Part I
Recurrences

1. Given $T(n) = T(n-3) + 3lg(n)$, guess $T(n) = O(nlgn)$. Show $T(n) \leq cnlg(n)$ for some constant $c > 0$.
$$T(n-3) \leq c(n-3)\,lg(n-3)$$
Plugging this back in, we have:
$$T(n) \leq c(n-3)\,lg(n-3) + 3lg(n)$$
$$T(n) \leq cnlg(n-3) - 3clg(n-3) + 3lg(n) \leq cnlg(n) - 3clg(n) + 3lg(n) = cnlg(n) - 3lg(n)(c-1)$$
For $c > 1$, we now have:
$$T(n) \leq cnlg(n) - 3lg(n)(c-1) \leq cnlg(n)$$

2. Given $T(n) = 4T\left(\frac{n}{3}\right) + n$, guess $T(n) = O\left(n^{log_3 4}\right)$. Show $T(n) \leq cn^{log_3 4}$ for some constant $c > 0$.
$$T\left(\frac{n}{3}\right) \leq c\left(\frac{n}{3}\right)^{log_3 4} \leq cn^{log_3 4}$$
Plugging this back in, we have:
$$T(n) \leq 4c\left(\frac{n}{3}\right)^{log_3 4} + n = 4cn^{log_3 4}\frac{1}{3^{log_3 4}} + n = 4cn^{log_3 4}\frac{1}{4} + n = cn^{log_3 4} + n$$
Our guess was not good enough, we will try again subtracting off a lower order term; $T(n) \leq cn^{log_3 4} - 3n$
$$T\left(\frac{n}{3}\right) \leq c\left(\frac{n}{3}\right)^{log_3 4} - 3\frac{n}{3} \leq cn^{log_3 4} - n$$
$$T(n) \leq cn^{log_3 4} + n - n = cn^{log_3 4}$$

3. Given $T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + T\left(\frac{n}{8}\right) + n$, guess $T(n) = O(n)$. Show $T(n) \leq cn$ for some constant $c > 0$.
$$T\left(\frac{n}{2}\right) \leq c\frac{n}{2}$$
$$T\left(\frac{n}{4}\right) \leq c\frac{n}{4}$$
$$T\left(\frac{n}{8}\right) \leq c\frac{n}{8}$$
Plugging this back in, we have:
$$T(n) \leq c\frac{n}{2} + c\frac{n}{4} + c\frac{n}{8} + n \leq 3c\frac{n}{2} + n$$
This is not good enough. Trying a stronger guess, lets try $T(n) \leq cn - n$
$$T\left(\frac{n}{2}\right) \leq c\frac{n}{2} - \frac{n}{2}$$
$$T\left(\frac{n}{4}\right) \leq c\frac{n}{4} - \frac{n}{4}$$
$$T\left(\frac{n}{8}\right) \leq c\frac{n}{8} - \frac{n}{8}$$
Plugging this back in, we have:
$$T(n) \leq c\frac{n}{2} + c\frac{n}{4} + c\frac{n}{8} - \frac{3n}{8} + n \leq \frac{3}{2}cn - \frac{1}{8}n \leq cn$$
Because c is an arbitrary constant, we can 'absorb' the $\frac{3}{2}$ into c.

4. Given $T(n) = 4T\left(\frac{n}{2}\right) + n^2$, guess $T(n) = O\left(n^2\right)$. Show $T(n) \leq cn^2$ for come constant $c > 0$. Using the masters method, case 2, we can make a guess of $\Theta\left(n^2 log(n)\right)$. That is to say, $T(n) \leq cn^2 log(n)$.
$$T\left(\frac{n}{2}\right) \leq c\left(\frac{n}{2}\right)^2 log\left(\frac{n}{2}\right) = \frac{1}{4}cn^2\left(log(n) - log(2)\right) = \frac{1}{4}cn^2\left(log(n) - 1\right) = \frac{1}{4}cn^2 log(n) - \frac{1}{4}cn^2$$
Plugging this back in:
$$T(n) \leq 4\left(\frac{1}{4}cn^2 log(n) - \frac{1}{4}cn^2\right) + n^2 = cn^2 log(n) - cn^2 + n^2 = cn^2 log(n) - n^2(c-1)$$
If we fix $c > 1$, then we can prove our bound:
$$T(n) \leq cn^2 log(n) - n^2(c-1) \leq cn^2 log(n) = \Theta\left(n^2 log(n)\right)$$