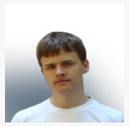
2.4.5 Introducing Dynamic Programming | Reply Sat, Feb 12, 2011 at 2:20 AM EST



syg96
182 posts

Problem:

About 25% of all SRM problems have the "Dynamic Programming" category tag. The DP problems are popular among problemsetters because each DP problem is original in some sense and you have to think hard to invent the solution for it. Since dynamic programming is so popular, it is perhaps the most important method to master in algorithm competitions.

The easiest way to learn the DP principle is by examples. The current recipe contains a few DP examples, but unexperienced reader is advised to refer to other DP tutorials to make the understanding easier. You can find a lot of DP examples and explanations in an excellent tutorial Dynamic Programming: From novice to advanced by Dumitru. The purpose of the recipe is to cover general DP aspects.

Solution

Tutorial (coins example)

So what is exactly the dynamic programming, how can we describe it?
There’s no clear definition for this technique. It can be rather characterized as an algorithmic technique that is usually based on a starting state of the problem, and a recurrent formula or relation between the successive states. A state of the problem usually represents a sub-solution, i.e. a partial solution or a solution based on a subset of the given input. And the states are built one by one, based on the previously built states.

Let’s now consider a very simple problem that will help to understand better the details that will be further discussed:
Given a list of n coins, their weights W1, W2, ..., Wn; and the total sum S. Find the minimum number of coins the overall weight of which is S (we can use as many coins of each type as we want), or report that it is not possible to select coins in such a way so that they sum up to S. This problem is a special case of the famous unbounded knapsack problem. For this problem a state, let’s call it (P) or (P)->k, would represent the solution for a partial sum (P), where P is not greater than S. k is minimal number of coins required to get exact overall weight P. The k value is usually called the result of corresponding state (P).

A dynamic programming solution would thus start with an initial state (0) and then will build the succeeding states based on the previously found ones. In the above problem, a state (Q) that precedes (P) would be the one for which sum Q is lower than P, thus representing a solution for a sum smaller than P. One starts with the trivial state (0), and then builds the state (P1), (P2), (P3), and so on until the final state (S) is built, which actually represents the solution of the problem. One should note that a state can not be processed until all of the preceding states haven't been processed – this is another important characteristic of DP technique.

The last, but not least, detail to discuss is about finding the relation between states that would allow us to build the next states. For simple problems this relation is quite easy to be observed, but for complex problems we may need to do some additional operations or changes to reach such a relation.
Let's again consider the sample problem described above. Consider a sum P of coin weights V1, V2, ..., Vj. The state with sum P can only be reached from a smaller sum Qi by adding a coin Ui to it so that Qi + Ui = P. Thus there is a limited amount of states which would lead to the succeeding state (P). The minimum number of coins that can sum up to P is thus equal to the number of coins of one of the states (Qi), plus one coin, the coin Ui.

Implementation-wise, the DP results are usually stored in an array. In our coin example the array "mink[0..S]" contains k values for states. In other words, mink[P] = k means that result of state (P) is equal to k. The array of DP results is calculated in a loop (often nested loops) in some order. The following piece of code contains recurrent equations for the problem, table of results (contents of array mink) and the solution itself.

```
/* Recurrent equations for DP:
  {k[0] = 0;
  {k[P] = min_i (k[P-Wi] + 1);   (for Wi <= P)
*/
/* Consider the input data: S=11, n=3, W = {1,3,5}
   The DP results table is:
  P = 0 |1 |2 |3 |4 |5 |6 |7 |8 |9 |10|11
  ------+--+--+--+--+--+--+--+--+--+--+--
  k = 0 |1 |2 |1 |2 |1 |2 |3 |2 |3 |2 |3
*/
// The implementation:
int n, S;                                       //n - number of coin types, S - desired overall weight
int wgt[MAXN];                                  //array of coin weights (W); for example: {1, 3, 5};
int mink[MAXW];                                 //array of DP results (k); look above for the example;

  mink[0] = 0;                                  //base of DP: 0 weight can be achieved by 0 coins
  for (int P = 1; P<=S; P++) {                  //iterate through all the states
    int minres = 1000000000;
    for (int i = 0; i<n; i++) if (wgt[i] <= P) { //suppose that the coin with weight wgt[i] is the last
      int tres = mink[P - wgt[i]] + 1;          //the number of coins with the coin is greater by one
      if (minres > tres) minres = tres;         //choose the minimal overall number of coins among all cases
    }
    mink[P] = minres;                           //store the result in mink array
  }
  int answer = mink[S];                         //the answer for the whole problem is the result for state (S)
```

Re: 2.4.5 Introducing Dynamic Programming (response to post by syg96) | Reply   Sat, Feb 12, 2011 at 2:20 AM EST

Tutorial (LCS example)

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Consider another problem: given two words, find the length of their longest common subsequence. For example, for two words "quetzalcoatl" and "tezcatlipoca" the longest subsequence has length 6, f.i. "ezaloa".

To solve the problem we introduce the set of subproblems: given a prefix of the first word and a prefix of the second word, find their LCS. Let the prefix of the first word has length i and the prefix of the second word has length j. As we see, the DP state is determined by two integer parameters: i and j. The state domain is therefore (i,j)->L, where i is the length of first word prefix, j is the length of second word prefix and L is the length of the longest common subsequence of these prefixes. The idea of solution is to take the solution for basic subproblem (0,0) and then add letters to the prefixes one-by-one until we reach the final state (n1,n2) which represents the problem for the full words.

Now let's derive the recurrent relations for DP results denoted as L[i,j]. Clearly, if one of the prefixes is empty, then the LCS must be zero. This is a base equation: L[i,0] = L[0,j] = 0. When i and j are positive then we have to treat several cases:
1. The last letter in the first word prefix is not used in the LCS. So it can be erased without changing the subsequence. The corresponding formula is L[i,j] = L[i-1,j].
2. The last letter in the second word prefix is unused. Similarly, the formula for the case is: L[i,j] = L[i,j-1]
3. Otherwise, last letters 's' and 't' of both prefixes are included in the common subsequence. Clearly, these letters must be equal. In such a case erasing both last letters will reduce LCS by exactly one. The corresponding formula is: L[i,j] = L[i-1,j-1] + 1 (only if 's' = 't').
Among all three cases we should choose the case which gives the maximal length of sequence.

Implementation-wise, the DP results are stored in two-dimensional array. The values of this array are calculated in two nested loops. It is important that the states are traversed in such order that parameter values are non-decreasing because the DP result for the state (i,j) depends on the results for states (i-1,j), (i,j-1), (i-1,j-1).

```
/* Recurrent relations for DP:
  {L[i,0] = L[0,j] = 0;
  |             {L[i-1,j],
  {L[i,j] = max|L[i,j-1],
                {L[i-1,j-1]+1   (only if last symbols are equal)
*/
/* Table of DP results:
   S|    t  e  z  c  a  t  l  i  p  o  c  a
T ji| 0  1  2  3  4  5  6  7  8  9 10 11 12
----+--------------------------------------
  0 | 0  0  0  0  0  0  0  0  0  0  0  0  0
q 1 | 0  0  0  0  0  0  0  0  0  0  0  0  0
u 2 | 0  0  0  0  0  0  0  0  0  0  0  0  0
e 3 | 0  0  1  1  1  1  1  1  1  1  1  1  1
t 4 | 0  1  1  1  1  1  2  2  2  2  2  2  2
z 5 | 0  1  1  2  2  2  2  2  2  2  2  2  2
a 6 | 0  1  1  2  2  3  3  3  3  3  3  3  3
l 7 | 0  1  1  2  2  3  3  4  4  4  4  4  4
c 8 | 0  1  1  2  3  3  3  4  4  4  4  5  5
o 9 | 0  1  1  2  3  3  3  4  4  4  5  5  5
a 10| 0  1  1  2  3  4  4  4  4  4  5  5  6
t 11| 0  1  1  2  3  4  5  5  5  5  5  5  6
l 12| 0  1  1  2  3  4  5  6  6  6  6  6  6
*/
```

```
// The implementation:
int n1, n2;                                             //lengths of words
char str1[1024], str2[1024];                            //input words
int lcs[1024][1024];                                    //DP results array

  for (int i = 0; i<=n1; i++)                           //iterate through all states (i,j)
    for (int j = 0; j<=n2; j++) {                       //in lexicographical order
      if (i == 0 || j == 0)
        lcs[i][j] = 0;                                  //the DP base case
      else {
        lcs[i][j] = max(lcs[i-1][j], lcs[i][j-1]);      //handle cases 1 and 2
        if (str1[i-1] == str2[j-1])
          lcs[i][j] = max(lcs[i][j], lcs[i-1][j-1] + 1); //handle case 3
      }
    }
  int answer = lcs[n1][n2];
```

Discussion

Comparison with memoization

There is another technique called memoization which is covered in detail by recipe "Optimizing recursive solution". Recursive solution with memoization is very similar to backward-style dynamic programming solution. Both methods solve recurrent equations, which means that they deal with state domain - set of states with some result defined. The results for some states are determined from base of recurrence. The results for other states depend on the results of previous states. The DP solution iterates through the states in some particular order set by coder, while memoization iterates through them in order of depth-first search. DP never calculates the DP result for any state twice, just like the recursive solution with full memoization. The memoization approach does not spend time on unnecessary states - it is a lazy algorithm. Only the states which influence the final answer are processed. Here are the pros and cons of memoization over DP:
1[+]. Sometimes easier to code.
2[+]. Does not require to specify order on states explicitly.
3[+]. Processes only necessary states.
4[-]. Works only in the backward-style DP.
5[-]. Works a bit slower than DP (by constant).

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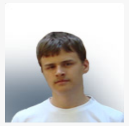
syg96

Most of DP problems can be divided into two types: optimization problems and combinatoric problems. The optimization problems require you to choose some feasible solution so that the value of goal function is minimized (or maximized). Combinatoric problems request the number of ways to do something or the probability of some event. Let's have a closer look at these problem types.

Optimization DP problem

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Optimization problem asks to choose the best feasible solution according to some goal function. Both coins and LCS examples are optimization-type. The recurrent equation looks like R[s] = min(F1(R[i], R[j], ..., R[k]), F2(R[u], R[v], ..., R[w]), ..., Fl(R[q], R[p], ..., R[z])), where R is the DP results array. Simply speaking, the result is chosen as the best = minimal among the several candidate cases. For each case the result is calculated from the results of previous DP states. For example in coins problem all the possible last coin cases are considered. Each of them yields one case in the recurrent formula. The result for the state is a minimum among all such cases. In LCS example there were three cases: first word last letter unused, second word last letter unused and both words last letter used.

It is often useful to fill the DP results array with neutral values before calculating anything. The neutral value is a result which does not affect the problem answer for sure. In case of minimization problem the neutral value is positive infinity: since it is greater than any number, all the recurrent formulas would prefer a case with finite value to such a neutral element. In other words, the state with neutral value result can be thought of as an impossible state. Note that for maximization problem negative infinity is a neutral element.

The DP states are often called DP subproblems because they represent some problem for input data which is subset of the whole problem input. For example, in LCS case each subproblem involves two arbitrary prefixes of the original two words. The DP method relies on the optimal substructure property: given the optimal solution for the whole problem, its partial solutions must be optimal for the subproblems. In the coins case it means that if the solution for whole problem with overall weight S is optimal and it contains coin with weight w, then the solution without w coin must also be optimal for the subproblem with overall weight (S - w).

Optimal substructure property is very important: if it does not hold and the optimal solution has the subsolution which is not optimal, then it would be discarded somewhere in the middle of DP on taking the minimum. Often the DP solution turns out to be theoretically wrong because it lacks the optimal substructure. For example there is a classical travelling salesman problem. Let the DP state domain be (k,l)->D where D is the minimal length of the simple path going through exactly k cities with 0-th city being the first one and l-th city being the last one. The optimal substructure property in such a DP does not hold: given the shortest tour its subpath with fixed last city and overall number of cities is not always the shortest. Therefore the proposed DP would be wrong anyway.

Combinatoric DP problem

The goal of combinatoric DP problem is to find number of ways to do something or the probability that the event happens. Often the number of ways can be big and only the reminder modulo some small number is required. The recurrent equation looks like R[s] = F1(R[i], R[j], ..., R[k]) + F2(R[u], R[v], ..., R[w]) + ... + Fl(R[q], R[p], ..., R[z]). The only difference from optimization case is the sum instead of minimum - and it changes a lot. The summation means that the different ways from F1, F2, ..., Fl cases altogether comprise the all the ways for state (s).

The example of combinatoric case is a modified coins problem: count the number of ways to choose coins so that their overall weight is equal to S. The state domain is the same: (P)->k where k is number of ways to choose coins so that their overall weight is exactly P. The recurrent equations are only a bit different in combinatoric problem.

```
/* Recurrent equations for DP:
  {k[0] = 1;
  {k[P] = sum_i (k[P-Wi]);   (for Wi <= P)
*/
/* Consider the input data: S=11, n=3, W = {1,3,5}
   The DP results table is:
  P = 0 |1 |2 |3 |4 |5 |6 |7 |8 |9 |10|11
  ------+--+--+--+--+--+--+--+--+--+--+--
  k = 1 |1 |1 |2 |3 |5 |8 |12|19|30|47|74
```

```
*/
```

There is also a neutral value for combinatoric problem. Since combinatoric problem uses summation, the neutral element is zero. The DP results in combinatoric case usually represents number of ways to do smth, so if the result is zero than there is no way to do it. The neutral result value means that the case is impossible. It may be useful to fill DP results array with zero values, though it is usually done automatically. In case of combinatorics it is important that each possible way is counted and that no way is counted more than once. The second condition is sometimes difficult to satisfy.

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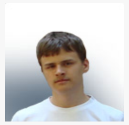

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Forward vs backward DP style

All the DPs described above are done in backward style. The schema is: iterate through all the states and for each of them calculate the result by looking backward and using the already known DP results of previous states. This style can also be called recurrent since it uses recurrent equations directly for calculation. The relations for backward-style DP are obtained by examining the best solution for the state and trying to decompose it to lesser states.

There is also forward-style DP. Surprisingly it is often more convenient to use. The paradigm of this style is to iterate through all the DP states and from each state perform some transitions leading forward to other states. Each transition modifies the currently stored result for some unprocessed states. When the state is considered, its result is already determined completely. The forward formulation does not use recurrent equations, so it is more complex to prove the correctness of solution strictly mathematically. The recurrent relations used in forward-style DP are obtained by considering one partial solution for the state and trying to continue it to larger states. To perform forward-style DP it is necessary to fill the DP results with neutral values before starting the calculation.

The first example will be combinatoric coins problem. Suppose that you have a partial solution with P overall weight. Then you can add arbitrary coin with weight Wi and get overall weight P+Wi. So you get a transition from state (P) to state (P+Wi). When this transition is considered, the result for state (P) is added to the result of state (P+Wi) which means that all the ways to get P weight can be continued to the ways to get P+Wi weight by adding i-th coin. Here is the code.

```
/* Recurrent relations (transitions) of DP:
  {k[0] = 1;
  {(P)->k ===> (P+Wi)->nk    add k to nk
*/
  //res array is automatically filled with zeroes
  res[0] = 1;                             //DP base is the same
  for (int p = 0; p<s; p++)               //iterate through DP states
    for (int i = 0; i<n; i++) {           //iterate through coin to add
      int np = p + wgt[i];                //the new state is (np)
      if (np > s) continue;               //so the transition is (p) ==> (np)
      res[np] += res[p];                  //add the DP result of (p) to DP result of (np)
    }
  int answer = res[s];                    //problem answer is the same
```

The second example is longest common subsequence problem. It is of maximization-type, so we have to fill the results array with negative infinities before calculation. The DP base is state (0,0)->0 which represents the pair of empty prefixes. When we consider partial solution (i,j)->L

we try to continue it by three ways:
1. Add the next letter of first word to the prefix, do not change subsequence.
2. Add the next letter of the second word to the prefix, do not change subsequence.
3. Only if the next letters of words are the same: add next letter to both prefixes and include it in the subsequence.
For each transition we perform so-called relaxation of the larger DP state result. We look at the currently stored value in that state: if it is worse that the proposed one, then it is replaced with the proposed one, otherwise it is not changed.
The implementation code and compact representation of DP relations are given below.

```
/* Recurrent relations (transitions) of DP:
  {L[0,0] = 0;
  |          /> (i+1,j)->relax(L)
  {(i,j)->L ==> (i,j+1)->relax(L)
             \> (i+1,j+1)->relax(L+1)  (only if next symbols are equal)
*/
void relax(int &a, int b) {                    //relaxation routine
  if (a < b) a = b;
}

  memset(lcs, -63, sizeof(lcs));               //fill the DP results array with negative infinity
  lcs[0][0] = 0;                               //set DP base: (0,0)->0
  for (int i = 0; i<=n1; i++)
    for (int j = 0; j<=n2; j++) {              //iterate through all states
      int tres = lcs[i][j];
      relax(lcs[i+1][j], tres);                //try transition of type 1
      relax(lcs[i][j+1], tres);                //try transition of type 2
      if (str1[i] == str2[j])                  //and if next symbols are the same
        relax(lcs[i+1][j+1], tres + 1);        //then try transition of type 3
    }
  int answer = lcs[n1][n2];
```

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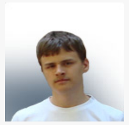

syg96
182 posts

Recovering the best solution for optimization problems

The optimization problem asks us to find the feasible solution with the minimal value of goal function, but DP finds only the goal function value itself. It does not produce the best solution along with the numerical answer. In practical usage the answer without a solution is useless, though in topcoder algorithm problems often only the answer is required. Anyway, it is useful to know how to reconstruct the best solution after DP.

In the most common case the each transition is an atomic improvement of some partial solution and recurrent equation is something like: R[s] = min(F1(R[u], u), F2(R[v], v), ..., Fk(R[w], w)). In other words, the result for the state is produced from a single best previous state plus some modification. In such a case the DP solution can be reconstructed trivially from the DP solution path. The DP solution path goes from some base DP state to some final state and consists of the states which represent all the partial solutions of the desired best solution. There are two ways to get this path.

The first way is to recalculate the DP from the end to the start. First we choose the final state (f) we want to trace the path from. Then we process the (f) state just like we did it in the DP: iterate through all the variants to get it. Each variant originates in a previous state (p). If the variant produces the result equal to DP result of state (f), then the variant if possible. There is always at least one possible variant to produce the DP result for the state, though there can be many of them. If the variant originating from (p) state is possible, then there is at least one best solution path going through state (p). Therefore we can move to state (p) and search the path from starting state to state (p) now. We can end path tracing when we reach the starting state.

Another way is to store back-links along with the DP result. For each state (s) we save the parameters of the previous state (u) that was continued. When we perform a transition (u) ==> (s) which produces better result than the currently stored in (s) then we set the back-link to (s) to the state (u). To trace the DP solution path we need simply to repeatedly move to back-linked state until the starting state is met. Note that you can store any additional info about the way the DP result was obtained to simplify solution reconstruction.

The first approach has a lot of drawbacks. It is usually slower, it leads to DP code being copy/pasted, it requires backward-style DP for tracing the path. It is good only in the rare case when there is not enough memory to store the back-links required in the second way. The second way is preferred since it is simple to use and supports both backward and forward style DP solutions. If the result for each DP state originates from more than one previous DP state, then you can store the links to all the previous states. The path reconstruction is easiest to implement in the recursive way in such a case.

For example of recovering the solution coins problem is again considered. Note that the DP code is almost the same except that the back-link and item info is set on the relaxation. The later part of tracing the path back is rather simple.

```
/* Consider the input data: S=11, n=3, W = {1,3,5}
   The DP results + back-links table is:
  P  = 0 |1 |2 |3 |4 |5 |6 |7 |8 |9 |10|11
  -------+--+--+--+--+--+--+--+--+--+--+--
mink = 0 |1 |2 |1 |2 |1 |2 |3 |2 |3 |2 |3
prev = ? |S0|S1|S0|S1|S0|S1|S2|S3|S4|S5|S6
item = ? |I0|I0|I1|I1|I2|I2|I2|I2|I2|I2|I2
*/

int mink[MAXW];                      //the DP result array
int prev[MAXW], item[MAXW];          //prev - array for back-links to previous state
int k;                               //item - stores the last item index
int sol[MAXW];                       //sol[0,1,2,...,k-1] would be the desired solution

  memset(mink, 63, sizeof(mink));    //fill the DP results with positive infinity
  mink[0] = 0;                       //set DP base (0)->0
  for (int p = 0; p<s; p++)          //iterate through all states
    for (int i = 0; i<n; i++) {      //try to add one item
      int np = p + wgt[i];           //from (P)->k we get
      int nres = mink[p] + 1;        //to (P+Wi)->k+1
      if (mink[np] > nres) {         //DP results relaxation
        mink[np] = nres;             //in case of success
        prev[np] = p;                //save the previous state
        item[np] = i;                //and the used last item
      }
    }
```

```
int answer = mink[s];
int cp = s;                          //start from current state S
while (cp != 0) {                    //until current state is zero
  int pp = prev[cp];                 //get the previous state from back-link
  sol[k++] = item[cp];               //add the known item to solution array
  cp = pp;                           //move to the previous state
}
```

END OF RECIPE

The coins tutorial was taken from Dumitru's DP recipe.

Re: 2.4.5 Introducing Dynamic Programming (response to post by syg96) | Reply   Sat, May 14, 2011 at 6:34 AM EDT

TarifEzaz
15 posts

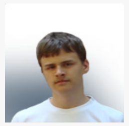
In your code for solving the LCS problem, you've written the following:

```
if (str1[i-1] == str2[j-1])
    lcs[i][j] = max(lcs[i][j], lcs[i-1][j-1] + 1); //handle case 3
```

Now according to the optimal subtructure of an LCS, if (i-1)th and (j-1)th character matches, then lcs[i][j] could only be lcs[i-1][j-1]+1. This happens because imagine: Zk is the LCS of prefixes Xi and Yj, where X and Y are the original sequence. Let zk, xi and yj, be the k-th, i-th and j-th character of Zk, Xi and Yj respectively. Now if zk != xi, then we could easily append xi to the end of zk to get a (k+1) length subsequence, which is a contradiction as we assumed Zk to be the LCS. On the other hand if zk = xi, then LCS Z(k-1) should be of length k-1 and should be constructed from X(i-1) and Y(j-1). If X(i-1) and Y(j-1) had a k-length subsequence then we would have taken that instead, causing a contradiction. I'm almost copy-pasting the idea that is described in CLR book :)

So I think the max operation is redundant. Thanks for your awesome recipe!

Re: 2.4.5 Introducing Dynamic Programming (response to post by TarifEzaz) | Reply   Sat, May 14, 2011 at 6:55 AM EDT


syg96
182 posts

Yes, you are right.
But I won't change the code. I think it is a bit simpler right now. I have described three theoretically possible cases, so let's DP just get the best value among all of them.
By the way, it is a very important principle of algorithmic programming competitions: the simpler the solution is, the better. Both in sense of theory and implementation=)

Re: 2.4.5 Introducing Dynamic Programming (response to post by syg96) | Reply   Thu, Feb 2, 2012 at 4:45 AM EST

yousuf_shawon
2 posts

In a problem lexicographically smallest lcs should be find out.
If two string s1="acbc", and s2 ="abczacb", then possible lcs are "acb","acc", "abc".
So lexicographically smallest is "abc".
How i find the lexicographically smallest lcs.
Please help.

Re: 2.4.5 Introducing Dynamic Programming (response to post by yousuf_shawon) | Reply Sat, Jul 12, 2014 at 9:43 AM EDT

rit2012008
23 posts

store back-link and when u have a chance that u can go Dp[i - 1][j] or Dp[i - 1][j - 1] then look last used value that which state have lexicographically smallest element :D