Chapter 8
Incompressible inviscid flow
-- Potential flow

（不可压缩理想流体流动）

Background:

All real fluids possess viscosity. However, there are some fluid flow cases, in which it is quite reasonable to neglect the effects of viscosity. Many engineering flows can be divided into two region a thin layer of near boundary flow being viscous; inviscid main flow stream. Thus it is useful to investigate the dynamics of an ideal fluid that is incompressible and has zero viscosity. The analysis of ideal fluid motion is simpler than for viscous flows since no shear stresses are present in inviscid flow

In this chapter we firstly consider the frictionless flow, and then turn to the more restrictive flow – frictionless and irrotational flow. In both cases, Euler's equation applies:

Question –

How to define mathematically the incompressible and inviscid fluids involved in this chapter ?

$$\rho = \text{constant} \quad \text{and} \quad \mu = 0$$

OR

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{and} \quad \mu = 0$$

Forces - for invisid flow, there is no shear stresses, the normal stress at a point is the same in all directions, and it is the negative of the thermodynamic pressure

$$\tau_{xy} = \tau_{yx} = \cancel{\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)}$$

$$\tau_{yz} = \tau_{zy} = \cancel{\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)}$$

$$\tau_{zx} = \tau_{xz} = \cancel{\mu\left(\frac{\partial u}{\partial z} + \frac{\partial W}{\partial x}\right)}$$

$$\sigma_{xx} = -p - \cancel{\frac{2}{3}\mu\nabla\cdot\vec{V}} + \cancel{2\mu\frac{\partial u}{\partial x}}$$

$$\sigma_{yy} = -p - \cancel{\frac{2}{3}\mu\nabla\cdot\vec{V}} + \cancel{2\mu\frac{\partial v}{\partial y}}$$

NS equation for impressible Newtonian Fluids

$$\rho\frac{Du}{Dt}=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\frac{Dv}{Dt}=\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\frac{Dw}{Dt}=\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

$$\Downarrow$$

Euler eqaution

$$\rho\frac{d\vec{V}}{dt}=\rho\vec{g}-\nabla p$$

Main subject

8.1 Unsteady Bernoulli's equation

8.2 Irrotational flows (FW 4.8)

8.2.1 Bernoulli's equation applied to irrotaional flows (FW4.9)

8.2.2 Velocity potential (FW4.9,FW8.1)

8.2.3 Stream function (FW4.7, FW8.1)

8.2.4 Relationship between stream function and velocity potential (FW8.1)

8.3 Elemental plane flows and their superposition

6.3.1 Elementary plane flows

6.3.2 Superposition of elementary plane flows

8.1 Unsteady Bernoulli's equation derived from differential form of Euler equation

Background - In the previous chapter, we established Bernoulli's equation for incompressible, frictionless (inviscid) along streamline but the flow needs to be steady.

Objective – To establish unsteady Bernoulli's equation along streamline.

Starting point - Euler equation. For incompressible inviscid (frictionless) flow, Navier-Stokes equations can be reduced to Euler equation (momentum equation),

$$\rho \frac{d\vec{V}}{dt} = \rho \vec{g} - \nabla p$$

Euler equation

$$\rho \frac{d\vec{V}}{dt} = \rho \vec{g} - \nabla p$$

If the z coordinate is directed vertically, we can use the gradient of coordinate to express the unit vector $\vec{k}$,

Then

$$\nabla z = \cancel{\frac{\partial z}{\partial x}\vec{i}} + \cancel{\frac{\partial z}{\partial y}\vec{j}} + \frac{\partial z}{\partial z}\vec{k} = \vec{k}$$

$$\rho \vec{g} = -\rho g\, \vec{k} = -\rho g \nabla z$$

Euler's equation can be written as

$$-g\ \nabla z - \frac{1}{\rho}\nabla p = \frac{d\vec{V}}{dt} \qquad (1)$$

- **Consider an elemental length along a streamline, which is given by**

$$d\vec{s} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$

- **Take dot product with $d\vec{s}$ about equation (1), we then obtain**

$$-\frac{1}{\rho}\nabla p \cdot d\vec{s} - g\nabla z \cdot d\vec{s} = \frac{d\vec{V}}{dt}\cdot d\vec{s} \qquad (2)$$

The products are the following

$$\nabla p \cdot d\vec{s} = (\frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k}) \cdot (dx\ \vec{i} + dy\ \vec{j} + dz\ \vec{k}) = dp$$

$$\nabla z \cdot d\vec{s} = (\vec{k}) \cdot (dx\ \vec{i} + dy\ \vec{j} + dz\ \vec{k}) = dz$$

$$\frac{d\vec{V}}{dt} \cdot d\vec{s} = \frac{dV_s}{dt} ds \ = V_s \frac{\partial V_s}{\partial s} ds + \frac{\partial V_s}{\partial t} ds = V_s\ dV_s + \frac{\partial V_s}{\partial t} ds$$

- **Substitute the derived three terms into equation 2 , we have**

$$-\frac{1}{\rho} \nabla p \cdot d\vec{s} - g \nabla z \cdot d\vec{s} = \frac{d\vec{V}}{dt} \cdot d\vec{s} \qquad (2)$$

- **Then, we have**

$$-\frac{dp}{\rho} - g dz = V_s\ dV_s + \frac{\partial V_s}{\partial t} ds$$

- Integrating the above equation along a streamline from point 1 and point 2 gives

$$\int_1^2 \frac{dp}{\rho} + \frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1) + \int_1^2 \frac{\partial V_s}{\partial t} ds = 0$$

For incompressible flow

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \int_1^2 \frac{\partial V_s}{\partial t}$$

This is the unsteady Bernoulli's equation, and it stands for the following conditions:

- **Incompressible flow**
- **Frictionless flow**
- **Flow along a streamline (the constants for different streamlines are different)**

Bernoulli's equation and Euler equation are consistent! Both are valid inviscid flow !!!!!

8.2 Irrotational flows

An irrotaional flow is one in which fluid elements moving in the flow field do not undergo any rotation, mathematically

$$\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} = 0$$

OR

$$\nabla \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = 0$$

OR

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = -\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

The assumption of irrotational flow may be valid for those regions of a flow in which viscous forces are negligible, **such a region exists outside the boundary layer** in the flow over a solid surface.

8.2.1 Bernoulli's equation for irrotaional flow

Starting point – Euler's equation
$$\rho \frac{d\vec{V}}{dt} = \rho \vec{g} - \nabla p$$

If z - axis directed vertically,
$$\vec{g} = -g\vec{k}$$
$$\nabla z = \frac{\partial z}{\partial x}\vec{i} + \frac{\partial z}{\partial y}\vec{j} + \frac{\partial z}{\partial z}\vec{k} = \vec{k}$$

then, it can be rewritten as
$$-g\ \nabla z - \frac{1}{\rho}\nabla p = \frac{d\vec{V}}{dt}$$

And
$$\frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} = \frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}$$

Then
$$-g\ \nabla z - \frac{1}{\rho}\nabla p = \frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}$$

Using the vector identity (标记) ***

$$(\vec{V}\cdot\nabla)\vec{V}=\frac{1}{2}\nabla(\vec{V}\cdot\vec{V})-\vec{V}\times(\nabla\times\vec{V})$$

For irrotational flow $\vec{\omega}=\frac{1}{2}\nabla\times\vec{V}=0$,

then

$$(\vec{V}\cdot\nabla)\vec{V}=\frac{1}{2}\nabla(\vec{V}\cdot\vec{V})=\frac{1}{2}\nabla(V^2)$$

Thus Euler's equation for irrotational flow can be re-written as

$$-g\ \nabla z-\frac{1}{\rho}\nabla p=\frac{\partial\vec{V}}{\partial t}+\frac{1}{2}\nabla(V^2)$$

During the interval dt , the fluid particle moves from the position $\vec{r}$ to position $\vec{r}+d\vec{r}$; the displacement of $d\vec{r}$ is an arbitrary infinitesimal displacement in any direction.

Taking the dot product of $d\vec{r}=dx\,\vec{i}+dy\,\vec{j}+dz\,\vec{k}$ and then

$$-g\ \nabla z\cdot d\vec{r}-\frac{1}{\rho}\nabla p\cdot d\vec{r}=\frac{\partial\vec{V}}{\partial t}\cdot d\vec{r}+\frac{1}{2}\nabla(V^2)\cdot d\vec{r}$$

$$-g\ \nabla z \cdot d\vec{r} - \frac{1}{\rho}\nabla p \cdot d\vec{r} = \frac{\partial \vec{V}}{\partial t}\cdot d\vec{r} + \frac{1}{2}\nabla(V^2)\cdot d\vec{r}$$

Where

$$\nabla z \cdot d\vec{r} = \left(\frac{\partial z}{\partial x}\vec{i} + \frac{\partial z}{\partial y}\vec{j} + \frac{\partial z}{\partial z}\vec{k}\right)\cdot\left(dx\ \vec{i} + dy\ \vec{j} + dz\ \vec{k}\right) = dz$$

$$\nabla p \cdot d\vec{r} = \left(\frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k}\right)\cdot\left(dx\ \vec{i} + dy\ \vec{j} + dz\vec{k}\right) = dp$$

$$\nabla(V^2)\cdot d\vec{r} = \left(\frac{\partial V^2}{\partial x}\vec{i} + \frac{\partial V^2}{\partial y}\vec{j} + \frac{\partial V^2}{\partial z}\vec{k}\right)\cdot\left(dx\ \vec{i} + dy\ \vec{j} + dz\ \vec{k}\right) = d(V^2)$$

Introduce the three terms into the above Euler's equation, we have

$$\frac{\partial \vec{V}}{\partial t}\cdot d\vec{r} + d\left(\frac{1}{2}V^2\right) + \frac{dp}{\rho} + g\ dz = 0$$

Bernoulli's equation for irrotational flows stands, if the flow

- Frictionless (inviscid)
- Irrotational

For steady flow, the above equation is written as

$$d\left(\frac{1}{2}V^2\right)+\frac{dp}{\rho}+g\ dz=0$$

For incompressible flow

$$\frac{p}{\rho}+\frac{V^2}{2}+gz=\text{constant}$$

This Bernoulli's equation can be used between any two points in the flow field, if the flow is

- Frictionless (inviscid)
- Irrotational
- Steady
- Incompressible

Bernoulli's equation has been extended to the entire flow field for irrotational flows.

Significance of BE for imcompressible irrotational flow:

N. S. equation:

$$\rho\frac{Du}{Dt}=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\frac{Dv}{Dt}=\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\frac{Dw}{Dt}=\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

$$\Downarrow$$

Bernoulli's equation:

$$\frac{p}{\rho}+\frac{V^2}{2}+gz=\text{constant}$$

8.2.2 Velocity potential

Definition

For an irrotational flow. i.e. $\nabla \times \vec{V} = 0$, a scalar function ϕ must exist, and $\vec{V} = \nabla \phi$

Where ϕ is termed velocity potential, and the flow is potential flow

- In a rectangular coordinate system, $\nabla = \vec{i}\dfrac{\partial}{\partial x} + \vec{j}\dfrac{\partial}{\partial y} + \vec{k}\dfrac{\partial}{\partial z}$

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y} \qquad w = \frac{\partial \phi}{\partial z}$$

- In a cylindrical coordinate system, $\nabla = \vec{i}\dfrac{\partial}{\partial r} + \vec{j}\dfrac{1}{r}\dfrac{\partial}{\partial \theta} + \vec{k}\dfrac{\partial}{\partial z}$

$$V_r = -\frac{\partial \phi}{\partial r} \qquad V_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} \qquad V_z = -\frac{\partial \phi}{\partial z}$$

*Question – what is the significance for introducing velocity potential?

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Answer. three velocity components can be represented by a single scalar, which may significantly reduce the solution process!!!!!!

It is noted that velocity potential satisfies irrotational flow condition exactly. For instance, in a rectangular coordinate system

$$\omega_z = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{\partial^2 \phi}{\partial y \partial x} - \frac{\partial^2 \phi}{\partial x \partial y} = 0$$

Potential lines (等势线):

Let $\phi = C1 = C2 \ldots\ldots = Cn$

Then a set of potential lines can e obtained !

8.2.3 Stream function

Definition

If the flow is described by only **two coordinates (i.e. two-dimensional),** the stream function $\psi(x, y, t)$ also exist as an alternate approach to describe the flow***.

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

For incompressible flow, stream function satisfies continuity equation exactly, that is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

*** The flow is not necessarily to be irrotational !!!!

Question – what is the significance for introducing stream function when we solve fluid flow problems ?

Physical significance of Stream function

(1) $\psi = \text{constant}$

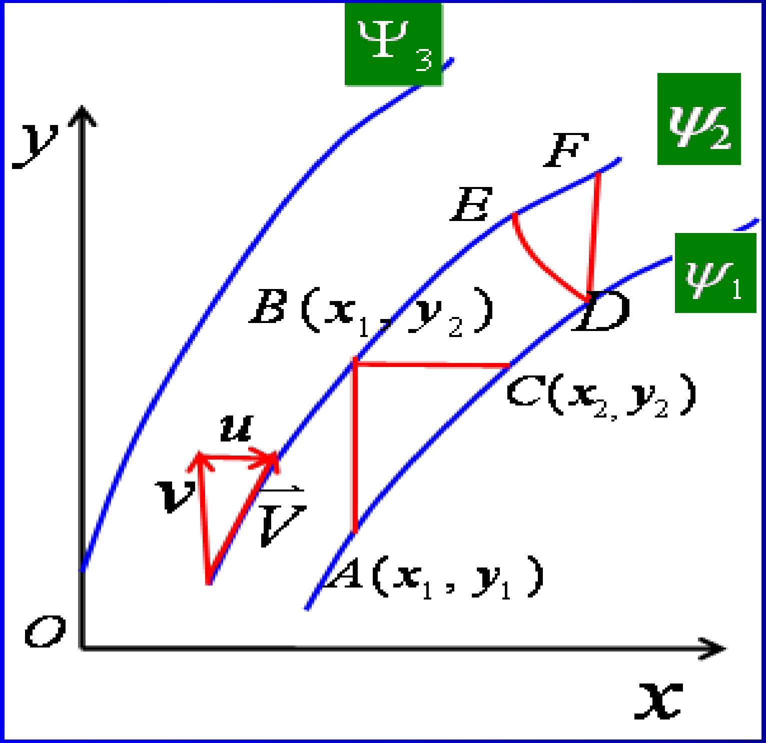
$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = -v\ dx + u\ dy = 0$$

$$\frac{u}{dx} = \frac{v}{dy}$$

$\psi = \text{constant}$ **represents a streamline, different values represent different streamlines.**

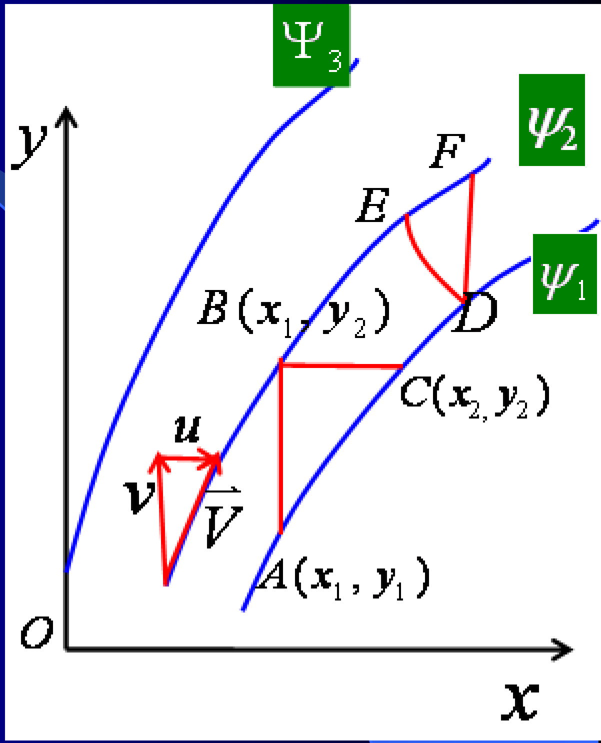
(2) ψ and volume flow rate

If the streamlines in a two-dimensional, incompressible flow field, at a given instant are shown in the figure, **the flowrate between streamlines** ψ_1 **and** ψ_2 **across the lines** *AB, BC, DE,* and *DF* **must be equal.**



$$Q_{AB} = Q_{BC} = Q_{DE} = Q_{DF}$$

(Since for incompressible flow volume flow rate is conserved over the entire control surface, $\sum \dot{Q}_{in} = \sum \dot{Q}_{out}$)



For a unit depth, the flow rate across AB is

$$Q_{AB} = \int_{y_1}^{y_2} u \times 1 \times dy = \int_{y_1}^{y_2} \frac{\partial \psi}{\partial y} dy$$

Along AB, $x =$ constant, and $d\psi = \partial\psi / \partial y \; dy$, thus

$$Q_{AB} = \int_{y_1}^{y_2} \frac{\partial \psi}{\partial y} dy = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$$

Similarly, we can obtain

$$Q_{BC} = \int_{x_1}^{x_2} v\,dx = -\int_{x_1}^{x_2} \frac{\partial \psi}{\partial x}\,dx = -\int_{\psi_2}^{\psi_1} d\psi = \psi_2 - \psi_1$$

Thus

$$Q_{AB} = Q_{BC} = \psi_2 - \psi_1$$

The volume flow through any line per unit depth in the flow field equals to the difference of stream function value at the two end points.

Section summary

Velocity potential exists only for **irrotational flow**. The **stream function** stands only for **2-D flows** but it is not restricted to irrotational flow and incompressible; for incompressible flow, steam function satisfies the continuity equation.

Suggested Reading on stream function:

---- It is used to reduce Navier-stokes equation!!!

FW 4.7 pp260 ---- to find how the momentum equation is reduced by introducing stream function?

8.2.4 Relationship between stream function and velocity potential

For **2-D incompressible** and **irrotational flow,** velocity potential and stream function exist.

(1) Both satisfy Laplace's equation

$$u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \psi}{\partial x}$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$u = \frac{\partial \phi}{\partial x}$$

$$v = \frac{\partial \phi}{\partial y}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

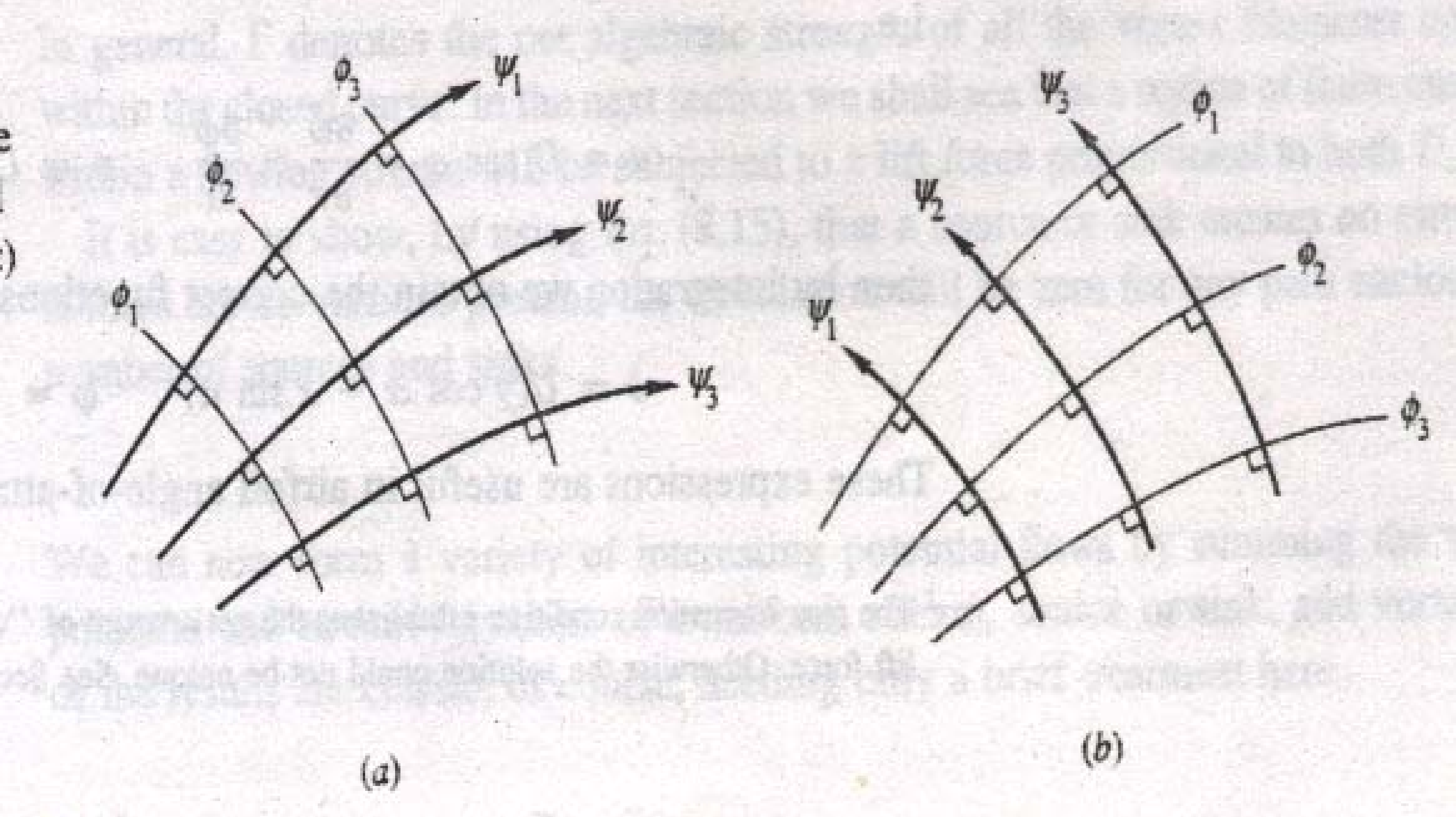
linear, homogeneous partial differential equation!!!!

In a polar coordinate system 极坐标下

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r}; \quad V_\theta = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$ （FW 4.101）

(2) Potential line and streamline are orthogonal

正交



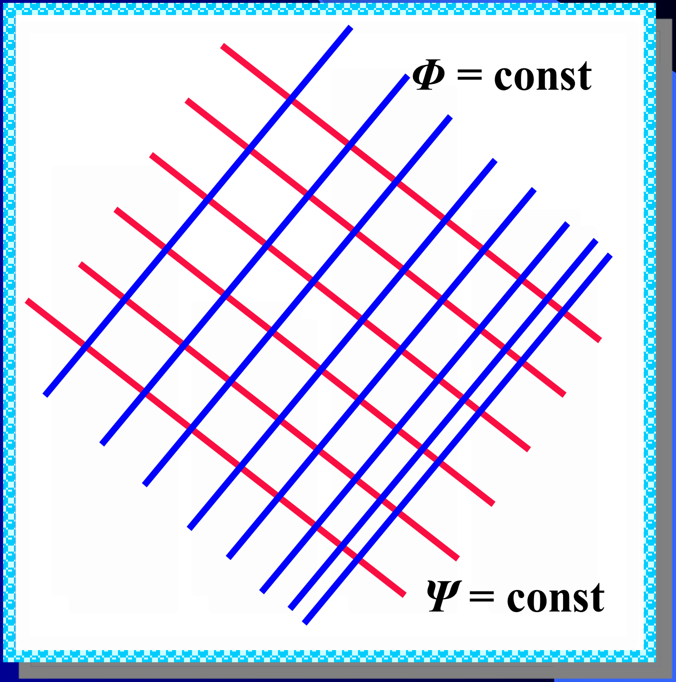
$$\psi = c \Rightarrow d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0 \Rightarrow \frac{dy}{dx}\Big|_{\psi=c} = -\frac{\partial \psi / \partial x}{\partial \psi / \partial y} = -(-\frac{v}{u}) = \frac{v}{u}$$

$$\phi = c \Rightarrow d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = 0 \Rightarrow \frac{dy}{dx}\Big|_{\phi=c} = -\frac{\partial \phi / \partial x}{\partial \phi / \partial y} = -\frac{u}{v}$$

Thus

$$\frac{dy}{dx}\Big|_{\psi=c} \cdot \frac{dy}{dx}\Big|_{\phi=c} = -1$$

Slope --- 斜率之积！



8.3 Elemental plane flows and their superposition （简单平面流动及其叠加）

For the (2-D) incompressible and irrotational flow:

(1) both velocity potential and stream function exist and satisfy Laplace's equation.

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0 \quad \text{and} \quad \frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}=0$$

(2) Since Laplace's equation is a linear, homogeneous partial differential equation (线形齐次偏微分方程), solutions may be superposed (added together) to develop more complex and interesting patterns of flow.

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \quad \text{and} \quad \frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$$

(3) Say ψ_1 and ψ_2 represent elementary flows, satisfy Laplace's equation, and $\psi_3 = \psi_1 + \psi_2$ describe more complex flows will also satisfy the Laplace's equation. Elementary plane flows are the building blocks in this superposition process.

Objective of superposition of elementary flows is to produce flow patterns similar to those of practical interest.

Question： Can we superpose the velocity of simple flows to obtain the velocity for a complex flow ? Why?

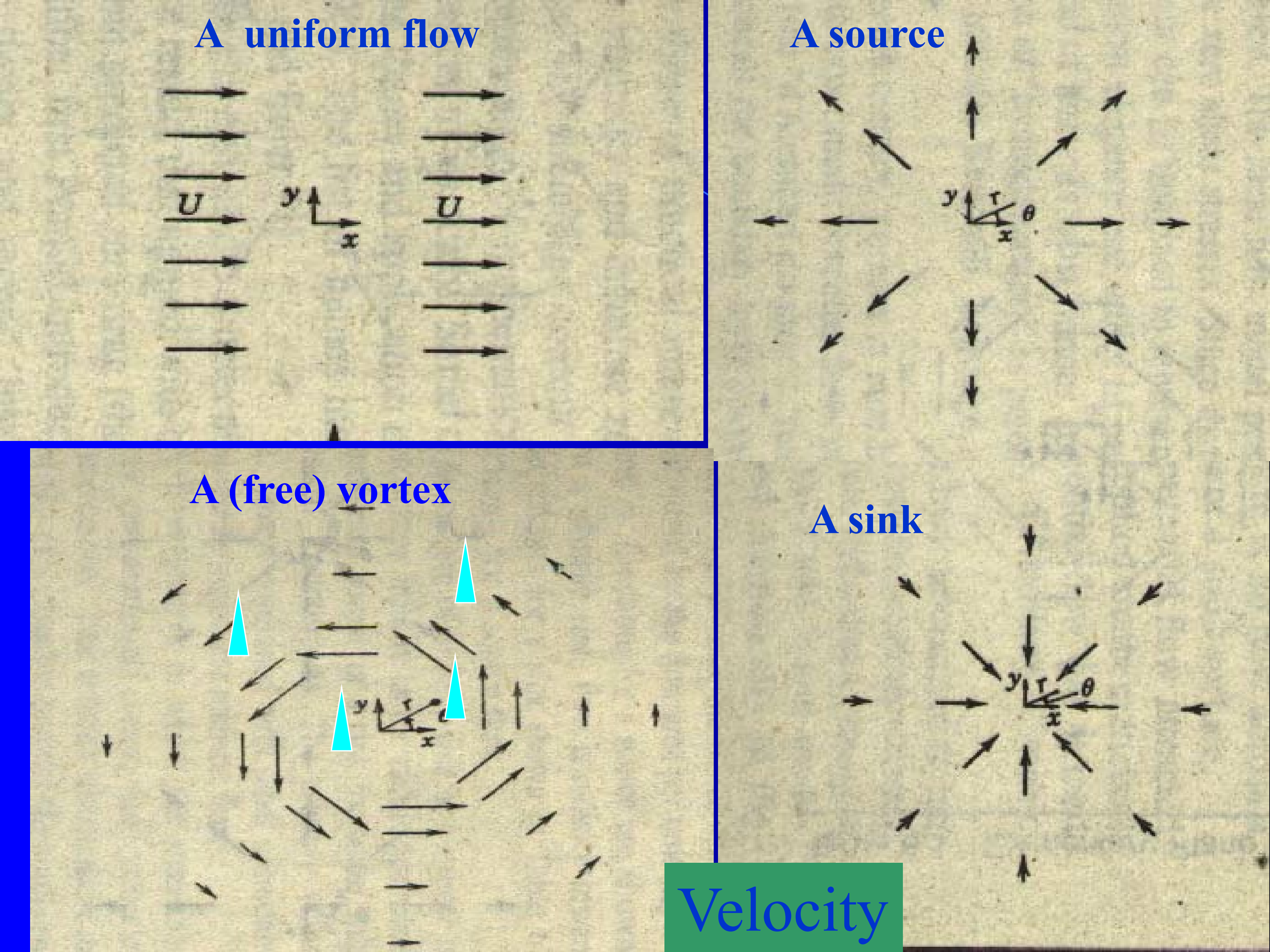
Mathematical basis of supperposition is Laplace equation. **For 2-D imcompressible and inviscid flow, velocity might not satisfy Laplace equation, thus we can not do superposition, and we can do superposition for stream function or velocity potential, since both satisfy the following**

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

For supperposition of elementary flows, we need firstly to obtain their velocity potentials or stream functions.

variety of potential flows (势流) can be constructed by superposing elementary flow patterns, such as *uniform flow, a source* or *sink*, and *free vortex* are the most commonly used ones for flow superposition!!

To superpose the flows, we need firstly to obtain their velocity potentials or stream functions.


A uniform flow
U
y
x
U
A source
y
r
θ
x
A (free) vortex
y
r
x
A sink
y
r
θ
x
Velocity

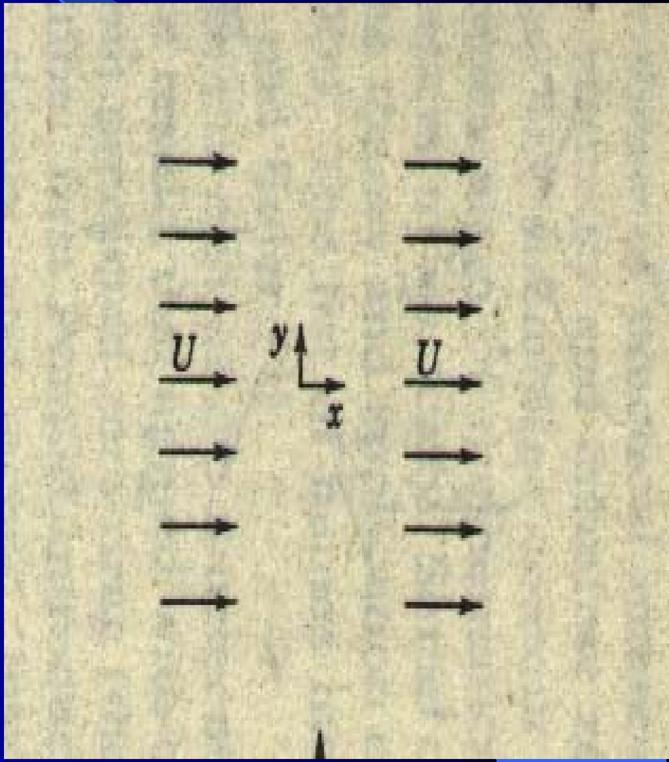
8.3.1 Elementary plane flows (简单平面流动)

- *1. 1-D Uniform flow（一元均匀流动）*

Given $\vec{V} = \vec{i}\,U;$ $\quad (U = \text{const})$

◆ *Velocity potential and stream function exist*

Since the velocity satisfy the following conditions



$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0, \text{ then } \phi \text{ exist;}$$

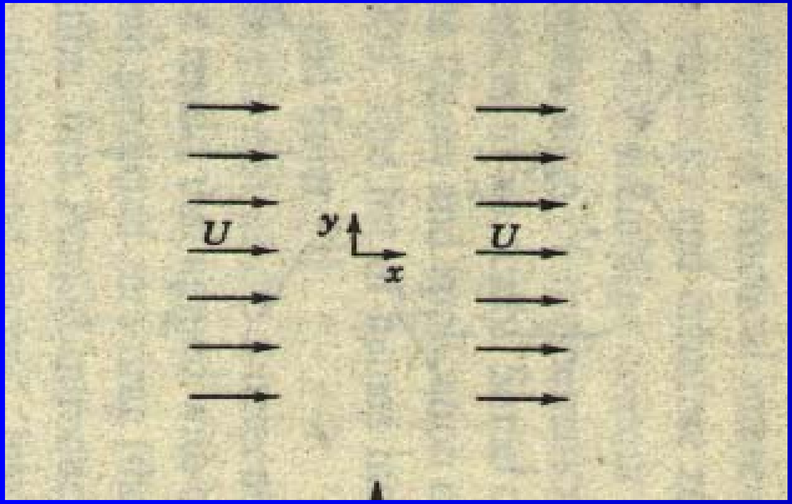
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \text{ then } \psi \text{ exist.}$$

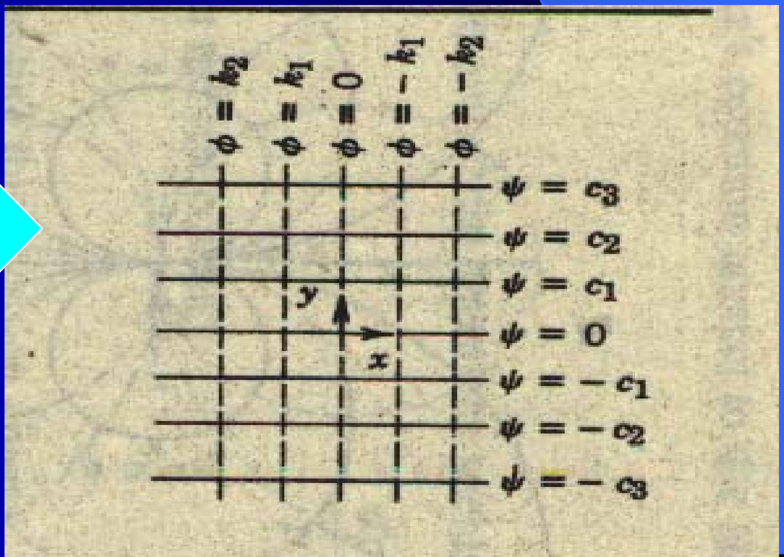
By definition $$u = U = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad v = 0 = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

◆ *Integrating the above two equations* gives the velocity potential and stream function, and they are given as

$$\psi = Uy \qquad \phi = Ux$$

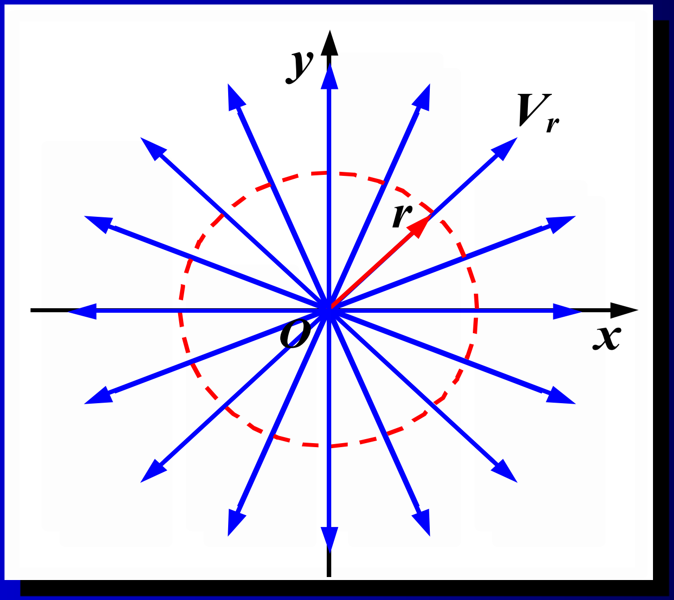
Clearly streamlines are horizontal straight lines, and the potential lines are vertical, both are orthogonal.





2. Source or sink （源或汇）

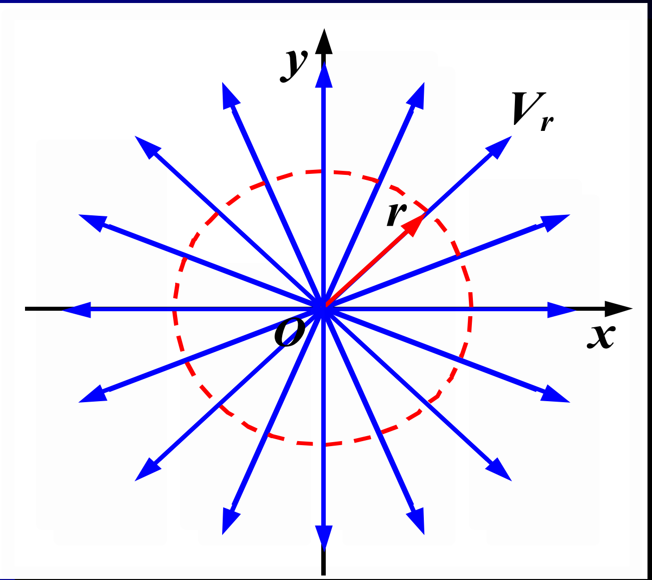
A simple source is a flow pattern in the xy plane in which flow is radially outward from the z axis and symmetrical in all directions.



Total flow rate with the length b in z-direction is given by Q,

Then we obtain the ***velocity components***

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r}; \quad V_\theta = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$ （FW 4.101）



$$V_r = \frac{Q}{2\pi r b} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r}; \quad V_\theta = 0 = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}; \quad V_z = 0$$

Velocity components satisfy both irrotational condition and continuity equation, thus ***velocity potential and stream function exist.*** (please prove this after class !!)

-- 参见Appendix D: Equations of motion in cylindrical coordinates **p854**

$$V_r = \frac{Q}{2\pi rb} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r}; \quad V_\theta = 0 = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}; \quad V_z = 0$$

Integrating the above equations, we obtain

$$\psi = m\theta; \qquad \phi = m \ln r$$

(m 源或汇的强度)

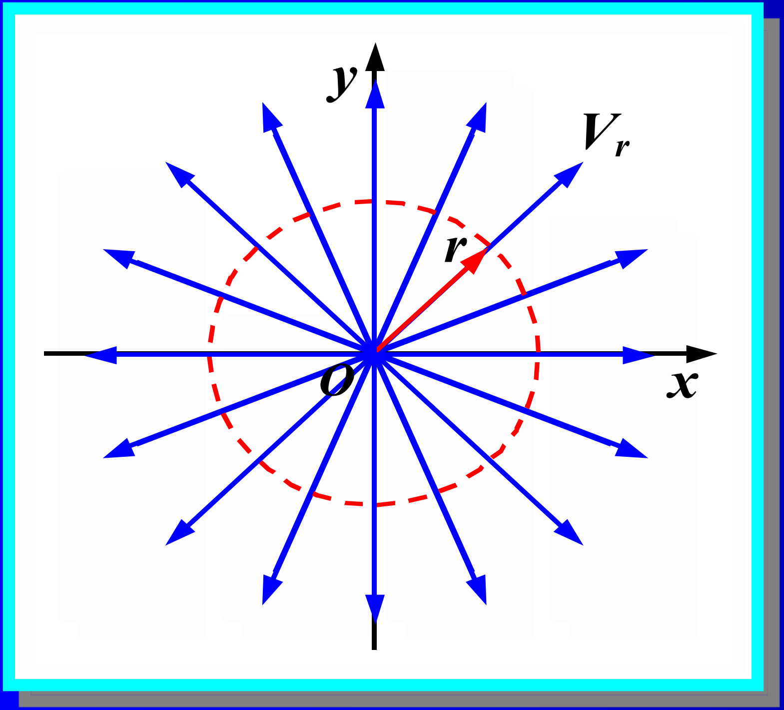
where m is called the strength of source or sink, $m = v_r r$

For a full-plane source, $m = v_r r = Q/(2\pi b)$;

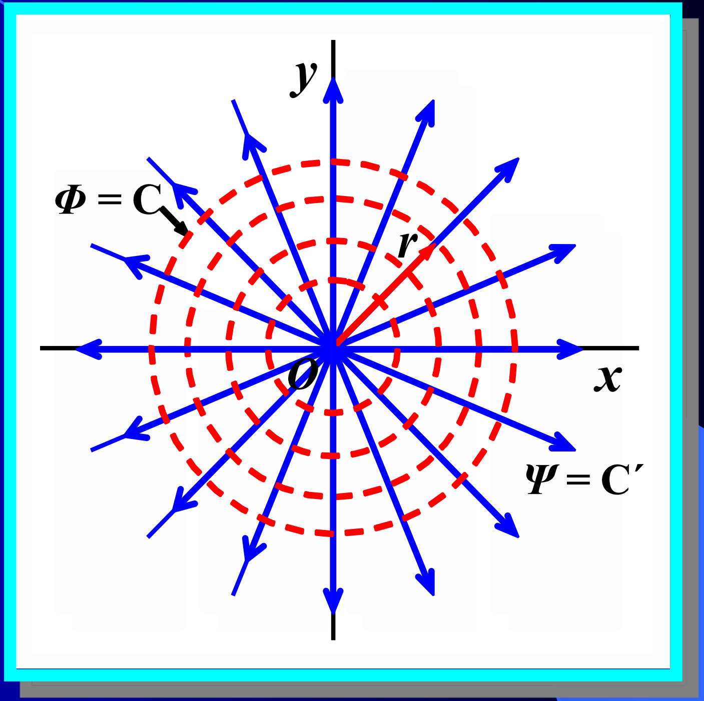
For a half-plane source, $m = v_r r = Q/(\pi b)$;

(1) Streamlines are spokes (射线) when θ is constant; **(2) potential lines are concentric circles (同心圆),** a group of circles of around the origin as r is constant, and both are orthogonal.

Source (源)



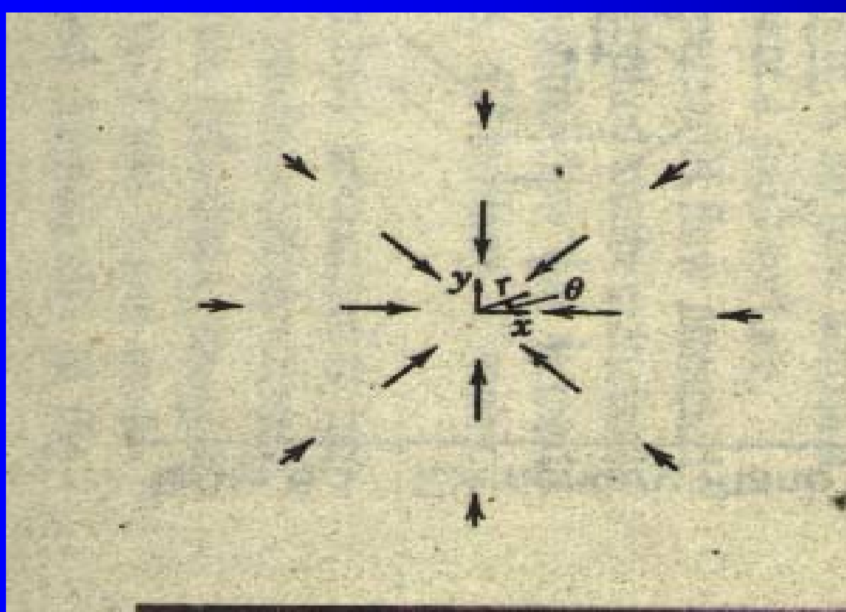
velocity



Stream lines and potential lines

A simple sink (汇) is a flow pattern in the xy plane of radially inward, which is a negative source. The ψ and ϕ for a sink are the negatives of the corresponding functions for a source flow.

$$\psi = -m\theta; \qquad \phi = -m \ln\ r$$



Singularity 奇点

$$V_r = \frac{Q}{2\pi rb} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\partial\phi}{\partial r};\ \ V_\theta = 0 = -\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\phi}{\partial\theta};\ \ V_z = 0$$

It is noted that the origin of either a sink or a source is a singular point, since the radial velocity approaches infinity as the radius approaches zero. Thus, while an actual flow may resemble a source or a sink for some values , sources and sinks have no exact physical counterparts at singularity. The primary value of the concept of sources and sinks is that when combined with other elementary flows, they produce flow patterns that adequately represent realistic flows.

3. Free vortex (irrotational vortex)

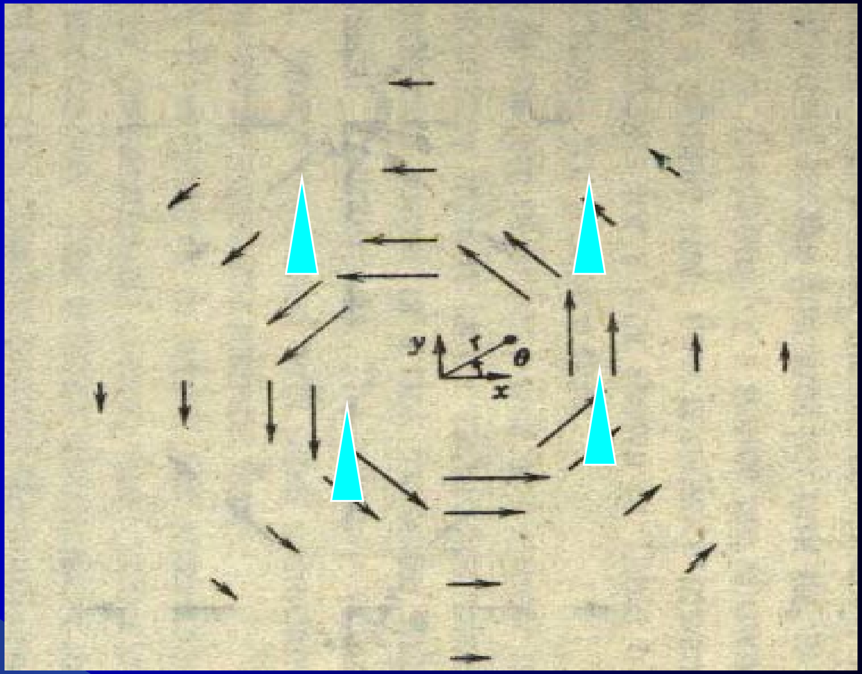
自由涡（无旋涡）

A flow pattern in which the streamlines are concentric circles is a vortex. Free vortex is a purely steady motion,

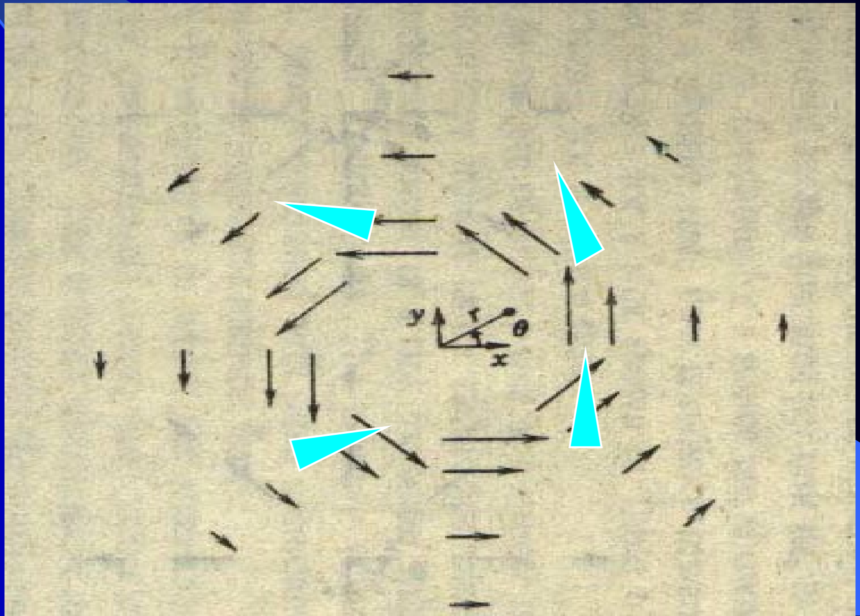
$$v_r = 0,\ v_z = 0$$

In an irrotational vortex, fluid particles do not rotate themselves as they move around the vortex center !!!!!

Free (irrotational) vortex flow



Rotational vortex flow



(1) Obtain velocity components;
(2) Obtain stream function / velocity potential

We consider free *(line)* vortex flow. In a cylindrical coordinate system

$$\omega_r = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)$$

$$\omega_\theta = \frac{1}{2}\left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)$$

$$\omega_z = \frac{1}{2}\left[\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

(FW Appendix D on page 851, equation D.11)

For irrotational flow, we have

$$\omega_r = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right) = 0$$

$$\omega_\theta = \frac{1}{2}\left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right) = 0$$

$$\omega_z = \frac{1}{2}\left[\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right] = 0$$

And as definition of free and line vortex, we have

$$v_r = v_z = 0$$

Then

$$\frac{\partial}{\partial r}(r v_\theta) = 0$$

$$\frac{\partial}{\partial r}(r v_\theta) = 0$$

To do the integration, we need to know whether

v_θ depends on θ ????

In a cylindrical coordinate system, continuity equation is written as

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta)+\frac{\partial v_z}{\partial z}=0$$

（FW equation D.2 of Page 854)

Since $v_r = v_z = 0$, thus we have

$$\frac{\partial}{\partial \theta}(v_\theta)=0$$

$$\Downarrow$$

$$v_\theta = v_\theta(r)$$

$$\frac{\partial}{\partial r}(rv_\theta) = 0 \xrightarrow{v_\theta = v_\theta(r)} \frac{d}{dr}(rv_\theta) = 0$$

Thus we have

$$\boxed{v_\theta . r = K = \text{constant}}$$

By definition

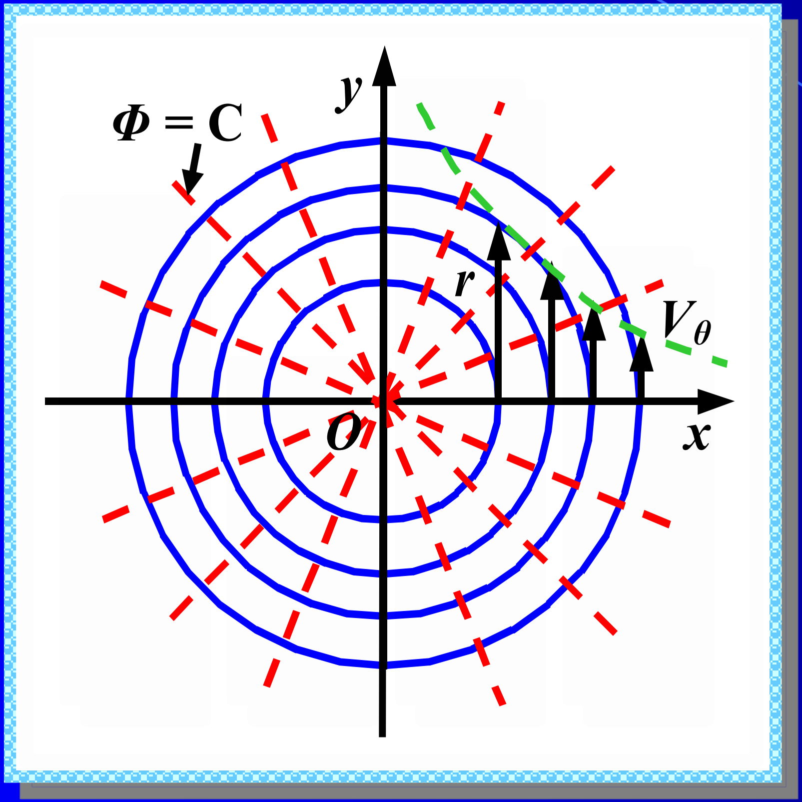
$$v_r = 0 = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r}; \qquad v_\theta = \frac{K}{r} = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

While K being constant, $K = rv_\theta$ representing strength of vortex (涡的强度).

Integrating the above equations, we have

$$\psi = -K \ln r, \quad \phi = K\theta$$

The streamlines are circles (constant r), and the potential lines are radial spokes (射线constant θ).



Potential and stream lines, and velocity of free vortex flow

6.3.2 Superposition of elementary plane flows

Mathematical basis: solutions of Laplace equation can be superposed!!

Objective of superposition of elementary flows is to produce flow patterns similar to the practical flow situation.

The procedure is as follows

(1) choose a coordinate system

(2) establish stream function / velocity potential for each elemental flow in the chosen coordinate system ***

(3) Obtain the combined stream function / velocity potential***

(4) obtain velocity using the relationship between the velocity components and the functions, and pressure using Bernoulli's equation.

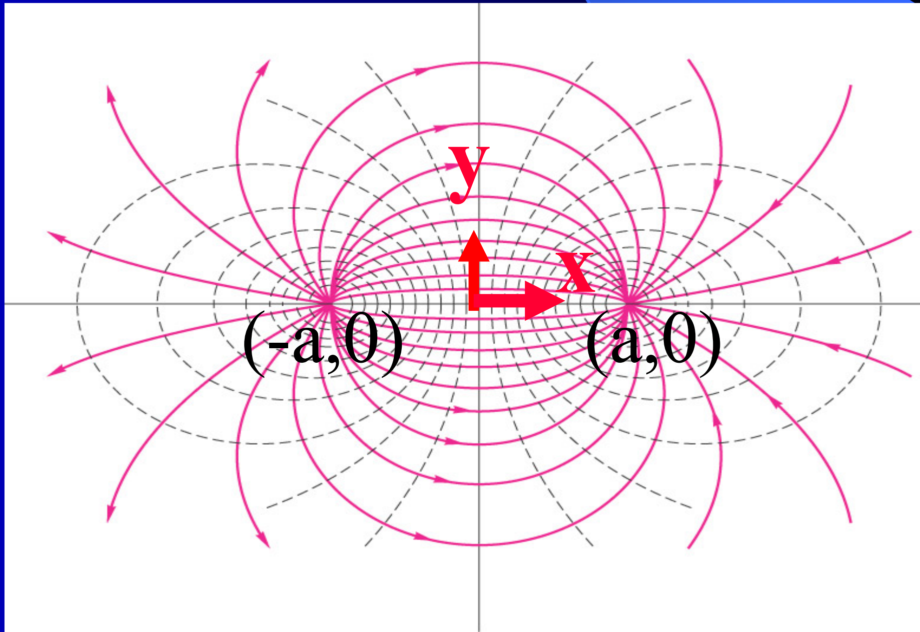
Examples

1. P534, fig 8.4: source plus equal sink

See eq. (8.15) of page535.

1) Choose a coordinate

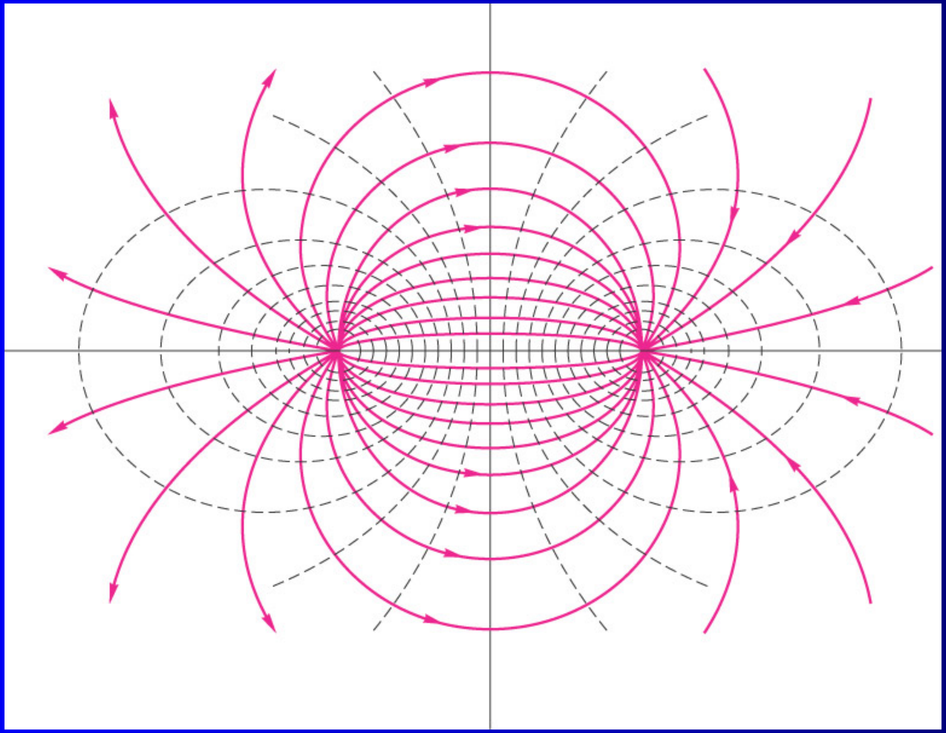
(注意：r 和θ的含义，m 的含义

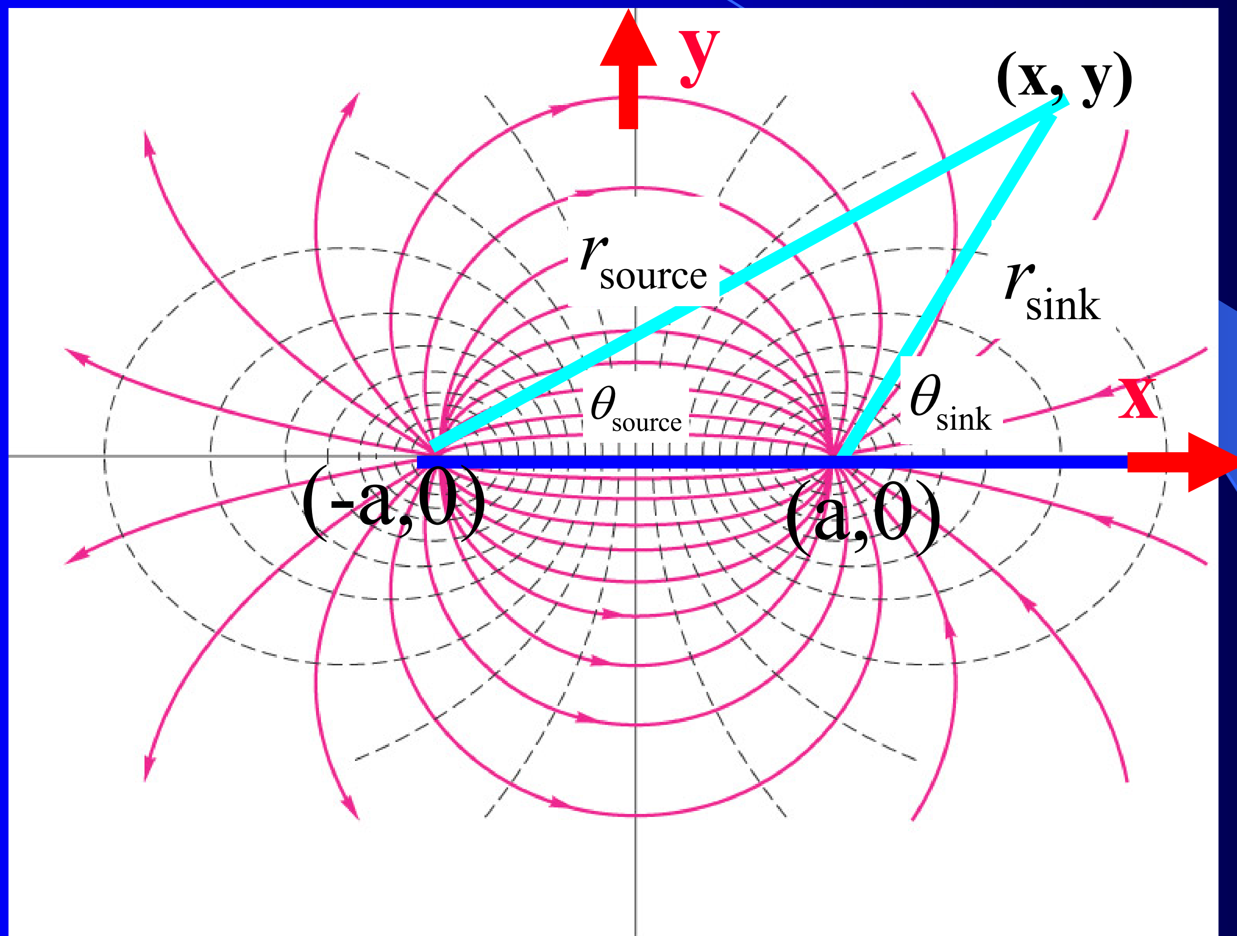


Step 2 and 3: Obtain stream function and velocity potential

$$\psi = \psi_{\text{source}} + \psi_{\text{sink}} = m\theta_{\text{source}} - m\theta_{\text{sink}};$$

$$\phi = \phi_{\text{source}} + \phi_{\text{sink}} = m\ln r_{\text{source}} - m\ln r_{\text{sink}}$$




y
(x, y)
r_{source}
r_{sink}
θ_{source}
θ_{sink}
x
(-a,0)
(a,0)

the combined stream function and velocity potential

$$\psi = \psi_{\text{source}} + \psi_{\text{sink}} = m\theta_{\text{source}} - m\theta_{\text{sink}}$$

$$= m\tan^{-1}\frac{y}{x+a} - m\tan^{-1}\frac{y}{x-a} = -m\tan^{-1}\frac{2ay}{x^2+y^2-a^2};$$

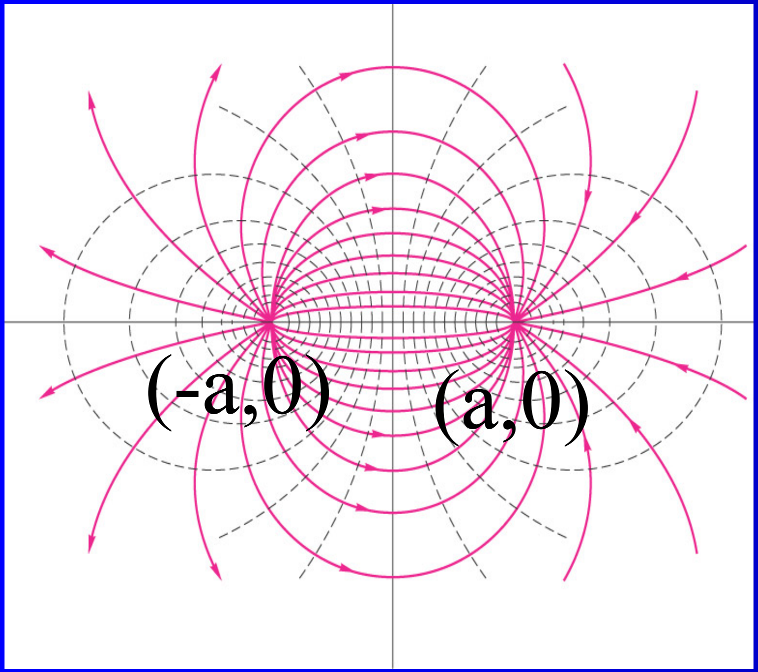
$$\phi = \phi_{\text{source}} + \phi_{\text{sink}}$$

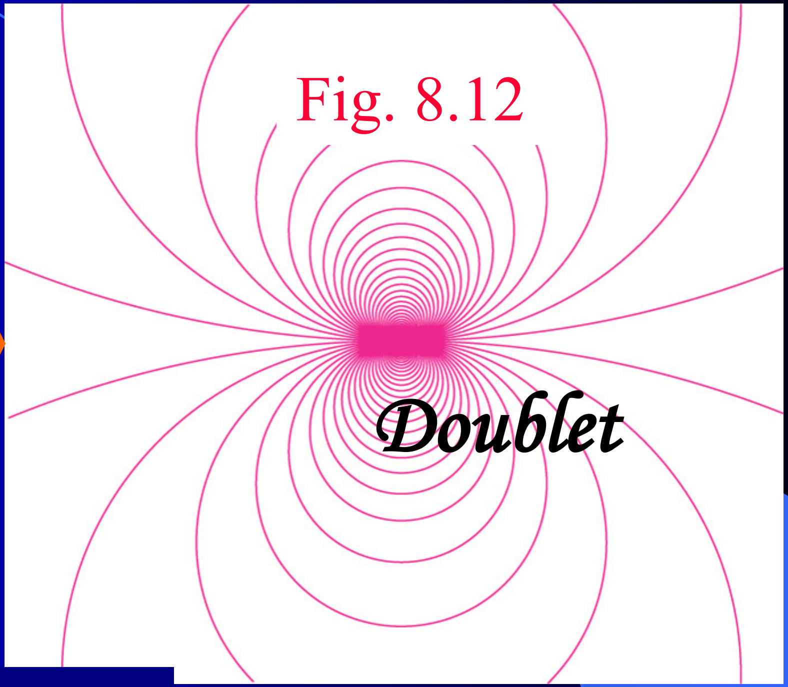
$$= m\ln r_{\text{source}} - m\ln r_{\text{sink}}$$

$$= \frac{1}{2}m\ln[(x+a)^2+y^2] - \frac{1}{2}m\ln[(x-a)^2+y^2]$$

$$= \frac{1}{2}m\ln\frac{[(x+a)^2+y^2]}{[(x-a)^2+y^2]} \qquad \text{(FW 8.15)}$$

2. Doublet (偶极子流动):---- Page 556





$$\begin{cases} \lambda = 2am = \text{constatnt} \\ a \to 0 \end{cases}$$

λ −−strength of doublet

偶极子强度

对式(8.15)求极限,得到偶极子表达式,即*page544-555*：方程(8.31)和（8.32）

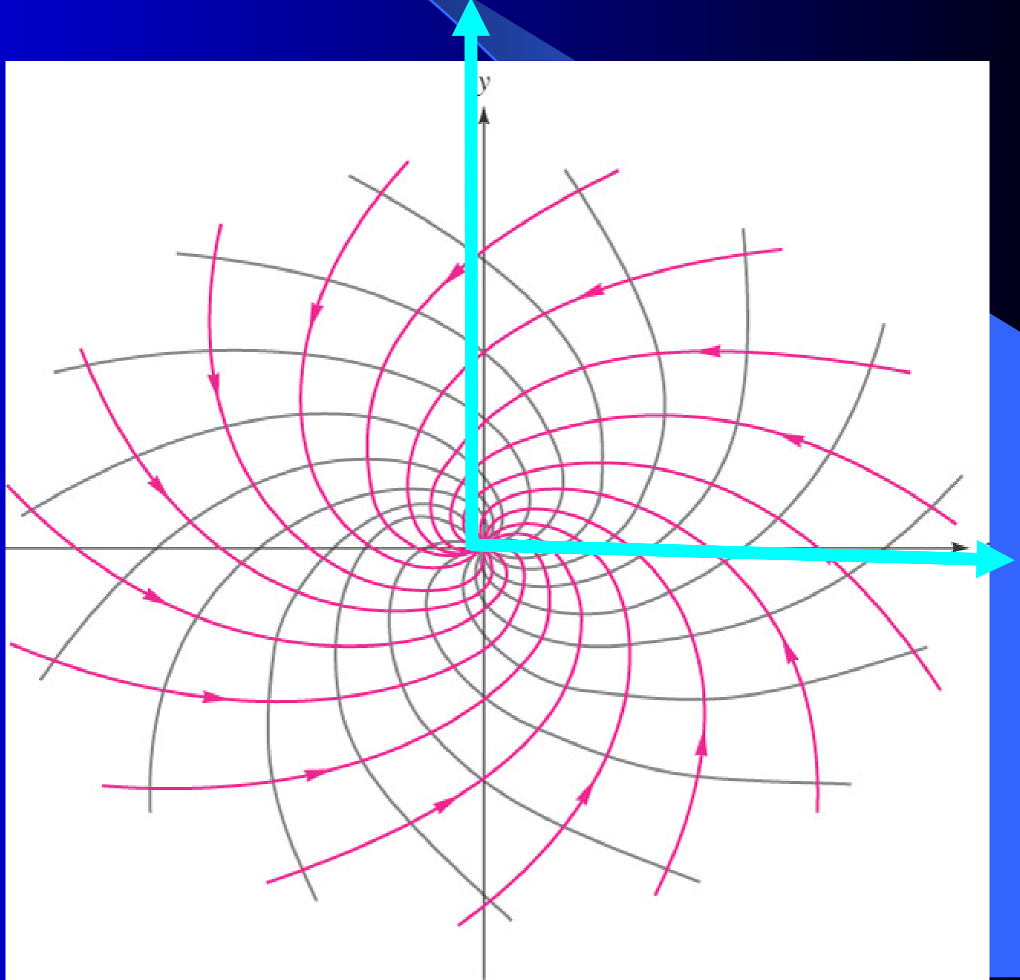
$$\psi_{\text{doublet}}=\lim_{\substack{a\to 0\\ 2am=\lambda}}\left(-m\tan^{-1}\frac{2ay}{x^2+y^2-a^2}\right)=-\frac{\lambda y}{x^2+y^2};$$

$$\phi_{\text{doublet}}=\lim_{\substack{a\to 0\\ 2am=\lambda}}\left(\frac{1}{2}m\ln\frac{[(x+a)^2+y^2]}{[(x-a)^2+y^2]}\right)=\frac{\lambda x}{x^2+y^2};$$

3. P535, fig 8.5(sink + a vortex):

simulating tornado or fast bathtub draining flow

Both centered at origin!



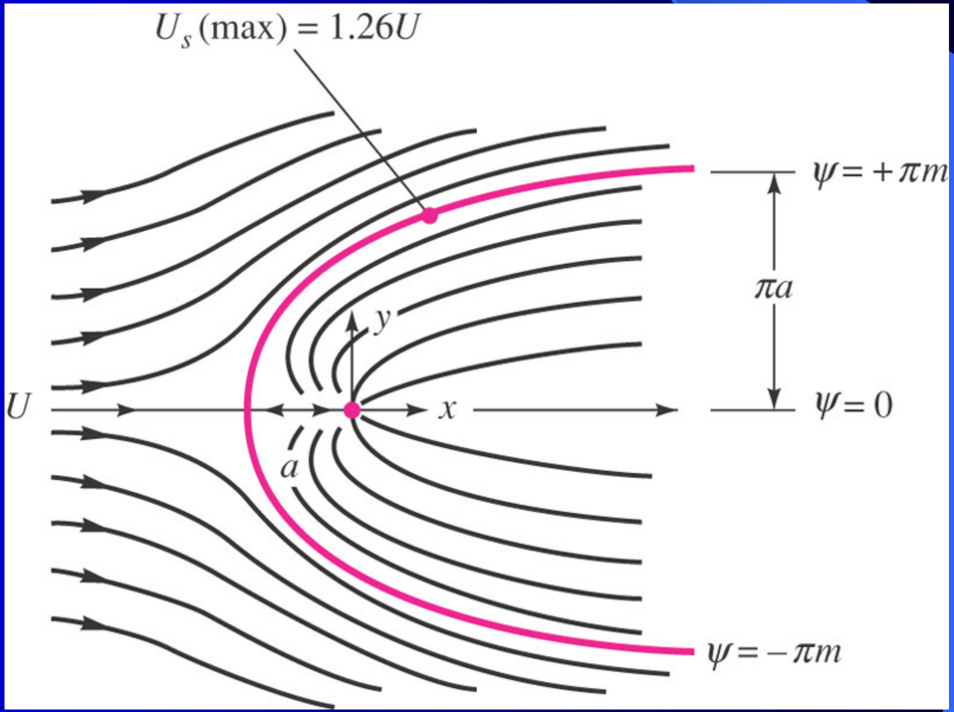
$$\psi = m\theta - K\ln r$$

$$\phi = m\ln r + K\theta$$

Where m is negative!

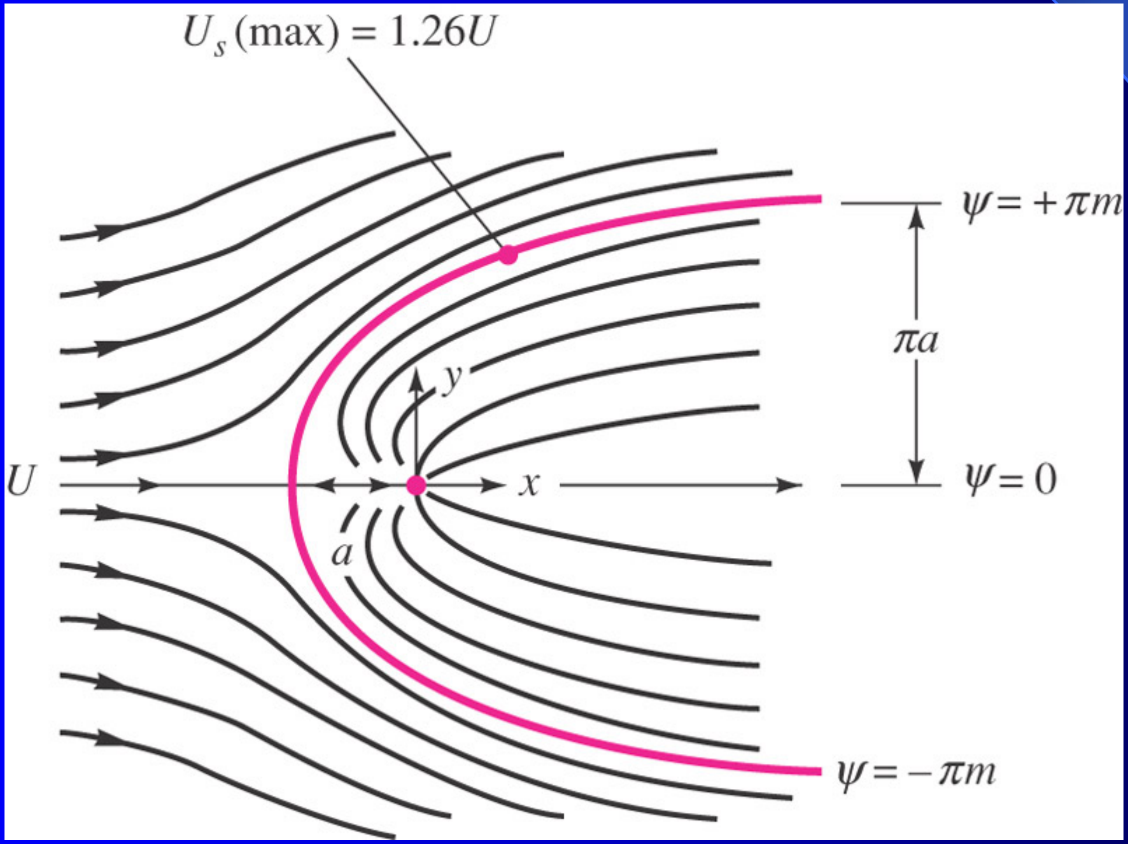
4. p536 fig 8.6 (uniform stream + sink at origin) simulating Rankine half-body flow(绕半椭圆体流动)

stagnation point



$$\psi = Uy + m\theta$$

$$= Ur\sin\theta + m\theta \qquad （m<0)$$



$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, v_\theta = -\frac{\partial \psi}{\partial r} \qquad (Eq4.101)$$

$$\text{if}\,\psi = Ur\sin\theta + m\theta$$

$$then, v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U\cos\theta + m/r$$

$$v_\theta = -\frac{\partial \psi}{\partial r} = -U\sin\theta$$

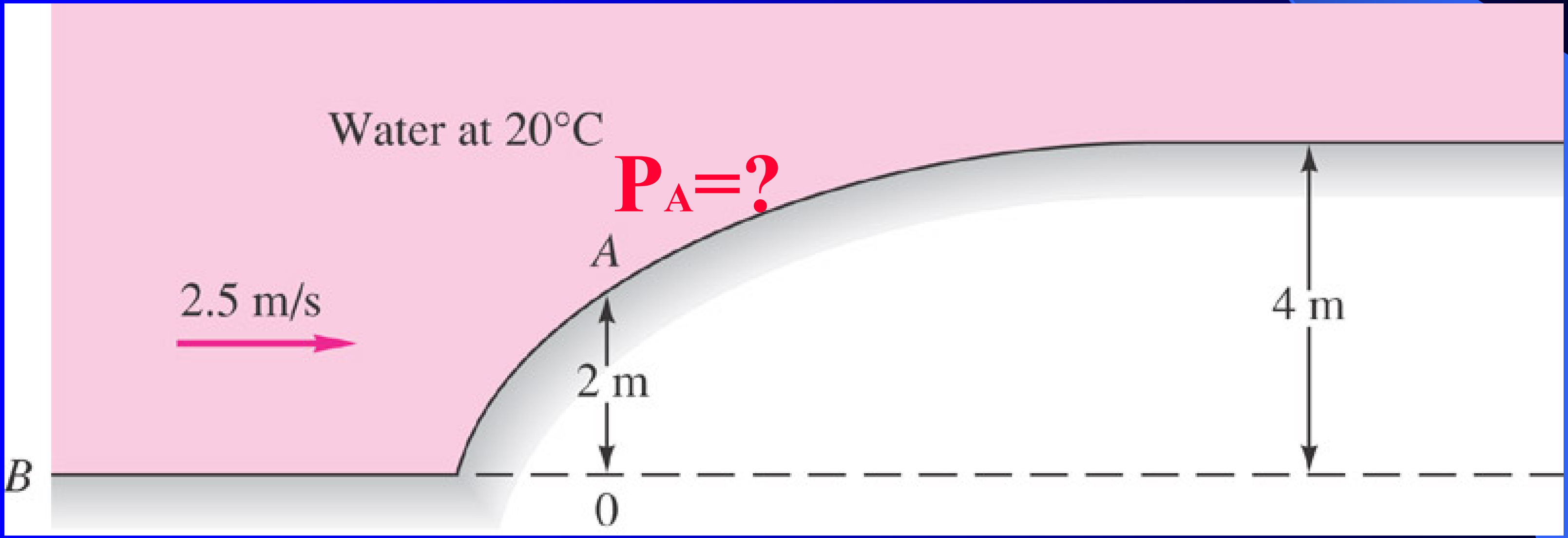
triangular relationship, $\underline{u = v_r \cdot \cos\theta - v_\theta \cdot \sin\theta,\ v = v_r \cdot \sin\theta + v_\theta \cdot \cos\theta}$

Then we obtain

$$\boxed{\begin{aligned} u &= v_r \cdot \cos\theta - v_\theta \cdot \sin\theta = \underline{U + m\cos\theta/r} \\ v &= v_r \cdot \sin\theta + v_\theta \cdot \cos\theta = \underline{m\sin\theta/r} \end{aligned}}$$

Equation 8.19 of page 536

5. p537, example 8.1 (绕半椭圆物体流动)

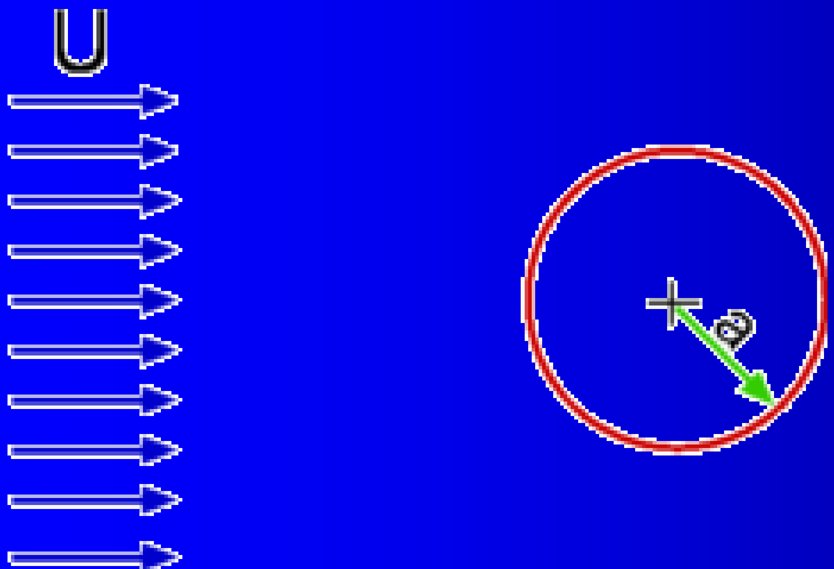


P_B=130KPa

6. Plane flow past a circular cylinder without circulation

(即柱体不旋转)

Page FW 546-553



drag and lift ???

It is approximated by a uniform flow + a doublet

Circulation (of velocity):

The counterclockwise line integral of velocity around a closed curve.

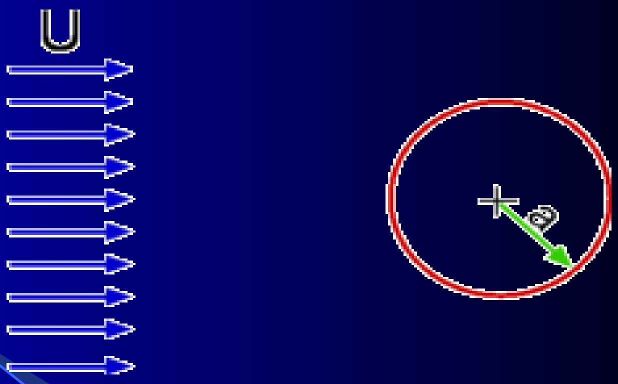
速度环量指切向速度沿圆周方向的积分，速度沿封闭曲线的积分叫速度环量。逆时针为＋。

$$\boxed{\Gamma = 2\pi r V_\theta}$$

Drag 阻力*:* the resultant fluid force on the body opposing to its movement and it is parallel to flow stream

Lift 升力*:* the resultant fluid force on the body perpendicular to the drag sustaining body weight

(Magnus effect --马格努斯效应)



Solution procedure:

- *Establish stream function/velocity potential* by flow superposition
- *Obtain solid surface flow velocity distribution*
- *Obtain solid surface pressure distribution*
- *Surface force: drag and lift*

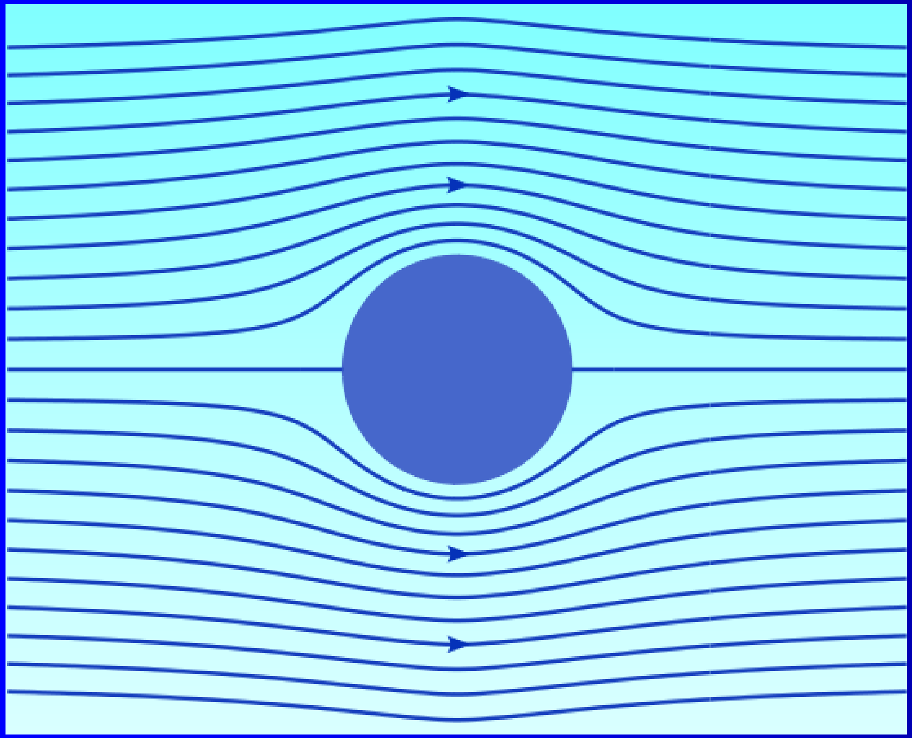
Velocity potential

$$\Phi = V_\infty r\cos\theta + \frac{\lambda}{r}\cos\theta$$

Stream function

$$\Psi = V_\infty r\sin\theta - \frac{\lambda}{r}\sin\theta$$

Taking $\lambda = 2am = \text{constant}$



P557-doublet
in polar coordinate (8.33)

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r}; \;\; V_\theta = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}; \;\; V_z = 0$$

velocity components

$$V_r = V_\infty \left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$V_\theta = -V_\infty \left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

$$V_r = V_\infty \left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$V_\theta = -V_\infty \left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

Velocity components on the surface of the cylinder are obtained by putting **r =a**

$$V_r = 0$$

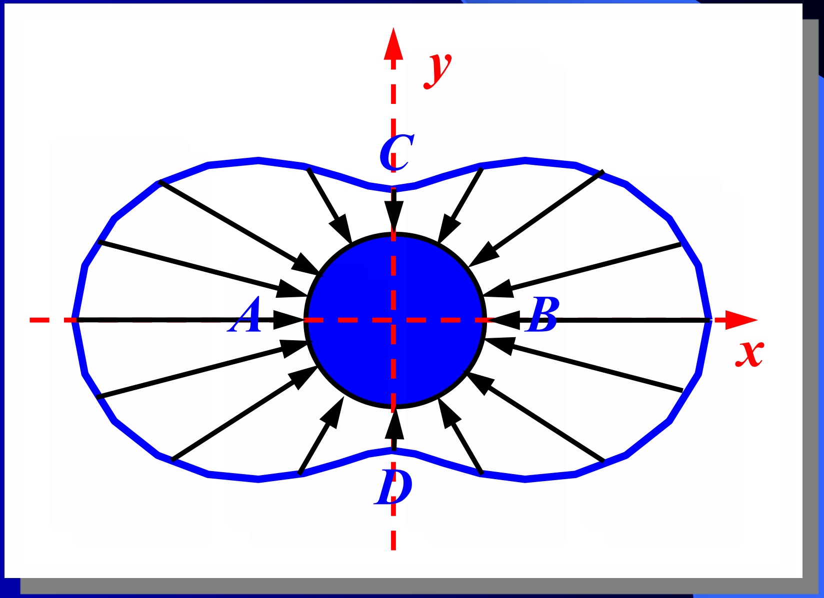
$$V_\theta = -2V_\infty \sin\theta$$

Surface pressure distribution is calculated from Bernoulli equation

$$p_s + \frac{1}{2}\rho V^2 = p_\infty + \frac{1}{2}\rho V_\infty^2$$

Where $$V = \sqrt{V_\theta^2 + V_r^2} = V_\theta = -2V_\infty \sin\theta$$

$$p_s = p_\infty + \frac{1}{2}\rho V_\infty^2 - \frac{1}{2}\rho\left(4V_\infty^2 \sin^2\theta\right)$$



Pressure coefficient, C_p,

$$C_p = \frac{p_s - p_\infty}{\frac{1}{2}\rho V_\infty^2}$$

leading to

$$C_p = 1 - 4\sin^2\theta$$

Drag,

$$D = -\int_0^{2\pi} p_s \cos\theta \; a \; d\theta$$

Lift,

$$L = -\int_0^{2\pi} p_s \sin\theta \; a \; d\theta$$

$$D = -\int_0^{2\pi} p_\infty a \cos\theta d\theta - \frac{1}{2}\rho U_\infty^2 a \int_0^{2\pi} \left(\cos\theta - 4\sin^2\theta\cos\theta\right) d\theta$$

$$= \quad -p_\infty a \left[\sin\theta\right]_0^{2\pi} - \frac{1}{2}\rho U_\infty^2 a \left[\sin\theta\right]_0^{2\pi} + \frac{1}{2}\rho U_\infty^2 a \left[\frac{4}{3}\sin^3\theta\right]_0^{2\pi}$$

$= -0 -0 + 0 = 0$

$$L = -\int_0^{2\pi} p_\infty a \sin\theta d\theta - \frac{1}{2}\rho U_\infty^2 a \int_0^{2\pi} \left(\sin\theta - 4\sin^3\theta\right) d\theta$$

$$= \quad -p_\infty a \left[\cos\theta\right]_0^{2\pi} - \frac{1}{2}\rho U_\infty^2 a \left[\cos\theta\right]_0^{2\pi} + \frac{1}{2}\rho U_\infty^2 a \left[\frac{4}{3}\cos^3\theta - 4\cos\theta\right]_0^{2\pi}$$

$= -0 -0 + 0 = 0$

7. *Plane flow past a circular cylinder with circulation* (FW page 548-)

（即柱体旋转）

(a uniform flow + a doublet + a vortex)

$D = 0$ (阻力为零：达郎伯尔佯论)

$\boxed{L = -b\rho U_\infty \Gamma}$ (升力: 库塔-儒可夫斯基定理)

达郎伯尔佯论(阻力为零)

D' Alembert's paradox:(1752)

"According to inviscid theory, the drag of any body of any shape immersed in a uniform stream is identically zero."

Solution application of inviscid flow are very restrictive. In the history of fluid mechanics, from D' Alembert's paradox, the people started to suspect the usefulness of inviscid flow solutions.

"d'Alembert paradox"(达郎伯尔佯论) was a famous example - a cylinder moving in the flow field would not experience a drag (of a uniform pressure distribution) is well known. Obviously this is not true, and it does not agree with the experiment. The paradox was puzzled the researchers for some years.

Because of the mathematical elegance and simplicity, study of potential flows have attracted many workers. The list of names include Bernoulli, Lagrange, d'Alembert, Euler, etc. Through the end of the nineteenth century, researchers on potential flows failed to produce results that agree with experiment. Potential flows produced body shapes with lift but zero drag.

In 1903, Prandtl introduced the boundary layer concept and changed the situation and reached a good agreement between the experiment and theoretical results: the viscous effects are confined to a thin boundary layer on the surface of a body. Even for real fluids, flow outside the boundary layer behaves as though the fluid had zero viscosity.

Potential flow theory predicted the lift:

------ By W. M. Kutta (1902) and N. Joukowski (1906)

库塔儒可夫斯基定理(升力)

According to such theory, the lift per unit depth (or length) of any cylinder of any shape immersed in a uniform stream equals $\rho U_\infty \Gamma$

where Γ is the total circulation contained within the body shape.

Direction of the lift is 90° from the flow stream direction, rotating opposite to the circulation.

Chapter summary

1. Objective – to obtain $\vec{V}$ and p

2. Relation of each section subjects

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

$$\rho \frac{d\vec{V}}{dt} = \rho \vec{g} - \nabla p$$

(Euler's equation)

$$\psi_3 = \psi_1 + \psi_2 + \ldots\ldots + \psi_n$$

or

$$\phi_3 = \phi_1 + \phi_2 + \ldots\ldots + \phi_n$$

p

$\vec{V}$

90

3. Reading suggested very strongly!!!!

Page 265-268“Generation of Rotationality”!!!!