Fluid Dynamics

Chapter 3
Integral Analysis and Relations for a Control Volume

控制容积积分方程/关系

Flow anlysis

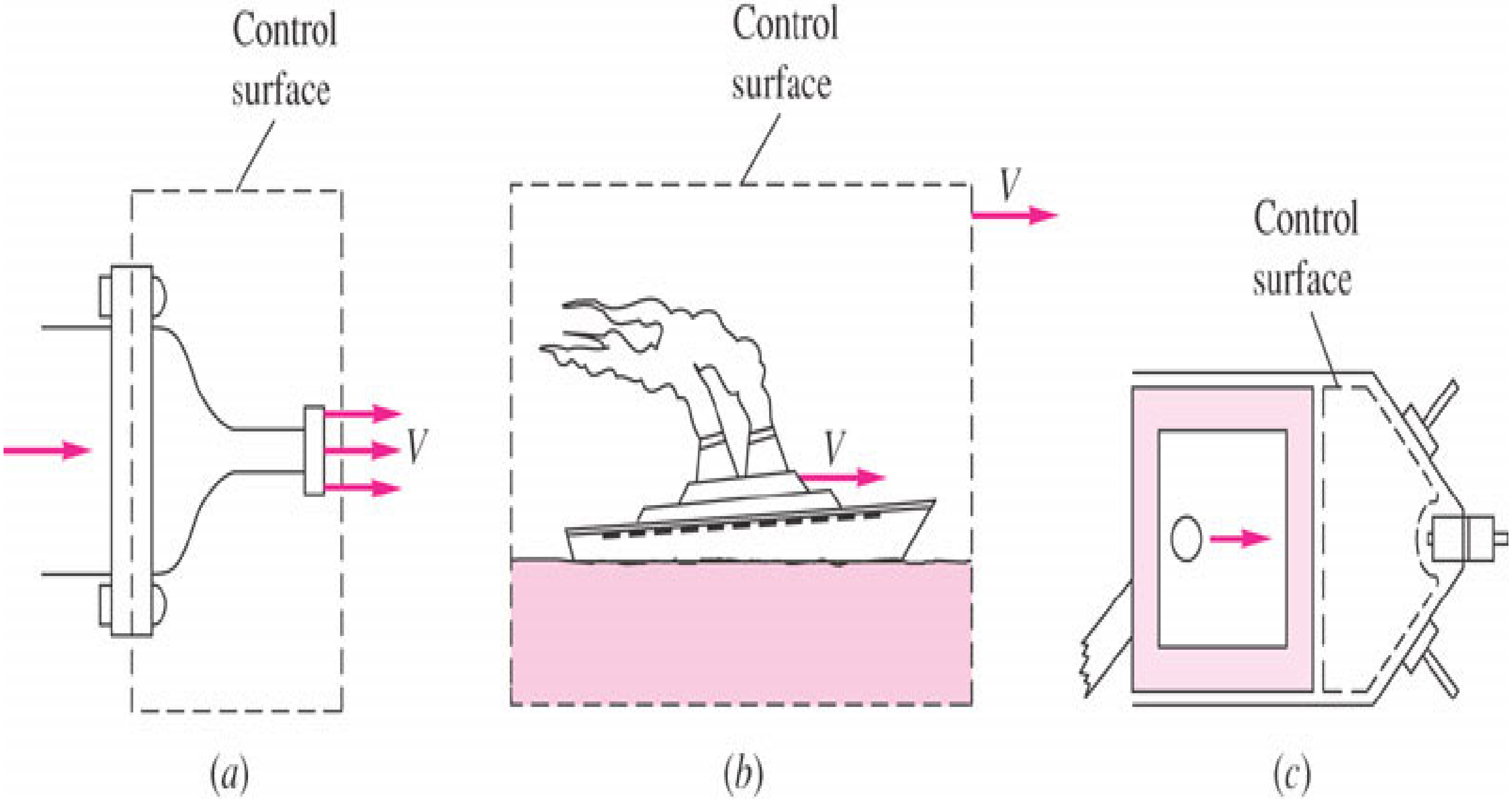
There are three different methods

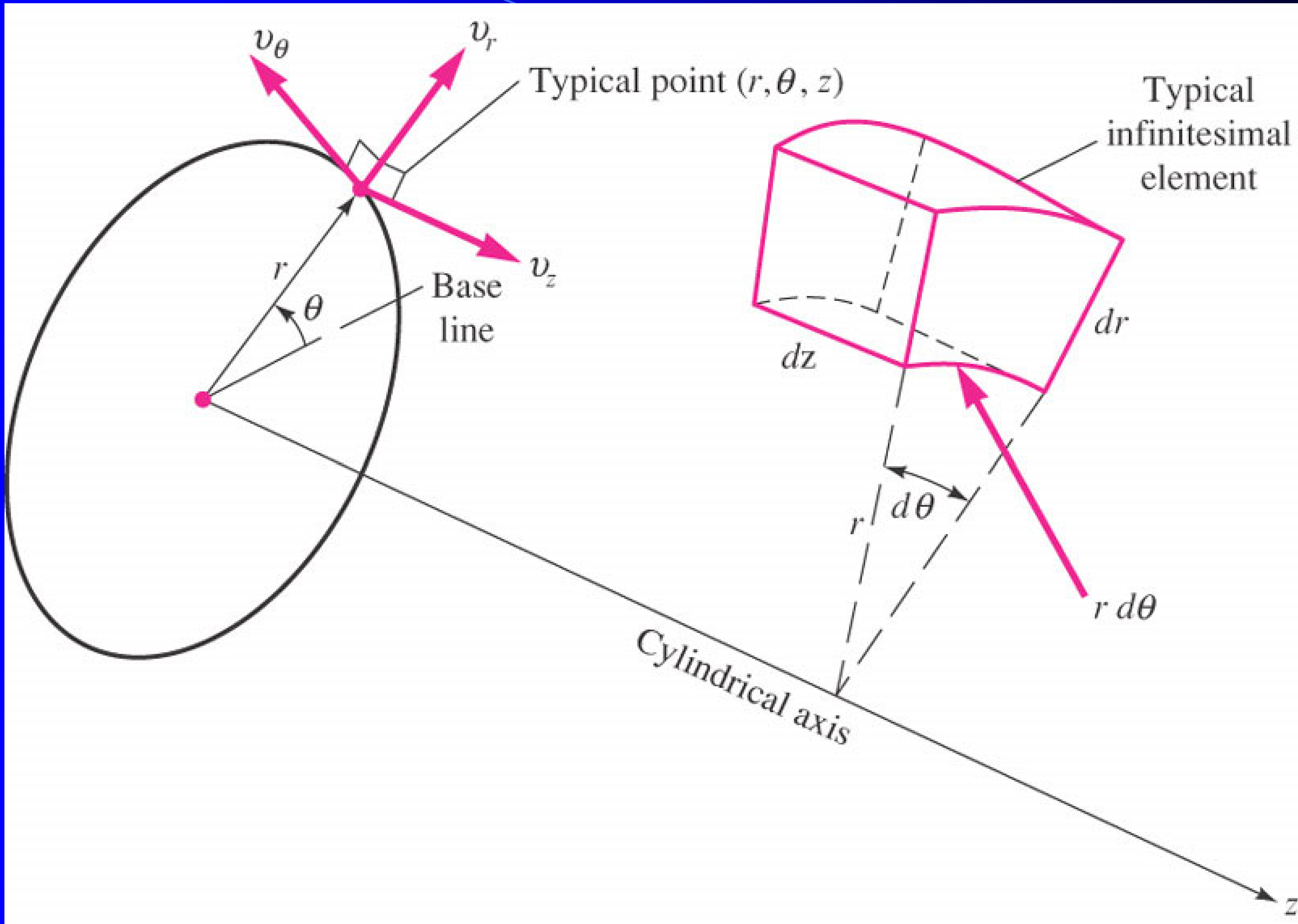
- *Integral analysis* (积分分析) *looking at* gross effect of fluid particles in *control volume or system* – To obtain some integral equations.
- *Differential analysis*（微分分析）looking at infinitesimal *system* or control volume (local individual behaviour) – To obtain differential equations
- *Dimensional analysis*（量纲分析）is used in experimental study of fluid flow to rearrange flow parameters and obtain dimensionless parameter groups, such as Re, Ma, etc. through which we can not obtain an exact flow solution.

Control Volume

Control volume is a region chosen by the analyst with open boundaries (control surface) through which mass, momentum, and energy are allowed to cross.

It is noted that the control volume can be fixed, moving, and deformable. It can be both finite and infinitesimal.


Control
surface
V
Control
surface
V
V
Control
surface
(a)
(b)
(c)



v_θ
v_r
Typical point (r, θ, z)
Typical
infinitesimal
element
r
θ
Base
line
v_z
dr
dz
$d\theta$
r
$r\,d\theta$
Cylindrical axis
z

It is noted that

all the previously established basic laws are applied only to a fixed quantity of mass, i.e. a system; there are no existing conservation equations suited for CV.

The Objective

of this chapter is to develop the basic conservation equations in integral form for control volumes.

(Conservation equations 守恒方程)

How to

establish conservation equations for CV

- Starting point is conservation equations for systems;
- Establish the general relation used as a bridge connecting the system and CV;
- Based on the "bridge" we shall establish the conservation equations for a control volume.

Main subjects

3.1 Basic physical laws for a system

3.2 Reynolds transport theorem** (The bridge!!)

3.3 Conservation of mass

3.4 Linear-momentum equation for inertial control volume

3.5 Moving control volume at constant velocity

3.6 Bernoulli's equation - application of continuity / momentum eq.

3.7 Angular-momentum equation

3.8 Energy equation

3.1 Basic physical laws for a system

1. Conservation of Mass

A system is composed of the same quantity of matter at all times. Mass of the system ,m , is conserved or constant:

$$m_{syst} = \text{constant}$$

$$\left.\frac{dm}{dt}\right|_{syst} = \dot{m} = 0$$

-- General form

For a finite volume of fluid body, it may not be a uniformly distributed body, thus to obtain the overall mass, an integration is required

$$m_{syst} = \int_{mass} dm = \int_{V} \rho \, dV = \text{constant}$$

$$\frac{d}{dt}\int_{sys} \rho dV = 0$$

--For non-uniform fluid

2. Newton's second law

For a system moving relative to an inertial reference frame （惯性参照系）**, Newton's law states that the sum of all external forces acting on the system is equal to the time change rate of linear momentum of the system.

** Inertial reference frame: the frame is stationary, moving at constant velocity, or not accelerating.

$$\sum \vec{F}\,|_{sys} = \frac{d(m\vec{v})}{dt}\,|_{sys} = \frac{d\vec{p}}{dt}\,|_{sys}$$

-- General form

For a fluid, overall linear momentum

$$\vec{P}_{syst} = \int_{mass} \vec{V}\; dm = \int_{V} \vec{V}\; \rho\; dV$$

Thus

$$\sum \vec{F}\,|_{sys} = \frac{d}{dt}\int_{sys} \vec{V} \rho dV$$

-- For non uniform fluid

3. Angular momentum conservation for a system

Angular momentum principle for a system states that the *variation angular momentum per unit time is equal to the sum of all torques (扭矩, 转矩) acting on the system.*

- Angular momentum of a system in general motion must be specified relative to an **inertial reference frame (非加速参考系).**
- Torque can be produced by surface and body forces, and input by shafts cross the system boundary.

$$\vec{T} = \frac{d\vec{H}}{dt}\Big|_{syst}$$

-- General form

For a fluid, overall angular momentum

$$\vec{H}_{syst} = \int_{mass} \vec{r} \times \vec{V}\, dm = \int_{V} \vec{r} \times \vec{V} \rho\, dV$$

-- For non uniform fluid

$$\vec{T} = \vec{r} \times \vec{F}_s + \int_{mass} \vec{r} \times \vec{g}\, dm + \vec{T}_{shaft}$$

Thus

$$\vec{r} \times \vec{F}_s + \int_{sys} \vec{r} \times \vec{g}\, dm + \vec{T}_{shaft} = \frac{d}{dt} \int_{sys} (\vec{r} \times \vec{V}) \rho dV$$

Surface force torque — Gravitational force torque — Angular momentum change

4. The first law of thermodynamics

The first law of thermodynamics is a statement of energy conservation for a system, quantity of heat transferred to a system from outside subtracts the work done by the system equals to change of the total energy of the system.

$$[dQ - dW]_{sys} = dE|_{sys}$$

Total energy consists of internal energy, potential energy, and kinetic energy, for a non-uniform mass system, it is given by

$$E_{sys} = \int_{sys} edm = \int_{sys} e\rho dV = \int_{sys} (\hat{u} + gz + v^2/2)\rho dV$$

Where $e = \hat{u} + \frac{1}{2}V^2 + gz$

Internal energy is stored in a system by molecular activity and molecular bonding forces, and for a single-phase pure substance, it is a function of T and p, $\hat{u}(T, p)$. ***Potential energy*** depends on the position of the system; ***Kinetic energy*** depends on the velocity of the system. Both are associated with the motion of the system, and they are kinematic properties.

Using time-derivative, the conservation equation can be rewritten

$$\left[\frac{dQ}{dt} - \frac{dw}{dt}\right]_{sys} = \left(\dot{Q} - \dot{w}\right)_{sys} = \frac{d}{dt}\int_{sys} (\hat{u} + gz + v^2/2)\rho dV$$

5. The second law of thermodynamics

The second law of thermodynamics also applies to fluid system, which says the entropy increases in any flow process.

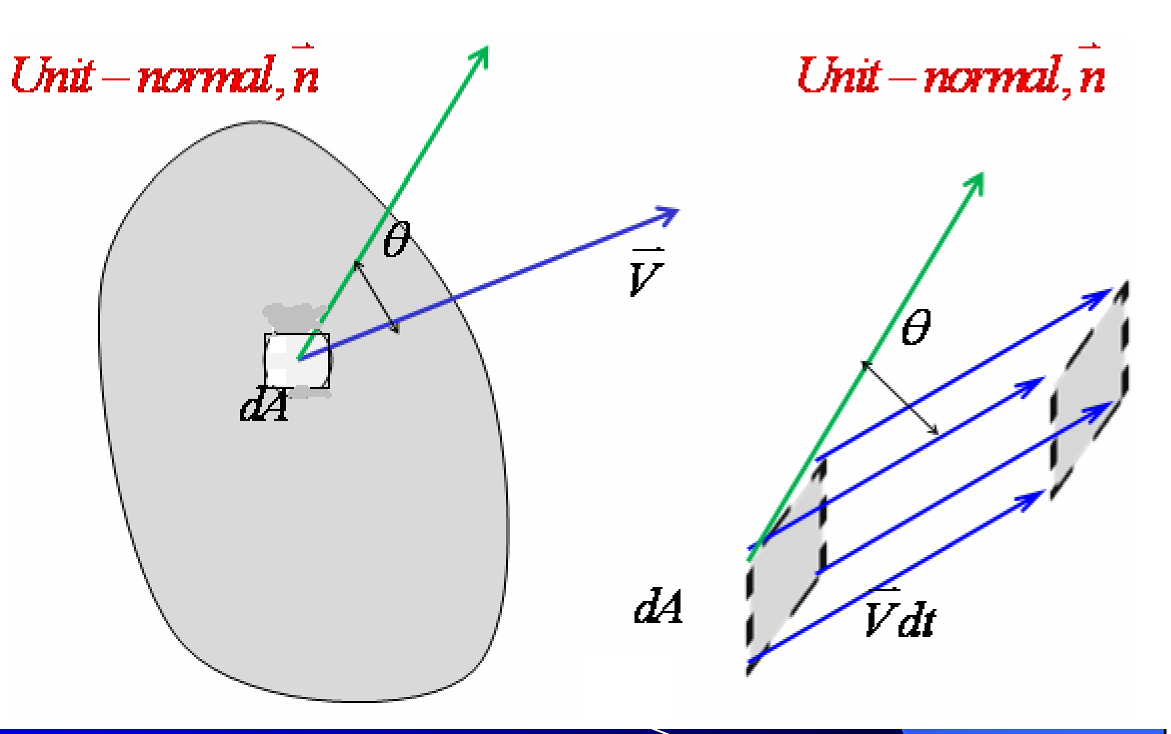
*3.2 Reynolds transport theorem*雷诺输运方程

– A relation between the system and control volume

Objective is to establish The Reynolds Transport Theorem (RTT).

(1) *Definitions* of volume flux (体积通量), volume flow rate (体积流量), mass flux, and mass flow rate.

Fluid particles through $d\vec{A}$ within dt forms a **parallelepiped**, with a bottom of dA and height $V\ dt \cos\theta$, thus the volume of fluid through dA is



柱体

$$dV = dA\,(V\ dt\ cos\ \theta) = dA.Vcos\ \theta.dt = (\vec{V}.\vec{n})\ dA\ dt$$

Volume flux 体积通量– refers to the volume of flow through $\vec{A}$ within dt

$$V = dt \int_A (\vec{V}.\vec{n})\, dA$$

Volume flow rate 体积流量– refers to the volume of flow through $\vec{A}$ per unit time

$$Q = \int_A (\vec{V}.\vec{n})\ \mathrm{dA} = \int_A V \cos\theta\ dA$$

Mass flux – refers to the mass of flow through the area $\vec{A}$ within dt

$$m = dt \int_{A} \rho(\vec{V}.\vec{n})\, dA$$

Mass flow rate – refers to the mass of flow through $\vec{A}$ per unit time

$$\dot{m} = \int_{A} \rho(\vec{V}.\vec{n})\, dA$$

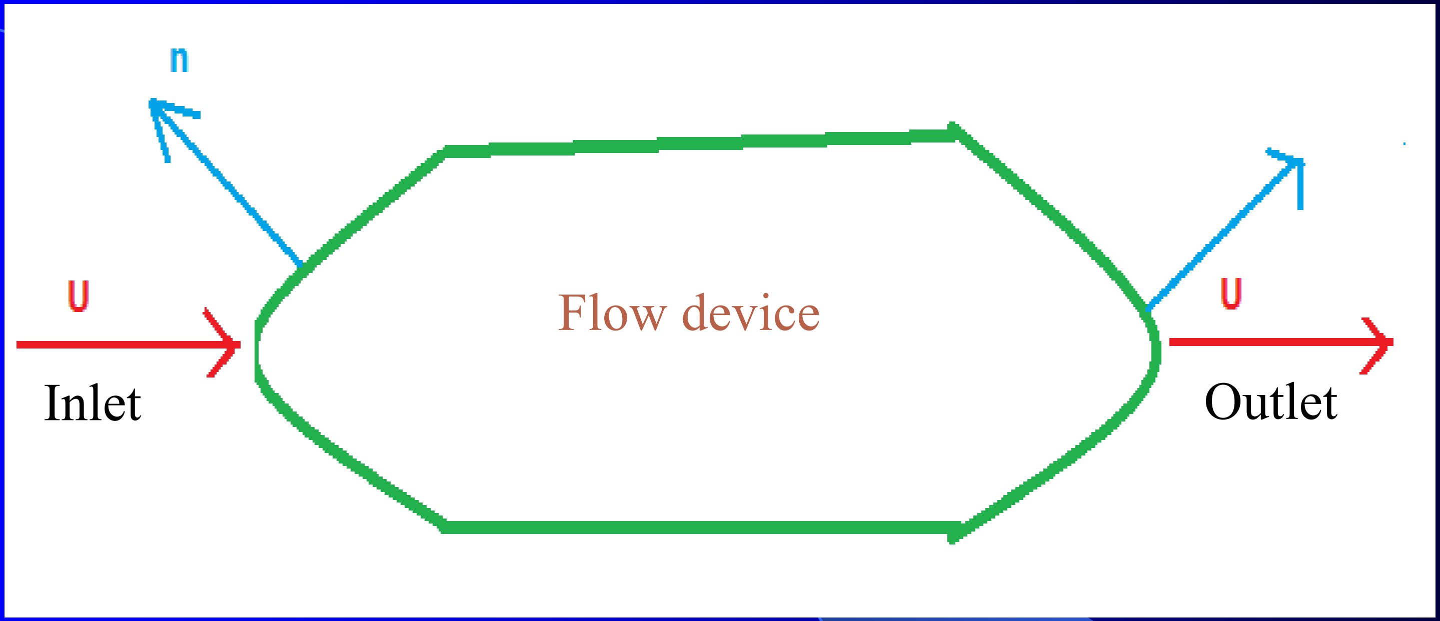
Signs at inlet and outlet surface:

$$Q = \int_A (\vec{V}.\vec{n})\ \mathrm{dA} = \int_A V\cos\theta\ dA;$$

$$m = \int_A \rho(\vec{V}.\vec{n})\ \mathrm{dA} = \int_A \rho V\cos\theta\ dA$$

- At inlet, $\theta \geq \pi/2$ and , $\cos\theta \leq 0$ thus both volume and mass flow rate (flux) are negative; at outlet, $\theta \leq \pi/2$ and , $\cos\theta \geq 0$ thus both are positive.
- At the inlet and outlet of a pipe， if the 1D-flow is incompressible and uniform, using the above equation, we have $Q_{out} = A\ u;\qquad Q_{in} = -A\ u$

$$\dot{m}_{out} = A\ \rho\ u;\qquad \dot{m}_{in} = -A\ \rho\ u$$


n
U
Inlet
Flow device
U
Outlet

(2) Extensive property (广义物性) of System

we have introduced **total (gross) properties** for a non-uniform fluid system

$$m_{syst} = \int_{mass} dm = \int_{V} \rho \, dV$$

$$\vec{P}_{syst} = \int_{mass} \vec{V} \, dm = \int_{V} \vec{V} \rho \, dV$$

$$\vec{H}_{syst} = \int_{mass} \vec{r} \times \vec{V} \, dm = \int_{V} \vec{r} \times \vec{V} \rho \, dV$$

$$E_{sys} = \int_{mass} e \, dm = \int_{V} e \rho \, dV$$

$$B_{sys} = \int_{mass} \beta \mathrm{d}m$$

Or

$$\beta = \frac{dB}{dm}$$

Gross property

广义物性
广义物量

B_{sys}

Property per unit mass (or intensive property)

β

单位质量物理量
或物量强度
或物量分布函数

We use the symbol ***B*** to represent any arbitrary ***extensive property*** of the system. The corresponding ***intensive property*** β (extensive property per unit mass) will be denoted by

$$\beta = \frac{dB}{dm}$$

$$B_{system} = \int_{\text{mass system}} \beta \, dm = \int_{\text{volume system}} \beta\rho \, dV$$

*** Question – do you know the relation between extensive and intensive property for a specific quantity? For example

$$B_{sys} = mass, \text{ and } \beta = ?$$

$$B_{sys} = \text{Linear momentum, and } \beta = ?$$

$$B_{sys} = \text{E}, \text{ and } \beta = ?$$

(3) Extensive property (广义物性) of CV

Use B to represent any arbitrary extensive property of the control volume, and the corresponding intensive property (extensive property per unit mass) will be denoted by β, then we have

$$B_{CV} = \int_{\mathrm{CV}} \beta \, dm = \int_{\mathrm{CV}} \beta\rho \, dV \quad \text{and} \quad \beta = \frac{dB_{CV}}{dm}$$

Question:

$B_{CV} = mass$, and $\beta = ?$

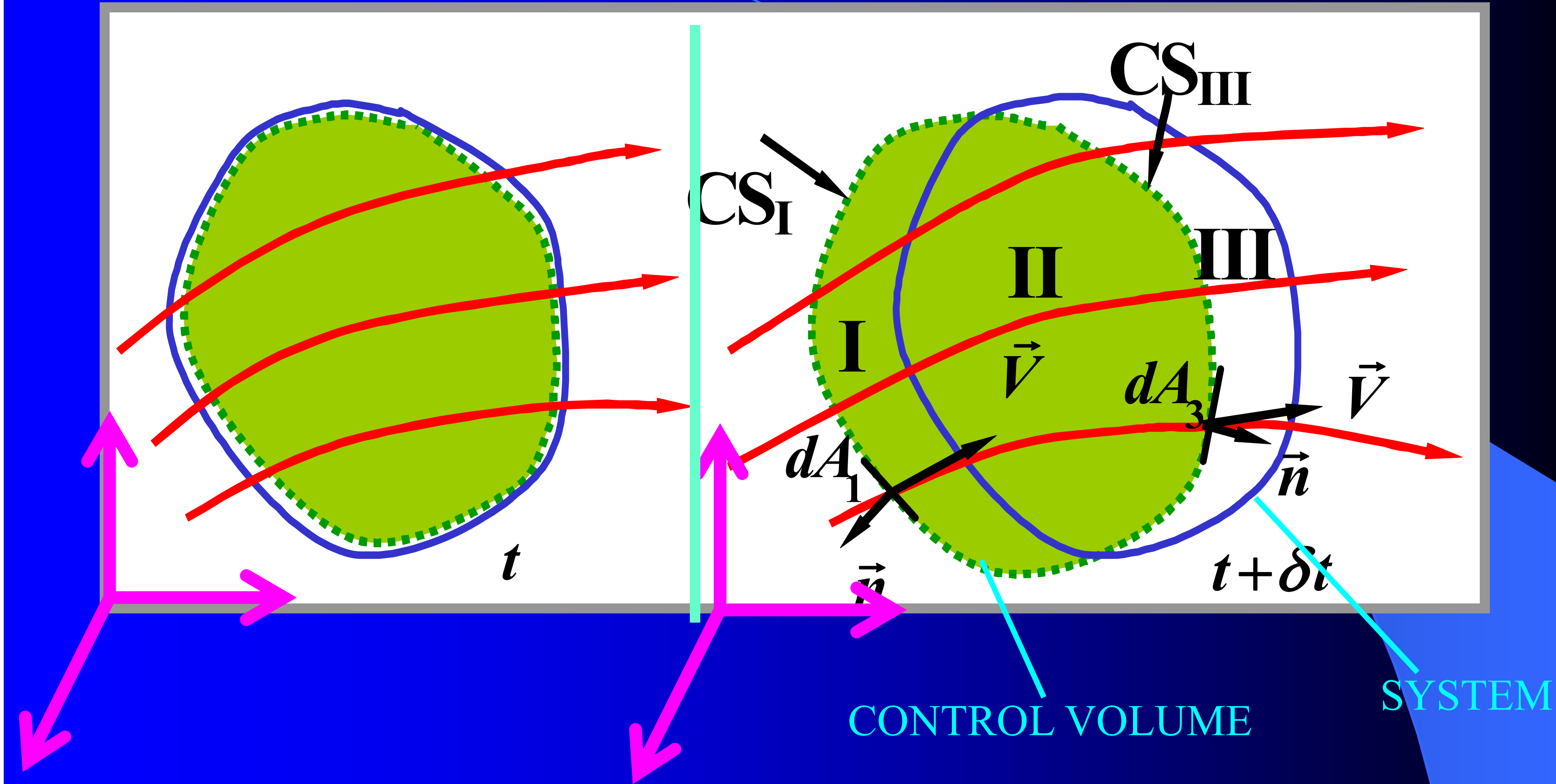
$B_{CV} =$ linear momentum, and $\beta = ?$

$B_{CV} =$ angular momentum, and $\beta = ?$

$B_{CV} =$ E, and $\beta = ?$

Reynolds's transport theorem

– A relation of time derivatives between the system and control volume



How to relate the rate of change of any arbitrary extensive property B, of the system to the time derivative of this property for the control volume ?

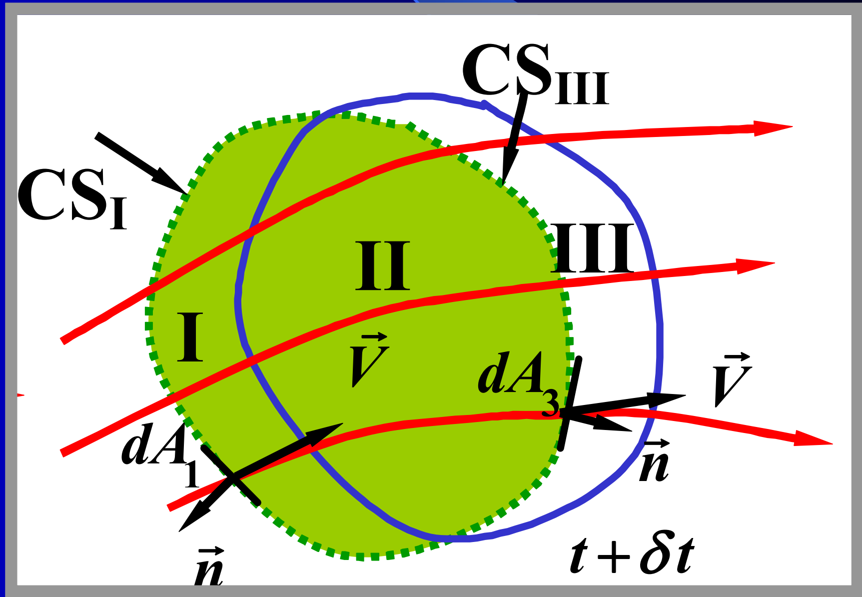
The boundaries of the system and control volume are shown at two different instants, t and $t+\Delta t$.

At t the boundaries of the system and the control volume coincide; at $t+\Delta t$, the system occupies region Ⅱ and Ⅲ. For the control volume, during Δt mass exchange occurred at its control surface, CS_I and CS_{III} The mass in region Ⅰ is the incoming flow through CS_I and that in region Ⅲ the outgoing flow through CS_{III}.

Change of Bsys per unit time

$$\frac{dB}{dt}\Big|_{sys} = \lim_{\Delta t \to 0} \frac{B_{sys}(t+\Delta t) - B_{sys}(t)}{\Delta t}$$ --------- (eq. 1)

- At $t+\Delta t$,the system occupies region II and III; at t the system and control volume coincide. Thus



$$B_{sys}(t) = B_{CV}(t) = (\int_{CV} \beta\rho dv)_{CV}$$

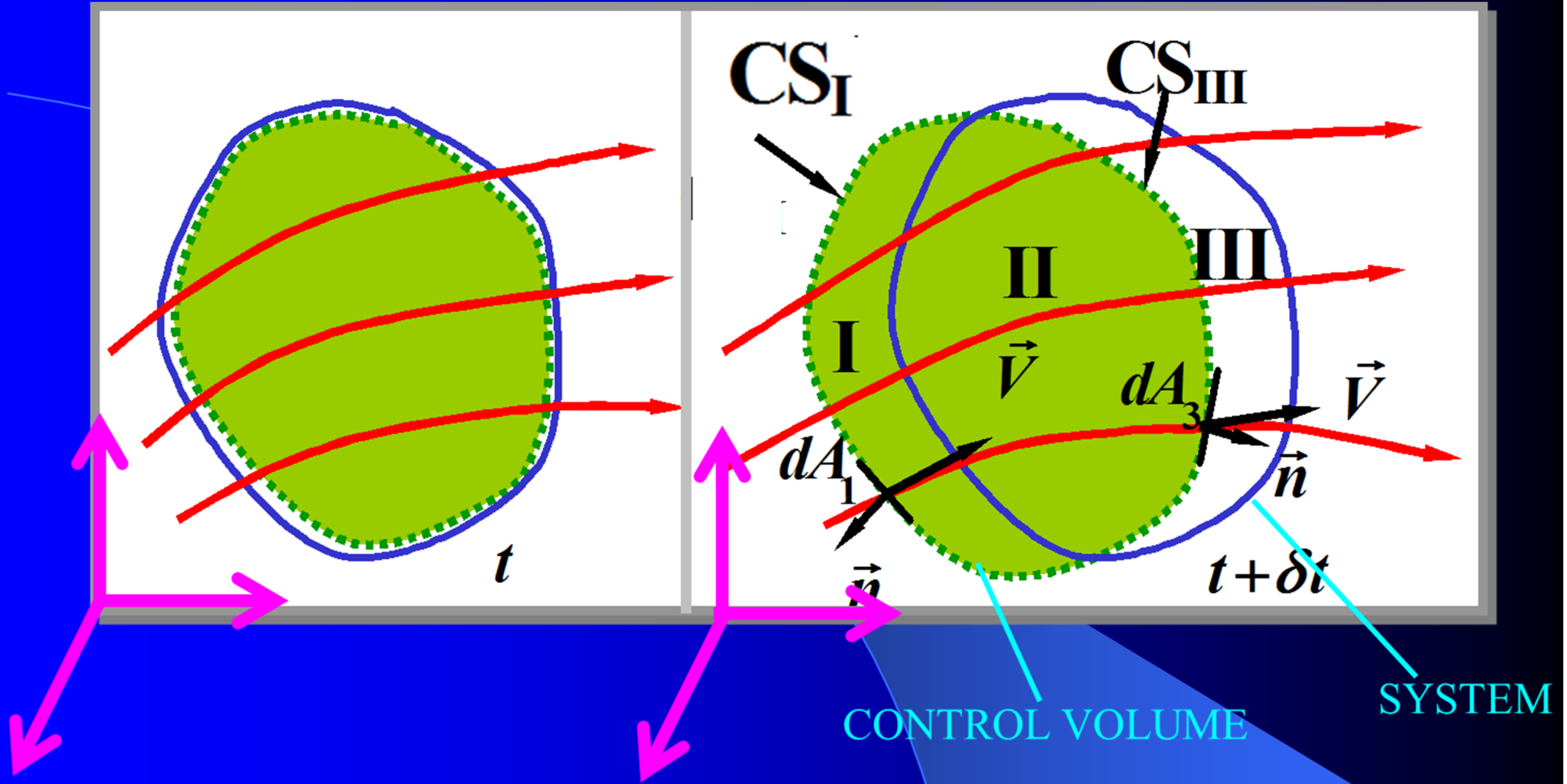
$$B_{sys}(t+\Delta t) = (B_{II} + B_{III})_{t+\Delta t} = (B_{CV} - B_{I} + B_{III})_{t+\Delta t}$$
$$= (\int_{CV} \beta\rho dv)_{t+\Delta t} - (\int_{I} \beta\rho dv)_{t+\Delta t} + (\int_{III} \beta\rho dv)_{t+\Delta t}$$

- Substitute the above two terms into the time-derivative equation of system, eq.1

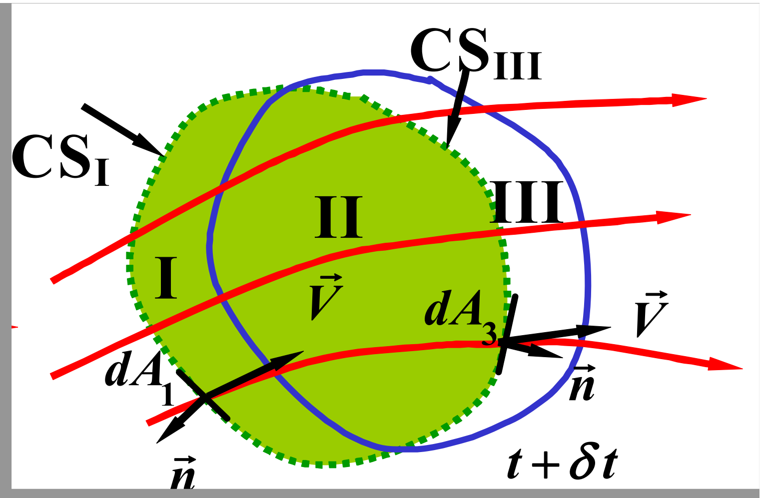
$$\frac{dB}{dt}\Big|_{sys} = \lim_{\Delta t \to 0} \frac{\left[\int_{CV} \beta\,\rho\, dV\right]_{t+\Delta t} + \left[\int_{III} \beta\,\rho\, dV\right]_{t+\Delta t} - \left[\int_{I} \beta\,\rho\, dV\right]_{t+\Delta t} - \left[\int_{CV} \beta\rho dV\right]_{t}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\left[\int_{CV} \beta\,\rho\, dV\right]_{t+\Delta t} - \left[\int_{CV} \beta\,\rho\, dV\right]_{t}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\left[\int_{III} \beta\,\rho\, dV\right]_{t+\Delta t}}{\Delta t} - \lim_{\Delta t \to 0} \frac{\left[\int_{I} \beta\,\rho\, dV\right]}{\Delta t}$$

(1) (2) (3)


CS_I
CS_III
III
II
I
$\vec{V}$
dA_3
$\vec{V}$
dA_1
$\vec{n}$
$\vec{n}$
t
$t+\delta t$
CONTROL VOLUME
SYSTEM

Reductions of each terms



Term 1

$$(1) = \lim_{\Delta t \to 0} \frac{B_{CV}(t+\Delta t) - B_{CV}(t)}{\Delta t} = \frac{dB_{CV}}{dt} = \frac{d}{dt}\int_{CV} \beta\, \rho\, dV$$

Term 2

$$(2) = \lim_{\Delta t \to 0} \frac{B_{III}(t+\Delta t)}{\Delta t}$$

The mass flux of fluid within the control volume *III* equals to the mass flux through the control surface CS_{III} within Δt. To evaluate the above term, we just need to obtain the flux of extensive property *B* through CS_{III} within Δt

Mass flux through the area $d\vec{A}$ within Δt

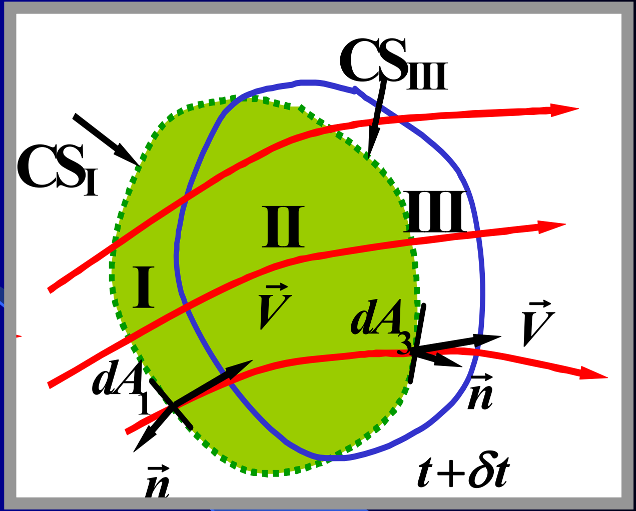
$$\rho V \cos\alpha\, dA\, \Delta t = \rho\, (\vec{V}.d\vec{A})\, \Delta t = \rho\, (\vec{V}.\vec{n})\, dA\, \Delta t$$

Flux of the extensive property *B*, through the area $d\vec{A}$ within Δt

$$\beta\rho\, (\vec{V}.\vec{n})\, dA\, \Delta t$$

Flux of the extensive property *B* through control surface CS_{III}

$$\int_{CS_{III}} \beta\rho(\vec{V}.d\vec{A})\ \Delta t. = \Delta t\int_{CS_{III}} \beta\rho\ (\vec{V}.\vec{n})dA$$



Thus

$$B_{III}\big|_{t=t+\Delta t} = \left|\Delta t \int_{CS_{III}} \beta\rho\ (\vec{V}.\vec{n})\ dA\right|$$

$$= \left|\Delta t \int_{CS_{III}} \beta\rho\ \mathrm{V}\cos\theta\ \mathrm{dA}\right|$$

$$= \Delta t \int_{CS_{III}} \beta\rho\ \mathrm{V}\cos\theta\ \mathrm{dA}$$

The extensive property, within the control volume equals to the absolute value of Flux of (the flux is positive at outlet and negative at inlet) through the area of CSIII within Δt

Finally
$$(2) = \lim_{\Delta t\to 0}\frac{B_{III}(t+\Delta t)}{\Delta t} = \int_{CS_{III}} \beta\rho\ \mathrm{V}\cos\theta\ \mathrm{dA} = \int_{CS_{III}} \beta\rho\ (\vec{V}.\vec{n})\ \mathrm{dA}$$

Term 3 - The extensive property B within control volume I equals to the absolute value of flux B passing through the area of CS_I within Δt, thus we have

$$B_I \big|_{t=t+\Delta t} = \left| \int_{CS_I} \beta\rho V \cos\theta \, dA \, \Delta t \right| = -\Delta t \int_{CS_I} \beta\rho V \cos\theta \, dA = -\Delta t \int_{CS_I} \beta\rho \, (\vec{V}.\vec{n}) \, dA$$

$$(\because \ \theta \geq \pi/2, \text{ and } \cos\theta \leq 0)$$

Thus

$$(3) = -\lim_{\Delta t \to 0} \frac{B_I)_{t=t+\Delta t}}{\Delta t} = \int_{CS_I} \beta\rho \, (\vec{V}.\vec{n}) \, dA$$

Since the entire control surface is constituted by CS_I and CS_{III}, $CS = CS_I + CS_{III}$ we have

$$(2)+(3) = \int_{CS_I} \beta\rho \, (\vec{V}.\vec{n}) \, dA + \int_{CS_{III}} \beta\rho \, (\vec{V}.\vec{n}) \, dA$$

$$= \int_{CS} \beta\rho \, (\vec{V}.\vec{n}) \, dA$$

Summary

$$\left.\frac{dB}{dt}\right|_{sys} = \lim_{\Delta t \to 0} \frac{\left[\int_{CV} \beta\,\rho\,dV\right]_{t+\Delta t} + \left[\int_{III} \beta\,\rho\,dV\right]_{t+\Delta t} - \left[\int_{I} \beta\,\rho\,dV\right]_{t+\Delta t} - \left[\int_{CV} \beta\rho dV\right]_{t}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\left[\int_{CV} \beta\,\rho\,dV\right]_{t+\Delta t} - \left[\int_{CV} \beta\,\rho\,dV\right]_{t}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\left[\int_{III} \beta\,\rho\,dV\right]_{t+\Delta t}}{\Delta t} - \lim_{\Delta t \to 0} \frac{\left[\int_{I} \beta\,\rho\,dV\right]}{\Delta t}$$

(1) (2) (3)

$$(1) = \lim_{\Delta t \to 0} \frac{B_{CV}(t+\Delta t) - B_{CV}(t)}{\Delta t} = \frac{dB_{CV}}{dt} = \frac{d}{dt} \int_{CV} \beta\ \rho\ dV$$

$$(2) + (3) = \int_{CS} \beta\rho\ (\vec{V}.\vec{n})\ \mathrm{dA}$$

Introducing the above termes (1), and (2)+(3) intro equation 1, it gives

Reynolds's theorem

$$\frac{d}{dt}(B_{syst}) = \frac{d}{dt}\left(\int_{SYS} \beta\rho\ \mathrm{dV}\right) = \frac{d}{dt}\left(\int_{CV} \beta\rho\ \mathrm{dV}\right) + \int_{CS} \beta\rho(\mathbf{\vec{V}}.\mathbf{\vec{n}})dA \qquad \text{(Frank white, eq. 3.12)}$$

Physical significance - Do you know the physical significance of each term in the above equation ?

Special cases:

- *For fixed control volume,* the volume elements do not vary with time, thus in the second term of equation (3.12), we can do the differentiation first,

$$\frac{d}{dt}\left(\int_{CV} \beta\ \rho\ dV\right) = \int_{CV} \frac{\partial}{\partial t}(\beta\ \rho)\ dV$$

we have

$$\frac{d}{dt}(B_{\text{syst}}) = \int_{CV} \frac{\partial}{\partial t}(\beta\rho)\ \mathrm{dV} + \int_{CS} \beta\rho(\vec{\mathbf{V}}.\vec{\mathbf{n}})dA \qquad \text{(Frank White,3.17)}$$

- *It is noted that*

① velocity $\vec{V}$ is flow velocity measured relative to the control volume, for moving CV, relative velocity $\vec{V}_r = \vec{V} - \vec{V}_{CV}$ should be used

(see FW Equation 3.16)

② the system and control volume coincided at instant t.

3.3 Conservation of mass ------ continuity equation

The time-derivatives of mass for a system and control volume satisfy the Reynolds's theorem at the time instant when they coincide.

$$\frac{d}{dt}(B_{\text{syst}}) = \frac{d}{dt}\left(\int_{CV} \beta\rho \, \mathrm{dV}\right) + \int_{CS} \beta\rho(\vec{\mathbf{V}}.\vec{\mathbf{n}})dA \qquad \text{(Frank white, eq. 3.12)}$$

To consider mass conservation, we set the extensive properties as follows

$$\beta = 1 \text{ and } B_{syst} = m$$

Then

$$\frac{dm_{syst}}{dt} = \frac{d}{dt}\left(\int_{CV} \rho \, dV\right) + \int_{CS} \rho \, (\vec{\mathbf{V}}.\vec{\mathbf{n}}) \, dA$$

Question ---- Can you do further reductions about this equation?

For a system, the mass must be conserved, we can obtain

$$\frac{d}{dt}\left(\int_{CV} \rho \, dV\right) + \int_{CS} \rho \, (\vec{\mathbf{V}} . \vec{\mathbf{n}}) \, dA = 0$$

Physical significance - The first term represents the rate of change of mass within the control volume; the second term represents the net rate of mass flux through the control surface. The equation of mass conservation says that for a control volume, total mass variation per unit time equals the net mass flow across the entire control surface per unit time.

Applications of mass conservation equation ---Special forms of

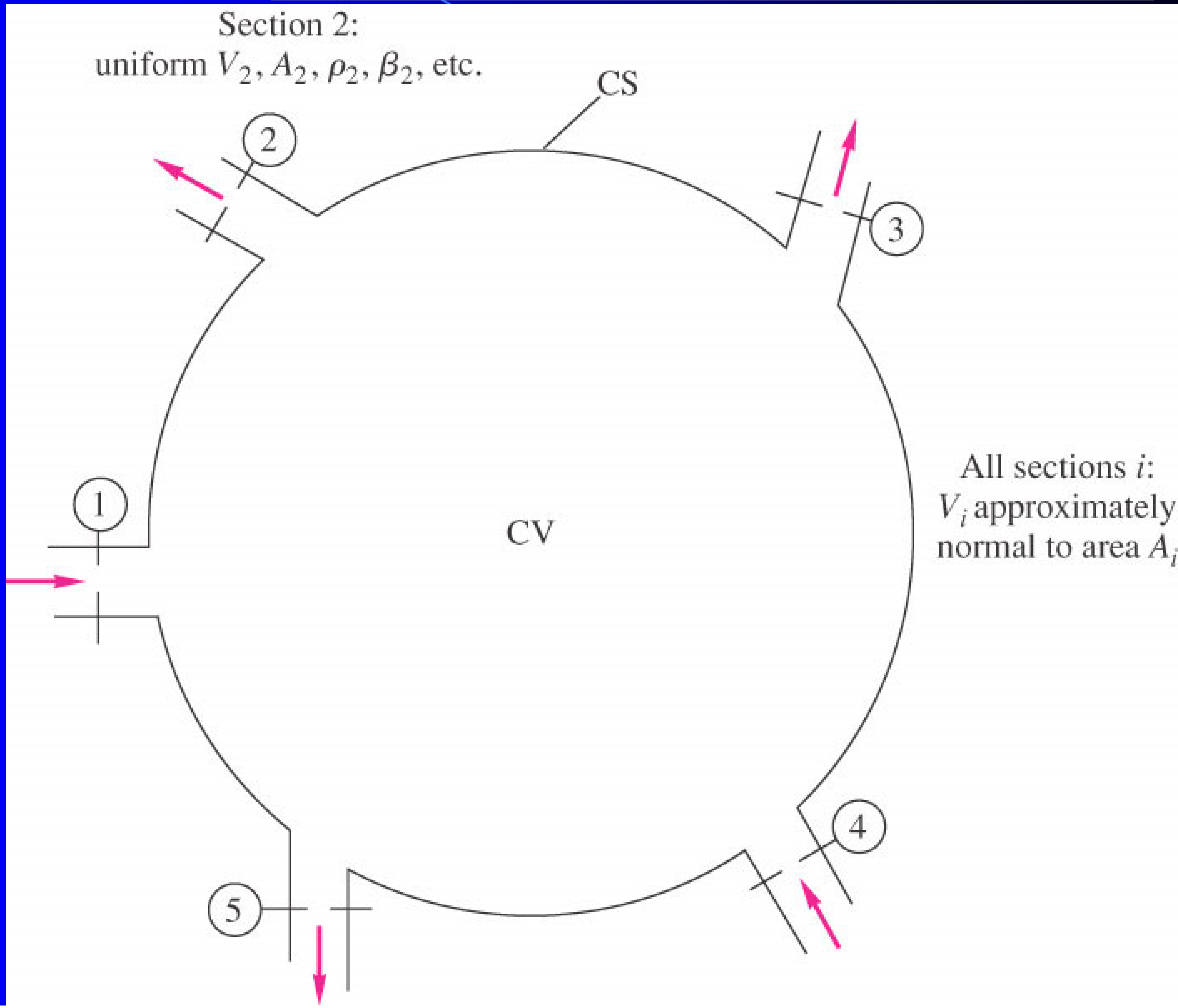
$$\frac{d}{dt}\left(\int_{CV} \rho \, dV\right) + \int_{CS} \rho \, (\vec{\mathbf{V}}.\vec{\mathbf{n}}) \, dA = 0$$

It is the most important to reduce the equation based on the given information of the specified problem.

关键点是：如何根据具体问题的条件简化这个质量守恒方程!!!!

(1) Reduce the flux term of the mass conservation equation for the specified CV.

$$\frac{d}{dt}\left(\int_{CV} \rho \, dV\right) + \int_{CS} \rho \, (\vec{\mathbf{V}}.\vec{\mathbf{n}}) \, dA = 0$$



$$\int_{CS} \rho\ (\vec{\mathbf{V}}.\vec{\mathbf{n}})\ dA = \int_{\text{port1}} \rho\ (\vec{\mathbf{V}}.\vec{\mathbf{n}})\ dA + \int_{\text{port2}} \rho\ (\vec{\mathbf{V}}.\vec{\mathbf{n}})\ dA + \int_{\text{port3}} \rho\ (\vec{\mathbf{V}}.\vec{\mathbf{n}})\ dA$$

$$+ \int_{\text{port4}} \rho\ (\vec{\mathbf{V}}.\vec{\mathbf{n}})\ dA + \int_{\text{port5}} \rho\ (\vec{\mathbf{V}}.\vec{\mathbf{n}})\ dA$$

$$= -\sum (\rho_i V_i A_i)_{in} + \sum_{out} (\rho_i V_i A)_{out}$$

$$= +\rho_2 V_2 A_2 + \rho_3 V_3 A_3 + \rho_5 V_5 A_5 - \rho_4 V_4 A_4 - \rho_1 V_1 A_1$$

$$\frac{d}{dt}\left(\int_{CV} \rho \, dV\right) + \int_{CS} \rho \, (\vec{\mathbf{V}}.\vec{\mathbf{n}}) \, dA = 0$$

(2) For fixed (not deformable) CV, its volume does not change with time, which gives

$$\int_{CV} \frac{\partial \rho}{\partial t} \, dV + \int_{CS} \rho \, (\vec{\mathbf{V}}.\vec{\mathbf{n}}) \, dA = 0$$

(fixed CV)

(3) For incompressible flow

$$\frac{d}{dt}\left(\int_{CV} \rho \, dV\right) + \int_{CS} \rho \, (\vec{\mathbf{V}}.\vec{\mathbf{n}}) \, dA = 0$$

(General form)

$\rho = \text{constant}$ gives $\rho \frac{d}{dt}\left(\int_{CV} dV\right) + \rho \int_{CS} (\vec{\mathbf{V}}.\vec{\mathbf{n}}) \, dA = 0$

Since $\int_{CV} dV = V$, then we rewrite the above equation

$$\frac{dV}{dt} + \int_{CS} (\vec{\mathbf{V}}.\vec{\mathbf{n}}) \, dA = 0$$

(For incompressibl flow)

If the control volume is fixed, the above equation can be further reduced

$$\int_{CS} (\vec{\mathbf{V}}.\vec{\mathbf{n}}) \, dA = 0$$

(For incompressible flow + fixed CV, but not necessarily steady)

$$\int_{CS} (\vec{\mathbf{V}}.\vec{\mathbf{n}})\, dA = 0$$

For incompressible flow of fixed control volume, the net volume flow rate/flux at the entire control surface is conserved.

For one-dimensional (pipe) flows, velocity are parallel to the surface outer normal at inlet $\vec{\mathbf{V}}.\vec{\mathbf{n}} \le 0$ and at outlet $\vec{\mathbf{V}}.\vec{\mathbf{n}} \ge 0$, we have

$$\int_{CS} (\vec{\mathbf{V}}.\vec{\mathbf{n}})\, dA = \sum [(\vec{V}.\vec{n})\, dA]_{in} + \sum [(\vec{V}.\vec{n})\, dA]_{out}$$

$$= - \sum Q_{in} + \sum Q_{out}$$

The conservation equation of volume flow is then written as

$$\sum Q_{in} = \sum Q_{out}$$

or

$$\sum (A_i\, u_i)_{in} = \sum (A_i\, u_i)_{out}$$

For 1D incompressible flow, the volume flow rate into a fixed control volume must be equal to the volume flow rate out of the control volume.

(4) **For steady flow** within the fixed Control Volume (not necessarily incompressible)

$$\int_{CV} \frac{\partial \rho}{\partial t}\, dV + \int_{CS} \rho\, (\vec{\mathbf{V}}.\vec{\mathbf{n}})\, dA = 0$$

(For fixed CV)

If the flow is steady, flow properties will not change with time, $\rho = \rho(x,.y,z)$

Leading to $\frac{\partial \rho}{\partial t} = 0$ The mass conservation equation can be written as

$$\int_{CS} \rho\, (\vec{\mathbf{V}}.\vec{\mathbf{n}})\, dA = 0$$

For steady flow of fixed control volume, the net mass flow rate at the entire control surface is conserved (zero).

For one-dimensional (pipe) flows, the conservation equation can be rewritten

$$\sum \dot{m}_{in} = \sum \dot{m}_{out}$$

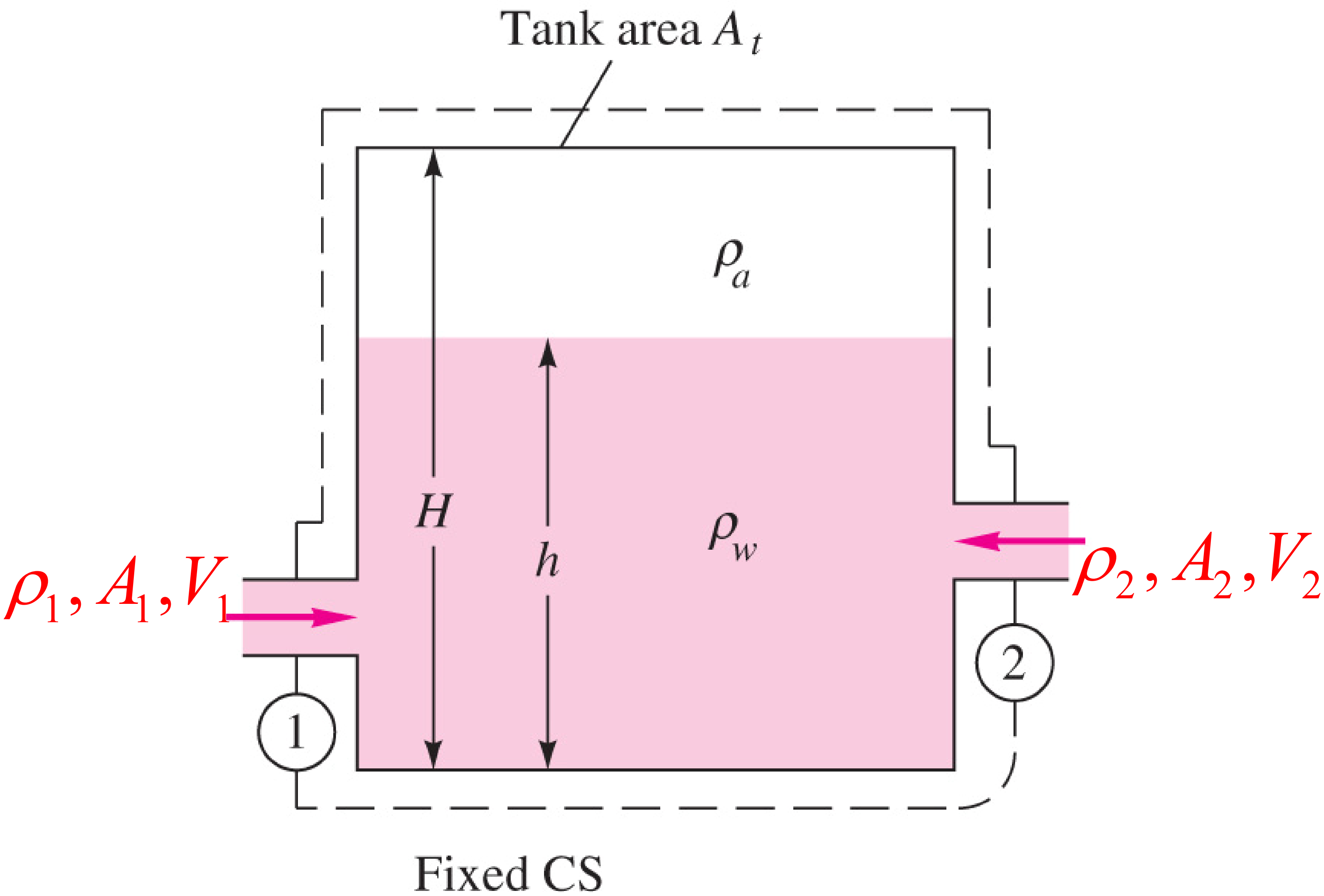
or

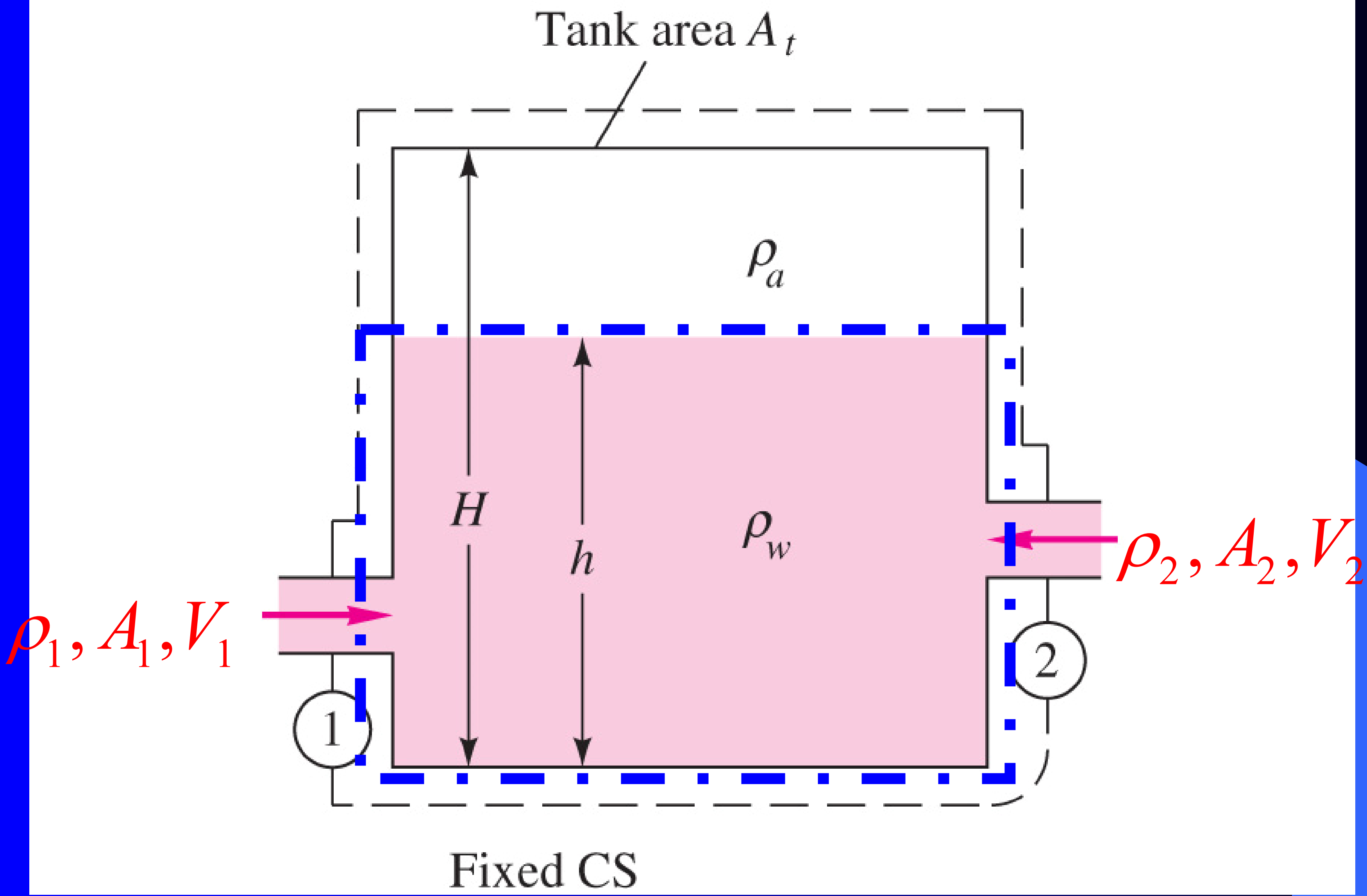
$$\sum (\rho_i A_i\, u_i\,)_{in} = \sum (\rho_i A_i\, u_i)_{out}$$

For 1D steady flow, the mass flow rate into a control volume must be equal to the mass flow rate out of the control volume (because the mass can not be accumulated in the control volume).

Example 3.5

$$dh / dt = ?$$





Tank area A_t
ρ_a
H
h
ρ_w
ρ_1, A_1, V_1
ρ_2, A_2, V_2
1
2
Fixed CS

Original mass conservation equation

$$\frac{d}{dt}\left(\int_{CV} \rho \, dV\right) + \int_{CS} \rho \, (\vec{\mathbf{V}}.\vec{\mathbf{n}}) \, dA = 0$$

Flux term:

$$\int_{CS} \rho \, (\vec{\mathbf{V}}.\vec{\mathbf{n}}) \, dA = \int_{1} \rho \, (\vec{\mathbf{V}}.\vec{\mathbf{n}}) \, dA + \int_{2} \rho \, (\vec{\mathbf{V}}.\vec{\mathbf{n}}) \, dA$$

$$= -\rho_1 V_1 A_1 - \rho_2 V_2 A_2$$

Unsteady term:

$$V = hA_t, \text{ then } dV = A_t dh$$

$$\frac{d}{dt}\left(\int_{CV} \rho \, dV\right) = \rho \frac{d}{dt}\left(\int_{CV} dV\right) = \rho A_t \frac{dh}{dt}$$

Introduce both terms into the original conservation eq, we have

$$\frac{dh}{dt} = \frac{\rho_1 A_1 V_1 + \rho_2 A_2 V_2}{\rho A_t}$$

3.4 Linear-momentum equation for inertial control volume

We consider a fixed control volume

- *Newton's second law for a system:*

$$[\vec{F}_S + \vec{F}_B]_{\text{on syst}} = \frac{d}{dt} \int_{v\,(\text{syst})} \vec{V} \rho \, dV$$

- *Reynolds's theorem*

$$\frac{d}{dt}(B_{\text{syst}}) = \int_{CV} \frac{\partial}{\partial t}(\beta\rho) \, \text{dV} + \int_{CS} \beta\rho(\vec{V}.\vec{n})dA$$

Substitute $\beta = \vec{V}$ and $B_{syst} = \vec{P} = \int_{m\,(\text{syst})} \vec{V} \, dm = \int_{v\,(\text{syst})} \vec{V} \rho \, dV$ into RT equation

$$\frac{d}{dt} \int_{v\,(\text{syst})} \vec{V} \rho \, dV = \int_{CV} \frac{\partial}{\partial t}(\vec{V}\rho) \, \text{dV} + \int_{CS} \vec{V} \rho(\vec{V}.\vec{n})dA$$

- *Combining the above two equations* to obtain

$$[\vec{F}_S + \vec{F}_B]_{\text{on syst}} = \int_{CV} \frac{\partial}{\partial t}(\vec{V}\rho)\ \mathrm{dV} + \int_{CS} \vec{V}\rho(\vec{V}.\vec{n})dA$$

- *The system and the control volume coincided at time* instant *t*, thus the forces acting on the both must be the same, then

$$[\vec{F}_S + \vec{F}_B]_{\text{on syst}} = [\vec{F}_S + \vec{F}_B]_{\text{on CV}}$$

- *Finally we obtain*

$$[\vec{F}_S + \vec{F}_B]_{\text{on CV}} = \int_{CV} \frac{\partial}{\partial t}(\vec{V}\rho)\ \mathrm{dV} + \int_{CS} \vec{V}\rho(\vec{V}.\vec{n})dA$$ (vector form

The vector summation of all forces (surface and body forces) acting on a non-accelerating control volume is equal to the rate of change of momentum within the control volume plus the net momentum *flux out through the control surface.*

Componental form in a rectangular coordinate system

$$[\vec{F}_S + \vec{F}_B]_{\text{on CV}} = \int_{CV} \frac{\partial}{\partial t}(\vec{V}\rho)\ \mathrm{dV} + \int_{CS} \vec{V}\rho(\vec{V}.\vec{n})dA$$

- Resolve force and velocity vector

$$\vec{F}_S = F_{Sx}\vec{i} + F_{Sy}\vec{j} + F_{Sz}\vec{k}$$

$$\vec{F}_B = F_{Bx}\vec{i} + F_{By}\vec{j} + F_{Bz}\vec{k}$$

$$\vec{V} = u\,\vec{i} + v\,\vec{j} + w\,\vec{k}$$

- Introducing the above terms into the vector form of equation, we obtain

$$F_{Sx} + F_{Bx} = \int_{CV} \frac{\partial}{\partial t}(u\rho)\ \mathrm{dV} + \int_{CS} u\rho(\vec{V}.\vec{n})dA$$

$$F_{Sy} + F_{By} = \int_{CV} \frac{\partial}{\partial t}(v\rho)\ \mathrm{dV} + \int_{CS} v\rho(\vec{V}.\vec{n})dA$$

$$F_{Sz} + F_{Bz} = \int_{CV} \frac{\partial}{\partial t}(w\rho)\ \mathrm{dV} + \int_{CS} w\rho(\vec{V}.\vec{n})dA$$

(componental form)

Question – Do we need to resolve $(\vec{V} \cdot \vec{n})$, why?

Answer is "No", it is a scalar, mass flow rate

$$\rho\,(\vec{V}\cdot\vec{n}\,)\,dA = \rho\ \mathrm{V}\ dA\cos\theta$$

Steps for using momentum conservation eq：

1） choose CV （unknown force as internal force)

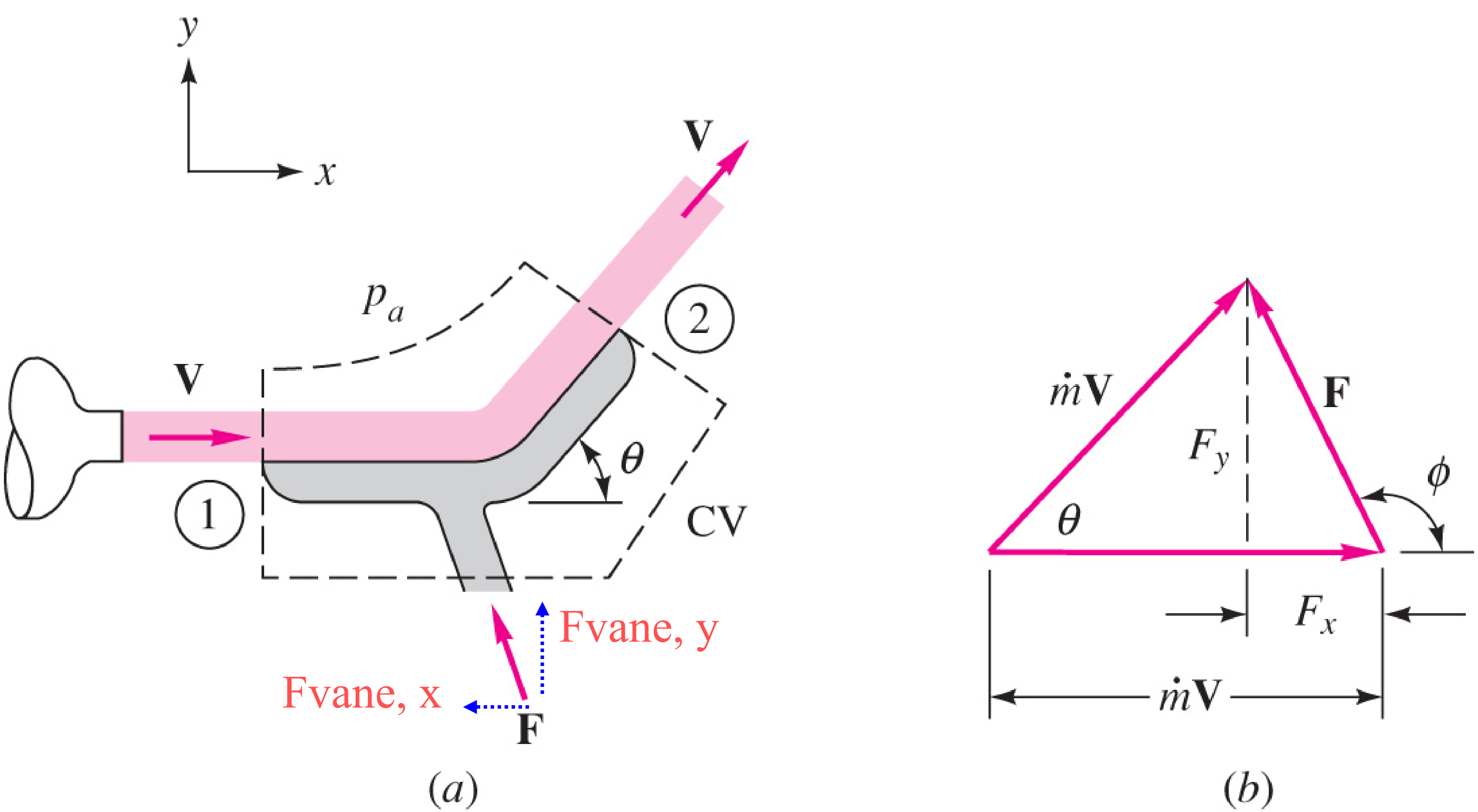
2） choose coordinate system (force/velocity)

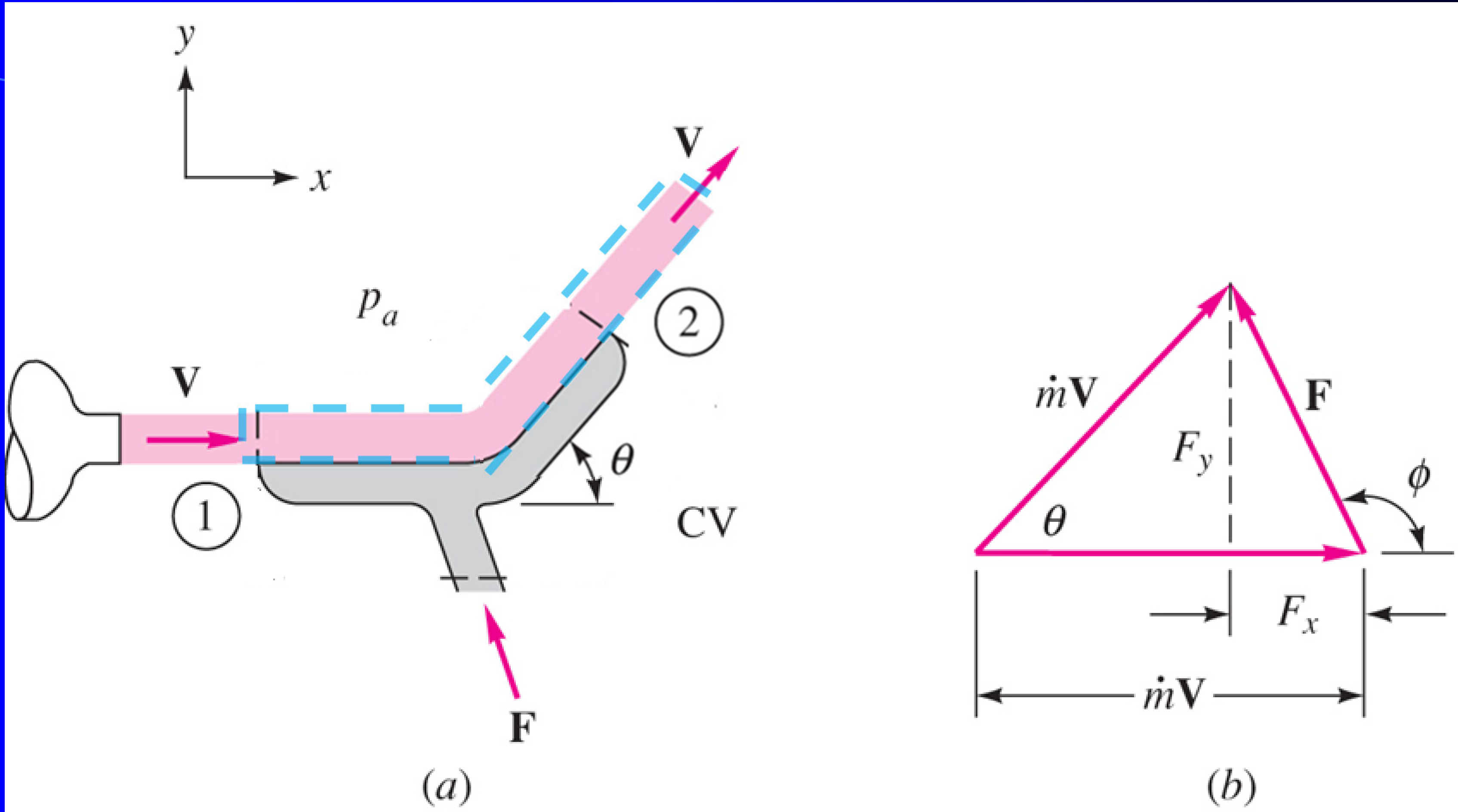
3） momentum + mass conservation equation

(a) mass flow rate/flux sign（+/-）

(b) force and velocity component（+ / -）

Page 163 Example 3.8





y
x
V
V
p_a
1
2
θ
CV
F
(a)
$\dot{m}\mathbf{V}$
F
F_y
θ
φ
F_x
$\dot{m}\mathbf{V}$
(b)

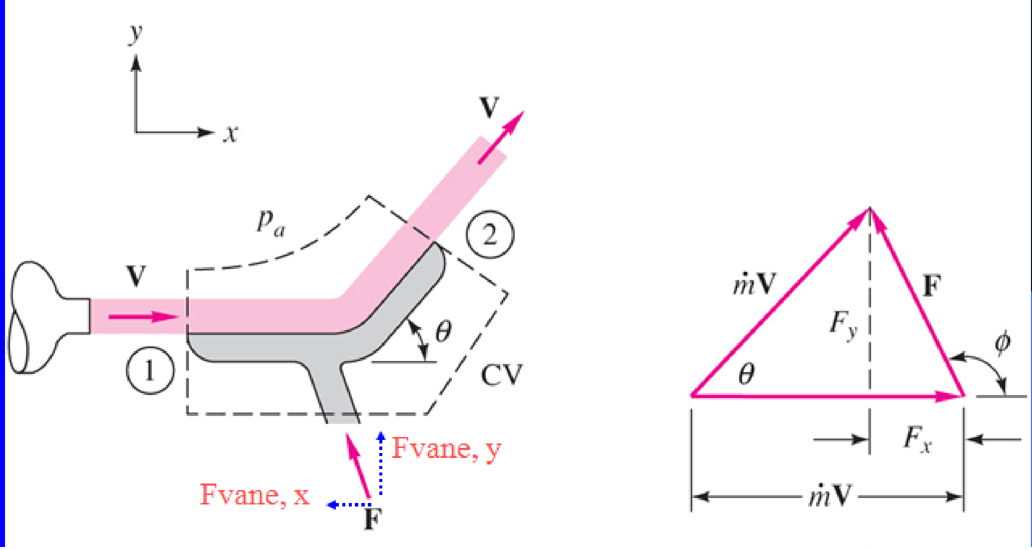
Original Momentum equation for fluid CV:

$$F_{Sx} + F_{Bx} = \int_{CV} \frac{\partial}{\partial t}(u\rho)\ \mathrm{dV} + \int_{CS} u\rho(\vec{V}.\vec{n})dA$$

$$F_{Sy} + F_{By} = \int_{CV} \frac{\partial}{\partial t}(v\rho)\ \mathrm{dV} + \int_{CS} v\rho(\vec{V}.\vec{n})dA$$

Mass conservvation

$$\rho V_1 A_1 = \rho V_2 A_2 = \rho V A = \dot{m}$$



$$-F_{\text{Vane, x}} + \cancel{F_{Sx}} + \cancel{F_{Bx}} = \int_{CV} \frac{\partial}{\partial t}\cancel{(u\rho)}\,\mathrm{dV} + \int_{CS} u\rho(\vec{V}.\vec{n})dA$$

$$F_{\text{Vane, y}} + \cancel{F_{Sy}} + \cancel{F_{By}} = \cancel{\int_{CV} \frac{\partial}{\partial t}(v\rho)\,\mathrm{dV}} + \int_{CS} v\rho(\vec{V}.\vec{n})dA$$

Mass conservvation

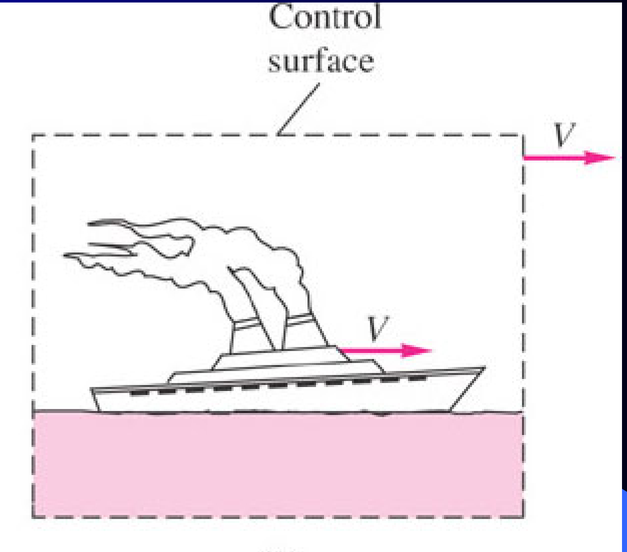
$$\rho V_1 A_1 = \rho V_2 A_2 = \rho VA = \dot{m}$$

$$\int_{CS_{1+2}} u\rho(\vec{V}.\vec{n})dA = u_2 m - u_1 m = (V\cos\theta - V)m$$

$$\int_{CS_{1+2}} v\rho(\vec{V}.\vec{n})dA = v_2 m - v_1 m = (V\sin\theta - 0)m$$

$$\Rightarrow \begin{cases} F_{vane,x} = mV(1-\cos\theta) \\ F_{vane,y} = mV\sin\theta \end{cases}$$

3.5 Moving control volume at constant velocity



(b)

Background – In some applications, it is much suited to choose a moving CV for flow analysis:

Analysis

- Since the control volume moves at a constant velocity and not accelerating (i.e inertial frame), thus the forces will not change in relative frame (coordinate system);

- In both mass and momentum conservation equations, we need to use the relative velocity of fluid to the CV (not absolute flow velocity), because we have assumed that the velocity is measured relative to control volume when deriving the Reynolds's theorem. It is given by

$$\vec{V}_r = \vec{V}_{abs} - \vec{V}_{CV}$$

Continuity equation

$$\int_{CV} \frac{\partial \rho}{\partial t}\, dV + \int_{CS} \rho\ (\vec{\mathbf{V}}_r . \vec{\mathbf{n}})\ dA = 0$$

Momentum equation

$$[\vec{F}_S + \vec{F}_B]_{\text{on CV}} = \int_{CV} \frac{\partial}{\partial t} (\vec{V}_r \rho)\ \mathrm{dV} + \int_{CS} \vec{V}_r \rho (\vec{V}_r . \vec{n}) dA$$

or

$$F_{Sx} + F_{Bx} = \int_{CV} \frac{\partial}{\partial t} (u_r \rho)\ \mathrm{dV} + \int_{CS} u_r \rho (\vec{V}_r . \vec{n}) dA$$

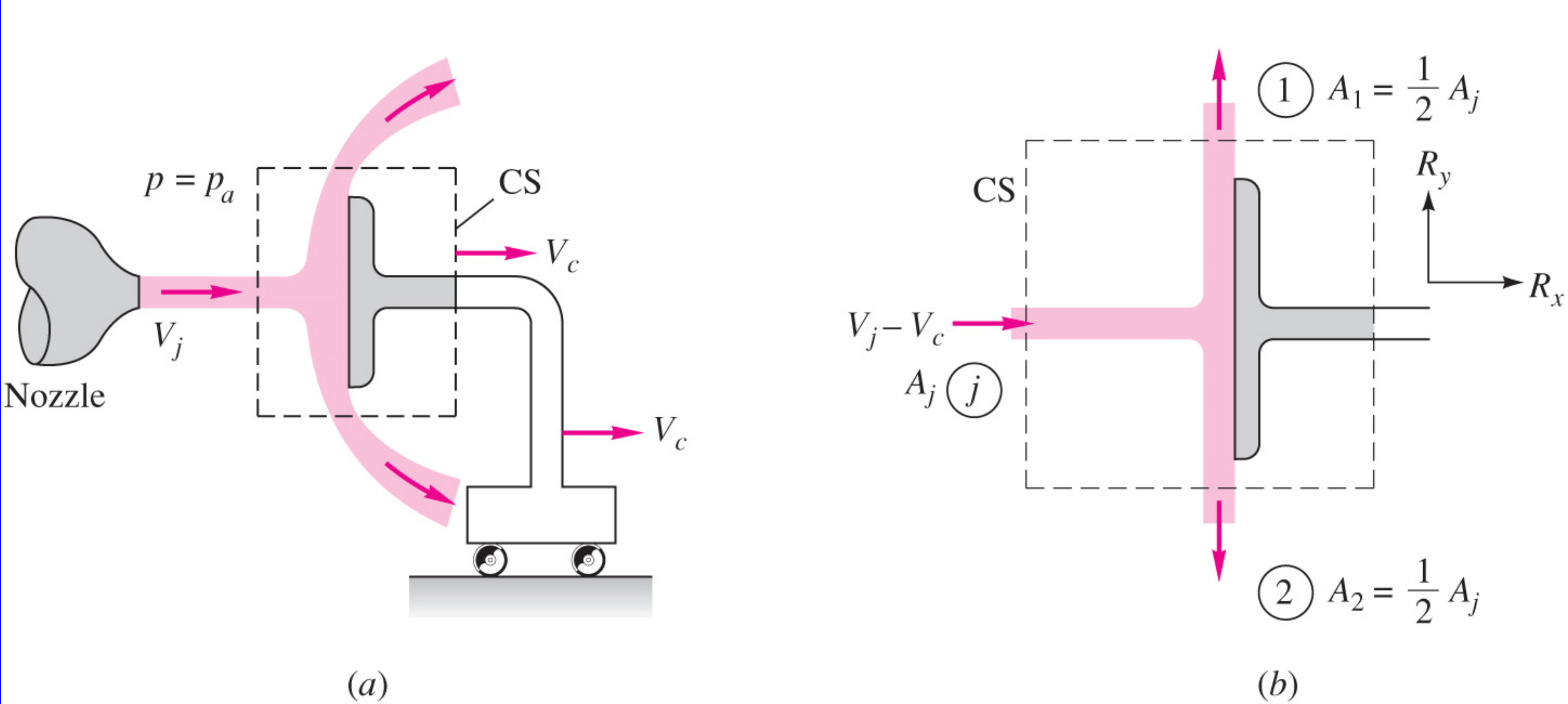
$$F_{Sy} + F_{By} = \int_{CV} \frac{\partial}{\partial t} (v_r \rho)\ \mathrm{dV} + \int_{CS} v_r \rho (\vec{V}_r . \vec{n}) dA$$

$$F_{Sz} + F_{Bz} = \int_{CV} \frac{\partial}{\partial t} (w_r \rho)\ \mathrm{dV} + \int_{CS} w_r \rho (\vec{V}_r . \vec{n}) dA$$

例题： FW example 3.9 pp164

例题： FW example 3.9 pp164

Find the force required to keep the plate moving at constant Vc?



$$\vec{V}_r = \vec{V}_{abs} - \vec{V}_{CV}$$

3.6 Bernoulli's equation

– application of continuity / momentum equation

Objective – Combine the continuity and momentum equations for an elemental control volume along streamline to obtain Bernoulli's equation.

Background knowledge –

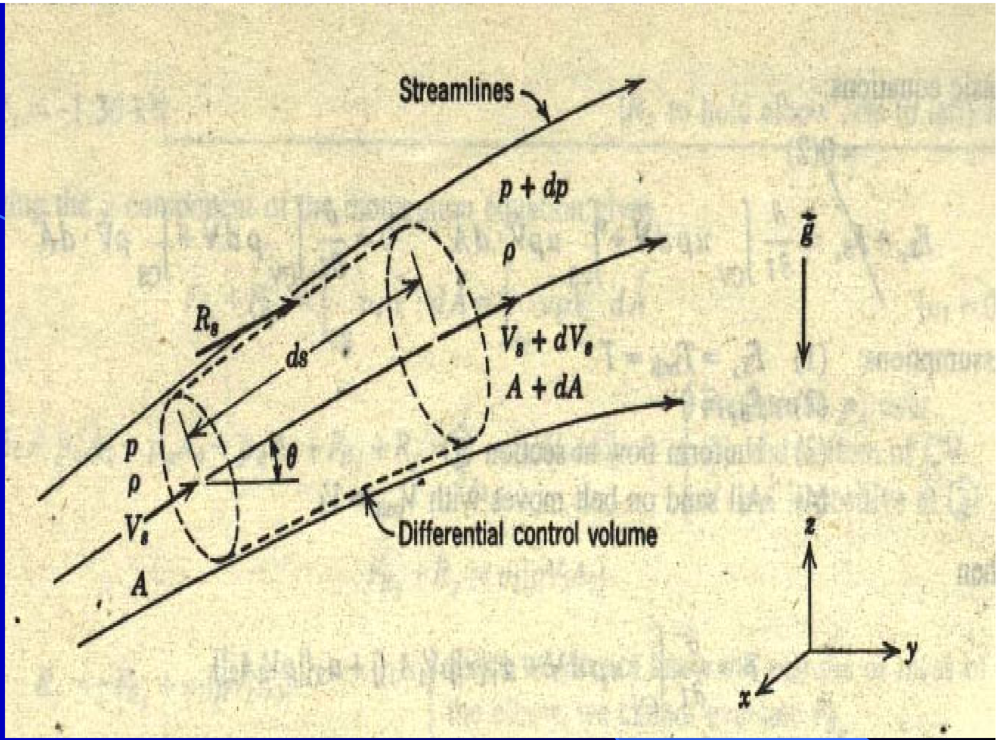
In the previous sections we have established the continuity and momentum conservation equation of integral form for control volume. Do they apply to any size of control volumes? Did we put any restrictions on the size of CV when establishing RT equation?

The control volumes chosen for analysis can be infinitesimal and finite. Application of the basic equations to elemental control volume leads to differential equations describing the relationships among properties in the flow field.

Problem description

- Consider a steady incompressible inviscid (frictionless) flow ;

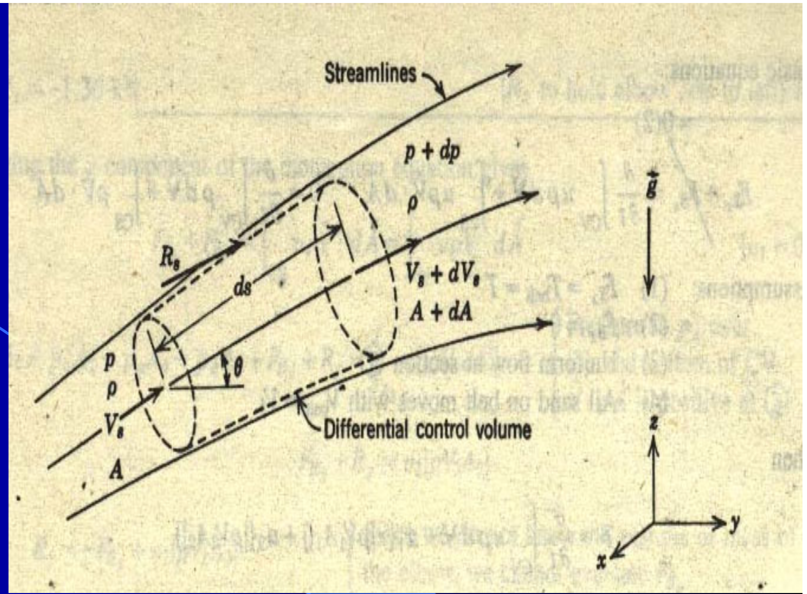
- The control volume chosen is fixed in space and bounded by streamlines, thus it is an element of a streamtube. The length of the elemental control volume is *ds*.



- Flow can only cross the control surface at the end sections. In streamline wise coordinate, these two surface are located at s and $s + ds$. Properties at the inlet section are denoted by p, ρ, V_s, and A. Properties at the outlet section expressed by the properties at the inlet section plus a differential increase.

Approach - apply the continuity equation and the (streamline-wise) component of the momentum equation to the control volume

1. Continuity equation



Basic equation $\frac{d}{dt}\left(\cancel{\int_{CV} \rho \, dV}\right) + \int_{CS} \rho \, (\vec{\mathbf{V}}.\vec{\mathbf{n}}) \, dA = 0$

Specific conditions

(1) Steady flow

(2) No flow across bounding streamlines

(3) Incompressible flow $\rho = cons\tan t$

Then we have $\frac{d}{dt}\left(\int_{CV} \rho \, dV\right) = \int_{CV} \frac{\partial \rho}{\partial t} \, dV = 0$

$$\int_{CS} \rho \, (\vec{\mathbf{V}}.\vec{\mathbf{n}}) \, dA = -\left|\rho \, V_S \, A\right| + \left|\rho(V_S + dV_S)(A + dA)\right| = 0$$

$$\rho \, V_S \, A = \rho(V_S + dV_S)(A + dA)$$

$$V_S dA + A \, dV_S + \cancel{dA \, dV_S} = 0$$

$$\boxed{V_S \, dA + A \, dV_S = 0}$$

2. Momentum equation in streamline-wise

Basic equation

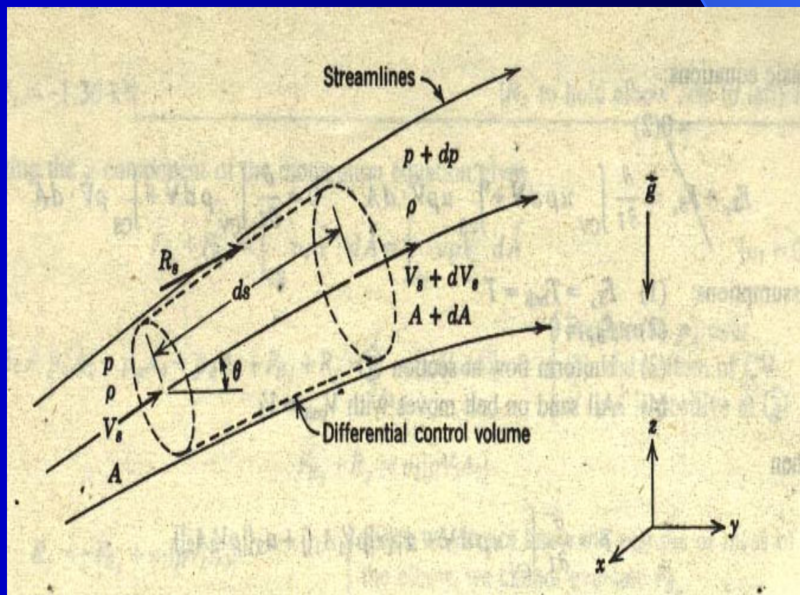
$$F_{Ss} + F_{Bs} = \cancel{\int_{CV} \frac{\partial}{\partial t}(u_s \rho)\, \mathrm{dV}} + \int_{CS} u_s \rho(\vec{V}.\vec{n})dA$$

Specific conditions

(1) steady flow

$$\int_{CV} \frac{\partial}{\partial t}(u_s \rho)\ \mathrm{dV}=0$$

(2) frictionless flow: tangential viscous force is zero, surface force only consists of pressure force



Pressure force in streamline direction consists three parts:

On the left end face $p\ A$

On the right end face $-(p+dp)(A+dA)$

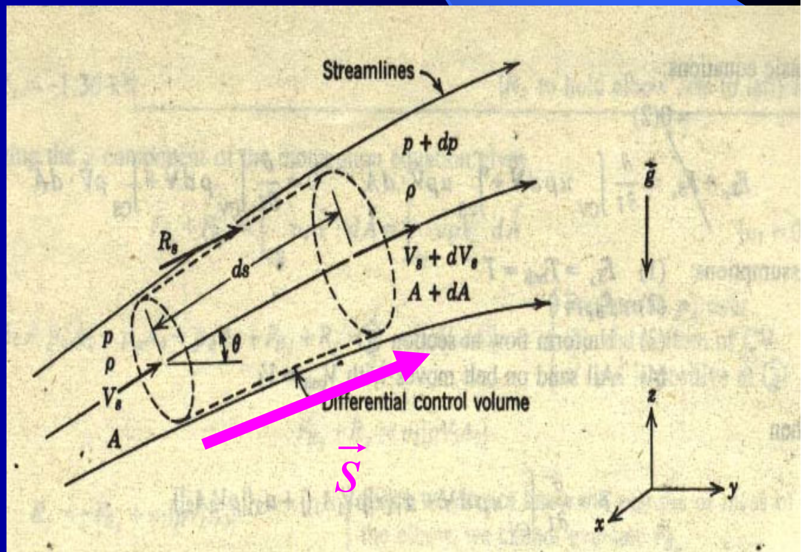
On the bounding stream surface $(p+dp/2)dA$

Where $p+dp/2$ is the averaged pressure acting on the stream surface (i.e. the averaged value of pressure acting on both end faces); dA denotes the area component of the stream surface in the streamline direction.

Resultant pressure force

$$F_{S_S} = pA-(p+dp)(A+dA)+(p+dp/2)dA$$

$$= p\ A-p\ A-p\ dA-A\ dp-dA\ dp+p\ dA+\frac{dp}{2}\ dA$$



$$F_{S_S} = -\ Adp-\frac{dp}{2}dA$$

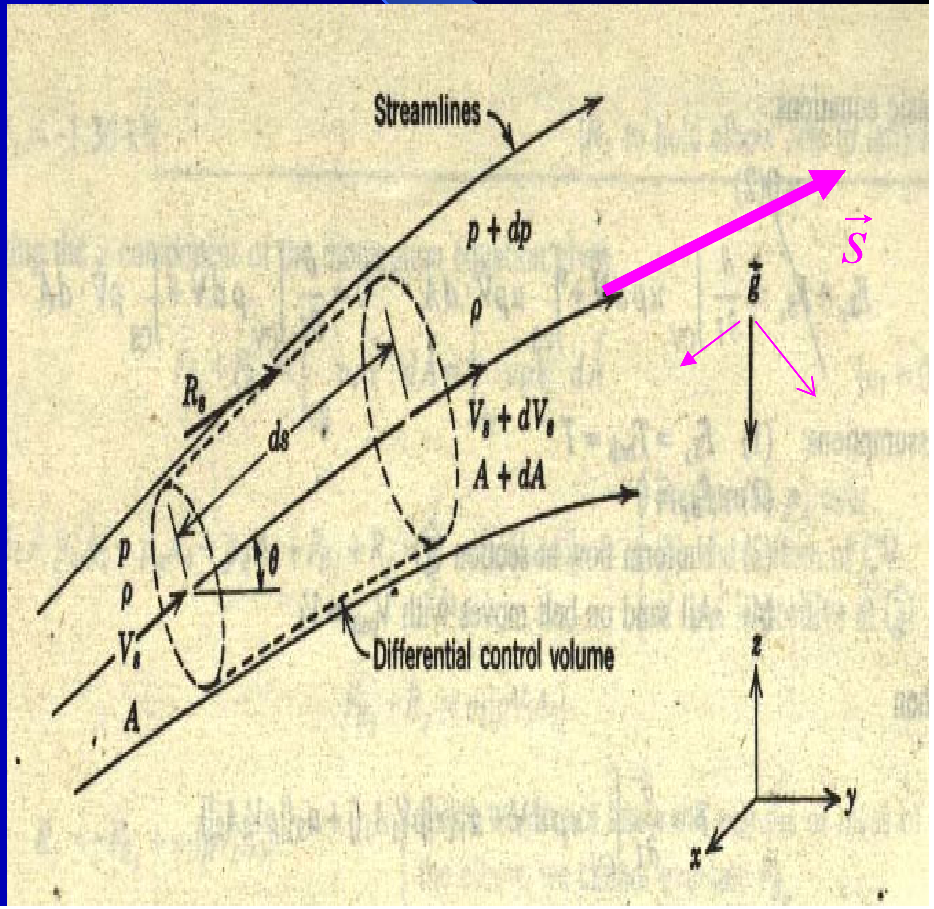
Body force

$$F_{B_S} = \rho g_s dV = \rho(-g \ \sin\theta) dV$$

Since $dV = 1/2(A + (A + dA))\, ds = (A + dA/2)\, ds$

and $\sin\theta \ ds = dz$, then

$$F_{B_S} = -\rho g \ (A + \frac{dA}{2})\, dz$$



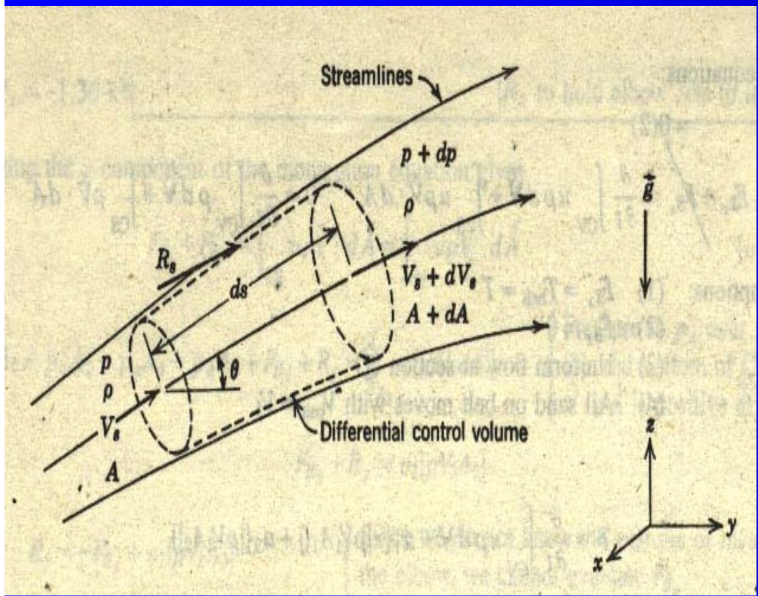
Momentum flux - There is no mass and momentum flux cross the stream surface

$$\int_{CS} u_S \rho(\vec{V}.\vec{n})dA = \int_{\text{lef facet}} u_S \rho(\vec{V}.\vec{n})dA + \int_{\text{right face}} u_S \rho(\vec{V}.\vec{n})dA$$

$$= \mathrm{V_S}\rho \int_{\text{lef facet}} (\vec{V}.\vec{n})dA + (V_S + dV_S)\rho \int_{\text{right face}} (\vec{V}.\vec{n})dA$$

$$= \mathrm{V_S}\rho(-V_S A) + (V_S + dV_S)\rho(V_S + dV_S)(A + dA)$$

$$= \mathrm{V_S}\rho(-V_S A) + (V_S + dV_S)\rho(V_S A + \cancel{\boxed{A\,dV_S + V_S\,dA}} + \cancel{dV_S\,\mathrm{dA}})$$



$(V_S\ dA + A\ dV_S = 0,$ arising from continuity equation)

Cancel and collect terms, and neglect the product of differentials, then obtain

$$\int_{CS} u_s \rho(\vec{V}.\vec{n})\, dA = \rho\, V_S\, A\, dV_S$$

Introducing all the reduced terms into the momentum equation

$$F_{S_s} + F_{B_s} = \int_{CS} u_s \rho(\vec{V}.\vec{n})dA$$

$$F_{B_S} = -\rho g \; (A + \frac{dA}{2}) \, dz$$

$$F_{S_S} = -\, A dp - \frac{dp}{2} dA$$

$$\int_{CS} u_s \rho(\vec{V}.\vec{n}) \; dA = \rho \; V_S \; A \; dV_S$$

$$-A\,dp - \cancel{\frac{1}{2}dp\,dA} - \rho g A\,dz - \cancel{\frac{1}{2}\rho g\,dA\,dz} = \rho V_S A\,dV_S$$

$$\frac{dp}{\rho} + g\,dz + d\left(\frac{V_S^2}{2}\right) = 0$$

Integrating the above equation, we then obtain

$$\frac{p}{\rho} + \frac{V_S^2}{2} + gz = \text{constant, or}$$

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant, or}$$

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

This is the so-called ***Bernoulli's equation***, named after Daniel Bernoulli (1700-1782), a Swiss mathematician

$$\frac{p}{\rho}+\frac{V^2}{2}+gz=\text{constant, or}$$

$$\frac{p_1}{\rho}+\frac{V_1^2}{2}+gz_1=\frac{p_2}{\rho}+\frac{V_2^2}{2}+gz_2$$

Restrictions of BE:

- Steady flow
- No friction
- Flow along a streamline (for different streamlines, the constants will be different)
- Incompressible flow

➢ *BE used in static and stagnation pressure definition & measurement*

- ***Definition of static pressure /stagnation pressure***

Static pressure (静压) = thermodynamic pressure. It is the pressure measured by means of a device, which is relatively static to the flow (theoretically there is **no influence** to the flow)

Stagnation pressure / total pressure（滞止压强、全压): **decelerate** the flow to zero velocity without friction and the corresponding pressure is stagnation pressure for the flow.

- ***Used in flow measurement:***

How to measure static and stagnation pressure? Relation of static and stagnation pressure?

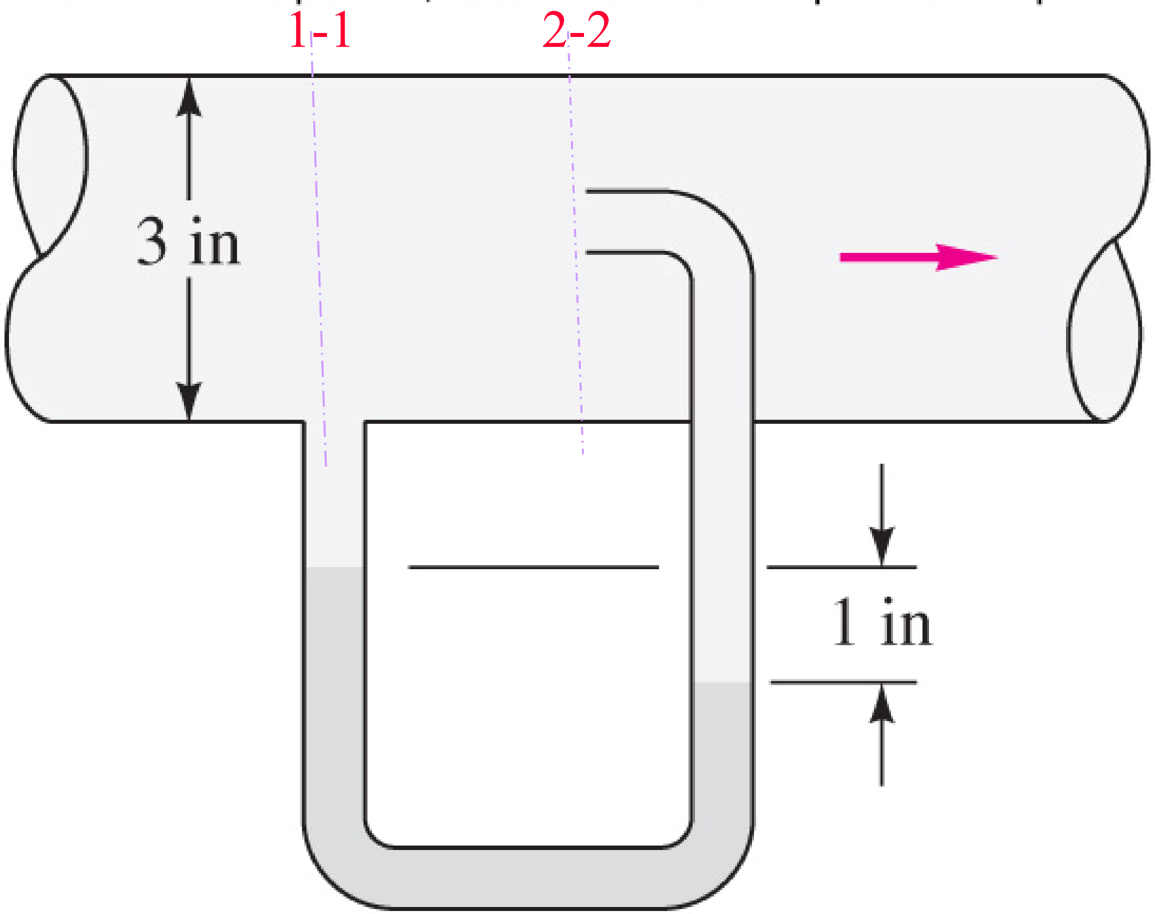
$$p_1 + \rho\frac{V_1^2}{2} + \rho g z_1 = p_2 + \rho\frac{V_2^2}{2} + \rho g z_2 = p_0 + \rho g z_0 + \rho\frac{0^2}{2}$$

If $z_1 = z_2 = z_0$, then

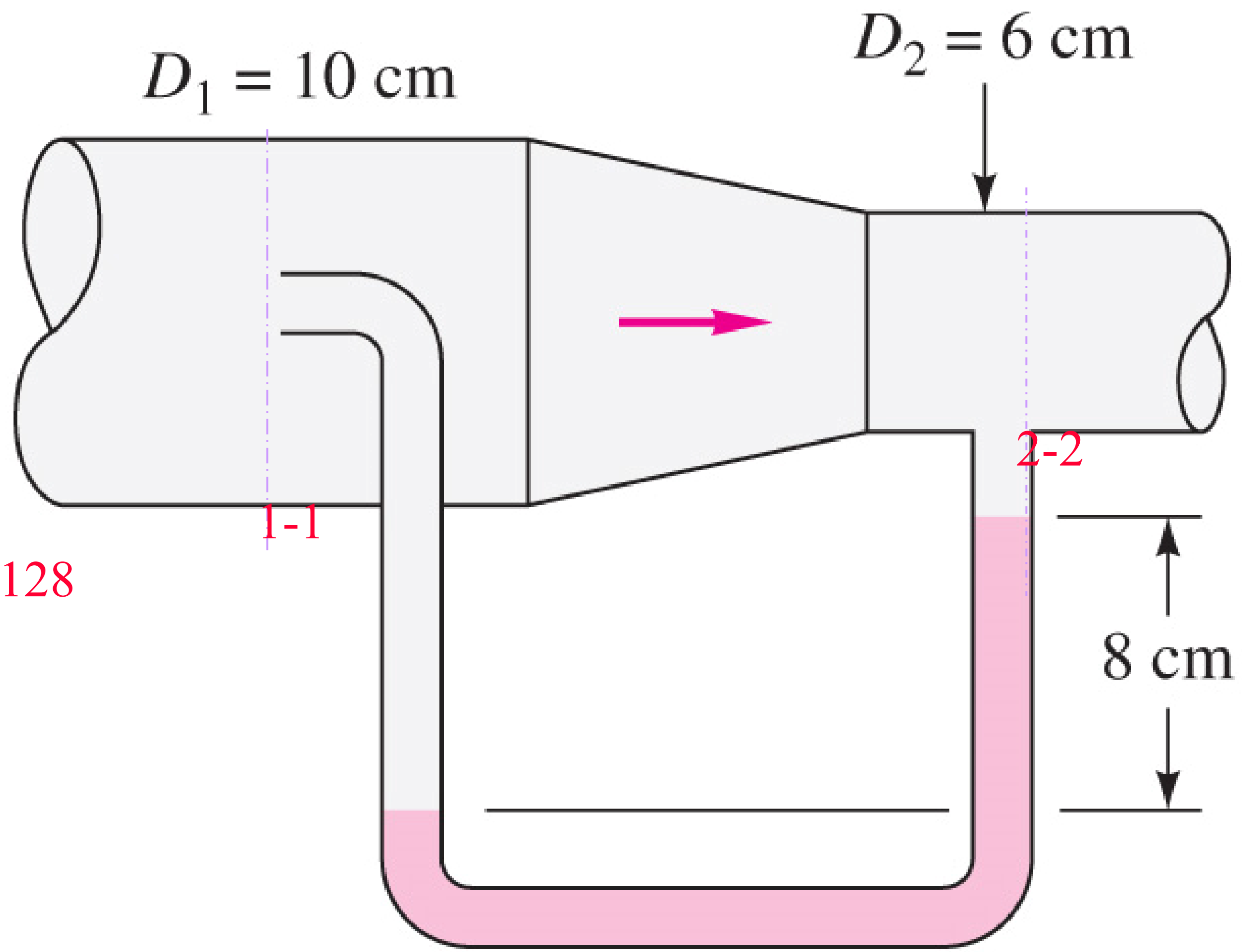
$$p_1 + \rho\frac{V_1^2}{2} = p_2 + \rho\frac{V_2^2}{2} = p_0$$

Example: Pitot tube (皮托管测静压全压附件)

Example: p3.127, 3.128 on FW page 219-320 （U型管两臂分别测静压和全压）



P3.127

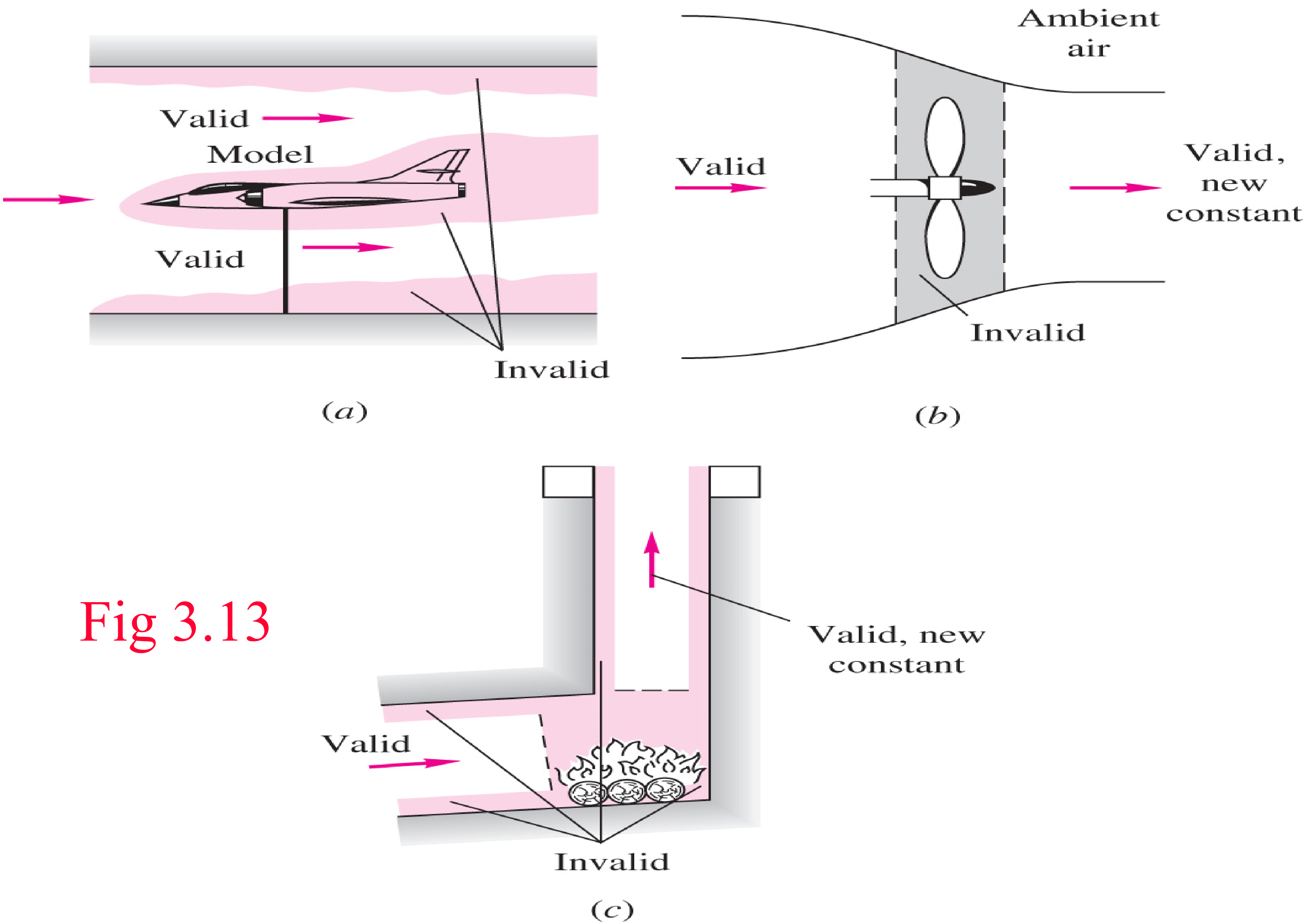


P3.128

➢ *Validity of Bernoulli equation*

- *For Straight pipe flow without friction* may be treated as one-dimensional flow along a streamline, and *B. E. can be applied.*
- *For viscous flow with considerable boundary layer effect* or *in flow separation region*, *B.E. shall not be applied.*
- *No work input and output*

(see fig 3.13 page 175-176 of FW)



Valid
Model
Valid
Invalid
(a)
Ambient air
Valid
Valid, new constant
Invalid
(b)
Valid, new constant
Valid
Invalid
(c)

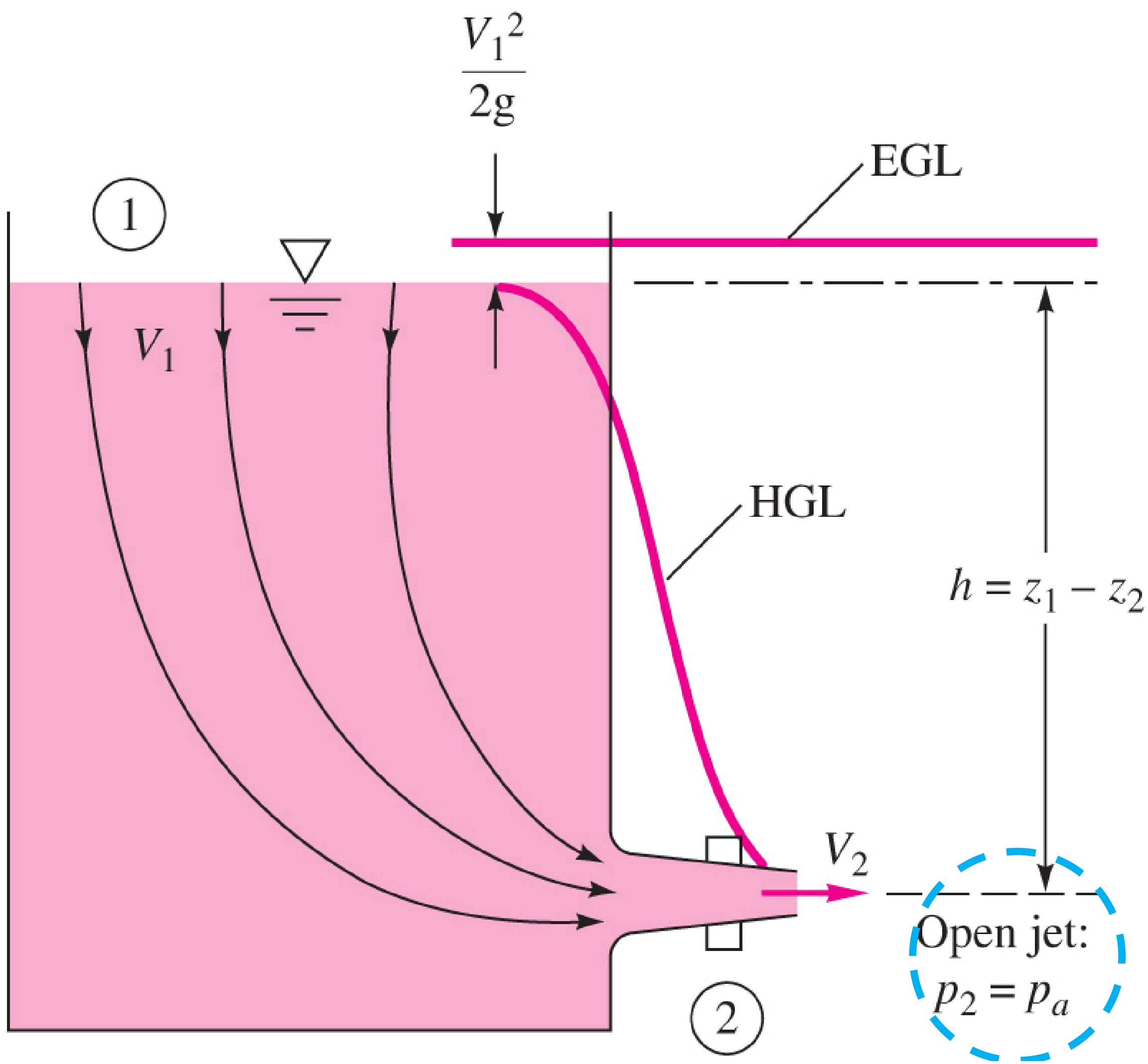
Fig 3.13

➢ Jet exit pressure equals atmospheric pressure

When a subsonic jet of liquid or gas exits from a duct into the free atmosphere, it immediately takes on the pressure of that atmosphere.

Page 178 ----
Example 3.13
Figure 3.13

Find the relation between nozzle velocity V2 and tank free surface height h. Assume steady frictionless flow.



$$\frac{\cancel{p_1}}{\rho g}+\frac{\cancel{V_1^2}}{2g}+z_1=\frac{\cancel{p_2}}{\rho g}+\frac{V_2^2}{2g}+z_2=C$$

$$p_1 = p_2 = p_{atm}$$

➢ *Hydraulic grade line and energy grade line*

Bernoulli's equation:

$$\frac{p_1}{\rho g}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{V_2^2}{2g}+z_2=\text{constant}$$

Energy grade for frictionless flow (inviscid)

$$\frac{p}{\rho g}+\frac{V^2}{2g}+z=\text{constant}$$

Hydraulic grade:

$$\frac{p}{\rho g}+z$$

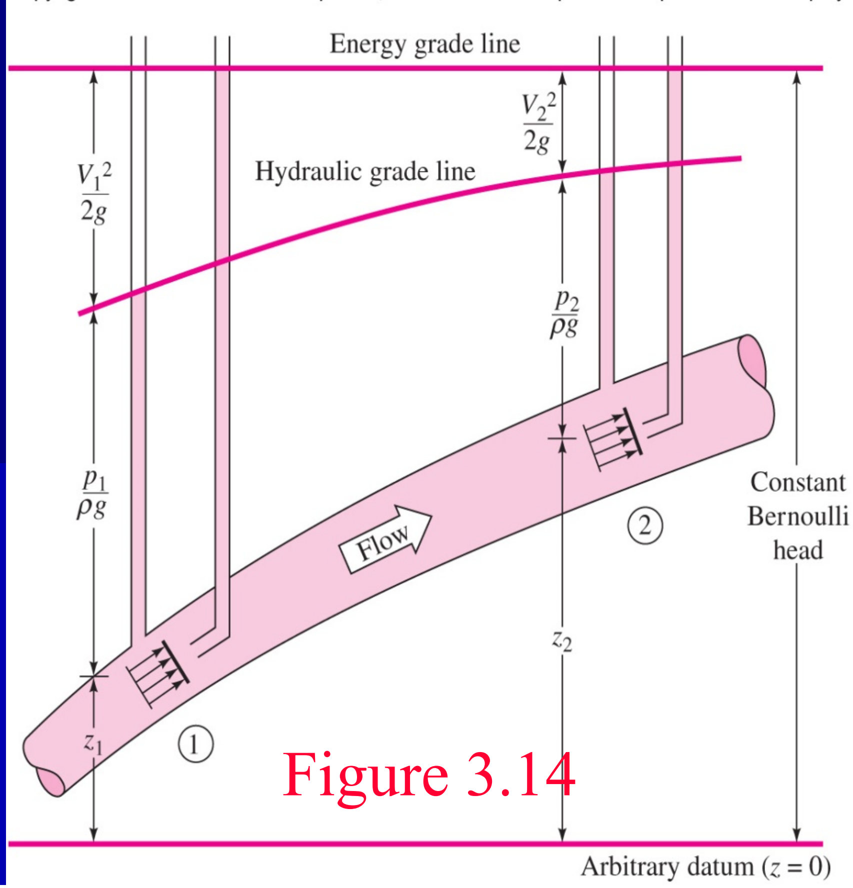


Figure 3.14

Pressure head Velocity head Elevation (Potential head)

$$\frac{p}{\rho g}+\frac{V^2}{2g}+z=\text{constant}$$

= Total mechanical head

Bernoulli's equation and energy equation!!!

➢ *Surface velocity condition for a large tank*

The problems involve liquid flow from a large tank or reservoir, if the outflow is small compared to the volume of the tank, the surface of the tank hardly moves. Thus in such problems, zero velocity at tank surface is assumed. In addition, the pressure at the top of the tank or reservoir is assumed to be atmospheric.

Page 179 -- Example 3.14

Consider the water siphon as as shown in fig.3.14. Assuming B.E is valid
(1)V2=?
(2)if tube d2=1cm, z1=60 cm, z2=-25cm, z3=90cm, z4=35cm, then obtain Q2=?

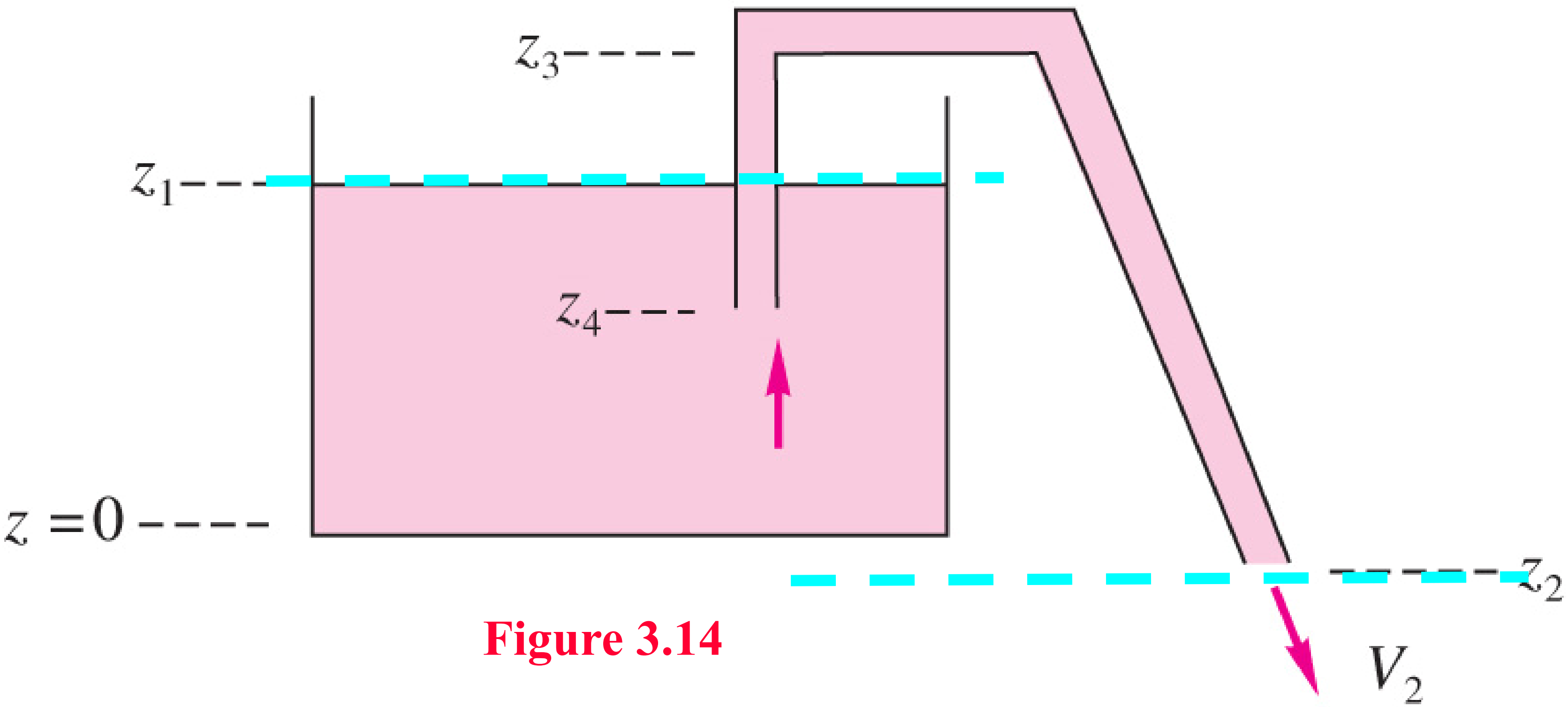


Figure 3.14

$$\frac{\cancel{p_1}}{\rho g}+\frac{\cancel{V_1^2}}{2g}+z_1=\frac{\cancel{p_2}}{\rho g}+\frac{V_2^2}{2g}+z_2=C$$

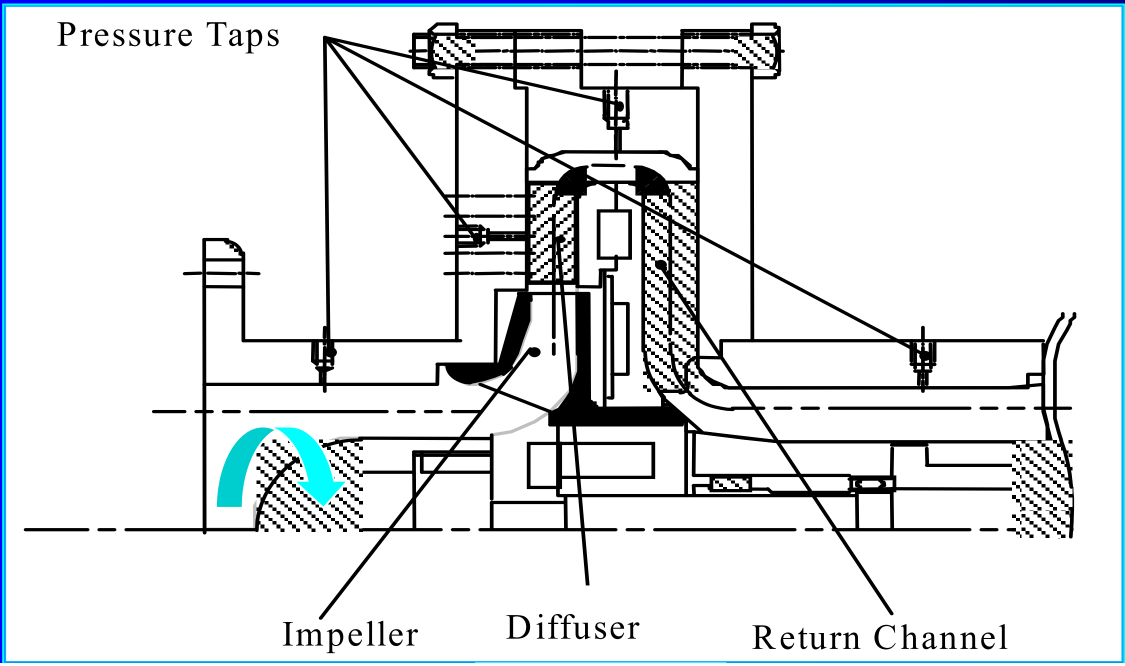
$$p_1 = p_2 = p_{atm}$$

$$V_2 = \sqrt{2g(z_1 - z_2)} = 4.18 m/s$$

$$Q_2 = V_2 A_2 = 321 \text{cm}^3/\text{s}$$

3.7 Angular-momentum equation

We have established continuity and linear momentum equations, and Bernoulli's equation, using these equations we can solve some engineering problems. But some applications involving torques (扭矩) or moments (力矩), such as rotating fluid-flow device, we need to use angular-momentum conservation analysis. In this section we shall develop a control volume expression for the angular momentum principle.



Shaft input torque = ?

It is noted that *angular momentum can be specified in two different ways:*

- *relative to an inertial (non-accelerating) reference frame* (参考系). Reference frames at rest, or moving with constant linear velocity, are inertial, the equation (a) can be used directly to develop the control volume form of the angular momentum principle.
- *relative to an non-inertial (accelerating) reference frame* (参考系). In many cases, it is convenient to use a accelerating / noninertial coordinate system. An example would be coordinates fixed to rocket during takeoff. It is noted rotating frame / coordinate system is noninertial one!!!!

(non-inertial coordinate system including both rotating and linear accelerating ones !!!)--------不讲解，请自学！！！）

3.7.1 Equation for inertial reference frame

（在惯性静止坐标系下，建立角动量守恒方程）

For a system, the angular momentum principle states that the rate of change of angular momentum is equal to the sum of all torques (=moment; 扭矩, 转矩) acting on the system

$$\vec{T}_{syst} = \frac{d\vec{H}}{dt}\Big|_{syst}$$

系统角动量守恒方程

Where $$\vec{H} = \int_{m\ syst} \vec{r}\times\vec{V}\, dm = \int_{v\ syst} \vec{r}\times\vec{V}\rho\, dV$$

$$\vec{T}_{syst} = [\vec{r}\times\vec{F}_s + \int_{mass} (\vec{r}\times\vec{g})\rho\, dV + \vec{T}_{shaft}]_{sys}$$

It is noted that angular momentum of a system in general motion must be specified relative to an inertial (non-accelerating) reference frame (参考系). Reference frames at rest, or moving with constant linear velocity, are inertial, the equation (a) can be used directly to develop the control volume form of the angular momentum principle.

Reynold's theorem is written

$$\frac{d}{dt}(B_{\text{syst}}) = \frac{d}{dt}\Big(\int_{CV} \beta\rho \ \mathrm{dV}\Big) + \int_{CS} \beta\rho(\vec{\mathbf{V}}.\vec{\mathbf{n}})dA$$

Let

$$B_{syst} = \vec{H} = \int_{\text{V syst}} \vec{r}\times\vec{V} \ \rho \ dV$$

$$\beta = \vec{r}\times\vec{V}$$

Then we obtain

$$\frac{d\vec{H}}{dt}\Big|_{syst} = \frac{d}{dt}\Big[\int_{CV} (\vec{r}\times\vec{V}) \ \rho \ dV\Big] + \int_{CS} (\vec{r}\times\vec{V}) \ \rho \ (\vec{V}.\vec{n}) \ dA$$

雷诺输运方程

Combine transport equation and angular conservation equation of a system

$$\Big[\vec{r}\times\vec{F}_s + \int_{mass} (\vec{r}\times\vec{g})\rho \ dV + \vec{T}_{shaft}\Big]_{sys} = \frac{d}{dt}\Big[\int_{CV} (\vec{r}\times\vec{V}) \ \rho \ dV\Big] + \int_{CS} (\vec{r}\times\vec{V}) \ \rho \ (\vec{V}.\vec{n}) \ dA$$

The system and control volume coincided at time t, we have

$$\vec{T}_{syst} = \vec{T}_{CV}$$

OR

$$[\vec{r}\times\vec{F}_s + \int_{mass} (\vec{r}\times\vec{g})\rho\, dV + \vec{T}_{shaft}]_{sys} = [\vec{r}\times\vec{F}_s + \int_{mass} (\vec{r}\times\vec{g})\rho\, dV + \vec{T}_{shaft}]_{CV}$$

Then

$$[\vec{r}\times\vec{F}_s + \int (\vec{r}\times\vec{g})\rho\, dV + \vec{T}_{shaft}]_{CV} = \frac{d}{dt}[\int_{CV} (\vec{r}\times\vec{V})\,\rho\, dV] + \int_{CS} (\vec{r}\times\vec{V})\,\rho\,(\vec{V}.\vec{n})\, dA$$

- The left side are all the torques acting on the control volume. Terms on the right denote the rate of change of angular momentum within the control volume, and the net rate of flux of angular momentum from the control volume.
- It is noted that all velocities are measured relative to the fixed inertial（non-accelerating）control volume!!!

Applications of angular momentum conservation equation for CV

控制容积角动量守恒方程的应用

1. Used to evaluate the shaft input torque for rotating machinery

$$[\vec{r}\times\vec{F}_s+\int(\vec{r}\times\vec{g})\rho\, dV+\vec{T}_{shaft}]_{CV}=\frac{d}{dt}[\int_{CV}(\vec{r}\times\vec{V})\,\rho\, dV]+\int_{CS}(\vec{r}\times\vec{V})\,\rho\,(\vec{V}.\vec{n})\, dA$$

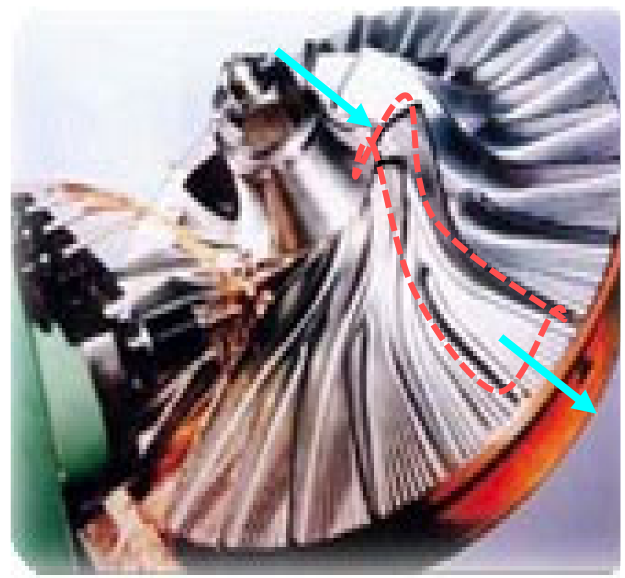
For rotating machinery, the above equation is often used in scalar form by considering only the component directed along axis of rotation.

If flow is frictionless and steady, effect of gravitational force can be neglected, then the shaft toque input in turbomachnery is given by

$$T_{shaft}=\int_{CS}(\vec{r}\times\vec{V})\,\rho\,(\vec{V}.\vec{n})\, dA$$

$$=\int_{\text{out}}(\vec{r}\times\vec{V})\,\rho\,(\vec{V}.\vec{n})\, dA+\int_{in}(\vec{r}\times\vec{V})\,\rho\,(\vec{V}.\vec{n})\, dA$$

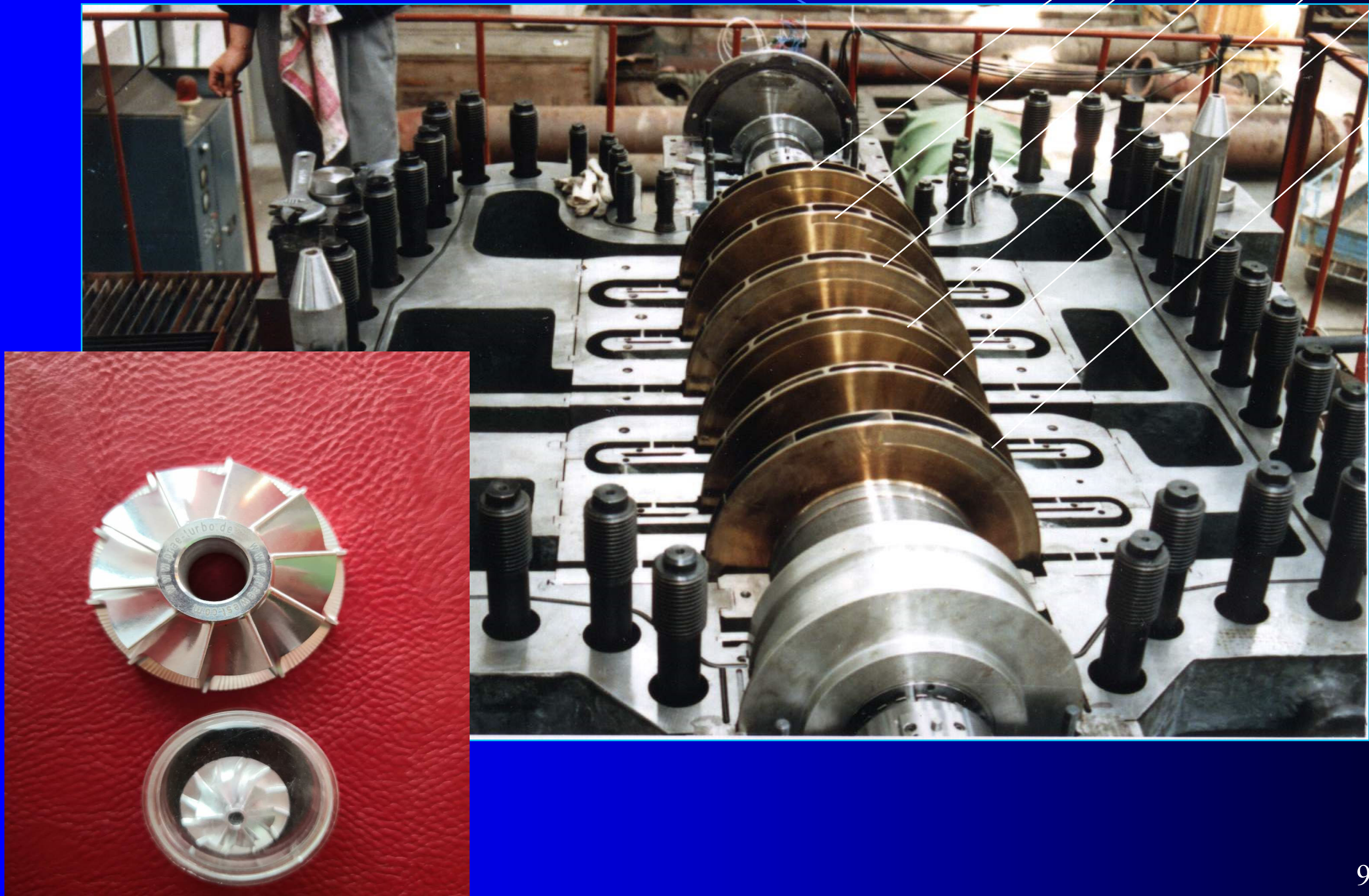
$$=(\vec{r}\times\vec{V})_{out}\int_{\text{out}}\rho\,(\vec{V}.\vec{n})\, dA+(\vec{r}\times\vec{V})_{in}\int_{in}\rho\,(\vec{V}.\vec{n})\, dA$$



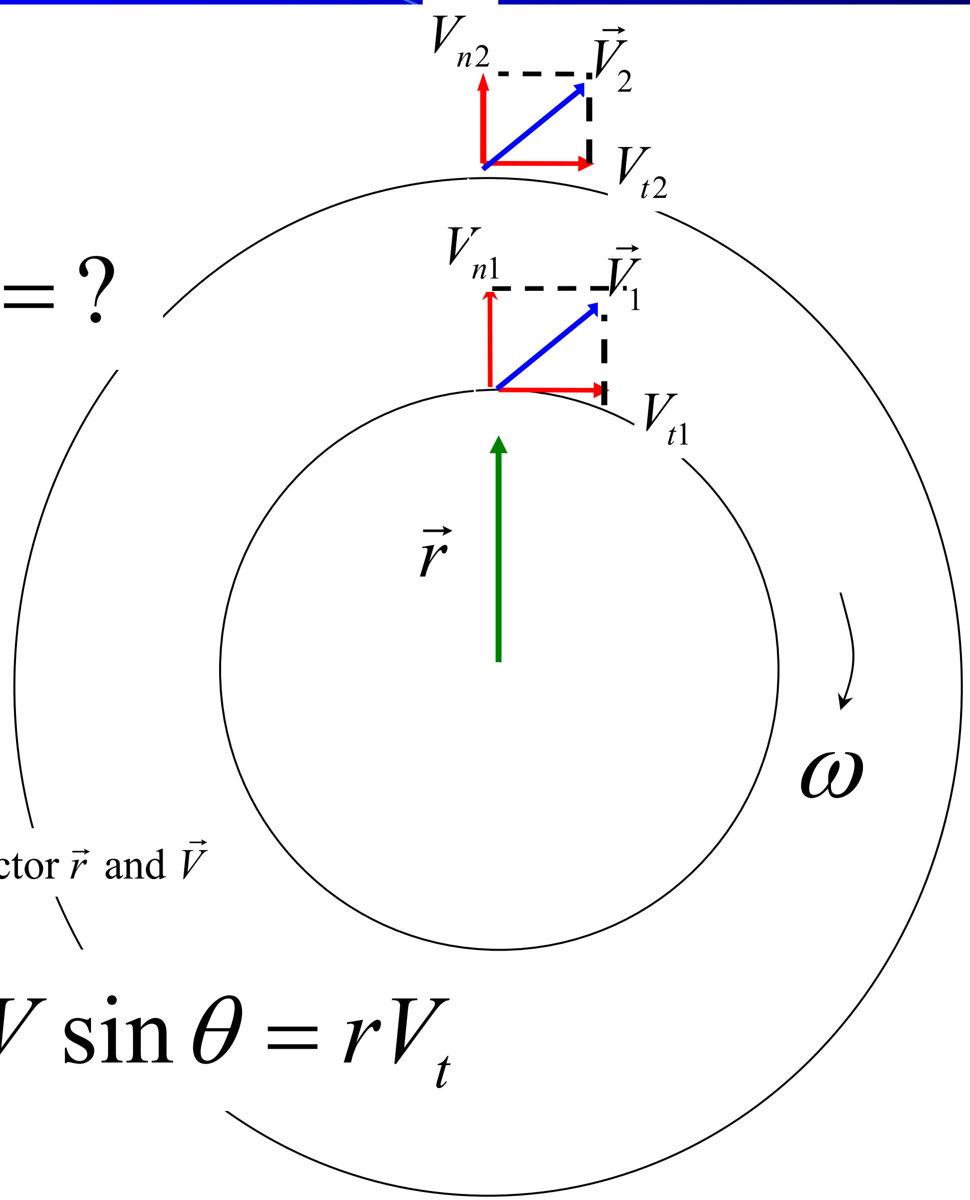
It is *Euler equation for turbomachinery.*

Example 3.18 on page 185

离心压缩机



How about moment produced by pressure force at inlet and outlet?



$$\vec{r} \times \vec{V} = ?$$

θ --- Angle between vector $\vec{r}$ and $\vec{V}$

$$\vec{r} \times \vec{V} = rV \sin\theta = rV_t$$

$$
\begin{aligned}
T_{shaft} &= \int_{CS} (\vec{r} \times \vec{V})\ \rho\ (\vec{V}.\vec{n})\ dA \\
&= \int_{\text{out}} (\vec{r} \times \vec{V})\ \rho\ (\vec{V}.\vec{n})\ dA + \int_{in} (\vec{r} \times \vec{V})\ \rho\ (\vec{V}.\vec{n})\ dA \\
&= (\vec{r} \times \vec{V})_{out} \int_{\text{out}} \rho\ (\vec{V}.\vec{n})\ dA + (\vec{r} \times \vec{V})_{in} \int_{in} \rho\ (\vec{V}.\vec{n})\ dA \\
&= r_1 v_{t1} \int_{in} \rho(\vec{v}.\vec{n}) dA + r_2 v_{t2} \int_{out} \rho(\vec{v}.\vec{n}) dA \\
&= \text{-} r_1 v_{t1} \dot{m}_1 + r_2 v_{t2} \dot{m}_2 \\
&= \dot{m}(r_2 v_{t2} - r_1 v_{t1})
\end{aligned}
$$

It is *Euler equation for turbomachinery*.

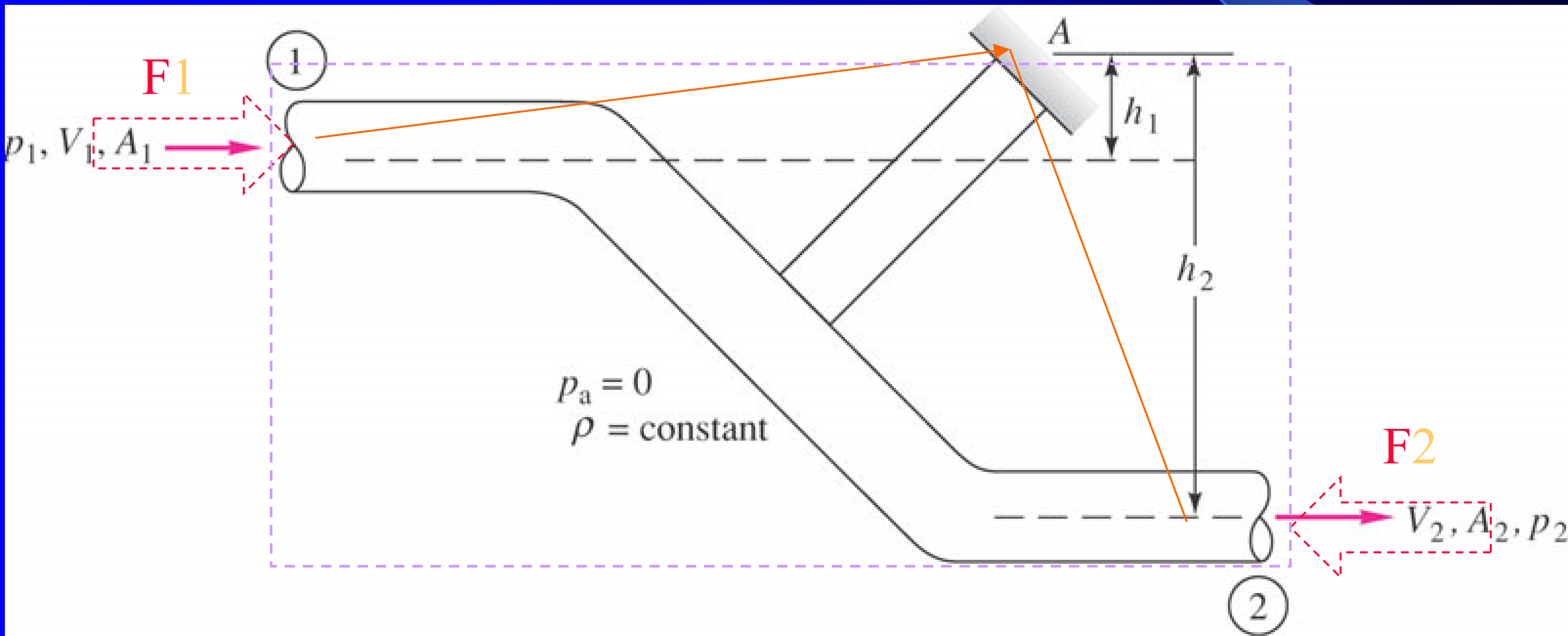
2. Used to calculate curved pipe (pipe bend) flow problems

Example 3.17 on page184

It is noted that a reference axis for rotating must be chosen when we use angular moment conservation equation for CV!!!!!

Example 3.17 on page 184-185 --- resistance torque at A

$$[T_A + \vec{r}\times\vec{F}_s + \int(\vec{r}\times\vec{g})\rho\, dV + \vec{T}_{shaft}]_{CV} = \frac{d}{dt}[\int_{CV}(\vec{r}_A\times\vec{V})\rho\, dV] + \int_{CS}(\vec{r}\times\vec{V})\rho(\vec{V}.\vec{n})\, dA$$

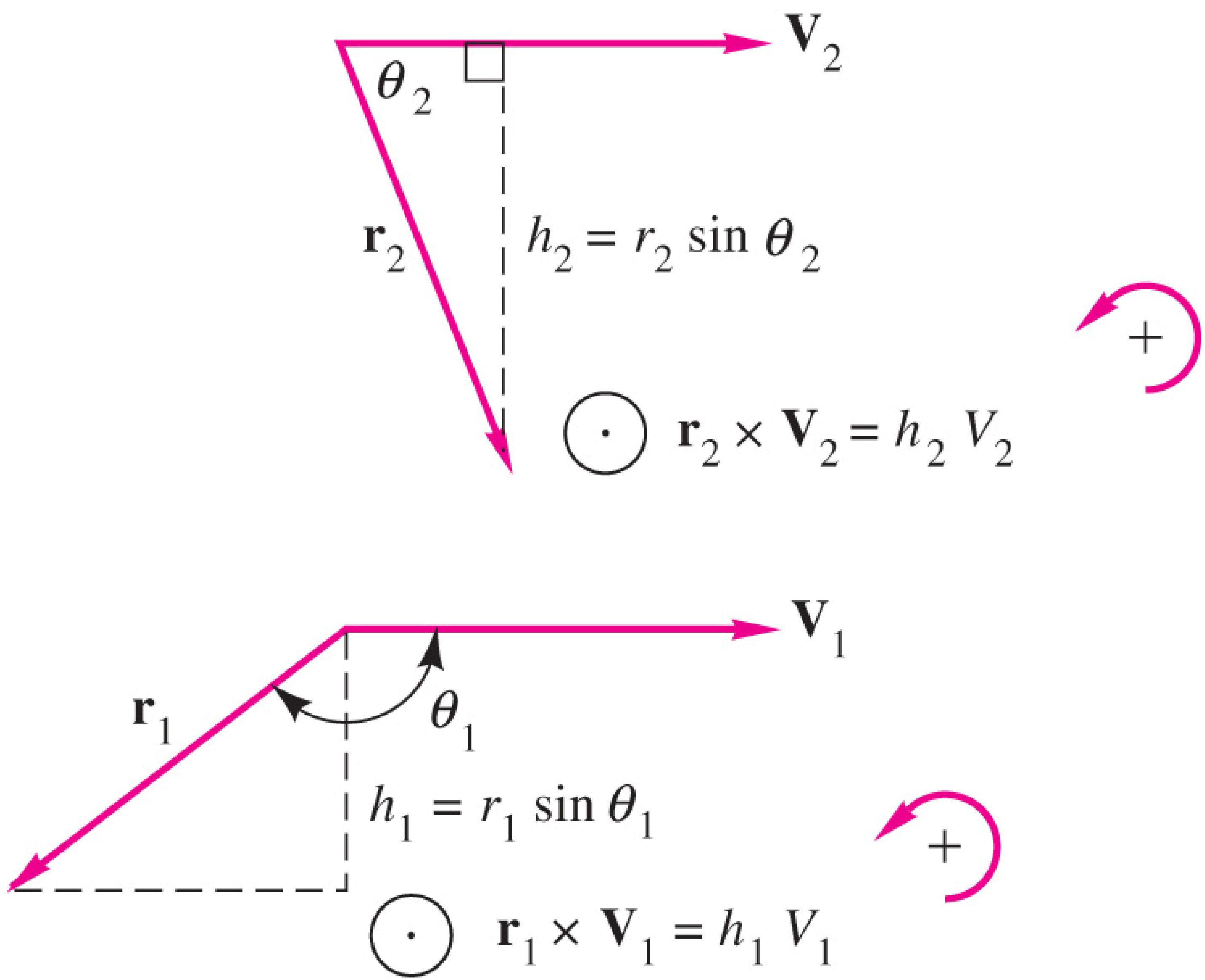


$$\int_{CS} (\vec{r}\times\vec{V})\,\rho\,(\vec{V}.\vec{n})\,dA$$

$$=(\vec{r}\times\vec{V})_1 \int_{CS1} \rho\,(\vec{V}.\vec{n})\,dA+(\vec{r}\times\vec{V})_2 \int_{CS2} \rho\,(\vec{V}.\vec{n})\,dA$$

$$=\mathrm{h}_1 V_1(-m_1)+\mathrm{h}_2 V_2(m_2)$$

$$\sum T = T_A + p_1 A_1 h_1 - p_2 A_2 h_2$$


$\mathbf{V}_2$
θ_2
$\mathbf{r}_2$
$h_2 = r_2 \sin\theta_2$
+
$\mathbf{r}_2 \times \mathbf{V}_2 = h_2 V_2$
$\mathbf{V}_1$
$\mathbf{r}_1$
θ_1
$h_1 = r_1 \sin\theta_1$
+
$\mathbf{r}_1 \times \mathbf{V}_1 = h_1 V_1$

3.8 Energy equation

We have established the linear/angular momentum equations for control volume. They can be used to calculate the forces and toque acting on the control volume. But we have not yet considered **frictional effect** on the motion, and **heat exchange** between the control volume and surroundings. **Heat exchange** with the surroundings is significant in gas turbine, since fluid temperature at the inlet of gas turbine can be 1500 ºC.

To consider all these effects, we need to establish energy conservation equation for control volume.

For a system energy conservation law is given

$$[\dot{Q}-\dot{W}]_{sys} = \frac{dE}{dt}\Big|_{syst} \qquad \text{------ } 1$$

Where

$$E_{syst} = \int_{V\ \mathrm{syst}} e\,\rho\,dV = \int_{\mathrm{V\ syst}} (\hat{u} + \frac{V^2}{2} + gz)\,\rho\,dV$$

Reynolds's theorem

$$\frac{dB}{dt}\Big|_{syst} = \frac{d}{dt}\int_{CV} \beta\rho\,dV + \int_{CS} \beta\,\rho\,(\vec{V}.\vec{n})\,dA$$

$$\frac{dE}{dt}\Big|_{syst} = \frac{d}{dt}\int_{CV} e\rho\,dV + \int_{CS} e\,\rho\,(\vec{V}.\vec{n})\,dA \qquad \text{------ } 2$$

At time instant t the system and CV coincide, then

$$\left[\dot{Q}-\dot{W}\right]_{syst} = \left[\dot{Q}-\dot{W}\right]_{CV} \qquad \text{------ } 3$$

Combine equation 1, 2, and 3, we obtain

$$[\dot{Q} - \dot{W}]_{CV} = \frac{d}{dt}\int_{CV} e\,\rho\,dV + \int_{CS} e\,\rho(\vec{V}.\vec{n})dA$$

Where $$e = \hat{u} + \frac{V^2}{2} + gz$$

It is the energy conservation equation for a control volume. It says thermal energy obtained from the surrounding subtracting work done by the CV equals the summation of the change rate of energy within CV and rate of energy flux across control surface of the CV.

Further reduction

$$[\dot{Q}-\dot{W}]_{CV}=\frac{d}{dt}\int_{CV} e\,\rho\, dV+\int_{CS} e\,\rho(\vec{V}.\vec{n})dA$$

- *work done per unit time by a CV*

Sign rule of $\dot{W}$: "+" for work done by the CV to the surroundings;
"-" for work done by the surrounding to the CV.

The rate of work done by the CV consists of normal stress work, shear stress work, and other work (electromagnetic but excluding gravity work), mathematically

$$\dot{W}=\dot{W}_{normal}+\dot{W}_{shear}+\dot{W}_{other}$$

Question – do we need to consider the work done by gravitational force?

$$e = \hat{u} + \frac{V^2}{2} + gz$$

It is noted that gravitational force is treated as an internal force, which is included in the total energy term

● *Normal force (stress) work*

The work done is given by a force

$$\delta W = \vec{F} . d\vec{s}$$

The **work** done per unit time **(i.e. power)** by the force

$$\dot{W} = \lim_{\Delta t \to 0} \frac{\delta W}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{F} . d\vec{s}}{\Delta t} = \vec{F} . \vec{V}$$

Work done by pressure force on fluid body in control volume **(施加在控制面上的正压力对CV做的功)**

$$\dot{W}_{normal} = \int_{CS} \vec{V} \cdot (-\vec{n} p dA)$$

Since the work output across the boundaries of the control volume is negative of the work done by the surrounding to the control volume, thus the work done by the control volume to the surroundings due to pressure force is **(CV对外界所做的功)**

$$\dot{W}_{normal} = -\int_{CS} \vec{V}\cdot(-\vec{n}pdA) = -\int_{CS} (-p)\ d\vec{A}\cdot\vec{V} = \int_{CS} p\ \vec{V}\cdot d\vec{A}$$

Shear force /stress work

At the boundaries of the control volume, the work can be done by the shear stresses. Shear force acting on an element of area of the control surface is

$$d\vec{F} = \vec{\tau}\ dA$$

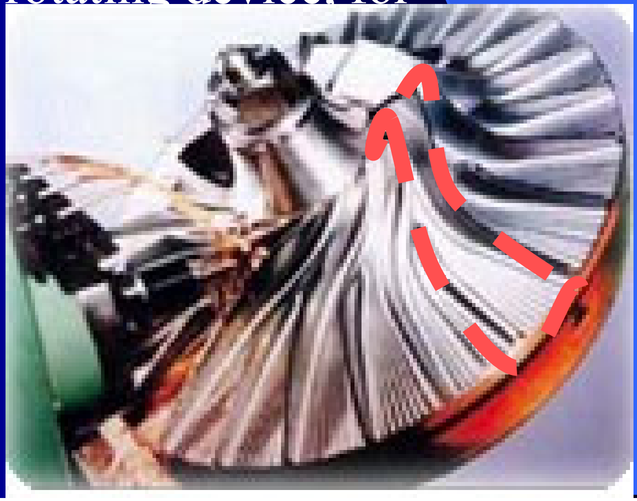
Work done by shear stresses to the CV per unit time across the entire control surface is

$$\int_{CS} (\vec{\tau}\ dA)\cdot\vec{V} = \int_{CS} \vec{\tau}\cdot\ \vec{V}\ dA$$

Work done per unit time by the control volume to the surrounding is just opposite in sign

$$\dot{W}_{\text{shear}} = -\int_{CS} \vec{\tau}\cdot\vec{V}\ dA$$

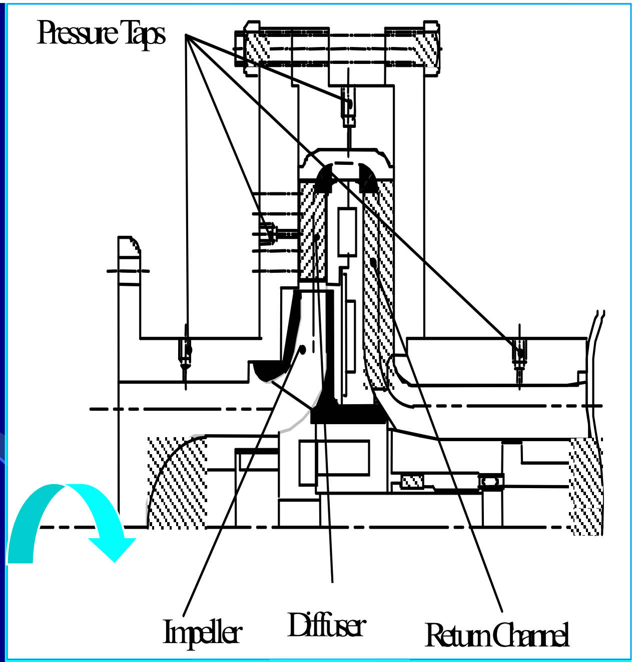
Rotating device. We consider the general case of rotating device, for example an impeller



They are rotating surface, inlet/outlet ports, and fixed solid boundary. The shaft inputs or outputs power to the fluid through the rotating surface. Mathematically

$$\dot{W}_{shear} = -\int_{CS} \vec{\tau}\cdot\vec{V}\, dA$$

$$= -\int_{A(\text{shaft})} \vec{\tau}\cdot\vec{V}\, dA - \int_{A(\text{other solid boundary})} \vec{\tau}\cdot\vec{V}\, dA - \int_{A(\text{ports})} \vec{\tau}\cdot\vec{V}\, dA$$



The first term is due to the rotation of the shaft

$$-\int_{A(shaft)} \vec{\tau}\cdot\vec{V}\, dA = \dot{W}_{\text{shaft}}$$

The second term is

$$\dot{W}_{\text{viscous}} = -\int_{A(\text{other solid boundary})} \vec{\tau}\cdot\vec{V}\, dA$$

The third term can made zero by proper choice of control surfaces. If we choose a control surface that cuts each port perpendicular to the flow, $\vec{\tau}$ will be perpendicular to $\vec{V}$, then

$$\dot{W}_{\text{port}} = -\int_{A(port)} \vec{\tau}\cdot\vec{V}\, dA = -\int_{A(port)} \tau\cdot V\cos\theta\, dA \approx 0$$

Other work not included in the above categories

For example, if the control surface is a streamline, the viscous-work (shear stress) term must be evaluated. $\dot{W}_{\text{other}}$

Having evaluated all the terms of $\dot{W}$ **,** we can re-write the expression

$$\dot{W} = \dot{W}_{shear} + \dot{W}_{normal}$$
$$= \dot{W}_{shaft} + \int_{CS} p\, \vec{V} \cdot d\vec{A} + \dot{W}_v$$

Introducing $\dot{W}$ into the energy conservation equation for CV

$$[\dot{Q} - \dot{W}]_{CV} = \frac{d}{dt} \int_{CV} e\, \rho\, dV + \int_{CS} e\, \rho (\vec{V}.\vec{n}) dA$$

Then we have

$$\dot{Q} - \dot{W}_{shaft} - \dot{W}_v - \int_{CS} p\, \vec{V} \cdot d\vec{A}_v = \frac{\partial}{\partial t} \int_{CV} e\rho\, dV + \int_{CS} e\rho \vec{V} \cdot d\vec{A}$$

$$\dot{Q} - \dot{W}_{shaft} - \dot{W}_v - \int_{CS} p\, \vec{V} \cdot d\vec{A} = \frac{\partial}{\partial t} \int_{CV} e\rho\, dV + \int_{CS} e\rho \vec{V} \cdot d\vec{A}$$

$$\int_{CS} p\vec{V} \cdot d\vec{A} = \int_{CS} pv\rho\vec{V} \cdot d\vec{A}$$

$$(\rho = 1/v)$$

$$e = \hat{u} + \frac{V^2}{2} + gz$$

Rearrange the above equation, we have

$$\dot{Q} - \dot{W}_{shaft} - \dot{W}_v = \frac{\partial}{\partial t} \int_{CV} (\hat{u} + \frac{V^2}{2} + gz)\rho dV + \int_{CS} (\hat{u} + pv + \frac{V^2}{2} + gz)\rho \vec{V} \cdot d\vec{A}$$

OR

$$\dot{Q} - \dot{W}_{shaft} - \dot{W}_v = \frac{\partial}{\partial t} \int_{CV} (\hat{u} + \frac{V^2}{2} + gz)\rho dV + \int_{CS} (h + \frac{V^2}{2} + gz)\rho \vec{V} \cdot d\vec{A}$$

(Frank White, eq. 3.67)

$$\dot{Q}-\dot{W}_{shaft}-\dot{W}_{v}=\frac{\partial}{\partial t}\int_{CV}(\hat{u}+\frac{V^2}{2}+gz)\rho dV+\int_{CS}(h+\frac{V^2}{2}+gz)\rho\ \vec{V}\cdot d\vec{A}$$

Enthalpy $h=\hat{u}+pv=c_pT$

Internal energy $\hat{u}=c_vT$

焓

内能

For one-dimensional flow, the Energy flux term over CS can be written as

$$\int_{CS}(h+\frac{V^2}{2}+gz)\rho\ \vec{V}\cdot d\vec{A}=\sum(h+\frac{V^2}{2}+gz)_{out}\dot{m}_{out}-\sum(h+\frac{V^2}{2}+gz)_{in}\dot{m}_{in}$$

(FW3.68)

$$\dot{Q}-\dot{W}_{shaft}-\dot{W}_{v}=\frac{\partial}{\partial t}\int_{CV}(\hat{u}+\frac{V^2}{2}+gz)\rho dV+\sum(h+\frac{V^2}{2}+gz)_{out}\dot{m}_{out}-\sum(h+\frac{V^2}{2}+gz)_{in}\dot{m}_{in}$$

One-Dimensional Steady Flow Energy Equation:

$$\dot{m}_{out}=\dot{m}_{out}=\dot{m}$$

$$\dot{Q}-\dot{W}_{shaft}-\dot{W}_{v}=(h+\frac{V^2}{2}+gz)_2\dot{m}-(h+\frac{V^2}{2}+gz)_1\dot{m} \qquad (3.69)$$

Introduce $h=\hat{u}+pv$ $=\hat{u}+p/\rho$ to above equation and divide both sides by $\dot{m}g$

Then, we obtain

$$(\frac{p_1}{\rho_1 g}+\frac{V_1^2}{2g}+z_1)=(\frac{p_2}{\rho_2 g}+\frac{V_2^2}{2g}+z_2)-\frac{\dot{Q}}{\dot{m}g}+\frac{\dot{W}_{\text{shaft}}}{\dot{m}g}+\frac{\dot{W}_{\text{viscous}}}{\dot{m}g}$$

$$\dot{W}_{\text{shaft}} = \dot{W}_{\text{turbine}} - \dot{W}_{\text{pump}}$$

$$h_{\text{friction}} = \dot{W}_{\text{viscous}} / \dot{m}g;$$

$$h_{\text{turbine}} = \dot{W}_{\text{turbine}} / \dot{m}g;$$

$$h_{\text{pump}} = \dot{W}_{\text{pump}} / \dot{m}g;$$

For adiabatic flow, then we have

$$\left(\frac{p_1}{\rho_1 g} + \frac{V_1^2}{2g} + z_1\right) = \left(\frac{p_2}{\rho_2 g} + \frac{V_2^2}{2g} + z_2\right) + h_{\text{friction}} - h_{\text{pump}} + h_{\text{turbine}} \tag{3.73}$$

$$(\frac{p_1}{\rho_2 g}+\frac{V_1^2}{2g}+z_1)=(\frac{p_2}{\rho_2 g}+\frac{V_2^2}{2g}+z_2)+h_{\text{friction}}-h_{\text{pump}}+h_{\text{turbine}} \quad (3.73)$$

$\frac{p_1}{\rho_2 g}+\frac{V_1^2}{2g}+z$ ------- Total (mechanical) head

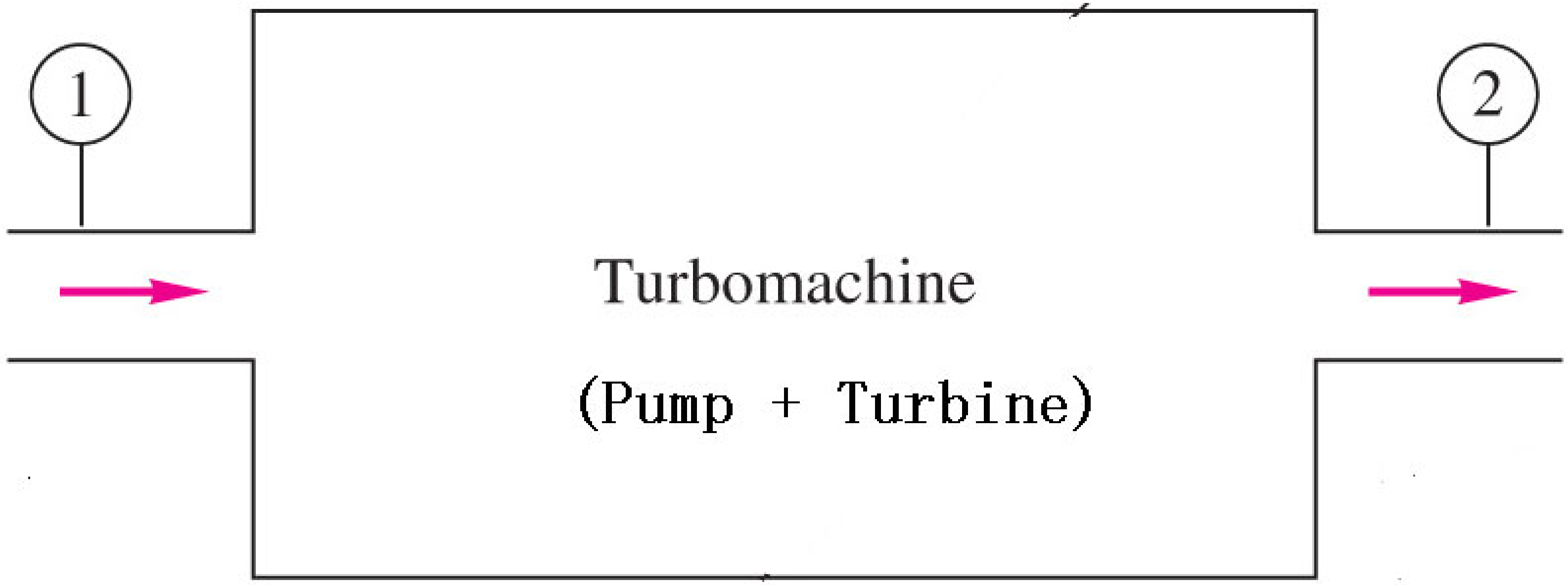
$\frac{p_1}{\rho_2 g}$ --------- Pressure head

$\frac{V_1^2}{2g}$ --------- Velocity head

z --------- Potential head

h_{friction} --------- Friction head loss

"HEAD" --- using length dimension to express energy , $\rho g h$



$$\left(\frac{p_1}{\rho_2 g}+\frac{V_1^2}{2g}+z_1\right)=\left(\frac{p_2}{\rho_2 g}+\frac{V_2^2}{2g}+z_2\right)+h_{\text{friction}}-h_{\text{pump}}+h_{\text{turbine}} \tag{3.73}$$

考虑此方程与Bernoulli's equation 的关系？

$$(\frac{p_1}{\rho_2 g}+\frac{V_1^2}{2g}+z_1)-(\frac{p_2}{\rho_2 g}+\frac{V_2^2}{2g}+z_2)$$

$$= h_{\text{friction}} = \text{Total (mechanical) head loss}$$

$$\cancel{\dot{Q}} - \dot{W}_{shaft} - \cancel{\dot{W}_v} = (h + \frac{V^2}{2} + gz)_2 \dot{m}_1 - (h + \frac{V^2}{2} + gz)_1 \dot{m}_2 \qquad (3.69)$$

Further, if the flow process is specified as

1) Steady / adiabatic;
2) one inlet / outlet;
3) Uniform
4) Frictionless

We then obtain the shaft power formula for rotating machine (旋转机械轴功率公式)

$$-\dot{W}_{shaft} = \dot{m}[(h + \frac{v^2}{2} + gz)_2 - (h + \frac{v^2}{2} + gz)_1]$$

Enthalpy	$h = \hat{u} + pv = c_p T$	焓
Internal energy	$\hat{u} = c_v T$	内能

Chapter Summary

Reading textbook page 199 for a brief summary of this chapter is strongly recommended.

Basic concepts (基本概念)

System; control volume and control surface;

Extensive property / total (gross) properties 广义物性;

Intensive property / property per unit mass单位质量物理量/物量强度/物量分布函数; volume flux (体积通量), volume flow rate (质量流量), mass flux (质量通量) and mass flow rate (质量流量);

Static pressure (静压) = thermodynamic pressure. It is the pressure measured by means of the device, which is relatively static to the flow (theoretically there is no influence to the flow)

Stagnation pressure（滞止压强、全压): decelerate the flow to zero velocity without friction and the corresponding pressure is stagnation pressure for the flow.

Basic equations (基本方程)

Reynolds transport theorem雷诺输运方程

$$\frac{d}{dt}(B_{\text{syst}}) = \frac{d}{dt}\left(\int_{SYS} \beta\rho\ \mathrm{dV}\right) = \frac{d}{dt}\left(\int_{CV} \beta\rho\ \mathrm{dV}\right) + \int_{CS} \beta\rho(\vec{\mathbf{V}}.\vec{\mathbf{n}})dA$$

(Frank white, eq. 3.12)

Conservation equations for CV控制容积守恒方程

$$\frac{d}{dt}\left(\int_{CV} \rho\ dV\right) + \int_{CS} \rho\ (\vec{\mathbf{V}}.\vec{\mathbf{n}})\ dA = 0$$

----- 质量守恒

$$F_{Sx} + F_{Bx} = \int_{CV} \frac{\partial}{\partial t}(u\rho)\ \mathrm{dV} + \int_{CS} u\rho(\vec{V}.\vec{n})dA$$

$$F_{Sy} + F_{By} = \int_{CV} \frac{\partial}{\partial t}(v\rho)\ \mathrm{dV} + \int_{CS} v\rho(\vec{V}.\vec{n})dA$$

$$F_{Sz} + F_{Bz} = \int_{CV} \frac{\partial}{\partial t}(w\rho)\ \mathrm{dV} + \int_{CS} w\rho(\vec{V}.\vec{n})dA$$

----- 动量守恒

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

----- 伯努力方程

$$[\vec{r}\times\vec{F}_s+\int(\vec{r}\times\vec{g})\rho\, dV+\vec{T}_{shaft}]_{CV}=\frac{d}{dt}[\int_{CV}(\vec{r}\times\vec{V})\,\rho\, dV]+\int_{CS}(\vec{r}\times\vec{V})\,\rho\,(\vec{V}.\vec{n})\, dA$$

-------角动量守恒方程

$$T_{shaft}=\int_{CS}(\vec{r}\times\vec{V})\ \rho\ (\vec{V}.\vec{n})\ dA=\dot{m}(r_2 v_{t2}-r_1 v_{t1})$$

------ 叶轮机械的欧拉方程（Euler equation for turbomachinery）

$$\dot{Q}-\dot{W}_{shaft}-\dot{W}_{other}=\frac{\partial}{\partial t}\int_{CV}(\hat{u}+\frac{V^2}{2}+gz)\rho dV+\int_{CS}(h+\frac{V^2}{2}+gz)\rho\ \vec{V}\cdot d\vec{A}$$

------ 能量守恒方程

Problem solving techniques (基本技能)

应用质量守恒、动量、角动量方程、能量方程求解应用问题；
伯努力方程+ U型求解管内流动问题。

关键点:1)选控制容积

2)根据题意简化方程