# Game Theory and Grundy Numbers

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Here we'll talk about finite 2-person impartial games with perfect information (** * **).

It means that in the games that'll be described the only difference between player 1 and player 2 is that player 1 plays first. Also, each player is completely aware of all the previous moves and about all the parameters of the current state of play and the game is bound to end after a certain finite number of moves.

The most famous game in this regard is the game of NIM.
Here, there are a set of piles of stones. Each pile contains some non-zero arbitrary number of stones. During any move, a player can choose a pile and can remove any number of stones from the selected pile.
The task is to find, if both players play optimally (i.e. out of all possible moves, they play the one which maximizes their chance of winning), who will win?
A player is said to loose if he is unable to perform any valid move in his turn. In this case a player looses if his move starts with zero piles. Also, if a player looses, the other wins.

Go to Top

In any game, the first thing that one needs to do is to represent any state of that game using some parameters. In doing so, it is always desirable that the number of parameters be minimal. Also, if we only need to know the final outcome of the game, any state will fall under 2 categories - winning or losing. This can be proven inductively and will be covered later.
Now, any state of NIM can be represented by the tuple where is the number of piles and is the number of stones in the pile.

It is always helpful to search for some invariant in the game. It is very hard to see what is the invariant for Nim, but if known, it can be proved easily.

Define a **good tuple** as a tuple such that the **xor** of all the values in the tuple is . Any tuple which is not good, is bad.

Now, the invariant for nim is:

**From any bad tuple, you cannot move to any bad tuple. From any good tuple, you always move to atleast one bad tuple.**

It shouldn't be very hard to see that the "all piles empty" is a **bad tuple**. Also, notice that you can always win if you are at a **good tuple**. Since, you can always force your opponent into a **bad tuple** and on his move, he will either loose or play to put you again into a **good tuple**.

Similarly, if you are in a **bad tuple**, you can never win if your opponent plays optimally.

This basically means that the first player wins if the **xor** of the number of stones in each pile is non-zero and looses otherwise.

Now we'll look into the theory of Grundy Numbers and come back to see how it applies to Nim and produces the above result, which might seem to be a bit out of the blue for now.

## Grundy Numbers

Ever game satisfying (*) can be converted into an equivalent Nim game using the Grundy Numbers. For now, you can imagine Grundy Numbers to be representation of the states in any game in the form of single non negative integers.

We calculate Grundy Number of any state through the following rules:
**Grundy Number of a loosing state is .
**Grundy Number of any other state is the  of the set of Grundy Numbers of all the states which can come up in the game with one valid move from the current state.

stands for Minimum Excluded Value.  of a set  is the minimum non negative integer which is not in .

Lets apply these rules to calculate the Grundy Number () of a single heap Nim game with  stones.

If ,
If ,
If ,
An easy induction will show you that .

**SUM OF GAMES**

If some games are played in parallel in a "super" game (i.e. upon his turn, a player may decide to move in any of the currently active game; when there is no valid move in any of the games, the player looses), then this "super" game is called the SUM of all the parallel "small" games.

If the "small" game have Grundy Numbers  then the "super" game has Grundy Number .

Now, the game of Nim can be considered as the sum many of single pile Nim games. Since, we know that each single piles' grundy number is equal to the number of stones in that pile, the overall grundy number is simply their xor. This is consistent with what we have done earlier.

Now, lets consider some simple games and find their Grundy Numbers.

**Game 1**:
Unlike Nim, here you can only remove 1,2 or 3 stones from a choosen pile (there is only one pile as of now).
Now, , , , , ,  and so on, i.e. .
Grundy numbers wasn't really necessary to figure out who wins in a single pile. But in case of multiple piles, it really helps to find out the grundy number of each pile and take the xor.

**Game 2**:
Consider an infinite grid, a coin is placed at , in each turn, a player can move the coin along decreasing  co-ordinate or decreasing  co-ordinate by as many steps as he desires with the only restriction being that neither  nor  become negative at any point.
This is basically a 2-pile Nim game with pile heights  and  respectively!