Chapter 9 Introduction to compressible flows

Motivation. All eight of previous chapters have been concerned with “low-speed” or “incompressible” flow, where the fluid velocity is much less than its speed of sound. In fact, we did not even develop an expression for the speed of sound of a fluid. That will be done in this chapter.

When a fluid moves at speeds comparable to its speed of sound, density changes become significant and the flow is termed *compressible*.

Such flows are difficult to obtain *in liquids, since high pressures of order 1000 atm are needed to generate sonic velocities. In gases, however, a pressure ratio of only 2:1 will likely cause sonic flow*. Thus compressible gas flow is quite common, and this subject is often called *gas dynamics*.

Probably the two most important and distinctive effects of compressibility on flow are

(1) *choking*, wherein the duct flow rate is sharply limited by the sonic condition, and

(2) *shock waves*, which are nearly discontinuous property changes in a supersonic flow.

The purpose of this chapter is to explain such compressible flow phenomena and to familiarize you with some calculations of compressible flow.

Main Subject

9.1 Introduction

9.2 Speed of Sound

9.3 Adiabatic and Isentropic Steady Flows

9.4 Isentropic Duct Flow With Area Changes

9.5 The Normal Shock Wave

9.6 Two-dimensional Supersonic flow (FW9.9)

9.1 Introduction: Nature of Compressible Flow

1. *Nature of Compressible Flow*

At high flow speeds compressibility become important, since such flows always involve large variations in velocity, consequently a large pressure variation; for gas flows these pressure changes are accompanied by significant variations in both density and temperature. Two additional variables, ρ and T, are encountered in treating compressible flow, two additional equations are needed. Both the energy equation and an equation of state must be applied to solve compressible flow problems together with the *momentum equation*, and *continuity equation*.

In compressible flow, friction drag and pressure drag can also cause losses, besides, shock often ocurrs and it shall cause additional losses.

What is shock wave?

When an object or a disturbance moves so fast that fluid particles cannot move out of the way, the molecular structure of the fluid permits the relative position of the molecules to move closer together to compress. Thus, as the mean free path of the molecules is reduced, layers of disturbances come together to form a shock wave. A pressure pulse is created, which grows larger and larger until it becomes a shock wave leading to a rise in temperature and pressure.

Main differences in treating compressible & incompressible flows

(1) Basic conservation equations :

- **Incompressible flows:** momentum equation, and continuity equation;
- **Compressible flows:** momentum equation, and continuity equation, plus energy equation and equation of state

(2) losses:

- **Incompressible flows:** losses mainly caused by friction drag and pressure drag;
- **Compressible flows:** friction drag and pressure drag cause losses, besides, shock shall cause additional losses.

2. Mach number and compressibility

Mach number.

It is an important quantity, which is introduced as the ratio of velocity and the speed of sound

$$M = \frac{V}{a} = \frac{\text{local flow speed}}{\text{local speed of sound}}$$

$$Ma^2 = \left(\frac{V}{a}\right)^2 = \frac{V^2}{dp/d\rho} = \frac{V^2}{E_v/\rho} = \frac{\rho V^2 L^2}{E_v L^2}$$

$$= \frac{\text{inertia force}}{\text{force due to elasticity}}$$

Mach number usually changes across the flow field, i.e. non-uniform Mach number distribution.

Flows for which M<1 are subsonic, while those with M>1 are supersonic. Flow fields have both subsonic and supersonic regions are termed transonic

$$\begin{cases} \text{subsonic flow} \quad M<1 \quad \text{(flow in machinery)亚音速} \\ \text{supersonic flow} \quad \mathrm{M}>1 \quad \text{(flow in engine, CONCORDE -2)超音速} \\ \text{transonic flow} \quad 0.8<\mathrm{M}<1.2\text{跨音速} \\ \text{hypersonic flow } M>3 \quad \text{(the space shuttle flies at} \\ \qquad\qquad\qquad \text{about Mach number 20 ）超超音速} \end{cases}$$

In subsonic flows, if M < 0.3, the flow can then be treated as in compressible. *Such a treatment has been proved to cause only very limited errors, which are acceptable in many cases.*

3. Perfect Gas

With constant specific heats, i.e.

$$p=\rho RT; \quad R=c_p - c_v=\text{constant};$$

$$k=c_p/c_v=\text{constant}$$

4. *Isentropic process*

The isentropic approximation is commonly used in compressible flows

$$Tds = dh - \frac{dp}{\rho}; \qquad dh = c_p dT$$

$$\int_1^2 ds = \int_1^2 c_p \frac{dT}{T} - R\int_1^2 \frac{dp}{p}$$

Then

$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1} = c_v \ln\frac{T_2}{T_1} - R\ln\frac{\rho_2}{\rho_1}$$

For isentropic process, we have

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{k/(k-1)} = \left(\frac{\rho_2}{\rho_1}\right)^k \qquad \text{(FW 9.9)}$$

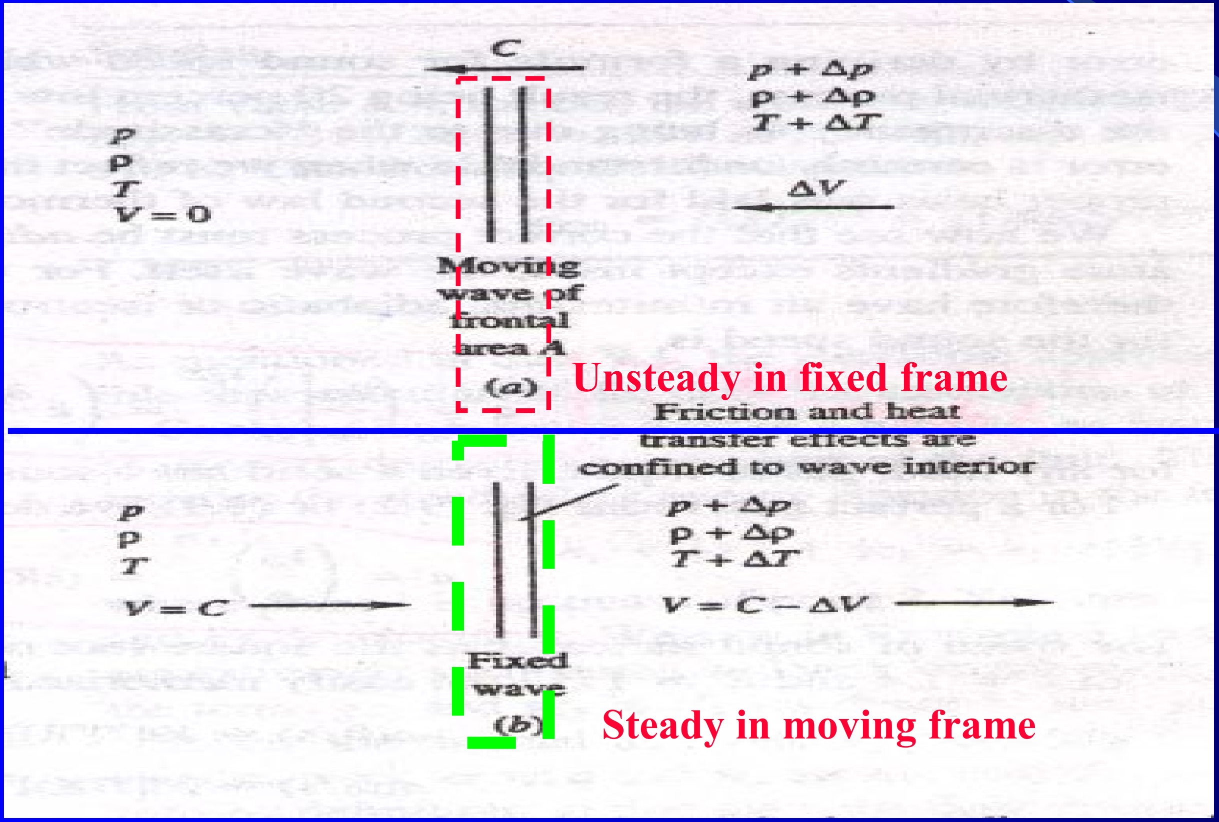
9.2 Speed of Sound

Speed of sound is defined on the basis of **sound wave propagation (传播) speed**.

Sound wave in gases are weak pressure disturbances in gas (flows) that do not cause any losses. Changes of Δp pressure, density, $\Delta\rho$ and velocity ΔV are infinitesimally small, and **the entropy in sound wave is constant.**

We consider a pressure wave (disturbance) of infinitesimal strength moves to the left in still fluid at speed C.

Velocity ahead the wave is zero, $V=0$; the **sound wave passing through the still fluid**, and has caused the fluid behind infinitesimal changes in velocity, pressure, density, and temperature, i.e. as shown in the Figure $\Delta V, \Delta p, \Delta \rho$, and ΔT.



$C = ?$

thickness $\approx 10^{-7}$ m

x

The wave propagation speed C can be obtained applying continuity equation, and momentum equation to a control volume that includes the wave.

How to choose a suitable control volume for analysis?

Choosing a absolute control volume, obviously the property inside the CV changes with time, the unsteady term in conservation equation will exist. To simplify the analysis, we deliberately choose a moving control volume fixed on the pressure disturbance, and its moving speed is constant in absolute frame denoted by C.

Thus in continuity and momentum equation, the velocity of flux term must be measured relative to velocity $\vec{C}$. That is

$$V_r\Big|_{left} = 0-(-C) = C, \text{ and } V_r\Big|_{right} = -\Delta V-(-C) = C-\Delta V$$

- ***Continuity equation***

Since the disturbance has infinitesimal length, thus we assume an uniform property distribution along the frontal area of wave, thus

$$\rho AC = (\rho + \Delta\rho)(A)(C - \Delta V)$$

Then

$$\Delta V = C\frac{\Delta\rho}{\rho + \Delta\rho} \qquad \text{(A)}$$

- ***Momentum equation***

Body force – it has no contribution in flow direction

Shear force – acting on the upper and downward side surface will contribute the fluid motion, but the thickness of the pressure pulse is very small (in order to 10^{-6}), the area where the shear stress acts is negligible, thus we can neglect the shear force.

Pressure force – acting on the left and right face of the CV, they are respectively $+pA$ and $-(p+\Delta p)A$

Basic equation

$$\sum F_x = \dot{m}_{out} V_{out} - \dot{m}_{in} V_{in}) = \rho CA(V_{out} - V_{in})$$

Then we obtain

$$pA - (p+\Delta p)A = (\rho AC)(C - \Delta V - C)$$

Or

$$\Delta p = \rho C \ \Delta V \qquad \text{(B)}$$

Combine equation (A) and (B), we have

$$C^2 = \frac{\Delta p}{\Delta \rho}(1 + \frac{\Delta \rho}{\rho})$$

This is the expression of ***(pressure) wave propagation speed.***

Speed of sound.

when the wave strength is infinitesimal, i.e. $\Delta\rho \to 0$, the **wave propagation speed is termed speed of sound** and it is written as

$$a^2 = \frac{\partial p}{\partial \rho}$$

When the changes in property is infinitesimal, the process need to be isentropic, then

$$a = \left(\frac{\partial p}{\partial \rho}\Big|_s\right)^{1/2}$$

(FW 9.15**)

It is noted that we did not assume the fluid are either gases or liquids, thus the above formula is suited for both.

For a perfect gas, using $a = (\frac{\partial p}{\partial \rho}|_s)^{1/2}$ and isentropic relation and equation of state, we can obtain

$$a = (kRT)^{1/2}$$ (FW 9.16)

Where $k = c_p / c_v$ --- ratio of specific heat;

$R = c_p - c_v$ --- gas constant

For a liquid or (a solid), compressibility is written as

$$K = \rho \frac{dp}{d\rho} \qquad (9.18)$$

$$a = \left(\frac{\partial p}{\partial \rho}\Big|_s\right)^{1/2} = \sqrt{\frac{K}{\rho}}$$

Where K is *bulk modulus* of material and represents the relation between the pressure change and density change.

9.3 Adiabatic and isentropic relations and critical parameters

（FW 9.3）

For compressible flow, ρ and T are not constants anymore.

two additional equations are needed. Both the energy equation and an equation of state must applied to solve compressible flow problems together with the momentum equation, and continuity equation. This increase the complexity.

Isentropic approximation is common in compressible-flow theory. The isentropic approximation greatly simplifies compressible-flow calculation. If it is non-isentropic, the assumption of adiabatic flow will also help to simplify compressible flow calculation.

Significance:

In some applications, if the flow is adiabatic or isentropic, the adiabatic and/or isentropic relations can then be alternatively used in combination with momentum and continuity equations to solve the compressible flows instead of solving the energy conservation equations, which can help to simplify the solutions to compressible flows.

- Adiabatic relation derived from energy equation (绝热)

$$\cancel{\dot{Q}} - \dot{W}_{shaft} - \dot{W}_{other} = \frac{\partial}{\partial t}\int_{CV}\left(u + \frac{V^2}{2} + gz\right)\rho dV + \int_{CS}\left(h + \frac{V^2}{2} + gz\right)\rho\, \vec{V}\cdot d\vec{A}$$

It includes Equation 9.21, 9.22~9.27 page 617-

- Isentropic relations derived based on （等熵）

$$s_2 - s_1 = C_v \ln\frac{T_2}{T_1} - R\ln\frac{\rho_2}{\rho_1} = 0$$

Isentropic relations include Equation (9.28a) and (9.28b) on page 619

- Bernoulli's equation (isentropic flow)

Equation (9.31) on page 619

Critical parameters 临界参数

A useful reference value for velocity is the critical speed, which is the flow speed at a Mach number of unity. Even if there is no point in a given flow field where the Mach number is equal to unity, such a condition is still useful as a reference condition.

generally using an asterisk to denote conditions at **M=1,** then by definition

$$\boxed{V^* = M^* c^* \equiv c^*}$$

From equation 9.28, we obtain equation 9.32 ~ 9. 33 on page 619-620

9.4 Isentropic Duct Flow With Area Changes (FW 9.4)

In this section, we look at the effect of area variation on fluid properties in isentropic compressible duct flow.

By combining isentropic- and/or adiabatic-flow relations with the equation of continuity we can study practical compressible-flow problems.

The duct flow is approximated as one-dimensional flow, if it is steady, the continuity equation is written as

$$\rho(x)V(x)A(x) = \text{const}$$

Differentiating the above equation gives

$$d\rho\ V(x)\ A(x) + \rho(x)\ d(V(x)\ A(X)) = d\rho\ V(x)\ \mathrm{A}(x) + \rho(x)\ dV(x)\ A(x) + \rho(x)\ V(x)\ dA(x) = 0$$

It reduces to

$$\frac{d\rho}{\rho}+\frac{dA}{A}+\frac{dV}{V}=0 \qquad \text{(A)}$$

Momentum equation for isentropic (inviscid, frictionless) flow is given by (see equation 9.31 on page 633)

$$\frac{dp}{\rho}+V\ dV=0 \qquad \text{(B)}$$

Speed of sound:

$$dp=a^2 d\rho \qquad \text{(C)}$$

Combine equations (A), (B), and (C), we have

$$\frac{dV}{V}=\frac{dA}{A}\frac{1}{Ma^2-1}=-\frac{dp}{\rho V^2}$$

$$\frac{dV}{V}=\frac{dA}{A}\frac{1}{Ma^2-1}=-\frac{dp}{\rho V^2}$$

Based on the above equation, we have seen that for the same duct geometry the property changes are of opposite sign for subsonic and supersonic inlet flow because of the term Ma^2-1

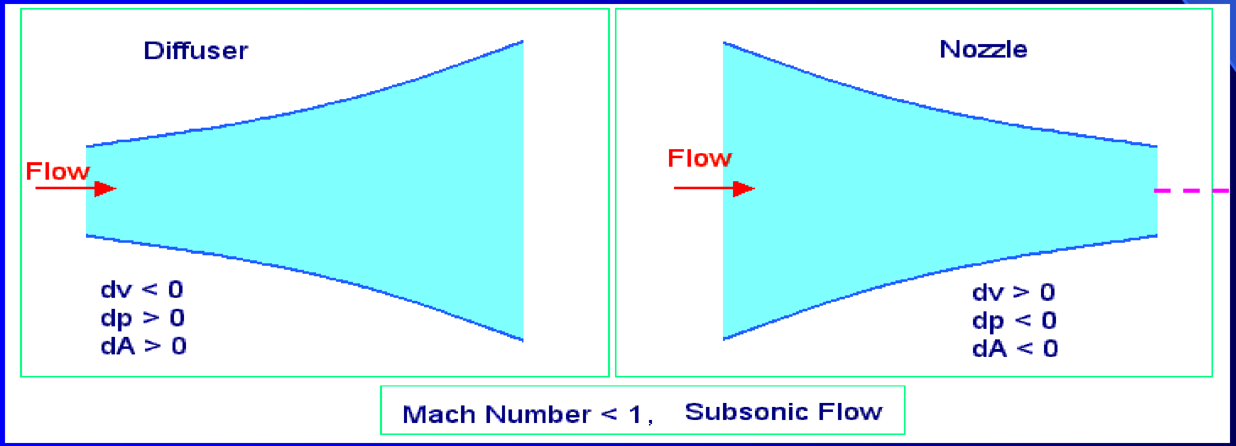
There are mainly two types of geometry variations in duct flow to produce an expected flow property change:

- ***Nozzle*** （喷咀） – a passage where flow velocity increases;
- ***Diffuser***（扩压器/管）— a passage where the flow velocity decreases and pressure increases.

For subsonic and supersonic inlet flow, the geometry of nozzle and diffuser are different, thus we shall look at them respectively.

(1) For subsonic flows, M<1, flow acceleration in a nozzle requires a passage of diminishing cross section; area must decrease to cause a velocity increase. A subsonic diffuser requires that the passage area increase to produce a velocity decrease. Maximum velocity, the speed of sound, can be reached at minimum area.

$$\frac{dV}{V} = \frac{dA}{A}\frac{1}{Ma^2 - 1} = -\frac{dp}{\rho V^2}$$

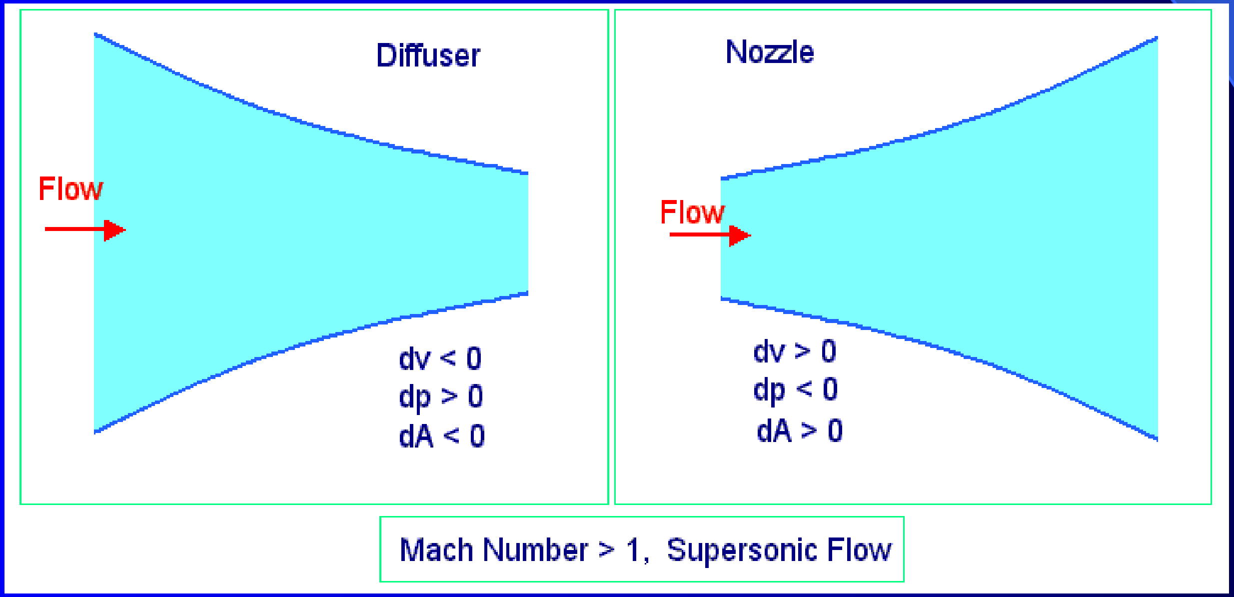


M=1 ?

Choked condition (堵塞工况)- When M=1, mass flow reaches maximum, the duct is said to be choked and can carry no additional mass flow unless the throat is widened.

(2) In supersonic flows, M>1, the effect of area change are different. Based on the above equation, a supersonic nozzle must be built with an area increase in the flow direction. Supersonic diffuser must be a converging channel.

$$\frac{dV}{V}=\frac{dA}{A}\frac{1}{Ma^2-1}=-\frac{dp}{\rho V^2}$$

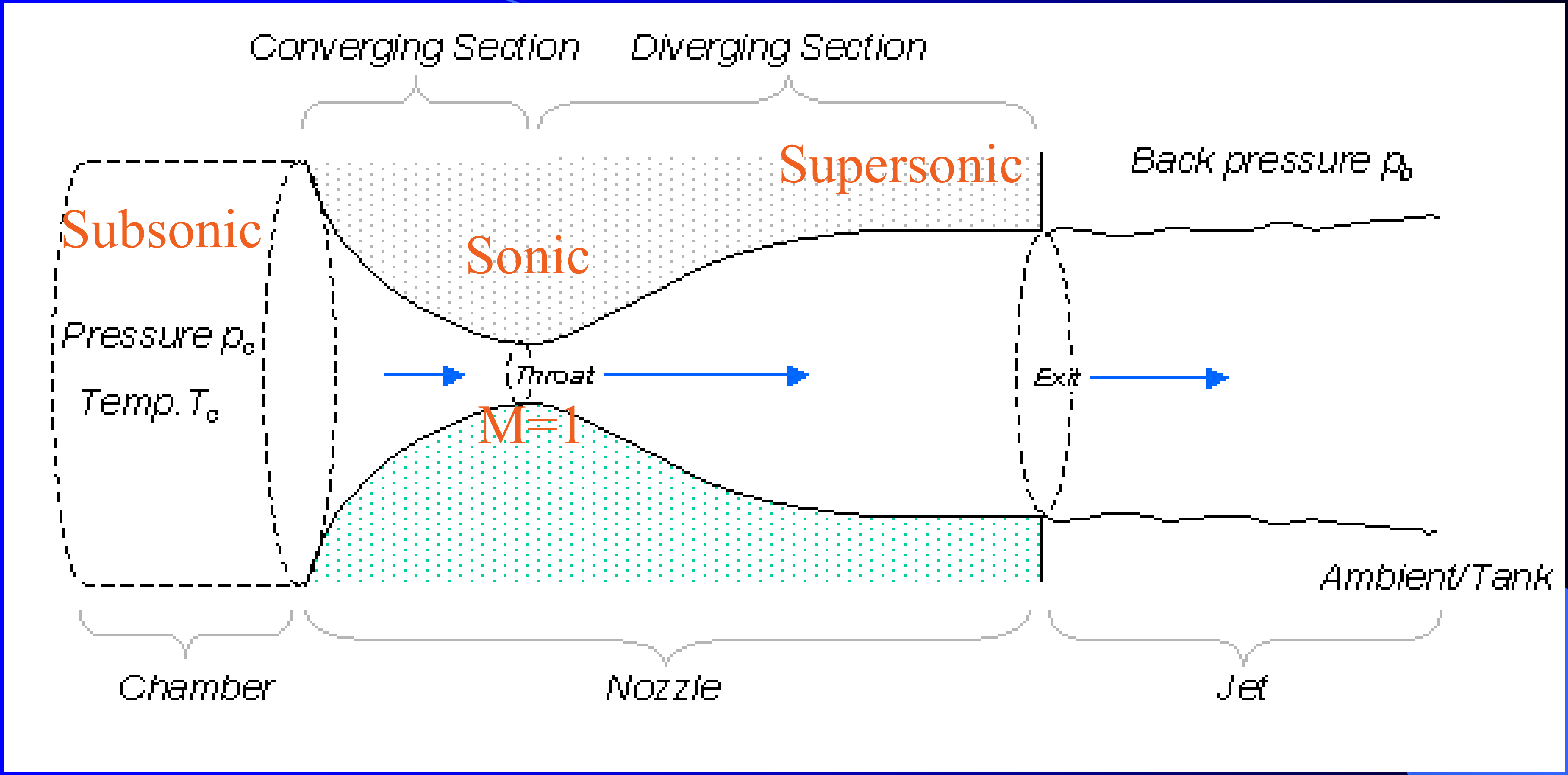


How to accelerate the subsonic flow to supersonic flow by means of a variable geometry duct ?

(3) From subsonic to supersonic flow

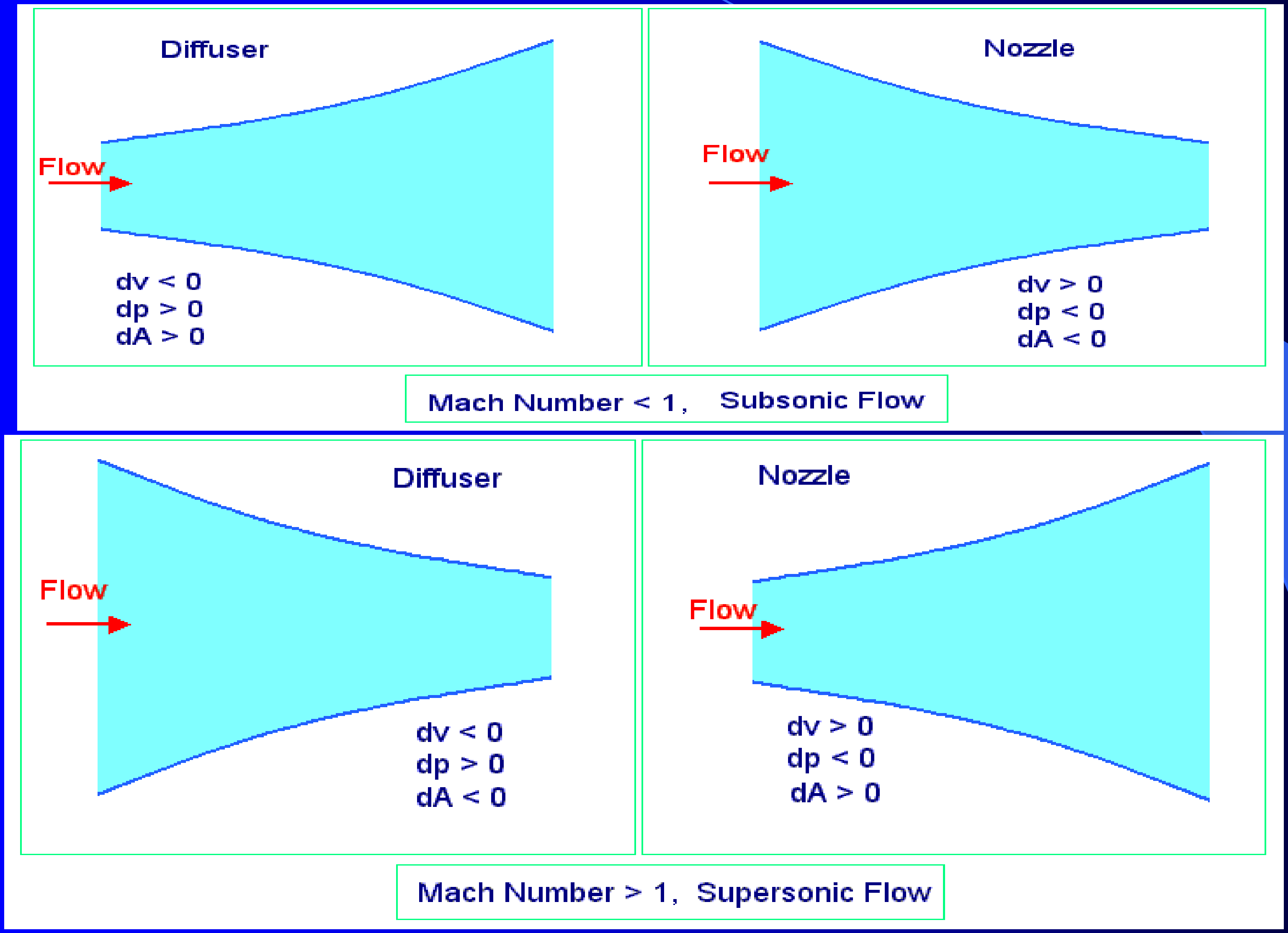
This requires first a subsonic converging nozzle. The flow will reach M=1 at the minimum area. If add a supersonic diverging nozzle downstream, flow can be accelerated to be supersonic.

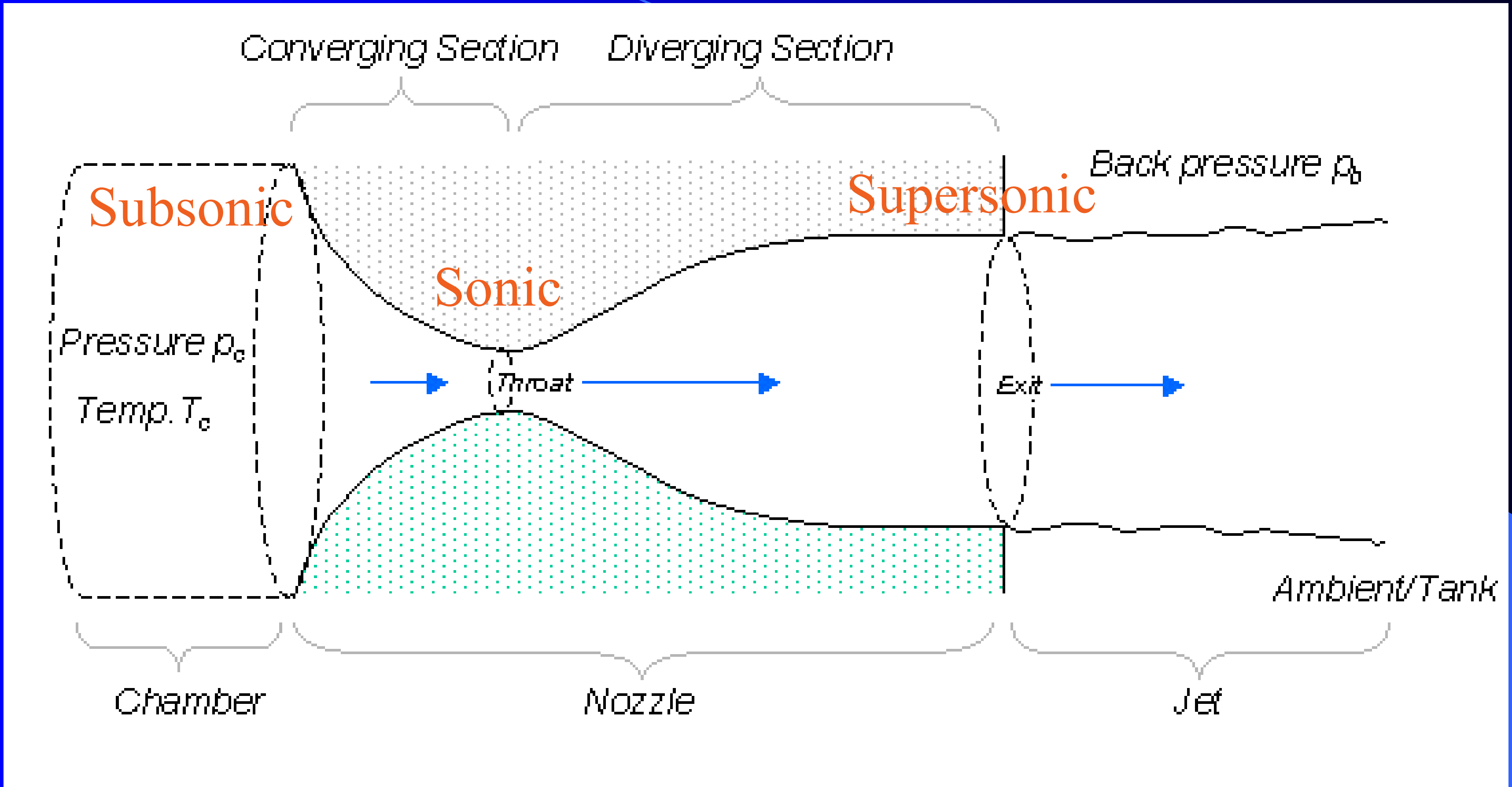
Can you draw such a configuration, through which the flow is accelerated from subsonic to supersonic?



Converging diverging nozzle configuration

CONCLUSION - nozzle and diffuser shapes are as a function of initial Mach number (i.e., inlet mach number).




Converging Section
Diverging Section
Back pressure p_b
Subsonic
Supersonic
Sonic
Pressure p_c
Temp. T_c
Throat
Exit
Ambient/Tank
Chamber
Nozzle
Jet

9.5 Shock waves

1. Definitions

Shock wave - A region of abrupt change (discontinuity) in pressure and density caused by supersonic flow around a body, such as strong pressure disturbance wave.

Shock wave is different from a single sound wave, but the wave supposed by many sound wave fronts in supersonic flow is shock wave :

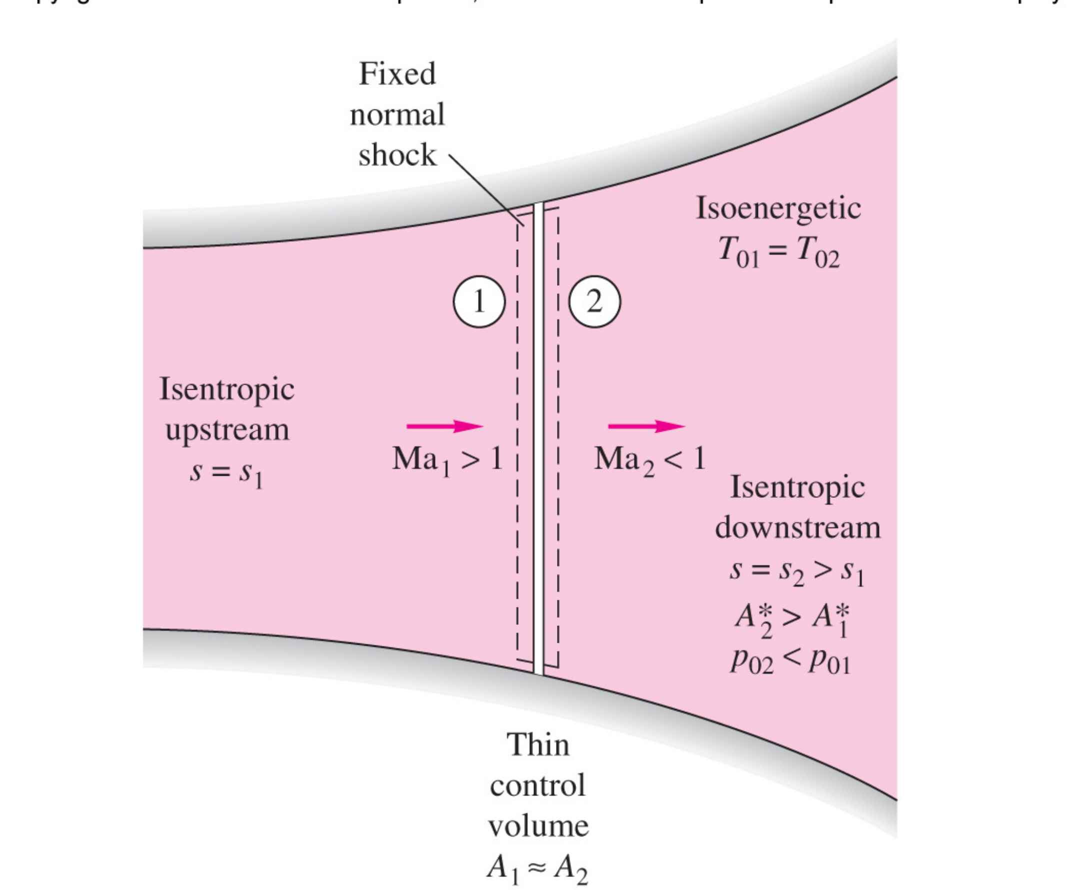
- Shock waves are not necessarily isentropic;
- Flow property changes (e.g. $\Delta p, \Delta\rho, \Delta V,$ and Δs) across shock wave are not necessarily small;
- Differential conservation equation is no longer valid.

Clearly, if the shock wave is of small intensity, $\Delta p, \Delta\rho, \Delta V,$ and Δs are negligible, shock wave are similar to sound wave, the derived isentropic relations can be used for flow analysis.

Classification:

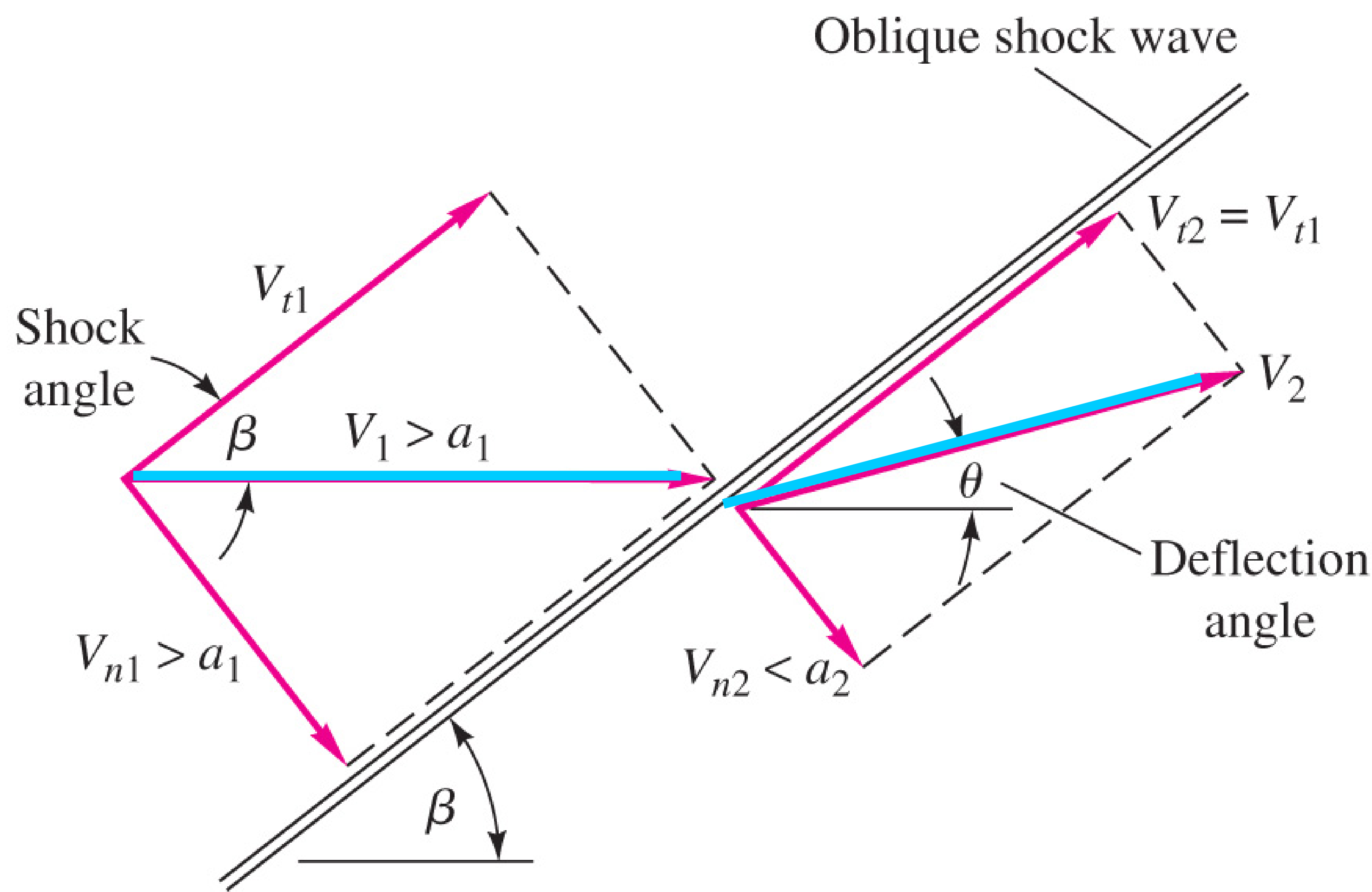
Normal shock （正激波）：a wave will not deflect（使偏离、折转） the flow stream, behind which the flow stream keeps the same direction as ahead.

(See figure 9.8 on page 629)

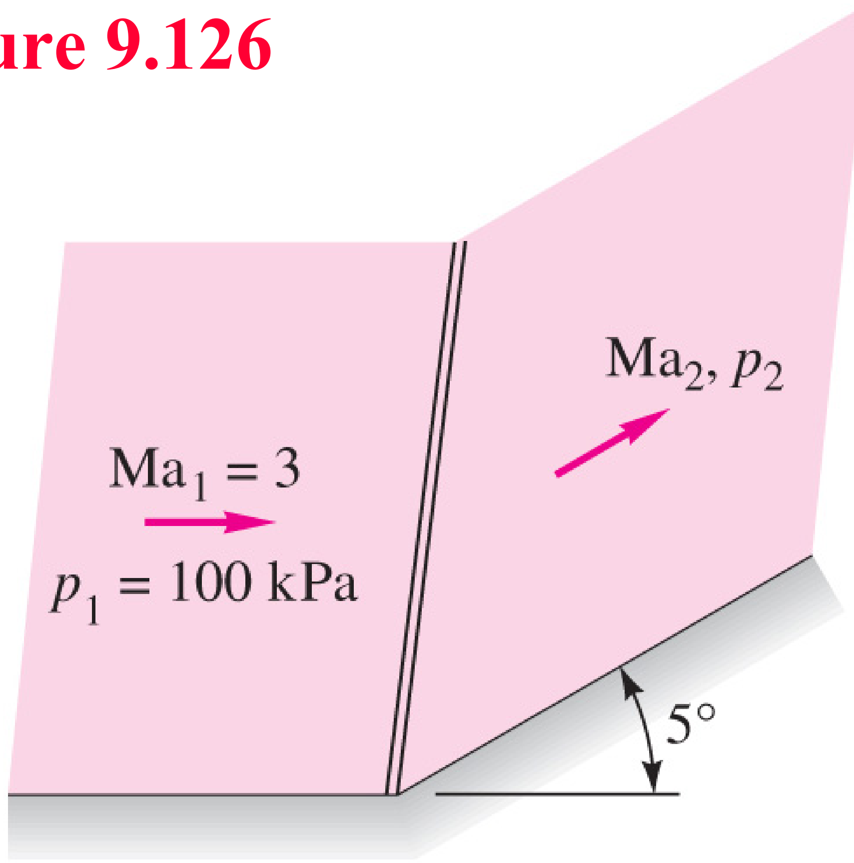


Oblique shock （斜激波）： a wave will deflect the flow stream with an angle, and the flow steam changes direction behind oblique shock

(See figure 9.20 on page 662)



Problem Figure 9.126

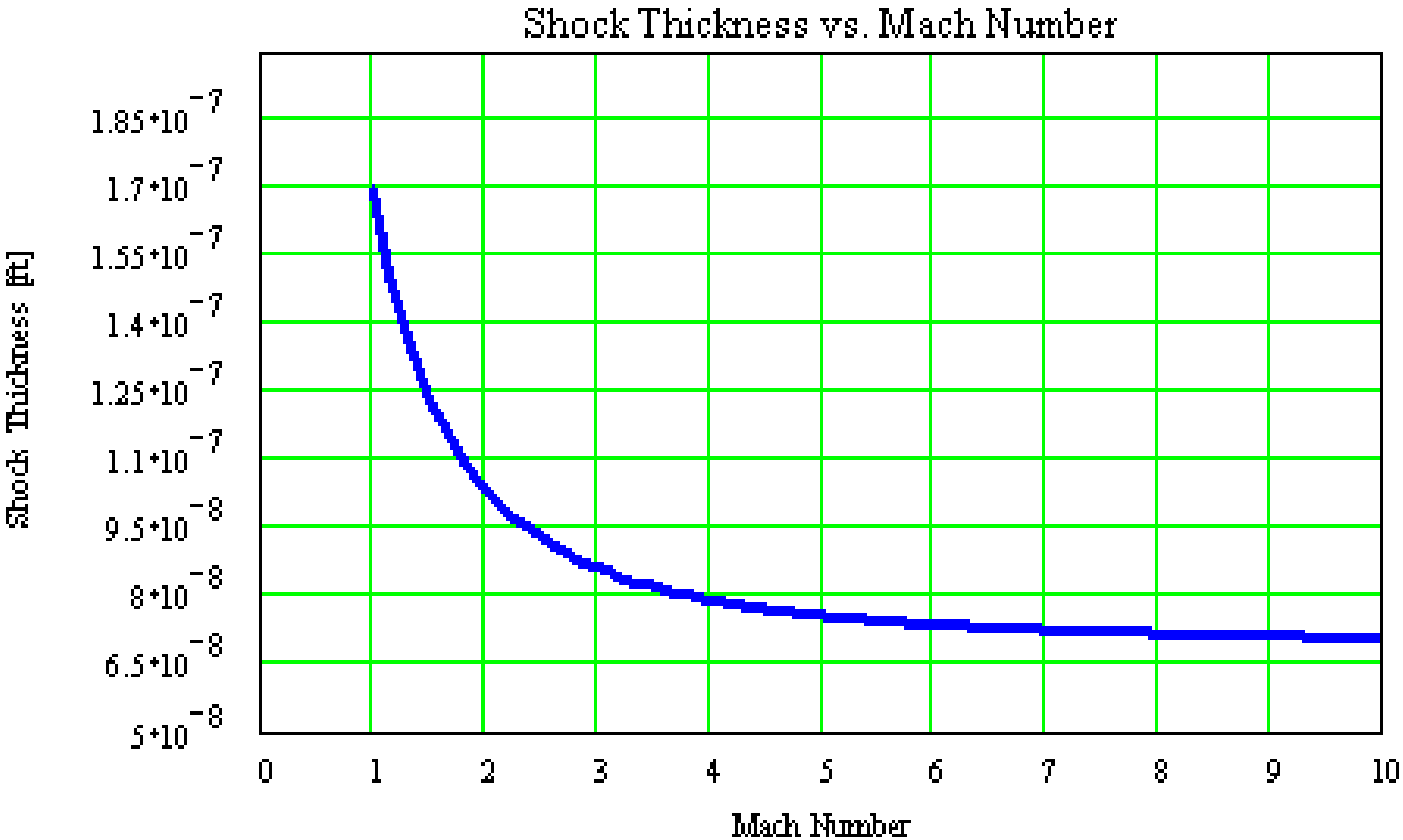


2. What is the nature of a shock wave?

When an object or a disturbance moves so fast that fluid particles cannot move out of the way, the molecular structure of the fluid permits the relative position of the molecules to move closer together to compress. Thus, as the mean free path of the molecules is reduced, layers of disturbances come together to form a shock wave. A pressure pulse is created, which grows larger and larger until it becomes a shock wave leading to a rise in temperature and pressure.

In short, as a disturbance approaches the speed of sound, gas molecules pack closer and closer together until a region of compressed fluid is formed, called a shock wave. A shock wave is actually a wave-front (波前) of compressed gas molecules with some unique properties.

Shock wave thickness tends to become smaller as Mach number is increased. For the example presented below，shock wave thickness is approximately 40% smaller at Mach = 10 than it was at Mach = 1.



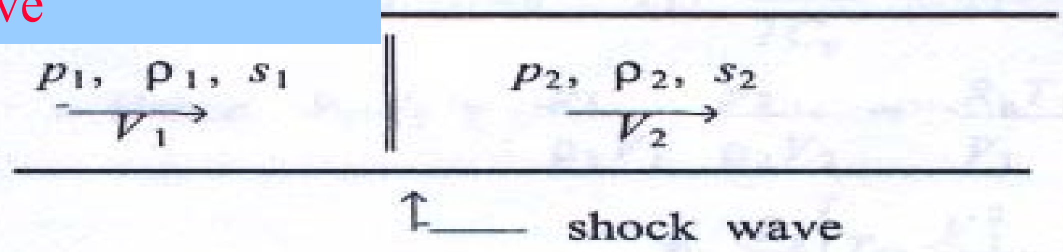
Shock Thickness vs. Mach Number
Shock Thickness [ft]
1.85*10^-7
1.7*10^-7
1.55*10^-7
1.4*10^-7
1.25*10^-7
1.1*10^-7
9.5*10^-8
8*10^-8
6.5*10^-8
5*10^-8
0
1
2
3
4
5
6
7
8
9
10
Mach Number

1ft= 0.3048m

3. Equations for normal shock waves (1-D steady flow)

Ahead of shock wave

Behind shock wave



- **Conservation of mass** $J \equiv V_1\rho_1 = V_2\rho_2$ (1)
- **Conservation of momentum** $p_1+\rho_1 V_1^2 = p_2+\rho_2 V_2^2$ (2)
- **Conservation of energy** $h_0 = h_1+\frac{V_1^2}{2} = h_2+\frac{V_2^2}{2}$ (3)

Perfect gas $$\frac{p_1}{\rho_1 T_1} = \frac{p_2}{\rho_2 T_2}$$

Constant c_p $$h = c_p T$$

(See equations 9.49a ~ 9.49e on page 630)

4. Equations for oblique shock waves

It is convenient to analyze the flow by breaking it up into normal and tangential components with respect to waves.

FW page 663: equation (9.80a ~ 9.85)

5. Shock wave refers to compression wave

Flow pressure increases after shock wave passing. Shock wave can not be an expansion (rarefaction shock) wave膨胀波,, which can possibly lead to a decrease in entropy (see the table on page 631)

$$\Delta s = s_2 - s_1 = C_v \ln \frac{T_2}{T_1} - R \ln \frac{\rho_2}{\rho_1}$$

9.6 Two-dimensional Supersonic flow (FW 9.9)

Flows for which $M<1$ are subsonic, while those with $M>1$ are supersonic. Flow fields have both subsonic and supersonic regions are termed transonic

Subsonic flow and supersonic flow behaves differently, some of the differences can be deduced quantitatively from propagation properties of a simple sound wave propagation in subsonic and supersonic flows.

To simplify our analysis, we consider a sound source moving at speed U in a still fluid, whose sound speed is a

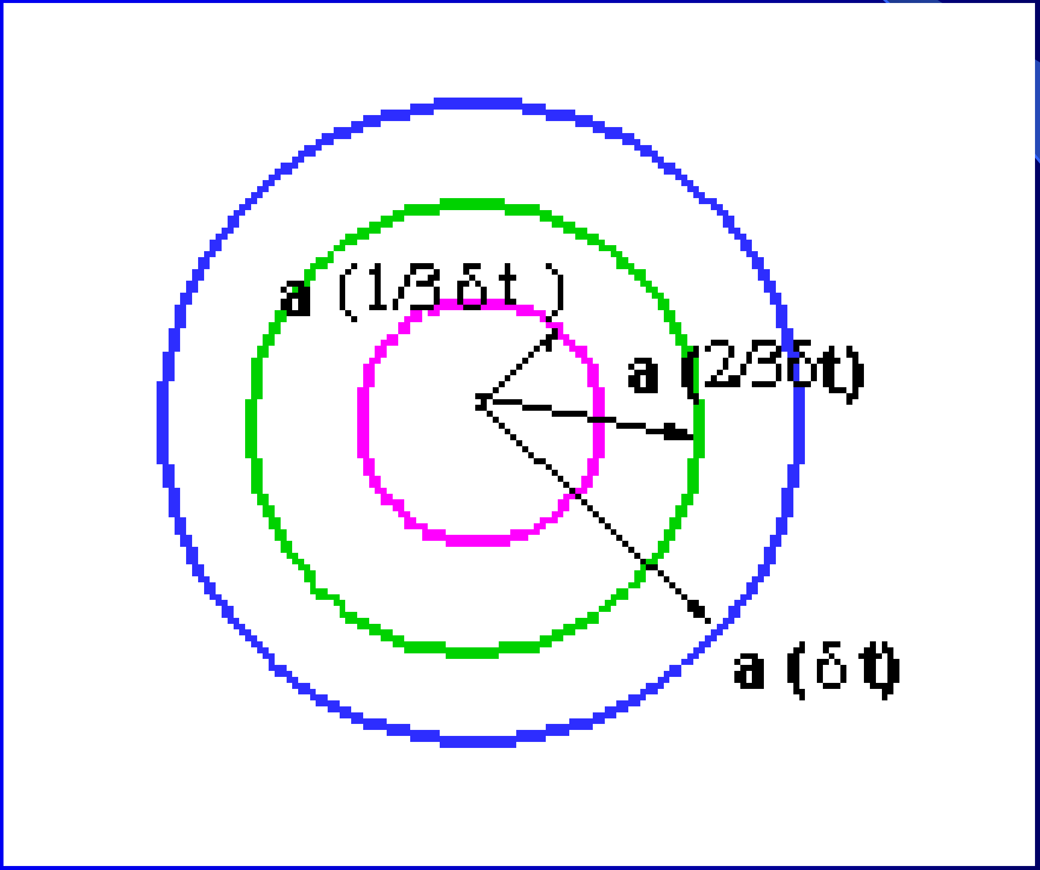
We assume

① sound source sends off spherical sound wave (pressure disturbances) at each time instant, $\frac{\delta t}{3}$

② and all the sound wave sent off at different time instant propagates in the flow field at speed of sound a.

Clearly, the propagation behavior is related to the speed of moving sound source.

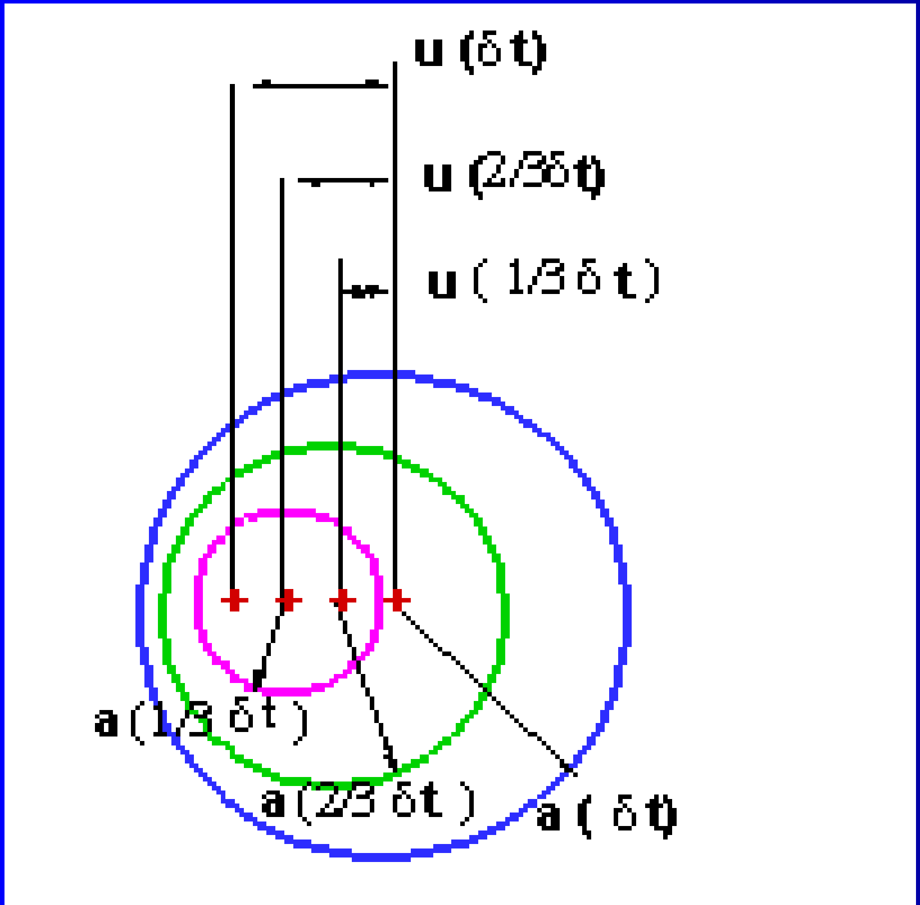
● **U=0**, the wave front (波前) at each time instant will form concentric circles, the sound wave propagates in the same way in all directions. *Staying at any location in the flow field we can hear the sound.*



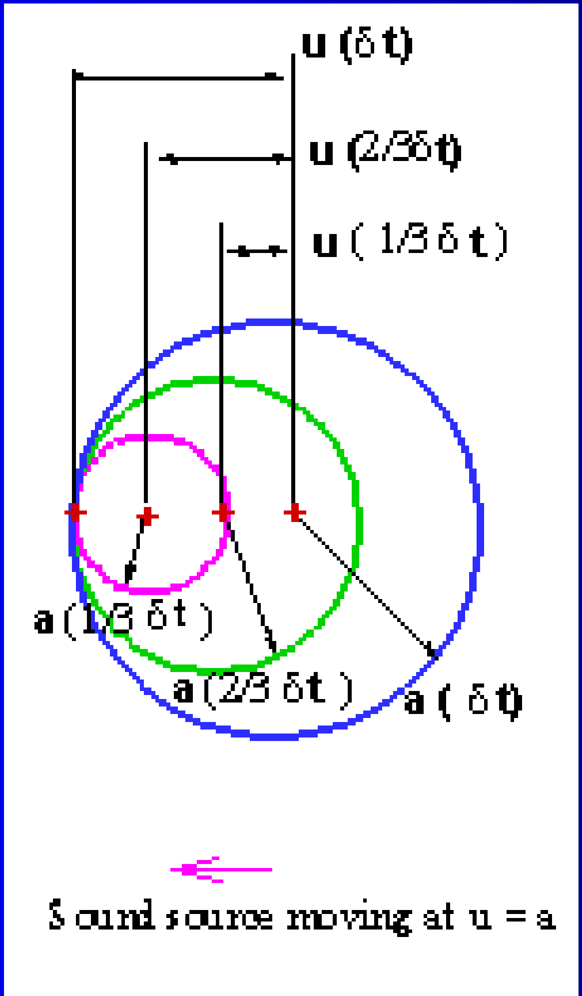
● **0<U<a** , The concentricity of wave pattern is lost.

Individual wave fronts are spherical, but each successive sound (sent off) emitted from a different position, $V\Delta t$ distant from the previous position.

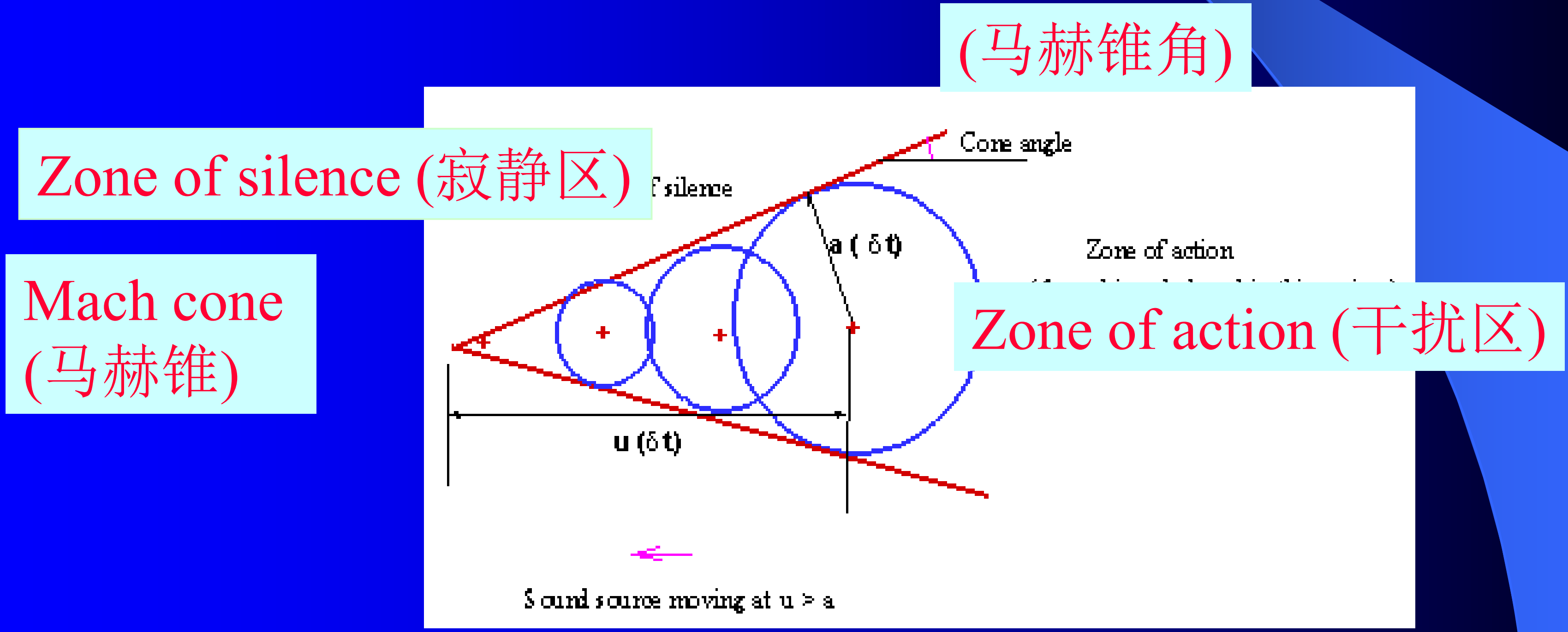
Thus a stationary observer hears more peaks per unit time as the source approaches. This is known as the **'Doppler effect'.**



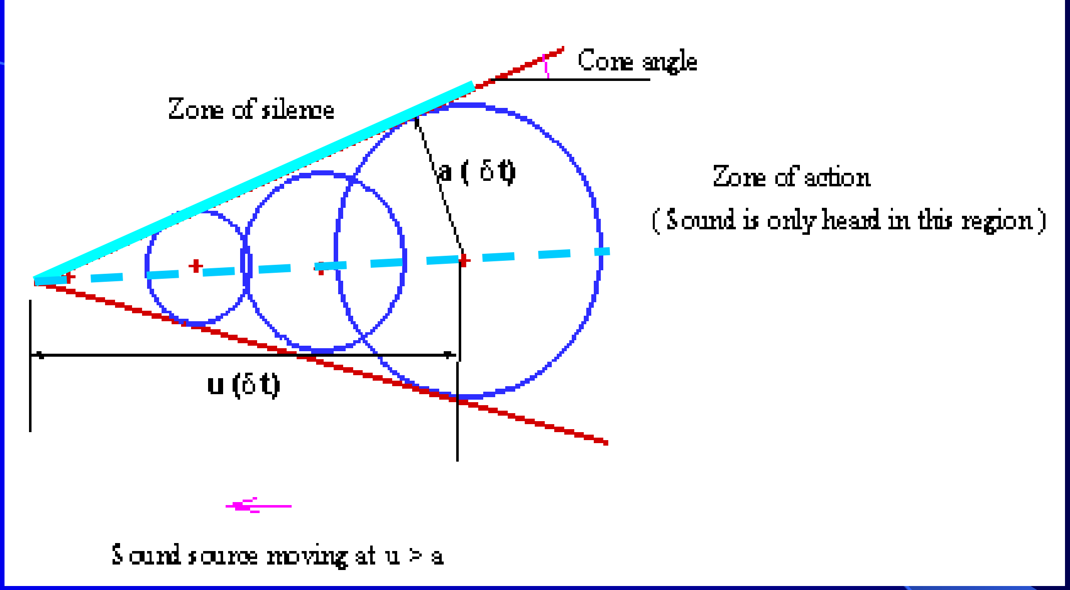
- **U=a,** the locus of leading surfaces of all waves will be at the **half plane downstream the sound source.** No sound wave can travel in front of the source. Consequently if you are in front of the source, you will not hear it approaching.



● **U>a,** In this case, the locus (圆图，轨迹) of leading surfaces of the sound waves will be a cone. Again, no sound will be heard in front of the cone. The region inside the cone is sometimes called the zone of action; the zone outside the cone called the zone of silence.



Mach wave: formed by wave fronts, wave front locus see Fig 9.19



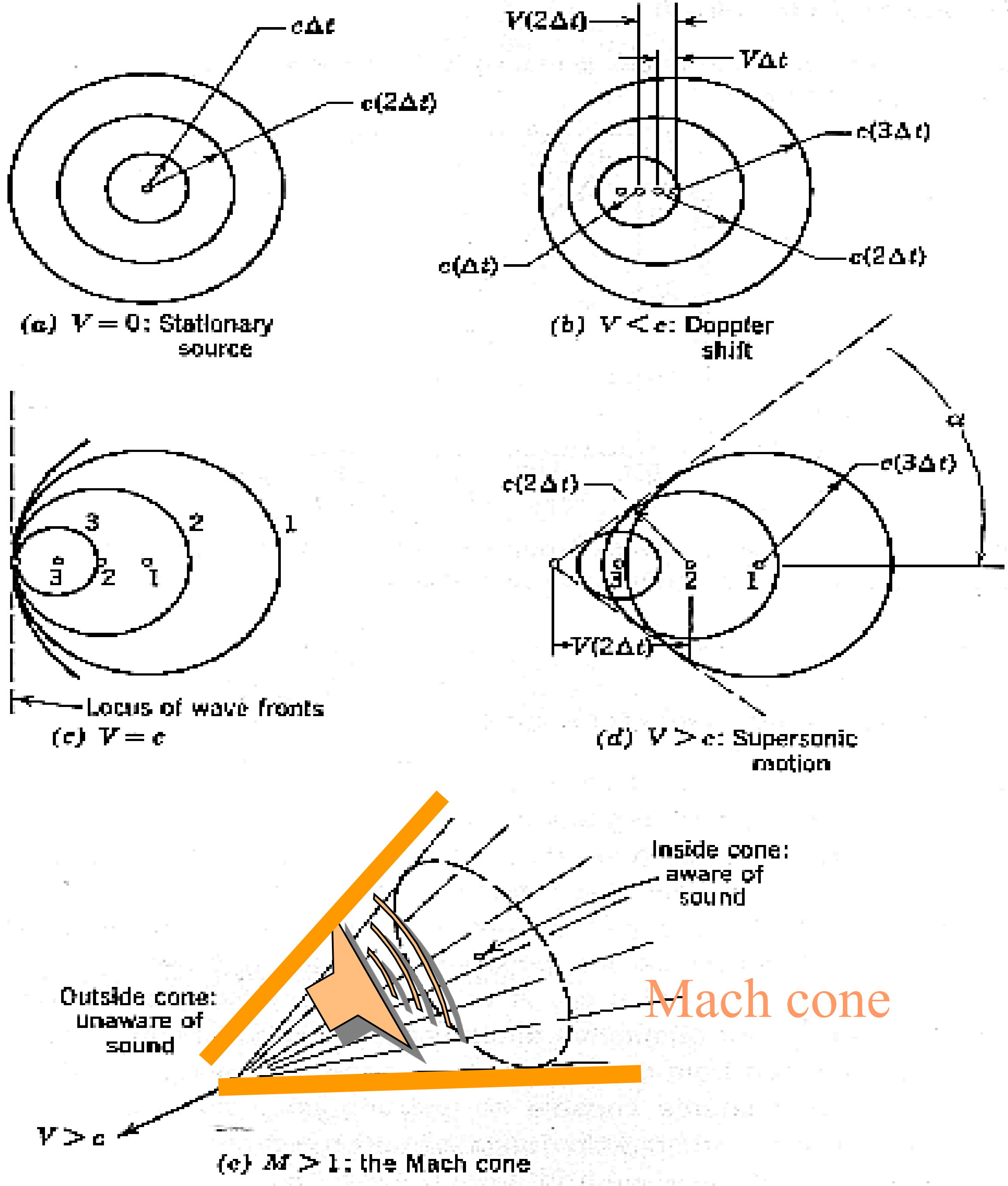
Cone angle – is related to the local Mach number in the flow field

$$\sin\alpha = \frac{a\ \delta t}{U\,\delta t} = \frac{1}{M}$$

$$\alpha = \sin^{-1}\left(\frac{1}{M}\right)$$

The cone is termed the *mach cone*. The sound wave front at the boundary of two zones has formed the so-called **Mach wave** of high-strength.

Propagation of sound waves from a moving source



As we hear the noise of the supersonic aircraft, it will disappear from our sight.