Lecture Notes on Markov Chains I

Week 2: Markov Chains I - Fundamental principles and transient states

Introduction to Markov Chains

Definition 1. *A **stochastic process** $\{X(t) : t \in T\}$ is a collection of random variables, where for each t in the index set T, $X(t)$ is a random variable. The index t often represents time, making $X(t)$ denote the state of the process at time t. The set T is known as the index set of the process. When T is countable, the process is said to be in discrete time. If T is an interval of the real line, the process is in continuous time.*

This broad definition encompasses the wide variety of processes that evolve over time, where the evolution is driven by some probabilistic rules. Examples include the number of customers in a store at any given time, the amount of rainfall accumulated in a day, or the stock price of a company.

Definition: Markov Chain

A Markov Chain is a stochastic process that satisfies the Markov property, meaning the future state depends only on the current state and not on the sequence of events that preceded it. Formally, for any set of states i, j and any time n, the transition probabilities satisfy:

$$P\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\} = P\{X_{n+1} = j | X_n = i\} = P_{ij}.$$

Examples of Markov Chains include:

- **Random Walk Model:** An individual moves along a line, stepping right with probability p and left with probability $1 - p$. The state space is the set of integers $\mathbb{Z}$.
- **Gambling Model:** A gambler wins 1 with probability p or loses 1 with probability $1-p$, stopping when they go broke or reach N. The state space is $\{0, 1, \ldots, N\}$, with 0 and N as absorbing states.
- **Weather Forecasting Model:** The weather is either "rain" or "no rain" with transition probability α for rain to rain and β for no rain to rain.

- **Communications System:** A system transmits digits 0 or 1, with each digit being transmitted correctly with probability p. This can be modeled as a two-state Markov chain.
- **Mood Model:** An individual's mood is modeled as cheerful, so-so, or glum, with transitions dependent only on the current state, forming a three-state Markov chain.

Transition Matrix

The transition matrix P encapsulates the probabilities of moving from one state to another in one time step. For a Markov chain with m states, P is an $m \times m$ matrix where the element P_{ij} represents the probability of transitioning from state i to state j.

Example: **Weather Forecasting Model**

Suppose the chance of rain tomorrow depends only on whether or not it is raining today. If it rains today (state 0), it will rain tomorrow with probability α; if it does not rain today (state 1), it will rain tomorrow with probability β. The transition matrix P is given by:

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix}$$

Example: **Communications System**

Consider a system transmitting digits 0 and 1. At each stage, there is a probability p that the digit will be transmitted correctly. Letting X_n denote the digit at stage n, we have a two-state Markov chain with transition matrix:

$$P = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

Example: **Mood Model**

On any given day, Gary's mood can be cheerful (C), so-so (S), or glum (G) with transition probabilities dependent on today's mood. The transition matrix P is:

$$P = \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}$$

Example: **Random Walk Model**

A model for an individual walking on a line who at each point either steps right with probability p or left with probability $1-p$. For state i, the transition probabilities to $i+1$ and $i-1$ are:

$$P_{i,i+1} = p = 1 - P_{i,i-1}$$

For a finite state space version of the **Random Walk Model**, with states $\{0, 1, 2, \ldots, N\}$ where N is a boundary, the transition probabilities for moving from state i to $i+1$ (right) with probability p and to $i-1$ (left) with probability $1-p$ can be represented as:

$$P = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1-p & 0 & p & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1-p & 0 & p \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

Note: This matrix is modified to include absorbing states at 0 and N for illustration purposes.

Example: **Gambling Model**

A gambler either wins \$1 with probability p or loses \$1 with probability $1-p$. With absorbing states at 0 and N (broke or target fortune), the transition probabilities for states i to $i+1$ and $i-1$ are similar to the random walk:

$$P_{i,i+1} = p = 1 - P_{i,i-1}$$

Absorbing states:

$$P_{00} = P_{NN} = 1$$

In the **Gambling Model**, with the gambler starting with a stake of i dollars and with absorbing states at 0 and N, the transition probabilities can be written as:

$$P = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ q & 0 & p & \cdots & 0 & 0 \\ 0 & q & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Where p is the probability of winning \$1 and $q = 1 - p$ is the probability of losing \$1. Here, states 0 and N are absorbing, meaning if the gambler reaches these states, they stay there indefinitely.

Chapman-Kolmogorov Equations

The *n-step transition probabilities*, denoted by $P_{ij}^{(n)}$, define the probability that a process in state i will transition to state j after n steps. Formally, for $n \geq 0$ and states i, j, we have:

$$P_{ij}^{(n)} = \Pr(X_{n+k} = j \mid X_k = i)$$

where $P_{ij}^{(1)} = P_{ij}$ corresponds to the one-step transition probability from state i to state j.

The Chapman-Kolmogorov equations describe how to compute the probabilities of transitioning from one state to another in a Markov chain over multiple steps. Given a Markov chain with states, the n-step transition probability from state i to state j, denoted as $P_{ij}^{(n)}$, is the probability of transitioning from i to j in n steps.

These probabilities can be computed using the Chapman-Kolmogorov equations, which are as follows:

$$P_{ij}^{(n+m)} = \sum_{k=0}^{\infty} P_{ik}^{(n)} P_{kj}^{(m)} \tag{1}$$

This equation states that the probability of moving from state i to state j in $n+m$ steps is the sum, over all possible intermediate states k, of the probability of moving from i to k in n steps and then from k to j in m steps.

In matrix notation, if $P^{(n)}$ denotes the matrix of n-step transition probabilities, then the Chapman-Kolmogorov equation can be expressed as:

$$P^{(n+m)} = P^{(n)} \cdot P^{(m)} \tag{2}$$

where the dot represents matrix multiplication. This formula is particularly useful for computing transition probabilities over multiple steps in a compact and efficient manner.

Example: Consider the weather as a two-state Markov chain, where state 0 represents rain and state 1 represents no rain. Given the transition probabilities $\alpha = 0.7$ for rain to rain and $\beta = 0.4$ for no rain to rain, we calculate the probability that it will rain four days from today, given that it is raining today.

The one-step transition probability matrix is:

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$$

To find the two-day transition probabilities, we compute $P^{(2)} = P^2$:

$$P^{(2)} = P \cdot P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} \cdot \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix}$$

For the four-day transition probabilities, we compute $P^{(4)} = (P^{(2)})^2$:

$$P^{(4)} = (P^{(2)})^2 = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix} \cdot \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix} = \begin{pmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{pmatrix}$$

Therefore, the probability that it will rain four days from today, given that it is raining today (i.e., the transition from state 0 to state 0 in four steps), is $P_{00}^{(4)} = 0.5749$

Example: Weather Dependence as a Four-State Markov Chain

Consider the weather as a four-state Markov chain with the following states based on the weather conditions of today and yesterday:

- State 0: It rained both today and yesterday.

- State 1: It rained today but not yesterday.
- State 2: It rained yesterday but not today.
- State 3: It did not rain either yesterday or today.

The transition probability matrix P is given by:

$$P = \begin{pmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{pmatrix}$$

Given that it rained on Monday and Tuesday, we have the initial condition as state 0. To find the probability that it will rain on Thursday, we need to compute the two-step transition probabilities by squaring the matrix P:

$$P^{(2)} = P^2 = \begin{pmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{pmatrix}^2$$

After computing the matrix multiplication, we get:

$$P^{(2)} = \begin{pmatrix} 0.49 & 0.12 & 0.21 & 0.18 \\ 0.35 & 0.20 & 0.15 & 0.30 \\ 0.20 & 0.12 & 0.20 & 0.48 \\ 0.10 & 0.16 & 0.10 & 0.64 \end{pmatrix}$$

To calculate the probability of rain on Thursday, we sum the probabilities of being in state 0 or 1 on Thursday:

$$P_{00}^{(2)} + P_{01}^{(2)} = 0.49 + 0.12 = 0.61$$

Thus, the probability that it will rain on Thursday, given that it rained on Monday and Tuesday, is 0.61.

Characteristics of Markov Chains

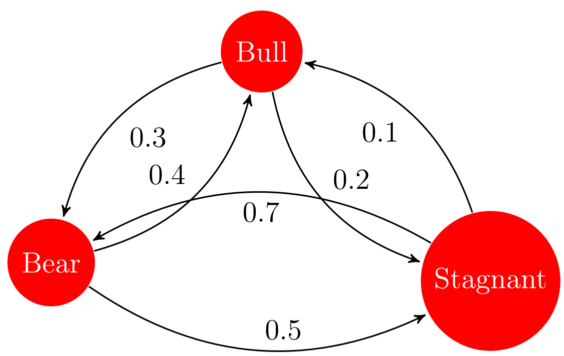
1. **Transition Probabilities**: The probabilities of moving from one state to another are called transition probabilities. They are typically represented in a matrix called the transition matrix.
2. **State Space**: The set of all possible states that the chain can be in. This could be a finite or countably infinite set.

3. **Time Structure**: Transitions occur at integer time steps.

Example 1 (Financial Market Dynamics)**.** *In financial market modeling, a Markov Chain captures the inherent volatility and unpredictability of the market. Here, the future state only depends on the current state, disregarding the historical path.*

Market States:

- ***Bull Market (Bull):*** *Optimistic phase with rising or expected-to-rise prices.*
- ***Bear Market (Bear):*** *Pessimistic phase with prolonged price declines.*
- ***Stagnant Market (Stagnant):*** *No clear trend; prices fluctuate within a narrow range.*



Transition Matrix P***:***

$$P = \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.5 \\ 0.1 & 0.7 & 0.2 \end{pmatrix}$$

Two-Step Transition P^2***:***

$$P^2 = \begin{pmatrix} 0.39 & 0.32 & 0.29 \\ 0.29 & 0.48 & 0.23 \\ 0.35 & 0.24 & 0.41 \end{pmatrix}$$

Three-Step Transition P^3***:***

$$P^3 = \begin{pmatrix} 0.352 & 0.352 & 0.296 \\ 0.360 & 0.296 & 0.344 \\ 0.312 & 0.416 & 0.272 \end{pmatrix}$$

Interpretation: *Transition matrices* P^2 *and* P^3 *offer insights into the market's longer-term dynamics. For instance, a Bull to Bear transition becomes more probable in three steps, highlighting the dynamic, unpredictable nature of financial markets modeled through Markov Chains.*

Let $P_{ij}^{(n)}$ be the probability that the system transitions from state i to state j in n steps. Then:

$$P_{ij}^{(n)} = P(S_{t+n} = j | S_t = i) = (P^n)_{ij}$$

Notes:

- $P_{ij}^{(n)}$ can be found using the (i, j)th element of the matrix P^n.
- The potential paths from i to j in n steps are up to m^{n-1} (m being the number of states).
- Matrix multiplication of P by itself n times accumulates all transition probabilities.

Another representation of this concept, considering any times $s < t < u$, is given by:

$$P_{ij}(s, u) = \sum_{k \in S} P_{ik}(s, t) \cdot P_{kj}(t, u) \tag{3}$$

Financial Market Dynamics with Chapman-Kolmogorov Equations Consider the previously discussed financial market model. If we're interested in the probability of transitioning from a Bull market to a Bear market over five steps, P^5, we can employ the Chapman-Kolmogorov equation. Using our already computed matrices P^2 and P^3, the equation becomes:

$$(P^5)_{\text{Bull, Bear}} = \sum_k (P^2)_{\text{Bull, } k} \cdot (P^3)_{k,\text{Bear}} \tag{4}$$

Using the above equation, we find that:

$$(P^5)_{\text{Bull, Bear}} \approx 0.3526$$

Here, k represents all possible market states (Bull, Bear, Stagnant). This methodology not only simplifies computations but also offers insights into multi-step transitions in financial markets, enabling better predictive models.

Example: Pensioner's Financial Model

Consider a pensioner who receives 2 (thousand francs) at the beginning of each month. The amount required for monthly expenses is independent of his current capital and equals i with probability P_i, for $i = 1, 2, 3, 4$, ensuring $\sum_{i=1}^{4} P_i = 1$. Should the pensioner's end-of-month capital exceed 3, the excess is given to his son. Assuming an initial capital of 5, we investigate the probability of the pensioner's capital dropping to 1 or less within the next four months.

Solution: We model the pensioner's financial status as a Markov chain, with the state representing the end-of-month capital. To focus on scenarios where the capital drops to 1 or less, state 1 signifies such occurrences. Given the pensioner's practice of giving away excess capital, our analysis is limited to states 1, 2, and 3. The transition probability matrix $Q = [Q_{i,j}]$ is defined as:

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ P_3 + P_4 & P_2 & P_1 \\ P_4 & P_3 & P_1 + P_2 \end{pmatrix}.$$

Considering $P_i = 1/4$ for $i = 1, 2, 3, 4$, the matrix simplifies to:

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}.$$

Upon squaring this matrix twice, we obtain:

$$Q^{(4)} = \begin{pmatrix} 1 & 0 & 0 \\ 222/256 & 13/256 & 21/256 \\ 201/256 & 21/256 & 34/256 \end{pmatrix}.$$

Given the pensioner's starting capital as 3, the probability of it reducing to 1 or less within four months is $Q^{(4)}_{3,1} = 201/256$.

Limit Distribution: Irreducibility, Aperiodicity, and Ergodicity

Limiting distribution, in the context of Markov chains, refers to the distribution to which the state probabilities converge as the number of steps (or time) goes to infinity. This concept is crucial for understanding the long-term behavior of stochastic processes.

Markov Chains, with their inherent ability to model complex stochastic systems, have significant applications across various domains, from finance and meteorology to social sciences. One of the most pivotal inquiries in the realm of Markov Chains pertains to their long-term behavior. Specifically, will the system stabilize into a steady state or equilibrium? This section delves into the notions that underpin this behavior, including the limit distribution and the properties of irreducibility, aperiodicity, and ergodicity.

Definition 2 (Limit Distribution). *A distribution π is called a limit distribution for a Markov Chain if*

$$\lim_{n\to\infty} P^n_{ij} = \pi_j$$

for every state i. The limit distribution provides insights into the enduring behavior of the system and is inherently linked to the properties discussed below.

The distribution π encapsulates the stable or steady-state probabilities associated with each state as the number of transitions grows indefinitely large. For a finite Markov Chain, the cumulative sum of all elements of π equals 1, emphasizing that π is a probability distribution.

Example 2 (Convergence of P to π). *For the given transition matrix P, let's inspect its powers:*

For P^5:

$$\begin{bmatrix} 0.34296 & 0.35264 & 0.3044 \\ 0.34664 & 0.33984 & 0.31352 \\ 0.33752 & 0.3648 & 0.29768 \end{bmatrix}$$

For P^{10}:

$$\begin{bmatrix} 0.34260 & 0.35183 & 0.30557 \\ 0.34251 & 0.35210 & 0.30539 \\ 0.34268 & 0.35159 & 0.30573 \end{bmatrix}$$

For P^{20}:

$$\begin{bmatrix} 0.34259 & 0.35185 & 0.30556 \\ 0.34259 & 0.35185 & 0.30556 \\ 0.34259 & 0.35185 & 0.30556 \end{bmatrix}$$

By the time we examine P^{50} and P^{100}, the matrix has stabilized to:

$$\begin{bmatrix} 0.34259 & 0.35185 & 0.30556 \\ 0.34259 & 0.35185 & 0.30556 \\ 0.34259 & 0.35185 & 0.30556 \end{bmatrix}$$

From the matrices above, we discern a clear trend: as we raise T to higher powers, the rows of the matrix are converging to the limit distribution π. This showcases the theoretical underpinning that, given certain conditions, the Markov Chain will stabilize to a unique long-term distribution.

Definition 3. *A Markov Chain is irreducible if it is possible to traverse from any state to any other state within a finite number of steps. Formally, for any states $i, j \in S$, there exists $n \geq 1$ such that $P_{ij}^{(n)} > 0$.*

Irreducibility plays a paramount role in systems like social networks, ensuring the flow of information across the entire network.

Example: Consider a Markov chain with state space $\{1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \end{pmatrix}.$$

This chain is irreducible because it is possible to move between any two states in at most 2 steps.

Definition The **period** of a state in a Markov chain is the greatest common divisor of all the lengths of paths that lead from the state back to itself. The period of state i is defined as $d(i) = \gcd\{n > 0 : P_{ii}^n > 0\}$.

Example:

Consider a Markov chain with four states, A, B, C, and D, and the transition matrix P given by:

$$P = \begin{pmatrix} 0 & 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0 \end{pmatrix}.$$

The powers of the transition matrix P are calculated as follows:

For P^2 (the matrix squared):

$$P^2 = \begin{pmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{pmatrix},$$

For P^3 (the matrix cubed):

$$P^3 = \begin{pmatrix} 0 & 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0 \end{pmatrix},$$

For P^4 (the matrix to the fourth power):

$$P^4 = \begin{pmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{pmatrix}.$$

These results indicate the probabilities of transitioning from one state to another after 2, 3, and 4 steps, respectively. The cyclic pattern, where P^2 and P^4 are identical, and similarly P and P^3 are identical, suggests a periodicity in the Markov chain. Specifically, this implies that the system exhibits a period of 2, as the transition probabilities return to their original configuration every 2 steps. Thus, each state in this Markov chain has a period of 2, meaning it is possible to return to the same state in multiples of 2 steps.

Definition 4. *A state of a Markov Chain exhibits aperiodicity if it doesn't revisit itself in a fixed pattern. Formally, a state i is aperiodic if the greatest common divisor of the set of steps n at which it returns to itself is one:* $\gcd\{n : P_{ii}^{(n)} > 0\} = 1$.

Aperiodicity is crucial in financial models to avoid deterministic cyclical behaviors, ensuring the model captures the nuances of real-world dynamics.

Definition 5. *A Markov Chain is termed ergodic if it embodies both irreducibility and aperiodicity. Ergodicity ensures the existence of a unique limit distribution π.*

Theorem (Convergence to a Limiting Distribution): Let $\{X_n, n \geq 0\}$ be an irreducible, aperiodic Markov chain with a finite or countably infinite state space S. If π is a stationary distribution for this chain, then for any initial state $i \in S$,

$$\lim_{n\to\infty} P_{ij}^{(n)} = \pi(j),$$

where $P_{ij}^{(n)}$ represents the n-step transition probability from state i to state j, and the stationary distribution π satisfies

$$\pi(j) = \sum_{i\in S} \pi(i)P_{ij},$$

for all $j \in S$, ensuring $\sum_{j\in S} \pi(j) = 1$.

This theorem illustrates that under conditions of irreducibility and aperiodicity, the distribution of states of the Markov chain converges to a stationary distribution π, which is independent of the initial state. The stationary distribution π is characterized by the property that the long-term behavior of the chain can be described as a weighted sum of its immediate one-step transitions, governed by the transition probabilities P_{ij}.

Example: Two-State Markov Chain

Consider a Markov chain with two states, 1 and 2, and transition matrix

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}.$$

Objective: Find the limiting distribution.

Solution:

The stationary distribution π satisfies $\pi P = \pi$ and $\pi_1 + \pi_2 = 1$. We solve

$$\begin{aligned} 0.9\pi_1 + 0.5\pi_2 &= \pi_1, \\ 0.1\pi_1 + 0.5\pi_2 &= \pi_2. \end{aligned}$$

This leads to $\pi_1 = \frac{5}{6}$ and $\pi_2 = \frac{1}{6}$.

Interpretation: The chain converges to a distribution where it is in state 1 with probability $\frac{5}{6}$ and in state 2 with probability $\frac{1}{6}$, regardless of the initial state.

Example: Weather Model

Consider a weather model with states "Sunny" (S) and "Rainy" (R), and transition matrix

$$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}.$$

Objective: Determine the long-term weather forecast.

Solution: Solving for π in $\pi P = \pi$ with $\pi_S + \pi_R = 1$ yields $\pi_S = 0.6$ and $\pi_R = 0.4$.

Conclusion: Long term, the forecast is sunny 60% of the time and rainy 40% of the time, regardless of initial weather.

Example: Three-State Markov Chain

Consider a Markov chain with states 1, 2, and 3, and transition matrix

$$P = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.1 & 0.6 & 0.3 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}.$$

Objective: Find the limiting distribution.

Solution: We find the stationary distribution $\pi = [\pi_1, \pi_2, \pi_3]$ by solving $\pi P = \pi$ subject to $\pi_1 + \pi_2 + \pi_3 = 1$. After solving, we interpret the probabilities of each state in the long term.

Example: Absorbing Markov Chain

Consider an absorbing Markov chain with states A (absorbing), 1, and 2, and transition matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}.$$

Objective: Probability of absorption starting from 1 and 2.

Solution: Calculate the fundamental matrix to find the absorption probabilities, leading to a clear understanding of long-term behavior towards the absorbing state.

Example: The Gambler's Problem

The Gambler's Problem is a classic scenario in stochastic processes, illustrating decision-making under uncertainty. A gambler has the opportunity to bet on the outcomes of a series of coin flips. If the coin comes up heads, the gambler wins as much money as they have bet; if it comes up tails, they lose their bet. The goal is to reach a certain amount of money, G, starting with an initial stake s. The game ends when the gambler reaches G or loses everything.

Objective: Determine the probability of reaching the goal G before going broke, given the initial stake s and the probability p of winning each bet.

Solution Approach:

Let $P(s)$ denote the probability of reaching the goal starting with s. The probabilities satisfy the following recursive relationship:

$$P(s) = pP(s+1) + (1-p)P(s-1), \quad 0 < s < G,$$

with boundary conditions $P(0) = 0$ and $P(G) = 1$.

This difference equation can be solved using methods for linear homogeneous recurrence relations with boundary conditions.

Example Solution:

Consider a simple case where $p = 0.5$ (a fair coin) and the goal $G = 4$. We seek the probability of reaching 4 starting from $s = 1$.

By solving the recurrence relation with the given boundary conditions, we find:

$$P(s) = \frac{s}{G},$$

for a fair game ($p = 0.5$). Therefore, the probability of reaching the goal of 4 starting with 1 is $P(1) = \frac{1}{4}$.

Interpretation: The Gambler's Problem showcases the use of stochastic models to evaluate probabilities of achieving certain states within a system governed by random processes. It highlights how initial conditions, transition probabilities, and boundary conditions play a crucial role in determining the outcome's likelihood.

Exercises

1. Given a Markov chain with four states and the transition matrix

$$P = \begin{pmatrix} 0.2 & 0.3 & 0.4 & 0.1 \\ 0.1 & 0.5 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 & 0.1 \\ 0.4 & 0.2 & 0.1 & 0.3 \end{pmatrix},$$

use the Chapman-Kolmogorov equation to calculate the five-step transition probabilities.

2. Consider a Markov chain with three states, where the transition probabilities are dependent on the parity of the step number (even or odd). For even steps, the transition matrix is

$$P_{\text{even}} = \begin{pmatrix} 0.6 & 0.4 & 0 \\ 0.3 & 0.7 & 0 \\ 0.5 & 0 & 0.5 \end{pmatrix},$$

and for odd steps,

$$P_{\text{odd}} = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}.$$

Calculate the six-step transition probability matrix.

3. A Markov chain models a system with three states A, B, and C with the following transition probabilities:

$$P = \begin{pmatrix} 0.5 & 0.25 & 0.25 \\ 0.4 & 0.4 & 0.2 \\ 0.3 & 0.3 & 0.4 \end{pmatrix}.$$

Using the Chapman-Kolmogorov equations, find the probability of transitioning from state A to state C in 8 steps.

4. For a periodic Markov chain with a period of 2 and four states, if the two-step transition matrix is

$$P^{(2)} = \begin{pmatrix} 0.1 & 0.3 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.3 & 0.3 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.3 & 0.2 & 0.3 & 0.2 \end{pmatrix},$$

calculate the four-step transition probabilities.

5. A Markov chain is defined with a state space of $\{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{pmatrix} 0.2 & 0.5 & 0.2 & 0.1 \\ 0.3 & 0.2 & 0.4 & 0.1 \\ 0.1 & 0.3 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.2 & 0.3 \end{pmatrix}.$$

Determine the transition probabilities from state 1 to all other states after 3 and 5 steps using Chapman-Kolmogorov equations.

Solutions

1

Given a Markov chain with four states and the transition matrix

$$P = \begin{pmatrix} 0.2 & 0.3 & 0.4 & 0.1 \\ 0.1 & 0.5 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 & 0.1 \\ 0.4 & 0.2 & 0.1 & 0.3 \end{pmatrix},$$

the five-step transition probability matrix calculated using the Chapman-Kolmogorov equation is

$$P^5 = \begin{pmatrix} 0.22362 & 0.35405 & 0.25318 & 0.16915 \\ 0.22393 & 0.35367 & 0.25284 & 0.16956 \\ 0.22363 & 0.35405 & 0.25317 & 0.16915 \\ 0.22388 & 0.35364 & 0.25371 & 0.16877 \end{pmatrix}.$$

2

Consider a Markov chain with three states, where the transition probabilities differ based on the parity of the step number—either even or odd. For even steps, the transition matrix is given by

$$P_{\text{even}} = \begin{pmatrix} 0.6 & 0.4 & 0 \\ 0.3 & 0.7 & 0 \\ 0.5 & 0 & 0.5 \end{pmatrix},$$

and for odd steps, the transition matrix is

$$P_{\text{odd}} = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}.$$

To calculate the six-step transition probability matrix, we alternate between multiplying the matrices for even and odd steps. This sequence of multiplications reflects the state transitions for a process that alternates in behavior with each step. The resulting six-step transition probability matrix, which is the product of three alternating multiplications of P_{even} and P_{odd}, is

$$P_{6-\text{step}} = P_{\text{even}} \cdot P_{\text{odd}} \cdot P_{\text{even}} \cdot P_{\text{odd}} \cdot P_{\text{even}} \cdot P_{\text{odd}} = \begin{pmatrix} 0.34851 & 0.37578 & 0.27571 \\ 0.348105 & 0.37434 & 0.277555 \\ 0.353025 & 0.373525 & 0.27345 \end{pmatrix}.$$

The elements of the six-step transition matrix represent the probabilities of transitioning from one state to another after six time steps, with the process's behavior changing at each step, from even to odd, and back again. This alternating dynamic is typical of systems where the conditions change periodically over time.

3

The eight-step transition probability matrix for the Markov chain is given by:

$$P^8 = \begin{pmatrix} 0.4137929 & 0.31034487 & 0.27586223 \\ 0.41379344 & 0.31034454 & 0.27586202 \\ 0.41379302 & 0.31034508 & 0.2758619 \end{pmatrix}$$

The probability of transitioning from state A to state C in 8 steps is approximately 0.2759.

4

The four-step transition probability matrix for the periodic Markov chain is:

$$P_{4-\text{step}} = P(2)^2 = \begin{pmatrix} 0.23 & 0.23 & 0.29 & 0.25 \\ 0.225 & 0.235 & 0.305 & 0.235 \\ 0.2125 & 0.2375 & 0.3125 & 0.2375 \\ 0.205 & 0.245 & 0.315 & 0.235 \end{pmatrix}$$

5

After applying the Chapman-Kolmogorov equations, the transition probabilities from state 1 to all other states after 3 and 5 steps are:

For 3 steps:

$$P^3 = \begin{pmatrix} 0.227 & 0.291 & 0.286 & 0.196 \end{pmatrix}$$

For 5 steps:

$$P^5 = \begin{pmatrix} 0.23831 & 0.28095 & 0.2841 & 0.19664 \end{pmatrix}$$