Analysis of Algorithms

Nicholai L'Esperance

February 28, 2019

1. Show that the following statements are true:

   (a) $\frac{n(n-1)}{2}$ is $O\left(n^2\right)$
   To solve this, we must prove there exists constants c, n_0 such that $f(n) \leqq cg(n)\ \forall\, n \geqq n_0$. To do this, we can pick c, and solve for n_0. Picking c=1

$$\frac{n(n-1)}{2} \leqq n^2$$
$$\frac{n^2}{2} - \frac{n}{2} \leqq n^2$$
$$\frac{1}{2} - \frac{1}{2n} \leqq 1$$
$$-\frac{1}{2n} \leqq \frac{1}{2}$$
$$-1 \leqq n$$

   So, $\frac{n(n-1)}{2}$ is in $O\left(n^2\right)$, which we prove with the existance of $c = 1$, $n_0 = -1$ satisfying the definition given above. Note: n must be greater than 1, by definition, but that still satisfies the condition.

   (b) $max\left(n^3, 10n^2\right)$ is $O\left(n^3\right)$
   To solve this, it is proven that both of the individual functions are in $O\left(n^3\right)$, and therefore the maximum of the two must also be in $O\left(n^3\right)$. First, $f(n) = n^3$:

$$f(n) \leqq cg(n)$$
$$n^3 \leqq cn^3$$

   Picking c =1, we can solve for n_0.

$$n^3 \leqq n^3\ \therefore n^3 \in O\left(n^3\right), c = 1, n_0\, is\, arbitrary$$

   Next, we examine$f(n) = 10n^2$:

$$f(n) \leqq cg(n)$$
$$10n^2 \leqq cn^3$$

   Picking c=1:

$$10n^2 \leqq n^3$$
$$10 \leqq n$$
$$\therefore 10n^2 \in O\left(n^3\right), c = 1, n_0 = 10$$

   Now, we can put this together for the final solution.

$$max\left(n^3, 10n^2\right) \in O\left(n^3\right), c = 1, n_0 = 10$$

   (c) $\sum_{i=1}^{n} i^k$ is $O\left(n^{k+1}\right)$ for integer k.
   First, we can note the this summation is less than a very similar summation.

$$\sum_{i=1}^{n} i^k \leqq \sum_{i=1}^{n} n^k = nn^k = n^{k+1}$$

Now, if we can show that n^{k+1} is in $O\left(n^{k+1}\right)$, we then also know$\sum_{i=1}^{n} i^k$ is in $O\left(n^{k+1}\right)$, as we have shown that $\sum_{i=1}^{n} i^k \leqq n^{k+1}$. Setting $f(n) = n^{k+1}$:

$$\sum_{i=1}^{n} i^k \leqq f(n)\,,\ f(n) \leqq cg(n)$$

$$n^{k+1} \leqq cn^{k+1}$$

Setting c = 1, we can see

$$\sum_{i=1}^{n} i^k \leqq n^{k+1} \leqq n^{k+1}$$

$$\sum_{i=1}^{n} i^k \in O\left(n^{k+1}\right),\ c = 1, n_0\, is\, arbitrary.$$

Thus, given the existance of a constant c and n_0 we have proven the summation lies within $O\left(n^{k+1}\right)$.

(d) If P(x) is any k^{th} degree polynomail with a positive leading coefficient, then p(n) is $O\left(n^k\right)$.
First, let us define our polynomial.

$$p(n) = a_n n^k + a_{n-1}n^{k-1} + a_{n-2}n^{k-2} + \ldots + a_1 n + a_0$$

To solve this, we can show that a larger funcion falls within order $O\left(n^k\right)$. First, we can divide out n^k.

$$p(n) = n^k\left(a_n + \frac{a_{n-1}}{n} + \frac{a_{n-2}}{n^2} + \ldots + \frac{a_1}{n^{k-1}} + \frac{a_0}{n^k}\right)$$

Because some of the coefficients could be negative, we can take the absolute value of each, guaranteeing we are either increasing the value of the function, or making no change at all.

$$p(n) = n^k\left(a_n + \frac{a_{n-1}}{n} + \frac{a_{n-2}}{n^2} + \ldots + \frac{a_1}{n^{k-1}} + \frac{a_0}{n^k}\right) \leqq n^k\left(|a_n| + \left|\frac{a_{n-1}}{n}\right| + \left|\frac{a_{n-2}}{n^2}\right| + \ldots + \left|\frac{a_1}{n^{k-1}}\right| + \left|\frac{a_0}{n^k}\right|\right)$$

We again again note, that this function is also less than or equal to a similar function without n terms in the denominators.

$$p(n) \leqq n^k\left(|a_n| + \left|\frac{a_{n-1}}{n}\right| + \left|\frac{a_{n-2}}{n^2}\right| + \ldots + \left|\frac{a_1}{n^{k-1}}\right| + \left|\frac{a_0}{n^k}\right|\right) \leqq n^k\left(|a_n| + |a_{n-1}| + |a_{n-2}| + \ldots + |a_1| + |a_0|\right)$$

We can now define a new constant that is the sum of all of the coefficients.

$$a = \sum_{i=0}^{n} |a_i|$$

Now, we must prove that $p(n) \leqq cg(n)\ \forall\, n \geqq n_0$ for $g(n) = n^k$.

$$p(n) \leqq n^k a \leqq cn^k$$

We can choose a value of c=a.

$$p(n) \leqq n^k \leqq n^k$$

This inequality holds true for all values of n. **Thus, $p(n) \in O\left(n^k\right)$ as proven by the existance of constants c=a, and n_0 arbitrary (where a is defined as the sum of the absolute value of the polynomial coefficients).**

2. Which function grows faster?

(a) $n^{log_2(n)}$; $(log_2(n))^n$
To compare growth rates, given functions f(n) and g(n), determine if the following limit goes to 0 or infinity.

$$\lim_{n\to\infty} = \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^{log_2(n)}}{(log_2(n))^n}$$

Taking the base n log of the top and bottom.

$$\lim_{n\to\infty} \frac{n^{log_2(n)}}{(log_2(n))^n} = \lim_{n\to\infty} \frac{log_2(n)}{log_n\left((log_2(n))^n\right)}$$

Using the properties of logs, factor out the exponent in the denominator.

$$\lim_{n\to\infty} \frac{log_2(n)}{log_n\left((log_2(n))^n\right)} = \lim_{n\to\infty} \frac{log_2(n)}{log_n\left((log_2(n))(log_2(n))(log_2(n))\ldots\right)}$$

$$= \lim_{n\to\infty} \frac{log_2(n)}{log_n(log_2(n)) + log_n(log_2(n)) + \ldots} = \lim_{n\to\infty} \frac{log_2(n)}{nlog_n(log_2(n))}$$

Change the base of the upper logarithm.

$$log_2(n) = \frac{log_n(n)}{log_n(2)} = \frac{1}{log_n(2)} \quad \therefore \lim_{n\to\infty} \frac{1}{log_n(2)\, nlog_n(log_2(n))}$$

All terms in the denominator go to infinity as n goes to infinity, thus

$$\lim_{n\to\infty} \frac{n^{log_2(n)}}{(log_2(n))^n} = \lim_{n\to\infty} \frac{1}{log_n(2)\, nlog_n(log_2(n))} = 0$$

Because our limit approaches 0 as n approaches infinity, we know that g(n) grows faster than f(n), or $\underline{(log_2(n))^n}$ **grows faster than $n^{log_2(n)}$.**

(b) $log_2\left(n^k\right)$; $(log_2(n))^k$
To compare growth rates, given functions f(n) and g(n), determine if the following limit goes to 0 or infinity.

$$\lim_{n\to\infty} = \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{log_2\left(n^k\right)}{(log_2(n))^k} = \lim_{n\to\infty} \frac{klog_2(n)}{(log_2(n))(log_2(n))\ldots(log_2(n))} = \lim_{n\to\infty} \frac{k}{(log_2(n))^{k-1}}$$

This limit is then easily evaluated

$$\lim_{n\to\infty} \frac{k}{(log_2(n))^{k-1}} = k \lim_{n\to\infty} \frac{1}{(log_2(n))^{k-1}} = k(0) = 0$$

Because our limit approaches 0 as n approaches infinity, we know that g(n) grows faster than f(n), or $\underline{(log_2(n))^k}$ **grows faster than $log_2\left(n^k\right)$.**

(c) $n^{log_2(log_2(log_2(n)))}$; $(log_2(n))!$
To compare growth rates, given functions f(n) and g(n), determine if the following limit goes to 0 or infinity.

$$\lim_{n\to\infty} = \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^{log_2(log_2(log_2(n)))}}{(log_2(n))!}$$

We note, for all positive values of x, x is greater than the value of $log_2(x)$

$$0 \leqq log_2(x) < x \;\forall\, x > 0$$

$$\therefore log_2(log_2(log_2(n))) < log_2(log_2(n)) < log_2(n) \;\; \forall\, n > 1$$

$$\therefore n^{log_2(log_2(log_2(n)))} < n^{log_2(n)} \;\forall\, n > 1$$

$$0 \leqq \lim_{n\to\infty} \frac{n^{log_2(log_2(log_2(n)))}}{(log_2(n))!} < \lim_{n\to\infty} \frac{n^{log_2(n)}}{(log_2(n))!}$$

Take the base n log of top and bottom.

$$\lim_{n\to\infty} \frac{log_2(n)}{log_n((log_2(n))!)}$$

Change the base of the top logarithm.

$$log_2(n) = \frac{log_n(n)}{log_n(2)} = \frac{1}{log_n(2)} \quad \therefore \lim_{n\to\infty} \frac{log_2(n)}{log_n((log_2(n))!)} = \lim_{n\to\infty} \frac{1}{log_n(2)\, log_n((log_2(n))!)}$$

All terms in the denominator go to infinity as n goes to infinity. Thus:

$$\lim_{n\to\infty} \frac{1}{log_n(2)\, log_n((log_2(n))!)} = 0$$

Finally, by the squeeze theorm, we know:

$$0 \leqq \lim_{n\to\infty} \frac{n^{log_2(log_2(log_2(n)))}}{(log_2(n))!} < \lim_{n\to\infty} \frac{n^{log_2(n)}}{(log_2(n))!} = 0 \;\therefore \lim_{n\to\infty} \frac{n^{log_2(log_2(log_2(n)))}}{(log_2(n))!} = 0$$

Because our limit approaches 0 as n approaches infinity, we know that g(n) grows faster than f(n), or $\underline{(log_2(n))!}$ **grows faster than $n^{log_2(log_2(log_2(n)))}$.**

(d) n^n; $n!$
To compare growth rates, given functions f(n) and g(n), determine if the following limit goes to 0 or infinity.

$$\lim_{n\to\infty} = \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^n}{n!} = \lim_{n\to\infty} \frac{n\cdot n\cdot n\cdot \ldots \cdot n}{n\cdot (n-1)\cdot (n-2)\cdot \ldots \cdot 2\cdot 1} = \lim_{n\to\infty} \frac{n^{n-1}}{(n-1)!}$$

Looking at the two sequences, it is clear that the denominator is larger than the numerator.

$$(n-1)! = \prod_{i=1}^{n-1} i$$

$$n^{n-1} = \prod_{i=1}^{n-1} n$$

$$\prod_{i=1}^{n-1} i \leqq \prod_{i=1}^{n-1} n$$

Thus, we can evaluate our limit.

$$\lim_{n\to\infty} \frac{n^n}{n!} = \lim_{n\to\infty} \frac{n^{n-1}}{(n-1)!} = \infty$$

Becuase our limit approaches infinity, we know that our function f(n) grows faster than g(n), or **n^n grows faster than $n!$.**

3. If $f_1(n)$ is $O(g_1(n))$ and $f_2(n)$ is $O(g_2(n))$ where f_1 and f_2 are positive functions of n, show that the function $f_1(n) + f_2(n)$ is $O(max(g_1(n), g_2(n)))$.
We know the following relations exist.

$$f_1(n) \leqq c_1 g_1(n)\ \forall n \geqq n_{0_1}$$

$$f_2(n) \leqq c_2 g_2(n)\ \forall n \geqq n_{0_2}$$

We can define a new constant c_3 that is the maximum of c_1 and c_2.

$$c_1 g_1(n) + c_2 g_2(n) \leqq c_3 g_1(n) + c_3 g_2(n) = c_3(g_1(n) + g_2(n))$$

Thus, we know

$$f_1(n) + f_2(n) \leqq c_1 g_1(n) + c_2 g_2(n) \leqq c_3(g_1(n) + g_2(n))\ \forall n \geqq max(n_{0_1}, n_{0_2})$$

We can then use another inequality to get our max.

$$c_3(g_1(n) + g_2(n)) \leqq c_3\cdot 2\cdot max(g_1(n), g_2(n))$$

Putting it all together, we can define our $n_0 = max(n_{0_1}, n_{0_2})$, and $c = 2c_3 = 2max(c_1, c_2)$

$$f_1(n) + f_2(n) \leqq c\cdot max(g_1(n), g_2(n))\ \forall n \geqq n_0$$

Thus we know that, given $f_1(n) \in O(g_1(n))$ and $f_2(n) \in O(g_2(n))$, **there exists constants c, and n_0 to satisfy our inequality proving $f_1(n) + f_2(n) \in O(max(g_1(n), g_2(n)))$.**

4. Prove or disprove: Any positive n is $O\left(\frac{n}{2}\right)$.
To solve this, we must prove there exists constants c, n_0 such that $f(n) \leqq cg(n)\ \forall\, n \geqq n_0$. To prove this in the arbitrary case, we define f(n) = bn, where b is an arbitrary positive constant.

$$f(n) \leqq cg(n)$$

$$bn \leqq c\frac{n}{2}$$

We can divide both sides by by, and rearrange.

$$n \leqq \frac{c}{2b}n$$

We select $c = \frac{1}{2b}$, giving us

$$n \leqq n$$

Thus, we have proven that every positive funcion $f(n) = bn$, where b is an arbitrary positive constant, exists in $O\left(\frac{n}{2}\right)$. Constant $c = \frac{1}{2b}$ and any arbitrary n_0 satisfy our defition $f(n) \leqq cg(n)\ \forall\, n \geqq n_0$.

5. Prove or disprove: 3^n is $O(2^n)$.
   To solve this, we must prove there exists constants c, n_0 such that $f(n) \leqq cg(n)\ \forall\, n \geqq n_0$.

$$f(n) \leqq cg(n)$$

$$3^n \leqq c2^n$$

$$\frac{3^n}{2^n} \leqq c$$

$$\left(\frac{3}{2}\right)^n \leqq c$$

$$log_{\frac{3}{2}}\left(\left(\frac{3}{2}\right)^n\right) \leqq log_{\frac{3}{2}}(c)$$

$$n \leqq log_{\frac{3}{2}}(c)$$

   In order for function $f(n) = 3^n$ to exist in $O(2^n)$, we must find constants c, n_0 that satisfy $f(n) \leqq cg(n)\ \forall\, n \geqq n_0$. However, the only solution to this inequality requires that n be less than $log_{\frac{3}{2}}(c)$. Thus, there exists no c, n_0 that can satisfy the condition, and we can say that 3^n **is not in** $O(2^n)$**.**