y : x \in \mathcal{M}, y \in \mathcal{N}\}$ is a closed subspace of $\mathcal{H}$.

   Proof. **INCOMPLETE** $\square$

6. Let $(X, \mathcal{S})$ be a measurable space, and let $E_n \in \mathcal{S}$, $n \in \mathbb{N}$, be a sequence of measurable sets. Prove that the set E consisting of all points $x \in X$ that belong to infinitely many of the sets E_n is measurable.

   This is effectively asking you to show that $\limsup(E_n)$ *is measurable, which has appeared on many past prelims.*

   Proof. From base principles, note that

   $$\{\sup_n f_n \geq c\} = \cup_n \{f_n \geq c\}.$$

   By the same reasoning, inf is measurable. The rest follows from the fact that

   $$\limsup_n f_n = \inf_n \sup_{k \geq n} f_k.$$

   $\square$

7. Let $(X, \mathcal{S}, \mu)$ be a measure space and let $f : X \to \mathbb{R}$ be an integrable function, i.e., $f \in L^1(X, \mu)$. Suppose that $E_n \in \mathcal{S}$, $n \in \mathbb{N}$, is a sequence of measurable sets such that $\lim_{n\to\infty} \mu(E_n) = 0$. Prove that

   $$\lim_{n\to\infty} \int_{E_n} f \; d\mu = 0.$$

Proof. Start by noting $f\chi_{E_n}$ is dominated by the integrable $f\chi_X$. Then applying DCT gives

$$\lim_{n\to\infty}\int_{E_n} f\ d\mu = \lim_{n\to\infty}\int_X f\chi_{E_n}\ d\mu = \int_X f\chi_\emptyset\ d\mu = 0.$$

$\square$

8. Let $(X, \mathcal{S})$ be a measure space, and let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of measures on $(X, \mathcal{S})$ such that $\mu_n(X) = 1$ for all $n \in \mathbb{N}$. Prove that $\lambda : \mathcal{S} \to [0, \infty]$ defined by

$$\lambda(F) := \sum_{n=1}^{\infty} \frac{\mu_n(F)}{2^n}, \qquad \text{for } F \in \mathcal{S}$$

is a measure on $(X, \mathcal{S})$ with $\lambda(X) = 1$.

Proof. Clearly,

$$\lambda(X) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Also, $\lambda(\emptyset) = 0$.

Since μ is a measure, for pairwise disjoint $\{A_i\}$, we can apply Fubini:

$$\begin{aligned}
\lambda\left(\cup_i A_i\right) &= \sum_n \frac{\mu_n\left(\cup_i A_i\right)}{2^n} \\
&= \sum_n \frac{\sum_i \mu_n(A_i)}{2^n} \\
&= \sum_i \sum_n \frac{\mu_n(A_i)}{2^n} \\
&= \sum_i \lambda(A_i).
\end{aligned}$$

$\square$

9. Let $f \in L^2[0, \infty)$ and let $G : (0, \infty) \to \mathbb{R}$ be defined by

$$G(t) = \int_0^{\infty} \frac{f(x)}{1 + tx}\ dx.$$

(a) Prove that $\lim_{t\to\infty} G(t) = 0$.

Proof. Let $||f||_2 < M$. By Hölder's,

$$
\begin{aligned}
G(t) &= \int_0^\infty \frac{f(x)}{1+tx}\, dx \\
&\leq \left(\int_0^\infty f(x)^2\, dx\right)^{1/2} \left(\int_0^\infty \frac{1}{1+tx}^2\, dx\right)^{1/2} \\
&\to_{\substack{1+tx=y \\ tdx=dy}} M \left(\int_1^\infty \frac{1}{ty^2}\, dy\right)^{1/2} \\
&= \frac{M}{\sqrt{t}} \to 0.
\end{aligned}
$$

Therefore $G(t) \to 0$. □

(b) Prove that G is continuous at every point of $(0, \infty)$.

Proof. **INCOMPLETE** □

10. Let $(X, \mathcal{S}, \mu)$ be a measure space and let $f : X \to [0, \infty)$ be a non-negative measurable function. Suppose that

$$
\int_X e^{sf(x)}\, d\mu(x) \leq e^{s^2}, \qquad \text{for every } s > 0.
$$

Prove that $\mu(\{x \in X : f(x) > t\}) \leq e^{-\frac{t^2}{4}}$, for every $t > 0$.

Proof. As the exponential is monotonic increasing and invertible, we have the equivalence $\{f(x) \geq t\}$ iff $\{e^{sf(x)} \geq e^{st}\}$. Then by Chebyshev,

$$
\begin{aligned}
\mu(\{f(x) \geq t\}) &= \mu(\{e^{sf(x)} \geq e^{st}\}) \\
&\leq \frac{1}{e^{st}} \int_X e^{sf(x)}\, d\mu \\
&\leq e^{s^2 - st}.
\end{aligned}
$$

Completing the square gives that $s^2 - st = \frac{1}{4}(2s - t)^2 - \frac{t^2}{4}$. Taking the infimum over s shows the minimum is clearly when $2s = t$. Thus

$$
\mu(\{f(x) \geq t\}) \leq \inf_s \{e^{\frac{1}{4}(2s-t)^2 - \frac{t^2}{4}}\} = e^{-\frac{t^2}{4}}.
$$

□

Summer 2021

1. Let $f : [0,1] \to \mathbb{R}$ be a bounded function. Suppose that for every $\varepsilon > 0$ there exists a continuous function $g : [0,1] \to \mathbb{R}$ such that

$$||f - g||_\infty = \sup_{x \in [0,1]} |f(x) - g(x)| < \varepsilon.$$

Prove that f is continuous on $[0,1]$.

Proof. Let $|x - y| < \delta$. By continuity, $|g(x) - g(y)| < \eta$. Hence

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - g(x)| + |g(x) - g(y)| + |g(y) - f(y)| \\ &< ||f - g||_\infty + \eta + ||f - g||_\infty \\ &< 2\varepsilon + \eta. \end{aligned}$$

$\square$

2. Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space and $\{e_n\}_{n=1}^\infty$ be some orthonormal basis of $\mathcal{H}$. Suppose $A, B : \mathcal{H} \to \mathcal{H}$ are two bounded linear operators such that

$$\sum_{n=1}^\infty ||Ae_n||^2 < \infty, \qquad \sum_{n=1}^\infty ||Be_n||^2 < \infty.$$

Prove that for any orthonormal basis $\{f_n\}_{n=1}^\infty$ of $\mathcal{H}$ the series $\sum_{n=1}^\infty \langle Af_n, Bf_n \rangle$ converges and

$$\sum_{n=1}^\infty \langle Af_n, Bf_n \rangle = \sum_{n=1}^\infty \langle Ae_n, Be_n \rangle.$$

Proof. By Cauchy-Schwarz and $2ab \leq a^2 + b^2$,

$$\begin{aligned} |\sum \langle Af_n, Bf_n \rangle| &\leq \sum |\langle Af_n, Bf_n \rangle| \\ &\leq \sum ||Af_n|| \; ||Bf_n|| \\ &\leq \frac{1}{2} \sum ||Af_n||^2 + \frac{1}{2} \sum ||Bf_n||^2, \end{aligned}$$

So the series in question converges.

Recall for ONB $\{e_n\}$, we have that $\langle x, y\rangle = \sum \langle x, e_n\rangle\langle e_n, y\rangle$. Therefore,

$$\begin{aligned}
\sum_n \langle Af_n, Bf_n\rangle &= \sum_n \langle f_n, A^*Bf_n\rangle \\
&= \sum_n \sum_k \langle f_n, e_k\rangle\langle e_k, A^*Bf_n\rangle \\
&= \sum_n \sum_k \langle B^*Ae_k, f_n\rangle\langle f_n, e_k\rangle \\
&= \sum_k \langle B^*Ae_k, e_k\rangle \\
&= \sum_k \langle Ae_k, Be_k\rangle.
\end{aligned}$$

□

3. Let $C[0,1]$ be the Banach space of all real-valued continuous functions on $[0,1]$ equipped with the supremum norm $||f||_\infty = \sup_{x\in[0,1]} |f(x)|$. Consider the subset $M[0,1] := \{f \in C[0,1] : \int_0^1 f(x)\ dx = 0\}$.

   (a) Prove that $M[0,1]$ is a closed subspace and then use this to show that $M[0,1]$ equipped with the same supremum norm is a Banach space itself.

   Proof. Let $\{f_n\}$ be in $M[0,1]$ such that $f_n \to f$. By continuity, there exists a finite number M such that $|f_n(x)| \leq M$ for all x and n. Thus $|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| \leq \varepsilon + M$ for n large enough. This integrable function dominates, so we can exchange integral and limit.

   $$\int_0^1 f\ dx = \lim_{n\to\infty} \int_0^1 f_x\ dx = 0.$$

   Thus f is in our space, so closed. □

   (b) Prove that $l : M[0,1] \to \mathbb{R}$ defined by

   $$l(f) = f(0) + \int_{1/2}^1 f(x)\ dx$$

   is a bounded linear functional and compute its norm. See Summer 2020 #2.

   Proof.

   $$|l(f)| \leq |f(0)| + \int_{1/2}^1 |f(x)|\ dx \leq ||f||_\infty + \frac{1}{2}||f||_\infty = \frac{3}{2}||f||_\infty.$$

   Thus $||l|| \leq \frac{3}{2}$.

Construct (draw a picture) f_n's as follows: $f_n(0) = 1$, linear on 0 to $1/n$, constant -1 on $1/n$ to $1/2-1/n$, linear on $1/2-1/n$ to $1/2+1/n$, constant 1 on $1/2+1/n$ to $1-1/n$, then linear from $1-1/n$ to 1 with $f_n(1) = -1$.

These functions tend towards the (discontinuous) f that looks like two rectangles, one negative from 0 to 1/2 and one positive 1/2 to 1. They're all symmetric, so integrate to 0. Yet, $l(f)$ is the $f(0)$ plus the area 1/2 to 1, so will be $1+1/2 = 3/2$. Thus equality. $\square$

(c) Let l be the linear functional from (b). Does there exist $f \in M[0,1]$ with $||f||_\infty \leq 1$ such that $l(f) = ||l||$?

Proof. It can never be that $l(f) = ||l||$, as we must either violate continuity or integrating to 0. $\square$

4. Let $\mathcal{H}$ be a Hilbert space. Suppose $\{x_n\}$ is a bounded sequence of vectors in $\mathcal{H}$ such that $\langle x_n, x_m \rangle = ||x_m||^2$ for all $n > m$. Prove that $\{x_n\}$ is convergent.

See Summer 2016 #5.

Proof. Note that $||x_m||^2 = |\langle x_n, x_m \rangle| \leq ||x_m||\ ||x_n||$ implies that $||x_m|| \leq ||x_n||$ for $m < n$. That is, $\{||x_m||\}$ is an increasing sequence. But increasing and bounded implies that $\{||x_m||\}$ is convergent, thus Cauchy. Hence

$$\begin{aligned} ||x_n - x_m||^2 &= \langle x_n - x_m, x_n - x_m \rangle \\ &= \langle x_n, x_n \rangle - \langle x_n, x_m \rangle - \langle x_m, x_n \rangle + \langle x_m, x_m \rangle \\ &= ||x_n||^2 - ||x_m||^2 - ||x_m||^2 + ||x_m||^2 \\ &= ||x_n||^2 - ||x_m||^2. \end{aligned}$$

So $\{||x_m||\}$ Cauchy implies $\{x_m\}$ Cauchy, hence convergent. $\square$

5. Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{M}$ and $\{B_n : n \in \mathbb{N}\} \subseteq \mathcal{M}$ are two sequences of measurable sets such that $B_n \subseteq A_n$ for all $n \in \mathbb{N}$. Prove that

$$\mu(\bigcup_{n=1}^\infty A_n) + \sum_{n=1}^\infty \mu(B_n) \leq \mu(\bigcup_{n=1}^\infty B_n) + \sum_{n=1}^\infty \mu(A_n)$$

Proof. Recall $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) - \mu(A_1 \cap A_2)$. This, along with $B_1 \subseteq A_1$, $B_1 \subseteq A_1 \Rightarrow B_1 \cap B_2 \subseteq A_1 \cap A_2 \Rightarrow \mu(B_1 \cap B_2) \leq \mu(A_1 \cap A_2)$ gives us

$$\begin{aligned} \mu(A_1 \cup A_2) - \mu(B_1 \cup B_2) &= \mu(A_1) + \mu(A_2) - \mu(B_1) - \mu(B_2) - \mu(A_1 \cap A_2) + \mu(B_1 \cap B_2) \\ &\leq \mu(A_1) + \mu(A_2) - \mu(B_1) - \mu(B_2) - \mu(A_1 \cap A_2) + \mu(A_1 \cap A_2) \\ &= \mu(A_1) + \mu(A_2) - \mu(B_1) - \mu(B_2). \end{aligned}$$

This gives our base case, justifying induction. Suppose that the statement is true for k many sets.

$$\begin{aligned}
\mu(\bigcup_{n=1}^{k} A_n \cup A_{n+1}) - \mu(\bigcup_{n=1}^{k} B_n \cup B_{n+1}) &\leq \mu(\bigcup_{n=1}^{k} A_n) + \mu(A_{n+1}) - \mu(\bigcup_{n=1}^{k} B_n) - \mu(B_{n+1}) \\
&\leq \sum_{n=1}^{k} \mu(A_n) + \mu(A_{n+1}) - \sum_{n=1}^{k} \mu(B_n) - \mu(B_{n+1}) \\
&= \sum_{n=1}^{k+1} \mu(A_n) - \sum_{n=1}^{k+1} \mu(B_n).
\end{aligned}$$

$\square$

6. Let $(X, \mathcal{M}, \mu)$ be a measure space and m be the Lebesgue measure on $\mathbb{R}$. Suppose $f : X \to [0, \infty)$ is a non-negative measurable function.

   (a) State the Fubini Theorem for the product space $X \otimes \mathbb{R}$.

   Proof. If $\int_X \int_{\mathbb{R}} |f(x,t)| \; d(\mu \otimes m) < \infty$, then

   $$\int_X \int_{\mathbb{R}} f(x,t) \; d(\mu \otimes m) = \int_{\mathbb{R}} \int_X f(x,t) \; d(m \otimes \mu).$$

   $\square$

   (b) Prove that

   $$(\mu \otimes m)(\{(x,t) : 0 \leq t < f(x)\}) = \int_X f(x) \; d\mu(x).$$

   Proof.

   $$\begin{aligned}
   (\mu \otimes m)(\{(x,t) : 0 \leq t < f(x)\}) &= \int_X \int_{\mathbb{R}} \chi_{\{0 \leq t < f(x)\}} \; d\mu \; dm \\
   &= \int_X \int_0^{f(x)} 1 \; d\mu \; dm \\
   &= \int_X f(x) \; d\mu.
   \end{aligned}$$

   $\square$

   (c) Suppose $\phi : \mathbb{R} \to [0, \infty)$ is a non-negative measurable function, and that $\Phi(x) := \int_0^x \phi(x) \; dm(t)$. Prove that

   $$\int_X \Phi(f(x)) \; d\mu(x) = \int_0^\infty \phi(t) \mu(\{x \in X : t < f(x)\}) \; dm(t).$$

Proof. Substitute $\phi(t)$ with $\phi(t)\chi_{\{t<f(x)\}}$ WLOG. Then,

$$\begin{aligned}\int_X \Phi(f(x))\, d\mu &= \int_X \int_0^\infty \phi(t)\chi_{\{t<f(x)\}}\, dm\, d\mu \\ &= \int_0^\infty \phi(t) \int_X \chi_{\{t<f(x)\}}\, d\mu\, dm \\ &= \int_0^\infty \phi(t)\mu(\{t<f(x)\})\, dm.\end{aligned}$$

$\square$

7. Let $(X, \mathcal{M}, \mu)$ be a measure space. For $p \in (0, \infty)$ denote (as usual) by $(L^p(X,\mu),\ ||\cdot||_p)$ the corresponding L^p space.

   (a) State the Hölder inequality for integrals.

   Proof. Where p and q satisfy $1/p + 1/q = 1$,

   $$\int |fg|\, d\mu \le \left(\int |f|^p\, d\mu\right)^{1/p} \left(\int |g|^q\, d\mu\right)^{1/q}$$

   $\square$

   (b) Suppose $0 < q < r < \infty$ and $0 < \theta < 1$. Let $p := (1-\theta)q + \theta r$. Prove that if $f \in L^q(X,\mu) \cap L^r(X,\mu)$, then $f \in L^p(X,\mu)$ and

   $$||f||_p^p \le ||f||_q^{(1-\theta)q}\ ||f||_r^{\theta r}.$$

   Proof. The trick here is to note that $1 = (1-\theta) + \theta = \frac{1}{1/(1-\theta)} + \frac{1}{1/\theta}$. We calculate

   $$\begin{aligned}\int |f|^p\, d\mu &= \int |f|^{(1-\theta)q}|f|^{\theta r}\, d\mu \\ &\le \left(\int (|f|^{(1-\theta)q})^{\frac{1}{1-\theta}}\, d\mu\right)^{1-\theta} \left(\int (|f|^{\theta r})^{\frac{1}{\theta}}\, d\mu\right)^{\theta} \\ &= \left(\int |f|^q\, d\mu\right)^{1-\theta} \left(\int |f|^r\, d\mu\right)^{\theta}\end{aligned}$$

   $\square$

   (c) Suppose $0 < q < r < \infty$ and $0 < \theta < 1$. Define $s \in (0, \infty)$ by $\frac{1}{s} = \frac{(1-\theta)}{q} + \frac{\theta}{r}$. Prove that if $f \in L^q(X,\mu) \cap L^r(X,\mu)$, then $f \in L^s(X,\mu)$ and

   $$||f||_s \le ||f||_q^{(1-\theta)}\ ||f||_r^{\theta}.$$

Proof. The trick here is similar to (b). Note that our condition implies that $1 = \frac{(1-\theta)s}{q} + \frac{\theta s}{r}$, thus $1 = \frac{1}{q/(1-\theta)s} + \frac{1}{r/\theta s}$.

$$\begin{aligned}
||f||_s &= \left(\int |f|^s \, d\mu \right)^{1/s} \\
&= \left(\int |f|^{s(1-\theta)} |f|^{s\theta} \, d\mu \right)^{1/s} \\
&\leq \left[\left(\int |f^{s(1-\theta)}|^{q/(1-\theta)s} \, d\mu \right)^{(1-\theta)s/q} \left(\int |f^{s\theta}|^{r/\theta s} \, d\mu \right)^{\theta s/r} \right]^{1/s} \\
&= \left(\int |f|^q \, d\mu \right)^{(1-\theta)/q} \left(\int |f|^r \, d\mu \right)^{\theta/r} \\
&= ||f||_q^{1-\theta} \, ||f||_r^{\theta}.
\end{aligned}$$

$\square$

8. Suppose $f : [1, \infty) \to \mathbb{R}$ is bounded and measurable, such that $\lim_{x\to\infty} \sqrt{x}|f(x)| = 0$. Prove that for any $p > 1/2$,

$$\lim_{n\to\infty} \int_1^\infty \frac{\sqrt{n} f(nx)}{x^p} \, dx = 0.$$

See Winter 2021 #8 and Winter 2019 #9.

Proof.

$$\begin{aligned}
\int_1^\infty \frac{\sqrt{n} f(nx)}{x^p} \, dx &= \int_1^\infty \frac{\sqrt{nx} f(nx)}{x^{p+1/2}} \, dx \\
&\to_{\substack{t=nx \\ dt=ndx}} \int_n^\infty \frac{\sqrt{t} f(t)}{(\frac{t}{n})^{p+1/2}} \, dt \\
&= \int_n^\infty \sqrt{t} f(t) \frac{n^{p-1/2}}{t^{p+1/2}} \, dt \\
&= \int_n^\infty \frac{\sqrt{t} f(t)}{t} \left(\frac{n}{t}\right)^{p-1/2} \, dt \\
&\leq \int_n^\infty \frac{f(t)}{\sqrt{t}} \, dt \\
&\leq \int_n^\infty f(t) \, dt \to 0.
\end{aligned}$$

This follows as $n/t \leq 1$ for $t \in (n, \infty)$. As our integral is bounded above by the tail end of a convergent "series," it is squeezed towards 0 in the limit. $\square$

Winter 2021

1. Consider the Banach space $l^\infty(\mathbb{N})$ consisting, as usual, of all vectors (x_n) whose coordinates form a bounded sequence, equipped with the norm

$$||(x_n)||_\infty = \sup_{n\in\mathbb{N}} |x_n|.$$

Let $C \subseteq l^\infty(\mathbb{N})$ be a subset of $l^\infty(\mathbb{N})$ consisting of all vectors $(x_n) \in l^\infty(\mathbb{N})$ such that $\lim_{n\to\infty} x_n$ exists. Prove that C is closed.

Proof. □

2. Let $K \subseteq l^\infty(\mathbb{N})$ be a subset of $l^\infty(\mathbb{N})$ given by

$$K = \{(x_n) \in l^\infty(\mathbb{N}) : |x_n| \le \frac{1}{n} \text{ for all } n \in \mathbb{N}\}.$$

Prove that K is compact.

Proof. The idea here is that the tail gets arbitrarily close to 0, so any ε-ball contains infinitely many terms in the sequence. Since there will only be finitely many terms left, K is totally bounded. □

3. Let $C[0,1]$ be the Banach space of all real-valued continuous functions on $[0,1]$ equipped with the supremum norm $||f||_\infty = \sup_{x\in[0,1]} |f(x)|$. Consider the subset $C_0[0,1] := \{f \in C[0,1] : f(0) = 0\}$.

   (a) Prove that $C_0[0,1]$ is a closed subspace and then use this to show that $C_0[0,1]$ is a Banach space itself.

   Proof. One can easily show this is in fact a subspace. Convergence in the sup norm is equivalent to uniform convergence, which I believe suffices. As C_0 is closed in the Banach space C, it is itself a Banach space. □

   (b) Prove that $l : C_0[0,1] \to \mathbb{R}$ defined by $l(f) = \int_0^1 f(x)\ dx$ is a bounded linear functional and compute its norm.

   Proof.
$$|l(f)| = |\int_0^1 f(x)\ dx| \le ||f||_\infty.$$
   Thus $||l|| \le 1$.

   Equality is attained by the sequence $f_n(x) = x^{1/n}$. Each $||f_n||_\infty = 1$, and $|l(f)| = \frac{n}{n+1}$. Thus $||l|| = 1$. □

   (c) Let l be the linear functional from (b). Prove that there exists no $f \in C_0[0,1]$ with $||f||_\infty$ such that $l(f) = ||l||$.

Proof. For $l(f) = ||l||$ to be true, the function f must look like the constant 1 function everywhere except $f(0)$. But to be continuous, there must be a region around the origin where f is not the constant 1 function. Thus the integral cannot be fully 1.

This contains two common themes in recent prelims: equality must be attained by a sequence, and there is some sort of give-and-take condition. i.e.: if f is continuous, it violates the integral requirement. If it satisfies the integral, it cannot be continuous. □

4. Let $\mathcal{H}$ be a complex Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator.

(a) Prove that if T is self-adjoint, then $||T|| = \sup\{|\langle Tx, x\rangle : ||x|| \leq 1|\}$.

Proof. This is a classic question where one direction is nearly trivial, but the other is assuredly not.

($\Leftarrow$) By Cauchy-Schwarz,

$$|\langle Tx, x\rangle| \leq ||T||\ ||x||^2.$$

Taking the supremum over x such that $||x|| \leq 1$ suffices.

($\Rightarrow$) Let $||x|| = ||y|| = 1$. While not obvious, introducing y makes the computations more manageable. Let $m = \sup\{|\langle Tx, x\rangle : ||x|| \leq 1|\}$, so that $|\langle Tu, u\rangle| \leq m\ ||u||^2$. This inequality can be seen by $u \mapsto \frac{u}{||u||}$.

$$\langle T(x \pm y), x \pm y\rangle = \langle Tx, x\rangle \pm 2|\langle Tx, y\rangle| + \langle Ty, y\rangle.$$

Thus we see by subtracting the $+$ and $-$ cases that

$$\begin{aligned}
|\langle Tx, y\rangle| &\leq \frac{1}{4}\,|\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle| \\
&\leq \frac{1}{4}\,(|\langle T(x+y), x+y\rangle| + |\langle T(x-y), x-y\rangle|) \\
&\leq \frac{1}{4}\left(m\ ||x+y||^2 + m\ ||x-y||^2\right) \\
&\leq \frac{m}{4}\left(||x+y||^2 + ||x-y||^2\right) \\
&\leq \frac{m}{4}\left(2||x||^2 + 2||y||^2\right) \\
&= m.
\end{aligned}$$

This takes advantage of the Parallelogram identity.

Now that we have $|\langle Tx, y\rangle| \leq \sup_{||x|| \leq 1|}\{|\langle Tx, x\rangle\}$, setting $y = \frac{||x||\ Tx}{||Tx||}$ gives $||x||\ ||Tx|| \leq m\ ||x||^2$, thus $||T|| = \frac{||Tx||}{||x||} \leq m$. □

(b) Prove that (a) is in general not true if T is not assumed to be self-adjoint.

Proof. Consider T being a 90° rotation in the plane. Then $\langle Tx, x\rangle = 0$ for all x. This suggests $||T|| = 0$. But this rotation preserves distances; it is an isometry, thus $||T|| = 1$. Contradiction. $\square$

5. Consider the real line $\mathbb{R}$ equipped with the usual Euclidean metric and the Lebesgue measure.

   (a) Prove that a finite intersection of sets which are open and dense in $\mathbb{R}$ is also open and dense in $\mathbb{R}$.

   Proof. (Intersection of open is open) A is open iff $A = Int(A)$. For $\{A_i\}_{i=1}^n$ open, let $x \in \cap A_i$. Then $x \in A_i$ for each i, implying $x \in Int(A_i)$, so there exists ε_i such that $\mathcal{B}_{\varepsilon_i}(x) \subseteq A_i$. For $\varepsilon := \min\{\varepsilon_i\}$, then $\mathcal{B}_\varepsilon(x) \subseteq \mathcal{B}_{\varepsilon_i}(x) \subseteq A_i$. Hence $\mathcal{B}_\varepsilon(x) \subseteq \cap A_i$, so $x \in Int(\cap A_i)$.

   (Intersection of dense is dense) $A \subseteq X$ is dense iff for all $\varepsilon > 0$, there exists $a \in A$ such that $d(x, a) < \varepsilon$. With this in mind, let $\{A_i\}_{i=1}^n$ be dense, with $x \in \cap A_i$. Then for all $\varepsilon_i > 0$, there exists $a_i \in A_i$ such that $d(x, a_i) < \varepsilon_i$. For $\varepsilon := \min\{\varepsilon_i\}$, then for all $a \in \cap A_i$ we have $d(x, a) \leq d(x, a_i) < \varepsilon$. Thus $\cap A_i$ is dense. $\square$

   (b) Prove that there exists a set with Lebesgue measure 0 which is a countable intersection of sets which are open and dense in $\mathbb{R}$.

   Proof. By the Lebesgue Regularization Theorem, for any measurable $A \subseteq \mathbb{R}$ there exists a G_δ set G such that $A \subseteq G$ and $\lambda(G \setminus A) = 0$. Unfortunately, this says nothing about density, so we must construct an example.

   Let $Q = \mathbb{Q} \cap [0, 1]$ and enumerate $q_n \in Q$ via the naturals. Let $A_\varepsilon = \bigcup_{q_n \in Q}(q_n - \frac{\varepsilon}{2^{n+1}},\ q_n + \frac{\varepsilon}{2^{n+1}})$. Each A_ε is open and dense, with $\lambda(A_\varepsilon) \leq \varepsilon$. Taking $A = \bigcap_{m=1}^\infty A_{1/m}$ then gives a countable intersection of open and dense sets such that $\lambda(A) < \varepsilon$. $\square$

6. Let $(X, \mathcal{M}, \mu)$ be a measure space.

   (a) State the Monotone Convergence Theorem.

   (b) Prove that the sequence $f_n : \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = -\frac{\chi_{[0,n]}(x)}{n}$ is monotonically increasing and converges almost everywhere, yet contradicts the conclusion of the Monotone Convergence Theorem. (Here $\chi_{[0,n]}$ denotes the characteristic/indicator function of the interval $[0, n]$.)

   Proof. Draw them!

   These functions tend a.e. to 0, yet they all integrate to -1. $\square$

   (c) Prove the following extension of Fatou's Lemma: Suppose $h : X \to [0, \infty)$ is an $\mathcal{M}$-measurable, integrable function, and $\{f_n\}_{n=1}^\infty$ is a sequence of $\mathcal{M}$-measurable functions so that $-h \leq f_n$ for all $n \in \mathbb{N}$. Prove that

$$\int (\liminf_n f_n)\ d\mu \leq \liminf_n \int f_n\ d\mu.$$

Proof. By the condition stated, $f_n + h \geq 0$, so apply Fatou's to this:

$$\int \liminf_n (f_n + h)\ d\mu \leq \liminf_n \int (f_n + h)\ d\mu.$$

Expanding and subtracting the integrals of h proves the problem. □

(d) Prove that the extended Fatou's Lemma (as stated in (c)) does not apply to the example in (b). Explain why.

Proof. If $-h \leq -\frac{1}{n}\chi_{[0,n]}$, then $h \geq \frac{1}{n}\chi_{[0,n]}$. We can bound this with $h \geq \frac{1}{n}\chi_{[n-1,n]}$, so since these indicators are disjoint, $h \geq \sum_{n=1}^{\infty} \frac{1}{n}\chi_{[n-1,n]}$. But this sum has infinite integral. □

7. Let $(X, \mathcal{M}, \mu)$ be a measure space and let (A_n) be a sequence of $\mathcal{M}$-measurable subsets of X. Recall that $\limsup A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$.

(a) Prove that $x \in \limsup A_n$ if and only if $\sum_{n=1}^{\infty} \chi_{A_n}(x) = \infty$.

(b) Let $f_n : X \to [0, \infty)$ be a sequence of non-negative $\mathcal{M}$-measurable functions. Prove that if $\sum_{n=1}^{\infty} \int_X f_n(x)\ d\mu < \infty$, then for μ a.e. $x \in X$ we have $\sum_{n=1}^{\infty} f_n(x) < \infty$.

Proof. Let $\sum_{n=1}^{\infty} \int_X f_n(x)\ d\mu = M < \infty$. By Chebyshev and Fubini's:

$$\mu\left(\{\sum_{n=1}^{\infty} f_n(x) > m\}\right) \leq \frac{1}{m}\int_X \sum_{n=1}^{\infty} f_n(x)\ d\mu = \frac{1}{m}M \to 0 \text{ as } m \to \infty.$$

□

(c) Prove that if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then $\mu(\limsup A_n) = 0$.

Proof. Classic Borel-Cantelli:

$$\mu(\limsup A_n) = \mu(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k) \leq \mu(\cup_{k=n}^{\infty} A_k) \leq \sum_{k=n}^{\infty} \mu(A_k) \to 0.$$

□

8. Suppose $f : [1, \infty) \to \mathbb{R}$ is bounded and measurable, such that $\lim_{x\to\infty} x|f(x)| = 0$. Prove that

$$\lim_{n\to\infty} \int_1^{\infty} \frac{nf(nx)}{x}\ dx = 0.$$

See Summer 2021 #8 and Winter 2019 #9.

Proof. (Freeman Slaughter)

$$\begin{aligned}
\int_1^\infty \frac{nf(nx)}{x}\ dx \to_{ndx=dt}^{nx=t} \int_n^\infty \frac{tf(t)}{(t/n)^2}\ \frac{1}{n}\ dt \\
= \int_n^\infty f(t)\frac{n}{t}\ dt \\
\le \int_n^\infty f(t)\ dt.
\end{aligned}$$

This follows as $t \in (n, \infty)$ implies $\frac{n}{t} \in (0, 1)$. ☐

Proof. (Alternative soln. by Travis Alvarez) Since f is bounded and $\lim_{x\to\infty} x|f(x)| = 0$, there exists some M such that $xf(x) \le M$ for all x. Thus

$$\frac{nf(nx)}{x} = \frac{nxf(nx)}{x^2} \le \frac{M}{x^2}.$$

This justifies applying DCT and swapping limit and integral. ☐

Summer 2022

1. Let $\mathcal{X}$ be a normed linear space. Suppose $\{x_n\}$ is a sequence of vectors in $\mathcal{X}$ such that $x_n \to x$ and $T_n : \mathcal{X} \to \mathcal{X}$ is a sequence of bounded linear operators such that $T_n \to T$ in operator norm. Prove that $T_n x_n \to Tx$.

   Proof. Apply the triangle inequality:

   $$\begin{aligned} ||T_n x_n - Tx|| &\leq ||T_n x_n - T_n x_m|| + ||T_n x_m - T_n x|| + ||T_n x - Tx|| \\ &\leq ||T_n||\ ||x_n - x_m|| + ||T_n||\ ||x_m - x|| + ||T_n - T||\ ||x|| \end{aligned}$$

   Each of the above can be sufficiently bounded by the appropriate ε. $\square$

2. Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator on a Hilbert space $\mathcal{H}$. Denote by $\mathrm{Ran}(T)$ the range of T. Suppose there exists $c > 0$ such that $c||x|| \leq ||Tx||$ for all $x \in \mathcal{H}$. Prove that $\mathrm{Ran}(T)$ is a closed subspace of $\mathcal{H}$.

   See Summer 2020 #3 and Winter 2015 #6 for more coercive operator problems.

   Proof. Let $\{x_n\}$ be a sequence in $\mathcal{H}$ such that $Tx_n \to y$. Then $c||x_n - x_m|| \leq ||Tx_n - Tx_m||$ shows that $\{Tx_n\}$ Cauchy implies $\{x_n\}$ Cauchy, so $x_n \to x$. Thus $Tx_n \to Tx$, meaning $Tx = y$. Since $y \in \mathrm{Ran}(T)$, thus $\mathrm{Ran}(T)$ is closed.

   In fact, stronger is true: it can be seen that T is injective. If $x, y \in \mathrm{Kern}(T)$, then $c||x - y|| \leq ||Tx - Ty|| = 0$ implies that $x = y$, so $\mathrm{Kern}(T) = \{0\}$.

   Adding this to the prelim problem gives that $\mathcal{H} = \mathrm{Ran}(T) \oplus \{0\}$. That is, for all y there exists a unique x such that $Tx = y$; thus T is invertible. $\square$

3. Let $C^1[-1, 1]$ be the set of all continuously differentiable real-valued function on $[-1, 1]$. Consider the following two norms on $C^1[-1, 1]$

   $$||f||_\infty = \sup_{x \in [-1,1]} |f(x)|, \qquad ||f||_1 = \sup_{x \in [-1,1]} |f(x)| + \sup_{x \in [-1,1]} |f'(x)|.$$

   Define a linear functional $T : C^1[-1, 1] \to \mathbb{R}$ by $Tf = f'(0)$.

   See Winter 2017 #3.

   (a) Prove that T is not bounded if $C^1[-1, 1]$ is equipped with the supremum norm $||f||_\infty$.

   Proof. Let $f_n(x) = \sin(\pi n x)$. Then $||f_n||_\infty = 1$ for all n. Yet $Tf_n = \pi n$, which increases without bound. $\square$

   (b) Prove that T is bounded if $C^1[-1, 1]$ is equipped with the norm $||f||_1$ defined above and compute its operator norm.

Proof. The following implies that $||T|| \leq 1$ under the $||\cdot||_1$ norm:

$$|Tf| = |f'(0)| \leq \sup_{x\in[-1,1]} |f'(x)| \leq \sup_{x\in[-1,1]} |f(x)| + \sup_{x\in[-1,1]} |f'(x)| = ||f||_1.$$

We will not be able to find an example with exact equality, so we aim to find an example with equality *in the limit*. For those using this to prepare, this treatment is becoming more frequent in the prelims.

Conversely, let $f_n(x) := x^n$. Then $||f_n||_1 = 1 + n$, so

$$||T|| = \sup_{f_n \neq 0} \frac{||Tf_n||}{||f_n||} \geq \frac{||Tf_n||}{||f_n||} = \frac{n}{1+n} \to 1.$$

As $||T|| \geq 1$ also, we have equality. □

4. Let $\mathcal{H}$ be a Hilbert space. Let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator, i.e., bounded linear operator satisfying $U^*U = UU^* = I$. Let $\mathcal{K} = \{x \in \mathcal{H} \ : \ Ux = x\}$ be the subspace of invariant vectors of U.

   (a) Prove that $\mathcal{K} = (\mathrm{Ran}(I-U))^\perp$

   Proof. Note the line of equivalences: $x \in \mathcal{K}$ iff $Ux = x$ iff $(I-U)x = 0$ iff $x \in \mathrm{Kern}(I-U)$. Thus $\mathcal{K} \perp \mathrm{Ran}(I-U)$. □

   (b) Prove that $\mathcal{H} = \mathcal{K} \oplus \overline{\mathrm{Ran}(I-U)}$.

   Proof. By Hilbert decomposition theorem,

$$\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp = \mathcal{K} \oplus (\mathrm{Ran}(I-U)^\perp)^\perp = \mathcal{K} \oplus \overline{\mathrm{Ran}(I-U)}.$$

□

   (c) Let $P : \mathcal{H} \to \mathcal{K}$ be the orthogonal projection onto $\mathcal{K}$. Prove that for each $x \in \mathcal{H}$ we have

$$Px = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} U^n x.$$

   Proof. By the above, we can decompose any x uniquely into $x = x_K + x_R$, where $x_K \perp x_R$. It is sufficient to determine the action of P on these parts.

   For $x \in \mathcal{K}$,

$$||Px - \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} U^n x|| = ||x - \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x|| = ||x - \lim_{N\to\infty} x|| = 0.$$

   For $x \in \overline{\mathrm{Ran}(I-U)}$, let $x = (I-U)y$. Then by telescoping sum,

$$
\begin{aligned}
||Px - \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} U^n x|| &= ||0 - \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} U^n (I-U)y|| \\
&= || \lim_{N\to\infty} \frac{1}{N}(Iy - U^{N+1}y)|| \\
&\le \lim_{N\to\infty} \frac{1}{N}(||y|| + ||U^{N+1}||\ ||y||) \\
&\le \lim_{N\to\infty} \frac{1}{N}(||y|| + ||U||^{N+1}\ ||y||) \\
&= \lim_{N\to\infty} \frac{1}{N}\ 2||y|| \qquad\qquad = 0.
\end{aligned}
$$

Recall that U unitary implies $||U|| = 1$. Thus, the two statements are equivalent. $\square$

5. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Suppose $\{A_n\}_{n=1}^{\infty}$ is a sequence of measurable sets such that $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$.

   (a) Prove that
   $$\mu(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n).$$

   Proof. Let $B_n = A_1 \setminus A_n$. Since the B_n's are increasing, we know from measure axioms that
   $$\mu\left(\cup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \mu(B_n).$$

   First, we prove a lemma:
   $$
   \begin{aligned}
   A_1 \setminus \cap_{n=1}^{\infty} A_n &= A_1 \cap \left(\cap_{n=1}^{\infty} A_n\right)^c \\
   &= A_1 \cap \cup_{n=1}^{\infty} A_n^c \\
   &= \cup_{n=1}^{\infty} (A_1 \cap A_n^c) \\
   &= \cup_{n=1}^{\infty} A_1 \setminus A_n.
   \end{aligned}
   $$

   This last union is the union of B_n's, so we aim to manipulate this feature.
   $$
   \begin{aligned}
   \mu(A_1) - \mu(\cap_{n=1}^{\infty} A_n) &= \mu(A_1 \setminus \cap_{n=1}^{\infty} A_n) \\
   &= \mu(\cup_{n=1}^{\infty} A_1 \setminus A_n) \\
   &= \mu(\cup_{n=1}^{\infty} B_n) \\
   &= \lim_{n\to\infty} \mu(B_n) \\
   &= \lim_{n\to\infty} \mu(A_1 \setminus A_n) \\
   &= \mu(A_1) - \lim_{n\to\infty} \mu(A_n).
   \end{aligned}
   $$

   Thus $\mu(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$. $\square$

(b) Let ν be another finite measure on $(X, \mathcal{M})$ such that $\nu(E) = 0$ whenever $E \in \mathcal{M}$ with $\mu(E) = 0$. Prove that for each ε there exists δ such that $\mu(E) < \delta$ implies $\nu(E) < \varepsilon$.

Proof. This condition is called "absolute continuity" and denoted $\nu << \mu$.

For contradiction, suppose that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mu(E) < \delta$ but $\nu(E) < \varepsilon$. Let $E_{n+1} \subseteq E_n$ with $\mu(E_n) = \frac{\delta}{n}$.

For $n \geq 2$, $\mu(E_n) < \delta$. Since $\mu(\cap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \frac{\delta}{n} = 0$, it must be that $\nu(\cap_{n=1}^{\infty} E_n) = 0$ also. But then $\mu(\cap_{n=1}^{\infty} E_n) < \delta$ while $\mu(\cap_{n=1}^{\infty} E_n) \not> \varepsilon$. □

6. Let $(X, \mathcal{M}, \mu)$ be a measure space. Recall that a set $E \subseteq X$ is said to be σ-finite if there exists a sequence $\{E_n\}_{n=1}^{\infty}$ of measurable sets with $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$ such that $E = \cup_{n=1}^{\infty} E_n$. Prove that if $f \in L^p(X, \mu)$ for $1 \leq p < \infty$, then the set $E := \{x \in X : f(x) \neq 0\}$ is σ-finite.

See Winter 2018 #7 for the special case of $p = 1$.

Proof. Let $E_n := \{|f| \geq \frac{1}{n}\}$. Clearly, $\cup_{n=1}^{\infty} E_n = E$.

To show finiteness:

$$
\begin{aligned}
\mu(E_n) &= \int_{\{\frac{1}{n} \leq |f|\}} 1 \; d\mu \\
&= n^p \int_{\{\frac{1}{n} \leq |f|\}} \left(\frac{1}{n}\right)^p \; d\mu \\
&\leq n^p \int_{\{\frac{1}{n} \leq |f|\}} |f|^p \; d\mu \\
&\leq n^p \int_X |f|^p \; d\mu \\
&< \infty.
\end{aligned}
$$

Thus for each n, we have $\mu(E_n) < \infty$. □

7. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f_n : X \to \mathbb{R}$ be a sequence of integrable functions and $f : X \to \mathbb{R}$ a measurable function.

(a) Prove that if for some $\delta > 0$ we have $\int_X |f_n(x) - f(x)| \; d\mu(x) \leq \frac{1}{n^{1+\delta}}$ for all $n \in \mathbb{N}$, then $f_n \to f$ pointwise a.e..

Proof. **INCOMPLETE** □

(b) Prove that $\int_X |f_n(x) - f(x)| \; d\mu(x) \leq \frac{1}{n}$ for all $n \in \mathbb{N}$ does not in general imply $f_n \to f$ pointwise a.e..

Proof. Consider the following modified typewriter sequence: $f_n = \frac{1}{2}\chi_{\{\frac{n-2^k}{2^k}, \frac{n+1-2^k}{2^k}\}}$ indexing over $2^k \leq n < 2^{k+1}$. This function converges to 0 in L^1, but not pointwise a.e. to any function.
See that

$$\int_X f_n(x)\ d\mu = \int_X \frac{1}{2}\chi_{\{\frac{n-2^k}{2^k}, \frac{n+1-2^k}{2^k}\}}\ d\mu = \frac{1}{2}\left(\frac{n+1-2^k}{2^k} - \frac{n-2^k}{2^k}\right) = \frac{1}{2^{k+1}} < \frac{1}{n}.$$

This follows as $n < 2^{k+1}$ implies $\frac{1}{2^{k+1}} < \frac{1}{n}$.

But, this sequence jumps back up to satisfy $f_n(x) = 1$ for infinitely many x, so cannot converge pointwise a.e. to the 0 function. $\square$

8. Suppose $f : [1, \infty) \to \mathbb{R}$ is a continuous function such that $\lim_{x\to\infty} |f(x)| = 0$. Prove that for any integrable function $g : [1, \infty) \to \mathbb{R}$ the following equality holds

$$\lim_{n\to\infty} \int_1^\infty f(n+x)g(x)\ dx = 0.$$

Proof. Since f is continuous, there exists some finite number M such that $|f(x)| \leq M$ for all x. Then $|f(n+x)g(x)| \leq Mg(x)$, with $\int Mg(x)\ dx < \infty$.

Since $f(n+x)g(x)$ is dominated by an integrable function, we are justified in exchanging the limit and integral. Hence

$$\lim_{n\to\infty} \int_1^\infty f(n+x)g(x)\ dx = \int_1^\infty \lim_{n\to\infty} f(n+x)g(x)\ dx = \int_1^\infty 0g(x)\ dx = 0.$$

$\square$

Winter 2023

1. Let $\mathbb{Q}$ be the set of rational numbers.

   a) Prove that $(\mathbb{Q}, d_1)$ is not complete, where $d_1(x, y) = |x - y|$.

   b) Prove that $(\mathbb{Q}, d_2)$ is complete, where $d_2(x, y) = \begin{Bmatrix} 0 & x = y, \\ 1 & x \neq y. \end{Bmatrix}$.

Proof. a) Consider truncations of e:

$$\left| \sum_{n=0}^{r} \frac{1}{n!} - \sum_{n=0}^{s} \frac{1}{n!} \right| = \left| \sum_{n=r+1}^{s} \frac{1}{n!} \right| \to 0,$$

so we have a Cauchy sequence of rational numbers tending to an irrational number.

b) Every countable space is complete under discrete metric.

□

2. Consider the Banach space $l^\infty(\mathbb{N})$ consisting of all vectors (x_n) whose coordinates form a bounded sequence, equipped with the supremum norm

$$||(x_n)||_\infty = \sup_{n \in \mathbb{N}} |x_n|.$$

Let $c_0(\mathbb{N}) \subseteq l^\infty(\mathbb{N})$ be a subset of $l^\infty(\mathbb{N})$ consisting of all vectors $(x_n) \in l^\infty(\mathbb{N})$ such that $\lim_{n\to\infty} x_n = 0$.

   a) Prove that $c_0(\mathbb{N})$ is a closed subspace of $l^\infty(\mathbb{N})$ and then use this to show that $c_0(\mathbb{N})$ is a Banach space itself.

   b) Prove that $f : c_0(\mathbb{N}) \to \mathbb{R}$ defined by $f((x_n)) = \sum_{n=1}^{\infty} \frac{x_n}{2^n}$ is a bounded linear functional and compute its norm.

Proof. a) Recall that a closed subspace of a Banach space is itself Banach!
Clearly c_0 is a subspace of l^∞. Let $x^{(k)} \in c_0$ converge to $y \in l^\infty$. Take $\varepsilon > 0$ and N_0 s.t. $\sup |x_j^{(k)} - y_j| < \varepsilon/2$ for all $k > N_0$. For each k, pick an N_1 s.t. $|x_j^{(k)}| < \varepsilon/2$ for all $j > N_1$. Thus, $|y_j| \leq |x_j^{(k)} - y_j| + |x_j^{(k)}| < \varepsilon$ for all $k > N_0$, $j > N_1$. Hence $y_j \to 0$, so $y \in c_0$.

b) We can see readily that $|f((x_n))| \leq ||(x_n)||_\infty$. For equality, we would *like* to take the sequence of all 1's, but this is not in c_0. We can find a sequence however that gets us arbitrarily close: let $x^{(k)} = (1, 1, ..., 1, 0, 0, ...)$ with every term after the k-th being 0. Then $|f(x^{(k)})| = 1 - 2^{-k}$, so taking $k \to \infty$ gives $||f|| = 1$.

□

3. Let $K \subseteq l^\infty(\mathbb{N})$ be a subset of $l^\infty(\mathbb{N})$ given by

$$K = \{(x_n) \in l^\infty(\mathbb{N}) : |x_n| \leq \frac{1}{n} \text{ for all } n \in \mathbb{N}\}.$$

Prove that K is compact.

Proof. See Winter 2021 #2. ☐

4. Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator. Suppose there exists a constant $\beta > 0$ such that $\langle Tx, x\rangle \geq \beta \,||x||^2$ for all $x \in \mathcal{H}$. Prove that for any bounded linear functional $f : \mathcal{H} \to \mathbb{C}$ there exists a unique $y \in \mathcal{H}$ such that $f(x) = \langle Tx, y\rangle$ for all $x \in \mathcal{H}$.

   Proof. Coercivity implies many useful properties, like invertibility. Here, the idea is to define a new inner product $(x, y) = \langle Tx, y\rangle$ then apply Riesz Representation Theorem - a Mishko classic. ☐

5. Consider the real line $\mathbb{R}$ with the usual Euclidean metric and the Lebesgue measure.

   a) Prove that if $\{U_n\}_{n\in\mathbb{N}}$ is a sequence of sets which are all open and dense in $\mathbb{R}$, then their intersection $\cap_{n=1}^{\infty} U_n$ cannot be empty.

   b) Prove that there exists a sequence of sets $\{U_n\}_{n\in\mathbb{N}}$ which are all open and dense in $\mathbb{R}$ whose intersection $\cap_{n=1}^{\infty} U_n$ has Lebesgue measure zero.

   Proof. See Winter 2021 #5. Baire Category Theorem! ☐

6. Let $(X, \mathcal{M}, \mu)$ be a measure space.

   a) Suppose $f : X \to [0, \infty]$ is an $\mathcal{M}$-measurable function which is integrable. Prove that $f(x) < \infty$ for μ a.e. $x \in X$.

   b) Let $f_n : X \to [0, \infty]$ be a sequence of non-negative $\mathcal{M}$-measurable functions. Suppose that $\sum_{n=1}^{\infty} \int_X f_n \, d\mu < \infty$. Prove that $\sum_{n=1}^{\infty} f_n(x) < \infty$ for μ a.e. $x \in X$.

   Proof. a) Suppose not: that there exists an interval where $f(x)$ is not finite. Then in this region, it cannot be integrable - contradiction.

   b) See Winter 2021 #7. Apply Fubini to get finiteness, then Chebyshev to show the measure is 0 in the limit.

   ☐

7. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{M}$-measurable subsets of X. Recall that $\limsup A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$.

   a) Prove that if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then $\mu(\limsup A_n) = 0$.

   b) Let $f_n : X \to \mathbb{R}$ be a sequence of $\mathcal{M}$-measurable functions. Suppose that for every $\varepsilon > 0$ we have $\sum_{n=1}^{\infty} \mu(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}) > \infty$. Prove that $f_n \to f$ pointwise μ a.e.

   c) Let $g_n : X \to \mathbb{R}$ be a sequence of $\mathcal{M}$-measurable functions. Suppose that $g_n \to g$ in measure, i.e., for every $\varepsilon > 0$ we have $\lim_{n\to\infty} \mu(\{x \in X : |g_n(x) - g(x)| \geq \varepsilon\}) = 0$. Prove that there exists a subsequence $\{g_{n_k}\}$ such that $g_{n_k} \to g$ pointwise μ a.e.

Proof. a) Winter 2021 #7; classic Borel-Cantelli! Think of the measure of the lim sup as the tail end of a convergent series.

b) Let $A_n = \{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}$. By part a) we see $\mu(\mu(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\})) = 0$.

c) Suppose $g_n \to g$ in μ. Let $n_1 = 1$ and define $n_j > n_{j-1}$ by $A_j = \{|g_{n_j} - g| \geq 1/j\}$ s.t. $\mu(A_j) \leq 2^{-j}$. Define $A = \limsup A_j$.

Then $\mu(A) = 0$ by part a), and for $x \notin A$ we have $x \notin \cup_{j=N}^{\infty} A_j$. That is, $|g_{n_j}(x) - g(x)| \leq 1/j$, so $g_{n_j} \to g$ on A^C.

□

8. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f_n, g_n : X \to \mathbb{R}$ be two sequences of $\mathcal{M}$-measurable functions such that $f_n \to f$ and $g_n \to g$ pointwise μ a.e. Suppose that

   a) $|f_n(x)| \leq g_n(x)$ for all $x \in X$,

   b) $\lim_{n\to\infty} \int_X g_n \, d\mu = \int_X g \, d\mu < \infty$.

   Prove that $\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$.

Proof. Apply Fatou to $g - f_n \geq 0$ to get

$$\limsup \int f_n \, d\mu \leq \int \limsup f_n \, d\mu.$$

Since $|f - f_n| \leq |f| + |f_n| \leq 2|g|$, we see that $f - f_n$ is dominated. Applying reverse Fatou gives

$$\limsup \int |f - f_n| \, d\mu \leq \int \limsup |f - f_n| \, d\mu = 0.$$

Thus,

$$\lim \left| \int f \, d\mu - \int f_n \, d\mu \right| \leq \limsup \int |f - f_n| \, d\mu = 0,$$

we have that

$$\int f \, d\mu = \lim \int f_n \, d\mu.$$

□

Summer 2023

1. Prove the following statements

   (a) If $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ is a Cauchy sequence that has a convergent subsequence, then $\{x_n\}$ is convergent

   Proof. $\square$

   (b) A subset $A \subseteq \mathbb{R}$ is bounded if and only if $\lim_{n\to\infty} a_n x_n = 0$ for all sequences $\{x_n\}_{n=1}^{\infty} \subseteq A$ and $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ with $\lim_{n\to\infty} a_n = 0$

   Proof. $\square$

2. Consider the Banach space $C[0,1]$ consisting of all continuous functions $f : [0,1] \to \mathbb{R}$, equipped with the supremum norm $||f||_\infty = \sup_{x\in[0,1]} |f(x)|$. Let $K : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function. Consider the operator $T : C[0,1] \to C[0,1]$ given by

$$Tf(x) = \int_0^1 K(x,y) f(y)\, dy.$$

   (a) Prove that T is bounded.

   Proof. $\square$

   (b) Find $||T||$. Justify your answer.

   Proof. $\square$

3. Let $\mathcal{H}$ be a Hilbert space. Recall that a sequence $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$ converges weakly to $f \in \mathcal{H}$ if $\lim_{n\to\infty}\langle f_n, g\rangle = \langle f, g\rangle$ for all $g \in \mathcal{H}$. Prove the following statements.

   (a) A sequence $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$ converges to $f \in \mathcal{H}$ if and only if $\lim_{n\to\infty} ||f_n|| = ||f||$ and $\{f_n\}$ converges weakly to f.

   Proof. $\square$

   (b) Let T be a bounded linear operator on $\mathcal{H}$. If $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$ converges weakly to $f \in \mathcal{H}$, then $\{Tf_n\}_{n=1}^{\infty}$ converges weakly to Tf.

   Proof. $\square$

4. Let $\mathcal{H}$ be a Hilbert space and let $T_n : \mathcal{H} \to \mathcal{H}$ be a sequence of bounded linear operators on $\mathcal{H}$ with $||T_n|| \leq 1$ for all $n \in \mathbb{N}$. Suppose that for every vector $x \in \mathcal{H}$ the following holds:

$$T_i^* T_j x = 0,$$

   for all $i, j \in \mathbb{N}$ with $i \neq j$.

(a) Prove that for every $i, j \in \mathbb{N}$ such that $i \neq j$, the ranges of T_i and T_j are orthogonal.

Proof. $\square$

(b) Prove that for every $x \in \mathcal{H}$ the sequence $\{T_n x\}$ is a Cauchy sequence.

Proof. $\square$

(c) Let $T : \mathcal{H} \to \mathcal{H}$ be defined by $Tx = \lim_{n\to\infty} T_n x$. Prove that T is bounded and $||T|| \leq 1$.

Proof. $\square$

5. Consider the real line $\mathbb{R}$ equipped with the usual Euclidean metric.

(a) Prove that if $A, B \subseteq \mathbb{R}$ are disjoint closed sets, then there exist disjoint open sets $U, V \subseteq \mathbb{R}$ such that $A \subseteq U$ and $B \subseteq V$.

Proof. $\square$

(b) Let m^* denote the Lebesgue outer measure on $\mathbb{R}$. Prove that for any two sets $A, B \subseteq \mathbb{R}$ such that $\inf_{a\in A, b\in B} |a - b| > 0$ we have

$$m^*(A \cup B) = m^*(A) + m^*(B).$$

Proof. $\square$

6. Assume you know the function

$$f_n(x) := \int_0^n t^{x-1} \left(1 - \frac{t}{n}\right)^n \, dt$$

is well defined for all $x > 0$ and $n \in \mathbb{N}$.

(a) Apply a *convergence theorem* to show that $\lim_{n\to\infty} f_n(x)$ exists for all $x > 0$.

Proof. $\square$

(b) Write down the *staetment of the convergence theorem* that you use in (a).

Proof. $\square$

7. Let $1 \leq p < \infty$ and $f \in L^p(\mathbb{R})$. Denote by m the Lebesgue measure on $\mathbb{R}$.

(a) Prove that $\lim_{n\to\infty} m(\{x \in \mathbb{R} : |f(x)| \geq n\}) = 0$.

Proof. $\square$

(b) Prove that the set $\{x \in \mathbb{R} : |f(x)| \neq 0\}$ is σ-finite.

Proof. ☐

(c) Prove that $m(\{x \in \mathbb{R} : |f(x)| = \infty\}) = 0$.

Proof. ☐

8. Denote by m the Lebesgue measure on $\mathbb{R}$. Let $E \subset \mathbb{R}$ be a measurable set with $m(E) < \infty$. Suppose $f : E \to \mathbb{R}$ is a measurable function, so that $f(x) > 0$ for a.e. $x \in E$. Prove that if $\{E_n\}$ is a sequence of measurable subsets of E, so that

$$\lim_{n\to\infty} \int_{E_n} f(x)\, dx = 0,$$

then $\lim_{n\to\infty} m(E_n) = 0$.

Proof. ☐