- QP-Spline-Path Optimizer

Tip: to read the equations in the document, you are recommended to use Chrome with [a plugin](https://chrome.google.com/webstore/detail/tex-all-the-things/cbimabofgmfdkicghcadidpemeenbffn) or copy the latex equation to [an online editor](http://www.hostmath.com/)

Quadratic programming + Spline interpolation

– 1. Objective function

— 1.1 Get path length

Path is defined in station-lateral coordination system. The **s** range from vehicle's current position to default planing path length.

— 1.2 Get spline segments

Split the path into **n** segments. each segment trajectory is defined by a polynomial.

— 1.3 Define function for each spline segment

Each segment ***i*** has accumulated distance d_i along reference line. The trajectory for the segment is defined as a polynomial of degree five by default.

$$l = f_i(s) = a_{i0} + a_{i1} \cdot s + a_{i2} \cdot s^2 + a_{i3} \cdot s^3 + a_{i4} \cdot s^4 + a_{i5} \cdot s^5 (0 \leq s \leq d_i)$$

— 1.4 Define objective function of optimization for each segment

$$cost = \sum_{i=1}^{n} \Big(w_1 \cdot \int\limits_{0}^{d_i} (f_i')^2(s) ds + w_2 \cdot \int\limits_{0}^{d_i} (f_i'')^2(s) ds + w_3 \cdot \int\limits_{0}^{d_i} (f_i''')^2(s) ds \Big)$$

— 1.5 Convert the cost function to QP formulation

QP formulation:

$$\begin{aligned} minimize & \frac{1}{2} \cdot x^T \cdot H \cdot x + f^T \cdot x \\ s.t. \qquad & LB \leq x \leq UB \\ & A_{eq}x = b_{eq} \\ & Ax \geq b \end{aligned}$$

Below is the example for converting the cost function into the QP formulaiton.

$$f_i(s) \begin{vmatrix} 1 & s & s^2 & s^3 & s^4 & s^5 \end{vmatrix} \cdot \begin{vmatrix} a_{i0} \\ a_{i1} \\ a_{i2} \\ a_{i3} \\ a_{i4} \\ a_{i5} \end{vmatrix}$$

And

$$f_i'(s) = \begin{vmatrix} 0 & 1 & 2s & 3s^2 & 4s^3 & 5s^4 \end{vmatrix} \cdot \begin{vmatrix} a_{i0} \\ a_{i1} \\ a_{i2} \\ a_{i3} \\ a_{i4} \\ a_{i5} \end{vmatrix}$$

And

$$f_i'(s)^2 = \begin{vmatrix} a_{i0} & a_{i1} & a_{i2} & a_{i3} & a_{i4} & a_{i5} \end{vmatrix} \cdot \begin{vmatrix} 0 \\ 1 \\ 2s \\ 3s^2 \\ 4s^3 \\ 5s^4 \end{vmatrix} \cdot \begin{vmatrix} 0 & 1 & 2s & 3s^2 & 4s^3 & 5s^4 \end{vmatrix} \cdot \begin{vmatrix} a_{i0} \\ a_{i1} \\ a_{i2} \\ a_{i3} \\ a_{i4} \\ a_{i5} \end{vmatrix}$$

then we have,

$$\int_0^{d_i} f_i'(s)^2 ds \int_0^{d_i} \begin{vmatrix} a_{i0} & a_{i1} & a_{i2} & a_{i3} & a_{i4} & a_{i5} \end{vmatrix} \cdot \begin{vmatrix} 0 \\ 1 \\ 2s \\ 3s^2 \\ 4s^3 \\ 5s^4 \end{vmatrix} \cdot \begin{vmatrix} 0 & 1 & 2s & 3s^2 & 4s^3 & 5s^4 \end{vmatrix} \cdot \begin{vmatrix} a_{i0} \\ a_{i1} \\ a_{i2} \\ a_{i3} \\ a_{i4} \\ a_{i5} \end{vmatrix} ds$$

extract the const outside the integration, we have,

$$\int_0^{d_i} f'(s)^2 ds \begin{vmatrix} a_{i0} & a_{i1} & a_{i2} & a_{i3} & a_{i4} & a_{i5} \end{vmatrix} \cdot \int_0^{d_i} \begin{vmatrix} 0 \\ 1 \\ 2s \\ 3s^2 \\ 4s^3 \\ 5s^4 \end{vmatrix} \cdot \begin{vmatrix} 0 & 1 & 2s & 3s^2 & 4s^3 & 5s^4 \end{vmatrix} ds \cdot \begin{vmatrix} a_{i0} \\ a_{i1} \\ a_{i2} \\ a_{i3} \\ a_{i4} \\ a_{i5} \end{vmatrix}$$

$$\begin{vmatrix} a_{i0} & a_{i1} & a_{i2} & a_{i3} & a_{i4} & a_{i5} \end{vmatrix} \cdot \int_0^{d_i} \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2s & 3s^2 & 4s^3 & 5s^4 \\ 0 & 2s & 4s^2 & 6s^3 & 8s^4 & 10s^5 \\ 0 & 3s^2 & 6s^3 & 9s^4 & 12s^5 & 15s^6 \\ 0 & 4s^3 & 8s^4 & 12s^5 & 16s^6 & 20s^7 \\ 0 & 5s^4 & 10s^5 & 15s^6 & 20s^7 & 25s^8 \end{vmatrix} ds \cdot \begin{vmatrix} a_{i0} \\ a_{i1} \\ a_{i2} \\ a_{i3} \\ a_{i4} \\ a_{i5} \end{vmatrix}$$

Finally, we have

$$\int_0^{d_i} f_i'(s)^2 ds = \begin{vmatrix} a_{i0} & a_{i1} & a_{i2} & a_{i3} & a_{i4} & a_{i5} \end{vmatrix} \cdot \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_i & d_i^2 & d_i^3 & d_i^4 & d_i^5 \\ 0 & d_i^2 & \frac{4}{3}d_i^3 & \frac{6}{4}d_i^4 & \frac{8}{5}d_i^5 & \frac{10}{6}d_i^6 \\ 0 & d_i^3 & \frac{6}{4}d_i^4 & \frac{9}{5}d_i^5 & \frac{12}{6}d_i^6 & \frac{15}{7}d_i^7 \\ 0 & d_i^4 & \frac{8}{5}d_i^5 & \frac{12}{6}d_i^6 & \frac{16}{7}d_i^7 & \frac{20}{8}d_i^8 \\ 0 & d_i^5 & \frac{10}{6}d_i^6 & \frac{15}{7}d_i^7 & \frac{20}{8}d_i^8 & \frac{25}{9}d_i^9 \end{vmatrix} \cdot \begin{vmatrix} a_{i0} \\ a_{i1} \\ a_{i2} \\ a_{i3} \\ a_{i4} \\ a_{i5} \end{vmatrix}$$

Please notice that we got a 6 x 6 matrix to represent the derivative cost of 5th order spline.

Similar deduction can also be used to calculate the cost of second and third order derivatives.

– 2 Constraints

— 2.1 The init point constraints

Assume that the first point is (s_0, l_0), (s_0, l_0') and (s_0, l_0''), where l_0 , l_0' and l_0'' is the lateral offset and its first and second derivatives on the init point of planned path, and are calculated from $f_i(s)$, $f_i'(s)$, and $f_i(s)''$.

Convert those constraints into QP equality constraints, using:

$$A_{eq}x = b_{eq}$$

Below are the steps of conversion.

$$f_i(s_0) = \begin{vmatrix} 1 & s_0 & s_0^2 & s_0^3 & s_0^4 & s_0^5 \end{vmatrix} \cdot \begin{vmatrix} a_{i0} \\ a_{i1} \\ a_{i2} \\ a_{i3} \\ a_{i4} \\ a_{i5} \end{vmatrix} = l_0$$

And

$$f_i'(s_0) = \begin{vmatrix} 0 & 1 & 2s_0 & 3s_0^2 & 4s_0^3 & 5s_0^4 \end{vmatrix} \cdot \begin{vmatrix} a_{i0} \\ a_{i1} \\ a_{i2} \\ a_{i3} \\ a_{i4} \\ a_{i5} \end{vmatrix} = l_0'$$

And

$$f_i''(s_0) = \begin{vmatrix} 0 & 0 & 2 & 3 \times 2s_0 & 4 \times 3s_0^2 & 5 \times 4s_0^3 \end{vmatrix} \cdot \begin{vmatrix} a_{i0} \\ a_{i1} \\ a_{i2} \\ a_{i3} \\ a_{i4} \\ a_{i5} \end{vmatrix} = l_0''$$

where i is the index of the segment that contains the s_0.

— 2.2 The end point constraints

Similar to the init point, the end point (s_e, l_e) is known and should produce the same constraint as described in the init point calculations.

Combine the init point and end point, and show the equality constraint as:

$$\begin{vmatrix} 1 & s_0 & s_0^2 & s_0^3 & s_0^4 & s_0^5 \\ 0 & 1 & 2s_0 & 3s_0^2 & 4s_0^3 & 5s_0^4 \\ 0 & 0 & 2 & 3\times 2s_0 & 4\times 3s_0^2 & 5\times 4s_0^3 \\ 1 & s_e & s_e^2 & s_e^3 & s_e^4 & s_e^5 \\ 0 & 1 & 2s_e & 3s_e^2 & 4s_e^3 & 5s_e^4 \\ 0 & 0 & 2 & 3\times 2s_e & 4\times 3s_e^2 & 5\times 4s_e^3 \end{vmatrix} \cdot \begin{vmatrix} a_{i0} \\ a_{i1} \\ a_{i2} \\ a_{i3} \\ a_{i4} \\ a_{i5} \end{vmatrix} = \begin{vmatrix} l_0 \\ l_0' \\ l_0'' \\ l_e \\ l_e' \\ l_e'' \end{vmatrix}$$

— 2.3 Joint smoothness constraints

This constraint is designed to smooth the spline joint. Assume two segments seg_k and seg_{k+1} are connected, and the accumulated **s** of segment seg_k is s_k. Calculate the constraint equation as:

$$f_k(s_k) = f_{k+1}(s_0)$$

Below are the steps of the calculation.

$$\begin{vmatrix} 1 & s_k & s_k^2 & s_k^3 & s_k^4 & s_k^5 \end{vmatrix} \cdot \begin{vmatrix} a_{k0} \\ a_{k1} \\ a_{k2} \\ a_{k3} \\ a_{k4} \\ a_{k5} \end{vmatrix} = \begin{vmatrix} 1 & s_0 & s_0^2 & s_0^3 & s_0^4 & s_0^5 \end{vmatrix} \cdot \begin{vmatrix} a_{k+1,0} \\ a_{k+1,1} \\ a_{k+1,2} \\ a_{k+1,3} \\ a_{k+1,4} \\ a_{k+1,5} \end{vmatrix}$$

Then

$$\begin{vmatrix} 1 & s_k & s_k^2 & s_k^3 & s_k^4 & s_k^5 & -1 & -s_0 & -s_0^2 & -s_0^3 & -s_0^4 & -s_0^5 \end{vmatrix} \cdot \begin{vmatrix} a_{k0} \\ a_{k1} \\ a_{k2} \\ a_{k3} \\ a_{k4} \\ a_{k5} \\ a_{k+1,0} \\ a_{k+1,1} \\ a_{k+1,2} \\ a_{k+1,3} \\ a_{k+1,4} \\ a_{k+1,5} \end{vmatrix} = 0$$

Use $s_0 = 0$ in the equation.
Similarly calculate the equality constraints for:

$$f'_k(s_k) = f'_{k+1}(s_0) f''_k(s_k) = f''_{k+1}(s_0) f'''_k(s_k) = f'''_{k+1}(s_0)$$

— 2.4 Sampled points for boundary constraint
Evenly sample **m** points along the path, and check the obstacle boundary at those points. Convert the constraint into QP inequality constraints, using:

$$Ax \geq b$$

First find the lower boundary $l_{lb,j}$ at those points (s_j, l_j) and $j \in [0, m]$ based on the road width and surrounding obstacles. Calculate the inequality constraints as:

$$\begin{vmatrix} 1 & s_0 & s_0^2 & s_0^3 & s_0^4 & s_0^5 \\ 1 & s_1 & s_1^2 & s_1^3 & s_1^4 & s_1^5 \\ ... & ... & ... & ... & ... & ... \\ 1 & s_m & s_m^2 & s_m^3 & s_m^4 & s_m^5 \end{vmatrix} \cdot \begin{vmatrix} a_{i0} \\ a_{i1} \\ a_{i2} \\ a_{i3} \\ a_{i4} \\ a_{i5} \end{vmatrix} \geq \begin{vmatrix} l_{lb,0} \\ l_{lb,1} \\ ... \\ l_{lb,m} \end{vmatrix}$$

Similarly, for the upper boundary $l_{ub,j}$, calculate the inequality constraints as:

$$\begin{vmatrix} -1 & -s_0 & -s_0^2 & -s_0^3 & -s_0^4 & -s_0^5 \\ -1 & -s_1 & -s_1^2 & -s_1^3 & -s_1^4 & -s_1^5 \\ ... & ...- & ... & ... & ... & ... \\ -1 & -s_m & -s_m^2 & -s_m^3 & -s_m^4 & -s_m^5 \end{vmatrix} \cdot \begin{vmatrix} a_{i0} \\ a_{i1} \\ a_{i2} \\ a_{i3} \\ a_{i4} \\ a_{i5} \end{vmatrix} \geq -1 \cdot \begin{vmatrix} l_{ub,0} \\ l_{ub,1} \\ ... \\ l_{ub,m} \end{vmatrix}$$