Lecture Notes

Week 4: Poisson Process

This week's content delves into two fundamental concepts: 1) the exponential distribution and its properties, and 2) the theory of counting processes. For a thorough understanding of these topics, please refer to Chapter 4 of "Introduction to Probability Models" by Sheldon M. Ross. This chapter serves as the foundation for today's lecture. Additional commentary and elaboration on Poisson processes can be found in the section below, tailored to enhance and supplement the material covered in class.

The literature supporting this week's discussions includes the following essential works by Sheldon M. Ross, which can be accessed through the provided Dropbox link:

- "Introduction to Probability Models," which offers comprehensive coverage on the probabilistic foundation for Poisson processes.
- "Stochastic Processes," presenting more advanced topics in the area.
- A third book pertinent to the applications of these processes in computer science, providing practical context to the theoretical framework.

1 Counting processes

Poisson processes are a fundamental concept in stochastic modeling, providing a rigorous mathematical framework for understanding events that occur randomly in time or space. Arrival times and counting processes such as Poisson processes find applications in a myriad of contexts, each with its own set of challenges and implications. For instance, in healthcare, modeling the arrival times of patients in an emergency room can be crucial for optimizing resource allocation and improving patient outcomes. Similarly, understanding the time intervals between bus arrivals at a specific stop can offer insights into public transportation scheduling and efficiency. In the realm of computer science, the arrival times of data packets in a network can be analyzed to optimize bandwidth and reduce latency. Businesses too can benefit; for example, modeling the arrival times of customers in a service queue, whether in a call center or a fast-food restaurant, can lead to enhanced service management. Natural events like earthquakes, floods, and forest fires also exhibit arrival times that can be modeled to better understand and predict these phenomena. In retail, the time between customer arrivals at a checkout counter can inform decisions about staffing and service speed. Social media platforms often scrutinize the timing of posts or mentions to understand user engagement or to detect trending topics. In manufacturing, arrival times of components on an assembly line can be critical for identifying bottlenecks and optimizing production. Financial markets are another fertile ground where the arrival times of buy/sell orders can shed light on market dynamics. Finally, in ecology, monitoring the arrival times of different species at a watering hole or feeding station can offer invaluable data for conservation efforts and ecological research.

Definition 1 (Counting Process)**.** *A counting process is a stochastic process $\{N(t), t \geq 0\}$ that represents the total number of events that have occurred up to time t. The function $N(t)$ satisfies the following properties:*

1. $N(0) = 0$ *(initial condition)*
2. $N(t)$ *is integer-valued for all* $t \geq 0$
3. $N(t)$ *is non-decreasing as* t *increases; that is, if* $s < t$*, then* $N(s) \leq N(t)$
4. *The function* $N(t)$ *is right-continuous, meaning that for each* t*,* $\lim_{s \to t^+} N(s) = N(t)$

In simpler terms, a counting process counts the number of times a certain event has occurred by any given time t. The count starts at zero and can only increase as time moves forward.

Consider a time interval T that we divide into n smaller intervals, each of length $\Delta t = \frac{T}{n}$. We are interested in counting the number of occurrences of a particular event within each small time interval Δt.

Initially, let's model this as a Bernoulli process. In each small time interval Δt, the event can either occur with probability p or not occur with probability $1 - p$.

$$P(\text{Event occurs in } \Delta t) = p \tag{1}$$

$$P(\text{Event does not occur in } \Delta t) = 1 - p \tag{2}$$

For large n and small Δt, we can relate p to a rate parameter λ as follows:

$$p = \lambda \Delta t \tag{3}$$

Now, let's consider the number of events X that occur in the entire interval T. The variable X is a sum of n independent Bernoulli random variables, each with success probability p. Therefore, X follows a binomial distribution:

$$X \sim \text{Binomial}(n, p) \tag{4}$$

The probability of observing exactly k events in T is given by:

$$P(X=k)=\binom{n}{k}p^k(1-p)^{(n-k)} \tag{5}$$

Substitute $p=\lambda\Delta t$ and $1-p=1-\lambda\Delta t$:

$$P(X=k)=\binom{n}{k}(\lambda\Delta t)^k(1-\lambda\Delta t)^{(n-k)} \tag{6}$$

The binomial coefficient can be expanded as:

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}=\frac{n(n-1)\cdots(n-k+1)}{k!} \tag{7}$$

Substitute this into the probability mass function:

$$P(X=k)=\frac{n(n-1)\cdots(n-k+1)}{k!}(\lambda\Delta t)^k(1-\lambda\Delta t)^{(n-k)} \tag{8}$$

The first limit becomes 1 as n gets larger and larger. The second limit turns into $\exp(-\lambda T)$ in the same way. These observations lead us to a key theorem about how the Bernoulli process evolves into a Poisson process.

Theorem 1 (Convergence from Bernoulli to Poisson)**.** *Let X be the number of events in a time interval T broken down into n smaller intervals. Each smaller interval has length $\Delta t=\frac{T}{n}$. If each interval has a Bernoulli-distributed event occurrence with probability $p=\lambda\Delta t$, then as n approaches infinity with $n\Delta t=T$ constant, the distribution of X turns into a Poisson distribution. Specifically,*

$$\lim_{n\to\infty}\binom{n}{k}p^k(1-p)^{(n-k)}=\frac{(\lambda T)^k}{k!}\exp(-\lambda T) \tag{9}$$

In this limit, X follows a Poisson distribution with parameter λT.

2 Poisson Processes

In the world of stochastic processes, the Poisson process holds a place of prominence for its mathematical elegance and wide-ranging applicability. It's a vital tool in various domains such as queuing theory, telecommunications, and even quantum physics. The Poisson process serves as a mathematical model for situations where events occur randomly in time or space.

2.1 Homogeneous Poisson Process

We start by introducing the most straightforward version of the Poisson process, the Homogeneous Poisson Process. In this variant, the rate at which events happen is constant over time, making it a natural extension of the Bernoulli process under limiting conditions.

Definition 2 (Homogeneous Poisson Process)**.** *Let* $(N(t) : t \geq 0)$ *be a counting process.* $N(t)$ *is said to be a Homogeneous Poisson Process with rate* $\lambda > 0$ *if the following conditions hold:*

1. $N(0) = 0$
2. *The increments are independent.*
3. *The number of events in any interval of length* t *follows a Poisson distribution with mean* λt*.*

The Homogeneous Poisson Process is uniquely characterized by its rate parameter λ, which tells us the average number of events per unit time. It's called 'homogeneous' because this rate is constant across time. This process provides a stochastic model for a variety of real-world phenomena where events occur continuously and independently at a constant average rate.

The concept of waiting times is crucial for understanding any stochastic process, and the Poisson process is no exception. In a Homogeneous Poisson Process, the waiting times between successive events are exponentially distributed.

Theorem 2 (Exponential Waiting Times)**.** *In a Homogeneous Poisson Process with rate* λ*, the time* T *until the first event occurs follows an exponential distribution with parameter* λ*, i.e.,*

$$P(T \leq t) = 1 - e^{-\lambda t}$$

This theorem can be derived from the properties of the Poisson process and provides essential insights into the behavior of the system. For example, it tells us that the process has no memory, meaning the time until the next event is independent of the past.

To prove that the waiting times are exponentially distributed in a Homogeneous Poisson Process, let's consider the probability that no event occurs in the interval $[0, t]$. According to the definition of a Homogeneous Poisson Process, the number of events $N(t)$ in any interval $[0, t]$ follows a Poisson distribution with mean λt. Therefore,

$$P(N(t) = 0) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t} \tag{10}$$

Now, the time T until the first event occurs is greater than t if and only if no event occurs in the interval $[0, t]$. Therefore,

$$P(T > t) = P(N(t) = 0) = e^{-\lambda t} \tag{11}$$

To find the distribution of T, we can find its cumulative distribution function (CDF), which is given by $P(T \leq t)$. The CDF is the complement of $P(T > t)$:

$$P(T \leq t) = 1 - P(T > t) = 1 - e^{-\lambda t} \tag{12}$$

Differentiating both sides with respect to t gives us the probability density function (PDF) of T:

$$f_T(t) = \frac{d}{dt} P(T \leq t) = \lambda e^{-\lambda t} \tag{13}$$

This is the PDF of an exponential distribution with rate parameter λ, completing the proof.

The memoryless property is a unique feature of the exponential distribution that has significant implications for the Poisson process. In mathematical terms, the memoryless property for an exponentially distributed random variable T with rate λ is described as follows:

$$P(T > s + t \,|\, T > s) = P(T > t) \quad \text{for all } s, t \geq 0 \tag{14}$$

This equation states that the probability that we have to wait an additional t time units given that we've already waited s time units is the same as if we had not waited at all.

To prove the memoryless property, we need to show that the conditional probability $P(T > s + t \,|\, T > s)$ equals $P(T > t)$.

Starting with the definition of conditional probability:

$$\begin{aligned} P(T > s + t \,|\, T > s) &= \frac{P(T > s + t \,\text{and}\, T > s)}{P(T > s)} \\ &= \frac{P(T > s + t)}{P(T > s)} \end{aligned}$$

We've used the fact that $T > s + t$ implies $T > s$, which allows us to simplify the numerator.

Now, we know that T is exponentially distributed with rate λ, so:

$$\begin{aligned} P(T > s + t \,|\, T > s) &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \\ &= e^{-\lambda t} \\ &= P(T > t) \end{aligned}$$

This completes the proof of the memoryless property.

Theorem 3 (Memoryless Property)**.** *The waiting times in a Homogeneous Poisson Process are memoryless, i.e., for any* $s, t \geq 0$,

$$P(T > s + t \,|\, T > s) = P(T > t)$$