Chapter 5 Dimensional analysis and similarity

量纲分析及相似

Three different ways for dealing with a fluid flow problem.

- ✓ ***Integral analysis*** considering gross effect of fluid particles in *control volume* – To obtain and solve some integral equations.
- ✓ ***Differential analysis*** for a infinitesimal *system* or control volume (local individual behaviour) – To obtain and solve differential equations
- ✓ ***Dimensional analysis*** is used in experimental study of fluid flow to identify the factors really involved in a physical phenomena, and to establish relationship between the parameters governing the flow. It is not a precise analysis and **does not provide quantitative results, but often provides some non-dimensional parameters governing the flow.**

Dimensional analysis is different from integral and differential analysis：

Integral and differential analysis are based on mathematical equations. Dimensional analysis is based on ***dimensional reasoning (dimensional homogeneity*** **量纲一致性** ***)***, that is, for an equation describing a physical situation, the two sides must be equal both numerically and dimensionally.

Main subjects

5.1 Overview of dimensional analysis

5.2 The principle of dimensional homogeneity

5.3 Buckingham Pi theorem

5.4 Dimensionless parameters and their physical significance

5.5 Flow similarity and model studies

5.1 Overview

Reasons for Dimensional Analysis (DA):

- *Experimental data-processing and correlation establishment requires DA.*

Very few real flows can be solved exactly by analytical methods alone, the development of fluid mechanics has depended heavily on experimental results. The real physical flow situation is approximated with a simplified and solvable mathematical model. Then experimental measurements, refinements in the analysis are made. Dimensional analysis can be used to correlate the experimental data to obtain minimum number of non-dimensional parameters governing the flows.

- ***To help achieve similarity in model study.***

Experimental testing of a full-size prototype is either impossible or too expensive, the only feasible way of attacking the problem is through testing in the laboratory. The model flow (模型流动) and prototype flow （原型流动）must be related by known scaling laws based on dimensional analysis. We shall investigate the conditions necessary to obtain this similarity of model and prototype flows.

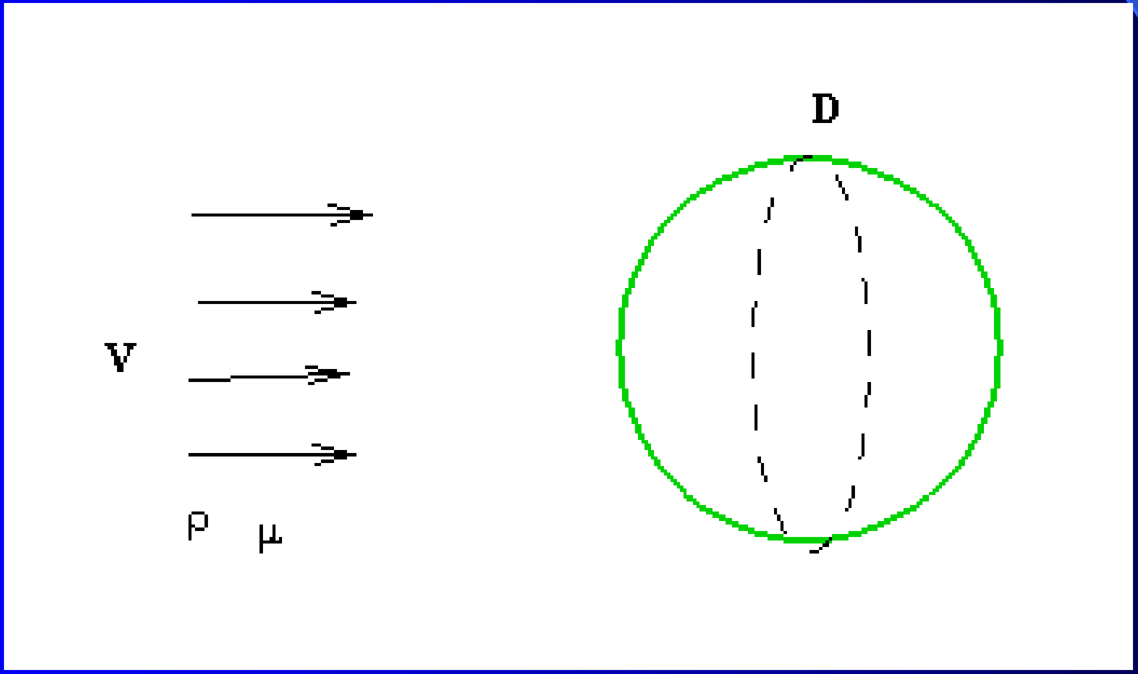
(The model is often tested in wind tunnel, water tunnel or ice tunnel)

- *Reduce the number of experimental testing.*

Experimental work in the laboratory is both time-consuming and expensive. The goal is to obtain the most information from the fewest experiments. Dimensional analysis is an important tool that often helps us to achieve this goal.

Example – using fewest experiments

Most phenomena in fluid mechanics depends on geometric and flow parameters in a complex way. For instance, we consider the drag force on a stationary smooth sphere immersed in a uniform stream.



To determine the drag experimentally, we need know how to conduct the experiments. The drag force, F, is then given by the symbolic equation

$$F = f(D, V, \rho, \mu)$$

Let us now imagine a series of experiments to determine the dependence of F on variable, D, V, μ and ρ.

To obtain a curve ($F \sim V$), we often need do tests at 10 different values of the independent variable (V) by fixing value the other parameters ($D, \rho,$ and μ).

Illustration

Item / Curve	variable	fixed	Num. of testing	
$F \sim V$	V	D, ρ, μ	10	
$F \sim V, D$	V, D	ρ, μ	10^2	
$F \sim V, D, \rho$	V, D, ρ	μ	10^3	
$F \sim V, D, \rho, \mu$	$F \sim V, D, \rho, \mu$	----	10^4	

Assuming each test takes a hour and we work 8 hours per day, the testing will require **2.5 years** to complete. Moreover, there also would be some difficulty in presenting the data.

Using Dimensional Analysis (DA)-

Fortunately we can obtain meaningful results with much less efforts by using dimensional analysis. All the data for the drag force on a smooth sphere can be represented by a functional relation between two nondimensional parameters (drag coefficient and Re) given by

$$\frac{F}{\rho V^2 D^2} = f_1\left(\frac{\rho V D}{\mu}\right)$$

Nature of DA

$$\frac{F}{\rho V^2 D^2} = f_1\left(\frac{\rho V D}{\mu}\right)$$

The relation of the function still must be determined experimentally. We could establish the nature of the function with only 10 tests, rather than 10^4 tests.

In doing experiment, we do not need to vary separately the values of fluid density and viscosity, and we need not make 10 spheres of different diameters either. *Instead we only need change the ratio,* $\rho VD/\mu$ *, This can be achieved by any convenient way, for example simply by changing the velocity.*

By means of DA (Dimensional Analysis), we can achieve the same goal with

- Less testing
- Easier setting up of model test
- Easier data-processing --- obtain correlation

Dimensional analysis is essential in experimental study:

- *The objective* is to obtain the non-dimensional parameters that governs the flow.
- *The nature* is to rearrange a number of dimensional variables into a smaller number of dimensionless groups.
- *Relies on* Buckingham Pi theorem

5.2 Principle of Dimensional Homogeneity (PDH)

Dimensional analysis is based on PDH.

If an equation truly expresses a proper relationship between variables in a physical process, it will be dimensionally homogeneous; i.e. each of its additive terms will have the same dimensions.

E.g.

$$\frac{p}{\rho}+\frac{V^2}{2}+gz=\text{const}$$

The following are noted

- Dimensional variables: all have dimensions and can be nondimensionalized by DA;
- Dimensional constants: all have dimensions but often used in non-dimensionlization of variables;
- Pure constants have no dimensions;
- Angles and revolutions are dimensionless.
- Counting numbers are dimensionless.

5.3 Buckingham Pi theorem

There are several methods of reducing a number of dimensional variables into a smaller number of dimensionless groups (eg. Rayleigh method, Buckingham theorem etc). The ***Buckingham Pi theorem*** allow us to develop the important nondimensional parameters quickly and easily from a number of dimensional variables.

Background knowledge

- Dimensions are properties that can be measured. Units are the standard elements we use to quantify these dimensions.
- Primary dimension: In fluid mechanics we are generally interested in four primary dimensions by which all quantities may be described, i.e.

$$L,\ M,\ t,\ \text{and}\ \theta$$

- Secondary units are used in general practice and they are made from combinations of these primary units. E.g. acceleration,

$$m/s^2$$

Primary dimension used in Fluid mechanics

Quantity	SI Unit	Dimension
length	Meter, M	L
mass	Kilogram, kg	M
time	Second, s	T
temperature	Kelvin, K	θ

Buckingham Pi theorem

Given a physical problem in which the dependent parameter is a function of $n-1$ independent parameters, we may express the relationship among the variables in functional form as

$$q_1 = f(q_2, q_3, \ldots\ldots q_n)$$

Alternatively, we can express the functional relationship in the form

$$g(q_1, q_2, q_3 \ldots\ldots q_n) = 0$$

Then the n parameters may be grouped into $n-j$ independent dimensionless ratios, given in a functional relation

$$G(\pi_1, \pi_2 \ldots\ldots \pi_{n-j}) = 0$$

OR

$$\pi_1 = G_1(\pi_2 \ldots\ldots \pi_{n-j})$$

$$\pi_1 = G_1(\pi_2 \ldots\ldots \pi_{n-j})$$

Where

n =	number of the original flow parameters;
n-j =	number of dimensionless flow parameters;
j =	number of flow parameters reduced through dimensional analysis, and it is taken as the number of primary dimension involved in all flow parameters.

It is noted that

- The theorem does not predict the functional relation of G or G_1. The functional relation among the $n-j$ independent dimensionless parameters must be determined experimentally.
- It is noted that the n-j parameters $\pi_1, \pi_2 \ldots\ldots \pi_{n-j}$ are not unique. One may be found by combination of others.

How to determine the dimensionless parameters for a flow problem?

Procedures:

Step 1. List all (dependent and independent)parameters involved, n

Step 2. Identify the involved fundamental (primary) dimensions, $MLT\theta$

Step 3. List the dimensions of all parameters in terms of primary dimensions

Step 4. Select j *repeating variables** from the list of parameters, and determine the structure of π

$$\pi = [\text{power product of repeating parameters}] \times [\text{one of the remaining paramters in turn}]$$

Where j represents the number of primary dimensions involved; repeating parameters are those will appear in each of the π parameters.

Step 5. Set up and solve the above dimensional equations to obtain all non-dimensional parameters $\pi_1, \pi_2 \ldots\ldots \pi_{n-j}$.

Step 6. Check to see whether or not each group obtained is dimensionless

***Repeating parameters (or scaling parameters) and remaining parameters (basic parameters)*

Scaling (or repeating) variables are these used to define dimensionless variables and will appear in all the dimensionless group, while the remained are basic parameters (variables)

There are different options for scaling parameters, page 307.

The selection of scaling variables is largely dependent on the user, but there are some guidelines:

- They must not form a dimensionless group among themselves, but adding one more will form a dimensionless quantity.
- Do not select output variables for your repeating or scaling variables;
- Select popular scaling variables and they will appear in all the dimensionless group.

Example -

The drag for a smooth sphere is given by the mathematical function

$$F = f(\rho, V, D, \mu)$$

Find the corresponding set of dimensionless groups.

The solution includes the above mentioned 5 steps:

1) List all (dependent and independent)parameters involved

F, V, D, ρ, μ (number of parameters involved, $n = 5$)

2) Count the fundamental (primary) dimensions

M, L, and t (3 primary dimensions) $n - j = 5 - 3$

3) List dimensions of all the parameters involved (in terms of primary dimensions)

F	V	D	ρ	μ
$\dfrac{ML}{t^2}$	$\dfrac{L}{t}$	L	$\dfrac{M}{L^3}$	$\dfrac{M}{Lt}$

4) Selecting repeating parameters (equals to the number of primary dimensions but including all primary dimensions)

$$\rho,\ V,\ D$$

and the remaining parameters are

$$F \text{ and } \mu$$

5) Setting up dimensional equations for $n - j = 5 - 3$ dimensionless group. Each is constituted by the power product of the repeating variable times one of the remaining parameter in turn

$$\pi_1 = \rho^a V^b D^c F = (\frac{M}{L^3})^a (\frac{L}{t})^b (L)^c (\frac{ML}{t^2}) = M^0 L^0 t^0$$

$$\pi_2 = \rho^d V^e D^f \mu = (\frac{M}{L^3})^d (\frac{L}{t})^e (L)^f (\frac{M}{Lt}) = M^0 L^0 t^0$$

Equating the exponents of M, l, and t results in

$$\begin{aligned} &M: a+1=0 \\ &L: -3a+b+c+1=0 \\ &t: -b-2=0 \end{aligned} \Rightarrow \begin{aligned} &a=-1 \\ &c=-2 \\ &b==-2 \end{aligned} \Rightarrow \pi_1 = \frac{F}{\rho V^2 D^2}$$

$$\begin{aligned} &M: d+1=0 \\ &L: -3d+e+f-1=0 \\ &t: -e-1=0 \end{aligned} \Rightarrow \begin{aligned} &d=-1 \\ &f=-1 \\ &e=-1 \end{aligned} \Rightarrow \pi_2 = \frac{\mu}{\rho VD}$$

5) Check the dimensions of π_1 and π_2

Original description of drag by physical parameters

$$F = f(\rho, V, D, \mu)$$

Rewritten as

$$\pi_1 = f(\pi_2)$$

$$\text{Where } \pi_1 = \frac{F}{\rho V^2 D^2}, \text{ and } \pi_2 = \frac{\mu}{\rho V D}$$

The function relation f must be determined experimentally.

5.4 Dimensionless parameters and their physical significance

The most commonly used non-dimensional parameters are introduced together with their physical significance, such as Reynolds number, Mach number, Fround number, Euler number.

- In fluid mechanics and related areas, several non-dimensional parameters are used very frequently, and each has been given the name of a scientist or engineer, who usually pioneered its use. Understanding the physical significance will help us to investigate the flow phenomena.

- Inertia forces are important in most fluid mechanics problems, and the non-dimensional parameters are defined as the ratio of the inertia force to each of the other force, thus we now write the expressions in terms of dimensions for the forces.

Forces in terms of dimension

- Forces in fluid mechanics can be classified into surface force and body force, involve inertia force, viscous force, pressure force, gravitational force, surface tension force, and elasticity force (惯性力、粘性力、压力、重力、表面张力、弹性力).

- They are expressed in terms of dimensions as follows

$$\text{Inertia force} = \rho L^3 \frac{V}{t} \propto \rho L^2 V^2 \qquad \text{B}$$

$$\text{Viscous force} = \tau A \propto \mu \frac{du}{dy} A \propto \mu \frac{V}{L} L^2 \propto \mu V L \qquad \text{S}$$

$$\text{Pressure force} = (\Delta p) A \propto (\Delta p) L^2 \qquad \text{S}$$

$$\text{Gravity force} = mg \propto g \rho L^3 \qquad \text{B}$$

Force due to elasticity

$$= E_v A \propto E_v L^2 \qquad \text{S}$$

(Where $E_v = \dfrac{dp}{d\rho / \rho}$, Modulus of elasticity or bulk modulus used to express compressibility for solids, liquids and also gases)

Dimensionless parameters

Reynolds number

In 1880s, Osborne Reynolds, the British engineer, studied the transition between laminar and turbulent flow regimes in a tube. He discovered that the parameter is a criterion by which the flow regime may be determined. Later experiments have shown that Reynolds number is a key parameter for other flow cases as well.

$$\mathrm{Re} = \frac{\text{inertia force}}{\text{viscous force}} = \frac{\rho V L}{\mu} = \frac{V L}{\nu}$$

Where L is a characteristic length of flow field geometry.

Euler number (pressure coefficient)

In aerodynamic study and other model testing, it is convenient to present data in dimensionless form.

$$Eu = \frac{\text{pressure force}}{\text{inertia force}} = \frac{\Delta p}{\frac{1}{2}\rho V^2} = \frac{p - p_\infty}{\frac{1}{2}\rho V^2}$$

（pressure coefficient: 压力系数）

Where Δp represent the local pressure subtracting the free stream (or inlet) pressure ρ and V are properties of the freestream flow.

For free-surface flow

$$Eu = \frac{p - p_v}{\rho V^2}$$

（Cavitation number 汽蚀数; Pv ---vapor pressure）

The ratio has been named after Leonhard Euler, the Swiss mathematician.

Froude number

$$Fr = \left(\frac{\text{inertia force}}{\text{gravity force}}\right)^{1/2} = \frac{V}{\sqrt{gL}}$$

OR

$$Fr = \frac{\text{inertia force}}{\text{gravity force}} = \frac{V^2}{gL}$$

(FW, page311,page308)

William Froude was a British naval architect, and discovered that the parameter was significant for flows with free surface effect.

Mach number

In the 1870s, the Austrian physicist Ernst Mach introduced the parameter

$$Ma = \frac{V}{a}$$

Where V is the flow speed and a is the local speed of sound. Mach number is a key parameter that describes compressibility effects in a flow.

(1)Mach number may be interpreted as a ratio of inertia forces and forces due to elasticity (resisting the compression)

$$Ma^2 = (\frac{V}{a})^2 = \frac{V^2}{dp/d\rho} = \frac{V^2}{E_v/\rho} = \frac{\rho V^2 L^2}{E_v L^2}$$

$$= \frac{\text{inertia force}}{\text{force due to elasticity}}$$

(2) For engineering applications the following rule can be used

Mach number ≤ 0.3 , incompressible flow;

Mach number ≥ 0.3, compressible flow.

(3) Mach number of gases, liquids and solids

$$Ma = \frac{V}{a}$$

For gas

$$a = \sqrt{\left.\frac{dp}{d\rho}\right|_s} = \sqrt{kRT}$$

For solid and liquid

$$a = \sqrt{E_v / \rho}$$

Where the Bulk compressibility Modulus or Modulus of elasticity is given by

$$E_v = \frac{dp}{d\rho / \rho}$$

Summary of Dimensionless parameters

Reynolds number

$$\mathrm{Re} = \frac{\text{inertia force}}{\text{viscous force}} = \frac{\rho V L}{\mu} = \frac{VL}{\nu}$$

Euler number
(pressure coefficient)

$$Eu = \frac{\text{pressure force}}{\text{inertia force}} = \frac{\Delta p}{\frac{1}{2}\rho V^2}$$

Froude number

$$Fr = \left(\frac{\text{inertia force}}{\text{gravity force}}\right) = \frac{V^2}{gL}$$

Mach number

$$Ma^2 = \left(\frac{V}{a}\right)^2 = \frac{V^2}{dp/d\rho} = \frac{V^2}{E_v/\rho} = \frac{\rho V^2 L^2}{E_v L^2} = \frac{\text{inertia force}}{\text{force due to elasticity}}$$

From above analysis we understand that the effect of forces (such as pressure force, gravity force, and viscous force, etc) can be represented by the corresponding non-dimensional parameters. Such a knowledge is very useful flow similarity and model studies.

5.5 Flow similarity and model studies

- *Experimental work plays very important roles in scientific studies.* But experimental testing of a full-size prototype is often impossible and expensive. The only feasible way of attacking the problem is through model testing in the laboratory. For instance research on moon-landing and mars-landing are all based on the earth model test, the three-gorge project also has required much laboratory model testing, before constructing a real-large-scale machine, small model testing were often needed. （模拟真实环境、模型可放大也可缩小、介质可以相同也可以不同）

  How to set up the laboratory testing? Can we set up the testing arbitrarily?

- ***Objective of model testing***

In model testing, the objective of model testing is to predict the prototype behavior from measured data on the model, we must be able to predict the forces, moments, and dynamic loads that would exist on the full-scale prototype based on the obtained data from model testing.

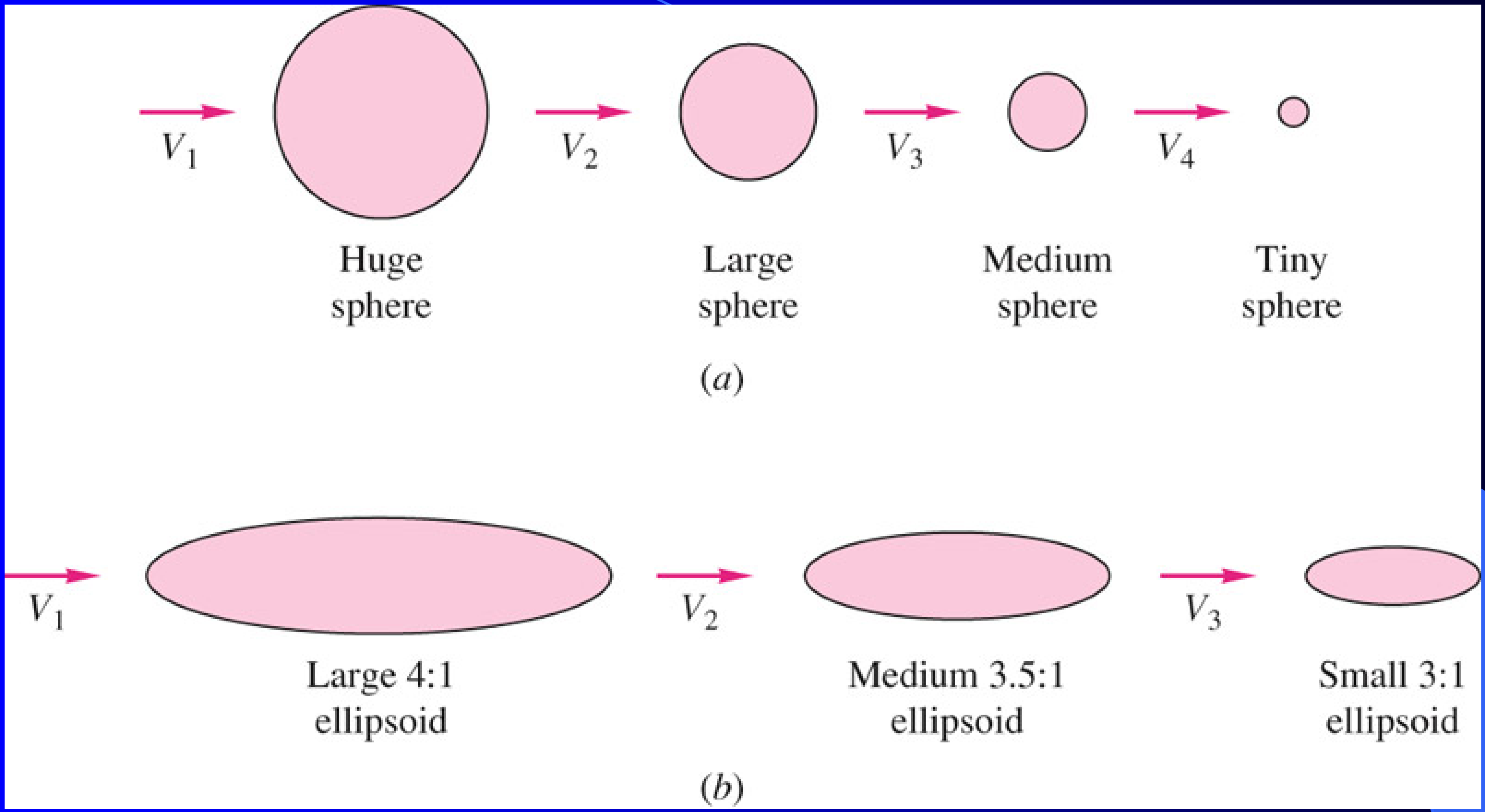
Obviously we cannot run a test on any model. The model flow and the prototype flow must be related by some scaling laws, i.e. they must be similar.

- *Condition of flow similarity* -

Geometric similarity, kinematic similarity, dynamic similarity

  - **Geometric similarity 几何相似**

    The most obvious requirement is that the model and ptototype must be geometrically similar. Geometric similarity requires that the model and prototype have the same shape, and that all the linear dimensions of the model and prototype must be related by a constant scale factor and all the angles are equal.



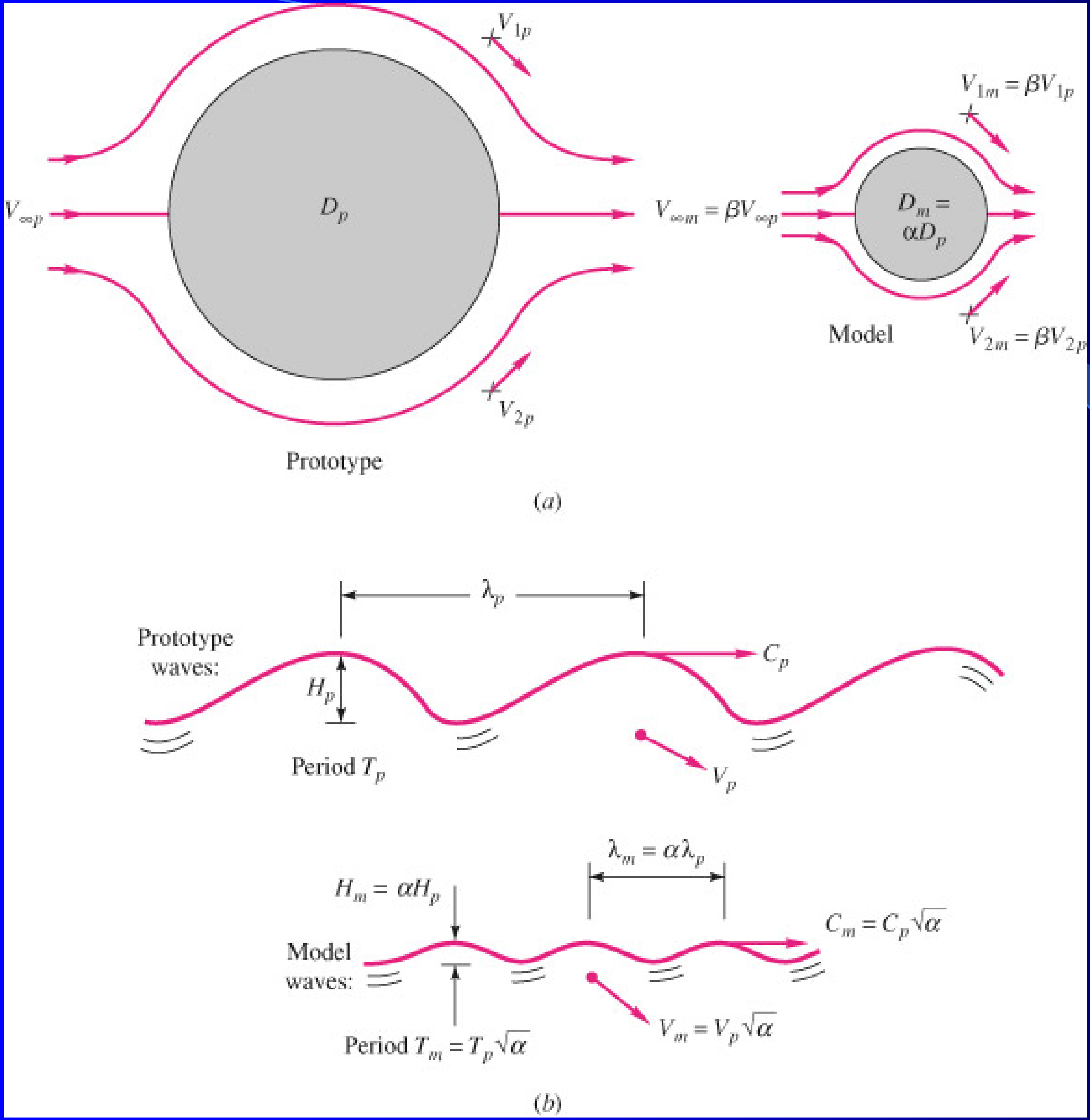
Geometric similarity

● Kinematic similarity 运动相似

Model and prototype flows must be kinematically similar. Two flows are kinematically similar when the velocities at corresponding points are in the same direction and are related in magnitude by a constant scale factor.

According to the definitions of streamlines, for kinematically similar flows, the streamlines patterns must also be related by the constant factor (similar). The geometric boundary can be treated as the bounding streamlines, thus flows that are kinematically similar *must be geometrically similar.*

Kinematic similarity requires that flow regimes must be the same for model and prototype

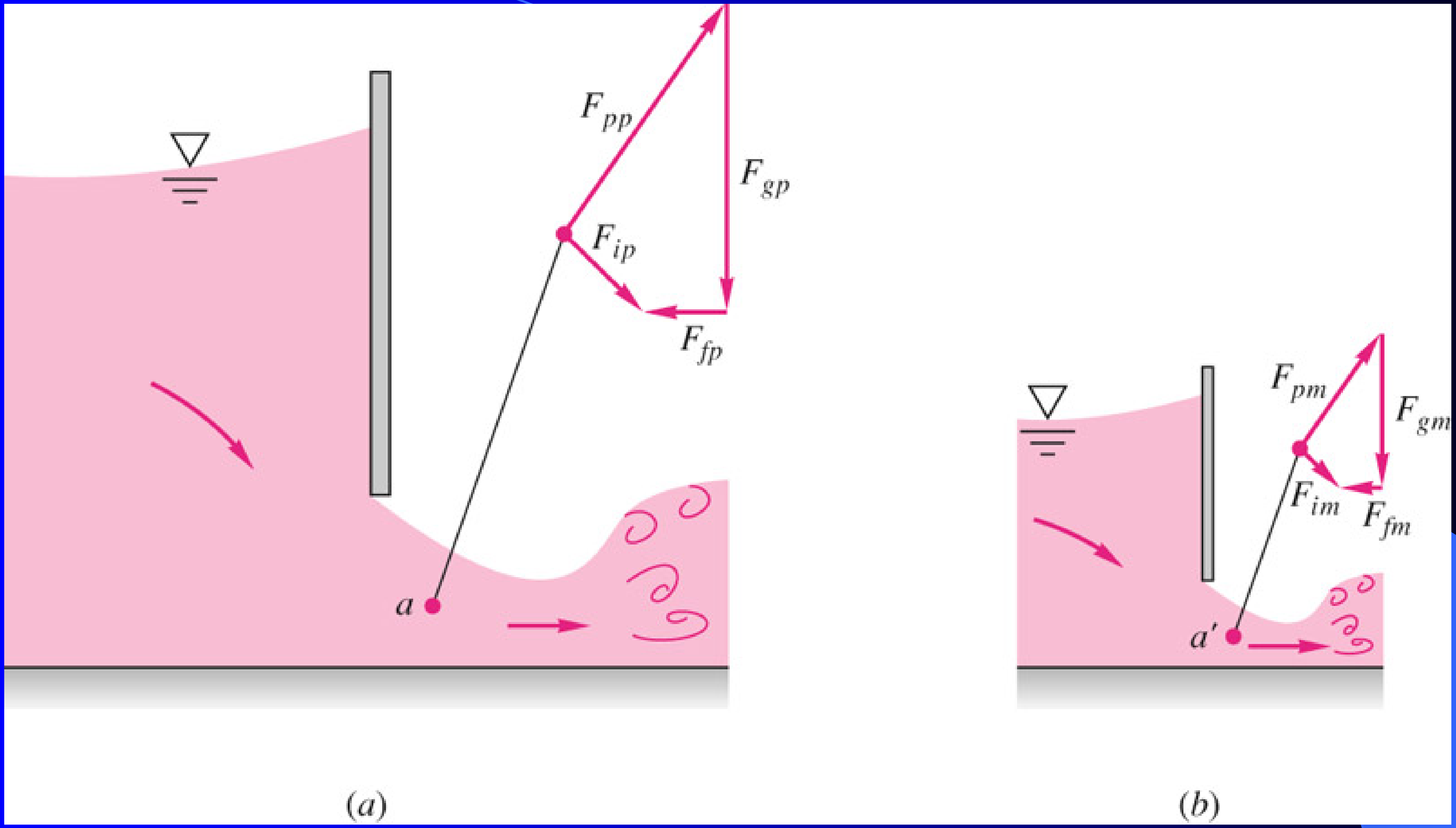


Kinematic similarity

- **Dynamic similarity 动力相似**

Motion is resulted from forces, if **the force diagrams are similar for two flows,** and the flows will be dynamically similar.

For dynamic similar flows, <u>each type of forces (e.g. gravity force, pressure force, and friction force) are</u> <u>parallel in direction, and are related in magnitude by a constant scale factor at all corresponding points for distributed forces.</u>



Dynamic similarity

If force diagrams are similar, then we have

$$\left(\frac{F_p}{F_i}\right)_{\text{proptype}} = \left(\frac{F_p}{F_i}\right)_{\text{model}}$$

$$\left(\frac{F_g}{F_i}\right)_{\text{proptype}} = \left(\frac{F_g}{F_i}\right)_{\text{model}}$$

$$\left(\frac{F_{viscous}}{F_i}\right)_{\text{proptype}} = \left(\frac{F_{viscous}}{F_i}\right)_{\text{model}}$$

$$\left(\frac{\vec{F}_{pressure}}{F_{inertia}}\right)_p; \quad \left(\frac{\vec{F}_{gravity}}{F_{inertia}}\right)_p; \quad \left(\frac{\vec{F}_{viscous}}{F_{inertia}}\right)_p;$$

$$\parallel \qquad \parallel \qquad \parallel$$

$$\left(\frac{\vec{F}_{pressure}}{F_{inertia}}\right)_m; \quad \left(\frac{\vec{F}_{gravity}}{F_{inertia}}\right)_m; \quad \left(\frac{\vec{F}_{viscous}}{F_{inertia}}\right)_m;$$

Eu

$$\frac{1}{\mathrm{Fr}}$$

$$\frac{1}{\mathrm{Re}}$$

Flow similarity 流动相似

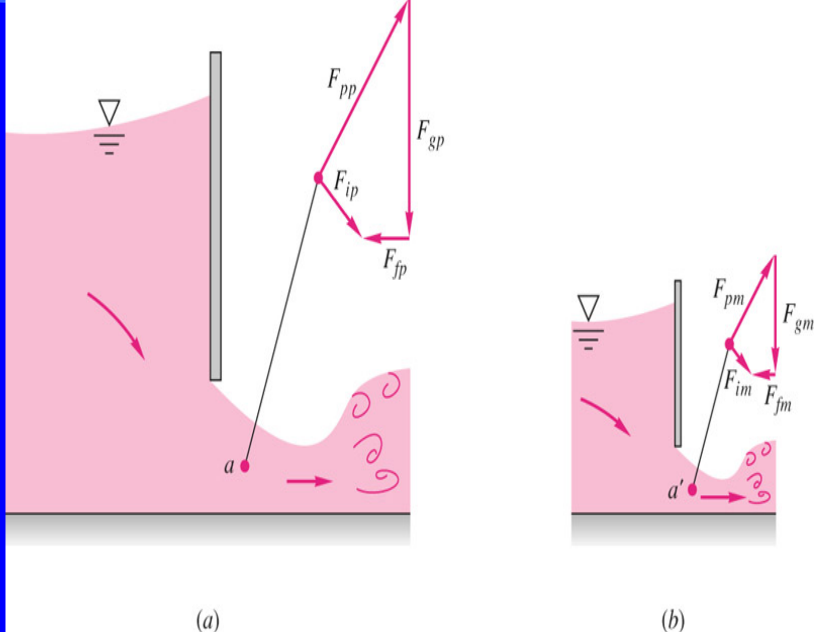
- The requirements for dynamic similarity are the most restrictive: two flows must be of both geometric and kinematic similarity to be dynamically similar. Conversely, <u>if the two flows are dynamically similar, they must be geometric and kinematically similar.</u>

- Flow similarity refers to dynamic similarity.

Complete / incomplete similarity

- ***Complete similarity*** - To establish the conditions required for complete dynamic similarity, all forces that are important in flow situation must be considered. Thus the effects of viscous forces, pressure forces, surface tension forces, ect. must be considered. For model studies, test condition must be established that all important forces are related by the same scale factor between model and prototype flows. What are the conditions for ensuring dynamic similarity between model and prototype flows?

  As dimensionless parameters can be viewed as ratio of forces, to achieve complete similarity between geometrically similar flows, we must duplicate (复制) all (not one) of these dimensionless parameters governing the flows.



$$\left(\frac{\vec{F}_{pressure}}{F_{inertia}}\right)_p; \quad \left(\frac{\vec{F}_{gravity}}{F_{inertia}}\right)_p; \quad \left(\frac{\vec{F}_{viscous}}{F_{inertia}}\right)_p;$$

$$\| \qquad \| \qquad \|$$

$$\left(\frac{\vec{F}_{pressure}}{F_{inertia}}\right)_m; \quad \left(\frac{\vec{F}_{gravity}}{F_{inertia}}\right)_m; \quad \left(\frac{\vec{F}_{viscous}}{F_{inertia}}\right)_m;$$

Eu

$$\frac{1}{\mathrm{Fr}}$$

$$\frac{1}{\mathrm{Re}}$$

For a simple flow, complete similarity can be produced.

Example – the drag force on a smooth sphere $F = f(D, V, \rho, \mu)$

Using Pi theorem we obtained the functional relation

Based on DA, we have

$$\frac{F}{\rho V^2 D^2} = f_1\left(\frac{\rho V D}{\mu}\right)$$

Complete similarity requires

$$\left(\frac{\rho V D}{\mu}\right)_{\text{model}} = \left(\frac{\rho V D}{\mu}\right)_{\text{prototype}}$$

$$\left(\frac{F}{\rho V^2 D^2}\right)_{\text{model}} = \left(\frac{F}{\rho V^2 D^2}\right)_{\text{prototype}}$$

Data obtained in the model study can be used to predict the drag on the full-scale prototype.

It is noted that

(1) the actual drag force on the object may be not the same in model and prototype flow, but its dimensionless value equals.

(2) The two tests can be run using different fluids, if it is necessary.

Incomplete similarity

In many model studies, to achieve complete similarity requires duplication of several dimensionless parameters. In some cases, complete similarity between model and prototype could not be maintained, because of conflict (discrepancy)矛盾 of the dimensionless parameters. We can only ensure the equality of the most important ratios to achieve an incomplete similarity.

Example - the resistance (drag force) of a ship cruising (巡航) in the sea

Resistance on a ship arises from viscous forces and surface wave resistance (gravity forces), their effect can be represented by Reynolds number and Froude number.

Complete similarity requires both Reynolds and Froude numbers be equal between the model and prototype flows

$$Fr_m = \frac{V^2{}_m}{gL_m} = Fr_p = \frac{V^2{}_p}{gL_p} \Rightarrow \frac{V_m}{V_p} = \left(\frac{L_m}{L_p}\right)^{1/2}$$

$$\mathrm{Re}_m = \frac{V_m L_m}{\nu_m} = \mathrm{Re}_p = \frac{V_p L_p}{\nu_p} \Rightarrow \frac{\nu_m}{\nu_p} = \frac{V_m L_m}{V_p L_p}$$

Combine the above two formulas, we have

$$\frac{\nu_m}{\nu_p} = \left(\frac{L_m}{L_p}\right)^{3/2}$$

For ship model tests, a typical length scale may be $L_m / L_p = 1/100$ then $\nu_m / \nu_p = 1/1000$ where ν_p represents the kinematic viscosity of water, from appendix of FW's book, we know mercury is the only liquid with kinematic viscosity less than water, but it is only about an order less.

There isn't such a fluid to keep equality of both non-dimensional parameters !!

How to solve the above problem? *An incomplete similarity can be achieved.* Since wave effect is difficult to be obtained analytically, we can firstly keep the equality of Froude number to obtain resistance of wave through the model test. Viscous effect may be estimated analytically, which then is used to modify the result of resistance obtained from the model test.

Example 5.3 on page 314

P5.73 on page 348 (作业题)