Principles of Applied Mathematics

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Applied Mathematics

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PREFACE

Principles of Applied Mathematics is the culmination of many years of experience and research right from the times when I was still just a high school student to now while am undertaking my post graduate studies in one of best universities in China.

With reference to the report made by UNEB on the performance of candidates in the year 2017, the examiners noted that students still had particular weaknesses in performing mathematic tasks with higher cognitive demands, such as taking real-world situations, translating them into mathematical terms, and interpreting mathematical aspects in real-world problems especially in the section of Mechanics.

This book is my effort to address these weaknesses by presenting a sound but simple treatment of applied mathematics at Advanced Level. It in great detail consists of three sections: Mechanics, Statistics and Numerical Methods as stipulated by the NCDC syllabus.

Mechanics itself is an endearing and very useful educational tool. It teaches the student to understand physical laws that are expressed in mathematical terms and to apply those principles in unfamiliar situations. It teaches how to work logically and provides an excellent training in problem-solving.

Applied mathematics requires the use of some basic knowledge of pure mathematics. Some proofs in the section of mechanics require knowledge of trigonometry while some questions like in numerical methods require knowledge of integration, especially when asked to find the error made in using the trapezium rule which requires finding the exact value of the integral. Also topics like centre of mass require the knowledge of finding the area under the curve and solids of revolution. The reader will find a number of scenarios where applied mathematics builds on the earlier concepts learnt in pure mathematics.

The worked examples in each section have been carefully selected to serve the needs of a wide range of students and teachers. At the end of each topic are self-evaluation exercises with answers to help the students to widen their experience and build their confidence. Some problems require more thought and application and might appear to be quite more demanding for average learners. They should therefore not rule out guidance from their teachers in such situations.

At the end of each section are examination past paper questions to help the learners get exposed on how questions are examined in various topics or subsections by UNEB. My decision not to group them according to their respective topics or subsections is to help the learners identify for themselves the topic or subsection where the question is coming from.

It is therefore my sincere hope that students, teachers as well as general readers will find this book a good, reliable and an indispensable guide to applied mathematics and I am therefore optimistic that it will meet the needs of readers at all levels.

Finally, all misfortunes, if any in this book are purely my responsibility because it is difficult to claim perfection. I will be glad for any comments or compliments that will be directed to me. For no one is a monopolist of knowledge and no scientific theory is born in vacuum. Each scientist builds on the work of his predecessors.

This edition gives me the opportunity to thank all those people who have suggested ways in which the book might be improved. I am particularly grateful to all the individuals who undertook the laborious task of assisting with proof reading and for the invaluable suggestions made throughout the preparation of this book.

Kawuma Fahad
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EXAMINATION TECHNIQUE

Problem solving

Problem solving is one of the most important aspects of your mathematics course so you will need to spend a large part of your private study time tackling problems to improve your understanding of a topic and your ability to answer questions.

Persevere with your problem solving: do not give in too easily, but on the other hand do not waste time on a problem that is not yielding to your strategy and on which you do not have any alternative strategies to try. Leaving a problem and coming back to it can often be fruitful, the break giving you new insight into its solution.

As you work through each topic, cover the solution to each worked example, try to answer the question yourself and then compare your working with that given. When you feel that you understand a topic, try to answer the self-evaluative exercises and the questions from examination past papers. This will build up your confidence in your problem-solving abilities.

Learning

Mathematics is a very demanding discipline. To understand some of the basic ideas and concepts will involve you in some hard thinking and for the first time in your intellectual development you may be aware of consciously directing your critical powers to discover the nature of any misunderstandings which you may have.

There are a large number of identities, formulae and equations in mathematics and most probably you may be provided with tables with a list of formulae for you to use. There is no reason why many of these formulae should not be committed to memory. If you try to memorise new results when you first see them it is not such a daunting task as it may appear and the effort pays handsome dividends when you do not constantly have to refer to a formulae book when solving problems. However, if you are not confident in your memorising abilities, regular use of formulae book will help you to get to know where in the book appears so that you can check and also put in mind the formulae that are not listed in the book for you to do your best to memorise them.

Answering the questions: Before you attempt to answer a question, read it all again carefully, jotting down points such as formulae and information relating to that question. These hints should help you when writing an answer.

1. Make sure that your writing is legible.
2. Draw a large clearly labelled diagram if appropriate
3. Present your solution in a neat, logical and concise way.
4. Show all your working; many marks are given for the working not the answers.
5. Solve the problem which has been set and not the one you think is being posed.
6. Do not do things you are not asked or, for example, do not do proofs unless specifically requested.
7. State any principles, results, formulae, etc. used.
8. Check any formulae you use with the formulae sheet, if provided.
9. Use and state the correct units, e.g. ms^{-2}, $N\ s$, etc.
10. Always do a rough estimate of any calculation to check that your answer is sensible.
11. When using a calculator, make sure that each calculation is shown clearly in your answer.
12. Give your final answer to the required degree of accuracy.
13. In questions saying ‘hence or otherwise’, try ‘hence’ first since it is usually easier and uses the suggestion given in the first part of the question.
14. If you get ‘stuck’, re-read the question carefully to check that you have not missed any important information or hints given in the question itself.
15. When you have completed your solution, re-read the question to check that you have answered all parts.

Examination discipline

It is important that you try to keep the times you have allocated to answering a question or section and that you answer the correct number of questions. If you answer less than the number required, you are limiting the number of marks available to you.

In short answer sections, which are often compulsory, if you can not see how to solve a problem fairly quickly, leave it and return to it later if you have time. A fresh look at a question often helps.

In longer question sections, do not overrun your time allocation on any question by more than a minute or so. Do not be lured into thinking ‘just a few more minutes and I’ll have the answer’. In most examinations, the first parts to many questions are easier than the later parts so it is usually easier to gain more marks by attempting all the questions required than by completing a question.

The Examination Board has different policies regarding candidates who have answered too many questions. This can vary from year to year so check the Board’s policy.

1. Marks all your questions and ignores your worst marks – hand in all your answers.
2. Ignores your last questions – cross out the questions you feel you have done badly, leaving only the correct number to be marked.

Avoid any kind of examination malpractice. This can be a disaster to your future if discovered.

Finally, there only remains for me to say good luck. But exams have little to do with luck. Luck goes to a prepared mind. If you have done the work and revised thoroughly, you will undoubtedly do well.

This book is dedicated to the following individuals some of which gave me an excellent background in the field of mathematics and others who gave me the inspiration to come up with this book.

1. Mr. Lubwama Hamza, currently the Head of Mathematics Department, Gombe Secondary School

2. Mr. Kaggwa John Bosco, currently an A-Level Mathematics and Chemistry teacher at Gombe Secondary School and Merry land High School.

3. Hajji Khawuka Muhammed, currently the Director of Studies at East High School, Ntinda, formerly a Mathematics teacher at Gombe Secondary School.

4. Mr. Ssebagala John Robert, Currently the Headteacher of Our Lady of Guadalupe Senior Secondary School, Maddu Gomba, formerly a Mathematics teacher at Gombe Secondary School

5. Mr. Ssembajjwe Lameck, currently a Mathematics teacher at Kawempe Muslim Secondary School.

6. Mr. Bbosa Ibrahim, currently a lecturer at Makerere University School of Computing, formerly a Mathematics teacher at Kawempe Muslim Secondary School.

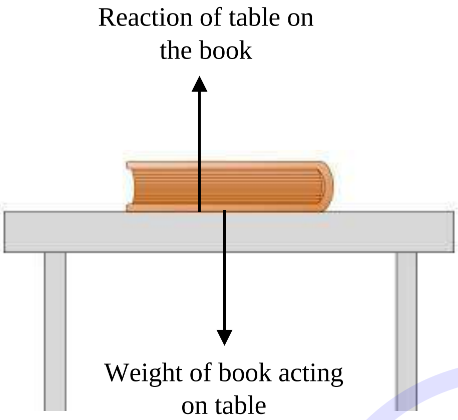
7. Mr. Mwesigwa Richards, the Director of Nsambya Hillside Schools. He was the master mind of the idea of me becoming an author.

Mechanics

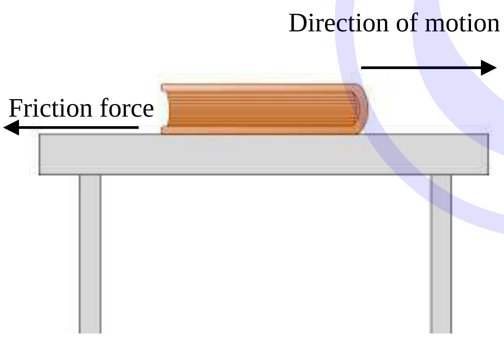
Introduction

Mechanics deals with motion and the action of forces on objects. You will encounter a variety of forces in mechanics. These force diagrams show some of the most common forces.

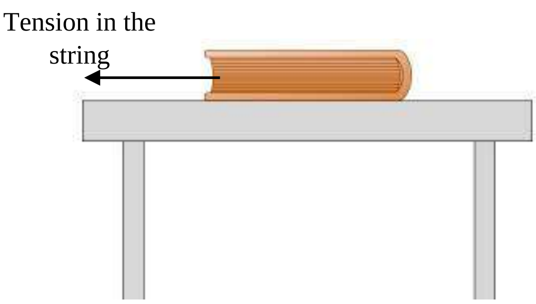
- The **weight** (or gravitational force) of an object acts vertically downwards.
- The **normal reaction** is the force which acts perpendicular to a surface when an object is in contact with the surface. In this example, the normal reaction is due to the weight of the book resting on the surface of the tabe.



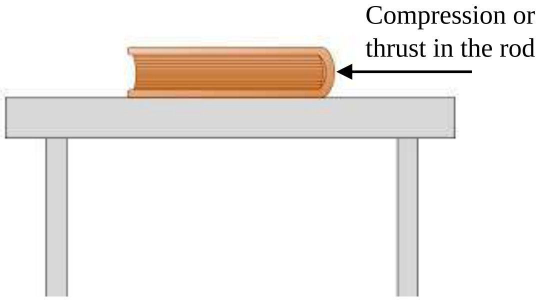
- The **friction** is a force which opposes the motion between two rough surfaces.



- If an object is being pulled along by a string, the force acting on the object is called the **tension** in the string



- If an object is being pushed along using a light rod, the force acting on the rod is called a **thrust** or **compression** in the rod.

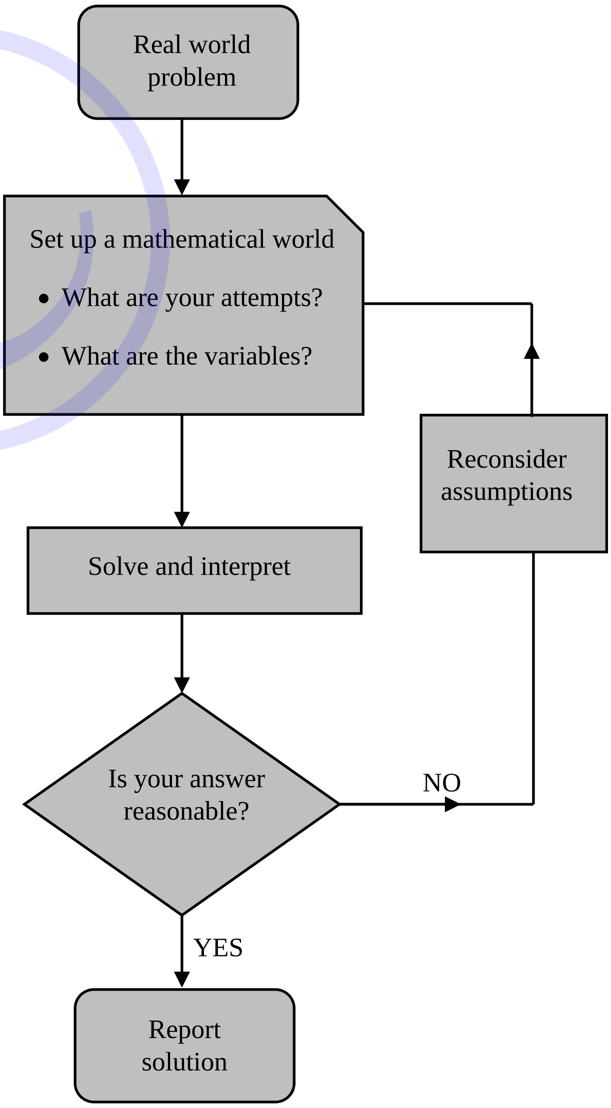


Mathematical modelling

Mathematical models can be constructed to simulate real-life situations, but in most cases it is necessary to simplify the problem by making assumptions so that it can be described using equations or graphs in order to solve it.

The solution to a mathematical model needs to be interpreted in the context of the original problem. It is possible your model may need to be refined and your assumptions reconsidered.

This flow chart summarises the mathematical modelling process.



Model	Modelling assumptions
Particle – Dimensions of the object are negligible	• mass of the object is concentrated at a single point • rotational forces and air resistnace can be ignored
Rod – All dimensions but one are negligible, like a pole or a beam	• mass is concentrated along a line • no thickness • rigid(does not bend or buckle)
Lamina – object with area but negligible thickness,like a sheet of paper	• mass is distributed across a flat surface
Uniform body – Mass is distributed evenly	• mass of the object is concentrated at a single point at the geometrical centre of the body – the **centre of mass**
Light object – Mass of the object is small compared to other masses,like a string or a pulley	• treat object as having zero mass • tension the same at both ends of a light string
Inextensible string – A string that does not stretch under load	• acceleration is the same in obejects connected by a taut inextensible string
Smooth surface	• assume that there is no friction between the surface and any object on it
Rough surface – if a surface is not smooth,it is rough	• objects in constact with the surface experience a frictional force if they are moving or are acted on by a force
Wire – Rigid thin length of metal	• treated as one dimensional
Smooth and light pulley – Most of the pulleys you consider will be smooth and light	• pulley has no mass • tension is the same on either side of the pulley
Bead – Particle with a hole in it for threading on a wire or string	• moves freely along a wire or string • tension is the same on either side of the bead
Peg – A suport from which a body can be suspended or rested	• dimensionless and fixed • can be rough or smooth as specified in question
Air resistance – Resistance experienced as an object moves through the air	• usually modelled as being negligible
Gravity – Force of attraction between all objects. Acceleration due to gravity is denoted by g	• assume that all objects with mass are attracted towards the Earth • Earth's gravity is uniform and acts vertically downwards • g is constant and is taken as 9.8 ms^{-2}, unless otherwise stated in the question

Drawing force diagrams

Drawing a clear force diagram is an essential first step in the solution of any problem in mechanics which is concerned with the action of forces on the body.

The following are important points to remember when drawing such a force diagram.

1. Make the diagram large enough to show clearly all the forces acting on the body and to enable any necessary geometry and trigonometry to be done.
2. Show only forces which are acting on the body being considered. A common fault is including forces which the body is applying to its surroundings (including other bodies).
3. Weight always acts on a body unless the body is described as light.
4. Contact with another object or surface gives rise to a normal reaction and sometimes friction
5. Attachment to another object (by a string, spring, hinge, etc.) gives rise to a force on the body at the point of attachment
6. Forces acting on a particle act at the same point. Forces acting on other bodies may act at different points.
7. Check that no forces have been omitted or included more than once.

Note that the drawing of force diagrams is a skill. It is essential to acquire during the study of mechanics. Although it does not appear in an examination question, it is often the first step in the solution of problems in this section.

KINEMATICS

Kinematics is the study of displacement, velocity and acceleration.

Displacement is the position of a point relative to the origin O. It is a vector and SI unit is metre (m)
Distance is the magnitude of the displacement. It is a scalar.

Velocity is the rate of change of displacement with respect to time. It is a vector and SI unit is metre per second (ms^{-1})
Speed is the magnitude of the velocity. It is a scalar.
Uniform velocity is constant speed in a fixed direction.
Average velocity is $\frac{\text{change in displacement}}{\text{time taken}}$
Average speed is $\frac{\text{change in distance}}{\text{time taken}}$

Acceleration is the rate of change of velocity with respect to time. It is a vector and SI unit is metre per second squared.
Negative acceleration is sometimes called **retardation**.
Uniform acceleration is constant acceleration in a fixed direction.

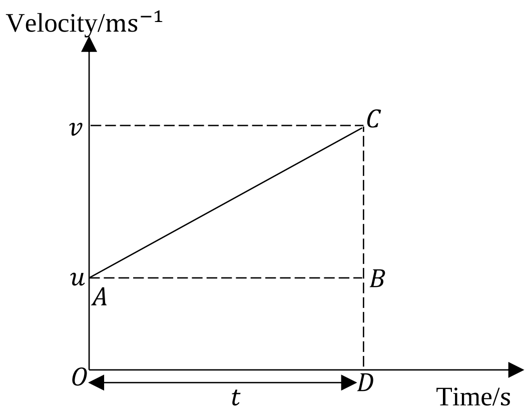
Equations of uniformly accelerated motion
If a body changes its velocity from u to v in a time t, its acceleration, $a = \frac{v-u}{t}$

$$\boldsymbol{v = u + at}$$

Since the acceleration is uniform, the average velocity during the time t is given by
Average velocity $= \frac{1}{2}$(initial velocity + final velocity)
$= \frac{1}{2}(u + v)$
The displacement during the time t is
$s =$ average velocity $\times$ time
$= \frac{1}{2}(u + v) \times t$
$= \frac{1}{2}(u + u + at) \times t$

$$\boldsymbol{s = ut + \frac{1}{2}at^2}$$

Alternatively;
Consider a body with initial velocity u and final velocity v as it moves from A to B.



Displacement = total area under velocity time graph
= Area of OABD + Area of ACB
Area of OABD $= ut$
Area of ABC $= \frac{1}{2} \times t \times (v - u)$ but $v = u + at$
$\Rightarrow$Area of ABC $= \frac{1}{2}at^2$
$\therefore$Total area $= ut + \frac{1}{2}at^2$
$\Rightarrow\ s = ut + \frac{1}{2}at^2$

Now, from $v = u + at,\ t = \frac{v-u}{a}$
But $s = \left(\frac{u+t}{2}\right) \times t \Rightarrow s = \left(\frac{u+v}{2}\right) \times \left(\frac{v-u}{a}\right)$
$s = \frac{uv-uv-u^2+v^2}{2a} \Rightarrow s = \frac{v^2-u^2}{2a}$

$$\boldsymbol{v^2 = u^2 + 2as}$$

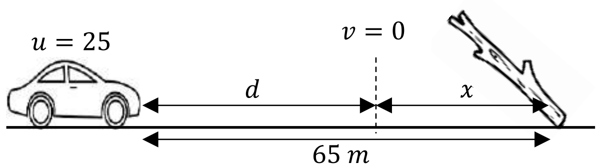
To solve problems using these equations of motion;
(a) Choose the positive direction
(b) List the five quantities (s, t, u, v, a), fill in known values and mark which are to be found
(c) Use the appropriate equation(s) to find the required unknown(s). If any three of the quantities are known, then the other two can always be found.

Note: These equations do not apply to acceleration which is not uniform.
Problems about non-uniform acceleration must be solved by graphical methods or by calculus

Example 1
A car is being driven along a road at a steady speed 25 ms^{-1} when the driver suddenly notices that there is a fallen tree blocking the road 65 m ahead. The driver immediately applies brakes giving the car a constant retardation of 5 ms^{-2}.
(a) How far in front of the tree does the car come to rest?
(b) If the driver had not reacted immediately and the brakes were applied one second later, with what speed would the car have hit the tree?

<u>Solution</u>

(a)



$u = 25, a = -5, s = d, v = 0$
$v^2 = u^2 + 2as$
$0 = 25^2 - 2(5)d$
$10d = 625$
$d = 62.5\ m$
Required distance, $x = 65 - d = 65 - 62.5 = 2.5\ m$
$\therefore$The car stops 2.5 m in front of the tree
(b) Distance travelled in 1 $s = v \times t = 25 \times 1 = 25\ m$

$$d + 200 = 44T + \frac{1}{2} \times \frac{1}{2}T^2$$

$$d + 200 = 44T + \frac{1}{4}T^2 \quad \text{.... (ii)}$$

Comparing (i) and (ii);

$$35T + \frac{1}{5}T^2 + 200 = 44T + \frac{1}{4}T^2$$

$$\frac{1}{20}T^2 + 9T - 200 = 0$$

$$T^2 + 180T - 4000 = 0$$

$$T^2 + 200T - 20T - 4000 = 0$$

$$T(T + 200) - 20(T + 200) = 0$$

$$(T + 200)(T - 20) = 0$$

$$T = -200 \text{ or } T = 20$$

$$\therefore T = 20 \text{ s}$$

From (i); $d = 35T + \frac{1}{5}T^2$

$$d = 35(20) + \frac{1}{5}(20)^2 = 700 + 80 = 780 \text{ m}$$

Distance before finishing post, $x = 1000 - d$

$$x = 1000 - 780 = 220 \text{ m}$$

Thus, car B passes car A 220 m before the finishing post.

When both cars arrive at the finishing post, $d = 1000$ m

Car A travels distance of 1000 m (1 km) while car B travels a distance of 1200 m

Let the times travelled by car A and car B be T_A and T_B respectively

For car A;

$$1000 = 35T_A + \frac{1}{5}T_A^2$$

$$T_A^2 + 175T_A - 5000 = 0$$

$$T_A^2 + 200T_A - 25T_A - 5000 = 0$$

$$T_A(T_A + 200) - 25(T_A + 200) = 0$$

$$(T_A + 200)(T_A - 25) = 0$$

$$T_A = -200 \text{ or } T_A = 25$$

$$\therefore T_A = 25 \text{ s}$$

For car B;

$$1200 = 44T_B + \frac{1}{4}T_B^2$$

$$T_B^2 + 176T_B - 4800 = 0$$

$$T_B^2 + 200T_B - 24T_B - 4800 = 0$$

$$T_B(T_B + 200) - 24(T_B + 200) = 0$$

$$(T_B + 200)(T_B - 24) = 0$$

$$T_B = -200 \text{ or } T_B = 24$$

$$\therefore T_B = 24 \text{ s}$$

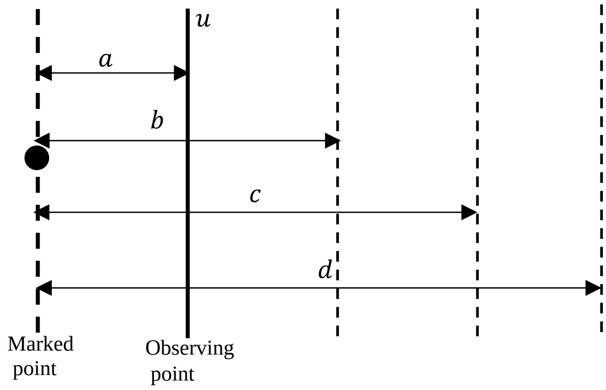
$T_A - T_B = 25 - 24 = 1$ s

Therefore, car B reaches the finishing post 1 s earlier than car A

Example 8

A particle is moving in a straight line and is observed to be at a distance a from a marked point initially, to be at a distance b after an interval of $n\ s$, to be at a distance c after $2n\ s$, and at a distance d after $3n\ s$. Prove that if the acceleration is uniform, $d - a = 3(c - b)$ and that the acceleration is equal to $\frac{c+a-2b}{n^2}$. Find also the initial velocity

Solution



Let the initial velocity observed at a distance a from the marked point be u and the uniform acceleration be A.

After n s, the particle has travelled a distance of $b - a$, a distance of $c - a$ after $2n$ s and a distance of $d - a$ after $3n$ s.

$$s = ut + \frac{1}{2}at^2$$

After $n\ s$,

$$b - a = nu + \frac{1}{2}An^2 \quad \text{...... (i)}$$

After $2n\ s$,

$$c - a = 2nu + \frac{1}{2}A(2n)^2$$

$$c - a = 2nu + 2An^2 \quad \text{.... (ii)}$$

After $3n\ s$,

$$d - a = 3nu + \frac{1}{2}A(3n)^2$$

$$d - a = 3nu + \frac{9}{2}An^2 \quad \text{..... (iii)}$$

(ii) – (i);

$$c - b = nu + \frac{3}{2}An^2$$

From (iii);

$$d - a = 3\left(nu + \frac{3}{2}An^2\right)$$

$$d - a = 3(c - b)$$

Eliminating u from (i) and (ii), the acceleration can be obtained.

(ii) – 2 (i);

$$(c - a) - 2(b - a) = (2nu + 2An^2) - (2nu + An^2)$$

$$c + a - 2b = An^2$$

$$A = \frac{c - 2b + a}{n^2}$$

Eliminating A from (i) and (ii), the initial velocity u can be obtained

4 (i) – (ii);

$$4(b - a) - (c - a) = (4nu + 2An^2) - (2nu + 2An^2)$$

$$4b - c - 3b = 2nu$$

$$u = \frac{4b - c - 3b}{2n}$$

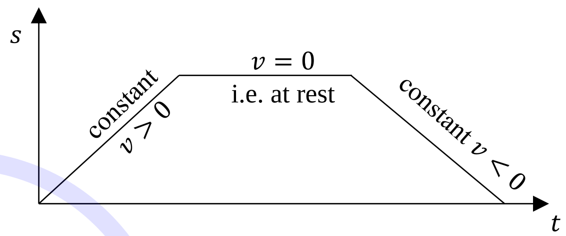
13. A train is timed between successive points A, B and C each 2 km apart. If it takes 100 s to travel from A to B and 150 s from B to C, find the retardation of the train, assuming that it remains uniform after the point A. Find also how far beyond C the train travels before it stops.
[Ans: $0.053\ \text{ms}^{-2}$, 0.82 km]
14. Two cars A and B are traveling along a straight path. The cars are observed to be side by side when they are at point P of the path and again at another point Q. Assuming that A and B moved with a uniform acceleration a_1 and a_2, prove that if their velocities are u_1 and u_2 respectively, the distance PQ is given by;
$$\frac{2(u_1 - u_2)(u_2 a_1 - u_1 a_2)}{(a_1 - a_2)^2}$$
15. A car A, moving with uniform velocity u_A along a straight road, passes a point X a car B, moving in the same direction with velocity u_B and uniform acceleration b. B overtakes A at a point Y
    (a) Prove that the time taken to reach Y is
    $$\frac{2(u_A - u_B)}{b}$$
    (b) Find the distance XY
    (c) After passing A but before reaching Y, find the distance between the cars at time T, and prove that the maximum distance between them is $\frac{(u_A - u_B)^2}{2b}$
    (d) At Y, A now accelerates with a uniform acceleration a and B changes to a uniform velocity, keeping the velocity it had on arrival at Y. If A overtakes B again after it has travelled a further distance equal to XY, prove that
    $$a = \frac{(2u_A - u_B)b}{u_A}$$
[Ans: (b) $\frac{2u_A(u_A - u_B)}{b}$ (c) $(u_A - u_B)T - \frac{1}{2}bT^2$]
16. Two trains P and Q, travel by the same route from rest at a station A to rest at a station B. Train P has a constant acceleration f for the first third of the time, constant speed for the second third and constant retardation f for the last third of the time. Train Q has a constant acceleration f for the first third of the distance, constant speed for the second third and constant retardation f for the last third of the distance. Show that the times taken by the two trains are in the ratio $3\sqrt{3} : 5$
17. A railway train is moving along a straight level track with a speed of $10\ \text{ms}^{-1}$ when the driver sights a signal which is green. As soon as the signal is sighted the train starts to accelerate. Given that the acceleration has a constant value of $f\ \text{ms}^{-2}$, show that the distance in metres moved by the train during the nth second after the signal is sighted is
$$\left(10 - \frac{f}{2} + nf\right)$$
Find the value of f given that the train travels 25 m during the 8th second after the signal is sighted.

Graphs in kinematics

You do not always have to use equations to describe mathematical models. Another method is to use graphs. There are two kinds of graph which are often useful in kinematics.
The first kind is a displacement-time graph and the other is a velocity-time graph.

Displacement-time graph

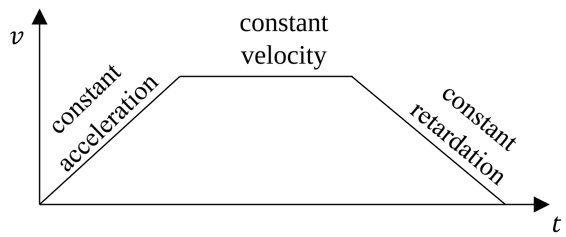
A displacement-time graph (or s-t graph) for a body moving in a straight line shows its displacement s from a fixed point on the line plotted against time t.
The velocity v of the body at a time t is given by the gradient of the s-t graph at t, since $v = \frac{ds}{dt}$
The s-t graph for a body moving with constant velocity is a straight line. The velocity v of the body is given by the gradient of the line.



The s-t graph for a body moving with variable velocity is a curve.
The velocity at any time may be estimated from the gradient of the tangent to the curve at that time.
The average velocity between two times may be estimated from the gradient of the chord joining them.

Velocity-time graph

A velocity-time graph (or v-t graph) for a body moving in a straight line shows its velocity v plotted against time t.
The acceleration a of the body at time t is given by the gradient of the v-t graph at t, since $a = \frac{dv}{dt}$
The displacement s in a time interval is given by the area under the v-t graph for that time interval, since $s = \int v\ dt$
The v-t graph for a body moving with uniform acceleration is a straight line. The acceleration a of the body is given by the gradient of the line.



The total displacement s of a body can be found from a v-t graph of the above type by calculating the area of the trapezium.
The v-t graph for a body moving with **variable acceleration is a curve**.

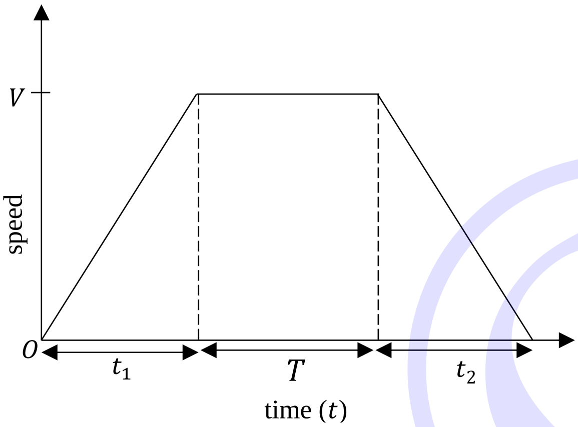
The acceleration a of the body at any time may be estimated from the gradient of the tangent to the curve at that time.
The displacement s of the body in a given time interval may be estimated by finding the area under the v-t in that interval by a numerical method.

Example

Two points P and Q are x metres a part in the same straight line. A particle starts from rest at P and moves directly towards Q with an acceleration of a ms^{-2} until it acquires a speed of V ms^{-1}. It maintains this speed for a time T seconds and is then brought to rest at Q under a retardation a ms^{-2}. Prove that $T = \frac{x}{V} - \frac{V}{a}$

Solution

Let t_1 and t_2 be the times for which the particle is accelerating and being retarded respectively.
Sketch the velocity-time graph



Using the definitions of acceleration and retardation

$$a = \frac{V}{t_1} \Rightarrow t_1 = \frac{V}{a}$$
$$a = \frac{V}{t_2} \Rightarrow t_2 = \frac{V}{a}$$

We know that the total distance travelled between P and Q is x m and this is represented by the area under the graph

$$x = \frac{1}{2}Vt_1 + VT + \frac{1}{2}Vt_2$$

Substituting for t_1 and t_2 gives

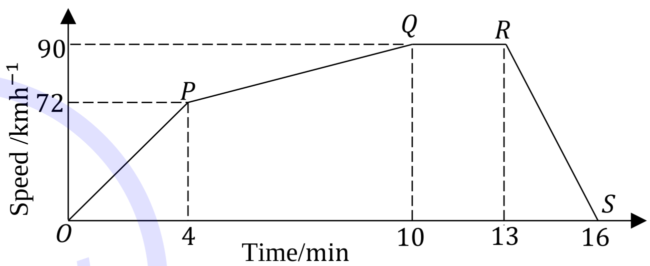
$$x = \frac{V^2}{2a} + VT + \frac{V^2}{2a}$$
$$x = \frac{V^2}{a} + VT$$
$$TV = x - \frac{V^2}{a}$$
$$T = \frac{x}{V} - \frac{V}{a}$$

Self-Evaluation exercise

1. A train travels between two stations 3.9 km apart in 6 minutes, starting and finishing at rest. During the first $\frac{3}{4}$ minute the acceleration is uniform, for the next $3\frac{3}{4}$ minutes the speed is constant and for the remainder of the journey the train retarded uniformly. Sketch a speed-time graph of the journey and hence, or otherwise, calculate
   (i) the maximum speed, in km/h, attained by the train
   (ii) the acceleration of the train during the first $\frac{3}{4}$ minute, stating the units

   [Ans: (i) 48 $km\ h^{-1}$ (ii) 64 $km/h/min$]
2. A motorist starting a car from rest accelerates uniformly to a speed of v m/s in 9 seconds. He maintains this speed for another 50 seconds and then applies the brakes and decelerates uniformly to rest. His deceleration is numerically equal to three times his previous acceleration
   (i) Sketch a velocity-time graph
   (ii) Calculate the time during which deceleration takes place
   (iii) Given that the total distance moved is 840 m, calculate the value of v.
   (iv) Calculate the initial acceleration

   [Ans: (ii) 3 s (iii) 15 ms^{-1} (i) $1\frac{2}{3}$ ms^{-2}]
3.



The figure above shows the speed/time graph for a train journey taking 16 minutes. The train reaches a speed of 72 $km\ h^{-1}$ after 4 minutes and 90 $km\ h^{-1}$ after a further 6 minutes. The train maintains a steady speed for the next three minutes and then decelerates to rest in the last 3 minutes. Calculate
   (i) the acceleration in ms^{-2} during those parts of the journey to OP and PQ
   (ii) the total length of the journey
   (iii) the average speed for the whole journey in $km\ h^{-1}$

   [Ans: (i) $\frac{1}{12}$ ms^{-2}; $\frac{1}{72}$ ms^{-2} (ii) 17250 m (iii) 64.7 $km\ h^{-1}$]
4. Trials are being undertaken on a horizontal road to test the performance of an electrically powered car. The car has a to speed V. In a test run, the car moves from rest with uniform acceleration a and is brought to rest with uniform retardation r.
   (a) If the car is to achieve top speed during a test run, by a velocity-time sketch, or otherwise, show that the length of the test run must be at least
   $$\frac{V^2(a+r)}{2ar}$$
   (b) Find the least time taken for a test run of length
      (i) $\frac{2V^2(a+r)}{9ar}$
      (ii) $\frac{2V^2(a+r)}{3ar}$

Vertical motion under gravity

When a body is projected vertically upwards, we regard the upward direction as the positive direction, and the body will experience a retardation or negative acceleration g.

If u is the initial velocity of projection, the equations for motion with uniform acceleration thus become

$$v = u - gt \;\ldots\ldots\; \text{(i)}$$

$$s = ut - \tfrac{1}{2}gt^2 \;\ldots\ldots\; \text{(ii)}$$

$$v^2 = u^2 - 2gs \;\ldots\ldots\; \text{(iii)}$$

At the highest point, it is clear that the velocity v must be zero, so that by putting $v = 0$ in equation (i), we get the time taken to reach the highest point.

$$0 = u - gt$$

$$t = \frac{u}{g}$$

If t is greater than u/g, v is negative and as t increases, v increases numerically, that is, after reaching the highest point the body begins to descend, and its speed increases.

Equation (iii) gives the greatest height by putting $v = 0$ i.e.

$$0 = u^2 - 2gs$$

$$s = u^2/2g$$

The greatest height attained is $u^2/2g$

The velocity on returning to the point from which it was projected is given by putting $s = 0$ in equation (iii), and then

$$v^2 = u^2$$

$$v = \pm u$$

The + sign gives the velocity on starting, and the – sign the velocity on returning to the point of projection. The magnitude is the same as that of the velocity of projection, but the body is now moving downwards.

The time of flight is obtained by putting $s = 0$ in equation (ii), and this gives

$$0 = ut - \frac{1}{2}gt^2$$

$$t = 0 \text{ or } t = \frac{2u}{g}$$

The two values of t corresponding to the height $s = 0$; the value of $t = 0$ obviously refers to the time of projection, while the value of $2u/g$ gives the time required to return to the point of projection i.e. the time of flight.

Note that this is twice the time to the greatest height. For any given height (less than the greatest) above the point of projection, equation (ii) will give two values of t, one the time taken to reach that height on way up, the other the time on way down.

If we require the time taken to reach a point below the point of projection (when the body is projected upwards), we need not find the time up and down to the point of projection, and then the time taken to reach the point below. We simply substitute the distance below the point of projection for s in equation (ii), giving it a negative sign.

Velocity due to falling a given vertical distance from rest

The positive direction is now taken downwards and the equation of motion is

$$v^2 = u^2 + 2gs$$

Since the body starts from rest, $u = 0$, and hence the velocity acquired in falling a distance h is given by

$$v^2 = 2gh$$

$$v = \sqrt{2gh}$$

This is also the velocity required to take the body to a height h when projected vertically upwards.

Example 1

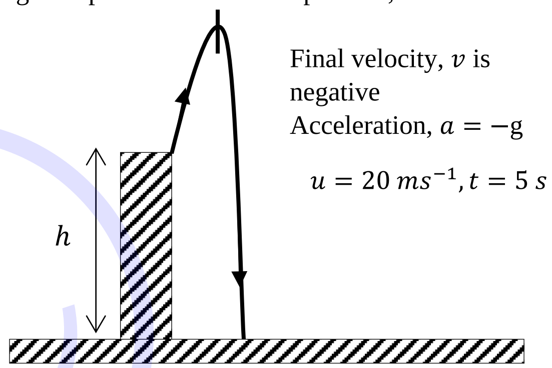
A stone is thrown vertically upwards with a speed of $20\ ms^{-1}$ from a point at a height h above the ground level. If the stone hits the ground 5 s later, find the

(i) velocity with which it hits the ground

(ii) the value of h

Solution

Taking the upward motion to be positive,



(i) From $v = u + at$

$$-v = u - gt$$

$$-v = 20 - 9.8 \times 5$$

$$v = 29\ ms^{-1}$$

(ii) Displacement, h below point of projection is negative.

From $s = ut + \frac{1}{2}at^2$

$$-h = ut - \tfrac{1}{2}gt^2$$

$$-h = 20 \times 5 - \tfrac{1}{2} \times 9.8 \times 5^2$$

$$h = 22.5\ m$$

Alternatively; using $v^2 = u^2 + 2as$

$$(-v)^2 = u^2 + 2(-g)(-h)$$

$$v^2 = u^2 + 2gh$$

$$29^2 = 20^2 + 2 \times 9.8 \times h$$

$$h = \frac{29^2 - 20^2}{19.6} = 22.5\ m$$

Example 2

A body is projected vertically upwards with a velocity $21\ ms^{-1}$. How long will it take to reach a point $280\ m$ below the point of projection?

Solution

$$s = ut - \frac{1}{2}gt^2$$

$s = -280, u = 21, g = 9.8$

$$-280 = 21t - 4.9t^2$$

$$4.9t^2 - 21t - 280 = 0$$
$$7t^2 - 30t - 400 = 0$$
$$7t^2 - 70t + 40t - 400 = 0$$
$$(t - 10)(7t + 40) = 0$$
$$t = 10 \text{ or } t = -40/7$$
$$\therefore t = 10\ s$$

$u = 9.8,\ s = -7,\ a = -9.8,\ v = ?$

$$v^2 = u^2 + 2as$$
$$v^2 = 9.8^2 + 2(-9.8)(-7)$$
$$v^2 = 233.24$$
$$v = 15.27\ ms^{-1}$$

The ball hits the ground at $15.27\ ms^{-1}$

Example 3

A boy throws a ball vertically upwards from a 7 m high roof.

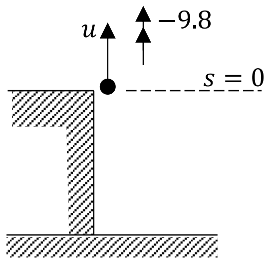
(a) If, after 2 seconds, he catches the ball on its way down again, with what speed was it thrown?

(b) What is the velocity of the ball when it is caught?

(c) If the boy fails to catch the ball, with what speed will it hit the ground?

Solution

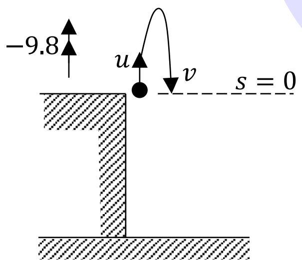
(a)



$a = -9.8,\ t = 2,\ s = 0,\ u = ?$

$$s = ut + \frac{1}{2}at^2$$
$$0 = 2u + \frac{1}{2}(-9.8)(4)$$
$$u = 9.8\ ms^{-1}$$

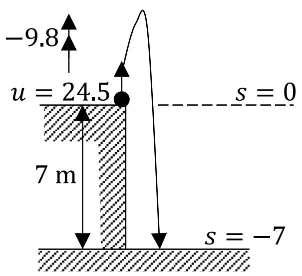
(b)



$$v^2 = u^2 + 2as$$
$$v^2 = 9.8^2 - 2(-9.8)(0)$$
$$v = 9.8\ ms^{-1}$$

The ball is falling at $9.8\ ms^{-1}$ when caught

(c)



The ground is 7 m below the point of projection so the final displacement of the ball is $-7\ m$. Working from the time when the ball was thrown gives

Example 4

A stone is projected vertically upwards from the top of a cliff 20 m high. After a time of 3 s, it passes the edge of the cliff on its way down. Calculate

(a) the speed of projection,

(b) the speed when it hits the ground,

(c) the times when it is 10 m above the top of the cliff,

(d) the time when it is 5 m above the ground

Solution

Take up as positive

(a) $s = 0,\ t = 3\ s,\ u = ?,\ v = ?,\ g = 9.8$

$$s = ut - \frac{1}{2}gt^2$$
$$0 = 3u - \frac{1}{2}(9.8)(3^2)$$
$$u = 14.7\ ms^{-1}$$

(b) $s = -20\ m,\ t = ?,\ u = 14.7\ ms^{-1},\ v = ?,\ a = -9.8$

$$v^2 = u^2 + 2as$$
$$v^2 = 14.7^2 + 2(-9.8)(-20)$$
$$v^2 = 608.09$$
$$v = 24.66\ ms^{-1}$$

(c) $s = 10\ m,\ t = ?,\ u = 14.7\ ms^{-1},\ a = -9.8$

$$s = ut + \frac{1}{2}at^2$$
$$10 = 14.7t + \frac{1}{2}(-9.8)t^2$$
$$4.9t^2 - 14.7t + 10 = 0$$
$$t = \frac{14.7 \pm \sqrt{(-14.7)^2 - 4(4.9)(10)}}{2(4.9)}$$
$$t = 1.96\ s \text{ or } t = 1.04\ s$$

(d) $s = -15\ m,\ t = ?,\ u = 14.7\ ms^{-1},\ a = -9.8\ ms^{-2}$

$$s = ut + \frac{1}{2}at^2$$
$$-15 = 14.7t - \frac{1}{2}(9.8)t^2$$
$$4.9t^2 - 14.7t - 15 = 0$$
$$t = \frac{14.7 \pm \sqrt{14.7^2 - 4(4.9)(-15)}}{2(4.9)}$$
$$t = 3.8\ s \text{ or } t = -0.8\ s$$
$$\therefore t = 3.8\ s$$

Example 5

A stone is thrown vertically upwards with a speed of $20\ ms^{-1}$. A second stone is thrown vertically upwards from the same point and with the same initial speed $20\ ms^{-1}$ but 2 s later than the first stone. Show that the two stones collide at a distance of 15.5 m above the point of projection.

Example 9

A body falls from rest from the top of a tower and during the last second, it falls $\frac{9}{25}$ of the whole distance. Find the height of the tower.

Solution

Let the height of the tower be h and the time taken to fall to the ground be t

$$s = ut + \frac{1}{2}at^2$$

$$h = 0 + \frac{1}{2}gt^2$$

$$h = 4.9t^2 \dots \text{(i)}$$

Distance covered in the last second is given by

$$s = s_t - s_{t-1}$$

$s_t = 0 + \frac{1}{2}gt^2 = \frac{1}{2}gt^2$

$s_{t-1} = 0 + \frac{1}{2}g(t-1)^2 = \frac{1}{2}g(t^2 - 2t + 1)$

$$\frac{9}{25}h = \frac{1}{2}gt^2 - \frac{1}{2}g(t^2 - 2t + 1)$$

$$\frac{9}{25}h = \frac{1}{2}gt^2 - \frac{1}{2}gt^2 + gt - \frac{1}{2}g$$

$$\frac{9}{25}h = 9.8t - 4.9 \text{ .. (ii)}$$

Comparing (i) and (ii);

$$\frac{9}{25} \times 4.9t^2 = 9.8t - 4.9$$

$$1.764t^2 - 9.8t + 4.9 = 0$$

$$t = \frac{9.8 \pm \sqrt{9.8^2 - 4(1.764)(4.9)}}{2(1.764)}$$

$$t = 5 \text{ or } t = \frac{5}{9}$$

$$\therefore t = 5\ s$$

From (i); $h = 4.9t^2$

$$h = 4.9(5^2) = 122.5\ m$$

Example 10

A particle projected vertically downwards descends 100 m in 4 s. Show that it describes the last 30 m in about 0.7 s.

Solution

Let the initial velocity of the particle be u. The particle is undergoing uniform acceleration.

$$s = ut + \frac{1}{2}at^2$$

$$100 = 4u + \frac{1}{2}(9.8)(16)$$

$$u = 5.4\ ms^{-1}$$

Let the time taken to travel 130 m from the beginning be T

$$130 = 5.4T + 4.9T^2$$

$$4.9T^2 + 5.4T - 130 = 0$$

$$T = \frac{-5.4 \pm \sqrt{5.4^2 - 4(4.9)(-130)}}{2(4.9)}$$

$$T = 4.63 \text{ or } T = -5.73$$

$$\therefore T = 4.63\ s$$

Time taken to describe the last 30 m is given by

$$t = 4.63 - 4 = 0.63\ s$$

Example 11

A particle is projected vertically upwards and at the same instant another is left to fall to meet it. Show that, if the particles have equal velocities when they impinge, one of them has travelled 3 times as far as the other.

Solution

Let the distance travelled by the particle moving downwards be x and be y for the particle moving upwards. Let the two particles collide with the same velocity V after time T. We are required to show that $x = 3y$ or $y = 3x$

For particle moving downwards,

$u = 0,\ a = g,\ v = V,\ s = x,\ t = T$

From $v^2 = u^2 + 2as$

$$V^2 = 0 + 2gx$$

$$V^2 = 2gx \text{ …. (i)}$$

Also, from $v = u + at$

$$V = 0 + gT$$

$$T = \frac{V}{g} \dots \text{(ii)}$$

For particle moving upwards,

$u = u,\ a = -g,\ v = V,\ s = y,\ t = T$

From $v^2 = u^2 + 2as$

$$V^2 = u^2 - 2gy \text{ …. (iii)}$$

Also, from$v = u + at$

$$V = u - gT$$

$$u = V + gT$$

But from (ii), $T = V/g$

$$u = V + g\left(\frac{V}{g}\right) = 2V$$

Substituting for u in (iii);

$$V^2 = (2V)^2 - 2gy$$

$$V^2 = 4V^2 - 2gy$$

$$2gy = 3V^2$$

From (i), $V^2 = 2gx$

$$2gy = 3(2gx)$$

$$y = 3x$$

$\therefore$ The particle projected upwards has travelled 3 times as far as one released from rest moving downwards.

(a) how long it takes to reach the highest point
(b) the distance it ascends during the third second of its motion

[Ans: (a) 3.57 s (b) 10.5 m]

22. A ball is thrown vertically upward with a speed of $14\ \text{ms}^{-1}$. Two seconds later a second ball is dropped from the same point. Find where the two balls meet.
[Ans: 11.0 m below their initial position]

23. In the last second of the motion of a ball dropped from rest at the top of a tower, it falls through a distance which is a fifth of the height of the tower. Find the height of the tower.
[Ans: 440 m]

24. A ball A falls vertically from the top of a tower 63 m high. At the same time as A begins to fall, another ball B is projected vertically upwards from the bottom of the tower with speed $21\ \text{ms}^{-1}$. The balls collide. Find the distance of the point where the balls collide from the bottom of the tower.
[Ans: 18.9 m]

25. A particle is projected vertically upwards from a point O with speed $u\ \text{ms}^{-1}$. The greatest height reached by the particle is 62.5 m above O. Find
(a) the value of u
(b) the total time for which the particle is 50 m or more above O
[Ans: (a) $35\ \text{ms}^{-1}$ (b) 3.2 s]

26. A ball is thrown vertically downward from the top of a tower with speed $18\ \text{ms}^{-1}$. It reaches the ground in 1.6 s. Find the height of the tower.
[Ans: 41 m]

27. A particle P is projected vertically upwards from a point X. Five seconds later, P is moving downwards with speed $10\ \text{ms}^{-}$. Find
(a) the speed of projection of P
(b) the greatest height above X attained by P during its motion
[Ans: (a) $39\ \text{ms}^{-1}$ (b) 78 m]

28. A ball is thrown vertically upwards with speed $21\ \text{ms}^{-1}$. It hits the ground 4.5 s later. Find the height above the ground from which the ball was thrown.
[Ans: 4.7 m]

29. A particle is projected vertically upwards from a point O with speed $u\ \text{ms}^{-1}$. Two seconds later it is still moving upwards and its speed is $\frac{1}{3}u\ \text{ms}^{-1}$. Find
(a) the value of u
(b) the time from the instant when the particle leaves O to the instant it returns to O
[Ans: (a) 29 (b) 6 s]

30. A ball A is thrown vertically downwards with speed $5\ \text{ms}^{-1}$ from the top of a tower block 46 m above the ground. At the same time as A is thrown downwards, another ball B is thrown vertically upwards from the ground with speed $18\ \text{ms}^{-1}$. The balls collide. Find the distance of the point where A and B collide from the point where A was thrown.
[Ans: 30 m]

31. A ball is released from rest at a point which is 10 m above a wooden floor. Each time the ball strikes the floor, it rebounds with three-quarters of the speed with which it strikes the floor. Find the greatest height above the floor reached by the ball
(a) the first time it rebounds from the floor
(b) the second time it rebounds from the floor.
Hint: Consider each bounce as a separate motion
[Ans: (a) 5.6 m (b) 3.2 m]

32. A particle P is projected vertically upwards from a point O with speed $12\ \text{ms}^{-1}$. One second after P has been projected from O, another particle Q is projected vertically upwards from O with speed $20\ \text{ms}^{-1}$. Find
(a) the time between the instant that P is projected from O and the instant when P and Q collide
(b) the distance of the point where P and Q collide from O
[Ans: (a) 1.4 s (b) 7.2 m]

33. A stone is dropped from the top of a building and two seconds later, another stone is thrown vertically downwards at a speed of $25\ \text{ms}^{-1}$. Both stones reach the ground at the same time. Find the height of the building.
[Ans: 155 m]

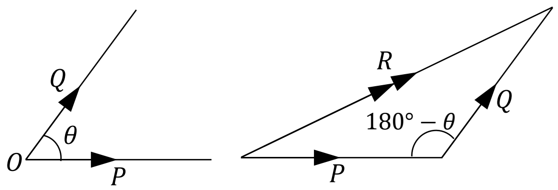
34. A ball is projected vertically upwards with speed $10\ \text{ms}^{-1}$ from a point X which is 50 m above the ground. T seconds after the first ball is projected upwards, a second ball is dropped from X. Initially the second ball is at rest. The balls collide 25 m above the ground. Find the value of T
[Ans: 1.2 s]

COMPONENTS AND RESULTANT OF FORCES

Coplanar forces are forces whose lines of action all lie in the same plane.
Concurrent forces act at the same point. If forces act on a particle, then they must be concurrent.

Resultant of forces

If two forces P and Q are concurrent, then they act at a point, O say. Their resultant R may be found by vector addition using the triangle law, i.e. $P + Q = R$

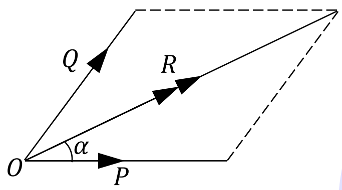


The magnitude of R is given by the cosine rule;

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

since $\cos(180° - \theta) = -\cos\theta$

The resultant R also acts at the point O as shown below;



The direction of R is given by the sine rule;

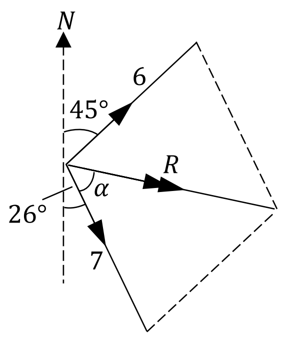
$$\sin\alpha = \frac{Q}{R}\sin\theta$$

If P and Q are at right angles, then $\theta = 90°$ and $R = \sqrt{P^2 + Q^2}$ and $\tan\alpha = \frac{Q}{P}$

Example 1

A particle is acted upon by forces in a horizontal plane. $6\ N$ in a direction NE, $7\ N$ in a direction $S\ 26°\ E$. Find the resultant force.

Solution



The magnitude R of the resultant force is given by

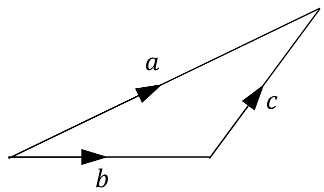
$$R = \sqrt{6^2 + 7^2 + 2 \times 6 \times 7\cos 109°}$$
$$R = \sqrt{36 + 49 - 27.35}$$
$$R = \sqrt{57.65} = 7.59\ N$$

If α is the angle between the direction of the resultant and the $7\ N$ force, then

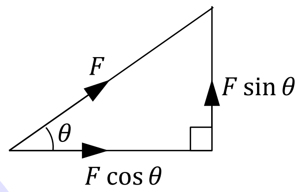
$$\sin\alpha = \frac{6}{7.59}\sin 109° = 0.7474$$
$$\alpha = 48.4°$$

Resolving a force

A single force can be split into two components or resolutes by the converse of the triangle law for vector addition. This process is a called resolving a force.

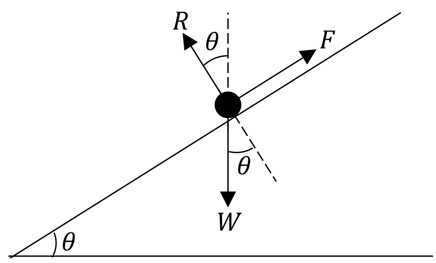


Although a force may be resolved in an infinite number of ways, the most useful way is when the components are perpendicular to each other



F is resolved into perpendicular components $F\cos\theta$ and $F\sin\theta$

Note: In problem solving, it is often necessary to resolve a force in one or more directions e.g. horizontally, vertically, parallel to the plane, perpendicular to the plane, etc.
For example, a particle is acted upon by the coplanar concurrent forces W, R and F as shown below.



The table below gives the components of forces in four directions.

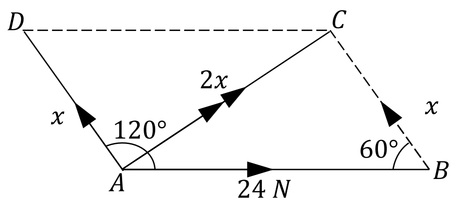
Direction / Force	Vertical	Horizontal	∥ to slope	⊥ to slope
$\boldsymbol{W}$	W	0	$W\sin\theta$	$W\cos\theta$
$\boldsymbol{R}$	$R\cos\theta$	$R\sin\theta$	0	R
$\boldsymbol{F}$	$F\sin\theta$	$F\cos\theta$	F	0

Resultant of a system of forces

To find the resultant of a system of coplanar concurrent forces:

1. Resolve each force in a stated direction and find the sum of the resolutes, P say, in that direction

Solution



By the cosine rule on triangle ABC

$$|AC|^2 = |AB|^2 + |BC|^2 - 2|AB||BC|\cos 60°$$
$$(2x)^2 = 24^2 + x^2 - 2 \times 24 \times x \times \frac{1}{2}$$
$$4x^2 = 516 + x^2 - 24x$$
$$3x^2 + 24x - 576 = 0$$
$$x^2 + 8x - 192 = 0$$

By the quadratic formula or completing the square;

$$(x+4)^2 - 16 - 192 = 0$$
$$(x+4)^2 = 208$$
$$x + 4 = \pm\sqrt{208}$$
$$x = -4 \pm 4\sqrt{13}$$
$$\text{Since } x > 0, x = -4 + 4\sqrt{3}$$

Example 8

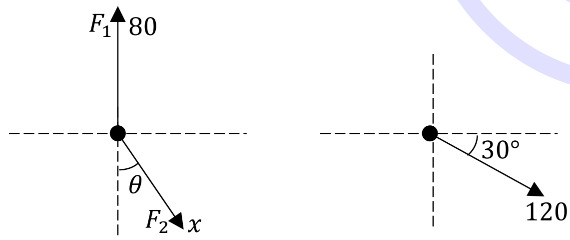
Two forces, F_1 N and F_2 N, are acting on a particle so that the resultant of the two forces has magnitude 120 N and acts on a bearing of 120°. It is further given that the F_1 acts due North and has magnitude 80 N.Calculate the

(a) magnitude of F_2

(b) the direction in which F_2 acts

Solution

Looking at two separate diagrams



Equating downward forces in the two diagrams;

$$x\cos\theta - 80 = 120\sin 30°$$
$$x\cos\theta = 80 + 60$$
$$x\cos\theta = 140 \quad \dots \text{(i)}$$

Equating forces to the right in the two diagrams

$$x\sin\theta = 120\cos 30°$$
$$x\sin\theta = 60\sqrt{3} \quad \dots\text{. (ii)}$$

Squaring and adding;

$$x^2\cos^2\theta = 19600$$
$$x^2\sin^2\theta = 10800$$
$$x^2(\cos^2\theta + \sin^2\theta) = 30400$$
$$x^2 = 30400$$
$$x = 40\sqrt{19} = 174\ N$$

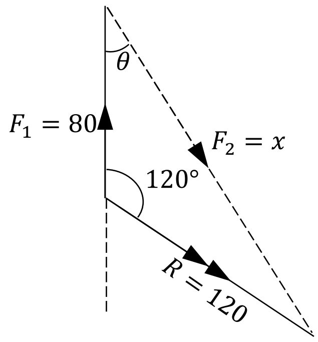
Dividing side by side;

$$\frac{x\sin\theta}{x\cos\theta} = \frac{60\sqrt{3}}{140}$$
$$\tan\theta = 0.742$$
$$\theta = 36.6°$$

Bearing $= 180° - \theta = 180° - 36.6° = 143.4°$

Alternative method by geometry

Looking at the triangle of forces



By using the cosine rule;

$$x^2 = 80^2 + 120^2 - 2 \times 80 \times 20\cos 120°$$
$$x^2 = 6400 + 14400 - 19200\cos 120°$$
$$x^2 = 30400$$
$$x = \sqrt{30400} = 40\sqrt{19} = 174\ N$$

Now by the sine rule;

$$\frac{\sin\theta}{120} = \frac{\sin 120°}{x}$$
$$\sin\theta = \frac{120\sin 120°}{40\sqrt{19}} = 0.596$$
$$\theta = 36.6°$$

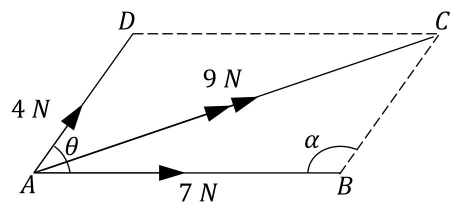
Bearing $= 180° - \theta = 180° - 36.6° = 143.4°$

Example 9

Find the angle between a force of 7 N and a force of 4 N if their resultant has a magnitude of 9 N

Solution

Since the magnitude of the resultant of the forces is greater than the magnitude of the larger force, the angle between the forces is acute. Let it be θ



From triangle ABC, using the cosine rule, we obtain

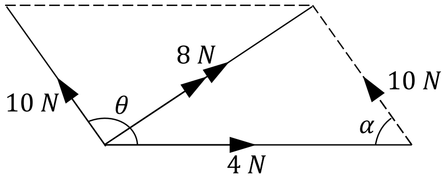
$$9^2 = 7^2 + 4^2 - 2 \times 7 \times 4\cos\alpha$$
$$81 = 49 + 16 - 56\cos\alpha$$
$$-56\cos\alpha = 16$$
$$\cos\alpha = 0.286$$
$$\alpha = 106.6°$$
$$\theta = 180° - \alpha = 180° - 106.6° = 73.4°$$

Example 10

Find the angle between a force of 10 N and a force of 4 N given that their resultant has magnitude 8 N

Solution

Since the magnitude of the resultant of the forces is smaller than the magnitude of the larger force, the angle between the forces is obtuse. Let it be θ



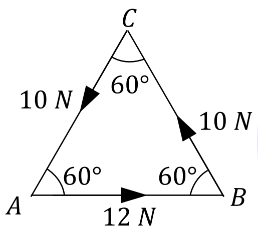
By cosine rule;

$$8^2 = 10^2 + 4^2 - 2 \times 4 \times 10 \cos\alpha$$
$$64 = 100 + 16 - 80\cos\alpha$$
$$-52 = -80\cos\alpha$$
$$\cos\alpha = 49.5°$$
$$\theta = 180° - \alpha = 180° - 49.5° = 130.5°$$

Example 11

ABC is an equilateral triangle. Forces of $12\ N$, $10\ N$ and $10\ N$ act along AB, BC and CA respectively, the direction of the forces being indicated by the order of the letters. Find the magnitude and direction of the resultant.

Solution



$\rightarrow$; $X = 12 - 10\cos 60° - 10\cos 60° = 2\ N$

$\uparrow$; $Y = 10\sin 60° - 10\sin 60° = 0$

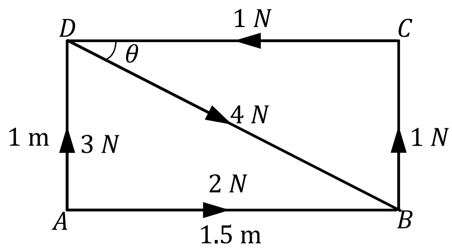
$$R = \sqrt{X^2 + Y^2} = \sqrt{2^2 + 0} = 2\ N$$

The resultant has a magnitude of $2\ N$ in the direction AB

Example 12

$ABCD$ is a rectangle with $AB = 1.5$ m and $AD = 1$ m. Forces of $2\ N$, $1\ N$, $1\ N$, $4\ N$ and $3\ N$ act along AB, BC, CD, DB and AD respectively. Calculate the magnitude and direction of the single force that could replace this system of forces.

Solution



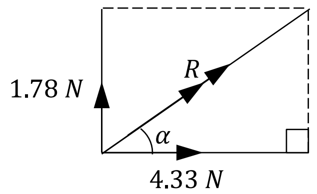
$$\tan\theta = \frac{1}{1.5} = \frac{2}{3} \Rightarrow \sin\theta = \frac{2}{\sqrt{13}} \text{ and } \cos\theta = \frac{3}{\sqrt{13}}$$

$$\rightarrow; X = 2 + 4\cos\theta - 1 = 2 + 4 \times \frac{3}{\sqrt{13}} - 1$$

$$X = 4.33\ N$$

$$\uparrow; Y = 3 + 1 - 4\sin\theta = 4 - 4 \times \frac{2}{\sqrt{13}}$$

$$Y = 1.78\ N$$



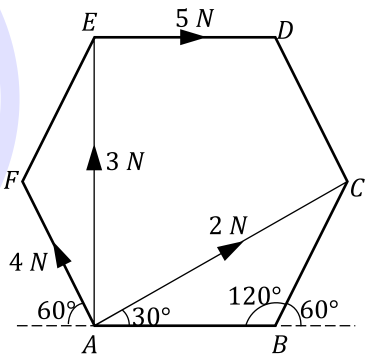
$$R = \sqrt{X^2 + Y^2} = \sqrt{4.33^2 + 1.78^2} = 4.68\ N$$
$$\tan\alpha = \frac{1.78}{4.33}$$
$$\tan\alpha = 0.411$$
$$\alpha = 22.35°$$

The single force is of magnitude $4.68\ N$ at $22.35°$ to the direction AB

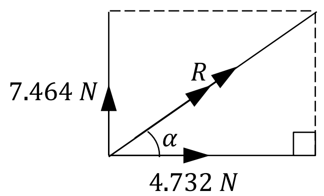
Example 13

$ABCDEF$ is a regular hexagon. Forces of $2\ N$, $3\ N$, $4\ N$ and $5\ N$ act along AC, AE, AF and ED respectively. Find the single force equivalent to this system stating both the magnitude and direction.

Solution



$$\rightarrow; X = 2\cos 30° + 5 - 4\cos 60° = 4.732\ N$$
$$\uparrow; Y = 2\sin 30° + 3 + 4\sin 60° = 7.464\ N$$



$$R = \sqrt{4.732^2 + 7.464^2} = 8.84\ N$$
$$\tan\alpha = \frac{7.464}{4.732} = 1.577$$
$$\alpha = 57.6°$$

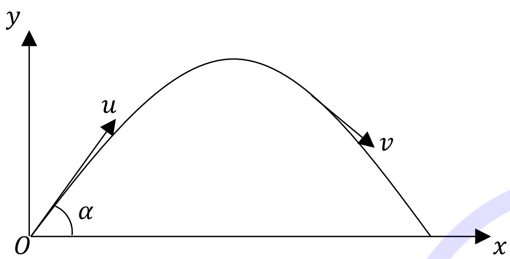
The single force equivalent to the system has magnitude $8.84\ N$ at $57.6°$ to the direction AB

PROJECTILES

A projectile is a particle which is given an initial velocity and then moves freely under gravity.
It is assumed that gravity is the only force acting on the particle i.e. air resistance is negligible.
If its initial velocity is vertical, then the particle will move in a straight line under gravity.
If its initial velocity is not vertical, the particle will move in a curve (a parabola).
Examples artillery shells, shot putts, high jumpers and balls in games such as tennis, football, basketball, volley ball, cricket, golf, only to mention but a few.

Analysis of results

Consider a particle projected with initial velocity u at an angle α to the horizontal and has velocity v at time t



Its flight can be analysed by considering horizontal and vertical motion separately and using the equations for uniform acceleration in a straight line, i.e.

$$v = u + at$$
$$s = ut + \frac{1}{2}at^2$$
$$v^2 = u^2 + 2as$$

	Horizontal motion	Vertical motion
u	$u_x = u\cos\alpha$	$u_y = u\sin\alpha$
a	$\ddot{x} = 0$	$\ddot{y} = -\mathrm{g}$
v	$v_x = u\cos\alpha$	$v_y = u\sin\alpha - \mathrm{g}t$
s	$x = (u\cos\alpha)t$	$y = (u\sin\alpha)t - \frac{1}{2}\mathrm{g}t^2$

Vector approach

If a particle is projected with initial velocity $a\mathrm{i} + b\mathrm{j}$, then its velocity at any time t can be expressed in the form

$$\mathrm{v} = a\mathrm{i} + (b - \mathrm{g}t)\mathrm{j}$$

and its position at any time t can be expressed in the form

$$\mathrm{r} = at\mathrm{i} + \left(bt - \frac{1}{2}\mathrm{g}t^2\right)\mathrm{j}$$

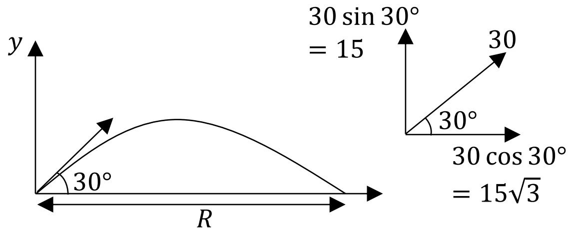
Any problem on projectiles may be solved using vector methods but in general it is unwise to do so unless the problem is phrased in vector terms.

Example 1

A particle is projected from ground level with speed $30\ \mathrm{ms}^{-1}$ at an angle of 30° to the horizontal. Calculate

(a) the time of flight
(b) the range

Solution



(a) Consider vertical motion
When the particle reaches the ground, $s = 0$
$u = 15$, $v = ?$, $a = -9.8$, $s = 0$, $t = ?$
Using $s = ut + \frac{1}{2}at^2$

$$0 = 15t + \frac{1}{2}(-9.8)t^2$$
$$0 = t(15 - 4.9t)$$
$$t = 0 \text{ or } t = 3.06\ \mathrm{s}$$

$t = 0$ is the starting time, $t = 3.06$ is the time of flight

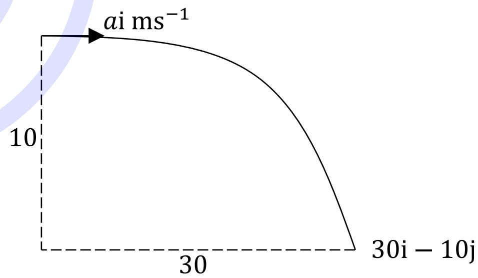
(b) The horizontal velocity is given by $15\sqrt{3}\ \mathrm{ms}^{-1}$ and it is constant. Since the particle travels for 3.06 s at this velocity,

$$\text{Range, } R = 15\sqrt{3} \times 3.06 = 79.5\ \mathrm{m}$$

Example 2

Initially a particle is at an origin O and is projected with a velocity $a\mathrm{i}\ \mathrm{ms}^{-1}$. After t seconds, the particle is at the point with position vector $(30\mathrm{i} - 10\mathrm{j})$ m. Find the values of t and a.

Solution



Considering vertical motion;
$u = 0$, $a = \mathrm{g}$, $s = 10$, $t = ?$

$$s = ut + \frac{1}{2}at^2$$
$$10 = \frac{1}{2}\mathrm{g}t^2$$
$$10 = 4.9t^2$$
$$t^2 = 2.04$$
$$t = 1.43$$

Considering horizontal motion;
$u = a$, $a = 0$, $s = 30$, $t = 1.43$

$$s = ut + \frac{1}{2}at^2$$
$$30 = at$$
$$30 = a \times 1.43$$
$$a = 20.98$$

$$t = \frac{x}{u\cos\alpha}$$

$$y = u\sin\alpha\left(\frac{x}{u\cos\alpha}\right) - \frac{gx^2}{2u^2\cos^2\alpha}$$

$$y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha$$

$$\boldsymbol{y = x\tan\alpha - gx^2(1+\tan^2\alpha)}$$

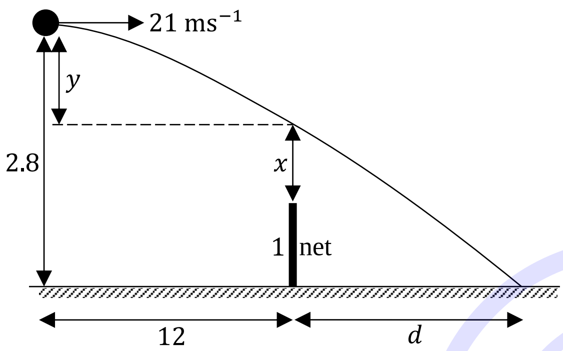
Example 3

A tennis ball is served horizontally with an initial speed of $21\ \text{ms}^{-1}$ from a height of 2.8 m.

(a) By what distance does the ball clear a net 1 m high situated 12 m horizontally from the server?

(b) How far behind the net does the netball land?

Solution



(a) Consider vertical motion from serving point to the net

$$u = 0, a = g, s = y, t = ?$$

Horizontal motion;

Time taken $= \frac{\text{Distance}}{\text{Speed}} = \frac{12}{21} = \frac{4}{7}$ s

Considering horizontal motion;

$$s = ut + \frac{1}{2}at^2$$

$$y = \frac{1}{2}\times 9.8\left(\frac{4}{7}\right)^2$$

$$y = 1.6\text{ m}$$

Now;

$$x + 1.6 + 1 = 2.8$$

$$x + 2.6 = 2.8$$

$$x = 0.2\text{ m}$$

∴ Required distance is 0.2 m

(b) Considering vertical motion from serving point to ground

$$u = 0, a = g, s = 2.8, t = 0$$

$$2.8 = 0 + \frac{1}{2}(9.8)t^2$$

$$t^2 = 0.571$$

$$t = 0.756\text{ s}$$

Horizontally;

$$12 + d = 21\times 0.756$$

$$12 + d = 15.876$$

$$d = 3.876\text{ m}$$

The distance behind the net where the balls is 3.876 m

Example 4

Two particles A and B are projected simultaneously under gravity; A from a point O on horizontal ground and B from a point 40 m vertically above O. B is projected horizontally with speed $28\ \text{ms}^{-1}$. If the particles hit the ground simultaneously at the same point, calculate

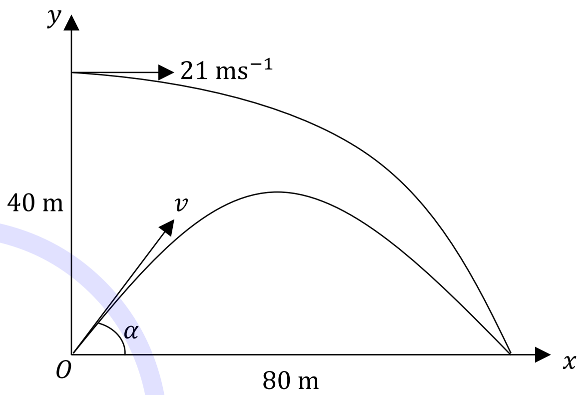
(a) the time taken for B to reach the ground and the horizontal distance it has then travelled

(b) the magnitude and direction of the velocity with which A is projected and the horizontal distance then travelled.

(c) Show that, just prior to hitting the ground, the directions of motions of A and B differ by about $18\frac{1}{2}°$

Solution

Take the horizontal and vertically upward displacements from O as x and y respectively.



(a)

For B: $\dot{x}_B = 28$, $\dot{y}_B = -gt$

$$x_B = 28t, y_B = 40 - \tfrac{1}{2}gt^2$$

When B strikes the ground, $y_B = 0$.

So $t^2 = \frac{40}{4.9}$

$$t = \frac{20}{7} = 2.86\text{ s}$$

At this time, $x_B = 28t = 28\left(\frac{20}{7}\right) = 80$ m

Therefore, B strikes the ground 2.86 s later having travelled a distance 80 m horizontally

(b) For A: Let the horizontal and vertically upwards components of the velocity of projection be $v\cos\alpha$ and $v\sin\alpha$.

Then $x_A = (v\cos\alpha)t$, $\quad y_A = (v\sin\alpha)t - \frac{1}{2}gt^2$.

Since A and B travel the same distance horizontally in the same time, in order to collide, they must have the same horizontal speed i.e.

$$v\cos\alpha = 28$$

When $t = \frac{20}{7}$, $y_A = 0$, so $v(\sin\alpha)t = \frac{1}{2}gt^2$

$$v\sin\alpha = \frac{9.8}{2}\left(\frac{20}{7}\right) = 14$$

and $y_B = -9.8\left(\frac{20}{7}\right) = -28$

$$\tan\alpha = \frac{1}{2}$$

$$\Rightarrow \alpha = 26.6°$$

$$t = \frac{16V}{g}$$

Considering vertical motion;

$u = 5V\sin\theta = 5V \times \frac{4}{5} = 4V,\ a = -g,\ s = -h$

$$s = ut + \frac{1}{2}at^2$$

$$-h = 4V\left(\frac{16V}{g}\right) - \frac{1}{2}g\left(\frac{16V}{g}\right)^2$$

$$-h = \frac{64V^2}{g} - \frac{1}{2}g\left(\frac{256V^2}{g^2}\right)$$

$$-h = \frac{64V^2}{g} - \frac{128V^2}{g}$$

$$h = \frac{64V^2}{g}$$

(ii) When P is directly level with O, considering vertical motion

$s = 0,\ u = 4V,\ a = -g,\ t = ?$

$$s = ut + \frac{1}{2}at^2$$

$$0 = 4Vt - \frac{1}{2}gt^2$$

$$0 = t\left(4V - \frac{1}{2}gt\right)$$

$$t = 0 \text{ or } t = \frac{8V}{g}$$

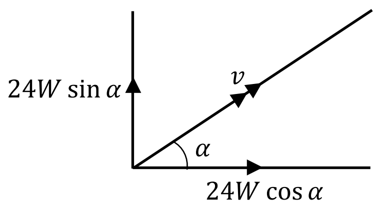
$$\therefore t = \frac{8V}{g}$$

Considering horizontal motion;

$s = OP = ?,\ t = \frac{8V}{g},\ u = 3V$

$$OP = 3V \times \frac{8V}{g} = \frac{24V^2}{g}$$

For the second particle;



Considering horizontal motion;

$u = 24W\cos\alpha,\ a = 0,\ t = T,\ s = \frac{48V^2}{g}$

$$\frac{48V^2}{g} = (24W\cos\alpha)T$$

$$T = \frac{2V^2}{gW\cos\alpha}$$

Considering vertical motion

$u = 24W\sin\alpha,\ a = -g,\ s = -\frac{64V^2}{g}$

$$-\frac{64V^2}{g} = (24W\sin\alpha)T - \frac{1}{2}gT^2$$

$$-\frac{64V^2}{g} = (24W\sin\alpha)\left(\frac{2V^2}{gW\cos\alpha}\right) - \frac{1}{2}g\left(\frac{2V^2}{gW\cos\alpha}\right)^2$$

$$-\frac{64V^2}{g} = \frac{48V^2}{g}\tan\alpha - \frac{g}{2}\left(\frac{4V^4}{g^2W^2\cos^2\alpha}\right)$$

$$-64V^2 = 48V^2\tan\alpha - \frac{2V^4}{W^2}\sec^2\alpha$$

$$-32 = 24\tan\alpha - \frac{V^2}{W^2}(1 + \tan^2\alpha)$$

$$-32 = 24W^2\tan\alpha - V^2(1 + \tan^2\alpha)$$

$$-32W^2 = 24W^2\tan\alpha - V^2 - V^2\tan^2\alpha$$

$$V^2\tan^2\alpha - 24W^2\tan\alpha + V^2 - 32W^2 = 0$$

When $\alpha = 45°$;

$$V^2\tan^2 45° - 24W^2\tan 45° + V^2 - 32W^2 = 0$$

$$V^2 - 24W^2 + V^2 - 32W^2 = 0$$

$$2V^2 - 56W^2 = 0$$

$$V^2 = 28W^2$$

$$W = \frac{V}{2\sqrt{7}}$$

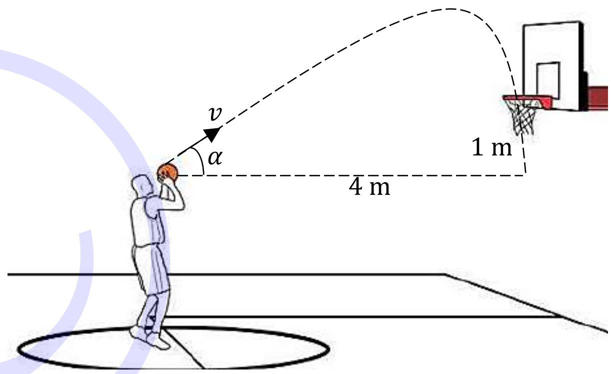
Now

$$28W^2\tan^2\alpha - 24W^2\tan\alpha + 28W^2 - 32W^2 = 0$$

$$28\tan^2\alpha - 24\tan\alpha - 4 = 0$$

$$7\tan^2\alpha - 6\tan\alpha - 1 = 0$$

Example 7



The motion of the ball in a successful free shot in basketball is illustrated above.

The ball is projected from a position, distance 4 m horizontally and 1 m vertically from the basket, with speed v ms^{-1} at an angle α to the horizontal. The ball falls into the basket.

(a) Show that v and α must satisfy

$$1 = 4\tan\alpha - \frac{78.4}{v^2}\sec^2\alpha$$

(b) Use this equation to find the required speed of projection when angle α equals 45°

(c) Also, use this equation to find the two possible trajectories when $v = 8.0$ ms^{-1}

For the ball to fall through the basket, the angle made with the vertical at the basket should be as small as possible. Which of your two solutions above would be preferred?

Solution

(a) Considering horizontal motion

$u = v\cos\alpha,\ s = 4\text{ m},\ a = 0,\ t = ?$

$$s = ut + \frac{1}{2}at^2$$

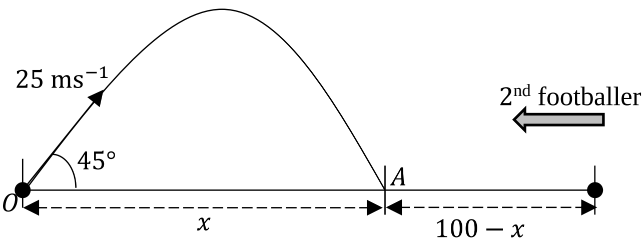
$$4 = (v\cos\alpha)t$$

$$t = \frac{4}{v\cos\alpha}$$

Example 17

Two footballers 100 m apart, stand facing each other. One of them kicks the ball from the ground such that the ball takes off at a velocity of 25 ms^{-1} at 45° to the horizontal. Find the speed at which the second footballer should run towards the first baller in order to trap the ball as it touches the ground, if he starts running at the instant the ball is kicked.

Solution



Considering the ball;

At the point A, the vertical displacement, $y = 0$

$\therefore$From $y = ut\sin\theta - \frac{1}{2}gt^2$

$$0 = 25t\sin 45° - \frac{1}{2}\times 9.8\times t^2$$

Either $t = 0$ i.e. at O or $25\sin 45° - \frac{1}{2}\times 9.8t = 0$

$$\therefore 4.9t = 25\times\sin 45°$$

$$t = 3.61\ s$$

From $x = ut\cos\theta$

$$x = 25\times 3.61\times\cos 45° = 63.82\text{ m}$$

Alternatively; as the ball touches the ground, it has travelled a distance equal to the range.

$$R = \frac{u^2\sin 2\theta}{g}$$

$$x = \frac{25^2\sin 90°}{9.8} = 63.82\text{ m}$$

Consider the second footballer;

He is supposed to travel a distance of

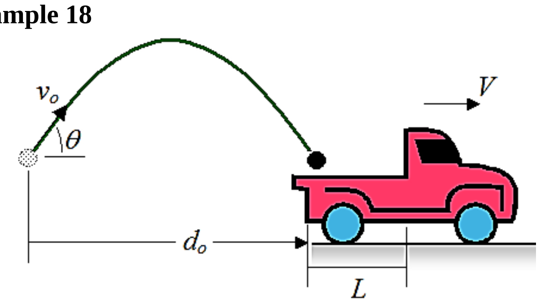
$$(100 - x) = 100 - 63.82 = 36.18\text{ m}$$

Since the second footballer starts running at the instant the ball is kicked and is supposed to run so as to trap the ball as it falls, he should take the same time as that taken by the ball to land.

$\Rightarrow$ He should take 3.61 s

$$\therefore\text{speed} = \frac{\text{Distance}}{\text{time}} = \frac{36.18}{3.61} = 10.02\text{ ms}^{-1}$$

Example 18



A ball is kicked at an angle $\theta = 45°$. It is intended that the ball lands in the back of a moving truck which has a trunk of length $L = 2.5$ m. If the initial horizontal distance from the back of the truck to the ball, at the instant of the kick is $d_o = 5$ m, and the truck moves directly away from the ball at a velocity of $V = 9\text{ ms}^{-1}$ as shown above, what is the maximum and minimum velocity so that the ball lands in the trunk? (Assume that the initial height of the ball is equal to the height of the ball at the instant it begins to enter the trunk)

Solution

When the ball begins to enter the trunk, the horizontal distance, d_x travelled by the ball is given by $d_x = (v_o\cos\theta)t$ which is also equal to the distance $d_o + Vt$ for minimum v_o and $d_o + L + Vt$ for maximum v_o where t is the total time.

Considering vertical motion;

$u = v_o\sin 45°$, $a = -9.8$, $s = 0$, $t = ?$

$$s = ut + \frac{1}{2}at^2$$

$$0 = (v_o\sin 45°)t - 4.9t^2$$

$$t = 0\text{ or }t = \frac{v_o\sin 45°}{4.9}$$

$$\therefore t = 0.1443v_o$$

The position of the two ends of the truck bed are given by

$$5 + 9t\ \text{ and } 7.5 + 9t$$

Considering horizontal motion;

For the ball to land on the back of the truck bed,

$$(v_o\cos 45°)t = 5 + 9t$$

$$v_o\cos 45°\times 0.1443v_o = 5 + 9(0.1443v_o)$$

$$0.102v_o^2 = 5 + 1.299v_o$$

$$0.102v_o^2 - 1.299v_o - 5 = 0$$

$$v_o = \frac{1.299\pm\sqrt{1.299^2 - 4(0.102)(-5)}}{2(0.102)}$$

$$v_o = 15.83\text{ or }v_o = -3.1$$

$$\therefore v_o = 15.83\text{ ms}^{-1}$$

For the ball to fall on the front of the trunk bed,

$$(v_o\cos 45°)t = 7.5 + 9t$$

$$v_o\cos 45°\times 0.1443v_o = 7.5 + 9(0.1443v_o)$$

$$0.102v_o^2 = 7.5 + 1.299v_o$$

$$0.102v_o^2 - 1.299v_o - 7.5 = 0$$

$$v_o = \frac{1.299\pm\sqrt{1.299^2 - 4(0.102)(-7.5)}}{2(0.102)}$$

$$v_o = 17.05\text{ or }v_o = -4.31$$

$$\therefore v_o = 17.05\text{ ms}^{-1}$$

The minimum velocity of the ball is 15.83 ms^{-1} and the maximum velocity is 17.05 ms^{-1}

Example 19

Two particles, A and B, are projected from the same fixed point O, with the same speed u ms^{-1}, at angles of elevation θ and 2θ respectively. It is further given that B is projected $\frac{2}{3}$ s after A and $\tan\theta = \frac{3}{4}$.

If A and B collide in the subsequent motion determine the value of u.

Solution

Suppose the collision takes place t seconds after A was projected.

For A collision, both particles must have the same x and y displacements, at time t and $t - \frac{2}{3}$

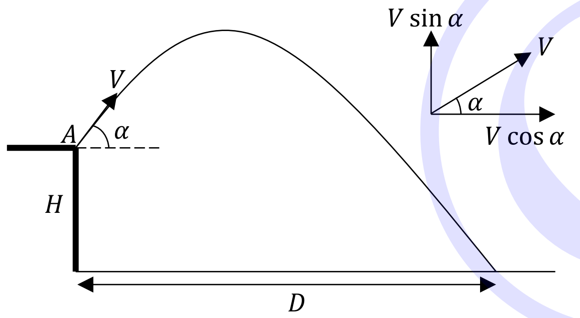
$$24T = 78.4$$
$$T = \frac{49}{15} = 3.26 \text{ s}$$
Finally, vertically for P;
$$v = u + at$$
$$v = 26\sin\theta - 9.8t$$
$$v = 26 \times \frac{12}{13} - 9.8 \times \frac{49}{15}$$
$$v = 24 - \frac{2401}{75}$$
$$v = -\frac{601}{75} = -8.01 \text{ ms}^{-1}$$
The negative sign implies that it is falling

Example 21

A particle is projected at an angle α above the horizontal, from a vertical cliff face of height H above level horizontal ground. It first hits the ground at a horizontal distance D, from the bottom of the cliff edge.

Assuming that air resistance can be ignored, show that the greatest height achieved by the particle from the level horizontal ground is
$$H + \frac{D\tan^2\alpha}{4(H + D\tan\alpha)}$$

Solution



Let T be the time taken to reach the maximum height.
$$v = u + at$$
$$0 = V\sin\alpha - gT$$
$$T = \frac{V\sin\alpha}{g}$$
Hence the maximum height above the ground is given by
$$H + ut + \frac{1}{2}at^2 = H + VT\sin\alpha - \frac{1}{2}gT^2$$
$$= H + V\left(\frac{V\sin\alpha}{g}\right)\sin\alpha - \frac{1}{2}g\left(\frac{V\sin\alpha}{g}\right)^2$$
$$= H + \frac{V^2\sin^2\alpha}{g} - \frac{1}{2}\frac{V^2\sin^2\alpha}{g}$$
$$= H + \frac{V^2\sin^2\alpha}{2g}$$
We need to get rid of V from the expression. Let τ be the flight time

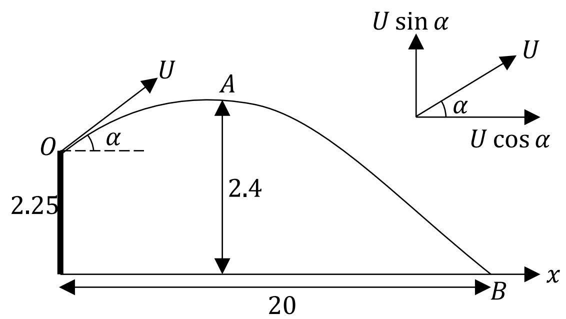
Horizontally;
$$D = (V\cos\alpha)\tau$$
$$\tau = \frac{D}{V\cos\alpha}$$
Vertically;
$$s = ut + \frac{1}{2}at^2$$
$$-H = (V\sin\alpha)\tau - \frac{1}{2}g\tau^2$$
$$-H = V\tau\sin\alpha - \frac{1}{2}g\tau^2$$
Substituting for τ;
$$-H = V\left(\frac{D}{V\cos\alpha}\right)\sin\alpha - \frac{1}{2}g\left(\frac{D}{V\cos\alpha}\right)^2$$
$$-H = D\tan\alpha - \frac{g}{2}\frac{D^2}{V^2\cos^2\alpha}$$
$$\frac{g}{2}\frac{D^2}{V^2\cos^2\alpha} = H + D\tan\alpha$$
$$\frac{2V^2\cos^2\alpha}{gD^2} = \frac{1}{H + D\tan\alpha}$$
$$V^2\left(\frac{2\cos^2\alpha}{gD^2}\right) = \frac{1}{H + D\tan\alpha}$$
$$V^2 = \frac{gD^2}{2\cos^2\alpha} \times \frac{1}{H + D\tan\alpha}$$
Finally substitute into $H + \frac{V^2\sin^2\alpha}{2g}$
$$= H + \frac{gD^2}{2\cos^2\alpha} \times \frac{1}{H + D\tan\alpha} \times \frac{\sin^2\alpha}{2g}$$
$$= H + \frac{D\sin^2\alpha}{4\cos^2\alpha\,(H + D\tan\alpha)}$$
$$= H + \frac{D\tan^2\alpha}{4(H + D\tan\alpha)}$$

Example 22

A tennis player standing on a level horizontal court serves the ball from a height of 2.25 m above the court. The ball reaches a maximum height of 2.4 m above the court and first hits the court at a horizontal distance of 20 m from the point where the player served the ball. The ball rises for T_1 s and falls for T_2 s. The ball is modelled as a particle moving through still air without any resistance.

(a) Show clearly that
$$\frac{T_2}{T_1} = 4$$
(b) Determine the magnitude and direction of the velocity of the ball
   (i) when it was first served.
   (ii) as it lands on the court.

Solution



Vertically, O to A;
$$v = u + at$$

$$0 = U\sin\alpha - gT_1$$
$$U\sin\alpha = gT_1$$

Vertically, O to A;

$$s = ut + \frac{1}{2}at^2$$
$$0.15 = (U\sin\alpha)T_1 - gT_1^2$$
$$\frac{3}{10} = 2(U\sin\alpha)T_1 - gT_1^2$$
$$\frac{3}{10} = 2(gT_1)T_1 - gT_1^2$$
$$\frac{3}{10} = gT_1^2$$
$$T_1^2 = \frac{3}{10g}$$

Vertically, A to B;

$$s = ut + \frac{1}{2}at^2$$
$$2.4 = \frac{1}{2}gT_2^2$$
$$4.8 = gT_2^2$$
$$gT_2^2 = \frac{24}{5}$$
$$T_2^2 = \frac{24}{5g}$$

Combining the results;

$$\frac{T_2^2}{T_1^2} = \frac{24/5g}{3/10g} = 16$$
$$\therefore \frac{T_2}{T_1} = 4$$

(b) (i)

Flight time, $T = T_1 + T_2 = \sqrt{\frac{3}{10g}} + \sqrt{\frac{24}{5g}} = \sqrt{\frac{3}{98}} + \sqrt{\frac{24}{49}}$

$$= \frac{\sqrt{6}}{14} + \frac{2}{7}\sqrt{6} = \frac{5}{14}\sqrt{6}$$

Horizontally from O to B;

$$20 = U\cos\alpha \times \frac{5}{14}\sqrt{6}$$

Vertically from O to B;

$$-2.25 = U\sin\alpha \times \frac{5}{14}\sqrt{6} - \frac{1}{2}g\left(\frac{5}{14}\sqrt{6}\right)^2$$
$$-2.25 = U\sin\alpha \times \frac{5}{14}\sqrt{6} - 3.75$$
$$1.5 = U\sin\alpha \times \frac{5}{14}\sqrt{6}$$

Dividing the equations;

$$\frac{U\sin\alpha \times \frac{5}{14}\sqrt{6}}{U\cos\alpha \times \frac{5}{14}\sqrt{6}} = \frac{1.5}{20}$$
$$\tan\alpha = \frac{3}{20}$$
$$\alpha = 4.29°$$

Hence $20 = U\cos\alpha \times \frac{5}{14}\sqrt{6}$

$$U = \frac{20 \times 14}{5\sqrt{6}\cos 4.29°}$$
$$U = 22.9\ \text{ms}^{-1}$$

(ii) Now using $v^2 = u^2 + 2as$ for OB

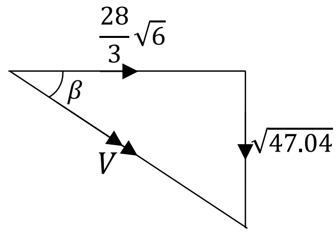
$$v^2 = (U\sin\alpha)^2 - 2 \times 9.8 \times (-2.25)$$
$$v^2 = U^2\sin^2\alpha + 44.1$$

From $1.5 = (U\sin\alpha) \times \frac{5}{14}\sqrt{6}$

$$U\sin\alpha = \frac{7}{10}\sqrt{6}$$

Thus, $v^2 = \frac{147}{50} + 44.1$

$$v^2 = 47.04$$
$$v = \sqrt{47.04}$$



Thus $V = \sqrt{47.04 + \frac{1568}{3}}$

$$V = 23.87\ \text{ms}^{-1}$$
$$\tan\beta = \frac{\sqrt{47.04}}{\frac{28}{3}\sqrt{6}}$$
$$\beta = 16.7°$$

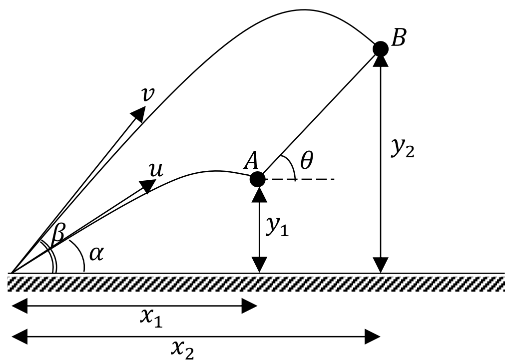
Example 23

Two particles are projected simultaneously from the same point with angles of projection α and β and initial speeds u and v. Show that at any time during their flight the line joining them is inclined to the horizontal at

$$\arctan\frac{u\sin\alpha - v\sin\beta}{u\cos\alpha - v\cos\beta}$$

Solution

Let the angle made by the line joining them to the horizontal be θ



Consider the motion of particle A;

Horizontal motion;

$$x_1 = uT\cos\alpha$$

Vertical motion;

$u = u\sin\alpha$, $a = -g$, $s = y_1$

$$y_1 = (u\sin\alpha)T - \frac{1}{2}gT^2$$
$$y_1 = T(u\sin\alpha - \frac{1}{2}gT)$$

Consider the motion of particle B;

Horizontal motion;

$$x_2 = vT\cos\beta$$

Vertical motion;

$$y_2 = (v\sin\beta)T - \frac{1}{2}gT^2$$

$$y_2 = T(v\sin\beta - \frac{1}{2}gT)$$

Now

$$\tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\tan\theta = \frac{T\left(v\sin\beta - \frac{1}{2}gT\right) - T\left(u\sin\alpha - \frac{1}{2}gT\right)}{vT\cos\beta - uT\cos\alpha}$$

$$\tan\theta = \frac{T\left[v\sin\beta - \frac{1}{2}gT - u\sin\alpha + \frac{1}{2}gT\right]}{T[v\cos\beta - u\cos\alpha]}$$

$$\tan\theta = \frac{v\sin\beta - u\sin\alpha}{v\cos\beta - u\cos\alpha} \times \frac{-1}{-1}$$

$$\tan\theta = \frac{u\sin\alpha - v\sin\beta}{u\cos\alpha - v\cos\beta}$$

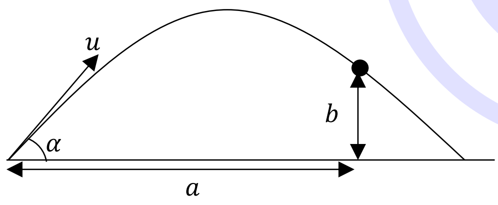
$$\theta = \tan^{-1}\frac{u\sin\alpha - v\sin\beta}{u\cos\alpha - v\cos\beta}$$

Example 24

A particle is projected from a point O with initial speed u to pass through a point which is at a horizontal distance a from O and a distance b vertically above the level of O. Show that there are two possible angles of projection. If these angles are α_1 and α_2, prove that $\tan(\alpha_1 + \alpha_2) = -(a/b)$

<u>Solution</u>

Let the angle of projection be α



$$s = ut + \frac{1}{2}at^2$$

Considering horizontal motion;

$$a = (u\cos\alpha)t$$

$$t = \frac{a}{u\cos\alpha}$$

Considering vertical motion;

$$b = (u\sin\alpha)t - \frac{1}{g}t^2$$

$$b = (u\sin\alpha)\left(\frac{a}{u\cos\alpha}\right) - \frac{1}{2}\left(\frac{a}{u\cos\alpha}\right)^2$$

$$b = a\tan\alpha - \frac{ga^2}{2u^2}\sec^2\alpha$$

$$b = a\tan\alpha - \frac{ga^2}{2u^2}(1 + \tan^2\alpha)$$

$$\frac{2u^2b}{ga^2} = \frac{2u^2}{ag}\tan\alpha - 1 - \tan^2\alpha$$

$$\tan^2\alpha - \frac{2u^2}{ag}\tan\alpha + 1 + \frac{2u^2b}{ga^2} = 0$$

$$\tan^2\alpha - \frac{2u^2}{ag}\tan\alpha + \left(\frac{ga^2 + 2u^2b}{ga^2}\right) = 0$$

This is a quadratic equation in terms $\tan\alpha$ hence two possible angles of projection.

Now;

$\tan\alpha_1 + \tan\alpha_2 = \frac{2u^2}{ag}$ and $\tan\alpha_1\tan\alpha_2 = \left(\frac{2u^2b+ga^2}{ga^2}\right)$

From compound angle formula;

$$\tan(\alpha_1 + \alpha_2) = \frac{\tan\alpha_1 + \tan\alpha_2}{1 - \tan\alpha_1\tan\alpha_2}$$

$$= \frac{(2u^2/ag)}{1 - \left(\frac{2u^2b + ga^2}{ga^2}\right)}$$

$$= \frac{(2u^2/ag)}{\left(\frac{ga^2 - 2u^2b - ga^2}{ga^2}\right)}$$

$$= \frac{2u^2}{ag} \div \frac{-2u^2b}{ga^2}$$

$$= \frac{2u^2}{ag} \times \frac{ga^2}{-2u^2b}$$

$$= -\frac{a}{b}$$

$$\therefore \tan(\alpha_1 + \alpha_2) = -\left(\frac{a}{b}\right)$$

Example 25

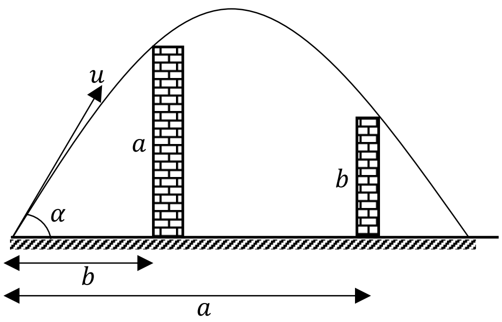
A ball is projected so as just to clear two walls, the first of height a at a distance b from the point of projection, and the second of height b at a distance a from the point of projection. Show that the range on a horizontal plane is

$$\frac{a^2 + ab + b^2}{a + b}$$

and that the angle of projection exceeds $\tan^{-1}3$

<u>Solution</u>

Let the speed of projection be u and the angle of projection be α



Using the trajectory equation

$$y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha$$

For clearance at first wall,

$$a = b\tan\alpha - \frac{gb^2}{2u^2}\sec^2\alpha$$

$$b\tan\alpha - a = \frac{gb^2}{2u^2}\sec^2\alpha \quad \ldots.\text{(i)}$$

For clearance at second wall,

$$b = a\tan\alpha - \frac{ga^2}{2u^2}\sec^2\alpha$$

$$\sin 2\theta = \frac{4}{5}$$
$$2\theta = 53.13°$$
$$\theta = 26.6°$$

(b) Now for the ratio of times, we can use

Time for no bounce throws, t is $\frac{u\sqrt{2}}{g}$

Total time for one bounce through, $T = t_1 + t_2$

$$T = \frac{2u\sin\theta}{g} + \frac{u\sin\theta}{g} = \frac{3u\sin\theta}{g} = \frac{3u\sin 26.6°}{g} = \frac{1.34u}{g}$$

The ratio of the times is given by;

$$\frac{t}{T} = \frac{u\sqrt{2}}{g} \div \frac{1.34u}{g} = \frac{u\sqrt{2}}{g} \times \frac{g}{1.34u} = 1.06$$

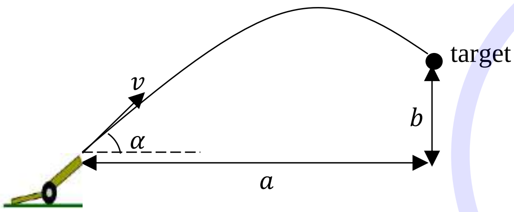
Example 27

The muzzle speed of a gun is v and it is desired to hit a small target at a horizontal distance a away and at a height b above the gun. Show that this is impossible if $v^2(v^2 - 2gb) < g^2a^2$, but that, if $v^2(v^2 - 2gb) > g^2a^2$, there are two possible elevations for the gun.

Show that if $v^2 = 2ag$ and $b = \frac{3}{4}a$, there is only one possible elevation, and find the time taken to hit the target.

<u>Solution</u>

Let the angle of elevation be α



The trajectory equation is $y = x\tan\alpha - \frac{gx^2(1+\tan^2\alpha)}{2v^2}$

For the shell to hit the target at $x = a$, $y = b$

$$b = a\tan\alpha - \frac{ga^2(1+\tan^2\alpha)}{2v^2}$$

$$\Rightarrow ga^2\tan^2\alpha - 2av^2\tan\alpha + 2v^2b + ga^2 = 0 \text{ .. (i)}$$

This a quadratic equation in $\tan\alpha$, which must have real solutions for the target to be hit.

(a) The target cannot be hit if the discriminant < 0

i.e. if $4a^2v^4 - 4(ga^2)(2v^2b + ga^2) < 0$

$$v^4 - 2gbv^2 - g^2a^2 < 0$$
$$v^2(v^2 - 2gb) < g^2a^2$$

(b) If the discriminant > 0, i.e. if $v^2(v^2 - 2gb) > g^2a^2$, then equation (i) has two real distinct solutions for $\tan\alpha$ and hence for the elevation

(c) If $v^2 = 2ga$ and $b = \frac{3}{4}a$, then (i) becomes

$$ga^2\tan\alpha - 4ga^2\tan\alpha + 3ga^2 + ga^2 = 0$$
$$\tan^2\alpha - 4\tan\alpha + 4 = 0$$
$$(\tan\alpha - 2)^2 = 0$$
$$\tan\alpha = 2$$

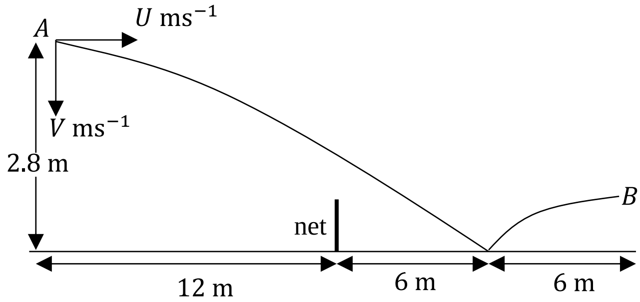
i.e. only one possible elevation, $\tan^{-1} 2$

Horizontal displacement, $a = (v\cos\alpha)t = \sqrt{2ga}\left(\frac{1}{\sqrt{5}}\right)t$

$$t = \frac{a\sqrt{5}}{\sqrt{2ga}} = \sqrt{\frac{5a}{2g}}$$

Time taken for the shell to reach the target is $\sqrt{\frac{5a}{2g}}$

Example 28



A ball is projected from a point A a height of 2.8 m above the ground with velocity components U and V vertically and horizontally respectively. It passes over a net at a horizontal distance of 12 m from A, landing 6 m behind the net. It bounces to a point B which is at a height of 0.75 m and at a distance of 6 m from the point of bounce. The ball takes 0.6 s to cover the 24 m. Assuming that there is no air resistance and the ball bouncing does not affect the horizontal velocity of the ball, find

(a) U and V

(b) the distance by which the ball just clears the net if it is 1 m high?

(c) the direction of the ball at B

<u>Solution</u>

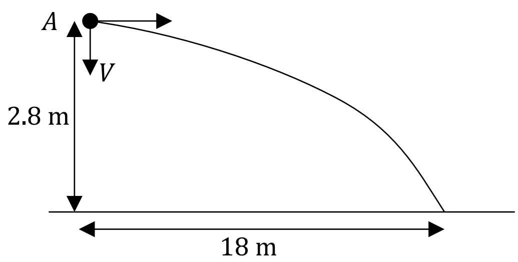
(a) Since the bouncing does not affect the horizontal velocity of the ball,

$$s_x = u_xt + \frac{1}{2}a_xt^2$$
$$24 = U \times 0.6$$
$$U = 40 \text{ ms}^{-1}$$

Considering the motion of the ball from the A to where it bounces.

Time taken is given by

$$\frac{\text{distance}}{\text{speed}} = \frac{s_x}{u_x} = \frac{18}{40} = 0.45 \text{ s}$$



Vertically;

$$s_y = u_yt + \frac{1}{2}a_yt^2$$
$$2.8 = V(0.45) + \frac{1}{2}(9.8)(0.45)^2$$
$$2.8 = 0.45V + 0.99225$$
$$0.45V = 1.80775$$
$$V = 4.02 \text{ ms}^{-1}$$

Force

A force is necessary to cause a body to accelerate. More than one force may act on a body. If the forces on a body are in **equilibrium**, i.e. balance out, then the body may be at rest or moving in a straight line at constant speed.
If there is a resultant force on the body, then the body will accelerate.
Force is a **vector**, i.e. has magnitude and direction.
S.I unit of force is the Newton (N).
1 Newton is the force needed to give a body of mass 1 kg an acceleration of 1 ms^{-2}

Mass and weight

Mass and weight are different.
The mass of a body is a measure of the matter contained in the body. A massive body will need a larger force to change its motion. The mass of a body may be considered to be constant, whatever the position of the body, provided that none of the body is destroyed or changed.
Mass is a scalar, i.e. it has magnitude only.
S.I unit of mass is the kilogram (kg)
The weight of the body is the force with which the earth attracts it. It is dependent upon the body's distance from the earth.
Weight is a vector, since it is a force.
S.I unit of weight is the Newton (N).
The weight, W, in newtons, and mass m, in kilograms are connected by the relation $W = mg$, where g is the acceleration due to gravity in ms^{-2}.

NEWTON'S LAWS OF MOTION

Newton's three laws are the fundamental basis of the study of mechanics at this level. Although there is no direct proof of these laws, predictions made using them agree very closely with observations.

1st law

Every body will remain at rest or continue to move in a straight line at constant speed unless an external force acts on it.

Consequences

- If a body has an acceleration, then there must be a force acting on it
- If a body has no acceleration, then the forces acting on it must be in equilibrium

2nd law

The rate of change of momentum of a moving body is proportional to the external force acting on it and takes place in the direction of that force.
So when an external force acts on a body of constant mass, the force produces an acceleration which is directly proportional to the force.

Consequences

- The basic equation of motion for constant mass is

  Force = **mass** × **acceleration**
  (in N) (in kg) (in ms^{-2})
- The force and acceleration of the body are both in the same direction
- A constant force on a constant mass gives a constant acceleration.

3rd law

If a body A exerts a force on B, then B exerts an equal and opposite force on A.

Consequences

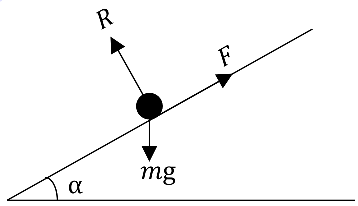
These forces between the bodies are often called reactions. In a rigid body, the internal forces occur as equal and opposite pairs and the net effect is zero. So only external forces need to be considered.

Problem solving

1. Draw a clear force diagram
2. If there is no acceleration, i.e. the body is either at rest or moving with uniform velocity, then the forces balance in each direction
3. If there is an acceleration
   (a) mark it on the diagram using $\xrightarrow{a}\!\!\blacktriangleright\!\!\blacktriangleright$
   (b) write down, if possible, an expression for the resultant force
   (c) use Newton's 2nd law i.e. write the equation of motion

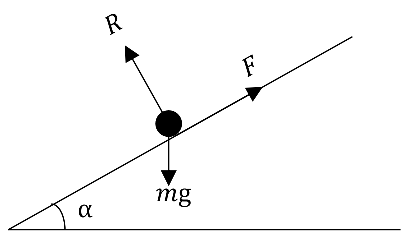
   force = mass × acceleration

Body at rest on a rough inclined plane



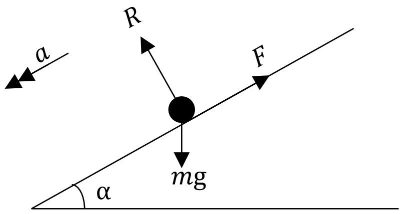
No acceleration so forces balance:
∥ to plane ⇒ $F = mg\sin\alpha$
⊥ to plane ⇒ $R = mg\cos\alpha$

Body sliding down rough plane at constant speed



∥ to plane ⇒ $F = mg\sin\alpha$
⊥ to plane ⇒ $R = mg\cos\alpha$

Body sliding down rough plane with acceleration



No $'a'$ $\perp$ to plane
$\Rightarrow R = mg\cos\alpha$
Resultant force down the plane is $mg\sin\alpha - F$
2nd law $\Rightarrow mg\sin\alpha - F = ma$

Example 1

A lift of mass 500 kg is lowered or raised by means of a metal cable attached to its top. The lift contains passengers whose total mass is 300 kg. The lift starts from rest and accelerates at a constant rate, reaching a speed of 3 ms^{-1} after moving a distance of 5 m. Find

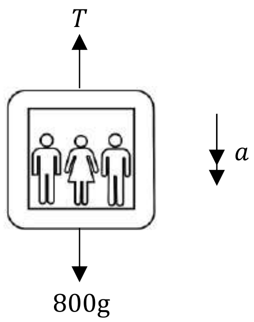
(a) the acceleration of the lift
(b) the tension in the cable if the lift is moving vertically downwards
(c) the tension in the cable if the lift is moving vertically upwards

<u>Solution</u>

(a) $u = 0, v = 3, s = 5, a = ?$

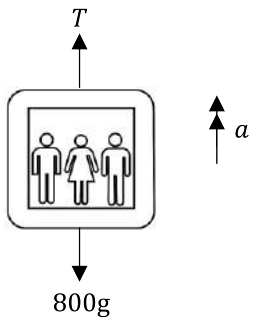
$$v^2 = u^2 + 2as$$
$$3^2 = 0 + 2 \times 5a$$
$$a = 0.9\ \text{ms}^{-2}$$

(b) Net downward force $= ma$



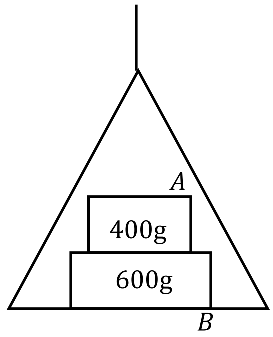
$$800\text{g} - T = 800a$$
$$T = 800(\text{g} - a)$$
$$T = 800(9.8 - 0.9) = 7120\ N$$

(c) Net upward force $= ma$



$$T - 800\text{g} = 800a$$
$$T = 800(\text{g} + a)$$
$$T = 800(9.8 + 0.9) = 8560\ N$$

Example 2



A light scale-pan is attached to a vertical light inextensible string. The scale pan carries two masses A and B. The mass of A is 400g and the mass of B is 600 g. A rests on top of B, as shown in the diagram above. The scale-pan is raised vertically, using the string, with acceleration 0.5 ms^{-2}.

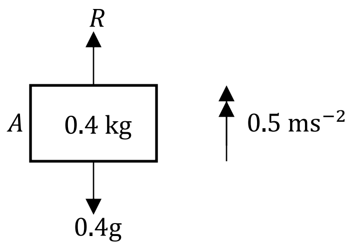
(a) Find the tension in the string
(b) Find the force exerted on mass B by mass A
(c) Find the force exerted on mass B by the scale-pan.

<u>Solution</u>

(a) Let the tension in the string be T
For the whole system
$R(\uparrow);\ T - (0.4 + 0.6)\text{g} = (0.4 + 0.6)a$

$$T - \text{g} = 1 \times 0.5$$
$$T = 10.3\ N$$

(b)

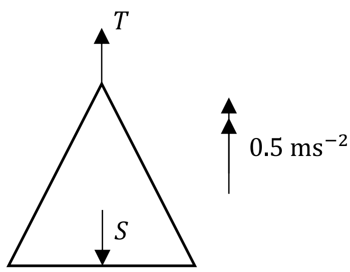


For A only:
$R(\uparrow);\ \ R - 0.4\text{g} = 0.4 \times 0.5$

$$R = 4.12\ N$$

So the force exerted on B by A is 4.12 N downwards

(c)



For scale-pan only:
Since it's light, its mass is zero.
$R(\uparrow);\ T - S = 0 \times 0.5$

$$T = S$$
$$S = 10.3\ N$$

So, the force exerted on B by the scale pan is $10.3\ N$

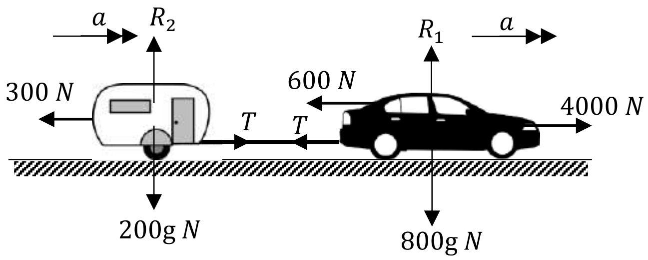
Example 3

A car of mass 800 kg pulls a trailer of mass 200 kg. The force produced by the engine is 4000 N. The resistances acting on the car and trailer are 600 N and 300 N respectively. Calculate:

(a) the acceleration of the car and trailer

(b) the tension in the tow bar

Solution



(a) Apply $F = ma$ to the car:

$$4000 - T - 600 = 800a$$
$$3400 - T = 800a \ldots.(i)$$

Apply $F = ma$ to the trailer:

$$T - 300 = 200a \ldots\ldots(ii)$$

Adding (i) and (ii0;

$$3100 = 1000a$$
$$a = 3.1\ \text{ms}^{-2}$$

(b) From (ii):

$$T - 300 = 200 \times 3.1$$
$$T = 920\ N$$

Example 4

A car of mass 1500 kg is towing a trailer of mass 500 kg by means of a light rigid horizontal tow-bar. The car is experiencing a constant air resistance of 300 N, while the corresponding constant air resistance on the trailer is 100 N. The car and trailer are modelled as particles.

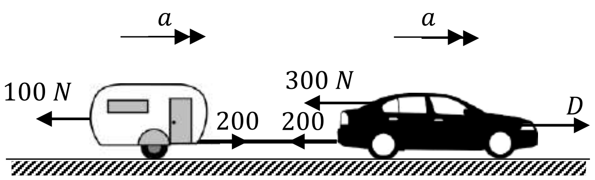
(a) Given the tension in the tow-bar is 200 N, calculate
 (i) the acceleration of the system.
 (ii) the driving force of the car.

Later in the journey, the car's driving force is removed and the car's brakes are applied, providing a constant breaking force of 400 N, on the car only. The air resistance on the car and trailer are unchanged.

(b) Determine
 (i) the deceleration of the system.
 (ii) the thrust in the tow-bar.

Solution

(a)



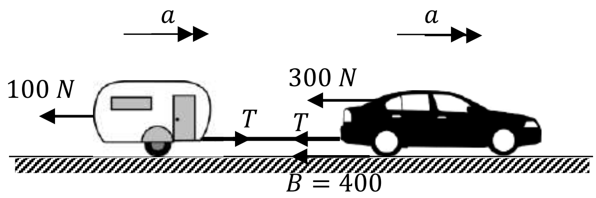
Trailer:

$$200 - 100 = 500a$$
$$100 = 500a$$
$$a = 0.2\ \text{ms}^{-2}$$

Car:

$$D - 300 - 200 = 1500a$$
$$D - 500 = 1500 \times 0.2$$
$$D = 800\ N$$

(b)



Trailer:

$$T - 100 = 500a$$

Car:

$$-300 - T - 400 = 1500a$$

Adding:

$$-800 = 2000a$$
$$a = -0.4\ \text{ms}^{-2}$$
$$T - 100 = 500(-0.4)$$
$$T = -100\ N$$

$\therefore$ Deceleration of $0.4\ \text{ms}^{-2}$ and thrust of $100\ N$

Example 5

A car of mass 1500 kg is towing a trailer of mass 1000 kg by means of a light inextensible rope. The car is experiencing a constant air resistance of 200 N, while the corresponding constant air resistance on the trailer is 300 N. The car and trailer are modelled as particles, with the tow rope remaining taut and horizontal throughout the motion.

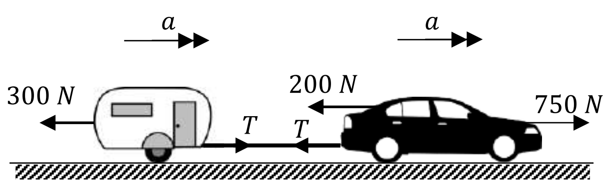
(a) Given that the driving force acting on the car is 750 N, determine
 (i) the acceleration of the system.
 (ii) the tension in the tow rope.

Later in the journey, the car and the trailer are ascending on a road which inclined at 5° to the horizontal. The air resistance on the car and trailer are unchanged.

(b) Assuming that the system now moves with constant speed, calculate
 (i) a new value for the tension in the tow rope.
 (ii) a new value for the driving force of the car.

Solution

(a)



Looking at the car and the trailer separately ($F = ma$)

Car:

$$750 - T - 200 = 1500a$$

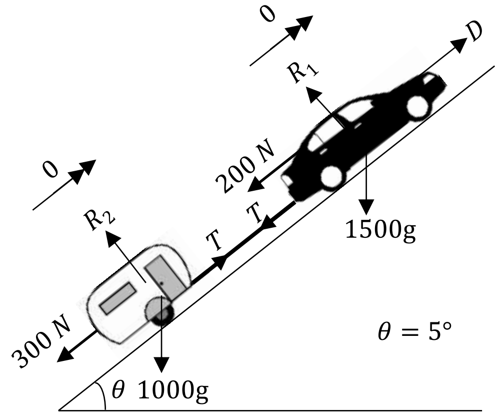
Trailer:

$$T - 300 = 1000a$$

Adding:

$$250 = 2500a$$
$$a = 0.1 \text{ ms}^{-2}$$
$$T - 300 = 1000 \times 0.1$$
$$T = 400\ N$$

(b)



Constant speed ⇒ equilibrium

Looking at the direction of motion, only for each object

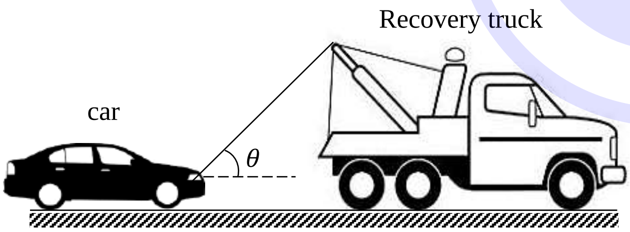
Trailer:

$$T = 300 + 1000\text{g} \sin\theta$$
$$T = 300 + 1000(9.8)(\sin 5°)$$
$$T = 1154\ N$$

Car:

$$D = T + 200 + 1500\text{g} \sin\theta$$
$$D = 1154 + 200 + 1500(9.8)(\sin 5°)$$
$$D = 2635\ N$$

Example 6



A recovery truck of mass 2800 kg is towing a car of mass 1200 kg along a straight horizontal road. The tow cable is inclined at an angle θ to the horizontal, where $\cos\theta = 0.75$, as shown in the figure above. The tow cable is modelled a light inextensible string and the two vehicles as particles.

The two vehicles were travelling at constant speed 12 ms^{-1} with the tow cable taut as they were travelling in an urban area. On leaving this area, the truck begins to accelerate uniformly bringing their speed to 27 ms^{-1} over a distance of 2.34 km.

(a) Calculate the acceleration of the truck and the car.

There is a constant resistance to the motion of the truck of 600 N, and a constant resistance to the motion of the car of 270 N.

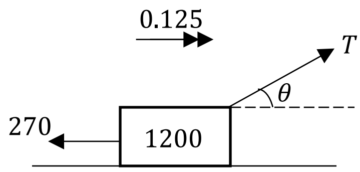
(b) For the part of the journey during which the two vehicles accelerate, determine
   (i) the force in the tow cable.
   (ii) the driving force of the truck

<u>Solution</u>

(a) By kinematics

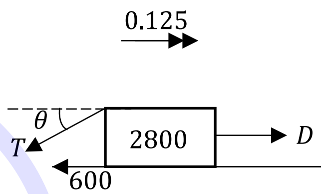
$$u = 12 \text{ ms}^{-1}, a = 3, s = 2340 \text{ m}, t = ?, v = 27 \text{ ms}^{-1}$$
$$v^2 = u^2 + 2as$$
$$27^2 = 12^2 + 2a(2340)$$
$$729 = 144 + 4680a$$
$$4680a = 585$$
$$a = \frac{1}{8} = 0.125 \text{ ms}^{-2}$$

(b) (i)



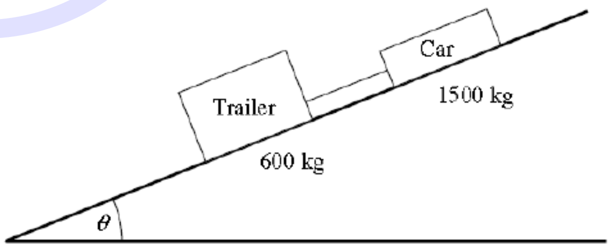
$$T\cos\theta - 270 = 1200a$$
$$\frac{3}{4}T - 270 = 1200 \times \frac{1}{8}$$
$$\frac{3}{4}T = 420$$
$$T = 560\ N$$

(ii)



$$D - T\cos\theta - 600 = 2800a$$
$$D - \frac{3}{4} \times 560 - 600 = 2800 \times \frac{1}{8}$$
$$D - 480 - 600 = 350$$
$$D = 1370\ N$$

Example 7



A trailer of mass 600 kg is connected to a car of mass 1500 kg by means of a light rigid tow bar. The car is moving up a line of greatest slope of a plane inclined at θ to the horizontal, where $\sin\theta = \frac{7}{25}$, as shown in the figure above. A constant resistance of magnitude 400 N acts on the car, and a constant resistance of magnitude 300 N acts on the trailer. The engine of the car produces a constant forward driving force of 8400 N. Determine the acceleration of the car and the tension in the tow bar.

<u>Solution</u>

Let the acceleration be a and the tension in the tow bar be T

CONNECTED PARTICLES

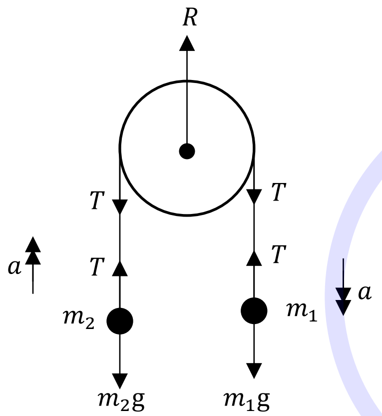
When particles are connected by a taut string the tension is the same throughout its length even if the string passes over a (frictionless) pulley. When the system is moving, the motion of each particle satisfies $F = ma$, the particles having accelerations of equal magnitude.

Example 1

Two particles of masses m_1 and m_2 kg are connected by a light inextensible string passing over a smooth fixed pulley. Find the resulting motion of the system and the tension in the string.

Solution

The tension is the same throughout the string. Let this be T N. Let also $m_1 > m_2$, then m_1 will move downwards and m_2 upwards, and since the string is inextensible, the upward acceleration of m_2 is equal to the downward acceleration of m_1. Let this acceleration be a ms^{-2}.



The resultant force on m_1 is downwards

$$m_1\text{g} - T = m_1 a \;\ldots.(i)$$

The resultant force on m_2 is upwards

$$T - m_2\text{g} = m_2 a \ldots..(ii)$$

Adding (i) and (ii);

$$(m_1 - m_2)\text{g} = (m_1 + m_2)a$$

$$a = \frac{m_1 - m_2}{m_1 + m_2}\text{g}$$

From (i);

$$T = m_1(\text{g} - a)$$

$$= m_1\left(1 - \frac{m_1 - m_2}{m_1 + m_2}\right)\text{g}$$

$$= \frac{2m_1 m_2}{m_1 + m_2}\text{g}$$

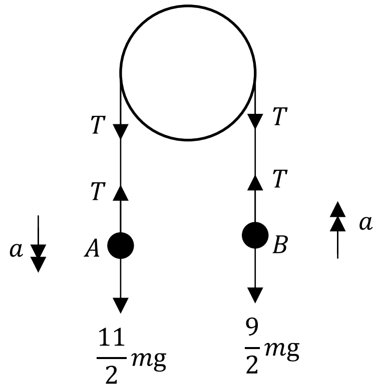
If the parts of the string not in contact with the pulley hang vertically, the force R on the pulley

$$= 2T = \frac{4m_1 m_2}{m_1 + m_2}\text{g}$$

Example 2

Two particles A and B are connected by a light inextensible string passing over a smooth fixed pulley. The masses of A and B are $\frac{11}{2}m$ and $\frac{9}{2}m$ respectively. With A and B hanging vertically, the system is released from rest with particle A a distance d above the floor. If a time t elapses before A hits the floor, show that $20d = t^2\text{g}$

Solution



For particle A; $\frac{11}{2}mg - T = \frac{11}{2}ma$ … (i)

For particle B; $T - \frac{9}{2}mg = \frac{9}{2}ma$ ….. (ii)

Adding (i) and (ii);

$$\frac{11}{2}mg - \frac{9}{2}mg = \frac{11}{2}ma + \frac{9}{2}ma$$

$$mg = 10ma$$

$$a = \frac{1}{10}\text{g}$$

Considering the motion of A;

$$u = 0, a = \frac{1}{10}\text{g}, t = t, s = d$$

$$s = ut + \frac{1}{2}at^2$$

$$d = 0 + \frac{1}{2} \times \frac{1}{10}\text{g}t^2$$

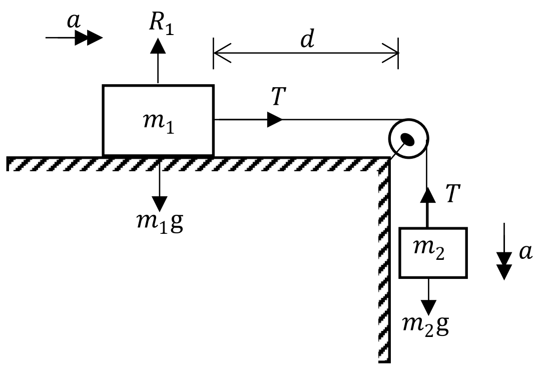
$$20d = \text{g}t^2$$

Example 3

A particle of mass m_1 lies on a smooth horizontal table and is connected to a freely hanging particle of mass m_2 by a light inextensible string passing over a smooth fixed pulley situated at the edge of the table. Initially the system is at rest with m_1 a distance d from the edge of the table.

Show that the acceleration of the system is $\frac{m_2\text{g}}{m_1+m_2}$ and that the time taken for m_1 to reach the edge of the table is $\sqrt{\frac{2d(m_1+m_2)}{m_2\text{g}}}$

Solution



Now the 3 kg mass is stopped by the ground, the string becomes slack and the 10 kg mass moves with velocity v, and subject to a retardation given by

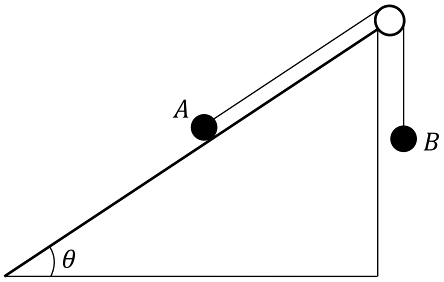
$$F = ma$$
$$0 - 2\text{g} = 10a$$
$$a = -\frac{\text{g}}{5} = -1.96\text{ ms}^{-2}$$

The time t s taken to up the plane and return to the point from which it began to move freely is given by

$$0 = 1.22t - \frac{1}{2}(1.96t^2)$$
$$t = \frac{1.22}{0.98} = 1.25\text{ s}$$

After this interval, the string again becomes taut.

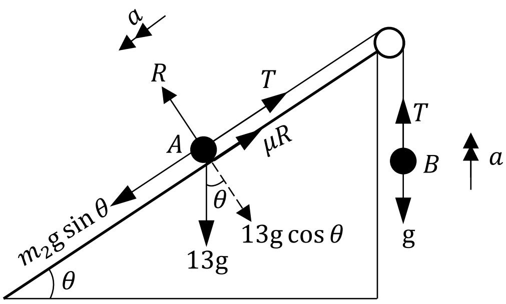
Example 10



The diagram above shows a body A of mass 13 kg lying on a rough inclined plane, coefficient of friction μ. From A, a light inextensible string passes up the line of greatest slope and over a smooth fixed pulley to a body B of mass m kg hanging freely. The plane makes an angle θ with the horizontal where $\sin\theta = \frac{5}{13}$. When $m = 1$ kg and the system is released from rest, B has an upward acceleration of a ms^{-2}; when $m = 11$ kg, and the system is released from rest, B has a downward acceleration of a ms^{-2}. Find a and μ.

<u>Solution</u>

When $m = 1$ kg;



For A:

$$13\text{g}\sin\theta - (T - \mu R) = 13a$$
$$13\text{g}\sin\theta - T - \mu(13\text{g}\cos\theta) = 13a$$
$$13\text{g}\left(\frac{5}{13}\right) - T - \mu(13\text{g})\left(\frac{12}{13}\right) = 13a$$
$$5\text{g} - T - 12\mu\text{g} = 13a \text{ .. (i)}$$

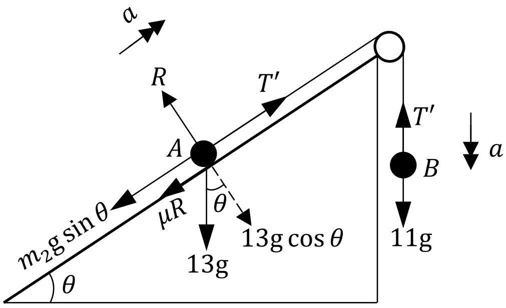
For B:

$$T - \text{g} = a \text{ ..... (ii)}$$

(i) + (ii);

$$4\text{g} - 12\mu\text{g} = 14a$$
$$2\text{g} - 6\mu\text{g} = 7a \text{ .... (iii)}$$

When $m = 11$ kg,



For A:

$$T' - (13\text{g}\sin\theta + \mu R) = 13a$$
$$T' - 13\text{g}\sin\theta - \mu(13\text{g}\cos\theta) = 13a$$
$$T' - 13\text{g}\left(\frac{5}{13}\right) - \mu(13\text{g})\left(\frac{12}{13}\right) = 13a$$
$$T' - 5\text{g} - 12\mu\text{g} = 13a \text{ ..... (iv)}$$

For B:

$$11\text{g} - T' = 11a \text{ ... (v)}$$

(v) + (iv);

$$6\text{g} - 12\mu\text{g} = 24a$$
$$3\text{g} - 6\mu\text{g} = 12a \text{ .... (vi)}$$

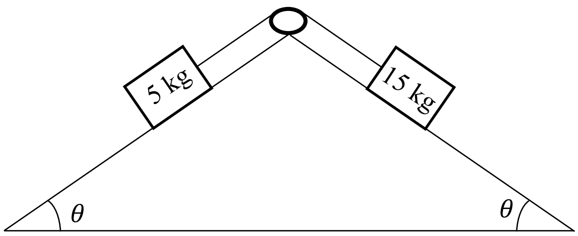
Solving (iii) and (vi);

$$2\text{g} - 6\mu\text{g} = 7a$$
$$\underline{-\ 3\text{g} - 6\mu\text{g} = 12a}$$
$$-\text{g} \qquad = -5a$$
$$a = \frac{\text{g}}{5} = 1.96\text{ ms}^{-2}$$

From (iii);

$$2\text{g} - 6\mu\text{g} = 7a$$
$$2\text{g} - 6\mu\text{g} = \frac{7\text{g}}{5}$$
$$-6\mu = -\frac{3}{5}$$
$$\mu = \frac{1}{10} = 0.1$$

Example 11



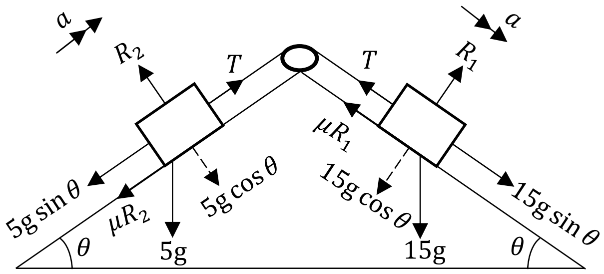
Masses of 5 kg and 15 kg are held at rest on inclined surfaces as shown in the diagram. The masses are connected by a light, taut, inextensible string passing over a smooth fixed pulley. The coefficient of friction between each mass and the surface with which it is in contact is 0.25. The inclination of the plane is such that $\sin\theta = \frac{3}{5}$. When the system is released from rest, the 15 kg mass accelerates down the slope.

(a) Find the magnitude of this acceleration and the tension in the string.

(b) Calculate the magnitude of the force on the pulley

Solution

(a)



For the 15 kg mass:

$$15g\sin\theta - (T + \mu R_1) = 15a$$

$$15g\left(\frac{3}{5}\right) - (T + 0.25 \times 15g\cos\theta) = 15a$$

$$9g - \left(T + 0.25 \times 15g \times \frac{4}{5}\right) = 15a$$

$$9g - T - 3g = 15a$$

$$6g - T = 15a \text{ ….. (i)}$$

For the 5 kg mass:

$$T - (5g\sin\theta + \mu R_2) = 5a$$

$$T - (5g\sin\theta + 0.25 \times 5g\cos\theta) = 5a$$

$$T - \left(5g \times \frac{3}{5} + 0.25 \times 5g \times \frac{4}{5}\right) = 5a$$

$$T - (3g + g) = 5a$$

$$T - 4g = 5a \quad \text{…. (ii)}$$

Adding (i) and (ii) gives;

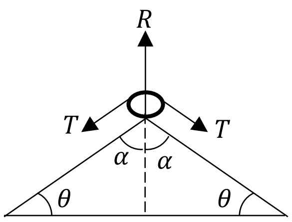
$$2g = 20a$$

$$a = \frac{g}{10} = 0.98 \text{ ms}^{-2}$$

From (ii); $T = 4g + 5a$

$$T = 4g + \frac{g}{2} = \frac{9g}{2} = 44.1\ N$$

(b) Looking at the diagram below showing only the forces acting on the pulley;



$$\alpha = 90° - \theta$$

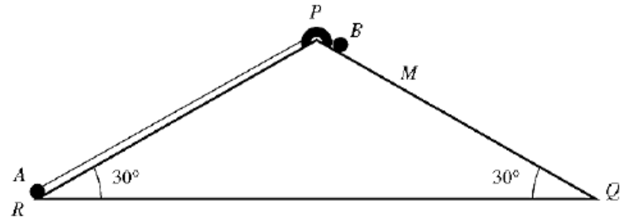
The force on the pulley is given by

$$R = 2T\cos\alpha$$

$$R = 2T\cos(90° - \theta)$$

$$R = 2T\sin\theta = 2 \times 44.1 \times \frac{3}{5} = 53.28\ N$$

Example 12

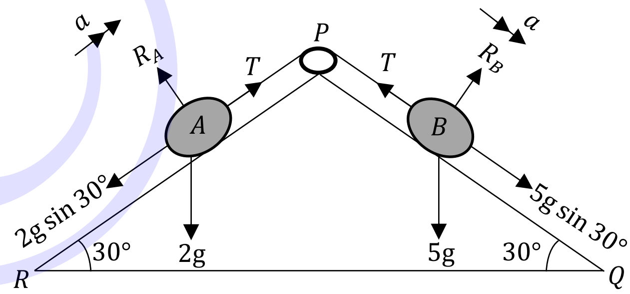


Two particles A and B have masses 2 kg and 5 kg, respectively. The particles are attached to the ends of a light inextensible string. The string passes over a small smooth pulley P which is fixed at the top of the cross section of a triangular prism RPQ, where $\measuredangle PRQ = \measuredangle PQR = 30°$. The string lies in the vertical plane which contains the pulley and lines of greatest slope of the inclined planes, PR and PQ, as shown in the figure above. When A is held at Q with the string taut, B is at P, on the line of greatest slope PQ. The point M, lies on PQ so that $PM : MQ = 1 : 3$.

The lines of greatest slope of the inclined planes, PR and PM, are smooth but the line of greatest slope MQ is rough. The system is released from rest with the string taut, when A is at Q and B is at P, on the line of greatest slope PQ. The system initially accelerates but due to the rough section MQ, B comes to rest as it reaches Q. Assuming that the string remains taut throughout the motion, show that the coefficient of friction between the B and MQ is 0.49.

Solution

Let's start by determining the acceleration of the system



For A:

$$T - 2g\sin 30° = 2a$$

For B:

$$5g\sin 30° - T = 5a$$

Adding;

$$7a = 3g\sin 30°$$

$$a = 2.1 \text{ ms}^{-2}$$

Now kinematics

Let $PQ = 4d$, then the smooth section is d and the rough section is $3d$

$u = 0,\ a = 2.1,\ s = d,\ v = ?$

$$v^2 = u^2 + 2as$$

$$v^2 = 2 \times 2.1 \times d$$

$$v^2 = 4.2d$$

Next the motion in the rough section

$u = \sqrt{4.2d},\ a = ?,\ s = 3d,\ v = 0$

$$v^2 = u^2 + 2as$$

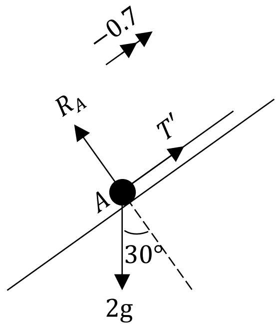
$$0 = 4.2d + 2a(3d)$$

$$0 = 4.2d + 6ad$$

$$6a = -4.2$$

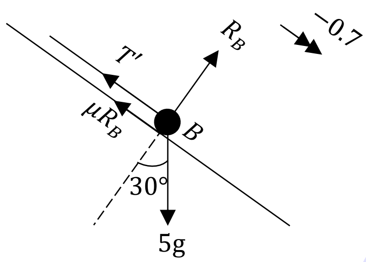
$$a = -0.7 \text{ ms}^{-2} \text{ (new deceleration)}$$

Looking at the dynamics of A, once B is on the rough section



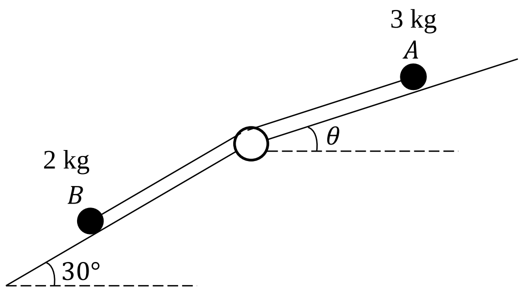
$$F = ma$$
$$T' - 2g \sin 30° = 2(-0.7)$$
$$T' - 9.8 = -1.4$$
$$T' = 8.4\ N \text{ (new tension)}$$

Dynamics of B, once in the rough section



$$F = ma$$
$$5g \sin 30° - T - \mu R_B = 5(-0.7)$$
$$24.5 - 8.4 - \mu(5g \cos 30°) = -3.5$$
$$20.6 = \mu(5g \cos 30°)$$
$$20.6 = 42.44\mu$$
$$\mu = 0.49$$

Example 13



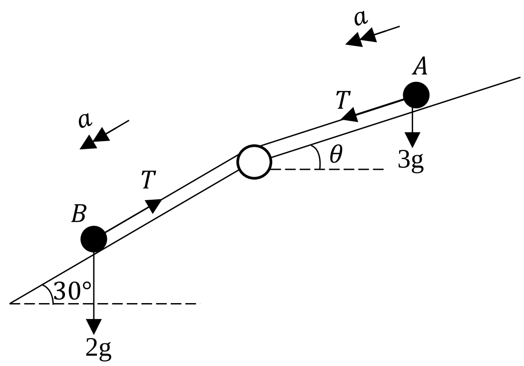
The diagram shows two slopes; the upper slope is inclined at an angle θ to the horizontal where $\sin\theta = \frac{1}{4}$, the lower slope at 30° to the horizontal. A particle A of mass 3 kg is held at rest on the upper slope. This particle is connected by a light inextensible string, passing over a smooth pulley, to a particle B of mass 2 kg on the lower slope.

(a) In the case where both slopes are smooth and A is released from rest, find
 (i) the acceleration of the particles
 (ii) the tension in the string
 (iii) the speed of each particle after travelling 0.63 m, assuming both particles remain on their respective slopes.

(b) In the case where the lower slope is smooth and the upper slope rough, the system is in limiting equilibrium. Find the coefficient of friction between particle A and the upper slope

Solution

(a) Let the acceleration of the particles be a and the tension in the string be T



For the 3 kg mass:

$$T + 3g \sin\theta = 3a$$
$$T + \frac{3g}{4} = 3a \quad \ldots.(i)$$

For the 2 kg mass:

$$2g \sin 30° - T = 2a$$
$$g - T = 2a \ldots.(ii)$$

Adding (i) and (ii);

$$\frac{7g}{4} = 5a$$
$$a = \frac{7g}{20} = 3.43 \text{ ms}^{-2}$$

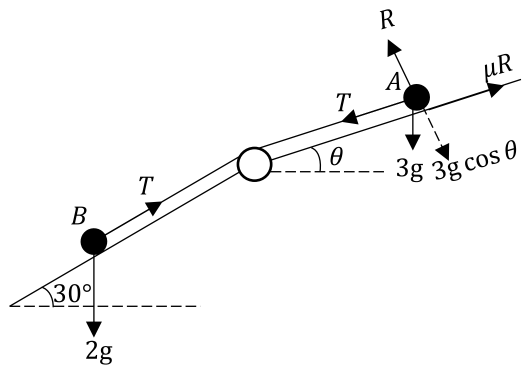
From (ii);

$$T = g - 2a = g - \frac{14g}{20} = \frac{6g}{20} = 2.94\ N$$

$u = 0$, $a = 3.43$, $s = 0.63$, $v = ?$

$$v^2 = u^2 + 2as$$
$$v^2 = 0^2 + 2 \times 3.43 \times 0.63$$
$$v = 2.08 \text{ ms}^{-1}$$

(b) When the system is in limiting equilibrium, the acceleration is 0



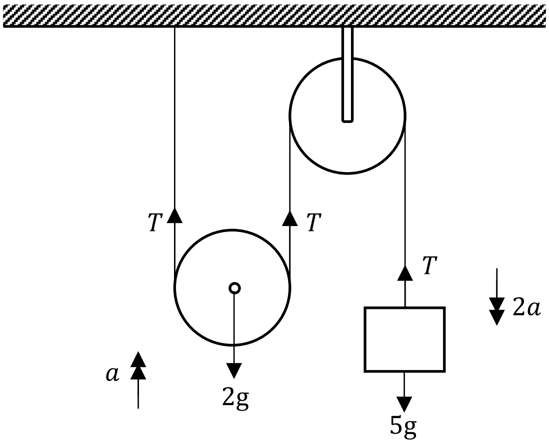
For the 2 kg mass:

Example 1

A string, with one end fixed, passes under a moveable pulley of mass 2 kg, over a fixed pulley and carries a 5 kg mass at its other end. Find the acceleration of the moveable pulley and the tension in the string.

Solution

Let the acceleration of the moveable pulley be a



For the mass:

$$5g - T = 5(2a)$$
$$5g - T = 10a \quad \ldots..(i)$$

For the moveable pulley:

$$2T - 2g = 2a$$
$$T - g = a \quad \ldots..(ii)$$

Adding (i) and (ii);

$$4g = 11a$$
$$a = \frac{4}{11}g \text{ ms}^{-2} \quad \text{or} \quad 3.56 \text{ ms}^{-2}$$

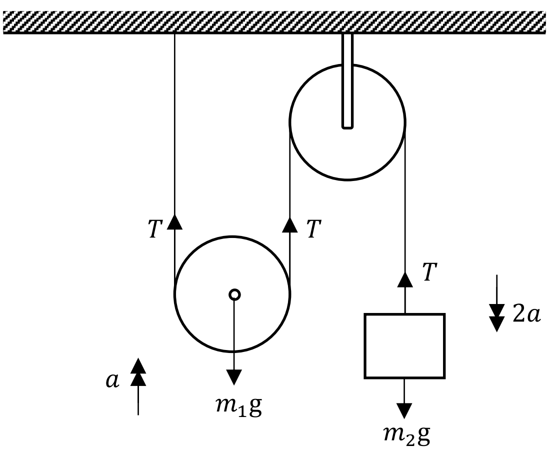
From (ii);

$$T = g + a = g + \frac{4}{11}g = \frac{92}{11}g\,N \quad \text{or} \quad 81.96\,N$$

Example 2

A string, with one end fixed, passes under a moveable pulley of mass m_1, over a fixed pulley, and carries a mass m_2 at its other end. With the system released from rest, show that the tension in the string is $\frac{3m_1m_2g}{4m_2+m_1}$ and that, after time t, the movable pulley has moved a distance $\frac{gt^2(2m_2-m_1)}{2(4m_2+m_1)}$

Solution



For moveable pulley mass m_1;

$$2T - m_1g = m_1a \quad \ldots.. (i)$$

For mass m_2;

$$m_2g - T = m_2(2a) \quad \ldots. (ii)$$

Multiplying (ii) by 2 to eliminate T

$$2m_2g - 2T = 4m_2a$$

Adding;

$$2m_2g - m_1g = 4m_2a + m_1a$$
$$a = \frac{(2m_2 - m_1)}{4m_2 + m_1}g$$

From (i);

$$2T = m_1(a + g)$$
$$2T = m_1\left(\frac{(2m_2 - m_1)}{4m_2 + m_1}g + g\right)$$
$$2T = m_1g\left(\frac{2m_2 - m_1 + 4m_2 + m_1}{4m_2 + m_1}\right)$$
$$2T = \frac{m_1g(6m_2)}{4m_2 + m_1}$$
$$T = \frac{3m_1m_2g}{4m_2 + m_1}$$

Now using kinematics;

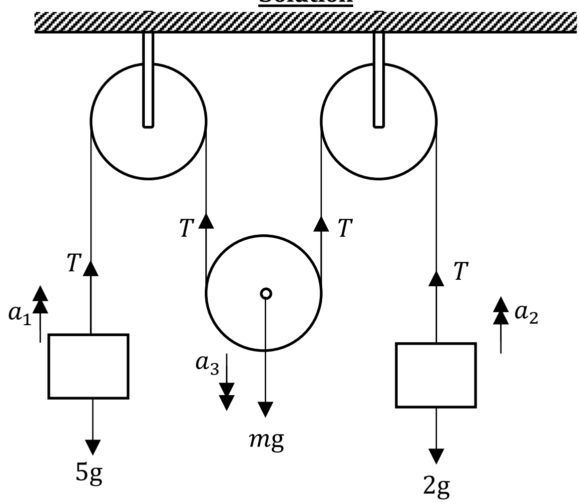
$u = 0,\ t = t,\ s = ?,\ a = \frac{(2m_2-m_1)}{4m_2+m_1}g$

$$s = ut + \frac{1}{2}at^2$$
$$s = 0 + \frac{1}{2} \times \left(\frac{2m_2 - m_1}{4m_2 + m_1}\right)gt^2$$
$$s = \frac{gt^2(2m_2 - m_1)}{2(4m_2 + m_1)}$$

Example 3

Masses of 5 kg and 2 kg are suspended from the ends of a string which passes over two fixed pulleys and under a movable pulley whose mass is m kg, the portions of the string not in contact with the movable pulley being vertical. Find the value of m in order that when the system is released, the movable pulley may remain at rest, and in this case the accelerations of the other masses and the tension of the string.

Solution



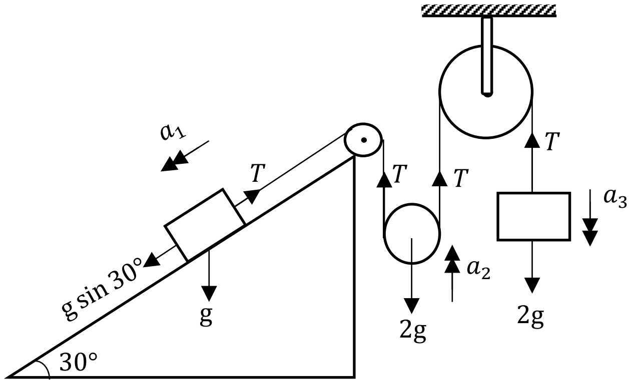
For the 5 kg mass: $\quad T - 5g = 5a_1 \ldots..(i)$

The string passes over a smooth pulley at the edge of the plane, under a second moveable pulley of mass 2 kg and over a third fixed pulley, and has a mass of 2 kg attached to the other end. Find the

(a) acceleration of the masses and the moveable pulley
(b) tension in the string

Solution

(a) Let the tension in the string be T and the accelerations be a_1, a_2 and a_3 in the respective directions as shown in the diagram.



Relationship between the accelerations

$$a_2 = \frac{1}{2}(a_1 + a_3)$$

For the 1 kg mass:

$$\text{g}\sin 30° - T = a_1$$
$$\frac{\text{g}}{2} - T = a_1 \dots (i)$$

For the moveable pulley:

$$2T - 2\text{g} = 2a_2$$
$$2T - 2\text{g} = a_1 + a_3 \dots (ii)$$

For the 2 kg mass:

$$2\text{g} - T = 2a_3 \;\dots .(iii)$$

$2(i) + (ii)$;

$$\text{g} - 2\text{g} = 2a_1 + (a_1 + a_3)$$
$$-\text{g} = 3a_1 + a_3 \dots (iv)$$

$(ii) + 2(iii)$;

$$4\text{g} - 2\text{g} = (a_1 + a_3) + 4a_3$$
$$2\text{g} = a_1 + 5a_3 \dots .(v)$$

$(v) - 5(iv)$;

$$7\text{g} = -14a_1$$
$$a_1 = -\frac{\text{g}}{2} = -4.9 \text{ ms}^{-2}$$

The acceleration of the 1 kg mass is 4.9 ms^{-2} up the slope.

From (iv);

$$a_3 = -\text{g} - 3a_1$$
$$a_3 = -\text{g} - 3 \times -\frac{\text{g}}{2}$$
$$a_3 = \frac{\text{g}}{2} = 4.9 \text{ ms}^{-2}$$

The acceleration of the 2 kg mass is 4.9 ms^{-2} downwards.

Now

$$a_2 = \frac{1}{2}(a_1 + a_3) = \frac{1}{2}\left(-\frac{\text{g}}{2} + \frac{\text{g}}{2}\right) = 0$$

The acceleration of the moveable pulley is 0 (stationary)

(b) From (i)

$$T = \frac{\text{g}}{2} - a_1 = \frac{\text{g}}{2} + \frac{\text{g}}{2} = \text{g } N$$

The tension in the string is 9.8 N

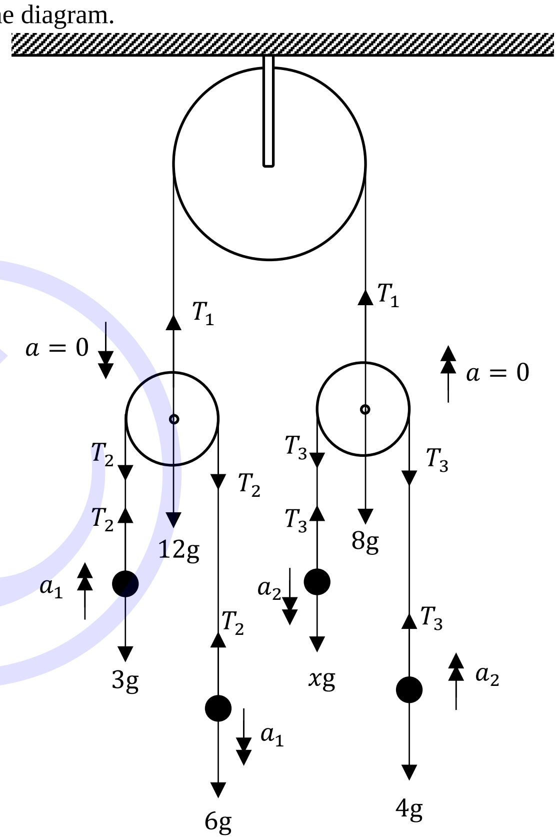
Example 6

Two pulleys, of masses 12 kg and 8 kg, are connected by a fine string hanging over a smooth fixed pulley. Over the former is hung a fine string with masses 3 kg and 6 kg at its ends, and over the latter a fine string with masses 4 kg and x kg. Determine x so that the string over the fixed pulley remains stationary, and find the tension in it.

Solution

Let the acceleration of masses on the 12 kg pulley be a_1 and that of the masses on the 8 kg pulley be a_2.

Let the tensions in the strings be T_1, T_2 and T_3 as shown in the diagram.



For the 12 kg pulley:

$$T_1 = 2T_2 + 12\text{g} \quad \dots \text{(i)}$$
$$T_2 - 3\text{g} = 3a_1 \;\dots\dots \text{(ii)}$$
$$6\text{g} - T_2 = 6a_1 \dots\dots \text{(iii)}$$

For the 8 kg pulley:

$$T_1 = 2T_3 + 8\text{g} \;\dots .\text{(iv)}$$
$$T_3 - 4\text{g} = 4a_2 \;\dots .\text{(v)}$$
$$x\text{g} - T_3 = xa_2 \;\dots .\text{(vi)}$$

Adding (i) and (ii);

$$3\text{g} = 9a_1$$
$$a_1 = \frac{\text{g}}{3}$$

From (ii);

$$T_2 = 3a_1 + 3\text{g} = 3\left(\frac{\text{g}}{3}\right) + 3\text{g} = 4\text{g}$$

From (i);

$$T_1 = 2T_2 + 12\text{g} = 2(4\text{g}) + 12\text{g} = 20\text{g}$$

pulley of mass 6 kg, over another fixed pulley and has a load of mass 3 kg attached to its other end. Find the acceleration of the moveable pulley and the tension in the string.

[Ans: $\frac{1}{9}\text{g ms}^{-2} \downarrow, \frac{8}{3}\text{g } N$]

15. A fixed pulley carries a string which has a mass of 4 kg attached at one end and light pulley A attached at the other. Another string passes over pulley A and carries a mass of 3 kg at one end and a mass of 1 kg at the other end. Find:
    (a) the acceleration of pulley A
    (b) the acceleration of 1 kg, 3 kg and 4 kg masses
    (c) the tensions in the strings

    [Ans: (a) $1.4 \text{ ms}^{-2} \uparrow$ (b) $7 \text{ ms}^{-2} \uparrow$, $4.2 \text{ ms}^{-2} \downarrow$, $1.4 \text{ ms}^{-2} \downarrow$ (c) $33.6\ N$, $16.8\ N$]

16. A fixed pulley carries a string which has a load of mass 7 kg attached to one end and light pulley attached to the other end. This light pulley carries another string which has a load of mass 4 kg at one end, and another load of mass 2 kg at the other end. Find the acceleration of the 4 kg mass and the tensions in the strings.

    [Ans: $\frac{9}{37}\text{g ms}^{-1} \downarrow, \frac{224}{37}\text{g } N, \ \frac{112}{37}\text{g } N$]

17. A string, with a particle A attached to one end passes over a fixed pulley, under a moveable pulley B, over another pulley, and has a particle C attached to its other end. The masses of A, B and C are $3m$, $4m$ and $4m$ respectively. Find the acceleration of A and the tension in the string.

    [Ans: $\frac{3}{19}\text{g ms}^{-2} \downarrow, \frac{48}{19}mg$]

18. To one end of a light string passing over a fixed pulley is attached to a particle of mass 8 kg and to the other end a light pulley. Over this pulley passes a light string to the ends of which are attached particles of masses 5 kg and 3 kg respectively. Find the acceleration of the 8 kg mass and the tension in the string attached to it.

    [Ans: $\frac{g}{31}$ ms^{-2}; $7\frac{23}{31}\text{g } N$]

19. One end of a light inextensible flexible string is attached to a mass of 9 kg, which is at rest on a smooth horizontal table. The string passes over the edge of the table and to its other end is attached a smooth light pulley. Over this pulley passes another similar string, to the ends of which are attached masses of 5 kg and 2 kg. If the system is released from rest with the hanging portions of the strings taut and vertical, show that the 9 kg mass moves along the table with acceleration $40\text{g}/103$.

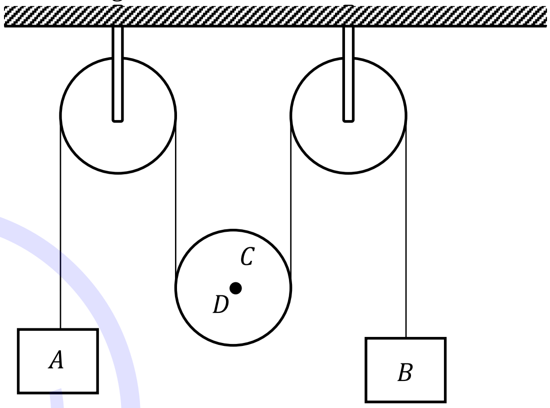
20. A particle of mass 2 kg rests on the surface of a rough plane which is inclined at 30° to the horizontal. It is connected by a light inelastic string passing over a light smooth pulley at the top of the plane, to a particle of mass 3 kg which is hanging freely. If the coefficient of friction between the 2 kg mass and the plane is $\frac{1}{3}$, find the acceleration of the system when it is released from rest and find the tension in the string. Find also the force exerted by the string on the pulley.

    [Ans: 2.8 ms^{-2}; $21\ N$; $36.4\ N$ at 30° to the vertical]

21. A particle A of mass 6 kg is connected by a light inextensible string passing over a fixed smooth pulley to a light smooth moveable pulley B. Two particles C and D of masses 2 kg and 1 kg are connected by a light inextensible string passing over the pulley B. When the system is moving freely, find the acceleration of the 1 kg mass and the tensions in the strings.

    [Ans: $\frac{11}{13}\text{g ms}^{-2}$; $\frac{24}{13}\text{g } N$; $\frac{48}{13}\text{g } N$]

22. The diagram below shows two particles A and B, of masses 3 kg and 5 kg, connected by a light inextensible string passing over two fixed smooth pulleys and a light smooth moveable pulley C, which carries a particle D of mass 6 kg.



The system is released from rest. Find the
    (a) acceleration of the particle A,
    (b) acceleration of pulley C
    (c) tension in the string

    [Ans: (a) $\frac{1}{9}\text{g ms}^{-2} \uparrow$ (b) $\frac{1}{9}\text{g ms}^{-2} \uparrow$ (c) $\frac{10}{3}\text{g } N$]

MOMENTUM

The momentum of a moving body is the product of its mass m and velocity v

$$\textbf{momentum} = \boldsymbol{mv}$$

It is a vector whose direction is that of the velocity.
SI unit of momentum is the newton second (Ns)

Impulse is the time effect of a force. It is a vector.
For a constant force F, acting for time t,

$$\text{impulse} = Ft$$

For a variable force F, acting for time T,

$$\text{impulse} = \int_0^T F\,dt$$

SI unit of impulse is the newton second (Ns)

Example 1

Find the momentum of a particle of mass 1.5 kg moving in a straight line at 5 ms^{-1}

Solution

$$\text{Momentum} = 1.5 \times 5 = 7.5\ Ns$$

Example 2

A constant force acts on a particle of mass 0.5 kg changing its speed from 3 ms^{-1} to 7 ms^{-1}, the force acting in the direction of motion. What is its impulse?

Solution

$$\text{Impulse, } I = Ft$$
$$\text{But } F = ma$$
$$I = mat$$
$$v = u + at \Rightarrow at = v - u$$
$$at = 7 - 3 = 4$$
$$\text{So, impulse} = 0.5 \times 4 = 2\ Ns$$

Relation between impulse and momentum

Consider a constant force F which acts for a time t on a body of mass m, thus changing its velocity from u to v. Because the force is constant, the body will travel with constant acceleration a where

$$F = ma$$
$$F = m\frac{v-u}{t}$$
$$Ft = mv - mu$$
$$\text{impulse} = mv - mu$$

The impulse of a force is equal to the change of momentum it produces.

Example 3

A golf ball of mass 0.06 kg resting on a tree is given a horizontal impulse of 1.8 Ns. Calculate the velocity v with which it moves off.

Solution

$$\text{Impulse} = \text{change in momentum}$$
$$1.8 = 0.06v - 0.06 \times 0$$
$$v = 30\ \text{ms}^{-1}$$

Example 4

A body moving with initial velocity $(2\text{i} - 7\text{j})\ \text{ms}^{-1}$ is given an impulse causing its velocity to change to $(-8\text{i} + \text{j})\ \text{ms}^{-1}$. The mass of the body is 0.5 kg. If the duration of the impulse is 0.2 seconds, calculate the magnitude of the constant force involved.

Solution

$$\text{Impulse} = mv - mu$$
$$= 0.5(-8\text{i} + \text{j}) - 0.5(2\text{i} - 7\text{j})$$
$$= -4\text{i} + 0.5\text{j} - \text{i} + 3.5\text{j}$$
$$= -5\text{i} + 4\text{j}$$
$$\text{Impulse} = Ft$$
$$-5\text{i} + 4\text{j} = F \times 0.2$$
$$F = (-25\text{i} + 20\text{j})\ N$$
$$|F| = \sqrt{(-25)^2 + 20^2} = \sqrt{1025} = 32$$

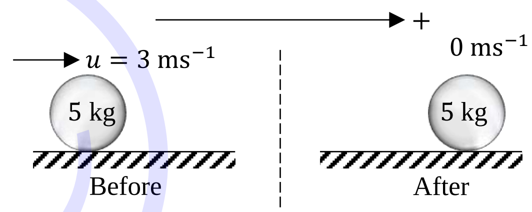
The magnitude of the constant force is 32 N

Example 5

A body of mass 5 kg is travelling at 3 ms^{-1}. It is brought to rest by an impulse. Find the magnitude of this impulse.

Solution

When no direction is specified, we can choose the positive direction to be shown to the right.



$$\text{Impulse} = mv - mu$$
$$= 5 \times 0 - 5 \times 3$$
$$= 0 - 15$$
$$\text{Impulse} = -15\ Ns$$

The magnitude of the impulse is 15 Ns. The negative sign means that the impulse acts in the opposite direction to the original motion.

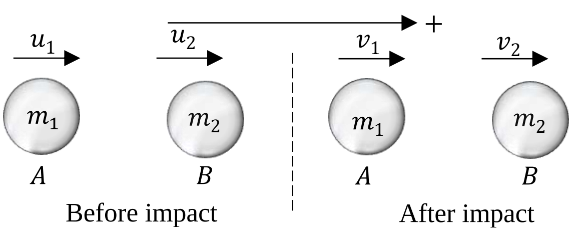
Conservation of momentum

The principle of conservation of momentum states that the total momentum of a system is constant in any direction provided no external force acts in that direction i.e.

initial momentum = final momentum

Collisions of two bodies

Consider two bodies A and B of masses m_1 and m_2 moving with velocities u_1 and u_2 respectively. When they collide head on, their respective velocities are now v_1 and v_2.



Example 7

A gun of mass 1000 kg can launch a shell of mass 1 kg with a horizontal velocity of 1200 ms^{-1}. What is the horizontal velocity of recoil of the gun.

<u>Solution</u>

$m_g = 1000$, $m_s = 1$, $v_s = 1200$, $v_g = ?$

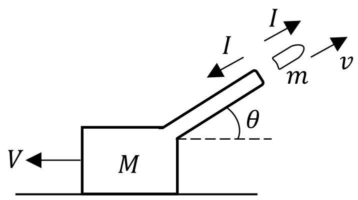
Using conservation of momentum;

$$m_s v_s - m_g v_g = 0$$
$$1 \times 1200 - 1000 v_g = 0$$
$$v_g = 1.2$$

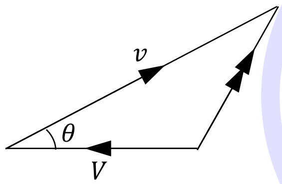
The gun recoils with a velocity 1.2 ms^{-1}

Example 8

A gun of mass M, whose barrel is at an angle of elevation θ, fires a shell of mass m and recoils horizontally with velocity V. The shell travels at speed v relative to the barrel of the gun.



Velocity triangle for the shell



Velocity of shell, relative to the ground, as it leaves the barrel is the resultant of v and V

For the gun ($\rightarrow$): $-I\cos\theta = -MV$

For the shell ($\rightarrow$): $I\cos\theta = m(v\cos\theta - V)$

For the shell ($\uparrow$): $I\sin\theta = mv\sin\theta$

By conservation of momentum ($\rightarrow$);

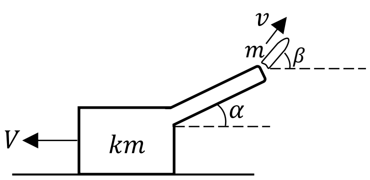
$$0 = m(v\cos\theta - V) - MV$$

Example 9

A gun of mass km fires a shell of mass m. The barrel of the gun is elevated at an angle α and the gun recoils horizontally. Show that the shell leaves the barrel at an angle β to the horizontal where

$$\tan\beta = \frac{k+1}{k}\tan\alpha$$

<u>Solution</u>

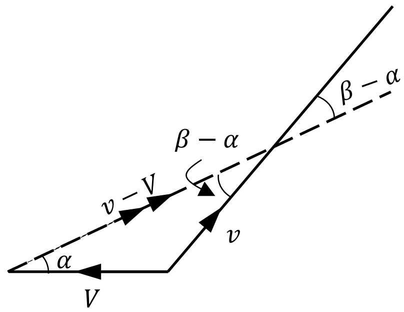


By conservation of momentum;

$$0 = mv\cos\beta - kmV$$

$$V = \frac{v\cos\beta}{k}$$

Using a velocity triangle;



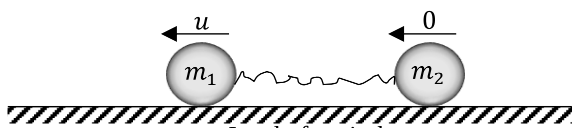
By the sine rule;

$$\frac{v}{\sin\alpha} = \frac{V}{\sin(\beta-\alpha)}$$
$$\frac{v}{\sin\alpha} = \frac{v\cos\beta}{k\sin(\beta-\alpha)}$$
$$k[\sin\beta\cos\alpha - \sin\alpha\cos\beta] = \sin\alpha\cos\beta$$
$$k\left[\frac{\sin\beta\cos\alpha}{\sin\alpha\cos\beta} - \frac{\sin\alpha\cos\beta}{\sin\alpha\cos\beta}\right] = \frac{\sin\alpha\cos\beta}{\sin\alpha\cos\beta}$$
$$k[\tan\beta\cot\alpha - 1] = 1$$
$$k\left[\frac{\tan\beta}{\tan\alpha} - 1\right] = 1$$
$$k\left[\frac{\tan\beta - \tan\alpha}{\tan\alpha}\right] = 1$$
$$k\tan\beta - k\tan\alpha = \tan\alpha$$
$$k\tan\beta = k\tan\alpha + \tan\alpha$$
$$\tan\beta = \frac{k+1}{k}\tan\alpha$$

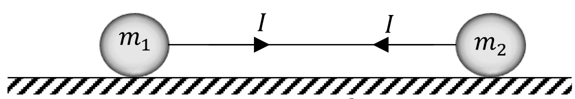
Impulses in strings

When a string jerks taut, impulses, which are equal in magnitude but opposite in direction, act at the two ends. If two particles are attached by a string which jerks taut, then the two particles will experience **equal and opposite impulses**.

Consider this system involving two masses m_1 and m_2



Just before jerk



At jerk

Just after jerk

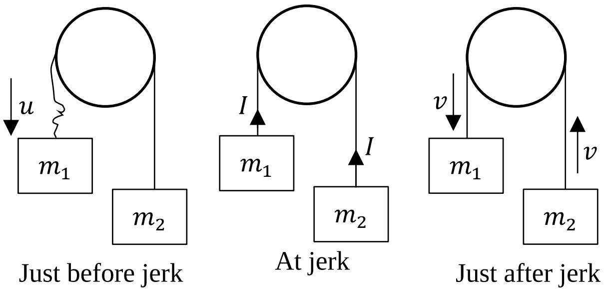
For mass m_1: $-I = m_1 v - m_1 u$

For mass m_2: $I = m_2 v - m_2 \times 0$

By conservation of momentum:

$$m_1 u = m_1 v + m_2 v$$

Impulse problems for other connected particles may be solved in the same way if the string is considered to be straight and the particles move in a straight line.



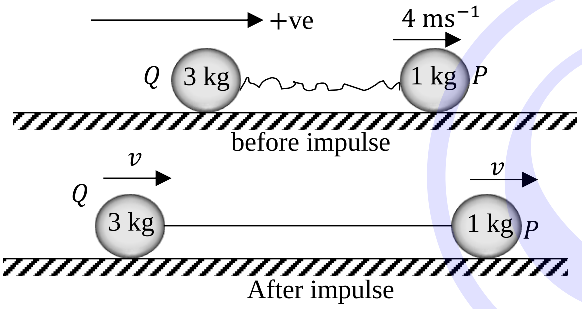
Example 10

Two bodies, P and Q, are lying on a smooth horizontal table. Their masses are 1 kg and 3 kg respectively. The two bodies are connected by a light inextensible string which is initially slack. Body P is projected horizontally with velocity 4 ms^{-1} away from Q. Find

(a) the speed of P and Q after the string becomes taut.

(b) The magnitude of the impulse due to the tension in the string

<u>Solution</u>



(a) By conservation of momentum;

Momentum before = momentum after

$$1 \times 4 + 0 = 3v + v$$

$$v = 1\ \mathrm{ms}^{-1}$$

So after impulse both bodies travel with speed 1 ms^{-1}

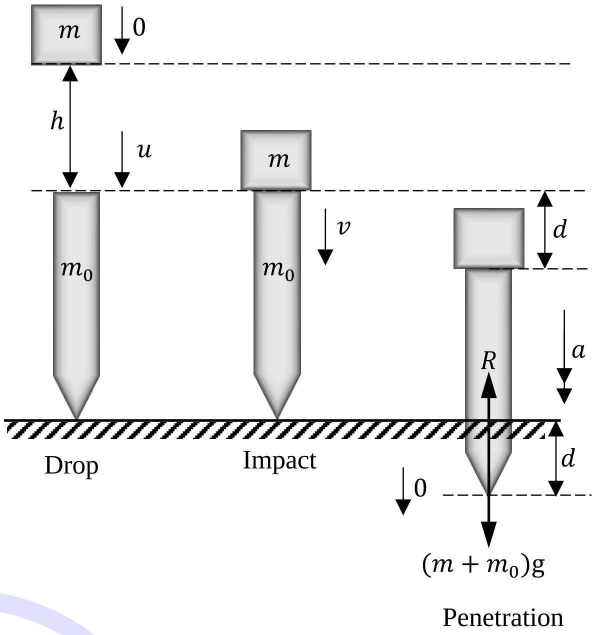
(b) Impulse = change in momentum of Q

$$= 3 \times v - 0$$

$$= 3 \times 1 = 3\ Ns$$

The pile driver

A pile driver consists of a hammer of mass m dropped from a height h onto a pile of mass m_0 whose point is just resting on the surface of the ground. Assume that the ground resists motion of the pile into the ground with a constant force R.



Phase 1: The drop

The hammer drops from rest a distance h and hits the pile with speed u. By conservation of energy, the gain in kinetic energy = loss of potential energy

$$\frac{1}{2}mu^2 = mgh$$

$$u = \sqrt{2gh}$$

Alternatively;

initial vel. = 0, final vel. = u, distance, $s = h$, $a =$ g

$$v^2 = u^2 + 2as$$

$$u^2 = 0^2 + 2gh$$

$$u = \sqrt{2gh}$$

Phase 2: The impact

Let v be the speed of the combined system just after the strike (but before resistance R starts). The momentum of the hammer just before it strikes is the same as the combined system (instantaneously) just after the strike.

$$m\sqrt{2gh} = (m + m_0)v$$

$$v = \frac{m}{m + m_0}\sqrt{2gh}$$

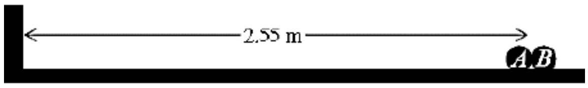
Phase 3: The penetration

The vertical momentum is not conserved as the resistance force acts to slow the hammer & pile. However, we can use the concept of energy to assert that the kinetic energy of the hammer & pile just after the strike is dissipated as the work done against the resistance and the loss of gravitational potential energy.

If we set the zero of the potential energy to be the point where the pile driver comes to rest, then

(a) Given that the speed of A immediately after the explosion is $12\ \text{ms}^{-1}$, determine the speed of B.

In the subsequent motion, A experiences no resistance or ground friction but B experiences constant ground friction.



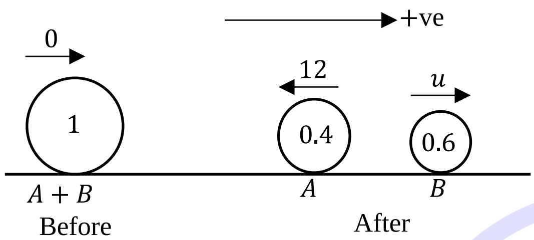
The explosion takes place 2.55 m away from a smooth vertical wall which is perpendicular to the direction of motion of A. A has a perfectly elastic collision with the wall, it rebounds and collides directly with B, 0.75 s after the explosion. All collisions are instantaneous.

(b) Show that the speed of B just before the two particles collide is $2.4\ \text{ms}^{-1}$.

(c) Calculate the coefficient of friction between the ground and B.

<u>Solution</u>

(a)



By conservation of momentum;

$$(1\times 0) = 0.4\times -12 + 0.6\times u$$
$$0 = -4.8 + 0.6u$$
$$0.6u = 4.8$$
$$u = 8\ \text{ms}^{-1}$$

(b) Particle A moves with constant speed of $12\ \text{ms}^{-1}$ for 0.75 s

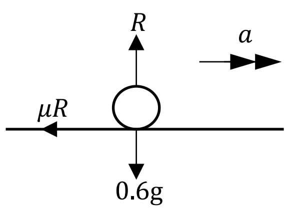
$$d_A = 12\times 0.75 = 9\ \text{m}$$
$$9 - 2\times 2.55 = 3.9\ \text{m (to the right of the explosion)}$$

Now kinematics for B; As friction is constant, deceleration must also be constant.

$$u = 8,\ a = ?,\ s = 3.9,\ t = 0.75,\ v = ?$$
$$s = \frac{u+v}{2}\times t$$
$$3.9 = \frac{8+v}{2}\times 0.25$$
$$v + 8 = 10.4$$
$$v = 2.4\ \text{ms}^{-1}$$

(c) Let us first obtain deceleration for B

$$v = u + at$$
$$2.4 = 8 + 0.75a$$
$$0.74a = -5.6$$
$$a = -\frac{112}{15}$$



$$F = ma$$
$$-\mu R = 0.6\times\left(-\frac{112}{15}\right)$$
$$\mu(0.6\text{g}) = 0.6\times\frac{112}{15}$$
$$\mu = \frac{16}{21} = 0.762$$

Example 25

A gun of mass 600 kg is free to move along a horizontal track and is connected by a light inelastic rope to an open truck containing sand whose total mass is 1490 kg. The truck is free to move along the same track as the gun. A shell of mass 10 kg is fired from the gun towards the truck and when it leaves the barrel has a horizontal velocity of $915\ \text{ms}^{-1}$ relative to the gun and parallel to the track. The shell lodges in the sand where it comes to relative rest before the rope tightens. Find

(a) the speeds of the gun and shell just after the shell leaves the barrel;

(b) the speed of the truck before the rope tightens when the shell is at relative rest inside the truck;

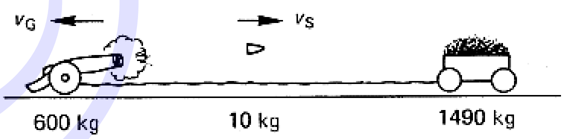
(c) the speed of the gun and truck just after the rope tightens;

(d) the loss in kinetic energy due to the rope tightening;

(e) the magnitude of the impulsive tension in the rope.

<u>Solution</u>

(a) After the shell leaves the gun let the speeds be v_S and v_G (in opposite directions).



By conservation of linear momentum;

$$100v_S = 600v_G$$
$$v_S = 60v_G\ \ldots\text{(i)}$$

Velocity of shell relative to the gun,

$${}_Sv_G = v_S - (-v_G) = v_S + v_G$$
$$v_S + v_G = 915\quad ..\text{(ii)}$$

Solving (i) and (ii);

$$60v_G + v_G = 915$$
$$v_G = \frac{915}{61} = 15\ \text{ms}^{-1}$$
$$v_S = 915 - v_G = 915 - 15 = 900\ \text{ms}^{-1}$$

The speeds of the gun and shell are $15\ \text{ms}^{-1}$ and $900\ \text{ms}^{-1}$ respectively.

(b) When the shell is at rest relative to the truck, let their common speed be v_T

By conservation of momentum:

$$10(900) = 1500v_T$$
$$v_T = 6\ \text{ms}^{-1}$$

Speed of the truck before the rope tightens is $6\ \text{ms}^{-1}$

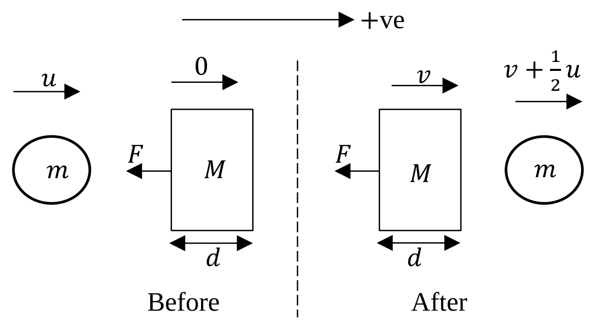
(c) Before rope tightens, total momentum in the direction of motion of the gun $= 600(15) - 1500(6) = 0$

Therefore total momentum after rope tightens is 0

Gun and truck are at rest after rope has tightened.

$$I = m\left(\frac{1}{2}u\right) - mu$$
$$-FT = -\frac{1}{2}mu$$
$$F\left(\frac{4d}{2u}\right) = \frac{1}{2}mu$$
$$F = \frac{3mu^2}{8d}$$

Second case: The block can move and the bullet now leaves with speed $\frac{1}{4}u$, relative to the block.



Let T' be the time to traverse through the moving block

$$s = \frac{u+v}{2} \times t$$
$$d = \frac{u + \frac{1}{4}u}{2} \times T'$$
$$d = \frac{5uT'}{8}$$
$$T' = \frac{8d}{5u}$$

Impulse on the bullet;

$$I = -FT' = -\left(\frac{3mu^2}{8d}\right)\left(\frac{8d}{5u}\right) = -\frac{3}{5}mu$$

Impulse = change in momentum

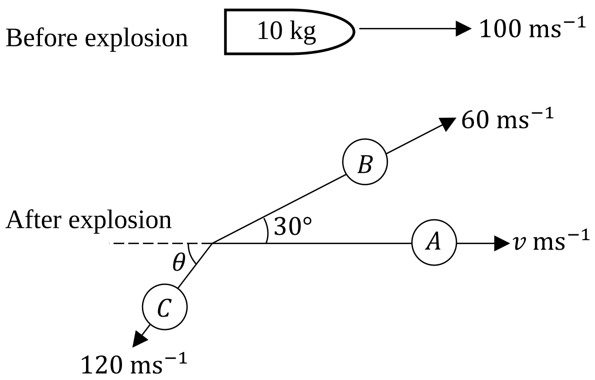
$$-\frac{3}{5}mu = m\left(v + \frac{1}{4}u\right) - mu$$
$$-\frac{3}{5}mu = mv + \frac{1}{4}mu - mu$$
$$-\frac{3}{5}mu = mv - \frac{3}{4}mu$$
$$v = \frac{3}{20}u$$

Finally, the impulse on the block is the negative of that on the bullet.

$$Mv = -\left(-\frac{3}{5}mu\right)$$
$$\frac{3}{20}Mu = \frac{3}{5}mu$$
$$M = 4m$$

Example 29

A shell of mass 20 kg is travelling horizontally at 100 ms^{-1} when it suddenly explodes into three pieces A, B, C of masses 12 kg, 6 kg and 2 kg respectively. The diagram shows the direction of travel of shell before the explosion and the directions of the three pieces A, B and C after the explosion.



Calculate

(a) the angle θ made by the direction of C with the backward horizontal direction

(b) the speed of A

<u>Solution</u>

Since no external impulse acts on the system, momentum is conserved in any chosen direction.

Using conservation of momentum at right angles to original direction of travel, we have;

$$6 \times 60\cos 60° - 2 \times 120 \sin\theta = 0$$
$$\sin\theta = \frac{3}{4}$$
$$\theta = \sin^{-1}\frac{3}{4} = 48.6°$$

Using conservation of momentum parallel to the original direction of travel, we have

$$6 \times 60\cos 30° + 12v - 2 \times 120\cos\theta = 20 \times 100$$
$$v = \frac{500}{3} + 20\cos\theta - 15\sqrt{3}$$

Since $\sin\theta = \frac{3}{4}$, $\cos\theta = \frac{\sqrt{7}}{4}$

$$v = \frac{500}{3} + 5\sqrt{7} - 15\sqrt{3}$$
$$v = 154 \text{ ms}^{-1}$$

Self-Evaluation exercise

1. A bullet of mass 0.04 kg travelling horizontally at 100 ms^{-1} hits a stationary block of wood of mass 8 kg, passes through it and emerges horizontally with a speed of 40 ms^{-1}. If the block is free to move on a smooth horizontal plane, find the speed with which it is moving after the bullet has passed through it.

   [Ans: 0.3 ms^{-1}]

2. A gun of mass 2000 kg fires horizontally a shell of mass 25 kg. The gun's horizontal recoil is controlled by a constant force of 8000 N which brings the gun to rest in 1.5 seconds. Find the initial velocity of the shell
   (a) relative to the gun
   (b) in the air

   [Ans: (a) 486 ms^{-1} (b) 480 ms^{-1}]

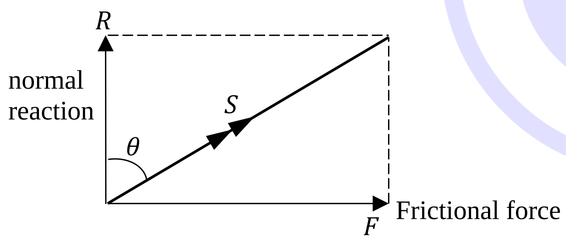
3. A particle of mass m, initially at rest, is subjected to an impulse I. In the ensuing motion the only force on the particle is a force directly opposing the motion of the particle and of magnitude k times the square of the

FRICTION

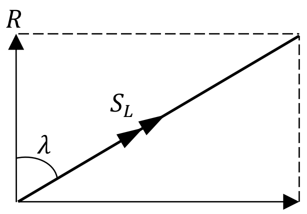
When one body slides or attempts to slide over another, **forces of friction** usually exist between the two surfaces in contact.
Forces of friction act between rough surfaces in contact. Smooth surfaces in contact are frictionless.
The following **experimental laws** describe the behaviour of frictional forces.

1. A frictional force only exists when one body **slides** or **tries to slide** over another
2. A frictional force always **opposes the tendency** of one body to slide over another
3. The magnitude of a frictional force may vary, always being just sufficient to prevent motion, until it reaches the **maximum value** called the **limiting value**.
4. The **limiting value** of the frictional force is μR, where μ is called the **coefficient of friction** and R is the normal reaction for surfaces in contact. μ is a measure of the degree of roughness of the two surfaces in contact and is different for different pairs of surfaces
5. When one body slides over another, the frictional force between them equals the limiting value μR. A consequence of laws (3) and (4) is that the frictional force F obeys the relation $F \le \mu R$

Angle of friction
The resultant S of the frictional force F and the normal reaction R is called the **total reaction**.



It makes an angle θ with the normal, where $\tan\theta = \frac{F}{R}$
The normal reaction R is constant, but frictional force F may vary.
As the frictional force F increases from zero to its maximum value F_L, the limiting value μR, the angle θ increases from zero to a maximum value λ, called the **angle of friction**.



Limiting friction, $F_L = \mu R$

$$\tan\lambda = \frac{\mu R}{R} = \mu$$
$$\lambda = \tan^{-1}\mu$$

When the frictional force has reached its limiting value; the direction of the total reaction S_L is at an angle λ to the normal reaction.
the magnitude of the total reaction is given by
$$\sqrt{R^2 + \mu^2 R^2} = R\sqrt{1+\mu^2} = R\sec\lambda$$

Problem solving:
The following points are important when solving problems involving a frictional force F.

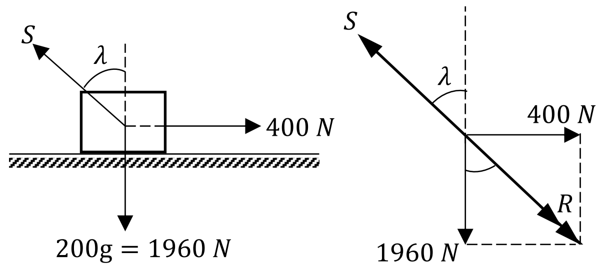
1. Draw a clear force diagram. Show the frictional force as F, don not use μR. Remember F tends to oppose motion.
2. In general $F \le \mu R$. If F has reached its limiting value, then $F = \mu R$ may be used in the solution.
3. Limiting equilibrium indicates that the body is at rest but on the point of moving and that $F = \mu R$
4. If λ is given and not μ, then it is often easier to solve the problem by considering the total reaction, rather than F and R separately. This is often the case in three force problems.

Example 1
Find the angle of friction if the coefficient of friction is $\frac{3}{7}$

<u>**Solution**</u>
$$\mu = \tan\lambda$$
$$\lambda = \tan^{-1}\mu = \tan^{-1}\frac{3}{7} = 23.2°$$

Example 2
A sledge of mass 200 kg is pulled along a horizontal snow field. The sledge travels at constant speed and the applied force is 400 N. Find the angle of friction and the resultant reaction.

<u>**Solution**</u>

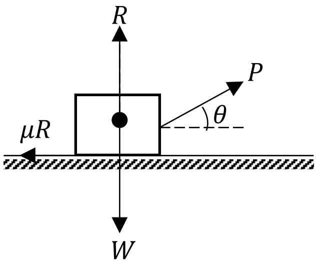


Using Pythagoras' theorem;
$$R^2 = 400^2 + 1960^2$$
$$R = 2000.4\ N$$
$$\text{So } S = 2000\ N$$
$$\tan\lambda = \frac{400}{1960}$$
$$\lambda = \tan^{-1}0.2041 = 11.53\,°$$

Example 3
A particle of weight W rests on a rough horizontal plane with which the angle of friction is λ. A force P inclined at an angle θ to the plane is applied to the particle until it is on

the point of moving. Find the value of θ for which the value of P will be least.

Solution



When the particle is in limiting equilibrium, $F = \mu R$

Resolving →; $P\cos\theta = \mu R$ … (i)

Resolving ↑; $R + P\sin\theta = W$…. (ii)

From (ii); $R = W - P\sin\theta$

Substituting in (i);

$$P\cos\theta = \mu(W - P\sin\theta)$$

But $\mu = \tan\lambda = \frac{\sin\lambda}{\cos\lambda}$

$$P\cos\theta = \frac{\sin\lambda}{\cos\lambda}(W - P\sin\theta)$$

$$P\cos\theta\cos\lambda = W\sin\lambda - P\sin\theta\sin\lambda$$

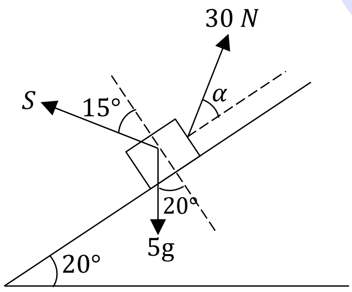
$$P\cos\theta\cos\lambda + P\sin\theta\sin\lambda = W\sin\lambda$$

$$P\cos(\theta - \lambda) = W\sin\lambda$$

$$P = \frac{W\sin\lambda}{\cos(\theta - \lambda)}$$

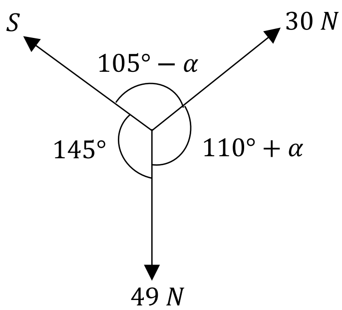
P will be least when $\cos(\theta - \lambda)$ is greatest, since W and λ are constant, i.e. when $\cos(\theta - \lambda) = 1$ and $\theta - \lambda = 0$. Therefore P is least when $\theta = \lambda$ and its value then is $W\sin\lambda$

Example 4



A mass of 5 kg is pulled at constant speed up a slope inclined at $20°$ to the horizontal. The angle of friction is $15°$ and the force pulling the mass up the slope is $30\ N$. Find the angle between the $30\ N$ force and the slope and the resultant reaction.

Solution



Applying Lami's theorem to find α;

$$\frac{\sin(105° - \alpha)}{49} = \frac{\sin 145°}{30}$$

$$\sin(105° - \alpha) = \frac{49\sin 145°}{30}$$

$$\sin(105° - \alpha) = 0.937$$

$$105° - \alpha = 69.6°$$

$$\alpha = 35.4°$$

Applying Lami's theorem to find S

$$\frac{S}{\sin(110° + \alpha)} = \frac{30}{\sin 145°}$$

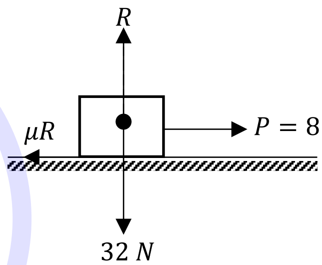
$$S = \frac{30}{\sin 145°} \times \sin 145.4°$$

$$S = 29.7\ N$$

Example 5

A small block of weight $32\ N$ is lying in rough contact on a horizontal plane. A horizontal force of P newtons is applied to the block until it is just about to move the block. If $P = 8$, find the coefficient of friction μ between the block and the plane.

Solution



Resolving →; $8 - \mu R = 0$

$$\mu R = 8$$

Resolving ↑; $R - 32 = 0$

$$R = 32$$

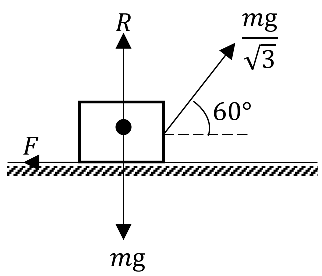
$$\mu = \frac{\mu R}{R} = \frac{8}{32} = \frac{1}{4}$$

Example 6

A particle of mass m rests on a rough horizontal plane and is pulled by a force of magnitude $\frac{mg}{\sqrt{3}}$ inclined at an angle $60°$ to the horizontal. Find the minimum value of μ, the coefficient of friction between the particle and the plane if the particle does not move.

Solution

Let the normal reaction and friction force be R and F respectively



Resolving vertically;

$$R + \frac{mg}{\sqrt{3}}\sin 60° = mg$$
$$R = mg - \frac{mg}{2} = \frac{mg}{2}$$

Resolving horizontally;

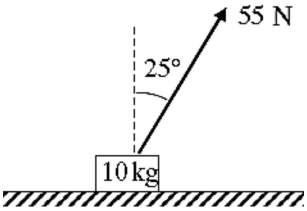
$$F = \frac{mg}{\sqrt{3}}\cos 60° = \frac{mg}{2\sqrt{3}}$$

Since the particle does not move, $F \leq \mu R$

$$\frac{mg}{2\sqrt{3}} \leq \mu\frac{mg}{2}$$
$$\frac{1}{\sqrt{3}} \leq \mu$$
$$\mu \geq \frac{1}{\sqrt{3}}$$

The minimum value of μ is $\frac{1}{\sqrt{3}}$

Example 7



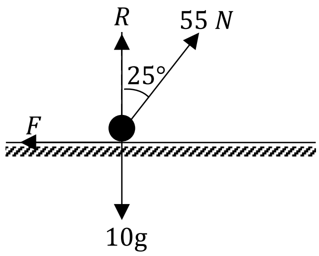
The figure above shows a small box of mass 10 kg pulled by a rope inclined at 25° to the vertical, along rough horizontal ground. When the tension in the rope is 55 N the box rests in equilibrium, on rough horizontal ground. Calculate

(a) the normal reaction between the box and the ground.

(b) the magnitude of the total force exerted by the ground on the box.

<u>**Solution**</u>

(a)



Resolving vertically; $R + 55\cos 25° = 10\text{g}$

$$R = 10\text{g} - 55\cos 25° = 48.2\ N$$

(b) Resolving horizontally; $F = 55\sin 25°$

$$F = 23.2\ N$$

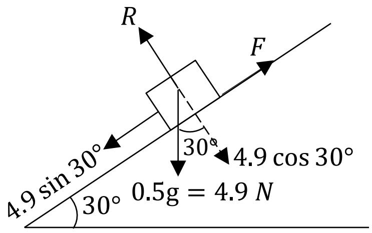
Total reaction is $\sqrt{F^2 + R^2}$

$$= \sqrt{23.2^2 + 48.2^2}$$
$$= 53.5\ N$$

Example 8

A book of mass 0.5 kg is placed on a slope inclined at 30° to the horizontal. If the book is just on the point of moving down the slope, calculate the value of the coefficient of friction.

<u>**Solution**</u>



Resolving ⊥ to the plane ↖;

$$R = 4.9\cos 30°$$
$$R = 4.24\ N$$

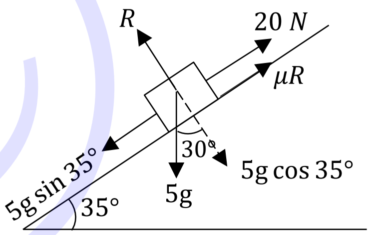
Resolving ∥ to the plane ↙;

$$F = 4.9\sin 30°$$
$$F \leq \mu R$$
$$4.24\mu = 2.45$$
$$\mu = 0.577$$

Example 9

A body of mass 5 kg lies on a rough plane which is inclined at 35° to the horizontal. When a force of 20 N is applied to the body, parallel to and up the plane, the body is found to be on the point of moving down the plane, i.e. in limiting equilibrium. Find μ, the coefficient of friction between the body and the plane.

<u>**Solution**</u>



Resolving ⊥ to the plane ↖;

$$R = 5\text{g}\cos 35°$$
$$R = 40.14\ N$$

Resolving ∥ to the plane ↙;

$$5\text{g}\sin 35° = 20 + \mu R$$
$$28.11 = 20 + 40.14\mu$$
$$40.14\mu = 8.11$$
$$\mu = 0.202$$

Example 10

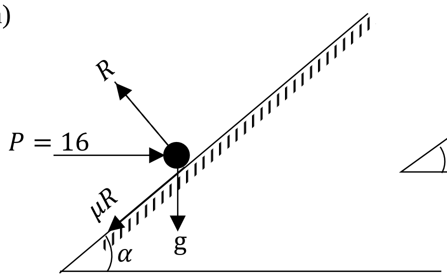
A particle of weight 8 N is resting in rough contact with a plane inclined at an angle α to the horizontal where $\tan\alpha = \frac{3}{4}$. The coefficient of friction between the particle and the plane is μ. A horizontal force P newtons is applied to the particle. When $P = 16$, the particle is on the point of slipping up the plane.

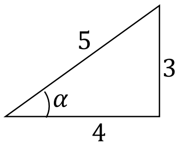
(a) Find μ

(b) Find the value of P such that the particle is just prevented from slipping down the plane

Solution

(a)





Resolving parallel and perpendicular to the plane involves μ in only one equation

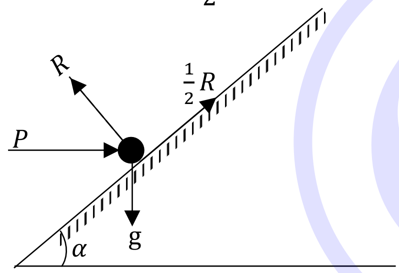
Resolving ↗;

$$16\cos\alpha = \mu R + 8\sin\alpha$$
$$16 \times \frac{4}{5} = \mu R + 8 \times \frac{3}{5}$$
$$8 = \mu R$$

Resolving ↖;

$$R = 16\sin\alpha + 8\cos\alpha$$
$$R = 16 \times \frac{3}{5} + 8 \times \frac{4}{5}$$
$$R = 16$$
$$\frac{\mu R}{R} = \frac{8}{16}$$
$$\mu = \frac{1}{2}$$

(b)



Resolving ↑; $R\cos\alpha + \frac{1}{2}R\sin\alpha = 8$

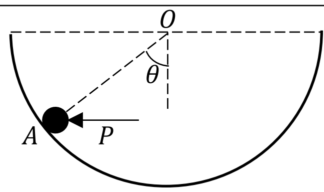
$$\frac{4R}{5} + \frac{3R}{10} = 8$$
$$\frac{11R}{10} = 8$$
$$R = \frac{80}{11}$$

Resolving →; $P + \frac{1}{2}R\cos\alpha = R\sin\alpha$

$$P + \frac{2}{5}R = \frac{3}{5}R$$
$$P = \frac{R}{5}$$
$$P = \frac{80}{11} \div 5 = \frac{16}{11} = 1.45\ N$$

Example 11

A fixed hollow hemisphere has centre O and is fixed so that the plane of the rim is horizontal. A particle A of mass m can move on the inside surface of the hemisphere. The particle is acted on by a horizontal force of magnitude P, whose line of action is in the vertical plane through O and A. The diagram shows the state when A is in equilibrium, the line OA making an acute angle θ with the vertical.



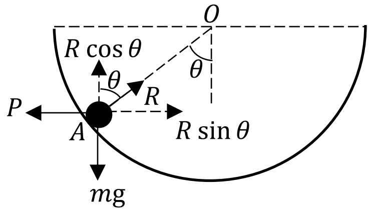
(a) Given that the inside surface of the hemisphere is smooth, find $\tan\theta$ in terms of P, m and g

(b) Given that instead the inside surface of the hemisphere is rough, with coefficient of friction μ between the surface and A, and that the particle is about to slip downwards, show that

$$\tan\theta = \frac{P + \mu m g}{mg - \mu P}$$

Solution

(a)



Resolving vertically;

$$P = R\sin\theta \quad \text{.... (i)}$$

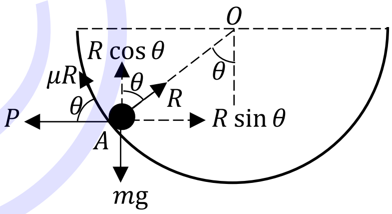
Resolving horizontally;

$$mg = R\cos\theta \quad \text{.... (ii)}$$

(i)÷ (ii);

$$\frac{P}{mg} = \tan\theta$$

(b)



Resolving horizontally, →

$$R\sin\theta = P + \mu R\cos\theta$$
$$R\sin\theta - \mu R\cos\theta = P$$
$$R(\sin\theta - \mu\cos\theta) = P \quad \text{... (i)}$$

Resolving vertically, ↑

$$\mu R\sin\theta + R\cos\theta = mg$$
$$R(\mu\sin\theta + \cos\theta) = mg \quad \text{.... (ii)}$$

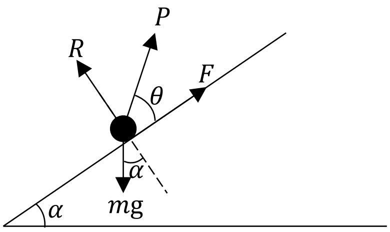
Dividing (i) by (ii);

$$\frac{(\sin\theta - \mu\cos\theta)}{(\mu\sin\theta + \cos\theta)} = \frac{P}{mg}$$

Dividing by $\cos\theta$;

$$\frac{\tan\theta - \mu}{\mu\tan\theta + 1} = \frac{P}{mg}$$
$$mg(\tan\theta - \mu) = P(\mu\tan\theta + 1)$$
$$mg\tan\theta - \mu mg = \mu P\tan\theta + P$$
$$mg\tan\theta - \mu P\tan\theta = P + \mu mg$$
$$\tan\theta\,(mg - \mu P) = P + \mu mg$$
$$\tan\theta = \frac{P + \mu mg}{mg - \mu P}$$

Example 12



Find the least force P required to just prevent the particle from sliding down the above inclined plane.

Solution

Resolving ∥ to the slope;

$$P\cos\theta + F = mg\sin\alpha$$
$$F = mg\sin\alpha - P\cos\theta$$

Resolving ⊥ to the slope;

$$R + P\sin\theta = mg\cos\alpha$$
$$R = mg\cos\alpha - P\sin\theta$$

Limiting friction, so $F = \mu R$

$$mg\sin\alpha - P\cos\theta = \mu(mg\cos\alpha - P\sin\theta)$$
$$mg\sin\alpha - P\cos\theta = \mu mg\cos\alpha - \mu P\sin\theta$$

Using $\mu = \tan\lambda = \frac{\sin\lambda}{\cos\lambda}$

$$mg\sin\alpha - P\cos\theta = mg\cos\alpha\frac{\sin\lambda}{\cos\lambda} - P\sin\theta\frac{\sin\lambda}{\cos\lambda}$$
$$mg\sin\alpha\cos\lambda - P\cos\theta\cos\lambda$$
$$= mg\cos\alpha\sin\lambda - P\sin\theta\sin\lambda$$
$$P(\cos\theta\cos\lambda - \sin\theta\sin\lambda)$$
$$= mg(\sin\alpha\cos\lambda - \cos\alpha\sin\lambda)$$
$$P\cos(\theta+\lambda) = mg\sin(\alpha-\lambda)$$
$$P = \frac{mg\sin(\alpha-\lambda)}{\cos(\theta+\lambda)}$$

P is a minimum when $\cos(\theta+\lambda)$ is a maximum i.e. 1

Hence, minimum $P = mg\sin(\alpha-\lambda)$

This occurs when $(\theta+\lambda) = 0$ i.e. $\theta = -\lambda$

Example 13

A particle P of weight W is in limiting equilibrium on a rough plane which is inclined at an angle α to the horizontal. Prove that the coefficient of friction between the particle and the plane is $\tan\alpha$

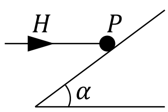


Fig (a)

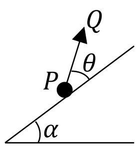


Fig (b)

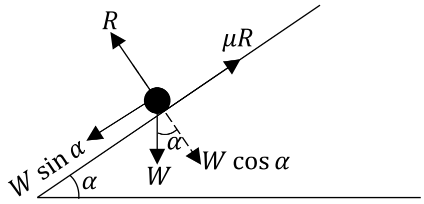
The figure (a) shows a horizontal force H, which is applied to P and acts in the vertical plane containing the line of greatest slope of the inclined plane which passes through P. If equilibrium is limiting with P on the point of moving up the plane, find H in terms of W and α

The figure (b) shows a force Q which is applied to P in the vertical plane containing the line of greatest slope through P. This force is inclined at an angle θ to this line. If equilibrium is limiting with P on the point of moving up the plane, show that

$$Q = \frac{W\sin 2\alpha}{\cos(\theta-\alpha)}$$

Find, in terms of α, the value of θ for which Q is least.

Solution



Resolving ⊥ to the slope;

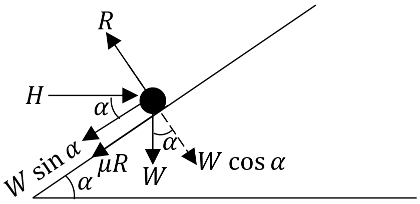
$$R = W\cos\alpha \text{ ….. (i)}$$

Resolving ∥ to the slope;

$$\mu R = W\sin\alpha \text{ ….. (ii)}$$

Dividing (ii) by (i);

$$\mu = \tan\alpha$$



Resolving ⊥ to the slope;

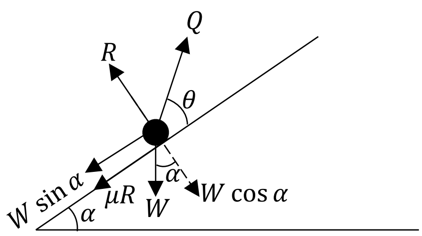
$$R = W\cos\alpha + H\sin\alpha$$

Resolving ∥ to the slope;

$$W\sin\alpha + \mu R = H\cos\alpha$$
$$W\sin\alpha + \mu(W\cos\alpha + H\sin\alpha) = H\cos\alpha$$

But $\mu = \tan\alpha = \frac{\sin\alpha}{\cos\alpha}$

$$W\sin\alpha + \frac{\sin\alpha}{\cos\alpha}(W\cos\alpha + H\sin\alpha) = H\cos\alpha$$
$$W\sin\alpha\cos\alpha + W\sin\alpha\cos\alpha + H\sin^2\alpha = H\cos^2\alpha$$
$$2W\sin\alpha\cos\alpha = H(\cos^2\alpha - \sin^2\alpha)$$
$$W\sin 2\alpha = H\cos 2\alpha$$
$$H = \frac{W\sin 2\alpha}{\cos 2\alpha}$$
$$H = W\tan 2\alpha$$



Resolving ⊥ to the slope;

$$R + Q\sin\theta = W\cos\alpha$$
$$R = W\cos\alpha - Q\sin\theta$$

Resolving ∥ to the slope;

$$Q\cos\theta = W\sin\alpha + \mu R$$
$$Q\cos\theta = W\sin\alpha + \mu(W\cos\alpha - Q\sin\theta)$$

But $\mu = \tan\alpha = \frac{\sin\alpha}{\cos\alpha}$

$$Q\cos\theta = W\sin\alpha + \frac{\sin\alpha}{\cos\alpha}(W\cos\alpha - Q\sin\theta)$$
$$Q\cos\theta\cos\alpha = W\sin\alpha\sin\alpha + W\sin\alpha\cos\alpha - Q\sin\theta\cos\alpha$$
$$Q(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = 2W\sin\alpha\sin\alpha$$

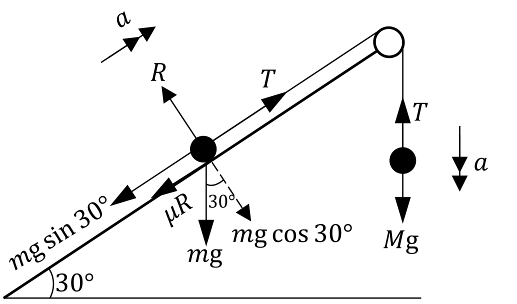
(b) Find the speed with which mass m is moving when mass M hits the horizontal plane

(c) For how much longer will mass m continue to move up the plane?

(d) What will happen then?

Solution

Let the initial acceleration of the system be a



(a) For mass m:

$$T - (mg\sin 30° + \mu R) = ma$$

$$T - \left(mg\sin 30° + \frac{1}{\sqrt{3}}mg\cos 30°\right) = ma$$

$$T - \left(\frac{mg}{2} + \frac{1}{\sqrt{3}}mg \times \frac{\sqrt{3}}{2}\right) = ma$$

$$T - mg = ma \quad \text{.... (i)}$$

For mass M:

$$Mg - T = Ma \quad \text{.... (ii)}$$

Adding (i) and (ii) gives;

$$Mg - mg = Ma + ma$$

$$(M - m)g = (M + m)a$$

$$a = \frac{(M - m)g}{M + m}$$

(b) When M hits the floor, m has travelled a distance of 1 m

$u = 0,\ a = \frac{(M-m)g}{M+m},\ s = 1,\ v = ?$

$$v^2 = u^2 + 2as$$

$$v^2 = 0 + 2 \times \frac{(M - m)g}{M + m} \times 1$$

$$v = \sqrt{\frac{2(M - m)g}{M + m}}\ \text{ms}^{-1}$$

(c) When M hits the floor, the string will become slack ($T = 0$) and mass m will move will move with a new acceleration a' until it comes to rest at some point on the plane.

$$0 - (mg\sin 30° + \mu R) = ma'$$

$$0 - \left(\frac{mg}{2} + \frac{1}{\sqrt{3}}mg\cos 30°\right) = ma'$$

$$0 - mg = ma'$$

$$a' = -g$$

Now using $v = u + at$

$$0 = \sqrt{\frac{2(M - m)g}{M + m}} - gt$$

$$t = \frac{1}{g}\sqrt{\frac{2(M - m)g}{M + m}} = \sqrt{\frac{2(M - m)}{(M + m)g}}\ s$$

(d) The limiting frictional force is given by

$$F_L = \mu R = \frac{1}{\sqrt{3}}mg\cos 30° = \frac{mg}{2}$$

This will be equal to the component of the weight down the plane i.e. $mg\sin 30°$

Then the particle will be at rest in limiting equilibrium on the point of moving down the plane.

Self-Evaluation exercise

1. A particle of weight 10 N rests in rough contact with a plane inclined at 30° to the horizontal and is just about to slip. Find the coefficient of friction between the plane and the particle.

   [Ans: $1/\sqrt{3}$]

2. A block of weight 10 N rests on a rough plane inclined at 30° to the horizontal. The coefficient of friction is $\frac{1}{2}$.
   Find the horizontal force required
   (a) to prevent the block from slipping down
   (b) to make just about to slide up the plane

   [Ans: (a) 0.6 N (b) 15.2 N]

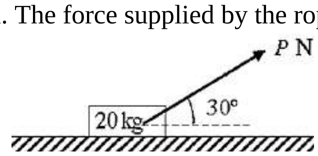
3. A particle P, of mass $7m$, is placed on a rough horizontal table, the coefficient of friction between P and the table being μ. A force of magnitude $2mg$, acting upwards at an acute angle α to the horizontal is applied to P and equilibrium is on the point of being broken by the particle sliding on the table. Given that $\tan\alpha = \frac{5}{12}$, find the value of μ

   [Ans: $\frac{8}{27}$]

4. A small object of weight $4W$ in rough contact with a horizontal plane is acted upon by a force inclined at 30° to the plane. When the force is of magnitude $2W$, the object is about to slip. Calculate the magnitude of the normal reaction and the coefficient of friction between the object and the plane.

   [Ans: Either $3W$, $1/\sqrt{3}$ or $5W$, $\sqrt{3}/5$]

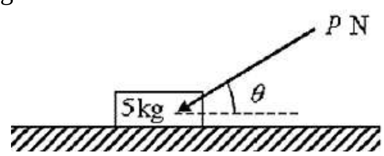
5. The figure below shows a small box of mass 20 kg, pulled along rough horizontal ground by a light inextensible rope, which is inclined at 30° to the horizontal. The force supplied by the rope is P N.



   Given further that the coefficient of friction between the box and the ground is 0.25 and that the box is on point of slipping, calculate the value of P.

   [Ans: 49.4 N]

6. The figure below shows a small box of mass 5 kg, pushed by a constant force P. The force pushing the box has magnitude P N and is inclined at θ to the horizontal.



WORK, ENERGY AND POWER

Introduction

Steven Kiprotich, a long-distance runner won a gold medal at the 2012 London summer Olympics becoming the second Ugandan to reach that milestone.

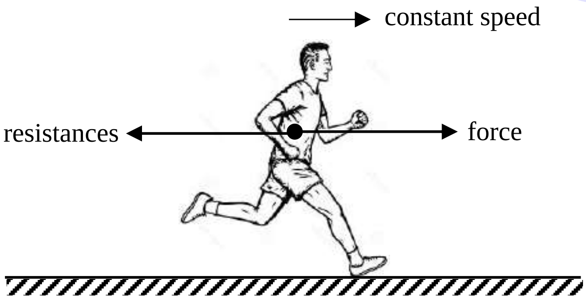
He clinched the Olympic gold with a winning time of 2:08:01 in hot, sunny, and humid conditions. This was the first Olympic medal for Uganda since 1996, the first Olympic gold medal since 1972, and the country's first ever in the marathon.

He completed the 42 km course in 2 hours, 8 minutes and 1 second. This gives him an average speed of 19.7 kmh^{-1} for the race.

$$\text{speed} = \frac{\text{distance}}{\text{time}} = \frac{42}{2.134} = 19.7$$

By the end of the race, Steven Kiprotich would have been very tired.

How could this "tiredness" be measured? To be able to answer this question, several assumptions will have to be made. Steven Kiprotich will have to be modelled as a particle travelling at constant speed. The course will be assumed straight and level. The resistances to motion are assumed constant.



Resolve →;

$$\text{force} - \text{resistances} = 0$$
$$\text{force} = \text{resistances}$$

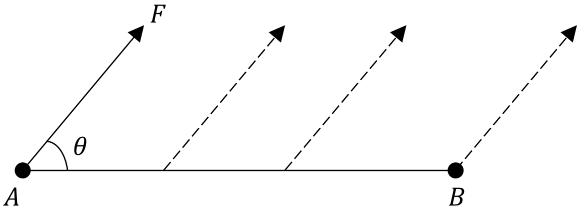
Since the resistances are constant, then the applied force must also be constant. As he covers more and more distance, he would become more and more tired. This is because he is applying the same force continuously, over a longer and longer distance. A measure of this effort could be the force multiplied by the distance covered.

$$\text{force} \times \text{distance}$$

In mechanics, this measurement is called the work done.

Work done

Work may be done by or against a force (often gravity). It is a scalar.



When a constant force F moves its point of application from A to B, the work done by F is $F\cos\theta \cdot AB$

SI unit of work is the joule (J)

Energy

Energy is the capacity to do work. It is a scalar. SI unit of energy is the joule (same as work).

A body possessing energy can do work and lose energy. Work can be done on a body and increase its energy i.e.

$$\text{Work done} = \text{change in energy}$$

Kinetic and potential energy

Kinetic and potential energy are types of mechanical energy.

Kinetic energy (KE) is due to a body's motion. The KE of a body of mass m moving with velocity v is $\frac{1}{2}mv^2$

Potential energy (PE) is due to a body's position. Gravitational potential energy is a property of height.

The PE of a body of mass m at a distance h,

(a) above an initial level is mgh

(b) below an initial level is $-mgh$

The initial level can be any level you choose and the potential energy at the initial level is zero.

Elastic PE, a property of stretched elastic strings and springs or compressed springs is $\frac{\lambda x^2}{l}$ where λ is the modulus of elasticity of the string, l is its natural length and x is the extension.

Mechanical energy

The mechanical energy (ME) of a particle (or body)

$= PE + KE$ of the particle (or body)

If a system includes one or more elastic strings, then

Total ME of system $= PE + KE +$ elastic PE

Mechanical energy is lost (as heat energy or sound energy) when we have

(a) resistances (friction) or

(b) impulses (collisions or strings jerking taut)

Conservation of mechanical energy

The total mechanical energy of a body (or system) will be conserved if

(a) no external force (other than gravity) cause work to be done

(b) none of the ME is converted into other forms.

Given these conditions:

$$PE + KE = \text{constant}$$
$$\text{or loss in } PE = \text{gain in } KE$$
$$\text{or loss in } KE = \text{gain in } PE$$

Note:

An alternative equation of the work-energy principle can be represented as below

$$\boldsymbol{KE_{\text{start}} + PE_{\text{start}} + W_{\text{in}} = KE_{\text{end}} + PE_{\text{end}} + W_{\text{out}}}$$

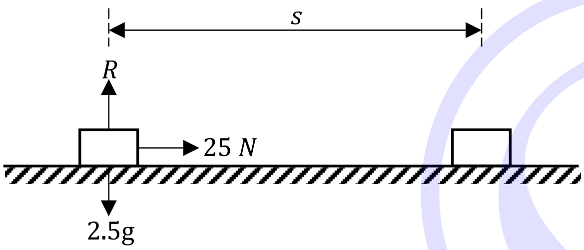
Sometimes work done by external forces (W_{in}) will increase the energy of the particle. In this formula, weight is not treated as an external force.

The work done to overcome resistances (W_{out}) will decrease the energy of the particle.

Example 1

A mass of 2.5 kg is pulled along a smooth horizontal surface by a 25 N force which acts horizontally. The work done by this force is 125 J. Determine the distance covered by the mass.

Solution



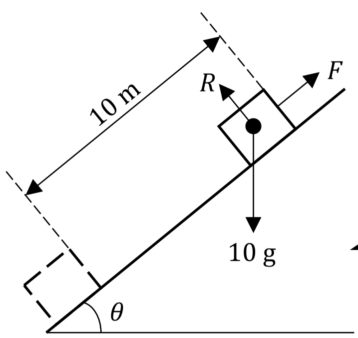
$W = 125$ J, $F = 25\ N$, $s = ?$

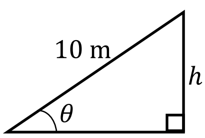
$$W = F \times s$$
$$125 = 25 \times s$$
$$s = 5 \text{ m}$$

Example 2

A mass of 10 kg is pulled up a smooth inclined plane for a distance of 10 m. The plane is inclined at $\sin^{-1}\frac{3}{5}$ to the horizontal. Calculate the work done against gravity.

Solution





$$\sin\theta = \frac{3}{5}$$

$$h = 10\sin\theta = 10 \times \frac{3}{5} = 6 \text{ m}$$

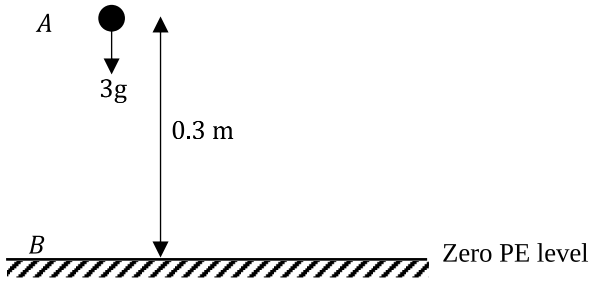
Work done against gravity $= mgh$

$$= 10 \times 9.8 \times 6$$
$$= 588 \text{ J}$$

Example 3

A toolbox, of mass 3 kg, is knocked off a table which is 0.3 m above the ground. Using conservation of energy, calculate the vertical speed with which the toolbox lands on the ground.

Solution

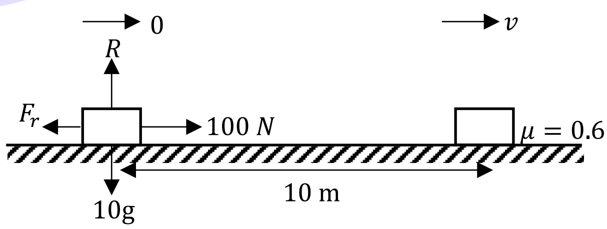


$$PE_A = KE_B$$
$$mgh = \frac{1}{2}mv^2$$
$$3 \times 9.8 \times 0.3 = \frac{1}{2} \times 3 \times v^2$$
$$v^2 = 5.88$$
$$v = 2.42 \text{ ms}^{-1}$$

Example 4

A 10 kg mass is pulled along a rough horizontal surface by a horizontal force of magnitude 100 N. The mass is initially at rest and it is pulled by the force for a distance of 10 m. The coefficient of friction between the mass and the surface is 0.6. Use the work-energy principle to find the final speed of the mass.

Solution



The work done by the 100 N force is needed to overcome the work against frictional force and to increase the kinetic energy of the mass.

$$W_{\text{in}} = KE_B + W_{\text{out}}$$
$$Fs = \frac{1}{2}mv^2 + F_r s$$
$$100 \times 10 = \frac{1}{2} \times 10v^2 + 10F_r$$
$$1000 = 5v^2 + 10F_r$$
$$F_r = \mu R = 0.6 \times 10g = 58.8\ N$$
$$1000 = 5v^2 + 588$$
$$v^2 = 82.4$$
$$v = 9.08 \text{ ms}^{-1}$$

bottom of the hill. The cyclist then ascends a vertical distance of 30 m to the top of another hill at point B. The speed of the cyclist at B is 12 ms^{-1}. The combined mass of the cyclist and his bike is 80 kg. The cyclist and his bike are modelled as a single particle subject to a constant non-gravitational resistance of 25 N, throughout the motion. Find the work done by the cyclist.

Solution

$$KE_{\text{start}} + PE_{\text{start}} + W_{\text{in}} = KE_{\text{end}} + PE_{\text{end}} + W_{\text{out}}$$

$$\tfrac{1}{2}(80)(10^2) + 80(9.8)(40) + W_{\text{in}}$$

$$= \tfrac{1}{2}(80)(12^2) + 80(9.8)(30) + 25 \times 600$$

$$4000 + 31360 + W_{\text{in}} = 5760 + 23520 + 15000$$

$$W_{\text{in}} = 8920 \text{ J}$$

The work done by the cyclist is 8920 J

Example 12

A car of mass 1500 kg is travelling up a hill on a straight road, with the engine of the car working at the constant rate of 13 kW for 1 minute. During this minute the car increases its speed from 7 ms^{-1} to 24 ms^{-1} and in addition to the work done against gravity, 80000 J of work is done against resistances to motion parallel to the direction of motion of the car. Calculate the vertical displacement of the car in this 1 minute interval.

Solution

$$\text{Power} = \frac{W_{\text{in}}}{\text{time}}$$

$$13000 = \frac{W_{\text{in}}}{60}$$

$$W_{\text{in}} = 780000 \text{ J}$$

Taking the level at which the speed of the car is 7 ms^{-1} as the zero potential level;

$$KE_A + PE_A + W_{\text{in}} = KE_B + PE_B + W_{\text{out}}$$

$$\frac{1}{2}(1500)(7^2) + 780000$$

$$= \frac{1}{2}(1500)(24^2) + 1500gh + 80000$$

$$36750 + 780000 = 432000 + 1500gh + 80000$$

$$1500gh = 304750$$

$$h = \frac{304750}{1500g} = 20.73 \text{ m}$$

Example 13

A particle P of mass 4 kg is moving on the line of greatest slope of a rough plane, inclined at an angle θ to the horizontal, where $\tan\theta = \frac{3}{4}$. The particle is projected up the plane with a speed u ms^{-1} from a point A on the plane, comes to instantaneous rest at a point B, and then slides back down the plane passing through A again. The coefficient of friction between the particle and the plane is $\frac{2}{7}$ and the distance AB is 2.5 m. Find

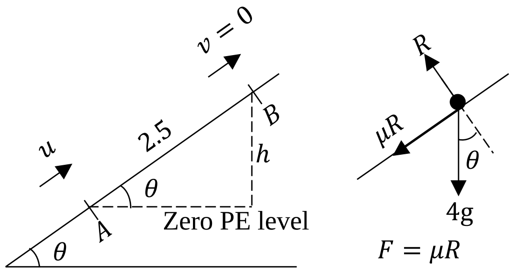
(a) the value of u.

(b) the speed of the particle as it passes through A again.

Solution

(a) $m = 4$, $\mu = \frac{2}{7}$

$$\tan\theta = \frac{3}{4} \Rightarrow \sin\theta = \frac{3}{5} \text{ and } \cos\theta = \frac{4}{5}$$



$$F = \mu R$$

$$= \mu(4g\cos\theta)$$

$$= \frac{2}{7} \times 4 \times 9.8 \times \frac{4}{5}$$

$$= 8.96$$

$$KE_A + PE_A + W_{\text{in}} = KE_B + PE_B + W_{\text{out}}$$

$$\frac{1}{2}mu^2 = mgh + \text{friction} \times 2.5$$

$$\frac{1}{2} \times 4u^2 = 4(9.8)(2.5\sin\theta) + 8.96 \times 2.5$$

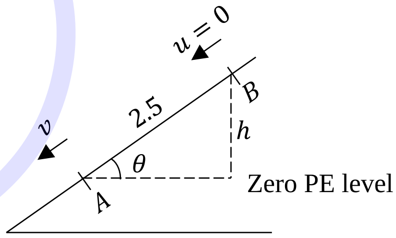
$$2u^2 = 4(9.8)(2.5)\left(\frac{3}{5}\right) + 22.4$$

$$2u^2 = 81.2$$

$$u^2 = 40.6$$

$$u = 6.37 \ \text{ms}^{-1}$$

(b)



$$KE_B + PE_B + W_{\text{in}} = KE_A + PE_A + W_{\text{out}}$$

$$mgh = \frac{1}{2}mv^2 + \text{friction} \times 2.5$$

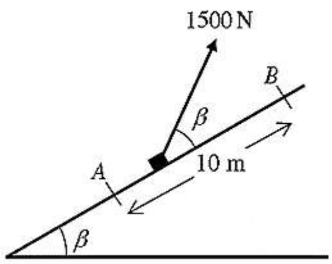
$$4(9.8)(2.5\sin\theta) = \frac{1}{2}(4)v^2 + 22.4$$

$$2v^2 = 58.8 - 22.4$$

$$v^2 = 18.2$$

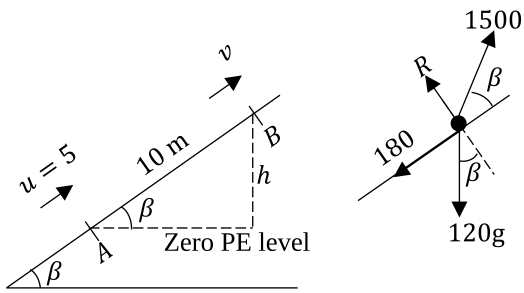
$$v = 4.27 \text{ ms}^{-1}$$

Example 14



The figure above shows a box of mass 120 kg being pulled up the line of greatest slope of the plane inclined at an angle β to the horizontal, by an electrically operated cable. The cable is supplying a constant tension of 1500 N and is inclined at an angle β to the plane. The box passes through the point A with speed 5 ms^{-1} and through the point B which is higher up the plane with speed $v\text{ ms}^{-1}$. The distance AB is 10 m. There is a constant non-gravitational resistance of 180 N acting on the box. The box is modelled as a particle and the cable as a light inextensible string. Given that $\tan\beta = \frac{3}{4}$, find the value of v.

Solution



$$\tan\beta = \frac{3}{4}\ ; \sin\beta = \frac{3}{5}; \cos\beta = \frac{4}{5}$$

$$h = 10\sin\beta = 10\times\frac{3}{5} = 6\text{ m}$$

By energies;

$$KE_A + PE_A + W_{\text{in}} = KE_B + PE_B + W_{\text{out}}$$

$\frac{1}{2}(120)(5^2) + 0 + 1500\cos\beta\times 10 = \frac{1}{2}(120)(v^2) + 120(9.8)(6) + 180(10)$

$$1500 + 12000 = 60v^2 + 7056 + 1800$$

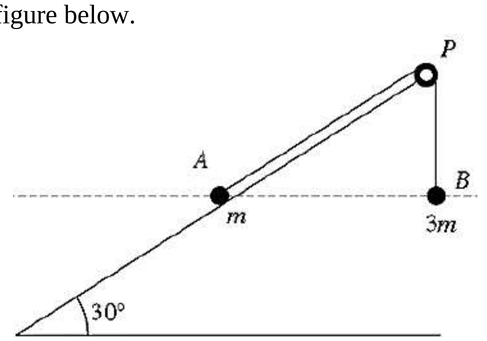
$$60v^2 = 4644$$

$$v^2 = 77.4$$

$$v = 8.8\text{ ms}^{-1}$$

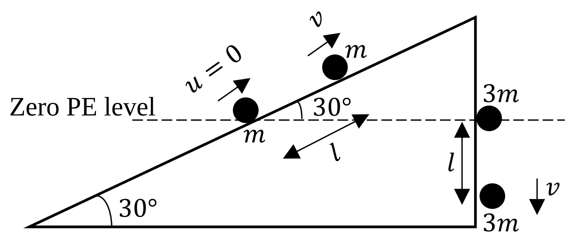
Example 15

Two particles A and B, of mass m and $3m$ respectively, are attached to each of the ends of a light inextensible string. The string passes over a smooth pulley P, at the top of a fixed smooth plane, inclined at 30° to the horizontal. Particle A is held at rest on the incline plane while B is hanging freely at the end of the incline plane vertically below P, as shown in the figure below.



The two particles, the pulley and the string lie in a vertical plane parallel to the line of greatest slope of the incline plane. The particles are released from rest, from the same horizontal level with the string taut. When B has fallen by a distance l, its speed is v, and A has not yet reach P. Ignoring air resistance, express v in terms of g and l.

Solution



Total ME on release = Total ME after both moving by l

$$0 = KE_A + KE_B + PE_A + PE_B$$

$$0 = \frac{1}{2}mv^2 + \frac{1}{2}(3m)v^2 + mgl\sin 30° + 3mg(-l)$$

$$0 = \frac{1}{2}mv^2 + \frac{3}{2}mv^2 + \frac{1}{2}mgl - 3mgl$$

$$0 = mv^2 + 3mv^2 + mgl - 6mgl$$

$$0 = 4mv^2 - 5mgl$$

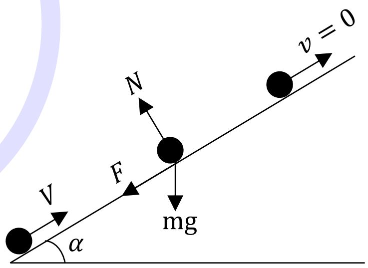
$$4v^2 = 5gl$$

$$v = \frac{1}{2}\sqrt{5gl}$$

Example 16

A particle of mass m is projected directly up a rough plane of inclination α with velocity V. If μ is the coefficient of friction between the particle and the plane, calculate how far up the plane the particle travels before coming to rest.

Solution



Let l be the distance along the slope the particle travels.

Friction force, $F = \mu R = \mu mg\cos\alpha$

At the bottom of the slope;

$$\text{Total energy} = \frac{1}{2}mV^2$$

At the top of the slope,

$$\text{total energy} = mgl\sin\alpha$$

Work done against friction $= \mu mgl\cos\alpha$

Initial total energy

= final total energy + work done against friction

$$\frac{1}{2}mV^2 = mgl\sin\alpha + \mu mgl\cos\alpha$$

$$V^2 = 2gl\sin\alpha + 2\mu gl\cos\alpha$$

$$V^2 = 2gl(\sin\alpha + \mu\cos\alpha)$$

$$l = \frac{V^2}{2g(\sin\alpha + \mu\cos\alpha)}$$

Power

Power is the rate at which a force does work. It is a scalar.
SI unit of power is the watt (W)

$$1 \text{ watt } (W) = 1 \text{ joule per second}$$

When a body is moving in a straight line with velocity v ms^{-1} under a tractive force F newtons, the power of the force is Fv watts.

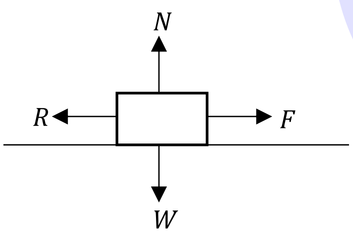
Moving vehicles

The power of a moving vehicle is supplied by its engine. The tractive force of an engine is the pushing force it exerts.
To solve problems involving moving vehicles:

(a) Draw a clear force diagram
Note: non-gravitational resistance means frictional force.
(b) Resolve forces perpendicular to the direction of motion
(c) If the velocity is
  (i) constant (vehicle moving with steady speed), then resolve forces parallel to the direction of motion
  (ii) not constant (vehicle accelerating), then find the resultant force acting and write down the equation of motion in the direction of motion
(d) Use power = tractive force × speed

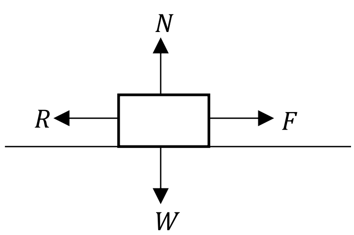
Common situations

1. Vehicles **on the level**
  (a) moving with **steady speed v**



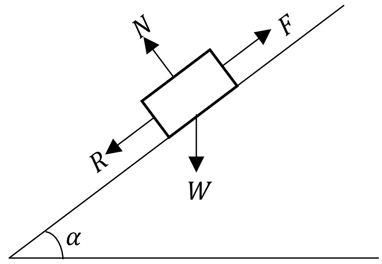
Resolve (↑); $N = W$
Resolve (→); $F = R$
Power, $P = Fv$

  (b) moving with **acceleration** a and instantaneous speed v



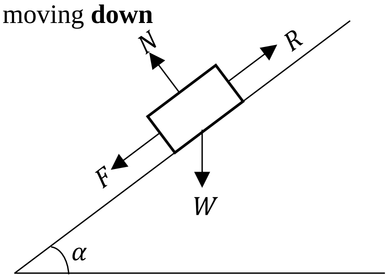
Resolve (↑); $N = W$
Equation of motion: $F - R = ma$
Power, $P = Fv$

2. Vehicles **on a slope** of angle α
  (a) moving with **steady speed v**
    (i) moving **up**



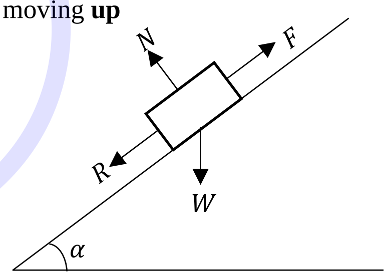
Resolve ⊥ to plane: $N = W\cos\alpha$
Resolve ∥ to plane: $F = R + W\sin\alpha$
Power, $P = Fv$

    (ii) moving **down**



Resolve ⊥ to plane: $N = W\cos\alpha$
Resolve ∥ to plane: $F + W\sin\alpha = R$
Power, $P = Fv$

  (b) moving with **acceleration** a and instantaneous speed v
    (i) moving **up**

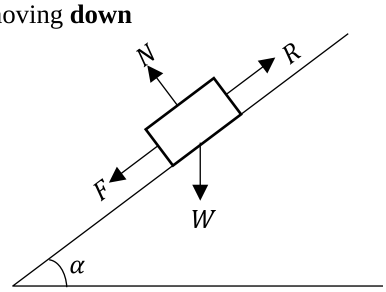


Resolve ⊥ to plane: $N = W\cos\alpha$
Equation of motion ∥ to plane:

$$F - R - W\sin\alpha = ma$$

Power, $P = Fv$
Note: If the vehicle is moving up the plane but retarding, then $F < R + W\sin\alpha$

    (ii) moving **down**



Resolve ⊥ to plane: $N = W\cos\alpha$
Equation of motion ∥ to plane:

$$F + W\sin\alpha - R = ma$$

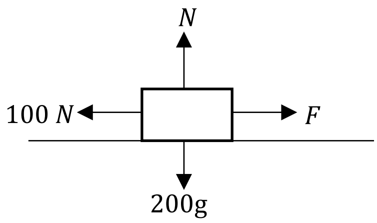
Power, $P = Fv$

Note: If the vehicle is retarding, using the engine for braking, then F acts up the plane and $F + R > W \sin \alpha$

Example 1

A train of total mass 200 kg is moving at a speed of 72 kmh^{-1} on a straight level track. If the non-gravitational resistance is 200 N, at what rate is the engine working.

Solution



$$v = \frac{72000}{3600} = 20 \text{ ms}^{-1}$$

Resolve (→); $F = 100\ N$

$$\text{Power} = Fv$$
$$= 100 \times 20$$
$$= 2000\ W$$

Example 2

A cyclist travels at a constant speed of 10 ms^{-1} along a straight horizontal road. The power the cyclist develops is 400 W and the mass of the cyclist and the bicycle totals 80 kg.

(a) Find the magnitude of the resistance forces at this speed

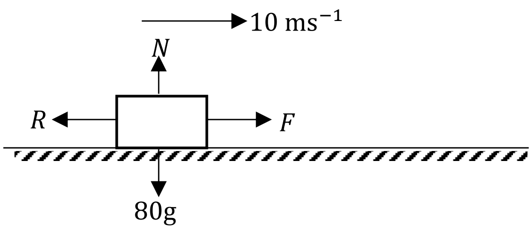
The resistance forces are found to be proportional to the cyclist's speed

(b) Write down the equation for the resistance forces

(c) Calculate the acceleration of the cyclist when his speed is 20 ms^{-1} if the power developed remains unchanged.

Solution

(a) $P = 400\ W$, $F = ?$, $v = 10 \text{ ms}^{-1}$



$$P = Fv$$
$$400 = F \times 10$$
$$F = 40\ N$$

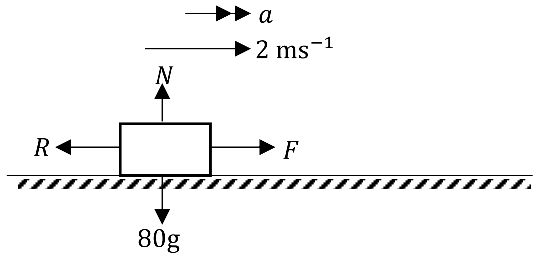
Since the speed is constant, the cyclist is in dynamic equilibrium ($a = 0$)

$$R = F = 40\ N$$

(b) $R \propto v$

$$R = kv$$
$$40 = k \times 10$$
$$k = 4$$
$$R = 4v$$

(c)



$$P = Fv$$
$$400 = F \times 2$$
$$F = 200\ N$$
$$R = 4v = 4 \times 2 = 8\ N$$
$$F = ma$$
$$200 - 8 = 80a$$
$$80a = 192$$
$$a = 2.4 \text{ ms}^{-2}$$

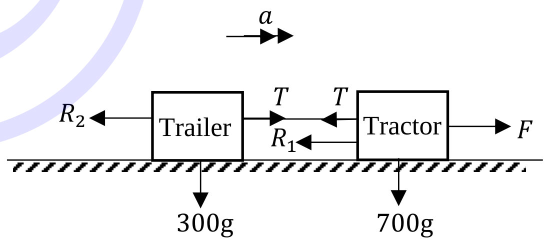
Example 3

A tractor of mass 700 kg pulls a trailer of mass 300 kg along a straight level road. The total resistance to motion is 1500 N and the tractor is using its full power of 30 kW.

(a) Show that when the tractor's speed is 36 kmh^{-1}, its acceleration is 1.5 ms^{-2}.

(b) Assuming that the resistive force is divided between the tractor and the trailer in the ratio of their masses, find the tension in the coupling between the tractor and the trailer when the speed is 36 kmh^{-1}

Solution



Total resistance, $R = R_1 + R_2 = 1500$

$$36\frac{km}{hr} = 36 \times \frac{1000}{3600} \text{ms}^{-1} = 10 \text{ ms}^{-1}$$
$$P = F \times v$$
$$30000 = F \times 10$$

Tractive force, $F = 3000\ N$

(a) Considering the motion of both the Tractor and the trailer;

$$F - R = ma$$
$$3000 - 1500 = 1000a$$
$$1500 = 1000a$$
$$a = 1.5 \text{ ms}^{-2}$$

(b) Taking the ratio of the masses of the tractor and the trailer

$$\frac{R_1}{R_2} = \frac{700}{300}$$
$$R_1 = \frac{7}{3}R_2$$

From $R_1 + R_2 = 1500$

$$\frac{7}{3}R_2 + R_2 = 1500$$
$$\frac{10R_2}{3} = 1500$$
$$R_2 = 450$$
$$R_1 = 1500 - R_2 = 1500 - 450 = 1050$$

Considering the separate motion of the tractor;

$$F - (T + R_1) = 700a$$
$$3000 - (T + 1050) = 700 \times 1.5$$
$$1950 - T = 1050$$
$$T = 900\ N$$

Alternatively, by considering the separate motion of the trailer;

$$T - R_2 = 300a$$
$$T - 450 = 300 \times 1.5$$
$$T = 900\ N$$

Example 4

A woman and her bike are modelled a single particle of combined mass 72 kg. The woman cycles with constant speed of 5 ms^{-1}, up a straight road, which lies on the line of greatest slope of a plane inclined at an angle θ to the horizontal, where $\tan\theta = \frac{2}{21}$. The total non-gravitational resistance experienced by the cyclist is assumed to be constant at 25 N.

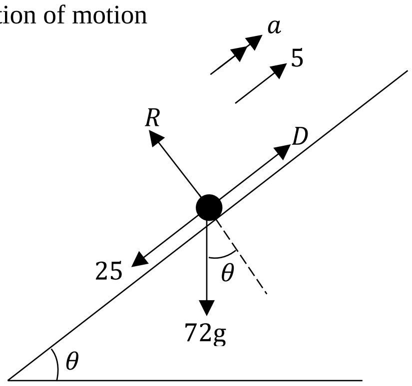
(a) Find the power generated by the woman when cycling up the hill.

The woman then turns her bike around at some point A on the road. She freewheels down the same road starting with a speed of 5 ms^{-1}. She passes through some point B on that road with a speed v ms^{-1}. The total non-gravitational resistance experienced by the cyclist is assumed to be the same as in part (a).

(b) Given that the distance AB is 180 m, find the value of v.

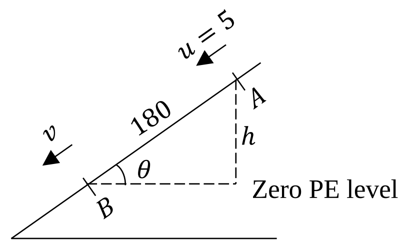
<u>Solution</u>

(a) Constant speed ⇒ zero acceleration ⇒ equilibrium in the direction of motion



$$D = 25 + 72\mathrm{g}\sin\theta$$
$$D = 25 + 72 \times 9.8 \times \frac{2}{21}$$
$$D = 92.2\ N$$
$$P = D \times v$$
$$P = 92.2 \times 5 = 461\ W$$

(b)



$$h = 180\sin\theta = 180 \times \frac{2}{21} = \frac{120}{7}$$
$$KE_A + PE_A + W_{\text{in}} = KE_B + PE_B + W_{\text{out}}$$
$$\frac{1}{2}(75)(5^2) + 72(9.8)\left(\frac{120}{7}\right) = \frac{1}{2}(72)v^2 + 25 \times 180$$
$$900 + 12096 = 36v^2 + 4500$$
$$36v^2 = 8496$$
$$v = 15.4\ \text{ms}^{-1}$$

Example 5

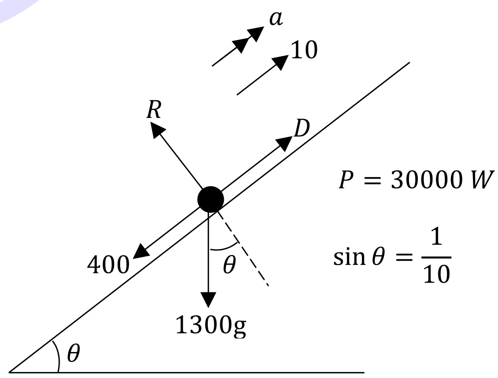
A car of mass 1300 kg is travelling on a straight road which lies on the line of greatest slope of a plane inclined at an angle θ to the horizontal, where $\sin\theta = \frac{1}{10}$. The total non-gravitational resistance experienced by the car is assumed to be a constant force of magnitude of 400 N. The engine of the car is working at the constant rate of 30 kW. The car is passing through the point A with a speed 10 ms^{-1} and continues to accelerate up the plane, passing through the point B with speed 30 ms^{-1}, 30 s after passing through A. Find

(a) the acceleration of the car at A.

(b) the distance AB.

<u>Solution</u>

(a)



$$\text{Power} = D \times v$$
$$30000 = D \times 10$$
$$D = 3000\ N$$

Resolving parallel to the plane:

$$D - (400 + 1300\mathrm{g}\sin\theta) = 1300a$$
$$3000 - \left(400 + 1300\mathrm{g} \times \frac{1}{10}\right) = 1300a$$
$$1326 = 1300a$$
$$a = 1.02\ \text{ms}^{-2}$$

(b) Power $= \frac{W_{\text{in}}}{\text{time}}$

Now the gain in kinetic energy and potential energy must be equal to the work in, supplied by the pump.

$$W_{\text{in}} = \text{Gain in } KE + \text{Gain in } PE$$

$$W_{\text{in}} = \frac{1}{2}mv^2 + mgh$$

$$\frac{W_{\text{in}}}{t} = \frac{1}{2}\left(\frac{m}{t}\right)v^2 + \left(\frac{m}{t}\right)gh$$

$$P = \frac{1}{2}\rho A v^3 + \rho v A g h$$

$$900 = \frac{1}{2} \times 1000A \times 20^3 + 1000 \times 20A \times 9.8 \times 10$$

$$900 = 4000000A + 1960000A$$

$$59600A = 9$$

$$A = 0.000151 \text{ m}^2$$

$$A = 0.000151 \times 100^2 = 1.51 \text{ cm}^2$$

Self-Evaluation exercise

1. The engine of a car is working at a constant rate of $6\ kW$ in driving the car along a straight horizontal road at constant speed of $54\ \text{kmh}^{-1}$. Find the resistance to the motion of the car.

   [Ans: $400\ N$]

2. A motor car of mass 800 kg is towing a trailer of mass 300 kg along a straight horizontal road. Resistances, which are constant are $600\ N$ for the car and $240\ N$ for the trailer.
   (a) Calculate the tractive force exerted by the motor and the tension in the coupling between the car and the trailer in each of the following cases
      (i) when both are travelling at constant velocity
      (ii) when both are acceleration at $2.5\ \text{ms}^{-2}$
   (b) Calculate the power developed by the motor when the car and are travelling at a constant velocity of $15\ \text{ms}^{-1}$

[Ans: (a) (i) $840\ N$, $240\ N$ (ii) $3590\ N$, $990\ N$ (b) $12.6\ kW$]

3. The frictional resistance to the motion of a car of mass 1000 kg is kv newtons, where $v\ \text{ms}^{-1}$ is its speed and k is constant. The car ascends a hill of inclination $\sin^{-1}\left(\frac{1}{10}\right)$ at a steady speed of $8\ \text{ms}^{-1}$, the power exerted by the engine being $9.76\ kW$.
   (a) Prove that the numerical value of k is 30.
   (b) Find the steady speed at which the car ascends the hill if the power exerted by the engine is $12.8\ kW$
   (c) When the car is travelling at this speed, the power exerted by the engine is increased by $2\ kW$. Find the immediate acceleration of the car.

   [Ans: (b) $10\ \text{ms}^{-1}$ (c) $0.2\ \text{ms}^{-2}$]

4. A pump raises water at a rate of 500 kg per minute through a vertical distance of 3 m. If the water is delivered at $2.5\ \text{ms}^{-1}$, find the power developed.

   [Ans: $276\ W$]

5. A car of mass 1200 kg is travelling along a straight horizontal road at a constant speed of $120\ \text{kmh}^{-1}$ against a resistance of $600\ N$.
   (a) Calculate the effective power being exerted
   Given that the resistance is proportional to the square of the velocity, calculate
   (b) the power required to go down a hill of 1 in 30 (along the slope) at a steady speed of $120\ \text{kmh}^{-1}$
   (c) the acceleration of the car up this hill with the engine working at $20\ kW$ at the instant when the speed is $80\ \text{kmh}^{-1}$.

   [Ans: (a) $20\ kW$ (b) $6.93\ kW$ (c) $0.201\ \text{ms}^{-2}$]

6. The mass of a car is 800 kg and the total resistance to its motion is constant and equal to a force of $320\ N$.
   (a) Find, in kW, the rate of working of the engine of the car when it is moving along a level road at a constant speed of $25\ \text{ms}^{-1}$.
   (b) What is the acceleration in ms^{-2}, when the car is moving along a level road at $20\ \text{ms}^{-1}$ with the engine working at $11\ kW$?
   (c) What is the maximum speed attained by the car with the engine working at this rate?

   [Ans: (a) $8\ kW$ (b) $0.2875\ \text{ms}^{-2}$ (c) $34.375\ \text{ms}^{-1}$]

7. (Take $g = 10\ \text{ms}^{-2}$) A car of mass 1000 kg is travelling up a slope, inclination $\sin^{-1} 0.05$, at a constant speed of $25\ \text{ms}^{-1}$. The engine is generating its maximum power of $1.2 \times 10^5\ W$. Show that, excluding gravity, the car is experiencing a resistance to motion of $4.5 \times 10^3\ N$
   Assuming that the resistance is proportional to the speed of the car determine:
   (a) the maximum speed of the car on the level road,
   (b) the maximum acceleration of the car when it is travelling at $10\ \text{ms}^{-1}$ on a level road.

   [Ans: (a) $26.4\ \text{ms}^{-1}$ (b) $10.4\ \text{ms}^{-2}$]

8. A car of mass 1000 kg moves with its engine shut off down a slope of inclination α, where $\sin\alpha = \frac{1}{20}$, a steady speed of $15\ \text{ms}^{-1}$.
   (a) Find the resistance, in newtons, to the motion of the car
   (b) Calculate the power delivered by the engine when the car ascends the same inclination at the same steady speed, assuming the resistance to motion is unchanged. [Take g as $10\ \text{ms}^{-2}$]

   [Ans: (a) $500\ N$ (b) $15\ kW$]

9. Water is being raised by a pump from a storage tank 4 m below the ground and delivered at $8\ \text{ms}^{-1}$ through a pipe at ground level. If the cross-sectional area of the pipe is $0.12\ \text{m}^2$, find the work done per second by the pump. (density of water = $1000\ \text{kgm}^{-3}$)

   [Ans: $68352\ W$]

10. A water pump raises 50 kg of water a second through a height of 20 m. The water emerges as a jet with speed $50\ \text{ms}^{-1}$.
    (a) Find the kinetic energy and the potential energy given to the water each second and hence the effective power developed by the pump

22. A locomotive of mass 48900 kg pulls a train of 8 trucks each of mass 9200 kg, up a straight slope inclined at 1° to the horizontal, at a constant speed of $8\ \mathrm{ms^{-1}}$. The total of the resistances to motion of the locomotive and its trucks is modelled at a constant force of magnitude 4000 N. Calculate the power generated by the locomotive.
    At a later instant the locomotive and trucks are travelling along a straight horizontal track at a speed of $20\ \mathrm{ms^{-1}}$, with the locomotive continuing to work at the same rate as before. With the same model for the total of the resistances as before, find the acceleration at this instant.
    [Ans: 203 kW, 0.050 $\mathrm{ms^{-2}}$]
23. A car of mass 1220 kg travels up a straight road which is inclined at an angle α to the horizontal, where $\sin\alpha = 0.05$. The resistances to motion are modelled as a constant force of magnitude 1400 N. the car travels a distance of 25.8 m whilst increasing its speed from $8\ \mathrm{ms^{-1}}$, at the point X, to $12\ \mathrm{ms^{-1}}$ at the point Y. Calculate the work done by the car's engine in travelling from X to Y
    The car's engine works at a constant rate of 40 kW. Calculate the time taken to travel from X to Y
    [Ans: 101 kJ, 2.52 s]
24. A car of mass 700kg descends a straight hill which is inclined at 3° to the horizontal. The car passes through the points P and Q with speeds of $12\ \mathrm{ms^{-1}}$ and $30\ \mathrm{ms^{-1}}$ respectively. The distance PQ is 500 m. Assuming there are no resistances to motion, calculate the work done by the car's engine for the journey from P to Q.
    Assuming further that the driving force produced by the car's engine is constant, calculate the power of the car's engine at P, at Q, and at the midpoint of PQ.
    [Ans: 81.4 kJ, 1,95 kW, 4.89 kW, 3.72 kW]
25. A car of mass 1050 kg moves along a straight horizontal road with its engine working at a constant rate of 25 kW. Its speed at point A on the road is $12\ \mathrm{ms^{-1}}$. Assuming that there is no resistance to motion, calculate the time taken for the car to travel from A until it reaches a speed of $20\ \mathrm{ms^{-1}}$.
    Assume now that there is a constant resistance to motion and that the car's engine continues to work at 25 kW. It takes 10.7 s for the car's speed to increase from $12\mathrm{ms^{-1}}$ to $20\ \mathrm{ms^{-1}}$. During this time, the car travels 179 m. Calculate the work done against the resistance hence find the magnitude of the resistance.
    Later the car moves up a straight hill inclined at 2° to the horizontal. The engine works at the 25 kW as before, and there is a constant resistance of the same magnitude as before. The car travels a distance of 393 m while its speed increases from $12\ \mathrm{ms^{-1}}$ to $20\ \mathrm{ms^{-1}}$. Calculate the time taken by the car to travel this distance.
    [Ans: 5.38 s, 133 kJ, 744 N; 22.8 s]
26. A car starts from rest and travels on a horizontal straight road. A resisting force acts on the car. By modelling the resisting force as a constant force of magnitude 750 N acting in the direction opposite to the motion of the car, calculate the maximum speed which the car can reach with its engine working at a constant rate of 30 kW.
    The maximum power of the car is 40 kW and the mass is 1250 kg. Calculate the maximum speed the car can attain after starting from rest and while travelling up a straight hill inclined at 3° to the horizontal, assuming that the resistance of 750 N continues to act.
    [Ans: 40 $\mathrm{ms^{-1}}$; 28.5 $\mathrm{ms^{-1}}$]
27. A car of mass 1200 kg travelling along a horizontal road experiences a resistance to motion of magnitude 150 N. The car is accelerating at $0.5\ \mathrm{ms^{-2}}$. Find the forward driving force acting on the car.
    A trailer of mass 800 kg is now attached to the car. When the car and trailer are travelling along the same road, the resistance to motion on the trailer has magnitude 500 N. The resistance on the car, and the forward driving force are the same as before the trailer was attached. Find the acceleration of the car and trailer, and find also the force in the tow-bar between the car and the trailer.
    [Ans: 750 N; 0.05 $\mathrm{ms^{-2}}$, 540 N]
28. The total mass of a cyclist and her bicycle is 120 kg. While pedaling, she generates power of 640 W. Her motion is opposed by road resistance of magnitude 16 N, and by air resistance of magnitude $8v$ N, where $v\ \mathrm{ms^{-1}}$ is her speed.
    (a) Find the cyclist's acceleration when she is riding along a horizontal road at a speed of $5\ \mathrm{ms^{-1}}$
    (b) Find the greatest speed that she can maintain on a horizontal road
    (c) When cycling down a hill, she finds that she can maintain a speed of $10\ \mathrm{ms^{-1}}$. Find the angle of inclination of the hill to the horizontal
    [Ans: (a) 0.6 $\mathrm{ms^{-2}}$ (b) 8 $\mathrm{ms^{-1}}$ (c) 1.5°]
29. A resistive force acts on a cyclist, as she free-wheels down a straight hill at constant speed. The cyclist and her machine are modelled as a particle of mass 70 kg, and the resistive force as a constant force. This constant force has magnitude 48 N and acts upwards in a direction parallel to the hill. Calculate the angle of inclination of the hill to the horizontal.
    The cyclist reaches the foot of the hill at a speed of $6\ \mathrm{ms^{-1}}$ and starts to pedal, travelling along a horizontal straight road. The cyclist works at a constant rate of 624 W. By modelling the resistive force as a constant horizontal force of magnitude 48 N, calculate the acceleration of the cyclist immediately after she starts pedaling. Show that her subsequent speed on the horizontal road cannot exceed $13\ \mathrm{ms^{-1}}$
    [Ans: 3.93°; 0.8 $\mathrm{ms^{-2}}$]

$$r = \int (8t^3\text{i} - 3t^2\text{j} + 2t\text{k})\,dt$$
$$r = 2t^4\text{i} - t^3\text{j} + t^2\text{k} + c$$

Substituting the boundary condition, $r = 2\text{i} + \text{j} - \text{k}$ when $t = 0$ gives

$$r = 2t^4\text{i} - t^3\text{j} + t^2\text{k} + 2\text{i} + \text{j} - \text{k}$$
$$r = 2(t^4 + 1)\text{i} + (1 - t^3)\text{j} + (t^2 - 1)\text{k}$$

When $t = 2$,

Velocity $= 8(2)^3\text{i} - 3(2)^2\text{j} + 2 \times 2\text{k}$

$= 64\text{i} - 12\text{j} + 4\text{k}\ \text{ms}^{-1}$

Acceleration$= 24(2)^2\text{i} + (1 - 2^3)\text{j} + (2^2 - 1)\text{k}$

$= 96\text{i} - 12\text{j} + 2\text{k}\ \ \text{ms}^{-2}$

Position vector $= 2(2^4 + 1)\text{i} + (1 - 2^3)\text{j} + (2^2 - 1)\text{k}$

$= 34\text{i} - 7\text{j} + 3\text{k}\ \text{m}$

Example 2

The position vector of a particle at time t seconds is given by

$$r = \sin 2t\,\text{i} + \cos 2t\,\text{j} \text{ metres}$$

(a) Find the velocity and acceleration as a function of t

(b) Show that the speed of the particle is constant and that its direction of motion is perpendicular to r

Solution

(a) $r = \sin 2t\,\text{i} + \cos 2t\,\text{j}$

$$v = \frac{dr}{dt} = 2\cos 2t\,\text{i} - 2\sin 2t\,\text{j}$$
$$a = \frac{dv}{dt} = -4\sin 2t\,\text{i} - 4\cos 2t\,\text{j}$$

(b) speed of particle is

$$|v| = \sqrt{(2\cos 2t)^2 + (2\sin 2t)^2}$$
$$= \sqrt{4(\cos^2 2t + \sin^2 2t)}$$
$$= \sqrt{4} = 2$$

The speed of the particle 2 ms^{-1} is constant

The scalar product can be used to show that the direction of motion is perpendicular to r.

$$v \cdot r = (2\cos 2t\,\text{i} - 2\sin 2t\,\text{j}) \cdot (\sin 2t\,\text{i} + \cos 2t\,\text{j})$$
$$v \cdot r = 2\cos 2t \sin 2t - 2\sin 2t \cos 2t$$
$$v \cdot r = 0$$

A scalar product of zero proves that v is perpendicular to r

Example 3

A particle P has position vector r at time t where

$$r = 2(1 + \cos 2t)\text{i} + 2(t - \sin t)\text{j} + 3\text{k}$$

Show that the speed of the particle at time t is $4\left|\sin\frac{t}{2}\right|$ and that the acceleration has constant magnitude.

Solution

$$r = 2(1 + \cos 2t)\text{i} + 2(t - \sin t)\text{j} + 3\text{k}$$
$$v = \dot{r} = -2\sin t\,\text{i} + 2(1 - \cos t)\text{j} + 0\text{k}$$
$$a = \ddot{r} = -2\cos t\,\text{i} + 2\sin t\,\text{j} + 0\text{k}$$
$$(\text{speed})^2 = |\dot{r}|^2 = 4\sin^2 t + 4(1 - \cos t)^2$$
$$= 4(\sin^2 t + 1 + \cos^2 t - 2\cos t)$$
$$= 8(1 - \cos t)$$
$$= 8(1 - \left(1 - 2\sin^2\frac{t}{2}\right)$$
$$= 16\sin^2\frac{t}{2}$$
$$\text{Speed} = \sqrt{16\sin^2\frac{t}{2}} = 4\left|\sin\frac{t}{2}\right|$$
$$(\text{Acceleration})^2 = 4\cos^2 t + 4\sin^2 t$$
$$= 4(\cos^2 t + \sin^2 t)$$
$$= 4$$

$|\text{Acceleration}| = 2$ which is a constant

Example 4

A particle moves so that its position vector at time t seconds is given by r metres, relative to a fixed origin O, where $r = 2t^3\text{i} - 9t\text{j}$. Calculate the speed of the particle when $t = 1$ s.

Solution

$$r = 2t^3\text{i} - 9t\text{j}$$
$$v = \frac{dr}{dt} = 6t^2\text{i} - 9\text{j}$$

when $t = 1$, $v = 6\text{i} - 9\text{j}$

$$\text{speed} = \sqrt{6^2 + 9^2} = \sqrt{117} = 10.82\ \text{ms}^{-1}$$

Example 5

The position vector r of a particle at time t is given by

$$r = (2t^2 - 4t + 15)\text{i} + (4t - 7)\text{j}$$

Find the velocity of the particle at time t and the initial velocity.

At time $t = t_1$, the direction of motion is at right angles to the original direction of motion. Find t_1 and the distance of the particle from its initial position at this time.

Solution

$$r = (2t^2 - 4t + 15)\text{i} + (4t - 7)\text{j}$$
$$v = \frac{dr}{dt} = (4t - 4)\text{i} + 4\text{j}$$

Initial velocity v_0 is when $t = 0$;

$$v_0 = -4\text{i} + 4\text{j}$$

Velocity v_{t_1} at time t_1 is $(4t_1 - 4)\text{i} + 4\text{j}$

If these velocities are perpendicular, $v_0 \cdot v_{t_1} = 0$

$$-4(4t_1 - 4) + 4(4) = 0$$
$$-16t_1 + 16 + 16 = 0$$
$$t_1 = 2$$

When $t = 0$, $r_0 = 15\text{i} - 7\text{j}$

When $t = 2$, $r_2 = 15\text{i} + \text{j}$

Displacement of particle $= r_2 - r_0 = 8\text{j}$

$\therefore$ Distance from the initial position $= 8$ units

Example 6

A particle of mass 500 g moves under a force F so that its position vector after t seconds is given by

$$r = (1 + t + 5t^2 - 3t^3)\text{i} + (3 - 4t)\text{j}$$

Find F and the magnitude of the force after 3 seconds

Solution

$$r = (1 + t + 5t^2 - 3t^3)\text{i} + (3 - 4t)\text{j}$$
$$\dot{r} = (1 + 10t - 9t^2)\text{i} - 4\text{ij}$$
$$\ddot{r} = (10 - 18t)\text{i}$$

But force = mass × acceleration

$$F = 0.5(10 - 18t)\text{i} = (5 - 9t)\text{i}$$

When $t = 3$, $F = -22\text{i}$

$$|F| = 22\ N$$

$$\text{r} = \frac{3t^2}{2}\text{i} - \frac{t^2}{2}\text{j} + \text{i} - 4\text{j}$$

$$\text{r} = \left(\frac{3t^2}{2} + 1\right)\text{i} - \left(\frac{t^2}{2} + 4\right)\text{j}$$

At $t = 2$; $\text{r} = 7\text{i} - 6\text{j}$

The position vector of P relative to O when $t = 2$ is $7\text{i} - 6\text{j}$

Example 14

A body of mass 3 kg moves along a curve under the action of a resultant force F N. At time t seconds, the position vector r m of the body is $\text{r} = 2t\text{i} + t^3\text{j} + t^2\text{k}$.

(a) Find an expression for F in terms of t

(b) Find an expression for $P(t)$, the power of the force at time t seconds.

(c) Calculate the work done by F between $t = 0$ and $t = 2$

(d) Verify that the work done calculated in (c) is equal to the change in kinetic energy of the body over the same interval

Solution

(a) $\text{r} = 2t\text{i} + t^3\text{j} + t^2\text{k}$

$$\text{v} = \frac{\text{dr}}{\text{d}t} = 2\text{i} + 3t^2\text{j} + 2t\text{k}$$

$$\text{a} = \frac{\text{dv}}{\text{d}t} = 6t\text{j} + 2\text{k}$$

Using $\text{F} = ma$;

$$\text{F} = 3(6t\text{j} + 2\text{k}) = 18t\text{j} + 6\text{k}$$

(b) $P(t) = \text{F} \cdot \text{v}$

$$= (18t\text{j} + 6\text{k}) \cdot (2\text{i} + 3t^2\text{j} + 2t\text{k})$$

$$= 54t^3 + 12t$$

The power of the force at time t seconds is $(54t^3 + 12t)$ Watts

(c) Work done is

$$\int_0^2 (\text{F} \cdot \text{v})\, dt = \int_0^2 (54t^3 + 12t)\, dt$$

$$= [13.5t^4 + 6t^2]_0^2$$

$$= 240 \text{ J}$$

The work done is 240 J

(d) At time $t = 0$, $\text{v} = 2\text{i}$

$$|\text{v}| = 2$$

$$KE = \frac{1}{2}(3)(4) = 6 \text{ J}$$

At time $t = 2$, $\text{v} = 2\text{i} + 12\text{j} + 4\text{k}$

$$|\text{v}| = \sqrt{164}$$

$$KE = \frac{1}{2}(3)(164) = 246 \text{ J}$$

Change in $KE = 246 - 6 = 240$ J

Example 15

A particle of mass 4 kg starts from rest at a point $(2i - 3j + k)$m. It moves with acceleration $a = (4i + 2j - 3k)$ ms^{-2} when a constant force $\boldsymbol{F}$ acts on it. Find the

(a) force $\boldsymbol{F}$

(b) velocity at any time t

(c) work done by the force $\boldsymbol{F}$ after 6 seconds

Solution

(a) $F = ma$

$$F = 4(4i + 2j - 3k) = (16i + 8j - 12k)\ N$$

(b) $v = \int a\, dt$

$$v = \int (4i + 2j - 3k)\, dt$$

$$v = (4ti + 2tj - 3tk) + \text{c}$$

At $t = 0$, $v = 0$

$$c = 0$$

$$\therefore v = (4ti + 2tj - 3tk) \text{ ms}^{-1}$$

(c) $\text{r} = \int v\, dt$

$$\text{r} = \int (4ti + 2tj - 3tk)\, dt$$

$$\text{r} = 2t^2 i + t^2 j - \frac{3}{2}t^2 k + c$$

At $t = 0$, $\text{r} = (2i - 3j + k)$

$$c = (2i - 3j + k)$$

$$\text{r} = 2t^2 i + t^2 j - \frac{3}{2}t^2 k + (2i - 3j + k)$$

$$\text{r} = (2t^2 + 2)i + (t^2 - 3)j + \left(1 - \frac{3}{2}t^2\right)k$$

At $t = 6$, the displacement d is given by;

$$\text{d} = [2(36) + 2]i + (36 - 3)j + \left[1 - \frac{3}{2}(36)\right]k$$

$$\text{d} = 74i + 33j - 53k$$

The work done by a constant force is given by $\text{F} \cdot \text{d}$

Work done $= (16i + 8j - 12k) \cdot (74i + 33j - 53k)$

$$= 16 \times 74 + 8 \times 33 + 12 \times 53$$

$$= 2084 \text{ J}$$

Self-Evaluation exercise

1. A particle moves in the xy plane such that its displacement from O at time t is given by $\text{r} = 3t^2\text{i} + (4\text{t} - 6)\text{j}$. Find vector expressions for the velocity and acceleration of the particle when $t = 4$.

[Ans:$24\text{i} + 4\text{j}$; 6i]

2. A particle moves in the xy plane such that it has an acceleration a at time t where $\text{a} = 2\text{i} - \text{j}$. Initially the particle is at rest at the point whose position vector is $3\text{i} + \text{j}$. Find the position vector of the particle at time t.

[Ans: $(t^2 + 3)\text{i} + \left(1 - \frac{1}{2}t^2\right)\text{j}$]

3. A particle is moving so that at any instant its velocity vector, v, is given by $\text{v} = 3t\text{i} - 4\text{j} + t^2\text{k}$. When $t = 0$, it is the point $(1, 0, 1)$. Find the position vector when $t = 2$. Find also the magnitude of the acceleration when $t = 2$.

[Ans: $7\text{i} - 8\text{j} + \frac{11}{3}\text{k}$; 5]

4. A particle moves in the xy plane such that its velocity at time t is given by $\text{v} = 3t^2\text{i} + (t - 1)\text{j}$. Find the acceleration vector and position vector of the particle when $t = 3$ if, initially, the particle is at the origin.

[Ans: $18\text{i} + \text{j}$; $27\text{i} + \frac{2}{3}\text{j}$]

5. At time t, the position vector of a particle of mass m is $(\cos t)\text{i} + t^2\text{j}$. Find the resultant force acting on the particle when $t = \pi$. [Ans: $((\text{i} + 2\text{j})m$]

But $a = \frac{dv}{dt}$

$$v = \int a\,dt$$

$$v = \int -\left(3 + \frac{5}{(t+1)^2}\right)dt$$

$$v = -\left(3t - \frac{5}{t+1}\right) + c$$

When $t = 4$, $v = 0$

$$0 = -\left(12 - \frac{5}{5}\right) + c$$

$$c = 11$$

$$\therefore v = -\left(3t - \frac{5}{t+1}\right) + 11$$

When $t = 0$,

$$v = -(0 - 5) + 11 = 16\text{ ms}^{-1}$$

Example 3

A particle P of mass 0.5 kg is moving along the positive x-axis. At time t seconds, P is moving under the action of a single force of magnitude $[4 + \cos(\pi t)]N$, directed away from the origin. When $t = 1$, the particle P is moving away from the origin with speed 6 ms^{-1}. Find the speed of P when $t = 1.5$

<u>Solution</u>

$$F = ma$$

$$0.5a = 4 + \cos(\pi t)$$

$$0.5\frac{dv}{dt} = 4 + \cos(\pi t)$$

$$\int 0.5\,dv = \int 4 + \cos(\pi t)\,dt$$

$$0.5v = 4t + \frac{\sin(\pi t)}{\pi} + c$$

When $t = 1$, $v = 6$

$$3 = 4 + c \Rightarrow c = -1$$

When $t = 1.5$,

$$0.5v = 6 - \frac{1}{\pi} - 1$$

$$v = 9.36\text{ ms}^{-1}$$

Example 4

A particle P moves on the positive x-axis. When the distance of P from the origin O is x metres, the acceleration of P is $(7 - 2x)$ ms^{-2}, measured in the positive x-direction. When $t = 0$, P is at O and is moving in the positive x-direction with speed 6 ms^{-1}. Find the distance of P from O when P first comes to instantaneous rest.

<u>Solution</u>

$$a = 7 - 2x$$

$$v\frac{dv}{dx} = 7 - 2x$$

$$\int v\,dv = \int 7 - 2x\,dx$$

$$\frac{1}{2}v^2 = 7x - x^2 + c$$

When $x = 0$, $v = 6$

$$\frac{1}{2}(36) = c \Rightarrow c = 18$$

$$\therefore \frac{1}{2}v^2 = 7x - x^2 + 18$$

When $v = 0$;

$$x^2 - 7x - 18 = 0$$

$$x^2 + 2x - 9x - 18 = 0$$

$$(x + 2)(x - 9) = 0$$

$$x = -2 \text{ or } x = 9$$

The distance from O when P comes to rest is 9 m

Example 5

A particle P is moving in a straight line. At time t seconds, P is at a distance x metres from a fixed point O on the line and is moving away from O with speed $\frac{10}{x+6}$ ms^{-1}.

(a) Find the acceleration of P when $x = 14$

Given that $x = 2$ when $t = 1$,

(b) find the value of t when $x = 14$

<u>Solution</u>

(a) $a = v\frac{dv}{dx}$

$$a = \frac{10}{x+6} \times \frac{-10}{(x+6)^2} = \frac{-10}{(x+6)^3}$$

When $x = 14$;

$$a = \frac{-10}{(14+6)^3} = -\frac{1}{80}\text{ ms}^{-2}$$

(b) $v = \frac{dx}{dt}$

$$\frac{dx}{dt} = \frac{10}{x+6}$$

$$\int_2^{14} x + 6\ dx = \int_1^T 10\,dt$$

$$\left[\frac{x^2}{2} + 6x\right]_2^{14} = \left[10t\right]_1^T$$

$$\frac{196}{2} + 6(14) - [\,2 + 12] = 10T - 10$$

$$10T = 178$$

$$T = 17.8\text{ s}$$

Example 6

A particle P of mass 0.5 kg moves along the positive x-axis under the action of a single force of magnitude F newtons. The force acts along the x-axis in the direction of x increasing. When P is x metres from the origin O, it is moving away from O with speed $\sqrt{(8x^{\frac{3}{2}} - 4)}$ ms^{-1}. Find F when P is 4 m from O.

<u>Solution</u>

$$v = \sqrt{(8x^{\frac{3}{2}} - 4)}$$

$$v^2 = (8x^{\frac{3}{2}} - 4)$$

$$2\frac{dv}{dx} = \frac{3}{2}\left(8x^{\frac{1}{2}}\right)$$

$$\frac{dv}{dx} = 12x^{\frac{1}{2}}$$

$$F = ma = 0.5 \times 12x^{\frac{1}{2}} = 3x^{\frac{1}{2}}$$

When $x = 4$, $F = 3 \times 4^{\frac{1}{2}} = 12$

Example 7

A particle P of mass m kg slides from rest down a smooth plane inclined at 30° to the horizontal. When P has moved a distance x metres down the plane, the resistance to the motion of P from non-gravitational forces has magnitude mx^2 newtons. Find

(a) the speed of P when $x = 2$

(b) the distance P has moved when it comes to rest for the first time

<u>Solution</u>

(a) Equation of motion along the plane;

$$mg\sin 30° - mx^2 = ma$$

$$\frac{g}{2} - x^2 = v\frac{dv}{dx}$$

$$\int \frac{g}{2} - x^2\,dx = \int v\,dv$$

$$\frac{gx}{2} - \frac{x^3}{3} + c = \frac{v^2}{2}$$

When $x = 0$, $v = 0 \Rightarrow c = 0$

$$\therefore \frac{gx}{2} - \frac{x^3}{3} = \frac{v^2}{2}$$

When $x = 2$;

$$g - \frac{8}{3} = \frac{v^2}{2}$$

$$v = 3.78$$

(b) When $v = 0$;

$$\frac{gx}{2} - \frac{x^3}{3} = 0$$

$$x^2 = \frac{3g}{2}$$

$$x = \sqrt{\frac{3g}{2}} = 3.83 \text{ m}$$

Example 8

A particle P is moving along the positive x-axis. At time $t = 0$, P is at the origin O. At time t seconds, P is x metres from O and has velocity $v = 2e^{-x}$ ms^{-1}in the direction of x increasing.

(a) Find the acceleration of P in terms of x.

(b) Find x in terms of t.

<u>Solution</u>

(a) $$a = v\frac{dv}{dt}$$

$$a = 2e^{-x} \times -2e^{-x}$$

$$a = -4e^{-x}$$

(b) $v = \frac{dx}{dt}$

$$\frac{dx}{dt} = 2e^{-x}$$

$$e^x dx = 2dt$$

$$\int e^x dx = \int 2dt$$

$$e^x + c = 2t$$

When $t = 0$, $x = 0 \Rightarrow c = -1$

$$e^x - 1 = 2t$$

$$e^x = 2t + 1$$

$$x = \ln(2t + 1)$$

Self-Evaluation exercise

1. A particle P is moving along the positive x-axis. When the displacement of P from the origin is x metres, the velocity of P is v ms^{-1} and the acceleration of P is $9x$ ms^{-2}. When $x = 2$, $v = 6$. Show that $v^2 = 9x^2$.

2. A particle P of mass 0.6 kg is moving along the x-axis in the positive direction. At time $t = 0$, P passes through the origin O with speed 15 ms^{-1}. At time t seconds the distance OP is x metres, the speed of P is v ms^{-1} and the resultant force acting on P has magnitude $\frac{12}{(t+2)^2}$ newtons. The resultant force is directed towards O.
   (a) Show that $v = 5\left(\frac{4}{t+2} + 1\right)$
   (b) Find the value of x when $t = 5$

   [Ans: (b) 50.1]

3. A particle P of mass 3 kg is moving along the horizontal x-axis. At time $t = 0$, P passes through the origin O moving in the positive x direction. At time t seconds, $OP = x$ metres and the velocity of P is v ms^{-1}. At time t seconds, the resultant force acting on P is $\frac{9}{2}(26 - x)$ N, measured in the positive x direction. For $t > 0$ the maximum speed of P is 32 ms^{-1}. Find v^2 in terms of x.

   [Ans: $v^2 = 3\left(26x - \frac{1}{2}x^2\right) + 10$]

4. A particle P is moving in a straight line. At time t seconds, the distance of P from a fixed point O on the line is x metres and the acceleration of P is $(6 - 2t)$ ms^{-2} in the direction of x increasing. When $t = 0$, P is moving towards O with speed 8 ms^{-1}
   (a) Find the velocity of P in terms of t.
   (b) Find the total distance travelled by P in the first 4 seconds.

   [Ans: (a) $v = 6t - t^2 - 8$ (b) 8 m]

5. A particle P of mass 0.6 kg is moving along the positive x-axis in the positive direction. The only force acting on P acts in the direction of x increasing and has magnitude $\left(3t + \frac{1}{2}\right)$ N, where t seconds is the time after P leaves the origin O. When $t = 0$, P is at rest at O.
   (a) Find an expression, in terms of t, for the velocity of P at time t seconds.

   The particle passes through the point A with speed $\frac{10}{3}$ ms^{-1}
   (b) Find the distance OA.

   [Ans: (a) $v = \frac{5}{2}t^2 + \frac{5}{6}t$ (b) 1.2 m]

RELATIVE MOTION

Resultant velocity

Since velocity is a vector quantity with both magnitude and direction, two velocities can be combined to form a single velocity. Such velocities are usually met in problems of crossing a river by a boat or swimmer, aircraft moving in the inflence of wind, etc.

Example 1

Find in vector form, the resultant velocity of the following set of velocities:

$$(2i - 5j)\text{ms}^{-1}\text{ , }(3i + 7j)\text{ ms}^{-1}\text{, }(-6i - 8j)\text{ ms}^{-1}$$

<u>**Solution**</u>

$$v_1 = \begin{pmatrix} 2 \\ -5 \end{pmatrix}; v_2 = \begin{pmatrix} 3 \\ 7 \end{pmatrix}; v_3 = \begin{pmatrix} -6 \\ -8 \end{pmatrix}$$

Resultant velocity, $v = v_1 + v_2 + v_3$

$$v = \begin{pmatrix} 2 \\ -5 \end{pmatrix} + \begin{pmatrix} 3 \\ 7 \end{pmatrix} + \begin{pmatrix} -6 \\ -8 \end{pmatrix} = \begin{pmatrix} -1 \\ -6 \end{pmatrix}$$

The resultant velocity is $(-i - 6j)$ ms^{-1}

Example 2

If the resultant of $(ai + bj)$ kmh^{-1} and $(bi - aj)$ kmh^{-1} is $(10i - 4j)$ kmh^{-1}, find the values of a and b

<u>**Solution**</u>

$$v_1 = \begin{pmatrix} a \\ b \end{pmatrix}; v_1 = \begin{pmatrix} b \\ -a \end{pmatrix}; v = \begin{pmatrix} 10 \\ -4 \end{pmatrix}$$

$$v_1 + v_2 = v$$

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} b \\ -a \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} a + b \\ b - a \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \end{pmatrix}$$

$$a + b = 10 \text{ …. (i)}$$

$$b - a = -4 \text{ …..(ii)}$$

(i) + (ii);

$$2b = 6$$

$$b = 3$$

Substituting for b in (i);

$$a + 3 = 10$$

$$a = 7$$

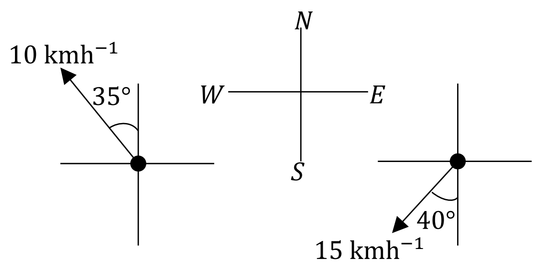
Example 3

Find by calculation the magnitude and direction of the resultant of these veocities

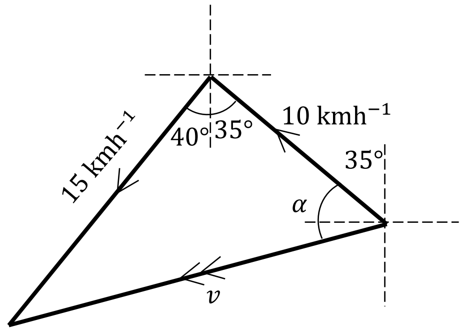
$$10\text{ kmh}^{-1}\ N\,35°\,W,\ 15\text{ kmh}^{-1}\ S\,40°\,W$$

<u>**Solution**</u>

Let us begin by showing the directions of the velocities using the compass direction as a frame of reference



Method 1: Velocity traingle



Using the cosine rule;

$$v^2 = 15^2 + 10^2 - 2(15)(10)\cos 75°$$

$$v^2 = 247.35$$

$$v = 15.73\text{ kmh}^{-1}$$

Using the sine rule;

$$\frac{15}{\sin\alpha} = \frac{v}{\sin 75°}$$

$$\sin\alpha = \frac{15\sin 75°}{15.73} = 0.921$$

$$\alpha = \sin^{-1} 0.921 = 67.1°$$

$$180 - (\alpha + 35°) = 180 - (67.1° + 35°) = 77.9°$$

Direction of the resultant velocity is $S\,77.9°\,W$

Method two: Vector method

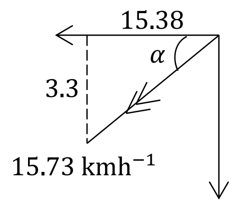
$$v_1 = \begin{pmatrix} -10\cos 55° \\ 10\sin 55° \end{pmatrix}; v_2 = \begin{pmatrix} -15\cos 50° \\ -15\sin 50° \end{pmatrix}; v = ?$$

$$v = v_1 + v_2$$

$$v = \begin{pmatrix} -10\cos 55° \\ 10\sin 55° \end{pmatrix} + \begin{pmatrix} -15\cos 50° \\ -15\sin 50° \end{pmatrix}$$

$$v = \begin{pmatrix} -15.38 \\ -3.3 \end{pmatrix}$$

$$|v| = \sqrt{(-15.38)^2 + (-3.3)^2} = \sqrt{247.43} = 15.73\text{ kmh}^{-1}$$



$$\tan\alpha = \frac{3.3}{15.38}$$

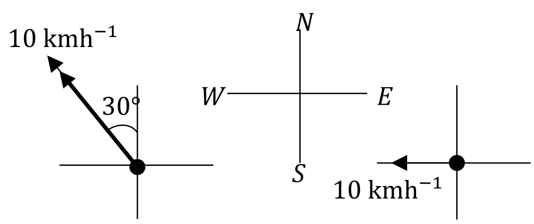
$$\alpha = 12.1°$$

The resultant force is 15.73 kmh^{-1} $S\,77.9°\,W$

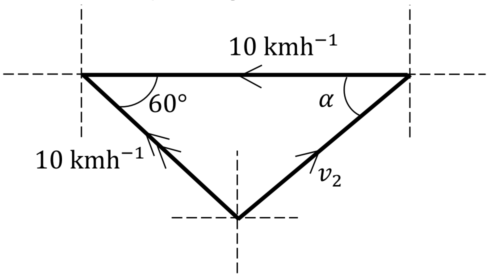
Example 4

The resultant of two velocities is a velocity of 10 kmh^{-1}, $N\,30°\,W$. If one of the velocities is 10 kmh^{-1} due west, find the magnitude and direction of the other velocity.

<u>**Solution**</u>



Method 1: Velocity triangle



Using the cosine rule;

$$v_2^2 = 10^2 + 10^2 - 2(10)(10)\cos 60°$$
$$v_2^2 = 100$$
$$v_2 = 10\text{ kmh}^{-1}$$

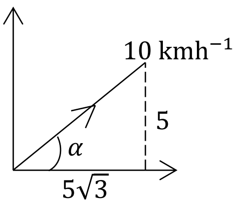
Using the sine rule;

$$\frac{10}{\sin\alpha} = \frac{v_2}{\sin 60°}$$
$$\sin\alpha = \frac{10\sin 60°}{10} = \sin 60°$$
$$\alpha = 60°$$

The other force is of magnitude 10 kmh^{-1} in the direction $N\ 30°\ E$

Method 2: Vector method

$$v_1 = \begin{pmatrix}-10\\0\end{pmatrix};\ ,\ v_2 = ?;\ v = \begin{pmatrix}-10\cos 60°\\10\sin 60°\end{pmatrix}$$
$$v = v_1 + v_2$$
$$v_2 = v - v_1 = \begin{pmatrix}-10\cos 60°\\10\sin 60°\end{pmatrix} - \begin{pmatrix}-10\\0\end{pmatrix}$$
$$v_2 = \begin{pmatrix}5\\5\sqrt{3}\end{pmatrix}$$
$$|v_2| = \sqrt{5^2 + \left(5\sqrt{3}\right)^2} = \sqrt{100}$$
$$|v_2| = 10\text{ kmh}^{-1}$$



$$\tan\alpha = \frac{5}{5\sqrt{3}} = \frac{1}{\sqrt{3}}$$
$$\alpha = 60°$$

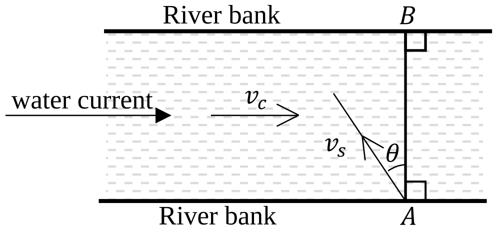
The magnitude of the other force is 10 kmh^{-1} in the direction $N\ 30°\ E$

Crossing a river

Consider a problem of crossing a river from a point on one bank to a point on the other bank. Assuming that the banks are parallel, we shall consider three cases.

Case 1:

If one is to cross from point A on one bank to a point B directly opposite to A on the other bank, course set by the boat or swimmer must be upstream.



In the figure, $\overrightarrow{v_c}$ is the velocity of water current and $\overrightarrow{v_s}$ the velocity of the boat or swimmer in still water.

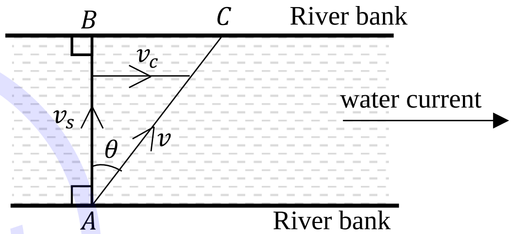
Resolving velocities; $v_s\sin\theta = v_c$

The resultant velocity in crossing the river is along AB and is given; $v_s\cos\theta$

It is instructive to note that in this case the boat or swimmer crosses the river by the shortest distance.

Case 2:

If the river is to be crossed by the shortest time possible (as quickly as possible), the course of the boat or swimmer is directly across (perpendicular) to the river bank, such that the water current carries the boat or swimmer down stream.



Resultant velocity,

$$v = \sqrt{{v_s}^2 + {v_c}^2}$$

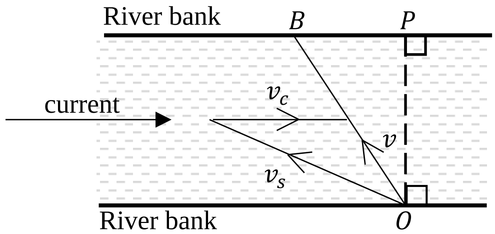
This resultant velocity is at an angle θ to the horizontal, where $\tan\theta = \frac{v_c}{v_s}$

Time taken to cross the river = $\frac{\overline{AB}}{v_s} = \frac{\overline{AC}}{v}$

The distance $\overline{BC}$, which the boat or swimmer moves down stream = $v_c \times t$

Case 3:

In order to cross the river and reach a point B on the other bank, the course set must be in such a direction that the resultant velocity of the boat is in the direction OB



$$v = v_c + v_s$$

Example 1

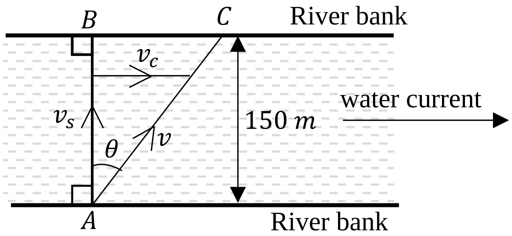
A boy who can swim at 2 ms^{-1} in still water wishes to swim across a river, 150 m wide. If the river flows at 1.5 ms^{-1}, find

(i) the time the boy takes for the crossing and how far downstream he travels, if he is to cross the river as quickly as possible.

(ii) the course that he must set in order to cross to a point exactly opposite the starting point, and the time taken for the crossing

Solution

(i)



$v_c = 1.5$ ms^{-1}, $v_s = 2$ ms^{-1}

Time taken = $\frac{\overline{AB}}{v_s} = \frac{150}{2} = 75\ s$

Distance down stream = $v_c \times t = 1.5 \times 75$

$= 112.5\ m$

(ii) Let the course be set at an angle θ

$v_s \sin\theta = v_c \Rightarrow \sin\theta = \frac{1.5}{3}$

$\therefore \theta = 48.6°$

The course set is therefore upstream at an angle of $(90 - 48.6) = 41.4°$ to the river bank

Speed across the river $= v_s \cos\theta = 2\cos 48.6°$

Time taken $= \frac{150}{2\cos 48.6} = 113.41\ s$

Example 2

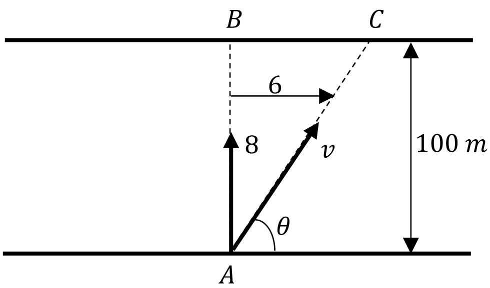
A boat is rowed with a velocity of 8 kmh^{-1} straight across a river which is flowing at 6 kmh^{-1}.

(a) Find the magnitude and direction of the resultant velocity of the boat

(b) If the river is 100 m wide, find how far down the river the boat will reach the opposite end

Solution

The component velocities of the boat are 8 kmh^{-1} and 6 kmh^{-1} at right angles.



(a)

Let v = resultant velocity

$v = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$ kmh^{-1}

Let θ be the direction this velocity makes with the bank.

$\tan\theta = \frac{8}{6}$

$\theta = \tan^{-1}\left(\frac{8}{6}\right) = 53.13°$

(b) If A is the point from which the boat starts and B is the point directly opposite on the other bank, C will be the point where the boat reaches the opposite bank.

$\tan\theta = \frac{BA}{BC}$

$BC = \frac{BA}{\tan\theta} = \frac{100}{\frac{8}{6}} = 75\ m$

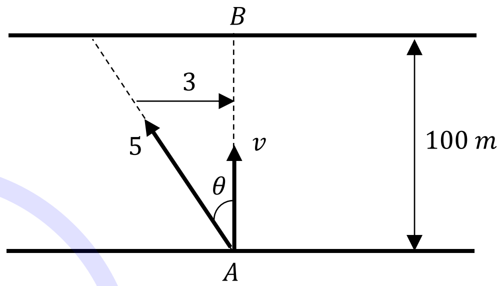
Hence the boat will be carried downstream a distance of $75\ m$.

Example 3

A stream is running at 3 kmh^{-1} and its width is 100 m. If a man can row a boat at 5 kmh^{-1}, find the direction in which he must row in order to go straight across the stream and the time it takes him to cross.

Solution

Let A be the point from which the man starts and AB is perpendicular to the banks.



Resultant velocity is along AB,

$v = \sqrt{5^2 - 3^2} = 4$ kmh^{-1}

$\cos\theta = \frac{4}{5}$

$\theta = \cos^{-1}\frac{4}{5} = 36.87°$

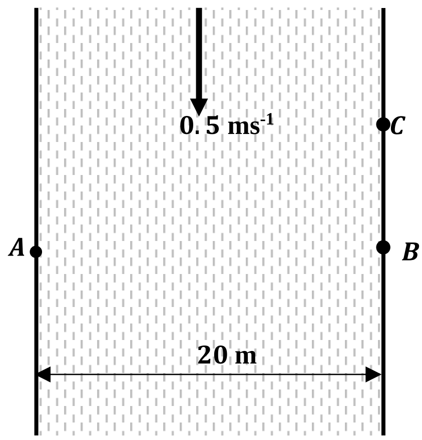
Direction is 36.87° to AB

Time taken to cross $= \frac{\text{distance}}{\text{velocity}}$

velocity $= 4$ kmh^{-1} $= \frac{4000}{3600} = \frac{10}{9}$ ms^{-1}

Time taken $= \frac{100}{\frac{10}{9}} = 90\ s$

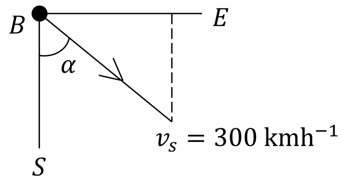
Example 4



A river is 20 m wide. A woman can swim at 2 ms^{-1} in still water. The current in the river flows at 0.5 ms^{-1}

(a) The woman swims at right angles to the river bank, starting at A.

(b) Let the pilot set the course in the direction $S\ \alpha°\ E$



The direction of the resultant velocity v must be in the direction BA

$$v = v_s + v_w$$

$$\begin{pmatrix} 0 \\ -v \end{pmatrix} = \begin{pmatrix} 300\sin\alpha \\ -300\cos\alpha \end{pmatrix} + \begin{pmatrix} -50\cos 30° \\ -50\sin 30° \end{pmatrix}$$

$$300\sin\alpha = 50\cos 30°$$

$$\sin\alpha = \frac{50\cos 30°}{300} = 0.144$$

$$\alpha = 8.3°$$

$$-v = -300\cos\alpha - 50\sin 30°$$

$$v = 300\cos 8.3° + 25 = 321.85$$

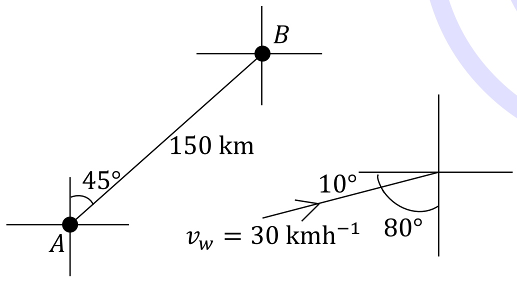
$$|v| = \sqrt{0^2 + (-321.85)^2} = 321.85\ \text{kmh}^{-1}$$

$$\text{Time} = \frac{\text{Distance}}{\text{speed}} = \frac{500}{321.85} = 1.55\ \text{h}$$

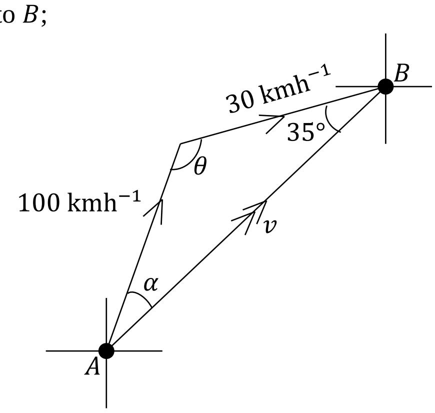
Example 7

Two heliports A and B are 150 km apart with B on a bearing 045° from A. A wind of 30 kmh^{-1} is blowing from a direction 260°. Assuming this wind remains constant throughout, find the time required for a helicopter to fly from A to B and back to A again, if the helicopter can fly at 100 kmh^{-1} in still air.

Solution



From A to B;



$$\frac{30}{\sin\alpha} = \frac{100}{\sin 35°}$$

$$\sin\alpha = \frac{30\sin 35°}{100} = 0.172$$

$$\alpha = 9.9°$$

$$\theta = 180° - (\alpha + 35°) = 180° - (9.9° + 35°) = 135.1°$$

$$v^2 = 30^2 + 100^2 - 2(30)(100)\cos 135.1°$$

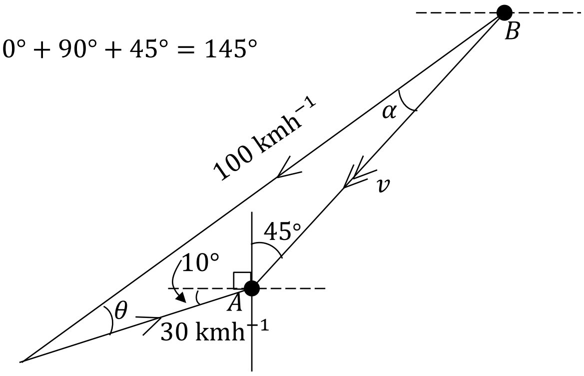
$$v = 123.09\ \text{kmh}^{-1}$$

Time from A to B is given by

$$T_{AB} = \frac{\text{Distance}}{\text{speed}} = \frac{150}{123.09} = 1.22\ \text{h}$$

From B to A;

$$10° + 90° + 45° = 145°$$



$$\frac{30}{\sin\alpha} = \frac{100}{\sin 145°}$$

$$\sin\alpha = \frac{30\sin 145°}{100} = 0.172$$

$$\alpha = 9.9°$$

$$\theta = 180° - (145° + \alpha) = 180° - (145° + 9.9°) = 25.1°$$

$$v^2 = 30^2 + 100^2 - 2(30)(100)\cos 25.1°$$

$$v^2 = 5466.59$$

$$v = 73.9\ \text{kmh}^{-1}$$

Time from B to A is given by

$$T_{BA} = \frac{\text{Distance}}{\text{speed}} = \frac{150}{73.9} = 2.03\ \text{h}$$

Total time from A to B and B to A again is given by

$$T = T_{AB} + T_{BA} = 1.22 + 2.03 = 3.25\ \text{h}$$

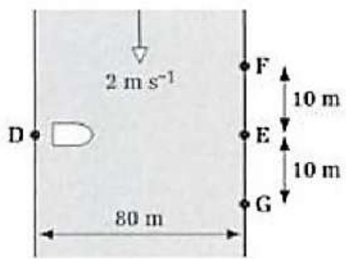
Therefore, the total time required is 3.25 hours or 3 hours and 15 minutes.

Self-Evaluation exercise

1. Calculate the magnitude and the direction of the resultant of the velocities 8 kmh^{-1} in a direction $N\ 80°\ W$ and 5 kmh^{-1} in a direction $S\ 25°\ W$

   [Ans: 10.5 kmh^{-1} in a direction $S\ 72.53°\ W$]

2. A boat can be rowed at a steady speed of 5 ms^{-1} in still water. The current in the river flows at 2 ms^{-1}.



   (a) The boat is rowed directly towards E.
      (i) How long will the boat take to cross the river?
      (ii) How far from E will the boat be when it reaches at the other side?

11. A man swims in a straight line due west from the end of a pier A to a buoy B, a distance of 2 km, and then back to the pier. The constant speed of the swimmer is $2\sqrt{3}$ kmh^{-1} in still water, and he experiences a current of 2 kmh^{-1} from 30° west of north. Find the directions in which the man must head on the outward and return journeys and the total time taken

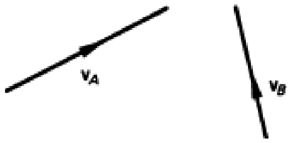
[Ans: $N\ 60°\ W$; $N\ 60°\ E$; 1 hour 30 minutes]

Relative velocity

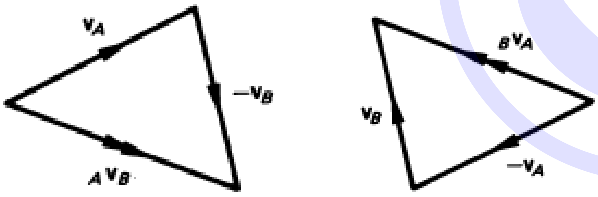
For a person in a moving vehicle, trees and buildings near the road appear to be moving in the opposite direction, yet those that are far appear to be moving in the same direction. If a car B moving with velocity v_B overtakes another car A moving with a velocity v_A, the passenger in car A sees car B apparently moving towards him. However, the person in car B sees a gradual catching up. In this case, car A appears to be stationary as B overtakes it .

The velocity car B appears to have to an observer in car A is called the velocity of B relative to A.

Therefore the velocity of B relative to A is the resultant velocity of B when A appears stationary.



The velocities of A and B can be represented on a velocity triangle as shown below



The velocity of A relative to B is given by

$$_A\boldsymbol{v}_B = \boldsymbol{v}_A - \boldsymbol{v}_B$$

The velocity of B relative to A is given by

$$_B\boldsymbol{v}_A = \boldsymbol{v}_B - \boldsymbol{v}_A$$

Hence, $_Av_B = -\,_Bv_A$

For three moving bodies A, B and C,

$$_A\boldsymbol{v}_B = \,_A\boldsymbol{v}_C - \,_B\boldsymbol{v}_C \text{ or } _A\boldsymbol{v}_B = \,_A\boldsymbol{v}_C + \,_C\boldsymbol{v}_B$$

Example 1

Joe rides his horse with velocity $\begin{pmatrix}6\\24\end{pmatrix}$ kmh^{-1} while Kim is riding her horse with velocity $\begin{pmatrix}5\\12\end{pmatrix}$ kmh^{-1}.

(a) Find Joe's velocity as seen by Kim

(b) What is Kim's velocity as seen by Joe?

Solution

$$v_J = \begin{pmatrix}6\\24\end{pmatrix};\ v_K = \begin{pmatrix}5\\12\end{pmatrix}$$

(a) $_Jv_K = v_J - v_K$

$$_Jv_K = \begin{pmatrix}6\\24\end{pmatrix} - \begin{pmatrix}5\\12\end{pmatrix} = \begin{pmatrix}1\\12\end{pmatrix} \text{ kmh}^{-1}$$

(b) $_Kv_J = v_K - v_J$

$$_Kv_J = \begin{pmatrix}5\\12\end{pmatrix} - \begin{pmatrix}6\\24\end{pmatrix} = \begin{pmatrix}-1\\-12\end{pmatrix} \text{ kmh}^{-1}$$

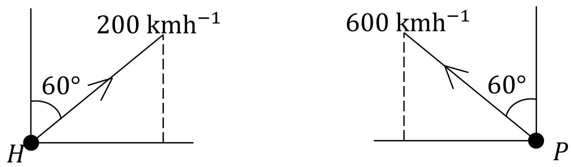
Alternatively;

$$_Kv_J = -\,_Jv_K = -\begin{pmatrix}1\\12\end{pmatrix} = \begin{pmatrix}-1\\-12\end{pmatrix} \text{ kmh}^{-1}$$

Example 2

A plane P flies at 600 kmh^{-1} on a bearing of 300° and a helicopter H flies at 200 kmh^{-1} on a bearing of 060°. Calculate the velocity of the plane relative to the helicopter.

Solution



Let us resolve the velocities in two components along the i and j directions.

$$v_P = \begin{pmatrix}-600\sin 60°\\600\cos 60°\end{pmatrix} = \begin{pmatrix}-519.6\\300\end{pmatrix}$$

$$v_H = \begin{pmatrix}200\sin 60°\\200\cos 60°\end{pmatrix} = \begin{pmatrix}173.2\\100\end{pmatrix}$$

$$_Pv_H = v_P - v_H$$

$$_Pv_H = \begin{pmatrix}-519.6\\300\end{pmatrix} - \begin{pmatrix}173.2\\100\end{pmatrix} = \begin{pmatrix}-692.8\\200\end{pmatrix}$$

$$_Pv_H = -692.8\text{i} + 200\text{j}$$

$$\left|\,_Pv_H\right| = \sqrt{(-692.8)^2 + (200)^2} = 721.1 \text{ kmh}^{-1}$$

True velocity

Suppose v_A, the true velocity of A, is known, and $_Av_B$, the velocity of A relative to B, is also known, then by using $_Av_B = v_A - v_B$, the true velocity of B can be found.

Example 1

To a man standing on the deck of a ship which is moving with a velocity of $(-6\text{i} + 8\text{j})$ kmh^{-1}, the wind seems to have a velocity of $(7\text{i} - 5\text{j})$ kmh^{-1}. Find the true velocity of the wind.

Solution

Let v_m and v_w be the velocities of the man and wind respectively, and $_wv_m$ be the velocity of the wind relative (as it appears) to the man

$_wv_m = \begin{pmatrix}7\\-5\end{pmatrix}$; $v_m = \begin{pmatrix}-6\\8\end{pmatrix}$; $v_w = ?$

$$_wv_m = v_w - v_m$$

$$v_w = \,_wv_m + v_m$$

$$v_w = \begin{pmatrix}7\\-5\end{pmatrix} + \begin{pmatrix}-6\\8\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix}$$

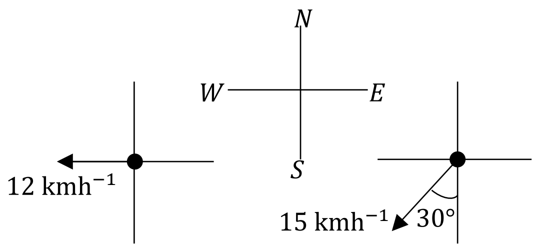
The true velocity of the wind is $(\text{i} + 3\text{j})$ kmh^{-1}

Example 2

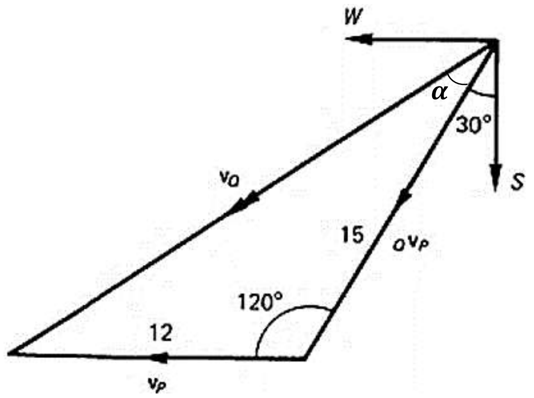
A ship P is moving due west at 12 kmh^{-1}. The velocity of a second ship Q relative to P is 15 kmh^{-1} in a direction 30° west of south. Find the velocity of ship Q

Solution

Let the velocities of P and Q be v_P and v_Q respectively, and the velocity of Q relative to P be ${}_Qv_P$.



Method 1: Velocity triangle



Using the cosine rule;

$$v_Q^2 = 12^2 + 15^2 - 2(12)(15)\cos 120°$$
$$v_Q^2 = 549$$
$$v_Q = 23.4 \text{ kmh}^{-1}$$

Using the sine rule;

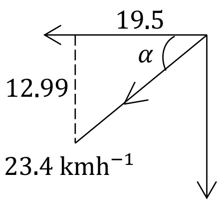
$$\frac{\sin\alpha}{12} = \frac{\sin 120°}{23.4}$$
$$\sin\alpha = \frac{12\sin 120°}{23.4}$$
$$\alpha = 26.3°$$

Hence the direction of motion of Q is $S\ 56.3°\ W$

Method 2: Vectors

$v_P = \begin{pmatrix}-12\\0\end{pmatrix}$; ${}_Qv_P = \begin{pmatrix}-15\cos 60°\\-15\sin 60°\end{pmatrix}$; $v_Q = ?$

$${}_Qv_P = v_Q - v_P$$
$$v_Q = {}_Qv_P + v_P$$
$$v_Q = \begin{pmatrix}-12\\0\end{pmatrix} + \begin{pmatrix}-15\cos 60°\\-15\sin 60°\end{pmatrix}$$
$$v_Q = \begin{pmatrix}-19.5\\-12.99\end{pmatrix}$$
$$|v_Q| = \sqrt{(-19.5)^2 + (-12.99)^2} = 23.4 \text{ kmh}^{-1}$$



$$\tan\alpha = \frac{12.99}{19.5} = 0.666$$
$$\alpha = 33.7°$$

Direction is $S\ 56.3°\ W$

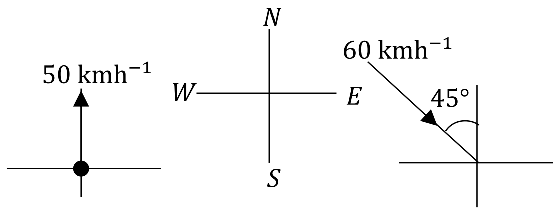
Example 3

To a motor-cyclist travelling due north at 50 kmh^{-1}, the wind appears to come from north west at 60 kmh^{-1}, what is the true velocity of the wind?

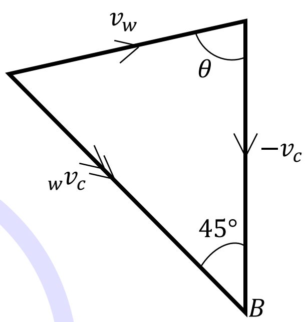
Solution

Let v_w and v_c represent the velocities of the wind and cyclist respectively, and ${}_wv_c$ the velocity of the wind relative to the cyclist

$v_c = 50\text{ kmh}^{-1}$, ${}_wv_c = 60\text{ kmh}^{-1}$, $v_w = ?$



Method 1: Velocity triangle



By the cosine rule;

$$v_w^2 = 60^2 + 50^2 - 2(50)(60)\cos 45°$$
$$v_w = 43.1 \text{ kmh}^{-1}$$

By the sine rule;

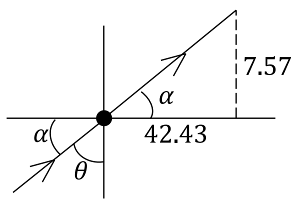
$$\frac{\sin\theta}{60} = \frac{\sin 45°}{43.1}$$
$$\sin\theta = \frac{60\sin 45°}{43.1}$$
$$\theta = 79.9°$$

∴ The true velocity of wind is 43.1 kmh^{-1} from $S\ 79.9°\ W$

Method 2: Vectors

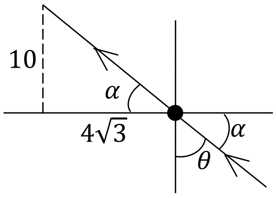
$v_c = \begin{pmatrix}0\\50\end{pmatrix}$; ${}_wv_c = \begin{pmatrix}60\cos 45°\\-60\sin 45°\end{pmatrix}$; $v_w = ?$

$${}_wv_c = v_w - v_c$$
$$v_w = {}_wv_c + v_c$$
$$v_w = \begin{pmatrix}60\cos 45°\\-60\sin 45°\end{pmatrix} + \begin{pmatrix}0\\50\end{pmatrix}$$
$$v_w = \begin{pmatrix}42.43\\7.57\end{pmatrix}$$
$$|v_w| = \sqrt{42.43^2 + 7.57^2} = 43.1 \text{ kmh}^{-1}$$



$$\tan\alpha = \frac{7.57}{42.43} = 0.178$$

$$|v_w| = \sqrt{\left(-4\sqrt{3}\right)^2 + 10^2} = \sqrt{148} = 12.17 \text{ kmh}^{-1}$$



$$\tan\alpha = \frac{10}{4\sqrt{3}} = 1.443$$

$$\alpha = \tan^{-1} 1.443 = 55.3°$$

$$\theta = 90° - \alpha = 90° - 55.3° = 34.7°$$

The true velocity of the wind is 12.2 kmh^{-1} blowing from $S\ 34.7°\ E$

Example 7

To a cyclist riding due north at 3 ms^{-1}, the wind appears to be blowing from the East. If the cyclist doubles his speed but does not change his direction, the wind appears to be blowing from $N\ 60°\ E$.

(a) Find the true wind speed and direction

(b) The cyclist now turns around and cycles due south at 3 ms^{-1}. Calculate the apparent wind direction

Solution

(a) When the cyclist rides at 3 ms^{-1}, let the magnitude of the apparent speed of wind be v

$$v_c = \begin{pmatrix} 0 \\ 3 \end{pmatrix};\ {}_wv_c = \begin{pmatrix} -v \\ 0 \end{pmatrix};\ v_w = ?$$

$${}_wv_c = v_w - v_c$$

$$v_w = {}_wv_c + v_c = \begin{pmatrix} -v \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -v \\ 3 \end{pmatrix}$$

When the cyclist doubles his speed i.e. to 6 ms^{-1}, let the magnitude of the apparent speed of wind be v'

$$v_c' = \begin{pmatrix} 0 \\ 6 \end{pmatrix};\ {}_wv_c' = \begin{pmatrix} -v'\sin 60° \\ -v'\cos 60° \end{pmatrix};\ v_w = ?$$

$${}_wv_c' = v_w - v_c'$$

$$v_w = {}_wv'_c + v_c'$$

$$v_w = \begin{pmatrix} -v'\sin 60° \\ -v'\cos 60° \end{pmatrix} + \begin{pmatrix} 0 \\ 6 \end{pmatrix} = \begin{pmatrix} -v'\sin 60° \\ 6 - \frac{1}{2}v' \end{pmatrix}$$

Since the true speed of wind does not change,

$$\begin{pmatrix} -v \\ 3 \end{pmatrix} = \begin{pmatrix} -v'\sin 60° \\ 6 - \frac{1}{2}v' \end{pmatrix}$$

$$-v = -v'\sin 60°$$

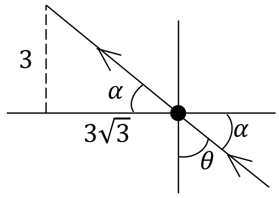
$$3 = 6 - \frac{1}{2}v'$$

$$v' = 6$$

$$v = v'\sin 60° = \frac{6\sqrt{3}}{2} = 3\sqrt{3}$$

$$\therefore v_w = \begin{pmatrix} -3\sqrt{3} \\ 3 \end{pmatrix}$$

$$|v_w| = \sqrt{\left(-3\sqrt{3}\right)^2 + 3^2} = \sqrt{36} = 6 \text{ ms}^{-1}$$



$$\tan\alpha = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\alpha = 30°$$

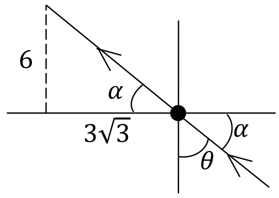
$$\theta = 90° - \alpha = 90° - 30° = 60°$$

The true wind speed is 6 ms^{-1} blowing from $S\ 60°\ E$

(b) When cyclist cycles dues south at 3 ms^{-1},

$$v_c = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

$${}_wv_c = v_w - v_c = \begin{pmatrix} -3\sqrt{3} \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ -3 \end{pmatrix} = \begin{pmatrix} -3\sqrt{3} \\ 6 \end{pmatrix}$$



$$\tan\alpha = \frac{6}{3\sqrt{3}} = \frac{2}{\sqrt{3}} = 1.155$$

$$\alpha = 49.1°$$

$$\theta = 90° - \alpha = 90° - 49.1° = 40.9°$$

The apparent wind direction is from $S\ 40.9°\ E$

Relative displacement

The displacement of A relative to B, is given by

$${}_Ar_B = r_A - r_B$$

The displacement of B relative to A, is given by

$${}_Br_A = r_B - r_A$$

Distance between A and $B = \left|{}_Ar_B\right| = \left|{}_Br_A\right|$

At time t, the respective displacements of A and B are given by

$$r_A(t) = r_A + v_At$$

$$r_B(t) = r_B + v_Bt$$

where r_A and r_B are the initial positions of A and B respectively

The displacement of A relative to B at time t is given by

$${}_Ar_B(t) = r_A(t) - r_B(t) = r_A + v_At - (r_B + v_Bt)$$

$${}_Ar_B(t) = r_A - r_B + (v_A - v_B)t$$

The displacement of B relative to A at time t is given by

$${}_Br_A(t) = r_B - r_A + (v_B - v_A)t$$

Least distance occurs when ${}_Ar_B.\ {}_Av_B = 0$ (or when $\left|{}_Ar_B\right|^2$ is a minimum)

Collision will occur if ${}_Av_B = k\ {}_Ar_B$, provided ${}_Av_B$ is a constant vector or if $r_A(t) = r_B(t)$

Example 1

Points A and B have position vectors of $2\text{i} + 3\text{j} - 5\text{k}$ and $(6\text{i} - \text{j} + 2\text{k})$ respectively. Determine

(a) ${}_Ar_B$

(b) the distance AB

Solution

(a) $r_A = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}; r_B = \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix}$

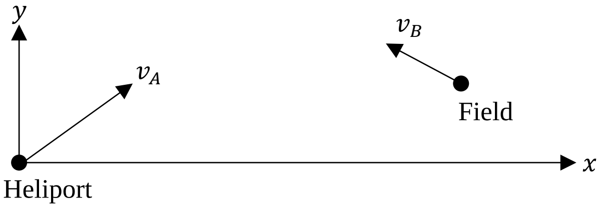
$$ {}_Ar_B = r_A - r_B = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} - \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ -7 \end{pmatrix} $$

$$ {}_Ar_B = -4\text{i} + 4\text{j} - 7\text{j} $$

(b) Distance $AB = |{}_Ar_B| = \sqrt{(-4)^2 + 4^2 + (-7)^2}$

$$ = \sqrt{16 + 16 + 49} = 9 \text{ units} $$

Example 2



A helicopter A leaves a heliport and flies with velocity $(10\text{i} + 4\text{j})\ \text{ms}^{-1}$. At the same time another helicopter B takes off from a field whose position vector relative to the heliport is $(36\text{i} + 2\text{j})$ m. The velocity of B is $(-8\text{i} + 3\text{j})\ \text{ms}^{-1}$

(a) Find, after t seconds,

(i) the position vector of A relative to the heliport

(ii) the position vector of B relative to the heliport

(iii) the displacement of B from A

(b) Explain what happens when $t = 2$

Solution

$$ v_A = \begin{pmatrix} 10 \\ 4 \end{pmatrix};\ r_A = \begin{pmatrix} 0 \\ 0 \end{pmatrix} $$

$$ v_B = \begin{pmatrix} -8 \\ 3 \end{pmatrix};\ r_B = \begin{pmatrix} 36 \\ 2 \end{pmatrix} $$

(a)

(i) $r_A(t) = r_A + v_At$

$$ = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 10 \\ 14 \end{pmatrix}t = \begin{pmatrix} 10t \\ 14t \end{pmatrix} $$

$$ = 10t\text{i} + 14t\text{j} \text{ m} $$

(ii) $r_B(t) = r_B + v_Bt$

$$ = \begin{pmatrix} 36 \\ 2 \end{pmatrix} + \begin{pmatrix} -8 \\ 3 \end{pmatrix}t = \begin{pmatrix} 36 - 8t \\ 2 + 3t \end{pmatrix} $$

$$ = (36 - 8t)\text{i} + (2 + 3t)\text{j} \text{ m} $$

(iii) ${}_Br_A(t) = r_B(t) - r_A(t)$

$$ = \begin{pmatrix} 36 - 8t \\ 2 + 3t \end{pmatrix} - \begin{pmatrix} 10t \\ 14t \end{pmatrix} = \begin{pmatrix} 36 - 18t \\ 2 - t \end{pmatrix} $$

$$ |{}_Br_A(t)| = \sqrt{(36 - 8t)^2 + (2 - t)^2} $$

$$ = \sqrt{1300 - 1300t + 325t^2} $$

(b) When $t = 2$;

$$ |{}_Br_A| = \sqrt{1300 - 1300(2) + 325(2)^2} = 0 $$

Since the displacement between the helicopters is 0, they collide

Self-Evaluation exercise

1. Bird A has a velocity of $(7\text{i} - 3\text{j} + 10\text{k})\ \text{ms}^{-1}$ and bird B has a velocity of $(6\text{i} - 17\text{k})\ \text{ms}^{-1}$, find the velocity of B relative to A.
[Ans: $(-\text{i} + 3\text{j} - 27\text{k})\ \text{ms}^{-1}$]

2. The body of particle A is $2(\text{i} - \text{j})\ \text{ms}^{-1}$ relative to particle B which moves with velocity $-4(\text{i} + \text{j})\ \text{ms}^{-1}$. Find the speeds of both particles
[Ans: speed of A $= 6.32\ \text{ms}^{-1}$; of $B = 5.66\ \text{ms}^{-1}$]

3. A particle with position vector $40\text{i} + 10\text{j} + 20\text{k}$ moves with a constant speed $5\ \text{ms}^{-1}$ in the direction of the vector $4\text{i} + 7\text{j} + 4\text{k}$. Find its distance from the origin after 9 seconds [Ans: 85 m]

4. A car is being driven at $20\ \text{ms}^{-1}$ on a bearing 040°. The wind is blowing from 330° with a speed of $10\ \text{ms}^{-1}$. Find the velocity of the wind as experienced by the driver of the car
[Ans: $25.2\ \text{ms}^{-1}$ from 018.1°]

5. A, B and C are three aircraft. A has a velocity $(200\text{i} + 170\text{j})\ \text{ms}^{-1}$. To the pilot of A, it appears that B has velocity $(50\text{i} - 270\text{j})\ \text{ms}^{-1}$. To the pilot of B it appears that C has a velocity $(50\text{i} + 170\text{j})\ \text{ms}^{-1}$. Find, in vector form, the velocities of B and C
[Ans: $(250\text{i} - 100\text{j})\ \text{ms}^{-1}$; $(300\text{i} + 70\text{j})\ \text{ms}^{-1}$]

6. An aircraft is moving at $250\ \text{kmh}^{-1}$ in a direction $N\ 60°\ E$. A second aircraft is moving at $200\ \text{kmh}^{-1}$ in a direction $N\ 20°\ W$. Find the velocity of the first aircraft as seen by the pilot of the second aircraft
[Ans: $292\ \text{kmh}^{-1}$ $S\ 77.9°\ E$]

7. To the pilot of an aircraft A, travelling at $300\ \text{kmh}^{-1}$ due south, it appears that an aircraft B is travelling at $600\ \text{kmh}^{-1}$ in a direction $N\ 60°\ W$. Find the true speed and direction of aircraft B
[Ans: $520\ \text{kmh}^{-1}$ west]

8. Jane is riding her horse at $5\ \text{kmh}^{-1}$ due north and sees Sasha riding her horse apparently with a velocity $4\ \text{kmh}^{-1}$, $N\ 60°\ E$. Find Sasha's true velocity
[Ans: $7.81\ \text{kmh}^{-1}$, $N\ 26.3°\ E$]

9. To a passenger on a boat which is travelling at $20\ \text{kmh}^{-1}$ on a bearing of 230°, the wind seems to be blowing from 250° at $12\ \text{kmh}^{-1}$. Find the true velocity of the wind. [Ans: $9.64\ \text{kmh}^{-1}$ from 024.8°]

10. To a jogger jogging at $12\ \text{kmh}^{-1}$ in a direction $N\ 10°\ E$, the wind seems to come from a direction $N\ 20°\ W$ at $15\ \text{kmh}^{-1}$. Find the true velocity of the wind
[Ans: $7.57\ \text{kmh}^{-1}$ from $N\ 72.5°\ W$]

11. A man walks with constant speed $6\ \text{kmh}^{-1}$ due north and to him, the wind appears to have a velocity $u_1(\sqrt{3}\text{i} - 3\text{j})\ \text{kmh}^{-1}$. Without changing speed, the man alters course so that he is walking in the direction of the vector $-\frac{\sqrt{3}}{2}\text{i} + \frac{1}{2}\text{j}$ and the velocity of the wind now appears to him to be u_2 i kmh^{-1}. Find u_1, u_2 and hence the actual velocity of the wind

[Ans: $u_1 = 1$; $u_2 = 4\sqrt{3}$; $(\sqrt{3}\mathrm{i} + 3\mathrm{j})$ kmh^{-1}]

12. To a bird flying due east at 10 ms^{-1}, the wind seems to come from the south. When the bird alters its direction of flight to $N\ 30°\ E$ without altering its speed, the wind seems to come from the north-west. Find the true velocity of the wind

[Ans: 10.6 ms^{-1} from $S\ 69.9°\ W$]

13. To an observer on a trawler moving at 12 kmh^{-1} in a direction $S\ 30°\ W$, the wind appears to come from $N\ 60°\ W$. To an observer on a ferry moving at 15 kmh^{-1} in a direction $S\ 80°\ E$, the wind appears to come from the north. Find the true velocity of the wind.

[Ans: 26.8 kmh^{-1} from $N\ 33.4°\ W$]

Interception and collision

Two bodies A and B, moving with constant velocities v_A and v_B respectively, will reach either a position of **interception** or of **closest approach**.

When solving such relative motion problems:

(a) Draw an initial sketch using the given information

(b) Imagine yourself to be on one of the bodies, B say
In an interception problem, B should be the body being intercepted

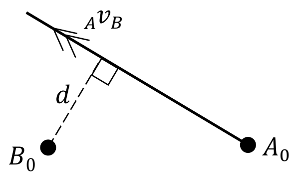
(c) State the relative velocity rule i.e. ${}_Av_B = v_A - v_B$

(d) Draw the correct velocity triangle for ${}_Av_B$

(e) To find the magnitude and direction of ${}_Av_B$, use the trigonometry of the velocity triangle
If it is an interception problem:
the time, t, at which interception occurs is given by

$$t = \frac{\text{initial distance apart}}{{}_Av_B}$$

If it is a closest approach problem:



(i) draw a displacement diagram showing the initial positions A_0 and B_0 of A and B respectively and ${}_Av_B$

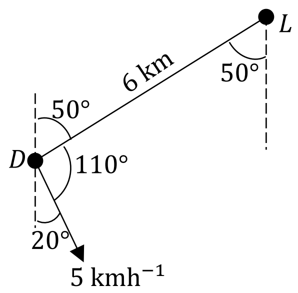
(ii) find d, the shortest distance between A and B during motion, by trigonometry or scale drawing

Example 1

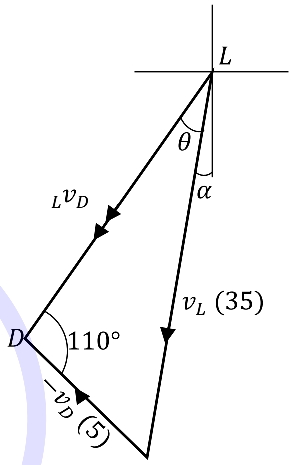
A dinghy in distress is 6 km $S\ 50°\ W$ of a lifeboat and drifting $S\ 20°\ E$ at 5 kmh^{-1}. In what direction should the lifeboat travel to reach the dinghy as quickly as possible if the maximum speed of the lifeboat is 35 kmh^{-1}. Find also the time it takes the lifeboat to reach the dinghy.

Solution

Imagine yourself on the dinghy D. The lifeboat will appear to travel directly towards you i.e. ${}_Lv_D$ will be in direction $S\ 50°\ W$



Method 1: Velocity triangle



In the velocity triangle; ${}_Lv_D = v_L - v_D$

$v_L = 35$ kmh^{-1}, $v_D = 5$ kmh^{-1}

Using the sine rule;

$$\frac{\sin\theta}{5} = \frac{\sin 110°}{35}$$

$$\sin\theta = \frac{5\sin 110°}{35} = 0.134$$

$$\theta = 7.7°$$

$$\alpha = 50° - 7.7° = 42.3°$$

The lifeboat must travel $S\ 42.3°\ W$

$$180° - (\theta + 110°) = 180 - (110° + 7.7°) = 62.3°$$

Using the cosine rule;

$${}_Lv_D^2 = v_L^2 + v_D^2 - 2(v_L)(v_D)\cos 62.3°$$

$${}_Lv_D^2 = 35^2 + 5^2 - 2(35)(5)\cos 62.3°$$

$${}_Lv_D = \sqrt{1087.31} = 32.97\ \text{kmh}^{-1}$$

Time taken, t for interception is given by

$$t = \frac{\text{initial distance apart}}{{}_Lv_D}$$

$$t = \frac{6}{32.97} = 0.182\ \text{h}$$

It takes the lifeboat 0.182 hours or 10 minutes 55 seconds to intercept or reach the dinghy.

$$T = \frac{\sqrt{(-5)^2 + 3^2}}{\sqrt{10^2 + (-6)^2}} = \frac{\sqrt{34}}{\sqrt{136}} = 0.5 \text{ h}$$

Collision will take place at 12 : 30 pm

After t hours, the position vector of A will be given by

$$r_A(t) = r_A + v_A t$$

After 0.5 hours,

$$r_A = \begin{pmatrix}1\\7\end{pmatrix} + \frac{1}{2}\begin{pmatrix}6\\2\end{pmatrix} = \begin{pmatrix}4\\8\end{pmatrix}$$

Position vector of collision point is (4i + 8j) km

The position vector of B could equally be used as $r_A = r_B$

Example 4

At 2 p.m, the position vectors r and velocity vectors v of three ships A, B and C are as follows:

$$r_A = (5\text{i} + \text{j})\text{km} \qquad v_A = (9\text{i} + 18\text{j}) \text{ kmh}^{-1}$$
$$r_B = (12\text{i} + 5\text{j})\text{km} \qquad v_B = (-12\text{i} + 6\text{j}) \text{ kmh}^{-1}$$
$$r_C = (13\text{i} + 3\text{j})\text{km} \qquad v_C = (9\text{i} + 12\text{j}) \text{ kmh}^{-1}$$

(a) Assuming that all the three ships maintain these velocities, show that A and B will collide and find when and where the collision takes place

(b) Find the position vector of C when A and B collide and find how far C is from the collision

(c) When the collision occurs, C immediately changes its course, but not its speed, and steams direct to the scene. When does C arrive?

Solution

(a) To show that A and B collide, we must show that

$$_Av_B = k\ _Ar_B$$

$$_Av_B = v_A - v_B = \begin{pmatrix}9\\12\end{pmatrix} - \begin{pmatrix}-12\\6\end{pmatrix} = \begin{pmatrix}21\\12\end{pmatrix}$$

$$_Ar_B = r_A - r_B = \begin{pmatrix}5\\1\end{pmatrix} - \begin{pmatrix}12\\5\end{pmatrix} = \begin{pmatrix}-7\\-4\end{pmatrix}$$

$$_Av_B = \begin{pmatrix}21\\12\end{pmatrix} = -3\begin{pmatrix}-7\\-4\end{pmatrix} = -3\ _Ar_B$$

$$_Av_B = -3\ _Ar_B$$

Collision will occur between ships A and B

Time taken for collision to occur is given by

$$T = \frac{\text{distance AB}}{\text{relative speed}} = \frac{|_Ar_B|}{|_Av_B|}$$

$$T = \frac{\sqrt{(-7)^2 + (-4)^2}}{\sqrt{21^2 + 12^2}} = \frac{\sqrt{65}}{\sqrt{585}} = \frac{1}{3} \text{ h} = 20 \text{ min}$$

Collision will take place at 12 : 20 pm

After t hours, the position vector of A will be given by

$$r_A(t) = r_A + v_A t$$

After 1/3 hours,

$$r_A = \begin{pmatrix}5\\1\end{pmatrix} + \frac{1}{3}\begin{pmatrix}9\\18\end{pmatrix} = \begin{pmatrix}8\\7\end{pmatrix}$$

Position vector of collision point is (8i + 7j) km

(b) A and B collide after $t = 1/3$ hours

After t hours, the position vector of C will be given by

$$r_C(t) = r_C + v_C t$$

After 1/3 hours,

$$r_C = \begin{pmatrix}13\\-3\end{pmatrix} + \frac{1}{3}\begin{pmatrix}9\\12\end{pmatrix} = \begin{pmatrix}13\\-3\end{pmatrix} + \begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}16\\1\end{pmatrix}$$

Position vector of C will be $(16\text{i} + \text{j})$ km

$$_Cr_A = r_C - r_A = \begin{pmatrix}16\\1\end{pmatrix} - \begin{pmatrix}8\\7\end{pmatrix} = \begin{pmatrix}8\\-6\end{pmatrix}$$

$$|_Cr_A| = \sqrt{8^2 + (-6)^2} = 10$$

Ship C is 10 km from the collision

(c) When C steams directly to the scene, the course of C will be in the direction of CA or CB.

Time to intercept where the collision takes place is given by

$$T' = \frac{\text{distance AC}}{\text{relative speed}} = \frac{|_Cr_A|}{|_Cv_A|}$$

$$_Cv_A = v_C - v_A = v_C \text{ since } v_A = 0$$

$$T = \frac{10}{\sqrt{9^2 + 12^2}} = \frac{10}{15} = \frac{2}{3} \text{ h} = 40 \text{ min}$$

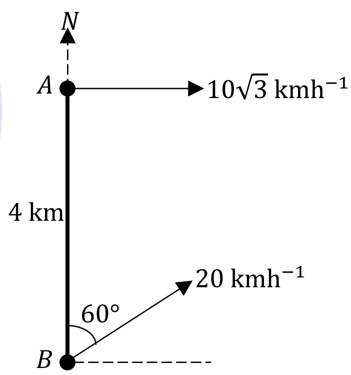
C will arrive at the scene after a further 40 minutes from when A and B collide.

C will therefore arrive at the scene at 3 p.m

Example 5

Ship A is 4 km due north of ship B. Ship A is travelling at $10\sqrt{3}$ kmh^{-1} due east while ship B is travelling at 20 kmh^{-1} on a bearing of 060°. If the two ships collide, find the time at which the ships collide and where the collision takes place.

Solution



Method 1: Vectors

$$v_A = \begin{pmatrix}10\sqrt{3}\\0\end{pmatrix};\ v_B = \begin{pmatrix}20\sin 60°\\20\cos 60°\end{pmatrix} = \begin{pmatrix}10\sqrt{3}\\10\end{pmatrix}$$

Take the position of B to be the origin;

$$r_B = \begin{pmatrix}0\\0\end{pmatrix} \text{ and } r_A = \begin{pmatrix}0\\4\end{pmatrix}$$

After time t hours,

$$r_A(t) = r_A + v_A t$$

$$r_A(t) = \begin{pmatrix}0\\4\end{pmatrix} + \begin{pmatrix}10\sqrt{3}\\0\end{pmatrix}t = \begin{pmatrix}10\sqrt{3}t\\4\end{pmatrix}$$

$$r_B(t) = r_B + v_B t$$

$$r_B(t) = \begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}10\sqrt{3}\\10\end{pmatrix}t = \begin{pmatrix}10\sqrt{3}t\\10t\end{pmatrix}$$

When the ships collide, $r_A(t) = r_B(t)$

$$\begin{pmatrix}10\sqrt{3}t\\4\end{pmatrix} = \begin{pmatrix}10\sqrt{3}t\\10t\end{pmatrix}$$

$$4 = 10t$$

$$t = 0.4 \text{ h}$$

The ships collide after 0.4 hours or 24 minutes

Distance $= S \times T = 20 \times 0.4 = 8$ km

Closest approach

If two bodies do not collide, then there will be an instant at which they are closer to each other than they are at any other instant.

Distance of closest approach

To find the closest distance of approach and the time taken, we can use two methods i.e. (a) vectors and (b) velocity triangle and the displacement diagram

(a) Vectors

If two bodies A and B are moving relative to one another, then they are closest to each if the following conditions are satisfied

(i) ${}_A r_B \cdot {}_A v_B = 0$. The displacement vector of A relative to be will be perpendicular to the velocity vector of A relative to B at time t

(ii) The displacement $|AB|$ at the time t when A and B are closest apart must be a minimum. By completing the square or differentiating with respect t, we can find the time taken for closest approach and hence the distance of closest approach.

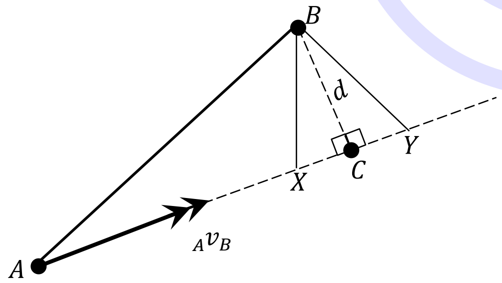
$$\frac{d\left|{}_A r_B\right|}{dt} = 0 \quad \text{or} \quad \frac{d(\overline{AB})}{dt} = 0$$

(b) Velocity triangle and displacement diagram

Draw a velocity triangle and use it to find ${}_A v_B$.

Draw a displacement diagram showing the initial positions of A and B respectively and ${}_A v_B$.

Find d, the shortest distance between A and B during motion, by trigonometry



The right-angled triangle ABC can be used to calculate the shortest distance, BC. The time to arrive at point C can be found using

$$\text{time} = \frac{\overline{AC}}{\text{speed of } A \text{ relative to } B} = \frac{\overline{AC}}{\left|{}_A v_B\right|}$$

Note: It is clear that BC gives the shortest distance. Look at either side of C, BX and BY will be longer than BC because they are hypotenuses of the triangles BCX and BCY respectively.

Example 1

Two particles P and Q are free to move in a horizontal plane. The particles move with constant velocities $(9\mathrm{i} + 6\mathrm{j})\ \mathrm{ms}^{-1}$ and $(5\mathrm{i} + 4\mathrm{j})\ \mathrm{ms}^{-1}$ respectively. At time $t = 4\ s$, the particles P and Q have position vectors, referred to the origin O, of $(96\mathrm{i} + 44\mathrm{j})$ m and $(100\mathrm{i} + 96\mathrm{j})$ m, respectively. Find

(a) the position vectors of P and Q at time $t = 0\ s$

(b) the position vector of Q relative to P at time t

(c) the time at which P and Q are nearest to each other and the length of PQ at this instant

Solution

(a) $v_P = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$; $v_Q = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

When $t = 4$, $r_P = \begin{pmatrix} 96 \\ 44 \end{pmatrix}$ and $r_Q = \begin{pmatrix} 100 \\ 96 \end{pmatrix}$

$$r_P(t) = r_P + v_P t$$

$$\begin{pmatrix} 96 \\ 44 \end{pmatrix} = r_P + 4\begin{pmatrix} 9 \\ 6 \end{pmatrix}$$

$$r_P = \begin{pmatrix} 96 \\ 44 \end{pmatrix} - \begin{pmatrix} 36 \\ 24 \end{pmatrix} = \begin{pmatrix} 60 \\ 20 \end{pmatrix}$$

$$r_P = (60\mathrm{i} + 20\mathrm{j})\ \mathrm{m}$$

$$r_Q(t) = r_Q + v_Q t$$

$$\begin{pmatrix} 100 \\ 96 \end{pmatrix} = r_Q + 4\begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$r_Q = \begin{pmatrix} 100 \\ 96 \end{pmatrix} - \begin{pmatrix} 20 \\ 16 \end{pmatrix} = \begin{pmatrix} 80 \\ 80 \end{pmatrix}$$

$$r_Q = (80\mathrm{i} + 80\mathrm{j})\ \mathrm{m}$$

(b) $r_P(t) = \begin{pmatrix} 60 \\ 20 \end{pmatrix} + \begin{pmatrix} 9 \\ 6 \end{pmatrix} t = \begin{pmatrix} 60 + 9t \\ 20 + 6t \end{pmatrix}$

$$r_Q(t) = \begin{pmatrix} 80 \\ 80 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \end{pmatrix} t = \begin{pmatrix} 80 + 5t \\ 80 + 4t \end{pmatrix}$$

$${}_Q r_P(t) = r_Q(t) - r_P(t) = \begin{pmatrix} 80 + 5t \\ 80 + 4t \end{pmatrix} - \begin{pmatrix} 60 + 9t \\ 20 + 6t \end{pmatrix}$$

$${}_Q r_P(t) = \begin{pmatrix} 20 - 4t \\ 60 - 2t \end{pmatrix}$$

$${}_Q r_P(t) = (20 - 4t)\mathrm{i} + (60 - 2t)\mathrm{j}$$

(c) Distance $PQ = \sqrt{(20 - 4t)^2 + (60 - 2t)^2}$

$$= \sqrt{400 - 160t + 16t^2 + 3600 - 240t + 4t^2}$$

$$= \sqrt{4000 - 400t + 20t^2}$$

$$= \sqrt{20(t^2 - 20t) + 4000}$$

By completing the square;

$$\overline{PQ} = \sqrt{20(t - 10)^2 + 2000}$$

Minimum distance occurs when $t - 10 = 0$

$$t = 10\ s$$

Least distance $= \sqrt{2000} = 44.7$ m

Alternatively;

Least distance occurs when ${}_Q r_P \cdot {}_Q v_P = 0$

$${}_Q v_P = v_Q - v_P = \begin{pmatrix} 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 9 \\ 6 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$$

$${}_Q r_P(t) = \begin{pmatrix} 20 - 4t \\ 60 - 2t \end{pmatrix}$$

$$\begin{pmatrix} 20 - 4t \\ 60 - 2t \end{pmatrix} \cdot \begin{pmatrix} -4 \\ -2 \end{pmatrix} = 0$$

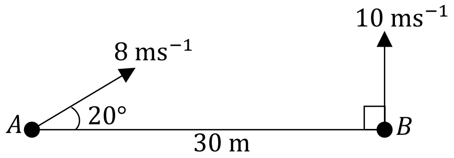
$$-4(20 - 4t) - 2(60 - 2t) = 0$$

$$-80 + 16t - 120 + 4t = 0$$

$$20t = 200$$

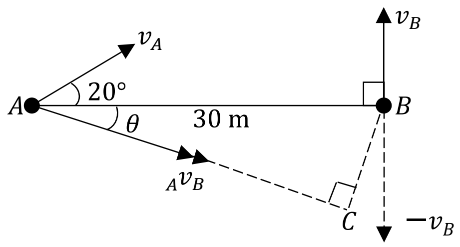
$$t = 10\ s$$

(a)

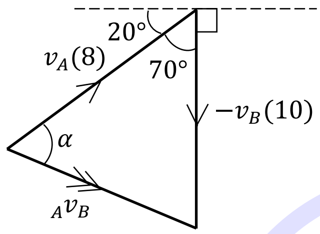


Solution

Initial sketch:



Velocity triangle



From the velocity triangle, ${}_Av_B = v_A - v_B$

Using the cosine rule;

$$\left|{}_Av_B\right|^2 = |v_A|^2 + |v_B|^2 - 2|v_A||v_B|\cos 70°$$

$$\left|{}_Av_B\right|^2 = 8^2 + 10^2 - 2(8)(10)\cos 70°$$

$$\left|{}_Av_B\right| = 10.5 \text{ ms}^{-1}$$

Using the sine rule;

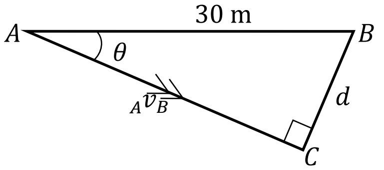
$$\frac{10.5}{\sin 70°} = \frac{10}{\sin \alpha}$$

$$\sin \alpha = \frac{10\sin 70°}{10.5} = 0.895$$

$$\alpha = 63.5°$$

$$\theta = \alpha - 20° = 63.5° - 20° = 43.5°$$

Displacement diagram;



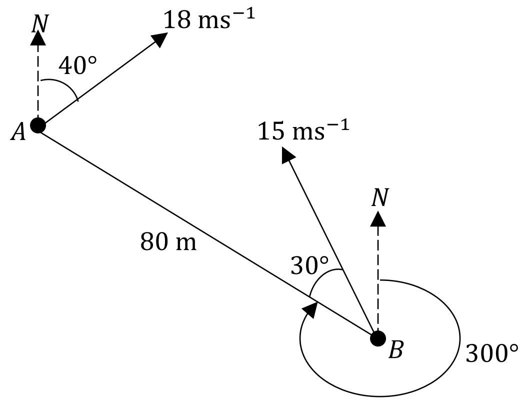
Distance of closest approach, d is given by $\overline{BC}$

$$d = 30\sin\theta = 30\sin 43.5° = 20.7 \text{ m}$$

Time taken to reach C is given by

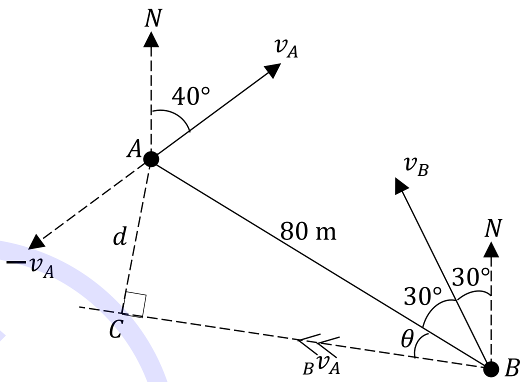
$$t = \frac{\overline{AC}}{\left|{}_Av_B\right|} = \frac{30\cos 43.5°}{10.5} = 2.07 \text{ s}$$

(b)

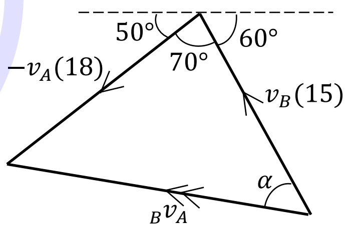


Solution

Initial sketch:



Velocity triangle:



From the velocity triangle; ${}_Bv_A = v_B - v_A$

Using the cosine rule:

$$\left|{}_Bv_A\right|^2 = 18^2 + 15^2 - 2(18)(15)\cos 70°$$

$${}_Bv_A = 19.1 \text{ kmh}^{-1}$$

Using sine rule:

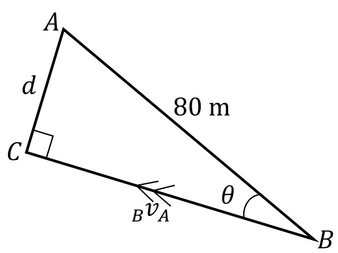
$$\frac{\left|{}_Av_B\right|}{\sin 70°} = \frac{|v_A|}{\sin\alpha}$$

$$\sin\alpha = \frac{18\sin 70°}{19.1} = 0.886$$

$$\alpha = 62.4°$$

$$\theta = \alpha - 30° = 62.4° - 30° = 32.4°$$

Displacement diagram;

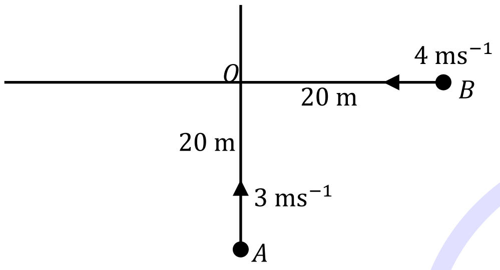


$$\sin\theta = \frac{d}{80}$$
$$d = 80\sin 32.4° = 42.9 \text{ m}$$
The closest distance of approach is 42.9 m
The time taken to reach is given by
$$t = \frac{\overline{BC}}{\left|{}_Bv_A\right|} = \frac{80\cos 32.4°}{19.1} = 3.54 \text{ s}$$

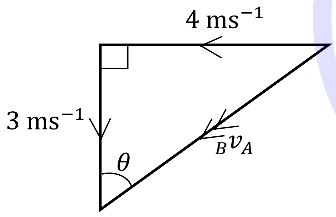
Example 4
Two straight roads, one running North-South and the other running East-West, intersect at a crossroads O. Two men A and B are cycling at steady speeds towards O. At a certain instant, A is 20 m from O and travelling due North at 3 ms^{-1} and B is 20 m from O and cycling due West at 4 ms^{-1}. Calculate the shortest distance apart of the two cyclists and the time which elapses before this is attained.

Solution

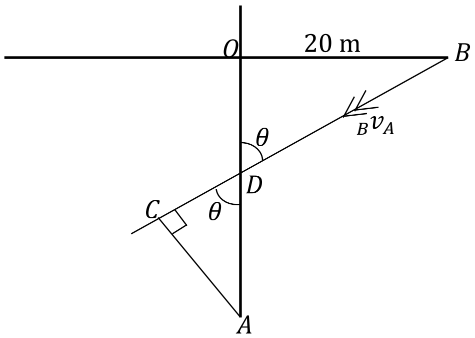
Initial sketch (situation seen by fixed observer)



Velocity triangle

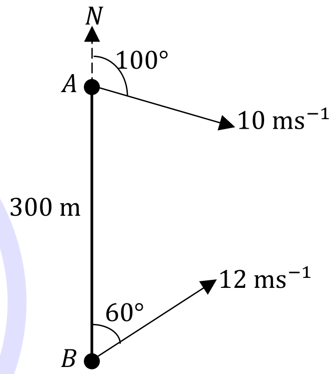


$$\left|{}_Bv_A\right| = \sqrt{3^2 + 4^2} = 5 \text{ ms}^{-1}$$
The direction of ${}_Bv_A$ is given by θ where;
$$\tan\theta = \tfrac{4}{3}, \sin\theta = \tfrac{4}{5} \text{ and } \cos\theta = \tfrac{4}{5}$$
Displacement diagram (as seen from A)



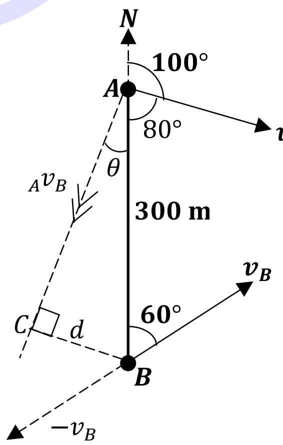
A sees B travelling along the direction BC. The cyclists are closest together when A sees B to be at a position C, where AC is perpendicular to BC. The shortest distance apart is therefore AC
Using this displacement diagram;
In $\Delta\, OBD$, $OD = 20$ m, $OD = \frac{20}{\tan\theta} = 20 \times \frac{3}{4} = 15$ m
$$AD = OA - OD = 20 - 15 = 5 \text{ m}$$
In $\Delta\, ACD$, $AC = AD\sin\theta = 5 \times \frac{4}{5} = 4$ m
Hence the shortest distance between the cyclists is 4 m
To find the time taken to reach the closest position, we first calculate distance BC
$$BC = BD + DC = \frac{20}{\sin\theta} + 5\cos\theta$$
$$BC = 20 \times \frac{5}{4} + 5 \times \frac{3}{5} = 25 + 3 = 28 \text{ m}$$
A sees B travelling along BC at ${}_Bv_A = 5 \text{ ms}^{-1}$
Time taken to travel the 28 m is given by
$$t = \frac{28}{5} = 5.6 \text{ s}$$
Hence, the cyclists are at the closest position after 5.6 s

Example 5
The diagram below shows the velocities and initial positions of two particles A and B. Find the closest distance between the particles and how long they take to reach that position

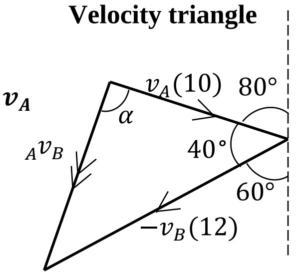


Solution

Initial sketch;



Velocity triangle



From the velocity triangle; ${}_Av_B = v_A - v_B$
Using the cosine rule;
$$\left|{}_Av_B\right|^2 = 12^2 + 10^2 - 2(12)(10)\cos 40°$$
$$\left|{}_Av_B\right| = 7.76 \text{ ms}^{-1}$$
Using the sine rule;
$$\frac{7.76}{\sin 40°} = \frac{12}{\sin\alpha}$$
$$\sin\alpha = \frac{12\sin 40°}{7.76} = 0.994$$

Closest distance, $d = 16\sin\theta = 16\sin 20° = 5.47$ km

Time taken to reach the position of closest approach is

$$t = \frac{\overline{BC}}{|{}_Bv_A|} = \frac{16\cos 20°}{24} = 0.626 \text{ h}$$

$$t = 0.626 \times 60 = 38 \text{ min}$$

The ships A and B are closest together at 10:38 a.m

Method 2: Vectors

$$v_A = \begin{pmatrix} 14\sin 29° \\ -14\cos 29° \end{pmatrix} = \begin{pmatrix} 6.79 \\ -12.24 \end{pmatrix}$$

$$v_B = \begin{pmatrix} 17\sin 50° \\ 17\cos 50° \end{pmatrix} = \begin{pmatrix} 13.02 \\ 10.93 \end{pmatrix}$$

(a) ${}_Bv_A = v_B - v_A$

$$ {}_Bv_A = \begin{pmatrix} 13.02 \\ 10.93 \end{pmatrix} - \begin{pmatrix} 6.79 \\ -12.24 \end{pmatrix} = \begin{pmatrix} 6.23 \\ 23.17 \end{pmatrix}$$

$$|{}_Bv_A| = \sqrt{6.23^2 + 23.17^2} = 24 \text{ kmh}^{-1}$$

Direction is $N\left(90° - \tan^{-1}\frac{23.17}{6.23}\right)E = N\,15°\,E$

The velocity of ship B relative to ship A is 24 kmh^{-1} on a bearing of $N\,15°\,E$

(b) Let ship A be at the origin at 10:00 a.m

$$r_A = \begin{pmatrix} 0 \\ 0 \end{pmatrix};\ r_B = \begin{pmatrix} -16\sin 35° \\ -16\cos 35° \end{pmatrix} = \begin{pmatrix} -9.18 \\ -13.11 \end{pmatrix}$$

$$ {}_Br_A = r_B - r_A = \begin{pmatrix} -9.18 \\ -13.11 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -9.18 \\ -13.11 \end{pmatrix}$$

After t hours past 10 a.m;

$$ {}_Br_A(t) = {}_Br_A + {}_Bv_A t$$

$$ {}_Br_A(t) = \begin{pmatrix} -9.18 \\ -13.11 \end{pmatrix} + \begin{pmatrix} 6.23 \\ 23.17 \end{pmatrix} t = \begin{pmatrix} -9.18 + 6.23t \\ -13.11 + 23.17t \end{pmatrix}$$

At closest approach, ${}_Br_A \cdot {}_Bv_A = 0$

$$\begin{pmatrix} -9.18 + 6.23t \\ -13.11 + 23.17t \end{pmatrix} \cdot \begin{pmatrix} 6.23 \\ 23.17 \end{pmatrix} = 0$$

$$6.23(-9.18 + 6.23t) + 23.17(-13.11 + 23.17t) = 0$$

$$-57.19 + 38.8t - 303.76 + 536.8t = 0$$

$$575.6t = 360.95$$

$$t = 0.627 \text{ h}$$

$$t = 0.627 \times 60 = 38 \text{ min}$$

The ships A and B are closest together at 10:38 a.m

$$ {}_Br_A = \begin{pmatrix} -9.18 + 6.23(0.627) \\ -13.11 + 23.17(0.627) \end{pmatrix} = \begin{pmatrix} -5.27 \\ 1.42 \end{pmatrix}$$

$$|{}_Br_A| = \sqrt{(-5.27)^2 + 1.42^2} = 5.46 \text{ km}$$

The closest distance between the ships is 5.46 km

Example 7

Two motorways intersect at a point O. Car A is on the first motorway travelling in the direction $8\mathrm{i} + 15\mathrm{j}$ while car B is on the second motorway travelling in the direction $3\mathrm{i} + 4\mathrm{j}$. The unit vectors i and j are due east and north respectively. At time $t = 0$, when car A is at the point O and has a constant speed of 85 kmh^{-1}, car B is still 2.5 km from O and travelling at 100 kmh^{-1}

(a) Write down, in terms of t where appropriate,
   (i) the velocity of B relative to A
   (ii) the position vector of each car relative to O
   (iii) the position vector of B relative to A

(b) Given that visibility is 2.5 km, show that the cars are within sight of each other for just over 11 minutes

Solution

(a) Unit vectors in the directions of the motorways are;

For A;

$$\frac{1}{\sqrt{8^2 + 15^2}}(8\mathrm{i} + 15\mathrm{j}) = \frac{8}{17}\mathrm{i} + \frac{15}{17}\mathrm{j}$$

For B;

$$\frac{1}{\sqrt{3^2 + 4^2}}(3\mathrm{i} + 4\mathrm{j}) = \frac{3}{5}\mathrm{i} + \frac{4}{5}\mathrm{j}$$

If B is still 2.5 km from O, the initial position vector of B relative to O is

$$r_B = 2.5\left(-\tfrac{3}{5}\mathrm{i} - \tfrac{4}{5}\mathrm{j}\right) = -1.5\mathrm{i} - 2\mathrm{j} \text{ km}$$

$v_A = 85\left(\frac{8}{17}\mathrm{i} + \frac{15}{17}\mathrm{j}\right) = 40\mathrm{i} + 75\mathrm{j}$ kmh^{-1}

$v_B = 100\left(\frac{3}{5}\mathrm{i} + \frac{4}{5}\mathrm{j}\right) = 60\mathrm{i} + 80\mathrm{j}$ kmh^{-1}

(i) Velocity of B relative to A is given by

$$ {}_Bv_A = v_B - v_A = \begin{pmatrix} 60 \\ 80 \end{pmatrix} - \begin{pmatrix} 40 \\ 75 \end{pmatrix} = \begin{pmatrix} 20 \\ 5 \end{pmatrix}$$

$$ {}_Bv_A = 20\mathrm{i} + 5\mathrm{j}$$

(ii) At time t, the position vectors of A and B relative to O are

$$r_A(t) = r_A + v_A t = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 40 \\ 75 \end{pmatrix} t = \begin{pmatrix} 40t \\ 75t \end{pmatrix}$$

$$r_B(t) = r_B + v_B t = \begin{pmatrix} -1.5 \\ -2 \end{pmatrix} + \begin{pmatrix} 60 \\ 80 \end{pmatrix} t = \begin{pmatrix} 60t - 1.5 \\ 80t - 2 \end{pmatrix}$$

(iii) The position vector of B relative to A is given by

$$ {}_Br_A(t) = r_B(t) - r_A(t)$$

$$ {}_Br_A(t) = \begin{pmatrix} 60t - 1.5 \\ 80t - 2 \end{pmatrix} - \begin{pmatrix} 40t \\ 75t \end{pmatrix} = \begin{pmatrix} 20t - 1.5 \\ 5t - 2 \end{pmatrix}$$

$$ {}_Br_A(t) = (20t - 1.5)\mathrm{i} + (5t - 2)\mathrm{j} \text{ km}$$

(b) $|{}_Br_A| = \sqrt{(20t - 1.5)^2 + (5t - 2)^2}$

$$= \sqrt{400t^2 - 60t + 2.25 + 25t^2 - 20t + 4}$$

$$= \sqrt{425t^2 - 80t + 6.25}$$

When $|{}_Br_A| = 2.5$ km

$$425t^2 - 80t + 6.25 = 6.25$$

$$425t^2 - 80t = 0$$

$$t(425t - 80) = 0$$

$$t = 0 \text{ and } t = \frac{80}{425} = 0.188 \text{ hours}$$

Therefore, cars are within sight of each other for 0.188 hours or 11.3 minutes

Self-Evaluation exercise

1. A particle Q is 200 m on a bearing of 040° from a particle P. Particle Q has a uniform speed of 25 ms^{-1} due south and particle P has a speed of 30 ms^{-1} on a bearing of 070°. Find the smallest distance between the particles and the time at which it occurs.
   [Ans: 4.74 m; 4.43 s]
2. An aeroplane has a speed of 300 kmh^{-1} due west. A helicopter is 50 km due south of the aeroplane and it travels 250 kmh^{-1} on a bearing of 320°. Calculate the closest distance between the aeroplane and the helicopter. How long does it take for the aircraft to reach this position?
   [Ans: 29.4 km; 0.171 h]

$$t = 0.7 \text{ h or } 42 \text{ min}$$

The motor boat reaches closest approach at 12:42 p.m

$${}_Br_Y = \begin{pmatrix} -15 - 6(0.7) \\ -20 + 8(0.7) \end{pmatrix} = \begin{pmatrix} -19.2 \\ -14.4 \end{pmatrix}$$

$$\left|{}_Br_Y\right| = \sqrt{(-19.2)^2 + (-14.4)^2} = 24 \text{ km}$$

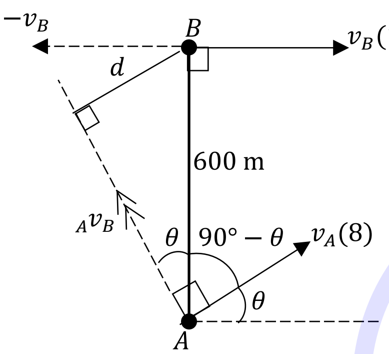
The motorboat gets as close as 24 km to the yacht

Example 2

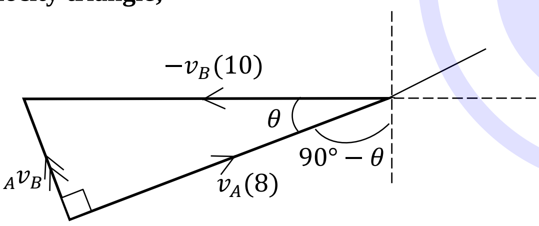
Motorboat B is travelling at a constant speed of 10 ms^{-1} due east and motorboat A is travelling at a constant speed of 8 ms^{-1}. Initially A and B are 600 m apart with A due south of B. Find the course that A should set in order to get as close as possible to B. Find this closest distance and the time taken for the situation to occur

Solution

Initial sketch;



Velocity triangle;



$$\cos\theta = \frac{8}{10} = 0.8$$

$$\theta = \cos^{-1} 0.8 = 36.9°$$

Direction $= 90° - \theta = 90° - 36.9° = 53.1°$

The motor boat A should set course in the direction $N\ 53.1°\ E$

$$\left|{}_Av_B\right|^2 + |v_A|^2 = |v_B|^2$$

$$\left|{}_Av_B\right| = \sqrt{10^2 - 8^2} = 6 \text{ ms}^{-1}$$

Closest distance,

$$d = 600\sin\theta = 600\sin 36.9° = 360.3 \text{ m}$$

Time taken to reach position,

$$t = \frac{600\cos\theta}{\left|{}_Av_B\right|} = \frac{600\cos 36.9°}{6} = 80 \text{ s}$$

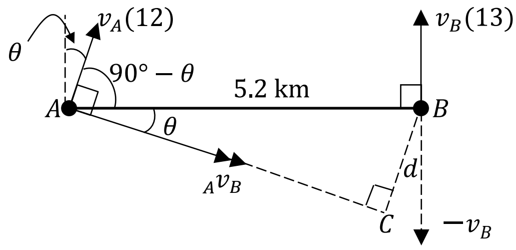
Example 3

At 8 a.m, two boats A and B are 5.2 km apart with A due west of B, and B is travelling due north at a steady 13 kmh^{-1}. If A travels with a constant speed of 12 kmh^{-1}, show that, for A to get as close as possible to B, A should set a course of $N\ \theta°\ E$ where $\sin\theta = \frac{5}{13}$

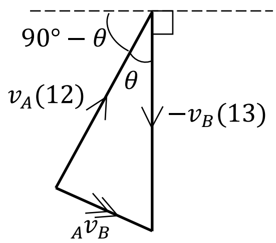
Find this closest distance and the time at which it occurs

Solution

Initial sketch:



Velocity triangle;



$$\left|{}_Av_B\right|^2 + |v_A|^2 = |v_B|^2$$

$$\left|{}_Av_B\right| = \sqrt{13^2 - 12^2} = 5 \text{ kmh}^{-1}$$

$$\sin\theta = \frac{\left|{}_Av_B\right|}{|v_B|} = \frac{5}{13}$$

Closest distance, $d = 5.2\sin\theta = 5.2 \times \frac{5}{13} = 2$ km

Time taken to reach C (closest approach);

$$t = \frac{\overline{AC}}{\left|{}_Av_B\right|} = \frac{5.2\cos\theta}{5} = \frac{5.2 \times \frac{12}{13}}{5} = 0.96 \text{ h}$$

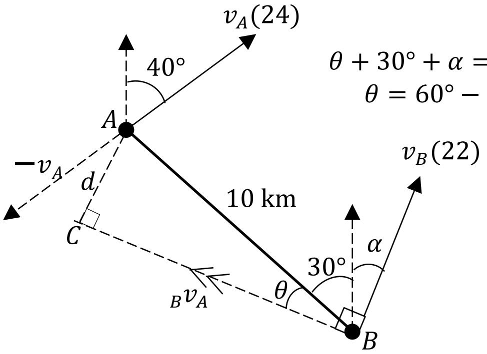
$$t = 0.96 \text{ h or } 57 \text{ min } 36 \text{ sec}$$

Therefore, closest approach occurs at 8:57 and 36 sec

Example 5

A ship A is moving with a constant speed of 24 kmh^{-1} in a direction $N\ 40°\ E$ and its initially 10 km from a second ship B, the bearing of A from B being $N\ 30°\ W$. If B moves with a constant 22 kmh^{-1}, find the course that B must set in order to pass as close as possible to A, the distance between the ships when they are closest, and the time taken for this to occur.

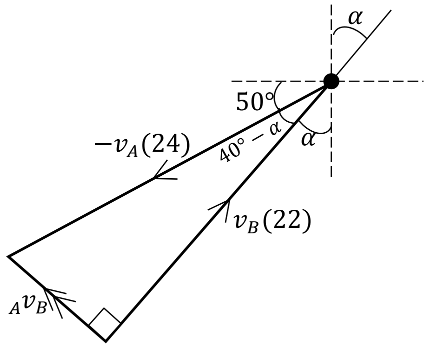
Solution



$$\theta + 30° + \alpha = 90°$$

$$\theta = 60° - \alpha$$

Velocity triangle;



$$\cos(40° - \alpha) = \frac{22}{24}$$
$$40° - \alpha = 23.6°$$
$$\alpha = 16.4°$$

B must set course in the direction $N\ 16.4°\ E$

$$|{}_Av_B| = \sqrt{24^2 - 22^2} = 9.6 \text{ kmh}^{-1}$$
$$\theta = 60° - \alpha = 60° - 16.4° = 43.6°$$

Closest distance between the ships;

$$d = 10\sin\theta = 10\sin 43.6° = 6.9 \text{ km}$$

Time taken for this to occur;

$$t = \frac{10\cos\theta}{9.6} = \frac{10\cos 43.6°}{9.6} = 0.754 \text{ h}$$

It occurs after approximately 45 min

Example 4

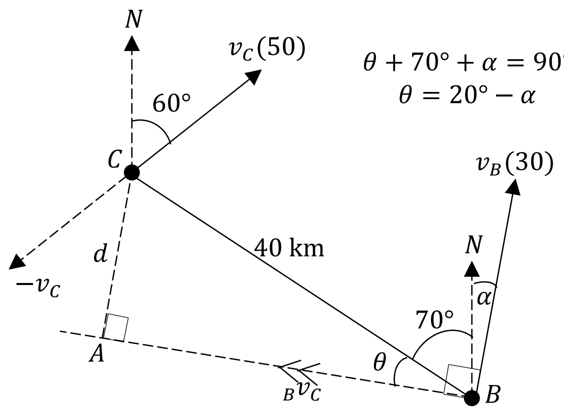
A battleship commander is informed that there is a lone cruiser positioned 40 km away from him on a bearing $N\ 70°\ W$. The guns on the battleship have a range of up to 8 km and the top speed of the battleship is 30 kmh^{-1}. The cruiser maintains a constant velocity of 50 kmh^{-1}, $N\ 60°\ E$. Show that whatever the course the battleship sets, it cannot get the cruiser within range of its guns.

Solution

To get the cruiser within range of its guns, the battleship must approach as close as possible with a distance d such that $d \leq 8$ km. It should thus set course in direction perpendicular to the relative path.

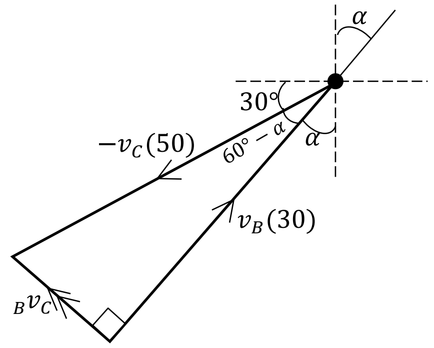
Let v_C and v_B represent the velocities of the cruiser and battleship respectively and ${}_Bv_C$ be the velocity of the battleship as seen by the cruiser

Initial sketch:



$$\theta + 70° + \alpha = 90°$$
$$\theta = 20° - \alpha$$

Velocity triangle;



$$\cos(60° - \alpha) = \frac{30}{50}$$
$$60° - \alpha = \cos^{-1}\frac{3}{5}$$
$$60° - \alpha = 53.1°$$
$$\alpha = 6.9°$$
$$\theta = 20° - \alpha = 20° - 6.9° = 13.1°$$

Distance of closest approach,

$$d = 40\sin\theta = 40\sin 13.1° = 9.1 \text{ km}$$

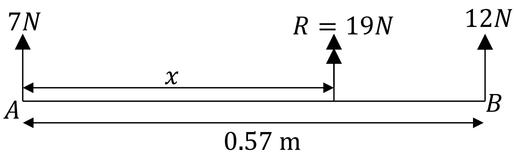
Since $d \geq 8$ km, the battleship cannot get the cruiser within range of its guns.

Self-Evaluation exercise

1. A particle at point A travels on a bearing of 060° at 12 ms^{-1}. A second particles starts at point B, which is due east of point A and has a maximum speed of 5 ms^{-1}. Find
   (a) the course that the second particle must set to get as close as possible to the first particle
   (b) the closest distance between the particles and the time at which this will occur
   [Ans: (a) 354.6° (b) 2.81 m, 2.74 s later]
2. Two bodies X and Y start at two points P and Q respectively. Point P is 100 m on a bearing of 150° away from Q. Body X moves with a constant speed of 30 ms^{-1} on a bearing of 090°. Body Y has a maximum speed of 22 ms^{-1}.
   (a) Determine the course for Y to get as close to X as possible
   (b) Calculate the time taken to reach this position and this distance between the two bodies.
   [Ans: (a) 132.8° (b) 1.45 s; 95.5 m]
3. Two objects P and Q are initially 20 m apart, with P being south west of Q. Object P has a speed of 16 ms^{-1} due north and Q can have a maximum speed of 10 ms^{-1}. How close can Q get to P?
   [Ans: 2.20 m]
4. A and B are two ships which, at 1200 hours, are at P and Q respectively where $PQ = 39$ km. A is steaming at 45 kmh^{-1} in a direction perpendicular to PQ and B is steaming on a straight course at 30 kmh^{-1} in such a direction as to approach A as closely as possible.
   (a) Show that B steams at an angle of $\sin^{-1}\frac{2}{3}$ with PQ
   (b) Find when the ships are closest together

The resultant of a set of parallel forces is parallel to the original forces. Its magnitude is the algebraic sum of the magnitude of the individual forces.
The location of the resultant may be found using the principle of moments.

Example 3



In the given diagram, find the distance x of the resultant R from the point A.

<u>Solution</u>

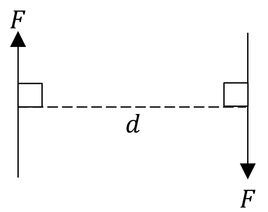
Moment of R about A is $19x$
Resultant moment of 7 N and 12 N forces about A is

$$(7 \times 0) + (12 \times 0.57) = 6.84$$

By the principle of moments:

$$19x = 6.84$$
$$x = 0.36 \text{ m}$$

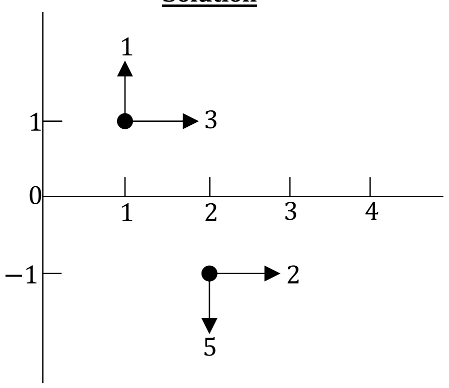
Couple
A couple is formed by two equal unlike parallel forces which are non-collinear.



It has zero resultant but it does have a moment.
The moment of the couple shown is Fd. This is constant about any point in the plane of the couple.

Example 4
Calculate the turning effect about O of the force $F_1 = 3\text{i} + \text{j}$ acting at the point $r_1 = \text{i} + \text{j}$ and the force $F_2 = 2\text{i} - 5\text{j}$ acting at the point $r_2 = 2\text{i} - \text{j}$

<u>Solution</u>



Total clockwise moment about O

$$= (3 \times 1) + (5 \times 2) = 13\ Nm$$

Total anticlockwise moment about O

$$= (1 \times 1) + (2 \times 1) = 3\ Nm$$

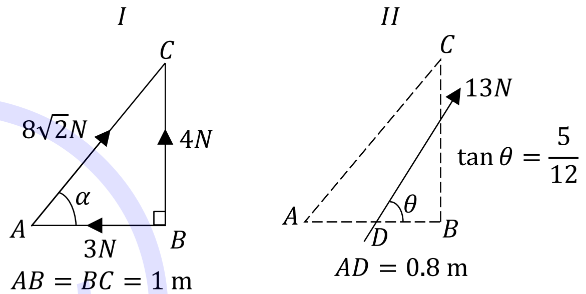
Resultant moment about $O = -13 + 3 = -10\ Nm$
i.e. a clockwise moment of 10 Nm

Equivalence
Systems of coplanar forces which produce exactly the same linear and turning effects on a rigid body are equivalent.
To establish that two coplanar system are equivalent, show that:

1. the total resolutes in two perpendicular directions are the same in both systems and
2. the resultant moment about a point in the plane containing the forces is the same in both systems

A known system of coplanar forces may be replaced by an equivalent, often simpler, system using (1) and (2).

Example 5
These two systems of forces are equivalent



	System I	System II
Resolve ∥ AB	$8\sqrt{2}\sin 45° - 3 = 5$	$13\sin\theta = 5$
Resolve ⊥ AB	$8\sqrt{2}\cos 45° + 4 = 12$	$13\cos\theta = 12$
Moments (A)	$4 \times 1 = 4$	$13 \times 0.8\sin\ \theta = 4$

Reduction to a force or a couple
The resultant of a system of coplanar forces, which is not in equilibrium, is either a **single force or a couple**. This resultant is equivalent to the original system of coplanar forces.

A system **reduces to a single force** if:
one or both of the total resolutes in two perpendicular directions is **non-zero**.

To locate the position of the line of action of the resultant force, take moments about any point in the plane and use the principle of moments.

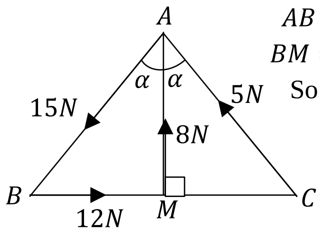
A system **reduces to a couple** if one of the following sets of conditions is satisfied.

SET I:
1. The total resolutes in two perpendicular directions are each zero, and
2. The resultant moment about any point in the plane is non-zero

SET II:
The resultant moments about three non-collinear points are each non-zero.

Example 6
ABC is an isosceles triangle with $AB = AC = 5$ cm and $BC = 6$ cm. M is the midpoint of BC. Forces of 15, 12, 5 and 8 N act along AB, BC, CA and MA respectively. Show that this system of forces is equivalent to a couple and find its moment.

Solution



$AB = AC = 5\text{ m} = 0.05\text{ m}$
$BM = MC = 3\text{ cm} = 0.03\text{ m}$
So $AM = 4\text{ cm} = 0.04\text{ m}$

Let $\angle MAB = \angle MAC = \alpha$

$$\text{So } \sin\alpha = \tfrac{3}{5};\ \cos\alpha = \tfrac{4}{5}$$

Resolve ∥ to BC:

$$12 - 15\sin\alpha - 5\sin\alpha$$
$$= 12 - 15 \times \frac{3}{5} - 5 \times \frac{3}{5} = 0$$

Resolve ⊥ to BC:

$$8 - 15\cos\alpha + 5\cos\alpha$$
$$= 8 - 15 \times \frac{4}{5} + 5 \times \frac{4}{5} = 0$$

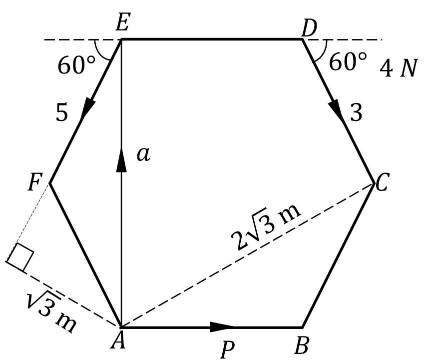
Moments about A:

$$12 \times 0.05\cos\alpha = 12 \times 0.05 \times \frac{4}{5} = 0.48\ Nm$$

Example 7
$ABCDEF$ is a regular hexagon of side 2 m. Forces of magnitude P, 3, 5 and Q newtons act along AB, DC, EF and AE respectively. Show that this system of forces is not in equilibrium.

(a) If the system is equivalent to couple, find its moment and the value of P and Q.
(b) If the system is equivalent to a single force through E, find P

Solution



The sum of the moments of forces about A

$$= 3 \times 2\sqrt{3} - 5 \times \sqrt{3} = \sqrt{3}$$

Since this sum is not zero, the forces are not in equilibrium.

(a) If the system is equivalent to a couple, it will have the same moment about any point in the plane.
∴ the moment of the couple is $\sqrt{3}\ Nm$
The sum of the components of the forces in the couple is zero in any direction,
∴ resolving in the directions AB and AE

$$\rightarrow \quad P + 3\cos 60° - 5\cos 60° = 0$$
$$\uparrow \quad Q - 3\sin 60° - 5\sin 60° = 0$$
$$P = 2\cos 60° = 1$$
$$Q = 8\sin 60° = 4\sqrt{3}$$

(b) If the system is equivalent to a single force through E, then the sum of the moments about E will be zero.
Taking moments about E;

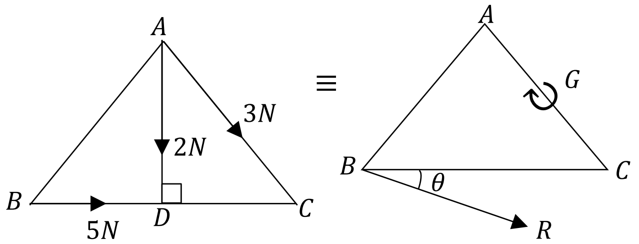
$$P \times 2\sqrt{3} - 3 \times \sqrt{3} = 0$$
$$P = \frac{3\sqrt{3}}{2\sqrt{3}} = 1.5$$

Reduction to a force and a couple
Any system of coplanar forces acting on a rigid body may be replaced by an equivalent system which consists of a **single force**, acting at a particular point in the plane of the forces, **together with a couple**.

Example 8
Forces 3 N and 5 N act along the sides $\overrightarrow{AC}$ and $\overrightarrow{BC}$ of an equilateral triangle, side a metres. A force of 2 N acts along the altitude $\overrightarrow{AD}$. Find the force at B and the couple which together are equivalent to this system.

Solution

Let R be the required single force at an angle θ to BC and G be the moment of the required couple.



Resolve along BC for original system: $5 + 3\cos 60°$
for new system: $R\cos\theta$
So $R\cos\theta = 5 + 3\cos 60°$ … [1]

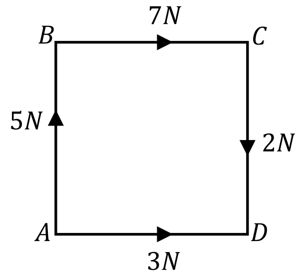
Resolve ⊥ to BC for original system: $2 + 3\cos 30°$
for new system: $R\sin\theta$
So $R\sin\theta = 2 + 3\cos 30°$ …..[2]

Solving [1] and [2];

$$\tan\theta = \frac{2 + 3\cos 30°}{5 + 3\cos 60°}$$
$$\theta = 35.3°$$

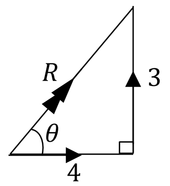
From [2];

$$R\sin 35.3° = 2 + 3\cos 30°$$
$$R = 7.96\ N$$



(a) Let X and Y be the resolutes of the resultant force in the directions AD and AB respectively.

$$X = (7-3)N = 4\ N$$
$$Y = (5-2) = 3\ N$$



Hence, $R = \sqrt{4^2+3^2} = 5\ N$

at an angle $\theta = \tan^{-1}\left(\frac{3}{4}\right)$ to AD

(b) Moments about A for the system of forces gives

$$G = (7\times 2) + (2\times 2)\ Nm$$
$$= 18\ Nm\text{, clockwise}$$

(c) Let R, the resultant of the forces, cut DA produced at P

Let $AP = x$ metres

Moments about A for the resultant force gives

$$G = 5\sin\theta\times x$$

Since the moments G must be equal

$$5\sin\theta\times x = 18$$
$$5\times\frac{3}{5}x = 18$$
$$x = 6\text{ m}$$

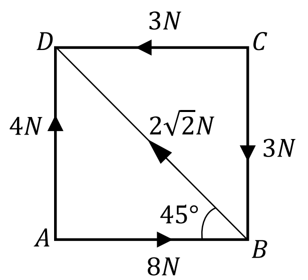
Hence, the resultant of the forces cuts DA produced at 6 m from A

Example 12

A square $ABCD$ of side 3 m has forces of magnitude 8, 3, 3, 4 and $2\sqrt{2}$ N acting along AB, CB, CD, AD and BD respectively. If the system is reduced to a force acting through A together with a couple, find the magnitude and direction of the force and the moment of the couple.

If the system is reduced to a single force acting through a point E on AB at a distance d metres from A, find d.

Solution



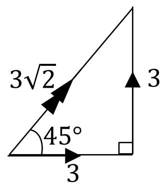
Suppose that the given system is equivalent to forces X and Y acting along AB and AD respectively, together with a couple of moment G in the sense $ABCD$.

Resolving in the directions of AB and AD:

$$\rightarrow X = 8 - 3 - 2\sqrt{2}\cos 45° = 3$$
$$\uparrow Y = 4 - 3 + 2\sqrt{2}\sin 45° = 3$$

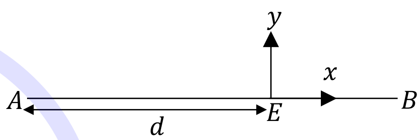
Taking moments about A:

$$G = 3\times 3 - 3\times 3 + 2\sqrt{2}\times 3\sin 45° = 6$$



Hence the system is equivalent to a force of $3\sqrt{2}\ N$ acting along AC together with a couple of moment $6\ Nm$ in the sense $ABCD$.

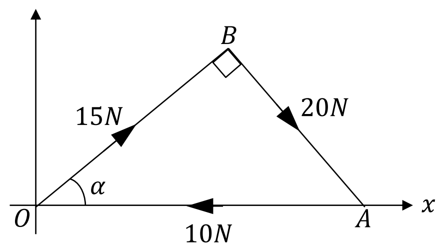
The single equivalent force through E has a moment G about A and components X and Y in the directions of AB and AD respectively.



Taking moments about A:

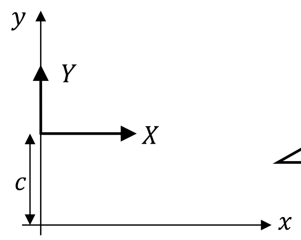
$$G = Yd$$
$$6 = 3d$$
$$d = 2$$

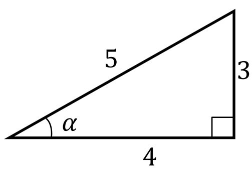
Example 13



Find the equation of the line of action of the resultant of the forces shown in the diagram, given that $\tan\alpha = \frac{3}{4}$ and that A is the point $(7, 0)$.

Solution





Let $X$and Y be the components of the resultant in the directions of the x- and y- axes respectively. Let the line of action cut the y-axis at the point $(0, c)$.

respectively. Find the single force equivalent to this system and where its line of action cuts AB.

[Ans: 8.84 N, 57.6° to AB. 2.32 m from BA produced]

23. Three forces are represented by the vectors $-2\text{i}-3\text{j}$, $3\text{i}+4\text{j}$ and $-\text{i}-\text{j}$. The forces act at the points $(2,0)$, $(0,3)$ and $(1,1)$ respectively. Show that the three forces combine to form a couple, and find the magnitude of the couple.

[Ans: 15 units]

24. The force 5j acts at the point $(2,3)$ and the force F acts at $(6,1)$. If the system forms a couple.
   (a) state the force F,
   (b) find the magnitude of the couple

[Ans: (a) -5j (b) 20 units]

25. $ABCD$ is a square of side 1 metre. Forces of magnitude 3 N, 6 N, 5 N and 4 N act along the sides AB, BC, DC and AD, the sense of each force being indicated by the order of the letters. Find
   (i) the magnitude of the resultant of the forces
   (ii) the total moment of the forces about A
   (iii) the perpendicular distance from A to the line of action of the resultant.

[Ans: (i) $2\sqrt{41}\,N$ (ii) 1 Nm anticlockwise (iii) $\frac{1}{2\sqrt{41}}$ m]

26. $ABCD$ is a square of side 2 m. Forces of magnitude 2 N, 1 N, 3 N, 4 N and $2\sqrt{2}\,N$ act along $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DA}$ and $\overrightarrow{BD}$ respectively. In order to maintain equilibrium, a force F, whose line of action cuts AD produced at E, has to be applied. Find
   (a) the magnitude of F,
   (b) the angle F makes with AD
   (c) the length AE

[Ans: (a) $\sqrt{10}\,N$ (b) 71.6° (c) 4 m]

27. $ABCDEF$ is a regular hexagon of length a. Forces having magnitudes $6W$, $10W$, $3W$, $6W$ and $5W$ act along the sides AB, BC, CD, DE and EF, the direction indicated by the order of the letters.
   (a) Prove that the resultant of this system of forces intersects AB produced at a point X distant $\frac{1}{8}a$ from B.
   (b) The magnitude of the resultant is R and its direction makes an angle θ with AB. Find R and θ.

[Ans: (b) $7W$, 81.8°]

28. ABCD is a square of side 3m. Forces of magnitude 1 N, 2 N, 3 N, s N, and t N act along the line AB, BC, CD, DA, and AC respectively, in each case the direction of the force being given by the order of the letters. Taking AB as horizontal and BC as vertical, find the values of s and t so that the resultant of the forces is a couple.

[Ans: $s=4$, $t=2\sqrt{2}$]

29. $OABC$ is a square of side 1 m. Forces of magnitude 2, 3, 4 and $5\sqrt{2}\,N$ act along OA, AB, CB and AC respectively in the directions indicated by the order of the letters and a couple of moment 7 Nm acts in the plane of the square in the sense $OCBA$. Find the magnitude and direction of the resultant of the system and the equation of its lines of action referred to OA and OC as axes. What is the magnitude and direction of the least force introduced at A if the resultant of the original system and this new force is to pass through O?

[Ans:$\sqrt{65}\,N$ at $\tan^{-1}8$ to OA; line of action $\frac{3}{8}$ m from O in direction OA and 3 m from O in direction CO, so $y=8x+3$; 3 N in direction BA]

30. A rectangle $ABCD$ has $AB=3$ cm and $BC=4$ cm. Forces, all measured in newtons and of magnitudes 2, 4, 6, 8 and k, act along AB, BC, CD, DA and AC respectively, the directions of each force being shown by the order of the letters. The resultant of the five forces is parallel to BD. Find k and show that the resultant has magnitude $\frac{5}{6}$ newtons. Find the distance from A of the line of action of the resultant.

[Ans: $k=\frac{35}{6}$; 43.2 cm]

31. Force of magnitude $2P$, P, $2P$, $3P$, $2P$ and P act along the sides AB, BC, CD, ED, EF and AF respectively of a regular hexagon of side $2a$ in the directions indicated by the letters. Prove that this system of forces can be reduced to a single force of magnitude $2P\sqrt{3}$ acting along AC together with a couple. Find the magnitude of the couple.
   Show that the system of forces can be reduced to a single force without a couple. If the line of action of this force cuts FA produced at X, calculate the length of AX.

EQUILIBRIUM

Particle in equilibrium

When a particle is in equilibrium under a system of coplanar concurrent forces, the following condition is satisfied.

The resultant of all forces in any direction must be zero

When solving problems about particles in equilibrium:

(a) Draw a clear force diagram
(b) Choose a direction for resolving, remembering that the component of a force in a direction perpendicular to itself is zero
(c) Resolve the forces acting in any chosen direction and equate to the resultant to zero
(d) If necessary, resolve the forces acting in another suitable direction and equate the resultant to zero

Example 1

An object is in equilibrium under the action of the following three forces:

$$\binom{2}{7}\ N,\ \binom{a}{-11}\ N \text{ and } \binom{4}{b}\ N$$

Calculate a and b

Solution

$$\binom{2}{7}+\binom{a}{-11}+\binom{4}{b}=\binom{0}{0}$$
$$\binom{a+6}{b-4}=\binom{0}{0}$$
$$a+6=0 \Rightarrow a=-6$$
$$b-4=0 \Rightarrow b=4$$

Example 2

A body is acted on by the following forces: $20\text{i}\ N$, $(-11\text{i}-8\text{j})\ N$, $(3\text{i}-14\text{j})N$ and $(-7\text{i}+17\text{j})\ N$. The addition of a fifth force P, brings about equilibrium. Find the force P

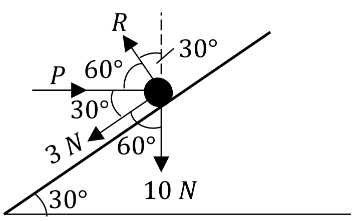
Solution

Let the force P be $x\text{i}+y\text{j}\ N$

$$\binom{20}{0}+\binom{-11}{-8}+\binom{3}{-14}+\binom{-7}{17}+\binom{x}{y}=\binom{0}{0}$$
$$\binom{3+x}{-5+y}=\binom{0}{0}$$
$$3+x=0 \Rightarrow x=-3$$
$$-5+y=0 \Rightarrow y=5$$

The fifth force is $(-3\text{i}+5\text{j})\ N$

Example 3



A particle on a slope is subject to the forces shown in the diagram above and is in equilibrium. Find the forces R and P.

Solution

Resolving vertically (to eliminate force P)

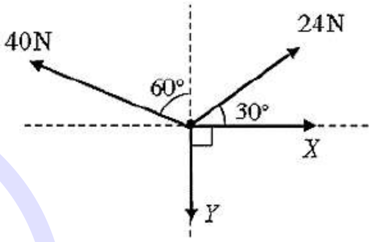
$$R\cos 30° - 10 - 3\cos 60° = 0$$
$$\frac{R\sqrt{3}}{2} = 10 + \frac{3}{2}$$
$$R\sqrt{3} = 23$$
$$R = \frac{23\sqrt{3}}{3}\ N$$

Resolving parallel to the slope (to eliminate R);

$$P\cos 30° - 3 - 10\cos 60° = 0$$
$$\frac{P\sqrt{3}}{2} = 3 + 5$$
$$P\sqrt{3} = 16$$
$$P = \frac{16\sqrt{3}}{3} N$$

Example 4

The four coplanar forces, shown in the figure below, are in equilibrium.



Determine the value of X and the value of Y.

Solution

Resolving →;

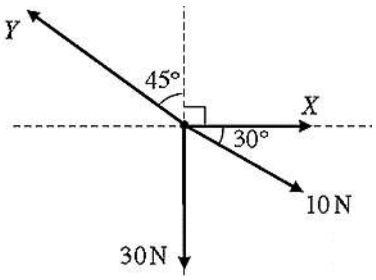
$$X + 24\cos 30° = 40\sin 60°$$
$$X + 12\sqrt{3} = 20\sqrt{3}$$
$$X = 8\sqrt{3} = 13.9\ N$$

Resolving ↑;

$$Y = 40\cos 60° + 24\sin 30°$$
$$Y = 20 + 12$$
$$Y = 32\ N$$

Example 5

The four coplanar forces, shown in the figure below, are in equilibrium.



Determine the value of X and the value of Y.

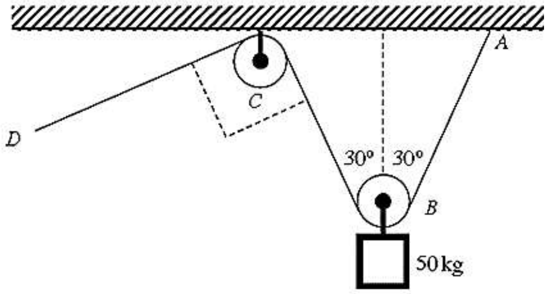
Solution

Resolving →; $X + 10\cos 30° = Y\sin 45°$

$$X + 5\sqrt{3} = \frac{\sqrt{2}}{2}Y$$

Resolving ↑; $Y\cos 45° = 30 + 10\sin 30°$

Example 9



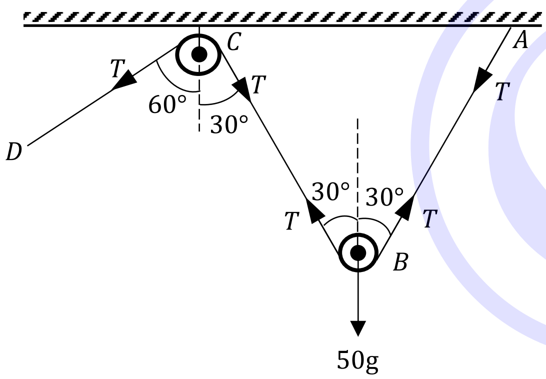
A light inextensible string passes through two smooth light pulleys at C and B, and the other end is tied at a point A on a fixed horizontal ceiling. A box of mass 50 kg is attached to the pulley at B. The string remains taut at all times by a force acting at D in the direction CD. The sections AB and BC are both inclined at 30° to the vertical and $\measuredangle DCB = 90°$. The system is in equilibrium with the points A, B, C and D lying on a vertical plane which is perpendicular to the ceiling.

(a) By considering the forces acting at B, find the tension in the string.

(b) Determine the magnitude and direction of the force exerted by the string on the pulley at C.

Solution

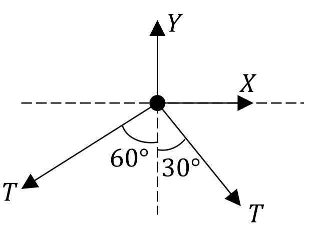
(a)



Looking at pulley at B (vertically);

$$T\cos 30° + T\cos 30° = 50\text{g}$$
$$2T\cos 30° = 50\text{g}$$
$$T = \frac{25\text{g}}{\cos 30°}$$
$$T = 283\ N$$

(b) Looking at the pulley at C

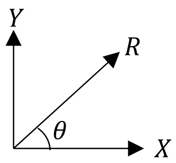


Resolving →; $T\sin 60° = X + T\sin 30°$

$$X = T(\sin 60° - \sin 30°)$$
$$X = 283(\sin 60° - \sin 30°) = 103.55\ N$$

Resolving ↑; $Y = T\cos 60° + T\cos 30°$

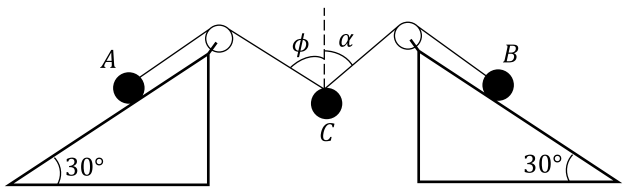
$$Y = T(\cos 60° + \cos 30°)$$
$$Y = 283(\cos 60° + \cos 30°) = 386.45\ N$$



$$R = \sqrt{103.55^2 + 386.45^2} = 400\ N$$
$$\tan\theta = \frac{Y}{X} = \frac{386.45}{103.55}$$
$$\theta = 75°$$

Example 10

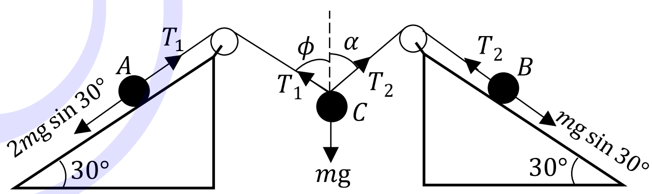
The diagram shows masses A and B each lying on smooth planes of inclination 30°



Light inextensible strings attached to A and B pass along the lines of greatest slope, over smooth pulleys and are connected to a third mass C hanging freely. The strings make angles of ϕ and α with the upward vertical as shown.

If A, B and C have masses $2m$, m and m respectively and the system rests in equilibrium, show that $\sin\alpha = 2\sin\phi$ and $\cos\alpha + 2\cos\phi = 2$. Hence find ϕ and α.

Solution



For particle A;

$$\nearrow T_1 = 2mg\sin 30°$$
$$T_1 = mg$$

For particle B;

$$\nwarrow T_2 = mg\sin 30°$$
$$T_2 = \frac{mg}{2}$$

For particle C;

Resolving →; $T_2\sin\alpha = T_1\sin\phi$

$$\frac{mg}{2}\sin\alpha = mg\sin\phi$$
$$\boldsymbol{\sin\alpha = 2\sin\phi}$$

Resolving ↑; $T_2\cos\alpha + T_1\cos\phi = mg$

$$\frac{mg}{2}\cos\alpha + mg\cos\phi = mg$$
$$\boldsymbol{\cos\alpha + 2\cos\phi = 2}$$

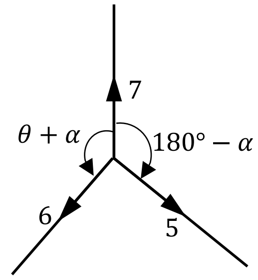
From $\sin\alpha = 2\sin\phi$

$$\sin^2\alpha = 4\sin^2\phi \ \ldots.\text{(i)}$$

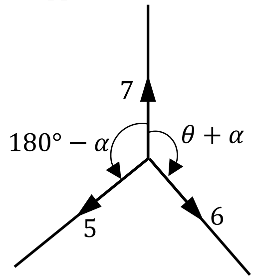
From $\cos\alpha + 2\cos\phi = 2$

$$\cos\alpha = 2 - 2\cos\phi$$
$$\cos^2\alpha = 4 - 8\cos\phi + 4\cos^2\phi \quad \ldots\text{(ii)}$$

Adding (i) + (ii) gives;



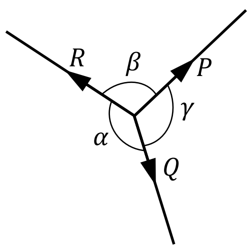
5 N force at 122.9° and 6 N force at 224.2°
Or (imagine the diagram flipped)



6 N force at 136° and 5 N force at 237.1°

Lami's theorem

The **third special** result is Lami's theorem which is a version of the sine rule.

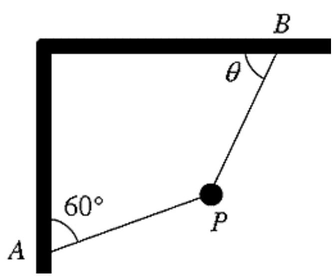


If three concurrent coplanar forces P, Q and R are in equilibrium and the angles between Q and R, R and P, P and Q are α, β and γ respectively as shown, then Lami's theorem states that:

$$\frac{\boldsymbol{P}}{\sin\boldsymbol{\alpha}} = \frac{\boldsymbol{Q}}{\sin\boldsymbol{\beta}} = \frac{\boldsymbol{R}}{\sin\boldsymbol{\gamma}}$$

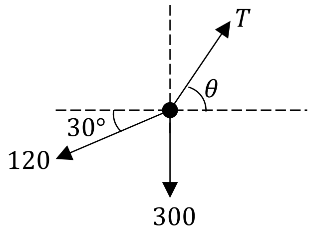
This is particularly useful when one of the forces and the angles between the pairs of forces are known.
It may also be used to solve the triangle of forces when this has been sketched.

Example 13



A particle P, of weight 300 N, is hanging in equilibrium by two light inextensible strings, AP and BP, which lie in the same vertical plane. It is further given that AP forms an angle of 60° with a vertical wall and BP forms an angle θ with a horizontal ceiling. Calculate the value of θ and the tension in BP, if the tension in AP is 120 N.

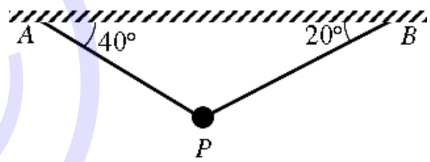
Solution



Using Lami's theorem;

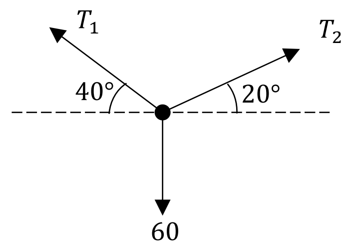
$$\frac{T}{\sin 60°} = \frac{120}{\sin(90° + \theta)} = \frac{300}{\sin(210° - \theta)}$$
$$120\sin(210° - \theta) = 300\sin(90° + \theta)$$
$$2[\sin 210°\cos\theta - \sin\theta\cos 210°]$$
$$= 5[\sin 90°\cos\theta + \sin\theta\cos 90°]$$
$$2\sin 210°\cos\theta - 2\sin\theta\cos 210° = 5\cos\theta$$
$$2\sin 210° - 2\tan\theta\cos 210° = 5$$
$$\tan\theta = \frac{2\sin 210° - 5}{2\cos 210°} = 3.464$$
$$\theta = 73.9°$$
$$T = \frac{120}{\sin(90° + \theta)} \times \sin 60°$$
$$T = \frac{120}{\sin 163.9°} \times \sin 60° = 374.75\ N$$

Example 14



A particle P of weight 60 N is suspended by two strings from a fixed horizontal ceiling. The particle hangs in equilibrium. The strings are light and inextensible and are inclined at 40° and 20° to the ceiling, as shown in the figure above. Find the tension in each of the two strings.

Solution



Using Lami's theorem;

$$\frac{T_1}{\sin 110°} = \frac{T_2}{\sin 130°} = \frac{60}{\sin 120°}$$
$$T_1 = \frac{60}{\sin 110°} \times \sin 110° = 65.1\ N$$
$$T_2 = \frac{60}{\sin 110°} \times \sin 130° = 53.1\ N$$

Example 15

One end of a string is fixed to a point A and the other is fastened to a small object of weight 8 N. The object is pulled

$\tan\theta = 2$

For equilibrium:

Resolving forces vertically, $R = W$

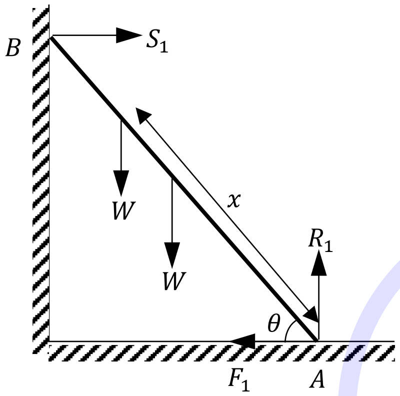
Resolving forces horizontally, $S = F$

Taking moments about A,

$$S(2l\sin\theta) = W(l\cos\theta)$$
$$S = \frac{W}{2\tan\theta} = \frac{W}{4}$$
$$F = S = \frac{W}{4}$$
$$F \leq \mu R$$
$$\frac{W}{4} \leq \mu W$$
$$\mu \geq \frac{1}{4}$$

The least value of μ is 0.25

(b) Let the particle be attached at a distance x from A



Resolving forces vertically; $R_1 = 2W$

Resolving forces horizontally, $S_1 = F_1$

Taking moments about A,

$$S_1(2l\sin\theta) = W(x\cos\theta) + W(l\cos\theta)$$
$$S_1 = \frac{W(x+l)}{2l\tan\theta} = \frac{W(x+l)}{4l}$$
$$F_1 = S = \frac{W(x+l)}{4l}$$
$$\frac{F_1}{R_1} = \frac{(x+l)}{2(4l)} = \frac{1}{8}\left(\frac{x}{l}+1\right)$$

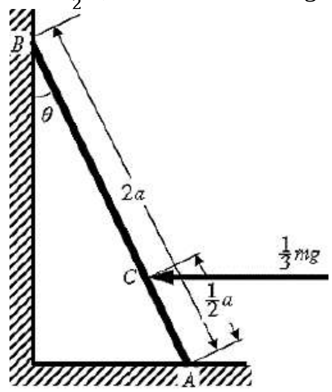
Given $\mu = \frac{5}{16}$, $\frac{F_1}{R_1} \leq \frac{5}{16}$

$$\frac{1}{8}\left(\frac{x}{l}+1\right) \leq \frac{5}{16}$$
$$\frac{x}{l}+1 \leq \frac{5}{2}$$
$$\frac{x}{l} \leq \frac{3}{2}$$
$$x = \frac{3}{2}l$$

Therefore, the highest point at which the particle can be attached is $\frac{3}{2}l$ from A

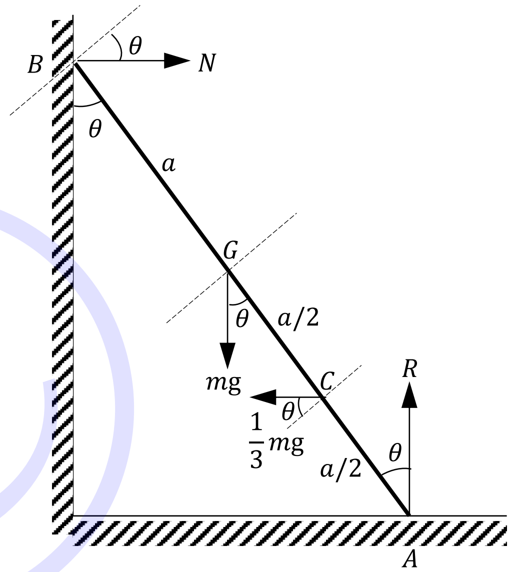
Example 18

A ladder of length $2a$ and mass m, has one end A on smooth horizontal ground and the other end B against a smooth vertical wall. The ladder is kept in equilibrium by a horizontal force of magnitude $\frac{1}{3}mg$ acting at a point C on the ladder, where $AC = \frac{1}{2}a$, as shown in the figure below.



The angle between the ladder and the vertical wall is θ. By modelling the ladder as a uniform rod show that $\tan\theta = \frac{1}{2}$.

Solution



$(\uparrow)$; $R = mg$

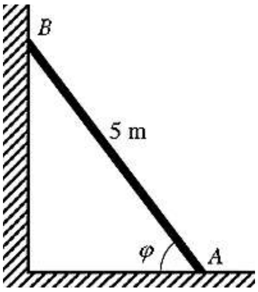
$(\rightarrow)$; $N = \frac{1}{3}mg$

Taking moments about A;

$$\left(\frac{1}{3}mg\cos\theta \times \frac{1}{2}a\right) + (mg\sin\theta \times a) = (N\cos\theta) \times 2a$$
$$\frac{1}{6}mga\cos\theta + mga\sin\theta = 2Na\cos\theta$$
$$\frac{1}{6}mga\cos\theta + mga\sin\theta = 2\left(\frac{1}{3}mg\right)a\cos\theta$$
$$\frac{1}{6}\cos\theta + \sin\theta = \frac{2}{3}\cos\theta$$
$$\cos\theta + 6\sin\theta = 4\cos\theta$$
$$6\sin\theta = 3\cos\theta$$
$$\tan\theta = \frac{1}{2}$$

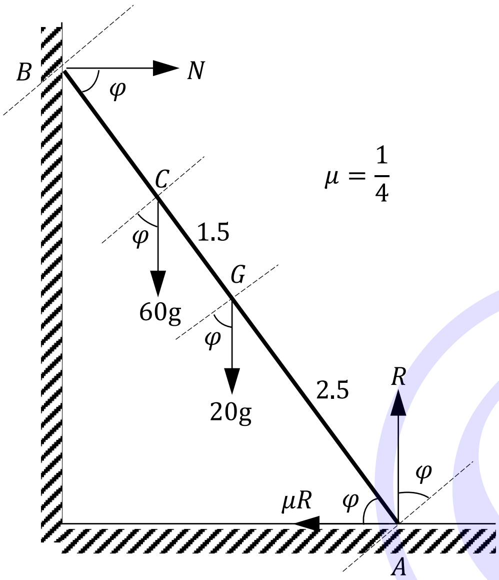
Example 19

The figure below shows a ladder AB resting in equilibrium with one end A on rough horizontal ground and the other end B against a smooth vertical wall. The ladder is modelled as a uniform rod of length 5 metres and mass 20 kg, and lies in a vertical plane perpendicular to the wall and the ground, inclined at an angle φ to the horizontal. When a person of mass 60 kg stands at a point C on the ladder, where $AC = 4$ metres the ladder is at the point of slipping.



Given that the coefficient of friction between the ladder and the ground is $\frac{1}{4}$, find the value of φ, to the nearest degree.

Solution



$(\uparrow)$; $R = 60\text{g} + 20\text{g} = 80\text{g}$

$(\rightarrow)$; $N = \mu R$

$\curvearrowright A$; $20\text{g}\cos\varphi \times 2.5 + 60\text{g}\cos\varphi \times 4 = N\sin\varphi \times 5$

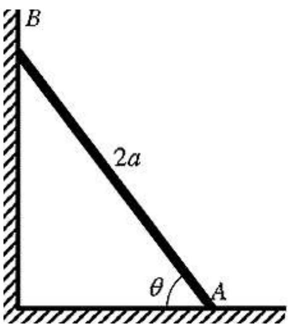
$$290\text{g}\cos\varphi = 5(\mu R)\sin\varphi$$

$$240\text{g}\cos\varphi = 5\left(\frac{1}{4} \times 80\text{g}\right)\sin\varphi$$

$$\tan\varphi = 2.9$$

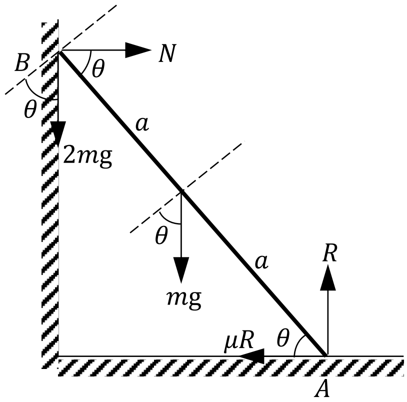
$$\varphi = \tan^{-1} 2.9 = 71°$$

Example 20



The figure above shows a uniform ladder AB of length $2a$ and of mass m resting with the end A on rough horizontal ground and the end B against a smooth vertical wall. The ladder is inclined at an angle θ to the ground. When a child of mass $2m$ is standing on the ladder at B, the ladder is about to slip. Given that the coefficient of friction between the ladder and the ground is $\frac{5}{12}$, find the value of θ.

Solution



$(\uparrow)$; $R = 3mg$

$(\rightarrow)$; $N = \mu R$

$\curvearrowright A$; $mg\cos\theta \times a + 2mg\cos\theta \times 2a = N\sin\theta \times 2a$

$$5mg\cos\theta = 2N\sin\theta$$

$$5mg\cos\theta = 2(\mu R)\sin\theta$$

$$5mg\cos\theta = 2 \times \frac{5}{12} \times 3mg\sin\theta$$

$$\cos\theta = \frac{1}{2}\sin\theta$$

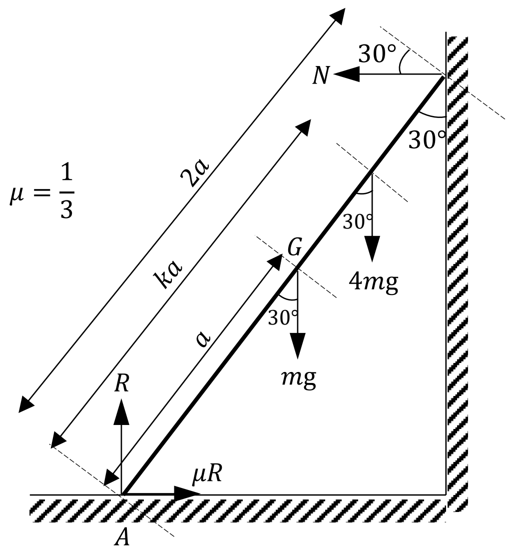
$$\tan\theta = 2$$

$$\theta = 63.4°$$

Example 21

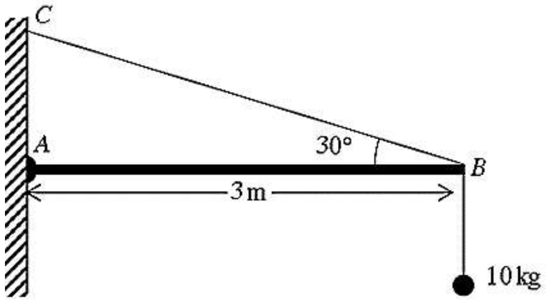
A uniform ladder AB of mass m and length $2a$ has one of its end A on rough horizontal ground and the other end B against a smooth vertical wall. The ladder lies in a vertical plane perpendicular to the wall and the ground, and makes an angle of 30° with the vertical wall. The coefficient of friction between the ladder and the ground is $\frac{1}{3}$. The greatest distance from A that a man of mass $4m$ can walk up this ladder is ka, where k is a positive constant. By modelling the man as a particle and the ladder as a uniform rod, determine the value of k.

Solution



Rods and Hinged joints

Example 29



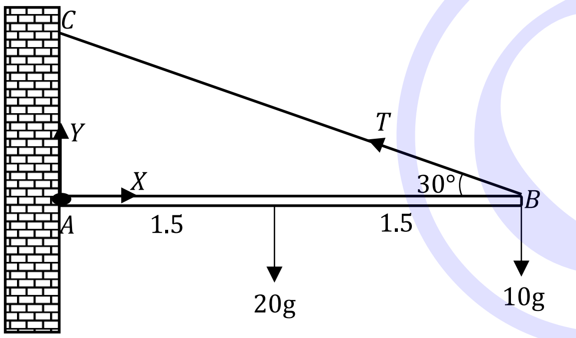
The figure above shows a uniform rod AB, of length 3 m and of mass 20 kg, smoothly hinged at the point A, which lies on a vertical wall. A particle, of mass 10 kg, is suspended from the end B of the rod. The rod is kept in a horizontal position by a light inextensible string BC, where C lies on the same wall vertically above A. The plane ABC is perpendicular to the wall and the angle ABC is 30°.

(a) Determine the tension in the string.

(b) Show that the reaction at the hinge has magnitude $98\sqrt{13}\ N$

Solution

(a)



$(\uparrow)$; $Y + T\sin 30° = 20\text{g} + 10\text{g}$

$(\rightarrow)$; $X = T\cos 30°$

$\curvearrowright A$; $(20\text{g} \times 1.5) + (10\text{g} \times 3) = T\sin 30° \times 3$

$$30\text{g} + 30\text{g} = \frac{3}{2}T$$

$$\frac{3}{2}T = 60\text{g}$$

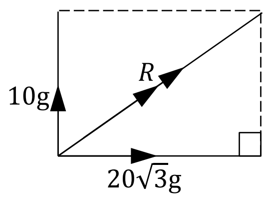
$$T = 40\text{g} = 392\text{ N}$$

(b) Using the vertical and horizontal equations with $T = 392$

$$X = T\cos 30° = \frac{40\text{g}\sqrt{3}}{2} = 20\text{g}\sqrt{3}$$

$$Y + T\sin 30° = 30\text{g}$$

$$Y = 30\text{g} - 40\text{g} \times \frac{1}{2} = 10\text{g}$$



Total reaction, $R = \sqrt{(10\text{g})^2 + \left(20\sqrt{3}\text{g}\right)^2}$

$$= \sqrt{100\text{g}^2 + 1200\text{g}^2}$$

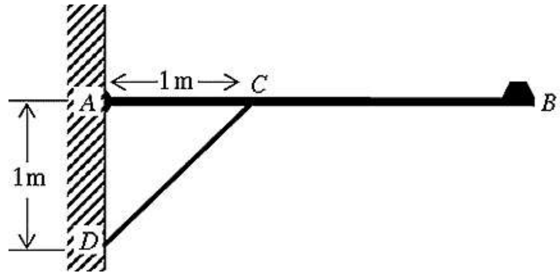
$$= \sqrt{1300\text{g}^2}$$

$$= \sqrt{100\text{g}^2} \times \sqrt{13}$$

$$= 10\text{g}\sqrt{13}$$

$$= 98\sqrt{13}\ N$$

Example 30



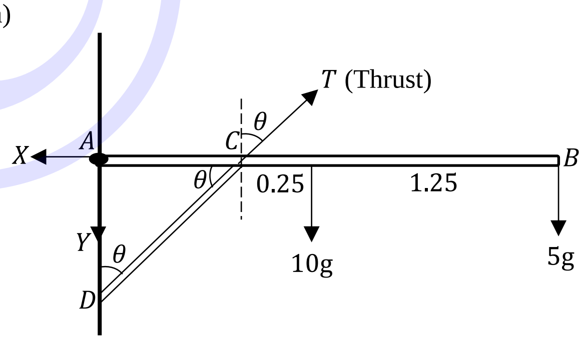
The figure above shows a uniform rod AB, of length 2.5 m and mass 10 kg, with one of its ends A smoothly hinged vertical wall. The rod is kept in equilibrium in a horizontal position by a light rigid strut DC, where D lies on the same wall vertically below A and C lies on the rod such that $AC = AD = 1$ m. A particle of mass 5 kg is placed at B. The plane ACD is perpendicular to the wall.

(a) Calculate the force exerted by the strut on the rod.

(b) Determine the magnitude and direction of the force exerted by the hinge on the rod AB.

Solution

(a)



ACD is right angled and isosceles, $\therefore\ \theta = 45°$

$(\uparrow)$; $T\sin\theta = Y + 10\text{g} + 5\text{g}$

$(\rightarrow)$; $X = T\cos\theta$

$\curvearrowright A$; $T\sin\theta \times 1 = 10\text{g} \times 1.25 + 5\text{g} \times 2.5$

$$T\sin 45° = 25\text{g}$$

$$T = 245\sqrt{2} = 346\ N$$

(b) $X = T\cos\theta = 245\sqrt{2}\cos 45° = 245$

$$T\sin\theta = Y + 15\text{g}$$

$$\left(245\sqrt{2}\right)\sin 45° = Y + 147$$

$$245 = Y + 147$$

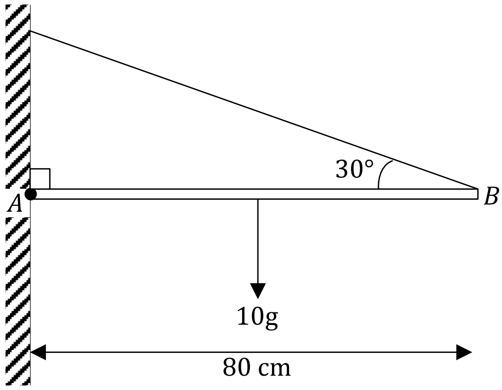
$$Y = 98$$

Determine
(a) the tension in the wire
(b) the normal reaction between the ground and the pipe
(c) the frictional force acting on the pipe
(d) the coefficient of friction between the ground and the pipe if the pipe is in limiting equilibrium

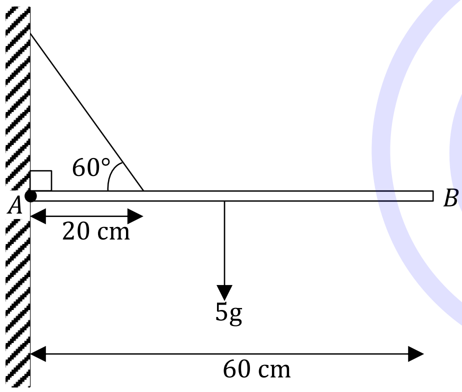
[Ans: (a) 10400 N (b) 9310 N (c) 1810 N (d) 0.195]

9. The diagrams below show uniform rods AB hinged at A being held in equilibrium by a piece of string.

(a)



(b)



For each one, determine
(i) the tension in the string
(ii) the magnitude and direction of the reaction at the hinge A

[Ans: (a) (i) 98 N (ii) 98 N at 30° from AB anticlockwise (b) (i) 84.9 N, 49 N at 30° from AB clockwise]

10. A uniform bar AB of weight W and length $4a$ rests on a rough peg at a point C, where $AC = a$. The bar is maintained in equilibrium, with A above B, by a force of magnitude $\frac{1}{2}W$ acting at A in a direction perpendicular to the bar.
(a) Find the angle which the bar makes with the horizontal
(b) Given that the bar is about to slip, find the coefficient of friction between the bar and the peg.

[Ans: 60°; $\frac{\sqrt{3}}{2}$]

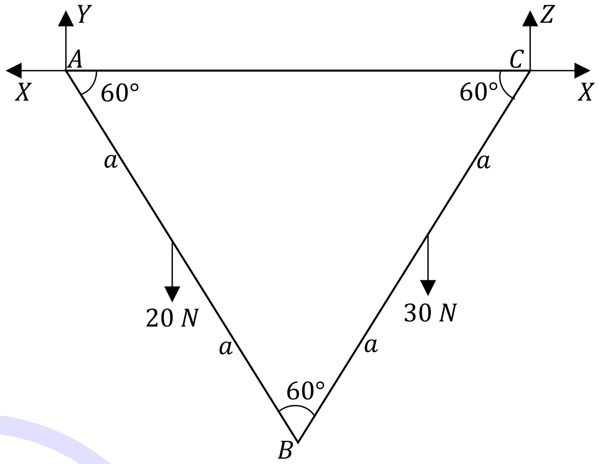
Jointed rods

Example 38

Two uniform rods AB and BC, each of length $2a$ and of mass 2 kg are smoothly hinged at B. The ends A and C are each smoothly hinged to two points in the same horizontal straight line and distance $2a$ apart. Find the horizontal and vertical components of the reaction at each hinge.

Solution

For the whole system



By symmetry, the horizontal reactions at A and C must be equal.

Let X and Y be the horizontal and vertical components of the reactions at A

Let X and Z be the horizontal and vertical components of the reaction at C.

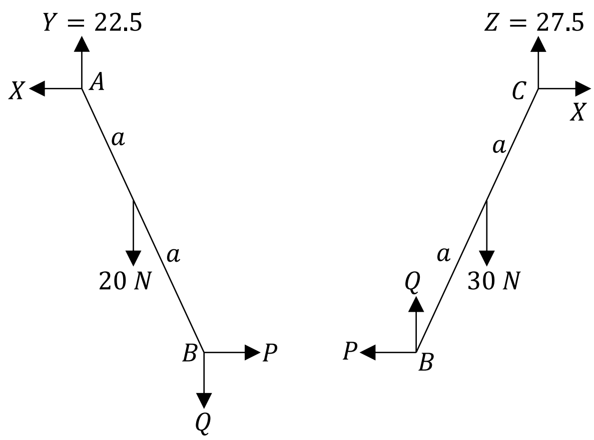
Resolve (↑); $Y + Z = 50$

Moments about A; $Z \times 2a = 20 \times \frac{1}{2}a + 30 \times \frac{3}{2}a$

$$Z = 27.5\ N$$
$$Y + 27.5 = 50$$
$$Y = 22.5\ N$$

For separate bodies

Let P and Q be the horizontal and vertical components of the reaction at B



Moments about B for BC;

$$30 \times \frac{1}{2}a + X \times 2a\sin 60° = Z \times 2a\cos 60°$$

$$T = \frac{W}{2}$$

(b) Let X, Y be the horizontal and vertical components of the action of the hinge at A on AC

Resolving horizontally for AC;

$$X = T\cos 30° = \frac{W}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}W$$

Resolving vertically for AC,

$$Y + W = W + T\sin 30°$$

$$Y = \frac{1}{2}T = \frac{1}{4}W$$

The resultant reaction R at A is given by

$$R = \sqrt{X^2 + Y^2} = W\sqrt{\frac{3}{16} + \frac{1}{16}} = \frac{1}{2}W$$

It is inclined to the horizontal at an angle

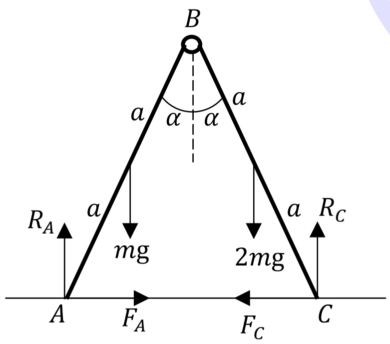
$$\tan^{-1}\frac{Y}{X} = \tan^{-1}\frac{1}{\sqrt{3}}$$

Example 41

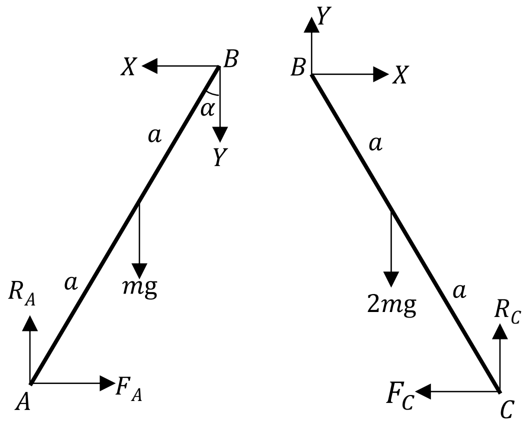
Two uniform rods AB, BC of equal lengths are freely joined together at B. The mass of AB is m and that of BC is $2m$. The rods stand in equilibrium in a vertical plane on a rough horizontal plane, and the angle $ABC = 2\alpha$. The coefficient of friction at A is μ_1, and the coefficient of friction at C is μ_2. Find the horizontal and vertical components of the reaction between the rods at B and prove that $\mu_1 \geq \frac{3}{5}\tan\alpha$ and $\mu_2 \geq \frac{3}{7}\tan\alpha$.

<u>Solution</u>

For the whole system



For separate bodies



For rod AB, taking moments about A,

$$X(2a\cos\alpha) = Y(2a\sin\alpha) + mg\,(a\sin\alpha)$$

$$X = \left(Y + \frac{mg}{2}\right)\tan\alpha \quad \ldots\text{(i)}$$

For rod BC, taking moments about C,

$$X(2a\cos\alpha) + Y(2a\sin\alpha) = 2mg(a\sin\alpha)$$

$$X = (mg - Y)\tan\alpha \;\ldots\text{. (ii)}$$

Adding (i) and (ii);

$$2X = \frac{3mg}{2}\tan\alpha$$

$$X = \frac{3mg}{4}\tan\alpha$$

Subtracting (ii) from (i);

$$0 = \left(2Y - \frac{mg}{2}\right)\tan\alpha$$

$$Y = \frac{mg}{4}$$

Thus the horizontal and vertical components of the reaction at B are $\frac{3}{4}mg\tan\alpha$ and $\frac{1}{4}mg$

Resolving horizontally for each rod;

$$X = F_A + F_C = \frac{3mg}{4}\tan\alpha$$

Resolving vertically for each rod;

$$R_A = mg + Y$$

$$R_A = mg + \frac{mg}{4} = \frac{5mg}{4}$$

$$R_C + Y = 2mg$$

$$R_C + \frac{mg}{4} = 2mg$$

$$R_C = \frac{7mg}{4}$$

At A, $\frac{F_A}{R_A} = \frac{3mg\tan\alpha}{5mg} = \frac{3}{5}\tan\alpha$

At C, $\frac{F_C}{R_C} = \frac{3mg\tan\alpha}{7mg} = \frac{3}{7}\tan\alpha$

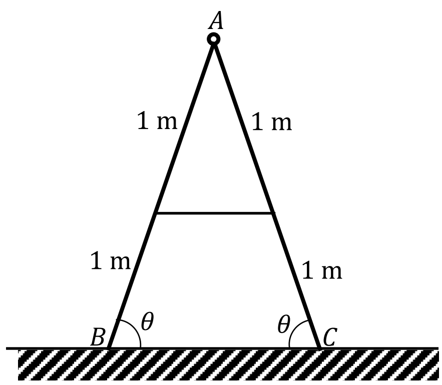
But, since the system is in equilibrium,

$$\mu_1 \geq \frac{F_A}{R_A} \text{ and } \mu_2 \geq \frac{F_C}{R_C}$$

Therefore,

$$\mu_1 \geq \frac{3}{5}\tan\alpha \text{ and } \mu_2 \geq \frac{3}{7}\tan\alpha.$$

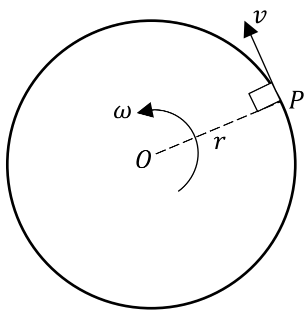
Example 42



A pair of steps can be modelled as a uniform rod AB, of mass 24 kg and length 2 m, freely hinged at A to a uniform rod AC, of mass 6 kg and length 2 m. The mid-points of the rods are joined by a light inextensible string. The rods rest in a

CIRCULAR MOTION

Consider a particle P of mass m moving in a **horizontal circle**, centre O, radius r, with **constant speed** v



The **linear velocity** v of P is directed along the tangent to the circle at P.

The constant **angular velocity** ω of P is $\boldsymbol{\omega = \frac{v}{r}}$

ω is measured in **radians per second** (rad s^{-1})

There is no acceleration along the tangent since the particle moves with constant speed around the circle.

The **acceleration** of P is in the direction $\overrightarrow{PQ}$ i.e. towards the centre of the circle.

$$\boldsymbol{a = \frac{v^2}{r} = \omega r^2}$$

By Newton's second law, this acceleration must be produced by a force which is also directed **towards the centre of the circle.**

So the **equation of motion** for the particle is

$$\textbf{force} = \boldsymbol{\frac{mv^2}{r}} \textbf{ or } \boldsymbol{m\omega^2 r}$$

This force may be the tension in a string, a frictional force, a gravitational force, etc.

The **time period** is the time taken for the object to complete one revolution. One revolution is equivalent to a rotation of 2π radians.

Example 1

Find the velocity and acceleration of a particle moving in a horizontal circle, radius 20 cm, at a constant angular velocity of 30 revolutions per minute.

Solution

Angular velocity, $\omega = 30$ revolutions per minute

$$= 30 \times \frac{2\pi}{60} \text{ rad s}^{-1}$$

$$= \pi \text{ rad s}^{-1}$$

Velocity, $v = \omega r = 0.2 \times \pi = 0.628 \text{ ms}^{-1}$

Acceleration, $a = \omega^2 r = \pi^2(0.2) = 1.97 \text{ ms}^{-2}$

Example 2

A car travels around a circular curve at 40 ms^{-1}. The radius of the curve is 20 m. Determine

(a) the car's angular speed

(b) the time the car takes to complete one circuit

Solution

(a) $v = 40 \text{ ms}^{-1}$; $\omega = ?$; $r = 20$ m

$$v = \omega r$$

$$40 = \omega \times 20$$

$$\omega = 2 \text{ rad s}^{-1}$$

(b) $\omega = 2 \text{ rad s}^{-1}$, $T = ?$

$$T = \frac{2\pi}{\omega}$$

$$T = \frac{2\pi}{\pi} = \pi = 3.14 \text{ s}$$

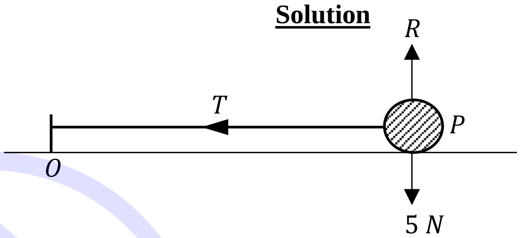
Motion in a horizontal circle

When a particle is moving in a horizontal circle, its speed is constant.

Example 1

A particle of mass 0.5 kg is attached by a light inextensible string, length 2 m, to a fixed point O on top of a smooth horizontal table. The particle is made to rotate in a horizontal circle with the string taut at a constant speed of 8 ms^{-1}. Calculate the tension in the string.

Solution



$$F = ma = \frac{mv^2}{r}$$

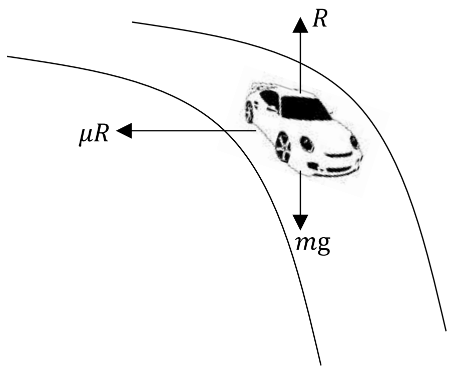
Equation of motion along PO is

$$T = 0.5 \times \frac{8^2}{2} = 16\ N$$

Example 2

A car of mass m moves in a circular path of radius 75 m round a bend in a road. The maximum speed at which it can move without slipping sideways on the road is 21 ms^{-1}. Given that this section of the road is horizontal, calculate the coefficient of friction between the car and the road.

Solution



$$R = mg$$

Friction force $= \mu R = \mu mg$

$$\mu mg = \frac{mv^2}{r}$$

$$\mu = \frac{v^2}{rg} = \frac{21^2}{75 \times 9.8} = 0.6$$

$$T_A - T_B = 3 \times 5 \times 9.8$$

$$T_A - T_B = 147 \quad \ldots \text{(i)}$$

Acceleration towards the centre;

$$T_A \sin\theta + T_B \sin\theta = m \times 0.6 \sin\theta \times \omega^2$$

$$T_A + T_B = 5 \times 0.6 \times 10^2$$

$$T_A + T_B = 300 \quad \ldots\ldots \text{(ii)}$$

(i) + (ii);

$$2T_A = 447$$

$$T_A = 223.5\ N$$

$$T_B = 300 - T_A = 300 - 223. = 76.5\ N$$

Note: The string AQ can never be slack but the string BQ could become slack. In order that it shall remain taut, its tension T_B must not become negative i.e. $T_B \geq 0$

Example 7

Two rigid light rods AB, BC each of length $\frac{1}{2}$ m are smoothly jointed at B and the rod AB is smoothly jointed at A to a fixed smooth vertical rod. The joint at B has a particle of mass 2 kg attached. A small ring, of mass 1 kg is smoothly jointed to BC at C and can slide on the vertical rod below A. The ring rests on a smooth horizontal ledge fixed to the vertical rod at a distance $\frac{\sqrt{3}}{2}$ m below A. The system rotates about the vertical rod with constant angular velocity 6 radians per second. Calculate

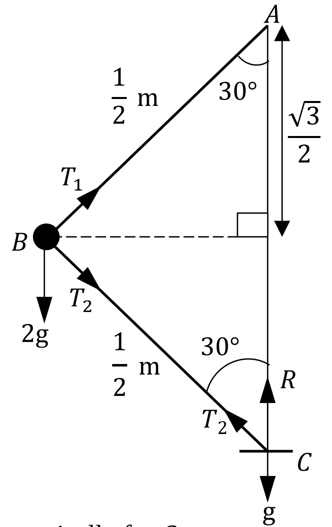
(a) the forces in the rods AB and BC

(b) the force exerted by the ledge on the ring

<u>Solution</u>

From trigonometry, the angles at A and C will be 30°

Let T_1 and T_2 be the forces in the rods AB and BC respectively and let R be the force exerted on the ring by the ledge.



Resolving vertically for C;

$$T_2 \cos 30° + R = \text{g}$$

$$\frac{T_2\sqrt{3}}{2} + R = \text{g} \quad \ldots\ldots \text{(i)}$$

Resolving vertically for B;

$$T_1 \cos 30° = T_2 \cos 30° + 2\text{g}$$

$$\frac{T_1\sqrt{3}}{2} = \frac{T_2\sqrt{3}}{2} + 2\text{g}$$

$$3T_1 = 3T_2 + 4\sqrt{3}\text{g}$$

$$3T_1 = 3T_2 + 67.9 \quad \ldots\ldots \text{(ii)}$$

Equation of motion for B;

$$T_1 \cos 60° + T_2 \cos 60° = 2 \times \frac{1}{4} \times 6^2$$

$$\frac{T_1}{2} + \frac{T_2}{2} = 18$$

$$T_1 + T_2 = 36 \quad \ldots\ldots \text{(iii)}$$

Solving (ii) and (iii);

$$3(36 - T_2) = 3T_2 + 67.9$$

$$108 - 3T_2 = 3T_2 + 67.9$$

$$6T_2 = 40.1$$

$$T_2 = 6.68\ N$$

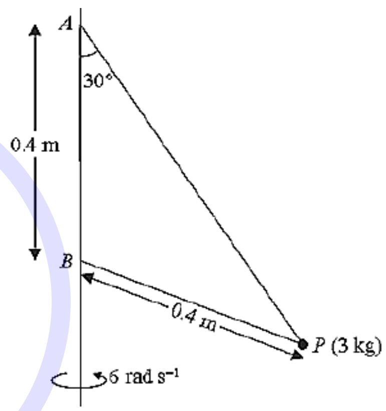
$$T_1 = 36 - T_2 = 36 - 6.68 = 29.32\ N$$

Substituting for T_2 in (i);

$$\frac{6.68\sqrt{3}}{2} + R = 9.8$$

$$R = 9.8 - \frac{6.68\sqrt{3}}{2} = 4.01\ N$$

Example 8

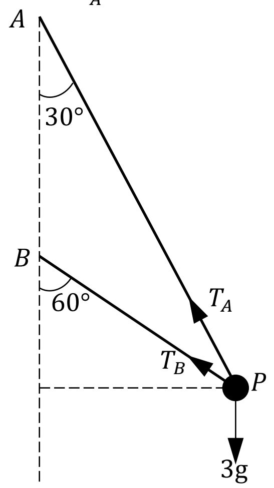


A particle P of mass 3 kg is attached by two light inextensible strings to two fixed points A and B on a fixed vertical pole. Both strings are taut and P is moving in a horizontal circle with constant angular speed 6 rad s^{-1}. String AP is inclined at 30° to the vertical. String BP has length 0.4 m and A is 0.4 m vertically above B, as shown above. Find the tension in

(i) AP (ii) BP.

<u>Solution</u>

Let the tension in AP be T_A and that in BP be T_B



$$\frac{5}{3}mg \times \frac{4}{5} = m \times \frac{8g}{9a} \times r$$

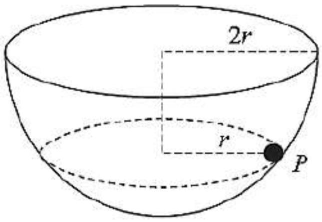
$$r = \frac{3}{2}a$$

$$\tan\alpha = \frac{r}{h} \Rightarrow h = \frac{r}{\tan\alpha}$$

$$h = \frac{3}{2}a \times \frac{4}{3} = 2a$$

The height of C above V is $2a$

Example 11

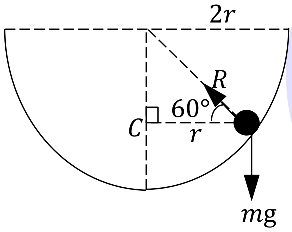


A hemispherical bowl of internal radius $2r$ is fixed with its circular rim horizontal. A particle P is moving in a horizontal circle of radius r on the smooth inner surface of the bowl, as shown in the figure above. Particle P is moving with constant angular speed ω. Show that $\omega = \sqrt{\frac{g\sqrt{3}}{3r}}$

Solution

$$\cos^{-1}\frac{r}{2r} = 60°$$

Let the mass of the particle be m



$$\uparrow \qquad R\sin 60° = mg$$

$$\leftarrow \qquad R\cos 60° = m\omega^2 r$$

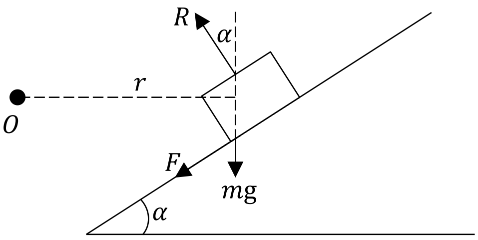
$$\tan 60° = \frac{g}{\omega^2 r}$$

$$\omega^2 = \frac{g}{r\tan 60°} = \frac{g}{r(\sqrt{3})}$$

$$\omega = \sqrt{\frac{g}{r(\sqrt{3})}} = \sqrt{\frac{g\sqrt{3}}{3r}}$$

Car rounding a bend on a banked track

(a) Car cornering at **maximum speed**



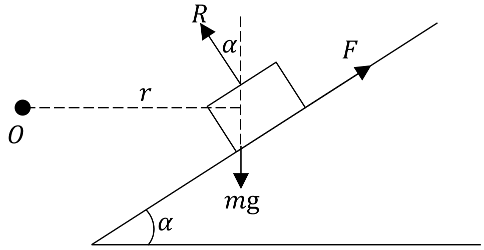
Resolve (↑);

$$R\cos\alpha = mg + F\sin\alpha$$

Equation of motion along $\overrightarrow{PO}$:

$$R\sin\alpha + F\cos\alpha = \frac{mv^2}{r}$$

(b) Car cornering at **minimum speed**



Resolve (↑);

$$R\cos\alpha + F\sin\alpha = mg$$

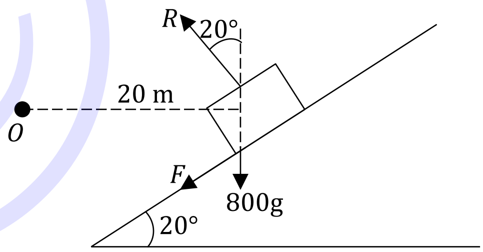
Equation of motion along $\overrightarrow{PO}$:

$$R\sin\alpha - F\cos\alpha = \frac{mv^2}{r}$$

Example 12

A car of mass 800 kg is driven at constant speed v ms^{-1} round a bend in a race track. Around the bend, the track is banked at 20° to the horizontal and the path followed by the car can be modelled as a horizontal circle of radius 20 m. The coefficient of friction between the car tyres and the track is 0.5. Given that the tyres do not slip sideways on the track, find the maximum value of v.

Solution



Resolving vertically (↑);

$$R\cos 20° = F\cos 70° + 800g \ \ldots \text{(i)}$$

Using Newton's second law (→);

$$R\cos 70° + F\cos 20° = 800\frac{v^2}{20} \quad \ldots\text{(ii)}$$

$$F = \mu R = 0.5R$$

From (i); $R\cos 20° = 0.5R\cos 70° + 800g$

$$R = \frac{800g}{(\cos 20° - 0.5\cos 70°)} = 10199.3\ N$$

From (ii);

$$40v^2 = 10199.3\cos 70° + 0.5(10199.3)\cos 20°$$

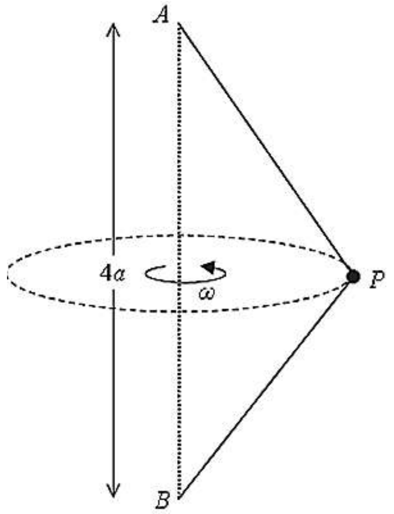
$$40v^2 = 8240.5$$

$$v = 14.38 \text{ ms}^{-1}$$

Example 13

A bend of a race track is modelled as an arc of a horizontal circle of radius 120 m. The track is not banked at the bend. The maximum speed at which a motorcycle can be ridden round the bend without slipping sideways is 28 ms^{-1}. The motorcycle and its rider are modelled as a particle and air resistance is assumed to be negligible.

of mass m. An identical string has one end attached to the fixed point B, where B is vertically below A and $AB = 4a$, and the other end attached to P, as shown below.

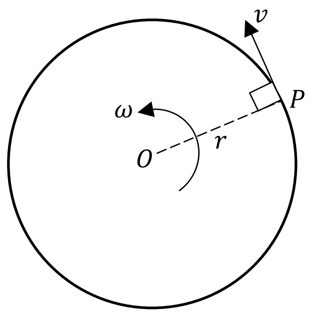


The particle is moving in a horizontal circle with constant angular speed ω, with both strings taut and inclined at 30° to the vertical. The tension in the upper string is twice the tension in the lower string. Find ω in terms of a and g.

[Ans: $\sqrt{\frac{3g}{2a}}$]

28. A particle moves with constant speed u in a horizontal circle of radius a on the inside of a fixed smooth hemispherical shell of radius $2a$. Show that $u^2\sqrt{3} = ag$

Motion in a vertical circle



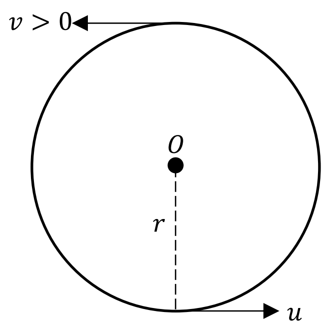
When a particle P, of mass m is moving in a vertical circle, centre O, radius r, its speed v is variable.
The particle P has an acceleration a in the direction $\overrightarrow{PO}$ i.e. towards the centre of the circle, given by $a = \frac{v^2}{r}$ or $\omega^2 r$. So this acceleration is variable too.
By Newton's 2nd law, the acceleration towards O must be produced by a force which is also directed towards O.
So the equation of motion for the particle along the radius $\overrightarrow{PO}$ is

$$\text{force} = \frac{mv^2}{r} \text{ or } m\omega^2 r$$

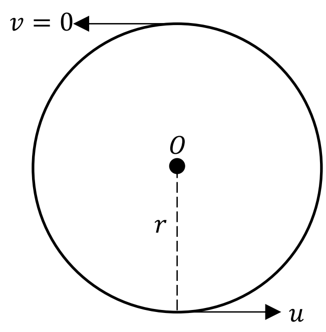
The variable force will be the resultant of the weight force mg and at least one other force.

Types of motion
There are two main types of motion in vertical circle for a particle with initial speed as described below.

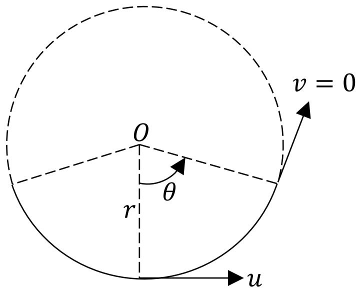
1. The particle **cannot leave the circular path**, e.g. a bead threaded on a vertical wire. (**motion restricted to circular path)**
   The particle can do one of these three things
   (i) **Complete the circle** if $v > 0$ at the top and $u^2 >$ 4gr



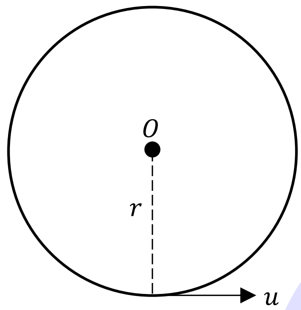
   (ii) Come to **rest** at top if $v = 0$ at the top and $u^2 = 4$gr



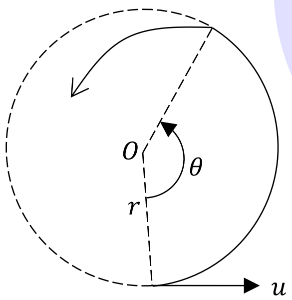
(iii) **Oscillate** if $v = 0$ for $0 < \theta < \pi$ and $u^2 < 4gr$



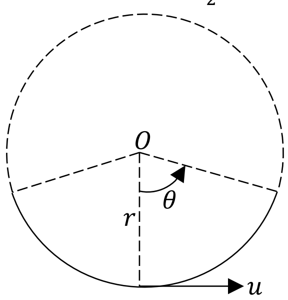
2. The particle **can leave the circular path** and become a projectile e.g. a particle attached to a string, a body moving on the inside/outside surface of a sphere, etc.
The particle can do one of these three things. (T is the tension in the string or normal reaction)
   (i) Complete the circle if $T \geq 0$ for all values of θ and $u^2 \geq 5gr$



   (ii) Become a projectile if $T = 0$ for $\frac{\pi}{2} < \theta \leq \pi$ and $2gr < u^2 < 5gr$



   (iii) Oscillate if $v = 0$ for $\theta \leq \frac{\pi}{2}$ and $u^2 \leq gr$

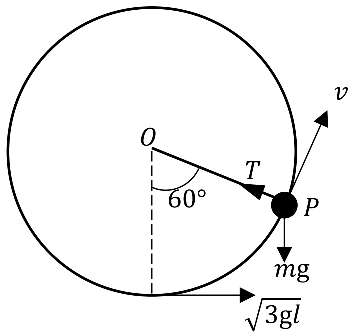


Problem solving:
When solving problems in which a particle P describes a vertical circle, centre O:
(a) Draw a clear force diagram
(b) Use conservation of mechanical energy i.e.
Initial $(PE + KE) = (PE + KE)$ at any point
(c) Write down the equation of motion for the particle along the radius $\overrightarrow{PO}$

Note: It is most important, when tackling a problem on vertical circular motion, to decide whether or not the particle can leave the circular path or is restricted to it. The special conditions that can be applied are different in the two cases.

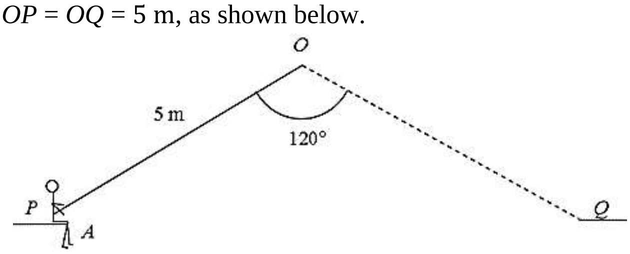
Example 1
A particle of mass m is suspended from a fixed point O by a light inextensible string of length l. When the particle is hanging freely in equilibrium, it is given a horizontal speed of $\sqrt{3gl}$. Find the tension in string when the angle between the string and the downward vertical is 60°

Solution



Height of P when the string is inclined at 60° is given by
$$h = l - l\cos 60° = l(1 - \cos 60°)$$
By conservation of mechanical energy;
$$\frac{1}{2}m\left(\sqrt{3gl}\right)^2 = \frac{1}{2}mv^2 + mgl(1 - \cos 60°)$$
$$\frac{1}{2}m(3gl) = \frac{1}{2}mv^2 + mgl\left(\frac{1}{2}\right)$$
$$3gl = v^2 + gl$$
$$v^2 = 2gl$$
Equation of motion along $\overrightarrow{PO}$;
$$T - mg\cos 60° = \frac{mv^2}{l}$$
$$T - mg\left(\frac{1}{2}\right) = \frac{m(2gl)}{l}$$
$$T = 2mg + \frac{mg}{2} = \frac{5}{2}mg$$

Example 2
A trapeze artiste of mass 60 kg is attached to the end A of a light inextensible rope OA of length 5 m. The artiste must swing in an arc of a vertical circle, centre O, from a platform P to another platform Q, where PQ is horizontal. The other end of the rope is attached to the fixed point O which lies in the vertical plane containing PQ, with ∠POQ = 120° and $OP = OQ$ = 5 m, as shown below.

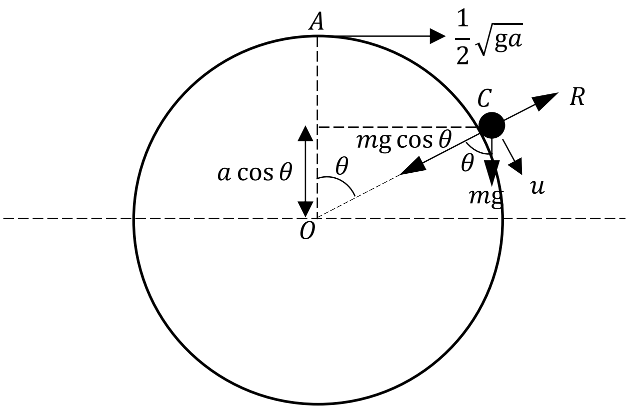


at the point C, where $\angle AOC = \theta$, and strikes the plane at the point P, as shown.

(a) Show that $\cos\theta = \frac{3}{4}$

(b) Find magnitude and direction of the velocity of the particle as it reaches P.

Solution



(a) Let the speed at C be u

Using conservation of mechanical energy;

Total ME at A = Total mechanical energy at C

$$\frac{1}{2}m\left(\frac{1}{2}\sqrt{ga}\right)^2 + mga = \frac{1}{2}mu^2 + mga\cos\theta$$

$$\frac{ga}{8} + ga = \frac{u^2}{2} + ga\cos\theta$$

$$\frac{9ga}{8} = \frac{u^2}{2} + ga\cos\theta$$

$$4u^2 = 9ga - 8ga\cos\theta$$

$$u^2 = \frac{ga}{4}(9 - 8\cos\theta)$$

Equation of motion along CO;

$$mg\cos\theta - R = \frac{mu^2}{a}$$

$$mg\cos\theta - R = \frac{mg}{4}(9 - 8\cos\theta)$$

When the particle leaves the surface, $R = 0$

$$mg\cos\theta = \frac{mg}{4}(9 - 8\cos\theta)$$

$$4\cos\theta = 9 - 8\cos\theta$$

$$12\cos\theta = 9$$

$$\cos\theta = \frac{9}{12} = \frac{3}{4}$$

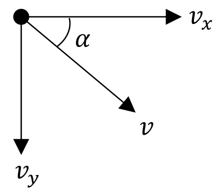
(b) Let the velocity of the particle as it reaches P be v at an angle α below the horizontal

At C, $u^2 = \frac{ga}{4}(9 - 8\cos\theta) = \frac{ga}{4}\left(9 - 8\left(\frac{3}{4}\right)\right) = \frac{3}{4}ga$

$$u = \sqrt{\frac{3ga}{4}}$$

$$(\rightarrow)\ u_x = u\cos\theta = \sqrt{\frac{3ga}{4}} \times \frac{3}{4} = \sqrt{\frac{27ga}{64}} = 2.033\sqrt{a}$$

$$(\downarrow)\ u_y = u\sin\theta = \sqrt{\frac{3ga}{4}} \times \frac{\sqrt{7}}{4} = \sqrt{\frac{21ga}{64}} = 1.792\sqrt{a}$$



$$h = a + a\cos\theta = a + \frac{3a}{4} = \frac{7a}{4}$$

$$v_y^2 = u_y^2 + 2gh$$

$$v_y^2 = \frac{21ga}{64} + 2g \times \frac{7a}{4} = \frac{245ga}{64}$$

$$v_y = \sqrt{\frac{245ga}{64}}$$

$$v^2 = v_y^2 + u_x^2 = \frac{245ga}{64} + \frac{27ga}{64} = \frac{272ga}{64}$$

$$v = \sqrt{\frac{272ga}{64}} = 6.45\sqrt{a}$$

$$\tan\alpha = \frac{v_y}{u_x} = \frac{\sqrt{\frac{245ga}{64}}}{\sqrt{\frac{27ga}{64}}} = \sqrt{\frac{245}{27}} = 3.012$$

$$\alpha = 72°$$

The velocity as it reaches P is $6.45\sqrt{a}$ at $72°$ below the horizontal

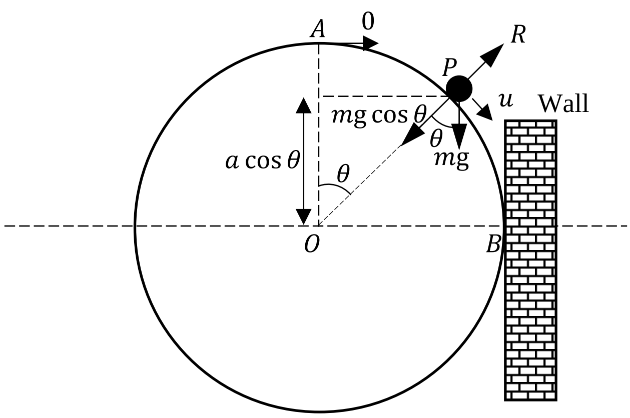
Example 10

A smooth sphere with centre O and radius a is fixed with a point B of its surface in contact with a vertical wall. A particle P of mass m rests at the highest point A of the sphere. It is slightly displaced so that it moves from rest towards the wall in the plane OAB. If at any instant in the subsequent motion, the line OP makes an angle θ with the line OA and the particle is still in contact with the sphere, find the expressions for the velocity of P at this point and for the reaction of the sphere upon it in terms of m, g and θ. Prove that the particle leaves the sphere when $\cos\theta = \frac{2}{3}$ and that its speed is then $\sqrt{(2ga/3)}$.

Show that P hits the wall at a height $\frac{1}{8}a(5\sqrt{5} - 9)$ above B

Solution

Let the velocity of the particle at this point be u



(a) At B, $mg\cos 60° + R = \frac{mv^2}{a}$

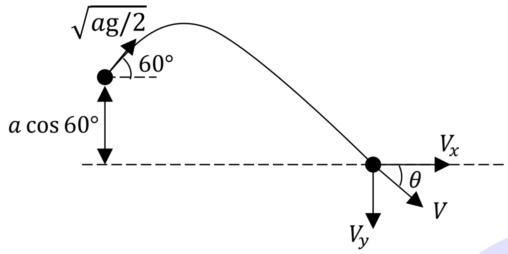
When the particle loses contact, $R = 0$

$$\frac{1}{2}mg = \frac{mv^2}{a}$$
$$v^2 = \frac{ag}{2}$$
$$v = \sqrt{\frac{ag}{2}}$$

(b) Conservation of mechanical energy between A and B

$$\frac{1}{2}mu^2 = mga\sin 30° + \frac{1}{2}m\left(\frac{ag}{2}\right)$$
$$u^2 = ag + \frac{ag}{2} = \frac{3ag}{2}$$
$$u = \sqrt{\frac{3ag}{2}}$$

(c)

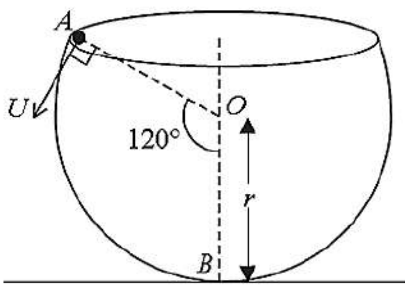


Horizontal velocity = $\sqrt{\frac{ag}{2}}\cos 60° = \frac{1}{2}\sqrt{\frac{ag}{2}} = \sqrt{\frac{ag}{8}}$

Initial vertical velocity = $\sqrt{\frac{ag}{2}}\sin 60° = \sqrt{\frac{3ag}{8}}$

$$V_y^2 = U_y^2 + 2gh$$
$$V_y^2 = \frac{3ag}{8} + 2g \times \frac{a}{2}$$
$$V_y^2 = \frac{11ag}{8}$$
$$V_y = \sqrt{\frac{11ag}{8}}$$
$$\tan\theta = \frac{V_y}{V_x} = \sqrt{\frac{11ag}{8}} \times \sqrt{\frac{8}{ag}} = \sqrt{11}$$
$$\theta = 73.2°$$

Example 12



A hollow sphere has internal radius r and centre O. A bowl with a plane circular rim is formed by removing part of the sphere. The bowl is fixed to a horizontal floor with the rim uppermost and horizontal. The point B is the lowest point of the inner surface of the bowl. The point A, where angle AOB = 120°, lies on the rim of the bowl, as shown above.

A particle P of mass m is projected from A, with speed U at 90° to OA, and moves on the smooth inner surface of the bowl. The motion of P takes place in the vertical plane OAB.

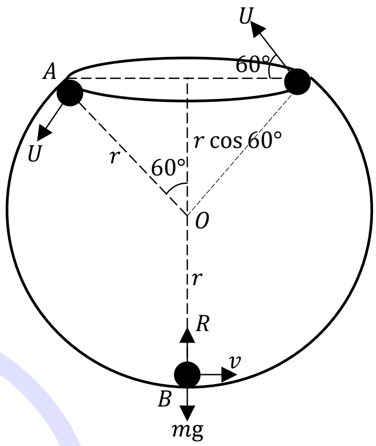
(a) Find, in terms of m, g, U and r, the magnitude of the force exerted on P by the bowl at the instant when P passes through B.

(b) Find, in terms of g, U and r, the greatest height above the floor reached by P.

Given that $U > \sqrt{2gr}$

(c) show that, after leaving the surface of the bowl, P does not fall back into the bowl.

Solution



Let the speed of the particle as it passes through B be v

(a) Using conservation of energy

$$\frac{1}{2}mv^2 = \frac{1}{2}mU^2 + mgr(1+\cos 60°)$$
$$v^2 = U^2 + 2gr\left(\frac{3}{2}\right)$$
$$v^2 = U^2 + 3gr$$

Equation of motion in the direction BO:

$$R - mg = \frac{mv^2}{r}$$
$$R - mg = \frac{m}{r}(U^2 + 3gr)$$
$$R = \frac{mU^2}{r} + 3mg + mg$$
$$R = 4mg + \frac{mU^2}{r}$$

(b) P leaves the bowl with speed U at 60° to the horizontal

Vertical component of velocity = $U\sin 60° = \frac{U\sqrt{3}}{2}$

$$v^2 = u^2 + 2as$$

$v = 0$, $u = \frac{U\sqrt{3}}{2}$; $s = ?$; $a = -g$

$$0 = \left(\frac{U\sqrt{3}}{2}\right)^2 - 2gs$$
$$s = \frac{3U^2}{8g}$$

Greatest height reached = $\frac{3}{2}r + \frac{3U^2}{8g}$

(c) When the ball falls to the level of the rim, the vertical displacement will be zero. Let t be the time taken to level of rim

ELASTICITY

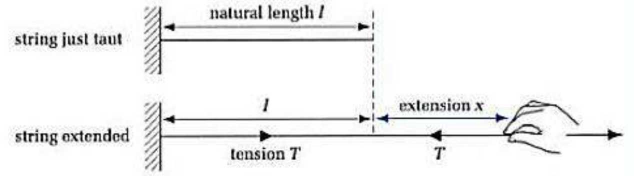
Hooke's law:

The extension in an elastic string is proportional to the tension provided the elastic limit is not exceeded.

$$\boldsymbol{T = kx}$$

The value of k varies from one string to another and is called the **stiffness** of the string.

The modulus of elasticity



For a particular material, the value of k is inversely proportional to the natural (i.e. unstretched) length of the string. This fact can be included in the formula for tension, giving

$$\boldsymbol{T = \frac{\lambda x}{l}}$$

where

$T =$ tension (N), $x =$ extension of the string or spring (m),

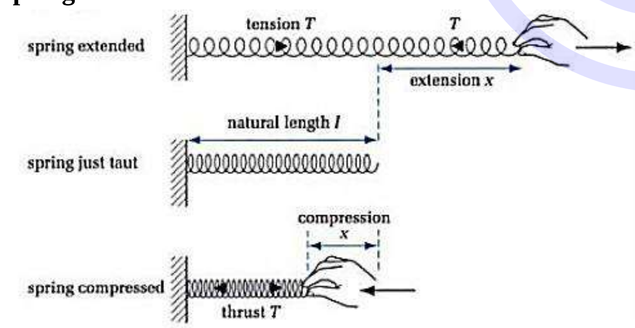
$\lambda =$ modulus of elasticity (N); $l =$ natural length (m)

When $x = l$, then total length is $2l$ and $\lambda = 2T$

Note:

This is the formula most often used for tension at this level in mathematics. The **modulus of elasticity, $\boldsymbol{\lambda}$,** is equal to the tension required to double the length of the string.

Springs



When springs are stretched, they are also subject to Hooke's law. When springs are **compressed**, they exert a **thrust** outward. Hooke's law still applies, but now T represents the thrust and x is the compression.

Example 1

An elastic rope is extended by 20% when it is used to secure goods on a lorry. If the tension in the rope is $140\ N$, find the modulus of elasticity.

Solution

$\lambda =?$, $T = 140\ N$

l and x are not known by since x is 20% of l, then

$$x = 0.2l$$

$$T = \frac{\lambda x}{l}$$

$$140 = \frac{\lambda \times 0.2l}{l}$$

$$140 = 0.2\lambda$$

$$\lambda = 700\ N$$

The modulus of elasticity is $700\ N$

Example 2

The spring in a suspension system exerts a thrust of 3 kN when it is compressed to a length of 17.5 cm. The modulus of elasticity is $24\ kN$. Find the natural length of the spring

Solution

$l =?$, $T = 3\ kN = 3000\ N$, $x = l - 0.175$ m, $\lambda = 24\ kN = 24000\ N$

$$T = \frac{\lambda x}{l}$$

$$3000 = \frac{24000(l - 0.175)}{l}$$

$$3000l = 24000(l - 0.175)$$

$$l = 8(l - 0.175)$$

$$l = 8l - 1.4$$

$$7l = 1.4$$

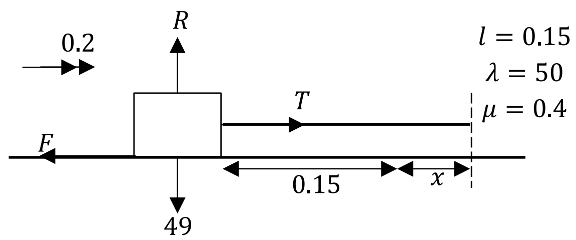
$$l = 0.2 \text{ m}$$

The natural length of the spring is 0.2 m or 20 cm

Example 3

A box of weight $49\ N$ is placed on a horizontal table. It is to be pulled along by a light elastic string with natural length 15 cm and modulus of elasticity $50\ N$. The coefficient of friction between the box and the table is 0.4. If the acceleration of the box is 20 cm s^{-1}and the string is pulled horizontally, what is the length of the string?

Solution



$$a = 20 \text{ cm s}^{-1} = 0.2 \text{ ms}^{-1}$$

Resolving ↑; $R = 49$

Box moving, $F = F_{max} = \mu R = 0.4 \times 49 = 19.6$

Resolving →; using Newton's 2nd law;

$$T - F = ma$$

$$T - 19.6 = 5(0.2)$$

$$T = 20.6$$

By Hooke's law;

$$T = \frac{\lambda x}{l}$$

$$20.6 = \frac{50x}{0.15}$$

$$x = 0.062 \text{ m}$$

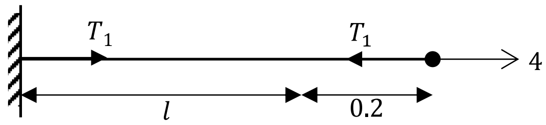
The length of the string is $0.15 + 0.0612 = 0.212$ m

Example 4

An elastic spring is fixed at one end. When a force of 4 N is applied to the end the spring extends by 0.2 m. If the spring hangs vertically supporting a mass of 1 kg at the free end, the spring is of length 2.49 m. Find the natural length and modulus of elasticity of the spring.

Solution

Let the natural length of the spring be l m

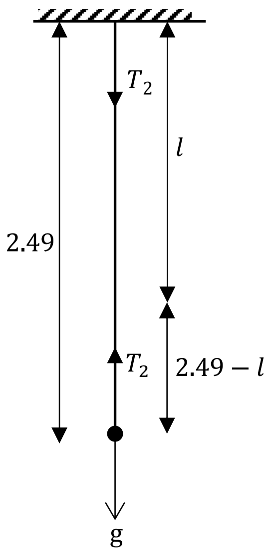


Extending force = Tension

$$T_1 = 4$$

From Hooke's law;

$$T_1 = \lambda\left(\frac{0.2}{l}\right)$$
$$4 = \frac{0.2\lambda}{l}$$
$$\lambda = 20l$$



$$T_2 = \lambda\left(\frac{2.49 - l}{l}\right)$$

But $T_2 = \mathrm{g} = 9.8$

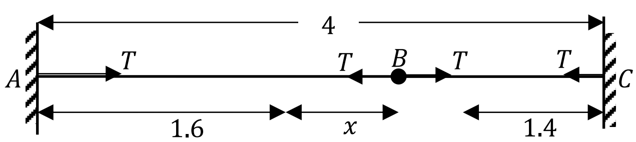
$$9.8 = 20l\left(\frac{2.49 - l}{l}\right)$$
$$9.8 = 49.8 - 20l$$
$$l = 2 \text{ m}$$
$$\lambda = 20l$$
$$\lambda = 20 \times 2 = 40\ N$$

The natural length is 2 m and modulus of elasticity is 40 N

Example 5

Two springs AB and BC are joined together end to end to form one long spring. The natural lengths of the separate springs are 1.6 m and 1.4 m and their moduli of elasticity are 20 N and 28 N respectively. Find the tension in the combined spring if it is stretched between two points 4 m apart.

Solution



Let the extension in spring AB be x, $AB = 1.6 + x$

Length of spring $BC = 4 - (1.6 + x) = (2.4 - x)$

Since the natural length of spring $BC = 1.4$ m

Extension in spring $BC = (2.4 - x) - 1.4 = 1 - x$

Since the point B is in equilibrium, the tensions in AB and BC are equal

Using Hooke's law;

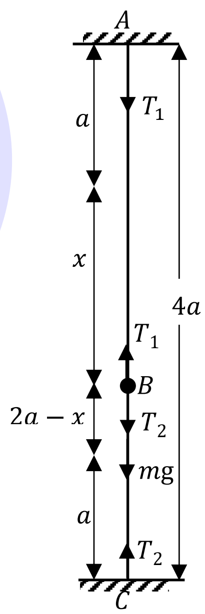
$$T = \frac{\lambda x}{l}$$
$$T_{AB} = 20\frac{x}{1.6} \text{ and } T_{BC} = 28\frac{(1 - x)}{1.4}$$
$$20\frac{x}{1.6} = 28\frac{(1 - x)}{1.4}$$
$$x = 1.6(1 - x)$$
$$x = \frac{1.6}{2.6} = 0.615$$
$$T = \frac{20x}{1.6} = \frac{20(0.615)}{1.6} = 7.69\ N$$

The tension in the spring is 7.69 N

Example 6

Two identical elastic strings AB and BC of natural length a and modulus of elasticity $2mg$ are fastened together at B. Their other ends A and C are fixed to two points $4a$ apart in a vertical line (A above C). A particle of mass m is attached at B. Find the height above C at which the particle rests in equilibrium.

Solution



Let x be the extension in AB.

Extension in $BC = (4a - 2a - x) = 2a - x$

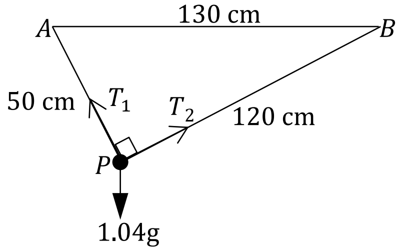
Since the particle at B is in equilibrium,

$$T_1 = T_2 + mg$$

Using Hooke's law

$$T_1 = \frac{\lambda x}{a} = \frac{2mgx}{a}$$
$$T_2 = \frac{\lambda(2a - x)}{a} = \frac{2mg(2a - x)}{a}$$
$$\frac{2mgx}{a} = \frac{2mg(2a - x)}{a} + mg$$
$$\frac{2x}{a} = 4 - \frac{2x}{a} + 1$$

5. A particle, P, of mass 1.04 kg is attached to two strings AP and BP. The points A and B are on the same horizontal level 130 cm apart. The string BP is inextensible and of length 120 cm. The particle hangs in equilibrium with $AP = 50$ cm and $APB = 90°$ as shown below.



(a) Find the tension in each of the of the strings AP and BP

(b) The string AP is elastic and its modulus of elasticity is 9.8 N. Find the natural length of the string AP

[Ans: (a) $T_1 = 9.4\ N$, 3.9 N (b) 25.5 cm]

6. Two springs AB and BC are fastened together at B. The ends A and C are fastened to two fixed points on a smooth horizontal table where AC is 2 m. AB and BC have natural lengths of 0.6 m and 0.8 m and the moduli of elasticity 2 N and 4 N respectively. Find the stretched lengths of AB and BC.

[Ans: 0.96 m; 1.04 m]

7. A body of mass 2.5 kg is attached to the end B of a light elastic string AB of natural length 2 m and modulus 5g N. The mass is suspended vertically in equilibrium by the string whose other end A is attached to a fixed point. Find the depth below A of B when the body is in equilibrium.

[Ans: 3 m]

8. A light elastic string, of unstretched length a and modulus of elasticity W, is fixed at one end to a point on the ceiling of a room. To the other end of the string is attached a particle of weight W. A horizontal force P is applied to the particle and in equilibrium it is found out that the string is stretched to three times its natural length. Calculate

(a) the angle the string makes with the horizontal

(b) the value of P in terms of W

[Ans: (a) 30° (b) $W\sqrt{3}$]

9. An elastic string of natural length $4l$ and modulus of elasticity 4mg is stretched between two points A and B which are on the same level, which are on the same level, where $AB = 4l$. A particle is attached to the midpoint of the string and hangs in equilibrium with both potions of the string making 30° with AB. What is the mass of the particle?

[Ans: $0.62m$]

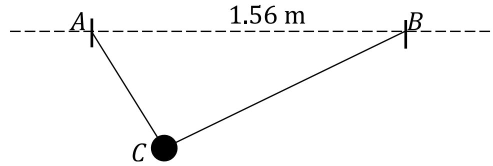
10. Two fixed points A and B on the same horizontal level are 20 cm apart. A light elastic string, which obeys Hooke's law, is just taut when its ends are fixed at A and B. A block of mass 5 kg is attached to the string at a point P where $AP = 15$ cm. The system is then allowed to take up its position of equilibrium with P below AB and it is found that in this position the angle APB is a right angle. If $\angle BAP = \theta$, show that the ratio of the extensions of AP and BP is

$$\frac{4\cos\theta - 3}{4\sin\theta - 1}$$

Hence show that θ satisfies the equation

$$\cos\theta\,(4\cos\theta - 3) = 3\sin\theta\,(4\sin\theta - 1)$$

11. Two points A and B are 1.56 m apart in a horizontal line. A body of mass 1.3 kg is suspended from A and B by two light elastic strings AC and BC. In the equilibrium position, AC is perpendicular to B and $\tan ABC = 5/12$.

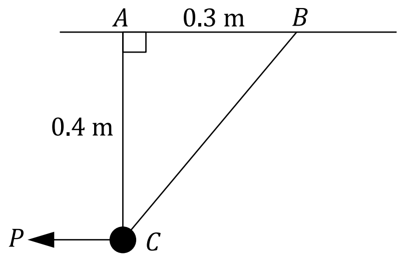


(a) Calculate, by resolving horizontally or otherwise, the ratio of the tensions in the two strings and find the magnitude of the tension in the string AC

(b) Given that the unstretched length of the string AC is 54 cm, calculate the modulus of elasticity of this string.

[Ans: (a) 12.5, 1.76 N (b) 106 N]

12. The diagram below shows a particle C of mass 2 kg suspended by two strings. The two strings are fixed to points A and B on a horizontal ceiling where $AB = 0.3$ m. The string AC is light and inextensible, with length 0.4 m while the string BC is light and elastic, with natural length 0.4 m and modulus 32 N. A horizontal force of magnitude P N holds the system in equilibrium with AC vertical.



(a) Show that the tension in BC is 8 N

(b) Find the value of P

(c) Find the tension in AC

[Ans: (b) 4.8 N (c) 13.2 N]

13. An elastic string of length 1.6 metres and modulus of elasticity 30 N is stretched between two horizontal points, P and Q, which are a distance 2.4 metres apart. A particle of mass m kg is then attached to the midpoint of the string, and rests in equilibrium, 0.5 metres below the line PQ. Find the value of m.

[Ans: 1.5]

Elastic potential energy

When a force stretches an elastic string, the force does work and the string is given potential energy. An object at the end of a stretched string will move if it is released. The energy which is converted into kinetic energy is called the **elastic potential energy** and can be abbreviated as EPE

Elastic potential energy (EPE), a property of stretched elastic strings and springs or compressed springs is given by

$$\boldsymbol{EPE = \frac{\lambda x^2}{2l}}$$

Derivation

The work done in stretching the string can be found by multiplying the average force by the distance moved in the direction of force.

As the extension increases from 0 to x, the force increases from 0 to $\frac{\lambda x}{l}$

So average force $= \left(0 + \frac{\lambda x}{l}\right) \div 2 = \frac{\lambda x}{2l}$ and distance moved is x

$$\therefore \text{Work done} = x \times \frac{\lambda x}{l} = \frac{\lambda x^2}{l}$$

This formula can also be derived using the area under a force-displacement graph or by integration.

When a variable force F moves an object a small distance δx, then work done is given by

$$\text{Work done} = F \times \delta x$$

When the force moves the object a total distance x, then the total work done is given by

Total work done $= \int_0^x F\,dx = \frac{\lambda}{l}\int_0^x x\,dx = \frac{\lambda x^2}{2l}$

This work done is stored as elastic potential energy

Example 1

An elastic string of natural length 2 m and modulus of elasticity 6 N is stretched until the extending force is of magnitude 4 N. How much work has been done and what is the final extension?

<u>Solution</u>

$$T = \frac{\lambda x}{l}$$

$$4 = \frac{6x}{2}$$

$$x = \frac{4}{3}\text{ m}$$

Work done $= EPE = \frac{\lambda x^2}{2l} = \frac{6}{2(2)} \times \left(\frac{4}{3}\right)^2 = \frac{8}{3}$ J

Example 2

The firing mechanism in a pinball machine contains a spring of natural length 4 cm whose modulus of elasticity is 10 N. The work done in compressing the spring is 0.05 J. By how much is the spring compressed?

<u>Solution</u>

$x =$?. $\lambda = 10\ N$, work done $= EPE = 10$ J

$$EPE = \frac{\lambda x^2}{2l}$$

$$0.05 = \frac{10x^2}{0.08}$$

$$x^2 = \frac{0.05 \times 0.08}{10} = 0.004$$

$$x = 0.02\text{ m}$$

The spring is compressed by 0.02 m or 2 cm

Conservation of energy and the work-energy principle

Many problems involving elastic strings and springs can be solved using the principle of conservation of energy and the work-energy principle.

In such problems;

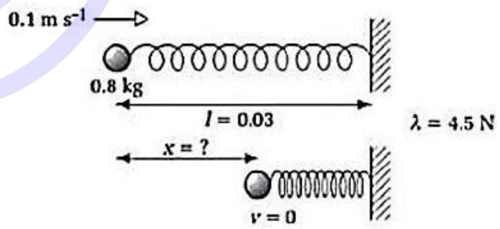
(a) Draw clear diagrams showing all the main details
(b) Decide on a zero level for potential energy
(c) Find expressions for the total mechanical energy at two points in motion
(d) Decide whether any external forces should be included. If no external forces are involved, use the principle of conservation of mechanical energy. If external forces are involved, use the work-energy principle.

Example 1

A toy engine has a mass of 800 g. When it reaches a buffer at the end of the line, it is travelling at 0.1 ms^{-1}. The buffer consists of a spring of natural length 3 cm and modulus of elasticity 4.5 N. Find the compression in the elastic spring when the engine comes to rest.

<u>Solution</u>

The engine is modelled as a particle with $m = 0.8$ kg and the buffer as a spring with $l = 0.03$ m and $\lambda = 4.5\ N$



Before engine meets buffer,

$$KE = \frac{1}{2}mv^2 \ ;\ EPE = 0$$

$$\text{Total } ME = \frac{1}{2}mv^2 = \frac{1}{2} \times 0.8 \times 0.1^2 = 0.004\text{ J}$$

When the engine comes to rest,

$$KE = 0,\quad EPE = \frac{\lambda x^2}{2l} = \frac{4.5x^2}{2(0.03)} = 75x^2$$

$$\text{Total } ME = 75x^2$$

Using the principle of conservation of mechanical energy;

$$75x^2 = 0.04$$

$$x = 0.0073\text{ m}$$

Example 2

A light horizontal spring, of natural length 0.25 m and modulus of elasticity 52 N, is fastened at one end to a point

$$v^2 = \frac{47}{0.5 \times 0.6}[0.56^2 - 0.28^2] - 2(9.8)(0.28)$$

$$v^2 = \frac{47}{0.3} \times \frac{147}{625} - \frac{686}{125}$$

$$v^2 = \frac{784}{25}$$

$$v = \frac{28}{5} = 5.6 \text{ ms}^{-1}$$

Example 13

A particle P of mass 12 kg is attached to the midpoint of a light elastic string of natural length 0.25 m and modulus of elasticity $\lambda\, N$. The ends of the string are attached to two fixed points A and B, where $AB = 0.8$ m and AB is horizontal. When P is held at the point M, where M is the midpoint of AB, the tension in the string is 216 N.

(a) Show that $\lambda = 360$.

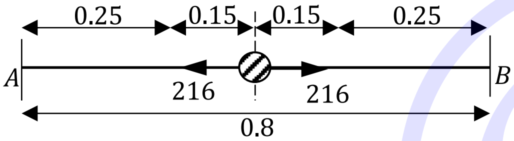
The particle is next held at the point C, where C is 0.3 m below M, and then it is released from rest.

(b) Find the initial acceleration of P.

(c) Calculate the speed of P as is passes through M.

Solution

(a)



By Hooke's law;

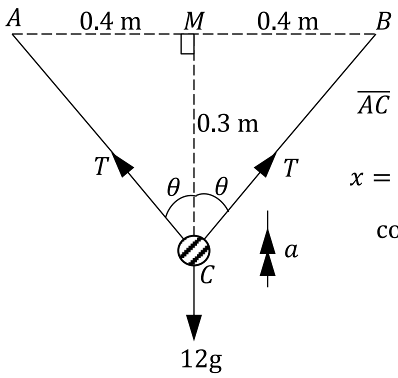
$$T = \frac{\lambda x}{l}$$

$$216 = \lambda \times \frac{0.15}{0.25}$$

$$216 = \frac{3}{5}\lambda$$

$$\lambda = 360\ N$$

(b)



$$\overline{AC} = \sqrt{0.3^2 + 0.4^2}$$

$$= 0.5 \text{ m}$$

$$x = 0.5 - 0.25 = 0.25$$

$$\cos\theta = \frac{0.3}{0.5} = \frac{3}{5}$$

Natural length $l = 0.25$, Extension, $x = 0.25$

$$T = \frac{\lambda x}{l} = \frac{360(0.25)}{0.25} = 360$$

$$F = ma$$

$$2T\cos\theta - 12\text{g} = 12a$$

$$2 \times 360 \times \frac{3}{5} - 12 \times 9.8 = 12a$$

$$432 - 117.6 = 12a$$

$$12a = 314.4$$

$$a = 26.2 \text{ ms}^{-2}$$

(c) By taking the level of AB as the zero potential level

$$KE_C + PE_C + EPE_C = KE_M + PE_M + EPE_M$$

$$0 - 12\text{g}(0.3) + 2\left[360 \times \frac{0.25^2}{2(0.25)}\right] = \frac{1}{2}(12)v^2 + 2\left[360 \times \frac{0.15^2}{2(0.25)}\right]$$

$$-\frac{882}{25} + 90 = 6v^2 + 32.4$$

$$v^2 = \frac{93}{25}$$

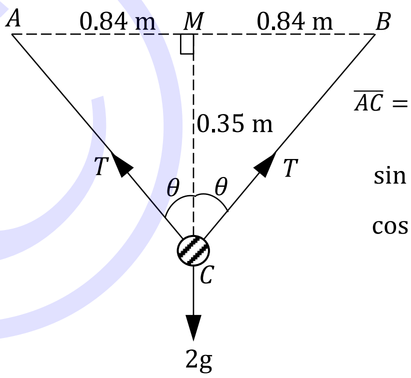
$$v = 1.93 \text{ ms}^{-1}$$

Example 14

A light elastic string, of natural length 1.42 m, has each of its two ends attached to two fixed points, A and B, where AB is horizontal and $AB = 1.68$ m. A particle, of mass 2 kg, is attached to the midpoint of the string, M. The particle is hanging in equilibrium at the point C, where MC is vertical and $MC = 0.35$ m. The particle is then held at M and released from rest. Calculate the speed of particle as it passes through C.

Solution

Let us start by finding the modulus of elasticity of the string



$$\overline{AC} = \sqrt{0.35^2 + 0.84^2}$$

$$= 0.91 \text{ m}$$

$$\sin\theta = \frac{0.84}{0.91} = \frac{12}{13}$$

$$\cos\theta = \frac{0.35}{0.91} = \frac{5}{13}$$

Resolving vertically;

$$2T\cos\theta = 2\text{g}$$

$$T \times \frac{5}{13} = \text{g}$$

$$\frac{\lambda x}{l} \times \frac{5}{13} = \text{g}$$

Extension in each string, $x = 0.91 - 0.71 = 0.2$

$$\frac{\lambda}{0.71} \times 0.2 \times \frac{5}{13} = \text{g}$$

$$\lambda = 90.45\ N$$

Now taking the level of AB as the zero potential level;

$$EPE_M = KE_C + PE_C + EPE_C$$

$$\frac{\lambda}{2l}x_M^2 = \frac{1}{2}mv_C^2 - mg \times \overline{MC} + \frac{\lambda}{2l}x_C^2$$

$$2\left[\frac{90.45}{2 \times 0.71}(0.13)^2\right] = \frac{1}{2} \times 2v_C^2 - 2\text{g}(0.35) + 2\left[\frac{90.45}{2 \times 0.71}(0.2)^2\right]$$

$$2.1531 = v^2 - 6.86 + 5.096$$

$$v^2 = 3.917$$

$$v = 1.98 \text{ ms}^{-1}$$

(b) Calculate the elastic energy in the string when the particle is in this position.

[Ans: (a) 84 N (b) 245 J]

4. A light elastic string has natural length a and modulus of elasticity $\frac{3}{2}mg$. A particle P of mass m is attached to one end of the string. The other end of the string is attached to a fixed point A. The particle is released from rest at A and falls vertically. When P has fallen a distance of $a + x$, where $x > 0$, the speed of P is v.

(a) Show that $v^2 = 2g(a + x) - \frac{3gx^2}{2a}$

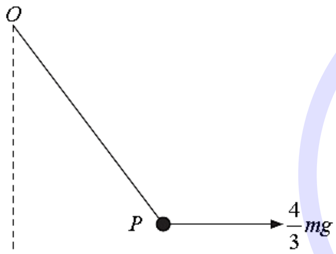
(b) Find the greatest speed attained by P as it falls

After release, P next comes to instantaneous rest at a point D

(c) Find the magnitude of the acceleration of P at D

[Ans: (b) $\frac{2}{3}\sqrt{6ag}$ (c) 2 g]

5. A particle P of mass m is attached to one end of a light elastic string, of natural length a and modulus of elasticity $3mg$. The other end of the string is attached to a fixed point O. The particle P is held in equilibrium by a horizontal force of magnitude $\frac{4}{3}mg$ applied to P. This force acts in the vertical plane containing the string, as shown below.



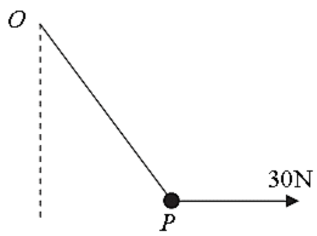
Find

(a) the tension in the string,

(b) the elastic energy stored in the string.

[Ans: (a) $\frac{5}{3}mg$ (b) $\frac{25}{54}mga$]

6. A particle P of weight 40 N is attached to one end of a light elastic string of natural length 0.5 m. The other end of the string is attached to a fixed point O. A horizontal force of magnitude 30 N is applied to P as shown below.



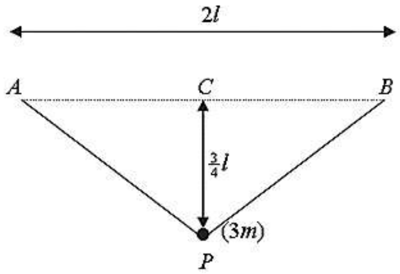
Given that the particle P is in equilibrium and the elastic energy stored in the string is 10 J, calculate the length OP

[Ans: 0.9 m]

7. A particle P of mass m is attached to one end of a light elastic spring, of natural length l and modulus of elasticity $2mg$. The other end of the spring is attached to a fixed point A on a rough horizontal plane. The particle is held at rest on the plane at a point B, where $AB = \frac{1}{2}l$, and released from rest. The coefficient of friction between P and the plane is 0.25. Find the distance of P from B when P first comes to rest.

[Ans: $0.75l$]

8. A small ball of mass $3m$ is attached to the ends of two light elastic strings AP and BP, each of natural length l and modulus of elasticity kmg. The ends A and B of the strings are attached to fixed points on the same horizontal level, with $AB = 2l$. The mid-point of AB is C. The ball hangs in equilibrium at a distance $\frac{3}{4}l$ vertically below C as shown below



(a) Show that $k = 10$

The ball is now pulled vertically downwards until it is at a distance $\frac{12}{5}l$ below C. The ball is released from rest.

(b) Find the speed of the ball as it reaches C.

[Ans: (b) $\sqrt{\left(\frac{184}{15}gl\right)}$]

9. A particle of mass 0.8 kg is attached to one end of a light elastic string of natural length 0.6 m. The other end of the string is attached to a fixed point A. The particle is released from rest at A and comes to instantaneous rest 1.1 m below A. Find the modulus of elasticity of the string.

[Ans: 41.4 N]

10. A particle P of mass 1.5 kg is attached to the mid-point of a light elastic string of natural length 0.30 m and modulus of elasticity λ newtons. The ends of the string are attached to two fixed points A and B, where AB is horizontal and $AB = 0.48$ m. Initially P is held at rest at the mid-point, M, of the line AB and the tension in the string is 240 N.

(a) Show that $\lambda = 400$

The particle is now held at rest at the point C, where C is 0.07 m vertically below M. The particle is released from rest at C.

(b) Find the magnitude of the initial acceleration of P.

(c) Find the speed of P as it passes through M.

[Ans: (b) 89.8 ms^{-2} (c) 2.3 ms^{-1}]

11. A light elastic string has natural length 5 m and modulus of elasticity 20 N. The ends of the string are attached to two fixed points A and B, which are 6 m apart on a horizontal ceiling. A particle P is attached to the midpoint of the string and hangs in equilibrium at a point which is 4 m below AB.

(a) Calculate the weight of P.

(b) the magnitude of the acceleration when the displacement is 0.1 m

Solution

(a) $T = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{2\pi}{T} = \frac{2\pi}{\pi/4} = 8$

Using $v^2 = \omega^2(a^2 - x^2)$

$$v^2 = 8^2[(0.5)^2 - (0.25)^2]$$

$$v = \sqrt{12} = 3.46 \text{ ms}^{-1}$$

(b) Using acceleration $= -\omega^2 x$

$$= -8^2(0.1)$$

$$= -6.4 \text{ ms}^{-2}$$

$\therefore$ The magnitude of the acceleration is 6.4 ms^{-2}

Example 2

A body describes simple harmonic motion with an amplitude of 5 cm and a period of $0.2\,s$. Find the acceleration and velocity of the body when the displacement is

(a) 5 cm (b) 3 cm and (c) 0 cm

Solution

Here $a = 5$ cm; $T = 0.2\,s$;

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.2} = 10\pi\, rad\, s^{-1}$$

If a particle executing SHM, has displacement x, then

Acceleration, $\frac{d^2x}{dt^2} = -\omega^2 x$; Velocity, $v = \omega\sqrt{a^2 - x^2}$

(a) When $x = 5$ cm $= 0.05$ m,

$$\frac{d^2x}{dt} = -(10\pi)^2 \times 0.05 = -5\pi^2\, ms^{-2}$$

$$v = 10\pi\sqrt{0.05^2 - 0.05^2} = 0$$

(b) When $x = 3$ cm $= 0.03$ m

$$\frac{d^2x}{dt} = -(10\pi)^2 \times 0.03 = -3\pi^2\, ms^{-2}$$

$$v = 10\pi\sqrt{0.05^2 - 0.03^2} = 0.4\pi\, ms^{-1}$$

(c) When $x = 0\, cm$,

$$\frac{d^2x}{dt} = -(10\pi)^2 \times 0 = 0$$

$$v = 10\pi\sqrt{0.05^2 - 0^2} = 0.5\pi\, ms^{-1}$$

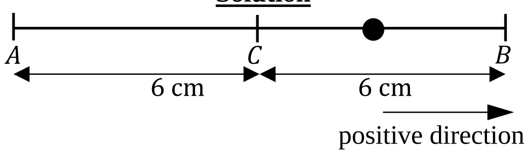
Example 3

An object performs simple harmonic motion at a rate of 20 oscillations per second between two points A and B which are 12 cm apart. If C is the midpoint of AB calculate the time take to travel directly

(a) from A to C

(b) from C to the midpoint of AB

Solution



$$a = 6 \text{ cm} = 0.06 \text{ m}$$

$$f = 20 \text{ oscillations per second}$$

$$T = \frac{1}{f} = \frac{1}{20} = 0.05 \text{ s}$$

(a) Time taken to travel from A to $C = \frac{1}{4}T$

Time taken to travel from A to $C = \frac{1}{4} \times 0.05$

$$= 0.0125 \text{ s}$$

(b) To find the time to move from C to the midpoint of CB, it is necessary to use one of the formulas for x in terms of t. As the question does not give the position of the object when $t = 0$, it can be assumed to be at C moving in the direction CB. The corresponding formula for x is

$$x = a\sin\omega t$$

$$x = \frac{1}{2}CB = \frac{1}{2} \times 0.06 = 0.03 \text{ m}$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.05} = 40\pi$$

$$0.03 = 0.06\sin 40\pi t$$

$$0.5 = \sin 40\pi t$$

$$40\pi t = \frac{\pi}{6}$$

$$t = \frac{1}{240}\, s = 0.0042 \text{ s}$$

Example 4

The velocities of a particle executing SHM are 4 cm s^{-1} and 3 cm s^{-1} when its distance from the mean position is 2 cm and 3 cm respectively. Calculate its amplitude and period

Solution

$v_1 = 4 \text{ cm s}^{-1} = 0.04 \text{ m s}^{-1}$, $x_1 = 2 \text{ cm} = 0.02$ m

$v_2 = 2 \text{ cm s}^{-1} = 0.02 \text{ m s}^{-1}$, $x_1 = 3 \text{ cm} = 0.03$ m

$$v_1 = \omega\sqrt{a^2 - x_1^2} \quad \text{……… (i)}$$

$$v_2 = \omega\sqrt{a^2 - x_2^2} \quad \text{…….. (ii)}$$

Squaring and dividing the equations;

$$\frac{v_1^2}{v_2^2} = \frac{a^2 - x_1^2}{a^2 - x_2^2}$$

$$v_1^2(a^2 - x_2^2) = v_2^2(a^2 - x_1^2)$$

$$v_1^2a^2 - v_1^2x_2^2 = v_2^2a^2 - v_2^2x_1^2$$

$$v_1^2a^2 - v_2^2a^2 = v_1^2x_2^2 - v_2^2x_1^2$$

$$a^2 = \frac{v_1^2x_2^2 - v_2^2x_1^2}{v_1^2 - v_2^2}$$

$$a^2 = \frac{(0.04 \times 0.03)^2 - (0.03 \times 0.02)^2}{0.04^2 - 0.03^2}$$

$$a = \sqrt{1.543 \times 10^{-3}} = 0.03928\, m$$

From (i); $\omega = \frac{v_1^2}{\sqrt{a^2 - x_1^2}} = \frac{0.04^2}{\sqrt{1.543 \times 10^{-3} - 0.02^2}}$

$$\omega = 0.0473 \text{ rad s}^{-1}$$

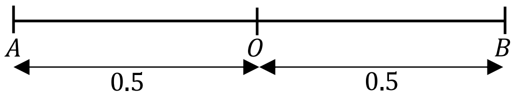
$$T = \frac{2\pi}{\omega} = \frac{2\pi}{0.0473} = 132.76 \text{ s}$$

Example 5

A particle is in simple harmonic motion about O. When it is 6 m from O, its speed is 4 ms^{-1}, and its deceleration is 1.5 ms^{-2}.

(a) Find the amplitude of the oscillation and the greatest speed as it oscillates of P when it is at a distance of 0.3 m from O, given that it comes to instantaneous rest at a distance 0.5 m from O.

Solution



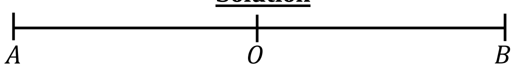
$T = 2\pi$, $a = 0.5$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$
$$v^2 = \omega^2(a^2 - x^2) = 1^2(0.5^2 - 0.3^2)$$
$$v^2 = 0.16$$
$$v = 0.4 \text{ ms}^{-1}$$

Example 11

A particle P is moving on a straight line with simple harmonic motion of period $\frac{\pi}{6}$ s. Given that the maximum speed of P is 12 ms^{-1} , find the speed of P 0.2 s after passing through the centre of the oscillation.

Solution



Period, $T = \frac{2\pi}{\omega}$
$$\frac{2\pi}{\omega} = \frac{\pi}{6}$$
$$\omega = 12$$

$$v_{max} = 12$$
$$\omega a = 12$$
$$12a = 12$$
$$a = 1$$

$$x = a \sin \omega t$$
$$\dot{x} = a\omega \cos \omega t$$
$$\dot{x} = 1 \times 12 \cos 12t$$
$$\dot{x} = 12 \cos 12t$$

When $t = 0.2$, $\dot{x} = 12 \cos 2.4$
$$\dot{x} = 8.8487$$
$$|\dot{x}| = 8.85 \text{ ms}^{-1}$$

Example 12

A particle P is moving on a straight line with simple harmonic motion of maximum speed 5 ms^{-1} and maximum acceleration 10 ms^{-2}. Calculate the speed of P when it is 2 m from the centre of the oscillation.

Solution

$$v_{max} = 5$$
$$\omega a = 5$$
$$|\ddot{x}|_{max} = 10$$
$$\omega^2 a = 10$$

$$\frac{a\omega^2}{a\omega} = \frac{10}{5}$$
$$\omega = 2$$
$$2a = 5$$
$$a = 2.5$$

Using $v^2 = \omega^2(a^2 - x^2)$
$$v^2 = 2^2(2.5^2 - 2^2)$$
$$v^2 = 4 \times \frac{9}{4}$$
$$v^2 = 9$$
$$v = 3 \text{ ms}^{-1}$$

Example 13

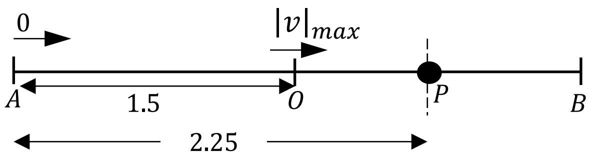
A particle is about to move in a straight line with simple harmonic motion. It is released from rest from a point A and travels directly to a point O, arriving there 0.75 s later with maximum speed $V \text{ ms}^{-1}$.

(a) Given that $AO = 1.5$ m, determine the value of V.

(b) Find the time it takes the particle to cover the first 2.25 m of the motion.

(c) Calculate the speed of the particle when is at a distance of 0.5 m from O.

Solution



(a) Period, $T = \frac{2\pi}{\omega}$
$$3 = \frac{2\pi}{\omega}$$
$$\omega = \frac{2\pi}{3}$$
$$v_{max} = \omega a = \frac{2\pi}{3} \times 1.5 = 3.14 \text{ ms}^{-1}$$

(b) $\overline{AP} = 2.25$

Taking the point A as the $+a$ amplitude
$$x = -0.75$$
$$\therefore x = a \cos \omega t$$
$$-0.75 = 1.5 \cos\left(\frac{2\pi}{3}t\right)$$
$$\cos\left(\frac{2\pi}{3}t\right) = -\frac{1}{2}$$
$$\frac{2\pi}{3}t = \frac{2\pi}{3}$$
$$t = 1 \text{ s}$$

Alternatively;

By taking displacement from O
$$OP = AP - AO = 2.25 - 1.5 = 0.75 \text{ m}$$
$$x = a \sin \omega t$$
$$0.75 = 1.5 \sin\left(\frac{2\pi}{3}t\right)$$
$$\sin\left(\frac{2\pi}{3}t\right) = \frac{1}{2}$$
$$\frac{2\pi}{3}t = \frac{\pi}{6}$$
$$t = \frac{3}{12} = 0.25 \text{ s}$$

Time to cover first 2.25 m of motion is given by
$$T_{AO} + T_{OP} = 0.75 + 0.25 = 1.0 \text{ s}$$

(c) When $x = 0.5$, $\quad v^2 = \omega^2(a^2 - x^2)$
$$v^2 = \left(\frac{2\pi}{3}\right)^2 [1.5^2 - 0.5^2]$$
$$v = 2.96 \text{ ms}^{-1}$$

Example 14

A particle P is moving on a straight line with simple harmonic motion of maximum speed 10 ms^{-1} and maximum acceleration 10 ms^{-2}. Calculate the distance of P from one of the endpoints of the oscillation 0.5 s after passing through the centre point of the motion.

$$\text{Using } v^2 = \omega^2(a^2 - x^2)$$
$$3^2 = 2^2(a^2 - 2^2)$$
$$a^2 - 4 = \frac{9}{4}$$
$$a^2 = \frac{25}{4}$$
$$a = 2.5$$

Modelling the particle to be at B, heading for C (B is $+a$)

$$x = a\cos\omega t$$
$$x = 2.5\cos 2t$$
$$2 = 2.5\cos 2t$$
$$\cos 2t = 0.8$$
$$2t = 0.6435$$
$$t = 0.3175$$
$$\therefore C \text{ to } B \text{ to } C,\ T = 2 \times 0.3175 = 0.6435 \text{ s}$$

C to O to A to O to C;

$$T = \text{Period} - 0.6435 = \pi - 0.6435 = 2.498 \text{ s}$$

The two possible values of T are 0.6435 s and 2.498 s

Example 18

A particle moves on the line Ox so that after time t, its displacement from O is x and $\frac{d^2x}{dt^2} = -9x$. When $t = 0$, $x = 4$ m and $\frac{dx}{dt} = 9$ ms^{-1}. Find

(a) the maximum displacement of the particle from O

(b) the position and velocity of the particle when $t = \pi/6\ s$

Solution

$\ddot{x} = -9x \Rightarrow \omega^2 = 9 \therefore \omega = 3$

At $t = 0$, $x = 4$ and $v = 9$

(a) Using $v^2 = \omega^2(a^2 - x^2)$

$$9^2 = 3^2(a^2 - 4^2)$$
$$a^2 - 16 = 9$$
$$a^2 = 25$$
$$a = 5$$

Maximum displacement from O is 5 m

(b) Using $x = a\sin(\omega t + \phi)$

$$x = 5\sin(3t + \phi)$$

When $t = 0$, $x = 4$

$$4 = 5\sin(3(0) + \phi)$$
$$\sin\phi = 0.8$$
$$\phi = 0.9273$$
$$x = 5\sin(3t + 0.9273)$$

When $t = \frac{\pi}{6}$;

$$x = 5\sin\left(3\left(\frac{\pi}{6}\right) + 0.9273\right) = 5\sin 2.498 = 3 \text{ m}$$
$$x = 5\sin(3t + 0.9273)$$
$$\dot{x} = 5(-3)\cos(3t + 0.9273)$$
$$\dot{x} = -15\cos(3t + 0.9273)$$

When $t = \frac{\pi}{6}$;

$$\dot{x} = -15\cos\left(3\left(\frac{\pi}{6}\right) + 0.9273\right)$$
$$= -15\cos 2.498$$
$$-12 \text{ ms}^{-1}$$
$$|v| = 12 \text{ ms}^{-1}$$

Alternatively; $\quad v = \omega\sqrt{a^2 - x^2}$

$$v = 3\sqrt{5^2 - 3^2} = 3(4) = \pm 12 \text{ ms}^{-1}$$
$$|v| = 12 \text{ ms}^{-1}$$

Example 19

A particle P moves along te x-axis. At time $t = 0$, P passes through the origin O, moving in the positive x-direction. At time t seconds, the velocity of P is v ms^{-1} and $OP = x$ metres. The acceleration of P is $\frac{1}{12}(30 - x)$ ms^{-2} measured in the positive x-direction. Given that the maximum speed of P is 10 ms^{-1}, find an expression for v^2 in terms of x.

Solution

$$\ddot{x} = \frac{1}{12}(30 - x)$$

For $v_{max}(10)$; $\ddot{x} = 0$

$$\frac{1}{12}(30 - x) = 0$$
$$x = 30$$

Now $\ddot{x} = v\frac{dv}{dx}$

$$v\frac{dv}{dx} = \frac{1}{12}(30 - x)$$
$$v\,dv = \frac{1}{12}(30 - x)\,dx$$
$$\int v\,dv = \frac{1}{12}\int(30 - x)\,dx$$
$$\frac{v^2}{2} = \frac{1}{12}\left(30x - \frac{x^2}{2}\right) + c$$

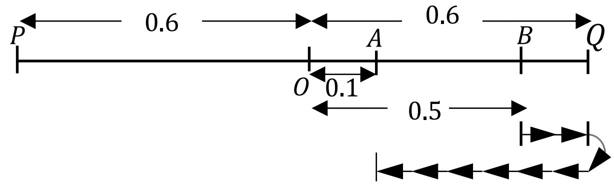
When $x = 30$, $v = 10$

$$\frac{10^2}{2} = \frac{1}{12}\left(30(10) - \frac{10^2}{2}\right) + c$$
$$c = 12.5$$
$$\frac{v^2}{2} = \frac{1}{12}\left(30x - \frac{x^2}{2}\right) + 12.5$$
$$v^2 = 25 + 5x - \frac{1}{12}x^2$$

Example 20

Three points O, A and B lie in that order on a straight line. A particle P is moving on this line with simple harmonic motion of period 3 s, amplitude 0.6 m and centre at O. It is further given that OA is 0.1 m and OB is 0.5 m. At a certain instant P is observed passing through B moving in the direction OB. Calculate the time when P reaches A.

Solution



Firstly; $\omega = \frac{2\pi}{T} = \frac{2\pi}{3}$

Time taken to travel from O to Q then Q to O is $\frac{1}{2}T = 1.5$ s

$$x = a\sin\omega t$$
$$x = 0.6\sin\left(\frac{2\pi}{3}t\right)$$

From O to A

$$0.1 = 0.6\sin\left(\frac{2\pi}{3}t\right)$$
$$\sin\left(\frac{2\pi}{3}t\right) = \frac{1}{6}$$
$$\frac{2\pi}{3}t = 0.1674$$
$$t = 0.08 \text{ s}$$

From O to B (or B to O)

$$0.5 = 0.6\sin\left(\frac{2\pi}{3}t\right)$$
$$\sin\left(\frac{2\pi}{3}t\right) = \frac{5}{6}$$
$$\frac{2\pi}{3}t = 0.9851$$
$$t = 0.47 \text{ s}$$

Forces producing S.H.M

A force directed towards a fixed point and proportional to the displacement from that point produces S.H.M

A simple example of a force producing S.H.M is the tension in a stretched elastic string or spring.

To show that the motion produced is S.H.M

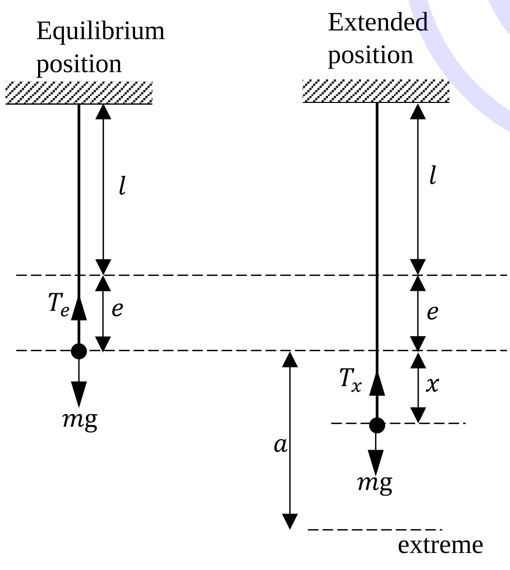
(a) Draw a clear force diagram showing the particle in equilibrium
(b) Use Hooke's law to find the static extension
(c) Draw a diagram showing the particle at a point between the equilibrium position and an extreme
(d) Write down the equation of motion measuring displacement from the equilibrium position
(e) Compare this equation with the basic equation of S.H.M i.e. $\frac{d^2x}{dt^2} = -\omega^2 x$
Once a motion has been shown to be S.H.M, then the equations of S.H.M can be used to find ω, v, a, etc.

Example 1

A particle of mass m hanging on the end of an elastic string of natural length l and modulus λ is pulled down a distance $a(< mgl/\lambda)$ below its equilibrium position and then released. Prove that the subsequent oscillations are simple harmonic, find the period of oscillations and state the amplitude.

Solution

Let e be the static tension in the equilibrium position



By Hooke's law;

$$T_e = \frac{\lambda e}{l}$$

Resolving vertically;

$$T_e = mg$$

$$\text{Thus } mg = \frac{\lambda e}{l}$$

In the extended position;

By Hooke's law;

$$T_x = \frac{\lambda(e+x)}{l}$$

Equation of motion is given by;

$$mg - T_x = m\frac{d^2x}{dt^2}$$

$$mg - \frac{\lambda(e+x)}{l} = m\frac{d^2x}{dt^2}$$

$$mg - \frac{\lambda e}{l} - \frac{\lambda x}{l} = m\frac{d^2x}{dt^2}$$

$$mg - mg - \frac{\lambda x}{l} = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} = -\frac{\lambda}{ml}x$$

The motion is in the form $\frac{d^2x}{dt^2} = -\omega^2 x$ hence S.H.M

$$\omega = \sqrt{\frac{\lambda}{ml}}$$

Period of oscillations, $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{ml}{\lambda}}$

Amplitude is a

Notes:

- If $a > e$, the mass will perform SHM as long as the string remains taut; when the string is not taut (not under tension), the mass will move freely under gravity.
- If a spring is used, then the mass will perform SHM for any a (as long as the mass does not try to go above the top of the spring)

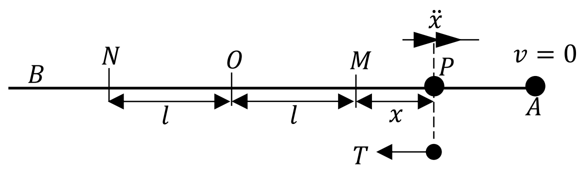
Example 2

A particle P of mass 0.2 kg is attached to one end of a light elastic string of natural length 0.8 m and modulus of elasticity 16 N. The other end of the string is attached to a fixed point A on a smooth horizontal surface on which P rests. With the string at natural length, P receives an impulse of magnitude 5 Ns, in the direction AP.

(a) Show that in the subsequent motion, while the string is taut, the motion of P is simple harmonic.
(b) Determine its amplitude of the motion.
(c) Find the time it takes P to travel between the extreme points of its motion.

Solution

(a)



Consider particle in an arbitrary position P with the string taut.

$$m\ddot{x} = -T$$

$$m\ddot{x} = -\frac{\lambda}{l}x$$

$$\ddot{x} = -\frac{\lambda}{ml}x$$

CENTRE OF MASS

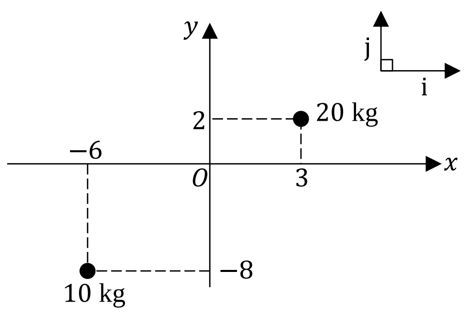
The **centre of mass** of a body is the point at which the mass of the body may be considered to be acting.
The **centre of gravity** of a body is the point through which the line of action of its weight acts.
The centre of mass and centre of gravity of a body coincide in a uniform gravitational field.
A **uniform body** has uniform density
Density is mass per unit area for laminas and mass per unit volume for solids.

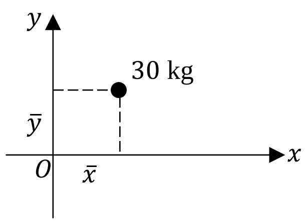
Centre of mass and centre of gravity
The terms centre of mass and centre of gravity are often used to mean the same thing. However, this is not strictly true. The centre of gravity can be different from the centre of mass. This is caused by the fact that the acceleration due to gravity can vary at different pints in space. This means that the weights for particles of the same mass will vary depending on their position. This effect will only be important for large objects, for example a planet. For small objects, the acceleration due to gravity will change so slightly over its length that this effect can be neglected.

Centre of mass of particle systems
Centre of mass is defined as the single point at which the total mass of the system can be placed and still have the same characteristics. The moments caused by the single mass must be the same as the moments caused by the separate masses.

Example 1
Masses of 10 kg and 20 kg are placed at the points with position vectors $-6\text{i} - 8\text{j}$ and $3\text{i} + 2\text{j}$ respectively. Find the position vector of their centre of mass.

Solution





$\bar{r} = \bar{x}\text{i} + \bar{y}\text{j}$

Mass (kg)	Weight	x-coordinate Distance from Oy	y-coordinate Distance from Ox
10	10g	-6	-8
20	20g	3	2
30	30g	$\bar{x}$	$\bar{y}$

Equating moments about Oy;

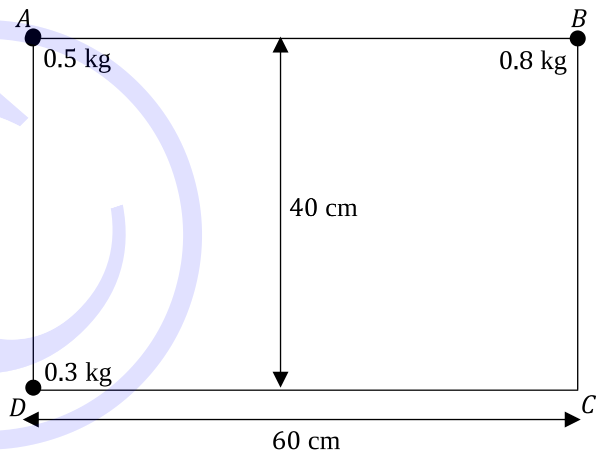
$$10\text{g}(-6) + 20\text{g}(3) = 30\text{g}\bar{x}$$
$$-60 + 60 = 30\bar{x}$$
$$\bar{x} = 0$$

Equating moments about Ox;

$$10\text{g}(-8) + 20\text{g}(2) = 30\text{g}\bar{y}$$
$$-80 + 40 = 30\bar{y}$$
$$-40 = 30\bar{y}$$
$$\bar{y} = -\frac{4}{3}$$

The position vector is $\bar{r} = -\frac{4}{3}\text{j}$

Example 2
Find the centre of mass of the following system of particles



Taking AD and DC as reference axes

Mass (kg)	Weight	x-coordinate Distance from AD	y-coordinate Distance from DC
0.5	0.5g	0	40
0.3	0.3g	0	0
0.8	0.8g	60	40
1.6	1.6g	$\bar{x}$	$\bar{y}$

Equating moments about AD;

$$0.5\text{g}(0) + 0.3\text{g}(0) + 0.8\text{g}(60) = 1.6\text{g}\bar{x}$$
$$\bar{x} = 30 \text{ cm}$$

Equating moments about DC;

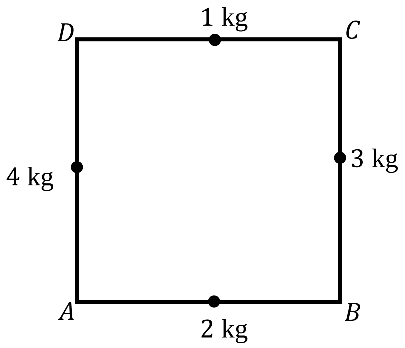
$$0.5\text{g}(40) + 0.3\text{g}(0) + 0.8\text{g}(40) = 1.6\text{g}\bar{y}$$
$$1.3\text{g}(40) = 1.6\text{g}\bar{y}$$
$$\bar{y} = \frac{1.3}{1.6} \times 40 = 32.5 \text{ cm}$$

Position is 30 cm from AD and 32.5 cm from DC.

Example 3
Four uniform rods AB, BC, CD and DA are each 4 m in length and have masses of 2 kg, 3 kg, 1 kg and 4 kg

respectively. If they are joined together to form a square framework $ABCD$, find the position of its centre of gravity.

Solution



Mass (kg)	Weight	x-coordinate Distance from AD	y-coordinate Distance from AB
2	2g	2	0
3	3g	4	2
1	g	2	4
4	4g	0	2
10	10g	$\bar{x}$	$\bar{y}$

Equating moments about AD;

$$10\bar{x} = 2(2) + 3(4) + 1(2) + 4(0)$$
$$\bar{x} = 1.8 \text{ m}$$

Equating moments about AB;

$$10\bar{y} = 2(0) + 3(2) + 1(4) + 4(2)$$
$$\bar{y} = 1.8 \text{ m}$$

The centre of gravity is at 1.8 m from AD and 1.8 m from AB

General formula for centre of mass

In the above examples, it should be noted that the acceleration due to gravity always cancels when moments are taken. An alternative way of calculating the centre of mass for masses m_1, m_2, m_3,……., m_n acting at the coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3), ….., (x_n, y_n) can be stated without including g.

In the x-y plane, the centre of mass $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots + m_nx_n}{m_1 + m_2 + m_3 + \cdots + m_n} = \frac{\sum m_ix_i}{\sum m_i}$$
$$\bar{y} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \cdots + m_ny_n}{m_1 + m_2 + m_3 + \cdots + m_n} = \frac{\sum m_iy_i}{\sum m_i}$$

Example 4

Four masses of 2 kg, 3 kg, 4 kg and 5 kg are placed at the coordinates $(0, 2)$, $(4, 4)$, $(4, 0)$ and $(1, 1)$ respectively. Determine the centre of mass of the system.

Solution

$$\bar{x} = \frac{2(0) + 3(4) + 4(4) + 5(1)}{2 + 3 + 4 + 5} = \frac{33}{14}$$
$$\bar{y} = \frac{2(2) + 3(4) + 4(0) + 5(1)}{2 + 3 + 4 + 5} = \frac{21}{14}$$

The centre of mass is at $\left(\frac{33}{14}, \frac{21}{14}\right)$ or (2.36, 1.5)

Example 5

Two masses of 2 kg and 4 kg are positioned at the coordinates (3, 2) and (−1, −2) respectively. Another mass of 4 kg is added to the system so that the centre of mass is located at (−1, 1). Determine the location of the 4 kg mass.

Solution

Let the position of the 4 kg mass be (a, b)

$$\frac{2(3) + 4(-1) + 4(a)}{2 + 4 + 4} = -1$$
$$2 + 4a = -10$$
$$a = -3$$
$$\frac{2(2) + 4(-2) + 4(b)}{2 + 4 + 4} = 1$$
$$-4 + 4b = 10$$
$$b = 3.5$$

The location of the 4 kg mass is (−3, 3.5)

Self-Evaluation exercise

1. Find the coordinates of the centre of gravity of four particles of mass 5 kg, 2 kg, 2 kg and 3 kg situated at $(3, 1)$, $(4, 3)$, $(5, 2)$ and $(-3, 1)$ respectively.
   [Ans: $(2, 1.5)$]
2. Three particles of mass 2 kg, 1 kg and 3 kg are situated at $(4, 3)$, $(1, 0)$ and (a, b) respectively. If the centre of gravity of the system lies at $(0, 2)$, find the values of a and b
   [Ans: $(-3, 2)$]
3. The rectangle $ABCD$ has $AB = 4$ cm and $AD = 2$ cm. Particles of mass 3 kg, 5 kg, 1 kg and 7 kg are placed at the points A, B, C and D respectively. Find the distance of the centre of gravity of the system from each of the lines AB and AD
   [Ans: 1 cm, 1.5 cm]
4. The rectangle $EFGH$ has $EF = 3$ m and $EH = 2$ m. Particles of mass 2 g, 3 g, 6 g and 1 gare placed at the mid-points of the sides EF, FG, GH and EH respectively. Find the distance of the centre of gravity of the system from each of the lines EF and EH
   [Ans: $\frac{4}{3}m$; $\frac{7}{4}m$]
5. A uniform rectangular lamina $ABCD$ is of mass $3M$; $AB = DC = 4$ cm and $BC = AD = 6$ cm. Particles each of mass M, are attached to the lamina at B, C and D. Calculate the distance of the centre of mass of the loaded lamina
   (a) from AB (b) from BC
   [Ans: (a) 3.5 cm (b) $\frac{5}{3}$ cm]
6. Masses p, q and r are placed at the points whose cartesian coordinates are respectively $(0, 0)$, $(20, 0)$ and $(0, 16)$. If $p : q : r = 1 : 4 : 3$, find the coordinates of the centre of mass of the three masses
   [Ans: $(10, 6)$]

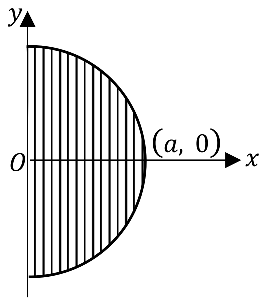
Finding centre of mass by integration

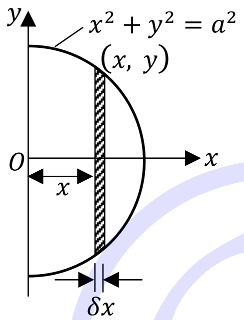
When an object can be divided into a small number of parts, the mass and centre of each part being known, the centre of mass of the whole object can be found by using $\sum m_n x_n = \bar{x}\sum m_n$ with similar expressions for $\bar{y}$ and $\bar{z}$ when the object is three dimensional.

Some bodies cannot be divided up in this way, but can be divided into a very large number of very small parts whose masses and centres of mass are known. In cases like this, it may be possible to evaluate $\sum m_n x_n$ and $\sum m_n$ by using integration.

Semicircular lamina

Consider a semicircular lamina, bounded by the y-axis and part of the curve with equation $x^2 + y^2 = a^2$, divided into vertical strips as shown. The x-axis is a line of symmetry so the centre of mass lies on it.





For one elementary strip of width δx,

the length is $2y$ so the area is approximately $2y\delta x$

the mass is approximately $2y\rho\delta x$ where ρ is the mass per unit area

the distance of the centre of mass from the y-axis is approximately x

Therefore, $mx \approx (2y\rho\delta x)x$ or $2xy\rho\delta x$

Now we can sum all the elements using $\sum mx = \bar{x}\sum m$ where

$$\sum_{0}^{a} mx = \sum_{0}^{a} 2xy\rho\delta x$$

As the width of the elemental strips approaches zero i.e. $\delta x \to 0$, the limit of

$$\sum_{0}^{a} 2xy\rho\delta x = \int_{0}^{a} 2xy\rho dx$$

and the limit of $\sum m$ is the mass of the semicircle i.e. $\frac{1}{2}\pi a^2\rho$

Now $x^2 + y^2 = a^2$

$$\Rightarrow y = \sqrt{a^2 - x^2}$$

$$2\rho\int_0^a x\sqrt{a^2 - x^2}\,dx = \frac{1}{2}\pi a^2\rho\bar{x}$$

$$2\left[-\frac{1}{3}(a^2 - x^2)^{\frac{3}{2}}\right]_0^a = \frac{1}{2}\pi a^2\bar{x}$$

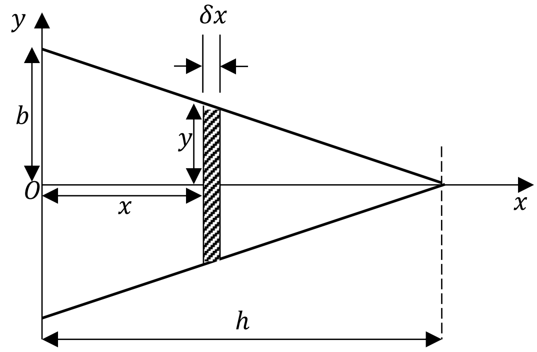
$$-\frac{2}{3}[0 - a^3] = \frac{1}{2}\pi a^2\bar{x}$$

$$\bar{x} = \frac{4a}{3\pi}$$

The centre of mass is at a distance of $4a/3\pi$ from the plane surface of the semicircular lamina

Triangular lamina

Consider a triangular lamina of height h and base $2b$



Comparing similar triangles

$$\frac{x}{h} = \frac{y}{b}$$

$$y = \frac{bx}{h}$$

Let the mas per unit area be ρ

Shape	Area	Mass	Distance from Oy
Small strip	$2y\delta x$ $= 2\left(\frac{bx}{h}\right)\delta x$	$2\rho\left(\frac{bx}{h}\right)\delta x$	x
Whole shape	bh	ρbh	$\bar{x}$

Equating moments;

$$\rho bh\bar{x} = \sum_0^h \left[2\rho\left(\frac{bx}{h}\right)\delta x \times x\right]$$

$$\rho bh\bar{x} = \frac{2b\rho}{h}\sum_0^h x^2\,\delta x$$

$$h^2\bar{x} = \sum_0^h x^2\,\delta x$$

As $\delta x \to 0$, $\sum_0^h x^2\,\delta x \to \int_0^h x^2\,dx$

$$h^2\bar{x} = \int_0^h x^2\,dx$$

$$h^2\bar{x} = \left[\frac{x^3}{3}\right]_0^h$$

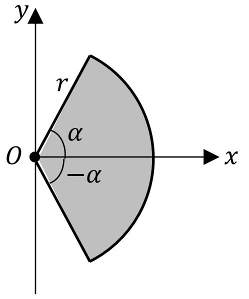
$$h^2\bar{x}. = \frac{h^3}{3} - 0$$

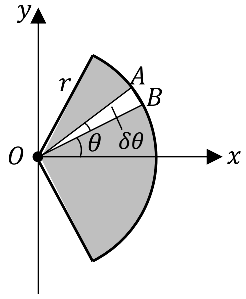
$$\bar{x} = \frac{h^3}{3} \times \frac{1}{h^2} = \frac{h}{3}$$

The centre of mass is $\frac{h}{3}$ from the base of the cone and $\frac{3}{4}h$ from the edge of the cone along the axis of symmetry.

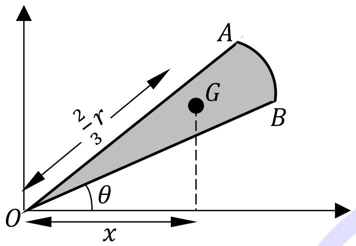
Uniform lamina in the shape of a sector of a circle

Consider a uniform lamina in the shape of a sector of a circle, centre O and radius r. The sector subtends an angle of 2α at O.





Element OAB subtends an angle $\delta\theta$ (in radians) at O. By considering this element as a triangle and by allowing θ to range from $-\alpha$ to $+\alpha$, the centre of mass can be obtained as follows.



OAB approximates a triangle so centre of mass is $\frac{2r}{3}$ from O.

Distance from y-axis is $\frac{2}{3}r\cos\theta$

Let the mas per unit area be ρ

Shape	Area	Mass	Distance from y-axis
Small triangle	$\frac{1}{2}r^2\delta\theta$	$\frac{1}{2}\rho r^2\delta\theta$	$\frac{2}{3}r\cos\theta$
Whole shape	$r^2\alpha$	$\rho r^2\alpha$	$\bar{x}$

Equating moments;

$$\rho r^2\alpha\bar{x} = \sum_{-\alpha}^{\alpha}\left[\frac{1}{2}\rho r^2\delta\theta \times \frac{2}{3}r\cos\theta\right]$$

$$\rho r^2\alpha\bar{x} = \frac{1}{3}\rho r^3\sum_{-\alpha}^{\alpha}\cos\theta\ \delta\theta$$

$$\alpha\bar{x} = \frac{r}{3}\sum_{-\alpha}^{\alpha}\cos\theta\ \delta\theta$$

As $\delta\theta \to 0$, $\sum_{-\alpha}^{\alpha}\cos\theta\ \delta\theta \to \int_{-\alpha}^{\alpha}\cos\theta\, d\theta$

$$\alpha\bar{x} = \frac{r}{3}\int_{-\alpha}^{\alpha}\cos\theta\, d\theta$$

$$\alpha\bar{x} = \frac{r}{3}[\sin\theta]_{-\alpha}^{\alpha}$$

$$\alpha\bar{x} = \frac{r}{3}\left(\sin\alpha - \sin(-\alpha)\right)$$

$$\alpha\bar{x} = \frac{2r\sin\alpha}{3}$$

$$\bar{x} = \frac{2r\sin\alpha}{3\alpha}$$

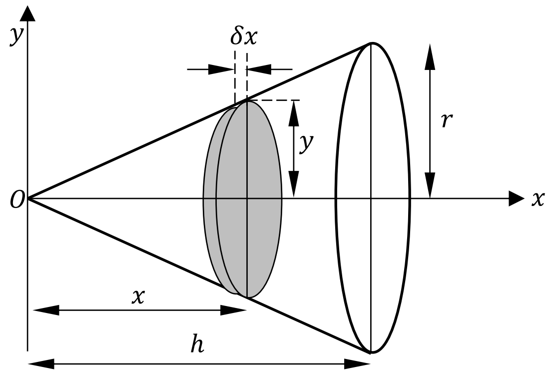
For semicircular lamina, $\alpha = \frac{\pi}{2}$

$$\bar{x} = \frac{2r\sin\frac{\pi}{2}}{3\left(\frac{\pi}{2}\right)} = 2r \times \frac{2}{3\pi} = \frac{4r}{3\pi}$$

The centre of mass is at a distance of $\frac{4r}{3\pi}$ from the centre along the axis of symmetry.

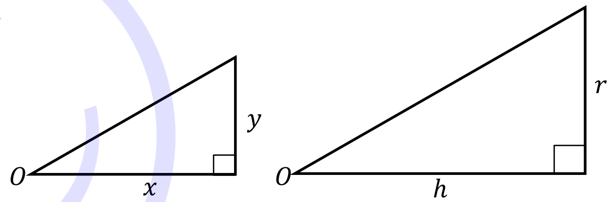
Solid circular right cone

Consider a right circular cone of radius r and height h.



Each cylinder has its centre of mass on the axis Ox. The radius of each cylinder is y and the thickness of each cylinder will be δx. So the volume of each cylinder will be $\pi y^2\delta x$.

Using similar triangles, y can be found in terms of x.



$$\frac{y}{x} = \frac{r}{h}$$

$$y = \frac{rx}{h}$$

Let the weight per unit volume be W

Shape	Volume	Weight	Distance from Oy
Small cylinder	$\pi y^2\delta x$ $= \pi\left(\frac{r^2x^2}{h^2}\right)\delta x$	$\pi W\left(\frac{r^2x^2}{h^2}\right)\delta x$	x
Whole shape	$\frac{1}{3}\pi r^2h$	$\frac{1}{3}\pi Wr^2h$	$\bar{x}$

Equating moments;

$$\frac{1}{3}\pi Wr^2h\bar{x} = \sum_{0}^{h}\left[\pi W\left(\frac{r^2x^2}{h^2}\right)\delta x \times x\right]$$

$$\frac{1}{3}\pi Wr^2h\bar{x} = \frac{\pi Wr^2}{h^2}\sum_{0}^{h}x^3\delta x$$

$$\frac{1}{3}h\bar{x} = \frac{1}{h^2}\sum_{0}^{h}x^3\delta x$$

As $\delta x \to 0$, $\sum_{0}^{h}x^3\delta x \to \int_{0}^{h}x^3\, dx$

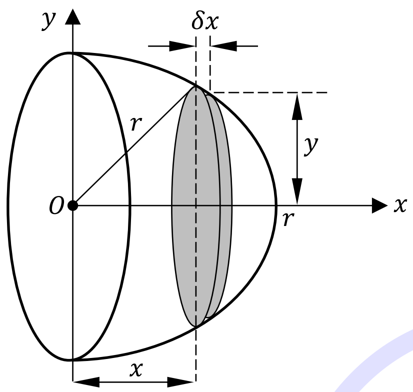
$$\frac{1}{3}h\bar{x} = \frac{1}{h^2}\int_{0}^{h}x^3\, dx$$

$$\frac{1}{3}h\bar{x} = \frac{1}{h^2}\left[\frac{x^4}{4}\right]_0^h$$
$$\frac{1}{3}h\bar{x} = \frac{1}{h^2}\left[\frac{h^4}{4}\right] - 0$$
$$\frac{1}{3}h\bar{x} = \frac{h^2}{4}$$
$$\bar{x} = \frac{h^2}{4} \times \frac{3}{h} = \frac{3h}{4}$$

The centre of mass is $\frac{3}{4}h$ from the vertex

Solid hemisphere

Consider a solid hemisphere of radius r



Each cylinder has its centre of mass on the axis Ox. The radius of each cylinder is y and the thickness of each cylinder will be δx. So the volume of each cylinder will be $\pi y^2 \delta x$.

Using the equation of a circle with centre O, y^2 can be found in terms of x

$$x^2 + y^2 = r^2$$
$$y^2 = r^2 - x^2$$

Let the weight per unit volume be W

Shape	Volume	Weight	Distance from Oy
Small cylinder	$\pi y^2 \delta x$ $= \pi(r^2 - x^2)\delta x$	$\pi W(r^2 - x^2)\delta x$	x
Whole shape	$\frac{2}{3}\pi r^3$	$\frac{2}{3}\pi r^3 W$	$\bar{x}$

Equating moments;

$$\frac{2}{3}\pi r^3 W\bar{x} = \sum_0^r [\pi W(r^2 - x^2)\delta x \times x]$$
$$\frac{2}{3}\pi r^3 W\bar{x} = \pi W \sum_0^r (xr^2 - x^3)\delta x$$
$$\frac{2}{3}r^3\bar{x} = \sum_0^r (xr^2 - x^3)\delta x$$

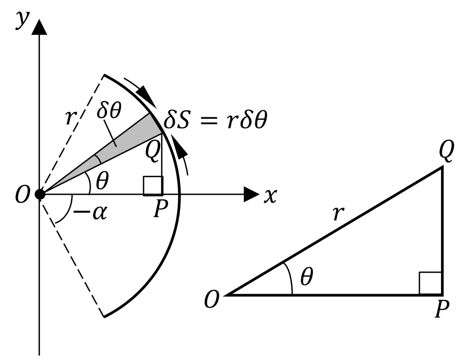
As $\delta x \to 0$, $\sum_0^r (xr^2 - x^3)\delta x \to \int_0^r (xr^2 - x^3)\,dx$

$$\frac{2}{3}r^3\bar{x} = \int_0^r (xr^2 - x^3)\,dx$$
$$\frac{2}{3}r^3\bar{x} = \left[\frac{x^2r^2}{2} - \frac{x^4}{4}\right]_0^r$$
$$\frac{2}{3}r^3\bar{x} = \left[\frac{r^4}{2} - \frac{r^4}{4}\right] - 0$$
$$\frac{2}{3}r^3\bar{x} = \frac{r^4}{4}$$
$$\bar{x} = \frac{r^4}{4} \times \frac{3}{2r^3} = \frac{3r}{8}$$

The centre of mass is at a distance of $\frac{3r}{8}$ from the centre along the axis of symmetry.

Circular arc

Consider a circular arc with radius r and angle 2α.



Each small length is $r\delta\theta$ long. The distance of the centre of mass from Oy will be OP since the length $r\delta\theta$ will be extremely small and so the centre and end of this arc will be negligible distance apart. This is calculated using trigonometry, $OP = r\cos\theta$.

Let the weight per unit length be W

Shape	Length	Weight	Distance from Oy
Small arc	$r\delta\theta$	$Wr\delta\theta$	$r\cos\theta$
Whole shape	$r(2\alpha)$	$2Wr\alpha$	$\bar{x}$

Equating moments;

$$(2Wr\alpha)\bar{x} = \sum_{-\alpha}^{\alpha}(Wr\delta\theta \times r\cos\theta)$$
$$2Wr\alpha\bar{x} = Wr^2 \sum_{-\alpha}^{\alpha}\cos\theta\,\delta\theta$$
$$2\alpha\bar{x} = r\sum_{-\alpha}^{\alpha}\cos\theta\,\delta\theta$$

As $\delta\theta \to 0$, $\sum_{-\alpha}^{\alpha}\cos\theta\,\delta\theta \to \int_{-\alpha}^{\alpha}\cos\theta\,d\theta$

$$2\alpha\bar{x} = r\int_{-\alpha}^{\alpha}\cos\theta\,d\theta$$
$$2\alpha\bar{x} = r[\sin\theta]_{-\alpha}^{\alpha}$$
$$2\alpha\bar{x} = r[\sin\alpha - \sin(-\alpha)]$$
$$2\alpha\bar{x} = 2r\sin\alpha$$
$$\bar{x} = \frac{r\sin\alpha}{\alpha}$$

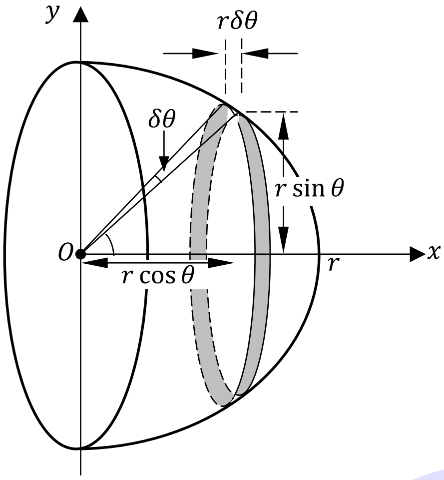
Semicircular arc

For a semicircle, $\alpha = \frac{\pi}{2}$

$$\bar{x} = \frac{r\sin\frac{\pi}{2}}{\frac{\pi}{2}} = r \times \frac{2}{\pi} = \frac{2r}{\pi}$$

Uniform hemispherical shell

Consider a uniform hemispherical shell of radius r. The shaded portion is a small circular element of the shell. This element is parallel to the plane face of the hemisphere and at a distance $r\cos\theta$ from it. The element may be considered to approximate to a circular ring of radius $r\sin\theta$ and thickness $r\delta\theta$. If θ is allowed to range from 0 to $\frac{\pi}{2}$, these elements together form the hemispherical shell.



Let the mass per unit area be ρ

Mass of whole shell $= 2\pi r^2\rho$

Mass of circular element $= 2\pi(r\sin\theta)(r\delta\theta)\rho$

Equating moments;

$$2\pi r^2\rho\bar{x} = \sum_{0}^{\pi/2}[2\pi(r\sin\theta)(r\delta\theta)\rho \times r\cos\theta]$$

$$2\pi r^2\rho\bar{x} = \pi r^3\rho\sum_{0}^{\pi/2}\sin 2\theta\ \ \delta\theta$$

$$2\bar{x} = r\sum_{0}^{\pi/2}\sin 2\theta\ \ \delta\theta$$

As $\delta\theta \to 0$, $\sum_0^{\pi/2}\sin 2\theta\,\delta\theta \to \int_0^{\pi/2}\sin 2\theta\,d\theta$

$$2\bar{x} = \int_0^{\pi/2}\sin 2\theta\,d\theta$$

$$2\bar{x} = r\left[-\frac{1}{2}\cos 2\theta\right]_0^{\pi/2}$$

$$2\bar{x} = \frac{r}{2}[-\cos\pi + \cos 0]$$

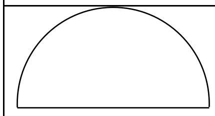
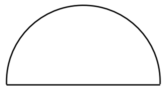
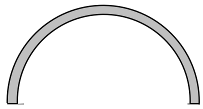
$$2\bar{x} = r$$

$$\bar{x} = \frac{r}{2}$$

The centre of mass is at $\frac{r}{2}$ from O (base) along the axis of symmetry.

Alternative method: We can use the theory of compound bodies to find the centre of mass of a hemispherical shell.

Note that if you use this method in an exam question which asks for a calculus technique, you would have to use calculus to prove the results for a solid hemisphere first. From a hemisphere of radius $r+\delta r$, we remove a hemisphere with radius r to form a hemispherical shell of radius r and thickness δr.

Shape	Mass	G above base
 $r+\delta r$	$\frac{2}{3}\pi(r+\delta r)^3\rho$	$\frac{3}{8}(r+\delta r)$
 r	$\frac{2}{3}\pi r^3\rho$	$\frac{3}{8}r$
	$\frac{2}{3}\pi(r+\delta r)^3\rho$ $-\frac{2}{3}\pi r^3\rho$	$\bar{y}$

$$\frac{2}{3}\pi(r+\delta r)^3\rho \times \frac{3}{8}(r+\delta r) - \frac{2}{3}\pi r^3\rho \times \frac{3}{8}r = \left(\frac{2}{3}\pi(r+\delta r)^3\rho - \frac{2}{3}\pi r^3\rho\right)\bar{y}$$

Now by expanding and ignoring $(\delta r)^2$ and higher terms;

$$\frac{1}{4}\pi(r^4 + 4r^3\rho \ldots. - r^4) = \frac{2}{3}\pi(r^3 + 3r^2\delta r \ldots. - r^3)\bar{y}$$

$$r^3\delta r \cong 2r^2\delta r\bar{y}$$

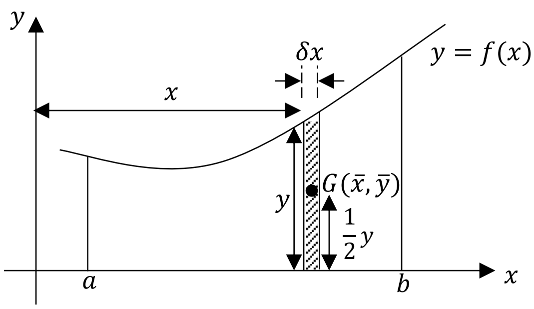
$$\text{As } \delta r \to 0,\ \bar{y} = \frac{1}{2}r$$

The centre of mass of mass of a hemispherical shell is on the line of symmetry, $\frac{1}{2}r$ from the base (centre).

To find the centre of mass of a non-uniform lamina

Situation 1:

Consider finding the position of G of a non-uniform lamina formed by the area enclosed by the curve/line $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$. Since the lamina is non-uniform (no line of symmetry), the position of G will have two coordinates i.e. $(\bar{x}, \bar{y})$



To find $\bar{x}$;

Strip area $= y\delta x$

Mass of strip $= \rho y\delta x$ where ρ is the mass per unit area

Mass of whole shape $= \int_a^b \rho y dx$

Equating moments;

Moment of whole shape = sum of moments

$$\bar{x}\int_a^b \rho y dx = \int_a^b \rho y dx \times x$$

$$\boldsymbol{\bar{x} = \frac{\int_a^b xy\,dx}{\int_a^b y\,dx}}$$

with equation $y = \sqrt{32 - 8x}$. The centre of mass of a uniform lamina whose shape is that of R, is denoted by G. Use integration to determine the exact coordinates of G

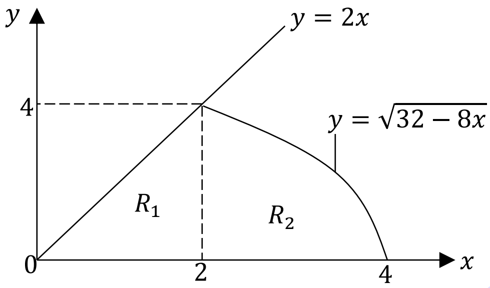
Solution

$y = 2x$ and $y = \sqrt{32 - 8x}$

$$2x = \sqrt{32 - 8x}$$
$$4x^2 = 32 - 8x$$
$$4x^2 + 8x - 32 = 0$$
$$x^2 + 2x - 8 = 0$$
$$x^2 - 2x + 4x - 8 = 0$$
$$(x - 2)(x + 4) = 0$$
$$x = 2 \text{ or } x = -4 \text{ (outside region)}$$

When $x = 2$, $y = 2(2) = 4$

When $y = 0$, $\sqrt{32 - 8x} = 0 \Rightarrow x = \frac{32}{8} = 4$



Area of $R_1 = \frac{1}{2} \times 2 \times 4 = 4$

Area of $R_2 = \int_2^4 (32 - 8x)^{\frac{1}{2}} dx$

$$= \left[-\frac{1}{12}(32 - 8x)^{\frac{3}{2}}\right]_2^4$$
$$= \frac{1}{12}\left[(32 - 8x)^{\frac{3}{2}}\right]_4^2$$
$$= \frac{1}{12}[64 - 0] = \frac{64}{12} = \frac{16}{3}$$

$$\bar{x} = \frac{\int_a^b xy\,dx}{\int_a^b y\,dx} = \frac{\int_2^4 x\sqrt{32 - 8x}\,dx}{16/3} = \frac{3}{16}\int_2^4 x(32 - 8x)^{\frac{1}{2}} dx$$

By substitution;

$$u = (32 - 8x)^{\frac{1}{2}}$$
$$u^2 = 32 - 8x$$
$$2u\frac{du}{dx} = -8x$$
$$dx = -\frac{1}{4}u\,du$$

x	u
2	4
4	0

$$8x = 32 - u^2$$
$$x = 4 - \frac{1}{8}u^2$$

$$\bar{x} = \frac{3}{16}\int_4^0 xu\left(-\frac{1}{4}udu\right) = \frac{3}{64}\int_4^0 u^2x\;du$$
$$= \frac{3}{64}\int_0^4 u^2\left(4 - \frac{1}{8}u^2\right) du$$
$$= \frac{3}{64}\int_0^4 4u^2 - \frac{1}{8}u^4\,du$$
$$= \frac{3}{512}\int_0^4 32u^2 - u^4\;du$$
$$= \frac{3}{512}\left[\frac{32}{3}u^3 - \frac{1}{5}u^5\right]_0^4$$
$$= \frac{3}{512}\left[\frac{2048}{3} - \frac{1024}{5}\right]$$
$$= \frac{14}{5}$$

$$\bar{y} = \frac{\int_a^b \frac{1}{2}y^2\,dx}{\int_a^b y\,dx} = \frac{\int_2^4 \frac{1}{2}(32 - 8x)\,dx}{16/3} = \frac{3}{16}\int_2^4 16 - 4x\,dx$$
$$= \frac{3}{16}[16x - 2x^2]_2^4$$
$$= \frac{3}{16}[(64 - 32) - (32 - 8)]$$
$$= \frac{3}{2}$$

Let the mass per unit area be ρ

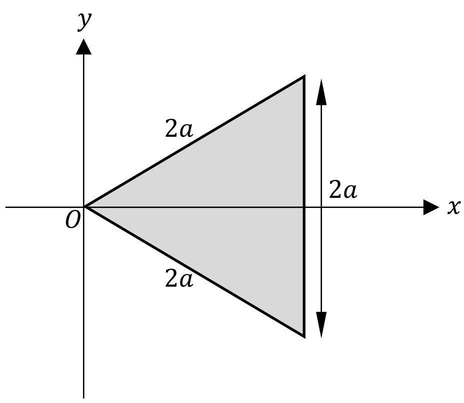
Shape	Mass	x	y
Triangle	4ρ	$2 - \frac{1}{3}(2) = \frac{4}{3}$	$\frac{1}{3}(4) = \frac{4}{3}$
Curved lamina	$\frac{16}{3}\rho$	$\frac{14}{5}$	$\frac{3}{2}$
Whole shape	$\frac{28}{3}\rho$	$\bar{x}$	$\bar{y}$

$$\frac{28}{3}\rho\bar{x} = 4\rho \times \frac{4}{3} + \frac{16}{3}\rho \times \frac{14}{5}$$
$$140\bar{x} = 80 + 224$$
$$\bar{x} = \frac{304}{140} = \frac{76}{35}$$
$$\frac{28}{3}\rho\bar{y} = 4\rho \times \frac{4}{3} + \frac{16}{3}\rho \times \frac{3}{2}$$
$$56\bar{y} = 32 + 48$$
$$\bar{y} = \frac{80}{56} = \frac{10}{7}$$

The centre of mass is $\left(\frac{76}{35}, \frac{10}{7}\right)$

Self-Evaluation exercise

1. Use integration to show that the centre of mass of a uniform solid hemisphere of base radius a is a distance $\frac{3}{8}a$ from the centre of the base of the hemisphere.
2.



The figure above shows a uniform lamina in the shape of an equilateral triangle of side length $2a$. The lamina

Standard results for centre of mass

Previously, integration was used to find formulas for the centre of mass of common shapes. These results can often be used straight from tables instead of being proved by integration each time. The table of standard results is shown below

Shape	Centre of mass
Circular arc, radius r and angle 2α	$\frac{r\sin\alpha}{\alpha}$ from the centre, O
Sector of a circle, radius r and angle 2α	$\frac{2r\sin\alpha}{3\alpha}$ from the centre, O
Semicircular lamina, radius r	$\frac{4r}{3\pi}$ from the centre, O
Hemispherical shell (hollow), radius r	$\frac{r}{2}$ from the centre, O
Solid hemisphere, radius r	$\frac{3r}{8}$ from the centre, O
Solid right circular cone, height h (tetrahedron, pyramid)	$\frac{3h}{4}$ from the vertex V or $\frac{h}{4}$ from the base
Hollow cone, height h	$\frac{h}{3}$ from the base or $\frac{2h}{3}$ from the vertex
Triangular lamina, height h	$\frac{h}{3}$ from the base or $\frac{2h}{3}$ from the vertex

These results can be used to determine the centre of mass of composite shapes

Composite bodies

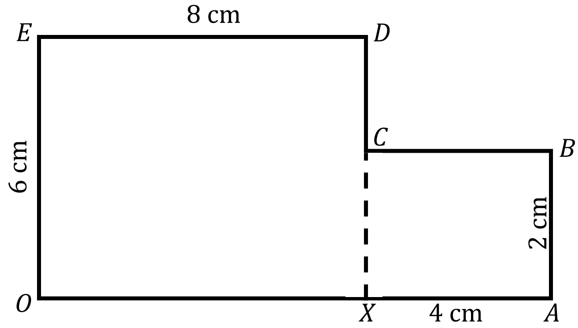
A composite is one made from two or more parts (usually standard).

To find the centre of mass of a composite body

(i) Draw a clear diagram s
(ii) Mark any lines of symmetry
(iii) Choose two axes at right angles to each other. If a line of symmetry exists choose this as an axis.
(iv) Divide the body into known (standard) bodies
(v) Tabulate the masses/weights and distances of centres of masses from the chosen axes.
Note: In a uniform lamina, mass/weight ∝ area
In a uniform solid, mass/weight ∝ volume
(vi) Take moments about the chosen axes
(vii) Use the principle of moments:
Moment of total mass/weight about an axis = sum of moments of separate masses about the same axis

This method can also be used to deal with a body from which a part has been removed.

Example 1



Find the centre of mass of the above uniform lamina taking OA and OE as axes

Solution

Divide the lamina into two rectangles $OXDE$ and $XABC$

Let the mass per unit area be M

Shape	Mass	Distance of $C.O.G$	
		from OE	From OA
$OXDE$	$48M$	4	3
$XABC$	$8M$	$(8+2)$ $=10$	1
$OABCDE$	$56M$	$\bar{x}$	$\bar{y}$

Moments about OE give;

$$48M\times 4+8M\times 10=56\times\bar{x}$$

$$\bar{x}=\frac{34}{7}\ cm$$

Moments about OA give;

$$48M\times 3+8M\times 1=56\times\bar{y}$$

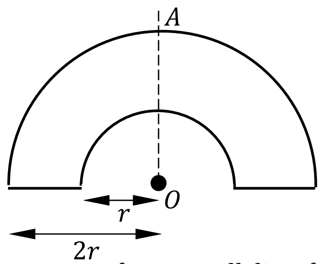
$$\bar{y}=\frac{19}{7}\ cm$$

The centre of mass is at a point which is $\frac{34}{7}\ cm$ from OE and $\frac{19}{7}\ cm$ from OA

Example 2

A semi-circle of radius r is cut out from a uniform semicircular laminar of radius $2r$. Find the position of the centre of mass of the resulting shape.

Solution



By symmetry, the centre of mass will lie of the axis of symmetry OA.

Mass of small semicircle $=\frac{1}{2}\pi r^2\rho$

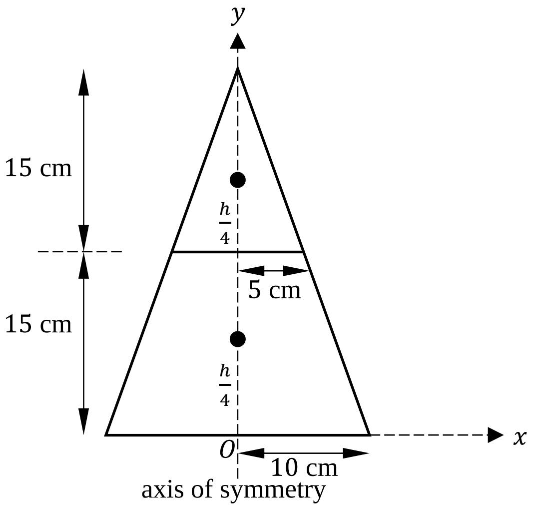
Mass of large semicircle $=\frac{1}{2}\pi(2r)^2\rho=\frac{1}{2}(4\pi r^2\rho)$

The mass of the compound shape is

$$M=\frac{1}{2}(4\pi r^2-\pi r^2)\rho=\frac{3}{2}\pi r^2\rho$$

and the centre of mass of a semicircle is $\frac{4r}{3\pi}$ from the centre

Solution



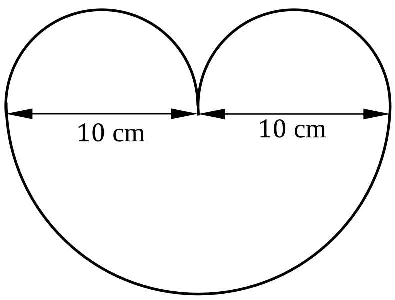
Let the weight per unit volume be W

Shape	Volume	Weight	Distance from Ox
Large cone	$\frac{1}{3}\pi(5^2)(30)$ $= 1000\pi$	$1000\pi W$	$\frac{h}{4} = \frac{30}{4}$ $= 7.5$
Small cone	$\frac{1}{3}\pi(5^2)(15)$ $= 125\pi$	$125\pi W$	$\frac{15}{4} + 15$ $= 18.75$
Whole shape	875π	$875\pi W$	$\bar{y}$

Equating moments;

$$875\pi W\bar{y} = 1000\pi W \times 7.5 - 125\pi W \times 18.75$$
$$875\bar{y} = 7500 - 2343.75$$
$$\bar{y} = 5.89\ cm$$

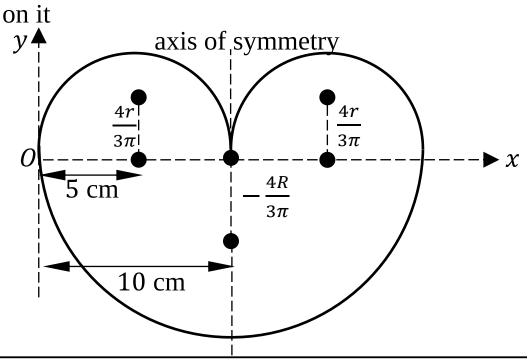
Example 6



Find the centre of gravity of the lamina made of three semicircles as shown above.

Solution

There is an axis of symmetry and so the centre of gravity must lie on it



Let the weight per unit area be W

Shape	Area	Weight	Distance from Ox
Small semicircle	$\frac{1}{2}\pi \times 5^2$ $= 12.5\pi$	$12.5\pi W$	$\frac{4r}{3\pi} = \frac{20}{3\pi}$
Small semicircle	12.5π	$12.5\pi W$	$\frac{4r}{3\pi} = \frac{20}{3\pi}$
Large semicircle	$\frac{1}{2}\pi \times 10^2$ $= 50\pi$	$50\pi W$	$-\frac{4R}{3\pi}$ $= -\frac{40}{3\pi}$
Whole shape	75π	$75\pi W$	$\bar{y}$

Equating moments;

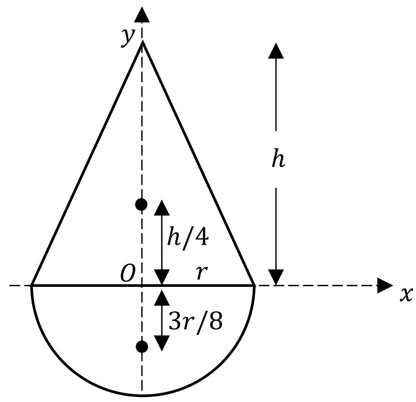
$$75\pi W\bar{y} = 12.5\pi W \times \frac{20}{3\pi} + 12.5\pi W \times \frac{20}{3\pi} + 50\pi W \times -\frac{40}{3\pi}$$
$$75\bar{y} = \frac{250}{3\pi} + \frac{250}{3\pi} - \frac{2000}{3\pi}$$
$$75\bar{y} = -\frac{1500}{3\pi}$$
$$\bar{y} = -\frac{500}{75\pi} = -\frac{20}{3\pi}$$

The centre of mass is $\frac{20}{3\pi}\ cm$ below the line Ox

Example 7

A body consists of a solid hemisphere of radius r joined to a solid right circular cone of base radius r and perpendicular height h. The plane surfaces of the cone and the hemisphere coincide and both solids are made of the same uniform material. Show that the centre of gravity of the body lies on the axis of symmetry at a distance $\frac{3r^2-h^2}{4(h+2r)}$ from the base of the cone.

Solution



Let the weight per unit volume be W

Shape	Volume	Weight	Distance from O
Cone	$\frac{1}{3}\pi r^2 h$	$\frac{1}{3}\pi r^2 hW$	$\frac{h}{4}$
Hemisphere	$\frac{2}{3}\pi r^3$	$\frac{2}{3}\pi r^3 W$	$-\frac{3r}{8}$
Whole shape	$\frac{1}{3}\pi r^2(h+2r)$	$\frac{1}{3}\pi r^2(h+2r)W$	$\bar{y}$

Equating moments;

$$\frac{1}{3}\pi r^2(h+2r)W\bar{y} = \frac{1}{3}\pi r^2 hW \times \frac{h}{4} + \frac{2}{3}\pi r^3 W \times -\frac{3r}{8}$$

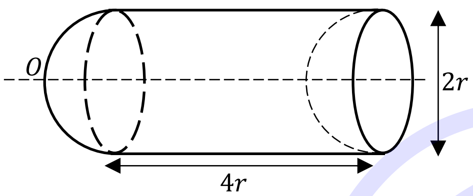
$$\frac{(h+2r)}{3}\bar{y} = \frac{h^2}{12} - \frac{r^2}{4}$$
$$\frac{(h+2r)}{3}\bar{y} = \frac{h^2-3r^2}{12}$$
$$\bar{y} = \frac{h^2-3r^2}{4(h+2r)}$$
$$\bar{y} = -\frac{3r^2-h^2}{4(h+2r)}$$

The centre of gravity lies at $\frac{3r^2-h^2}{4(h+2r)}$ below the base of the cone. (within the hemisphere or below Ox)

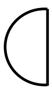
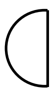
Example 8

A uniform right circular solid cylinder has a radius r and length $4r$. A solid hemisphere of radius r is cut from one end of the cylinder, the plane face of which is one of the plane faces of the cylinder. The hemisphere so removed is now attached by its plane face to the uncut plane face of the cylinder thus forming a new solid. Find the position of the centre of mass of the new solid.

Solution



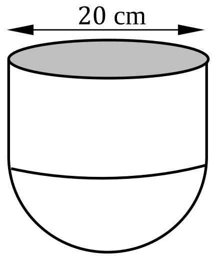
Let O be the point on the line of symmetry at the extreme end of the solid.
Let $\bar{x}$ be the distance of the centre of mass of the new solid from O.
Let the mass per unit volume be M

Shape	Mass	Distance of G from O
	$4\pi r^3 M$	$r + 2r = 3r$
	$\frac{2}{3}\pi r^3 M$	$\frac{5}{8}r$
	$\frac{2}{3}\pi r^3 M$	$r + 4r - \frac{3}{8}r = \frac{37}{8}r$
	$4\pi r^3 M$	$\bar{x}$

Moments about O give;

$$4\pi r^3 M \times \bar{x} = (4\pi r^3 M \times 3r) + \left(\frac{2}{3}\pi r^3 M \times \frac{5}{8}r\right) - \left(\frac{2}{3}\pi r^3 M \times \frac{37}{8}r\right)$$
$$4\bar{x} = 12r + \frac{5}{12}r - \frac{37}{12}r$$
$$4\bar{x} = \frac{112r}{12}$$
$$\bar{x} = \frac{7}{3}r$$

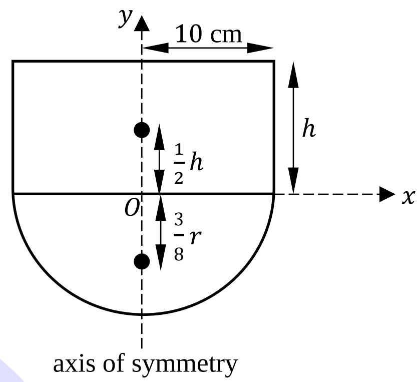
Example 9



A child's toy is formed by gluing a cylinder onto a hemisphere. The centre of mass of the toy lies on the glued edge. Calculate the height of the cylinder.

Solution

Let the height of the cylinder be h



Let the weight per unit volume be W

Shape	Volume	Weight	Distance from Ox
Cylinder	$\pi(10^2)h = 100\pi h$	$100\pi hW$	$\frac{h}{2}$
Hemisphere	$\frac{2}{3}\pi(10^3) = \frac{2000}{3}\pi$	$\frac{2000}{3}\pi W$	$-\frac{3r}{8} = -\frac{30}{8} = -3.75$
Whole shape	$100\pi h + \frac{2000}{3}\pi$	$\left(100\pi h + \frac{2000}{3}\pi\right)W$	$\bar{y} = 0$

Equating moments;

$$\left(100\pi h + \frac{2000}{3}\pi\right)W \times 0 = 100\pi hW \times \frac{h}{2} + \frac{2000}{3}\pi W \times -3.75$$
$$0 = 50h^2 - 2500$$
$$h^2 = 50$$
$$h = \sqrt{50} = 7.07 \text{ cm}$$

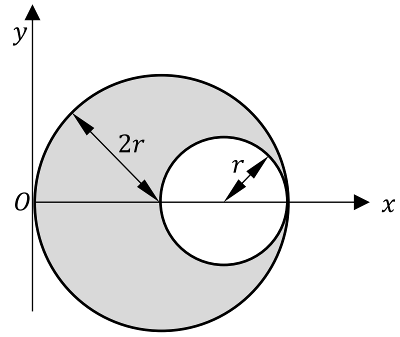
Self-Evaluation exercise

1. A circular lamina, made of a uniform material, has its centre at the origin and a radius of 6 units. Two smaller circles are cut from this circle, one of radius 1 unit and centre $(-1, -3)$ and the other of radius 3 units and centre $(1, 2)$. Find the coordinates of the centre of gravity of the remaining shape.

   [Ans: $\left(-\frac{4}{13}, -\frac{15}{26}\right)$]

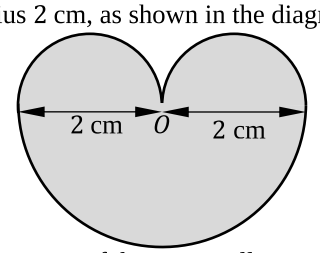
2. A uniform lamina is formed by removing a circular disc of radius r from a circular disc of radius $2r$ as shown.

Find the position of the centre of gravity of the lamina with respect to the axes Ox and Oy as shown.



[Ans: $\bar{x} = \frac{5r}{3}$; $\bar{y} = 0$]

3. A badge is cut from a uniform thin sheet of metal. The badge is formed by joining the diameters of two semicircles, each of radius 1 cm, to the diameter of a circle of radius 2 cm, as shown in the diagram.

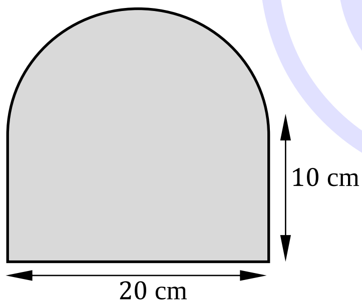


The point of contact of the two smaller semicircles is O. Determine, in terms of π, the distance from O of the centre of mass of the badge.

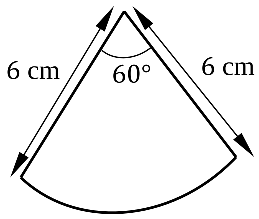
[Ans: $\frac{4}{3\pi}$ cm]

4. Determine the centre of mass of the uniform laminas shown below

(a)

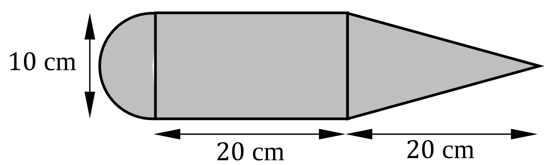


(b)



[Ans: (a) 9.07 cm from bottom of shape (b) $12/\pi$ from vertex]

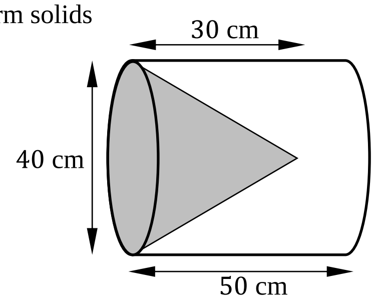
5. A spike is made from a hemisphere, a cylinder and a cone. Determine the centre of mass of the spike



[Ans: 28 cm from vertex of the cone]

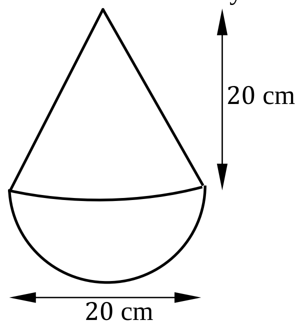
6. Calculate the position of the centre of mass of the uniform solids

(a)



Cone drilled out of a cylinder

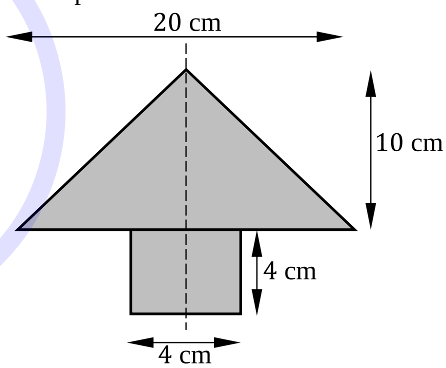
(b)



Solid cone and solid hemisphere joined

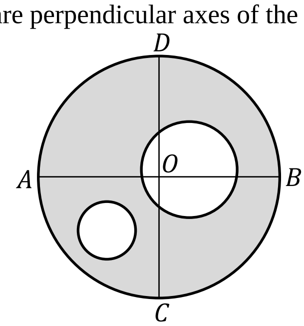
[Ans: (a) 29.375 cm from open end (b) 0.625 cm above joined faces]

7. A child's spinner is made from a right circular cone and a cylinder of the same material. Find the centre of gravity of the spinner



[Ans: 2.29 cm above joint]

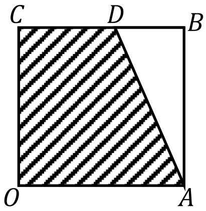
8. The figure below shows a plate of uniform thickness, circular in shape, which has two circular holes drilled through it. The radius of the plate is 80 mm and AOB and COD are perpendicular axes of the plate.



One hole, of radius 20 mm, has its centre on OB at a distance of 5 mm from O; the other, of radius 10 mm, has its centre at a distance of 40 mm from OA and 20 mm from OC. Show that the centre of mass of the plate is located on the axis COD and find its distance from O.

[Ans: $\frac{40}{59}$ mm from O]

9. The diagram shows a square $OABC$ of side a. The mid-point of BC is D. Triangle ADB is cut off.



With respect to OA and OC as axes, find the coordinates of the centre of mass of a uniform lamina in the form of the figure $OADC$.

[Ans: $\left(\frac{7a}{18}, \frac{4a}{9}\right)$]

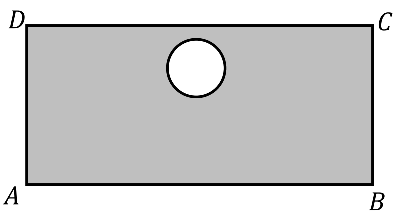
10. A uniform solid right circular cone has its top removed by cutting the cone by a plane parallel to its base, leaving a truncated cone of height h, the radii of its ends being r and $4r$. Show that the distance of the centre of gravity of the truncated cone from its broader end is $\frac{9}{28}h$

11. A can in the form of a circular cylinder, without a lid, is made of a thin metal sheeting of uniform thickness and with a mass per unit area of 1 g/cm^2. The radius of the can is 10 cm and its height is 20 cm. The can is placed with its base on a horizontal plane and is half-filled with a liquid of density 1.5 g/cm^3. Calculate the height of the centre of gravity of the can together with the liquid, above the base of the can.

[Ans: 5.75 cm]

12. Prove that the centre of gravity of a uniform circular arc subtending an angle 2θ at the centre of a circle of radius a is $a\sin\theta/a$ from the centre. Deduce that the centre of gravity of a uniform sector bounded by that arc and the radii to its extremities is $\frac{2}{3}a\sin\theta/\theta$ from the centre. Show also that the centre of gravity of a segment of a circular lamina cut off from a chord subtending a right angle at the centre of the circle is $\frac{2}{3}\sqrt{2a}/(\pi-2)$ from the centre.

13. The diagram, shows a uniform lamina formed by drilling a circular hole of radius 2 cm in a rectangle $ABCD$ with length $AB = 8\pi$ cm and the breadth $BC = 12$ cm. The centre of the hole is 9 cm from AB and is equidistant from AD and BC. Find the centre of mass of the lamina from AB

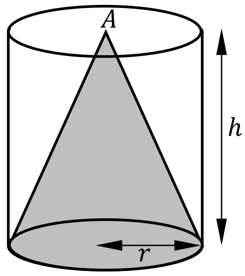


[Ans: 5.87 cm]

14. The base of a uniform solid hemisphere has radius $2a$ and its centre at O. A uniform solid S is formed by removing, from the hemisphere, the solid hemisphere of radius a and centre O. Determine the position of the centre of mass of S.

[Ans: on the axis of symmetry $\frac{45a}{56}$ from O]

15. A solid uniform cylindrical piece of metal, of height h and radius r, has a cone shape removed from it as shown in the diagram. The base of the cone is of radius r and its height is h



Show that the centre of gravity of the resulting solid is at a distance of $\frac{3h}{8}$ from the point A measured along the axis of symmetry

16. A solid frustum of a uniform cone is of thickness h, and the radii of its end-faces are a and b ($a > b$). A cylindrical hole is bored through the frustum. The axis of the hole coincides with the axis of the cone, and the radius of the hole is equal to that of the smaller face of the frustum. Show that the centre of gravity of the solid thus obtained is at a distance from the larger face of the frustum equal to

$$\frac{h(a+3b)}{4(a+2b)}$$

17. A hollow conical vessel made of a thin sheet of metal is closed at its base. If it is cut across by a plane parallel to the base at half the perpendicular height, and the upper cone removed, prove that the distance from the base of the centre of gravity of the remainder of the vessel is

$$\frac{2lh}{3(3l+4r)}$$

where h is the perpendicular height of the original vessel, l is its slant height, and r the radius of the cone

18. A four-sided plane lamina has the shape of a rectangle $ABCD$ surmounted by a triangular part BEC, the side BE of which is a prolongation of AB. If the lengths of AB, AD, BE are a, b, c respectively, find how far the centre of gravity of the lamina is from AB and AD. Also prove that the centre of gravity will lie of BC if $c = a\sqrt{3}$

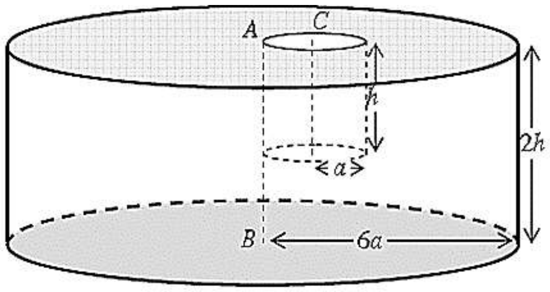
[Ans: $\frac{1}{3}b\frac{3a+c}{2a+c}$; $\frac{3a^2+3ac+c^2}{3(2a+c)}$]

19. A frustum of a cone has its circular ends of radii r_1 and r_2 and at a distance h apart. Its curved surface is covered with a thin uniform material. Show that the height of the centre of gravity of the covering material above the end of radius r_2 is

$$\frac{h(2r_1+r_2)}{3(r_1+r_2)}$$

20. A piece of cardboard is in the form of a rectangle, $ABCD$, where $AB = 5$ cm and $BC = 8$ cm. A piece of

cylinder. The resulting composite solid is shown in the figure below.

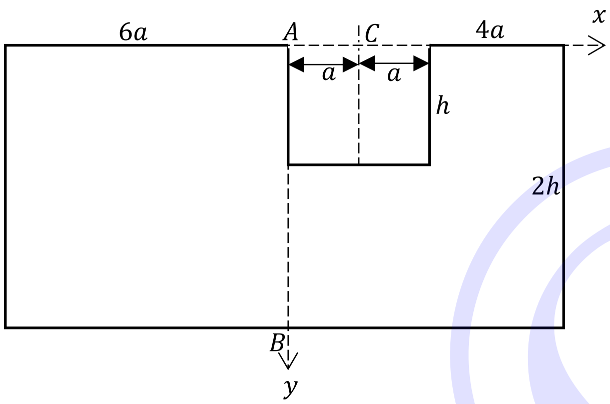


(a) Show that the centre of mass of the composite is at a vertical distance $\frac{143}{142}h$ from the plane face containing A.

(b) The composite is freely suspended from A and hangs in equilibrium with the axis AB inclined at an angle $\tan^{-1} 13$ to the horizontal. Express a in terms of h.

Solution

(a)



Let the weight per unit volume be W

Shape	Weight	x	y
Large cylinder	$\pi(6a)^2(2h)W$ $= 72\pi a^2hW$	0	h
Small cylinder	πa^2hW	a	$\frac{1}{2}h$
Whole shape	$71\pi a^2hW$	$\bar{x}$	$\bar{y}$

Equating moments;

$$71\pi a^2hW \times \bar{y} = 72\pi a^2hW \times h - \pi a^2hW \times \frac{1}{2}h$$

$$71\bar{y} = 72h - \frac{h}{2}$$

$$71\bar{y} = \frac{143}{2}h$$

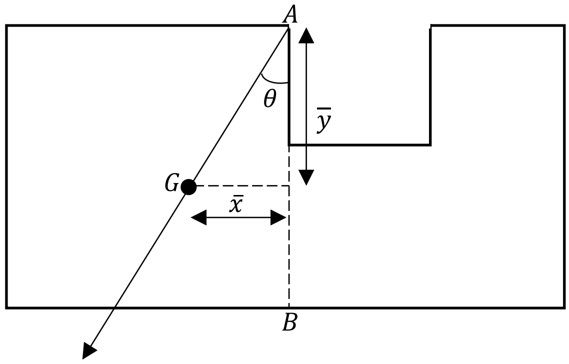
$$\bar{y} = \frac{143}{142}h$$

(b) Also,

$$71\pi a^2hW \times \bar{x} = 72\pi a^2hW \times 0 - \pi a^2hW \times a$$

$$71\bar{x} = -a$$

$$\bar{x} = -\frac{a}{71}$$



Vertical (Weight)

Now $\tan^{-1} 13$ to the horizontal is the same as $\tan^{-1}\frac{1}{13}$ to the vertical

$$\frac{\bar{x}}{\bar{y}} = \tan\theta$$

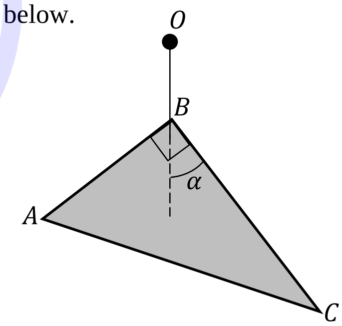
$$\frac{a/71}{\frac{143}{142}h} = \frac{1}{13}$$

$$\frac{a}{71} = \frac{11}{142}h$$

$$a = \frac{11}{2}h$$

Self-Evaluation exercise

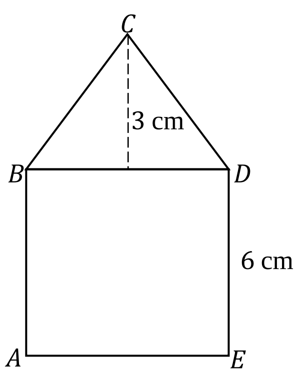
1. A uniform triangular lamina ABC is in equilibrium, suspended from a fixed point O by a light inelastic string attached to the point B of the lamina, as shown in the diagram below.



AB = 45 cm, $BC = 60$ cm and angle $ABC = 90°$. Calculate the angle θ between BC and the downward vertical

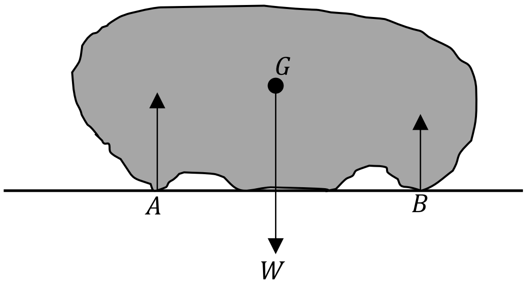
[Ans: 36.9°]

2. A thin uniform plate is formed by a square of side 6 cm being surmounted by an isosceles triangle whose vertical height is 3 cm as shown below



Toppling bodies

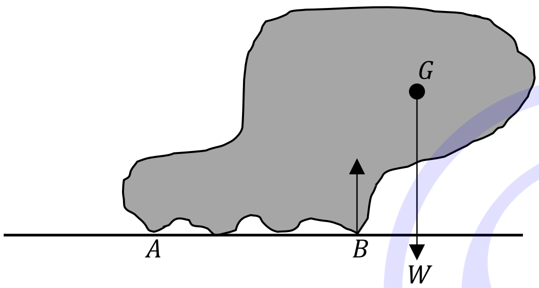
When a body rests in equilibrium on a plane, it will be **stable** provided the line of action of the weight force lies within the extreme points of the contact between the body and the plane



Stable equilibrium

The body will **topple** if the line of action of the weight force lies outside one of these extreme points of contact. It will topple about the point of contact nearest to the line of action of its weight

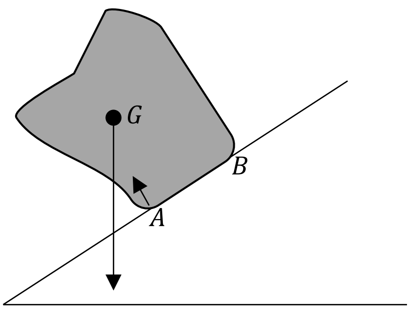
*



Body topples about B

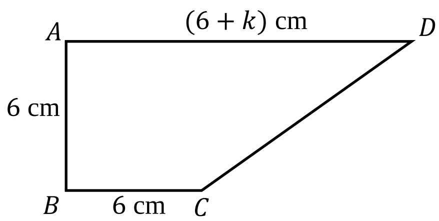
If the body is on a rough inclined plane, it will **topple** if the line of action of the weight force lies outside the lower point of contact of the body with the plane.

On an inclined plane, the equilibrium of a body may be broken by sliding rather than toppling unless the plane is rough enough to prevent the body from sliding.



Body topples about A

Example 1



The figure below shows the cross section of a solid uniform right prism which in the shape of a right-angled trapezium $ABCD$. It is further given that $AB = BC = 6$ cm, $AD = (6 + k)$ cm and $\measuredangle DAB = \measuredangle ABC = 90°$.

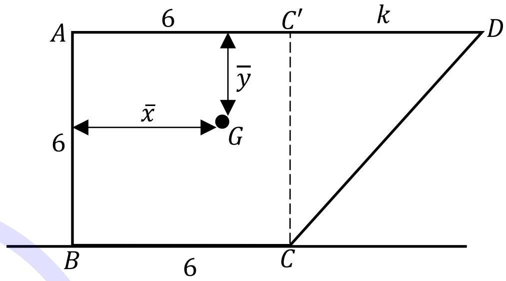
(a) By treating $ABCD$ as a uniform lamina, find in terms of the constant k the position of the centre of mass of $ABCD$, relative to the vertex A.

(b) The prism is resting with $ABCD$ perpendicular to a horizontal surface and the face which contains BC, in contact with this horizontal surface. Calculate the greatest value of k which allows the prism **not** to topple.

(c) The prism is the placed on a rough plane inclined at θ to the horizontal, with BC lying on the line of greatest slope of the plane. The value of k is taken to be $3\sqrt{6}$. Given the prism is about to topple, determine the exact value of $\tan\theta$.

Solution

(a) Let the mass per unit area be M



Shape	Mass	Distance of $C.O.G$	
		from AB	from AD
$ABCC'$	$36M$	3	3
$CC'D$	$3kM$	$6+\frac{1}{3}k$	$\frac{1}{3}(6) = 2$
$ABCD$	$(36+3k)M$	$\bar{x}$	$\bar{y}$

Equating moments about AB;

$$(36+3k)M\bar{x} = 36M \times 3 + 3kM \times \left(6+\frac{1}{3}k\right)$$

$$(k+12)\bar{x} = 36 + k(6+\frac{1}{3}k)$$

$$3(k+12)\bar{x} = 108 + 18k + k^2$$

$$\bar{x} = \frac{k^2+18k+108}{3(k+12)}$$

Equating moments about AD;

$$(36+3k)M\bar{y} = 36M \times 3 + 3kM \times 2$$

$$(k+12)\bar{y} = 36 + 2k$$

$$\bar{y} = \frac{2k+36}{k+12}$$

(b) For no toppling, $\bar{x} \le 6$

$$\frac{k^2+18k+108}{3(k+12)} \le 6$$

$$k^2 + 18k + 108 \le 18k + 216$$

$$k^2 \le 108$$

$$k \le \sqrt{108}$$

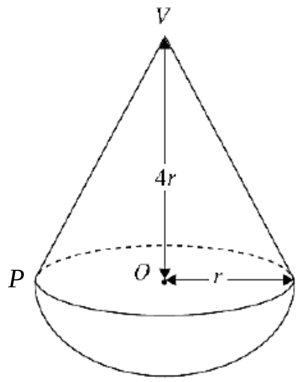
$$k \le 6\sqrt{3}$$

$$\therefore k_{max} = 6\sqrt{3}$$

$$\bar{y}(2+\tan\theta)=\frac{1}{4}h\tan\theta-\frac{3}{4}r$$
$$\bar{y}(2+\tan\theta)=\frac{1}{4}(r\tan\theta)\tan\theta-\frac{3}{4}r$$
$$4\bar{y}(2+\tan\theta)=r\tan^2\theta-3r$$
$$4\bar{y}(2+\tan\theta)=r(\tan^3\theta-3)$$
$$\bar{y}=\frac{\tan^2\theta-3}{8+4\tan\theta}r$$

(b) To return to the upright position, the centre of mass of S must be inside the hemisphere, i.e. $\bar{y}<0$
As $0<\theta<90°$, $8+4\tan\theta>0$
$$\therefore \tan^2\theta-3<0$$
$$\tan^2\theta<3$$
$$\tan\theta<+\sqrt{3}$$
$$0°<\theta<60°$$

Example 5



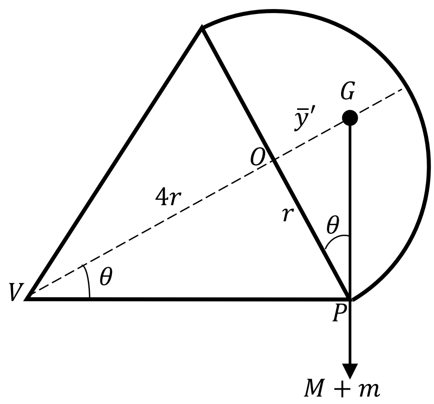
A uniform solid S, consists of a hemisphere of radius r and mass M, and a right circular cone of radius r, height $4r$ and mass m. The centre of the plane face of the hemisphere is at O and this plane face coincides with the plane face at the base of the cone, as shown in the figure above. The point P lies on the circumference of the base of the cone. S is placed on a horizontal surface, so that VP is in contact with the surface, where VP is the vertex of the cone. Given that S remains in equilibrium in that position, show that $m \leq 10M$.

Solution

Shape	Weight	Distance from O
Cone	M	$\frac{1}{4}(4r)=r$
Hemisphere	m	$-\frac{3}{8}r$
Whole shape	$M+m$	$\bar{y}$

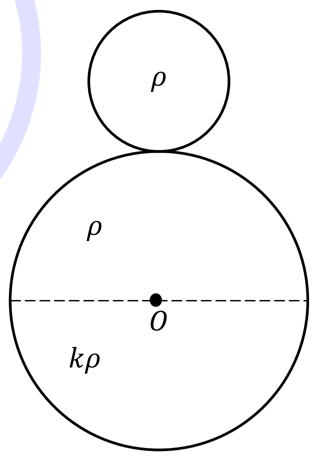
Equating moments about O;
$$(M+m)\bar{y}=Mr-\frac{3}{8}mr$$
$$8(M+m)\bar{y}=(8M-3m)r$$
$$\bar{y}=\frac{8M-3m}{8(M+m)}r$$

Now looking at the object in equilibrium.
Note that in the above calculation we took $\bar{y}$ to be positive in the cone, so if we take positive in the hemisphere
$$\bar{y}'=\frac{(3m-8M)}{8(M+m)}r$$



Comparing similar triangles;
ΔVOP; $\tan\theta=\frac{r}{4r}=\frac{1}{4}$ and ΔPOG; $\tan\theta=\frac{OG}{r}$
$$\frac{OG}{r}=\frac{1}{4}$$
$$|OG|=\frac{1}{4}r$$
$$\Rightarrow \bar{y}'\leq\frac{1}{4}r$$
$$\frac{(3m-8M)}{8(M+m)}r\leq\frac{1}{4}r$$
$$3m-8M\leq 2M+2m$$
$$m\leq 10M$$

Example 6



The diagram above shows a child's toy which consists of a head and a body. The head is a uniform sphere of radius a and density ρ. The head is rigidly attached to the body, which is spherical, of radius $2a$ and centre O. The upper half of the body is a uniform hemisphere of density ρ and this is rigidly attached to the lower half of the body which is a uniform hemisphere of density $k\rho$. The common plane surface of the two hemispheres is perpendicular to the axis of symmetry of the toy.

(a) Find the distance of the centre of mass of the toy from O

(b) The toy is now placed on a horizontal plane and rests with its axis of symmetry vertical. Show that when it is displaced to a different position, it will always return to the vertical position provided that $k > 2$

Solution

(a) Density is mass per unit volume (mass = density × vol)

Shape	Volume	Mass	Distance from O
Head (upper shere)	$\frac{4}{3}\pi a^3$	$\frac{4}{3}\pi a^3\rho$	$3a$
Upper hemisphere	$\frac{2}{3}\pi(2a)^3 = \frac{16}{3}\pi a^3$	$\frac{16}{3}\pi a^3\rho$	$\frac{3(2a)}{8} = \frac{3a}{4}$
Lower hemisphere	$\frac{2}{3}\pi(2a)^3 = \frac{16}{3}\pi a^3$	$\frac{16}{3}\pi a^3 k\rho$	$-\frac{3a}{4}$
Whole shape		$\frac{4}{3}\pi a^3\rho(5+4k)$	$\bar{h}$

Equating moments about O;

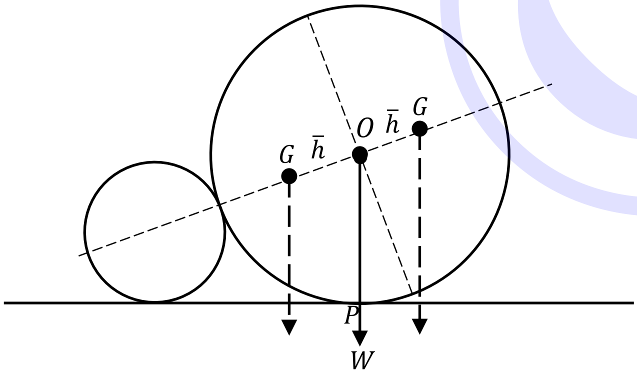
$\frac{4}{3}\pi a^3\rho(5+4k)\times\bar{h}$

$$= \frac{4}{3}\pi a^3\rho\times 3a + \frac{16}{3}\pi a^3\rho\times\frac{3a}{4} + \frac{16}{3}\pi a^3 k\rho\times -\frac{3a}{4}$$

$$(5+4k)\bar{h} = 3a + 3a - 3ka$$

$$\bar{h} = \frac{3a(2-k)}{5+4k}$$

(b) Let the weight of the child's toy be W and be displaced to the position shown. Now look at the two positions of G i.e. one with in the lower hemisphere and the other within the upper hemisphere.



It is clear that for the toy to return to its vertical position, the point of action of its weight must lie on the left of P i.e. $\bar{h} < 0$

$$\frac{3a(2-k)}{5+4k} < 0$$

$$3a(2-k) < 0$$

$$2 - k < 0$$

$$2 < k$$

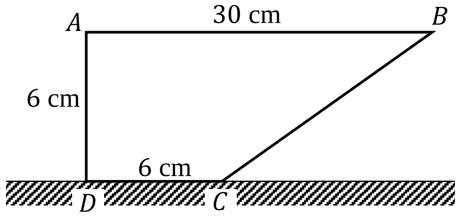
$$\therefore k > 2$$

Thus, body will always return to vertical position provided $k > 2$

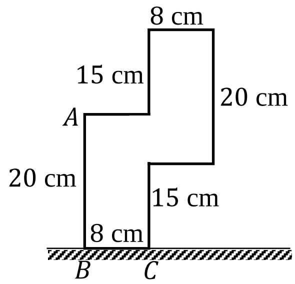
Self-Evaluation exercise

1. Find the position of the centre of mass of the laminas shown and determine whether the laminas will topple

(a)



(b)

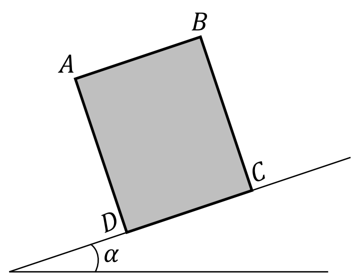


[Ans: (a) 10.3 cm from AD and 3.67 cm from D; topple (b) 8 cm from AB and 17.5 cm from BC, does not topple (just about to topple)]

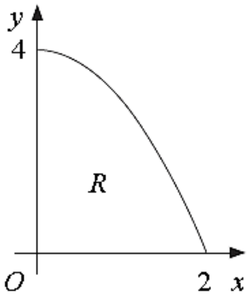
2. A toy consists of a solid hemisphere of radius a to which is glued a solid circular cylinder of radius a and height $2a$ so that the plane end of the hemisphere is in complete contact with a plane end of the cylinder. The cylinder is made of uniform material of density ρ, and the hemisphere is made of a uniform material with density $k\rho$. The toy is designed so that if placed on a horizontal table with the hemisphere downwards and then tilted on one side, it will return to the vertical position. Show that $k > 8$. The toy is placed on a desk of slope α where $\sin\alpha = \frac{1}{8}$, sufficiently rough to prevent slipping. It rests in equilibrium with the hemisphere in contact with the desk. Find an expression giving the (acute) angle β made by its axis of symmetry with the vertical. Hence deduce that $k \geq 13.5$

[Ans: $\sin\beta = \frac{3+k}{3(k-8)}$]

3. A thin uniform rectangular metal plate $ABCD$ of mass M rests on a rough plane inclined at an angle α to the horizontal. The plate lies in a vertical plane containing a line of greatest slope of the inclined plane, with the edge CD in contact with the plane and C further up the plane than D, as shown.

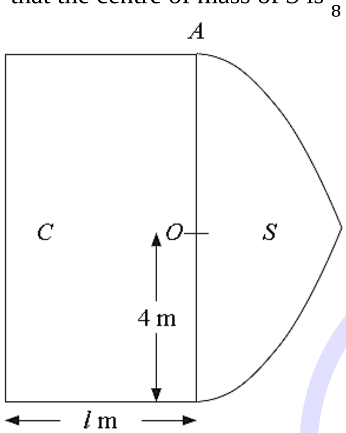


5. The region R is bounded by part of the curve with equation $y = 4 - x^2$, the positive x-axis and the positive y-axis, as shown below.



The unit of length on both axes is one metre. A uniform solid S is formed by rotating R through 360° about the x-axis.

(a) Show that the centre of mass of S is $\frac{5}{8}$ m from O.

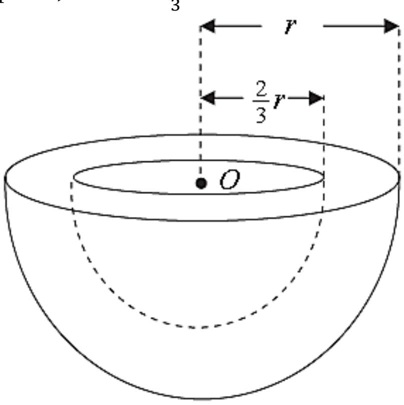


The figure above shows a cross section of a uniform solid P consisting of two components, a solid cylinder C and the solid S. The cylinder C has radius 4 m and length l metres. One end of C coincides with the plane circular face of S. The point A is on the circumference of the circular face common to C and S. When the solid P is freely suspended from A, the solid P hangs with its axis of symmetry horizontal.

(b) Find the value of l.

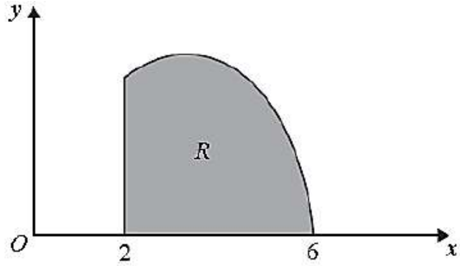
[Ans: (b) $l = \frac{2\sqrt{3}}{3}$]

6. A bowl, B consists of a uniform solid hemisphere, of radius r and centre O, from which is removed a solid hemisphere, of radius $\frac{2}{3}r$ and centre O as shown below.



Show that the centre of mass of B from O is $\frac{65}{152}r$

7. The shaded region R is bounded by the curve with equation $y = \frac{1}{2}x(6 - x)$, the x-axis and the line $x = 2$, as shown below. The unit of length on both axes is 1 cm. A uniform solid P is formed by rotating R through 360° about the x-axis.



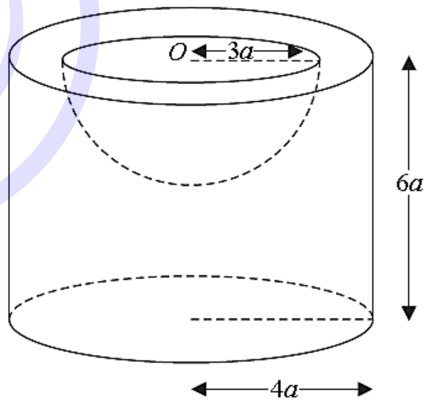
(a) Show that the centre of mass of P is, to 3 significant figures, 1.42 cm from its plane face.

The uniform solid P is placed with its plane face on an inclined plane which makes an angle θ with the horizontal. Given that the plane is sufficiently rough to prevent P from sliding and that P is on the point of toppling,

(b) find the angle θ.

[Ans: (b) 70.5°]

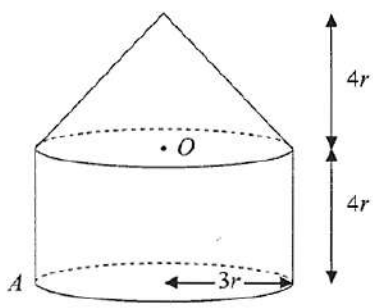
8. A uniform right circular solid cylinder has radius $4a$ and height $6a$. A solid hemisphere of radius $3a$ is removed from the cylinder forming a solid S. The upper plane face of the cylinder coincides with the plane face of the hemisphere. The centre of the upper plane face of the cylinder is O and this is also the centre of the plane face of the hemisphere, as shown below.



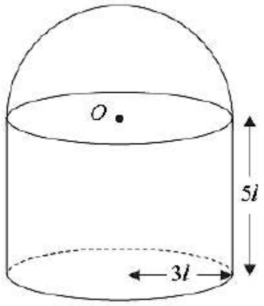
Find the distance from O to the centre of mass of S.

[Ans: $\frac{357}{104}a$]

9. A toy is formed by joining a uniform solid right circular cone of base radius $3r$ and height $4r$, to a uniform solid cylinder, also of radius $3r$ and height $4r$. The cone and the cylinder are made from the same material as shown below. The centre of this plane face is O



hemisphere coincides with a circular end of the cylinder and has centre O, as shown below



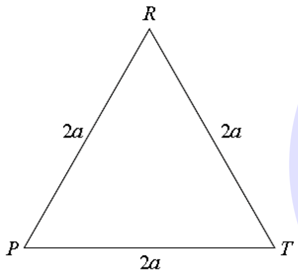
The density of the hemisphere is **twice** the density of the cylinder.

(a) Find the distance of the centre of mass of the solid from O.

(b) The solid is now placed with its circular face on a plane inclined at an angle $\theta°$ to the horizontal. If the plane is sufficiently rough to prevent the solid from slipping and the solid is on the point of toppling, find the value of θ

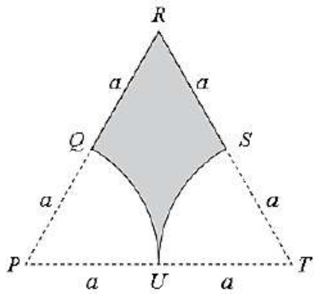
[Ans: (a) $\frac{8}{9}l$ (b) $36.1°$]

14. The figure below shows a uniform equilateral triangular lamina PRT with sides of length $2a$.



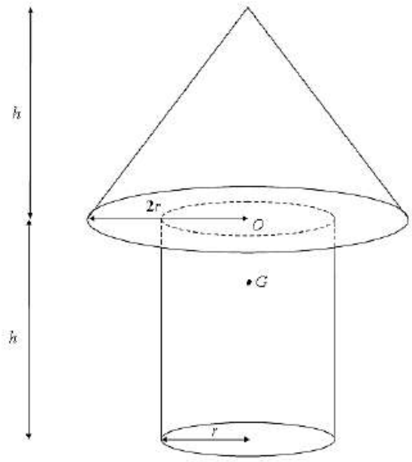
(a) Using calculus, prove that the centre of mass of PRT is at a distance $\frac{2\sqrt{3}}{3}a$ from R.

(b) The circular sector PQU, of radius a and centre P, and the circular sector TUS, of radius a and centre T, are removed from PRT to form the uniform lamina $QRSU$ shown in below



Show that the distance of the centre of mass from U of $QRSU$ is $\frac{2a}{3\sqrt{3}-\pi}$

15. A model tree is made by joining a uniform solid cylinder to a uniform solid cone made of the same material. The centre O of the base of the cone is also the centre of one end of the cylinder, as shown below. The radius of the cylinder is r and the radius of the base of the cone is $2r$. The height of the cone and the height of the cylinder are each h. The centre of mass of the model is at the point G.

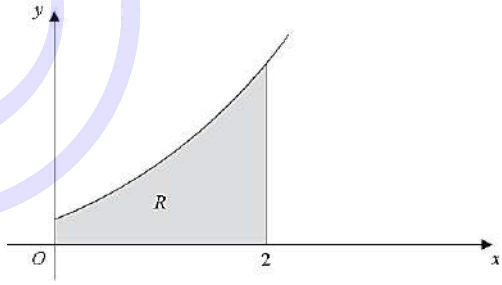


(a) Show that OG = $\frac{1}{14}h$

(b) The model stands on a desk top with its plane face in contact with the desk top. The desk top is tilted until it makes an angle θ with the horizontal, where $\tan\theta = \frac{7}{26}$. The desk top is rough enough to prevent slipping and the model is about to topple. Find r in terms of h.

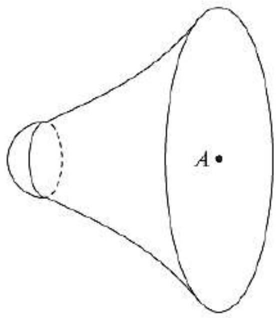
[Ans: (b) $r = \frac{1}{4}h$]

16. The shaded region R is bounded by the curve with equation $y = (x+1)^2$, the x-axis, the y-axis and the line with equation $x = 2$, as shown in the figure below.



The region R is rotated through 2π radians about the x-axis to form a uniform solid S.

(a) Use algebraic integration to find the x-cordinate of the centre of mass of S



A uniform solid hemisphere is fixed to S to form a solid T. The hemisphere has the same radius as the smaller plane face of S and its plane face coincides with the smaller plane face of S, as shown above. The mass per unit volume of the hemisphere is 10 times the mass per unit volume of S. The centre of the

EXAMINATION QUESTIONS

SECTION A

1. A uniform rod AB of length 3 m and mass 8 kg is freely hinged to a vertical wall at A. A string BC of length 4 m attached to a point C on the wall, keeps the rod in equilibrium. If C is 5 m vertically above A, find the;
   (a) tension in the string
   (b) magnitude of the normal reaction at A
   [2019, No.2: Ans: (a) $31.36\ N$ (b) $56.535\ N$]
2. A stone is thrown vertically upwards with velocity $16\ \text{ms}^{-1}$ from a point H metres above the ground level. The stone hits the ground 4 seconds later. Calculate the;
   (a) value of H
   (b) velocity of the stone as it hits the ground
   [2019, No. 5: Ans: (a) 14.4 m (b) $23.2\ \text{ms}^{-1}$]
3. A particle is moving with Simple Harmonic motion (SHM). When the particle is 15 m from the equilibrium, its speed is $6\ \text{ms}^{-1}$. When the particle is 13 m from the equilibrium, its speed is $9\ \text{ms}^{-1}$. Find the amplitude of the motion.
   [2019, No. 8: Ans: 16.426 m]
4. A stone is thrown vertically upwards with a velocity of $21\ \text{ms}^{-1}$. Calculate the:
   (a) maximum height attained by the stone
   (b) time the stone takes to reach the maximum height.
   [2018, No. 1: Ans: (a) 22.5 m (b) 2.143 s]
5. A particle of mass 15 kg is pulled up a smooth slope by a light inextensible string parallel to the slope. The slope is 10.5 m long and inclined at $\sin^{-1}\left(\frac{4}{7}\right)$ to the horizontal. The acceleration of the particle is $0.98\ \text{ms}^{-2}$. Determine the:
   (a) tension in the string
   (b) work done against gravity when the particle reaches the end of the slope.
   [2018, No. 4: Ans: (a) $98.7\ N$ (b) $882\ J$]
6. In an equilateral triangle PQR, three forces of magnitude $5N$, $10N$ and $8N$ act along the sides PQ, QR and PR respectively. Their directions are in the order of the letters. Find the magnitude of the resultant force.
   [2018, No. 7: Ans: $16.09\ N$]
7. A particle is projected from a point O with speed 20 m/s at an angle of $60°$ to the horizontal. Express in vector form its velocity $\boldsymbol{v}$ and its displacement $\boldsymbol{r}$, from O at any time t seconds.
   [2017, No. 1: Ans: $v = \begin{pmatrix} 10 \\ 10\sqrt{3} - 9.8t \end{pmatrix}$; $r = \begin{pmatrix} 10t \\ 10\sqrt{3}t - 4.9t^2 \end{pmatrix}$]
8. In a square $ABCD$, three forces of magnitudes $4N$, $10N$ and $7N$ act along AB, AD and CA respectively. Their directions are in the order of the letters. Find the magnitude of the resultant force.
   [2017, No.4: Ans: $5.14\ N$]
9. The engine of a lorry of mass $5{,}000$ kg is working at a steady rate of 350 kW against a constant resistance force of $1{,}000\ N$. The lorry ascends a slope of inclination $\theta°$ to the horizontal. If the maximum speed of the lorry is $20\ \text{ms}^{-1}$, find the value of θ.
   [2017, No.8: Ans: 19.68]
10. A ball is projected vertically upwards and it returns to its point of projection 3 seconds later. Find the:
    (a) speed with which the ball was projected
    (b) greatest height reached
    [2016, No. 1: Ans: (a) $14.7\ \text{ms}^{-1}$ (b) 11.025 m]
11. A body of mass 4 kg is moving with an initial velocity of $5\ \text{ms}^{-1}$ on a plane. The kinetic energy of the body is reduced by 16 Joules in a distance of 40 metres. Find the deceleration of the body.
    [2016, No. 4: Ans: $-0.1\ \text{ms}^{-2}$]
12. A particle of mass 2 kg rests in limiting equilibrium on a rough plane inclined at $30°$ to the horizontal. Find the value of the coefficient of friction.
    [2016, No. 7: Ans: 0.5774]
13. Find the magnitude and direction of the resultant of the forces
    $\begin{pmatrix} -3 \\ 1 \end{pmatrix} N$, $\begin{pmatrix} 4 \\ 2 \end{pmatrix} N$, and $\begin{pmatrix} 1 \\ 2 \end{pmatrix} N$
    [2015, No. 1: Ans: $5.83\ N$ at $30.96°$ to the horizontal]
14. Two cyclists A and B are 36 m apart on a straight road. Cyclist B starts from rest with an acceleration of $6\ \text{ms}^{-2}$ while A in pursuit of B with velocity of $20\ \text{ms}^{-1}$ and acceleration of $4\ \text{ms}^{-2}$. Find the times when A overtakes B.
    [2015, No. 4: Ans: $2\ s$]
15. A fixed hollow hemisphere has centre O and is fixed so that the plane of the rim is horizontal. A particle A of weight $30\sqrt{2}\ N$ can move on the inside surface of the hemisphere. The particle is acted upon by a horizontal force P, whose line of action is in a vertical plane through O and A. OA makes an angle of $45°$ with the vertical. If the coefficient of friction between the particle and the hemisphere is 0.5 and the particle is just about to slip downwards, find the
    (a) normal reaction
    (b) value of P
    [2015, No. 8: Ans: (a) $40N$ (b) $14.14\ N$]
16. A particle starts from rest at the origin $(0, 0)$. Its acceleration in ms^{-2} at time t seconds is given by $a = 6ti - 4j$. Find its speed when $t = 2$ seconds.
    [2014, No. 2: Ans:$14.42\ \text{ms}^{-1}$]
17. Forces of $\begin{pmatrix} 1 \\ 4 \end{pmatrix} N$, $\begin{pmatrix} 3 \\ 6 \end{pmatrix} N$, $\begin{pmatrix} -9 \\ 1 \end{pmatrix} N$ and $\begin{pmatrix} 5 \\ -3 \end{pmatrix} N$ act at the points having position vectors $(3i - j)$ m, $(2i + 2j)$ m, $(-i - j)$ m and $(-3i + 4j)$ m respectively. Show that the forces reduce to a couple.
    [2014, No. 5]
18. A bullet of mass 50 grammes travelling horizontally at $80\ \text{ms}^{-1}$ hits a block of wood of mass 10 kg resting on

and moves uniformly for 15 minutes. It is then brought to rest at a constant retardation of $\frac{5}{3}$ ms^{-2} at station B.

[2003, No. 8: Ans: 23212.8 m]

52. A driver of a car travelling at $72\ \text{kmh}^{-1}$ notices a tree which has fallen across the road, 800 m ahead, and suddenly reduces the speed to$36\ \text{kmh}^{-1}$ by applying the brakes. For how long did the driver apply the brake?

[2002, No. 3: Ans: 53.33 s]

53. The resistance to the motion of a lorry of mass, m kg is $\frac{1}{200}$ of its weight. When travelling at $108\ \text{kmh}^{-1}$ on a level road and ascends a hill its engine fails to work. Find how far up the hill (in km) the lorry moves before it comes to rest. Give your answer correct to **one** decimal place.

[2002, No. 6: Ans: $\frac{90000}{g(1+200\sin\alpha)}$]

54. A particle moves in the $x-y$ plane such that its position vector at any time, t is given by

$$\text{r} = (3t^2-1)\text{i} + (4t^3+t-1)\text{j}$$

Find:

(a) its speed,

(b) the magnitude of acceleration,

after $t = 2$

[2002, No. 8: Ans: (a) 50.448 units(b) 48.374 units]

55. A particle of mass 5 kg is placed on a smooth plane inclined at $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ to the horizontal. Find the magnitude of the force acting horizontally, required to keep the particle in equilibrium and the normal reaction to the plane.

[2001, No. 5: Ans: 56.58 N]

56. A, B and C are points on a straight road such that $\overline{AB} = \overline{BC} = 20$ m. A cyclist moving with uniform acceleration passes A and then notices that it takes him 10 s and 15 s to travel between A and B, and A and C respectively. Find:

(i) his acceleration,

(ii) the velocity with which he passes A

[2001, No. 7: Ans: (i) 0.27 ms^{-2} (ii) 0.65 ms^{-1}]

57. An inextensible string attached to two scale pans A and B, each of weight 20 gm, passes over a smooth fixed pulley. Particles of weight 3.8 N and 5.8 N are placed on pans A and B respectively. Find the reaction of the scale pan holding the 3.8 N weight, if the system is released from rest. [Take g = 10 ms^{-2}].

[2001, No. 8: Ans: 4.56 N]

58. A boat travelling at 5 ms^{-1} in the direction 030° in still water is blown by wind moving at $8\ \text{ms}^{-1}$ from the bearing of 150°. Calculate the true speed and course the boat will be steered.

[2000, No. 4: Ans: 7 ms^{-1} on a bearing of 111.8°]

59. A force acting on a particle of mass 15 kg moves it along a straight line with a velocity of $10\ \text{ms}^{-1}$. The rate at which work is done by the force is 50 watts. If the particle starts from rest, determine the time it takes to move a distance of 100 m.

[2000, No. 7: Ans: $10\sqrt{6}$ s]

60. A particle executing simple harmonic motion about a point O, has speeds of $3\sqrt{3}\ \text{ms}^{-1}$ and $3\ \text{ms}^{-1}$ when at distances of 1 m and 0.268 m respectively from the end point. Find the amplitude of the motion.

[2000, No. 8: Ans: 2 m]

SECTION B

1. Car A is 80 m North West of point O. Car B is 50 m $N30°E$ of O. Car A is moving at $20\ \text{ms}^{-1}$ on a straight road towards O. Car B is also moving at $10\ \text{ms}^{-1}$ on another straight road towards O. Determine the;

(a) initial distance between the two cars.

(b) velocity of A relative to B

(c) shortest distance between the two cars as they approach O.

[2019, No. 9: Ans: (a) 82.642 m (b) 19.91 ms^{-1}, $E15.98°S$ (c) 9.7 m]

2. A force $\boldsymbol{F} = (2t\,\boldsymbol{i} + \boldsymbol{j} - 3t\,\boldsymbol{k})\,N$ acts on a particle of mass 2 kg. The particle is initially at a point $(0, 0, 0)$ and moving with a velocity $(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$ ms^{-1}. Determine the;

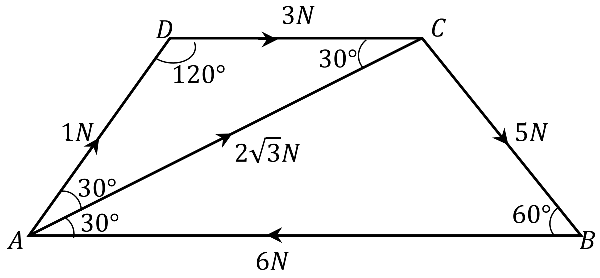
(a) magnitude of the acceleration of the particle after 2 s.

(b) velocity of the particle after 2 s

(c) displacement of the particle after 2 s

[2019, No. 12: Ans: (a) 2.55 m^{-2} (b) $3i + 3j - 4k$ ms^{-1} (c) $\frac{10}{3}i + 5j - 4k$]

3. The diagram below shows a trapezium $ABCD$, $AD = DC = CB = 1$ metre and $AB = 2$ metres. Forces of magnitude $1N$, $3N$, $5N$, $6N$ and $2\sqrt{3}N$ act in the directions AD, DC, CB, BA and AC respectively.



(a) Calculate the magnitude of the resultant force and the angle it makes with side AB.

(b) Given that the line of action of the resultant force meets AB at X, find the distance AX.

[2019, No. 15: Ans: (a) $2\sqrt{3}N$ at 30° below AB (b) 6.5 m]

4. A particle of mass 4 kg starts from rest at a point $(2i - 3j + k)$ m. It moves with acceleration $a = (4i + 2j - 3k)$ ms^{-2} when a constant force $\boldsymbol{F}$ acts on it. Find the

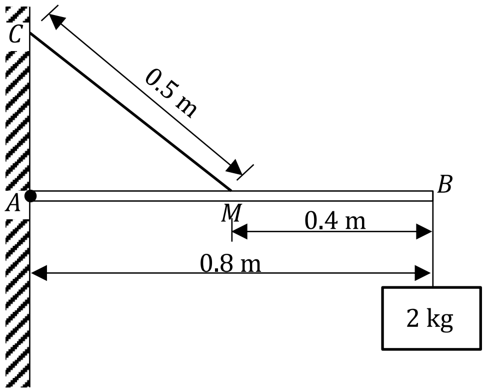
(a) force $\boldsymbol{F}$

(b) velocity at any time t

(c) work done by the force $\boldsymbol{F}$ after 6 second

[2018, No. 10: Ans: (a) $\begin{pmatrix} 16 \\ 8 \\ -12 \end{pmatrix} N$ (b) $\begin{pmatrix} 4t \\ 2t \\ -3t \end{pmatrix} \text{ms}^{-1}$ (c) $2084\ J$]

5. The figure below shows a uniform beam of length 0.8 metres and mass 1 kg. The beam is hinged at A and has a load of mass 2 kg attached at B.



The beam is held in a horizontal position by a light inextensible string of length 0.5 metres. The string joins the mid-point M of the beam to a point C vertically above A. Find the:

(a) tension in the string

(b) magnitude and direction of the force exerted by the hinge.

[2018, No. 13: Ans: (a) 81.67 N (b) 68.21 N]

6. At 10: 00 a.m, ship A and ship B are 16 km apart. Ship A is on a bearing $N35°E$ from ship B. Ship A is travelling at $14\ \text{kmh}^{-1}$ on a bearing $S29°E$. Ship B is travelling at $17\ \text{kmh}^{-1}$ on a bearing $N50°E$. Determine the;

(a) velocity of ship B relative to ship A

(b) closest distance between two ships and the time when it occurs.

[2018, No. 16: Ans: (a) $24\ \text{kmh}^{-1}$, $N15°E$ (b) 5.46 km; 10: 38 a.m]

7. A particle of mass 3 kg is acted upon by a force $F = 6i - 36t^2j + 54tk$ Newtons at a time t. At time $t = 0$, the particle is at the point with a position vector $i - 5j - k$ and its velocity is $3i + 3j$ m/s. Determine the

(a) position vector of the particle at time $t = 1$ second

(b) distance of the particle from the origin at time $t = 1$ second.

[2017, No. 10: Ans: (a) $5i - 3j + 2k$ (b) 6.164]

8. A non-uniform rod AB of mass 10 kg has its centre of gravity at a distance $\frac{1}{4}AB$ from B. The rod is smoothly hinged at A. It is maintained in equilibrium at $60°$ above the horizontal by a light inextensible string tied at B and a right angle to AB. Calculate the magnitude and direction of the reaction at A.

[2017, No. 13: Ans: 85.75 N in the direction $E68.2°N$]

9. At 12 noon, a ship A is moving with constant velocity of $20.4\ \text{kmh}^{-1}$ in the direction $N\theta°E$, where $\tan\theta = \frac{1}{5}$. A second ship B is 15 km due north of A. Ship B is moving with constant velocity of $5\ \text{kmh}^{-1}$ in the direction $S\alpha°W$, where $\tan\alpha = \frac{3}{4}$. If the shortest distance between the ships is 4.2 km, find the time to the nearest minute when the distance between the ships is shortest.

[2017, No. 16: Ans: 12: 35 p.m]

10. Five forces of magnitudes $3N$, $4N$, $4N$, $3N$ and $5N$ act along the lines AB, BC, CD, DA and AC respectively, of a square $ABCD$ of side 1 m. The direction of the forces is given by the order of the letters. Taking AB and AD as reference axes; find the

(a) magnitude and direction of the resultant force

(b) point where the line of action of the resultant force cuts the side AB

[2016, No. 10: Ans: (a) 5.196 N at 60.8° with AB (b) 1.764 m from A]

11. A particle starts from rest at a point $(2, 0, 0)$ and moves such that its acceleration at any time $t > 0$ is given by

$\boldsymbol{a} = [16\cos 4ti + 8\sin 2tj + (\sin t - 2\sin 2t)k]\ \text{ms}^{-2}$

Find the:

(a) speed when $t = \frac{\pi}{4}$

(b) distance from the origin when $t = \frac{\pi}{4}$

[2016, No. 13: Ans: (a) $4.062\ \text{ms}^{-1}$ (b) 4.248 m]

12. A particle of mass 2 kg moving with Simple Harmonic Motion (SHM) along the x-axis, is attracted towards the origin O by a force of $32x$ Newtons. Initially the particle is at rest at $x = 20$. Find the

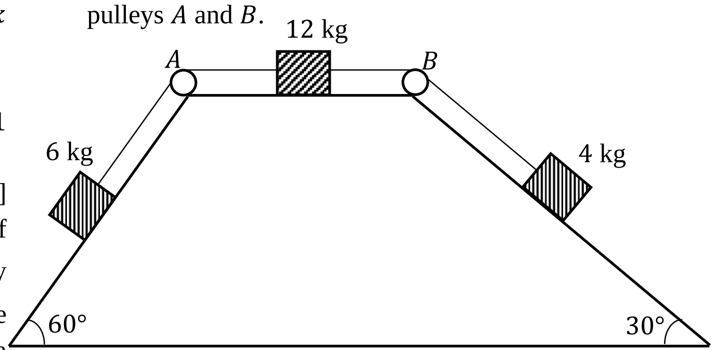
(a) amplitude and period of oscillation

(b) velocity of the particle at any time, $t > 0$

(c) speed when $t = \frac{\pi}{4}\ s$

[2016, No. 16: Ans: (a) 20 m; $\frac{\pi}{2}\ s$ (b) $v = -80\sin 4t$ (c) 0]

13. The diagram below shows a 12 kg mass on a horizontal rough plane. The 6 kg and 4 kg masses are on rough planes inclined at angles of 60° and 30° respectively. The masses are connected to each other by light inextensible strings passing over light smooth fixed pulleys A and B.



The planes are equally rough with coefficient of friction $\frac{1}{12}$. If the system is released from rest, find the;

(a) acceleration of the system

(b) tensions in the strings

[2015, No. 11: Ans: (a) $0.738\ \text{ms}^{-2}$ (b) 44.04 N, 25.4 N]

14. A ball is projected from a point A and falls at a point B which is in level with A and at a distance of 160 m from

power it develops is approximately 2.25 kW (to 3 significant figures).

(b) A car of mass 1000 kg, has a maximum speed of 150 kmh^{-1} on a level rough road and the engine is working at 60 kW.

(i) Calculate the coefficient of friction between the car and the road if all the resistance is due to friction

(ii) Given that the tractive force remains unaltered and the non-gravitational resistance in both cases varies as the square of the speed, find the greatest slope on which a speed of 120 kmh^{-1} could be maintained.

[2005, No. 13: Ans; (b) (i) 0.1469 (ii) 3.03°]

44. A particle P moving with a constant velocity $2i + 3j + 8k$, passes through a point with position vector $6i - 11j + 4k$. At the same instant, a particle Q passes through a point with position vector $i - 2j + 5k$, moving with constant velocity $3i + 4j - 7k$. Find the

(a) position and velocity of Q relative to P at that instant.

(b) shortest distance between P and Q in the subsequent motion.

(c) time that elapses before the particles are nearest to one another.

[2005, No. 14: Ans: (a) $\begin{pmatrix}1\\1\\-15\end{pmatrix}$; $\begin{pmatrix}-5\\9\\1\end{pmatrix} + \begin{pmatrix}1\\1\\-15\end{pmatrix}t$ (b) 10.32 units (c) $\frac{11}{227}$ s]

45. A uniform ladder of length $2l$ and weight W rests in a vertical plane with one end against a rough vertical wall and the other against a rough horizontal surface, the angles of friction at each end being $\tan^{-1}\frac{1}{3}$ and $\tan^{-1}\frac{1}{2}$ respectively.

(a) If the ladder is in limiting equilibrium at either end, find θ, the angle of inclination of the ladder to the horizontal.

(b) A man of weight 10 times that of the ladder begins to ascend it, how far will he climb before the ladder slips?

[2005, No. 16: Ans: (a) 39.8° (b) $\frac{1}{2}$ the ladder]

46. A car of mass M kg has an engine which works at a constant rate of $2H$ kW. The car has a constant speed of V ms^{-1} along a horizontal rod.

(a) Find in terms of M, H, V, g and θ, the acceleration of the car when travelling:

(i) up a road of inclination θ with a speed of $\frac{3}{4}V$ ms^{-1}

(ii) down the same road with a speed of $\frac{3}{5}V$ ms^{-1}, the resistance to the motion of the car apart from gravitational force, being constant.

(b) If the acceleration in (a)(ii) above is 3 times that of (a)(i) above, find the angle of inclination θ, of the road.

(c) If the car continues directly up the road, in case (a)(i) above, show that its maximum speed is $\frac{12}{13}V$ ms^{-1}.

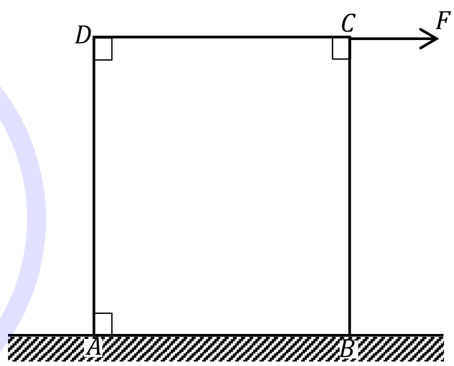
[2004, No. 13: Ans: (a) (i) $\frac{2000H - 3mvg\sin\theta}{3mv}$ (ii) $\frac{4000H + 3mvg\sin\theta}{3mv}$ (b) $\sin^{-1}\left(\frac{2000H}{12mvg}\right)$]

47. (a) A non-uniform ladder AB, 10 m long and mass 8 kg, lies in limiting equilibrium with its lower end A resting on a rough horizontal ground and the upper end B resting against a smooth vertical wall. If the centre of gravity of the ladder is 3 m from the foot of the ladder and the ladder makes an angle of 30° with the horizontal, find the:

(i) coefficient of friction between the ladder and the ground.

(ii) reaction at the wall

(b) The diagram below shows a cross-section $ABCD$ of a uniform rectangular block of base, 8 cm and height, 10 cm resting on a rough horizontal table.



An increasing force, F, parallel to the table is applied on the upper edge. If the coefficient of friction between the block and the table is 0.7, show that the block will tilt before sliding.

[2004, No. 15: Ans: (a) (i) $\frac{3\sqrt{3}}{10}$ (ii) $\frac{12g\sqrt{3}}{5}$ N (b) show that $\mu R \geq F$]

48. Two planes A and B are both flying above the Pacific Ocean. Plane A is flying on a course of 010° at a speed of 300 kmh^{-1}. Plane B is flying on a course of 340° at 200 kmh^{-1}. At a certain time, plane B is 40 km from plane A. Plane A is then on bearing 060°. After what time will they come closest together, and what will be their minimum distance apart?

(Give your answer correct to **1** decimal place)

[2004, No. 16: Ans: 14.5 minutes; 8.1 km]

49. A particle of weight 24 N is suspended by a light inextensible string from a light ring. The ring can slide along a rough horizontal rod. The coefficient of friction between the rod and the ring is $\frac{1}{3}$. A force of P newtons acting upwards on the particle at 45° to the horizontal,

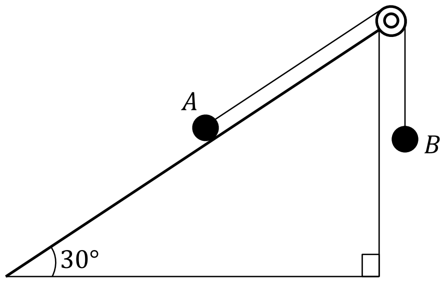
keeps the system in equilibrium with the ring at a point of sliding. Find the

(i) value of P

(ii) tension in the string

[2003, No. 14: Ans: (i) $6\sqrt{2}\ N$ (ii) $6\sqrt{10}\ N$]

50. Blocks A and B of masses 2 and 3 kg respectively are connected by a light inextensible string passing over a smooth pulley as shown below.



Block A is resting on a rough plane is inclined at 30° to the horizontal while block B hangs freely. When the system is released from rest, block B travels a distance of 0.75 m before it attains a speed of 2.25 ms^{-1}. Calculate the

(i) acceleration of the blocks,

(ii) coefficient of friction between the plane and block A

(iii) reaction of the pulley on the string

[2003, No. 11: Ans: (i) 3.375 ms^{-2} (ii) 0.16 (iii) 33.4 N]

51. Two particles P and Q initially at positions $3\text{i} + 2\text{j}$ and $13\text{i} + 2\text{j}$ respectively begin moving. Particle P moves with a constant velocity of $2\text{i} + 6\text{j}$ while particle Q moves with a constant velocity of 5j, the units being in metres and metres per second respectively.

(a) Find the

(i) time the two particles are nearest to each other

(ii) bearing of particle P from Q when they are nearest to each other.

(b) Given that after half the time the two particles are moving closest to each other, particle P reduces its speed to half its original speed, in the direction to approach particle Q, and the velocity of Q remains unchanged, find the direction of particle P.

[2003, No. 16: Ans: (a) (i) 4 s (ii) 333.43° (b) $N50.8°E$]

52. (a) A particle of mass 3 kg is attached to the lower end B of an inextensible string. The upper end A of the string is fixed to a point on the ceiling of a roof. A horizontal force of 22 N and an upward vertical force of 4.9 N act upon the particle making it to be in equilibrium, with the string making an angle α to the vertical. Find the value of α and the tension in the string.

(b) A non-uniform rod of mass 9 kg rests horizontally in equilibrium supported by two light inextensible strings tied to the rods of the rod. The strings make angles of 50° and 60° with the rod. Calculate the tensions in the strings

[2002, No. 12: Ans: (a) 45°, 33 N (b) 46.92 N; 60.34 N]

53. (a) The velocities of two ships P and Q are $\text{i} + 6\text{j}$ and $-\text{i} + 3\text{j}\ \text{kmh}^{-1}$ respectively. At a certain instant the displacement between the two ships is $7\text{i} + 4\text{j}$ km. Find the:

(i) relative velocity of ship P to Q

(ii) magnitude of displacement between ships P and Q after 2hours

(b) The position vectors of two particles are: $\text{r}_1 = (4\text{i} - 2\text{j})t + (3\text{i} + \text{j})t^2$ and $\text{r}_2 = 10\text{i} + 4\text{j} + (5\text{i} - 2\text{j})t$ respectively. Show that the two particles will collide. Find their speeds at the time of collision.

[2002, No. 15: Ans: (a) (i) $2\text{i} + 3\text{j}\ \text{kmh}^{-1}$ (ii) 14.87 km (b) 16.245 kmh^{-1}; 5.385 kmh^{-1}]

54. A particle is projected from level ground towards a vertical pole, 4 m high and 30 m away from the point of projection. It just passes the pole in one second. Find:

(a) its initial speed and angle of projection,

(b) the distance beyond the pole where the particle will fall.

[2002, No. 16: Ans: (a) 31.29 ms^{-1}; 16.5° (b) 24.42 m]

55. An object of mass 5 kg is initially at rest at a point whose position vector is $-2\text{i} + \text{j}$. If it is acted upon by a force, $\text{F} = 2\text{i} + 3\text{j} - 4\text{k}$, find

(i) the acceleration,

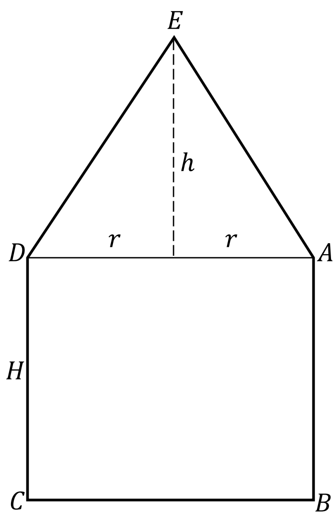
(ii) its velocity after 3 s,

(iii) its distance from the origin after 3 s.

[2001, No. 13: Ans: (i) $\frac{1}{5}(2\text{i} + 3\text{j} - 4\text{k})$ (ii) $\frac{1}{5}(6\text{i} + 9\text{j} - 12\text{k})$ (iii) 5.166 m]

56. (a) Prove that the centre of mass of a solid cone is ¼ of the vertical height from the base.

(b) The figure $ABCDE$ below shows a solid cone of radius r, height h, joined to a solid cylinder of the same material with the same radius and height H.



If the centre of mass of the whole solid lies in the plane of the base of the cone where the two solids are joined, find H. If instead $H = h$ and $r = ½h$, find the angle AB makes with the horizontal, if the body is hang from A.

[2001, No. 16: Ans: (b) 32.01°]

57. (a) A mass oscillates with S.H.M of period one second. The amplitude of the oscillation is 5 cm. Given that the particle begins from the centre of the motion, state the relationship between the displacement x of the mass at any time, t.
Hence find the first times when the mass is 3 cm from its end position.
(b) A particle of mass M is attached by means of light strings AP and BP of the same natural length a m and moduli of elasticity mg and $2mg$ N respectively, to the points A and B on a smooth horizontal table. The particle is released from the mid-point of $\overline{AB}$, where $\overline{AB} = 3a$ m. Show that the motion of the particle is S.H.M with period

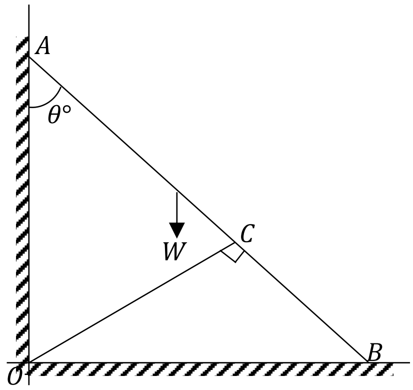
$$T = \left(\frac{4\pi^2 a}{3g}\right)^{\frac{1}{2}}$$

[2001, No. 14: Ans: (a) $x = 0.05\sin 2\pi t$; 0.066, 0.434]

58. A particle moving with an acceleration given by $a = 4e^{-3t}i + 12\sin t\, j - 7\cos t\, k$ is located at the point (5, -6, 2) and has a velocity, $v = 11i - 8j + 3k$ at time $t = 0$. Find the
(i) magnitude of the acceleration when $t = 0$,
(ii) velocity at any time t,
(iii) displacement at any time, t.

[2000, No. 11: Ans: (i) 8.06 (ii) $\frac{1}{3}\left((37 - 4e^{-3t})i + (4 - 112\cos t)j + (3 - 7\sin t)k\right)$ (iii) $r = \left(\frac{41}{9} + \frac{37}{3}t + \frac{4e^{-3t}}{9}\right)i + (-6 + 4t - 12\sin t)j + (-5 + 3t + 7\cos t)k$]

59. The diagram below shows a uniform rod AB of weight W and length l resting at an angle θ against a smooth vertical wall at A. The other end B rests on a smooth horizontal table. The rod is prevented from slipping by an inelastic string OC, C being a point on $\overline{AB}$ such that $\overline{OC}$ is perpendicular to $\overline{AB}$ and O on the point of intersection of the wall and the table. Angle AOB is 90°.



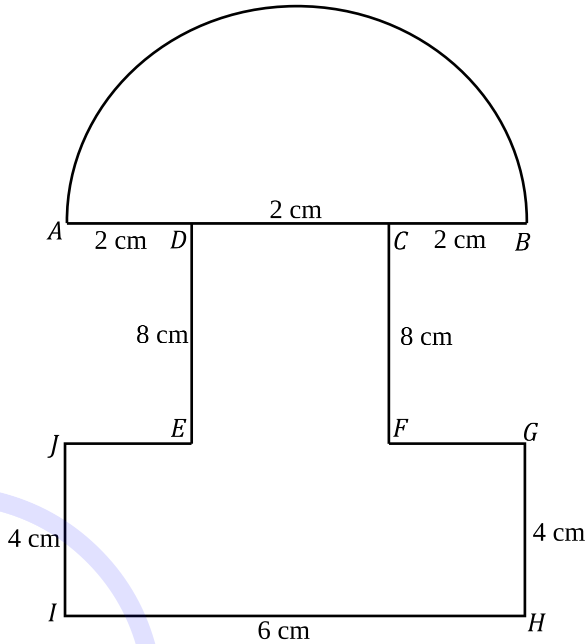
Find the
(i) tension in the string,
(ii) reactions at A and B in terms of θ and W

[2000, No. 13: Ans: (i) $\frac{W}{2}\tan^2\theta$ (ii) $\frac{W\tan^2\theta}{4}$]

60. Six forces, 9 N, 5 N, 7 N, 3 N, 1 N and 4 N act along the sides PQ, QR, RS, ST, TU and UP of a regular hexagon of side 2 m, their directions being indicated by the order of the letters. Taking PQ as the reference axis, express each of the forces in vector form. Hence find the
(i) magnitude and direction of the resultant of the forces
(ii) distance from P, where the line of action of the resultant cuts PQ

[2000, No. 16: Ans: (i) 8.9 N at 43° with PQ (ii) 7.43 m]

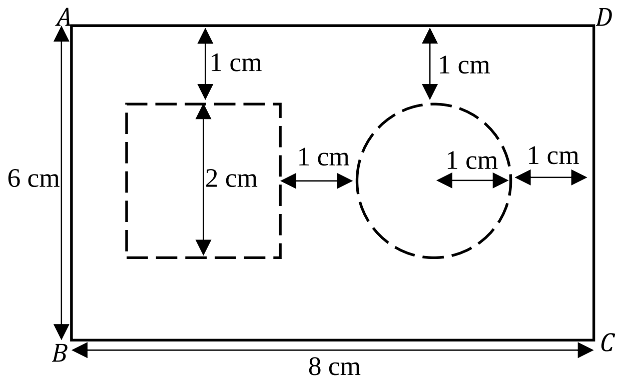
61.



The figure $ABCDEFGHIJ$ shows a symmetrical composite lamina made up of a semi-circle, radius 3 cm, a rectangle $CDEF$ 2 cm × 8 cm and another rectangle $GHIJ$ 6 cm × 4 cm.
Find the distance of the centre of gravity of this lamina from IH.
If the lamina is suspended from H, by means of a peg through a hole, calculate the angle of inclination of HG to the vertical.

[1998, March: No. 16: Ans: 6.716 cm ; 24.07°]

62.



$ABCD$ is uniform rectangular sheet of cardboard of length 8 cm and width 6 cm. A square and a circular hole are cut off from the cardboard as shown above. Calculate the position of the centre of gravity of the remaining sheet. [1999, No. 16: Ans: 3.944 cm from AB 2.825 cm from BC]

Statistics

INTRODUCTION

Statistics is the science of investigating, collecting, organizing, analyzing, interpreting, and presenting data to assist in making more effective decisions. Statistics allow a researcher to make reliable inferences from data observed in any research setting, and make conclusions regarding population(s). In general, statistics as a field consists of two subdivisions: descriptive statistics and inferential statistics.
If mathematics is the handmaiden of science, statistics is its whore: all that scientists are looking for is a quick fix without the encumbrance of a meaningful relationship. Statisticians are second-class mathematicians, third-rate scientists and fourth rate thinkers. They are the hyenas, jackals and vultures of the scientific ecology: picking over the bones and carcasses of the game that the big cats, the biologists, the physicists and the chemists, have brought down.
Statistics tells us how to evaluate evidence, how to design experiments, how to turn data into decisions, how much credence should be given to whom to what and why, how to reckon chances and when to take them. Statistics deals with the very essence of the universe: chance and contingency are its discourse and statisticians know the vocabulary. If you think that statistics has nothing to say about what you do or how you could do it better, then you are either wrong or in need of a more interesting job.

Probability
The world around us is full of phenomena that we perceive as random or unpredictable. Probability is concerned with events which occur by chance. Examples include occurrence of accidents, errors of measurements, production of defective and non-defective items from a production line, and various games of chance, such as drawing a card from a well-mixed deck, flipping a coin, or throwing a symmetrical six-sided die. In each case we may have some knowledge of the likelihood of various possible results, but we cannot predict with any certainty the outcome of any particular trial. Probability is a mathematical area which has developed over the past three or four centuries. One of the early uses was to calculate the odds of various gambling games. Its usefulness for describing errors of scientific and engineering measurements was soon realised. Probability has so many practical uses, ranging from quality control and quality assurance to communication theory in electrical engineering.

Importance of statistics
In recent days, we hear talking about statistics from a common person to highly qualified person. It only shows how statistics has been intimately connected with wide range of activities in daily life. Students in the biological, physical and social sciences often face the study of statistics with mixed emotions, they realise that work in their fields require some understanding of statistics. It indicates the importance of statistics.
Statistics is widely used in the policy-making process in such diverse field as law, medicine, public health, public administration, health-care administration, politics, education, agricultural science, criminology, economic development, regional planning, and the like. Let us examine the importance of statistics in some fields as follows:

Statistics and Medicine
There are a number of applications of statistics in the medical field (health sciences) which include but not limited to epidemiology, screening for disease, forensic medicine, health-care planning, creation of expert systems for diagnosis, clinical research and selection of drugs and vaccines to develop.

Statistics in Government: Statistics are extensively used as a basis for governmental plans and policies. The government of every country collects numerical data in relation to its people, its economy and its socio-economic conditions as a routine job. Every government makes an estimate of its receipts and expenditure. Export and import, production, consumption, price variation, labour, crime, etc., all require some statistical treatment for their compilation.

Statistics and Economics: Statistics is gaining an ever increasing importance in the field of economics. Statistics and economics are so interrelated to each other that the new disciplines like econometrics and economic statistics have been developed. Statistics has become indispensable for description, comparison and correlation of economic data.

Statistics and Economic Planning: For the proper utilization of natural resources, labour, etc., statistics play a vital role. In economic planning, we need al such type of statistics and that is why it is essential to collect data on overall resources of the community including physical, financial and human. In Uganda, the Uganda Bureau of Statistics (UBOS) was established to provide all type of information at state level which is useful in economic planning of the country.

Statistics in Business and Commerce: In the past, business was small. Now the picture has completely changed, industrial and commercial units have expanded their size, competition has grown and thus the problems of business organisations have increased. In such situations, it is very essential to know about past as well as future. Market conditions are changing their shape rapidly and therefore, it is necessary for a businessman to use some scientific approach to maintain his position in the market. To earn better price in a competitive market, it is necessary to watch the quality of the product, which is possible by effective quality control. This is a statistical analysis. For increasing sales, market research is necessary i.e. study the habits, tastes, need and other allied matters of the society. A study on these points cannot be done unless we have past figures. In absence of these figures, a trial and error method is adopted which entails a lot of waste.
Many business organisations have their own research and development department, which is responsible for collection of such data. These departments also prepare charts, graphs and other statistical analyses for the purpose.

DESCRIPTIVE STATISTICS

Descriptive statistics is the term given to the analysis of data that helps describe, show or summarize data in a meaningful way such that, for example, patterns might emerge from the data. Descriptive statistics do not, however allow us to make conclusions beyond the data we have analysed or reach conclusions regarding any hypotheses we might have made. They are simply a way to describe our data.

Descriptive statistics are very important because if we simply presented our raw data, it would be hard visualize what the data was showing, especially if there was a lot of it. Descriptive statistics therefore enables us to present the data in a more meaningful way, which allows simpler interpretation of the data.

When we use descriptive statistics, it is useful to summarize our group of data using a combination of tabulated description (i.e. tables), graphical description (i.e. graphs and charts).

Data classification

In order to present and analyse data in a logical and meaningful way, it is necessary to understand some of the natural forms that they can take. There are various ways of classifying data and are as follows

a) By source: Data can be described as either primary or secondary, depending on their source.

b) By measurement; Data can be measurement in either numeric (or quantitative) or non-numeric (qualitative) terms.

c) By preciseness: Data can either be measured precisely (described as discrete) or only ever be approximated to (described as continuous)

d) By number of variables: Data can consist of measurements of one or more variable for each subject or item. Univariate is the name given to a set of data consisting of measurements of just one variable, bivariate is used for two variables, and for two or more variables the data is described as multivariate.

Discrete data

Discrete data can be described as data that can be measured precisely. One way of obtaining discrete data is by counting. For example:

i. the number of components produced from an assembly line over a number of consecutive shifts:

$45, 51, 44, 44, 43, 50, 46, 43, \ldots$ etc.

ii. the number of employees working in various offices of a company

$12, 32, 8, 13, 8, 6, 11, 24, \ldots$ etc.

Discrete data can also be obtained from situations where counting is not involved.

For example:

iii. shoe sizes of a sample of people:

$8, 10, 10, 6\frac{1}{2}, 9, 9, 9\frac{1}{2}, 8\frac{1}{2}, \ldots$ etc.

iv. weekly wages in thousands of shillings for a set of workers

$121.45, 162.85, 133.37, 108.32, \ldots$ etc.

A particular characteristic of discrete data is the fact that possible data values progress in definite steps, i.e. shoe sizes are measured as 6 or $6\frac{1}{2}$ or 7 or $7\frac{1}{2}$ … etc. or there are 1 or 2 or 3 … etc people (and not 3.5 or 4.67

Continuous data

The most significant characteristic of continuous data is the fact that they cannot be measured precisely; their values can only be approximated to. Examples of continuous data are dimensions (lengths, heights); weights; areas and volumes; temperatures; times.

Although continuous values cannot be identified exactly, they are often recorded as if they were precise and this is normally acceptable. For example:

i. diameters (in mm) of a sample of screws from a production run:

$4.11, 4.10, 4.10, 4.10, 4.15, 4.09, 4.12, \ldots$ etc.

ii. weights (in gm) of the contents of a selection of cans of baked beans:

$446.8, 447.0, 446.8, 447.2, 447.1, ..$ etc.

Frequency distributions

This is concerned with the organisation and presentation of ‘numeric (univariate)’ data. It describes how numeric data can be organized into frequency distributions of various types.

Raw statistical data

Before the data obtained from a statistical survey or investigation have been worked on, they are known as raw data. The table below gives an example of a set of raw data.

Hours worked in one week by employees in a company’s production department

46.3	45.1	45.6	46.1	45.0	43.5	39.2	39.2
39.2	42.3	39.6	38.9	44.4	43.4	43.8	43.2
44.2	43.5	42.0	42.4	42.4	42.8	42.9	42.9
41.3	40.0	39.6	42.1	39.8	44.3	46.2	46.2

Simple frequency distributions

Some sets raw data contain a limited number of data values, even though there may be many occurrences of each value. In this type of situation, the standard form into which the data is organized is known as a simple frequency distribution.

A simple frequency distribution consists of a list of data values, each showing the number of items having that value (called the frequency). This type of structure is normally applicable to discrete raw data (i.e. where values have usually been obtained by counting), since data values are quite likely to be repeated many times. A simple frequency distribution is not normally suitable for continuous data, since the likelihood of repeated values is small.

Sometimes limits and boundaries will coincide, as in the structure ‘12 and up to 13’, ‘13 and up to 14’, ‘14 and up to 15’, … etc.

c) Class widths (or lengths or intervals)

There are the numerical differences between the lower and upper class boundaries (and not class limits)

d) Class mid-points (class mark)

These are situated in the centre of the classes. They can be identified as being mid-way between the upper and lower boundaries (or limits).

For example:

Class 10 to 19 has a midpoint of 14.5 (not 15); class width= 10 (19.5 − 9.5)
Class 20 to 29 midpoint = 24.5; class width = 10
Class 30 to 39 midpoint = 34.5; class width = 10
etc.

However:

Class 30 and up to 40 has a midpoint = 35; class width = 10
Class 40 and up to 50 has a midpoint = 45; class width = 10
etc.

Cumulative frequency distributions

Any frequency distribution can be adopted to form what is known as a cumulative frequency distribution. Whereas ordinary frequency distributions describe a particular class of values according to how may items lie within it, cumulative frequency distributions describe the number of items that have values either above or below a particular level. Cumulative frequency distributions come in two different forms described as follows

a) Less than distributions

Here, a set of item values is listed (normally class ‘upper boundaries’), with each one showing the number of items in the distribution having values less than this. For example, let us transform an ordinary frequency distribution into a cumulative (less than) distribution.

Ordinary frequency distribution:

Miles travelled	Number of salesmen
100 and up to 200	3
200 and up to 300	5
300 and up to 400	2
400 and up to 500	8
500 and up to 600	2

Cumulative frequency distribution:

Miles travelled	Number of salesmen
Less than 200	3
Less than 300	8
Less than 400	10
Less than 500	18
Less than 600	20

Note carefully, in the case of transformed distribution, how:

i. the description of classes need to be changed
ii. the frequencies are accumulated

b) More than distributions

Here, a set of item values is listed (normally class ‘lower boundaries’), with each one showing the number of items in the distribution having values greater than this. The table below shows the distribution from a) transformed into a cumulative (more than) distribution.

Miles travelled	Number of salesmen
More than 100	20
More than 200	17
More than 300	12
More than 400	10
More than 500	2

Note again how:

i. the distributions of classes need to be changed
ii. the frequencies are formed by accumulating ‘in reverse’

Formation of grouped frequency distributions

Given a set of raw statistical data, there is no single grouped frequency distribution which is uniquely correct in representing them; many different structures of classes could be set up to describe the data. However governing principles such as the class width and starting class can be important to specify the frequency distribution we are supposed to arrive at.

Example 2

Thirty AA batteries were tested to determine how long they would last. The results, to the nearest minute were as follows:

423	369	387	411	393	394
371	377	389	409	392	408
431	401	363	391	405	382
400	381	399	415	428	422
396	372	410	419	386	390

Construct a grouped frequency distribution with class width 10, starting with the class 360 − 369

Solution

Battery life (minutes)	Tally	Frequency (f)
360 − 369	//	2
370 − 379	///	3
380 − 389	~~////~~	5
390 − 399	~~////~~ //	7
400 − 409	~~////~~	5
410 − 419	////	4
420 − 429	///	3
430 − 439	/	1
Total		**30**

Statistical measures

Statistical measures describe the basic analysis of univariate data (data obtained from measuring just one attribute). The measures themselves are split into various groups. We shall discuss two groups of these measures.

a) Measures of central tendency
b) Measures of variation (or spread)

a) Measures of central tendency

These are statistical constants which give us an idea about the concentration of the values in the central part of the distribution.

Generally, a measure of central tendency of a statistical series is the value of the variable which is representative of the entire distribution. The following are the measures of central tendency discussed in this book.

i. Arithmetic mean or simply mean
ii. Median
iii. Mode

i. The arithmetic mean/mean

The arithmetic mean of a set of values is defined as 'the sum of values' divided by 'the number of values'.

$$\text{Arithmetic mean} = \frac{\text{the sum of all values}}{\text{the number of values}}$$

The arithmetic mean is normally abbreviated to just the 'mean'

Example 3

The mean of the values 12, 8, 25, 26 and 10 is calculated as

$$\frac{12+8+25+26+10}{5} = \frac{81}{5} = 16.2$$

Formula for the mean of a set of values

The mean of a set of values x_1, x_2, …., x_n is calculated as follows

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{\sum x}{n}$$

Example 4

To calculate the mean or the set: 43, 75, 50, 51, 51, 47, 40, 48.

Here, $n = 10$ and $\sum x = 502$

$$\text{Therefore: } \bar{x} = \frac{\sum x}{n} = \frac{502}{10} = 50.2$$

Formula for the mean of a frequency distribution

The mean for a frequency distribution is calculated using the formula

$$\text{Mean, } \bar{x} = \frac{\sum fx}{\sum f}$$

Note: For a grouped frequency distribution, x is the class mid-point

Example 5

To calculate the mean of the following simple distribution

Number of vehicles (x)	0	1	2	3	4	5
Number of days (f)	2	5	11	4	4	1

the normal layout for calculation is

x	f	fx
0	2	0
1	5	5
2	11	22
3	4	12
4	4	16
5	1	5
$\sum$	27	60

Thus:

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{60}{27} = 2.2$$

Hence, the mean number of vehicles is 2.2

Example 6

The following data relates to the number of successful sales made by the salesman employed by a large computer firm in a particular quarter.

Number of sales	Number of salesmen
$0-4$	1
$5-9$	14
$10-14$	23
$15-19$	21
$20-24$	15
$25-29$	6

Calculate the mean number of sales

<u>Solution</u>

The standard layout and calculations are shown as follows:

Number of sales	Number of salesmen (f)	Class mid-point (x)	fx
$0-4$	1	2	2
$5-9$	14	7	98
$10-14$	23	12	276
$15-19$	21	17	357
$20-24$	15	22	330
$25-29$	6	27	162
$\sum$	80		1225

Here $\sum fx = 1225$ and $\sum f = 80$

$$\therefore \text{ mean number of sales, } \bar{x} = \frac{\sum fx}{\sum f} = \frac{1225}{80} = 15.3$$

Manipulation of the mean formula

There are some circumstances where the simple manipulation of the formula

$$\bar{x} = \frac{\sum x}{n}$$

to become

$$\sum x = n\bar{x}$$

ii. The median

The median of a set of data is the value of that item which lies exactly half way along the set (arranged into size order).

Note:

1. When a set of data contains an even number of items, there is no unique middle or central value. The convention in this situation is to use the mean of the middle two items to give a (practical) median.
2. For a set with an odd number (n) of items, the median can be precisely identified as the value of the $\left(\frac{n+1}{2}\right)$th term. Thus in a size ordered set of 15 items, the median would be the $\frac{15+1}{2}$th $=$ the 8th item along.

Example 11

(a) The median of 43, 75, 48, 51, 51, 47, 50 is determined by size ordering the set as

43, 47, 48, 50, 51, 51, 75

Then the median = middle term = 50

(b) The median of 2, 4, 6, 1, 2, 3, 3, 2 is found by size-ordering the set as

1, 2, 2, 2, 3, 3, 4, 6

noticing that there is an even number of items which gives

$$\text{median} = \text{mean of middle two } = \frac{2+3}{2} = 2.5$$

Median for a simple frequency distribution

When there is a large number of discrete items in a data set, but the range of values is limited, a simple frequency distribution will probably have been compiled.

To calculate the median for a simple(discrete) frequency distribution, the following procedure should be followed

1. Calculate the value of $\frac{\sum f+1}{2}$ (identifying the central item)
2. Form a F (cumulative frequency) column
3. Find that F value which first exceeds $\frac{\sum f+1}{2}$
4. The median is that x-value corresponding to the F value identified in step 3

Note: Sometimes $\sum f$ is replaced by N for convenience.

Example 12

Calculate the median for the following distribution of delivery times of orders sent out from a firm.

Delivery time (days)	Number of orders
0	4
1	8
2	11
3	12
4	21
5	15
6	10
7	4
8	2
9	2
10	1
11	1

Solution

Delivery time (x)	Number of orders (f)	Cum f (F)
0	4	4
1	8	12
2	11	23
3	12	35
4	21	56
5	15	71
6	10	81
7	4	85
8	2	87
9	2	89
10	1	90
11	1	91

The median is the $\frac{N+1}{2}$th value $= \frac{91+1}{2} = 46$ th term

The first F value to exceed 46 is $F = 56$

The median is thus 4 days

Median for a grouped frequency distribution

There are two methods commonly employed for estimating the median:

(a) using an interpolation formula

(b) graphically using a cumulative frequency curve (O-Give)

Interpolation in this context is a simple mathematical technique which estimates an unknown value by utilizing immediately surrounding known values.

Estimating the median by formula

The procedure for estimating the median (by formula) for a grouped frequency distribution is:

1. Form a cumulative frequency (F) column
2. Find the value of $\frac{N}{2}$ (where $N = \sum f$)
3. Find that F value that first exceeds (or equal to), which identifies the median class m
4. Calculate the median using the following interpolation formula

$$\textbf{median} = \boldsymbol{L_m} + \left(\frac{\frac{\boldsymbol{N}}{\boldsymbol{2}} - \boldsymbol{F_b}}{\boldsymbol{f_m}}\right)\boldsymbol{c}$$

where:

L_m = lower bound of the median class

F_b = cumulative frequency of class immediately before the median class

f_m = actual frequency of median class

c = median class width

Example 13

Calculate the median or the following data which represents the ages of a set of 130 representatives who took part in a statistical survey

Age in years	Number of representatives
20 and under 25	2
25 and under 30	14
30 and under 35	29
35 and under 40	43
40 and under 45	33
45 and under 50	9

Solution

Class	f	F
$20 -< 25$	2	2
$25 -< 30$	14	16
$30 -< 35$	29	45
$35 -< 40$	43	88
$40 -< 45$	33	121
$45 -< 50$	9	130

$$\frac{N}{2} = \frac{130}{2} = 65$$

The median class is the class that has the first F greater than 65. Here it is $35 -< 40$.

$L_m = 35, f_m = 43, c = 5$

$$\text{median} = L_m + \left(\frac{\frac{N}{2} - F_b}{f_m}\right)c$$

$$= 35 + \left(\frac{65 - 45}{43}\right) \times 5$$

$$= 37.33 \text{ years}$$

Estimating the median graphically

The procedure for estimating the median graphically for a grouped frequency distribution is:

1. Form a cumulative frequency distribution
2. Draw up a cumulative frequency curve by plotting class upper bounds against cumulative frequency and join the points with a smooth curve
3. The middle number of the distribution is located on the cumulative frequency axis and the corresponding value of the variable is the median.

Note: The quartiles, percentiles or deciles can be found in a similar way.

iii. The mode

The mode of a set of data is that value which occurs most often or, equivalently, has the largest frequency.

Example 14

(a) The mode of the set 2, 1, 3, 3, 1, 1, 2, 4 is 1, since this value occurs most often

(b) The mode of the following simple discrete frequency distribution

x	4	5	6	7	8	9	10
f	2	5	21	18	9	2	1

is 6 since this value has the largest frequency (of 21)

The mode for grouped data

For a grouped frequency distribution, the mode (in line with the mean and median cannot be determined exactly and so must be estimated. The technique used is one of interpolation, similar to that used to estimate the median of a frequency distribution. There are two methods that can be used to estimate the mode:

(a) using an interpolation formula

(b) graphically, using a histogram

Formula for the mode of a grouped frequency distribution

An estimate of the mode of a grouped frequency distribution can be obtained using the following procedure.

1. Determine the modal class
   (a) For frequency distribution with **equal class widths**, modal class is the class with the highest frequency.
   (b) For frequency distribution with **unequal class widths**, modal class is the class with the highest frequency density.
2. Calculate D_1 = difference between the largest frequency (or frequency density) and the frequency (or frequency density) immediately preceding it
3. Calculate D_2 = difference between the largest frequency (or frequency density) and the frequency (frequency density) immediately following it.
4. Use the following interpolation formula

$$\textbf{Mode} = \boldsymbol{L} + \left(\frac{\boldsymbol{D_1}}{\boldsymbol{D_1 + D_2}}\right)\boldsymbol{c}$$

where:

L =lower bound of modal class

c = class width of modal class

It should be noted that when the frequency distribution has unequal class widths, we use differences in the frequency densities instead of the differences in the frequencies

Example 15

Calculate the mode of the following distribution of ages

Age (years)	Number of employees
$20 - 25$	2
$25 - 30$	14
$30 - 35$	29
$35 - 40$	43
$40 - 45$	33
$45 - 50$	9

Solution

The highest frequency is 43 and thus the modal class is $35 - 40$

$$D_1 = 43 - 29 = 14$$

$$D_2 = 43 - 33 = 10$$

Lower class bound of modal class is 35

Class width of modal class is 5

$$\text{Mode} = L + \left(\frac{D_1}{D_1 + D_2}\right)c$$

$$= 35 + \left(\frac{14}{14 + 10}\right) \times 5$$
$$= 37.92 \text{ years}$$

Example 16

The frequency distribution below shows the ages of 240 students admitted to a certain university.

Age (years)	Number of students
$18 - < 19$	24
$19 - < 20$	70
$20 - < 24$	76
$24 - < 26$	48
$26 - < 30$	16
$30 - < 32$	6

Calculate the modal age of the students

Solution

Age (years)	f	Class width c	Frequency density, $\frac{f}{c}$
$18 - < 19$	24	1	24
$19 - < 20$	70	1	70
$20 - < 24$	76	4	19
$24 - < 26$	48	2	24
$26 - < 30$	16	4	4
$30 - < 32$	6	2	3

Modal class is $19 - < 20$

$D_1 = 70 - 24 = 46$, $D_2 = 70 - 19 = 51$, $c = 1$

$$\text{Mode} = L + \left(\frac{D_1}{D_1 + D_2}\right) c$$

$$\text{Mode} = 19 + \left(\frac{46}{46 + 51}\right) \times 1 = 19.5 \text{ years}$$

Example 17

Given below is the distribution of 140 candidates obtaining marks X or higher in a certain examination (all marks are given in whole numbers)

X	Cumulative frequency
10	140
20	133
30	118
40	100
50	75
60	45
70	25
80	9
90	2
100	0

Calculate the mean, median and mode of the distribution

Solution

This cumulative frequency distribution is a 'more than' distribution i.e. > 10, > 20, > 30 and so on. Thus the class limits will be $10 - 19$, $20 - 29$, $30 - 39$, and so on.

The respective frequencies are calculated as $140 - 133 = 7$, $133 - 118 = 15$, $118 - 100 = 18$, and so on.

Class	f	Class boundaries	x	F (*less than*)	fx
$10 - 19$	7	$9.5 - 19.5$	14.5	7	101.5
$20 - 29$	15	$19.5 - 29.5$	24.5	22	367.5
$30 - 39$	18	$29.5 - 39.5$	34.5	40	621
$40 - 49$	25	$39.5 - 49.5$	44.5	65	1112.5
$50 - 59$	30	$49.5 - 59.5$	54.5	95	1635
$60 - 69$	20	$59.5 - 69.5$	64.5	115	1290
$70 - 79$	16	$69.5 - 79.5$	74.5	131	1192
$80 - 89$	7	$79.5 - 89.5$	84.5	138	591.5
$90 - 99$	2	$89.5 - 99.5$	94.5	140	189
Total	140				7100

$$\text{Mean, } \bar{x} = \frac{\sum fx}{\sum f} = \frac{7100}{140} = 50.7$$

$$\frac{N}{2} = \frac{140}{2} = 70$$

The first F above 70 is 95, thus median class is $50 - 59$

$$\text{median} = L_m + \left(\frac{\frac{N}{2} - F_b}{f_m}\right) c$$

$$\text{median} = 49.5 + \left(\frac{70 - 65}{30}\right) \times 10 = 51.2$$

The modal frequency is 30, thus the modal class is $50 - 59$

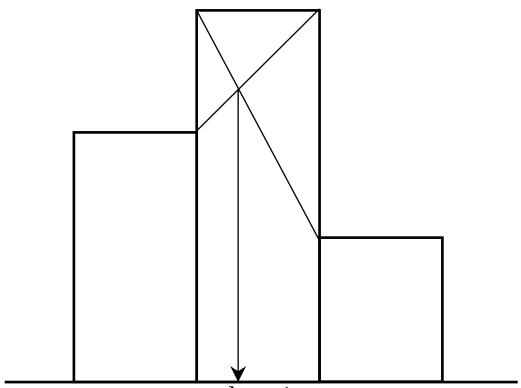
$D_1 = 30 - 25 = 5$, $D_2 = 30 - 20 = 10$

$$\text{Mode} = L + \left(\frac{D_1}{D_1 + D_2}\right) c$$

$$= 49.5 + \left(\frac{5}{5 + 10}\right) \times 10 = 52.8$$

Graphical estimation of the mode

The graphical equivalent of the mode extrapolation formula is to construct a histogram. Identify the histogram bar with the highest frequency (frequency density for classes with unequal class width) and draw two lines as shown below. The mode estimate is the x-value corresponding to the intersection of lines.



mode estimate

b) Measure of dispersion

The measures of central tendency give us an idea of the concentration of the observations about the central part of the distribution. If we know the average alone, we cannot form a complete idea about the distribution as will be clear from the following example.

Consider the series (i) 7, 8, 10, 11 (ii) 3, 6, 9, 12, 15 (iii) 1, 5, 9, 13, 17.

In all these cases, we see that n, the number of observations is 5 and the mean is 9. If we are given that the mean of 5 observations is 9, we cannot form an idea as to whether it is the average of first series or second series or third series. Thus we see that the measures of central tendency are inadequate to give us a complete idea of the distribution. They must be supported and supplemented by some other measures. One such measure is **dispersion**.

Measures of dispersion describe how spread out or scattered a set or distribution of numeric data is. There are different bases on which the spread of data can be measured.

The following are the measures of dispersion

i. Range
ii. Interquartile range and quartile deviation
iii. Standard deviation and variance

The range

The range is the simplest measure of dispersion available in statistics analysis.

The range is defined as the numerical difference between the smallest and largest values of the items in a set or distribution.

Example 18

The values below represent the number of students who sat for A-level exams in the year 2019 in ten different schools.

161	161	163	167	162
168	170	172	165	166

Find the range for this data.

Solution

$$\text{Range} = 172 - 161 = 11 \text{ students}$$

Standard deviation and variance

A procedure for calculating the standard deviation is now described and at the same time, demonstrated using a set of values 2, 4, 6 and 8.

Step 1: Calculate the mean

$$\bar{x} = \frac{2+4+6+8}{4} = 5$$

Step 2: Find the sum of the squares of deviations of items from the mean

$$(2-5)^2 + (4-5)^2 + (6-5)^2 + (8-5)^2$$
$$= (-3)^2 + (-1)^2 + (1)^2 + (3)^2$$
$$= 9 + 1 + 1 + 9$$
$$= 20$$

Step 3: Divide this sum by the number of items and take the square root.

$$\sqrt{\frac{20}{4}} = \sqrt{5} = 2.24$$

This value so obtained, 2.24, is the standard deviation.

This procedure can be summarized as follows.

Standard deviation for a set of values

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n}}$$

In other words, the standard deviation can be defined as 'the root of the mean of the squares of deviations from the common mean' of a set of values.

Note:

If the mean is not a whole number, the calculations could involve some awkward, decimal bound work. An example of this follows:

Example 19

Find the standard deviation of 6, 11, 14, 10, 8, 11 and 9

Solution

The layout of calculations is shown in the table for convenience

x	$x-\bar{x}$	$(x-\bar{x})^2$
6	-3.857	14.876
11	1.143	1.306
14	4.143	17.164
10	0.143	0.020
8	-1.857	3.448
11	1.143	1.306
9	-0.857	0.734
69		38.854

$$\bar{x} = \frac{69}{7} = 9.857$$

$$\sum(x-\bar{x})^2 = 38.854$$

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n}} = \sqrt{\frac{38.854}{7}} = 2.36$$

In such cases, a computational formula is generally preferred since it involves less awkward arithmetic.

Computational formula of standard deviation of a set

$$s = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2}$$

From example 19,

$\sum x^2 = 719$ and $\bar{x} = \frac{69}{7}$

$$s = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2} = \sqrt{\frac{719}{7} - \left(\frac{69}{7}\right)^2} = \sqrt{\frac{272}{49}} = 2.36$$

Both formulas will always yield the same value for the standard deviation.

Example 20

Calculate the mean and standard deviation of the values

43, 75, 48, 51, 51, 47, 50, 47, 40, 48

Solution

x	x^2
43	1849
75	5625
48	2304
51	2601
51	2601
47	2209
50	2500
47	2509
40	1600
48	2304
$\sum x = 500$	$\sum x^2 = 25802$

$n = 10, \sum x = 500, \sum x^2 = 25802$

$$\bar{x} = \frac{\sum x}{n} = \frac{500}{10} = 50$$

$$s = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2} = \sqrt{\frac{25802}{10} - 50^2} = 8.96$$

Hint:

When calculating $\sum x^2$, a student may choose to square and add automatically using the calculator. This will ensure the least effort for the information needed.

Standard deviation for a frequency distribution

For large sets of data, a frequency distribution is normally compiled and the computational formula for the standard deviation for a set is adopted as follows:

$$s = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$$

When the assumed mean is given, then the standard deviation is given by

$$s = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2}$$

Example 21

The data below relates the number of successful sales made by the salesmen employed by a large computer firm in a particular quarter.

Number of sales	Number of salesmen
0 – 4	1
5 – 9	14
10 – 14	23
15 – 19	21
20 – 24	15
25 – 29	6

Calculate the mean and standard deviation of the number of sales.

Solution

Class	f	x	fx	fx^2
0 – 4	1	2	2	4
5 – 9	14	7	96	686
10 – 14	23	12	276	3312
15 – 19	21	17	357	6069
20 – 24	15	22	330	7260
25 – 29	6	27	162	4374
$\sum$	80		1225	21705

$$\text{Mean, } \bar{x} = \frac{1225}{80} = 15.3 \text{ sales}$$

$$\text{Standard deviation, } s = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$$

$$= \sqrt{\frac{21705}{80} - \left(\frac{1225}{80}\right)^2}$$

$$= 6.1 \text{ sales}$$

Example 22

The table below shows the weight in kg of 100 boys in a certain school

Weight (kg)	Frequency
60 – 62	8
63 – 65	10
66 – 68	45
69 – 71	30
72 – 74	7

Using the assumed mean of 67, calculate the mean and standard deviation

Solution

Weight	f	x	d	fd	fd^2
60 – 62	8	61	−6	−48	288
63 – 65	10	64	−3	−30	90
66 – 68	45	67	0	0	0
69 – 71	30	70	3	3	270
72 – 74	7	73	6	6	252
$\sum$	100			54	900

$$\text{Mean, } \bar{x} = a + \frac{\sum fd}{\sum f} = 67 + \frac{54}{100}$$

$$= 67 + 0.54 = 67.54 \text{ points}$$

$$\text{Standard deviation, } s = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2}$$

$$s = \sqrt{\frac{900}{100} - \left(\frac{54}{100}\right)^2}$$

$$= \sqrt{9 - 0.2916}$$

$$= \sqrt{8.7084} = 2.951$$

Variance

In many situations, it is the square of the standard deviation, s^2, which is important. We call s^2 the variance.

The formulae for variance can be given by removing the square roots from the formula for standard deviation i.e.

$$Q = L_Q + \left[\frac{P_Q - F_b}{f_Q}\right]c$$

(a) For Q_1: Position is $\frac{N}{4} = \frac{1303}{4} = 325.75$

So Q_1 lies in class $500 - 1000$

$L_Q = 500, F_b = 210, f_Q = 184, c = 500$

$$Q_1 = 500 + \left[\frac{325.75 - 210}{184}\right] \times 500 = 814.5$$

For Q_3: Position is $\frac{3N}{4} = \frac{3(1303)}{4} = 977.25$

So Q_1 lies in class $2000 - 2500$

$L_Q = 2000, F_b = 974, f_Q = 1151, c = 500$

$$Q_3 = 2000 + \left[\frac{977.25 - 974}{177}\right] \times 500 = 2009.2$$

The semi-interquartile range is given by

$$\frac{Q_3 - Q_1}{2} = \frac{2009.2 - 814.5}{2} = 597.35$$

(b) For 60th percentile,

Position is $\frac{60}{100}N = \frac{60}{100} \times 1303 = 781.8$

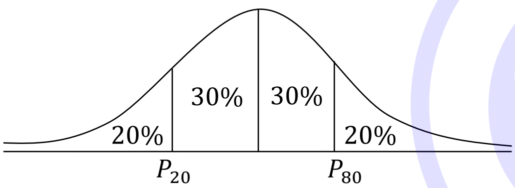
So P_{60} lies in class $1500 - 2000$

$L_Q = 1500, F_b = 626, f_Q = 348, c = 500$

$$P_{60} = 1500 + \left[\frac{781.8 - 626}{348}\right] \times 500$$

$$P_{60} = 223.9$$

(c) The middle 60% range can be represented by the diagram below of a frequency curve.



Middle 60% range $= P_{80} - P_{20}$

To get P_{80}, position is $\frac{80}{100}N = \frac{80}{100} \times 1303 = 1042.4$

So P_{80} lies in class $2000 - 2500$

$L_Q = 2000, F_b = 974, f_Q = 177, c = 500$

$$P_{80} = 2000 + \left[\frac{1042.4 - 974}{177}\right] \times 500 = 2193.2$$

To get P_{20}, position is $\frac{20}{100}N = \frac{20}{100} \times 1303 = 260.6$

So P_{20} lies in class $500 - 1000$

$L_Q = 500, F_b = 210, f_Q = 187, c = 500$

$$P_{20} = 500 + \left[\frac{260.6 - 210}{184}\right] \times 500 = 637.5$$

$\therefore$ The middle 60% range $= 2193.2 - 637.5 = 1555.7$

(d) To get D_9, position is $\frac{9}{10}N = \frac{9}{10} \times 1303 = 1172.7$

So D_9 lies in class $2500 - 3000$

$L_Q = 2500, F_b = 1151, f_Q = 83, c = 500$

$$D_9 = 2500 + \left[\frac{1172.7 - 1151}{83}\right] \times 500 = 2630.7$$

Finding population mean and variance given sample parameters

The population mean is denoted by μ and its variance σ^2 whereas the sample mean is denoted by $\bar{x}$ and its variance s^2.

If we can calculate the sample mean, $\bar{x}$ and variance s^2 (or if they have been given), then we can use sample parameters to obtain the population parameters. The population parameters are known as unbiased estimates.

a) The unbiased estimate for the population mean, μ is $\hat{\mu}$

$$\text{where } \hat{\mu} = \bar{x} = \frac{\sum x}{n} = \frac{\sum fx}{\sum f}$$

Hence the population mean is the same as the sample mean.

b) The unbiased estimate for population variance, σ^2 is $\hat{\sigma}^2$

$$\text{where } \hat{\sigma}^2 = \frac{n}{n-1}s^2$$

and

$$s^2 = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2$$

Thus,

$$\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2\right]$$

$$\text{or } \hat{\sigma}^2 = \frac{1}{n-1}\left[\sum x^2 - \frac{(\sum x)^2}{n}\right]$$

Example 27

The number of cars caught speeding on a certain length of motorway is 7.2 per day, on average. Speed cameras are introduced and the results shown in the following table are those from a random selection of 40 days after this.

Number of cars caught speeding	4	5	6	7	8	9	10
Number of days	5	7	8	10	5	2	3

Calculate the unbiased estimates for the population mean and variance for the number of cars per day caught speeding after the cameras were introduced.

Solution

x	4	5	6	7	8	9	10	Total
f	5	7	8	10	5	2	3	40
fx	20	35	48	70	40	18	30	261
fx^2	80	175	288	490	320	162	300	1815

$$\text{Mean, } \bar{x} = \frac{\sum fx}{\sum f} = \frac{261}{40} = 6.525$$

The unbiased estimate of the population mean is 6.525 per day.

$$\text{Variance, } s^2 = \frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2 = \frac{1815}{40} - 6.525^2 = 2.8$$

The unbiased estimate of the population variance, σ^2 is given by

$$\hat{\sigma}^2 = \frac{n}{n-1}s^2 = \frac{40}{39} \times 2.8 = 2.871 \text{ per day}$$

Graphical representation and interpretation of data

Frequency polygon

This is a line graph drawn from class frequencies plotted against class marks.

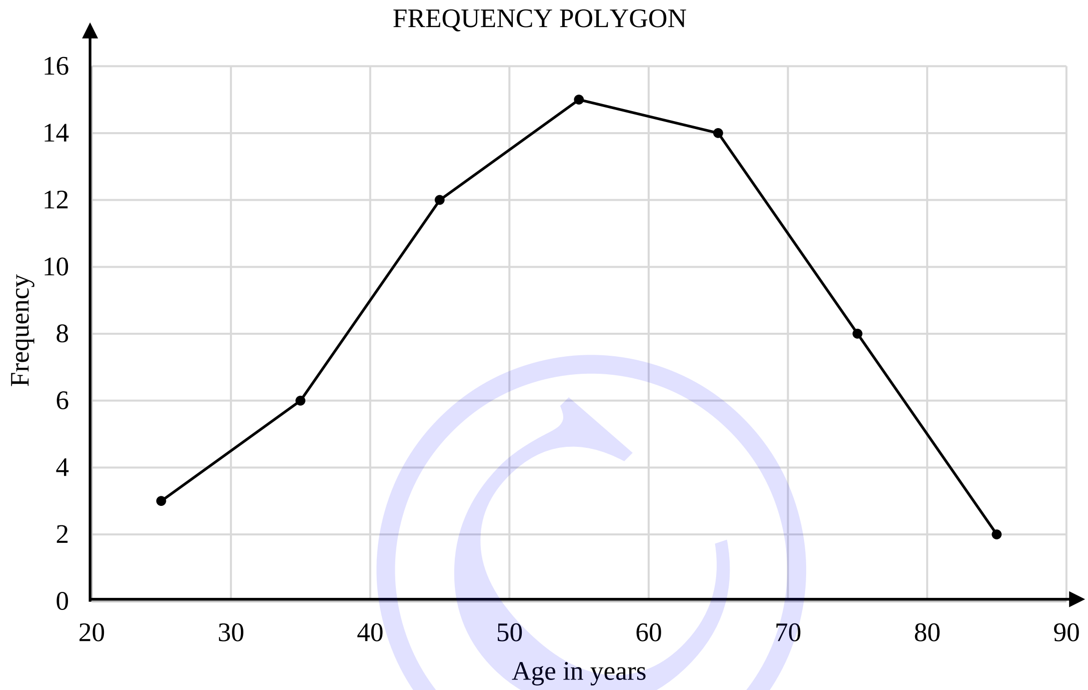
A frequency polygon may be also be drawn by joining the midpoints of the tops of the rectangles which would form a histogram. In this case, we say that the frequency polygon has superimposed the histogram.

Example 28

The table below (right) the age distribution of 60 members

Age in years	No. of members
20 – 30	3
30 – 40	6
40 – 50	12
50 – 60	15
60 – 70	14
70 – 80	8
80 – 90	2

Construct a frequency polygon for the data



Histogram

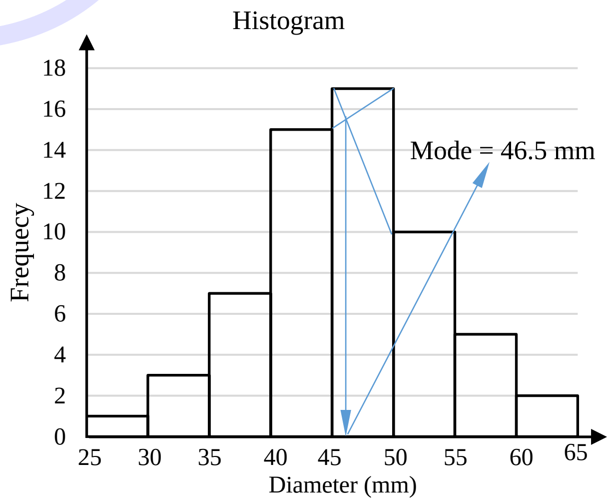
Histograms are used to illustrate continuous data. The columns in a histogram may have different widths and the area of each column is proportional to the frequency. Unlike bar charts, there are no gaps between the columns because where one class ends, the next begins.

Continuous data with equal class widths

A sample of 60 components is taken from a production line and their diameters, d mm, recorded. The resulting data are summarized in the table below.

Diameter	Frequency
$25 \le d < 30$	1
$30 \le d < 35$	3
$35 \le d < 40$	7
$40 \le d < 45$	15
$45 \le d < 50$	17
$50 \le d < 55$	10
$55 \le d < 60$	5
$60 \le d < 65$	2

Construct a histogram for the data above and use it to estimate the modal diameter



The boundaries are 25, 30, 35, 40, 50, 55, 60 and 65. The width of each class is 5.

The area of each column is proportional to the class frequency. In this example, the class widths are equal so the height of each column is also proportional to the class frequency.

The column representing $45 \le d < 50$ is the highest and this tells you that this is the modal class.

Continuous data with unequal class widths When the class widths are unequal, we use

$$\text{frequency density} = \frac{\text{frequency}}{\text{class width}}$$

Example 29

The frequency distribution below shows the ages of 240 students admitted to a certain university.

Age (years)	Number of students
18−< 19	24
19−< 20	70
20−< 24	76
24−< 26	48
26−< 30	16
30−< 32	6

(a) Calculate the mean age of the students

(b) (i) Draw a histogram for the data

(ii) Use the histogram to estimate the modal age

Solution

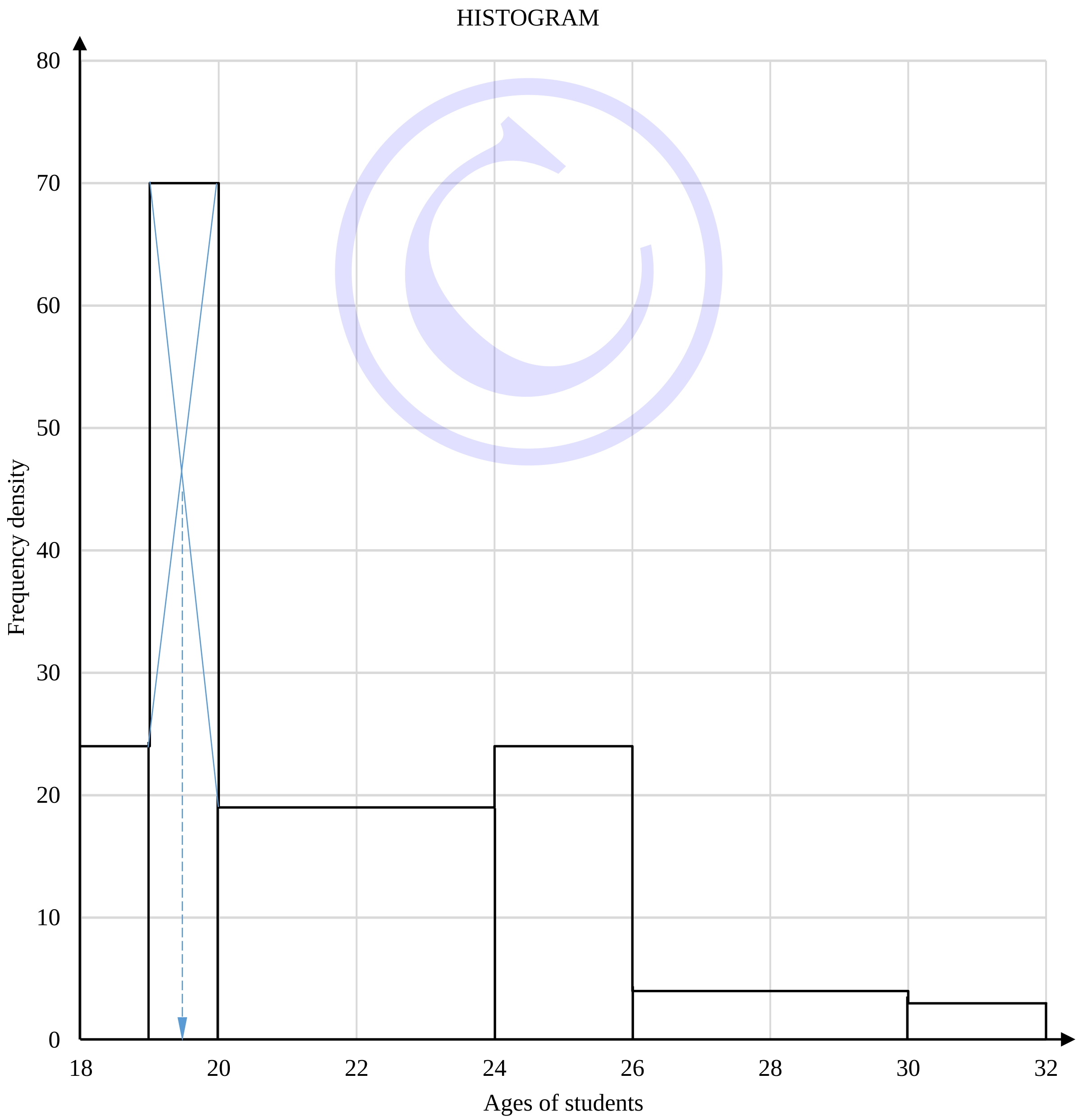
Let c = class width, x = class mark, frequency density = $\frac{f}{c}$

Age	f	c	$\frac{f}{c}$	x	fx
18−< 19	24	1	24	18.5	444
19−< 20	70	1	70	19.5	1365
20−< 24	76	4	19	22	1672
24−< 26	48	2	24	25	1200
26−< 30	16	4	4	28	448
30−< 32	6	2	3	31	186
Total	240				5315

(a) The mean age is given by

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{5315}{240} = 22.15 \text{ years}$$

(b) (ii) Modal age is 19.5 years



Cumulative frequency curve (O-Give)

A cumulative frequency curve is obtained by plotting the upper boundary of each class against the cumulative frequency. The points are joined by a smooth curve.

The cumulative frequency curve can be used to estimate the median, quartiles, percentiles and deciles as discussed in the previous sections.

Example 30

The table below is the frequency distribution of marks obtained in a test by 200 students.

Mark	f
10 −	18
20 −	34
30 −	58
40 −	42
50 −	24
60 −	10
70 −	6
80 − 90	8

(a) Draw a cumulative frequency polygon to illustrate the data.

(b) Estimate the median from the graph and by calculation

(c) How many students would fail if the pass mark is 45?

(d) If the top 10% of students are to be given a grade I, what is the lowest mark which will achieve this?

Solution

The cumulative frequency distribution is:

Mark	F
10 −	18
20 −	52
30 −	110
40 −	152
50 −	176
60 −	186
70 −	192
80 − 90	200

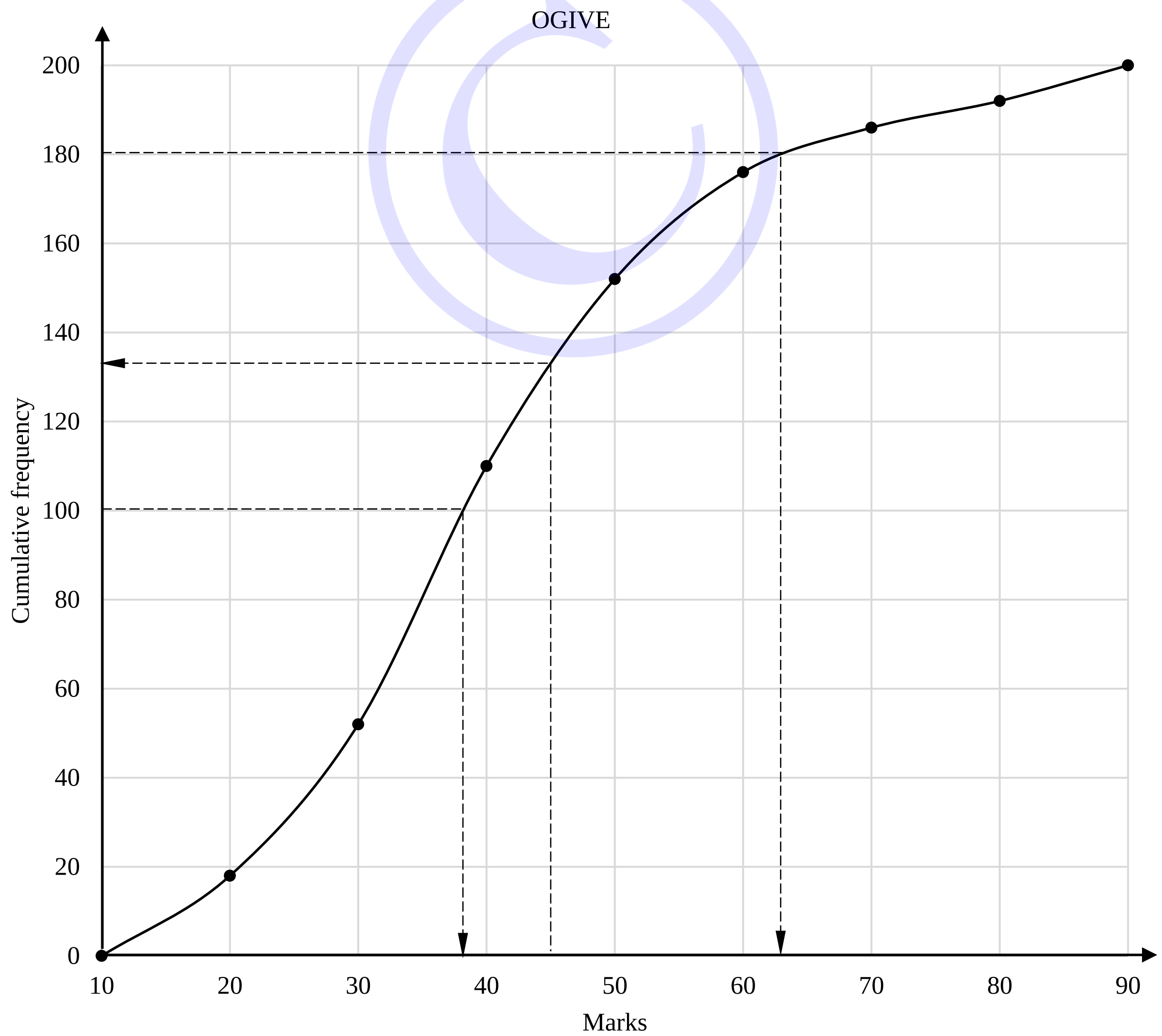
(b) From the graph, the median ≈ 38 marks

(c) $200 - 134 = 66$

66 students would fail if the pass mark was 45

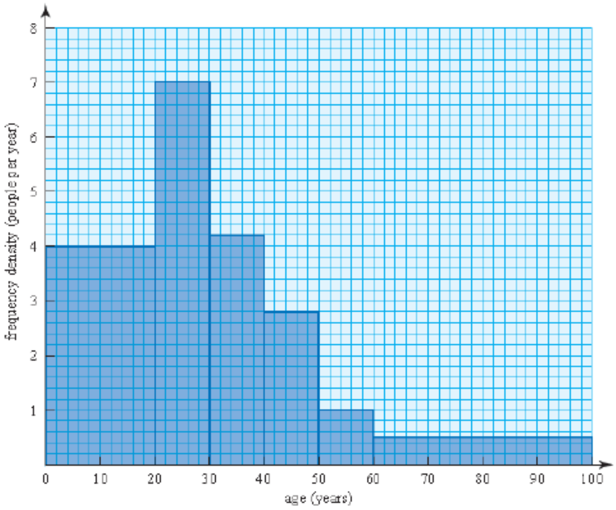
(d) $\frac{10}{100} \times 200 = 20$ i.e. top 20 students $\Rightarrow F = 180$

63 is the lowest mark for grade I



Example 31

A random sample of people were asked how old they were when they first met their partner. The histogram represents this information.



(a) What is the modal age group?

(b) How many people took part in the survey?

(c) Calculate the mean age that a person first met their partner

(d) Draw a cumulative frequency curve for the data and use the curve to estimate the middle 60% range of the age.

Solution

(a) The bar with the greatest frequency density represents the modal age group.
So the modal age group is $20-<30$

(b) Frequency density $= \frac{\text{frequency}}{\text{class width}}$

Frequency = frequency density × class width

Age	$f.d.$	c	f	x	fx	F
$0-<20$	4	20	80	10	800	80
$20-<30$	7	10	70	25	1750	150
$30-<40$	4.2	10	42	35	1470	192
$40-<50$	2.8	10	28	45	1260	220
$50-<60$	1	10	10	55	550	230
$60-<100$	0.5	40	20	80	1600	250
$\sum$			250		7430	

The total number of people is 250

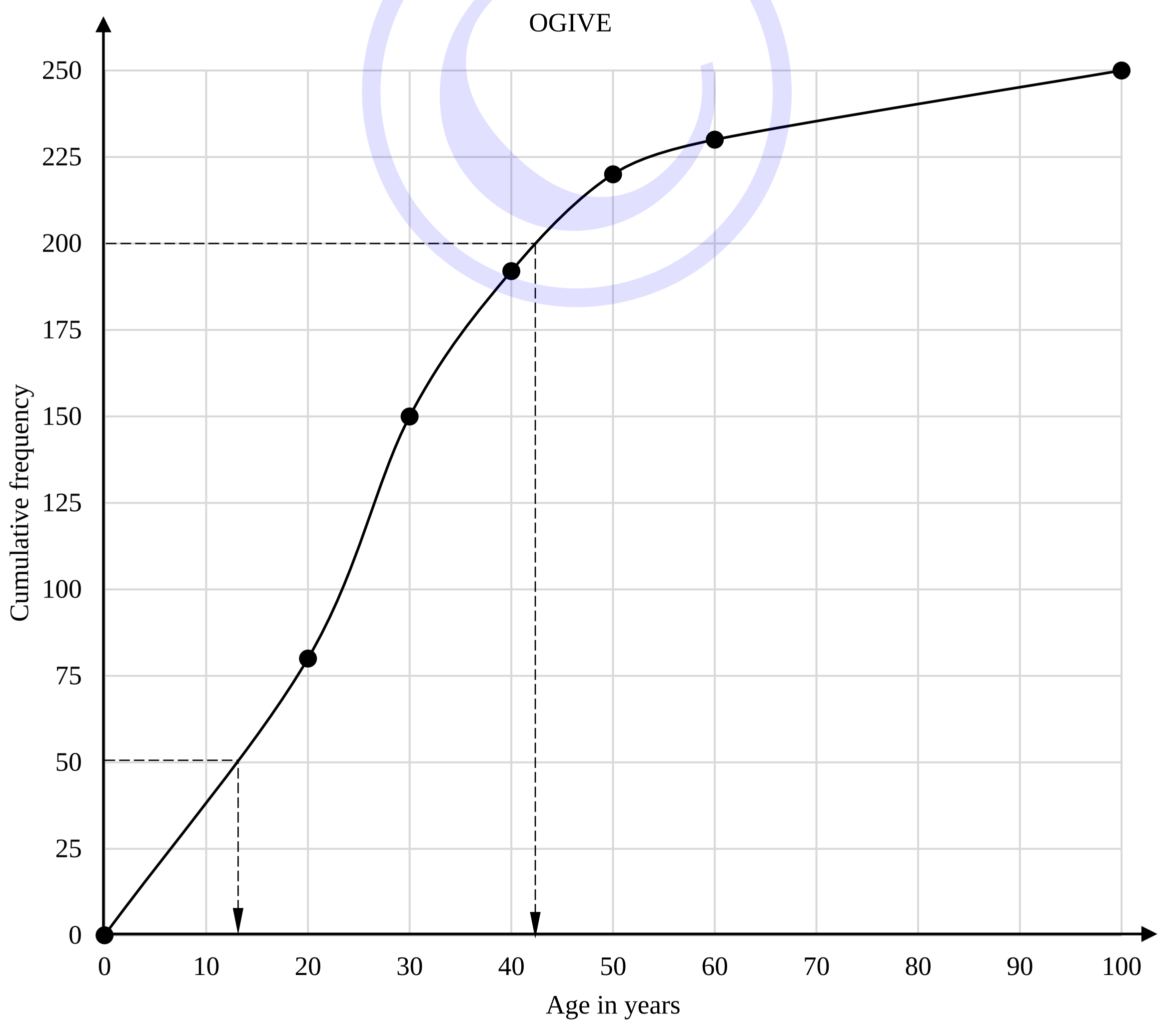
(c) Mean $= \frac{\sum fx}{\sum f} = \frac{7430}{250} = 29.7$ years

(d) Middle 60% range implies P_{20} to P_{80}

$$P_{20} = \left(\frac{20}{100} \times 250\right)^{th} = 50^{th}\text{value} = 43$$

$$P_{80} = \left(\frac{80}{100} \times 250\right)^{th} = 200^{th}\text{value} = 13$$

Middle 60% age range is $43 - 13 = 30$



9. The table shows the marks, collected into groups, of 400 candidates in an examination. The maximum mark was 99.

Marks	Number of candidates
$0-9$	10
$10-19$	26
$20-29$	42
$30-39$	66
$40-49$	83
$50-59$	71
$60-69$	52
$70-79$	30
$80-89$	14
$90-99$	6

(a) Compile the cumulative frequency table and draw the cumulative frequency curve. Use your curve to estimate:
   (i) the median
   (ii) the 20th percentile
(b) If the minimum mark for Grade A was at 74, estimate from your curve the percentage of candidates obtaining Grade A.

[Ans: (a) (i) 46.25 (ii) 29.80 (b) 9.125%]

10. A visit to Mama Babies Home revealed the following about the ages, in weeks of the babies under their care:

Age (in weeks)	Number of children
$11-16$	2
$17-22$	8
$23-28$	16
$29-34$	15
$35-40$	16
$41-46$	1

Calculate the:
(a) median age of the babies
(b) interquartile range

[Ans: (a) 29.7 (b) 11.25]

11. An inspection of 34 aircraft assemblies revealed a number of missing rivets as shown in the following table.

Marks	Number of candidates
$0-2$	4
$3-5$	9
$6-8$	11
$9-11$	6
$12-14$	2
$15-17$	1
$18-20$	0
$21-23$	1

Draw a cumulative frequency curve. Use this curve to estimate the median and the quartiles of the distribution.

[Ans: 6.6, 4, 9.25]

12. 30 specimens of sheet steel are tested for tensile strength, measured in kN m^{-2}. The table gives the distribution of the measurements.

Tensile strength	Number of specimens
$405-415$	4
$415-425$	3
$425-435$	6
$435-445$	10
$445-455$	5
$455-465$	2

Draw a cumulative diagram of this distribution. Estimate the median and the 10th and 90th percentiles.

[Ans: 437, 412.5, 453 kN m^{-2}]

13. The following table summarises the masses, measured to the nearest gram, of 200 animals of the same species.

Mass (g)	Frequency
$70-79$	7
$80-84$	30
$85-89$	66
$90-94$	57
$95-99$	27
$100-109$	13

(a) Calculate estimates of median and upper quartile of the distribution
(b) Estimate the number of animals whose actual masses are less than 81 g.
(c) Calculate the mean and standard deviation of the distribution

[Ans: (a) 89.27 g, $Q_3 = 93.62$ g (b) 16 (c) 89.725 g, 6.52 g]

14. A train travelled from one station to another. The times, to the nearest minute, it took for the journey were recorded over a certain period. The times are shown in the table.

Time for journey (minutes)	Frequency
$15-16$	5
$17-18$	10
$19-20$	35
$21-22$	10

Calculate the 5% and 95% interpercentile range.

[Ans: 6.2]

15. The weights of a random sample of a variety of apples, in grams, is summarized in the table below.

Weight (grams)	Frequency
$120 \le w < 140$	6
$140 \le w < 150$	16
$150 \le w < 160$	30
$160 \le w < 170$	36
$170 \le w < 180$	30
$180 \le w < 190$	0
$190 \le w < 220$	1

(a) Calculate the mean and standard deviation of these weights
(b) Calculate the median and quartiles of these weights

[Ans: (a) 161, 13 (b) $Q_1 = 152.6$, $Q_2 = 162.2$, $Q_3 = 170.4$]

INDEX NUMBERS

Index numbers provide a standardised way of comparing the values, over time, of commodities such as prices, volume of output and wages. They are used extensively, in various forms, in Business, Commerce and Government.

An **index number** measures the percentage change in the value of some economic commodity over a period of time.
It is always expressed in terms of a base of 100
Examples of typical index number values are:
125 (an increase of 25%), 90 (a decrease of 10%), 300 (an increase of 200%)

Simple index number construction
Suppose that the price of a kilogram of sugar was 3000 in January and 3600 in April. We can calculate as follows

$$\text{Percentage increase} = \frac{3600-3000}{3000} \times 100 = 20$$

In other words, the price of a kilogram of sugar rose by 20% from January to April.
To put this into index number form, the 20% increase is added to the base of 100, giving 120. This is then described as follows:
"The price index of a kilogram of sugar in April was 120 (January = 100)"
Note that any increase must always be related to sometime related, otherwise it is meaningless. Index numbers are no exception, hence the (January = 100) in the above statement, which:
(i) gives the starting point (January) over which the increase in price is being measured
(ii) emphasises the base value (100) of the index number

Some notation
Prices and quantities (since they are commonly quoted indices) have their own special letters, p and q respectively. In order to bring in the idea of time, the following standard convention is used.
Index number notation

$$p_0 = \text{price at base time point}$$
$$p_1 = \text{price at some other time point}$$
$$q_0 = \text{quantity at base time point}$$
$$q_1 = \text{quantity at some other point}$$

In the example above, time point 0 was January and time point 1 was April, i.e. $p_0 = 3000$ and $p_1 = 3600$.
Note:
It is also convenient on occasions to label index numbers themselves in a compact way. There is no standard form for this but an example is given below

$$I_{1985/1983} = 97$$

which is translated as
"*the index for 1985, based on 1983(as 100), is 97*"

Index relatives
An index relative (sometimes just called a relative) is the name given to an index number which measures the change in a single distinct commodity.
The following shows the method of calculating a price and quantity relative

$$\text{Price relative: } I_P = \frac{p_1}{p_0} \times 100$$
$$\text{Quantity relative: } I_Q = \frac{q_1}{q_0} \times 100$$

Example 1
The table below gives details of prices and quantities sold of two particular items in a department store over two years.

Item	1984		1985	
	Price (shs)	Number sold	Price (shs)	Number sold
A	438	37	462	18
B	322	26	384	45

Calculate the price and quantity relatives for 1985 (1984 = 100) for both items:

Solution

Base year = 1984, Current year = 1985
For commodity A:

$$\text{Price relative: } I_P = \frac{p_1}{p_0} \times 100 = \frac{462}{438} \times 100 = 105.5$$
$$\text{Quantity relative: } I_Q = \frac{q_1}{q_0} \times 100 = \frac{18}{37} \times 100 = 48.6$$

For commodity B:

$$\text{Price relative: } I_P = \frac{p_1}{p_0} \times 100 = \frac{384}{322} \times 100 = 119.3$$
$$\text{Quantity relative: } I_Q = \frac{q_1}{q_0} \times 100 = \frac{45}{26} \times 100 = 173.1$$

Example 2
The wage of nurses in Uganda in 1995 was shs 20,000. The wage of the same nurses in 1997 was increased by shs 25000. Using 1995 as the base year, calculate the nurses' wage index for 1997.

Solution

$$W_1 = 20{,}000 + 25{,}000 = 45{,}000$$
$$W_0 = 20{,}000$$
$$\text{Wage index} = \frac{W_1}{W_0} \times 100 = \frac{45000}{20000} \times 100 = 225$$

Therefore, the nurses wage increased by 125% in 1995

Simple aggregate price index
This is a simple method for constructing index numbers. In this, the total of current year prices for various commodities is divided by the corresponding base year total and multiplying the result by 100

Simple aggregate price index = $\frac{\sum p_1}{\sum p_0} \times 100$

where
$\sum p_1$ = the sum of commodity prices in the current year
$\sum p_0$ = the sum of commodity prices in the base year

Example 3

Calculate the price index number for 2003, taking the year 2000 as the base year

Commodity	Price in the year 2000	Price in the year 2003
A	600	800
B	500	600
C	700	1000
D	1200	1600
E	1000	1500

Solution

$$S.A.P.I = \frac{\sum p_1}{\sum p_0} \times 100 = \frac{800+600+1000+1600+1500}{600+500+700+1200+1000} \times 100$$

$$= \frac{5500}{4000} \times 100 = 137.5$$

This shows that there is an increase of 37.5% in the prices of the commodities

Example 4

Data chip manufactures and sells three computer chip models; the basic, financial and scientific. The respective retail prices are 950, 3500 and 7000 in 1994; 150, 1800 and 2500 in 1998; 80, 600 and 1250 in 2002. Calculate the simple aggregate price index for 1998 and 2002 taking 1994 as the base year.

Solution

Chip Model	Retail price		
	1994	1998	2002
Basic	950	150	80
Financial	3500	1800	600
Scientific	7000	2500	1250
$\sum$	11450	4450	1930

Simple aggregate price index for 1998

$$= \frac{4450}{11450} \times 100 = 38.86$$

Simple aggregate price index for 2002

$$= \frac{1930}{11450} \times 100 = 16.86$$

Conclusion:

Since the index in the base period, 1994 is 100, the difference in the indices for 1994 and 1998 indicates that the average price of the three models is declined by;

$$100 - 38.86 = 61.14\%$$

The decline in price from 1994 to 2002 is $100 - 16.86 = 83.14\%$

Simple average price index

This price index is constructed giving equal weights to all commodities and it is called the simple price index number. In this case, all the commodities are given equal importance. The average price is calculated simply by dividing the sum of prices by the number of commodities.

$$\textbf{Simple average price index} = \frac{\sum \frac{p_1}{p_0} \times 100}{n}$$

Example 5

Construct by simple average of price relative method the price index of 2004, taking 1999 as the base year from the following data

Item	A	B	C	D	E	F
Price (in 1999)	6000	5000	6000	5000	2500	2000
Price (in 2004)	8000	6000	7200	7500	3750	3000

Solution

Item	Price in 1999 (p_0)	Price in 2004 (p_1)	Price relatives $\left(\frac{p_1}{p_0} \times 100\right)$
A	6000	8000	133.33
B	5000	6000	120.00
C	6000	7200	120.00
D	5000	7500	150.00
E	2500	3750	150.00
F	2000	3000	150.00
			823.33

$$\text{Simple price index} = \frac{\sum \frac{p_1}{p_0} \times 100}{n} = \frac{823.33}{6} = 137.22$$

Example 6

Find the simple price index for 2001 taking 1996 as the base year from the following data

Commodity	Wheat	Rice	Sugar	Ghee	Meat
Price (1996)	1200	2000	1200	4000	8000
Price (2001)	1600	2500	1600	6000	9600

Solution

Commodity	Price in 1996 (p_0)	Price in 2001(p_1)	Price relatives
Wheat	1200	1600	133.33
Rice	2000	2500	125.00
Sugar	1200	1600	133.33
Ghee	4000	6000	150.00
Meat	8000	9600	120.00
$\sum$			661.66

$$\text{Simple price index} = \frac{\sum \frac{p_1}{p_0} \times 100}{n} = \frac{661.66}{6} = 132.33$$

Example 7

The table below shows the average price in shillings of a kilogram of sugar during the years 1983 – 1988

Year	1983	1984	1985	1986	1987	1989
Price	110	120	130	150	165	185

(a) Using 1983 as the base, find the price index corresponding to all years. By how much would a family have reduced their consumption of sugar in 1988 if they had to spend the same amount of money as they did in 1983?

(b) Using 1986 as the base, find the retail price index for the given years

Example 19

The table below shows the prices for six items and the number of units of each consumed by a typical family in 1995 and 2005.

Item	1995		2005	
	Price	Quantity	Price	Quantity
Bread (loaf)	770	50	1980	55
Eggs(dozen)	1850	26	2980	20
Milk (litre)	880	102	1980	130
Apples (500 g)	1460	30	1750	40
Juice (300ml)	1580	40	1700	41
Coffee (400 g)	4400	12	4750	12

Determine the weighted price index and interpret the result

Solution

Using the Laspeyres calculation

Item	p_0	q_0	p_1	q_1	q_0p_1	q_0p_0
Bread	770	50	1980	55	99000	38500
Eggs	1850	26	2980	20	77480	48100
Milk	880	102	1980	130	201960	89760
Apples	1460	30	1750	40	52500	43800
Juice	1580	40	1700	41	68000	63200
Coffee	4400	12	4750	12	57000	52800
$\sum$					555940	336160

Weighted Price index = $\frac{\sum q_0p_1}{\sum q_0p_1} \times 100$

$= \frac{555940}{336160} \times 100 = 165.4$

This result shows an increase of 65.4% in the prices of the items between 1995 and 2005

Using the Paasche calculation:

Item	p_0	q_0	p_1	q_1	q_1p_1	q_1p_0
Bread	770	50	1980	55	108900	42350
Eggs	1850	26	2980	20	59600	37000
Milk	880	102	1980	130	257400	114400
Apples	1460	30	1750	40	70000	58400
Juice	1580	40	1700	41	69700	64780
Coffee	4400	12	4750	12	57000	52800
$\sum$					622600	369730

Weighted price index = $\frac{\sum P_1q_1}{\sum P_0q_1} \times 100$

$= \frac{622600}{369730} \times 100 = 168.4$

This result indicates that there has been an increase of 68.4% in the prices of the items between 1995 and 2005

Value index

The value index number compares the value of a commodity in the current year, with its value in the base year.

The value of the commodity is the price of the commodity and the quantity. So the value index number is the sum of the value of the commodity of the current year divided by the sum of its value in the chosen base year. The formula is as follows:

$$\textbf{Value index} = \frac{\sum q_1p_1}{\sum q_0p_0} \times 100$$

Example 20

We can calculate the value index number of the items from example 16 as follows;

$\sum q_1p_1 = 622600$ and $\sum q_0p_0 = 336160$

Value index $= \frac{\sum q_1p_1}{\sum q_0p_0} \times 100 = \frac{622600}{336160} \times 100 = 185.2$

Example 21

The table below shows some selected items consumed by a certain low income earner in 1990 and 1995.

	1990		1995	
Item	Price (shs)	Qty (kg)	Price (shs)	Qty (kg)
Rice	700	20	700	30
Millet	1500	10	1600	10
Beans	150	5	200	7

Calculate the value index for the year 1995 using 1990 as the base.

Solution

Item	p_0	q_0	p_1	q_1	p_0q_0	p_1q_1
Rice	700	20	700	30	14000	21000
Millet	1500	10	1600	10	15000	16000
Beans	150	5	200	70	750	14000
					29750	51000

Value index $= \frac{\sum q_1p_1}{\sum q_0p_0} \times 100 = \frac{51000}{29750} \times 100 = 171.4$

Some uses of index numbers

Index numbers are used to reflect general economic conditions over a period of time, for example, the retail price index measures changes in the cost of living.

In particular, they can be used by Government to decide on tax changes, subsides to industries or regions or national retirement pension increases.

Trade unions often use national cost of living across national boundaries, regions or professions.

Insurance companies use various cost indices to index-link house (building or contents) policies.

28. Calculate the cost of living index number from the following data.

Groups	Index number (I)	Weight (W)
Food	360	60
Clothing	295	5
Fuel and Light	287	7
House rent	110	8
Miscellaneous	315	20

[Ans: 322.64]

29. From the following data, calculate the composite index price number for the groups combined.

Group	Weight	Index Number
Food articles	50	241
Liquor and tobacco	2	221
Fuel, power, etc.	3	204
Industrial raw materials	16	256
Manufactured items	29	179

[Ans: 223.91]

30. The average price of mustard oil per litre in the year 1984 to 1988 are given below:

Year	1984	1985	1986	1987	1988
Price	295	275	300	225	250

[Ans: 98.3, 91.7, 100, 75, 83.3]

31. Find the cost of living index number from the following data.

Commodity	Base Price (£)	Current Price (£)
Rice	35	42
Wheat	30	35
Maize	40	38
Fish	105	120

[Ans: 111.5]

32. The table below gives the average wholesale price in Chinese Yen (¥) for three commodities during the years $1985 - 90$.

Commodity / Price (¥)	A	B	C
1985	25.3	17.3	7.8
1986	30.8	14.5	5.4
1987	33.4	4.9	6.7
1988	35.5	5.7	5.6
1989	35.3	17.1	7.2
1990	36.0	11.6	10.2

Construct the index numbers for all the years using price relative method taking 1985 as the base year.

[Ans: 100, 92, 82, 82, 111, 113]

33. The price relatives and weights of a set of commodities are given in the following table

Commodity	A	B	C	D
Price relative	120	127	125	119
Weight	$2w_1$	w_2	w_1	$w_2 + 3$

If the index for the set is 122 and the sum of the weights is 40, find w_1 and w_2.

[Ans: 7, 8]

34. Find the weighted price index number from the following data:

Commodity	Weight	Current price	Base price
Cloth	13	250	225
Wheat	18	26	22
Rice	25	32	26
Potato	8	65	70

[Ans: 115.5]

35. Given below are the data on prices of some consumer goods and the weights attached to various items.

Items	Unit	Price		Weight
		1988	1989	
Wheat	kg	0.50	0.75	2
Milk	litre	0.60	0.75	3
Egg	dozen	2.00	2.40	4
Sugar	kg	1.80	2.10	8
Shoes	pair	8.00	10.00	1

Compute price index numbers for the year 1989 (Base $1988 = 100$), using

(i) simple average,

(ii) weighted average of price relatives.

[Ans: (i) 127.4 (ii) 123.3]

36. Find the weighted index number, using the following data:

Items	Index	Weight
Food	152	48
Clothing	110	5
Rent	130	10
Fuel and lighting	100	12
Miscellaneous	80	15

[Ans: 128.29]

37. Calculate the Laspeyres' and Paasche's index numbers from the following data:

Items	Base year		Current year	
	Price (£)	Quantity	Price (£)	Quantity
Rice	40	6.0	30	7.0
Meat	90	4.0	50	5.0
Tea	90	0.5	40	1.5

[Ans: Laspeyres' index $= 62.02$, Paasche's index $= 60.12$]

38. From the following table, calculate the Paasche's quantity index number for 1989 with 1971 as base:

Items	Quantity		Value
	1971	1989	1989
A	54	250	540
B	93	75	825
C	18	56	448
D	6	8	56
E	23	47	141

[Ans: 144.12]

REGRESSION AND CORRELATION

Regression analysis
Regression analysis is a mathematical measure of the average relationship between two or more variables in terms of the original units of data.
In regression analysis there are two types of variables. The variable whose value is influenced or is to be predicted is called dependent variable and the value which influences the values or is used for prediction, is called independent variable. In regression analysis, independent variable is also known as regressor or predictor or explanatory variable while the dependent variable is also known as regressed or explained variable.

Line of regression
If the variables in a bivariate distribution are related, we will find that the points in the scatter diagram will cluster around a line called a line of regression.
The line of regression is the line which gives the best estimate to the value of one variable for any specific value of the other variable. Thus the line of regression is the line o "**best fit**" and is obtained by the principles of least squares.
Let us suppose that in the bivariate distribution (x_i, y_i); i =1, 2, … ,n. y is depended variable and x is independent variable. Let the line of regression of y on x be

$$y = a + bx$$

The regression line passes through the point $(\bar{x}, \bar{y})$ where

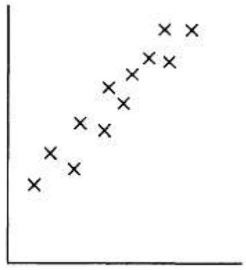
$$\bar{x} = \frac{\sum x}{n} \text{ and } \bar{y} = \frac{\sum y}{n}$$

Finding the equation of the line of best fit
The equation of the line of best fit can be obtained in the similar way we obtain an equation of a line.
Since the line of best fit is generally a line that passes through the point $(\bar{x}, \bar{y})$, then $\bar{y} = a + b\bar{x}$ gives an equation for the line.
To find the constants a and b, we choose any other point (x_1, y_1) on the line such that $y_1 = a + bx_1$.
Thus the equation of the line of best fit can be obtained from the above two equations.
Alternatively, one might choose any two points (x_1, y_1) and (x_2, y_2) lying on the line of best fit to obtain its gradient then its equation. Thus calculating the mean point $(\bar{x}, \bar{y})$ is not a must.

Note:
We can estimate the value of y for a given value of x or a value of x for a given value of y by substituting in the equation of the line of best fit.
If we have not been asked to determine the equation of the line of best fit, then we can estimate the required values from the graph directly.
If you don't wish to calculate the mean point $(\bar{x}, \bar{y})$, then you can predict by use of eyes where the line of best fit can pass.

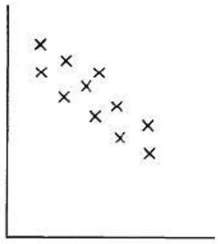
Scatter diagrams and correlation
If a change in one variable is matched by a similar proportional change in another variable, than the technique used to measure the degree of association is called correlation.
Relationships can sometimes be shown to exist between the most obscure of variates. Justification is perhaps based only on the fact that both variates change with respect to time in a similar manner.
Before quantifying the interdependence of two variables, it is often useful to plot the paired observations on a scatter diagram. By inspection if a straight line can be drawn to fit the data reasonably well, then there exists a strong linear correlation between the variables.
The three types of correction can be described below

(a) Positive correlation
Both variables increase together



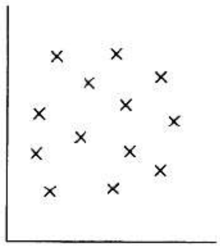
The points lie close to a straight line which has a positive gradient.

(b) Negative correlation
As one variable increases, the other decreases.



The points lie close to a straight line, which has a negative gradient.

(c) Zero correlation/ No correlation
The is no connection between the two variables.



There is no pattern to the points

$$\sum d^2 = 9$$

$$\rho = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6(9)}{8(63)} = 0.893$$

Example 5

The table below shows scores by 10 students (A to J) in Physics and Mathematics tests

Student	Mathematics (x)	Physics (y)
A	28	30
B	20	20
C	40	40
D	28	28
E	21	22
F	31	35
G	36	35
H	29	27
I	33	31
J	24	23

Calculate the rank correlation coefficient for the data and comment on your result.

Solution

Student	R_x	R_y	d	d^2
A	6.5	5	1.5	2.25
B	10	10	0	0
C	1	1	0	0
D	6.5	6	0.5	0.25
E	9	9	0	0
F	4	2.5	1.5	2.25
G	2	2.5	-0.5	0.25
H	5	7	-2	4
I	3	4	-1	1
J	8	8	0	0
			$\sum$	10

$$\rho = 1 - \frac{6\sum d^2}{n(n^2-1)}$$
$$= 1 - \frac{6(10)}{10(10^2-1)}$$
$$= 1 - \frac{60}{990}$$
$$= 0.939$$

There is a very high/strong positive correlation between the two scores.

Example 6

Nine gymnasts performed in a gymnastic competition. Their names were Arnold (A), Brian (B), Christian (C), Denis (D), Enock (E), Fabian (F), Gordon (G), Harry (H) and Ian (I).

Rank	1	2	3	4	5	6	7	8	9
Judge 1	D	C	E	B	F	A	I	H	G
Judge 2	D	E	F	C	I	B	A	G	H

(a) Calculate Spearman's rank correlation coefficient for this data

(b) Test whether or not the judges are generally in agreement, at the 1% level of significance.

Solution

(a) Rewrite the table in a more user-friendly form

Gymnast	Judge 1 rank	Judge 2 rank	d	d^2
A	6	7	-1	1
B	4	6	-2	4
C	2	4	-2	4
D	1	1	0	0
E	3	2	1	1
F	5	3	2	4
G	9	8	1	1
H	8	9	-1	1
I	7	5	2	4
			$\sum$	20

$$\rho = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6(20)}{9(80)} = 0.833$$

(b) The critical value for $n = 9$, at 1% significance is 0.83
As $0.833 > 0.83$, there is evidence that the judges are in general agreement.

Example 7

The table below shows the heights to the nearest cm and masses to the nearest kg of 10 students, A to J

Student	Mass (kg)	Height (cm)
A	53	148
B	68	172
C	57	156
D	66	139
E	64	163
F	63	158
G	64	168
H	58	151
I	57	144
J	68	170

Calculate the rank correlation coefficient and comment on your result.

Solution

Student	R_M	R_H	d	d^2
A	10	8	-2	4
B	1.5	1	0.5	0.25
C	7.5	6	1.5	2.25
D	9	10	-1	1
E	3	4	-1	1
F	4	5	-1	1
G	5	3	2	4
H	6	7	-1	1
I	7.5	9	-1.5	2.25
J	1.5	2	-0.5	0.25
			$\sum$	17

$$\rho = 1 - \frac{6\sum d^2}{n(n^2-1)}$$
$$= 1 - \frac{6(17)}{10(10^2-1)}$$

PROBABILITY THEORY

Probability (or chance) is a way of describing the likelihood of different possible outcomes occurring as a result of some experiment.

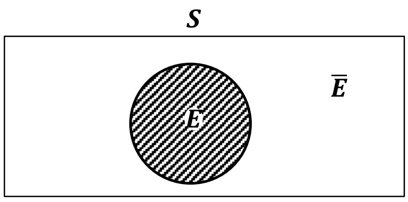
Terminologies used

An experiment has a finite number of outcomes, called the outcome set S.

An event E of an experiment is defined to be a subset of the outcome set S.

The complement of E, E', is the subset of S where E does not occur.

Venn diagram showing outcome set S and its subsets E and $\bar{E}$



Two events of the same experiment are mutually exclusive if they cannot occur simultaneously.

Two events are independent if the occurrence of one has no effect on the occurrence of the other.

For example, if a die is thrown once, then two different scores, e.g. 2 and 3, cannot occur simultaneously, so they are mutually exclusive events. If the die is thrown again, the second score is independent of the first.

If an experiment has $n(S)$ equally likely outcomes and $n(E)$ of them are the event E, then the theoretical probability of event E occurring is

$$P(E) = \frac{n(E)}{n(S)}$$

Note: $0 \le P(E) \le 1$

If the outcomes S has only n different possible events E_1, E_2, ….., E_n, then

$$P(E_1) + P(E_2) + \cdots + P(E_n) = \sum_{i=1}^{n} P(E_i) = 1$$

and $P(E') = 1 - P(E)$

Probabilities of 0 and 1

The two extremes of probability are certainty at one end of the scale and impossibility at the other. Here are examples of certain and impossible events.

Experiment	Certain event	Impossible event
Rolling a single die	The result is in the range 1 to 6 inclusive	The result is a 7
Tossing a coin	Getting either heads or tails	Getting neither heads nor tails

Certainty

As you can see from the table above, for events that are certain, the number of ways that the event can occur, $n(E)$ in the formula, is equal to the number of possible events, $n(S)$.

$$\frac{n(E)}{n(S)} = 1$$

So the probability of an event which is certain is one.

Impossibility

For impossible events, the number of ways that the event can occur, $n(E)$, is zero.

$$\frac{n(E)}{n(S)} = \frac{0}{n(S)} = 0$$

So the probability of an event which is impossible is zero.

Typical values of probabilities might be something like 0.3 or 0.9. If you arrive at probability values of, say, -0.4 or 1.7, you will know that you have made a mistake since these are meaningless.

Example 1

A fair die is thrown. List the possible outcomes.

What is the probability of scoring:

(a) a multiple of 3, (b) not a multiple of 3?

<u>Solution</u>

$S =$ {possible scores with a die} $= \{1, 2, 3, 4, 5, 6\}$

(a) $E =$ {multiple of 3 scores} $= \{3, 6\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

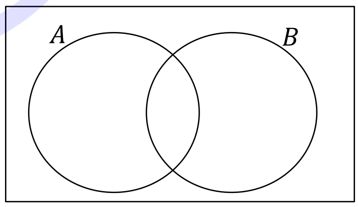
(b) $P(E') =$ {not multiple of 3 scores}

$$P(E') = 1 - P(E) = 1 - \frac{1}{3} = \frac{2}{3}$$

Intersection of events

If A and B are two events of the sample space S, then the intersection of events is denoted as $A \cap B$ and contains sample points common to both A and B.

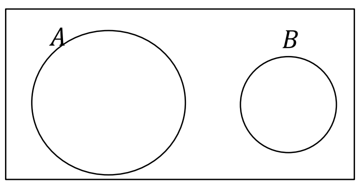
Addition rule



If A and B are two events of the same experiment, then the probability of A or B or both occurring is $P(A \text{ or } B)$ or $P(A \cup B)$ given by

$$P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If A and B are mutually exclusive, then $P(A \cap B) = 0$



So $P(A \cup B) = P(A) + P(B)$

If A and B are independent events, then

$$P(A \cap B) = P(A) \cdot P(B)$$

So $P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$

Conditional probability

If A and B are two events (not necessarily from the same experiment), then the conditional probability that A will occur given that B has occurred is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

If A and B are mutually exclusive, then $P(A|B) = 0$

Two events A and B are independent, if

$$P(A) = P(A|B) \text{ and } P(B) = P(B|A)$$

Multiplication rule

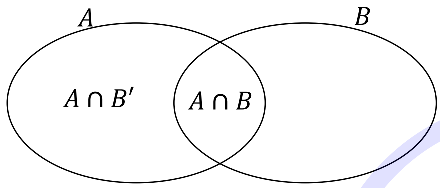
If A and B are any two events, then the probability that both A and B occur is

$$\begin{aligned} P(A \cap B) &= P(A) \times P(A|B) \\ &= P(B) \times P(B|A) \end{aligned}$$

Interaction with the set theory

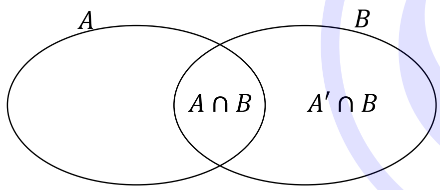
The following results can be deduced from the set theory for any two events A and B

(a) Result 1



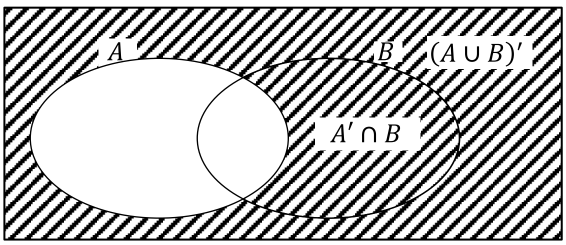
$$P(A) = P(A \cap B) + P(A \cap B')$$

(b) Result 2



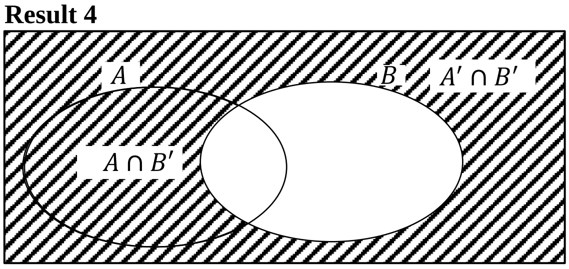
$$P(B) = P(A \cap B) + P(A' \cap B)$$

(c) Result 3



$$P(A') = P(A' \cap B) + P(A' \cap B')$$
$$P(A' \cap B') = P(A \cup B)'$$

(d) Result 4



$$P(B') = P(A \cap B') + P(A' \cap B')$$

(e) Result 5

$$P(A' \cup B') = P(A \cap B)'$$

The contingency table

The alternative way of recalling the results is by using the contingency table as shown below.

Event	**Event**		**Total**
	$\boldsymbol{B}$	$\boldsymbol{B'}$	
$\boldsymbol{A}$	$P(A \cap B)$	$P(A \cap B')$	$\boldsymbol{P(A)}$
$\boldsymbol{A'}$	$P(A' \cap B)$	$P(A' \cap B')$	$\boldsymbol{P(A')}$
Total	$\boldsymbol{P(B)}$	$\boldsymbol{P(B')}$	**1**

$$P(A) = P(A \cap B) + P(A \cap B')$$
$$P(A') = P(A' \cap B) + P(A' \cap B')$$
$$P(B) = P(A \cap B) + P(A' \cap B)$$
$$P(B') = P(A \cap B') + P(A' \cap B')$$
$$1 = P(A) + P(A')$$
$$1 = P(B) + P(B')$$

Example 2

The events A and B are such that $P(A) = 0.43$, $P(B) = 0.48$, $P(A \cup B) = 0.78$. Show that the events A and B are neither mutually exclusive nor independent.

<u>Solution</u>

A and B are mutually exclusive if $P(A \cup B) = P(A) + P(B)$

$$P(A \cup B) = 0.78$$
$$P(A) + P(B) = 0.43 + 0.48 = 0.91$$
$$\neq P(A \cup B)$$

$\therefore$ A and B are not mutually exclusive events, in which case

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= 0.91 - 0.78 = 0.13 \end{aligned}$$

If A and B are independent events, then $P(A \cap B) = P(A) \times P(B)$

$$P(A) \times P(B) = 0.43 \times 0.48 = 0.2064$$
$$\neq P(A \cap B)$$

A and B are neither mutually exclusive nor independent

Example 3

A bag contains 10 red balls, 9 blue balls and 5 white balls. Three balls are taken from the bag at random without replacement.

(a) Find the probability that all the three balls are of the same colour

(b) Find also the probability that all the three balls are of different colours

<u>Solution</u>

A and B are neither mutually exclusive nor independent

(a) With an obvious notation we have,

$10R$, $9B$, $5W$, a total of 24 balls

3 balls are drawn at random without replacement.

P(3 balls of the same colour)

$$\begin{aligned} &= P(R_1 \cap R_2 \cap R_3) + P(B_1 \cap B_2 \cap B_3) \\ &\quad + P(W_1 \cap W_2 \cap W_3) \\ &= \frac{10}{24} \times \frac{9}{23} \times \frac{8}{22} + \frac{9}{24} \times \frac{8}{23} \times \frac{7}{22} + \frac{5}{24} \times \frac{4}{23} \times \frac{3}{22} \\ &= \frac{107}{1012} \end{aligned}$$

(b) P(balls all of different colour)

$$\begin{aligned} &P(R_1 \cap W_2 \cap B_3) + P(R_1 \cap B_2 \cap W_3) \\ &\quad + P(W_1 \cap R_2 \cap B_3) + P(W_1 \cap B_2 \cap R_3) \\ &\quad + P(B_1 \cap R_2 \cap W_3) + P(B_1 \cap W_2 \cap R_3) \end{aligned}$$

$= P(E_1)P(\overline{E_2})P(\overline{E_3}) + P(\overline{E_1})P(E_2)P(\overline{E_3})$
$+ P(\overline{E_1})P(\overline{E_2})P(E_3)$

(Since E_1, E_2 and E_3 are independent)

$= 0.5 \times 0.4 \times 0.2 + 0.5 \times 0.6 \times 0.2$
$+ 0.5 \times 0.4 \times 0.8$
$= 0.26$

(b) At least two hits can be registered in the following mutually exclusive ways:
(i) $E_1 \cap E_2 \cap \overline{E_3}$ happens, (ii) $E_1 \cap \overline{E_2} \cap E_3$ happens, (iii) $\overline{E_1} \cap E_2 \cap E_3$ happens (iv) $E_1 \cap E_2 \cap E_3$ happens

Required probability
$= P(E_1 \cap E_2 \cap \overline{E_3}) + P(E_1 \cap \overline{E_2} \cap E_3)$
$+ P(\overline{E_1} \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$
$= 0.5 \times 0.6 \times 0.2 + 0.5 \times 0.4 \times 0.8 + 0.5 \times 0.6 \times 0.8$
$+ 0.5 \times 0.6 \times 0.8$
$= 0.06 + 0.16 + 0.24 + 0.24 = 0.70$

Example 14
Two dice are thrown. The scores on the dice are added.
(a) What is the probability of a score of 4?
(b) What is the most likely outcome?

Solution

		First die					
		1	**2**	**3**	**4**	**5**	**6**
Second die	**1**	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

(a) Number of events with a sum of $4 = 3$
Total number of events $= 36$
Probability of a score of $4 = \frac{3}{36} = \frac{1}{12}$
(b) The most likely outcome is a score of 7

Problems involving an 'at least' situation
$P($ happening of at least one of the events)
$= 1 - P($none of the events happens)
or equivalently,
$P($none of the given events happens)
$= 1 - P($at least one of them happens)

Example 15
(a) Find the probability of obtaining at least one six when five dice are thrown
(b) Find the probability of obtaining at least one six when n dice are thrown
(c) How many dice must be thrown so that the probability of obtaining at least one six is at least 0.99?

Solution

(a) In one throw, $P(6) = \frac{1}{6}$ and $P(\text{not } 6) = \frac{5}{6}$
When five dice are thrown,
$P(\text{at least one six}) = 1 - P(\text{no sixes})$
$= 1 - \left(\frac{5}{6}\right)^5$
$= 0.598$

(b) When n dice are thrown,
$P(\text{at least one six}) = 1 - \left(\frac{5}{6}\right)^n$

(c) You need to find n such that

$$1 - \left(\frac{5}{6}\right)^n \geq 0.99$$

$$\left(\frac{5}{6}\right)^n \leq 0.01$$

Take logs to the base 10 of both sides

$$n \log\left(\frac{5}{6}\right) \leq \log 0.01$$

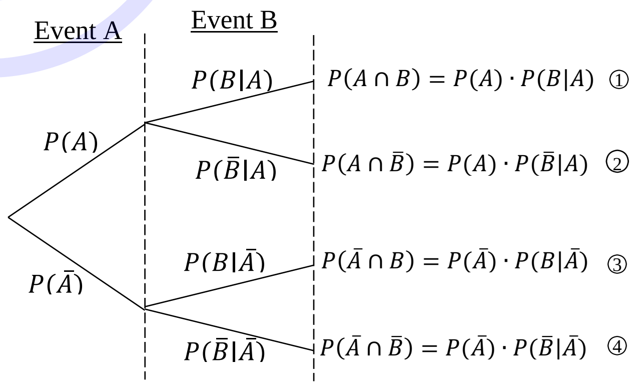
Divide both sides by $\log\left(\frac{5}{6}\right)$. Since $\log\left(\frac{5}{6}\right)$ is negative, this will reverse the inequality sign.

$$n \geq \frac{\log 0.01}{\log\left(\frac{5}{6}\right)}$$

$$n \geq 25.3$$

The least value of n is 26

Probability tree diagrams
Tree diagrams are useful for organizing and visualizing the different possible outcomes of a sequence of events. For each possible outcome of the first event, we draw a line where we write down the outcome and state of the world if that outcome happened. Then, for each possible outcome of the second event, we do the same thing.
Tree diagrams are very helpful for analyzing dependent events. A tree diagram allows you to show how each possible outcome of one event affects the probability of the other events.
Tree diagrams are not so useful for independent events since we can multiply the probabilities of separate events to get the probability of the combined event.



Remember:
(a) The total probability for any one set of 'branches' $= 1$
(b) The sum of final probabilities (intersections) $= 1$
(c) The tree shows conditional probabilities for $(B|A)$ etc.
If $P(A|B)$ is required, then:
(i) Look in the final column for the intersections containing B; in this case $P(B) = 1 + 3$
(ii) Find the term giving $A \cap B$, in this case 1,

$$\text{Then } P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{1+3}$$

Easier to work: P(both the same type)

$$= P_1P_2 + M_1M_2 + W_1W_2$$

$$= \frac{6}{15}\times\frac{4}{12}+\frac{4}{15}\times\frac{3}{12}+\frac{5}{13}\times\frac{5}{12}$$

$$= \frac{24+12+25}{180} = \frac{61}{180}$$

$\therefore P$(both different) $= 1-\frac{61}{180} = \frac{119}{180}$

Self-Evaluation exercise

1. The events A and B are such that $P(A\cap B') = 0.25$, $P(A) = 2P(B)$ and $P(A\cup B) = 0.45$. Determine
   (a) $P(A\cap B)$
   (b) $P(A\cup B')$
   [Ans: (a) 0.15 (b) 0.95]
2. The chances of winning of two race-horses are $1/3$ and $1/6$ respectively. What is the probability that at least one will win when the horses are running
   (a) in different races, and
   (b) in the same race?
   [Ans: (a) $8/18$ (b) $1/2$]
3. A bag contains just 10 balls, of which 5 are red and 5 are black. One ball is drawn at random from the bag and replaced; a second ball is drawn at random and replaced and then a third ball is drawn at random. By means of a tree diagram, or otherwise, show that the probability of drawing 2 black balls and one red ball is $\frac{3}{8}$.
   Event A is that the 3 balls drawn include at least 1 red ball and at least 1 black ball. Event B is that the 3 balls drawn include at least 2 black balls. Find $P(A)$ and $P(B)$ and show that the events A and B are independent.
   Given that event C is that the first 2 balls drawn are of the same colour, ascertain whether events B and C are mutually exclusive.
   [Ans: $P(A) = \frac{3}{4}$; $P(B) = \frac{1}{2}$; independent; not mutually exclusive]
4. A bag contains blue, yellow and red discs and these show on their face a single whole number. The probability of drawing of a blue disc is $\frac{1}{2}$, the probability of drawing a yellow disc is $\frac{1}{3}$ and the probability of drawing a red disc is $\frac{1}{6}$. $\frac{5}{12}$ of the blue discs show an even number, $\frac{5}{8}$ of the yellow discs show an even number and $\frac{1}{4}$ of the red discs show an even number. A disc is drawn at random from the bag.
(a) Determine the probability that the disc will show an even number
(b) Given that the disc that was drawn was showing an even number, find the probability that the disc was not red
   [Ans: (a) $\frac{11}{24}$ (b) $\frac{10}{11}$]
5. The events E and F are such that $P(E) = 0.5$, $P(F|E) = 0.6$ and $P(E'\cap F') = 0.4$. Determine the value of
   (a) $P(E\cap F)$
   (b) $P(E|F)$
   (c) $P(F'|E')$
   [Ans: (a) 0.3 (b) 0.75 (c) 0.8]
6. The events A and B are such that $P(A) = P(B) = p$ and $P(A\cup B) = 0.84$. Given that A and B are independent events, determine the value of p.
   [Ans: $p = 0.6$]
7. Mass-produced ceramic tiles are inspected for defects. The probability that a tile has air bubbles is 0.0015. If a tile has air bubbles the probability that it is also cracked is 0.55 while the probability that a tile free from air bubbles is cracked is 0.0055.
   (a) What is the probability that a tile selected at random is cracked?
   The probability that a tile is discoloured is 0.0065. Given that discolouration is independent of the other two defects,
   (b) find the probability that a tile selected at random has no defects.
   [Ans: (a) 0.00632 (b) 0.987]
8. The events A and B satisfy $P(A) = x$, $P(B) = y$, $P(A\cup B) = 0.6$ and $P(B|A) = 0.2$.
   (a) Show clearly that $4x + 5y = 3$
   (b) The events B and C are mutually such that $P(B\cup C) = 0.9$ and $P(C) = x + y$. Find the value of x and y
   (c) Show that A and B are independent events
   [Ans: (b) $x = 0.5$, $y = 0.2$]
9. The events A and B are such that $P(A) = \frac{1}{2}$, $P(A'|B) = \frac{1}{3}$, $P(A\cup B) = \frac{3}{5}$, where A' is the event 'A does not occur'.
   (a) Using a Venn diagram, or otherwise, determine $P(B|A')$, $P(B\cap A)$ and $P(A/B')$.
   (b) The event C is independent of A and $P(A\cap C) = \frac{1}{8}$. Determine $P(C|A')$
   (c) State, with a reason in each case, whether
      (i) A and B are independent
      (ii) A and C are mutually exclusive
   [Ans: (a) $1/5$, $1/5$, $3/7$ (b) $1/4$ (c) (i) not independent (ii) not mutually exclusive]
10. Three boxes A, B and C contain coins. Box A contains 3 gold coins, Box B contains 2 gold coins and 2 silver coins. Box C contains 4 gold coins and 1 silver coin. A box is selected at random and 2 coins are selected. Find the probability that box C was selected, if both coins selected were gold.
    [Ans: $\frac{18}{53}$]
11. At a college course 75% of the students are male and 25% are female. It is further known that 60% of the male students own a bike and 40% of the female students own a bike. A student is selected at random. Given that the student selected owns a bike, determine the probability that the student is female.
    [Ans: $\frac{2}{11}$]
12. A, B and C are independent witnesses of an event which is known to have occurred. A speaks the truth three times out of four, B four times out of five and C five times out of six. What is the probability that the

Example 4

The discrete random variable X has probability function given by

$$p(x) = \begin{cases} \left(\frac{1}{2}\right)^x & x = 1,2,3,4,5, \\ c & x = 6, \\ 0 & \text{otherwise,} \end{cases}$$

where c is a constant

Determine the value of c and hence the mode and mean of X

Solution

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	c

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + c = 1$$

$$\frac{31}{32} + c = 1$$

$$c = \frac{1}{32}$$

Mode is the value of x with the highest probability

Mode = 1, since $P(X = 1) = \frac{1}{2}$

Mean, $E(X) = 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{8}\right) + 4\left(\frac{1}{1}\right) + 5\left(\frac{1}{32}\right) + 6\left(\frac{1}{32}\right)$

$$= \frac{63}{32} = 1.97$$

Variance (Var)

The variance of a probability distribution associated with the random variable X is

$\text{Var}[X] = E[(X - \mu)^2]$ where $\mu = E[X]$

Computational formula:

$\text{Var}[X] = E[X^2] - (E[X])^2$

Properties of Var

If a and b are constants:

$\text{Var}[a] = 0$

$\text{Var}[aX] = a^2\text{Var}[X]$

$\text{Var}[aX + b] = a^2\text{Var}[X]$

Example 5

The random variable X has probability distribution

x	1	2	3	4	5
$P(X = x)$	0.10	p	0.20	q	0.30

Given that $E(X) = 3.5$, find

(a) the value of p and q

(b) $\text{Var}(X)$

(c) $\text{Var}(3 - 2X)$

Solution

(a) $\sum_{all\ x} P(X = x) = 1$

$$0.10 + p + 0.20 + q + 0.30 = 1$$

$$p + q = 0.4 \quad \ldots \text{(i)}$$

$E(X) = \sum_{all\ x} xP(X = x)$

$$0.10 + 2p + 0.60 + 4q + 1.50 = 3.5$$

$$2p + 4q = 1.3$$

$$p + 2q = 0.65 \quad \ldots \text{(ii)}$$

(ii) – (i);

$$q = 0.25$$

Using (i); $p + 0.25 = 0.4$

$$p = 0.15$$

(b) $E(X^2) = 1^2(0.10) + 2^2(0.15) + 3^2(0.20) + 4^2(0.25) + 5^2(0.30) = 14$

$\text{Var}(X) = 14 - 3.5^2 = 1.75$

(c) $\text{Var}(3 - 2X) = 4\text{Var}(X) = 7.00$

Example 6

The discrete random variable X has probability function

$$P(X = x) = \begin{cases} k(2 - x), & x = 0,1,2 \\ k(x - 2), & x = 3, \\ 0 & \text{otherwise} \end{cases}$$

where k is a positive constant

(a) Show that $k = 0.25$

(b) Find $E(X)$ and show that $E(X^2) = 2.5$

(c) Find $\text{Var}(3X - 2)$

Solution

(a)

x	0	1	2	3
$P(X = x)$	$2k$	k	0	k

$$2k + k + 0 + k = 1$$

$$4k = 1$$

$$k = 0.25$$

(b)

x	0	1	2	3
$P(X = x)$	0.5	0.25	0	0.25
$xP(X = x)$	0	0.25	0	0.75
$x^2P(X = x)$	0	0.25	0	2.75

$E(X) = \sum xP(X = x) = 0 + 0.25 + 0 + 0.75 = 1$

$$E(X^2) = 0 + 0.25 + 0 + 2.25 = 2.5$$

(c) $\text{Var}(3X - 2) = 3^2\text{Var}(X)$

$$= 9(2.3 - 1^2) = 13.5$$

Example 7

The discrete random variable X has probability function

$$P(X = x) = \begin{cases} \frac{kx}{x^2 + 1}, & x = 2, 3 \\ \frac{2kx}{x^2 - 1}, & x = 4, 5 \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that the value of k is $\frac{20}{33}$

(b) Find the probability that X is less than 3 or greater than 4

(c) Find $F(3.2)$

(d) Find (i) $E(X)$ (ii) $\text{Var}(X)$

Solution

(a) When $x = 2$, $P(X = 2) = \frac{2k}{2^2+1} = \frac{2k}{5}$

When $x = 3$, $P(X = 3) = \frac{3k}{10}$

When $x = 4$, $P(X = 4) = \frac{8k}{15}$

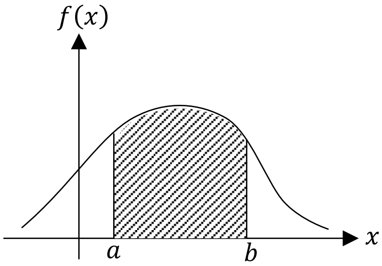
CONTINUOUS RANDOM VARIABLES

A random variable X is said to be continuous if it can take all possible values between certain limits. A continuous random variable can be measured to any desired degree of accuracy. Examples include age, height, weight, etc.

The probability density function (p.d.f) of the continuous random variable X is a function $f(x)$ such that

$$P(a \le X \le b) = \int_a^b f(x)\ dx$$

Since $P(X = a) = P(X = b) = 0$, the values $P(a < X < b)$, $P(a \le X < b)$ and $P(a < X \le b)$ are also given by this integral.
It follows that we can now think of probabilities as areas under the graph of $f(x)$.



As probabilities are never negative, $f(x) \ge 0$ for all values of x. Since the sum of the probabilities of all possible outcomes of an experiment is 1, the total area under the graph must be 1 i.e.

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

Mean and variance of a continuous random variable
For a discrete random variable X, $E(X) = \mu = \sum x\, p(x)$ and $\text{Var}(X) = \sigma^2 = \sum (x - \mu)^2\, p(x) = \sum x^2 p(x) - \mu^2$. To obtain similar formulae for the mean and variance of a continuous random variable X, let us consider the probability that the value of X lies between x and $x + \delta x$. Provided that δx is small, this probability is given approximately by $f(x)\delta x$. Thus we get

$$E(X) = \lim_{\delta x \to 0} \sum x f(x)\, \delta x = \int_{-\infty}^{\infty} x f(x)\, dx$$

Applying similar arguments to the variance, we have:
For a continuous random variable X,

$$E(X) = \mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$Var(X) = \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

Mode
A mode or modal value of a continuous random variable X with probability density function $f(x)$ is a value of x for which $f(x)$ takes a maximum value. Thus the modes of a distribution are the x-coordinates of the maximum points on the graph of $f(x)$.
Mode is thus the solution of

$$f'(x) = 0 \text{ and } f''(x) < 0$$

provided it lies in $[a, b]$

Median
Median is the point which divides the entire distribution in two equal parts. In case of a continuous distribution, median is the point which divides the total area into two equal parts. Thus if m is the median, then

$$\int_a^m f(x)\, dx = \int_m^b f(x)\, dx = \frac{1}{2}$$

Thus solving

$$\int_a^m f(x)\, dx = \frac{1}{2} \text{ or } \int_m^b f(x)\, dx = \frac{1}{2}$$

for m, we get the value of median.
If there is more than one interval, we must test for the reasonable interval where the median lies.

Quartiles and Deciles
The lower quartile Q_1 and the upper quartile Q_3 are given by the equations

$$\int_a^{Q_1} f(x)\, dx = \frac{1}{4} \text{ and } \int_a^{Q_3} f(x)\ dx = \frac{3}{4}$$

D_i, the i th decile is given by

$$\int_a^{D_i} f(x)\, dx = \frac{i}{10}$$

Example 1
Let X be a continuous random variable with p.d.f.

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ -ax + 3a, & 2 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Determine the constant a
(b) Compute $P(X \le 1.5)$

Solution

(a) Constant 'a' is determined from the consideration that total probability is unity, i.e.

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\int_0^1 ax\, dx + \int_1^2 a\, dx + \int_2^3 (-ax + 3a)\, dx = 1$$

$$a\left|\frac{x^2}{2}\right|_0^1 + a\left|x\right|_1^2 + a\left|-\frac{x^2}{2} + 3x\right|_2^3 = 1$$

$$\frac{a}{2} + a + a\left[\left(-\frac{9}{2} + 9\right) - (-2 + 6)\right] = 1$$

$$\frac{a}{2} + a + \frac{a}{2} = 1$$

$$2a = 1$$

$$a = \frac{1}{2}$$

(b) $P(X \le 1.5) = a\int_0^1 x\, dx + \int_1^{1.5} a\, dx$

$$= a\left|\frac{x^2}{2}\right|_0^1 + a\left|x\right|_1^{1.5}$$

$$= \frac{a}{2} + 0.5a$$

$$= a$$

$$= \frac{1}{2}$$

$$k\left|\frac{9x^2}{2}-\frac{x^4}{4}\right|_0^3=1$$
$$\frac{81k}{4}=1$$
$$k=\frac{4}{81}$$

(b) $E(X)=\int_0^3 xf(x)\,dx$
$$=\frac{4}{81}\int_0^3(9x^2-x^3)\,dx$$
$$=\frac{4}{81}\left|3x^3-\frac{x^5}{5}\right|_0^3$$
$$=4\left(1-\frac{3}{5}\right)$$
$$=1.6$$

(c) $f(x)=\frac{4}{81}(9x-x^3)$
$$f'(x)=\frac{4}{81}(9-3x^2)=\frac{4}{27}(3-x^2)$$
$$f'(x)=0$$
$$\frac{4}{27}(3-x^2)=0$$
$$x=\sqrt{3}=1.73$$
Hence the mode of the distribution is 1.73

(d) If m is the median, $\int_0^m f(x)\,dx=\frac{1}{2}$
$$\frac{4}{81}\int_0^m(9x-x^3)\,dx=\frac{1}{2}$$
$$\frac{4}{81}\left|\frac{9x^2}{2}-\frac{x^4}{4}\right|_0^m=\frac{1}{2}$$
$$\frac{1}{81}(18m^2-m^4)=\frac{1}{2}$$
$$36m^2-2m^4=81$$
$$2m^4-36m^2+81=0$$
$$m^2=\frac{36\pm\sqrt{36^2-4(2)(81)}}{4}$$
$$m^2=9\pm\frac{9}{2}\sqrt{2}$$
$$m^2=15.364 \text{ or } m^2=2.636$$
Since $0<m<3$,
$$m^2=2.636$$
$$m=1.62$$
Hence the median of the distribution is 1.62.

Example 6

A probability density function is defined as
$$f(x)=\begin{cases}\frac{3}{4}x(2-x) & 0\le x\le 2\\ 0 & \text{otherwise}\end{cases}$$
Find $P(|X-\mu|<\sigma)$

<u>Solution</u>

Before we can attempt the solution we must first understand what is meant by the probability statement '$P(|X-\mu|<\sigma)$'. $X-\mu$ is contained between two vertical lines which mathematically means whatever is the result of subtracting μ from X, always take it as positive. It is known as the modulus of $X-\mu$.

The statement $|X-\mu|<\sigma$ can therefore be written as
$$-\sigma<X-\mu<\sigma$$
or adding μ to each term
$$\mu-\sigma<X<\mu+\sigma$$
and so we require the area between $\mu-\sigma$ and $\mu+\sigma$.

$$\text{Mean}\quad \mu=\int_0^2\frac{3}{4}x^2(2-x)\,dx$$
$$=\frac{3}{4}\left[\frac{2x^3}{3}-\frac{x^4}{4}\right]_0^2$$
$$=\frac{3}{4}\left[\frac{16}{3}-4\right]=1$$
$$\text{Variance}\quad \sigma^2=\int_0^2\frac{3}{4}x^3(2-x)\,dx-1^2$$
$$=\frac{3}{4}\left[\frac{2x^4}{4}-\frac{x^5}{5}\right]-1$$
$$=\frac{3}{4}\left(8-\frac{32}{5}\right)-1$$
$$=\frac{1}{5}$$
Standard deviation, $\sigma=\sqrt{0.2}=0.447$
$$P(|X-\mu|<\sigma)=\int_{1-0.447}^{1+0.447}\frac{3}{4}x(2-x)\,dx$$
$$=\frac{3}{4}\left[x^2-\frac{x^3}{3}\right]_{0.553}^{1.447}$$
$$=\frac{3}{4}[1.084-0.2491]$$
$$=0.626$$

Cumulative distribution functions

Another function used when modelling continuous probability distributions is defined as follows:

The cumulative distribution (c.d.f) of the continuous random variable X is the function $F(x)$ such that
$$F(x)=P(X\le x)$$
If X has probability density function $f(x)$, then
$$F(t)=P(X\le t)=\int_{-\infty}^{t}f(x)\,dx$$
It follows that: $F'(x)=f(x)$

If M, Q_1, Q_3 are the median, lower and upper quartiles of x, then
$$F(M)=\frac{1}{2},\quad F(Q_1)=\frac{1}{4},\quad F(Q_3)=\frac{3}{4}$$

Remarks:

1. $0\le F(x)\le 1, -\infty<x<\infty$
2. From analysis, we know that
$$F'(x)=\frac{d}{dx}F(x)=f(x)\ge 0$$
$F(x)$ is non-decreasing function of x
3. $F(-\infty)=\lim_{x\to-\infty}F(x)=\lim_{x\to-\infty}\int_{-\infty}^{x}f(x)\,dx=\int_{-\infty}^{-\infty}f(x)\,dx=0$
$F(+\infty)=\lim_{x\to\infty}F(x)=\lim_{x\to-\infty}\int_{-\infty}^{x}f(x)\,dx=\int_{-\infty}^{\infty}f(x)\,dx=1$
4. $F(x)$ is a continuous function of x on the right
5. The discontinuities of $F(x)$ are at the most countable
6. It may be noted that

(b) Proceed as follows;

If $1 \le x \le 3$

$$F_1(x) = \int_1^x \frac{1}{12}(x-1)\,dx = \int_1^x \frac{1}{12}x - \frac{1}{12}dx$$
$$= \left[\frac{1}{24}x^2 - \frac{1}{12}x\right]_1^x$$
$$= \left(\frac{1}{24}x^2 - \frac{1}{12}x\right) - \left(\frac{1}{24} - \frac{1}{12}\right)$$
$$= \frac{1}{24}x^2 - \frac{1}{12}x + \frac{1}{24}$$

If $3 < x \le 6$

$$F_1(3) = \frac{1}{24}\times 3^2 - \frac{1}{12}\times 3 + \frac{1}{24} = \frac{1}{6}$$
$$F_2(x) = \frac{1}{6} + \int_3^x \frac{1}{6}dx = \frac{1}{6} + \left[\frac{1}{6}x\right]_3^x$$
$$= \frac{1}{6} + \left(\frac{1}{6}x - \frac{1}{2}\right)$$
$$= \frac{1}{6}x - \frac{1}{3}$$

If $6 < x \le 10$,

$$F_2(6) = \frac{1}{6}\times 6 - \frac{1}{3} = \frac{2}{3}$$
$$F_3(x) = \frac{2}{3} + \int_6^x \frac{5}{12} - \frac{1}{24}x\,dx = \frac{2}{3} + \left[\frac{5}{12}x - \frac{1}{48}x^2\right]_6^x$$
$$= \frac{2}{3} + \left[\left(\frac{5}{12}x - \frac{1}{48}x^2\right) - \left(\frac{5}{2} - \frac{3}{4}\right)\right]$$
$$= \frac{5}{12}x - \frac{1}{48}x^2 - \frac{13}{12}$$

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{24}x^2 - \frac{1}{12}x + \frac{1}{24} & 1 \le x \le 3 \\ \frac{1}{6}x - \frac{1}{3} & 3 < x \le 6 \\ -\frac{1}{48}x^2 + \frac{5}{12}x - \frac{13}{12} & 6 < x \le 10 \\ 1 & x > 10 \end{cases}$$

(c) From the construction of $F(x)$, it is evident that the median lies in the 'middle section'

$$F(x) = \frac{1}{2}$$
$$\frac{1}{6}x - \frac{1}{3} = \frac{1}{2}$$
$$\frac{1}{6}x = \frac{5}{6}$$
$$x = 5$$

The median is 5

(d) Using the C.D.F

$P(2 < X < 4) \cup P(5 < X < 9)$

$$= [P(X<4) - P(X<2)] + [P(X<9) - P(X<5)]$$
$$= F(4) - F(2) + F(9) - F(5)$$
$$= [F(4) + F(9)] - [F(2) + F(5)]$$
$$= \left[\left(\tfrac{1}{6}(4) - \tfrac{1}{3}\right) + \left(-\tfrac{1}{48}(9)^2 + \tfrac{5}{12}(9) - \tfrac{13}{12}\right)\right] - \left[\left(\tfrac{1}{24}(2)^2 - \tfrac{1}{12}(2) + \tfrac{1}{24}\right) + \left(\tfrac{1}{6}(5) - \tfrac{1}{3}\right)\right]$$
$$= \left(\frac{1}{3} + \frac{47}{48}\right) - \left(\frac{1}{24} + \frac{1}{2}\right)$$
$$= \frac{37}{48}$$

(e) $E(X) = \int_a^b xf(x)\,dx$

$$E(X) = \int_1^3 x \times \frac{1}{12}(x-1)\,dx + \int_3^6 x \times \frac{1}{6}dx + \int_6^{10} x\left(\frac{5}{12} - \frac{1}{24}x\right)dx$$
$$E(X) = \left[\frac{1}{36}x^3 - \frac{1}{24}x^2\right]_1^3 + \left[\frac{1}{12}x^2\right]_3^6 + \left[\frac{5}{24}x^2 - \frac{1}{72}x^3\right]_6^{10}$$
$$E(X) = \left(\frac{3}{4} - \frac{3}{8}\right) - \left(\frac{1}{36} - \frac{1}{24}\right) + \left(3 - \frac{3}{4}\right) + \left(\frac{125}{6} - \frac{125}{9}\right) - \left(\frac{15}{2} - 3\right)$$
$$E(X) = \frac{3}{8} + \frac{1}{72} + \frac{9}{4} + \frac{125}{18} - \frac{9}{2} = \frac{61}{12}$$

Example 13

A random variable X has probability density function $f(x)$ given by

$$f(x) = \begin{cases} \frac{1}{3} & 0 \le x < 1 \\ \frac{2}{7}x^2 & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $E(X)$

(b) Specify fully the cumulative distribution function of X

(c) Find the median of X

Solution

(a) $E(X) = \int_0^1 \frac{x}{3}dx + \int_1^2 \frac{2x^3}{7}dx$

$$= \left[\frac{x^2}{6}\right]_0^1 + \left[\frac{2x^4}{28}\right]_1^2$$
$$= \frac{1}{6} + \left(\frac{32}{28} - \frac{2}{28}\right)$$
$$= \frac{26}{21}$$

(b) $F(x) = \int_0^x \frac{1}{3}dt$

$$= \left[\frac{t}{3}\right]_0^x = \frac{x}{3}$$
$$\int_0^1 \frac{1}{3}dx + \int_1^x \frac{2t^2}{7}dt = \left[\frac{x}{3}\right]_0^1 + \left[\frac{2t^3}{21}\right]_1^x$$
$$= \frac{1}{3} + \frac{2x^3}{21} - \frac{2}{21}$$
$$= \frac{2x^3}{21} + \frac{5}{21}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{3} & 0 \le x < 1 \\ \frac{2x^3}{21} + \frac{5}{21} & 1 \le x \le 2 \\ 1 & x > 2 \end{cases}$$

(c) $F(m) = \frac{1}{2}$

It is clear that the median lies between 1 and 2

$$\frac{2m^3}{21} + \frac{5}{21} = \frac{1}{2}$$
$$\frac{2m^3}{21} = \frac{11}{42}$$
$$m^3 = \frac{11}{4}$$
$$m = 1.4$$

Example 14

A random variable X has cumulative distribution function

$$F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{6}x^2 & 0 \leq x \leq 2 \\ -2 + 2x - \frac{1}{3}x^2 & 2 \leq x \leq 3 \\ 1 & x \geq 3 \end{cases}$$

Find

(a) $P(X \leq 1.5)$

(b) the median

(c) the probability density function $f(x)$

(d) Sketch $F(x)$ and $f(x)$

Solution

(a) $P(X \leq 1.5) = F(1.5) = \frac{1}{6}(1.5)^2 = 0.375$

(b) The median is the value m such that $F(m) = \frac{1}{2}$

Since $F(0) = 0$ and $F(2) = \frac{2}{3}$, m lies in the interval from 0 to 2 and thus $F(m) = \frac{1}{6}m^2$

$$\frac{1}{6}m^2 = \frac{1}{2}$$

$$m = \sqrt{3}$$

Hence the median of the distribution is 1.73

(c) The probability density function $f(x) = F'(x)$

For $0 \leq x \leq 2$, $f(x) = \frac{d}{dx}\left(\frac{1}{6}x^2\right) = \frac{1}{3}x$

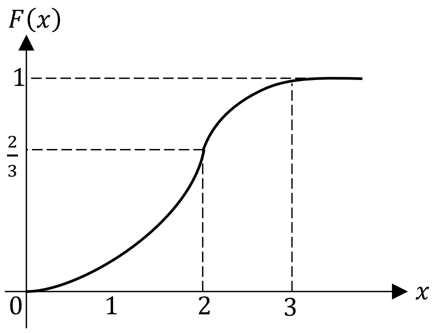
For $2 \leq x \leq 3$, $f(x) = \frac{d}{dx}\left(-2 + 2x - \frac{1}{3}x^2\right)$

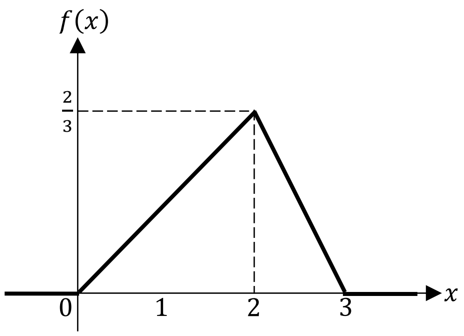
$$= 2 - \frac{2}{3}x$$

Otherwise, $f(x) = 0$

$$\therefore f(x) = \begin{cases} \frac{1}{3}x & 0 \leq x \leq 2 \\ 2 - \frac{2}{3}x & 2 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

(d)





Self-Evaluation exercise

1. For the following density function

$$f(x) = cx^2(1-x), \quad 0 < x < 1$$

Find

(i) constant c

(ii) mean

[Ans: (i) $c = 12$ (ii) 0.6]

2. A continuous distribution of a variable X in the range $(-3, 3)$ is defined by

$$f(x) = \begin{cases} \frac{1}{16}(3+x)^2 & -3 \leq x \leq -1 \\ \frac{1}{16}(6-2x^2) & -1 \leq x \leq 1 \\ \frac{1}{16}(3-x)^2 & 1 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Find the mean and variance of the above distribution

[Ans: 0, 1]

3. A continuous variable X is distributed at random between two values, $x = 0$ and $x = 2$, and has a probability density function of $ax^2 + bx$. The mean is 1.25.

(a) Show that $b = \frac{3}{4}$ and find the value of a

(b) Find the variance of X

(c) Verify that the median value of X is approximately 1.3

(d) Find the mode

[Ans: (a) $-3/16$ (b) $19/80$ (d) 2]

4. The continuous random variable X has probability density function given by

$$f(x) = \begin{cases} k(1+x^2) & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

(a) Find the value of k

(b) Determine $E(X)$ and $\text{Var}(X)$

(c) A is the event $X > \frac{1}{2}$; B is the event $X > \frac{3}{4}$. Find

(i) $P(B)$

(ii) $P(B|A)$

[Ans: (a) $3/8$ (b) 0, 0.4 (c) (i) $85/512$ (ii) $85/152$]

5. A continuous random variable X has the probability density function defined by

$$f(x) = \begin{cases} \frac{1}{3}cx, & 0 \leq x < 3 \\ c, & 3 \leq x \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

where c is a positive constant.

Find

(a) the value of c

(b) the mean of X

(c) the value, a, for there to be a probability of 0.85 that a randomly observed value of X will exceed a

[Ans: (a) $2/5$ (b) $13/5$ (c) $3/2$]

6. The continuous random variable X has probability density function $f(x)$, given by

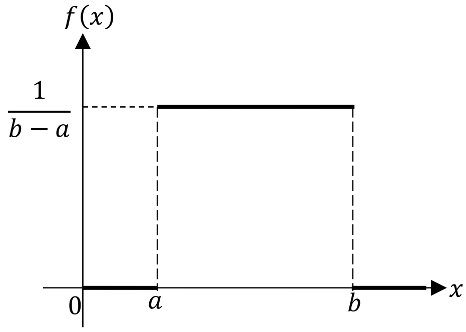
$$f(x) = \begin{cases} k(x^2 - 2x + 3) & 0 \leq x < 2 \\ \frac{1}{3}k & 2 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

THE UNIFORM (RECTANGULAR) DISTRIBUTION

It is common to describe distributions by the shapes of the graphs of their p.d.fs, U-shaped, J-shaped, etc.

The uniform (rectangular) distribution is particularly simple since its p.d.f is constant over a range of values and zero elsewhere.

In the figure below, X may take values between a and b, and zero elsewhere. Since the area under the graph must be 1, the height is $\frac{1}{b-a}$. The term 'uniform distribution" can be applied to both discrete and continuous variables so in the continuous case it is often written as 'uniform(rectangular)'



The continuous random variable X has a uniform distribution in the interval $a \leq x \leq b$ if the probability density function is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

The mean and variance of the uniform (rectangular) distribution.

$$E(X) = \int_a^b x\, f(x)dx = \int_a^b \frac{1}{b-a} x\, dx = \frac{1}{b-a}\int_a^b x\, dx$$

$$= \frac{1}{b-a}\left[\frac{1}{2}x^2\right]_a^b = \frac{1}{2} \times \frac{1}{b-a}[b^2 - a^2]$$

$$= \frac{(b-a)(b+a)}{2(b-a)}$$

$$E(X) = \frac{a+b}{2}$$

$$E(X^2) = \int_a^b x^2\, f(x)dx = \int_a^b \frac{1}{b-a} x^2\, dx$$

$$= \frac{1}{b-a}\int_a^b x^2\, dx$$

$$= \frac{1}{b-a}\left[\frac{1}{3}x^3\right]_a^b$$

$$= \frac{1}{3} \times \frac{1}{b-a}[b^3 - a^3]$$

$$= \frac{b^3 - a^3}{3(b-a)}$$

$$= \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)} = \frac{b^2 + ab + a^2}{3}$$

$$\text{Var}(X) = E(X^2) - (E(X)^2)$$

$$= \frac{b^2 + ab + a^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ba + b^2}{4}$$

$$= \frac{4b^2 + 4ab + 4a^2 - (3a^2 + 6a + 3b^2)}{12}$$

$$= \frac{a^2 + b^2 - 2ab}{12}$$

$$\text{Var}(X) = \frac{(b-a)^2}{12}$$

Note:

$$a^3 \pm b^3 \equiv (a \pm b)(a^2 \mp ab + b^2)$$

Example 1

A junior gymnastics league is open to children who are at least five years old but have not yet had their ninth birthday. The age, X years, of a member is modelled by the uniform (rectangular) distribution over the range of possible values between five and nine. Age is measured in years and decimal parts of a year rather than just completed years. Find

(a) the p.d.f. $f(x)$ of X
(b) $P(6 \leq X \leq 7)$
(c) $E(X)$
(d) $\text{Var}(X)$

<u>Solution</u>

(a) The p.d.f. $f(x) = \frac{1}{9-5} = \frac{1}{4}$ for $5 \leq x < 9$

(b) $P(6 \leq X \leq 7) = \int_5^7 f(x)\ dx = \int_6^7 \frac{1}{4} dx$

$$= \left[\frac{x}{4}\right]_6^7 = \frac{7}{4} - \frac{6}{4} = \frac{1}{4}$$

(c) $E(X) = \int_{-\infty}^{\infty} xf(x)\, dx = \int_5^9 \frac{x}{4} dx$

$$= \left[\frac{x^2}{8}\right]_5^9 = \frac{81}{8} - \frac{25}{8} = 7$$

(d) $\text{Var}(X) = \int_{-\infty}^{\infty} x^2 f(x)\, dx - [E(X)]^2$

$$= \int_5^9 \frac{x^2}{4} dx - 7^2$$

$$= \left[\frac{x^3}{12}\right]_5^9 - 49$$

$$= \frac{729}{12} - \frac{125}{12} - 49$$

$$= 1.33$$

Example 2

The radius r of a circle is a continuous random variable R uniformly distributed in the interval $7 \leq r \leq 11$

(a) State the mean and variance of R

The circumference of the circle is denoted by C

(b) Determine, in terms of π, the mean and variance of C

(c) Calculate $P(C > 53)$

The area of the circle is denoted by A

(d) Find the mean of A

<u>Solution</u>

(a) $R \sim U(7, 11)$

$$E(R) = \frac{a+b}{2} = \frac{7+11}{2} = 9$$

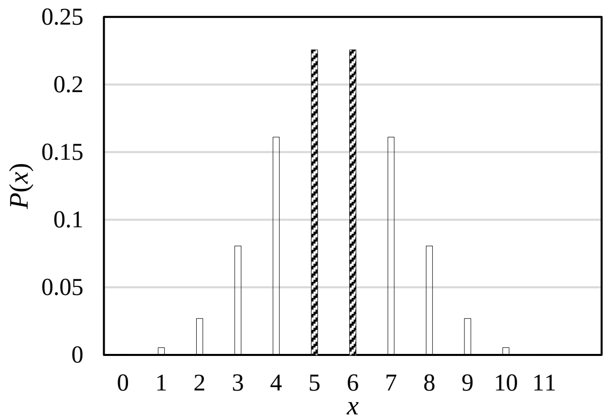
$$Var(R) = \frac{(b-a)^2}{12} = \frac{(11-7)^2}{12} = \frac{4}{3}$$

(b) $E(C) = E(2\pi R) = 2\pi E(R) = 2\pi \times 9 = 18\pi$

$\text{Var}(X) = \text{Var}(2\pi R) = (2\pi)^2 \text{Var}(R)$

$$= 4\pi^2 \times \frac{4}{3} = \frac{16}{3}\pi^2$$

(c) $P(C > 53) = P(2\pi R > 53)$



$n = 11 \quad p = 0.5$

Note:
You can check the above property with the other values of n where the probability of success is 0.5.

Example 1
Seeds have a probability of germinating of 0.9. If six seeds are sown, what is the probability of five or more seeds germinating?

Solution

This is a binomial situation with $n = 6$ and
$P(\text{success}) = P(\text{germination}) = 0.9$

$$P(X = x) = \binom{6}{x}(0.9)^x(0.1)^{n-x}$$

$P(5 \text{ or more germinates}) = P(X = 5 \text{ or } 6)$

$$= P(X = 5) + P(X = 6)$$

$$= \binom{6}{5}(0.5)^5(0.1)^1 + \binom{6}{6}(0.9)^6(0.1)^0$$

$$= 0.88$$

Example 2
Ten coins are thrown simultaneously. Find the probability of getting at least seven heads.

Solution

$p =$ probability of getting a head $= \frac{1}{2}$

$q =$ probability of not getting a head $= \frac{1}{2}$

The probability of getting x heads in a random throw of 10 coins is

$$p(x) = \binom{10}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{10-x}$$

The probability of getting at least seven heads is given by

$$P(X \geq 7) = p(7) + p(8) + p(9) + p(10)$$

$$= \binom{10}{7}\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^3 + \binom{10}{8}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2 + \binom{10}{9}\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^1$$

$$+ \binom{10}{10}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^0$$

$$= \frac{120}{1024} + \frac{45}{1024} + \frac{10}{1024} + \frac{1}{1024}$$

$$= \frac{176}{1024}$$

Example 3
The probability that a marksman will hit a target is $\frac{5}{6}$. He fires 9 shots. Calculate the probability that he will hit the target

(a) at least 7 times
(b) no more than 6 times

Solution

$P(\text{hitting target}) = P(\text{success}) = \frac{5}{6}$

$P(\text{not hitting target}) = P(\text{failure}) = \frac{1}{6}$

Making the nine shots to be the independent events the probability distribution of X, 'the number of targets hit' is $B\left(9, \frac{5}{9}\right)$ and so $P(X = x) = \binom{9}{x}\left(\frac{5}{6}\right)^x\left(\frac{1}{6}\right)^{9-x}$

(a) $P(\text{at least 7 targets hit}) = P(X = 7) + P(X = 8) + P(X = 9)$

$$= \binom{9}{7}\left(\frac{5}{6}\right)^7\left(\frac{1}{6}\right)^2 + \binom{9}{8}\left(\frac{5}{6}\right)^8\left(\frac{1}{6}\right)^1 + \binom{9}{9}\left(\frac{5}{6}\right)^9\left(\frac{1}{6}\right)^0$$

$$= 0.822$$

(b) $P(\text{no more than 6 targets})$

$$= P(X = 0) + P(X = 1) + P(X = 2) + \cdots + P(X = 6)$$

$$= 1 - [P(X = 7) + P(X = 8) + P(X = 9)]$$

$$= 1 - 0.822 = 0.178$$

Example 4
An irregular six faced die is thrown and the expectation that in 10 throws it will give five even numbers is twice the expectation that it will give four even numbers. How many times in 10,000 sets of 10 throws each, would you expect it to give no even number.

Solution

$$P(X = x) = \binom{10}{x}p^x q^{10-x}\,; x = 0, 1, 2, 3, \ldots, 10$$

$$\binom{10}{5}p^5q^5 = 2\binom{10}{4}p^4q^6$$

$$252p^5q^5 = 420p^4q^6$$

$$3p = 5q$$

$$3p = 5(1 - p)$$

$$3p = 5 - 5p$$

$$8p = 5$$

$$p = \frac{5}{8}$$

$$q = \frac{3}{8}$$

$$P(X = x) = \binom{10}{x}\left(\frac{5}{8}\right)^x\left(\frac{3}{8}\right)^{10-x}$$

Hence the required number of times that in 10,000 sets of 10 throws each, we get no even number

$$= 10{,}000 \times P(X = x) = 10{,}000 \times \left(\frac{3}{8}\right)^{10} \approx 1$$

Example 5
It is given that the discrete random variable X satisfies

$$X \sim B(n, p)$$

Given further that $P(X = 2) = P(X = 3)$, show that

$$E(X) = 3 - p$$

just a smaller probability than can be expressed in the four decimal places used in this table.

$$P(X > 9) = P(X = 10) + P(X = 11)$$
$$= 0.0007 + 0.0000$$
$$= 0.0007$$

(e) Here, you want to find the probability equal to 3 and 5 and everything in between. In other words, you want the probabilities for $X = 3$, $X = 4$ and $X = 5$. Reading off from the tables and adding;

$$P(3 \leq X \leq 5) = P(X = 3) + P(X = 4) + P(X = 5)$$
$$= 0.1774 + 0.2365 + 0.2207$$
$$= 0.6346$$

Cumulative binomial probabilities

We shall compare the use of the binomial tables to obtain the individual probabilities and the cumulative table to obtain probabilities $P(X \geq r)$. Let us follow the example below.

Example 13

By estimates, 20% of Ugandans have no health insurance. Randomly sample $n = 15$ Ugandans. Let X denote the number in the sample with no health insurance. What is the probability that exactly 3 of the 15 sampled have no health insurance?

<u>Solution</u>

Here $n = 15$, $p = 0.20$. We are required to find $P(X = 3)$. Using the binomial tables

				x		
n	r	0.01	0.05	0.10	0.15	**0.20**
15	0	0.8601	4633	2059	0874	0352
	1	0.1303	3658	3432	2312	1319
	2	0.0092	0307	2669	2856	2309
	3	0.0004	0049	1285	2184	**2501**
	4		0006	0428	1156	1876
	5			0105	0449	1032
	6			0019	0132	0430
	7			0003	0030	0138
	8				0001	0035
	9					0007
	10					0001
	11					
	12					
	13					
	14					
	15					

$$P(X = 3) = 0.2501$$

What is the probability that **at most one** has no health insurance?

<u>Solution</u>

"At most one" means either 0 or 1 of those sampled have no health insurance. That is, we need to find

$$P(X \leq 1) = P(X = 0) + P(X = 1)$$

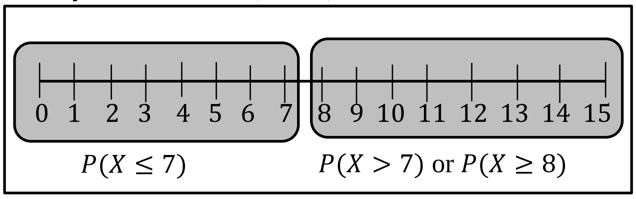
From the binomial table,

$$P(X \leq 1) = 0.0352 + 0.1319 = 0.1671$$

What is the probability that **more than seven** have no health insurance?

<u>Solution</u>

"More than seven" in the sample means 8, 9, 10, 11, 12, 13, 14, 15. As the following diagram illustrates, there are two ways to calculate $P(X > 7)$.



$$P(X \leq 15) = 1$$

We would calculate $P(X > 7)$ by adding up $P(X = 8)$, $P(X = 9)$, up to $P(X = 15)$. Alternatively, we could calculate $P(X > 7)$ by finding $P(X \leq 7)$ and subtracting from 1. But to find $P(X \leq 7)$, we would still have to add up $P(X = 0)$, $P(X = 1)$, up to $P(X = 7)$. Either way, it becomes readily apparent that answering this question is going to involve more work than the previous two questions. It thus becomes clearly helpful to use an alternative to the binomial tables or the binomial p.d.f. This alternative involves cumulative binomial probabilities.

To find the cumulative binomial probabilities,

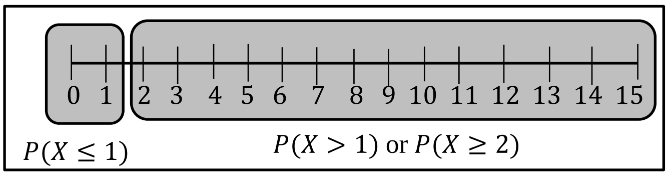
- Find n, the number in the sample, in the first column on the left
- Find the column containing p, the probability of success
- Find the r in the second column on the left for which you want to find $P(X \geq r)$
- If you want to find $F(r) = P(X \leq r)$, then subtract the value read off from the table from 1 i.e. calculate $P(X \leq r) = 1 - P(X \geq r)$

Let us try this in our health insurance example

What is the probability that **at most 1** of the 15 has no health insurance?

<u>Solution</u>

If we use the cumulative binomial distribution,



$$P(X \leq 15) = 1$$

$$P(X \leq 1) = 1 - P(X \geq 2) = 1 - 0.8329 = 0.1671$$

To find $P(X \geq 2)$,

- Find $n = 15$, in the first column on the left
- Find the column containing $p = 0.20$

- Find the 2 in the second column on the left, since we want to find $P(X \geq 2)$.

		x				
n	r	0.01	0.05	0.10	0.15	**0.20**
15	**1**	0.1399	5367	7941	9126	**9648**
	2	0.0096	1710	4510	6814	**8329**
	3	0.0004	0362	1841	3958	6020
	4		0055	0556	1773	3518
	5		0006	0127	0617	1642
	6		0001	0022	0168	0611
	7			0003	0036	0181
	8				0006	**0042**
	9				0001	0008
	10					0001
	11					
	12					
	13					
	14					

Now, all we need to do is read the probability value where $p = 0.20$ column and the $(n = 15, r = 2)$ row intersect.

What is the probability that **more than 7** have health insurance?

Solution

As we discussed previously, we can calculate $P(X > 7)$ by finding $P(X \geq 8)$directly from the cumulative table.

The good news is that the cumulative binomial probability table makes it easy to determine $P(X \geq 8)$. To find $P(X \geq 8)$ using the table, we

- find $n = 15$ in the first column on the left
- find the column containing $p = 0.20$
- find the 8 in the second column on the left, since we want to find $P(X \geq 8)$

Now, all we need to do is read the probability value where the $p = 0.20$ column and the $(n = 15, r = 8)$ row intersect.

Thus, $P(X \geq 8) = 0.0042$

$$\therefore P(X > 7) = 0.0042$$

What is the probability that **at least 1** has no health insurance?

Solution

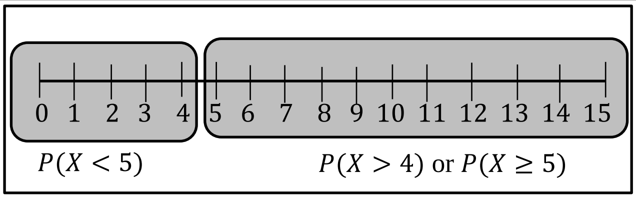
We can find $P(X \geq 1)$ by reading it directly from the cumulative binomial tables i.e.

$$P(X \geq 1) = 0.9648$$

What is the probability that **fewer than 5** have no health insurance?

Solution

"Fewer than 5" means 0, 1, 2, 3, or 4. That is, $P(X < 5) = 1 - P(X \geq 5)$ and $P(X \geq 5)$ can be readily found using the cumulative binomial table.



$$P(X \leq 15) = 1$$

Read the probability value where the $p = 0.20$ column and the $n = 15, r = 5)$ row intersect.

$$P(X \geq 5) = 0.1642$$

$$P(X < 5) = 1 - P(X \geq 5) = 1 - 0.1642 = 0.8358$$

Note:

We have now taken a look at an example involving all of the possible scenarios….at most, more than, exactly, at least, and fewer than… of the kinds of binomial probabilities that you might need to find.

Have you noticed that p, the probability of success, in the binomial table only goes up to 0.50?. What happens if your p equals 0.60 or 0.70? All you need to do in that case is to turn the problem on its head, for example, suppose you have $n = 10$ and $p = 0.60$, and you are looking for the probability of at most 3 successes. Just change the definition of a success into a failure, and vice versa i.e. finding the probability of at most 3 successes is equivalent to 7 or more failures with the probability of failing being 0.40.

Example 14

In January 2020, it was reported that 70% of the people infected with the novel corona virus recovered from the infection after treatment. If 10 patients were randomly selected from a hospital in China, what is the probability that at least four recovered?

Solution

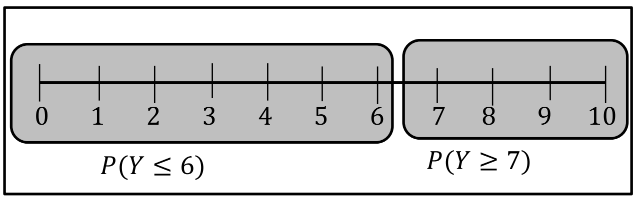
Let X be the number of patients who recovered, then $X \sim B(10, 0.70)$

Let Y be the number of patients who did not recover, then $Y \sim B(10, 0.30)$

We are interested in $P(X \geq 4)$ and we can't use the cumulative binomial table because it only goes up to 0.50.

The good news is that we can rewrite $P(X \geq 4)$ as a probability statement in terms of Y.

$$P(X \geq 4) = P(-X \leq -4) = P(10 - X \leq 10 - 4)$$
$$= P(Y \leq 6)$$



$$P(Y \leq 10) = 1$$

$$P(Y \leq 6) = 1 - P(Y \geq 7)$$

Reading off the probability value where $p = 0.30$ column and the $(n = 10, r = 7)$ intersect,

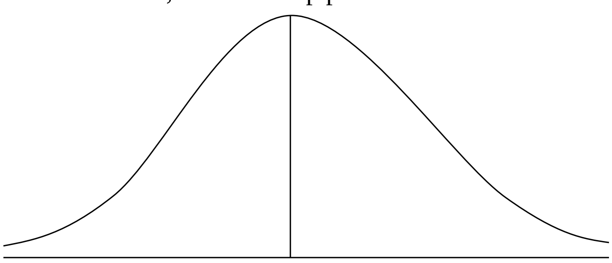
NORMAL DISTRIBUTION

A random variable X is said to have a normal distribution with parameters μ (called mean) and σ^2 (called variance) if its density function is given by the probability law:

$$f(x:\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \quad -\infty < x < \infty, \sigma > 0$$

Remarks:

1. A random variable X with mean μ and variance σ^2 and following the normal law is expressed by $X \sim N(\mu, \sigma^2)$
2. If $X \sim N(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma}$ is a standard normal variate with $E(Z) = 0$ and Var $(Z) = 1$ and we write $Z \sim N(0,1)$
3. The p.d.f. of a standard normal variate Z is given by
$$\varphi(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \quad -\infty < x < \infty,$$
and the corresponding distribution function, denoted by $\Phi(z)$ is given by
$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \varphi(u)\,du$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\,du$$
We shall prove below two important results on the distribution function $\Phi(z)$ of the standard normal variate.
Result 1: $\Phi(-z) = 1 - \Phi(z)$
Proof:
$$\Phi(-z) = P(Z \le -z) = P(Z \ge z) \text{ (By symmetry)}$$
$$= 1 - P(Z \le z)$$
$$= 1 - \Phi(z)$$
Result 2: $P(a \le X \le b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$
where $X \sim N(\mu, \sigma^2)$
Proof: $P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right)$
$$= P\left(Z \le \frac{b-\mu}{\sigma}\right) - P\left(Z \le \frac{a-\mu}{\sigma}\right)$$
$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$
4. The graph of $f(x)$ is famous 'bell-shaped' curve. The top of the bell is directly above the mean μ. For large values of σ, the curve tends to flatten out and for small values of σ, it has a sharp peak.



$x = \mu$

Characteristics of Normal distribution and Normal Probability curve

The normal probability curve with mean μ and standard deviation σ is given by the equation

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \quad -\infty < x < \infty$$

(i) The curve is bell shaped and symmetrical about the line $x = \mu$
(ii) Mean, median and mode of the distribution coincide
(iii) As x increases numerically, $f(x)$ decreases rapidly, the maximum probability occurring at the point $x = \mu$, and given by $\frac{1}{\sigma\sqrt{2\pi}}$
(iv) Since $f(x)$ being the probability, can never be negative, no portion of the curve lies below the x-axis.
(v) x-axis is an asymptote to the curve
(vi) The points of inflexion of the curve are given by
$$\left[x = \mu \pm \sigma, f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}}\right]$$

Calculating probabilities

Example 1

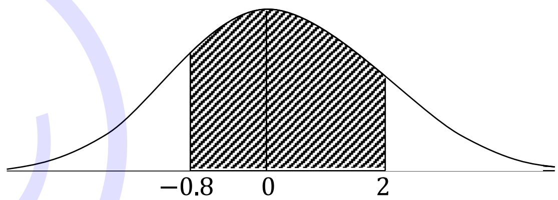
X is a random variable with mean 30 and standard deviation 5. Find the probabilities that

(a) $26 \le X \le 40$
(b) $X \ge 45$
(c) $|X - 30| > 5$

Solution

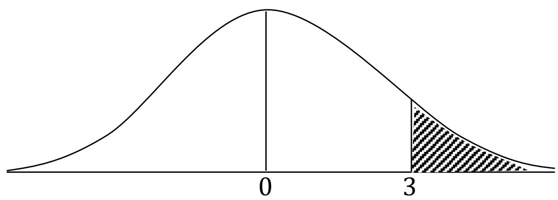
Here $\mu = 30$ and $\sigma = 5$

(a) When $X = 26$, $Z = \frac{X-\mu}{\sigma} = \frac{26-30}{5} = -0.8$
and when $X = 40$, $Z = \frac{40-30}{5} = 2$



$$P(26 \le X \le 40) = P(-0.8 \le Z \le 2)$$
$$= P(-0.8 \le Z \le 0) + P(0 \le Z \le 2)$$
$$= P(0 \le Z \le 0.8) + P(0 \le Z \le 2)$$
$$= 0.2881 + 0.4772$$
$$= 0.7653$$

(b) When $X = 45$, $Z = \frac{45-30}{5} = 3$



$$P(X \ge 45) = P(Z \ge 3) = 0.5 - P(0 \le Z \le 3)$$
$$= 0.5 - 0.4987$$
$$= 0.0013$$

(c) $P(|X - 30| \le 5) = P(25 \le X \le 35)$
$$= P\left(\frac{25-30}{5} \le Z \le \frac{35-30}{5}\right)$$
$$= P(-1 \le Z \le 1)$$
$$= 2P(0 \le Z \le 1)$$
$$= 2 \times 0.3413$$
$$= 0.6826$$
$$P(|X - 30| > 5) = 1 - P(|X - 30| \le 5)$$
$$= 1 - 0.6826$$
$$= 0.3174$$

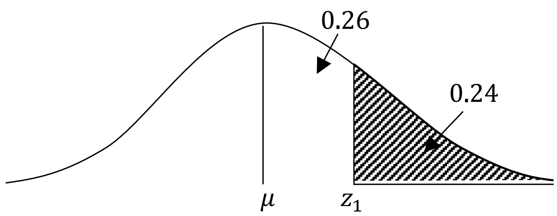
$$= 0.5 - 0.4772$$
$$= 0.0228$$

(ii) $P(X \le 20) = 1 - P(X \ge 20)$
$$= 1 - 0.0228$$
$$= 0.9772$$

(iii) $P(0 \le X \le 12) = P\left(\frac{0-12}{4} < Z < \frac{12-12}{4}\right)$
$$= P(-3 \le Z \le 0)$$
$$= P(0 \le Z \le 3)$$
$$= 0.4987$$

(b) When $X = x'$, $Z = \frac{x'-12}{4} = z_1$

then, we are given
$$P(X > x') = 0.24$$
$\Rightarrow P(Z > z_1) = 0.24$ i.e. $P(0 < Z < z_1) = 0.26$

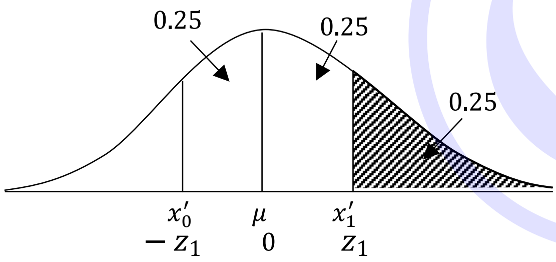


From normal tables, $z_1 = 0.71$
$$\frac{x' - 12}{4} = 0.71$$
$$x' = 12 + 4 \times 0.71 = 14.84$$

(c) We are given

$P(x_0' < X < x_1') = 0.50$ and $P(X > x_1') = 0.25$

The points x_0' and x_1' are located as shown below



When $X = x_1'$, $Z = \frac{x_1'-12}{4} = z_1$

and when $X = x_0'$, $Z = \frac{x_0'-12}{4} = -z_1$
$$P(Z > z_1) = 0.25$$
$$\Rightarrow P(0 < Z < z_1) = 0.25$$
From critical tables, $z_1 = 0.674$
$$\frac{x_1' - 12}{4} = 0.674$$
$$x_1' = 12 + 0.674 \times 4 = 14.7$$
$$\frac{x_0' - 12}{4} = -0.674$$
$$x_0' = 12 - 4 \times 0.674 = 9.3$$

Example 13

The mean yield for one-acre plot is 662 kilos with a standard deviation 32 kilos. Assuming normal distribution, how many one-acre plots in a batch of 1000 plots would you expect to have yield

(a) over 700 kilos

(b) below 650 kilos

(c) what is the lowest yield of the best 100 plots?

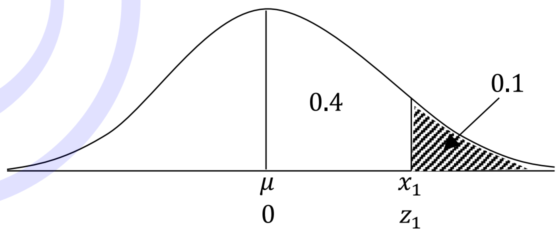
Solution

Let the random variable X denotes the yield (in kilos) for one-acre plot, then we are given that $X \sim N(\mu, \sigma^2)$ where $\mu = 662$ and $\sigma = 32$.

(a) The probability that a plot has a yield over 700 kilos is given by
$$P(X > 700) = P\left(Z > \frac{700 - 662}{32}\right)$$
$$= P(Z > 1.19)$$
$$= 0.5 - P(0 \le Z \le 1.19)$$
$$= 0.5 - 0.3830$$
$$= 0.1170$$
Hence in a batch of 1000 plots, the expected number of plots with yield over 700 kilos is
$$1000 \times 0.117 = 117$$

(b) Required number of plots with yield below 650 kilos is given by $1000 \times P(X < 650)$
$$P(X < 650) = P\left(Z < \frac{650 - 662}{32}\right)$$
$$= P(Z < -0.38)$$
$$= P(Z > 0.38) \quad \text{(By symmetry)}$$
$$= 0.5 - P(0 \le Z \le 0.38)$$
$$= 0.5 - 0.1480$$
$$= 0.352$$
Number of plots below 650 kilos $= 1000 \times 0.352 = 352$

(c) The lowest yield, say, x_1 of the best 100 plots is given by
$$P(X > x_1) = \frac{100}{1000} = 0.1$$



When $X = x_1$, $Z = \frac{x_1 - \mu}{\sigma} = \frac{x_1 - 662}{32} = z_1$ ... *
$$P(Z > z_1) = 0.1$$
$$\Rightarrow P(0 \le Z \le z_1) = 0.4$$
From Normal probability tables, $z_1 = 1.28$

Substituting in (*), we get
$$\frac{x_1 - 662}{32} = 1.28$$
$$x_1 = 662 + 1.28 \times 32 = 702.96$$
Hence the best 100 plots have yield over 702.96 kilos

Example 14

In an examination it is laid down that a student passes if e secures 30% or more marks. He is placed in the first, second or third division according as he secures 60% or more parks, between 45% and 60% marks and marks between 30% and 45% respectively. He gets distinction in case he secures 80% or more marks. It is noticed from the result that 10% of the students failed in the examination, whereas 5% of them obtained distinction. Calculate the percentage of students placed in the second distribution. (Assume normal distribution of marks)

(c) Given that the weight of a marmalade jar is between 249 and 253 grams, determine the probability that the jar weighs more than 250 grams.

[Ans: (a) $\sigma = 2.6$ (b) 0.5825 (c) 0.7145]

15. The length of an engine part must be between 4.81 cm and 5.20 cm. In mass production, it is found that 0.8% are too short and 3% are too long.
(a) If these lengths are normally distributed, find the mean μ and standard deviation σ
Each part costs \$4 to produce; those that turn out to be too long are shortened, at an extra cost of \$2; those that turn out to be too short have to be scrapped.
(b) Find the expected total cost of producing 100 parts that meet the specification.
[Ans: (a) $\mu = 5.03$ cm, $\sigma = 0.09$ cm (b) \$409.27]

16. A sample of 100 apples is taken from a load. The apples have the following distribution of sizes.

Diameter to nearest cm	6	7	8	9	10
Frequency	11	21	38	17	13

(a) Determine the mean and standard deviation of these diameters.
(b) Assuming that the distribution is approximately normal with this mean and this standard deviation, find the range of size of apples for packing, if 5% are to be rejected as too small and 5% are to be rejected as too large.
[Ans: (a) 8 cm, 1.16 cm (b) 6.09 to 9.91 cm]

17. A machine packs flour into bags which nominally contain 1 kg but there is a variation in the actual weight (kg), which is described by a normal random variable of mean μ and variance σ^2. Previous investigations indicate that $\sigma = 0.03$ kg and that the probability that a bag is underweight is 0.02.
(a) Find the value of μ at which the machine is operating.
An attempt is made to improve the machine with hope that, while it operates with the same value of μ, σ will be reduced.
(b) Find the value of σ which is required to ensure that the probability that a bag is underweight is 0.001.
(c) Assuming this improved value of σ to have been achieved, show that the new probability that a random bag will weigh more than 1.1 kg is just less than 0.03
[Ans: (a) 1.06 (b) 0.020]

18. The lengths of pine needles, in cm, are normally distributed. It is further given that 11.51% of these pine needles are shorter than 6.2 cm and 3.59% are longer than 9.5 cm. Find the mean and standard deviation of the length of these pine needles.
[Ans: $\mu = 7.52$, $\sigma = 1.1$]

19. The weights, W grams, of shaving from canisters are normally distributed with a mean of 125 and a standard deviation of 4.
(a) Determine the probability the weight of one such canister will be between 127 and 132 grams.
(b) Find the value of b, so that $P(b < W < 128) = 0.7672$.
[Ans: (a) 0.2684 (b) 115]

20. The random variable Y is normally distributed with mean μ and variance σ^2. Given that $P(Y < 48) = P(Y > 57) = 0.0668$, find the value of $P(50.1 < Y < 55.8)$.
[Ans: 0.6524]

21. The weights of packs of cheese, in grams, are thought to be normally distributed with a standard deviation of 4.
(a) Find the mean weight of a pack of cheese, if 95.5% of these packs are heavier than 248 grams.
Using the value of the mean obtained in (a),
(b) determine the probability that a randomly chosen pack of cheese weighs between 250 and 256 grams
(c) determine the probability that a randomly chosen pack of cheese weighs more than 248 grams then its actual weight is less than 256 grams.
Ten packs of cheese are selected at random.
(d) Calculate the probability that exactly 6 of these packs weigh over 248 grams.
[Ans: (a) $\mu = 255$ (b) 0.4931 (c) 0.5798 (d) 0.000653]

22. Adult men in a certain country have heights which are normally distributed with a mean of 1.81 m and a standard deviation of 0.05 m.
(a) Estimate the number of men in a random sample of 100 men whose heights, when measured to the nearest 0.01 m, exceed 1.90 m
(b) Find the probability that a man selected at random will have a height, when measured to the nearest 0.01 m, less than 1.72 m
(c) Find the height which is exceeded by 90% of the men.
[Ans: (a) 4 men (b) 0.0446 (c) 1.746 m]

23. Find the mean and variance of the normally distributed variable X in each of the following cases.
(a) $P(X > 3) = 0.5, P(3 < X < 5) = 0.1$
(b) $P(X > 20) = P(X < 10) = 0.05$
(c) $P(X < 60) = 0.01, P(X > 90) = 0.3$
(d) $P(X > 10) = 0.1, P(9 < X < 10) = 0.2$
[Ans: (a) 3, 62.3 (b) 15, 9.24 (c) 84.5, 111 (d) 8.31, 1.74]

24. A machine is producing curtain rails whose lengths are normally distributed with mean 2.4 m.
(a) If 5% of the rails are longer than 2.42 m, find the standard deviation of the distribution.
(b) If the machine is adjusted so that 95% of the rails have lengths between 2.38 m and 2.42 m, find the new standard deviation of the distribution.
[Ans: (a) 0.0122 (b) 0.0102]

25. In an examination, 30% of the candidates fail and 10% achieve distinction. Last year the pass mark (out of 200) was 84 and the minimum mark required for a distinction was 154. Assuming that the marks of the candidates were normally distributed, estimate the mean mark and the standard deviation.
[Ans: 104, 38.8]

Normal approximation to the binomial distribution

The normal distribution may be used to approximate the binomial distribution when n is large $(n > 20)$and p is not too big or small. If n is very large, then the approximation is good even if p is near to 0 or 1.

If $X \sim B(n, p)$, then $X \sim N(np, npq)$ since $E(X) = np$ and $Var(X) = npq$.

Continuity correction

To compensate for the change from a discrete distribution (the binomial) to a continuous distribution (the normal), a continuity correction is made.

The discrete integer value a in the binomial distribution becomes the class interval $[(a - 0.5)$ up to $(a + 0.5)]$ in the normal distribution i.e. the discrete variable 3 becomes the class interval 2.5 up to 3.5.

Continuity correction table

If $P(X = a)$ use $P(a - 0.5 < X < a + 0.5)$
If $P(X \geq a)$ use $P(X \geq a - 0.5)$
If $P(X > a)$ use $P(X > a + 0.50$
If $P(X \leq a)$ use $P(X \leq a + 0.5)$
If $P(X < a)$ use $P(X < a - 0.5)$
If $P(a < X < b)$ use $P(a + 0.5 < X < b - 0.5)$
If $P(a \leq X \leq b)$ use $P(a - 0.5 \leq X \leq b + 0.5)$

Example 21

The discrete random variable X takes integer and is to be approximated by a normal distribution. Apply a continuity correction to the following probabilities.

(a) $P(X = 25)$
(b) $P(X \leq 7)$
(c) $P(X < 10)$
(d) $P(X > 5)$
(e) $P(X \geq 3)$
(f) $P(17 \leq X \leq 20)$
(g) $P(18 < X < 30)$
(h) $P(28 < X \leq 40)$
(i) $P(23 \leq X < 35)$

Solution

(a) $P(X = 25) = P(24.5 < X < 25.5)$
(b) $P(X \leq 7) = P(X \leq 7.5)$
(c) $P(X < 10) = P(X \leq 9.5)$
(d) $P(X > 5) = P(X \geq 5.5)$
(e) $P(X \geq 3) = P(X \geq 2.5)$
(f) $P(17 \leq X \leq 20) = P(16.5 \leq X < 20.5)$
(g) $P(18 < X < 30) = P(18.5 \leq X < 29.5)$
(h) $P(28 < X \leq 40) = P(28.5 \leq X < 40.5)$
(i) $P(23 \leq X < 35) = P(22.5 \leq X < 34.5)$

Example 22

The discrete random variable X has a probability distribution

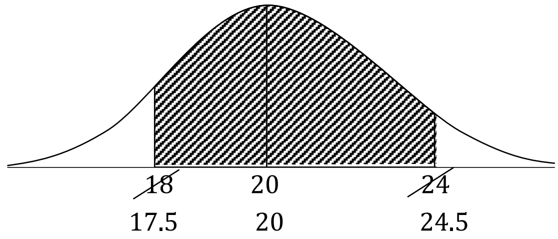
$$X \sim B(160, 0.125)$$

Find $P(18 \leq X < 25)$

Solution

Calculate the mean and variance
$E(X) = np = 160 \times 0.125 = 20$
$\text{Var}(X) = np(1 - p) = 20 \times 0.875 = 17.5$
Approximate by normal, $Y \sim N(20, 17.5)$

$$P(18 \leq X < 25) = P(18 \leq X \leq 24)$$
$$= P(17.5 \leq Y \leq 24.5)$$
$$= P\left(\frac{17.5 - 20}{\sqrt{17.5}} < Z < \frac{24.5 - 20}{\sqrt{17.5}}\right)$$
$$= P(-0.598 < Z < 1.076)$$



$$= P(-0.598 < Z < 0) + P(0 < Z < 1.076)$$
$$= 0.2251 + 0.3591$$
$$= 0.5842$$

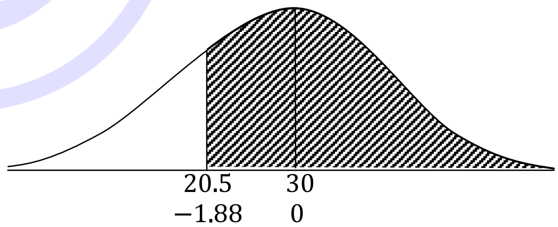
Example 23

A machine manufacturing nails makes approximately 15% that are outside set tolerance limits. If a random sample of 200 is taken, find the probability that more than 20 will be outside the tolerance limits.

Solution

Let X be the random variable 'number of nails outside limits', then $X \sim B(200, 0.15)$
$E(X) = 200 \times 0.15 = 30$
$\text{Var}(X) = 30 \times 0.85 = 25.5$
So $X \sim N(30, 25.5)$
We require $P(X > 20.5)$ using the continuity correction

$$P(X > 20.5) = P\left(Z > \frac{20.5 - 30}{5.05}\right)$$
$$= P(Z > -1.88)$$



$$= 0.5 + P(-1.88 < Z < 0)$$
$$= 0.5 + 0.4699$$
$$= 0.9699$$

Example 24

It has been established over a long period of time, that in Enzo's restaurant, 30% of the orders are vegetarian. Find the probability that in a given day with 80 orders, there will be more than 30 vegetarian orders.

Solution

Let X = number of vegetarian orders

$$X \sim B(80, 0.3)$$

$E(X) = np = 80 \times 0.3 = 24$
$\text{Var}(X) = np(1 - p) = 24 \times 0.7 = 16.8$
Approximate by $Y \sim N(24, 16.8)$

$$P(X > 30) = P(X \geq 31) = P(Y > 30.5)$$
$$= P\left(Z > \frac{30.5 - 24}{\sqrt{16.8}}\right)$$

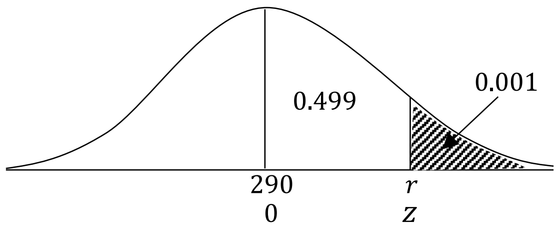
$E(X) = np = 500 \times \frac{2}{50} = 290$

$\text{Var}(X) = np(1-p) = 290 \times \frac{21}{50} = 121.8$

Let $X \sim N(290, 121.8)$

$$P(X > r) = P\left(Z > \frac{r + 0.5 - 290}{\sqrt{121.8}}\right) < 0.001$$

$$P\left(Z > \frac{r - 289.5}{\sqrt{121.8}}\right) < 0.001$$



From the critical tables, $z_{0.499} = 3.1$

$$\frac{r - 289.5}{\sqrt{121.8}} > 3.1$$

$$r > 289.5 + 3.1\sqrt{121.8}$$

$$r > 323.71$$

Hence the least integer value of r is 324

Example 28

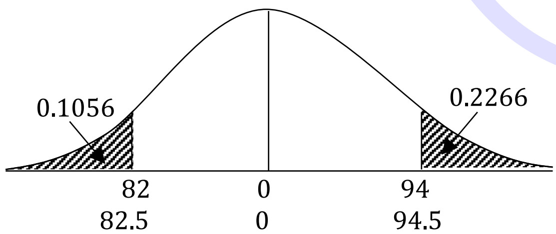
The discrete random variable $X \sim B(n, p)$. The value of n and the value of p are such that X can be approximated by a normal distribution.

- Using a normal approximation, the probability that X is at most 82 is 0.1056
- Using the same normal approximation, the probability that X is less than 95 is 0.7734

Determine the value of n and the value of p

Solution

$X \sim B(n, p)$ is approximated by $Y \sim N\big(np, np(1-p)\big)$



$$P(X \le 82) = 0.1056$$

$$P(Y < 82.5) = 0.1056$$

$$P\left(Z < \frac{82.5 - \mu}{\sigma}\right) = 0.1056$$

$$\frac{82.5 - \mu}{\sigma} = -1.25$$

$$82.5 - \mu = -1.25\sigma \ldots (i)$$

$$P(X < 95) = 0.7734$$

$$P(X \le 94) = 0.7734$$

$$P(Y < 94.5) = 0.7734$$

$$P\left(Z < \frac{94.5 - \mu}{\sigma}\right) = 0.7734$$

$$\frac{94.5 - \mu}{\sigma} = 0.75$$

$$94.5 - \mu = 0.75\sigma \ldots (ii)$$

Solving simultaneously;

$(ii) - (i)$;

$$12 = 2\sigma$$

$$\sigma = 6$$

From (ii);

$$\mu = 94.5 - 0.75\sigma = 94.5 - 0.75(6) = 90$$

Finally;

$$\sigma = \sqrt{np(1-p)} = 6$$

$$\mu = np = 90$$

$$np(1-p) = 36$$

$$90(1-p) = 36$$

$$1 - p = 0.4$$

$$p = 0.6$$

$$n = \frac{90}{p} = \frac{90}{0.6} = 150$$

Example 29

The random variable $X \sim B(200, 0.2)$. Use a suitable approximation to estimate

(a) $P(X < 45)$

(b) $P(25 \le X < 35)$

(c) $P(X = 42)$

[Ans: (a) 0.7881 (b) 0.163 (c) 0.0636]

Solution

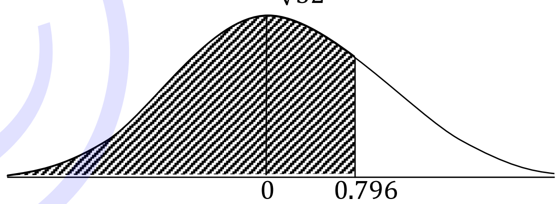
$n = 200, p = 0.2, q = 0.8$

Mean, $\mu p = 200 \times 0.2 = 40$

Variance, $npq = 40 \times 0.8 = 32$

(a) $P(X < 45) = P(X \le 44.5)$

$$P(X \le 44.5) = P\left(Z < \frac{44.5 - 40}{\sqrt{32}}\right) = P(Z < 0.796)$$



$$P(Z < 0.796) = 0.5 + P(0 < Z < 0.796)$$

$$= 0.5 + 0.2871$$

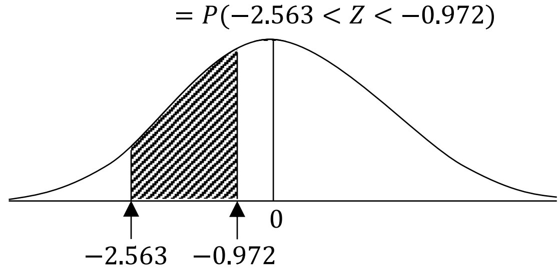
$$= 0.7871$$

(b) $P(25 \le X < 35) = P(25.5 \le X \le 34.5)$

$P(25.5 \le X \le 34.5)$

$$= P\left(\frac{25.5 - 40}{\sqrt{32}} < Z < \frac{34.5 - 40}{\sqrt{32}}\right)$$

$$= P(-2.563 < Z < -0.972)$$



$P(-2.563 < Z < -0.972)$

$$= P(0 < Z < 2.563) - P(0 < Z < 0.972)$$

$$= 0.4952 - 0.3346$$

$$= 0.1606$$

(c) $P(X = 42) = P(41.5 < X < 42.5)$

$$P(41.5 \le X \le 42.5) = P\left(\frac{41.5 - 40}{\sqrt{32}} < Z < \frac{42.5 - 40}{\sqrt{32}}\right)$$

$$= P(0.265 < Z < 0.442)$$

SAMPLING DISTRIBUTIONS

Introduction

In everyday life, people who are working with the same information arrive at different ideas/decisions based on the same information in a given population.
A population can be defined as a collection of all items.
Populations can be at least partially described by population parameters such as mean, variance, proportion, etc.
Because populations are often very large (maybe infinite, like the output of a process) or otherwise hard to investigate, we often have no way to investigate the exact values of the parameters.
Statistics or point estimators and interval estimation are the used to estimate the population parameters.
An estimator is calculated using a function that depends on the information taken from a sample from the population.

Point estimation

Point estimation involves using a statistic from a random sample to find an estimator for the corresponding population parameter.
A statistic is a random variable that is a function of the sample which contains no unknown quantities/parameters.
Examples include

- A calculation based solely on observations from a given sample
- A calculation based only on date from a sample
- A calculation based on known observations from a sample

An **unbiased estimator** is a sample statistic whose expectation is equal to the population parameter.
The **best** or **most efficient estimator** is the unbiased estimator which has the smallest variance.
Some important results are shown in the table below

	Sample statistic	Best estimator for population
Mean	$\bar{x}$	$\bar{x}$ for μ
Variance	s^2	$\frac{ns^2}{n-1}$ for σ^2

The unbiased estimate of the population mean $\bar{X}$ or μ is given by $\bar{x}$ or $\hat{\mu}$ where

$$\boxed{\bar{\boldsymbol{x}} = \frac{\sum \boldsymbol{x}}{\boldsymbol{n}}}$$

The unbiased estimate of the population variance σ^2 is $\hat{\sigma}^2$ where

$$\hat{\sigma}^2 = \frac{n}{n-1}s^2$$

and s^2 is the sample variance.
The sample variance is given by

$$s = \left[\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2\right] \text{ or } s = \frac{\sum(x-\bar{x})^2}{n}$$

Substituting

$$\boxed{\hat{\boldsymbol{\sigma}}^2 = \frac{\boldsymbol{n}}{\boldsymbol{n}-\boldsymbol{1}}\left[\frac{\sum \boldsymbol{x}^2}{\boldsymbol{n}} - \left(\frac{\sum \boldsymbol{x}}{\boldsymbol{n}}\right)^2\right]}$$

or

$$\boxed{\hat{\boldsymbol{\sigma}}^2 = \frac{\boldsymbol{1}}{\boldsymbol{n}-\boldsymbol{1}}\left[\sum \boldsymbol{x}^2 - \frac{(\sum \boldsymbol{x})^2}{\boldsymbol{n}}\right]}$$

or

$$\boxed{\hat{\boldsymbol{\sigma}}^2 = \frac{\boldsymbol{1}}{\boldsymbol{n}-\boldsymbol{1}}\left[\sum \boldsymbol{x}^2 - \boldsymbol{n}\bar{\boldsymbol{x}}^2\right]}$$

or

$$\boxed{\hat{\boldsymbol{\sigma}}^2 = \frac{\boldsymbol{n}}{\boldsymbol{n}-\boldsymbol{1}}\left[\frac{\sum(\boldsymbol{x}-\bar{\boldsymbol{x}})^2}{\boldsymbol{n}}\right] = \frac{\sum(\boldsymbol{x}-\bar{\boldsymbol{x}})^2}{\boldsymbol{n}-\boldsymbol{1}}}$$

Example 1

A computer company repairs large numbers of PCs and wants to estimate the mean time to repair a particular fault. Five pairs are chosen at random from the company's records and the times taken, in seconds, are

205 310 405 195 320

Calculate unbiased estimates of the mean and the variance of the population of repair times from which this sample has been taken.

Solution

$\sum x = 205 + 310 + 405 + 195 + 320 = 1435$

$\sum x^2 = 205^2 + 310^2 + 405^2 + 195^2 + 320^2 = 442575$

$$\bar{x} = \frac{\sum x}{n} = \frac{1435}{5} = 287$$

$$\hat{\sigma}^2 = \frac{1}{n-1}\left[\sum x^2 - n\bar{x}^2\right]$$

$$\hat{\sigma}^2 = \frac{1}{4}(442575 - 5(287)^2) = 7682.5$$

Example 2

The table below gives the frequency distribution of the masses of 113 animals, recorded to the nearest gram.

Mass(g)	41	42	43	44	45	46
Frequency	3	17	36	38	15	4

Calculate the unbiased estimate of the mean and variance of the given frequency distribution

Solution

x	f	fx	fx^2
41	3	123	5043
42	17	714	29988
43	36	1548	66564
44	38	1672	73568
45	15	675	30375
46	4	184	8464
Total	113	4916	214002

Unbiased estimate of the mean is

$$\mu = \bar{x} = \frac{\sum fx}{\sum f} = \frac{4916}{113} = 43.5\text{g}$$

$$\text{Variance, } s = \frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2$$

$$s = \frac{214002}{113} - 43.5^2 = 1.57$$

Unbiased estimate of the variance

$$\hat{\sigma}^2 = \frac{n}{n-1}s^2 = \frac{113}{112} \times 1.573 = 1.46$$

Example 3

A fishing crew recorded the masses in kilograms of 200 fish of a particular species that were caught on their traveler. The results are summarized in the table below. The weights given are mid-class values.

Weight of fish (kg)	Number of fish in class
0.5	21
1.25	32
1.75	33
2.25	24
2.75	18
3.5	21
4.5	16
5.5	12
7.0	11
10.5	12

Assuming that these fish are a random sample from the population of this species, estimate

(a) the mean mass, in kilograms, of a fish of this species

(b) the variance of masses of this species

Solution

x	f	fx	fx^2
0.5	21	10.5	5.25
1.25	32	40	50
1.75	33	57.75	101
2.25	24	54	121.5
2.75	18	49.5	136.1
3.5	21	73.5	257.2
4.5	16	72	324
5.5	12	66	363
7.0	11	77	539
10.5	12	126	1323
Total	200	626.25	3220.1875

(a) To find the mean of the sample, use the formula

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{626.25}{200} = 3.13$$

Since the sample mean is an unbiased estimator of the population mean, the estimate of the mean mass of all the fish in this species is 3.13 kg

(b) An unbiased estimate of the population variance is given by

$$\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$$

$$= \frac{200}{199}\left[\frac{3220.1875}{200} - 3.13^2\right]$$

$$= 6.33 \text{ kg}$$

Interval estimation

Interval estimation involves using the data from a random sample to find an interval within which an unknown population parameter is expected to lie with a given degree of confidence (probability).

The interval is called a **confidence interval** and the two extreme values are called the **confidence limits**.

If a sample has been drawn from a large population with mean μ and variance σ^2, i.e. there is no significant difference between the sample mean $(\bar{x})$ and population mean (μ), then

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

Confidence limits for μ are given by

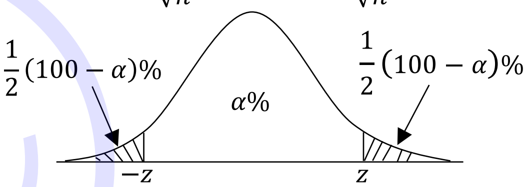
$$|Z| < z \quad \text{i.e.} \quad \left|\frac{\bar{x} - \mu}{\sigma/\sqrt{n}}\right| \leq z$$

$$-z < \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} < z$$

$$-z \cdot \frac{\sigma}{\sqrt{n}} < \bar{x} - \mu < z \cdot \frac{\sigma}{\sqrt{n}}$$

$$z \cdot \frac{\sigma}{\sqrt{n}} > \mu - \bar{x} > -z \cdot \frac{\sigma}{\sqrt{n}}$$

$$\bar{x} + z \cdot \frac{\sigma}{\sqrt{n}} > \mu > \bar{x} - z \cdot \frac{\sigma}{\sqrt{n}}$$



Case 1:

If $\bar{x}$ is the mean of a random sample of size n from $N(\mu, \sigma^2)$, where σ^2 is known, then a symmetric $\alpha\%$ confidence interval for μ is given by

$$\bar{x} \pm z\frac{\sigma}{\sqrt{n}}$$

$$\text{i.e. } \bar{x} - z\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z\frac{\sigma}{\sqrt{n}}$$

where z is the $\frac{1}{2}(100 - \alpha)\%$ point of $N(0, 1)$

$\frac{\sigma}{\sqrt{n}}$ is called the standard error of the mean

Case 2:

If a large sample $(n \geq 30)$ from any distribution has a sampling distribution which is approximately normal $N\left(\mu, \frac{\sigma^2}{n}\right)$, where μ is the mean and σ^2, the variance of the parent population, both unknown, then a symmetric $\alpha\%$ confidence interval for μ is

$$\bar{x} - z\frac{\hat{\sigma}}{\sqrt{n}} < \mu < \bar{x} + z\frac{\hat{\sigma}}{\sqrt{n}}$$

where $\bar{x}$, $\hat{\sigma}^2$ are the mean and variance of the sample and z is the $\frac{1}{2}(100 - \alpha)\%$ point of $N(0, 1)$.

$$= 2 \times 0.3849$$
$$= 0.7698$$

(b) Since the width is measured 16 times, $n = 16$

$$x \sim N\left(w, \frac{0.5^2}{16}\right)$$
$$P(|\bar{x} - w| < 0.3) = P\left(|Z| < \frac{0.3}{0.125}\right)$$
$$= P(|Z| < 2.4)$$
$$= 2 \times P(0 < Z < 2.4)$$
$$= 2 \times 0.4918$$
$$= 0.9836$$

(c) For 98% confidence limits, $\frac{\alpha}{2} = 0.01 \Rightarrow z = 2.326$

$$35.6 \pm 2.326 \times \frac{0.5}{\sqrt{16}}$$
$$35.6 \pm 0.3$$
$$(35.3, 35.9)$$

Example 13

A random sample of the daily sales (in \$) of a small company is taken and, using tables of the normal distribution, a 99% confidence interval for the mean daily sales is found to be

$$(123.5, 154.7)$$

Find a 95% confidence interval for the mean daily sales of the company.

Solution

$$\bar{x} = \frac{1}{2}(123.5 + 154.7) = 139.1$$

For 99% confidence interval, $\frac{\alpha}{2} = 0.005$, $z = 2.576$

$$\bar{x} + z\frac{\sigma}{\sqrt{n}} = 154.7$$
$$139.1 + 2.576\frac{\sigma}{\sqrt{n}} = 154.7$$
$$\frac{\sigma}{\sqrt{n}} = \frac{15.6}{2.576} = 6.056$$

For the 95% confidence interval, $\frac{\alpha}{2} = 0.025$, $z = 1.96$

$$\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}}$$
$$139.1 \pm 1.96 \times 6.056$$
$$139.1 \pm 11.87$$

So 95% $CI = (127.23,\ 150.97)$

How large a sample do you need?

You are now in a position to start to answer the question of how large a sample needs to be. The answer depends on the precision you require, and the confidence level you are prepared to accept.

Example 14

The officer plans to take, secretly, a random sample of n sacks, find the total weight of the coal inside them and thereby estimate the mean weight of the coal per sack. He wants to present this figure correct to the nearest kilogram with 95% confidence. What value of n should he choose?

Solution

The 95% confidence interval for the mean is given by

$$\bar{x} - 1.96\frac{\sigma}{\sqrt{n}} \quad \text{to} \quad \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}$$

and so, since $\sigma = 1.5$, the inspector's requirement is that

$$\frac{1.96 \times 1.5}{\sqrt{n}} \leq 0.5$$
$$\frac{1.96 \times 1.5}{0.5} \leq \sqrt{n}$$
$$n \geq 34.57$$

So the inspector needs to take 35 sacks

Example 15

The mean muscular endurance score of a random sample of 60 subjects was found to be 145 with a standard deviation of 40.

(a) Construct a 95% confidence interval for the true mean.

(b) Assuming that the sample size is large enough for normal approximation, what size of sample is required to estimate the mean within 5 of the true mean with a 95% confidence?

Solution

(a) We are given: $n = 60$, $\bar{x} = 145$ and $s = 40$

95% confidence limits for true mean (μ) are:

$$\bar{x} \pm 1.96\frac{s}{\sqrt{n}} \quad (\sigma^2 = s^2 \text{, since sample is large})$$
$$= 145 \pm 1.96 \times \frac{40}{\sqrt{60}}$$
$$= 145 \pm 10.12$$
$$= 134.88, 155.12$$

Hence 95% confidence interval for μ is (134.88, 155.12)

(b) $|\bar{x} - \mu| < 5 = E$, $z = 1.96$, $\hat{\sigma} = s = 40$

$$E = z \times \frac{\hat{\sigma}}{\sqrt{n}}$$
$$n = \left(\frac{z \times \hat{\sigma}}{E}\right)^2 = \left(\frac{1.96 \times 40}{5}\right)^2 = 245.86 \approx 246$$

Example 16

The guaranteed average life of a certain type of electric light bulbs is 1000 hours with a standard deviation of 125 hours. It is decided to sample the output so as to ensure that 90% of the bulbs do not fall short of the guaranteed average by

EXAMINATION QUESTIONS

SECTION A

1. The table below shows the masses of bolts bought by a carpenter.

Mass (grams)	98	99	100	101	102	103	104
No. of bolts	8	11	14	20	17	6	4

Calculate the;
(a) median mass
(b) median mass of the bolts.

[2019, No. 1: Ans: (a) 101 (b) 100.7625]

2. A discrete random variable X has the following probability distribution:

x	0	1	2	3	4	5
$P(X = x)$	0.11	0.17	0.2	0.13	p	0.09

Find the;
(a) value of p
(b) expected value of X

[2019, No. 4: Ans: (a) 0.3 (b) 2.61]

3. The amount of meat sold by a butcher is normally distributed with mean 43 kg and standard deviation 4 kg. Determine the probability that the amount of meat sold is between 40 kg and 50 kg.

[2019, No. 7: Ans: 0.7333]

4. Two events A and B are such that $P(A|B) = \frac{2}{5}$, $P(B) = \frac{1}{4}$ and $P(A) = \frac{1}{5}$. Find
(a) $P(A \cap B)$
(b) $P(A \cup B)$

[2018, No. 2: Ans: (a) 0.1 (b) 0.35]

5. The price index of an article in 2000 based on 1998 was 130. The price index for the article in 2005 based on 2000 was 80. Calculate the:
(a) price index of the article in 2005 based on 1998
(b) price of the article in 1998 if the price of the article was 45,000 in 2005.

[2018, No. 5: Ans: (a) 104 (b) 43,269.23]

6. A biased coin is such that a head is three times as likely to occur as a tail. The coin is tossed 5 times. Find the probability that at most two tails occur.

[2018, No. 8: Ans: 0.8965]

7. The probability that a patient suffering from a certain disease recovers is 0.4. If 15 people contracted the disease, find the probability that:
(a) more than 9 will recover
(b) between five and eight will recover

[2017, No. 2: Ans: (a) 0.0338 (b) 0.3837]

8. A box A contains 1 white ball and 1 blue ball. Box B contains only 2 white balls. If a ball is picked at random, find the probability that it is:
(a) white.
(b) from box A given that it is white.

[2017, No. 5: Ans: (a) $\frac{3}{4}$ (b) $\frac{1}{3}$]

9. The table below shows the retail price (Shs) and amount of each item bought weekly by a restaurant in 2002 and 2003.

Item	**Price (Shs)**		**Amount bought**
	2002	**2003**	
Milk (per litre)	400	500	200
Eggs (per tray)	2,500	3,000	18
Cooking oil (per litre)	2,400	2,100	2
Baking flour (per packet)	2,000	2,200	15

(a) Taking 2002 as the base year, calculate the weighted aggregate price index.
(b) In 2003, the restaurant spent Shs 450,000 on buying these items. Using the weighted aggregate price index obtained in (a), calculate what the restaurant could have spent in 2002.

[2017, No. 7: Ans: (a) 119.65 (b) 376,096.95]

10. The table below shows the values of two variables P and Q.

$\boldsymbol{P}$	14	15	25	20	15	7
$\boldsymbol{Q}$	30	25	20	18	15	22

Calculate the rank correlation coefficient between the two variables.

[2016, No. 2: Ans: −0.4714]

11. A continuous random variable X has a cumulative distribution function:

$$F(x) = \begin{cases} 0, & x \leq 0 \\ \lambda x^3, & 0 \leq x \leq 4 \\ 1, & x \geq 4 \end{cases}$$

Find the
(a) value of the constant λ
(b) probability density function, $f(x)$

[2016, No. 5: Ans: (a) $\frac{1}{64}$ (b) $f(x) = \begin{cases} \frac{3}{64}x^2 & 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$]

12. A bag contains 5 Pepsi Cola and 4 Mirinda bottle tops. Three bottle tops are picked at random from the bag one after the other without replacement. Find the probability that the bottle tops picked are of the same type.

[2016, No. 8: Ans: 0.1667]

13. The table below shows the mass of boys in a certain school.

Mass (kg)	15	20	25	30	35
Number of boys	5	6	10	20	9

Calculate the mean mass.

[2015, No.3: Ans: 27.2 kg]

14. Events A and B are independent. $P(A) = x$, $P(B) = x + 0.2$ and $P(A \cup B) = 0.65$. Find the value of x.

[2015, No. 5: Ans: 0.3]

15. The marks in an examination were found to be normally distributed with mean 53.9 and standard deviation 16.5. 20% of the candidates who sat this examination failed. Find the pass mark for the examination.

[2015, No. 7: Ans: 40%]

[2004, No. 5: Ans: 0.0439]

48. Eight applicants for a certain job obtained the following marks in aptitude and written tests:

Applicant	A	B	C	D	E	F	G	H
Aptitude test	33	45	15	42	45	35	40	48
Written test	57	60	40	75	58	48	54	68

Calculate a rank correlation coefficient of the applicants' performance in the two tests. Comment on your result.

[2004, No. 7: Ans: 0.78]

49. Events A and B are such that $P(A) = \frac{1}{2}$, $P(B) = \frac{3}{8}$ and $P\left(\frac{A}{B}\right) = \frac{7}{12}$. Find:
   (i) $P(A \cap B)$
   (ii) $P(B|\bar{A})$

[2003, No. 1: Ans: (i) $\frac{7}{32}$ (ii) $\frac{5}{16}$]

50. In an examination, scaling is done such that candidate A who had originally scored 35% gets 50% and candidate B with 40% gets 65%. Determine the original mark for candidate C whose new mark is 80%.

[2003, No. 4: Ans: 45%]

51. The table below shows the marks scored in a mathematics examination by students in a certain school.

Marks	Number of students
$30 - 39$	12
$40 - 49$	16
$50 - 59$	14
$60 - 69$	10
$70 - 79$	8
$80 - 89$	4

   (i) Draw a histogram and use it to estimate the mode.
   (ii) Calculate the mean score

[2003, No. 7: Ans: (i) 46 (ii) 54.19]

52. On a certain day, fresh fish from lakes: Kyoga, Victoria, Albert and George were supplied to one of the central markets of Kampala in the ratios 30%, 40%, 20% and 10% respectively. Each lake had an estimated ratio of poisoned fish of 2%, 3%, 3% and 1% respectively. If a health inspector picked a fish at random,
   (a) what is the probability that the fish was poisoned?
   (b) given that the fish was poisoned, what is the probability that it was from lake Albert?

[2002, No. 1: Ans: (a) 0.025 (b) 0.24]

53. The chance that a person picked from a Kampala street is employed is 30 in every 48. The probability that that person is a university graduate given that he is employed is 0.6. Find the:
   (a) probability that a person picked at random from the street is a university graduate and is employed
   (b) number of people that are not university graduates and are employed from a group of 120 people.

[2002, No. 4: Ans: (a) 0.375 (b) 30]

54. The table below shows the cumulative distribution of the age (in years) of 400 students of a girls' secondary school.

Age (in years)	Cumulative Frequency
< 12	0
< 13	27
< 14	85
< 15	215
< 16	320
< 17	370
< 18	395
< 19	400

Plot an ogive for the data and use it to estimate the:
   (a) median age,
   (b) 20th to 80th percentage range

[2002, No. 7: Ans: (a) 14.9 (b) 2.1]

55. Two events A and B are neither independent nor mutually exclusive. Given that $P(A) = \frac{1}{3}$, $P(A) = \frac{1}{2}$ and $P(A \cap B') = \frac{1}{3}$, find
   (i) $P(A' \cup B')$
   (ii) $P(A'|B')$

[2001, No. 1: Ans: (i) $\frac{5}{6}$ (ii) $\frac{1}{2}$]

56. A coin is biased so that a head is twice as likely to occur as a tail. If the coin is tossed 15 times, determine the,
   (i) expected number of heads,
   (ii) probability of getting at most 2 tails

[2001, No. 4: Ans: (i) 10 (ii) 0.0793]

57. A physics student measured the times taken in seconds for a trolley to run down slopes of varying gradients and obtained the following results:

35.2, 34.5, 33.5, 29.3, 30.9, 31.8

Calculate the mean time and standard deviation.

[2001, No. 6: Ans: 32.53, 2.065]

58. A family plans to have three children.
   (i) Write down the possible sample space and construct its probability distribution table
   (ii) Given that X is the number of boys in family, find the expected number of boys.

[2000, No. 1: Ans: (ii) 1.5]

59. Two balls are randomly drawn without replacement from a bag containing 10 white and 6 red balls. Find the probability that the second ball drawn is
   (i) red given that the first one was white,
   (ii) white

[2000, No. 3: Ans: (i) 0.4 (ii) 0.375]

60. On a certain farm, 20% of the cows are infected by a tick disease. If a random sample of 50 cows is selected from the farm, find the probability that not more than 10% of the cows are infected.

[2000, No. 6: Ans: 0.0558]

SECTION B

1. The table below shows the marks obtained in a Mathematics test by a group of students.

Marks	No. of students
$5-<15$	5
$15-<25$	7
$25-<35$	19
$35-<45$	17
$45-<55$	7
$55-<65$	4
$65-<75$	2
$75-<85$	3

(a) Construct a cumulative frequency curve (Ogive) for the data.
(b) Use your Ogive to find the;
(i) range between the 10^{th} and 70^{th} percentiles.
(ii) probability that a student selected at random scored below 50 marks.

[2019, No. 10]

2. Two events A and B are such that $P(B)=\frac{1}{8}$, $P(A\cap B)=\frac{1}{10}$ and $P(B|A)=\frac{1}{3}$. Determine the;
(a) $P(A)$
(b) $P(A\cup B)$
(c) $P(A|\bar{B})$

[2019, No. 13: Ans: (a) 0.3 (b) 0.325 (c) 8/35]

3. A biased die with faces labelled 1, 2, 2, 3, 5 and 6 is tossed 45 times. Calculate the probability that 2 will appear;
(a) more than 18 times
(b) exactly 11 times

[2019, No. 16: (a) 0.1339 (b) 0.0568]

4. The frequency distribution below shows the ages of 240 students admitted to a certain university.

Age (years)	**Number of students**
$18-<19$	24
$19-<20$	70
$20-<24$	76
$24-<26$	48
$26-<30$	16
$30-<32$	6

(a) Calculate the mean age of the students
(b) (i) Draw a histogram for the given data
(ii) Use the histogram to estimate the modal age

[2018, No. 9: Ans: (a) 22.15 (b) (ii) 19.5]

5. The table below shows the number of red and green balls put in three identical boxes A, B and C.

Boxes	A	B	C
Red balls	**4**	**6**	**3**
Green balls	**2**	**7**	**5**

A box is chosen at random and two balls are then drawn from it successively without replacement. If the random variable X is "the number of green balls drawn",
(a) draw a probability distribution table for X
(b) calculate the mean and variance of X

[2018, No. 15: Ans: (b) 0.9979, 0.462]

6. A random variable X has a normal distribution where $P(X>9)=0.9192$ and $P(X<11)=0.7580$. Find:
(a) the values of the mean and standard deviation
(b) $P(X>10)$

[2018, No. 12.: Ans: (a) 10.33, 0.9254 (b) 0.6368]

7. A discrete random variable X has a probability distribution given by

$$P(X=x)=\begin{cases} kx, & x=1,2,3,4,5. \\ 0, & \text{Otherwise} \end{cases}$$

where k is a constant.
Determine;
(a) the value of k
(b) $P(2<X<5)$
(c) Expectation, $E(X)$
(d) Variance, Var(X)

[2017, No. 9: Ans: (a) $\frac{1}{15}$ (b) $\frac{7}{15}$ (c) $\frac{11}{3}$ (d) $\frac{14}{9}$]

8. The times taken for 55 students to have their lunch to the nearest minute are given in the table below.

Time(minutes)	3 – 4	5 – 9	10 – 19	20 – 29	30 – 44
No. of students	2	7	16	21	9

(a) Calculate the mean time for the students to have lunch
(b) (i) Draw a histogram for the given data
(ii) Use your histogram to estimate the modal time for the students to have lunch.

[2017, No. 12: Ans: (a) 20.65 (b) (ii) 22]

9. The number of cows owned by residents in a village is assumed to be normally distributed. 15% of the residents have less than 60 cows. 5% of the residents have over 90 cows.
(a) Determine the values of the mean and standard deviation of the cows.
(b) If there are 200 residents, find how many have more than 80 cows.

[2017, No. 15: Ans: (a) 11.19; 71.59 (b) 45]

10. The data below shows the length in centimetres of different calendars produced by a printing press. A cumulative frequency distribution was formed.

Length(cm)	<20	<30	<35	<40	<50	<60
Cumulative frequency	4	20	32	42	48	50

(a) Construct a frequency distribution table
(b) Draw a histogram and use it to estimate the modal length
(c) Find the mean length of the calendars

[2016, No. 9: Ans: (b) 33.5 (c) 31.7]

11. A newspaper vendor buys 12 copies of a sports magazine every week. The probability distribution for the number of copies sold in a week is given in the table below.

Number of copies sold	9	10	11	12
Probability	0.2	0.35	0.30	0.15

(a) Estimate the

(ii) $P(\pi/3 < X < 3\pi/4)$

(b) Show that the mean, μ of the distribution is $1+\frac{\pi}{4}$

[2008, No. 12: Ans: (a) (i) $\frac{2}{\pi}$ (ii) 0.6982]

36. (a) Sixty students sat for a mathematics contest whose pass mark was 40 marks. Their scores in the contest were approximately normally distributed. 9 students scored less than 20 marks while 3 scored more than 70 marks. Find the:

(i) mean score and standard deviation of the contest

(ii) probability that a student chosen at random passed the contest

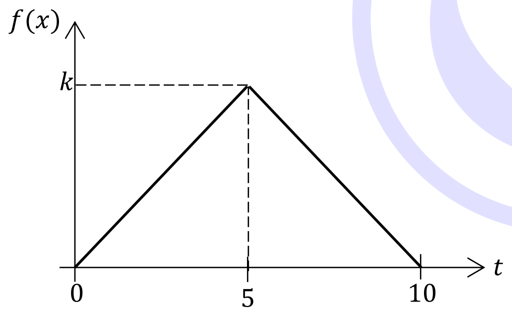
(b) The times a machine takes to print each of the 10 documents were recorded in minutes as given below:

16.5, 18.3, 18.5, 16.6, 19.4, 16.8, 18.6, 16.0, 20.1, 18.2.

If the times of printing the documents are approximately normally distributed with variance of 2.56 minutes, find the 80% confidence interval for the mean time of printing the documents.

[2008, No. 15: Ans: (a)(i) 39.32, 18.65 (ii) 0.4856 (b) (17.25, 28.55)]

37. The departure time, T of pupils from a certain day primary school can be modelled as in the diagram below, where t is the time in minutes after the final bell at 5: 00 p.m.



Determine the:

(a) value of k,

(b) equations of the p.d.f.

(c) $E(T)$

(d) probability that a pupil leaves between 4 and 7 minutes after the bell

[2007, No. 9: Ans: (a) $\frac{1}{5}$ (b) $f(t)=\begin{cases}\frac{1}{25}t\ ; & 0\le t<5\\ \frac{1}{25}(10-t)\ ; & 5\le t\le 10\\ 0 & \text{otherwise}\end{cases}$

(c) 5 (d) 0.5]

38. Below are marks scored by 8 students A, B, C, D, E, F, G and H in Mathematics, Economics and Geography in the end of term examinations.

	A	B	C	D	E	F	G	H
Mathematics	52	75	41	60	81	31	65	52
Economics	50	60	35	65	66	66	69	48
Geography	35	40	60	54	63	63	55	72

Calculate the rank correlation coefficients between the performances of the students in:

(i) Mathematics and Economics

(ii) Geography and Mathematics

Comment on the significance of Mathematics in the performance of Economics and Geography

[Spearman, $\rho = 0.86$, Kendalls, $\tau = 0.79$ based on 8 observations at 1% level of significance]

[2007, No. 12: Ans: (i) 0.8512 (ii) 0.1905]

39. (a) A box contains 7 red balls and 6 blue balls. Three balls are selected at random without replacement. Find the probability that:

(i) they are of the same colour

(ii) at most two are blue

(b) Two boxes P and Q contain white and brown cards. P contains 6 white cards and 4 brown cards. Q contains 2 white cards and 3 brown cards. A box is selected at random and a card is selected. Find the probability that:

(i) a brown card is selected

(ii) box Q is selected given that the card is white.

[2007, No. 15: Ans: (a)(i) 0.1923 (ii) 0.9301 (b)(i) 0.5 (ii) 0.4]

40. (a) Among the spectators watching a football match, 80% were the home team's supporters while the rest were the visitor team's supporters. If 2500 of the spectators are selected at random, what is the probability that there were more than 540 visitors in this sample?

(b) The times a factory takes to make units of a product are approximately normally distributed. A sample of 49 units of the product was taken and found to take an average of 50 minutes with a standard deviation of 2 minutes.

Calculate the 99% confidence limits of the mean time of making all units of the product.

[2006, No. 9: Ans: (a) 0.0215 (b) (49.264, 50.736)]

41. The table below is the distribution of weights of a group of animals.

Mass (kg)	Frequency
$21-25$	10
$26-30$	20
$31-35$	15
$36-40$	10
$41-50$	30
$51-65$	45
$66-75$	5

(a) Draw a cumulative frequency curve and use it to estimate the semi-interquartile range.

(b) Find the:

(i) mode,

(ii) standard deviation of the weights.

[2006, No. 12: Ans: (a) 24 (b)(i) 28.83 (ii) 11.77]

42. A continuous random variable X has a probability density function given by

[2004, No. 14: Ans: (a) 160.9, 5.6 (c) (i) 161 (ii) 10]

49. Given the cumulative distribution function,

$$F(x) = \begin{cases} \frac{x^2 - 1}{2} - x \ ; 1 \le x < 2; \\ 3x - \frac{x^2}{2} \ ; 2 \le x < 3; \\ 1 \ ; \ x \ge 3 \end{cases}$$

(a) Find
   (i) the p.d.f.
   (ii) $P(1.2 < x < 2.4)$
   (iii) the mean of x
(b) Sketch $f(x)$

[2003, No. 10: Ans: (a) (i) $f(x) = \begin{cases} x - 1 & ; \ 1 \le x < 2 \\ 3 - x & ; \ 2 \le x < 3 \\ 0 & \text{elsewhere} \end{cases}$ (ii) 0.8 (iii) 2]

50. In a school of 800 students, their average weight is 54.5 kg and standard deviation 6.8 kg. If the weights of all the students in the school assume a normal distribution, find the
   (i) probability that a student picked at random weighs 52.8 or less kg.
   (ii) number of students who weigh over 75 kg
   (iii) weight range of the middle 56% of the students of the school

[2003, No. 13: Ans: (i) 0.4013 (ii) 1 (iii) $49.25 - 59.75$]

51. The table below shows the percentage of sand, y, in the soil at different depths x (in cm).

Soil depth (x)	Percentage of sand, (y)
35	86
65	70
55	84
25	92
45	79
75	68
20	96
90	58
51	86
60	77

(a) (i) Plot a scatter diagram for the data. Comment on the relationship between the depth of the soil and the percentage of sand in the soil.
   (ii) Draw a line of best fit through the points of the scatter diagram. Use it to estimate the
   - percentage of sand in the soil at the depth of 31 cm
   - depth of soil with 54% sand

(b) Calculate the rank correlation coefficient between the percentage of sand in the soil and the depth of soil.

[2003, No. 15: Ans: (b) -0.9485]

52. A pair of dice is tossed 180 times, determine the probability that a sum of 7 appears:
   (a) exactly 40 times,
   (b) between 25 and 35 inclusive times.

[2002, No. 10: Ans: (a) 0.0108 (b) 0.7286]

53. (a) A random variable X has the probability density function

$$f(x) = \begin{cases} \frac{1}{b-a} \ , & a \le x \le b; \\ 0 \ \ , & \text{else where} \end{cases}$$

Show that the variance of X is $\frac{(b-a)^2}{12}$

(b) During rush hours, it was observed that the number of vehicles departing for Entebbe from Kampala old taxi park take on a random variable X with a uniform distribution over the interval $[x_1, x_2]$. If in one hour, the expected number of vehicles leaving the stage is 12, with variance of 3, calculate the:
   (i) values of x_1 and x_2
   (ii) probability that at least 11 vehicles leave the stage.

[2002, No. 11: Ans: (b) (i) $x_1 = 9, x_2 = 15$ (ii) $\frac{2}{3}$]

54. The times taken by a group of students to solve a mathematical problem are given below:

Time (min)	No. of students
$5 - 9$	5
$10 - 14$	14
$15 - 19$	30
$20 - 24$	17
$25 - 29$	11
$30 - 34$	3

(a) Draw a histogram for the data. Use it to estimate the modal time for solving a problem
(b) Calculate a mean time and standard deviation of solving a problem.

[2002, No. 14: Ans: (a) 17.3 (b) 18.5, 5.99]

55. (a) Bag A contains 2 green and 2 blue balls, while bag B contains 2 green and 3 blue balls. A bag is selected at random and two balls drawn from it without replacement. Find the probability that the balls drawn are of different colours.

(b) A fair die is rolled 6 times. Calculate the probability that
   (i) a 2 or 4 appears on the first throw,
   (ii) four 5 s will appear in the six throws.

[2001, No. 10: Ans: (a) $\frac{19}{30}$ (b) (i) $\frac{1}{3}$ (ii) 0.008]

56. The table below shows the weights to the nearest kg of 150 patients who visited a certain health unit during a certain week:

Weight (kg)	No. of patients (f)
$0 - 19$	30
$20 - 29$	16
$30 - 39$	24
$40 - 49$	32
$50 - 59$	28
$60 - 69$	12
$70 - 79$	8

Numerical Methods

INTRODUCTION

Numerical methods are techniques where mathematics problems are formulated as a series of arithmetic and logical operations. An example is if we have a function $y = f(x)$, we can approximate the derivative as

$$\frac{dy}{dx} \cong \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

If you recall from calculus, the definition of the derivative of the function is

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

In numerical methods, we just drop the limit and just use a Δx that is really very small and in so doing we can approximate the derivative by a couple of simple arithmetic operations i.e. addition, subtraction, function evaluation and division. The process is implemented usually with an algorithm, which is a series of steps and a strategy.

Why study Numerical methods

1. Numerical methods help us to solve problems that are difficult/ tedious or impossible to solve analytically.
   Some simple equations can be solved analytically i.e. $x^2 + 4x + 3 = 0$.
   Analytic solution roots = $\frac{-4\pm\sqrt{4^2-4(1)(3)}}{2(1)}$
   $$x = -1 \text{ and } x = -3$$
   Many other equations have no analytical solutions i.e. $x^9 - 2x^5 + 5 = 0$, $x = e^{-x}$, $x = \cos x$, etc.
2. Numerical methods form the "guts" of most analysis software in engineering and science. They underly finite element analysis used in structures and mechanical design, computational fluid dynamics, electric circuit modelling, only to mention but a few.
3. Studying numerical methods can enhance your understanding of higher-level mathematics. (calculus, differential equations, linear algebra, etc.).
   Just like the numerical algorithm for the derivative example I provided at the beginning, you will be taking a series of such higher-level mathematical concepts and break them down in a series of arithmetic operations and in so doing, that can enhance your understanding of how those higher-level mathematical concepts actually work.

Examples of Numerical Methods

- Finding the roots of non-linear equations, a case where we have an equation $f(x) = 0$ and we would like to find x. Sometimes that is not possible to do analytically and we need a numerical method to do that. The numerical methods to solve $f(x) = 0$ that are covered in this book are graphical method, general iterative method (fixed point iteration) and the Newton's method.
- Solving systems of algebraic equations from different mathematical models that may include linear systems and non-linear systems of equations. These may be impossible to solve analytically.
  For example, there is an analytic solution for the motion of two objects interacting gravitationally but no solution for three or more objects. This is why we can prove that two objects orbit in ellipses but the motion of three or more bodies is unpredictable. This inability is the infamous "three body problem". It shows up in atoms as well; we can analytically describe the shape of electron orbitals and energy levels in individual hydrogen atoms (1 proton + 1 electron = 2 bodies), but not easy for other elements.
  The numerical method in this case will describe how individual pieces interact with neighbours, then assemble a large number of pieces and the resulting equations are relatively simple to solve since they have to be solved in the context of their neighbours. Now we can bring to bear a large body of knowledge regarding linear algebra and matrices to solve the problems.
- Fitting mathematical models to experimental data. This can be done by curve fitting and how to assess that a model you fit to assess a set of data is an appropriate model for that data.
  In this book we have considered modelling mathematical situations/models by representing algorithms by flow charts or flow diagrams and assessing the purpose of the flow diagrams.
- Interpolation and extrapolation that allows us to estimate intermediate values and also values that are at the extreme ends of a table data.
- Numerical integration which includes integrating data directly and integrating functions. Classic examples of integrals that cannot be integrated analytically include $\int e^{x^2}dx$, $\int_0^{\pi}\sqrt{1+\cos^2 x}\,dx$, etc. Of course, the integrals exist that is obviously the area under the curves described by the functions but it is impossible using the analytical methods you learn in calculus to determine the above integrals.

Numerical methods generally give an approximation of the true/exact solution and the method to be used depends on the following questions.

- How good is our approximation? (**Error analysis**)
- How fast and efficient is our method? (**Algorithm design, Convergence rate**)
- Does our method always work? (**Convergence**)

There are so many other numerical methods that are beyond the scope of this book such as numerical differentiation, solutions to ordinary differential equations, boundary value problems, least squares method among others.

Numerical Analysis

This is the application of an appropriate numerical method to a physical problem in a systematic manner to arrive at a solution and to make interpretation of the said solution with the help of the various fast and efficient computing tools.
Thus, an Applied mathematician needs to understand the numerical methods very well such that he can be able to apply the methods to solve problems and model systems.
He therefore needs to understand enough theory to understand the techniques, know their strengths and weaknesses, and how to implement them using MATLAB and other programming software.

ERRORS

No measurement made is ever exact. The accuracy (correctness) and precision (number of significant figures) of a measurement are always limited by a degree of refinement of the apparatus used, by the skill of the observer, and by the basing physics in the experiment.
In doing experiments, we are trying to establish the best values for certain quantities, or trying to validate a theory. We also give a range of possible true values based on our limited number of measurements.
Also numerical solutions are an approximation and thus need to be examined closely. One way of doing this is establishing the error, that is how far an approximation is from the true value.
Error in computation is the difference between the exact value and the approximated value.

Error = Approximate value – Exact value

Since we are usually interested in the magnitude or absolute of the error, we define

Absolute error = |Approximate value – Exact value|

The absolute error can sometimes be misleading. Even if the absolute error is small in magnitude, the approximation may be grossly inaccurate if the exact value is even smaller in magnitude. For this reason, it is preferable to measure accuracy in terms of the relative error.

$$\textbf{Relative error} = \frac{\textbf{Absolute error}}{\textbf{Exact value}}$$

The relative error is usually expressed as a percentage

$$\textbf{Percentage error} = \frac{\textbf{Absolute error}}{\textbf{Exact value}} \times \mathbf{100\%}$$

Types of errors

We shall discuss basically three types of errors i.e. random errors, round-off errors and truncation errors

Random errors

Random errors arise from the fluctuations that are most easily observed by making multiple trials of a given measurement. For example, if you were to measure the period of a pendulum many times with a stop watch, you would find that your measurements were not always the same.
The main source of these fluctuations would probably be the difficulty of judging exactly when the pendulum came to a given point in its motion, and in starting and stopping the stop watch at the time that you judge.

Round-off errors

Round-off error is the difference between a rounded-off numerical value and the actual value. A rounded quantity is represented by a numeral with a fixed number of allowed digits, with the last digit set to the value that produces the smallest difference between the rounded quantity and the actual quantity.
Rounding can produce a value that is easier to deal with than the actual value, especially if the actual value contains a lot of digits. Rounding can also be done to indicate the relative precision of a value. For example, the irrational number pi equals approximately 3.14, rounded to two decimal places or three significant digits.

Truncation errors (Round by chop)

These errors occur when an infinite process is terminated at a certain point or approximated by a finite one. Examples include

- Taylor's series where terms of higher derivatives are ignored.
- Approximate calculation of derivatives

The Taylor series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

If the formula is used to calculate $e^{0.3}$ we get

$$e^{0.3} = 1 + 0.3 + \frac{0.3^2}{2!} + \cdots + \frac{0.3^n}{n!} + \cdots$$

Where do we stop? How many terms do we include? Theoretically the calculation will never stop. There are always more terms to add on. If we do stop after a finite number of terms, we will not get the exact answer. For example, if we do take the first four terms as the approximation, we get

$$r = e^{0.3} \approx 1 + 0.3 + \frac{0.3^2}{2!} + \frac{0.3^3}{3!} = \bar{r}$$

For this calculation, the truncation error is

$$\text{Truncation error } = r - \bar{r}$$

Note:

The truncation error is dependent on the specific numerical method or algorithm used to solve a problem and it is independent of the round-off error.

The following example illustrates the two rounding rules i.e. truncation and round-off to 1 decimal place.

x	Truncation	Error	Round-to nearest	Error
1.649	1.6	0.049	1.6	0.049
1.650	1.6	0.050	1.7	0.050
1.651	1.6	0.051	1.7	0.049
1.699	1.6	0.099	1.7	0.001
1.749	1.7	0.049	1.7	0.049
1.750	1.7	0.050	1.8	0.050

Limits of accuracy (Error bound)

A basic question associated with any approximation is "How good is the approximation?". Thus we introduce the term "error bound" an upper bound on the size of the error.
It is important to realize that although the absolute value of the error may be considerably smaller than the error bound, it can never be larger. In general, the smaller the error bound, the better the approximation.

Tolerance, abbreviated as TOL, is often used as synonym for error bound.

Sometimes the degree of accuracy needed in the approximations is specified by saying that it must be accurate to a given number of decimal places.

One says that x an approximation of a quantity X, is accurate to n decimal places if

$$|x - X| < \frac{1}{2} \times 10^{-n}$$

This means that the true value of X lies between $X - \frac{1}{2}(10^{-n})$ and $X + \frac{1}{2}(10^{-n})$

Thus, the error bound can be obtained from the maximum and minimum values of the function from the formula

$$\text{Error bound} = \frac{1}{2}(\text{Maximum value} - \text{Minimum value})$$

Range of values within which the exact/true value lies

The range of values is obtained by adding and subtracting the error bound to and from the true value respectively.

Maximum value = True value + Error bound

Minimum value = True value – Error bound

Example 1

Given $x = 2.2255$, $y = 0.449$ correct to the given number of decimal places. State the maximum possible errors in the value of x and y. Hence determine the

(a) absolute error

(b) limits within which the value of quotient $\frac{x}{y}$ lies giving your answers to 2 decimal places

Solution

Maximum possible error in the values x and y is given by

$$\text{Error} = \frac{1}{2} \times 10^{-n}$$

For $x = 2.2255$, 4 d.p.

$$\Delta x = \frac{1}{2} \times 10^{-4} = 0.00005$$

For $y = 0.449$, 3 d.p.

$$\Delta y = \frac{1}{2} \times 10^{-3} = 0.0005$$

(a) Maximum value $= \frac{x_{max}}{y_{min}} = \frac{2.22555}{0.4485}$

$= 4.96221$

Minimum value $= \frac{x_{min}}{y_{max}} = \frac{2.22545}{0.4495}$

$= 4.95095$

Absolute value in $\frac{x}{y} = \frac{1}{2}$(max. value-min. value)

$$= \frac{1}{2}(4.96221 - 4.95095)$$

$$= 0.00563$$

(b) Limits within which the exact value of x/y lies

$$= \text{Working value} \pm \text{Error}$$

$$= \frac{x}{y} \pm \text{Error}$$

$$= \frac{2.2255}{0.449} \pm 0.00563$$

$$= 4.95657 \pm 0.00563$$

Hence the limits are 4.95 and 4.96 (2 d.p.)

Example 2

The radius of a sphere is measured as (2.1 ± 0.5) cm. Calculate the limits within which its surface area lies.

Solution

$r = 2.1$, $\Delta r = 0.5$, Let A = surface area

$$A = 4\pi r^2 = 4\pi(2.1)^2 = 55.418 \text{ cm}^2$$

$$A_{max} = 4\pi r_{max}^2 = 4\pi(2.6)^2 = 84.949 \text{ cm}^2$$

$$A_{min} = 4\pi r_{min}^2 = 4\pi(1.6)^2 = 32.170 \text{ cm}^2$$

Absolute error in $A = \frac{1}{2}(84.949 - 32.170) = 26.2895$

Limits within which the surface area lies are given by

$$55.418 \pm 26.2895 \text{ cm}^2$$

$$\text{Or } [29.1285 \text{ cm}^2, 81.7075 \text{ cm}^2]$$

Example 3

Given that the values $x = 4$, $y = 6$ and $z = 8$ each has been approximated to the nearest integer. Find the maximum and minimum values of

(a) $\frac{y}{x}$

(b) $\frac{z-x}{y}$

(c) $(x + y)z$

Solution

(a) $\left(\frac{y}{x}\right)_{max} = \frac{y_{max}}{x_{min}} = \frac{6.5}{3.5} = 1.85714$

$\left(\frac{y}{x}\right)_{min} = \frac{y_{min}}{x_{max}} = \frac{5.5}{4.5} = 1.2222$

(b) $\left(\frac{z-x}{y}\right)_{max} = \frac{z_{max}-x_{min}}{y_{min}} = \frac{8.5-3.5}{5.5} = 0.90909$

$\left(\frac{z-x}{y}\right)_{min} = \frac{z_{min}-x_{max}}{y_{max}} = \frac{7.5-4.5}{6.5} = 0.46154$

(c) $[(x + y)z]_{max} = (x_{max} + y_{max})z_{max}$

$= (4.5 + 6.5)8.5 = 93.5$

$[(x + y)z]_{min} = (x_{min} + y_{min})z_{min}$

$= (3.5 + 5.5)7.5 = 67.5$

Example 4

Given that $A = 2.5$, $B = 1.710$, $C = 16.01$, correct to the given number of decimal places. State the maximum possible errors in the values of A, B and C. Hence determine the limits within which the value of $\frac{AB}{C-B}$ lies giving your answer to 3 decimal places.

Solution

$\Delta A = 0.05$, $\Delta B = 0.0005$, $\Delta C = 0.005$

$$\left(\frac{AB}{C - B}\right)_{max} = \frac{(AB)_{max}}{(C - B)_{min}} = \frac{A_{max}B_{max}}{C_{min} - B_{max}}$$

$$= \frac{2.55 \times 1.7105}{16.005 - 1.7105} = 0.30514$$

$$= 0.305 \text{ (3 d.p)}$$

$$\left(\frac{AB}{C - B}\right)_{min} = \frac{(AB)_{min}}{(C - B)_{max}} = \frac{A_{min}B_{min}}{C_{max} - B_{min}}$$

$$= \frac{2.45 \times 1.7095}{16.015 - 1.7095} = 0.2928$$

$$= 0.293 \text{ (3 d.p)}$$

$\therefore$ The limits are [0.293, 0.305]

Function, z	Powers	$\frac{\Delta z}{z}$
x^2y	$m=2, n=1$	$\frac{2\lvert\Delta x\rvert}{x}+\frac{\lvert\Delta y\rvert}{y}$
xy^2	$m=1, n=2$	$\frac{\lvert\Delta x\rvert}{x}+\frac{2\lvert\Delta y\rvert}{y}$
$\frac{x^2}{y}$	$m=2, n=-1$	$\frac{2\lvert\Delta x\rvert}{x}+\frac{\lvert\Delta y\rvert}{y}$
$\frac{x}{y^2}$	$m=1, n=-2$	$\frac{\lvert\Delta x\rvert}{x}+\frac{2\lvert\Delta y\rvert}{y}$

Do you notice the relationship in the maximum relative errors in the table above?
It agrees with the fact that the maximum relative error of the product or quotient of two variables is the sum of the relative errors of the individual variables.

Example 9

Given that $x = 4.52 \pm 0.02$, $A = 2.0 \pm 0.2$, $y = 3.0 \pm 0.6$ and $z = xy^2/\sqrt{A}$. Find the range of values within which lies.

Solution

$\Delta x = 0.02$, $\Delta A = 0.2$ and $\Delta y = 0.6$

$$z = \frac{xy^2}{\sqrt{A}} = \frac{4.52(3.02)^2}{\sqrt{2.0}} = 29.15$$

The maximum relative error in the product /quotient is the sum of the relative errors of the individual numbers

$$\frac{\Delta z}{z} = \frac{\Delta x}{x} + 2\frac{\Delta y}{y} + \frac{1}{2}\frac{\Delta A}{A}$$

The second relative error, $(\Delta y/y)$ is multiplied by 2 because the power of y is 2.

The third relative error, $(\Delta A/A)$ is multiplied by $\frac{1}{2}$ since a square root is power of one half.

$$\frac{\Delta z}{29.15} = \frac{0.02}{4.52} + \frac{2(0.6)}{3.0} + \frac{1}{2}\left(\frac{0.2}{2.0}\right)$$

$$\frac{\Delta z}{29.15} = 0.4544$$

$$\Delta z = 13.25$$

$$\therefore z = 29.15 \pm 13.25$$

Example 10

Find the maximum absolute error in the expression $\frac{\sqrt{A}}{B^2C^3}$, given that $A = 2.8$, $B = 6.4$ and $C = 3.4$, if all values are rounded-off to the given number of decimal places.

Solution

$$\text{Let } z = \frac{\sqrt{A}}{B^2C^3} = A^{\frac{1}{2}}B^{-2}C^{-3}$$

$$z = \frac{\sqrt{2.8}}{(6.4)^2(3.4)^3} = 0.001$$

$\Delta A = 0.05$, $\Delta B = 0.05$, $\Delta C = 0.05$

$$\frac{\Delta z}{z} = \frac{1}{2}\frac{\Delta A}{A} + 2\frac{\Delta B}{B} + 3\frac{\Delta C}{C}$$

$$\frac{\Delta z}{0.001} = \frac{1}{2}\left(\frac{0.05}{2.8}\right) + 2\left(\frac{0.05}{6.4}\right) + 3\left(\frac{0.05}{3.4}\right)$$

$$\frac{\Delta z}{0.001} = 0.0687$$

$$\Delta z = 0.0000687$$

The maximum absolute error in the expression is 0.0000687

Example 11

A physical quantity X is given by $X = \frac{A^2B^3}{C\sqrt{D}}$.

If the percentage errors of measurement in A, B, C and D are 4%, 2%, 3% and 1% respectively, the calculate the percentage error in X.

Solution

$$X = \frac{A^2B^3}{C\sqrt{D}} = \frac{A^2B^3}{CD^{\frac{1}{2}}}$$

$$\frac{\Delta X}{X} = 2\frac{\Delta A}{A} + 3\frac{\Delta B}{B} + \frac{\Delta C}{C} + \frac{1}{2}\frac{\Delta D}{D}$$

$$\frac{\Delta X}{X} = 2(0.04) + 3(0.02) + 0.03 + \frac{1}{2}(0.01)$$

$$\frac{\Delta X}{X} = 0.175$$

$$\frac{\Delta X}{X} \times 100 = 17.5\%$$

$\therefore$ The percentage error in X is 17.5%

Example 12

If $x = 5.94$ and $y = 3.012$ are each rounded off to the given number of decimal places, calculate the:

(a) percentage error in $\frac{y^2}{x}$

(b) limits within which $\frac{y^2}{y-x}$ is expected to lie.

Give your answers correct to 3 decimal places.

Solution

$\Delta x = 0.005$, $\Delta y = 0.0005$

(a) Let $z = \frac{y^2}{x}$

$$\frac{\Delta z}{z} = 2\frac{\Delta y}{y} + \frac{\Delta x}{x}$$

$$\frac{\Delta z}{z} = 2\left(\frac{0.0005}{3.012}\right) + \frac{0.005}{5.94} = 0.001174$$

$$\frac{\Delta z}{z} \times 100 = 0.1174$$

Percentage error in $\frac{y^2}{x} = 0.117\%$

(b) Let $z = \frac{y^2}{y-x}$

$$\frac{\Delta z}{z} = 2\frac{\Delta y}{y} + \left|\frac{\Delta y - \Delta x}{y - x}\right|$$

$$\frac{\Delta z}{z} = 2\frac{\Delta y}{y} + \frac{\Delta y}{|y-x|} + \frac{\Delta x}{|y-x|}$$

$$\frac{\Delta z}{z} = 2\left(\frac{0.0005}{3.012}\right) + \frac{0.0005}{2.928} + \frac{0.005}{2.928}$$

$$\frac{\Delta z}{z} = 0.0022$$

$$\Delta z = 0.0022z$$

$$z = \frac{3.012^2}{3.012 - 5.94} = -3.0984$$

$$|\Delta z| = 0.0022 \times 3.0984 = 0.00682$$

$$\frac{\Delta z}{z} = |\Delta x| \tan x$$

(c) $z = \tan x$

$$\Delta z = \tan(x + \Delta x) - \tan x$$
$$\Delta z = \frac{\sin(x + \Delta x)}{\cos(x + \Delta x)} - \frac{\sin x}{\cos x}$$
$$\Delta z = \frac{\cos x \sin(x + \Delta x) - \sin x \cos(x + \Delta x)}{\cos x \cos(x + \Delta x)}$$
$$\Delta z = \frac{\cos x[\sin x \cos \Delta x + \sin \Delta x \cos x] - \sin x[\cos x \cos \Delta x - \sin x \sin \Delta x]}{\cos x \cos(x+\Delta x)}$$
$$\Delta z = \frac{\cos x \sin x \cos \Delta x + \cos^2 x \sin \Delta x - \sin x \cos x \cos \Delta x + \sin^2 x \sin \Delta x}{\cos x \cos(x+\Delta x)}$$
$$\Delta z = \frac{\cos^2 x \sin \Delta x + \sin^2 x \sin \Delta x}{\cos x \cos(x + \Delta x)}$$
$$\Delta z = \frac{(\cos^2 x + \sin^2 x) \sin \Delta x}{\cos x \cos(x + \Delta x)}$$
$$\Delta z = \frac{\sin \Delta x}{\cos x \cos(x + \Delta x)}$$
$$\Delta z = \frac{\Delta x \left(\frac{\sin \Delta x}{\Delta x}\right)}{\cos x \cos(x + \Delta x)}$$

Assuming $\Delta x \approx 0$, $\frac{\sin \Delta x}{\Delta x} = 1$, $\cos(x + \Delta x) = \cos x$

$$\Delta z = \frac{\Delta x}{\cos^2 x}$$
$$|\Delta z| = \left|\frac{\Delta x}{\cos^2 x}\right|$$
$$\leq \frac{|\Delta x|}{\cos^2 x}$$
$$\frac{\Delta z}{z} = \frac{|\Delta x|}{\cos^2 x} \div \tan x$$
$$\frac{\Delta z}{z} = \frac{|\Delta x|}{\cos^2 x} \times \frac{\cos x}{\sin x}$$
$$\frac{\Delta z}{z} = \frac{|\Delta x|}{\sin x \cos x}$$

6. Other functions:

(a) Getting formulas using partial derivatives

The general method of generating formulas for propagating errors involves the total differential of a function. Suppose $z = f(x, y, ...)$ where the variables x, y, etc. must be independent variables, the total differential is then

$$dz = \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy + \cdots$$

We treat $dx = \Delta x$ as the error in x, and likewise for other differentials dz, dy, etc.

The general result is

$$\Delta z = \left|\frac{\partial f}{\partial x}\right| \Delta x + \left|\frac{\partial f}{\partial y}\right| \Delta y + \cdots$$

(b) Getting formula's using Taylor's series

Consider a polynomial function $f(x)$ of any degree. If a function $z = f(x)$ is approximated as $Z = f(X)$, then

$$\Delta z = Z - z$$
$$\Delta z = f(X) - f(x)$$
$$\Delta z = f(x + \Delta x) - f(x)$$

From Taylor's series;

$$f(x + \Delta x) = f(x) + \Delta x f'(x) + (\Delta x)^2 \frac{f''(x)}{2!} + \cdots$$

Neglecting the terms involving $(\Delta x)^2$ and higher powers of Δx

$$f(x + \Delta x) = f(x) + \Delta x f'(x)$$

Now

$$\Delta z = f(x) + \Delta x f'(x) - f(x)$$
$$\Delta z = \Delta x\, f'(x)$$
$$|\Delta z| = |\Delta x\, f'(x)|$$
$$\left|\frac{\Delta z}{z}\right| = \left|\frac{\Delta x\, f'(x)}{f(x)}\right|$$

Note:

1. If n is an exact number and $z = x^n$, then
$$dz = nx^{n-1}dx$$
Equivalently; $\Delta z = nx^{n-1}\Delta x$
$$\left|\frac{\Delta z}{z}\right| = \left|\frac{nx^{n-1}\Delta x}{x^n}\right|$$
$$\boldsymbol{\left|\frac{\Delta z}{z}\right| = |n| \frac{|\Delta x|}{x}}$$
This is easier to remember: The relative error gets multiplied by $|n|$ when you raise x to the nth power.
There is a very important case here when $n = -1$. In this case the rule says, the relative error is unchanged if you take the reciprocal of the quantity. (This, incidentally, is why multiplication and division are treated exactly in the same way.)
2. If $z = x^m y^n$, then $dz = mx^{m-1}y^n dx + ny^{n-1}x^m dy$
If $dz \approx \Delta z$
$$\Delta z = mx^{m-1}y^n \Delta x + ny^{n-1}x^m \Delta y$$

Example 14

The period of oscillation is measured to be $T = 0.20 \pm 0.01$ s. Find the range of values within which the frequency, f lies if the frequency is the reciprocal of the period.

Solution

$T = 0.20$, $\Delta T = 0.01$

$$f = \frac{1}{T} = \frac{1}{0.20} = 5.0 \text{ s}^{-1}$$

Writing $f = T^{-1}$, we can see from the power rule that

$$\frac{\Delta f}{f} = \frac{\Delta T}{T}$$
$$\frac{\Delta f}{5.0} = \frac{0.01}{0.20}$$
$$\Delta f = 0.25 \text{ s}^{-1}$$
$$\therefore f = 5.0 \pm 0.25 \text{ s}^{-1}$$

Example 15

The radius of a circle is $x = (3.0 \pm 0.2)$ cm. Find the range of values with in which the circumference lies.

Solution

Circumference, $c = 2\pi x = 2\pi(3.0) = 18.85$ cm

$\Delta c = 2\pi\Delta x = 2\pi(0.2) = 1.26$ cm

$$\therefore c = (18.85 \pm 1.26) \text{ cm}$$

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v} = \frac{v+u}{uv}$$

$$f = \frac{uv}{u+v} = \frac{(50.1)(20.1)}{(50.1+20.1)} = 14.3 \text{ cm}$$

$$\frac{\Delta f}{f} = \frac{\Delta u}{u} + \frac{\Delta v}{v} + \frac{\Delta u + \Delta v}{u+v}$$

$$\frac{\Delta f}{f} = \frac{0.5}{50.1} + \frac{0.2}{20.1} + \frac{0.5+0.2}{50.1+20.1}$$

$$\frac{\Delta f}{f} = 0.0299$$

$$\Delta f = 0.0299 \times f = 0.0299 \times 14.3 = 0.428 \text{ cm}$$

$$\therefore f = (14.3 \pm 0.4) \text{ cm}$$

or $[13.9 \text{ cm}, 14.7 \text{ cm}]$

Alternatively:

$$f = \frac{uv}{u+v} = \frac{(50.1)(20.1)}{(50.1+20.1)} = 14.3 \text{ cm}$$

$$f_{max} = \frac{(uv)_{max}}{(u+v)_{min}} = \frac{u_{max}v_{max}}{u_{min}+v_{min}} = \frac{50.6 \times 20.3}{49.6 + 19.9}$$

$$= 14.78 \text{ cm}$$

$$f_{min} = \frac{(uv)_{min}}{(u+v)_{max}} = \frac{u_{min}v_{min}}{u_{max}+v_{max}} = \frac{49.6 \times 19.9}{50.6 + 20.3}$$

$$= 13.92 \text{ cm}$$

Absolute error in f,

$$\Delta f = \frac{1}{2}(f_{max} - f_{min}) = \frac{1}{2}(14.78 - 13.92) = 0.43$$

Range of values within which the exact value lies is given by Working value $\pm$ Absolute error

$$f = (14.3 \pm 0.4) \text{ cm}$$

or $[13.9 \text{ cm}, 14.7 \text{ cm}]$

Example 21

Find the maximum possible percentage error in the measurement of force, F on an object of mass, m travelling at a velocity, v in a circle of radius, r, if $m = (4.0 \pm 0.1)$ kg, $v = 10 \pm 0.1 \text{ ms}^{-1}$ and $r = 8.0 \pm 0.2$ m. It is further given that the expression for the force F is given by

$$F = \frac{mv^2}{r}$$

Solution

$$\frac{\Delta F}{F} = \frac{\Delta m}{m} + 2\frac{\Delta v}{v} + \frac{\Delta r}{r}$$

$$\frac{\Delta F}{F} = \frac{0.1}{4.0} + 2\left(\frac{0.1}{10}\right) + \frac{0.2}{8.0}$$

$$\frac{\Delta F}{F} = 0.07$$

Maximum possible percentage error in measurement of force is given by

$$\frac{\Delta F}{F} \times 100 = 7\%$$

Alternatively;

$$F = \frac{mv^2}{r} = \frac{4.0 \times 10^2}{8.0} = 50$$

$$F_{max} = \frac{(mv^2)_{max}}{r_{min}} = \frac{m_{max}v_{max}^2}{r_{min}} = \frac{4.1 \times 10.1^2}{7.8} = 53.6206$$

$$F_{min} = \frac{(mv^2)_{min}}{r_{max}} = \frac{m_{min}v_{min}^2}{r_{max}} = \frac{3.9 \times 9.9^2}{8.2} = 46.6145$$

Absolute error in F,

$$\Delta F = \frac{1}{2}(F_{max} - F_{min}) = \frac{1}{2}(53.6206 - 46.6145)$$

$$= 3.50305$$

Maximum possible percentage error in measurement of force is given by

$$\frac{\Delta F}{F} \times 100 = \frac{3.50305}{50} \times 100 = 7.01\%$$

Example 22

The density of a cylindrical rod was calculated by using the formula

$$\rho = \frac{4m}{\pi D^2 l}$$

The percentage errors in the measurements of m, D and l are 1%, 1.5% and 0.5% respectively. Calculate the maximum percentage error in the value of the density.

Solution

$$\text{Density,} \quad \rho = \frac{4m}{\pi D^2 l}$$

$$\left(\frac{\Delta\rho}{\rho}\right)_{max} = \frac{\Delta m}{m} + 2\frac{\Delta D}{D} + \frac{\Delta l}{l}$$

$$But\, \frac{\Delta m}{m} = 0.01, \frac{\Delta D}{D} = 0.015\ and\, \frac{\Delta l}{l} = 0.005$$

Maximum percentage error in calculated value of density

$$\left(\frac{\Delta\rho}{\rho}\right)_{max} \times 100 = 100[0.01 + 2(0.015) + 0.005]$$

$$= 4.5\%$$

Important rules of counting significant figures

- All the non-zero digits are significant
- All the zeros between two non-zero digits are significant, no matter where the decimal point is, if at all.
- If the number is less than 1, the zero(s) on the right of the decimal point but to the left of the first non-zero digit are not significant. In $0.\underline{00}2308$, the underlined zeros are not significant.
- The terminal or trailing zero (s) in a number without a decimal point are not significant. Thus $123 \text{ m} = 12300 \text{ cm} = 123000 \text{ mm}$ has three significant figures, the trailing zero(s) being not significant.
- The trailing zero(s) in a number with a decimal point are significant. Thus numbers 3.500 or 0.06900 have four significant figures each.
- For a number greater than 1, without any decimal, the trailing zero(s) are not significant.

LINEAR INTERPOLATION

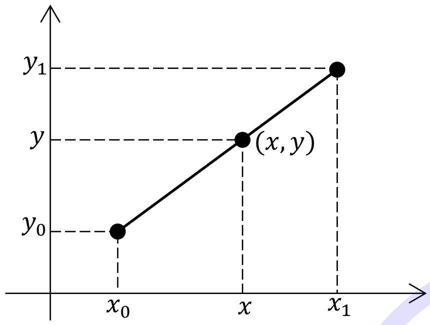
Interpolation is the method of finding a point between two points on a line or curve. More precisely,

- If we want to find coordinates of a point between two given points, then we use the linear interpolation
- If we want to find coordinates of a point that is not between two given points, then we use the linear extrapolation.

In other words, the linear interpolation is used to fill the gaps in a collection of points.

Linear interpolation between two known points

If two known points are given by the coordinates (x_0, y_0) and (x_1, y_1), the linear interpolant is the straight line between these points.



For a value x in the interval (x_0, x_1), the value y along the straight line is given by from the equation of slopes which can be derived geometrically from the figure above

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

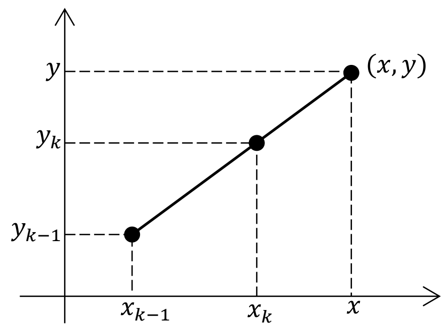
Solving this equation for y, which is the unknown value at x gives

$$y = y_0 + (x - x_0)\frac{y_1 - y_0}{x_1 - x_0} = \frac{y_0(x_1 - x) + y_1(x - x_0)}{x_1 - x_0}$$

LINEAR EXTRAPOLATION

Extrapolation is a type of estimation, beyond the original observation range, the value of a variable on the basis of its relationship with another variable. It is similar to interpolation, which produces estimates between known observations, but extrapolation is subject to greater uncertainty and a higher risk of producing meaningless results.

Linear extrapolation means creating a tangent line at the end of the known data and extending it beyond that limit.



If the two data points nearest the point x, to be extrapolated are (x_{k-1}, y_{k-1}) and (x_k, y_k), equating the slopes gives

$$\frac{y - y_{k-1}}{x - x_{k-1}} = \frac{y_k - y_{k-1}}{x_k - x_{k-1}}$$

And thus the linear extrapolation gives the function

$$y = y_{k-1} + (x - x_{k-1})\frac{y_k - y_{k-1}}{x_k - x_{k-1}}$$

Example 1

The table below shows temperatures of a patient recorded at the given times:

Time X (a.m)	8.00	m	8.06	8.08	8.12
Temperature Y (°C)	47	44	42	40	n

Use linear interpolation to find the values of m and n

<u>Solution</u>

To find m, extract

8.00	m	8.06
47	44	42

$$\frac{8.06 - 8.00}{42 - 47} = \frac{m - 8.00}{44 - 47}$$
$$0.036 = m - 8.00$$
$$m = 8.036$$
$$\therefore m = 8.00 \text{ a.m}$$

To find n, extract

8.06	8.08	8.12
42	40	n

$$\frac{8.12 - 8.06}{n - 42} = \frac{8.08 - 8.06}{40 - 42}$$
$$\frac{0.06}{n - 42} = -0.01$$
$$0.06 = -0.01n + 0.42$$
$$0.01n = 0.36$$
$$n = 36$$
$$\therefore n = 36°\text{C}$$

Example 2

The table below shows x and $f(x)$:

x	50.24	48.11	46.93	44.06
$f(x)$	4.116	7.621	9.043	11.163

Use linear interpolation or extrapolation to estimate:

(a) x when $f(x) = 8.614$

(b) $f^{-1}(51.07)$

<u>Solution</u>

(a) Extracting and using linear interpolation

48.11	x	46.93
7.621	8.641	9.043

$$\frac{46.93 - 48.11}{9.043 - 7.621} = \frac{x - 48.11}{8.461 - 7.621}$$
$$-0.8298 = \frac{x - 48.11}{0.84}$$
$$x - 48.11 = -0.697$$
$$x = 47.413$$

(b) Extracting and using linear extrapolation

51.07	50.24	48.11
y	4.116	7.621

$$\frac{48.11 - 51.07}{7.621 - y} = \frac{50.24 - 51.07}{4.116 - y}$$

LOCATION OF ROOTS

Many equations cannot be solved exactly, but various methods of finding approximate solutions exist. The most commonly used methods have two main part:

(a) finding an initial approximate value
(b) improving this value by an iterative process

Initial values
The roots of $f(x) = 0$ can be located approximately by either a graphical or an algebraic method.

Algebraic method
Find two values a and b such that $f(a)$ and $f(b)$ have different signs. At least one root must lie between a and b if $f(x)$ is continuous.
If more than one root is suspected between a and b, sketch a graph of $y = f(x)$.

Example 1
Show that the equation $e^x = 5\sin x$ has a root between $x = 0$ and $x = 1.5$.

Solution

First rearrange the equation in the form $f(x) = 0$

$$e^x - 5\sin x = 0$$
$$f(x) = e^x - 5\sin x$$
$$f(0) = e^0 - 5\sin 0 = 1$$
$$f(1.5) = e^{1.5} - 5\sin 1.5 = -0.506$$
$$f(0) \cdot f(1.5) = -0.506 < 0$$

Since $f(0) \cdot f(1.5) < 0$ or there is change of sign, there exists a root between 0 and 1.5.

Example 2
Show that the equation $x^2 - 3x + 1 = 0$ has a root between $x = 2$ and $x = 3$.

Solution

Let $f(x) = x^2 - 3x + 1$

$$f(2) = 2^2 - 3(2) + 1 = -1$$
$$f(3) = 3^2 - 3(2) + 1 = 4$$
$$f(1) \cdot f(2) = -4 < 0$$

Since $f(1) \cdot f(2) < 0$ or there is a change of sign, there is a root between 2 and 3

Graphical method
Either plot (or sketch) the graph $y = f(x)$. The real roots are at the points where the curve cuts the x-axis.
Or rewrite $f(x) = 0$ in the form $F(x) = G(x)$. Plot (or sketch) $y = F(x)$ and $y = G(x)$. The real roots are at the points where these graphs intersect.

Example 1
By drawing the graphs of $y = \ln x$ and $y = 2 - x$ in the interval $1 \le x \le 2$, find the solution of the equation $\ln x + x - 2 = 0$, correct to 1 decimal place.

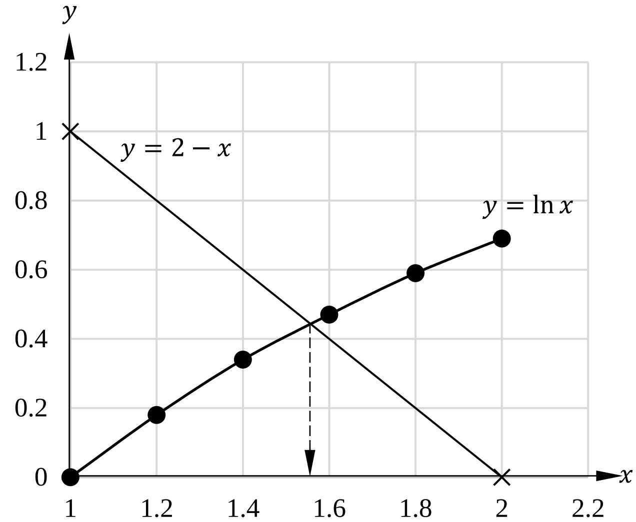
Solution

$y = \ln x$

x	1.0	1.2	1.4	1.6	1.8	2.0
$\ln x$	0	0.182	0.336	0.470	0.588	0.693

$y = 2 - x$

x	1	2
$2 - x$	1	0



At the point of intersection of the graphs of $y = \ln x$ and $y = 2 - x$, the approximate value of x is 1.6
Hence if $\ln x + x - 2 = 0$, then $x \approx 1.6$

Example 2
Locate an approximate value for the root of

$$f(x) = x + e^x = 0$$

Solution

Let us find a range within which the root of the equation lies.

$$f(0) = 0 + e^0 = 1$$
$$f(-1) = -1 + e^{-1} = -0.632$$

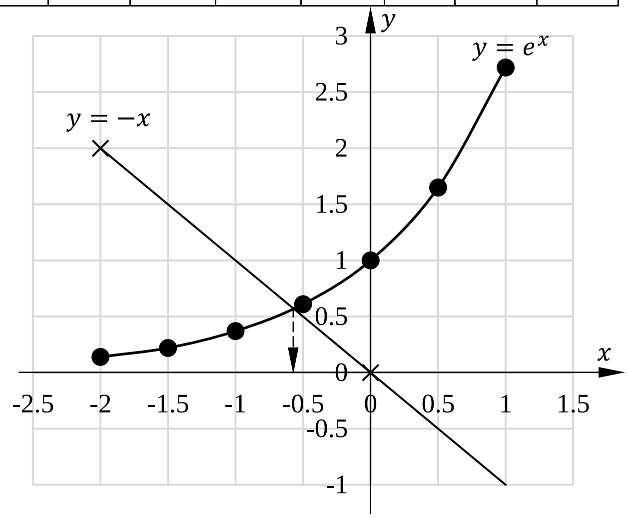
The root lies between -1 and 0.

$$e^x = -x$$

We can thus plot the graphs $y = e^x$ and $y = -x$ in the range $-2 \le x \le 1$

$y = e^x$

x	-2	-1.5	-1.0	-0.5	0	0.5	1.0
e^x	0.14	0.22	0.37	0.61	1.0	1.65	2.72

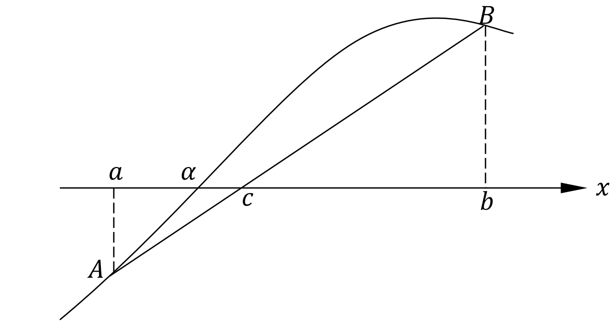


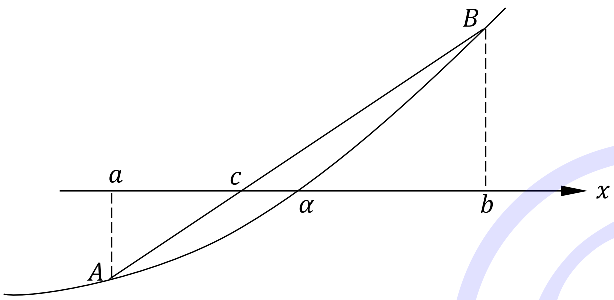
$\therefore$ The root is approximately -0.6

Further linear interpolation

When it has been shown that an equation $f(x) = 0$ has a root α in the interval $a < x < b$, then linear interpolation can be used to estimate the value of α.

Let us suppose that A and B are the points on the curve $y = f(x)$ at which $x = a$ and $x = b$ respectively. If the straight line joining A to B cuts the x-axis at $(c, 0)$, then c is taken as an approximate value of the root α.





By using similar triangles

a	c	b
$f(a)$	0	$f(b)$

By linear interpolation,

$$\frac{f(b) - f(a)}{b - a} = \frac{0 - f(a)}{c - a}$$

$$c = a + \frac{af(a) - bf(a)}{f(b) - f(a)}$$

$$\therefore \alpha \approx a + \frac{af(a) - bf(a)}{f(b) - f(a)}$$

Note:

A better approximation of the root can be obtained by using linear interpolation more than once by calculating $f(\alpha)$ and taking note of the change in the sign.

If there is change in sign between $f(a)$ and $f(\alpha)$, then a better approximation of the root lies between a and α, otherwise if the change in sign is between $f(\alpha)$ and $f(b)$, then a better approximation of the root lies between α and b.

Example 3

Show that the equation $x^3 - 7x - 12 = 0$ has a root between 3 and 4. Use linear interpolation to estimate the root correct to one decimal place.

Solution

Let $f(x) = x^3 - 7x - 12$

$f(3) = 3^3 - 7(3) - 12 = -6$

$f(4) = 4^3 - 7(4) - 12 = 24$

Since $f(3) < 0$ and $f(4) > 0$, then $f(3) \cdot f(4) < 0$, the equation has a root between 3 and 4.

Let the approximate root be α

3	α	4
-6	0	24

By linear interpolation;

$$\frac{4 - 3}{24 + 6} = \frac{\alpha - 3}{0 + 6}$$

$$\alpha = 3 + \frac{6}{30} = 3.2$$

Example 4

Given the equation

$$\tan x = 4x^2 - 1$$

(a) Show that the above equation has a root α, which lies between 1.4 and 1.5

(b) Use linear interpolation twice, to find, correct to 3 decimal places two approximations of α

Solution

(a) Rewriting the equation as; $\tan x - 4x^2 + 1 = 0$

Let $f(x) = \tan x - 4x^2 + 1$

$f(1.4) = \tan 1.4 - 4(1.4)^2 + 1 = -1.0421 < 0$

$f(1.5) = \tan 1.5 - 4(1.5)^2 + 1 = 6.1014 > 0$

Since $f(1.4) \cdot f(1.5) < 0$, the root α lies between 1.4 and 1.5.

(b) Let the approximations of α be x_1 and x_2

1.4	x_1	1.5
-1.0421	0	6.1014

$$\frac{x_1 - 1.4}{0 + 1.0421} = \frac{1.5 - 1.4}{6.1014 + 1.0421}$$

$$x_1 = 1.4 + \frac{1.0421(0.1)}{7.1435}$$

$$x_1 = 1.4 + 0.0146$$

$$x_1 = 1.415 \text{ (3 d.p)}$$

Now $f(1.415) = \tan 1.415 - 4(1.415)^2 + 1 = -0.6423$

Since $f(1.415) < 0$, the next approximation x_2 of the root α lies between 1.415 and 1.5 such that $f(1.415)f(1.5) < 0$

1.415	x_2	1.5
-0.06423	0	6.1014

$$\frac{x_2 - 1.415}{0 + 0.6423} = \frac{1.5 - 1.415}{6.1014 + 0.6423}$$

$$x_1 = 1.415 + \frac{0.6423(0.085)}{6.7437}$$

$$x_1 = 1.415 + 0.0081$$

$$x_1 = 1.423 \text{ (3 d.p)}$$

Note:

When the function involves usage of a trigonometric function and the values are not stated as degrees, always remember to change your calculator in radians when doing the computation otherwise you will obtain wrong values.

Example 9

It is required to find an approximate root of the equation $\ln(x-1) - 9 + x^2 = 0$ by use of a graph in the range $0 \leq x \leq 3.5$ and then use linear interpolation once to find a better approximation of the root correct to 3 decimal places.

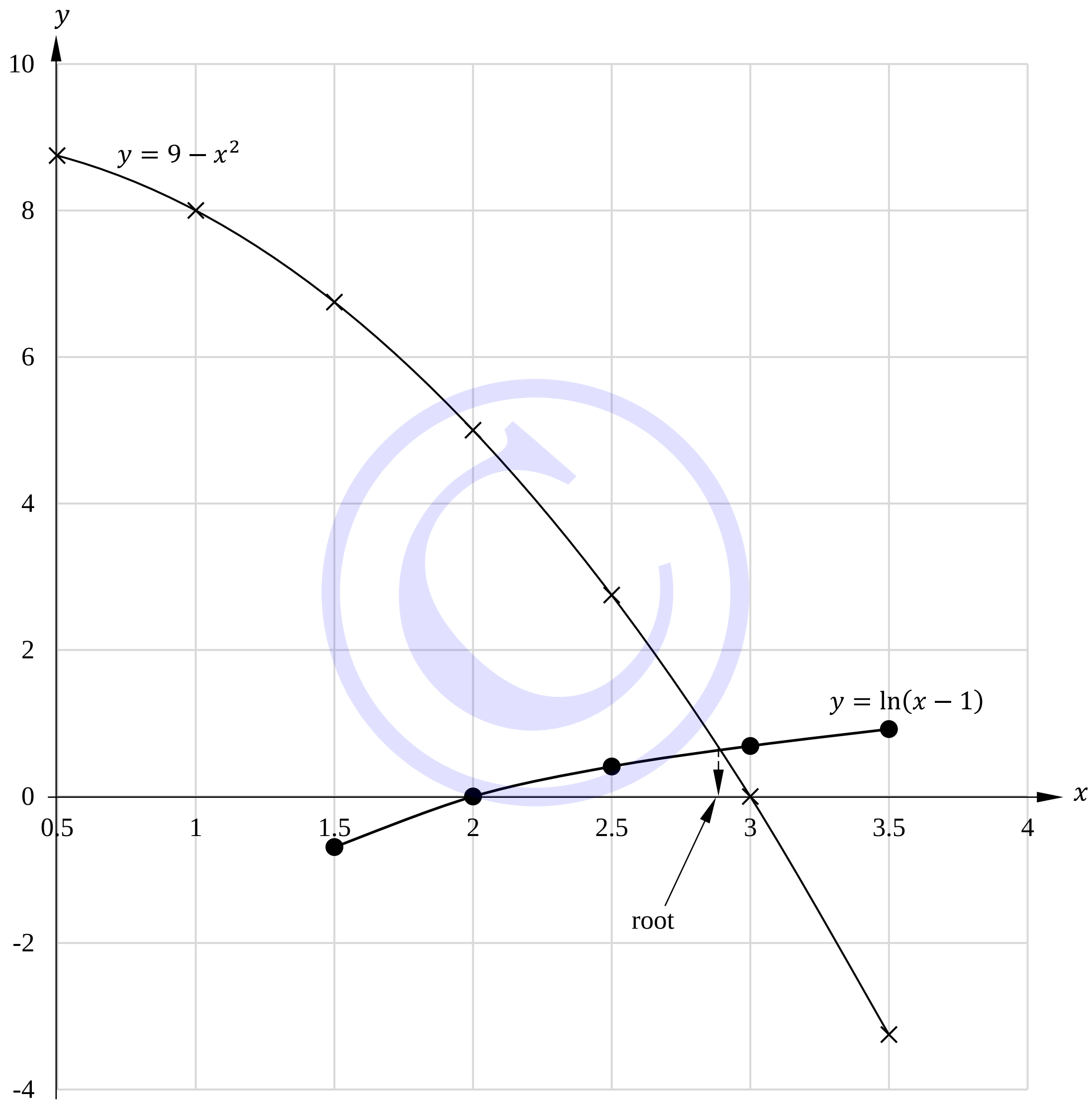
Solution

Rearranging the equation in the form $f(x) = g(x)$

$$\ln(x-1) = 9 - x^2$$

Let $y_1 = \ln(x-1)$ and $y_2 = 9 - x^2$

x	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5
$y_1 = \ln(x-1)$	−	−	−	−0.693	0	0.405	0.693	0.916
$y_2 = 9 - x^2$	9	8.75	8	6.75	5	2.75	0	−3.25



From the graph, the approximate root of the equation is 2.9

$$f(2.9) = \ln(2.9-1) - 9 + 2.9^2 = 0.0519$$

$$f(2.8) = \ln(2.8-1) - 9 + 2.8^2 = -0.5722$$

The root lies lies between 2.8 and 2.9

2.8	α	2.9
−0.5772	0	0.0519

$$\frac{2.9-2.8}{0.0519+0.5772} = \frac{\alpha-2.8}{0+0.5772}$$

$$\alpha - 2.8 = 0.0918$$

$$\alpha = 2.8918$$

$\therefore$ The root is 2.892 (3 d.p)

Solution

Let $f(x) = x^2 - 8x + 10$

$$f(1) = (1)^2 - 8(1) + 10 = 3$$
$$f(2) = (2)^2 - 8(2) + 10 = -2$$
$$f(1) \cdot f(2) = -6 < 0$$

Since $f(1) \cdot f(2) < 0$, then there is a root between 1 and 2.

Now, rearranging the equation $x^2 - 8x + 10 = 0$ in the form $x = g(x)$ yields an iterative formula i.e.

$$x^2 - 8x + 10 = 0$$
$$x^2 = 8x - 10$$
$$\frac{x^2}{x} = \frac{8x}{x} - \frac{10}{x}$$
$$x = 8 - \frac{10}{x}$$
$$x_{n+1} = g(x_n)$$
$$x_{n+1} = 8 - \frac{10}{x_n}$$

$x_0 = 2$,

$$x_1 = 8 - \frac{10}{2} = 3$$
$$x_2 = 8 - \frac{10}{3} = 4.667$$
$$x_3 = 8 - \frac{10}{4.667} = 5.857$$

Comment: The values x_1, x_2 and x_3 do not yield convergence to the root as $|x_{n+1} - x_n|$ keeps on increasing.

Example 4

The iterative formulae,

(a) $x_{n+1} = \dfrac{2x_n^3 + 10}{3x_n^2}$ and (b) $x_{n+1} = \dfrac{10}{x_n^2}$

can be obtained by rearranging the equation $x^3 - 10 = 0$. Starting from $x_0 = 2$, find the values of x_1, x_2 and x_3, which are produced by these iterative formulae.

Use the method that converges to find $(10)^{1/3}$, correct to four significant figures.

Solution

$x_0 = 2$

Using formula (a):

$$x_1 = \frac{2(2)^3 + 10}{3(2)^2} = 2.1667$$
$$x_2 = \frac{2(2.1667)^3 + 10}{3(2.1667)^2} = 2.1545$$
$$x_3 = \frac{2(2.1545)^3 + 10}{3(2.1545)^2} = 2.1544$$

Using formula (b):

$$x_1 = \frac{10}{2^2} = 2.5$$
$$x_2 = \frac{10}{2.5^2} = 1.6$$
$$x_3 = \frac{10}{1.6^2} = 3.90625$$

From $x^3 - 10 = 0$, $x = (10)^{\frac{1}{3}}$

From formula (a); $10^{1/3} = 2.154$ (3 s.f.)

Example 5

Show that the equation $x^2 - 5x + 1 = 0$ can be arranged as $x = \frac{x^2+1}{5}$, or alternatively, as $x = 5 - \frac{1}{x}$.

Hence write down two possible iterative formulae which might be used or solving this quadratic, and starting from $x_0 = 0.2$, find the values of x_1, x_2 and x_3 produced by each of these iterative formulae.

Solution

$$x^2 - 5x + 1 = 0 \qquad\qquad x^2 - 5x + 1 = 0$$
$$x^2 = 5x - 1 \qquad\qquad 5x = x^2 + 1$$
$$\frac{x^2}{x} = \frac{5x}{x} - \frac{1}{x} \qquad\qquad \frac{5x}{5} = \frac{x^2+1}{5}$$
$$x = 5 - \frac{1}{x} \qquad\qquad x = \frac{x^2+1}{5}$$
$$\therefore x_{n+1} = 5 - \frac{1}{x_n} \qquad\qquad \therefore x_{n+1} = \frac{x_n^2+1}{5}$$

$x_0 = 0.2$

Using $x_{n+1} = \frac{x_n^2+1}{5}$,

$$x_1 = \frac{0.2^2+1}{5} = 0.208$$
$$x_2 = \frac{0.208^2+1}{5} = 0.2087$$
$$x_3 = \frac{0.2087^2+1}{5} = 0.2087$$

Using $x_{n+1} = 5 - \frac{1}{x_n}$

$$x_1 = 5 - \frac{1}{0.2} = 0$$
$$x_2 = 5 - \frac{1}{0} = \text{undefined}$$

Example 6

The equation $x^2 - 3 = 0$ (for the square root $\alpha = \sqrt{3}$) can be written equivalently in the form $x = g(x)$ in many different ways, for example

I	II	III
$g(x) = \frac{1}{2}\left(x + \frac{3}{x}\right)$	$g(x) = \frac{3}{x}$	$g(x) = 2x - \frac{3}{x}$

Discuss the convergence (or nonconvergence) behaviour of the iteration $x_{n+1} = g(x_n)$, $n = 0, 1, 2, \dots$, for each of these three iteration functions.

Solution

To determine convergence, we need to find the value of $|g'(x)|$ near the root.

We shall choose the value near the root, $x_0 = 1.5$, as the root lies between 1 and 2

For function I:

$$g'(x) = \frac{1}{2}\left(1 - \frac{3}{x^2}\right)$$
$$g'(1.5) = \frac{1}{2}\left(1 - \frac{3}{1.5^2}\right) = -\frac{1}{6}$$
$$|g'(x)| = \frac{1}{6} < 1$$

Function I converges to the root

For function II:

$$g'(x) = -\frac{3}{x^2}$$
$$g'(1.5) = -\frac{3}{1.5^2} = -\frac{4}{3}$$
$$|g'(1.5)| = \frac{4}{3} > 1$$

Function II does not converge to the root

For function III:

$$g'(x) = 2 + \frac{3}{x^2}$$
$$g'(1.5) = 2 + \frac{3}{1.5^2} = \frac{10}{3}$$
$$|g'(x)| = \frac{10}{3} > 1$$

Function III does not converge

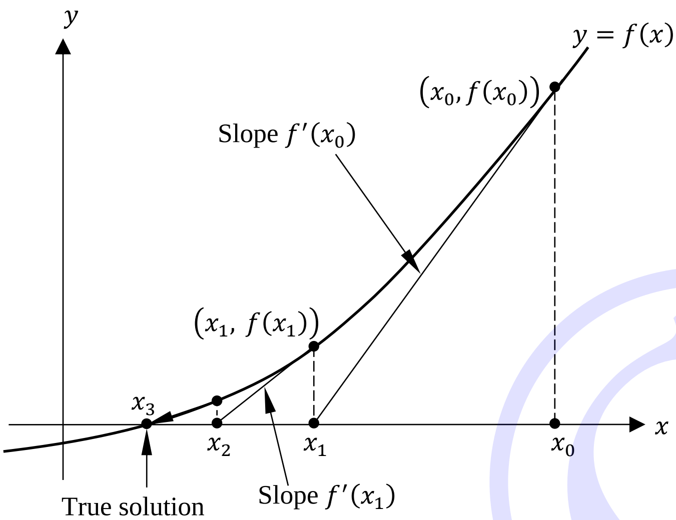
Newton-Raphson method

Newton's method also called the Newton-Raphson method usually converges much faster than the general iteration method we saw above.

We can derive Newton's method graphically, or by a Taylor series. We again want to construct a sequence $x_0, x_1, x_2, \dots$ that converges to the root $x = \alpha$.

Graphically

To find a root of $f(x) = 0$, an initial approximation x_0 is given, and the tangent line to the function at x_0 is drawn. The tangent line will approximately follow the function down to the x-axis toward the root. The intersection point of the line with the x-axis is an approximate root, but probably not exact if $f(x)$ curves. Therefore, this step is iterated.



From the geometric picture, we can develop an algebraic formula for Newton's method. The tangent line at x_0 has slope given by the derivative $f'(x_0)$. One point on the tangent line is $(x_0, f(x_0))$.

The point-slope formula for the equation of a line is

$$\frac{y - f(x_0)}{x - x_0} = f'(x_0)$$

$$y - f(x_0) = f'(x_0)(x - x_0)$$

So that looking for the intersection point of the tangent line with the x-axis is the same as substitution $y = 0$ in the line.

$$0 - f(x_0) = f'(x_0)(x - x_0)$$

$$x - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Solving for x gives an approximation for the root, which we call x_1. Next, the entire process is repeated, beginning with x_1, to produce x_2, and so on, yielding the following iterative formula.

Newton's method

$x_0 =$ initial approximation

$$\boldsymbol{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

for $n = 0, 1, 2, \dots\dots$

Taylor's series

It given by the following formula

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)h^2}{2!} + \frac{f'''(x)h^3}{3!} + \cdots$$

If we expand $f(x_{n+1})$ about the point $f(x_n)$, we have

$$f(x_{n+1}) = f(x_n) + (x_{n+1} - x_n)f'(x) + (x_{n+1} - x_n)^2\frac{f''(x)}{2!} + \cdots$$

where $h = (x_{n+1} - x_n)$

If $(x_{n+1} - x_n)$ is small, then the higher order terms will be much smaller and can be dropped. Since we are interested in finding $f(x) = 0$, we can assume $f(x_{n+1}) = 0$

$$0 = f(x_n) + (x_{n+1} - x_n)f'(x)$$

$$\boldsymbol{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Note:

The Newton-Raphson method fails if $f'(x_n)$ is near zero. When the derivative is close to zero, the tangent line is nearly horizontal and hence may overshoot the desired root (numerical difficulties). We choose another starting point in such a case.

Example 7

Find the Newton's method formula for the equation $x^3 + x - 1 = 0$.

<u>Solution</u>

$f(x) = x^3 + x - 1$, $f'(x) = 3x^2 + 1$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 + x_n - 1}{3x_n^2 + 1}$$

$$= \frac{x_n(3x_n^2 + 1) - (x_n^3 + x_n - 1)}{3x_n^2 + 1}$$

$$= \frac{3x_n^3 + x_n - x_n^3 - x_n + 1}{3x_n^2 + 1}$$

$$x_{n+1} = \frac{2x_n^3 + 1}{3x_n^2 + 1}$$

Example 8

Find the solution of $x + e^x = 0$ near $x = -1$ to three decimal places

<u>Solution</u>

Since the solution is required to 3 d.p, then $|x_{n+1} - x_n| < 0.5 \times 10^{-3}$

$f(x) = x + e^x$, $f'(x) = 1 + e^x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

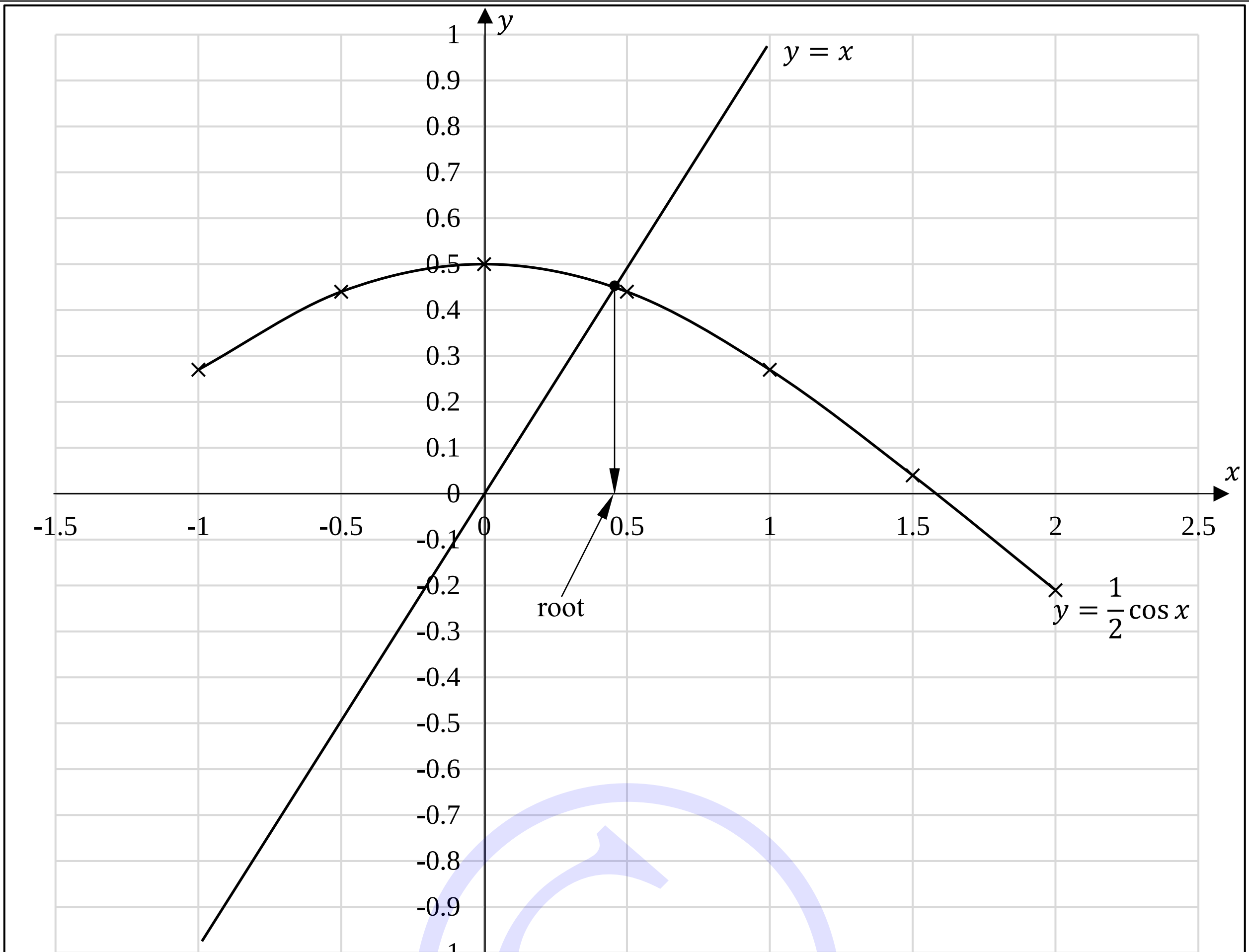
$$x_{n+1} = x_n - \frac{x_n + e^{x_n}}{1 + e^{x_n}}$$

$$x_{n+1} = \frac{x_n(1 + e^{x_n}) - (x_n + e^{x_n})}{1 + e^x}$$

$$x_{n+1} = \frac{x_n + x_ne^{x_n} - x_n - e^{x_n}}{1 + e^{x_n}}$$

$$x_{n+1} = \frac{e^{x_n}(x_n - 1)}{1 + e^{x_n}}$$

$x_0 = -1$



From the graph, $x = 0.45$

$$f(x) = x - \frac{1}{2}\cos x \qquad f'(x) = 1 + \frac{1}{2}\sin x$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{\left[x_n - \frac{1}{2}\cos x_n\right]}{\left[1 + \frac{1}{2}\sin x_n\right]}$$

$$x_{n+1} = x_n - \frac{[2x_n - \cos x_n]}{[2 + \sin x_n]}$$

$$x_{n+1} = \frac{2x_n + x_n \sin x_n - 2x_n + \cos x_n}{2 + \sin x_n}$$

$$x_{n+1} = \frac{x_n \sin x_n + \cos x_n}{2 + \sin x_n}$$

$x_0 = 0.45$, $TOL = 0.5 \times 10^{-4}$

$$x_1 = \frac{0.45 \sin 0.45 + \cos 0.45}{2 + \sin 0.45} = 0.45018$$

$$|x_1 - x_0| = |0.45018 - 0.45| = 1.84 \times 10^{-4} > TOL$$

$$x_2 = \frac{0.45018 \sin 0.45018 + \cos 0.45018}{2 + \sin 0.45018} = 0.45018$$

$$|x_2 - x_1| = |0.45018 - 0.45018| = 0 < TOL$$

$\therefore$ The root $x = 0.4502$ (4 d.p.)

Note: We can generate the original expression/equation from the Newton Raphson's expression by writing the expression in the form $x = g(x)$ and replacing x_{n+1} and x_n by x i.e.

$$x_{n+1} = \frac{x_n \sin x_n + \cos x_n}{2 + \sin x_n}$$

$$x = \frac{x \sin x + \cos x}{2 + \sin x}$$

$$2x + x\sin x = x\sin x + \cos x$$

$$\therefore x = \frac{1}{2}\cos x$$

Solution

Using 5 ordinates in the interval from $x = 0$ to $x = 2$ gives 4 strips of width, $h = \frac{2-0}{4} = 0.5$

n	x_n	y_n (correct to 5 d.p)	
0	0	1.00000	
1	0.5		0.80000
2	1.0		0.50000
3	1.5		0.30769
4	2.0	0.20000	
$\sum$		1.20000	1.60769

$$\int_0^2 \frac{1}{1+x^2}\,dx \approx \frac{1}{2}h\{(y_0 + y_4) + 2(y_1 + y_2 + y_3)\}$$
$$\approx \frac{1}{2}(0.5)[1.20000 + 2(1.60769)]$$
$$\approx 1.103845$$
$$\approx 1.1038 \text{ (4 d.p)}$$

Exact value:

This integral can be evaluated directly as follows:

$$\int_0^2 \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right]_0^2 = \tan^{-1} 2 = 1.1072 \text{ (4 d.p)}$$

Error = Exact value – Approximate value

$$= 1.1072 - 1.1038 = 0.0034$$

Reduction in the error

The error may be reduced by **increasing the number of strips or ordinates** used.

For example, we can obtain a good approximation of the integral from example 2 by using 11 ordinates i.e. 10 strips of width $h = 0.2$.

n	x_n	y_n (correct to 5 d.p)	
0	0	1.00000	
1	0.2		0.96154
2	0.4		0.86207
3	0.6		0.73529
4	0.8		0.60976
5	1.0		0.50000
6	1.2		0.40984
7	1.4		0.33784
8	1.6		0.28090
9	1.8		0.23585
10	2.0	0.20000	
$\sum$		1.20000	4.93309

$$\int_0^2 \frac{1}{1+x^2}\,dx \approx \frac{1}{2}h\{(y_0 + y_{10}) + 2(y_1 + \cdots + y_9)\}$$
$$\approx \frac{1}{2}(0.2)[1.20000 + 2(4.93309)]$$
$$\approx 1.106618$$
$$\approx 1.1066 \text{ (4 d.p)}$$

Error = Exact value – Approximate value

$$= 1.1072 - 1.1066$$
$$= 0.0006$$

Using 21 ordinates, the value 1.1070 is obtained. However, rounding errors in the values of y_n will limit further improvements in accuracy.

Note: We cannot reduce the error by changing $(b - a)$ because this would require us to change the interval over which we are integrating.

Example 4

Use the trapezium rule with 5 ordinates to estimate $\int_0^4 xe^{-x}\,dx$ correct to four decimal places.

Calculate the percentage error in estimating the above integral by the trapezium rule.

Solution

$$h = \frac{4-0}{4} = 1$$

n	x_n	y_n (correct to 5 d.p)	
0	0	0.00000	
1	1		0.36788
2	2		0.27067
3	3		0.14936
4	4	0.07326	
$\sum$		0.07326	0.78791

$$\int_0^4 xe^{-x}\,dx \approx \frac{1}{2}h\{(y_0 + y_4) + 2(y_1 + y_2 + y_3)\}$$
$$\approx \frac{1}{2}(1)[0.07326 + 2(0.78791)]$$
$$\approx \frac{1}{2} \times 1.64908$$
$$\approx 0.82454$$
$$\approx 0.8245 \text{ (4 d.p)}$$

The exact value of the integral can be obtained using integration by parts

$$\int_0^4 xe^{-x}\,dx = \left[-xe^{-x}\right]_0^4 - \int_0^4 -e^{-x}\,dx$$
$$= \left[-xe^{-x}\right]_0^4 + \int_0^4 e^{-x}\,dx$$
$$= -4e^{-4} + \left[-e^{-x}\right]_0^4$$
$$= -4e^{-4} + (-e^{-4} + e^0)$$
$$= 1 - 5e^{-4}$$
$$= 0.9084 \text{ (4 d.p)}$$

Error = Exact value – Approximate value

$$= 0.9084 - 0.8245$$
$$= 0.0839$$

Percentage error $= \frac{\text{error}}{\text{exact value}} \times 100 = \frac{0.0839}{0.9084} \times 100$

$$= 9.24\%$$

Example 5

(a) Use the trapezium rule with 6 equally spaced strips to find an estimate, correct to 3 decimal places, for

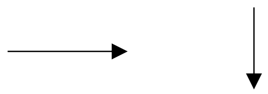
$$\int_0^{1.2} \sin^2 x\,dx$$

FLOW CHARTS

A flow chart is a diagram that sequentially represents an algorithm or process.
An **algorithm** is just a detailed sequence of simple steps that are needed to solve a problem.
The flow chart shows the steps as boxes of various kinds, and their order by connecting the boxes with arrows. This diagrammatic representation illustrates a solution model to a given problem.
Flow charts are used in analyzing, designing, documenting or managing a process or a program in various fields.

Common symbols

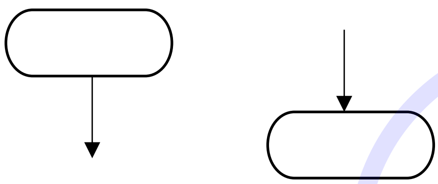
1. Flow line



This shows the process' order of operation. A line coming from one symbol and pointing at another.

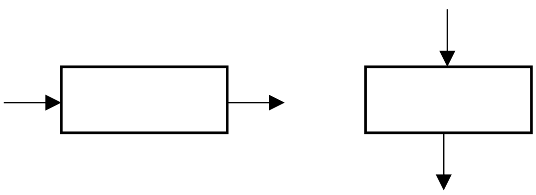
2. Terminal

Represented as an oval or rounded (fillet) rectangle.



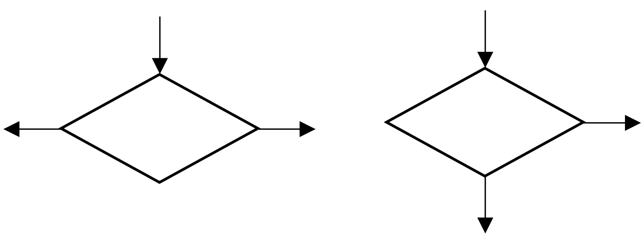
Indicates the beginning and ending of a program or sub-process.
They usually contain the word "Start" or "End/Stop" or another phrase signaling the start or the end of a process.
Note: Only one flow line is used in conjunction with the terminal symbol.

3. Process



It is represented as a rectangle. It represents the general processing operation, usually an assignment statement or data movement instructions.
Note: Only one flow line should come out from the process symbol.

4. Decision



It is represented as a diamond(rhombus).
It shows a conditional operation that determines which one of the two paths the program will take. The operation is commonly a yes/no question or true/false test.
Note: Only one flow line should enter a decision symbol, but two, one for each possible answer should leave the decision symbol.

5. Input/output

Represented as a parallelogram.
Indicates the process of inputting and outputting data as in entering data or displaying results.

Rules for drawing a flow chart

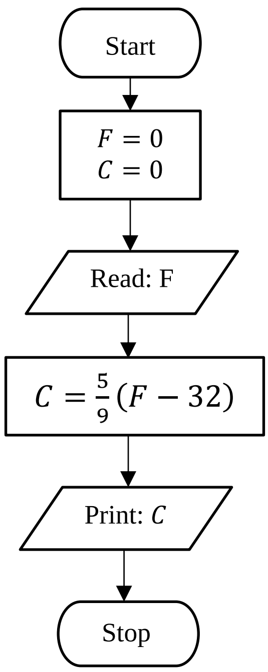
- It should contain only one start and one end symbol
- It should be drawn clearly and neatly. This means crossing of lines should be avoided and symbols should be drawn having exact shape.
- It should be as simple as possible
- The branches of decision box must be labelled.
- Only standard symbols should be used
- The assignment statements are denoted with the left arrow ← indicating the passing of the specified values/variables to the variable on the left i.e.
  $A \leftarrow 20$ would declare value 20 in variable A
  $x_n \leftarrow x_{n+1}$ would declare value x_{n+1} in variable x_n
  $n \leftarrow n+1$ would declare value $n+1$ in variable n
  and so on.

Note:
It might be useful to test the validity of the flow chart by passing through it with a simple test data.

Example 1

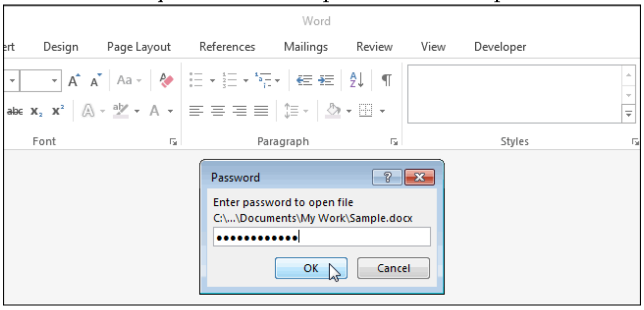
Draw a flow chart to convert temperature given in Fahrenheit to Celsius.

Solution



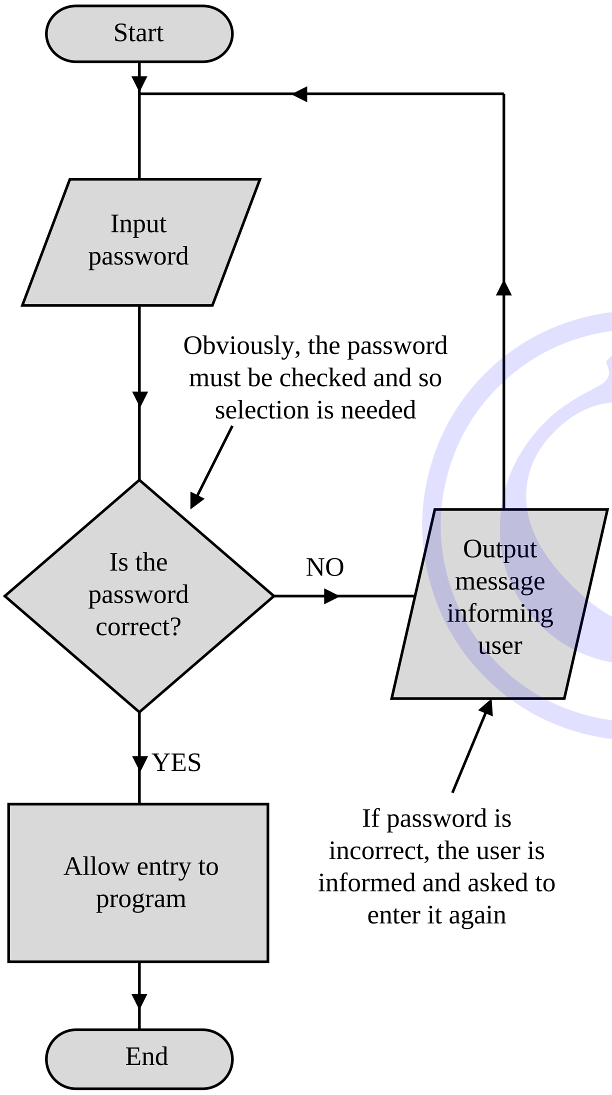
Example 2

A common request for user input is to enter a password.



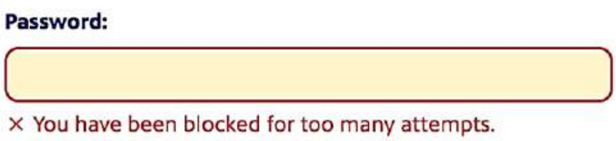
Here is a flow diagram to authenticate a user's password.

When a user enters a password, it has to be confirmed that it is the same as the one stored; it has to be authenticated.



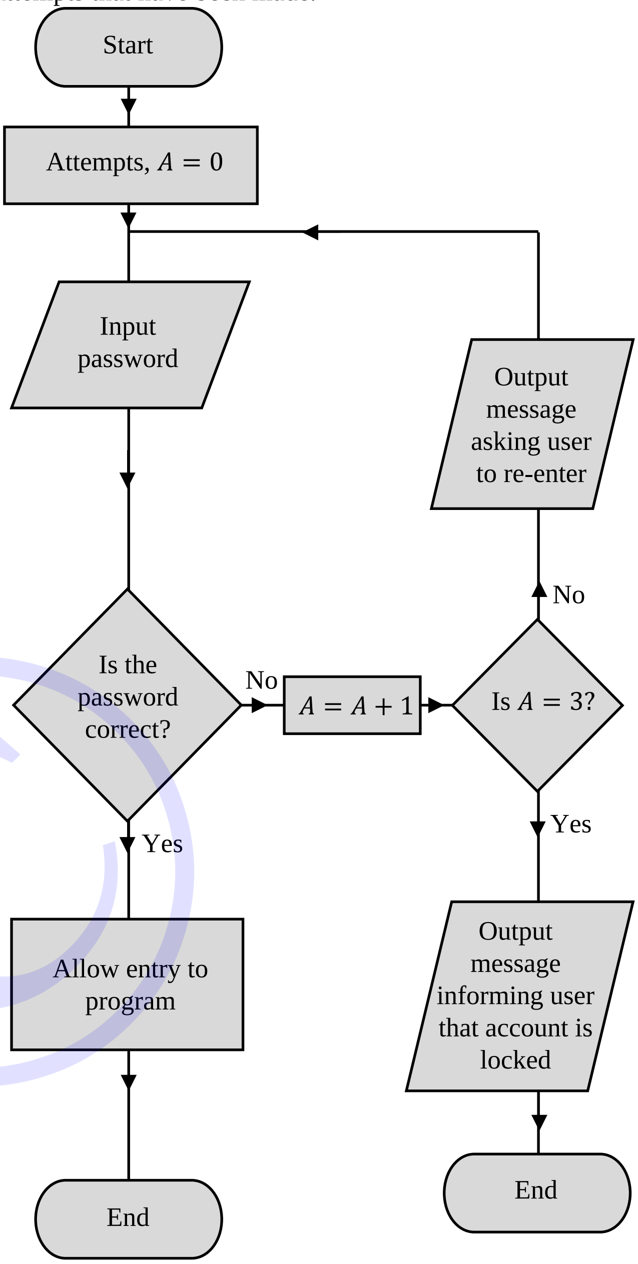
In this flow chart, there is iteration. If the password is incorrect, then in this particular algorithm the user is asked to enter it again, and again, and again, forever or until it is correct.

Now assume that the user is given three attempts and then the account is locked.



Here is a flow chart to authenticate a password and lock the account after three incorrect attempts.

The algorithm will have to keep account of the number of attempts that have been made.



In this algorithm, we have used a container called 'Attempts' to keep a count of the number of attempts that have been made.

When incorrect attempts are made, the value of 'Attempts' changes. It doesn't keep the same value throughout the algorithm, it can change because it is a variable.

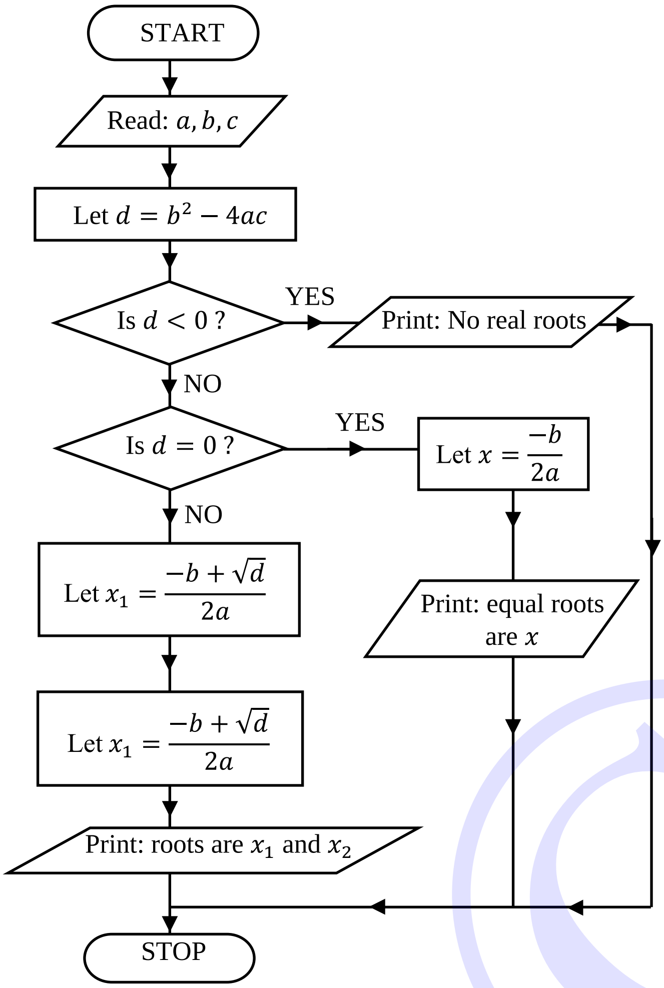
At first attempt, it is changed to 1, on the second attempt to 2 and to 3 on third attempt.

If three attempts are made, then the container 'Attempts' equals 3 and if there is still no correct password then the account is locked.

Containers like 'Attempts' are used in algorithms to store values that can change as the algorithm is running. As the values they contain can change, they are called **variables**.

Example 7

The flow chart can be used to find the roots of an equation of the form $ax^2 + bx + c = 0$.



Iteration method algorithm

Iteration method is one of the most popular approaches to find the real roots of non-linear functions. Both the algorithm and flow chart require one initial guess and modification approach to find the root of the function.

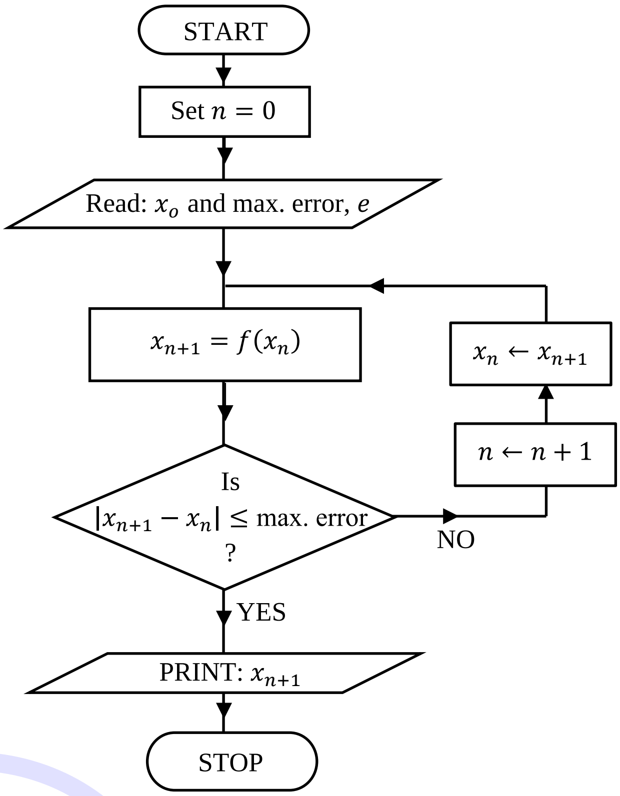
The general iteration method algorithm/flow chart work in such a way that modifications alongside iteration are progressively continued with newer and fresher approximations of the initial approximation.

These algorithm and flow chart presented below and the iteration method itself are used to determine the roots of functions in the form of an infinite series such as geometric series, arithmetic series, Taylor's series, and others.

1. Start
2. Read the values of x_0 and e
   Here x_0 is the initial is the initial approximation
   e is the absolute error or the desired degree of accuracy (tolerance), also the stopping criteria.
3. Calculate $x_1 = g(x_0)$
4. If $|x_1 - x_0| \leq e$, go to step 6
5. Else, assign $x_1 = x_0$ and go to step 3.
6. Display x_1 as the root.
7. Stop

Note: In some cases, the stopping criteria might be specified by the number of iterations rather than the error/tolerance.

Iteration method flow chart



Testing and evaluating algorithms

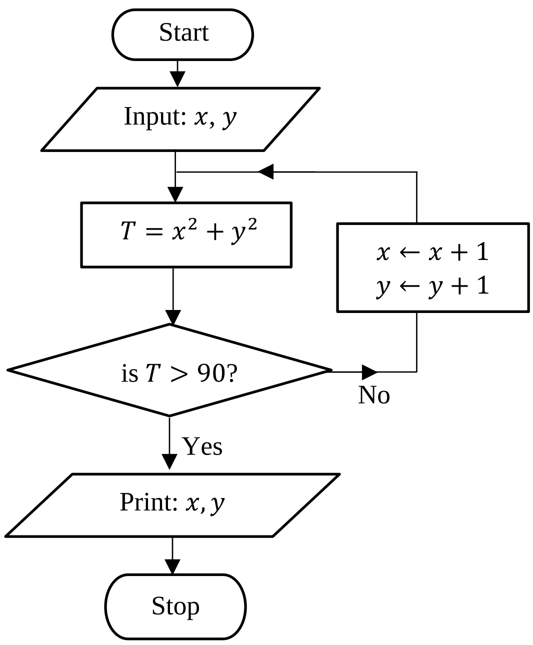
One way to test short programs/flow charts is to perform what is known as a **dry run**.

A dry run involves creating a trace table, containing all the variables a flow chart contains. Whenever the value of a variable changes, the change is indicated in the trace table.

By performing a dry run, it can formalize what exactly an algorithm is designed to do or establish the purpose of the flow chart.

Example 8

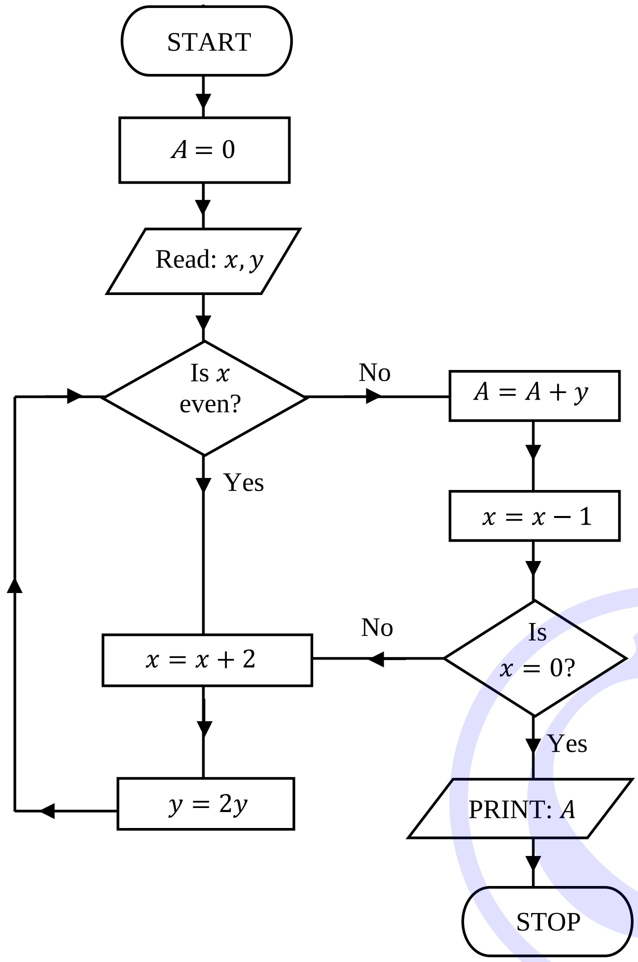
Starting with $x = 1$ and $y = 3$, use the flow chart to find the values printed.



(b) The purpose of the flow chart is to compute and print the sum of the cubes of the first 12 natural numbers.

Example 11

Study the flow chart below and answer the questions that follow.



(a) Given that $x = 54$ and $y = 63$, perform a dry run for the flow chart.

(b) State the purpose of the flow chart.

Solution

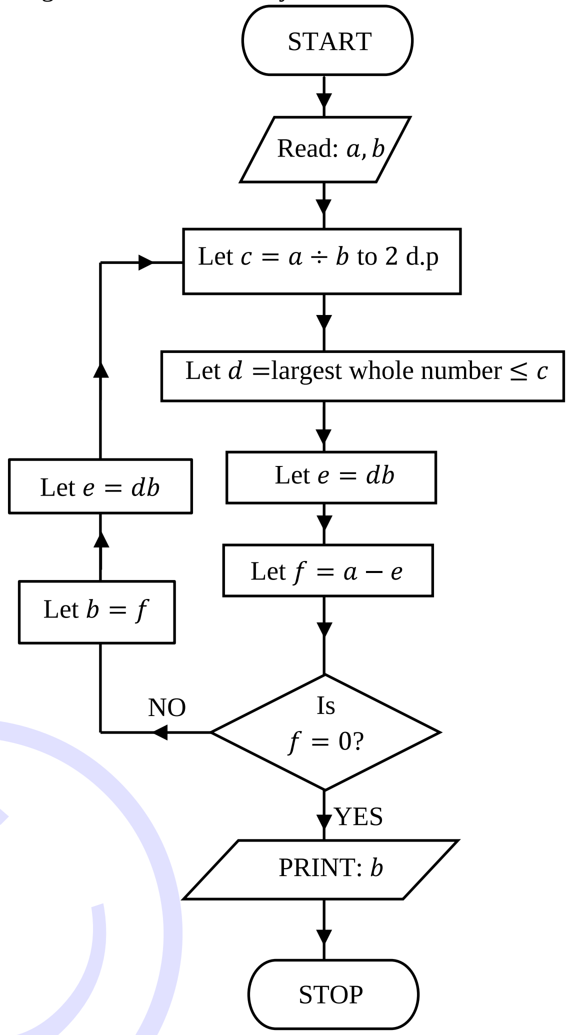
(a)

a	x	y	even?	$x = 0$?
0	54	63	Y	
	27	126	N	
126	26			N
	13	252	N	
378	12			N
	6	504	Y	
	3	1008	N	
1386	2			N
	1	2016	N	
3402	0			Y

(b) The purpose of the flow chart is to obtain the product xy

Example 12

An algorithm is described by the flow chart below.



(a) Given that $a = 645$ and $b = 255$, complete the table below to show the results obtained at each step.

a	b	c	d	e	f	$f = 0$?

(b) Explain how your solution to part (a) would be different if you had been given $a = 645$ and $b = 255$

(c) State the purpose of the flow chart.

Solution

(a)

a	b	c	d	e	f	$f = 0$?
645	255	2.53	2	510	135	No
255	135	1.89	1	135	120	No
135	120	1.13	1	120	15	No
120	15	8	8	120	0	Yes

The answer is 15

(b) The first row would be

255, 645, 0.45, 0, 0, 255, No

But the second row would be the same as the first row above, and the solution thereafter would be the same

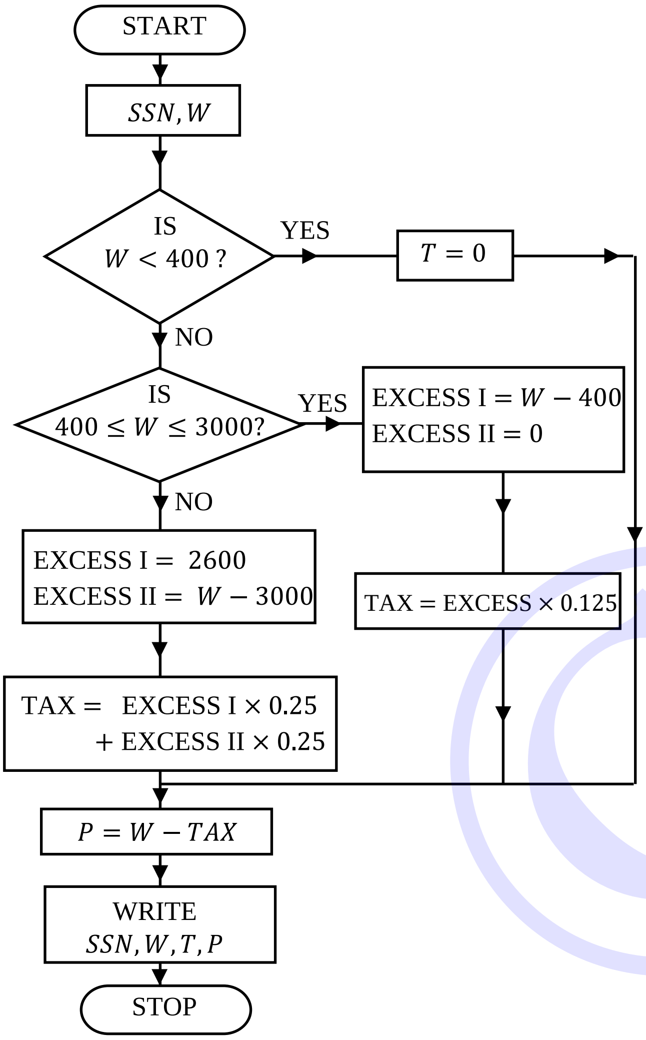
(c) The flow chart finds the highest common factor (H.C.F) of a and b.

(b) Tax on US S$ 25000 is US $ 4450

(c) Tax free sum is US $ 8000

Example 15

The flow chart below shows the social security numbers (SSN) and the monthly wage (W shillings) of an employee. represents the net pay.



Copy and complete the following table.

SSN	W	T	P
280 – 04	380	……	…….
180 – 34	840	……	…….
179 – 93	4500	……	…….
80 – 66	5500	……	…….
385 – 03	8000	……	…….

Solution

SSN	W	T	P
280 – 04	380	0	380
180 – 34	840	55	785
179 – 93	4500	700	3800
80 – 66	5500	970	4610
385 – 03	8000	1575	6425

Example 16

(a) Show that the iterative formula based on Newton Raphson's method for finding the natural logarithm of a number N is given by

$$x_{n+1} = \frac{e^{x_n}(x_n - 1) + N}{e^{x_n}}, n = 0, 1, 2, \ldots.$$

(b) Draw a flow chart that:

 (i) reads N and the initial approximation x_0 of the root

 (ii) computes and prints the natural logarithm after four iterations and gives the natural logarithm to three decimal places.

(c) Taking, $N = 10$, $x_0 = 2$, perform a dry run for the flow chart, give your root correct to three decimal places.

Solution

(a) Let $x = \ln N \Rightarrow e^x = N$

$e^x - N = 0$

Let $f(x) = e^x - N$, $f'(x) = e^x$

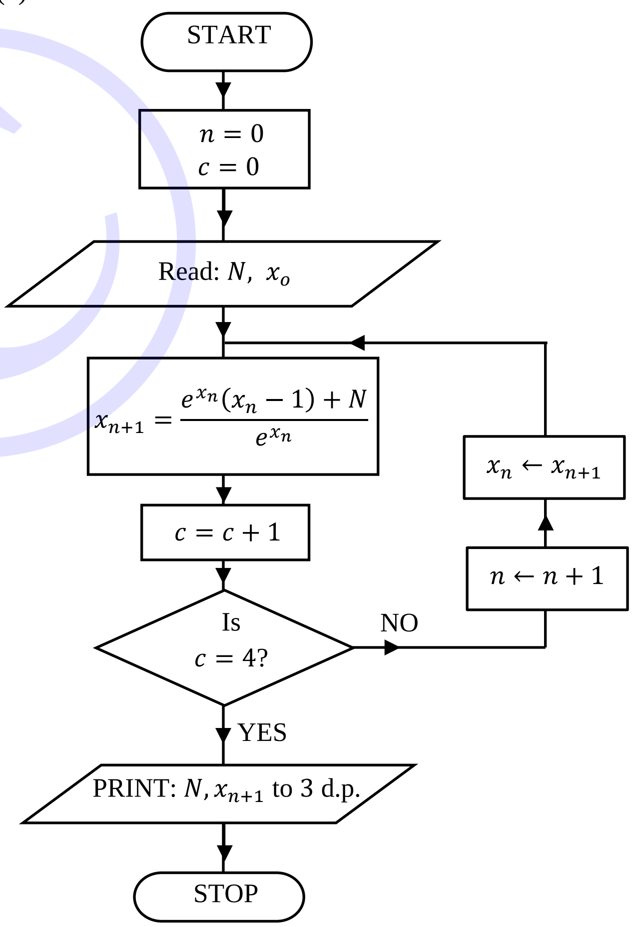
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{e^{x_n} - N}{e^{x_n}}$$

$$x_{n+1} = \frac{x_n e^{x_n} - e^{x_n} + N}{e^{x_n}}$$

$$x_{n+1} = \frac{e^{x_n}(x_n - 1) + N}{e^{x_n}}, n = 0, 1, 2, 3, \ldots$$

(b) Let c = counter for the number of iterations.



(c)

n	x_n	x_{n+1}	c
0	2.0	2.3533	1
1	2.3533	2.3039	2
2	2.3039	2.3026	3
3	2.3026	2.3026	4

The root is 2.303 (3 d.p)

(c) $x_0 = 2.0, N = 39.0, c = 4$

n	x_n	x_{n+1}	c
0	2.0	2.71875	1
1	2.71875	2.524236	2
2	2.524236	2.499375	3
3	2.499375	2.4989994	4

$\therefore N = 39.0, x_{n+1} = 2.499$ (3 d.p.)

Conclusion: Why Algorithms/flow charts?

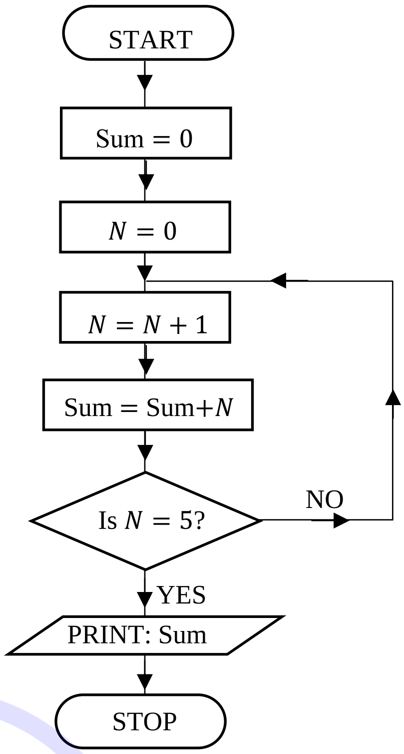
Algorithms run our world. In every area, algorithms are used to decide what action should be taken in a particular circumstance and as computers can consider all the possibilities far more quickly than a human brain, they are becoming more important to the running to the world. Here are just a few examples. Computer programs are implemented algorithms.

- Algorithms are used in the programming of calculators with a number of functions to solve various problems.
- In a game of chess, when each player has made 3 moves, there are over 9 million possible moves available; after 4 moves there are over 288 billion possible moves. Computers have the ability to consider all these moves far more quickly than humans. That is why no chess grandmaster has beaten a top computer chess algorithm since 2005.
- Algorithms are used by financial organisations to trade shares on the stock market. A computer following an algorithm can decide which deal to make far more quickly than a human and a split-second difference can be worth of million dollars.
- Closely guarded algorithms are used for internet searches to make them quicker and the results more relevant to the user. They will even auto complete the search terms based on previous searches.
- Algorithms are used to control automatic-pilot systems in airplanes. You have probably been piloted by an algorithm!
- Algorithms are used in economics by revenue authorities to calculate different taxes for different products and by banks in Automatic Teller Machines (ATMs).

Much as algorithms are useful in our daily lives, they must be correct enough to perform tasks without error. Errors in algorithms can be disastrous and can cost companies very big losses. Algorithms should as well be efficient to use as little time as possible.

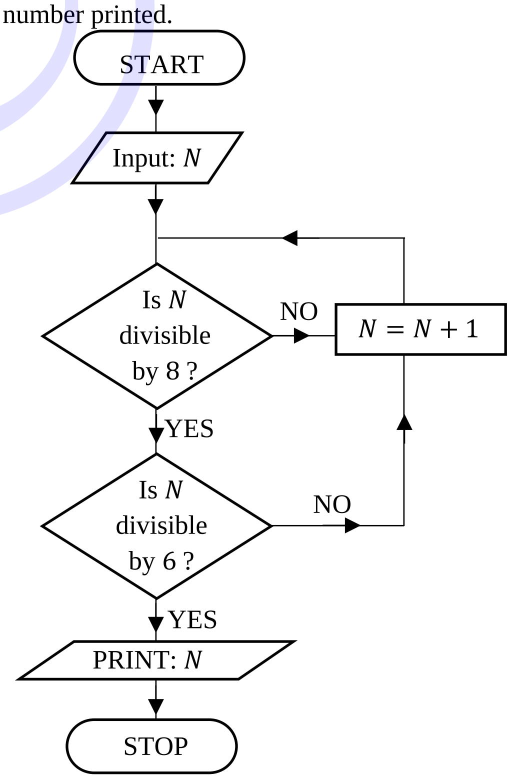
Self-Evaluation exercise

1. Study the flow chart below and answer the questions that follow.



 (a) Perform a dry run for the flow chart
 (b) State the purpose of the flow chart
 [Ans: (b) Prints the sum of numbers from 0 to 5]

2. Starting with $N = 60$, use the flow chart to find the number printed.



 [Ans: 72]

3. Draw a flow chart that computes and prints the sum of the cubes of the first ten natural numbers.
4. Construct a flow chart that computes and prints the sum of the squares of the first 10 natural numbers.

EXAMINATION QUESTIONS

SECTION A

1. Use the trapezium rule with seven ordinates to estimate

$$\int_0^3 [(1.2)^x - 1]^{1/2}\, dx \quad \text{correct to 2 decimal places}$$

[2019, No. 3: Ans: 1.58]

2. The table below shows the commuter bus fares from stage A to stages B, C, D and E.

Stage	A	B	C	D	E
Distance (km)	0	12	16	19	23
Fare (Shs)	0	1300	1700	2200	2500

(a) Jane boarded from A and stopped at a place 2 km after E. How much did she pay?

(b) Okello paid Shs 2000. How far from A did the bus leave him?

[2019, No. 6: Ans: (a) shs 2650 (b) 17.8 km]

3. The table below shows how T varies with S

T	-2.9	-0.1	2.9	3.1
S	30	20	12	9

Use linear interpolation/extrapolation to estimate the value of

(a) T when $S = 26$

(b) S when $T = 3.4$

[2018, No. 3: Ans: (a) -1.78 (b) 4.5]

4. Two numbers A and B have maximum possible errors e_a and e_b respectively.

(a) Write an expression for the maximum possible error in their sum

(b) If $A = 2.03$ and $B = 1.547$, find the maximum possible error in $A + B$.

[2018, No. 6: Ans: (a) $|e_a| + |e_b|$ (b) 0.0055]

5. The table below gives values of x and the corresponding values of $f(x)$

x	0.1	0.2	0.3	0.4	0.5	0.7
$f(x)$	4.21	3.83	3.25	2.85	2.25	1.43

Use linear interpolation/extrapolation to find

(a) $f(x)$ when $x = 0.6$

(b) the value of x when $f(x) = 0.75$

[2017, No. 3: Ans: (a) 1.84 (b) 0.9]

6. Given that $y = \frac{1}{x} + x$ and $x = 2.4$ correct to **one** decimal place, find the limits within which y lies.

[2017, No. 6: Ans: [2.7582, 2.8755]]

7. The table below shows the values of $f(x)$ for given values of x.

x	0.4	0.6	0.8
$f(x)$	-0.9613	-0.5108	-0.2231

Use linear interpolation to determine $f^{-1}(-0.4308)$ correct to **2** decimal places.

[2016, No. Ans: 0.66]

8. Use the trapezium rule with 4 sub-intervals to estimate

$$\int_0^{\frac{\pi}{2}} \cos x\, dx$$

correct to **three** decimal places.

[2016, No. 3: Ans: 0.987]

9. Use the trapezium rule with five subintervals to estimate

$$\int_2^4 \frac{5}{(x+1)}\, dx$$

Give your answer correct to **3** decimal places.

[2015, No. 2: Ans: 2.558]

10. The table below shows the values of a function $f(x)$ for given values of x.

x	$f(x)$
9	2.66
10	2.42
11	2.18
12	1.92

Use linear interpolation or extrapolation to find:

(a) $f(10.4)$

(b) the value of x, corresponding to $f(x) = 1.46$

[2015, No. 6: Ans: (a) 2.324 (b) 13.769]

11. Use the trapezium rule with four sub-intervals to estimate

$$\int_{0.2}^{1.0} \left(\frac{2x+1}{x^2+x}\right) dx, \text{ correct to } \textbf{two} \text{ decimal places}$$

[2014, No. 3: Ans: 2.20]

12. Given the table below;

x	0	10	20	30
y	6.6	2.9	-0.1	-2.9

Use linear interpolation to find;

(a) y when $x = 16$

(b) x when $y = -1$

[2014, No.6: Ans: (a) 1.1 (b) 23.2]

13. The table below shows the values of a function $f(x)$.

x	1.8	2.0	2.2	2.4
$f(x)$	0.532	0.484	0.436	0.384

Use linear interpolation to find the value of

(a) $f(2.08)$

(b) x corresponding to $f(x) = 0.5$

[2013, No. 2: Ans: (a) 0.465 (b) 1.9]

14. Find the approximate value of $\int_0^2 \frac{1}{1+x^2} dx$ using the trapezium rule with 6 ordinates. Give your answer to 3 decimal places.

[2013, No. 5: Ans: 1.105]

15. Use the trapezium rule with four sub-intervals to estimate:

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x} dx$$

Give your answer correct to **three** decimal places

[2012, No. 3: Ans: 1.013]

16. The table below the cost y in shillings for hiring a motorcycle for a distance of x kilometres.

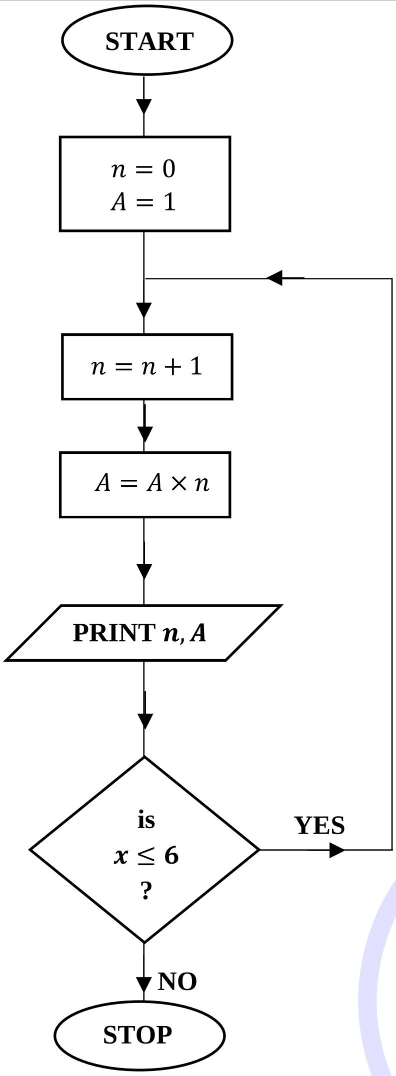
Distance (x km)	10	20	30	40
Cost (shs. y)	2800	3600	4400	5200

Use linear interpolation or extrapolation to calculate the:

(a) cost of hiring the motorcycle for a distance of 45 km

SECTION B

1. (a) Show that the equation $x - 3\sin x = 0$ has a root between 2 and 3.
   (b) Show that the Newton-Raphson iterative formula for estimating the root of the equation in (a) is given by
   $$x_{n+1} = \frac{3(\sin x_n - x_n \cos x_n)}{1 - 3\cos x_n}, n = 0, 1, 2, \dots$$
   Hence find the root of the equation correct to **2** decimal places.
   [2019, No. 11: Ans: (b) 2.28]
2. (a) Given that $y = e^x$ and $x = 0.62$ correct to two decimal places, find the interval within which the exact value of y lies.
   (b) Show that the maximum possible relative error in $y\sin^2 x$ is
   $\left|\frac{\Delta y}{y}\right| + 2\cot x\,|\Delta x|$, where Δx and Δy are errors in x and y respectively.
   Hence find the percentage error in calculating $y\sin^2 x$ if $y = 5.2 \pm 0.05$ and $x = \frac{\pi}{6} \pm \frac{\pi}{360}$.
   [2019, No. 14: Ans: (a) $(1.84966, 1.86825)$ (b) 3.985%]
3. (a) Use the trapezium rule with 6-ordinates to estimate the value of $\int_0^{\frac{\pi}{2}}(x + \sin x)\,dx$, correct to **three** decimal places
   (b) (i) Evaluate $\int_0^{\frac{\pi}{2}}(x + \sin x)\,dx$, correct to **three** d.ps
   (ii) Calculate the error in your estimation in (a) above
   (iii) Suggest how the error may be reduced
   [2018, No. 11: Ans: (a) 2.225 (b) (i) 2.234 (ii) 0.009]
4. (a) Draw on the same axes the graphs of the curves $y = 2 - e^{-x}$ and $y = \sqrt{x}$ for $2 \le x \le 5$.
   (b) Determine from your graphs the interval within which the root of the equation $e^{-x} + \sqrt{x} - 2 = 0$ lies.
   Hence, use Newton-Raphson's method to find the root of the equation correct to **3** decimal places.
   [2018, No.14: Ans: (b) Between 3.9 and 4; 3.921]
5. A student used the trapezium rule with five sub-intervals to estimate
   $$\int_2^3 \frac{x}{x^2 - 3}\,dx \quad \text{correct to } \textbf{three} \text{ decimal places}$$
   Determine;
   (a) the value the student obtained
   (b) the actual value of the integral
   (c) (i) the error the student made in the estimate
   (ii) how the student can reduce the error
   [2017, No. 11: Ans: (a) 0.917 (b) 0.896 (c) (i) 0.021]
6. By plotting graphs of $y - x$ and $y = 4\sin x$ on the same axes, show that the root of the equation $x - 4\sin x = 0$ lies between 2 and 3.
   Hence use Newton's Raphson's method to find the root of the equation correct to **3** decimal places.
   [2017, No. 14: Ans: 2.475]
7. Given the equation $x^3 - 6x^2 + 9x + 2 = 0$;
   (a) show that the equation has a root between -1 and 0
   (b) (i) show that the Newton Raphson formula for approximating the root of the equation is given by
   $$x_{n+1} = \frac{2}{3}\left[\frac{x_n^3 - 3x_n^2 - 1}{x_n^2 - 4x_n + 3}\right]$$
   (ii) use the formula b(i) above, with an initial approximation of $x_0 = -0.5$, to find the root of the given equation correct to **two** decimal places.
   [2016, No. 11: Ans: (b) (ii) -0.20]
8. The numbers x and y are approximated by X and Y with errors Δx and Δy respectively.
   (a) Show that the maximum relative error in xy is given by
   $$\left|\frac{\Delta x}{X}\right| + \left|\frac{\Delta y}{Y}\right|$$
   (b) If $x = 4.95$ and $y = 2.013$ are each rounded off to the given number of decimal places, calculate the
   (i) percentage error in xy,
   (ii) limits within which xy is expected to lie. Give your answer to **three** decimal places.
   [2016, No. 14: Ans: (b)(i) 0.126% (ii) $(9.952, 9.977)$]
9. The numbers $A = 6.341$ and $B = 2.6$ have been rounded to the given number of decimal places.
   (a) Find the maximum possible error in AB
   (b) Determine the interval within which $\frac{A^2}{B}$ can be expected to lie. Give your answer correct to **3** decimal places.
   [2015, No. 10: Ans: (a) 0.31835 (b) $(15.171, 15.770)$]
10. (a) Show that the iterative formula based on Newton Raphson's method for approximating the root of the equation $2\ln x - x + 1 = 0$ is given by
    $$x_{n+1} = x_n\left(\frac{2\ln x_n - 1}{x_n - 2}\right), n = 0,1,2 \dots\dots$$
    (b) Draw a flow chart that:
    (i) reads the initial approximation x_0 of the root
    (ii) computes and prints the root correct to two decimal places, using the formula in (a).
    (c) Taking $x_0 = 3.4$, perform a dry run to find the root of the equation.
    [2015, No. 14: Ans: (c) 3.51]
11. The numbers x and y are approximated with possible errors of Δx and Δy respectively.
    (a) Show that the maximum absolute error in the quotient $\frac{x}{y}$ is given by
    $$\frac{y\Delta x + x\Delta y}{y^2}$$
    (b) Given that $x = 2.68$ and $y = 0.9$ are rounded to the given number of decimal places, find the interval within which the exact value of $\frac{x}{y}$ is expected to lie.
    [2014, No. 11: Ans: (b) $(2.8158, 3.1588)$]
12. (a) Show that the Newton-Raphson formula for finding the root of the equation $x = N^{\frac{1}{5}}$ is given by
    $$x_{n+1} = \frac{4x_n^5 + N}{5x_n^4}, n = 0,1,2, \dots$$
    (b) Construct a flow chart that
    (i) reads N and the first approximation x_0,



(i) Perform a dry run for the flow chart,

(ii) State the purpose of the flow chart,

(iii) Write down the relationship between n and A

(b) Draw a flow chart that reads and prints the mean of the first twenty counting numbers and perform a dry run of your chart.

[2006, No. 10: Ans: (a)(ii) compute and print the factorial of whole numbers up to 7 (iii) $A = n!$]

30. (i) Use the trapezium rule to estimate the area of $y = 3^x$ between the x-axis, $x = 1$ and $x = 2$, using five sub-intervals. Give your answer correct to 4 significant figures.

(ii) Find the exact value of $\int_1^2 3^x\, dx$

(iii) Find the percentage error in calculations (i) and (ii) above.

[2005, No. 10: Ans: (i) 5.4834 (ii) 5.4614 (iii) 0.4028%]

31. (a) Use the trapezium rule to estimate the area of $y = 5^{2x}$ between the x-axis, $x = 0$ and $x = 1$, using five sub-intervals. Give your answer correct to 3 decimal places.

(b) Find the exact value of:

$$\int_0^1 5^{2x}\, dx$$

(c) Determine the percentage error in the two calculations in (a) and (b) above.

[2004, No. 10: Ans: (a) 7.712 (b) 7.456 (c) 3.43%]

32. (a) Use a graphical method to find a first approximation to the real root of $x^3 - 3x + 4 = 0$

(b) Use the Newton-Raphson method to find the root of the equation correct to 2 decimal places.

[2004, No. 12: Ans: 2.20]

33. (i) Show that the equation $x = \ln(8 - x)$ has a root between 1 and 2

(ii) Use the Newton's Raphson method to find the approximate root of $x = \ln(8 - x)$ correct to 3 decimal places.

[2003, No. 9: Ans: (ii) 1.821]

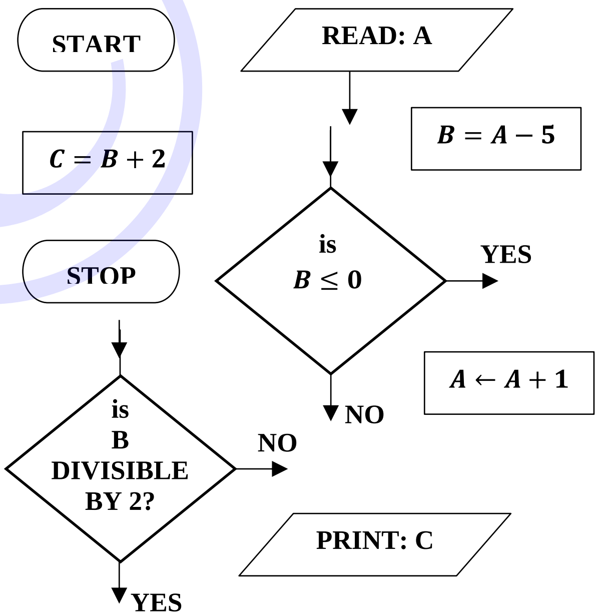
34. (i) Determine the iterative formula for finding the fourth root of a given number N

(ii) Draw a flow chart that reads N and the initial approximation, X_0, computes and prints the fourth root of N correct to 3 decimal places

(iii) Perform a dry run for $N = 150.10$ and $X_0 = 3.200$

[2003, No. 12: Ans: (i) $\frac{3}{4}\left(x_n + \frac{N}{3x_n^3}\right)$ (iii) 3.500]

35. Given below are parts of a flow chart not arranged in order.



(a) Rearrange them and draw a complete logical flow chart

(b) State the purpose of the flow chart

(c) Perform a dry run of your rearranged flow chart by copying and completing the table below:

A	B	C
46	…..	…..
77	…..	…..
120	…..	…..
177	…..	…..

BINOMIAL PROBABILITIES (DISTRIBUTION) B(*n*,*x*), INDIVIDUAL TERMS *Pr*

n	*r*						*x*					
		0.01	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
2	0	0.9801	9025	8100	7225	6400	5625	4900	4225	3600	3025	2500
	1	0.0198	0950	1800	2550	3200	3750	4200	4550	4800	4950	5000
	2	0.0001	0025	0100	0225	0400	0625	0900	1225	1600	2025	2500
3	0	0.9703	8574	7290	6141	5120	4219	3430	2746	2160	1664	1250
	1	0.0294	1354	2430	3251	3840	4219	4410	4436	4320	4084	3750
	2	0.0003	0071	0270	0574	0960	1406	1890	2389	2880	3341	3750
	3		0001	0010	0034	0080	0156	0270	0429	0640	0911	1250
4	0	0.9606	8145	6561	5220	4096	3164	2401	1785	1296	0915	0625
	1	0.0388	1715	2916	3685	4096	4219	4116	3845	3456	2995	2500
	2	0.0006	0135	0486	0975	1536	2109	2646	3105	3456	3675	3750
	3		0005	0036	0115	0256	0469	0756	1115	1536	2005	2500
	4			0001	0005	0016	0039	0081	0150	0256	0410	0625
5	0	0.9510	7738	5905	4437	3277	2373	1681	1160	0778	0503	0312
	1	0.0480	2036	3280	3915	4096	3955	3602	3124	2592	2059	1562
	2	0.0010	0214	0729	1382	2048	2637	3087	3364	3456	3369	3125
	3		0011	0081	0244	0512	0879	1323	1811	2304	2757	3125
	4			0004	0022	0064	0146	0284	0488	0768	1128	1562
	5				0001	0003	0010	0024	0053	0102	0185	0312
6	0	0.9415	7351	5314	3771	2621	1780	1176	0754	0467	0277	0156
	1	0.0571	2321	3543	3993	3932	3560	3025	2437	1866	1359	0938
	2	0.0014	0305	0984	1762	2458	2966	3241	3280	3110	2780	2344
	3		0021	0146	0415	0819	1318	1852	2355	2765	3032	3125
	4		0001	0012	0055	0154	0330	0595	0951	0382	1861	2344
	5			0001	0004	0015	0044	0102	0205	0369	0609	0938
	6					0001	0002	0007	0018	0041	0083	0156
7	0	0.9321	6983	4783	3206	2097	1335	0824	0490	0280	0152	0078
	1	0.0659	2573	3720	3960	3670	3115	2471	1848	1306	0872	0547
	2	0.0020	0406	1240	2097	2753	3115	3177	2985	2613	2140	1641
	3		0036	0230	0617	1147	1730	2269	2679	2903	2918	2734
	4		0002	0026	0109	0287	0577	0972	1442	1935	2388	2734
	5			0002	0012	0043	0115	0250	0466	0774	1172	1641
	6				0001	0004	0013	0036	0084	0172	0320	0547
	7						0001	0002	0006	0016	0037	0078
8	0	0.9227	6634	4305	2725	1678	1001	0576	0319	0168	0084	0039
	1	0.0746	2793	3826	3847	3355	2670	1977	1373	0896	0548	0312
	2	0.0026	0515	1488	2376	2936	3115	2965	2587	2090	1569	1094
	3	0.0001	0054	0331	0839	1468	2076	2541	2786	2787	2568	2188
	4		0004	0046	0185	0459	0865	1361	1875	2322	2627	2734
	5			0004	0026	0092	0231	0467	0808	1239	1719	2188
	6				0002	0011	0038	0100	0217	0413	0703	1094
	7					0001	0004	0012	0033	0079	0164	0312
	8							0001	0002	0007	0017	0039
9	0	0.9135	6302	3874	2316	1342	0751	0404	0207	0101	0046	0020
	1	0.0830	2985	3874	3679	3020	2253	1556	1004	0605	0339	0176
	2	0.0034	0629	1722	2597	3020	3003	2668	2162	1612	1110	0703
	3	0.0001	0077	0446	1069	1762	2336	2668	2716	2508	2119	1641
	4		0006	0074	0283	0661	1168	1715	2194	2508	2600	2461
	5			0008	0050	0165	0389	0735	1181	1672	2128	2461
	6			0001	0006	0028	0087	0210	0424	0743	1160	1641
	7					0003	0012	0039	0098	0212	0407	0703
	8						0001	0004	0013	0035	0083	0176
	9								0001	0003	0008	0020
10	0	0.9044	5987	3487	1969	1074	0563	0282	0135	0060	0025	0010
	1	0.0914	3151	3874	3474	2684	1877	1211	0725	0403	0207	0098
	2	0.0042	0746	1937	2759	3020	2816	2335	1757	1209	0763	0439
	3	0.0001	0105	0574	1298	2013	2503	2668	2522	2150	1665	1172
	4		0010	0112	0401	0881	1460	2001	2377	2508	2384	2051
	5		0001	0015	0085	0264	0584	1029	1536	2007	2340	2461
	6			0001	0012	0055	0162	0368	0689	1115	1596	2051
	7				0001	0008	0031	0090	0212	0425	0746	1172

Where a space in the table is empty the probability is less than 0.00005.

BINOMIAL PROBABILITIES (DISTRIBUTION) B(*n*,*x*), INDIVIDUAL TERMS *Pr*

n	*r*	0.01	0.05	0.10	0.15	0.20	*x* 0.25	0.30	0.35	0.40	0.45	0.50
10	8					0001	0004	0014	0043	0106	0229	0439
	9							0001	0005	0016	0042	0098
	10									0001	0003	0010
11	0	0.8953	5688	3138	1673	0859	0422	0198	0088	0036	0014	0005
	1	0.0995	3293	3835	3248	2362	1549	0932	0518	0266	0125	0054
	2	0.0050	0867	2131	2866	2953	2581	1998	1395	0887	0513	0269
	3	0.0002	0137	0710	1517	2215	2581	2568	2254	1774	1259	0806
	4		0014	0158	0536	1107	1721	2201	2428	2365	2060	1611
	5		0001	0025	0132	0388	0803	1321	1830	2207	2360	2256
	6			0003	0023	0097	0268	0566	0985	1471	1931	2256
	7				0003	0017	0064	0173	0379	0701	1128	1611
	8					0002	0011	0037	0102	0234	0462	0806
	9						0001	0005	0018	0052	0126	0269
	10								0002	0007	0021	0054
	11										0002	0005
12	0	0.8864	5404	2824	1422	0687	0317	0138	0057	0022	0008	0002
	1	0.1074	3413	3766	3012	2062	1267	0712	0368	0174	0075	0029
	2	0.0060	0988	2301	2924	2835	2323	1678	1088	0639	0339	0161
	3	0.0002	0173	0852	1720	2362	2581	2397	1954	1419	0923	0537
	4		0021	0213	0683	1329	1936	2311	2367	2128	1700	1208
	5		0002	0038	0193	0532	1032	1585	2039	2270	2225	1934
	6			0005	0040	0155	0401	0792	1281	1766	2124	2256
	7				0006	0033	0115	0291	0591	1009	1489	1934
	8				0001	0005	0024	0078	0199	0420	0762	1208
	9					0001	0004	0015	0048	0125	0277	0537
	10							0002	0008	0025	0068	0161
	11								0001	0003	0010	0029
	12										0001	0002
15	0	0.8601	4633	2059	0874	0352	0134	0047	0016	0005	0001	
	1	0.1303	3658	3432	2312	1319	0668	0305	0126	0047	0016	0005
	2	0.0092	1348	2669	2856	2309	1559	0916	0476	0219	0090	0032
	3	0.0004	0307	1285	2184	2501	2252	1700	1110	0634	0318	0139
	4		0049	0428	1156	1876	2252	2186	1792	1268	0780	0417
	5		0006	0105	0449	1032	1651	2061	2123	1859	1404	0916
	6			0019	0132	0430	0917	1472	1906	2066	1914	1527
	7			0003	0030	0138	0393	0811	1319	1771	2013	1964
	8				0005	0035	0131	0348	0710	1181	1647	1964
	9				0001	0007	0034	0116	0298	0612	1048	1527
	10					0001	0007	0030	0096	0245	0515	0916
	11						0001	0006	0024	0074	0191	0417
	12							0001	0004	0016	0052	0139
	13								0001	0003	0010	0032
	14										0001	0005
20	0	0.8179	3585	1216	0388	0115	0032	0008	0002			
	1	0.1652	3774	2702	1368	0576	0211	0068	0020	0005	0001	
	2	0.0159	1887	2852	2293	1369	0669	0278	0100	0031	0008	0002
	3	0.0010	0596	1901	2428	2054	1339	0716	0323	0123	0040	0011
	4		0133	0898	1821	2182	1897	1304	0738	0350	0139	0046
	5		0022	0319	1028	1746	2023	1789	1272	0746	0365	0148
	6		0003	0089	0454	1091	1686	1916	1712	1244	0746	0370
	7			0020	0160	0545	1124	1643	1844	1659	1221	0739
	8			0004	0046	0222	0609	1144	1614	1797	1623	1201
	9			0001	0011	0074	0271	0654	1158	1597	1771	1602
	10				0002	0020	0099	0308	0686	1171	1593	1762
	11					0005	0030	0120	0336	0710	1185	1602
	12					0001	0008	0039	0136	0355	0727	1201
	13						0002	0010	0045	0146	0366	0739
	14							0002	0012	0049	0150	0370
	15								0003	0013	0049	0148
	16									0003	0013	0046
	17										0002	0011
	18											0002

If the probability of success in a single trial is x the probability Pr of exactly r successes in n independent trials is given by the binomial or Bernoulli distribution B(n, x_p):

$$Pr = \binom{n}{r} x^r (1-x)^{n-r}$$

NORMAL DISTRIBUTION N(0,1) $\phi(Z)$

Z	0	1	2	3	4	5	6	7	8	9	SUBTRACT 1	2	3	4	5	6	7	8	9
0.0	0.3989	3989	3989	3988	3986														
						3984	3982	3980	3977	3973	0	1	1	1	1	2	2	2	3
0.1	0.3970	3965	3961	3956	3951						0	1	1	2	2	3	3	4	4
						3945	3939	3932	3925	3918	1	1	2	3	3	4	5	6	6
0.2	0.3910	3902	3894	3885	3876						1	2	3	4	4	5	6	7	8
						3867	3857	3847	3836	3825	1	2	3	4	5	7	8	9	10
0.3	0.3814	3802	3790	3778	3765						1	2	4	5	6	7	8	10	11
						3752	3739	3725	3712	3697	1	3	4	6	7	8	10	11	13
0.4	0.3683	3668	3653	3637	3621	3605	3589	3572	3555	3538	2	3	5	6	8	10	11	13	14
0.5	0.3521	3503	3485	3467	3448	3429	3410	3391	3372	3352	2	4	6	8	10	11	13	15	17
0.6	0.3332	3312	3292	3271	3251	3230	3209	3187	3166	3144	2	4	6	8	10	13	15	17	19
0.7	0.3123	3101	3079	3056	3034	3011	2989	2966	2943	2920	2	5	7	9	11	14	16	18	21
0.8	0.2897	2874	2850	2827	2803	2780	2756	2732	2709	2685	2	5	7	10	12	14	17	19	22
0.9	0.2661	2637	2613	2589	2565	2541	2516	2492	2468	2444	2	5	7	10	12	14	17	19	22
1.0	0.2420	2396	2371	2347	2323	2299	2275	2251	2227	2203	2	5	7	10	12	14	17	19	22
1.1	0.2179	2155	2131	2107	2083	2059	2036	2012	1989	1965	2	5	7	10	12	14	17	19	22
1.2	0.1942	1919	1895	1872	1849	1826	1804	1781	1758	1736	2	5	7	9	12	14	16	18	21
1.3	0.1714	1691	1669	1647	1626	1604	1582	1561	1539	1518	2	4	7	9	11	13	15	18	20
1.4	0.1497	1476	1456	1435	1415	1394	1374	1354	1334	1315	2	4	6	8	10	12	14	16	18
1.5	0.1295	1276	1257	1238	1219	1200	1182	1163	1145	1127	2	4	6	8	9	11	13	15	17
1.6	0.1109	1092	1074	1057	1040	1023	1006	0989	0973	0957	2	3	5	7	8	10	12	14	15
1.7	0.0940	0925	0909	0893	0878	0863	0848	0833	0818	0804	2	3	5	6	8	9	11	12	14
1.8	0.0790	0775	0761	0748	0734						1	3	4	6	7	8	10	11	13
						0721	0707	0694	0681	0669	1	3	4	5	7	8	9	10	12
1.9	0.0656	0644	0632	0620	0608	0596	0584	0573	0562	0551	1	2	4	5	6	7	8	10	11
2.0	0.0540	0529	0519	0508	0498	0488	0478	0468	0459	0449	1	2	3	4	5	6	7	8	9
2.1	0.0440	0431	0422	0413	0404	0396	0387	0379	0371	0363	1	2	3	4	4	5	6	7	8
2.2	0.0355	0347	0339	0332	0325	0317	0310	0303	0297	0290	1	1	2	3	4	4	5	6	6
2.3	0.0283	0277	0270	0264	0258	0252	0246	0241	0235	0229	1	1	2	2	3	4	4	5	5
2.4	0.0224	0219	0231	0208	0203	0198	0194	0189	0184	0180	0	1	1	2	2	3	3	4	4
2.5	0.0175	0171	0167	0163	0158	0154	0151	0147	0143	0139	0	1	1	2	2	2	3	3	4
2.6	0.0136	0132	0129	0126	0122	0119	0116	0113	0110	0107	0	1	1	1	2	2	2	2	3
2.7	0.0104	0101	0099	0096	0093	0091	0088	0086	0084	0081	0	1	1	1	1	2	2	2	3
2.8	0.0079	0077	0075	0073	0071	0069	0067	0065	0063	0061									
2.9	0.0060	0058	0056	0055	0053	0051	0050	0048	0047	0046									
3.	0.0044	0033									1	2	3	4	5	6	7	8	9
			0024	0017							1	1	2	2	3	4	4	5	5
					0012	0009	0006	0004	0003	0002									

The functions tabled are:

$\phi(Z) = \sqrt{(\frac{1}{2\pi})}\exp(-\frac{1}{2}Z^2)$, where $\phi(Z)$ is the probability density of the standardized normal distribution N(0,1)

CRITICAL POINTS OF THE NORMAL DISTRIBUTION Z_p

P	Q	z	P	Q	z	P	Q	z
.00	.50	0.000	.460	.040	1.751	.490	.010	2.326
.05	.45	0.126	.462	.038	1.774	.491	.009	2.366
.10	.40	0.253	.464	.036	1.799	.492	.008	2.409
.15	.35	0.385	.466	.034	1.825	.493	.007	2.457
.20	.30	0.524	.468	.032	1.852	.494	.006	2.512
.25	.25	0.674	.470	.030	1.881	.495	.005	2.576
.30	.20	0.842	.472	.028	1.911	.496	.004	2.652
.35	.15	1.036	.474	.026	1.943	.497	.003	2.748
.40	.10	1.282	.476	.024	1.977	.498	.002	2.878
.45	.05	1.645	.478	.022	2.014	.499	.001	3.090
.450	.050	1.645	.480	.020	2.054	.4995	.0005	3.291
.452	.048	1.665	.482	.018	2.097	.4999	.0001	3.719
.454	.046	1.685	.484	.016	2.144	.49995	.00005	3.891
.456	.044	1.706	.486	.014	2.197	.49999	.00001	4.265
.458	.042	1.728	.488	.012	2.257	.499995	.000005	4.417

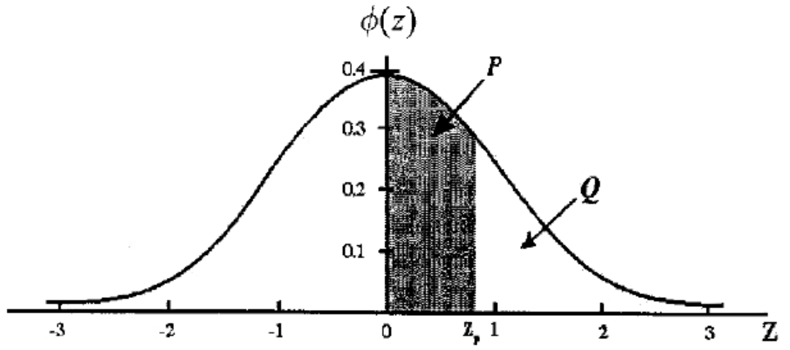
CUMULATIVE NORMAL DISTRIBUTION $P(z)$

z	0	1	2	3	4	5	6	7	8	9	ADD 1	2	3	4	5	6	7	8	9
0.0	0.0000	0040	0080	0120	0160	0199	0239	0279	0319	0359	4	8	12	16	20	24	28	32	36
0.1	0.0398	0438	0478	0517	0557	0596	0636	0675	0714	0753	4	8	12	16	20	24	28	32	36
0.2	0.0793	0832	0871	0910	0948	0987	1026	1064	1103	1141	4	8	12	15	19	22	27	31	35
0.3	0.1179	1217	1255	1293	1331	1368	1406	1443	1480	1517	4	8	11	15	19	22	26	30	34
0.4	0.1554	1591	1628	1664	1700	1736	1772	1808	1844	1879	4	7	11	14	18	22	25	29	32
0.5	0.1915	1950	1985	2019	2054	2088	2123	2157	2190	2224	3	7	10	14	17	21	24	27	31
0.6	0.2257	2291	2324	2357	2389	2422	2454	2486	2517	2549	3	6	10	13	16	19	23	26	29
0.7	0.2580	2611	2642	2673							3	6	9	12	15	19	22	25	28
					2704	2734	2764	2794	2823	2852	3	6	9	12	15	18	21	24	27
0.8	0.2881	2910	2939	2967	2995	3023					3	6	8	11	14	17	20	22	25
							3051	3078	3106	3133	3	5	8	11	13	16	19	22	24
0.9	0.3159	3186	3212	3238	3264	3289					3	5	8	10	13	16	18	21	23
							3315	3340	3365	3389	2	5	7	10	12	15	17	20	22
1.0	0.3413	3438	3461	3485	3508						2	5	7	10	12	14	17	19	22
						3531	3554	3577	3599	3621	2	4	7	9	11	13	15	18	20
1.1	0.3643	3665	3686	3708							2	4	6	8	11	13	15	17	19
					3729	3749	3770	3790	3810	3830	2	4	6	8	10	12	14	16	18
1.2	0.3849	3869	3888	3907	3925						2	4	6	8	10	11	13	15	17
						3944	3962	3980	3997	4015	2	4	5	7	9	11	13	14	16
1.3	0.4032	4049	4066	4082	4099	4115	4131	4147	4162	4177	2	3	5	6	8	10	11	13	14
1.4	0.4192	4207	4222	4236	4251	4265	4279	4292	4306	4319	1	3	4	6	7	8	10	11	13
1.5	0.4332	4345	4357	4370	4382	4394	4406	4418	4429	4441	1	2	4	5	6	7	8	10	11
1.6	0.4452	4463	4474	4484	4495	4505	4515	4525	4535	4545	1	2	3	4	5	6	7	8	9
1.7	0.4554	4564	4573	4582	4591	4599	4608	4616	4625	4633	1	2	3	3	4	5	6	7	8
1.8	0.4641	4649	4656	4664	4671	4678	4686	4693	4699	4706	1	1	2	3	4	4	5	6	6
1.9	0.4713	4719	4726	4732	4738	4744	4750	4756	4761	4767	1	1	2	2	3	4	4	5	5
2.0	0.4772	4778	4783	4788	4793	4798	4803	4808	4812	4817	0	1	1	2	2	3	3	4	4
2.1	0.4821	4826	4830	4834	4838	4842	4846	4850	4854	4957	0	1	1	2	2	2	3	3	4
2.2	0.4861	4864	4868	4871	4875	4878	4881	4884	4887	4890	0	1	1	1	2	2	2	3	3
2.3	0.4893	4896	4898	4901	4904	4906	4909	4911	4913	4916	0	0	1	1	1	2	2	2	2
2.4	0.4918	4920	4922	4925	4927	4929	4931	4932	4934	4936	0	0	1	1	1	1	1	2	2
2.5	0.4938	4940	4941	4943	4945	4946	4948	4949	4951	4952									
2.6	0.4953	4955	4956	4957	4959	4960	4961	4962	4963	4964									
2.7	0.4965	4966	4967	4968	4969	4970	4971	4972	4973	4974									
2.8	0.4974	4975	4976	4977	4977	4978	4979	4979	4980	4981									
2.9	0.4981	4982	4982	4983	4984	4984	4985	4985	4986	4986									
3.0	0.4987	4990	4993	4995	4997	4998	4998	4999	4999	5000									

The table gives $P(z) = \int_0^z \phi(z)dz$

If the random variable Z is distributed as the standard normal distribution N(0,1) then:

1. $P(o < Z < z_p) = P(\text{Shaded Area})$
2. $P(Z > Z_p) = Q = \frac{1}{2} - P$
3. $P(Z > |Z_p|) = 1 - 2P = 2Q$



REFERENCES

1. S. Chandler, L. Bostock (1996), Mechanics and Statistics for Advanced Level
2. Thorning, A. J. and D.W.S. (1996), Understanding Mechanics, Oxford University Press, United Kingdom
3. J. Crawshaw and J. Chambers, A concise course in Advanced Level Statistics, Fourth Edition, Nelson Thornes (Publishers) Ltd.
4. Core Mathematics for Advanced Level by L. Bostock and S. Chandler
5. Understanding statistics by Graham Upton and Ian Cook
6. Business Mathematics and Statistics, 6th Edition 2008 by Andre Francis
7. [Cambridge International AS & A Level] Douglas Quadling - Advanced Level Mathematics Mechanics 1 (2002, Cambridge University Press)
8. [Complete Advanced Level Mathematics] Martin Adams, June Haighton, Jeff Trim - Complete Advanced Level Mathematics Mechanics (1999, Stanley Thornes (Publishers) Ltd)
9. [IIT JEE] M K Sinha - Understanding Mechanics for JEE Main and Advanced (Bansal Classes Kota)
10. Cambridge-International-AS-and-A-Level-Mathematics-Statistics by Steve Jobbs and Jane Miller
11. Cambridge-International-AS-and-A-Level-Mathematics-Statistics by Sophie Goldie
12. Edexcel-AS-and-A-level-Mathematics-Statistics-and-Mechanics-Year-1-AS, Pearson 2017
13. Humphrey Topping- A Shorter Intermediate Mechanics, SI Edition Longman 1971
14. S.C. Gupta, V.K. Kapoor - Fundamentals of Mathematical Statistics (A Modern Approach), 10th Edition (2000, Sultan Chand & Sons)
15. Macmillan Work Out Series – A-Level Statistics by Alan Ball & Geoff Buckwell, Second edition 1991
16. Work-Out-Applied-Mathematics-A-Level by R. Haines and B. Haines, Macmillan press, 1986
17. Applied Mathematics, E.D. Hodge and B.G.J. Wood, Blackie and Son Ltd
18. Applied Mathematics I and II, L. Bostock and S. Chandler, Stanley Thornes (Publishers) Ltd.
19. Applied Mathematics for Advanced Level, H. Mulholland and H.G. Phillips. Butterworths.
20. Advanced level Applied mathematics. C.G. Lambe. The English Universities Press Ltd.
21. A-Level Applied Mathematics A.J. Francis Bell & Hyman.
22. Statistics and Probability. S.E Hodge and M. L Seed. Blackie and Son Ltd.
23. Modern Applied Mathematics. J.C. Turner, Hodder and Stoughton.
24. Fundamentals of Statistics. H. Mulhollahd and C.R. Jones. Butterworths
25. Advanced General Statistics. B.C. Erricker. Hodder and Stoughton.
26. Mathematics A-Level course companion, Letts Study Aids.
27. Advanced Level Statistics. A. Francis. Stanley Thornes (Publishers) Ltd.
28. Edexcel AS and A level Further Mathematics Decision Mathematics 1, Pearson Education Ltd, 2017.
29. Numerical Analysis by Timothy Sauer, Second Edition 2012, Pearson Education Inc.
30. National Curriculum Development Centre Mathematics and Physics syllabus, Volume 2, 2013
31. Uganda National Examination Board Examination Past Papers

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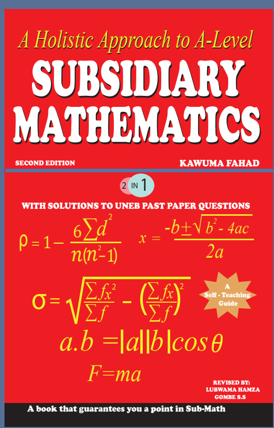

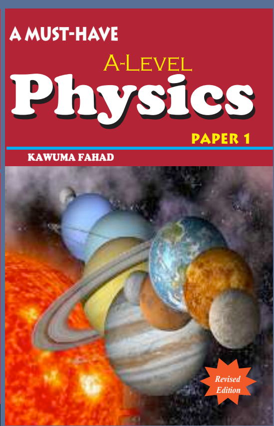

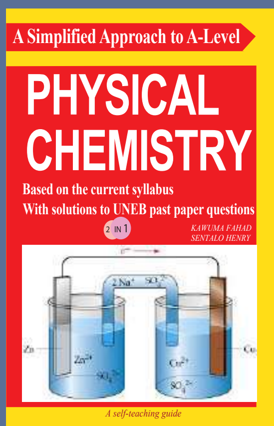

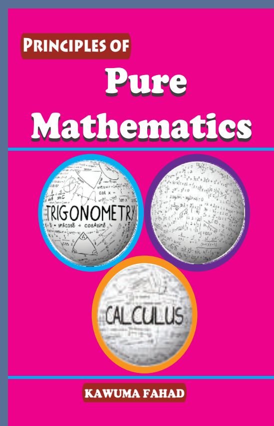

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