Stochastic Methods SP 2024

Assignment 2, Due: 22th March (9.00am)

Name: ______________________________

1: A spider hunting a fly moves between locations 1 and 2 according to a Markov chain with transition matrix

$$\begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix},$$

starting in location 1. The fly, unaware of the spider, starts in location 2 and moves according to a Markov chain with transition matrix

$$\begin{bmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{bmatrix}.$$

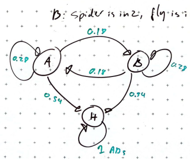
The spider catches the fly and the hunt ends whenever they meet in the same location.

Show that the progress of the hunt, except for knowing the location where it ends, can be described by a three-state Markov chain where one absorbing state represents hunt ended and the other two that the spider and fly are at different locations.

a) Obtain the transition matrix for this chain.

$S = \{1, 2, H\}$

state:
H: they both in same pos.
A: Spider is in 1, fly is in 2
B: spider is in 2, fly is in 1



• $p(A \to B) = p_S(1 \to 2) \cdot p_F(2 \to 1) = 0.3 \cdot 0.6 = 0.18$

• $p(B \to A) = 0.3 \cdot 0.6 = 0.18$

• $p(B \to B) = 0.7 \cdot 0.4 = 0.28$

• $p(A \to A) = 0.7 \cdot 0.4 = 0.28$

• $p(A \to H) = p_S(1) \cdot p_F(2 \to 1) + p_S(1 \to 2) \cdot p_F(2) = (0.7 \cdot 0.6) + (0.3 \cdot 0.4) = 0.42 + 0.12 = 0.54$

• $p(B \to H) = (0.3 \cdot 0.4) + (0.7 \cdot 0.6) = 0.54$

$$\begin{array}{c} \\ A \\ B \\ H \end{array} \begin{array}{c} \begin{array}{ccc} A & B & H \end{array} \\ \begin{pmatrix} 0.28 & 0.18 & 0.54 \\ 0.18 & 0.28 & 0.54 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

b) Find the probability that at time n the spider and fly are both at their initial locations.

For n steps theres no algorithm to compute it.

For infinite steps, since the Pha and Phb are going to always be 0, it means that its NOT irreducible, which means its not ergotic, which means that there is not stationary convergence probability.

2: The Absolute Value of the Simple Random Walk. Consider a random walk $\{S_n, n \geq 1\}$, where $S_n = \sum_1^n X_i$, and it is said to be a simple random walk if for some $p, 0 < p < 1$,

$$P(X_i = 1) = p,$$

$$P(X_i = -1) = q = 1 - p.$$

In the simple random walk, the process always either goes up one step (with probability p) or down one step (with probability q). Now consider $|S_n|$, the absolute value of the simple random walk. The process $\{|S_n|, n \geq 1\}$ measures at each time unit the absolute distance of the simple random walk from the origin. Show that $|S_n|$ is itself a Markov chain and determine the probability that $|S_n| = i$.

Hint: Reflect on the symmetry of the random walk and consider the implications of the process's independent increments.

q^n ... p^n

3: A graph consisting of a central vertex, labeled 0, and rays emanating from that vertex is called a star graph (see Figure 1). Let r denote the number of rays of a star graph and let ray i consist of n vertices, for $i = 1, \ldots, r$. Suppose that a particle moves along the vertices of the graph so that it is equally likely to move from whichever vertex it is presently at to any of the neighbors of that vertex, where two vertices are said to be neighbors if they are joined by an edge. Thus, for instance, when at vertex 0 the particle is equally likely to move to any of its r neighbors. The vertices at the far ends of the rays are called *leafs*. What is the probability that, starting at node 0, the first leaf visited is the one on ray i, $i = 1, \ldots, r$?

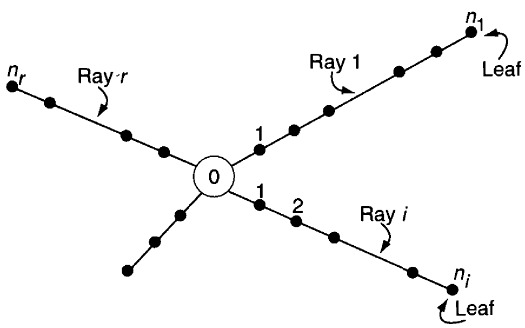


Figure 1: A star graph.

4: Consider a Markov chain with states indexed by i, where each state's dynamics are governed by the probability f_i of the chain returning to state i after leaving it. States are classified based on f_i as follows:

$$\text{State } i \begin{cases} \text{is } \textit{recurrent} & \text{if } f_i = 1, \\ \text{is } \textit{transient} & \text{if } f_i < 1. \end{cases} \tag{1}$$

(a) In a branching process, identify which states are transient and which are recurrent, providing mathematical justification for your classification.

(b) The classification into transient and recurrent states can be determined by analyzing the expected number of visits, E_i, starting from state i:

$$E_i = \sum_{n=0}^{\infty} P(\text{visiting state } i \text{ after } n \text{ steps}), \tag{2}$$

Prove that if a state is recurrent, then the expected number of visits E_i is infinite. Prove that if a state is transient, then the expected number of visits E_i is finite.

5: Consider a particle that moves along a set of $m+1$ nodes, labeled $0, 1, \ldots, m$, that are arranged around a circle. At each step, the particle is equally likely to move one position in either the clockwise or counterclockwise direction. That is, if X_n is the position of the particle after its nth step, then:

$$P(X_{n+1} = i+1 | X_n = i) = P(X_{n+1} = i-1 | X_n = i) = \frac{1}{2}$$

where $i+1 = 0$ when $i = m$, and $i-1 = m$ when $i = 0$. Suppose now that the particle starts at 0 and continues to move around according to the above rules until all the nodes $1, 2, \ldots, m$ have been visited. What is the probability that node i, $i = 1, \ldots, m$, is the last one visited?

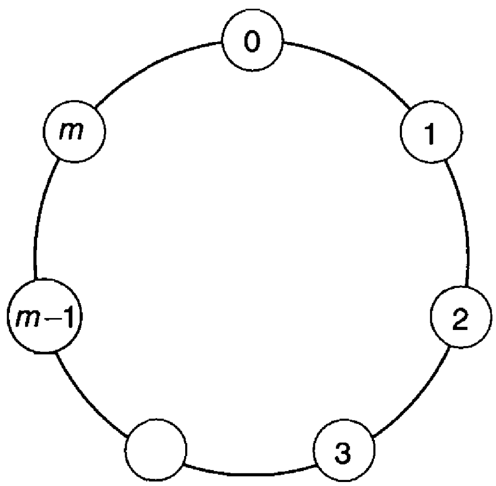


Figure 2: A particle moving around a circle.