SHINING UCE MATHEMATICES

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1. SET THEORY

1.1 Definition:

A set is a collection of objects or members, which are related in some way. Sets are denoted by capital letters, e.g. set **A**, **B**, **M** etc.

1.2 Terms used:

1.2.1 Member (element) of a set

The objects in a set are called members or elements of the set. Members

$\notin$

Consider set **A** = {1, 3, 4, 6}. Then 1 **A** **A** **A** **A**. Whereas 7, 8, 9 etc are not members of set **A**, i.e. 7$\notin$**A**, 8$\notin$**A**, 9$\notin$**A**, etc.

1.2.2 Subsets

Set **A** is said to be a subset of set **B** if every element of set **A** is also in set **B**. E.g. given that **B** = {1, 2, 3, 4, 5, 6, 7, 8} and **A** = {1, 3, 5, 7}. Here every element of set **A** is also in set **B**. Therefore, set **A** is a subset of **B**. The symbol $\subset$ or $\supset$ $\not\subset$
Therefore $A \subset B$ or $B \supset A$ but $B \not\subset A$ because not all elements of **B** are in A.

1.2.3 Empty set (null set)

An empty set is a set with no elements. It is at time called null set. The

<u>Note:</u>

The empty set { } is not the same as {0}. This is because the set {0} has one element which is 0 whereas the set { } has no element.

1.2.4 Finite sets

The set is called finite if the elements of the set can be counted.

Example

Consider the following sets:

D = {days of the week}
F = {factors of 12}
G = {whole numbers greater than 5 but less than 11}

We can list all the members of these sets.

D = [Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, and Sunday}
F = {1, 2, 3, 4, 6, and 12}
G = {6, 7, 8, 9, and 10}

1.2.5 Infinite sets

These are sets with unlimited number of elements.

Example

Given the following sets:

W = {whole numbers}
R = {real numbers}
M = {multiple of 3}

Here, we cannot list all the members of these sets.

...}
R = { ..-2, -1, 0, 1, 2
M = {3, 6, 9, 12 ..}

All members of these sets cannot be exhausted so they are infinite sets.

1.2.6 Number of elements in a set

The number of element in a finite set can be counted. The number of elements of set **A** is denoted by $n(A)$ and it is the total number of elements in set **A**.

Example

Find the number of elements in the following sets.

R = {1, 2, 3, 4, 5, 6, 7, 8, 12}
B = {2, 4, 6, 8, 9}

Solution

$n(A) = 9$
$n(A) = 5$

Example

Given that set B = {factors of 24}

a) Write out set B in full

b) Find $n(B)$

Solution

a) $B = \{1,2,3,4,6,8,12,24\}$

b) $n(B) = 8$

Example

Given that set N = {natural numbers from 2 to 11}

a) Write out set N in full

b) Find $n(N)$

Solution

a) $N = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

b) $n(N) = 10$

1.3 Equal sets

Two or more sets are equal if they contain the same elements.

E.g. **A** = [1, 3, 5, 7} and **B** = {1, 3, 5, 7} are equal sets. Here $A \subset B$ and also $B \subset A$

1.3.1 Equivalent sets

Two or more sets are said to be equivalent if they contain the same number of elements. E.g. set **A** = {a, e, i, o, u} and **B** = {2, 4, 6, 8, 10}. Sets **A** and **B** contain the same number of elements which is 5. We therefore say that they are equivalent sets.

1.3.2 Union of sets $(\cup)$

The union of two sets is the set of all elements that are members of either set. The symbol for union is $\cup$.

Example

Given that: M = {1, 2, 3, 4} and N = [3, 4, 6, 7}

i) List $M \cup N$

ii) Find $n(M \cup N)$

Solution

i) $M \cup N = \{1,2,3,4,6,7\}$

ii) $n(M \cup N) = 6$

1.3.3 Intersection of sets $(\cap)$

The intersection of two sets or more sets is the set of elements that are in both sets.

Example

Given two sets: A = {-1, 0, 4, 5, 6, 7} and B = {-1, 6, 8, 10}

Find:

i) $n(A \cup B)$

ii) $n(A \cap B)$

Solution

i) $(A \cup B) = \{-1,0,4,5,6,7,8,10\} \therefore n(A \cup B = 8$

ii) $(A \cap B) = \{-1,6\} \therefore n(A \cap B) = 2$

1.3.4 Disjoint set

When the intersection of the two sets is empty, the two sets are called disjoint sets. E.g. given that **P** = {1, 3, 5, 7} and **Q** = {2, 4, 6, 8}. Here $(P \cap Q) = \{\}$

1.3.5 Complement of set

Consider two sets: **A** = {a, b, c, d} and **B** = {a, b, c, d, e, f}.

Members which are present in B and present in A is called complement of **A** denoted by **Á** or **Â**. From the two sets above:

$A' = \{e, f\} \therefore n(A') = 2$

$Also$:

$A' \cap B = \{e, f\} \therefore n(A' \cap B) = 2$

1.3.6 The universal set (ε)

This is a set that contains all the members of an item or object under consideration. It is denoted by the symbol Ɛ.

1.3.7 The Venn diagram

The Venn diagram is used to simplify solving problems in sets or used to illustrate sets.

Example

Given that set A = {1, 2, 3, 4, 5} and B = {2, 4, 6, 8}. Use a Venn diagram to find:

a) i) $A \cap B$

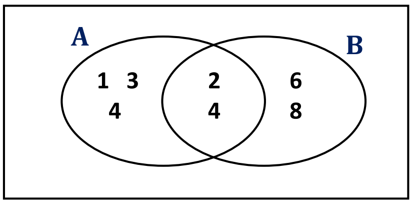
ii) $n(A \cap B)$

b) i) $A \cup B$

ii) $n(A \cup B)$

Solution

a)



i. $A \cap B = \{2,4\}$

ii. $n(A \cap B) = 2$

b) i) $A \cup B = \{1,2,3,4,5,6,8\}$

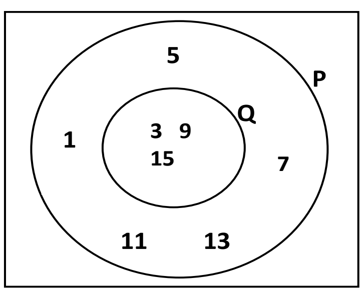
ii) $n(A \cup B) = 7$

Example

Given that P = {1, 3, 5, 7, 9, 11, 13, 15} and Q = {3, 9, 15}. Illustrate this information in a Venn diagram.

Solution

Since $Q \subset P$, Q is drawn inside P.



$P \cap Q = \{3,9,15\} = Q$

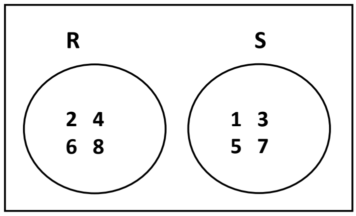
$P \cup Q = \{1,3,5,7,9,11,13,15\} = P$

Example

Given that, R = {2, 4, 6, 8} and S = {1, 3, 5, 7}. Show this information in a Venn diagram

Solution

Here R and S are disjoint sets i.e. $R \cap S = \{\ \}$, *so R and S are drawn separately as shown below.*



1.3.8 Solving problems using Venn diagram

The following examples will illustrate how a Venn diagram can be used to solve certain mathematical problem.

Example

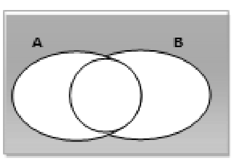
Use Venn diagram to show that:

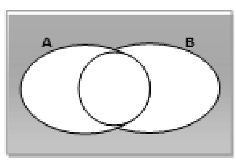
a) $(A \cap B)' = A' \cup B'$

b) $(A \cup B)' = A' \cap B'$

Solution

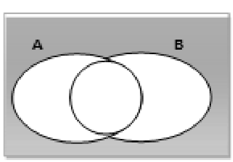
a) $(A \cap B)'$ $\qquad$ $A' \cup B'$

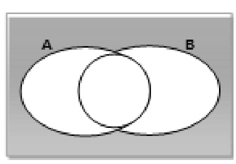




$$\therefore (A \cap B)' = A' \cup B'$$

b) $(A \cup B)'$ $\qquad$ $A' \cap B'$





$$\therefore (A \cup B)' = A' \cap B'$$

Example

There are 40 men working in a company. 38 of them own either a car or a house or both. 33 men own a car, 24 of whom also own a house. Represent this information in a Venn diagram and use it to state:

a) the number of men who do not own a house.
b) the number of men who own a house but do not own a car.
c) the number of men who own neither a car nor a house.

Solution

Let, C represent those who own a car

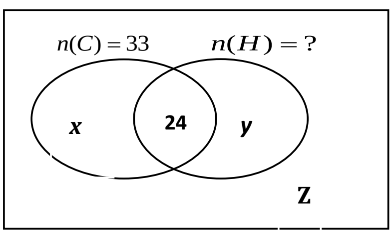
H represents those who own a house

And let: x be the number of men who own a car only.

y be the number of men who own a house only

Z be the number of men who own neither a car nor a house.

$Given:$
$n(\varepsilon) = 40$
$n(C \cup H) = 38, n(C) = 33$
$n(C \cap H) = 24, n(H) = ?$



From the diagram:

$x + 24 = 33$

$\therefore x = 33 - 24 = 9$

$n(C \cup H) = x + 24 + y = 38$

$\Rightarrow 9 + 24 + y = 38$

$\therefore y = 5$

a) Number of men who do not own a house $= n(H') = x + z = 9 + 5 = \underline{\underline{14}}$

b) Number of men who own a house but not a car = y = 5

c) $n(C' \cup H') = z = \underline{\underline{2}}$

1.4 Three sets problem

So far we have seen how to represent two sets in a Venn diagram. We shall also use Venn diagram to represent or solve problems that may involve three or more sets.

Example

Given that A = {1, 2, 3, 8, 9, 10}, B = {3, 4, 5, 6, 7, 8, 9} and C = {7, 8, 9, 10, 11}

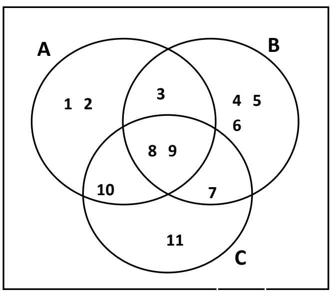
Represent the above information on a Venn diagram, hence fine:

a) $n(A \cap B \cap C)$

b) $n(A \cup B \cup C)$

c) $n(A \cap B \cap C')$

Solution



Side work

$A \cap B \cap C = \{8,9\} \Rightarrow n\{A \cap B \cap C\} = 2$

$A \cap B \cap C' = \{3\} \Rightarrow n\{A \cap B \cap C'\} = 1$

$A \cup B \cup C = \{1,2,3,4,5,6,7,8,9,10,11\}$

$\Rightarrow \{A \cap B \cap C\} = 11$

$n(A \cap B \cap C) = 2$

$n(A \cup B \cup C) = 11$

$n(A \cap B \cap C') = 1$

Example

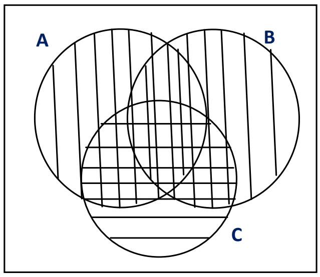
On the Venn diagram, shade the following:

a) $(A \cup B) \cap C$

b) $(A \cap C) \cup B$

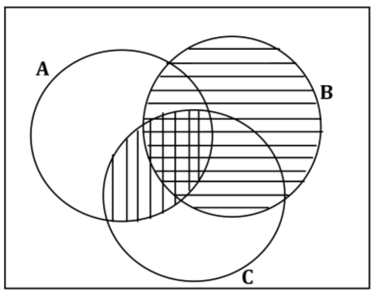
Solution

a) For $(A \cup B) \cap C$



$A \cup B$ *is the shaded vertically and C is shaded horizontally.*
$\therefore (A \cup B) \cap C$ *is the area shaded both ways.*

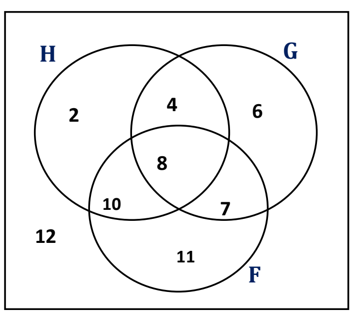
b) For $(A \cap C) \cup B$



$A \cap C$ is shaded vertically and B is shaded horizontally.
$\therefore (A \cap C) \cup B$ is the area shaded both ways.

Example

The Venn diagram below represents the members of students taking History (H), Geography (G) and French (F) in a certain class.



a) How many take history?
b) How many take French and Geography?
c) How many students take all the three subjects?
d) How many students are there altogether?
e) Write down:
   i. $n(H \cap F)$
   ii. $n(F \cup G)$
   iii. $n(H \cap F \cap G')$
   iv. $n(H \cup F \cup G)'$

Solution

a) Number taking history = 2 + 4 + 8 + 10 = 24

b) Number taking French and geography = 8 + 7 = 15

c) $n(H \cap F \cap G) = 8$

d) $n(\varepsilon) = 2 + 4 + 6 + 8 + 7 + 10 + 11 + 12 = 60$

There are 60 students altogether.

e) i) $n(H \cap F) = 10 + 8 = \underline{\underline{18}}$

ii) $n(F \cup G) = 4 + 6 + 8 + 7 + 10 + 11 = \underline{\underline{46}}$

iii) $n(H \cap F \cap G') = \underline{\underline{10}}$

iv) $n(H \cup F \cup G)' = \underline{\underline{12}}$

Example

50 students were asked whether they liked football (F), volleyball (V) or basketball (B). 8 like football and volleyball, 11 liked basketball and volleyball, 19 liked football and basketball, 6 liked basketball only, 7 liked football only and 8 liked volleyball only.

How many liked:

a) All the three games if three of the students liked none of the games.

b) Basketball and football only.

Solution

$n(\varepsilon) = 50$

$n(B \cap F' \cap V') = 6$

$n(V \cap B' \cap F') = 8$

$n(F \cap B' \cap V') = 7$

$n(F \cup V \cup B)' = 3$

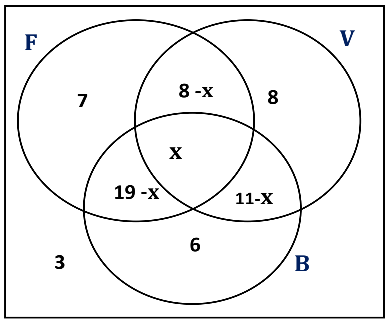
$let, n(F \cap V \cap B) = x$

$n(F \cap V) = 8$

$n(B \cap V) = 11$

$n(F \cap B) = 19$

$n(\varepsilon) = 50$



a) *For all the three games:*

$$7 + 8 - \cancel{x} + 8 + \cancel{x} + 19 - x + 11 - x + 6 + 3 = 50$$

$$62 - 2x = 50$$

$$-2x = 50 - 62 = {}^{-}12$$

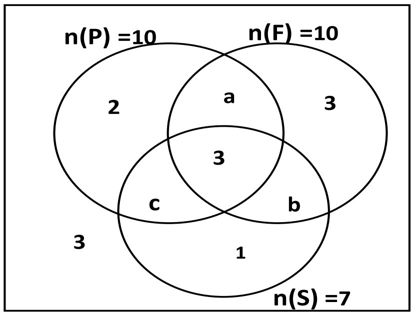
$$\therefore x = \frac{{}^{-}12}{{}^{-}2} = 6$$

Therefore, 6 students liked all the three games.

b) *Basketball and football only:* $= 19 - x = 19 - 6 = \underline{\underline{13}}$

Example

The Venn diagram below shows representation of members of community council to three different committees of Finance (F), Production (P), and Security (S)



a) Determine the value of a, b and c.

b) Find:

i. The total number of members who make up the community council

ii. Number of members who belong to more than one committee.

Solution

For Production:

$$a + c + 2 + 3 = 10$$
$$a + c = 10 - 5$$
$$\therefore a + c = 5 \text{.......................................}(1)$$

For Finance:

$$a + b + 3 + 3 = 10$$
$$a + b = 10 - 6$$
$$\therefore a + b = 4 \text{.......................................}(2)$$

For Security:

$$b + c + 3 + 1 = 7$$
$$b + c = 7 - 4$$
$$\therefore b + c = 3 \text{.......................................}(3)$$

From equation (1): $a = 5 - c$ *and substituting for a from (1) in equation (2):*

$$\Rightarrow 5 - c + b = 4$$
$$\therefore b - c = -1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Solving (3) and (4) simultaneously

$$b + c = 3$$
$$\underline{b - c = {}^{-}1}$$
$$\therefore 2b = 2 \Rightarrow b = 1$$
$$c = 3 - b \Rightarrow c = 2$$
$$a = 5 - c \Rightarrow a = 3$$
$$\therefore \underline{\underline{a = 3, b = 1, c = 2}}$$

a) i) The total number of members making up the community council

$$= 2 + a + 3 + 3 + c + b + 1 + 3$$
$$= 2 + 3 + 3 + 3 + 2 + 1 + 1 + 3$$
$$= \underline{\underline{18}}$$

ii) Number of members who belong to more than one committee

$$= a + c + b + 3$$
$$= 3 + 2 + 1 + 3$$
$$= \underline{\underline{9}}$$

1.5 Set builder notation

A set can be described using symbols rather than words, for instance; $A = \{x : x \in N, x < 6\}$

A

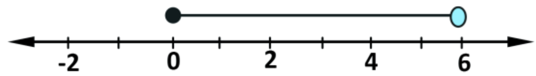
number and x is less than 6. This notation is called set builder notation. Such a set can be represented on a number line.

Example

Given that, $B = \{x : x \in R, 0 \le x < 6\}$, where **R** is a Real number. Show the solution set of **B** on a number line.

Solution

***B** is the set of real number x such that x is greater or equal to 0 and less than 6. This can be represented on a number line as below.*



Note:

The solid cycle at 0 indicates that 0 is included in the solution set of B while the empty cycle at 6 indicates that 6 is not included in the solution set of B.

Example

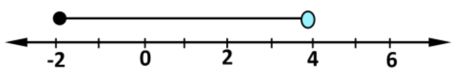
Show the following sets on the number line

a) $C = \{x : x \in R, -2 \leq x < 4\}$ Where R is a real number.

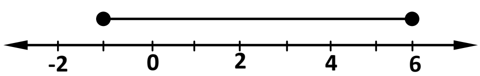
b) $D = \{y : y \in W, 1 \leq x \leq 6\}$Where W is a whole number. Find n(D)

Solution

a)



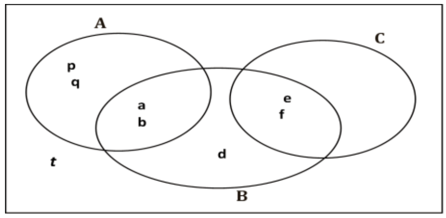
b)



$$D = \{1,2,3,4,5,6\} \Rightarrow n(D) = 6$$

1.6 Miscellaneous exercise

1. Study the Venn diagram below



a) Write down the members of the universal set.
b) Write down the members of the following sets.
   i) **A**
   ii) **B**
   iii) **C**
   iv) $A \cap C$

c) Find:

   i) $n(A)+n(B)+n(C)$
   ii) $n(A \cup B \cup C)$

2. Given that: n(A) = 22, n(B) = 22, n(A *n* B *n* C) = 5, n(A *n* B) = 11, n(C *n* A) = 7,n(B n C) = 9, and n(A *u* B *u* C) = 40. Find n(C).

3. A certain class was asked whether they liked science and history. Twice as many liked science as liked history. Eight said they liked both subjects and nine pupils said they did not like either subject. If there were 46 pupils in the class, use a Venn diagram to work out how many pupils liked science.

4. Given that $A = \{x : -2 \leq x \leq 1\}, and, B = \{x : 0 < x \leq 5\}$.

   a) Represent $(A \cap B)$ on a number line.
   b) State $(A \cap B)$

5. Given the sets:

   A = {all natural numbers less than 30}
   B = {All prime numbers between 10 and 30}

   Find:

   a) $(A \cap B')$
   b) $n(A' \cap B)$ Where **B'** stands for complement of set **B**.

6. In a form 3 class, the teacher told the students to bring a pen, a pencil and a ruler to class. The following she found that of the 40 students, only 12 had brought all the three instruments, 5 students

   student had only a pen, 2 students had only a pencil, and no student had only a ruler.

   a) Draw a Venn diagram to illustrate this information and find out how many students had at least a pencil and a ruler.

   b) The students who had less than two instruments were put in detention. How many students were put in detention?

7. In a certain school, a sample of 100 students was picked randomly. In this sample, it was found out that 78 students play netball (N), 82 play volleyball (V),53 play tennis (T) and 2 do not play any of the three games. All those that play tennis also play volleyball. 48 play all the three games.

   a) Represent the given information on a Venn diagram.

   b) How many students play both netball and volleyball but not tennis?

8. Of the 80 senor five students that passed Math (M) in Teso Integrated S.S; 45 passed Physics (P), 60 passed Chemistry (C), 5 passed Biology (B) and M only, 5 passed M only. Those who passed P, C, B and M equal to those who passed only B, C, and M. The number of students who passed M and C only equal to those who passed M, B and P only and are 5 less than those who passed all the 4 subjects.

   a) Represent the above information on a Venn diagram.

   b) Find the number of those who passed:

      i. all the four Subjects.
      ii. only three subjects.

   c) A student is selected at random. Find the probability that the student;

      i. passed by 2 subjects.
      ii. did not pass Biology.

2. EQUATIONS OF STRAIGHT LINES

2.1 Coordinates:

A point can be described by a number pair. Consider point A described by a pair of numbers on the Cartesian plane as shown below.

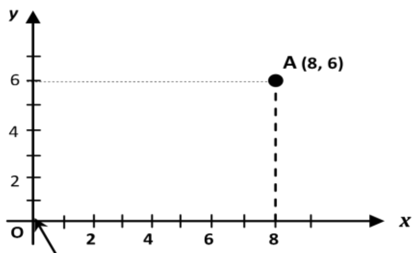


Fig 1

The first number i.e. 8 is the horizontal displacement of point A from the origin O. This is known as the x-coordinates. The second number i.e. 6 is the vertical displacement of point A from the origin. This is known as the y-coordinate.

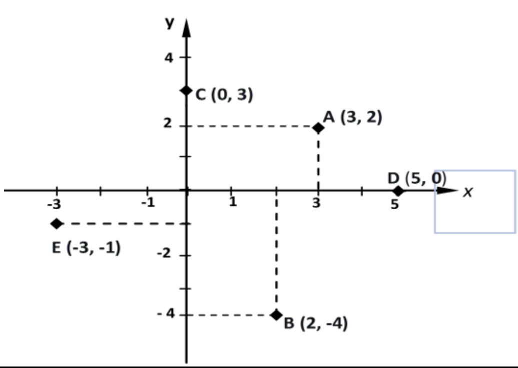
Pont **A** is located by moving 8 units along the x-axis and 6 units by moving along the y-axis. Any point can be located in this way.

Example

Draw the Cartesian pane and locate the following points.

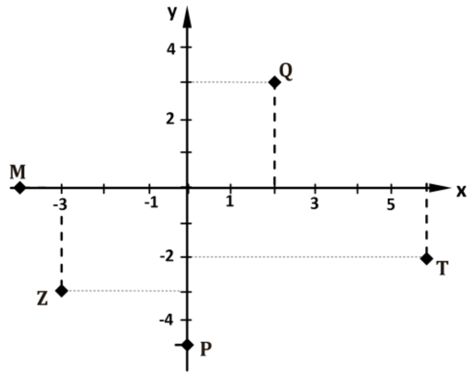
A (3, 2), B (2, -4),C (0, 3), D (5, 0), and E (-3, -1)

Solution



Example

Obtain the coordinates of the following points from the Cartesian plane below.



Solution

$Q(2, 3), T(6, -2), P(0, {}^{-}5),$
$Z(-3, -3), M({}^{-}4, 0)$

2.2 The gradient of a straight line

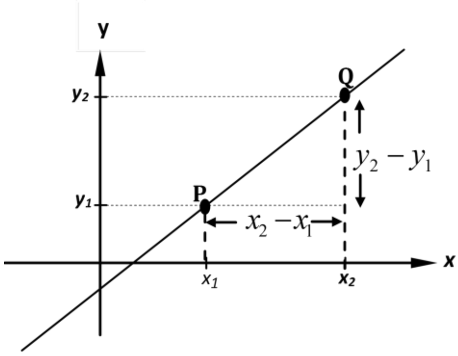
Definition:

Gradient is the measure of steepness (slope). Another name for gradient is slope.

Gradient id defined as the ratio of the vertical distance to the horizontal distance, i.e.

$$Gradient(slope) = \frac{vertical\ distance}{horizontal\ distance}$$

Generally, consider a line passing through at least two points, $P(x_1, y_1)$ *and* $Q(x_2, y_2)$ as shown in the figure below.



$$Gradient\ of\ the\ line\ PQ = \frac{vertical\ distance}{horizontal\ distance}$$

$$= \frac{change\ in\ y-coordinates}{change\ in\ x-coordiates} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$Gradient, m = \frac{y_2 - y_1}{x_2 - x_1} \text{ Or } Gradient, m = \frac{y_1 - y_2}{x_1 - x_2}$$

The letter ***m*** is used to denote the gradient.

Example

Find the gradient of the line passing through the following points.

a) A (4, 1) and B (6, 5)
b) P (3, 5) and Q (9, 8)
c) D (0, 7) and C (4,0)
d) J (0, 0) and E (6, 0)
e) N (0, 0) and K (0, 8)

Solution

a) $A(\overset{x_1}{4}, \overset{y_1}{1}),\ B(\overset{x_2}{6}, \overset{y_2}{5})$

$$Grdadient, m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-1}{6-4} = \frac{4}{2} = \underline{\underline{2}}$$

b) $P(\overset{x_1}{3}, \overset{y_1}{5}),\ Q(\overset{x_2}{9}, \overset{y_2}{8})$

$$Grdadient, m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-5}{9-3} = \frac{3}{6} = \underline{\underline{1/2}}$$

c) $\overset{x_1}{D}(0,\overset{y_1}{7}),\ C(\overset{x_2}{4},\ \overset{y_2}{0})$

$$Grdadient, m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-7}{4-0} = \underline{\underline{-7/4}}$$

d) $J(\overset{x_1}{0},\overset{y_1}{0}),\ E(\overset{x_2}{6},\ \overset{y_2}{0})$

$$Grdadient, m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-0}{6-0} = \frac{0}{6} = \underline{\underline{0}}$$

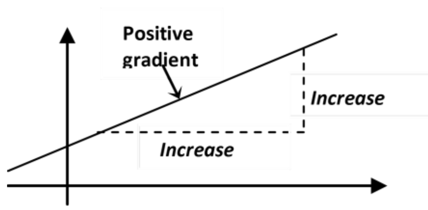
e) $N(\overset{x_1}{0},\overset{y_1}{0}),\ K(\overset{x_2}{0},\ \overset{y_2}{8})$

$$Grdadient, m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-0}{0-0} = \frac{8}{0}, does\ not\ exist$$

2.3 Nature of the gradient:

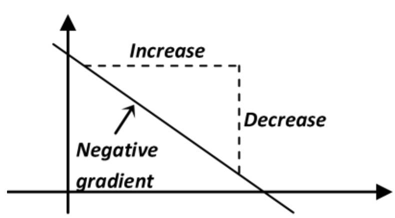
2.3.1 Positive gradient:

If an increase in the x-coordinate causes increase in the y-coordinate, then the line slopes upwards from left to right, the gradient therefore is positive.



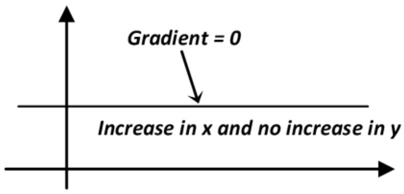
2.3.2 Negative gradient:

If an increase in the x-coordinate causes a decrease in the y-coordinate, the line slopes downwards from left to right, the gradient is therefore negative.



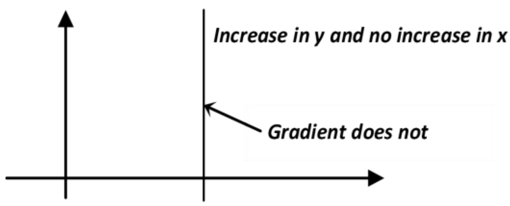
2.3.3 Zero gradient:

If for the increase in x coordinate there is no increase in the y coordinate, the line therefore runs horizontally and the gradient is zero i.e.



2.3.4 Undefined gradient:

If there is no change in the x-coordinate while there is increase in the y coordinate, the line therefore runs vertically, and the gradient in this case is undefined.

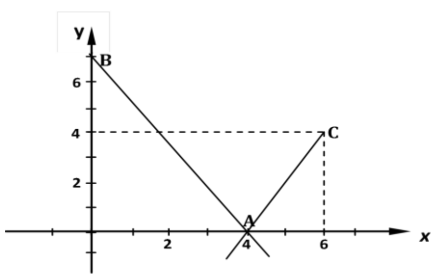


For example, consider the line passing through points B (3, 0) and P (3, 7). There is change in the *y* coordinates but no change in the *x* coordinate; hence, the gradient does not exist. i.e.

$$Gradient = \frac{7-0}{3-3} = \frac{7}{0}, does\ not\ exist\ \text{since}\ zero\ can't\ be\ divided\ by\ any\ number$$

Example

Consider the figure below.



Determine the gradient of the following line segment.

i) AB

ii) AC

Solution

i. $\textit{Coordinates of poits}; A(\overset{x_1}{4}, \overset{y_1}{0}), B(\overset{x_2}{0}, \overset{y_2}{7})$

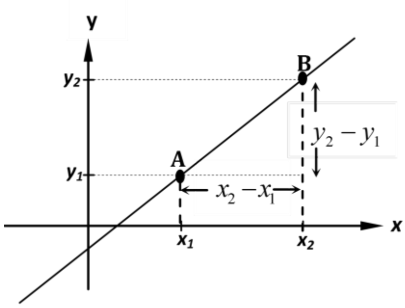
$$\textit{Gradient of } AB\ m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-0}{0-4} = \underline{\underline{-7/4}}$$

ii. $\textit{Coordinates of poits}; A(\overset{x_1}{4}, \overset{y_1}{0}), C(\overset{x_2}{6}, \overset{y_2}{4})$

$$\textit{Gradient of } AC\ m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-0}{6-4} = \underline{\underline{2}}$$

2.4 The length of a straight line

Consider the line AB on the Cartesian plane with coordinates; $A(x_1, y_1)$ *and* $B(x_2, y_2)$



By Pythagoras theorem;

$$AB^2 = AC^2 + CB^2$$

$$\therefore \textit{length } AB = \sqrt{AC^2 + CB^2},\ \textit{but } AC = x_2 - x_1, \textit{and} CB = y_2 - y_1$$

$$\therefore \textit{Length } AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The length AB of a straight line is denoted by the symbol $|AB|$ *or* $\overline{AB}$.

Example

Find the length of the straight line joining each of the following pair of points.

a) B (1, 3) and D (4, 7)
b) T (-7, -1 and Q (-1, -4)
c) N (0, 0) and P (4, -7)

Solution

a) $B(\overset{x_1}{1}, \overset{y_1}{3}), D(\overset{x_2}{4}, \overset{y_2}{7})$

$$Length\ BD = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|BD| = \sqrt{(4-1)^2 + (7-1)^2} = \sqrt{3^2 + 4^2} = \sqrt{25}$$

$$= \underline{\underline{5 units}}$$

b) $T(\overset{x_1}{^-7}, \overset{y_1}{^-1}), Q(\overset{x_2}{^-1}, \overset{y_2}{^-4})$

$2 < x < 6, \quad {}^-3 \le x \le 4$

c) $N(\overset{x_1}{0}, \overset{y_1}{0}), P(\overset{x_2}{4}, \overset{y_2}{^-7})$

$$Length\ NP = \sqrt{(4-0)^2 + ({}^-7-0)^2} = \sqrt{16+49} = \sqrt{65}$$

$$\therefore |NP| = \underline{\underline{8.06 units}}$$

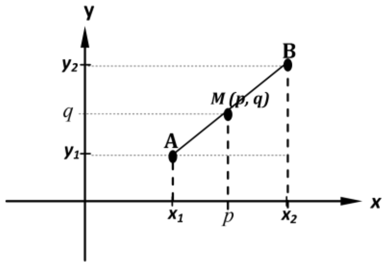
2.4.1 The midpoint of a straight line

If **A** and **B** have coordinates $A(x_1, y_1)$ *and* $B(x_2, y_2)$ respectively, the coordinates of the midpoint M of the line joining AB is given by:

$$M\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$$

Proof:

Let **M** be the midpoint of the line joining **AB** and let its coordinates be (p, q).



For horizontal displacement

$AC = CD$

$p - x_1 = x_2 - p$

$p + p = x_2 + x_1$

$2p = x_2 + x_1$

$\therefore\ p = \dfrac{x_2 + x_1}{2}$

For vertical displacement

$DE = EB$

$q - y_1 = y_2 - q$

$q + q = y_2 + y_1$

$2q = y_2 + y_1$

$\therefore q = \dfrac{y_2 + y_1}{2}$

Coordinates of M the midpoint of AB = (p, q). Therefore the coordinates of the midpoint of a straight line joining any two points is given by:

$$M = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$$

Example

Find the coordinates of the midpoint of a straight line joining each of the following pairs of points.

a) (-2, 1) and (6, 5)
b) (-2, 6) and (-8, -5)
c) (3, 8) and (1, 2)

Solution

a) $(\overset{x_1}{^-2}, \overset{y_1}{1}),\ and\ (\overset{x_2}{6}, \overset{y_2}{5})$

$$\text{The midpoint}, M = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right) = \left(\frac{6-2}{2}, \frac{5+1}{2}\right) = \left(\frac{4}{2}, \frac{6}{2}\right) \quad \therefore \underline{\underline{M = (2, 3)}}$$

b) $(\overset{x_1}{^-2}, \overset{y_1}{6}),\ and\ (\overset{x_2}{^-8}, \overset{y_2}{^-5})$

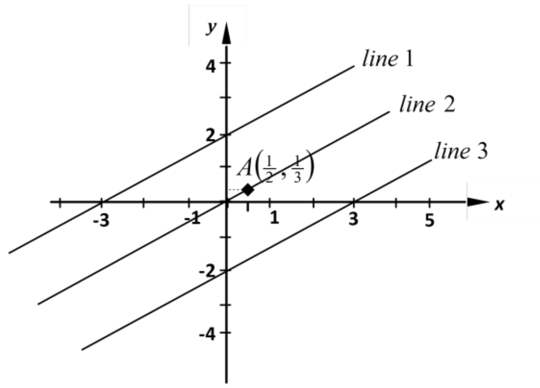
$$\text{The midpoint}, M = \left(\frac{^-8 + {}^-2}{2}, \frac{^-5 + 6}{2}\right) = \left(\frac{^-10}{2}, \frac{1}{2}\right) \quad \therefore \underline{\underline{M = \left(^-5, 0.5\right)}}$$

c) $(\overset{x_1}{3}, \overset{y_1}{8}),\ and\ (\overset{x_2}{1}, \overset{y_2}{2})$

2.4.2 Parallel and perpendicular lines

a) Parallel lines:

Lines are said to be parallel if they do not cross the path of one another or if they do not meet even though they have been extended indefinitely. Parallel lines have the same gradient. Consider the following lines on the Cartesian plane.



<u>*Gradient of line* 1:</u>

Considering coordinates of $x-$ *and* $y-$intercept *respectively*

i.e $(^-3, 0)$ *and* $(0, 2)$

$$m_1 = \frac{2-0}{0-^-3} = {}^2\!/_3$$

<u>*Gradient of line* 2:</u>

Considering the origin $(0, 0)$ *and point* $A\left(\frac{1}{2}, \frac{1}{3}\right)$

$$m_2 = \frac{{}^1\!/_3 - 0}{{}^1\!/_2 - 0} = \frac{1}{3} \times \frac{2}{1} = {}^2\!/_3$$

The gradients of these lines are the same. Hence, they are parallel.

Example

Using the points A (2, 4), B (8, 7), C (5, -1) and D (19, 5), show that line AB is parallel to line CD.

Solution

$A(2, 4) \quad C(5, -2)$
$B(8, 7) \quad D(19,5)$

$4x + 3y$

Since the gradient of AB is equal to the gradient of CD; the lines AB and CD are therefore parallel.

b) Perpendicular lines.

A line is said to be perpendicular to another line if they meet at right angle to one another. The product of the gradients of a pair of perpendicular lines is **-1**.

Consider two lines $\mathbf{L_1}$ and $\mathbf{L_2}$ and let their gradients be $\boldsymbol{m_1}$ and $\boldsymbol{m_2}$ respectively. If $\mathbf{L_1}$ is perpendicular to $\mathbf{L_2}$, then;

$$\boxed{m_1 \times m_2 = {}^-1}$$

Example

Given three points A (2, 4), C (5, -7) and D (19, 5). Prove that the line AC is perpendicular to the line CD.

Solution

$A(2, 4) \quad C(5, -2) \quad D(19,5)$

$$Gradient\ of\ \ AC\ m_1 = \frac{{}^-2 - 4}{5 - 2} = -6/3 = -2$$

$$Gradient\ of\ \ CD\ m_2 = \frac{5 - {}^-2}{19 - 5} = \frac{7}{14} = 1/2$$

$$So\ m_1 \times m_2 = \tfrac{1}{2} \times -2 = -1$$

Since $m_1 \times m_2 = {}^-1$, *the two lines AC and CD are therefore perpendicular to one another.*

2.4.3 Sketching a straight line

When sketching a straight line, we simply need two points lying on the line. The two points are plotted and are joined using a ruler.

Example

A line passes through points (0, 2) and (2, 4). Find:

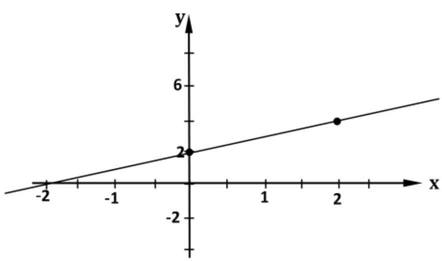
a) the gradient of the line
b) the length of the line
c) Hence sketch the line.

Solution

a) $\overset{x_1}{(0},\overset{y_1}{2)}, \textit{and } (\overset{x_2}{2}, \overset{y_2}{4})$

$\sum f = 40$

$$\begin{aligned} \textit{Length of the line} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(2-0)^2 + (4-2)^2} = \sqrt{2^2 + 2^2} \\ &= \sqrt{8} \\ &= \underline{\underline{2\sqrt{2}\,\textit{units}}} \end{aligned}$$



2.4.4 The equation of a straight line

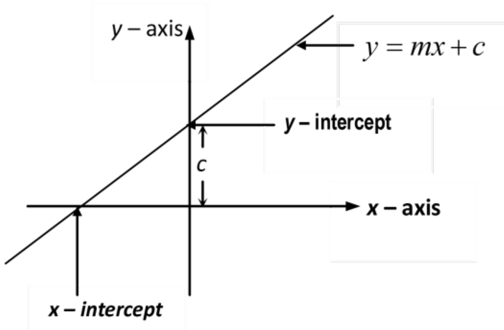
The equation of a straight line is in the form given below:

$y = mx + c$

where m – *gradient of the line*

c – *is the* y – *intercept*

When the above equation is drawn on the Cartesian graph, it may appear as shown below.



Y intercept is where the line cuts the y axis and the x intercept is where the line cuts the x axis.

a) Obtaining the equation of the line given the gradient and the y – intercept

Example

Find the equation of a line whose gradient is 3 and y intercept is -2. Hence, sketch the line.

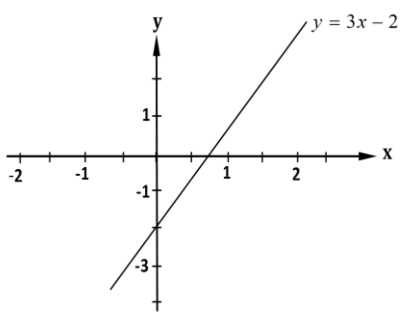
Solution

$From: y = mx + c, m = 3, c = -2$

$\therefore \underline{\underline{y = 3x - 2}}$

To sketch a line, we need only two points.

x	0	1
y	-2	0



Example

Obtain the gradient and y intercept of the line whose equation is $4x-3y-9=0$

Solution

First, we have to express this equation in the gradient-intercept form of the equation of a straight line

$$4x-3y-9=0,\ can\ be\ \text{expressed}\ as;$$
$$3y=4x-9$$
$$\therefore y=\tfrac{4}{3}x-3$$
$$comparing\ this\ with\quad y=mx+c$$
$$\Rightarrow m=\tfrac{4}{3},\ and\ c=-3$$
$$\therefore gradient\ m=\tfrac{4}{3},\ and\ y-\text{intersecpt},\ c=-3$$

b) Obtaining the equation of a line given the gradient and a point on the line

Example

A straight line with gradient 3 passes through the point A (3, -4). Find the equation of the line.

Solution

Method 1

$$From: y=mx+c, m=3, c=?$$
$$\therefore y=3x+c$$
$$considering\ \text{point}\ A(3,-4), x=3, y=-4$$
$$\Rightarrow -4=3(3)+c. \Leftrightarrow c=-13$$
$$\therefore \underline{\underline{y=3x-13}}$$

Method 2

By using any other point on the line.

Let B (x, y) represents any general point on the line. So now we have two points lying on the line i.e.

$$A\,(\overset{x_1}{3},\overset{y_1}{-4})\qquad B\,(\overset{x_2}{x},\ \overset{y_2}{y})$$

$$Gradient = \frac{y_2 - y_1}{x_2 - x_1},\ but\ m = 3$$

$$\Rightarrow 3 = \frac{y - (^{-}4)}{x - 3} = \frac{y + 4}{x - 3}$$

$$\Rightarrow \frac{3}{1} = \frac{y + 4}{x - 3}$$

$$y + 4 = 3(x - 3)$$

$$y = 3x - 9 - 4$$

$$\therefore \underline{\underline{y = 3x - 13}}$$

c) Obtaining the equation of a line given two points on it

Example

Find the equation of the straight line which passes through points A (1, -1) and B (2, 3).

Solution

Method 1

Let $P(x, y)$ be the general point on the line. So now we have three points lying on the line, i.e. $A(1,-1), B(2,3)$ and $P(x, y)$

Gradient of AB = gradient of BP

$$\frac{3 - (^{-}1)}{2 - 1} = \frac{y - 3}{x - 2}$$

$$\frac{4}{1} = \frac{y - 3}{x - 2}$$

$$4x - 8 = y - 3$$

$$\therefore \underline{\underline{y = 4x - 5}}$$

Method 2

$$A(\overset{x_1}{1}, \overset{y_2}{-1}) \quad B(\overset{x_2}{2}, \overset{y_2}{3})$$

$$From\ y = mx + c,\ m = ?, c = ?$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-(^-1)}{2-1} = 4$$

$$\Rightarrow y = 4x + c$$

$$considering\ po\text{int}\ A(1,\ ^-1), x = 1, y = -1$$

$$\Rightarrow -1 = 4(1) + c \Leftrightarrow c = -5$$

$$\therefore \underline{\underline{y = 4x - 5}}$$

d) Obtaining the equation of a line which is parallel to a given line and passing through a point

Example

Write down the equation of a straight line passing through;

i. the point (5, 11) and parallel to the x axis.
ii. the point (0, -1) and parallel to the line $3x - 2x + 5y = 0$

Solution

i. *Gradient of line parallel to x –axis is zero*

$$From\ y = mx + c, m = 0, x = 5, y = 11$$

$$\Rightarrow 11 = 5(0) + c \Leftrightarrow c = 11$$

$$\therefore \underline{\underline{y = 11}}$$

ii.

$$For\ the\ line\ 3y - 2y + 5 = 0$$

$$2y = 3x + 5 \Rightarrow y = \frac{3}{2}x + \frac{5}{2}$$

Therefore the gradient of this line is $= {}^3\!/_2$. *The gradient of the line parallel to this line is also* $= {}^3\!/_2$, *now from;*

$$y = mx + c, m = {}^3\!/_2,\ for\ p\text{int}(0,-1), x = 0, y = {}^-1$$

$$\Rightarrow {}^-1 = \frac{3}{2}(0) + c \Leftrightarrow c = -1$$

$$\therefore \underline{\underline{y = \frac{3}{2}x - 1}}$$

e) Obtaining the equation of a straight line which is perpendicular to a given line and passes through a given point

Example

Write down the equation of a straig2ht line passing through the point (0, -2) and perpendicular to the line $4y = 2x + 3$

Solution

$Line\,1: 4y = 2x + 3$

$y = \frac{1}{2}x + \frac{3}{4}, has\ gradient\ m_1 = \frac{1}{2}$

Let m_2 be the gradient of the line which is perpendicular to line 1

$Line\ 2: y = m_2 x + c$

$but\ m_1 \times m_2 = -1$

$\Rightarrow \frac{1}{2} \times m_2 = -1 \therefore m_2 = -2$

$\therefore y = -2x + c, considering\ po\text{int}\ (0, -2), x = 0, y = -2$

$\Rightarrow -2 = -2(0) + c. \Leftrightarrow c = -2$

$\therefore \underline{\underline{y = -2x - 2\ or\ y + 2x + 2 = 0}}$

Example

Find the equation of the line which is a perpendicular bisector of the line passing through points A (5, 4) and B (3, 8).

Solution

Let $\mathbf{m_1}$ be the gradient of the line passing through A (5, 4), B (3, 8)

$$m_1 = \frac{8-4}{3-5} = -2$$

Let $\mathbf{m_2}$ be the gradient of the line that bisects the line at 90^0

$\therefore m_1 \times m_2 = -1$

$\Rightarrow {}^{-}2m_2 = -1$

$\therefore m_2 = \frac{1}{2}$

Line 2 passes through the midpoint of AB. Let M be the midpoint of AB, then;

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{5+3}{2}, \frac{4+8}{2}\right) = (4,\ 6)$$

$From;\ y = mx + c, M(4,\ 6)$

$m = \frac{1}{2}, x = 4, y = 6$

$\Rightarrow 6 = \frac{1}{2}(4) + c \Rightarrow c = 4$

$\therefore\ \underline{\underline{y = \frac{1}{2}x + 4 \quad or \quad 2y - x - 8 = 0}}$

2.4.5 Intersection of lines

If two or more lines intersect (meet), then at the point of intersection, they have the same coordinates. In order to find the coordinates of the point of intersection, we solve simultaneously the two equations.

Example

Find the coordinates of the point where the following pairs of line intersect.

a) $x + 2y = 2 \text{ and } 3x - 2y = 14$

b) $y = 2x - 1 \text{ and } y = x + 1$

Solution

a) Solving the two equations simultaneously,

$$\begin{array}{r} x + 2y = 2 \\ {}^{+}\ 3x - 2y = 14 \\ \hline 4x = 16 \end{array}$$

$$\therefore x = 4, y = \frac{2 - x}{2} = \frac{2 - 4}{2} = -1$$

$\therefore$ *The coordinates of point of intersection is* $(4, -1)$

b) For this:

$y = 2x - 1, \quad y = x + 1$

$\Rightarrow 2x - 1 = x + 1 \Leftrightarrow 2x - x = 1 + 1$

$\therefore x = 2, \quad y = 2 + 1 = 1$

$\therefore$ *The point of intersection is* $(2, 3)$

2.4.6 Intersection of a line and a curve

If a line and a curve intersect, then at the point of intersection they have the same coordinates. A curve may intersect with a line at several points and to find the coordinates of the points of intersection, we equate the two equations when y is the subject in both equations.

NB

The curve has the equation in the form $ax^n + bx^{n-1} + ... + c$ where *n, a, b* and *c* are constants. E.g. $2x^2 + 1, 3x^3 + 2x^2 + x + 4, e.t.c$

Example

Find the coordinates of points of intersection of the curve $y = x^2 - 3$ and the line $y = 5x - 9$.

Solution

$curve: y = x^2 - 3....................................(1)$
$line: \quad y = 5x - 9....................................(2)$

Equating the two equations:

$x^2 - 3 = 5x - 9$
$x^2 - 5x + 6 = 0$
$x^2 - 2x - 3x + 6 = 0$
$(x-3)(x-2) = 0$
$\therefore x = 3, x = 2$
$From\ eqn(2), i.e. y = 5x - 9$
$when\ x = 3, y = 5 \times 3 - 9 = 6$
$coordinates\ of\ this\ point\ is\ (3, 6)$
$when\ x = 2, y = 5 \times 2 - 9 = 1$
$coordinates\ of\ this\ point\ is\ (2, 1)$

Therefore the coordinates of the points of intersection are (2, 1) and (3, 6)

Example

Given the curve $y = 2x^2 + 3x$ and the line $y = 5x + 4$, determine the coordinates of the points of intersection of the curve and the line.

Solution

$curve: y = 2x^2 + 3x$..................................(1)
$line: \quad y = 5x + 4$......................................(2)

$Equ(1) = equ(2)$

$2x^2 + 3x = 5x + 4$

$2x^2 + 3x - 5x - 4 = 0$

$2x^2 - 2x - 4 = 0 \Leftrightarrow x^2 - x - 2 = 0$

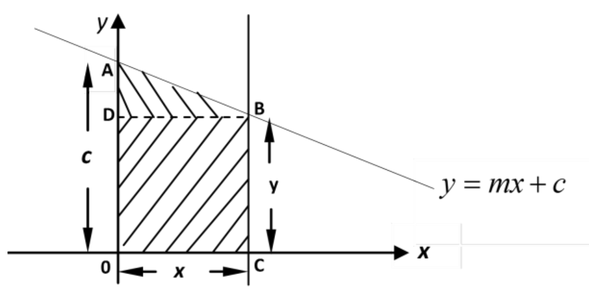
$(x-2)(x+1) = 0 \Rightarrow x = 2, x = -1$

$From\ equ(2), i.e. y = 5x + 4$
$when\ x = 2, y = 5 \times 2 + 4 = 14 \therefore coordinates\ (2, 14)$
$when, x = -1, y = 5(^-1) + 4 = -1 \therefore coordinates\ (-1, -1)$

Therefore, coordinates of points of intersection are (2, 14) and (-1, -1)

2.4.7 Area enclosed by the line(s) and the x – and y – axis

Consider the area enclosed by the lines AB, BC and the x and y axis as shown below.



The area of the shaded part can be divided up into two figures i.e. rectangle OCBD and triangle ABD.

$Area\ of\ OABC = Area\ of\ rectangle\ ODBC + Area\ of\ triangle\ ABD$

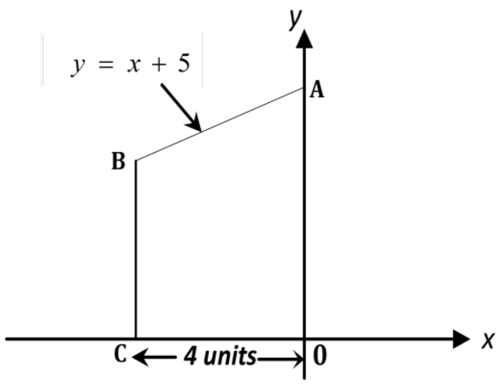
$$= l \times w + \tfrac{1}{2}bh, but\ h = c - y$$

$$= xy + \tfrac{1}{2}x(c - y)$$

NB:

Point A is the y intercept of the equation $y = mx + c$. It therefore has coordinates A (0, c). The line CB and AB intersect at point B. therefore the coordinates of B is obtained by considering the intersection of lines BC and AB.

Example



In the diagram above, the equation of the line AB is $y - 5 = x$ and C is 4 units from O. Find the area of OABC.

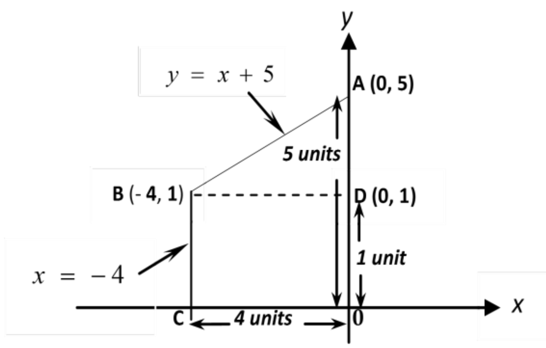
Solution

The equation $y = x + 5$ *cuts the y –axis at A. Therefore A is the y – intercept of* $y = x + 5$ *i.e. A = 5. Coordinates of A is (0, 5)*

Equation of the line ***BC*** *is* $x = {}^{-}4$ *. Line 1 and line 2 intersect at B. the coordinates of point B can be obtained by considering the intersection the two lines.*

$$Line1 : y = x + 5.................................(1)$$
$$Line2 : x = -4....................................(2)$$

Substituting for x from equation (1) in equation (2)



$y = 5 - 4 = 1$

$\therefore coordinates\ of\ B\ is\ (^-4, 1)$

$Area\ of\ OABC = Area\ of\ rectangle\ ODBC + Area\ of\ triangle\ ABD$

$= l \times w + \frac{1}{2}bh \quad h = 1, w = 4units, b = 4units, h = 5 - 1 = 4units$

$= 1 \times 4 + \frac{1}{2} 4 \times 4$

$= 4 + 8$

$= \underline{\underline{12 sq\ units}}$

Example

Find the point of intersection of the lines, $y = 2x - 3,$ *and* $y = -x - 3.$

Calculate the area of triangle enclosed between the two lines and the x axis.

Solution

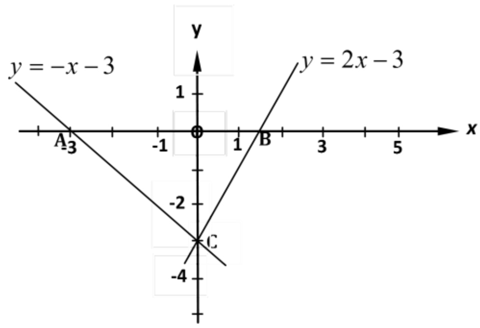
First, sketch the two lines on the same diagram.

$For: y = 2x - 3$

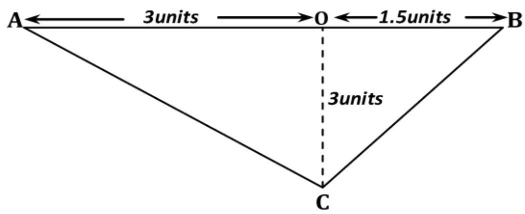
x	*0*	³⁄₂
y	*-3*	*0*

$For: y = -x - 3.$

x	*0*	*3*
y	*-3*	*0*



Extracting the triangle ABC:



$$Area\ of\ \ ABC = \tfrac{1}{2}bh,\ h = 3units, b = 3 + 1.5 = 4.5units$$

$$= \tfrac{1}{2} \times 3 \times 4.5$$

$$= \underline{\underline{6.75sq\ units}}$$

2.4.8 Miscellaneous exercise

1. A line of gradient $\frac{7}{9}$ passing through the point Q (3, 4) cuts the y axes at point P. find the coordinates of P.
2. Show that the points (3x, -2y) and (2x, y) and (0, 7y) lie on a straight line.

3. Two points P (5, 2) and Q (2, 4) are in a plane.

   Find:

   a) Coordinates of M, the midpoint of $\overrightarrow{PQ}$

   b) $|OM|$

4. Given the line $3y + 2x - 6 = 0$.
   a) What is the gradient of this line?
   b) Obtain the x and y intercept and hence sketch the line.

5. a) A straight line passes through the origin and the point (1, -1). Find the equation of the line.
   b) A straight line of gradient -1 passes through the point (3, -1)
      i. Determine the equation of the line.
      ii. Through which point does the line cut the y axis.

6. a) A line passes through the points (a, 0) and (0, b). Find the equation of the line.
   b) Given that a line, **L** passing through the point (0, 2) is perpendicular to the line $2y = 5x + 3$, find the point of intersection of the line L with the line $2x = 3y - 5$.

7. Sketch the lines $y = 4x, and\ 2y + 3x = 3$. hence calculate the area of a triangle enclosed by the two lines and the y axis.

8. Obtain the coordinates of points of intersection of the curve $y = 2x^2 - 3x$ and the line $y = 3(2x - 3)$.

9. a) Find the equation of the line that passes through **H** (1, 5) and is perpendicular to the line $x + 5y = 1$.
   b) Given that, $5x - 10y - 30 = 0$ is an equation of a straight line. Find the coordinates of its x intercept.
   c) The line through A (***a***, 2) and B (3, 6) is parallel to the line whose equation is $y = 4x - 5$. Find the value of ***a***.

10. a) Determine the area of the figure enclosed by the x axis, y axis and the line $2x + y = 8$.
    b) The points M (1, 3), N (5, 11), A (0, -3) and B (4, y) are such that MN is parallel to AB
    c) Given two points; D (3, 5) and E (6, 2). Find the gradient of the line, which joints these two points and the distance between these two points.

3 GRAPHS OF QUADRATIC FUNCTIONS (ax2+bx+c)

3.1 Introduction:

Quadratic function is of the form $ax^2 + bx + c$, where ***a, b***, and ***c*** are integers.

The graph of such a function is a curve and the curve is called a **parabola**.

The curve has one turning point called **vertex**, which may be either minimum or maximum depending on the nature of the quadratic function.

The curve is symmetrical about a line, which is parallel to the y axis and passes through its vertex.

The graph of $ax^2 + bx + c$ is ***u*** shaped, i.e. faces up if ***a*** is negative. In this case, the vertex is minimum. It is ***n*** shaped i.e. faces upside down if ***a*** is positive and in this case, the vertex is maximum. In other words, the graph of quadratic function faces up if the coefficient of $\boldsymbol{x^2}$ is positive and is upside down if the coefficient of $\boldsymbol{x^2}$ is negative.

3.2 Sketching graph of Quadratic function

The following examples will illustrate how to draw the graph of quadratic function.

Example

Draw the graph of the function $y = x^2 + 5x - 6$ for $-7 \leq x \leq 2$ and hence state the coordinates of the vertex.

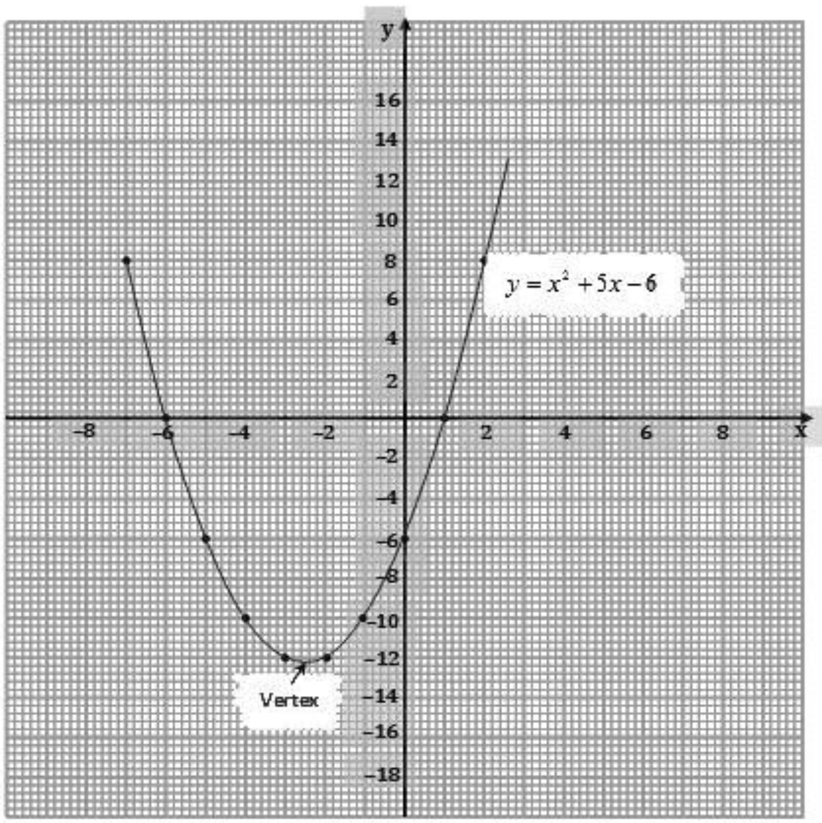
Solution

x	–7	–6	–5	–4	–3	–2	–1	0	1	2
x^2	49	36	25	16	9	4	1	0	1	4
$5x$	–35	–30	–25	–20	–15	–10	–5	0	5	10
–6	–6	–6	–6	–6	–6	–6	–6	–6	–6	–6
y	8	0	–6	–10	–12	–12	–10	–6	0	8

∗ *First, construct table of values for x from –7 to 2. The table is drawn as below.*

∗ *Next, choose a suitable scale and plot the values of **y** against those of the corresponding values of **x**.*

∗ *Lastly, join the various points you have plotted with a continuous smooth curve, using **free hand.** You then obtain the curve below.*



∗ *The curve has a minimum value at y = –12.4 and x = –2.3. So the coordinates of the vertex is (–2.3, –12.4).*

From the graph, the curve cuts the x axis at two distinct points, i.e. at 6 and 1. At these points, the y coordinates are zero.

$$So\ \ from\ \ \ y = x^2 + 5x - 6,\ if\ \ y = 0$$

$$\Rightarrow \ \ x^2 + 5x - 6 = 0$$

$$\therefore x = -6\ and\ x = 1\ are\ the\ solutions\ to\ the\ quadratic\ equation\ above$$

This implies that we can also use graphical mean to solve a given quadratic equation.

Example

Draw the graph of $y = 2 + 2x - x^2$ for values of x from 2 to 4.

From the graph find:

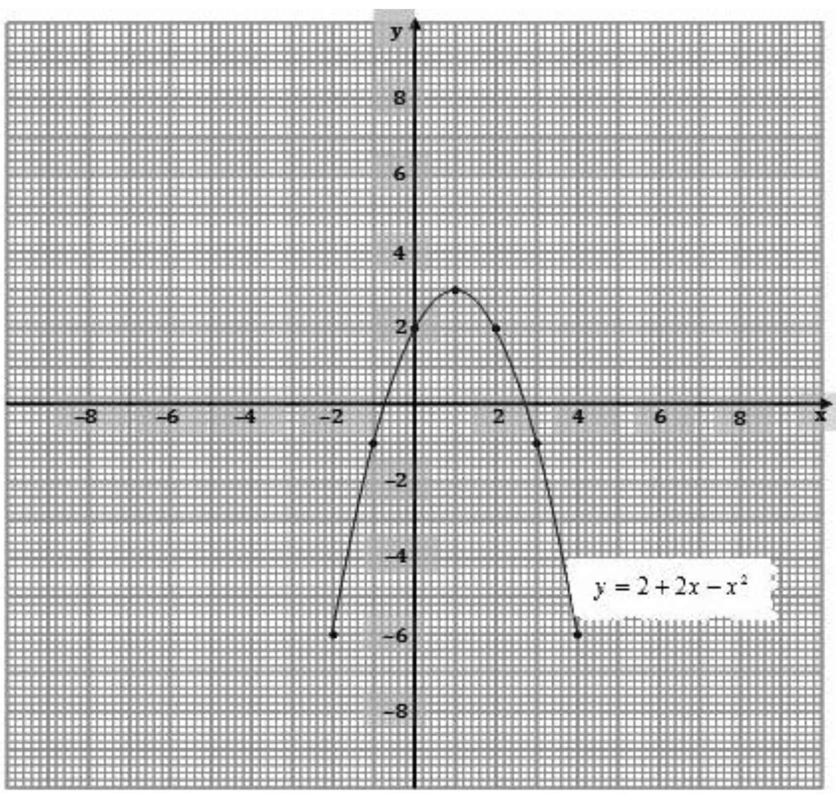
a) the maximum of $y = 2 + 2x - x^2$

b) the value of x for which y is greatest, hence state the coordinates of the vertex (maximum point).

c) the range of values of x for which y is positive.

Solution

x	2	1	0	1	2	3	4
2	2	2	2	2	2	2	2
$2x$	4	2	0	2	4	6	8
x^2	4	1	0	1	4	9	16
$-x^2$	4	1	0	1	4	9	16
$y = 2 + 2x + x^2$	6	1	2	3	2	1	6



From the graph:

a) *The maximum value of* $y = 2 + 2x - x^2$ *is 3.*

b) *The value of x for which y is greatest is 1. Hence the coordinates of vertex is (1, 3).*

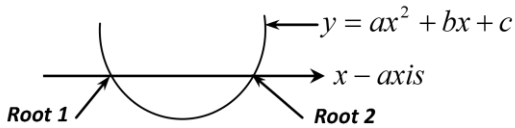
c) *The curve cuts the x –axis at* $x = -0.8$ *and,* $x = 2.8$. *Therefore,* ***y*** *is positive for all parts of the curve above the x –axis i.e. when* $x > -0.8$ *and,* $x < 2.8$. *In other word,* ***y*** *is positive over the range* $-0.8 < x < 2.8$

3.3 Solving quadratic equations by graphical method

To obtain the solution (root) of the quadratic equation $ax^2 + bx + c = 0$ graphically, you have to draw the graph of $y = ax^2 + bx + c$ and then read the values of x at the points where the graph cuts the x axis.

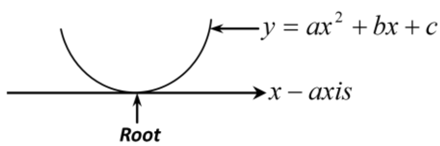
3.4 Nature of the graph (curve)

1 If the curve of the quadratic function cuts the x axis at two distinct points as depicted below:



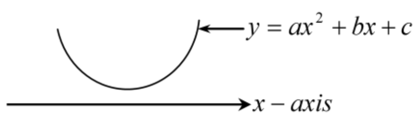
Then the related equation has two distinct roots. The two values of x at these points therefore give the solution the quadratic equation.

2 If the curve just touches the x axis at only one point as shown below:



Then the related equation has one root which is repeated at the point where the curve touches the x axis.

3 However, if the curve does not cut or touch the x axis at all, then the related equation has no solution (has no root). Such a curve may appear as depicted below.



Example

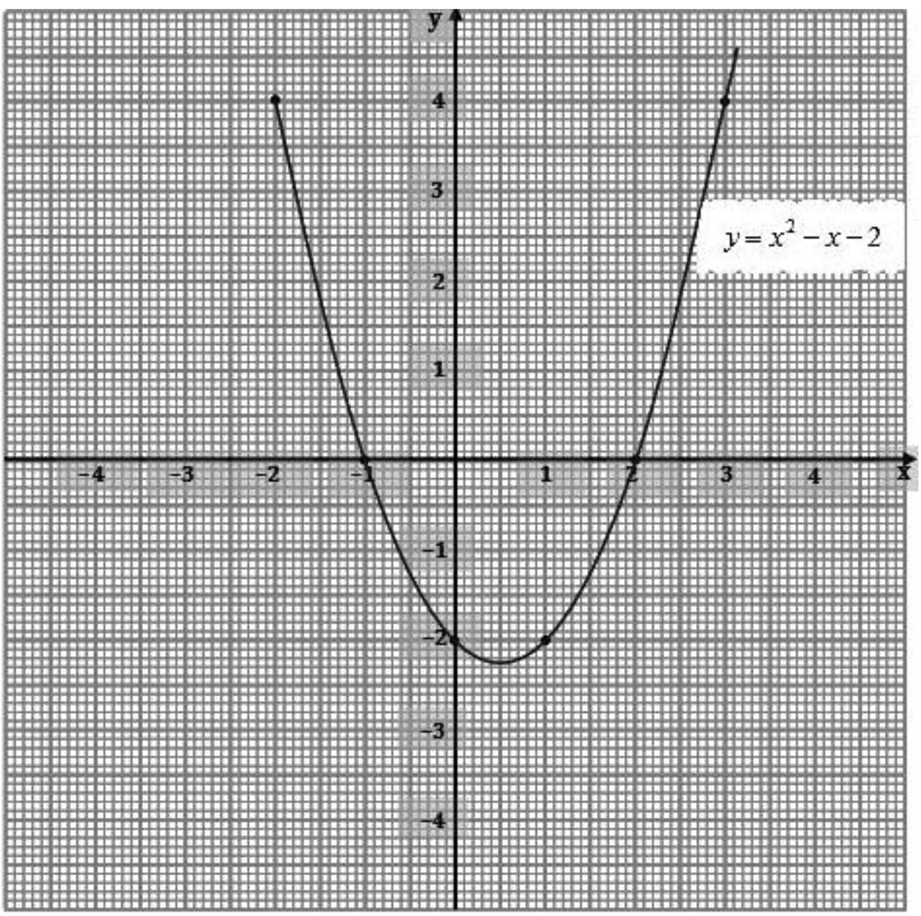
Solve the quadratic equation $x^2 - x - 2 = 0$ by graphical method.

Solution

∗ *Let* $y = x^2 - x - 2$

∗ *Select the range of values of x for which you want to obtain the corresponding values of y, say from –2 to 3.*

x	2	1	0	1	2	3
x^2	4	1	0	1	4	9
$-x$	2	1	0	1	2	3
2	2	2	2	2	2	2
y	4	0	2	2	0	4



$*$ *From the graph, the curve cuts the x –axis at –1 and 2.* $\therefore x = -1$ *and* $x = 2$

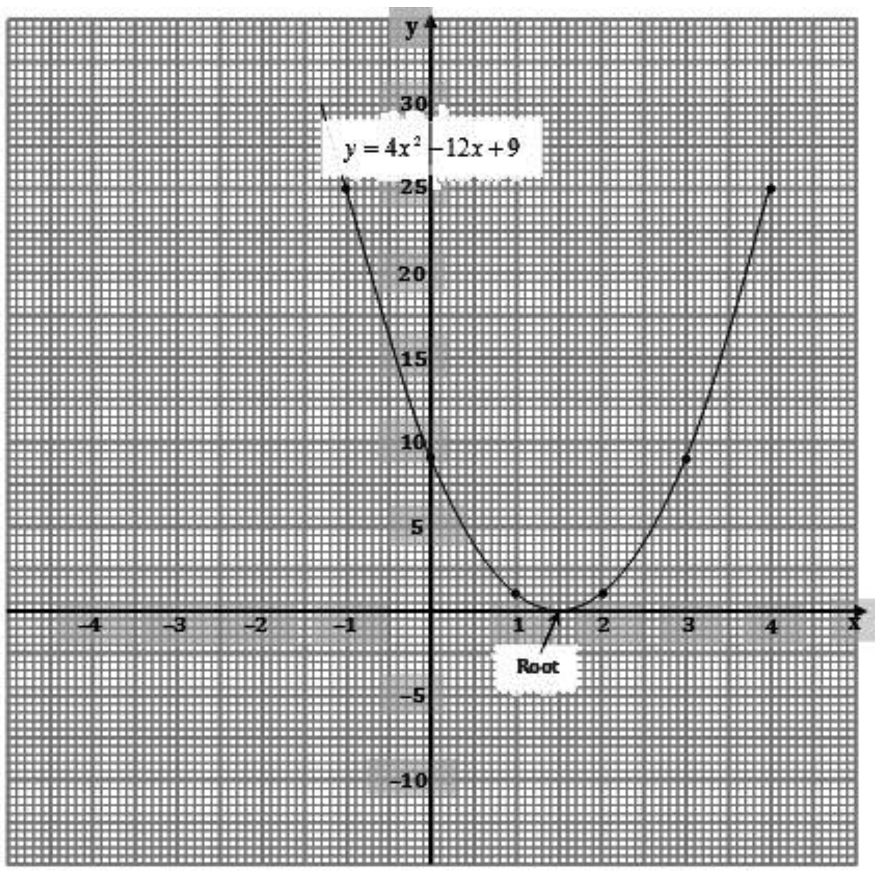
Example

Use graphical method to find the roots of the equation $4x^2 - 12x + 9 = 0$

Solution

$*$ *Let* $y = 4x^2 - 12x + 9$

x	1	0	1	2	3	4
$4x^2$	4	0	4	16	36	64
$-12x$	12	0	12	24	36	48
9	9	9	9	9	9	9
y	25	9	2	1	9	25



$*$ *From the graph, the curve does not cut the x –axis but rather touches it where x = 1.5.Ttherefore the root of the equation is 1.5 (twice i.e. repeated)*

NB: you can check this by factorization method.

4 Obtaining the coordinates of the point of intersection of a line and a curve by graphical method

Here the graph of the line $y = mx + c$ and the curve $y = ax^2 + bx + c$ should be plotted on the same graph paper and the coordinates of the points where the two graphs meet give the solution to the two equations. Remember that at the point of intersection, the two equations arte the same, i.e. $ax^2 + bx + c = mx + c$

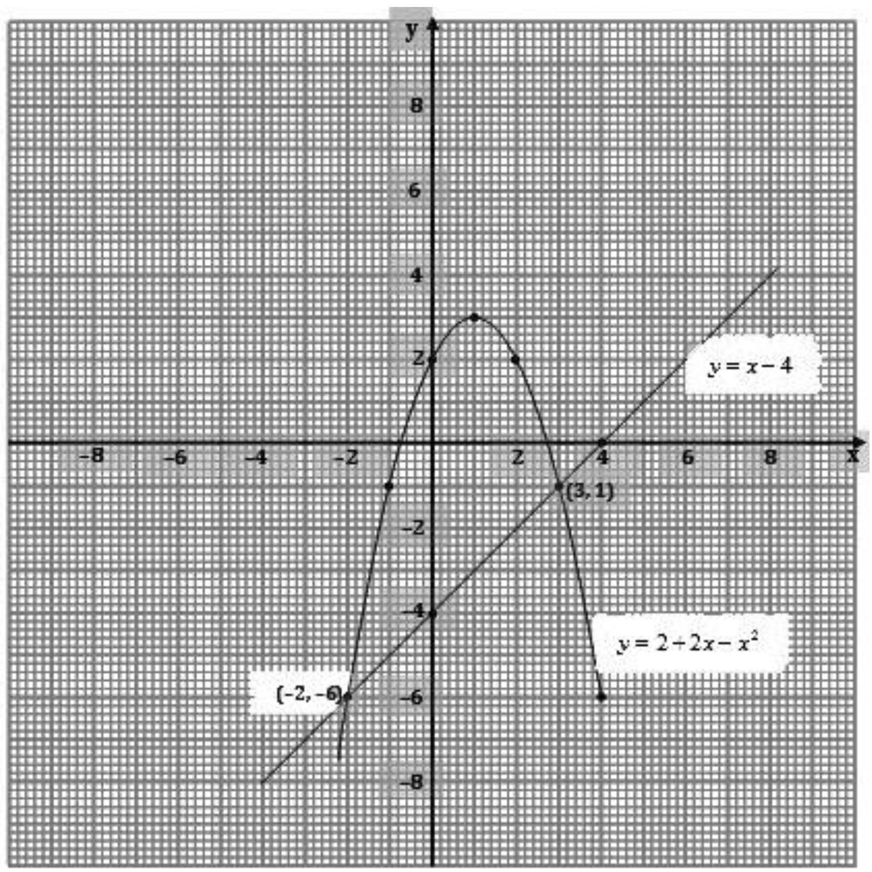
Example

Draw a graph of $y = 2 + 2x - x^2$ *and* $y = x - 4$ on the same graph paper for $-2 \le x \le 4$

Use your graph to solve $2 + 2x - x^2 = x - 4$

Solution

For $y = 2 + 2x - x^2$



x	–2	–1	0	1	2	3	4
2	**2**	**2**	**2**	**2**	**2**	**2**	**2**
$2x$	**–4**	**–2**	**0**	**2**	**4**	**6**	**8**
$-x^2$	**–4**	**–1**	**0**	**–1**	**–4**	**–9**	**–16**
y	**–6**	**–1**	**2**	**3**	**2**	**–1**	**–6**

For $y = x - 4$

x	2	1	0	1	2	3	4
$y = x - 4$	6	5	4	3	2	1	0

The two graphs meet at (–2, –6) and (3, 1). Therefore the solution to the equation $2 + 2x - x^2 = x - 4$ *are* $x = -2$ *and* $x = 3$

Exercise

1. Use graphical method to solve:

   a) the equation $x^2 - 4x + 1 = 0$ *for* $-3 \leq x \leq 3$

   b) the simultaneous equation $y = x^2 - 4x + 1$ *and* $y = 8 - 2x$

2. Draw the graph of $y = 2x^2 - 3x - 7$ taking values of x from 2 to 4. Use your graph to solve:

   a) the equation $2x^2 - 3x - 7 = 0$

   b) the equation $2x^2 - 3x - 12 = 0$

   c) the simultaneous equations $y = 2x^2 - 3x - 7$ *and* $y = 2 - 1x$

3. Draw the graph of the function $y = x^2 - 5x + 4$ for the domain $-1 \leq x \leq 6$. From the graph, solve the following equations:

   i. $x^2 + 4 = 5x$

   ii. $x^2 - 5x = 1$

   iii. $x^2 - 6x + 2 = 0$

4. (a) Copy and complete the table below for $y = 2x^2 - 3x - 7$

x	2	1	0	1	2	3	4
$2x^2$	8	2	0	2			
$-3x$	6	3	0			9	12
7	7	7	7	7	7	7	7
y	7	2	7				

(b) Use a scale of 1cm to 1unit on the x axis and of 1cm to 2units on the y axis to draw the graph of $y = 2x^2 - 3x - 7$

(c) Use your graph to find:

i. the value of y when x = 3.4

ii. the values of x when y = 7

iii. the minimum value of y.

4 INEQUALITIES

4.1 Introduction:

An inequality is an expression that contains the following symbols:

a)

b)

c) $\leq$

d) $\geq$

The following are some examples of inequalities:

a) $2x-3 \leq 10$

b) $2x-2 < 4x+6$

c) $x^2+2x+1>0$

d) $\frac{3}{8}(x+1) \geq (4+2x)$

4.2 Solving Inequalities:

When solving inequalities, the following rules may be put into consideration:

1. Multiplication on both sides of the inequality symbol by a positive number does not change the order of the inequality. E.g.

$$2x-3 < 4x+6, \text{ } \textit{multiplying both sides by} 2$$
$$\Rightarrow 2(2x-3) < 2(4x-6) \Rightarrow 4x-6 < 8x+12$$

2. Adding a constant on both sides of the inequality symbol leaves the order of the inequality unchanged. E.g.

$$2x-3 < 4x+6, \text{ } \textit{adding a } 5 \textit{ on both sides}$$
$$\Rightarrow 2x-3+5 < 4x-6+5 \Rightarrow 4x+2 < 4x+11$$

3. Subtraction of a constant from both sides of the inequality symbol leaves the order of the inequality unchanged. E.g.

$$3x+1 \leq 4x-5, \text{ } \textit{subtracting a } 4 \textit{ from both sides}$$
$$\Rightarrow 3x+1-4 \leq 4x-5-4 \Rightarrow 3x-3 \leq 4x-11$$

4. Multiplying or dividing both sides of the inequality symbol by a negative reverses the order of the inequality. E.g.

$$3x+1 \le 4x-5,\ \textit{multiplying both sides by}\ ^{-}2$$
$$\Rightarrow {}^{-}2(3x+1) \ge {}^{-}2(4x-5)$$

Similarly:

$$3x+1 \le 4x-5,\ \textit{dividing both sides by}\ ^{-}4$$
$$\Rightarrow \frac{1}{^{-}4}(3x+1) \ge \frac{1}{-4}(4x-5)$$

4.3 Linear inequalities in one unknown

A linear inequality is an inequality for which the highest power of the variable is a one.

Example

Find the value of x for which:

a) $x-1 \le 5$

b) $x+2 > 6$

c) $2x-3 \le 13-6x$

d) $5+\frac{3}{x} \ge 8$

e) $\frac{1}{4}(x-1) < \frac{x}{3}$

Solution

a) $x-1 \le 5$

Collecting like terms

$\Rightarrow x-\cancel{1}+\cancel{1} \le 5+1$

$\therefore\ \underline{\underline{x \le 6}}$

b) $x+2 > 6$

$\Rightarrow x+2-2 > 6-2$

$\therefore \underline{\underline{x > 4}}$

c) $2x-3\le 13-6x$

$\Rightarrow 2x-3+3\le 13+3-6x$

$\Leftrightarrow 2x\le 16-6x$

$2x+6x\le 16-6\cancel{x}+6\cancel{x}$

$8x\le 16$

$\therefore x\le {}^{16}\!/_{8} \;\Rightarrow\; \underline{\underline{x\le 2}}$

d) $5+\dfrac{3}{x}\ge 8$

$\Rightarrow \dfrac{3}{x}\ge 8-5 \Leftrightarrow \dfrac{3}{x}\ge 3$

$\cancel{x}\times\dfrac{3}{\cancel{x}}\ge 3\times x$

$\dfrac{3}{3}\ge\dfrac{\cancel{3}}{\cancel{3}}x \Rightarrow \underline{\underline{1\ge x \text{ or } x\le 1}}$

e) $\dfrac{1}{4}(x-1)<\dfrac{x}{3}$

Cross-multiplying:

$\Rightarrow 3(x-1)<4x$

$\Rightarrow 3x-9<4x \Leftrightarrow 3x-4x<9$

$-x<9$

Multiplying all through by -1 *and reversing the order of the inequality i.e.*

$-1(-x)>-1(9)$

$\therefore \underline{\underline{x>-9 \text{ or } -9<x}}$

Example

Solve the following inequalities

a) $-6<2x-4<2$

b) $3\ge\dfrac{7-x}{2}\ge 1$

c) $0<2-\dfrac{3x}{4}\le\dfrac{1}{2}$

d) $-2 \leq \frac{5-3x}{4} \leq \frac{1}{2}$

Solution

a) $-6 < 2x - 4 < 2$

Addig 4 to both sides of the inequality symbols

$\Rightarrow -6+4 < 2x-4+4 < 2+4$

$-2 < 2x < 6$, *dividing all through by 2*

$\Rightarrow \frac{-2}{2} < \frac{2}{2}x < \frac{6}{2}$

$\therefore \underline{\underline{-1 < x < 3}}$

b) $3 \geq \frac{7-x}{2} \geq 1$

Multiplying both sides of the inequality symbol by 2

$2 \times 3 \geq \frac{(7-x)}{\cancel{2}} \times \cancel{2} \geq 1 \times 2$

$6 \geq 7 - x \geq 2$

$6-7 \geq -x \geq 2-7$

$-1 \geq -x \geq -5$

Dividing all through by -1 *and reversing the inequality symbol, we have;*

$\frac{-1}{-1}x \leq \frac{-1}{-1}x \leq \frac{-5}{-1}$

$\therefore \underline{\underline{1 \leq x \leq 5}}$

c) $0 < 2 - \frac{3x}{4} \leq \frac{1}{2}$

$\Rightarrow 0 < \frac{8-3x}{4} \leq \frac{1}{2}$, *and multiplying althrough by 4*

$0 < 8 - 3x \leq 2 \Leftrightarrow -8 < -3x \leq -6$

Dividing all through by -3 *and reversing the inequality symbol*

$\frac{-8}{-3} > x \geq \frac{-6}{-3}$

$\therefore \underline{\underline{8/3 > x \geq 2 \text{ or } 2 \leq x < \frac{8}{3}}}$

d) $$-2 \le \frac{5-3x}{4} \le \frac{1}{2}$$

$$-2 \times 4 \le 5 - 3x \le \frac{1}{2} \times 4$$

$$-8 \le 5 - 3x \le 2$$

$$-13 \le -3x \le -3$$

$$\tfrac{-13}{-3} \ge x \ge \tfrac{-3}{-3}$$

$$\therefore \underline{\underline{13/3 \ge x \ge 1 \quad or \quad 1 \le x \le 13/3}}$$

Example

The temperature readings on the Fahrenheit (F) and Celsius (C) scale are related by the equation:

$C = \frac{5}{9}(F - 32)$. What range of F corresponds to $30 \le C \le 40$?

Solution

$$30 \le C \le 40$$

$$\Rightarrow 30 \le \frac{5}{9}(F - 32) \le 40$$

$$30 \times 9 \le 5(F - 32) \le 40 \times 9$$

$$\frac{270}{5} \le F - 32 \le \frac{360}{5}$$

$$54 \le F - 32 \le 72$$

$$\therefore \underline{\underline{86 \le F \le 104}}$$

4.4 Building up linear Inequality

We can form linear inequality from statement.

Example

The result of the sum of two numbers is less than ten. If one of the numbers is six, write down an inequality statement.

Solution

Let the number be x. Sum of x and $6 = x + 6$, *bet this must be less than* 10

$\Rightarrow x + 6 < 10$

Example

The sum of three consecutive integers is less than 99. Write down the inequality for this statement.

Solution

$$\text{Let the first integer be } n$$
$$\therefore 2^{nd} \text{ integer} = n+1$$
$$3^{rd} \text{ integer} = n+2$$
$$\text{But their sum must be less than } 99.$$
$$\Rightarrow n+(n+1)+(n+2)<99$$
$$n+n+n+1+2<99$$
$$\therefore \underline{\underline{3n+3<99}}$$

Example

The difference between two numbers is 30 and their sum is at most 68. What are the largest values the two numbers can have?

Solution

$$\text{Let } x \text{ be the larger number, then the smaller number} = x-30$$
$$\text{But their sum is less or equal to } 68$$
$$\Rightarrow x+(x-30)\le 68$$
$$x+x-30\le 68$$
$$2x\le 68+30$$
$$\tfrac{2}{2}x\le \tfrac{98}{2}$$
$$\therefore x\le 49$$
$$\text{Therefore the maximum the two numbers can take are } 49 \text{ and } 19$$

4.5 Quadratic Inequalities involving one unknown

Quadratic inequality is an inequality for which the highest power of the unknown is two.

Solving Quadratic Inequality

When solving quadratic inequality, we have to be very careful with the conditions of the inequality to be fulfilled.

Consider $(x+2)(x-1)<0$. Here there are two conditions under which the above given quadratic inequality can be less than zero.

1st condition:

$x+2<0 \;\; and \; x-1>0$

With this condition, if $(x+2)$ is negative i.e. $(x+2<0)$ and $(x-1)$ is positive i.e. $(x-1>0)$

Then multiplying negative by positive you obtain a negative and negative, as we all know is less than zero.

$$So \;\; for \;\; x+2<0 \Rightarrow x<-2$$
$$and \;\; for \;\; x-1>0 \Rightarrow x>1$$

2nd condition:

$x+2>0 \;\; and \; x-1<0$

$$So \;\; for \;\; x+2>0 \Rightarrow x>-2$$
$$and \;\; for \;\; x-1<0 \Rightarrow x<1$$

Example

Solve the following inequalities

a) $(x+2)(x-4)<x^2-6$

b) $2x^2-9x+10<0$

c) $p^2+2p+1\geq 0$

Solution

a) $(x+2)(x-4)<x^2-6$

$$x^2-4x+2x-8<x^2-6$$
$$\cancel{x}^2-\cancel{x}^2-2x-8<-6$$
$$\Rightarrow -2x<-6+8$$
$$\tfrac{-2}{-2}x>\tfrac{2}{-2}$$
$$\therefore x>-1 \quad or \;\; -1<x$$

b) $2x^2 - 9x + 10 < 0$

$2x^2 - 4x - 5x + 10 < 0$

$2x(x-2) - 5(x-2) < 0$

$(2x-5)(x-2) < 0$

Side work
sum $= -9$
product $= 20$
factors $: (-4, -5)$

1^{st} *condition* $: (2x-5) < 0$ *and* $(x-2) > 0$

For $2x - 5 < 0 \Rightarrow 2x < 5 \therefore x < 5/2$, *and for* $x - 2 > 0 \Rightarrow x > 2$

$\therefore \underline{\underline{x < 5/2 \text{ and } x > 2}}$

2^{nd} *condition* $: (2x-5) > 0$ *and* $(x-2) < 0$

For $2x - 5 > 0 \Rightarrow 2x > 5 \therefore x > 5/2$, *and for* $x - 2 < 0 \Rightarrow x < 2$

$\therefore \underline{\underline{x > 5/2 \text{ and } x < 2}}$

c) $p^2 + 2p + 1 \geq 0$

$p^2 + p + p + 1 \geq 0$

$(p+1)(p+1) \geq 0$

$p + 1 \geq 0$

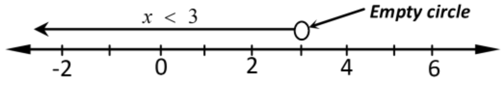
$\therefore \underline{\underline{p \geq -1}}$

Side work
sum $= 2$
product $= 1$
factors $: (1 \quad 1)$

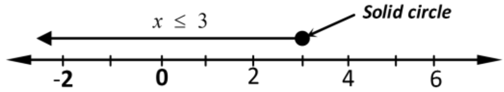
4.6 Representing inequality on a number line

When representing inequality on a number line, a line is drawn with an arrow on it representing the range of values of the solution to the inequality as follows:

1. If the inequality has < or > symbol, such as $x < 3$. In this case, an arrow is drawn beginning from 3 with an empty circle at its tail showing that 3 does not form part of the solution to the inequality.



2. But if the inequality has $\leq$ *or* $\geq$ *symbol e.g.* $x \leq 3$. In this case 3 is included in the range of value of the solution to the inequality, so a solid circle is drawn at the tail of an arrow i.e.



Example

Show these inequalities on number lines.

a) $x < 1$

b) $a > -2$

c) $y \leq 6$

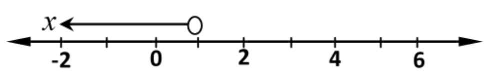
d) $-3 \leq x < 1$

e) $-2 \leq y \leq 4$

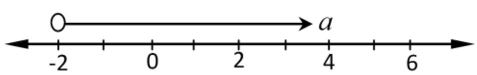
f) $0 < q < 5$

Solution

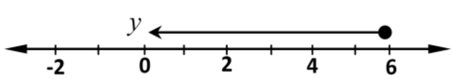
a) $x < 1$



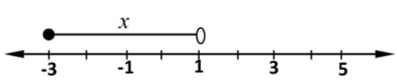
b) $a > -2$



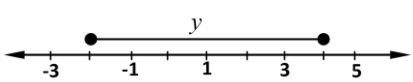
c) $y \leq 6$



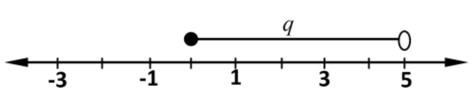
d) $-3 \leq x < 1$



e) $-2 \leq y \leq 4$



f) $0 < q < 5$



Example

Solve the following inequalities and illustrate their solution on number lines.

a) $15 - 4t \geq 5$

b) $10 \leq 20 - 5y \leq 15$

c) $3x^2 + x - 10 \geq 0$

Solution

a) $15 - 4t \geq 5$

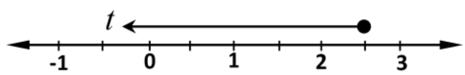
$\Rightarrow 15 - 5 - 4t \geq 0$

$10 - 4t \geq 0$

$\frac{10}{4} \geq \frac{4}{4}t$

$\therefore \underline{\underline{2.4 \geq t}}$

The solution set is; $\{t : t \leq 2.5\}$



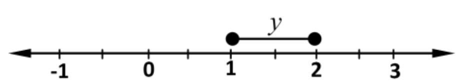
b) $10 \leq 20 - 5y \leq 15$

$\Rightarrow 10 - 20 \leq -5y \leq 15 - 20$

$-10 \leq -5y \leq -5$

$\frac{-10}{-5} \geq \frac{-5}{-5}y \geq \frac{-5}{-5}$

$\therefore \underline{\underline{2 \geq y \geq 1 \quad or \quad 1 \leq y \leq 2}}$



c) $3x^2 + x - 10 \geq 0$

$3x^2 + 6x - 5x - 10 \geq 0$

$3x(x+2) - 5(x+2) \geq 0$

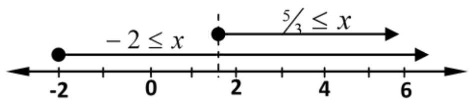
$(3x-5)(x+2) \geq 0$

Here there are two conditions for which $(3x-5)(x+2)\geq 0$

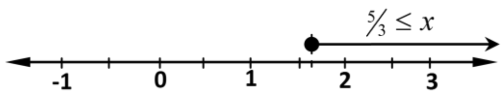
1st condition:

For $(3x-5)(x+2)$ *to be greater or equal to 0,*
$(3x-5)\geq 0$ *and also* $(x+2)\geq 0$

So for $(3x-5)\geq 0 \Rightarrow x \geq 5/3$
and for $(x+2)\geq 0 \Rightarrow x \geq -2$



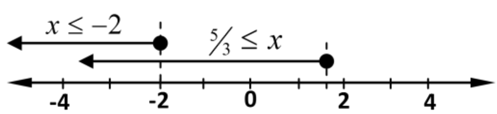
The intersection of the two inequalities gives the final solution to the inequalities. The final solution is as drawn below.



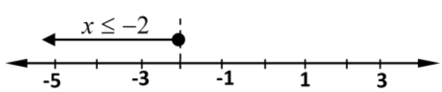
2nd condition:

For $(3x-5)(x+2)$ *to be greater or equal to 0 also,* $(3x-5)\leq 0$ *and also* $(x+2)\leq 0$ *i.e.* $(3x-5)$ *and* $(x+2)$ *should be negative, so that if you multiply negative by negative, you obtain a positive which is greater than zero*

So for $(3x-5)\leq 0 \Rightarrow x \leq 5/3$
and for $(x+2)\leq 0 \Rightarrow x \leq -2$



The intersection of the two inequalities gives the final solution to the inequalities. The final solution is as drawn below.



4.7 Linear inequality in two unknowns

This is when there are two variables involved in a linear inequality. Such inequality can be shown on a Cartesian graph.

4.8 Showing the inequality on a graph

If we are to show an inequality on a Cartesian graph, we consider the inequality as if it was an equation and then plot the line corresponding to it. We then shade the unwanted region. The unwanted region is the region on the Cartesian graph for which the inequality has no solution.

Note:

When drawing the line, we look for the line; we look for the inequality symbol and treat the line as follows;

i) If the inequality symbol is < or >, we use a broken line meaning that it is not part of the solution to inequality.

ii) If the inequality symbol is $\leq or \geq$, we use a solid (continuous) line meaning that the line forms part of the solution to the inequality.

4.9 Steps involved in drawing the inequality graphs

- Form an equation for the boundary line for each region by replacing the inequality symbol with equal sing.
- Find the intercepts for each of the boundary lines.
- Join the two intercepts for each line with a continuous or a broken line depending on the nature of the inequality symbol.
- Choose a test point on one side of the boundary line and check whether the test point satisfies the inequality.
- Lastly, you can now shade out the unwanted region with the help of a test point.

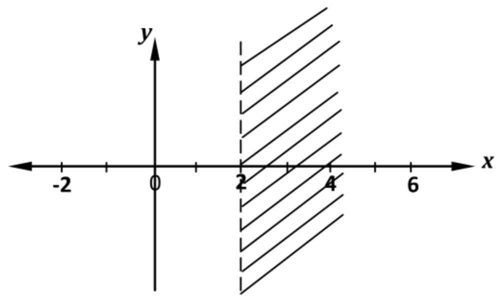
Example

Show the region $x < 2$ on a graph by shading out the unwanted region.

Solution

The inequality $x < 2$ *can be treated as a line* $x = 2$

Next, sketch the line $x = 2$ *using a broken line because of the symbol* <



Let $x = 1$ *be the test point. Substituting for 1 in the inequality* $x < 2 \Rightarrow 1 < 2$*. Now what do you say. Is 1 less than 2? The answer is yes, so shade out the unwanted region (which is to your right).*

Example

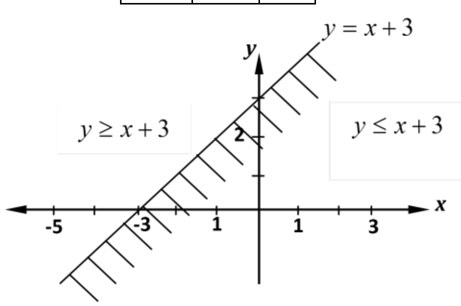
Show the region for which $y \geq x + 3$ by shading out the unwanted region.

Solution

First draw $y = x + 3$

Intercepts:

x	0	-3
y	3	0



Let the point (0, 0), be the test point.

From $y \geq x + 3$

Is $0 \geq 0 + 3$*?, the answer is NO. So the unwanted region is to the right of the line* $y = x + 3$

Example

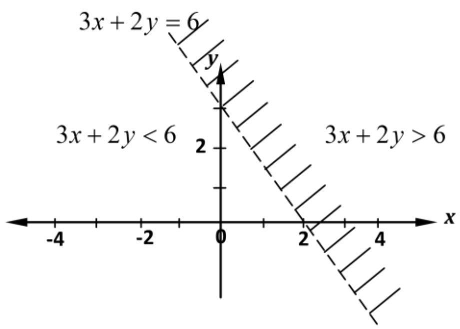
Show the region for which $3x + 2y < 6$ by shading out the unwanted region.

Solution

Boundary line is $3x + 2y = 6 \Rightarrow y = 3 - \frac{3}{2}x$

Intercepts:

x	0	2
y	3	0



Test point (0, 0):

From $3x + 2y < 6$

Is $3(0) + 2(0) < 6 \Rightarrow$ *yes. So the unwanted region is to the right of the line* $3x + 2y = 6$

Example

By shading the unwanted regions, show the region, which satisfies the following inequalities.

$$x + y \leq 3$$
$$y > x - 4$$
$$x \geq 0$$

Hence, find the area of the wanted region.

Solution

$For : x + y \leq 3$

$y \leq 3 - x$

$Boundary\ line\ is\ y = 3 - x$

Intercept

x	0	3
y	3	0

Test point (4, 0)

$From\ x + y \leq 3$

$Is\ 4 + 0 \leq 3 \Rightarrow No. So\ the\ unwanted\ region\ is\ to\ the\ right\ of\ the\ line\ x + y = 3$

$For : y > x - 4$

$y = x - 4$

$Boundary\ line\ is\ y = x - 4$

Intercept

x	0	4
y	-4	0

Test point (1,-2)

$From\ y > x - 4$

$Is\ -2 > 1 - 4 \Rightarrow yes. So\ the\ unwanted\ region\ is\ below\ the\ line\ y = x - 4$

$For : x > 0$

$x = 0$

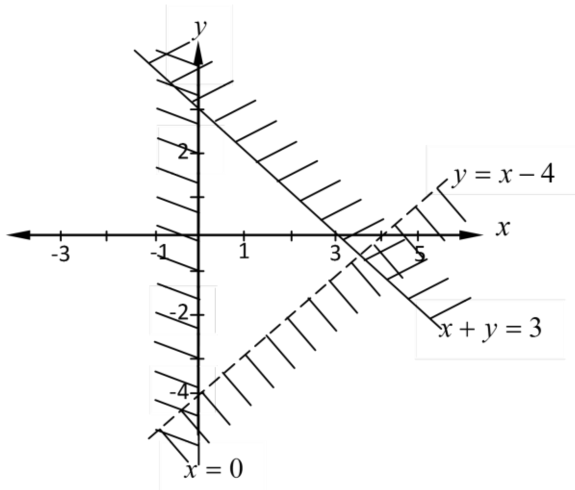
$Boundary\ line\ is\ x = 0$

Test point (-1, 0)

$Form: x > 0, \text{ using the point } (-1, 0) \Rightarrow x = -1, y = 0.$

$Is\ -1 > 0 \Rightarrow No, so, shade\ the\ left\ hand\ side\ of\ the\ line\ x = 0$

$Boundary\ line\ is\ x = 0$



The wanted region is a triangle with base 7 units and height h unknown. To find h, we consider the intersection of the

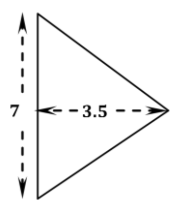
$y = 3 - x\ and\ y = x - 4$

$$\Rightarrow 3 - x = x - 4$$

$$2x = 7$$

$$\therefore x = 7/2 = 3.5$$

$$Hence\ h = 3.5 units$$



$$Area\ of\ the\ wanted\ region = \tfrac{1}{2}bh = \tfrac{1}{2} \times 7 \times 3.5$$

$$= \underline{\underline{12.25 sq units}}$$

4.10 Miscellaneous exercise

1. Using a number line, determine the values of x satisfying the given pairs of inequalities.

   i. $1 \le x \le 7, \quad 2 \le x \le 10$

   ii. $^{-}1 < x < 4, \quad 0 \le x < 8$

   iii. $2 < x < 6, \quad ^{-}3 \le x \le 4$

   iv. $^{-}3 < x < 2, \quad 0 < x < 5$

   v. $3 \le x \le 9, \quad 7 < x < 11$

2. Solve the following inequalities.

   i. $3x - 5 \ge 4$

   ii. $2y - \frac{1}{4} \ge 7$

   iii. $2(3x+1) \ge 4x - 3$

   iv. $3x + 4 \le 6x - 7$

3. Represent on suitable number line the solution set of each of the following inequalities.

   i. $3x^2 + 7x - 20 \le 0$

   ii. $3x^2 + 7x - 20 > 0$

   iii. $\dfrac{x-2}{4x-2x^2} < \dfrac{2}{3}$

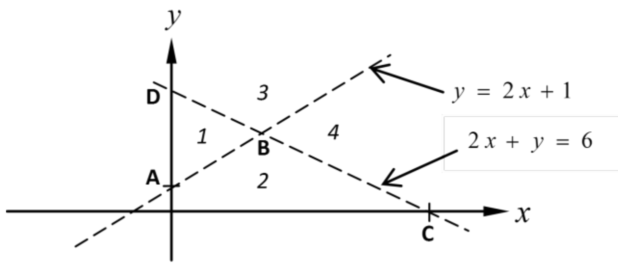
4. Show the regions that satisfy the given inequalities.

   i. $y > 2x - 1, \quad y < 3 - x$

   ii. $y > 3x - 2, \quad y < -2, \quad 4x + 3y + 12 \ge 0$

   iii. $y > 0, \quad 0 < x < 4, \quad x + y \le 6$

5. Study the figure below and use it to answer the questions that follow.



a) State the coordinates of the points A, B, C, and D.

b) Write down the inequalities satisfied by regions labeled 1, 2, 3, and 4.

6. Show on a Cartesian graph the region satisfying the inequalities:

$$y - x - 2 < 0, \quad x + 3y - 6 \le 0, \quad x \ge 0, \quad y \le 0$$

a) List the points with integer coordinates that are contained in the region.

b) What is the area of the figure that forms the wanted region?

c) For which vertex of the figure that forms the wanted region is $4x + 3y$:

i. Greatest

ii. Least.

5 GROUPED DATA

5.1 (Data Collection/Display)

5.1.1 Introduction:

When dealing with large items in the data, we break them into classes. The number of individual entries of the data falling into a class is the class frequency.

The top and bottom values of the classes are called class limits. The top value is the upper class limit and the lower value is the lower class limit.

5.1.2 Frequency Distribution

The following example will illustrate how to obtain the frequency distribution for grouped data.

Example

The following marks were scored by S.4 students in a certain school.

50	53	31	56	38
33	39	51	38	41
69	57	63	50	54
40	41	45	48	64
59	61	55	36	52

Form a frequency distribution starting with 30   43as the first class and using classes of equal length.

To draw up a frequency distribution table for any group data, we need to understand the following:

5.1.3 Lower class and upper class boundary

a) *For classes with no decimal places*

To get the lower class boundary, we subtract 0.5 from lower class limit. For the class 30 43 above, the lower class boundary = 30 0.5 = 29.5.

To get the upper class boundary, we add 0.5 to the upper class limit. For the class 30 34, its upper class

b) *For classes with one decimal place*

To get the lower class boundary, we subtract 0.05 from the lower class limit and to get the upper class boundary, we add 0.05 to the upper class limit.

5.1.4 Class marks

This is the midpoint of the class limit. For the class 30 34, its class mark is 32, obtained as follows. $Class\ mark = \frac{30+34}{2} = 32$

5.1.5 Class size or class interval or class width.

Class width is the upper class boundary minus lower class boundary. For the class 30 34, its class width is 5, obtained as follows. $Class\ width = 34.5 - 29.5 = 5$

We can now construct the frequency table for the above marks.

Solution

Frequency table

Marks (Class)	***Class boundary***	***Tally***	***Frequency (f)***	***Mid-mark (x)***
30 –34	*29.5 – 34.5*		*2*	*32*
35 – 39	*34.5 – 39.5*		*4*	*37*
40 – 44	*39.5 – 44.5*		*3*	*42*
45 – 49	*44.5 – 49.5*		*2*	*47*
50 – 54	*49.5 – 54.5*		*6*	*52*
55 – 59	*54.5 – 59.5*		*4*	*57*
60 – 64	*59.5 – 64.5*		*3*	*62*
65 – 69	*64.5 – 69.5*		*1*	*67*

5.2 Mean of Grouped Data

The mean for grouped data is calculated from the expression below.

$$Mean\ \bar{x} = \frac{\sum fx}{\sum f}$$

Where : x is the class mark or midmark

f is the frequncy

Example

The table below shows percentage marks gained in a mathematics test.

43	40	49	80	76	46	60	55	58	55
75	79	70	83	82	56	67	63	67	63
69	53	73	61	48	58	60	75	73	69
77	62	66	54	53	63	73	49	59	78

Copy and complete the table below.

Class	Tally	Frequency(f)	Mid-mark(x)	*fx*
40 44				
45 49				
50 54				
55 59				
60 64				
65 69				
70 74				
75 79				
80 85				

Find the mean mark.

Solution

Class	*Tally*	*Frequency (f)*	*Mid-mark (x)*	*fx*
40 –44		*2*	*42*	*84*
45 – 49		*4*	*47*	*188*
50 – 54		*3*	*52*	*156*
55 – 59		*6*	*57*	*342*
60 – 64		*7*	*62*	*434*
65 – 69		*5*	*67*	*335*
70 – 74		*4*	*72*	*288*
75 – 79		*6*	*77*	*462*
80 – 85		*3*	*82.5*	*247.5*
		$\sum f = 40$		$\sum fx = 2536.5$

$$Mean\ \bar{x} = \frac{\sum fx}{\sum f}, \quad \sum fx = 2536.5, \sum f = 40$$

$$= \frac{2536.5}{40}$$

$$= \underline{\underline{63.4}}(1dp)$$

5.2.1 Calculating mean of grouped data using assumed mean (Working mean/working zero)

The assume mean may be used to make work easier and quicker when finding the mean of a distribution, especially if the values are large.

Here, we choose any value within the mean mark of the distribution to be the mean called assumed mean denoted by **A**, and then calculate the deviation **d** of the assumed mean from the class mark from the expression below:

$$d = x - A$$ *Where x is the class mark (mean mark)*

The mean of the distribution is therefore given by:

$$Mean = A + \frac{\sum fd}{\sum f}$$

Example

The table below shows the weight in kilogram of 100 boys.

Weight (kg)	70 72	73 75	76 78	79 81	82 84
Frequency	8	10	45	30	7

Calculate the mean using assumed mean (A = 77)

Solution

Weight (kg)	***Frequency (f)***	***Mid value (x)***	***Deviation (d = x – A)***	*fd*
70 – 72	*8*	*71*	*–6*	*–48*
73 – 75	*10*	*74*	*–3*	*–30*
76 – 78	*45*	*77*	*0*	*0*
79 – 81	*3*	*80*	*3*	*90*
82 – 84	*7*	*83*	*6*	*42*
	$\sum f = 100$			$\sum fd = 54$

$$\begin{aligned} Mean &= A + \frac{\sum fd}{\sum f}, \quad A = 77 \\ &= 77 + \frac{54}{100} \\ &= \underline{\underline{77.54}} \end{aligned}$$

5.2.2 The Mode of Grouped Data

The mode of a set of data is the value that occurs most frequently.

Modal class:

This is the class with the highest frequency.

The mode of a grouped data is calculated from the formula below:

$$Mode \quad = \quad L_1 + \left(\frac{D_1}{D_1 + D_2} \right) c$$

$Where: L_1 = lower\ class\ boundary\ of\ \text{mod}al\ class$

$D_1 = difference\ between\ highest\ frequency\ and\ that\ before\ it$

$D_2 = difference\ between\ highest\ frequency\ and\ that\ after\ it$

$c = class\ width$

5.2.3 The Median of Grouped Data

Median is the value that divides a distribution into two equal parts with equal frequencies. The median for a grouped data is calculated from the formula below:

$$Median = L_1 + \left(\frac{\frac{N}{2} - F_b}{f_m} \right) c$$

$Where: L_1 = lower\ class\ boundary\ of\ median\ class$

$N = total\ number\ of\ observatio\ n$

$F_b = cumulative\ frequency\ before\ the\ median\ class$

$f_m = frequency\ of\ median\ class$

$c = class\ width$

NB:

The median class is the class that contains the median.

5.3 Quartiles

These are quantities that divide a set of data into four equal parts. The quartiles are:

5.3.1 Lower quartile (Q1)

This is a value that divides 25% way through the distribution when the observations are arranged in order of magnitude.

$$Q_1 = \frac{25}{100} \times N$$

Where N = total number of observation

5.3.2 Second quartile (Q2)

This is the same as the median. It divides the observation into two equal parts. I.e. it divides 5% 0f the distribution when the distribution is arranged in order of magnitude.

$$Q_2 = \frac{50}{100} \times N$$

5.3.3 Upper quartile (Q3)

It is the value that divides 75% way through the distribution when the observations are arranged in order.

$$Q_3 = \frac{75}{100} \times N$$

5.3.4 Inter–quartile range

This is the difference between the upper quartile and lower quartile.

$$Inter-quartile = Q_3 - Q_1$$

5.3.5 Semi Inter–quartile range

This is half of the inter quartile range.

$$Semi\ \text{inter}-quartile = \frac{Q_3 - Q_1}{2}$$

Example

The table below shows the marks observed in the end of year exams in mathematics by S.3 students of Kakungulu Memorial School in 2011.

Marks out of 100%	**Number of students (f)**	**Class marks (x)**	fx
10 14	8		
15 19	6		
20 24	14		
25 29	18		
30 34	10		
35 39	12		
40 44	8		
45 49	6		
	$\sum f = ...$		$\sum fx = ...$

a) Copy and complete the table below. Hence, calculate the mean mark and mode for all the students.

b) Obtain the cumulative distribution table. Use the table to find the median mark.

Solution

a)

Marks	*Frequency (f)*	*Class marks (x)*	fx	*Class boundaries*	*Cumulative frequency (F)*
10 – 14	*8*	*12*	*96*	*9.5 – 14.5*	*8*
15 – 19	*6*	*17*	*102*	*14.5 – 19.5*	*14*
20 – 24	*14*	*22*	*308*	*19.5 – 24.5*	*28*
25 – 29	*18*	*27*	*486*	*24.5 – 29.5*	*46*
30 – 34	*10*	*32*	*320*	*29.5 – 34.5*	*56*
35 – 39	*12*	*37*	*444*	*34.5 – 39.5*	*68*
40 – 44	*8*	*42*	*336*	*39.5 – 44.5*	*76*
45 – 49	*6*	*47*	*282*	*44.5 – 49.5*	*82*
	$\sum f = 82$		$\sum fx =$		

$$Mean = \frac{\sum fx}{\sum f}, \quad \sum fx = 2374, \sum f = 82$$

$$= \frac{2374}{82}$$

$$= \underline{\underline{28.95}}$$

$$Mode = L_1 + \left(\frac{D_1}{D_1 + D_2}\right)c \quad L_1 = 24.5,\ D_1 = 18 - 14 = 4,\ D_2 = 18 - 10 = 8,\ c = 5$$

$$\therefore Mode = 24.5 + \left(\frac{4}{4+8}\right) \times 5$$

$$= 24.5 + 1.67$$

$$= \underline{\underline{26.17}}$$

b) *From the cumulative frequency which is equal to 82; the median item* $= {}^{82}\!/_{2}$ *= 42nd item. Therefore, the class containing 42 is 25 – 29.*

$$\begin{aligned} Median &= L_1 + \left(\frac{\frac{N}{2} - F_b}{f_m}\right)c \quad L_1 = 24.5, N = 82, F_b = 28, f_m = 18, c = 5 \\ &= 24.5 + \left(\frac{42 - 28}{18}\right) \times 5 \\ &= \underline{\underline{28.4}} \end{aligned}$$

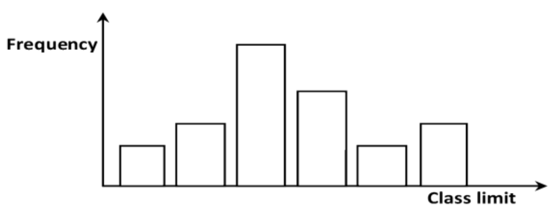
5.4 Presentation of Data

Once data have been collected, they can be displayed in various ways, which makes it easier to interpret and compare the data. Below are some of the ways of presenting data:

* Bar charts
* Pie charts
* Histogram
* Frequency polygon
* Cumulative frequency curve (Orgive)

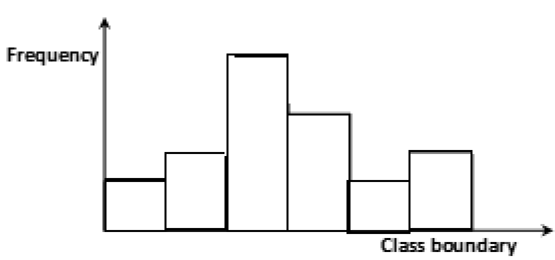
5.4.1 Bar chart (graph)

This is a graph where frequencies are plotted against class limits. The shape of the graph appears as depicted below.



5.4.2 Histogram

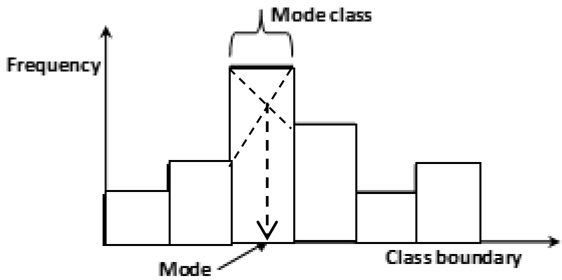
This is a graph where frequencies are plotted against class boundary.



5.4.3 Using histogram to estimate the mode

Mode can be estimated from the histogram as below:

- ∗ Draw a histogram of the data.
- ∗ Identify the class with the highest frequency i.e. the longest rectangle of the histogram.
- ∗ Join the corners of the rectangle of the modal class to the corners of the adjacent rectangles opposite to these corners of the rectangle of the modal class just as illustrated below.



- ∗ Now where the two dotted lines meet; the value along the class boundary (x axis) corresponding to this point gives the mode of the data.

Example

The table below shows the weight of S.4 students of academic year 2011 in Teso Integrated S.S who went for medical checkup at Fredica Hospital in Ngora.

a) Use the data recorded below to plot a histogram and use it to estimate the modal weight.

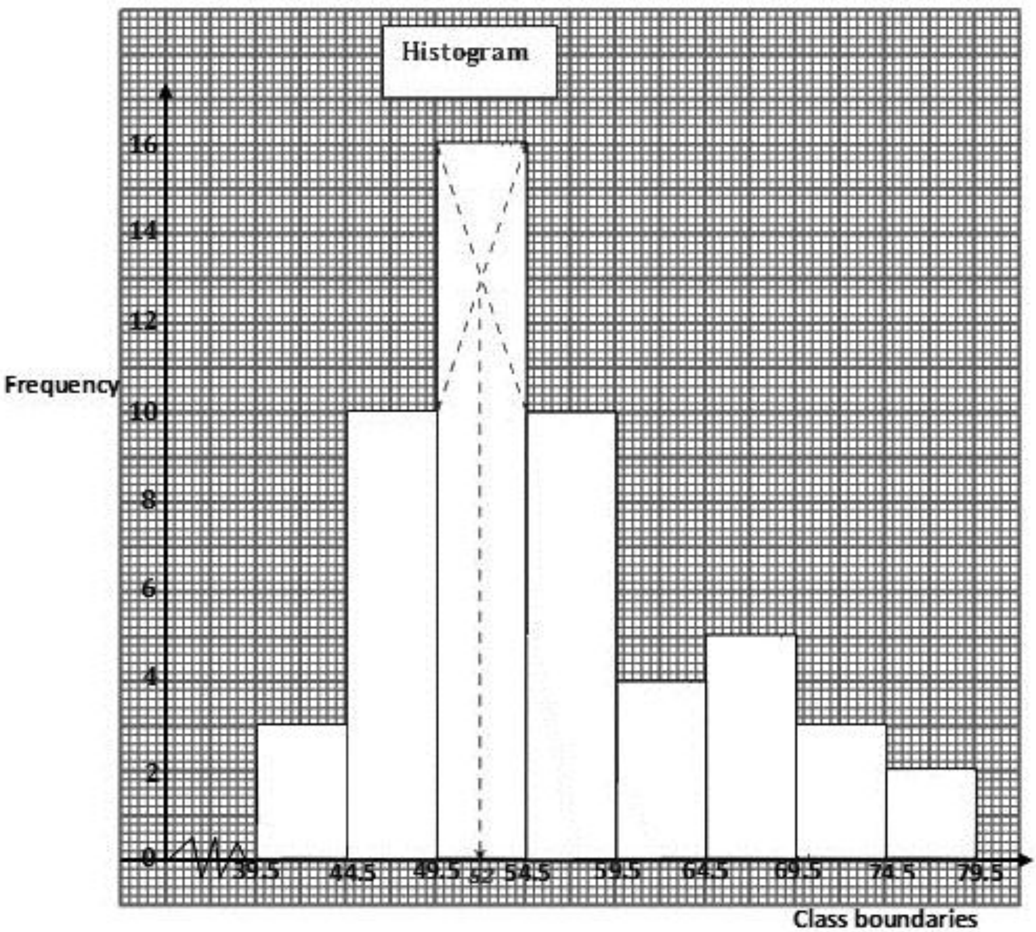
Weight (kg)	Number of students
40 44	3
45 49	10
50 54	16
55 59	10
60 64	4
65 69	5
70 74	3
75 79	2

b) What is the mean weight of these students?

Solution

a)

Marks	***Frequency (f)***	***Mid-mark (x)***	fx	***Class boundaries***
40 – 44	*3*	*42*	*126*	*39.5 – 44.5*
45 – 49	*10*	*47*	*470*	*44.5 – 49.5*
50 – 54	*16*	*52*	*832*	*49.5 – 54.5*
55 – 59	*10*	*57*	*570*	*54.5 – 59.5*
60 – 64	*4*	*62*	*248*	*59.5 – 64.5*
65 – 69	*5*	*67*	*335*	*64.5 – 69.5*
70 – 74	*3*	*72*	*216*	*69.5 – 74.5*
75 – 79	*2*	*77*	*154*	*74.5 – 79.5*
	$\sum f = 53$		$\sum f = 2951$	

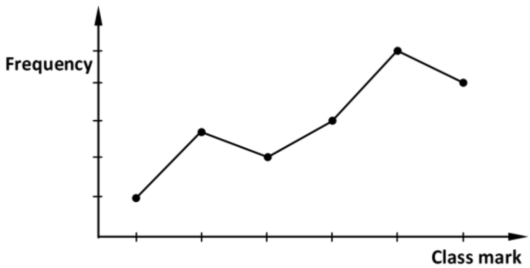


b) *Mean:*

$$Mean = \frac{\sum fx}{\sum f}, \quad \sum fx = 2951, \sum f = 53$$
$$= \frac{2951}{53}$$
$$= \underline{\underline{55.68\,kg}}$$

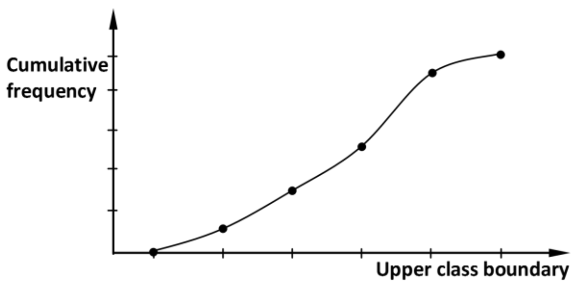
5.4.4 Frequency polygon

This is obtained by plotting frequency against class marks and then joining the consecutive points using a straight line as sketched below.



5.4.5 Cumulative frequency curve (Ogive)

This is obtained by plotting cumulative frequency against upper class boundaries and then joining the consecutive points using a smooth curve as illustrated below.



Example

The table below shows the weight of some patients recorded from a certain health clinic.

Weight (kg)	50 54	55 59	60 64	65 69	70 74	75 79	80 84
Number of patients	3	5	4	10	12	14	8

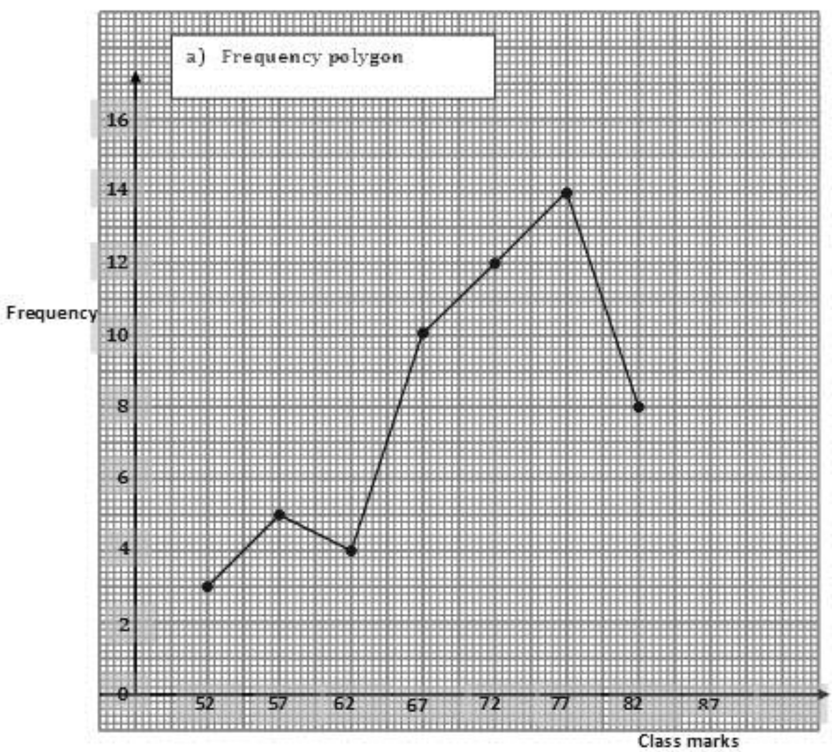
Using the above information, construct:

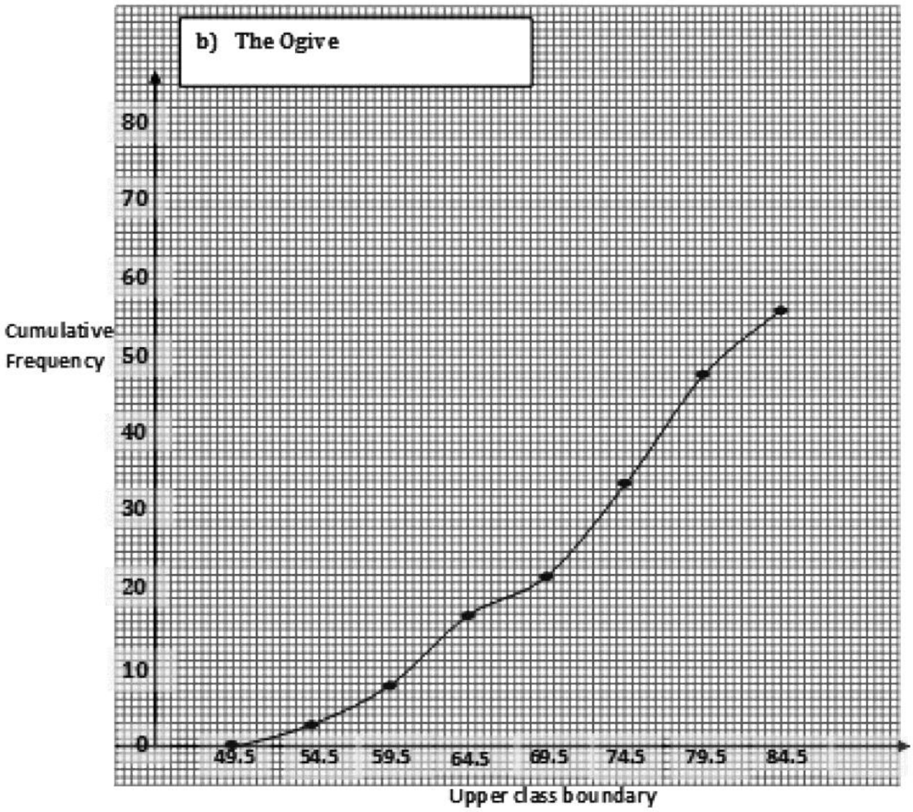
a) A frequency polygon and,

b) An Ogive

Solution

Weight (Kg)	*Class boundaries*	*Mid-mark (x)*	*Frequency (f)*	*Cumulative frequency*
50 – 54	*49.5 – 54.5*	*52*	*3*	*3*
55 – 59	*54.5 – 59.5*	*57*	*5*	*8*
60 – 64	*59.5 – 64.5*	*62*	*4*	*12*
65 – 69	*64.5 – 69.5*	*67*	*10*	*22*
70 – 74	*69.5 – 74.5*	*72*	*12*	*34*
75 – 79	*74.5 – 79.5*	*77*	*14*	*48*
80 – 84	*79.5 – 84.5*	*82*	*8*	*56*





5.5 Using the Ogive to obtain the median and quartiles

An Ogive can be used to obtain the median and quartiles. The following example shall therefore illustrate how to obtain the median and quartiles from an Ogive.

Example

The table below shows the mass, measured to the nearest kg of 50 boys.

Mass (kg)	60 64	65 69	70 74	75 79	80 84	85 89
Frequency	2	6	12	14	10	6

a) From a cumulative frequency table and use, it to draw a cumulative frequency curve.

b) Use your curve to estimate:

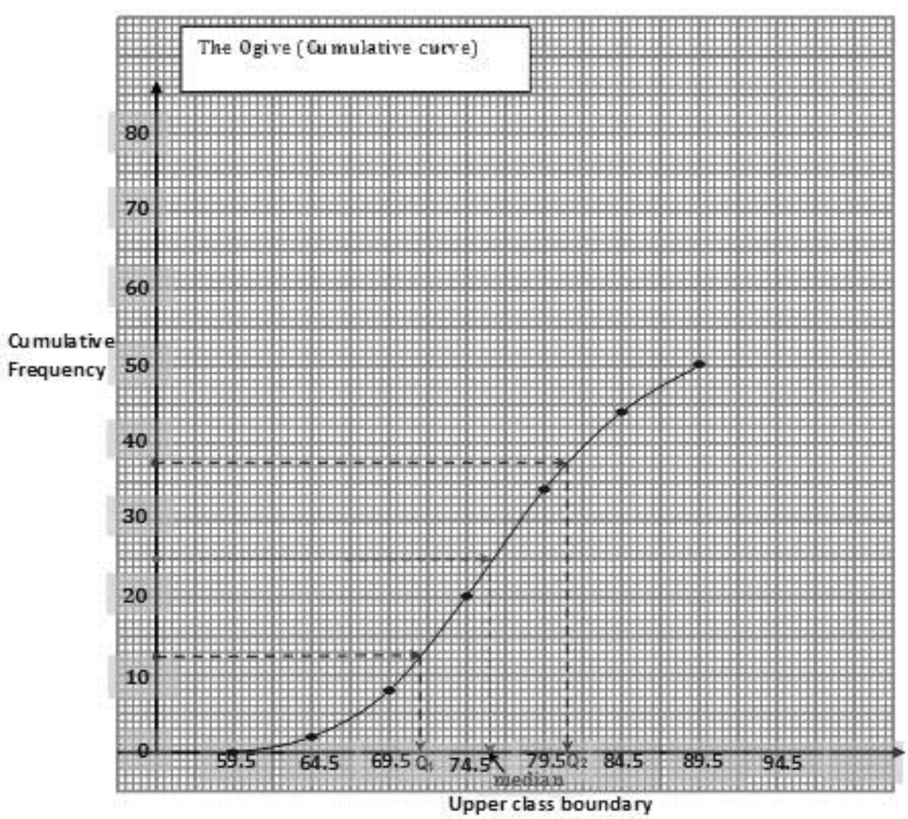
i) Median

ii) Upper and lower quartiles and hence calculate the inter quartile range.

Solution

a)

Mass (Kg)	*Class boundaries*	*Frequency (f)*	*Cumulative frequency*
60 – 64	*59.5 – 64.5*	*2*	*2*
65 – 69	*64.5 – 69.5*	*6*	*8*
70 – 74	*69.5 – 74.5*	*12*	*20*
75 – 79	*74.5 – 79.5*	*14*	*34*
80 – 84	*79.5 – 84.5*	*10*	*44*
85 – 89	*84.5 – 89.5*	*6*	*50*



b) i) ***Median****:*

The median corresponds to 50% reading, i.e.

$$\frac{50}{100} \times total \ \ frequency = \frac{50}{100} \times 50 = 25$$

From the graph, median = 76.0kg

ii) ***Quartiles:***

The upper quartile Q_3 is the value that corresponds to 75% i.e. the value corresponding to;

$$\frac{75}{100} \times total \ \ frequency = \frac{75}{100} \times 50 = 37.5$$

From the graph; Q_3 = 81.0kg

The lower quartile Q_1 is the value that corresponds to 25% reading, i.e.

$$\frac{25}{100} \times total \ \ frequency = \frac{25}{100} \times 50 = 12.5$$

From the graph, Q_1 = 71.5kg

$$\therefore Inter - quartile \ range \ = \ Q_3 - Q_1 \ = \ 81.0 - 71.5 \ = \ \underline{\underline{9.5\,kg}}$$

Exercise

1. The table below shows the weights of workers in Kawempe division council.

52	36	76	51	62	67	70	50
45	49	54	58	53	74	64	56
50	80	70	57	64	64	43	78
84	71	85	72	78	46	42	75
81	72	69	49	66	48	65	88

a) Form a grouped frequency distribution table for the weight of workers using an interval of 10kg starting with 30 39.

b) Calculate the mean weight of the class.

c) Draw a histogram and use it to estimate the mode.

2. The table below shows the heights of 150 students in Layibi College who participated in inter house competition during a certain week.

Height	10 19	20 29	30 39	40 49	50 59	60 69	70 79
Number of students	30	16	24	32	28	12	8

i. Calculate the mean and modal height

ii. Plot an Ogive for the above data use it to estimate the median.

iii. Estimate the lower and upper quartiles from the graph.

3. At 5:30 am, a daily school bus leaves Wobulenzi town for Kampala. The times taken to cover the journey were recorded over a period of time and were recorded as shown in the table below.

a) Calculate the mean time

b) Draw an Ogive and use it to estimate the median.

Time (mins)	**Frequency**
80 84	10
85 89	15
90 94	35
95 99	40
100 104	28
105 109	15
110 114	4
115 119	2
120 124	1

4. The frequency distribution table below shows the weights of 100 patients from Mulago hospital measured to the nearest tenth.

Use the table above to:

Weight (kg)	10 14	15 19	20 24	25 29	30 34	35 39
Number of patients	5	9	12	18	25	6

i. Calculate the mean using assumed mean of 27.

ii. Calculate the median and modal weight.

iii. Draw a cumulative frequency curve to represent the information.

5. The table below shows the marks obtained by 250 students in chemistry test.

Marks	**Number of students**
44.0 47.9	3
48.0 51.9	1
52.0 55.9	17
56.0 59.9	50
60.0 63.9	45
64.0 67.9	46
68.0 71.9	23
72.0 75.9	9

Use the above table to calculate the:

i. Mean mark

ii. Median

iii. Modal mark

6 VECTORS

6.1 Definition

A vector is a quantity that has both direction and magnitude. The following are some examples of a vector quantity.

* Forces
* Velocity
* Acceleration
* Momentum, etc.

6.2 Representation of a vector

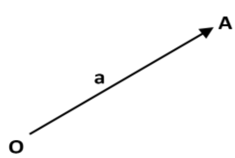
a) Graphical representation

Graphically a vector is represented using a straight line and an arrow on it.

The length of the straight line represents the magnitude of the vector and the arrow represents it direction.

b) Symbolic representation

Analytically, a vector is represented using a bolt letter or letter with an arrow or a bar above it, e.g. $\vec{a}$ or $\underset{\sim}{a}$ or a, $\overrightarrow{AO}$ or $\underset{\sim}{AO}$ or AO.



c) Specifying direction (sing convention)

The vector *OA* runs from O to A, therefore it is given a positive sign, i.e. OA = *a*, but the vector AO runs in the opposite direction to *OA*, it is therefore given a negative sign i.e. *AO* = -*a*.

6.3 Vector classification

Vector can be given as:

* Column vectors
* Displacement vector
* Position vectors

a) Column vector

Under equation of a straight line you learnt that a point in a plane can be specified using a number pair as (x, y), where *x* is the number of units along the *x-axis* from the origin and y is the number of units along the *y-axis* from the origin. These two numbers (x, y) are called Cartesian coordinate. The point **P**(x, y) can be represented as $\begin{pmatrix} x \\ y \end{pmatrix}$. Therefore, $P = \begin{pmatrix} x \\ y \end{pmatrix}$

This is known as column vector.

b) Displacement vectors

These vectors represent motion with respect to a fixed point. Consider points **A** (1, 2) and **B** (8, 6) in the figure below.

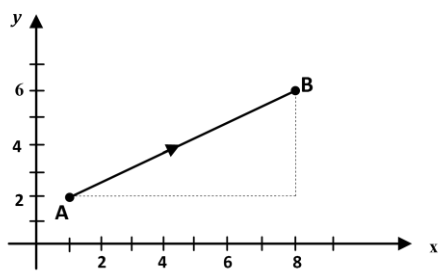


Fig1

Motion from **A** to **B** involves making (8-1) =7 steps parallel to the x-axis, followed by (6-2) = 4 steps parallel to the y-axis. The displacement vector **AB** is thus given by:

$$\vec{AB} = \begin{pmatrix} 8-1 \\ 6-2 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

c) Position vector

When the displacement is from the origin **O** (0, 0) to any point **P** (x, y), the corresponding vector **OP** is known as position vector.

The position vector of **P** therefore is: $\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$

Example

Describe the displacement made by the following vector in term of letters given:

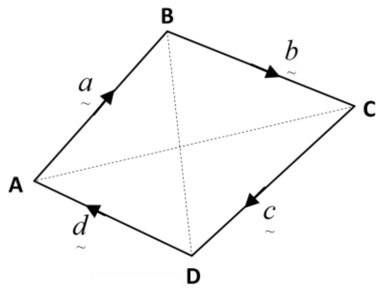


Fig 2

i. ***AB***

ii. ***BA***

iii. ***DA***

iv. ***AD***

v. ***BC***

vi. ***CB***

Solution

i. **AB =** $\underset{\sim}{a}$

ii. **BA = -AB =** $-\underset{\sim}{a}$

iii. **DA =** $\underset{\sim}{d}$

iv. **AD = -DA =** $-\underset{\sim}{d}$

v. **B C =** $\underset{\sim}{b}$

vi. **CB = -BC =** $-\underset{\sim}{b}$

6.4 Mid-point of a Vector

Consider vectors $\vec{OA}$ and $\vec{OB}$ with position vectors $\underset{\sim}{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ respectively.

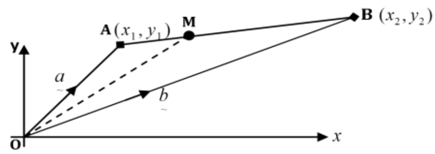


Fig.3

The midpoint of $AB = M\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$. Therefore the position vector

$\vec{OM}$ of the midpoint of vector $\vec{AB}$ is therefore given by:

$$\vec{OM} = \begin{pmatrix} \frac{x_2 + x_1}{2} \\ \frac{y_2 + y_1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x_2 + x_1 \\ y_2 + y_1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \frac{1}{2}\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{a}$$

$$\therefore \vec{OM} = \frac{1}{2}\left(\underset{\sim}{b} + \underset{\sim}{a}\right)$$

6.5 Operations on vectors

a) Vector addition

Vectors are added by adding the respective components.

Consider two vectors: $\underset{\sim}{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$.

$$\therefore \underset{\sim}{a} + \underset{\sim}{b} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$$

b) Vector subtraction

Vectors are subtracted by subtracting respective components.

Let $\underset{\sim}{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, then:

$$\therefore \underset{\sim}{a} - \underset{\sim}{b} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}$$

c) Multiplying a vector by a scalar

Let $\underset{\sim}{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and k be a scalar quantity, then:

$$k\underset{\sim}{a} = k\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} kx_1 \\ ky_1 \end{pmatrix}$$

Here we see that when we multiply a scalar by a vector, we still obtain a vector quantity.

Example

Given that $\underset{\sim}{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \underset{\sim}{b} = \begin{pmatrix} ^-4 \\ ^-1 \end{pmatrix}, and\ \underset{\sim}{c} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$.

Find:

Solution

i. $\underset{\sim}{a} + \underset{\sim}{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} ^-4 \\ ^-1 \end{pmatrix} = \begin{pmatrix} 3+^-4 \\ 2+^-1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} ^-1 \\ 1 \end{pmatrix}}}$

ii. $\underset{\sim}{a} - \underset{\sim}{c} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 3-0 \\ 2-5 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 3 \\ ^-2 \end{pmatrix}}}$

iii. $\underset{\sim}{b} - \underset{\sim}{c} + \underset{\sim}{a} = \begin{pmatrix} ^-4 \\ ^-1 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} ^-4-0+3 \\ ^-1-5+2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} ^-1 \\ ^-4 \end{pmatrix}}}$

iv. $4\underset{\sim}{a} - \underset{\sim}{b} = 4\begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} ^-4 \\ ^-1 \end{pmatrix} = \begin{pmatrix} 12-^-4 \\ 8-^-1 \end{pmatrix} = \begin{pmatrix} 12+4 \\ 8+1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 16 \\ 9 \end{pmatrix}}}$

6.6 Magnitude of a vector

If vector $\underset{\sim}{a} = \begin{pmatrix} x \\ y \end{pmatrix}$, then magnitude of $\underset{\sim}{a}$ is denoted by $|\underset{\sim}{a}|$ and is defined as:

$$|\underset{\sim}{a}| = \sqrt{x^2 + y^2}$$

This is the same as the length of the line of vector ***OA***. Graphically,

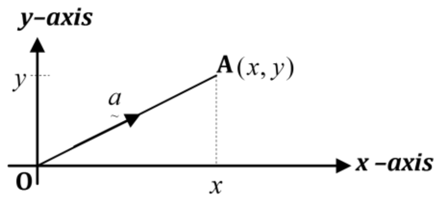


Fig.4

By Pythagoras theorem, the length of **OA** is denoted by $|\overrightarrow{OA}| = |\underset{\sim}{a}|$ and it is given by:

$$|a| = \sqrt{x^2 + y^2}$$

Example

Given that $\overrightarrow{OP} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}, and, \overrightarrow{PQ} = \begin{pmatrix} 5 \\ ^-1 \end{pmatrix}$, where O is the origin.

Determine:

i. The position vector of **Q**

ii. $|\overrightarrow{OQ}|$

Solution

i. $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} \Rightarrow \overrightarrow{OQ} = \overrightarrow{PQ} + \overrightarrow{OP}$

$$\Rightarrow \overrightarrow{OQ} = \begin{pmatrix} 5 \\ ^{-}1 \end{pmatrix} + \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 5+3 \\ 7-1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 8 \\ 6 \end{pmatrix}}}$$

ii. $\left|\overrightarrow{OQ}\right| = \sqrt{x^2 + y^2} = \sqrt{8^2 + 6^2} = \sqrt{100} = \underline{\underline{10}}$

6.7 Equality of vectors:

If vectors $\underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, and\ \underset{\sim}{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, then if $\underset{\sim}{a} = \underset{\sim}{b}$ it follows that, $a_1 = b_1$ and $a_2 = b_2$

Example

Given that $\underset{\sim}{a} = \begin{pmatrix} 2+3x \\ 4 \end{pmatrix}, and\ \underset{\sim}{b} = \begin{pmatrix} 6-4x \\ 8-2y \end{pmatrix}$. Solve for x and y if $\underset{\sim}{a} = \underset{\sim}{b}$.

Solution

If $\underset{\sim}{a} = \underset{\sim}{b}$ *then:*

$$\begin{pmatrix} 2+3x \\ 4 \end{pmatrix} = \begin{pmatrix} 6-4x \\ 8-2y \end{pmatrix}$$

$$\Rightarrow 2 + 3x = 6 - 4x, and, 4 = 8 - 2y$$

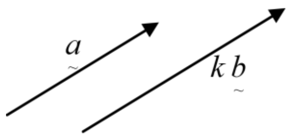
$$\therefore \underline{\underline{x = 4/7}}, and, \underline{\underline{y = 1/2}}$$

6.8 Parallel vectors:

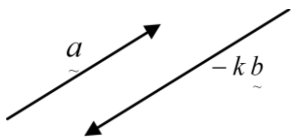
Two lines are said to be parallel if they do not meet and run in the same direction. Vectorally, we say that two vectors are parallel only if one can be expressed as a scalar multiple of the other. I.e. if vector $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$ then

$$\underset{\sim}{a} = k\,\underset{\sim}{b}$$

Here $\underset{\sim}{a}$ has been expressed as a scalar product of $\underset{\sim}{b}$. Diagrammatically, they appear as below.



If k is negative, then it reverses the direction of the vector ***kb*** but still this vector is parallel to vector ***a***. i.e.



Example

Given that P = (1, 1), Q = (3, 4), R = (8, 5) and S = (6, 2). Show that ***PQ*** and ***SR*** are parallel and deduce that P, Q, R, and S are vertices of a parallelogram.

Solution

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix}3\\4\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$$

$$\overrightarrow{SR} = \overrightarrow{OR} - \overrightarrow{OS} = \begin{pmatrix}8\\5\end{pmatrix} - \begin{pmatrix}6\\2\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$$

Clearly $\overrightarrow{PQ} = 1\times\begin{pmatrix}2\\3\end{pmatrix} = 1\times\overrightarrow{SR}$, *where 1 is a scalar.*

$$\Rightarrow \overrightarrow{PQ} \text{ is parallel to } \overrightarrow{SR}$$

To deduce that PQRS *is a parallelogram, we have to show that* $\overrightarrow{PS}$ *is also parallel to* $\overrightarrow{QR}$ *and that* $\left|\overrightarrow{PS}\right| = \left|\overrightarrow{QR}\right|$

$$\overrightarrow{PS} = \overrightarrow{OS} - \overrightarrow{OP} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = \begin{pmatrix} 8 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$\therefore \overrightarrow{PS} = \overrightarrow{QR} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \Rightarrow \overrightarrow{PS} \text{ is parallel to } \overrightarrow{QR}$$

$$\left|\overrightarrow{PS}\right| = \sqrt{5^2 + 1^2} = \sqrt{26}$$

$$\left|\overrightarrow{QR}\right| = \sqrt{5^2 + 1^2} = \sqrt{26}$$

Since $\overrightarrow{PS} = \overrightarrow{QR}$ *and* $\left|\overrightarrow{PS}\right| = \left|\overrightarrow{QR}\right|$ *, PQRS is a parallelogram.*

6.9 Collinear points

6.9.1 Definition:

These are points that lie on the same line. The idea of parallel vectors may be used to test if any *three* given points are collinear. Consider three points A, B and C. to show that points A, B, and C are collinear, we have to show that vector *AB* and *AC* are parallel. If *AB* and *AC* are parallel and are taken with respect to a common point A, then the points A, B and C are collinear.

Example

The coordinates of P, Q and R are (1, 2), (9, 2) and (5, 2) respectively. Find *PQ* and *QR* and show that P, Q and R are collinear.

Solution

If P, Q, and R are collinear, then $\overrightarrow{PQ} = k\,\overrightarrow{PR}$

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = -\overrightarrow{OP} + \overrightarrow{OQ},\ \overrightarrow{OP} = \begin{pmatrix}1\\2\end{pmatrix}, \overrightarrow{OQ} = \begin{pmatrix}9\\2\end{pmatrix}$$

$$= \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= \begin{pmatrix}9\\2\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix}$$

$$= \begin{pmatrix}8\\0\end{pmatrix}$$

$$\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR} = -\overrightarrow{OP} + \overrightarrow{OR},\ \overrightarrow{OP} = \begin{pmatrix}1\\2\end{pmatrix}, \overrightarrow{OR} = \begin{pmatrix}5\\2\end{pmatrix}$$

$$= \overrightarrow{OR} - \overrightarrow{OP}$$

$$= \begin{pmatrix}5\\2\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix}$$

$$= \begin{pmatrix}4\\0\end{pmatrix}$$

$$\overrightarrow{PQ} = \begin{pmatrix}8\\0\end{pmatrix} = 2\begin{pmatrix}4\\0\end{pmatrix},\ but \begin{pmatrix}4\\0\end{pmatrix} = \overrightarrow{PR}$$

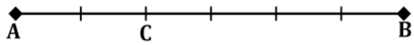
$$\overrightarrow{PQ} = 2\,\overrightarrow{PR} \quad where \quad k = 2$$

Therefore, P, Q *and* R *are collinear points.*

6.9.2 Proportional division of a line

a) Internal division of a line:

Consider the line segment AB below divided into six equal parts.



Point **C** is such that it is 2 units from **A** and it is 4 units from **B**. The ratio of ***AC*** to ***BC*** is equal to 2 to 4, i.e.

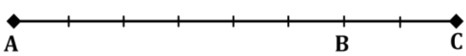
$$AC : CB = 2 : 4$$
$$\frac{AC}{CB} = \frac{2}{4} = \frac{1}{2}$$
$$\therefore AC : CB = 1 : 2$$

Point **C** is therefore said to divide ***AB*** internally in the ratio1: 2.

Generally a point **P** is said to divide a line segment ***AB*** internally in the ratio $s : t$ if the point **P** lies between points **A** and **B** such that $\frac{AP}{PB} = \frac{s}{t}$.

b) External division:

Consider the figure below where AB is produced to point C



Point C is outside the internal AB, i.e. it is external. *AB* is divided up into six equal units and *BC* is equal to 3 of these. Taking the direction from *AB* as positive, then C to B is negative. Thus

$$AC : CB = 9 : -3$$
$$\frac{AC}{CB} = \frac{9}{^{-}3} = \frac{3}{^{-}1}$$
$$\therefore AC : CB = 3 : -1$$

In this case, we say that C divides *AB* externally in the ratio 3:1 or simply C divides *AB* in the ration 3: -1.

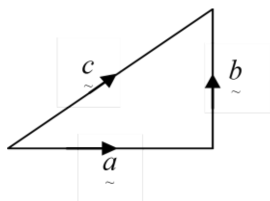
Generally if point P lies on *AB* produced and that $AP : PB = s : -t$ we say that P divides *AB* externally in the ratio $s : t$.

6.9.3 Vector Geometry

Here we shall extend the concept we have learnt so far to the combination of vectors geometrically.

Addition:

If ***a***, ***b*** and ***c*** are the displacement vectors ***a*** followed by ***b*** are equivalent to ***c***, i.e.

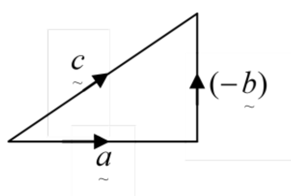


Then:

$$\underset{\sim}{c} = \underset{\sim}{a} + \underset{\sim}{b}$$

Subtraction:

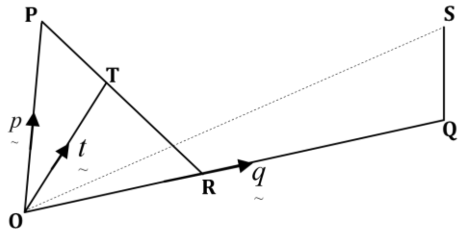
If we replace $\underset{\sim}{b}$ by $(-\underset{\sim}{b})$ i.e.



Then: $\underset{\sim}{c} + (^{-}\underset{\sim}{b}) = \underset{\sim}{a}$

$$\underset{\sim}{c} - \underset{\sim}{b} = \underset{\sim}{a}$$

Example



In the diagram above, $\overrightarrow{OP} = \underset{\sim}{p}, \overrightarrow{OQ} = \underset{\sim}{q}, and\ \overrightarrow{OT} = \underset{\sim}{t}$. **R** and **T** are the mid-points of $\overrightarrow{OQ}\ and\ \overrightarrow{PR}$ respectively.

a) Express ***t*** in terms of $\underset{\sim}{p}\ and\ \underset{\sim}{q}$.

b) Given that $\overrightarrow{OP}$ is parallel to $\overrightarrow{QS}$ such that $\overrightarrow{OP} = 2\overrightarrow{QS}$, find $\overrightarrow{QS}$ in terms of $\underset{\sim}{p}\ and\ \underset{\sim}{q}$.

c) Taking **O** as the origin, and **P** (0, 8) and **Q** (6, 4), determine the lengths of $\overrightarrow{OS}\ and\ \overrightarrow{PS}$.

Solution

$$\overrightarrow{OR} = \overrightarrow{RQ} = \frac{1}{2}\overrightarrow{OQ} = \frac{1}{2}\underset{\sim}{q}$$

R and T are midpoints of $\overrightarrow{OQ}\ and\ \overrightarrow{PR}$ *respectively*

a) $\overrightarrow{OT} = \overrightarrow{OP} + \overrightarrow{PT} = \underset{\sim}{t}$

$$but\ \overrightarrow{PT} = \frac{1}{2}\overrightarrow{PR}\ and\ \overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR} = \frac{1}{2}\underset{\sim}{q} - \underset{\sim}{p}$$

$$\therefore PT = \frac{1}{2}\left[\frac{1}{2}\underset{\sim}{q} - \underset{\sim}{p}\right]$$

$$\therefore \underset{\sim}{t} = \underset{\sim}{p} + \frac{1}{2}\left[\frac{1}{2}\underset{\sim}{q} - \underset{\sim}{p}\right] = \underset{\sim}{p} + \frac{1}{4}\underset{\sim}{q} - \frac{1}{2}\underset{\sim}{p}$$

$$\therefore \underline{\underline{\underset{\sim}{t} = \frac{1}{2}\underset{\sim}{p} + \frac{1}{4}\underset{\sim}{q}}}$$

b) $\overrightarrow{OP} = 2\overrightarrow{QS} \Rightarrow \overrightarrow{QS} = \frac{1}{2}\overrightarrow{OP} = \frac{1}{2}\underset{\sim}{p}$

$$Also\ \overrightarrow{QS} = \overrightarrow{QO} + \overrightarrow{OS} = -\overrightarrow{OQ} + \overrightarrow{OS}$$

$$\Rightarrow \overrightarrow{OS} = \overrightarrow{QS} + \overrightarrow{OQ}$$

$$\therefore \overrightarrow{OS} = \frac{1}{2}\underset{\sim}{p} + \underset{\sim}{q}$$

c) ***O** as the origin implies that **O** (0, 0), **P** (0, 9) and **Q** (6, 4). If **O** is the origin then;*

$$\underset{\sim}{p} = \overrightarrow{OP} = \begin{pmatrix} 0 \\ 8 \end{pmatrix} and\ \underset{\sim}{q} = \overrightarrow{OQ} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$\overrightarrow{OS} = \frac{1}{2}\underset{\sim}{p} + \underset{\sim}{q} = \frac{1}{2}\begin{pmatrix} 0 \\ 8 \end{pmatrix} + \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 6 \\ 4 \end{pmatrix} \therefore \overrightarrow{OS} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

$$Length, of\ \overrightarrow{OS} = \left|\overrightarrow{OS}\right| = \sqrt{6^2 + 8^2} = \sqrt{100} = \underline{\underline{10 units}}$$

For PS:

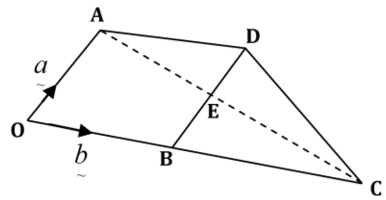
$$\overrightarrow{PS} = \overrightarrow{PO} + \overrightarrow{OS} = -\overrightarrow{OP} + \overrightarrow{OS}$$

$$\Rightarrow \overrightarrow{PS} = \overrightarrow{OS} - \overrightarrow{OP} = \frac{1}{2}\underset{\sim}{p} + \underset{\sim}{q} = \underset{\sim}{q} - \frac{1}{2}\underset{\sim}{p}$$

$$\therefore \overrightarrow{PS} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ 8 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

$$Length\ of\ \overrightarrow{PS} = \left|\overrightarrow{PS}\right| = \sqrt{6^2 + 0^2} = \sqrt{36} = \underline{\underline{6 units}}$$

Example



In the figure above **AD** is parallel to **OB** and **OA** is parallel to **BD**. 3**OC** = 5**OB**. E is the point where **AC** meets **BD. AC**: **EC** = 3: 2

Find:

i. in terms of the vectors ***a*** and ***b***, the vectors ***AC***, ***DC***, ***ED***, ***AE*** and **OE**.

ii. the ratio ***BE***: ***ED***

Solution

Using the above diagram: $\overrightarrow{OA} = \underset{\sim}{a}, \overrightarrow{OB} = \underset{\sim}{b}$

$$3\overrightarrow{OC} = 5\overrightarrow{OB}$$

$$\Rightarrow \overrightarrow{OC} = \frac{5}{3}\overrightarrow{OB} = \frac{5}{3}\underset{\sim}{b}$$

$$AE : EC = 3:2$$

$$\Rightarrow \frac{AE}{EC} = \frac{3}{2} \Rightarrow 2AE = 3EC$$

$$\overrightarrow{AE} + \overrightarrow{EC} = \overrightarrow{AC}, but\ \overrightarrow{AE} = \frac{3}{2}\overrightarrow{EC}$$

$$\therefore \overrightarrow{AE} = \frac{2}{5}\overrightarrow{AC}$$

i. *In terms of* $\underset{\sim}{a}$ *and* $\underset{\sim}{b}$:

<u>**For AC:**</u>

$$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{OC} - \overrightarrow{OA}$$

$$\therefore \overrightarrow{AC} = \underline{\underline{\frac{5}{3}\underset{\sim}{b} - \underset{\sim}{a}}}$$

<u>For DC</u>:

$$\vec{DC} = \vec{DB} + \vec{BC}$$

Since AD is parallel to OC and OA is parallel to BD, this implies that:

$$\vec{AD} = \vec{OB} = \underset{\sim}{b}$$

$$\vec{OA} = \vec{BD} = \underset{\sim}{a} \Rightarrow \vec{DB} = {}^{-}\underset{\sim}{a}$$

$$Also: \vec{OC} = \vec{OB} + \vec{BC}$$

$$\Rightarrow \vec{BC} = \vec{OC} - \vec{OB}, but\ \vec{OC} = \frac{5}{3}\underset{\sim}{b}$$

$$\therefore \vec{BC} = \frac{5}{3}\underset{\sim}{b} - \underset{\sim}{b} = \frac{2}{3}\underset{\sim}{b}$$

$$so: \vec{DC} = \vec{DB} + \vec{BC} = -\underset{\sim}{a} + \frac{2}{3}\underset{\sim}{b}$$

$$\therefore \vec{DC} = \frac{2}{3}\underset{\sim}{b} - \underset{\sim}{a}$$

<u>For ED</u>:

$$\vec{ED} = \vec{EA} + \vec{AD}$$

$$= \frac{3}{5}\vec{AC}$$

$$= \frac{-3}{5}\vec{AC}, but\ \vec{AC} = \frac{5}{3}\underset{\sim}{b} - \underset{\sim}{a}$$

$$\frac{-3}{5}\left(\frac{5}{3}\underset{\sim}{b} - \underset{\sim}{a}\right) = \frac{3}{5}\underset{\sim}{a} - \underset{\sim}{b}, and\ \vec{AD} = \underset{\sim}{b}$$

$$\Rightarrow \vec{ED} = \frac{3}{5}\underset{\sim}{a} - \underset{\sim}{b} + \underset{\sim}{b}$$

$$\therefore \vec{ED} = \frac{3}{5}\underset{\sim}{a}$$

***<u>For AE</u>*:**

$$\vec{AE} = \frac{3}{5}\vec{AC} = \frac{3}{5}\left(\frac{5}{3}\underset{\sim}{b} - \underset{\sim}{a}\right)$$

$$\therefore \vec{AE} = \underset{\sim}{b} - \frac{3}{5}\underset{\sim}{a}$$

*<u>For **OC**</u>:*

$$\vec{OE} = \vec{OA} + \vec{AE} = \underset{\sim}{a} + \underset{\sim}{b} - \frac{3}{5}\underset{\sim}{a} = \frac{5\underset{\sim}{a} - 3\underset{\sim}{a} + 5\underset{\sim}{b}}{5}$$

$$\therefore \underline{\underline{\vec{OE} = \frac{1}{5}\left(2\underset{\sim}{a} + 5\underset{\sim}{b}\right)}}$$

ii. *Ratio* ***BE:ED***

$$We\ have\ \underset{\sim}{ED} = \frac{3}{5}\underset{\sim}{a}$$

$$and\ \vec{BD} = \vec{BE} + \vec{ED} \Rightarrow \underset{\sim}{a} = \vec{BE} + \frac{3}{5}\underset{\sim}{a}$$

$$\Rightarrow \vec{BE} = \underset{\sim}{a} - \frac{3}{5}\underset{\sim}{a} = \frac{2}{5}\underset{\sim}{a}$$

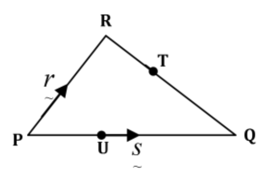
$$\therefore \vec{BD} = \underset{\sim}{a} \quad and \quad \vec{ED} = \frac{3}{5}\underset{\sim}{a}$$

$$So\ \frac{BE}{ED} = \frac{\frac{2}{5}\cancel{\underset{\sim}{a}}}{\frac{3}{5}\cancel{\underset{\sim}{a}}} = \frac{2/5}{3/5} = \frac{2}{3}$$

$$\Rightarrow \underline{\underline{BE : ED = 2:3}}$$

Example

In the figure below, $\vec{PQ} = \underset{\sim}{s}, \vec{PR} = \underset{\sim}{r}, 2\vec{QT} = 3\vec{TR}\ and PU : UQ = 2:3$



a) Find in terms of vectors $\underset{\sim}{r}\ and\ \underset{\sim}{s}$ the vectors:

i. **QR**
ii. **QT**
iii. **PT**

b) Show that UT is parallel to PR.

Solution

$$\underset{\sim}{PQ} = \underset{\sim}{s}, \ and \ \underset{\sim}{PR} = \underset{\sim}{r}$$

$$PU : UQ = 2 : 3. \quad Total \ ratio = 5$$

$$\Rightarrow PU = \tfrac{2}{5} PQ, \ and \ \ UQ = \tfrac{3}{5} \underset{\sim}{PQ}$$

$$= \frac{2}{5}\underset{\sim}{s} \qquad\qquad = \frac{3}{5}\underset{\sim}{s}$$

$$Also \ 2\overrightarrow{QT} = 3\overrightarrow{TR} \Rightarrow \frac{QT}{TR} = \frac{3}{2}$$

$$\therefore QT : TR = 3 : 2. \quad Total \ ratio = 5$$

$$\Rightarrow \underset{\sim}{QT} = \frac{3}{5}\underset{\sim}{QR}, \ and \ \underset{\sim}{QT} + \underset{\sim}{TR} = \underset{\sim}{QR} \Rightarrow \underset{\sim}{TR} = \frac{2}{5}\underset{\sim}{QR}$$

a) Now in terms of the vectors $\underset{\sim}{r}$ *and* $\underset{\sim}{s}$

i.
$$\underset{\sim}{QR} = \underset{\sim}{QP} + \underset{\sim}{PR}$$
$$= -\underset{\sim}{s} + \underset{\sim}{r}$$
$$= \underline{\underline{\underset{\sim}{r} - \underset{\sim}{s}}}$$

ii.
$$\underset{\sim}{QT} = \frac{3}{5}\underset{\sim}{QR}$$
$$= \underline{\underline{\frac{3}{5}\left(\underset{\sim}{r} - \underset{\sim}{s}\right)}}$$

iii.
$$\underset{\sim}{PT} = \underset{\sim}{PQ} + \underset{\sim}{QT}$$
$$= \underset{\sim}{s} + \frac{3}{5}\left(\underset{\sim}{r} - \underset{\sim}{s}\right)$$
$$= \frac{5\underset{\sim}{s} + 3\left(\underset{\sim}{r} - \underset{\sim}{s}\right)}{5}$$
$$= \frac{2\underset{\sim}{s} + 3\underset{\sim}{r}}{5}$$
$$= \underline{\underline{\frac{1}{5}\left(2\underset{\sim}{s} + 3\underset{\sim}{r}\right)}}$$

b) *If* $\underset{\sim}{UT}$ *is parallel to* $\underset{\sim}{PR}$ *then* $\underset{\sim}{UT} = k\,\underset{\sim}{PR}$ *where* k *is a scalar.*

$$\underset{\sim}{UT} = \underset{\sim}{UQ} + \underset{\sim}{QT} \quad but\ \underset{\sim}{UQ} = \frac{3}{5}\underset{\sim}{s}\ qnd\ \ \underset{\sim}{QT} = \frac{3}{5}\left(\underset{\sim}{r} - \underset{\sim}{s}\right)$$

$$\Rightarrow \underset{\sim}{UT} = \frac{3}{5}\underset{\sim}{s} + \frac{3}{5}\left(\underset{\sim}{r} - \underset{\sim}{s}\right) = \frac{3s + 3r - 3s}{5}$$

$$\therefore \underset{\sim}{UT} = \frac{3}{5}\underset{\sim}{r} \quad but\ \underset{\sim}{r} = \underset{\sim}{PR}$$

$$\Rightarrow \underset{\sim}{UT} = \frac{3}{5}\underset{\sim}{PR},\ where\ k = \frac{3}{5}$$

$$Hence\ \underset{\sim}{UT}\ is\ \ parallel\ \ to\ \ \underset{\sim}{PR}$$

6.9.4 Miscellaneous exercise

1. a) Find the value of x and y if $\begin{pmatrix} -9 \\ -3 \end{pmatrix} - 3\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

   b) Given that the vectors $\underset{\sim}{p} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, and\ \underset{\sim}{q} = \begin{pmatrix} 3 \\ ^{-}1 \end{pmatrix}$

   Find t if $t\left|\underset{\sim}{p}\right| = \left|\underset{\sim}{q}\right|$ where t is a scalar.

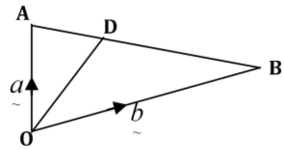
   c) Given that $\underset{\sim}{a} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \underset{\sim}{b} = \begin{pmatrix} 4 \\ ^{-}8 \end{pmatrix}, and\ \underset{\sim}{c} = \begin{pmatrix} 7 \\ ^{-}5 \end{pmatrix}$.

   Find:

   i. $\frac{1}{2}\left(\underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{a}\right)$

   ii. $3\left(\underset{\sim}{b} - \underset{\sim}{c} - \underset{\sim}{a}\right)$

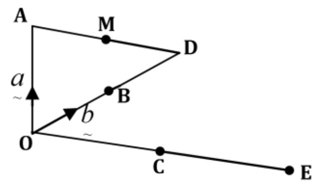
2.



In the diagram above $\vec{OA} = \underset{\sim}{a}, \vec{OB} = \underset{\sim}{b}; 3\vec{AD} = \vec{AB}$.

Find $\vec{OD}$ in terms of $\underset{\sim}{a}$ *and* $\underset{\sim}{b}$.

3.



In the figure above**,** $\vec{OA} = \underset{\sim}{a}, \vec{OB} = \underset{\sim}{b}$, *and* $3\vec{OB} = 2\vec{BD}$; M is the point on ***AD*** such that $MD : AM = 1:2, OC = 3CE = 3AM$.

i. Express the vectors ***AD***, ***BM*** and ***DC*** in terms of $\underset{\sim}{a}$ *and* $\underset{\sim}{b}$.

ii. Show that $\vec{AD} : \vec{OC} = 3:8$.

4. In a triangle **OPQ**, **X** is a point such that $\vec{OX} = \frac{2}{3}\vec{OP}$ and **Y** is the midpoint of ***PQ***.

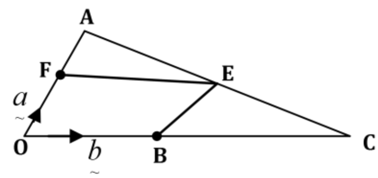
The point Z on ***OQ*** is such that $\vec{OQ} = \vec{QZ}$. Given that $\vec{OP} = \underset{\sim}{p}$ *and* $\vec{OQ} = \underset{\sim}{q}$.

a) Determine in terms of $\underset{\sim}{p}$ *and* $\underset{\sim}{q}$, the vector

i. **OX**
ii. **OY**
iii. **OZ**
iv. **XY**
v. **YZ**

b) Hence or otherwise that **X**, **Y** and **Z** lie on a straight line. State the ratio of the lengths ***XY*** and ***YZ***.

5. In the diagram below, $\overrightarrow{OA} = \underset{\sim}{a}, \overrightarrow{OB} = \underset{\sim}{b}, \overrightarrow{OB} : \overrightarrow{BD} = 1:3, 3\overrightarrow{OF} = 2\overrightarrow{OA}$ and E divides ***AC*** in the ratio 3:2.



Express the following vectors in terms of $\underset{\sim}{a}$ *and* $\underset{\sim}{b}$.

i. **BC**

ii. **CA**

iii. **BE**

iv. **FE**

7 MATRICES

7.1 Introduction:

Consider the information given below in the post-primary football tournament last season in Kampala district. The result for three schools, Kakungulu Memorial School, Kibuli SS, were as shown below.

School	**P**	**W**	**D**	**L**
Kakungulu M.S	10	4	4	2
Kibuli SS	8	6	1	1
St Peter S.S	8	2	3	3

The above information can shortly be written as

$$\begin{pmatrix} 10 & 4 & 4 & 2 \\ 8 & 6 & 1 & 1 \\ 8 & 2 & 3 & 3 \end{pmatrix}$$

The numbers above are arranged in a rectangular form. Such an arrangement of numbers is what is known as matrix.

Definition

A matrix is arrays of numbers in rectangular form with large brackets around them.

OR:

A matrix is a collection of information (numbers) stored in rows and columns.

7.2 Common terms used

Below are some of the frequently used terms:

1. Entry (an element)

This is a number within the matrix. At times it is known as component. Consider the matrix below.

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

The numbers: 2, 3, 4 and 5are the elements of the above matrix.

2. Rows of a matrix $(\rightarrow)$

These are the lines of numbers that goes across the page. Considering the above matrix i.e.

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

(2 3) forms the first row and (4 5) forms another row. Therefore, the matrix above has two rows.

3. Columns of a matrix $(\downarrow)$

These are the lines of numbers that go down the page. Considering $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ forms the first column and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ forms another column.

So the above matrix has two columns also.

NB:

A matrix is represented with upper case letters. In identifying matrix, one has to use the position of row $(\rightarrow)$ and column $(\downarrow)$. For example:

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then; ***a*** is element of 1^{st} row and 1^{st} column

b is element of 1^{st} row and 2^{nd} column

c is element of 2^{nd} row and 1^{st} column

d is element of 2^{nd} row and 2^{nd} column.

4. Order of a matrix

This refers to the number of rows and columns in a given matrix and it is given by

$Order = Row \times Column$ Consider the matrix below:

$$A = \begin{array}{cc} & \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \end{array}$$

The matrix above has 2 rows and 2 columns. Therefore the order of the above matrix above is 2×2.

Example

State the order of the following given matrices.

a. $\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$, is a 2×2 matrix 2 rows and 2 columns

b. $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$, is a 2×3matrix i.e. 2 rows and 3 columns

c. $\begin{pmatrix} 3 & 5 \\ 1 & ^{-}2 \\ 1 & 13 \end{pmatrix}$, is a 3×2 matrix i.e. 3 rows and 2 columns

d. $(1 \quad 0)$, is 1×2 a matrix i.e. 1 row and 2 columns

e. $\begin{pmatrix} 4 \\ ^{-}3 \\ 21 \\ ^{-}7 \\ 62 \end{pmatrix}$, is a 5×1matrix i.e. 5 rows and 1column.

NB:

The number of rows is denoted by **m** and column by **n**. When stating the order of the matrix, the number of rows is written (stated) first. This is followed by the number of columns. I.e.

$$Order = m \times n$$

5. Leading diagonal (major diagonal)

This is a line of numbers that runs diagonally from the top left-hand corner to the bottom right-hand corner for the matrix with equal numbers of rows and columns (i.e. a square matrix). Consider the matrix below.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & ^{-}1 \end{pmatrix}$$

This matrix has the same number of rows and columns. It is a 3×3 matrix. Its leading diagonal is what has been enclosed in the loop below.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & ^{-}1 \end{pmatrix}$$

1, 5 and -1 are the entries of the leading diagonal.

6. Minor diagonal

This is a line of numbers that runs diagonally from the bottom left-hand corner to the top right-hand corner. For the matrix above its minor diagonal is what has been enclosed in the loop.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & ^{-}1 \end{pmatrix}$$

1, 5 and 3 are the entries of the minor diagonal.

7.3 Types of matrices

1. Column matrix

   This is a matrix with only one column. E.g.

$$A = \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix}$$

2. Row matrix

   This is a matrix with only one row. E.g.

$$C = \begin{pmatrix} 1 & 0 & 4 \end{pmatrix}$$

3. Zero matrix

   This is a matrix all its elements zeros. E.g.

$$X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

4. Equal matrix

   Two or more matrices are equal if and only if their corresponding elements are equal and are of same order.

Example

Given matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 10 \\ 6 & ^{-}4 \end{pmatrix}$ Therefore if matrix A is equal to matrix B, then;

$$a = 4, b = 10$$
$$c = 6, d = -4$$

5. Square matrix

   This is a matrix with equal number of rows and columns. E.g.

$$A = \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix} Order = 2 \times 2$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 4 & 6 \\ 2 & 5 & 4 \end{bmatrix} Order = 3 \times 3$$

6. Identity matrix

   This is a square matrix with 1 as an element in the leading (major) diagonal and zeros elsewhere. At times, it is known as unit matrix and it is denoted by the letter **I**.

   Some few examples of identity matrix include the following:

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which is a 2×2 unit matrix

$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which is a 3×3 unit matrix

$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, which is 4×4 unit matrix

Example

1. Three sales girls sold the following numbers of bottles of lotion on a certain day.

   * Liz sold 9 bottles of Dear heart, 13 of Razac and 6 of Venus.
   * Suzi sold 8 bottles of Movit lotion, 7 of Razac and 10 of Venus.

∗ Aber sold 15 bottles of Movit lotion, 1 of Dear heart and 18 of Razac.

Show this information in a 3x4 matrix.

Solution

$$\begin{array}{c} \\ Liz \\ Suzi \\ Aber \end{array} \begin{array}{c} \begin{array}{cccc} DearHart & Razac & Venus & Movit \end{array} \\ \begin{bmatrix} 9 & 13 & 6 & 0 \\ 0 & 7 & 10 & 8 \\ 1 & 18 & 0 & 15 \end{bmatrix} \end{array}$$

The above matrix can shortly be written as

$$\begin{pmatrix} 9 & 13 & 6 & 0 \\ 0 & 7 & 10 & 8 \\ 1 & 18 & 0 & 15 \end{pmatrix}$$

2. The table below shows the number of times that three couples attended various types of entertainment in one year.

Type of entertainment	**Couple**		
	The	The	The
Cinema	7	2	5
Dance	1	2	9
Play	5	8	1
Circus	0	3	2

a) Write down the information in the table in the form of a matrix and state the order of the matrix

b) s the order of this matrix?

c) Write as a row matrix, the number of times the plays have been attended, and state the order of the matrix.

Solution

a) $\begin{pmatrix} 7 & 2 & 5 \\ 1 & 2 & 9 \\ 5 & 8 & 1 \\ 0 & 3 & 2 \end{pmatrix}$ The order is 4×3

b) $\begin{pmatrix} 2 \\ 2 \\ 8 \\ 3 \end{pmatrix}$ The order is 4×1

c) $\begin{pmatrix} 5 & 8 & 1 \end{pmatrix}$ *The order is* 1×3

7.4 Matrix Algebra

1. Addition and subtraction of matrices

When adding two or more matrices, their corresponding elements (components) are added together. The method of subtraction follows the same pattern as that of addition.

NB:

- We can only add or subtract matrices if they are of the same order. Such matrices are said to be compatible for addition or subtraction.
- When the orders of the matrices are different, then addition or subtraction is impossible. Such matrices are said to be incompatible for addition or subtraction.

Example

Given matrices, $A = \begin{pmatrix} 2 & 3 & 1 \\ {}^{-}1 & 2 & 2 \\ 0 & 3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 2 & 2 \\ 1 & 1 & {}^{-}1 \end{pmatrix}$

Work out;

a) A+ B

b) A B

Solution

a) $$A+B=\begin{pmatrix}2&3&1\\{}^{-}1&2&2\\0&3&1\end{pmatrix}+\begin{pmatrix}0&2&4\\1&2&2\\1&1&-1\end{pmatrix}=\begin{pmatrix}2+0&3+2&1+4\\{}^{-}1+1&2+2&2+2\\0+1&3+1&1+({}^{-}1)\end{pmatrix}=\underline{\underline{\begin{pmatrix}2&5&5\\0&4&4\\1&4&0\end{pmatrix}}}$$

b) $$A-B=\begin{pmatrix}2&3&1\\{}^{-}1&2&2\\0&3&1\end{pmatrix}-\begin{pmatrix}0&2&4\\1&2&2\\1&1&-1\end{pmatrix}=\begin{pmatrix}2-0&3-2&1-4\\{}^{-}1-1&2-2&2-2\\0-1&3-1&1-({}^{-}1)\end{pmatrix}=\underline{\underline{\begin{pmatrix}2&1&{}^{-}3\\{}^{-}2&0&0\\{}^{-}1&2&2\end{pmatrix}}}$$

Example

Given that matrices C, D and E are:

$$C=\begin{pmatrix}4&9&1\\0&3&5\end{pmatrix}, D=\begin{pmatrix}2&{}^{-}4&4\\7&{}^{-}2&1\end{pmatrix}, E=\begin{pmatrix}5&0&{}^{-}2\\{}^{-}3&1&7\end{pmatrix}$$

Find:

a) $C+(D-E)$
b) $C+D+E$
c) $C-(D+E)$

Solution

a) $$C+(D-E)=\begin{pmatrix}4&9&1\\0&3&5\end{pmatrix}+\begin{pmatrix}2&{}^{-}4&4\\7&{}^{-}2&1\end{pmatrix}-\begin{pmatrix}5&0&{}^{-}2\\{}^{-}3&1&7\end{pmatrix}=\begin{pmatrix}4+2-5&9+({}^{-}4)-0&1+4-({}^{-}2)\\0+7-({}^{-}3)&3+({}^{-}2)-1&5+1-7\end{pmatrix}$$

$$\therefore C+D-E=\underline{\underline{\begin{pmatrix}1&5&7\\10&0&{}^{-}1\end{pmatrix}}}$$

b) $$C+D+E=\begin{pmatrix}4&9&1\\0&3&5\end{pmatrix}+\begin{pmatrix}2&{}^{-}4&4\\7&{}^{-}2&1\end{pmatrix}+\begin{pmatrix}5&0&{}^{-}2\\{}^{-}3&1&7\end{pmatrix}=\begin{pmatrix}4+2+5&9+({}^{-}4)+0&1+4+({}^{-}2)\\0+7+({}^{-}3)&3+({}^{-}2)+1&5+1+7\end{pmatrix}$$

$$\therefore C+D+E=\underline{\underline{\begin{pmatrix}11&5&3\\4&2&13\end{pmatrix}}}$$

c) $$C-(D+E)=C-D-E=\begin{pmatrix}4&9&1\\0&3&5\end{pmatrix}-\begin{pmatrix}2&{}^{-}4&4\\7&{}^{-}2&1\end{pmatrix}-\begin{pmatrix}5&0&{}^{-}2\\{}^{-}3&1&7\end{pmatrix}$$

$$\Rightarrow C-(D+D)=\begin{pmatrix}4-2-5&9-({}^{-}4)-0&1-4-({}^{-}2)\\0-7-({}^{-}3)&3-({}^{-}2)-1&5-1-7\end{pmatrix}$$

$$\therefore C-D-E=\underline{\underline{\begin{pmatrix}{}^{-}3&13&{}^{-}1\\{}^{-}4&4&{}^{-}3\end{pmatrix}}}$$

Example

Given that:

$$P = \begin{pmatrix} 4 & -6 \\ 0 & 10 \end{pmatrix}, Q = \begin{pmatrix} ^-3 & 5 \\ 1 & 0 \\ 7 & ^-2 \end{pmatrix}$$

Find;

i. $P + Q$

ii. $Q - P$

Solution

Here matrix addition and subtraction is impossible because the two matrices are of different orders.

7.5 Multiplication of matrices

a) Multiplication by a scalar (a number)

Consider the matrix $M = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. Then;

$$M + M + M + M + M = \begin{pmatrix} 1+1+1+1+1 & 2+2+2+2+2 \\ 2+2+2+2+2 & 4+4+4+4+4 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix}$$

This is the same as multiplying each entry of M by 5.ie

$$M + M + M + M + M = 5M = 5\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 5\times 1 & 5\times 2 \\ 5\times 2 & 5\times 4 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix}$$

In general therefore, to multiply a matrix by a real number (a scalar); we multiply each element in the matrix by the number.

Example

Given that matrix

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}, and, B = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}. \text{ Find } 5A + 4B.$$

Solution

$$5A+4B=5\begin{pmatrix}1&3&5\\2&4&6\end{pmatrix}+4\begin{pmatrix}3&2&1\\1&0&1\end{pmatrix}=\begin{pmatrix}5\times1&5\times3&5\times5\\5\times2&5\times4&5\times6\end{pmatrix}+\begin{pmatrix}4\times3&4\times2&4\times1\\4\times1&4\times0&4\times1\end{pmatrix}$$

$$5A+4B=\begin{pmatrix}5&15&25\\10&20&30\end{pmatrix}+\begin{pmatrix}12&8&4\\4&0&4\end{pmatrix}=\begin{pmatrix}5+12&15+8&25+4\\10+4&20+0&30+4\end{pmatrix}=\underline{\underline{\begin{pmatrix}17&23&29\\14&20&34\end{pmatrix}}}$$

Example

Given that $P=\begin{pmatrix}4&^-6\\0&10\end{pmatrix}$ and $Q=\begin{pmatrix}3&9\\^-3&0\end{pmatrix}$

Find:

i. $2P$

ii. $\frac{1}{3}Q$

iii. $\frac{1}{2}P-\frac{1}{3}Q$

Solution

i. $2P=2\begin{pmatrix}4&^-6\\0&10\end{pmatrix}=\begin{pmatrix}2\times4&2\times^-6\\2\times0&2\times10\end{pmatrix}=\underline{\underline{\begin{pmatrix}8&^-12\\0&20\end{pmatrix}}}$

ii. $\frac{1}{3}Q=\frac{1}{3}\begin{pmatrix}3&9\\^-3&0\end{pmatrix}=\begin{pmatrix}\frac{1}{3}\times3&\frac{1}{3}\times9\\\frac{1}{3}\times^-3&\frac{1}{3}\times0\end{pmatrix}=\underline{\underline{\begin{pmatrix}1&3\\^-1&0\end{pmatrix}}}$

iii.

$$\frac{1}{2}P-\frac{1}{3}Q=\frac{1}{2}\begin{pmatrix}4&^-6\\0&10\end{pmatrix}-\frac{1}{3}\begin{pmatrix}3&9\\^-3&0\end{pmatrix}=\begin{pmatrix}2&^-3\\0&5\end{pmatrix}-\begin{pmatrix}1&3\\^-1&0\end{pmatrix}$$

$$=\begin{pmatrix}2-1&^-3-3\\0-(^-1)&5-0\end{pmatrix}$$

$$=\underline{\underline{\begin{pmatrix}1&^-6\\1&5\end{pmatrix}}}$$

b) Multiplication of two or more matrices

To multiply two or more matrices together, we multiply the first number in the row matrix by the first number in the column matrix, the second number in the row matrix by the second number in the column matrix and so on and add then add the products together.

NB:

For multiplication of two matrices to be possible, it is essential that the <u>*number of columns in*</u> <u>*the first*</u> matrix should be the same as the <u>*number of rows in the second*</u> matrix.

Example

Given that the matrix: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 6 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$

Find:

i. **AB**

ii. **BA**

Solution

i. $$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 6 & 5 \end{pmatrix}\begin{pmatrix} 7 \\ 8 \end{pmatrix} = \begin{pmatrix} 1\times 7 + 2\times 7 \\ 3\times 7 + 4\times 8 \\ 6\times 7 + 5\times 8 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 23 \\ 53 \\ 82 \end{pmatrix}}}$$

ii. $$BA = \begin{pmatrix} 7 \\ 8 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 6 & 5 \end{pmatrix}$$

This matrix multiplication is not compatible i.e. multiplication in this case is impossible because the number of columns in the first matrix is not equal to the number of rows in the second matrix.

Generally, for any two matrices ***A*** *and B; AB ≠ BA except when of the matrices is an identity matrix.*

Example

Work out the following

a. $$\begin{pmatrix} 2 & 4 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 3 & 8 & ^-5 \\ 1 & 4 & ^-1 \end{pmatrix}$$

b. $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 & 2 \\ 3 & 1 & 4 \end{pmatrix}$$

Solution

a. $$\begin{pmatrix}2 & 4 & 0\end{pmatrix}\begin{pmatrix}1 & 2 & 0\\ 3 & 8 & ^-5\\ 1 & 4 & ^-1\end{pmatrix} = \begin{pmatrix}2\times1+4\times3+0\times1 & 2\times2+4\times8+0\times4 & 20+4\times^-5+0\times^-1\end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix}14 & 36 & ^-20\end{pmatrix}}}$$

b. $$\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}5 & 6 & 2\\ 3 & 1 & 4\end{pmatrix} = \begin{pmatrix}1\times5+2\times3 & 1\times6+2\times1 & 1\times2+2\times4\\ 3\times5+4\times3 & 3\times6+4\times1 & 3\times2+4\times4\end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix}11 & 8 & 10\\ 27 & 22 & 22\end{pmatrix}}}$$

7.6 Application of matrix multiplication

The concept of matrix multiplication is widely applied in day-to-day arithmetic. The following examples illustrate the application of matrix multiplication.

Example

1. If $\begin{bmatrix}4 & 1\\ x & ^-1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}4\\ 8\end{bmatrix}$, determine the values of x and y.

Solution

$$\begin{bmatrix}4 & 1\\ x & ^-1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}4x+y\\ x^2-y\end{bmatrix} = \begin{bmatrix}4\\ 8\end{bmatrix}$$

$4x + y = 4$.........................(1)

$x^2 - y = 8$.........................(2)

From equation (2), $y = x^2 - 8$ *and substituting for y in equation (1), we obtain:*

$$4x + x^2 - 8 = 4$$
$$x^2 + 4x - 12 = 0$$

$$sum = 4$$
$$product = {}^{-}12$$
$$factor : 6, {}^{-}2$$
$$\therefore x^2 + 4x - 12 = x^2 + 6x - 2x - 12 = 0$$
$$\Rightarrow x(x+6) - 2(x+6) = 0$$
$$\Rightarrow (x-2)(x+6) = 0$$
$$\therefore x = 2, x = {}^{-}6$$

$$For : x = 2, y = (2)^2 - 8 = {}^{-}4$$
$$\therefore \underline{\underline{x = 2, y = {}^{-}4}}$$

$$For : x = {}^{-}6, y = ({}^{-}6)^2 - 8 = 28$$
$$\therefore \underline{\underline{x = {}^{-}6, y = 28}}$$

2. Find the values of p and q given that

$$\begin{pmatrix} 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ p & 2 \\ 10 & q \end{pmatrix} = \begin{pmatrix} 39 & 25 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ p & 2 \\ 10 & q \end{pmatrix} = \begin{pmatrix} 4 + 3p + 20 & 3 + 6 + 2q \end{pmatrix} = \begin{pmatrix} 39 & 25 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} 3p + 24 & 2q + 9 \end{pmatrix} = \begin{pmatrix} 39 & 25 \end{pmatrix}$$
$$3p + 24 = 39 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$
$$2q + 9 = 25 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From equation (1):

$$3p = 39 - 24 = 15$$
$$\Rightarrow p = {}^{15}\!/_{3}$$
$$\therefore p = 5$$

From equation (2):

$$2q = 25 - 9 = 16$$
$$\Rightarrow q = {}^{16}\!/_{2}$$
$$\therefore q = 8$$

3. Jack went to buy 3 pencils, 2 rulers and 4 ball point pens. Okot went to buy 1 pencil, 1 ruler and 8 ball point pens. In Gulu, pencils and rulers cost shs 60 each and ballpoint pen costs shs 20 each. In Pader, pencils costs and rulers cost shs 80 each and ballpoint pen costs shs 30 each.

$$\begin{array}{c} Jack \\ Okot \end{array} \begin{pmatrix} 3 & 2 & 4 \\ 1 & 1 & 8 \end{pmatrix} \begin{array}{c} \begin{array}{cc} Gulu & Pader \end{array} \\ \begin{pmatrix} 60 & 80 \\ 60 & 80 \\ 20 & 30 \end{pmatrix} \end{array}$$

From the above items of jack and Okot as a 2x3 matrix and the cost in two towns as a 3x2 matrix, find the cost of items in:

i. Gulu

ii. Pader

Solution

$$\begin{pmatrix} 3 & 2 & 4 \\ 1 & 1 & 8 \end{pmatrix} \begin{pmatrix} 60 & 80 \\ 60 & 80 \\ 20 & 30 \end{pmatrix} = \begin{pmatrix} 180+120+80 & 240+160+120 \\ 60+60+160 & 80+80+240 \end{pmatrix} = \begin{pmatrix} 380 & 520 \\ 280 & 400 \end{pmatrix}$$

i. *The cost of items in Gulu* $= 380 + 280 = \underline{\underline{660/-}}$

ii. *The cost of item in Pader* $= 520 + 400 = \underline{\underline{920/-}}$

4. Peter when shopping and bought 5 books (**B**) for shs 100each, 3 rubber (**R**) for shs50@ and 10 pens (**P**) for shs 200@. How much money did Peter spend?

Solution

The items Peter bought can be written as row matrix as below.

$$\begin{array}{c} \begin{matrix} B & R & P \end{matrix} \\ \begin{pmatrix} 5 & 3 & 10 \end{pmatrix} \end{array}$$

And the cost of the items is written as a column matrix.

$$\begin{array}{c} B \\ R \\ P \end{array}\begin{pmatrix} 100 \\ 50 \\ 20 \end{pmatrix}$$

The amount Peter spent $= \begin{pmatrix} 5 & 3 & 10 \end{pmatrix}\begin{pmatrix} 100 \\ 50 \\ 20 \end{pmatrix}$

$$= 500 + 150 + 2000$$

$$= \underline{\underline{2650/-}}$$

5. In the football league, a win (**W**) earns 3 points, a draw (**D**) only 1point and a loss (L) 0 point. The results for two football clubs in the English premier league, Man U and Arsenal are given in the following table.

Club	**P**	**W**	**L**
Man U	11	5	4
Arsenal	7	9	4

Use matrix multiplication to find the number of points each scored.

Solution

The matrix from the above information is

$$\begin{array}{c} M \\ A \end{array}\begin{pmatrix} 11 & 5 & 4 \\ 7 & 9 & 4 \end{pmatrix}$$

The number of points for W, D and L can be written as a column matrix as

$$\begin{array}{c} W \\ D \\ L \end{array}\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

$$\therefore \begin{array}{c} M \\ A \end{array}\begin{pmatrix} 11 & 5 & 4 \\ 7 & 9 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 33+5+0 \\ 21+9+0 \end{pmatrix} = \begin{pmatrix} 38 \\ 30 \end{pmatrix}$$

Thus, Man-U scored 38 points and Arsenal scored 30 points.

6. Jane wants to go shopping and buy 3 writing pads (**W**), 4exercise books (**B**) and 5 ball pens (**P**). Her sister Eva wants to buy 2 writing pads, 6 exercise books and 3 ball pens. In Entebbe (**E**), writing pads cost shs 800, exercise books cost shs 300 and ball pens cost shs 150 each. In Kampala (**K**), writing pads cost shs 700, exercise books cost shs 500 and ball pens cost shs 100 each. Use matrix multiplication to find out where it is better to shop.

Solution

The items Jane and Eva want can be shown in a 2×2 *matrix*

$$\begin{array}{c} \\ Jane \\ Eva \end{array}\begin{array}{c} \begin{array}{ccc} W & B & P \end{array} \\ \begin{pmatrix} 3 & 4 & 5 \\ 2 & 8 & 3 \end{pmatrix} \end{array}$$

Their costs can be shown in a 3×2 *matrix*

$$\begin{array}{c} \\ W \\ B \\ P \end{array}\begin{array}{c} \begin{array}{cc} E & K \end{array} \\ \begin{pmatrix} 800 & 700 \\ 300 & 500 \\ 150 & 100 \end{pmatrix} \end{array}$$

We multiply them together thus:

$$\begin{pmatrix} 3 & 4 & 5 \\ 2 & 8 & 3 \end{pmatrix}\begin{pmatrix} 800 & 700 \\ 300 & 500 \\ 150 & 100 \end{pmatrix} = \begin{pmatrix} 2400+1200+750 & 2100+2000+500 \\ 1600+2400+450 & 1400+4000+300 \end{pmatrix} = \begin{pmatrix} 4350 & 4600 \\ 4450 & 5700 \end{pmatrix}$$

Jane would spend shs 4350 in Entebbe and shs 4600 in Kampala

Eva would spend shs 4450 in Entebbe and shs 5700 in Kampala

7.7 Determinant of a matrix

For this course, we shall restrict ourselves to the determinant of a 2×2 matrices only.

Definition:

The determinant of a 2×2 matrix is the difference between the products of the leading diagonal and the minor diagonal. It is denoted as ***det***.

Consider the matrix below:

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, a 2×2 matrix.

Its determinant is denoted by ***det A*** or $|A|$ and is defined as:

$$\boxed{\det A = |A| = ad - bc}$$

Example

a) Given that $D = \begin{pmatrix} 1 & 3 \\ 3 & {}^{-}2 \end{pmatrix}$ Find ***det D***

Solution

$$\det D = (1 \times -2) - (3 \times 3) = {}^{-}2 - 9 = \underline{\underline{{}^{-}11}}$$

b) Given that $A = PB$. Find $\det A$, if $P = \begin{pmatrix} 2 & {}^{-}3 \\ 0 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ {}^{-}1 & 0 \end{pmatrix}$

Solution

$$A=\begin{pmatrix}2 & ^{-}3\\ 0 & 5\end{pmatrix}\begin{pmatrix}1 & 3\\ ^{-}1 & 0\end{pmatrix}=\begin{pmatrix}2+3 & 6+0\\ 0-5 & 0+0\end{pmatrix}=\begin{pmatrix}5 & 6\\ ^{-}5 & 0\end{pmatrix}$$

$$\Rightarrow \det A=(5\times 0)-(^{-}5\times 6)=0+30$$

$$\therefore \underline{\underline{\det A=30}}$$

c) Given that $P=\begin{pmatrix}a & 0\\ 3 & a\end{pmatrix}$ and $Q=\begin{pmatrix}4 & 1\\ 7 & 2\end{pmatrix}$. If ***det P* = *det Q***, find the possible values of a.

Solution

$$P=\begin{pmatrix}a & 0\\ 3 & a\end{pmatrix}, \det P=(a\times a)-(3\times 0)=a^2$$

$$Q=\begin{pmatrix}4 & 1\\ 7 & 2\end{pmatrix}, \det Q=(4\times 2)-(7\times 1)=8-7=1$$

But $\det P=\det Q$

$$\Rightarrow a^2=1$$

$$\therefore a=\pm\sqrt{1}$$

$$\therefore \underline{\underline{a=1, a=^{-}1}}$$

Note:

Some matrices have ***det*** = 0. Such matrices are called ***singular matrices*** where as those for which ***det*** ***non-singular matrices***.

7.8 Inverse of a matrix

For this sub-topic, we shall also restrict ourselves to the inverse of a 2×2 matrices only.

Introduction:

If **A** is a $n\times n(2\times 2)$ matrix and **I** is also a $n\times n(2\times 2)$ identity matrix, then;

$$IA = AI = A$$

Example

1. If $A = \begin{pmatrix} 1 & 4 \\ 3 & 10 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then;

$$AI = \begin{pmatrix} 1 & 4 \\ 3 & 10 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 3 & 10 \end{pmatrix}$$

$$\therefore AI = A$$

Definition:

The inverse of a matrix A is the matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$

Example

Find the inverse of matrix $A = \begin{pmatrix} 8 & 3 \\ 5 & 2 \end{pmatrix}$

Solution

From the definition, i.e. $AA^{-1} = I$

$$Let\ A^{-1} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow AA^{-1} = \begin{pmatrix} 8 & 3 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 8a+3c & 8b+3d \\ 5a+2c & 5b+2d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$8a + 3c = 1$..................(1)

$5a + 2c = 0$..................(2)

$8b + 3d = 0$..................(3)

$5b + 2d = 1$..................(4)

$$2(8a + 3c = 1)$$
$$\underline{3(5a + 2c = 0)}$$

$$16a + 6c = 2$$
$$-\underline{(15a + 6c = 0)}$$
$$a = 2$$
$$c = -5$$

$$2(8b + 3d = 0)$$
$$\underline{3(5b + 2d = 1)}$$

$$16b + 6d = 0$$
$$-\underline{(15b + 6d = 3)}$$
$$b = -3$$
$$d = 8$$

$$\therefore A^{-1} = \begin{pmatrix} 2 & ^{-}3 \\ ^{-}5 & 8 \end{pmatrix}$$

Example

2. Find $\mathbf{B^{-1}}$, given that $B = \begin{pmatrix} 5 & 3 \\ 6 & 4 \end{pmatrix}$

Solution

$$Let\ B^{-1} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

From $BB^{-1} = I$

$$\Rightarrow \begin{pmatrix} 5 & 3 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 5x+3z & 5y+3w \\ 6x+4z & 6y+4w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$5x+3z=1$..................(1) $\qquad$ $5y+3w=0$..................(3)

$6x+4z=0$..................(2) $\qquad$ $6y+4w=1$..................(4)

$$4(5x+3z=1)$$
$$\underline{3(6x+4z=0)}$$

$$20x+12z=4$$
$$-\underline{(18x+12z=0)}$$
$$2x=4 \Rightarrow x=2$$
$$z=-3$$

$$4(5y+3w=0)$$
$$\underline{3(6y+4w=1)}$$

$$20y+12w=0$$
$$-\underline{(18y+12w=3)}$$
$$2y=-3 \Rightarrow y = {}^{-3}\!/_2$$
$$w = {}^5\!/_2$$

$$\therefore B^{-1} = \underline{\underline{\begin{pmatrix} 2 & {}^{-3}\!/_2 \\ {}^{-}3 & {}^5\!/_2 \end{pmatrix}}}$$

7.9 Obtaining the inverse of a matrix from its adjoin and its determinant

Consider the matrix $B^{-1} = \begin{pmatrix} 2 & {}^{-3}\!/\!_2 \\ {}^{-}3 & {}^{5}\!/\!_2 \end{pmatrix}$ of the matrix $B = \begin{pmatrix} 5 & 3 \\ 6 & 4 \end{pmatrix}$ above.

$$B^{-1} = \begin{pmatrix} 2 & {}^{-3}\!/\!_2 \\ {}^{-}3 & {}^{5}\!/\!_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \times \frac{2}{2} & {}^{-3}\!/\!_2 \\ {}^{-}3 \times \frac{2}{2} & {}^{5}\!/\!_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} \times 4 & \frac{1}{2} \times {}^{-}3 \\ \frac{1}{2} \times {}^{-}6 & \frac{1}{2} \times 5 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 4 & {}^{-}3 \\ {}^{-}6 & 5 \end{pmatrix}$$

$$\therefore B^{-1} = \frac{1}{2}\begin{pmatrix} 4 & {}^{-}3 \\ {}^{-}6 & 5 \end{pmatrix}$$

The matrix $\begin{pmatrix} 4 & {}^{-}3 \\ {}^{-}6 & 5 \end{pmatrix}$ is similar to the matrix $B = \begin{pmatrix} 5 & 3 \\ 6 & 4 \end{pmatrix}$. The only difference is that the position of the elements in the leading diagonal i.e. 4 and 5 have been interchanged and the signs of the elements in the minor diagonals have i.e. 6 and 3 have been changed.

The matrix $\begin{pmatrix} 4 & {}^{-}3 \\ {}^{-}6 & 5 \end{pmatrix}$ is what is known as the adjoin of the matrix **B**.

’ outside the bracket of the matrix $\mathbf{B}^{-1}$ is the determinant of the matrix **B**.

Generally, to obtain the adjoin of 2×2 matrix; you simply need to alter the position of the elements in the major diagonal and the change the signs of the elements in the minor diagonal.

Consider the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the adjoin of matrix **A** is the matrix

$$\begin{pmatrix} d & {}^{-}b \\ {}^{-}c & a \end{pmatrix}$$

The determinant of matrix **A** (***det A***) is $ad - bc$. The inverse of the matrix **A** is therefore obtained from:

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & {}^{-}b \\ {}^{-}c & a \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & {}^{-}b \\ {}^{-}c & a \end{pmatrix}$$

Example

Find the inverse of matrix $A = \begin{pmatrix} 8 & 3 \\ 5 & 2 \end{pmatrix}$

Solution

$A = \begin{pmatrix} 8 & 3 \\ 5 & 2 \end{pmatrix}$

$\det A = 8 \times 2 - 5 \times 3 = 1$

$Adjoin, A = \begin{pmatrix} 2 & -3 \\ -5 & 8 \end{pmatrix} \qquad \Rightarrow A^{-1} = \frac{1}{1}\begin{pmatrix} 2 & -3 \\ -5 & 8 \end{pmatrix}$

$$\therefore A^{-1} = \begin{pmatrix} 2 & -3 \\ -5 & 8 \end{pmatrix}$$

NB:

- If the determinant of the matrix is equal to 1, then the inverse of the that matrix is equal to its adjoin.
-
  zero.

Example

Find the inverse of the matrix $\begin{pmatrix} 39 & 91 \\ 51 & 119 \end{pmatrix}$

Solution

Let $C = \begin{pmatrix} 39 & 91 \\ 51 & 119 \end{pmatrix}$

$$\det C = 33 \times 119 - 51 \times 91 = 4641 - 4641 = 0$$

Therefore, the matrix $\begin{pmatrix} 39 & 91 \\ 51 & 119 \end{pmatrix}$ *has no inverse because it can't be divided by zero I.e. it is a singular matrix.*

Example

Given that $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

Find the matrix:

a) P such that $AB = P$

b) P^{-1}

c) $(A+B)^{-1}$

d) $(B-A)^{-1}$

Solution

a) $AB = P$

$$AB = P = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2+2 & 2+4 \\ 1+3 & 1+6 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 4 & 7 \end{pmatrix}$$

$$\therefore P = \begin{pmatrix} 4 & 6 \\ 4 & 7 \end{pmatrix}$$

b) $\det P = 4 \times 7 - 4 \times 6 = 4 \quad Adjo\text{int } of\ P = \begin{pmatrix} 7 & -6 \\ -4 & 4 \end{pmatrix}$

$$\therefore P = \frac{1}{4}\begin{pmatrix} 7 & -6 \\ -4 & 4 \end{pmatrix} = \begin{pmatrix} 7/4 & -3/2 \\ -1 & 1 \end{pmatrix}$$

c) $A + B = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 2 & 5 \end{pmatrix}$

$$\det(A+B) = 3 \times 5 - 2 \times 6 = 9,\ Adjoin(A+B) = \begin{pmatrix} 5 & -3 \\ -2 & 3 \end{pmatrix}$$

$$\Rightarrow (A+B)^{-1} = \frac{1}{9}\begin{pmatrix} 5 & -3 \\ -2 & 3 \end{pmatrix}$$

$$\therefore (A+B)^{-1} = \begin{pmatrix} 5/9 & -1/3 \\ -2/9 & 1/3 \end{pmatrix}$$

d)

$$B-A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$$

$$\det(B-A) = (^{-}1 \times {}^{-}1) - (0 \times {}^{-}1) = 1,$$

$$Adjoin(B-A) = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow (B-A)^{-1} = \frac{1}{1}\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$\therefore (B-A)^{-1} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

7.10 Solving simultaneous equation by matrix method

Consider the simultaneous equation below:

$$ax + dy = m$$
$$cx + dy = n$$

Where ***a, b, c, d, m*** and ***n*** are constants.

In matrix form, the above equation can be written as;

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\otimes)\cdot$$

The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is known as the coefficient matrix. Equation $\otimes$ is then multiplied by the adjoin of the coefficient matrix *from the left* i.e.

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix}$$

$$\begin{pmatrix} da-bc & db-db \\ ac-ac & ad-cb \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} dm-bn \\ an-cm \end{pmatrix}$$

$$\begin{pmatrix} da-bc & 0 \\ 0 & ad-cb \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} dm-bn \\ an-cm \end{pmatrix}$$

$(da-bc)x = dm-bn.......................(1)$

$(ad-cb) = an-cm........................(2)$

From equation (1): $x = \dfrac{(dm-bn)}{(da-bc)}$

From equation (2): $y = \dfrac{(an-cm)}{(ad-cb)}$

Example 1

Write the simultaneous equation below in matrix form and hence solve it

$7x+9y=3$

$5x+7y=1$

Solution

$$\begin{pmatrix} 7 & 9 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 7 & -9 \\ -5 & 7 \end{pmatrix}\begin{pmatrix} 7 & 9 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 & -9 \\ -5 & 7 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 49-45 & 0 \\ 0 & -45+49 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 21-9 \\ -15+7 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ -8 \end{pmatrix}$$

$4x = 12...............(1)$

$4y = -8..............(2)$

From equation (1): $x = \dfrac{12}{4} = 3$

From equation (2): $y = \dfrac{-8}{4} = -2$

$\therefore \underline{\underline{x=3, y=-2}}$

Example

Solve the following pairs of simultaneous equation by matrix method

a) $$2x + y = 6$$
$$x - y = 3$$

b) $$2y - 11 = 3x$$
$$2y + 5x = 3$$

c) $$x + y = 3$$
$$2x - 2y + 1 = 0$$

Solution

a) $$2x + y = 6$$
$$x - y = 3$$

$$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} -2-1 & 0 \\ 0 & -1-2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6-3 \\ -6+6 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -9 \\ 0 \end{pmatrix}$$

$$-3x = -9$$
$$-3x = 0$$
$$\therefore \underline{\underline{x = 3, y = 0}}$$

b) $$2y - 11 = 3x \qquad \Longrightarrow \qquad 2y - 3x = 11$$
$$2y + 5x = 3 \qquad\qquad\qquad 2y + 5x = 3$$

$$\begin{pmatrix} 2 & -3 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 11 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 3 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 11 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 10+6 & 15-15 \\ 4-4 & 6+10 \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 55+9 \\ -22+6 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 64 \\ -16 \end{pmatrix}$$

$$16y = 64$$
$$16x = -16$$
$$\therefore \underline{\underline{x = -1, y = 4}}$$

c) $x + y = 3$
$2x - 2y = -1$

$$\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -2-2 & 0 \\ 0 & -2-2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6+1 \\ -6-1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ -7 \end{pmatrix}$$

$$-4x = -5$$
$$-4y = -7$$
$$\therefore \underline{\underline{x = 5/4, y = 7/4}}$$

7.11 Miscellaneous exercise:

1. a) The table below shows information from Uganda nation football
league.

Club	**P**	**W**	**Pts**
URA	8	3	4
Villa	4	3	6
KCC	3	2	1

Write the information given in the table above in matrix form and state the order of the matrix.

b) Write the members of the set {3, 4, 1, 8, 2}

2. Given the following matrices:

$$A=\begin{pmatrix}1 & 2\\ 2 & 3\end{pmatrix}, B=\begin{pmatrix}2 & 1 & 3\\ 1 & 0 & 4\end{pmatrix}, C=\begin{pmatrix}1 & 2 & 4\\ 3 & 1 & 2\\ 4 & 0 & 1\end{pmatrix}$$

Workout where possible the following:

i. **AB**

ii. **BA**

iii. **BC**

3. Find x and y if: $5\begin{pmatrix}x & 3\\ 7 & 3\end{pmatrix}-2\begin{pmatrix}x & 4\\ y & 2\end{pmatrix}=\begin{pmatrix}18 & 7\\ 29 & 11\end{pmatrix}$

4. In a national soccer league, the results of two soccer clubs, Gulu united and KCC were as shown in the table below.

Club	**Won**	**Drawn**	**Lost**
Gulu united	8	3	4
KCC	3	2	1

If three points are awarded when a match is won, 1point when it is a draw, and no point if it is lost, use matrices to find the total number of points obtained by each club.

5. Mrs. Lukyamuzi bought 2kg of meat at shs 3500per kilogram, 3 packets of unga maize meal at shs 2000 per packet and 4 loaves of bread at shs 875 per loaf. At the same time and at the same store, Mrs. Frank bought 3kg of meat, 2 packets of unga maize meal and 5 loaves of bread. On a different day, the two ladies bought the same quantity of food items from the store where the prices were shs 3750 per kilogram of meat, shs 2500 per packet of unga maize meal and shs 1000 per loaf of bread. Use matrix method to find how much each lady spent at each place.

6. Four students: Kelly, Liz, Musa, and Adong went to a stationary shop.

   * Kelly bought 4 pens, 6 counter books, and 1 graph book
   * Liz bought 10 pens and 5 counter books.
   * Musa bought 3 pens and 3 graph books.
   * Adong bought 5 pens, 2 counter books, and 8 graph books.

   The costs of a pen, a counter book, and a graph book were shs 400, shs 1200, and shs 1000 respectively.

   a) i Write a 4x3 matrix for the items bought by the four students

      ii. Write a 3x1 matrix for the costs of each item.

   b) Use the matrices in a) to calculate the amount of money spent by each student.

   c) If each student was to buy 4 pens, 20 counter books and 6 graph books, how much money would be spent by all the four students?

7. Given that $P = \begin{pmatrix} 2 & -2 \\ -3 & 4 \end{pmatrix}$. Find matrix Q such that $PQ = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

8. Solve the following pairs of simultaneous equation using matrix method

   a) $3x + 5y = 1$
      $2x - y = -8$

   b) $4x + y = 2$
      $x - y = 8$

   c) $m - 2n = -4$
      $m - n = -1$

   d) $p + 2q = 11$
      $2p - q = 2$

   e) $3x - y = 11$
      $2x - 3y = 5$

   f) $4x = 3y + 1$
      $3x + y + 1 = 0$

8 FUNCTIONS

8.1 Introduction:

Under mapping, we saw that many members are mapped to many image points. But under this topic, we shall see the relationship that maps a single object to a single image.

Definition:

A function is a relation which maps a single object onto a single image. In other words, it is a rule that assigns to each element x in set **A**, one element known as $f(x)$ in **B**.

Suppose set **A** has elements $\{x_1, x_2, x_3\}$ and set **B** has elements $\{y_1, y_2, y_3\}$ and that each element of **A** is mapped to one element in set **B** as shown below.

$$\begin{array}{ccc} X & & Y \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} & \begin{matrix} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{matrix} & \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \end{array}$$

Set **A** is called the *domain* of the function and set **B** is called the *range* of the function. Each member of X has one and only one corresponding member in Y, f is therefore a function and it is written as:

$$f : A \to B$$

Meaning that, the input values of f come from **A**, and the output values of f are stored in **B**. The action of a function on an element is denoted by: $A \to f(x)$

Consider the function: $x \to 3x + 4$. This can be represented as follows;

a) $f(x) \to 3x + 4$, which is read as a function which maps x onto $3x + 4$

b) $f : x \rightarrow 3x + 4$

c) $f(x) = 3x + 4$. We can let $y = f(x)$. $\therefore\ y = 3x + 4$. In this case, the value of y can only be obtained when x is known. In other words, y depends on x and hence it is known as the *dependent variable*. x, on the other hand does not depend on y. It is therefore known as *independent variable*.

Example

A function $f(x)$ is defined by: $f(x) = 3x + 3$, find $f(5)$

Solution

$f(x) = 3x + 3$

$f(5) = f(x = 5) = ?$

In order to obtain $f(5)$, we have to substitute x in the expression $3x + 3$ with 5 and then simplify it. I.e.

$f(5) = 3(5) + 3 = 15 + 3$

$\therefore\ f(5) = \underline{\underline{18}}$

Example

Given that $f(x) = \dfrac{1}{x}$.

Find:

i. $f(a)$

ii. $f(x + h)$

Solution

i. $f(x) = \dfrac{1}{x}\ \therefore\ f(x = a) = \dfrac{1}{\underline{\underline{a}}}$

ii. $f(x = x + h) = \dfrac{1}{\underline{\underline{x + h}}}$

Example

Given that $f(x)=x^2+3x-9$

Find:

i. $f(1)$

ii. $f(^-5)$

Solution

$$f(x)=x^2+3x-9$$

i. $\Rightarrow f(x=1)=1^2+3(1)-9$

$$=1+3-9$$

$$=\underline{\underline{^-5}}$$

ii. $f(^-5)=(^-5)^2+3(^-5)-9$

$$=1+3-9$$

$$=\underline{\underline{^-5}}$$

Example

A function is defined by the formula $f(x)=3x+1$. If $f(a)=19$, find the value of a.

Solution

$$f(x)=3x+1$$

$$f(a)=3(a)+1=3a+1, but: f(a)=19$$

$$\Rightarrow 3a+1=19 \Leftrightarrow 3a=18$$

$$\therefore a={}^{18}\!/\!_{3}=\underline{\underline{6}}$$

Example

Given that $f(x)=\frac{2}{2x^2-6}$. Find the value of p for which $f(p)=1$.

Solution

$$f(x)=\frac{2}{2x^2-6}$$

$$\Rightarrow f(p)=\frac{2}{2p^2-6}=1 \Leftrightarrow 2=2p^2-6$$

$$\therefore 2p^2=8 \Leftrightarrow p^2=4$$

$$\Rightarrow p=\pm\sqrt{4}=\pm 2$$

$$\therefore \underline{\underline{p=2,\ p={}^-2}}$$

Example

Given that $g(x)=ax^2+b, g({}^-2)=3, and, g(1)={}^-3$. Find the value of a and b

Solution

$$g(x)=ax^2+b$$

For g (-2):

$$g({}^-2)=a({}^-2)^2+b$$

$$\therefore g({}^-2)=4a+b \quad g({}^-2)=3$$

$$\Rightarrow 4a+b=3\text{............................}(1)$$

For g (1):

$$g(1)=a(1)^2+b$$

$$\therefore g(1)=a+b \quad g(1)={}^-3$$

$$\Rightarrow a+b={}^-3\text{............................}(2)$$

$$Equation(1)-equation(2)$$

$$\begin{array}{l} 4a+b=3 \\ \underline{-(a+b={}^-3)} \end{array}$$

$$3a=6 \Rightarrow a=\frac{6}{3}$$

$$\therefore \underline{\underline{a=2}}$$

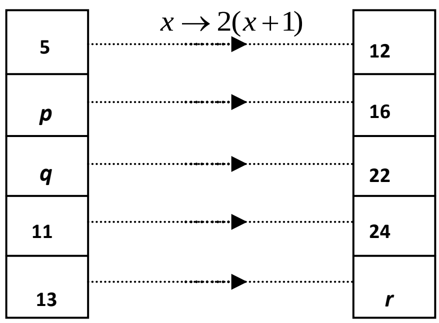
From (2):

$$2+b={}^{-}3.$$

$$\therefore \underline{\underline{b={}^{-}5}}$$

Example

Find the unknown values in the arrow diagrams for the mapping.



Solution

We know the range and the domain: x *is the domain and* $range = 2(x+1)$

$$\therefore\ f(x)=2(x+1)$$

To prove: $f(5)=2(5+1)=12$

$$f(p)=2(p+1), but\ \ from\ the\ diagram,\ f(p)=16$$

$$\Rightarrow 16 = 2(p+1)$$

$$\therefore 2p+2=16$$

$$2p=14$$

$$\therefore \underline{\underline{p=7}}$$

$$f(q)=2(q+1), but\ \ from\ the\ diagram,\ f(q)=22$$

$$\Rightarrow 2(q+1)=22$$

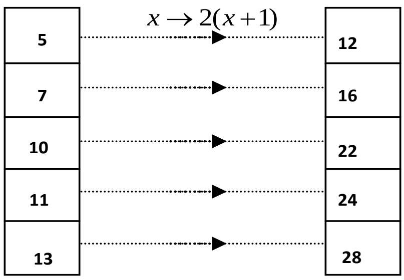
$$q+1=11$$

$$\therefore \underline{\underline{q=10}}$$

$f(x) = 2(x+1)$

$f(13) = 2(13+1) = 28, but,\ f(13) = r$

$\therefore \underline{\underline{r = 28}}$



Example

Given that $f(x) = \dfrac{2}{x+2} + \dfrac{3x+4}{x^2-4}$. Express the $f(x)$ in the form $f(x) = \dfrac{Px}{x^2+Q}$, and hence state the value of P and Q.

Solution

$$f(x) = \frac{2}{x+2} + \frac{3x+4}{x^2-4}$$

Factorizing $x^2 - 4 = (x+2)(x-2)$ *from difference of two squares.*

$$\therefore\ f(x) = \frac{2}{x+2} + \frac{3x+4}{(x+2)(x-2)}, \Rightarrow L.c.m = (x+2)(x-2)$$

$$\therefore\ f(x) = \frac{2(x-2)+3x+4}{(x+2)(x-2)} = \frac{2x-4+3x+4}{x^2-4}$$

$$\therefore\ f(x) = \frac{5x}{x^2-4}$$

Since $f(x) = \dfrac{Px}{x^2+Q}$*,by comparing with* $f(x) = \dfrac{5x}{x^2-4}$

$$\Rightarrow P = 5, and, Q = {}^{-}4$$

8.2 Undefined functions:

A function is undefined if the denominator is equal to zero. This is because we cannot divide something by zero. Therefore the function $f(x)=\frac{1}{x}$ is undefined when $x=0$ i.e. $f(x)=\frac{1}{0}$ which cannot be divided. In *math error*

Example

Given that $f(x)=\frac{1}{1-x}$

i. Find $f(2)$

ii. Find the value of x for which $f(x)$ is not defined.

Solution

i. $f(x)=\frac{1}{1-x}$

$$\therefore f(2)=\frac{1}{1-2}=\frac{1}{^-1}=\underline{\underline{^-1}}$$

ii. $f(x)$ *is undefined if* $1-x=0$

$$\Rightarrow 1-x=0$$

$$\therefore \underline{\underline{x=1}}$$

Example

Find the value of x for which $f(x)=\frac{5x+6}{4-x^2}$ is not defined.

Solution

$$f(x)=\frac{5x+6}{4-x^2}$$

$f(x)$ is undefined if, $4-x^2=0$

$$\Rightarrow x^2=4 \Leftrightarrow x=\pm\sqrt{4}=\pm 2$$

$$\therefore \underline{\underline{x=2, or, x={}^-2}}$$

8.3 Inverse of a function

8.3.1 Introduction:

If a function f maps elements of **A** onto elements of **B** i.e.

$f: A \to B$

$$\begin{array}{ccc} A & & B \\ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} & \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} & \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \end{array}$$

Then the function that maps elements of **B** back onto elements of **A** i.e.

$$\begin{array}{ccc} A & & B \\ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} & \begin{array}{c} \longleftarrow \\ \longleftarrow \\ \longleftarrow \end{array} & \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \end{array}$$

Is known as the inverse function of f and it is denoted as f^{-1}.

8.3.2 Obtaining the inverse function:

The following examples will illustrate how to obtain the inverse of a function.

Example

Given that $f(x) = 4x + 8$.

Find:

a) $f^{-1}(x)$

b) $f^{-1}(1)$

Solution

a) ***Step 1****: let* $f(x) = y \therefore y = 4x + 8$

Step 2*: make x the subject*

$$y - 8 = 4x$$

$$\therefore x = \frac{y-8}{4}$$

Step 3*: replace x with* $f^{-1}(x)$ *and y with x. i.e.*

$f^{-1}(x) = \frac{x-8}{4}$ *This is the inverse function of* $f(x)$

b) $f^{-1}(1) = \frac{(1)-8}{4} = \frac{^{-}7}{4}$

Example

Given that $g(x) = 2x - 17$.

Find:

a) $g^{-1}(x)$
b) $g^{-1}(3)$
c) $g^{-1}(^{-}4)$

Solution

a) $g(x) = 2x - 17$

Let $y = g(x) \Leftrightarrow y = 2x - 17$

$$y + 17 = 2x$$

$$\therefore x = \frac{y+17}{2}$$

$$x = g^{-1}(x), y = x$$

$$\therefore g^{-1}(x) = \frac{x+17}{2}$$

b) $g^{-1}(3) = \frac{3+17}{2} = 10$

c) $g^{-1}(^{-}4) = \frac{17-4}{2} = \frac{13}{2} = 6.5$

Example

Obtain the inverse of the function $f(x)=\frac{x+1}{x-1}$, hence find $f^{-1}({}^{-}2)$

Solution

$$f(x)=\frac{x+1}{x-1}$$

$$Let\ \ y = f(x) \Leftrightarrow y=\frac{x+1}{x-1}$$

$$y(x-1)=(x+1)$$

$$yx-y=x+1$$

$$yx-x-y=1$$

$$x(y-1)-y=1$$

$$x(y-1)=1+y$$

$$\therefore x=\frac{y+1}{y-1}$$

Replacing x *with* $f^{-1}(x)$ *and* y *with* x

$$\underline{\underline{f^{-1}(x)=\frac{x+1}{x-1}}}$$

Hence $f^{-1}(x={}^{-}2)=\frac{{}^{-}2+1}{{}^{-}2+1}=\frac{{}^{-}1}{{}^{-}3}=\underline{\underline{\frac{1}{3}}}$

8.4 Composite Functions

8.4.1 Introduction:

A composite function is a function, which is composed of a sequence of simple functions.

For example, the function $fg(x)$ where f and g are functions is known as a composite function. The result is image of x under g first followed by f.

Example

If $f(x)=x^2, and, g(x)=x+1$, find:

a) i) $gf(x)$
   ii) $gf(x)$, when $x=3$

b) i) $fg(x)$

ii) $fg(x)$, when $x = 3$

Solution

a) i) $gf(x) = g\left[f(x)\right], but \ f(x) = x^2$

$= g(x^2), and, g(x) = x + 1$

$= (x)^2 + 1$

$\therefore gf(x) = x^2 + 1$

ii) *When* $x = 3$:

Method 1:

$gf(x = 3) = (3)^2 + 1 = \underline{\underline{10}}$

Method 2:

$f(x) = x^2$

$\therefore \ f(3) = (3)^2 = 9$

$\Rightarrow gf(3) = g(9), but : g(x) = x + 1$

$= 9 + 1$

$= \underline{\underline{10}}$

b) i) $fg(x) = f\left[g(x)\right], but : g(x) = x + 1$

$= f(x + 1), and : f(x) = x^2$

$= (x + 1)^2$

$\therefore \ fg(x) = x^2 + 2x + 1$

ii) If $x = 3$:

Method 1:

$fg(x = 3) = (3)^2 + 2(3) + 1$

$= 9 + 6 + 1$

$= \underline{\underline{16}}$

Method 2:

$$g(x) = x + 1$$
$$\therefore\ g(3) = 3 + 1 = 4$$
$$\Rightarrow\ fg(3) = f(4), but: f(x) = x^2$$
$$= (4)^2$$
$$= \underline{\underline{16}}$$

8.5 Composite function linked with matrices

In the above example, we saw that:

$$gf(x) = x^2 + 1, and, gf(3) = 10,$$
$$fg(x) = x^2 + 2x + 1, and, fg(3) = 16.$$

We can now conclude that for any two functions f and g :

$$fg(x) \neq gf(x)$$

This is similar to what we saw with matrices. For instance, if **A** and **B** are two matrices, then: $AB \neq BA$

Example

The functions f and g are given by:

$$f(x) = \frac{8}{x-1}, \text{where } x \neq 1 \text{ and } g(x) = 2x + 1$$

Find:

a) fg
b) g^2
c) $(fg)^{-1}$
d) $(gf)^{-1}$
e) $f^{-1}(x)g^{-1}(x)$

Solution

a) $fg(x) = f[g(x)], but : g(x) = 2x + 1$

$$= f(2x+1), and : f(x) = \frac{8}{x-1}$$

$$= \frac{8}{(2x+1)-1}$$

$$= \frac{8}{2x}$$

$$\therefore \underline{\underline{fg(x) = \frac{4}{x}}}$$

b) $g^2(x) = gg(x) = g[g(x)], but : g(x) = 2x + 1$

$$= g(2x+1)$$

$$= 2(2x+) + 1$$

$$= 4x + 2 + 1$$

$$= \underline{\underline{4x + 3}}$$

c) $(fg(x))^{-1} = ?\, but \quad fg(x) = \dfrac{4}{x}$

Let $y = fg(x) \therefore y = \dfrac{4}{x} \Rightarrow x = \dfrac{4}{y}$

Replacing x with $(fg(x))^{-1}$ *and y with x, we obtain:*

$$\underline{\underline{(fg(x))^{-1} = \frac{4}{x}}}$$

In this case, the composite function fg *and its inverse* $(fg)^{-1}$ *are the same.*

d) $(gf)^{-1} = ?\, Find\ \ gf\ first.$

$$gf(x) = g[f(x)], f(x) = \frac{8}{x-1}$$

$$= g\left[\frac{8}{x-1}\right], and : g(x) = 2x + 1$$

$$= 2\left(\frac{8}{x-1}\right) + 1$$

$$\therefore gf(x) = \frac{16}{x-1} + 1$$

$$y = gf(x) \therefore y = \frac{16}{x-1} + 1$$

Making x the subject,

$$y-1=\frac{16}{x-1}\Leftrightarrow(y-1)(x-1)=16$$

$$\therefore x-1=\frac{16}{y-1}$$

$$\Rightarrow x=\frac{16}{y-1}+1$$

Replacing x with $(gf)^{-1}$ *and y with x, we obtain:*

$$\underline{\underline{(gf(x))^{-1}=\frac{16}{x-1}+1}}$$

e) $f^{-1}(x)g^{-1}(x)=?$ first find $f^{-1}(x), and, g^{-1}(x)$

For $f^{-1}(x)$:

$$f(x)=\frac{8}{x-1}.let, y=f(x)$$

$$\Rightarrow y=\frac{8}{x-1}$$

$$\therefore x=\frac{8}{y}+1$$

$$\therefore f^{-1}(x)=\frac{8}{x}+1$$

For $g^{-1}(x)$:

$$g(x)=2x+1: let, y=g(x)$$

$$\Rightarrow y=2x+1$$

$$\therefore x=\frac{y-1}{2}$$

$$\therefore g^{-1}(x)=\frac{x-1}{2}$$

$$\Rightarrow f^{-1}g^{-1}(x)=f^{-1}\left[g^{-1}(x)\right]$$

$$= f^{-1}\left(\frac{x-1}{2}\right)$$

$$= \frac{8}{\left(\frac{x-1}{2}\right)} + 1$$

$$= \frac{8 \times 2}{x-1} + 1$$

$$\therefore\ f^{-1}g^{-1}(x) = \underline{\underline{\frac{16}{x-1} + 1}}$$

Now compare $(gf)^{-1}$ *with*, $f^{-1}g^{-1}$, *they are the same, aren't they?*

8.6 Conclusion:

If $g(x)$ and $f(x)$ two functions, then:

$$(gf)^{-1} = f^{-1}g^{-1}$$

Example

If $g(x) = 2x, and,\ f(x) = x + 3$. Find $gf(x)$ and hence, evaluate $gf(2)$

Solution

$$g(x) = 2x,\ f(x) = x + 3$$

$$\begin{aligned} gf(x) &= g[f(x)] \\ &= g(x+3) \\ &= 2(x+3) \\ &= \underline{\underline{2x+6}} \end{aligned}$$

Hence $gf(2) = 2(2) + 6 = 4 + 6 = \underline{\underline{10}}$

Example

Given that $f(x) = x^3 + 3$ and $g(x) = x - 1$.

Find the value of ***a*** such that:

$$fg(a) = gf(a)$$

Solution

$f(x)=x^3+3, g(x)=x-1$

$fg(x)= f[g(x)]$

$= f(x-1)$

$=(x-1)^2+3$

$=x^2-2x+1+3$

$\therefore\ fg(x)=x^2-2x+4$

$\Rightarrow\ fg(a)=a^2-2a+4$..........................(1)

Also: $gf(x)=g[f(x)]$

$=g(x^2+3)$

$=(x^2+3)-1$

$\therefore gf(x)=x^2+2$

$\Rightarrow gf(a)=a^2+2$..............................(2)

Since $fg(a)=gf(a)$

$\Rightarrow a^2-2a+4=a^2+2$

$\therefore -2a+4=2$

$\therefore 4-2=2a \Leftrightarrow 2=2a$

$\therefore \underline{\underline{a=1}}$

8.7 Miscellaneous exercise

1. Given that: $f(x)=x^2+3x-9, g(x)=x^2-4x-2, and, h(x)=3x^2-3x+5$
   Find:

   i. $f(2)$
   ii. $g(^-1)$
   iii. $h(^-3)$

2. i) If $f(x)=x^2+5x+c, and, f(^-6)=0$, find c.

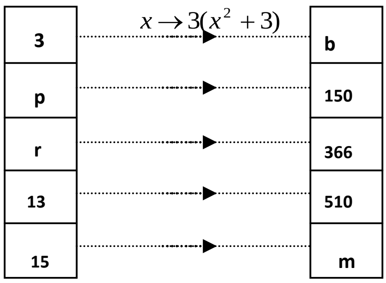
   ii) Given that $g(x)=x^2-bx$, find the value of b if $g(3)={}^-3$

   iii) $g(x)=ax^2+5x-3$. If $g(1)=9$, find a.

3. a) Given that $f(x) = ax - 7$ and $f(8) = 17$, find the value of:
   i. a
   ii. $f(4)$

   b) Given that $f(x) = ax^2 + bx$, $f(1) = 5$ and $f(2) = 14$. Find the values of a and b.

4. Find the unknown value in the arrow diagram below



5. Given that $f(x) = \frac{1}{2}(3x + 5)$. Find $f^{-1}(x)$ and hence evaluate $f^{-1}(10)$.

6. Given that $f(x) = 2x^2 + 9, and, g(x) = x + 1$.
   Find:
   a) $fg(x)$
   b) $fg(^{-}1)$
   c) $gf(5)$
   d) $fgf(x)$
   e) $g^{-1}f(3)$

7. Given the functions $f(x) = \frac{x+3}{2}$ and $g(x) = \frac{1-2x}{5}$. Determine the value of x for which $fg(x) = \frac{9 + 24x + 8x^2}{10}$

8. Two functions f and g are defined as: $f(x) = x^2 and, g(x) = 5x - 4$. Find the value of x for which $fg(x) = 5|gf(x)|$.

9. For the following functions, find the value of x for which $f(x)$ is undefined.

   a) $\dfrac{x+2}{2x-4}$

   b) $\dfrac{1}{1-x^2}$

   c) $\dfrac{5x+6}{9-x^2}$

   d) $\dfrac{3x+1}{x^2+3x-40}$

   e) $\dfrac{4x-9}{20x^2-x-1}$

10. Given that $f(x) = \dfrac{2}{x+3} + \dfrac{4}{x^2-9}$. Express $f(x)$ in the form $\dfrac{qx+c}{x^2+r}$ and hence, find the value of x for which $f(x)$ is not defined.

9 BUSINESS MATHEMATICS 2

Topics dealt with under business mathematics 2 include the following:

- Currency
- Compound interest formula
- Depreciation
- Hire purchase
- Taxation

9.1 Currency

9.1.1 Introduction:

The medium for business transaction is called currency. Thus, *currency of a country* means the particular type of money in use in that country. Different countries have different types of currencies as shown in the table below.

Country	**Currency**
Uganda	Shilling (**Ush**)
Kenya	Shilling(**Ksh)**
Tanzania	Shilling(**Tsh**)
Ethiopia	Ethiopian Birr
South Africa	South African Rand
Nigeria	Naira (**₦**)
Britain	Sterling Pound (**UK£)**
Europe	**⁰⁄**
Japan	Japanese Yen (**¥**)
India	Indian Rupee
Canada	Canadian Dollar (**C$**)
USA	US Dollar (**US$**)
Sweden	Swedish Kronor (**Kr**)
France	French Francs (**FF**)

9.1.2 Currency conversion:

It is often necessary to exchange the currency of one country for those of other countries. Such exchange is what is known as currency conversion.

Currency conversion is usually done through the following institutions:

- Central bank of a country
- Commercial Banks
- Foreign Exchange (Forex) Bureaus
- Some big hotels

Conversion between various currencies is usually done using currency conversion tables. The figures given in the tables are called exchange rates and they give the equivalent of one currency to units of other currencies.

For instance, the table below shows the exchange rates that were produced by the Central Bank of Uganda and published in the daily monitor in July 2002.

Central Bank Uganda

Exchange Rates

Currency	**Buying**	**Selling**	**Mean**
1US Dollar	2090	3108	2099
1Sterlin Pound	3410	3470	3440
1Euro	2800	3180	2990
1Ksh	26.6	29	27.8
1Tsh	1.2	1.7	1.45
1South African Rand	200	280	240
1Canadian Dollar	1300	1700	1500
1Rwandan Franc	2.5	3.5	3.0
1Sudanese Pound	500	700	600

Example

Use the mean exchange rates in the table above to convert each of the following currencies to the stated equivalent.

a) 150 US Dollars (US$) to Ush

b) 85 Euros to Ush

c) 3050 Ush to Ksh

d) Ush 2000 to US$

e) Ush 6500 to sterling pound (UK£)

Solution

a) $150US\$ \longrightarrow Ush$

$$1US\$ = 2099Ush$$

$$\therefore 150\$ = 2099 \times 150 = \underline{\underline{314850Ush}}$$

b) $85Euros \longrightarrow Ush$

$$1Euro = 2990Ush$$

$$\therefore 85Euros = 2990 \times 85 = \underline{\underline{254150Ush}}$$

c) $3050Ush \longrightarrow Ksh$

$$1Ksh = 27.8Ush$$

$$\Rightarrow 1Ush = \frac{1}{27.8}Ksh$$

$$\therefore 3050Ush = \frac{1}{27.8} \times 3050 = \underline{\underline{109.71Ksh}}$$

d) $Ush2000 \longrightarrow US\$$

$$1\$ = 2099Ush$$

$$\Rightarrow 1Ush = \frac{1}{2099}\$$$

$$\therefore 2000Ush = \frac{1}{2099} \times 2000 = \underline{\underline{0.953\$}}$$

e) $Ush6500 \rightarrow UK\,£$

1£ = 3440 Ush

$$1Ush = \frac{1}{3440}\,pound$$

$$\therefore 6500 = \frac{1}{3440} \times 6500 = \underline{\underline{1.89\,pounds}}$$

Example

Using the exchange rates given in the table above, determine how many Euros are worth 1UK£.

Solution

$$First : Euro \rightarrow Ush$$

$$1Euro = 2990Ush$$

$$\therefore 1Ush = \frac{1}{2990}\,Euro$$

$$Next : Pound \rightarrow Ush$$

$$1pound = 3440Ush$$

$$\therefore 1Ush = \frac{1}{3440}\,pound$$

$$\Rightarrow \frac{1}{3440}\,pound = \frac{1}{2990Euro}$$

$$\therefore 1pound = \frac{1}{2990} \times 3440 = \underline{\underline{1.15Euros}}$$

Example

If the exchange rate for French Franc to Sterling pound is *1£ = 9.00* Francs and *1£ = $1.53* (American Dollars). Find how many American dollars one can get in exchange for 1,000 Francs.

Solution

1£ = 9.00Francs and 1£ = 1.53$

£? = 1,000Francs

$1Franc = \frac{1}{9}£ \therefore 1{,}000Francs = \frac{1}{9} \times 1{,}000£$

But 1£ = 9.0Francs = 1.53$

$\therefore 1Franc = \frac{1.53}{9}\$$

$\Rightarrow 1000Francs = \frac{1.53}{9.0} \times 1000\$ = \underline{\underline{170\$}}$

Example

If the exchange rate of a Kenya shilling to Uganda shilling is $1Ksh = 24Ush$ and an American dollar to Uganda shilling is $1\$ = Ush1{,}950,$ how many American dollars one would get in exchange for Ksh 9,750?

Solution

$1Ksh = 24Ush$

$\$1 = 1950Ush$

$9750\,Ksh = ?\$$

$1Ush = \frac{1}{24}Ksh = \frac{1}{1950}\$$

$\therefore 1Ksh = \frac{24}{1950}\$$

$\Rightarrow 9750\,Ksh = \frac{24}{1950} \times 9750 = \underline{\underline{120\$}}$

Example

A television set costs British pound sterling 220£. Given the exchange rates:

1US$ = 0.75£ and 1US$ = Ush 1,800. Determine the cost of the T.V set in Uganda shillings.

Solution

Cost of T.V set = £220

1$ = 0.75£ and 1$ = Ush 1800

$\Rightarrow 1£ = \dfrac{1800}{0.75}Ush$

$\therefore 220£ = \dfrac{1800}{0.75} \times 22 = \underline{\underline{528{,}000Ush}}$

Example

A musical tape costs pounds Sterling (£) 8.95. Given that $1.56 = £1.00 and Ush 1045 =1$. Find the equivalent cost of the musical tape in:

i. US dollars
ii. Uganda shillings

Solution

Cost of musical tape = £8.95
1.56$ = £1.00 and Ush 1045 = 1.0$

i. In US dollars:

£1 = $1.56

$\therefore 8.95£ = 1.56 \times 8.95 = \underline{\underline{13.962\$}}$

ii. In Uganda shillings:

$1\$ = 1045Ush$
$\therefore 13.962\$ = 1045 \times 13.962 = \underline{\underline{14590.29Ush}}$

Example

Covert 250 US dollars ($) to pound sterling (£) if;

1 US$ = Ush 980 and 1£ = Ush 1750.

Solution

$\$250 \rightarrow £ = ?$

$1\$ = 980Ush$ and $1£ = 1750Ush$

$\therefore 250\$ = 250 \times 980Ush = 24500Ush$

But $1£ = 1750Ush$

$\therefore 1Ush = \frac{1}{1750}£$

$\Rightarrow 24500Ush = \frac{24500}{1750} = 140£$

$\therefore 250\$ = 140£$

9.2 Compound Interest Formula

In S.2, you learnt how to calculate compound interest using *step–by–step* method were the amount at the end of the year is taken to be the principle for the next year.

Some times when calculating compound interest the, the time interval through which the principle is compounded is many and thus the *step–by–step* method proves to be tedious. An easier way in this case is to use the compound interest formula, which is given below.

9.2.1 Compound interest formula:

Amount ***A*** of investment of a principal ***P*** at a compound interest at a rate ***r*** % per annum (p.a) after ***n*** years can be computed using the formula:

$$A = P\left(1 + \frac{r}{100}\right)^n$$

Example

Find the compound interest in 10 years on Shs.1, 050,000 at a rate of 8% p.a.

Solution

$$A = P\left(1+\frac{r}{100}\right)^n, \ p = 1{,}050{,}000Ush, r = 8\%, n = 10yrs$$

$$\therefore A = 1050000\left(1+\frac{8}{100}\right)^{10}$$

$$= 1050000(1.08)^{10}$$
$$= 2266{,}871shs$$

$$Interest = Amount - \Pr incipal$$

$$= 2266871 - 1050000$$
$$= \underline{\underline{1{,}216{,}871shs}}$$

Example

A man invested 900,000shs at 18% compound interest. Find the amount of investment after 2 years.

Solution

$$A = P\left(1+\frac{r}{100}\right)^n, \ p = 900{,}000Ush, r = 18\%, n = 2yrs$$

$$\therefore A = 900000\left(1+\frac{18}{100}\right)^{2}$$

$$= 900000(1.18)^{10}$$
$$= \underline{\underline{1{,}253{,}160shs}}$$

Example

A certain amount of money was invested at a compound interest rate 10% for 5 years. Given that at the end of the period, the owner received Shs. 500,000.

Find the amount originally deposited.

Solution

$$Amount, A = P\left(1+\frac{r}{100}\right)^n, p = ?, r = 100\%, n = 5yrs, A = 500{,}000shs$$

$$\Rightarrow 500{,}000 = P\left(1+\frac{r}{100}\right)^5$$

$$\Rightarrow 500{,}000 = P(1.1)^5$$

$$\Rightarrow 500{,}000 = 1.61051p$$

$$\therefore P = \frac{500{,}000}{1.61051} = \underline{\underline{310{,}461shs}}$$

Example

Juma deposited 10 million on his savings account at the bank at a compound interest rate of 55 per annum. Determine the number of years the money will take to exceed 15 million.

Solution

$$A = P\left(1+\frac{r}{100}\right)^n, p = 10million, r = 5\%, n = ?$$

$$P\left(1+\frac{r}{100}\right)^n > A$$

$$10\left(1+\frac{5}{100}\right)^n > 15$$

$$10(1.05)^n > 15$$

$$\Rightarrow (1.05)^n > \frac{15}{10}$$

Taking log₁₀ on both sides,

$$n\log 1.05 > \log 1.5$$

$$n > \frac{\log 1.5}{\log 1.05}$$

$$n > 8.3104yrs$$

$$\sin ce : n > 8.3104years$$

$$\therefore \underline{\underline{n = 9years}}$$

9.3 Hire Purchase

9.3.1 Definition:

This is a system in which a customer purchases (buys) an item but pays a certain amount first known as *deposit* and the remaining amount is then paid in parts known as *installments* over an agreed period of time.

The Hire Purchase price (H.P) is given by:

H.P = Deposit + Total Installments

9.3.2 Advantages of Hire purchase

* Allows low income earners to enjoy expensive goods.
* The customers enjoy the goods while paying for them.

9.3.3 Disadvantages of Hire purchase

* Hire purchase price is higher than the cash price.
* The customer is issued with a final receipt after making payment for the last installment.
* If the customer fails to pay or fails to complete the last installment, the goods are confiscated and the customer may be required to pay a fine.

Example

The cash price of an electric cooker is Shs. 400,000. If one buys the electric cooker on hire purchase, he has to pay a deposit of Shs. 100,000 and then pay installments of Shs. 15,000 per month for 24 months. Determine:

a) The hire purchase value of the cooker

b) How much more one would pays under hire purchase terms than cash terms?

Solution

a) *Hire purchase price = Deposit + Total amount payable in installments*

$$= 100000 + 15000 \times 24$$
$$= 100000 + 360000$$
$$= \underline{\underline{460{,}000 shs}}$$

b) *H.P price exceeds cash price of the cooker by an amount equal to:*

$$= 460{,}000 - 400{,}000$$
$$= \underline{\underline{60{,}000 shs}}$$

Example

The deposit for an office chair in hire purchase term is indicated as Shs. 150,000. The balance for the office chair is payable in 15 equal monthly installments of shs. 30,000.

A customer who defaults on an installment is charged a penalty of 10% of the defaulted installment payable next month together with the installments due.

Mr. Mwanje bought an office chair on hire purchase and paid the deposit.

a) If Mr. Mwanje defaulted on the 3rd and the 10th months, calculate the penalty charges he had to pay.

b) If Mr. Mwanje had paid cash for the chair and was given allowed 15% cash discount, how much money would he have saved?

Solution

a) *Amount paid in penalty for one month* $= \frac{10}{100} \times 30{,}000 = 3{,}000 shs$

Amount paid in penalty for two months $= 2 \times 3000 = \underline{\underline{6{,}000 shs}}$

b) *If there was no defaulting, the total amount that Mr. Mwanje was to pay for the office chair* $= 150{,}000 + 30{,}000 \times 15 = \underline{\underline{60{,}000 shs}}$

Total amount paid by Mr. Mwanje through hire purchase
$= 600{,}000 + 6{,}000 = 606{,}000 shs$

Therefore the amount of money that could have been saved
$= 606{,}000 - 510{,}000 = \underline{\underline{96{,}000 shs}}$

Example

The following is an advertisement for executive office furniture set:

EXECUTIVE OFFICE FURNITURE

CASH TERMS: *Shs. 1,500,000*

HIRE PURCHASE TERMS:

Either: *i) Deposit 10% of the value and pay Shs. 130,000 monthly for 12 months*

Or: *ii) Deposit 10% of the value and pay Shs. 40,000 for 40 weeks.*

a) Calculate the total amount of money one would pay for the furniture:

i. On a monthly hire purchase
ii. On a weekly hire purchase

b) If the cost of the office furniture is 20% below the cash price, calculate the profit made on:

i. A monthly hire purchase and the percentage profit.
ii. A weekly hire purchase and the percentage profit.

Solution

a) i) Total amount payable on monthly hire purchase:

$$= \left(\frac{10 \times 1{,}500{,}000}{100}\right) + (130{,}000 \times 12)$$
$$= 150{,}000 + 1{,}560{,}000$$
$$= \underline{\underline{1{,}710{,}000 shs}}$$

ii) Total amount payable on weekly hire purchase:

$$= \left(\frac{10 \times 1{,}500{,}000}{100}\right) + (40{,}000 \times 40)$$
$$= 150{,}000 + 1{,}600{,}000$$
$$= \underline{\underline{1{,}750{,}000 shs}}$$

b) *Cost price* **(C.P)** $= \left(\dfrac{80 \times 1{,}500{,}000}{100}\right) = 1{,}200{,}000 shs$

Selling price through weekly hire purchase $= 1{,}750{,}000 shs$

Selling price through monthly hire purchase $= 1{,}710{,}000 shs$

i. *Profit on weekly H.P* $= 1{,}750{,}000 - 1{,}200{,}000 = \underline{\underline{550{,}000 shs}}$

$$\% profit = \frac{550{,}000 \times 100}{1{,}200{,}000} = \underline{\underline{45.8\%}}$$

ii. *Profit on monthly hire purchase*

$= 1{,}710{,}000 - 1{,}200{,}000 = \underline{\underline{510{,}000 shs}}$

$$\% profit = \frac{510{,}000 \times 100}{1{,}200{,}000} = \underline{\underline{42.5\%}}$$

9.4 Taxation

9.4.1 Introduction:

Taxation is the means by which the central government of any country raises fund for running its services such as:

- Defence
- Health
- Education

Taxes are levied annually on all individuals and companies who earn income either by employment or through business. This tax collected is known as revenue.

In Uganda the institution known as Uganda Revenue Authority (URA) is mandated to collect taxes and review the tax rule if necessary.

There are two major categories of tax, namely; direct tax and indirect tax. Two examples of direct tax include:

i) Income tax

ii) Pay as you earn (PAYE)

9.4.2 Income Tax:

This is the tax levied on income generated by an individual, companies, partnership, and sole proprietors.

9.4.3 Common terms:

a. Gross income
This is the total amount of money which an individual or company earns.

b. Taxable income

Before tax is calculated, some deduction in the form of personal allowance is made from the gross income. Tax is then calculated from the remaining amount. This remaining amount is known as *taxable income*.

c. Net income
This is the amount of money left after tax has been deducted.

d. Tax free income

This is any amount of money earned by an individual and is not taxed, e.g.

* Personal allowance
* Medical allowance
* Children allowance
* Electricity allowance, etc.

Taxable income is therefore calculated from:

Taxable income = Gross income –Tax-free income.

9.4.4 Tax rates:

Individual taxpayers are assessed using graduated scale rates. For instance, the table below shows the tax income rates applicable in Uganda in a certain year.

Taxable income (Ush), p.a	**Tax rate**
Shs. 1,560,000 and below	No tax
Shs. 1,560,000 to Shs. 2,820,000 (next 1,260,000)	10% of the amount by which the taxable income exceeds shs. 1,560,000.
Shs.2,820,000 to 4,920,000 (next 2,100,000)	15% of the amount by which by which the taxable income exceeds shs. 2,820,000.
Shs. 4,920,001 and above	25% of the amount by which the taxable Income exceeds shs. 4,920,000.

Example

In a certain country, the income tax is levied as follows:

from it before it is subjected to taxation. This includes family relief and insurance value.

The allowances are as follows:

- Married man Shs. 1,800
- Unmarried man Shs. 1,200
- Each child below 11years Shs. 500
- Each child above 11 but 18 years Shs. 700
- Insurance premium Shs.1,200

Peter earns shs. 64000. He is married with 3 children of ages between 11 and 18 years and 2 children below eleven years. Given that, he is insured and has claimed transport allowances of shs. 1,700.

Calculate:

a) His taxable income.

b) The income tax he pays under the income tax rates below:

Taxable income	**Rate (%)**
0 10,000	10
10,001 20,000	25
20,001 30,000	30
30,001 40,000	45
40,000 and above	50

Solution

a) ***Taxable income = Gross income – Tax free income***

Total allowances

$= (700 \times 3) + 1800 + 1700 + (500 \times 2) + 1200 = 7{,}800 shs$

Gross income $= 64{,}000 shs$

$\therefore Taxable, Income = 64{,}000 - 7{,}800 = \underline{\underline{56{,}200 shs}}$

b) *Taxable income in the first row = shs 10,000*

Therefore income tax $= \frac{10}{100} \times 10000 = 1{,}000 shs$

c) *Taxable income in the second row* $= 20000 - 10000 = 10{,}000 shs$

Therefore income tax $= \frac{25}{100} \times 10000 = 2{,}5000 shs$

d) *Taxable income in the third row* $= 30000 - 20000 = 10{,}000 shs$

Income tax $= \frac{30}{100} \times 10000 = 3{,}000 shs$

e) *Taxable income in the 4th row* $= 40000 - 30000 = 10{,}000shs$

Income tax $= \dfrac{45}{100} \times 10000 = 4{,}500shs$

f) *Taxable income in the 5th row* $= 56200 - 40000 = 16{,}2000shs$

Income tax $= \dfrac{50}{100} \times 16200 = 8{,}100shs$

e) *Total income tax that he pays*

$= 1000 + 2500 + 3000 + 4500 + 8100 = \underline{\underline{19{,}100shs}}$

Example

The table below shows the tax income on taxable income of citizens in the working class of a certain country.

Income (Shs) per annum	**Tax rate (%)**
1st Shs. 80,000	7.5
Next Shs. 80,000 (80,001 160,000)	12.5
Next Shs. 80,000 (160,001 240,000)	20.0
240,001 320,000	30.0
320,001 400,000	36.5
400,001 480,000	45.0

e is Shs. 964,000. The following are the allowances including insurance accrued to him. Housing Shs. 14,000 per month.

i. Marriage, one tenth of his gross annual income

ii. Medical Shs. 50, 700 per annum

iii. Transport Shs. 10,000 per month

iv. He has to pay an insurance premium of shs. 68,900 per annum

v. Family allowances for only four children at the following rates: Shs 3,400 for each child above the age of 18, Shs 4,200 for each child above 10 but below 18 years and Shs. 5,400 for each child below 9 years. Given that he has a family of five children with three of them below the age of 8, one 16 years and the elder child 20 years.

Determine:

a) His taxable income

b) The income tax he pays annually as a percentage of his gross annual income.

Solution

a) ***Taxable income = Gross income – Tax free income.***

Allowances:

- *Housing* $=14500\times 12=174{,}000/-\ p.a$
- *Marriage allowance* $=\frac{1}{10}\times 964000=964{,}000 shs$
- *Medical* $=50{,}700 shs$
- *Transport* $=10000\times 12=120{,}000/-\ p.a$
- *Insurance* $=68{,}900 shs$
- *Family allowances will be due to the three children below the age of 8 and one of 16 years* $=5400\times 3+4200=20{,}400 shs$

Total tax-free income

$=174000+96400+50700+120000+68900+20400\ =530{,}400 shs$

Taxable income $=964{,}000-530{,}400=\underline{\underline{433{,}600 shs}}$

b)

Income (Shs) per annum	***Tax rate (%)***	***Income tax***
1st Shs. 80,000	*7.5*	$\frac{7.5}{100}\times 80{,}000=6{,}000 shs$
80,001 – 1 0,000	*12.5*	$\frac{12.5}{100}\times(160{,}000-80{,}000)=10{,}000 shs$
160,001 – 240,000	*20.0*	$\frac{20}{100}\times(240{,}000-160{,}000)=16{,}000 shs$
240,001 – 320,000	*30,0*	$\frac{30}{100}\times(320{,}000-240{,}000)=24{,}000 shs$
320,001 – 400,000	*36.5*	$\frac{36.5}{100}\times(400{,}000-320{,}000)=29{,}200 shs$
400,001 – 433,600	*45.0*	$\frac{45}{100}\times(433{,}600-400{,}000)=15{,}120 shs$

Total income tax
$= 6{,}000 + 10{,}000 + 16{,}000 + 29{,}200 + 15{,}120 = 100{,}320 shs$

The income tax he pays as a percentage of his gross annual income

$$= \frac{100{,}320}{964{,}000} \times 100$$
$$= \underline{\underline{10.4\%}}$$

9.5 Miscellaneous exercise

1. A camera costs pound sterling £9.50 in UK and US dollars $9.80 in USA. Given that £1.26 = $2. In which country would one prefer to buy the camera and how much pound sterling would one save?

2. When the exchange rate was Ush 1,860 to 1 US$, a tourist who was living in Uganda changed Ush 24,950 at the airport bank. If a commission of Ush 260 was charged, how many dollars did he get?

3. A trader imported an item worth 10,000 Yen from Japan. If this item was subjected to 25% import tax in Uganda, how much was it worth in Uganda shillings if the exchange rate was Ush 1,449.36 to 100 Japanese yen at that time.

4. Hudson wants to go for holidays in USA and he needs dollars. The selling rate is Ush 1,800 for 1dolar. How many dollars will he get for720, 000 Ush?

5. The crested Forex Bureau is offering the following rates for pound sterling:
   Buy at Ush 1,600
   Sell at Ush 1,700

   a) How many pounds do you get for Ush 10,000
   b) How many shillings would you get for £5000

6. Stella borrows £2,000 for 3 years at a compound interest rate of 5% per annum. How much money does she replay altogether?

7. Mr. Opio deposited 1.321 million shillings in his bank account at a compound interest rate of 7.5% per annum. Determine the number of years his money will take to accumulate to 1.75 million shillings?

8. Mr. Lwanga and Mr. Okot were each given Uganda shillings 980,000 at the beginning of 1999. Mr. Lwanga exchanged his money to US dollars and then banked it on his foreign currency account at a compound interest rate of 2% per annum while Mr. Okot banked his money without exchanging it at a compound interest rate of 12% per annum. The exchange rates in 1999 and 2000 were, Ush 1,250 and Ush 1,500 to a US dollar respectively. If Okot withdrew Shs. 120,000 at the end of 2000:

   a) Calculate the amount of money (in Ush) each man had in the bank at the end of 200

   b) Who had more money and by how much?

9. Mr. Omona borrowed Shs. 500,000,000 from stanbic bank at 5% p.a compound interest. After some years he paid back Shs. 578,813,000 without any additional charge. Find the number of years for which he borrowed the money.

10. Mr. Ben borrowed 14.8m to boost his business at a bank rate of 12% compound interest p.a. Mr. Ben has to repay the loan and interest within two years. He is to repay these bank dues in six equal installments. Calculate:

    a) Total amount Mr. Ben paid to the bank

    b) Interest Mr. Ben paid to the bank

    c) The amount of money Mr. Ben paid per installment.

11. The price of a modern mobile phone is marketed at Shs. 60,000. If one pays cash, he gets a discount of 5%, but if one buys at a hire purchase terms, he pays a deposit of half the market price and pays the rest in monthly installments for 15 months Shs. 2,100 each month.

    a) If Mr. Okello opted to pay the phone on cash terms, how much would he pay?

    b) If Jane opted to buy the phone on hire purchase terms, how much more would she pay than Okello?

12. The deposit for an office chair in hire purchase shop is indicated by sh. 1,400. The balance is payable in equal installments of shs. 210. A customer who defaults on an installment is charged a penalty of 10% of the defaulted installment payable in the next month together with installment due. Mr. Ojok bought a chair on hire purchase terms and paid a deposit. If Mr. Ojok defaulted twice in the 3rd and 10th months.

    a) i) Calculate the penalty charges that he paid.

    ii) What was the total cost of the chair?

    b) If Mr. Ojok had paid, cash for the chair and was allowed 15% cash discount. How much money would he have saved?

13. The following advert appeared in the Sunday Vision.

USED COMPUTERS

Price: *Shs. 500,000*

Terms: **(1) Cash:** *4% discount*

(2) Hire purchase: *deposit 45% of marked price then equal monthly installments of 10% of the marked price for 7 months or Shs. 5,500 per week for 6 weeks.*

Find how much a customer saves by paying cash other than hire purchase on:

a) Monthly basis.

b) Weekly basis.

14.

allowances

Type of allowance	**Amount (shs)**
Legally married teacher	10,000
Each child under 10 years	2,500
PTA	50,000
Head of department/subject	10,000
Class teacher	5,000
Housemaster/mistress	5,000
Unmarried teacher	6,000
Each child above 10 years	2,000

Mr. Birungi and Mr. Serubiri are senor teachers in a certain school. Mr. Birungi is married with two children under the age of 10 years and one child above 10 years. He is also a class teacher and head of commerce department.

Mr. Serubiri is single but has two children under the age of 10 years and is also a house master and a class teacher.

The gross income at the end of the month are each subjected to a PAYE (pay as you earn) which has the following rate for the 1st shs 10,000 taxable income, the tax is 20% while the rest is taxed at 15% at the end of the month.

gross income was shs. 130,000.

a) Calculate the taxable income for each teacher

b) Calculate the tax paid as a percentage of the gross income for each teacher.

15. James works with a certain NGO. James is paid a monthly salary of Shs 750,000. The NGO gives some allowances to each of her employees earning Shs. 500,000 and above according to the following schedule.

Allowance	**Rate**
Medical	monthly salary exceeds shs. 500,000
Lunch	monthly salary exceeds shs. 500,000
Transport	monthly salary exceeds shs. 500,000
Child allowance (per child aged 11 and below)	monthly salary exceeds shs. 500,000
Child allowance (per child aged 12 and below 19 years)	monthly salary exceeds shs. 500,000
Electricity	Shs. 20,000
Per spouse/wife	Shs. 25.000

James has three wives, 6 children aged below 12 years and 3 children aged 12 and above but under 19. Determine the total amount James receives from the NGO before taxation

16. The table below shows tax rates for employees of a certain firm.

Total monthly income	**Rate**
Below shs. 130,000	2%
Shs. 130,000 shs.199,000	20% by which the total monthly income exceeds shs. 130,000
Shs. 200,000 shs.299,000	30% by which the total monthly income exceeds shs. 130,000
Shs. 300,000 and above	40% by which the total monthly income exceeds shs. 300,000

What tax does an employer whose monthly income is shs. 1,200,000 pay?

17. The table below shows the tax structure on taxable income of employees in a certain company.

Income per month	**Tax rate (%)**
0 40,000	Free
40,001 100,000	10.0
100,001 200,000	16.5
200,001 350,000	23.5
350,001 510,000	32.0
Above shs. 510,000	40.0

An employee earns shs. 9,000,000 per month. His allowances include:

- Marriage allowance = one fifteenth of his gross monthly income.
- Water and electricity = shs. 180,000 p.a
- Relief and insurance = shs. 15,000 per month
- Housing allowance = shs. 40,000 per month
- Medical allowance = shs. 36,00 per month
- Transport allowance = shs.300,000 p.a
- Family allowance for four children only, as given below: *for children in the age 0 – 10 years, shs. 12,500 per child, 10 –18 years shs. 8,250 per child and over 18 years, shs. 5,000 per child.*

a)
that he has 3 children two of whom are age 0 10 years and the other 13 years.

b) Calculate the percentage of his gross income that goes to tax.

18. In Ghana, tax is levied on Government employees after deducting allowances as follows:

Amount (Shs)	**Rate**
1^{st}150,000	5%
Next 100,000	7.5%
Next 150,000	10%
Next 200,000	15%
Next 200,000	25%
Next 200,000	40%
Extra amount	45%

The following are the entitled allowances:

- Electricity: Shs. 480,000 per annum.
- Housing: Shs. 80,000 per annum.
- Medical care: Shs. 840, 000 per annum.
- Child care: (only two children below 16 years), shs. 15,000 per child

Given that, an unmarried employee paid shs. 150,000 as monthly income tax;

Calculate:

a) His monthly taxable income.

b) His gross monthly income.

c) His monthly net income.

10 PROBABILITY

10.1 Introduction:

Probability is used to tell how likely or unlikely a future will occur. The concept of probability is used to frequently answer some questions in our daily life. Some of these questions include:

1. Will Uganda Crane qualify for the 2012 Africa cup of nation?
2. Which party will win the 2016 presidential election in Uganda?
3. Will he undergo a successful heart operation?

To be in position to predict correctly how likely or unlikely a future event will occur, an experiment should be carried out randomly about the event and the experimental results analyzed and appropriate conclusion made out of it.

10.2 Common terms used:

1. Outcome of an experiment.

   What comes out or what results what results when an experiment has been performed is what is known as outcome.

2. Sample space or probability space.

   This is a set of possible outcomes of an experiment. A sample space is denoted by the letter **S**. The number of outcome of ***S*** is denoted by ***n(S).*** For instance, consider tossing a coin twice. The following are the possible outcome of an experiment: HH, HT, TH, TT, where H and T stand for head and tail respectively. Thus the sample space S = {HH, HT, TH, TT} and n(S) = 4

3. An event.

   An event is a subset of sample space consisting of sample point of interest. It is denoted by the symbol **E**. the number of items of an event is denoted by **n (E)**

The probability of an event denoted as **P (E)** is defined as:

$$P(E) = \frac{n(E)}{n(S)}$$

Consider tossing a coin twice as in the previous case. Here the event could be getting two heads i.e. {HH}, at least a head i.e. {HH, HT, and TH}

10.3 Range of probability measure

a) A sample space is an event with probability 1,i.e.

$$P(S) = \frac{n(S)}{n(S)} = 1$$

b) An empty set Ø or{ } is an event with probability 0, since n(Ø)=0

$$P(\phi) = \frac{n(\phi)}{n(S)} = \frac{0}{n(S)} = 0$$

c) Since $0 \le n(E) \le n(S)$

$$\Rightarrow \frac{0}{n(S)} \le \frac{n(E)}{n(S)} \le \frac{n(S)}{n(S)}, \quad but\ P(E) = \frac{n(E)}{n(S)}$$
$$\therefore 0 \le P(E) \le 1$$

So the probability of any event lies between zero (0) and 0ne.

Example

Three coins are tossed. State the possibility space and use it to find:

a) P(3 heads)

b) P(2 heads)

c) P(at most one head)

Solution

The sample space S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

a) *3 heads ={HHH}*

$$n(3\ heads) = 1, n(S) = 8$$

$$\therefore P(3\ heads) = \frac{n(3\ heads)}{n(S)} = \frac{1}{8}$$

b) *2 heads = {HHT, HTH, THH}*

$$n(2\ heads) = 3$$

$$\therefore P(2\ heads) = \frac{n(2\ heads)}{n(S)} = \frac{3}{8}$$

c) *At most one head = {HTT,THT, TTH, TTT}*

$$n(at\ most\ one\ heads) = 4$$

$$\therefore P(at\ most\ one\ head) = \frac{n(at\ most\ one\ head)}{n(S)} = \frac{4}{8} = \frac{1}{2}$$

Example

A fair die is rolled once. Calculate the probability of getting:

a) An even number

b) A prime number

c) A score of 6

d) A score of 10

Solution

A die is numbered 1 to 6. If it is rolled once, the possible outcomes are:

Sample space S = {1, 2, 3, 4, 5, 6}, n(S) = 6

a) *An even number = {2, 4, 6}*

$$n(even\ number) = 3$$

$$\therefore P(even\ number) = \frac{3}{6} = \frac{1}{2}$$

b) *A prime or odd number = {1, 2, 3, 5}*

$n(prime\ or\ odd\ number) = 4$

$\therefore P(prime\ or\ odd\ number) = \frac{4}{6} = \frac{2}{3}$

c) *A score 6 = {6}*

$n(score\ 6) = 1$

$\therefore P(score\ 6) = \frac{1}{6}$

d) *Score 10 = { }*

$n(score\ 10) = 0$

$\therefore P(score\ 10) = \frac{0}{6} = 0$

Example

A bag contains 10 bottle tops of which 3 are of drink **A**, 5 are of drink **B** and 2 are of drink **C**. If a bottle top is picked from the bag at random, what is the probability that it is:

a) a drink A?

b) a drink B?

c) a drink C?

Solution

$Let\ A\ represent\ drink\ A$

$B\ represent\ drink\ B$

$C\ represent\ drink\ C$

$Sample\ space\ S = \{A, A, A, B, B, B, B, B, C, C\}$

$\therefore n(S) = 10$

a) *Outcome for drink A = {A, A, A}*

$n(drink\ A) = 3$

$\therefore P(drink\ A) = \frac{3}{10}$

b) *Outcome for drink B = {B, B, B, B, B}*

$$n(drink\ B) = 5$$

$$\therefore P(drink\ B) = \frac{5}{10} = \frac{1}{2}$$

c) *Outcome for drink C = {C, C}*

$$n(drink\ C) = 2$$

$$\therefore P(drink\ C) = \frac{2}{10} = \frac{1}{5}$$

Example

A number is to be selected at random from numbers 1 to 20 inclusive. Determine the probability that the numbers selected at random is:

a) prime

b) divisible by 3

c) an even number

d) divisible by either 2 or 3

e) divisible by 2 but not divisible by 3

Solution

Sample space S = {1, 2, 3, 4, 5,6,7,8,9,10,11,12,13,14,15,16, 17, 18, 19, 20}

n(S) =20

a) *Prime numbers = {2, 3, 5, 7, 11, 13, 17, 19}*

$$n(\text{Primes}) = 8$$

$$\therefore P(\text{Primes}) = \frac{8}{20} = \frac{2}{5}$$

b) *Numbers divisible by 3 = {3, 6, 9, 12, 15, 18}*

$$n(numbers\ divisible\ by\ 3) = 6$$

$$\therefore P(numbers\ divisible\ by 3) = \frac{6}{20} = \frac{3}{10}$$

c) *Even numbers = {2, 4, 6, 8, 10, 12 ,14 ,16, 18, 20}*

$$n(even\ numbers) = 10$$

$$\therefore P(even\ numbers) = \frac{10}{20} = \frac{1}{2}$$

d) *Numbers divisible by 2 or 3 = {2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20}*

$$n(numbers\ divisible\ by\ 2\ or\ 3) = 13$$

$$\therefore P(numbers\ divisible\ by\ 2\ or\ 3) = \frac{13}{20}$$

e) *Numbers divisible by 2 but not 3 = {2, 4, 8, 10, 14, 16, 20}*

$$n(numbers\ divisible\ by\ 2\ but\ not\ 3) = 7$$

$$\therefore P(numbers\ divisible\ by\ 2\ but\ not\ 3) = \frac{7}{20}$$

Example

The table below shows the length of words in certain chapter of physics textbook.

Number of letters	Frequency
1	30
2	80
3	40
4	60
5	70
6	80
7	90
8	50

If a word is selected at random from the words in the chapter, find the probability that:

a) the word selected has 6 letters

b) a word with 5 or 8 letters is selected

c) a word with less than 8 letters is selected

d) a word with at least 4 letters is selected

e) a word with 9 letters is selected

Solution

n(S) = 30 + 80 + 40 + 60 + 70 + 80 + 90 + 50 = 500

a) n(a word with 6 letters) = 80

$$\therefore P(a\ word\ \ selected\ \ has\ 6\ letters) = \frac{80}{500} = \frac{4}{25}$$

b) n(a word with 5 or 8 letters) = 70 + 50=120

$$\therefore P(a\ word\ \ selected\ \ has\ 5\ or\ 8\ letters) = \frac{120}{500} = \frac{5}{25}$$

c) n(a word with less than 8 letters) = 30 + 80 + 40 + 60 +70 + 80 + 90 = 450

$$\therefore P(a\ word\ \ selected\ \ has\ less\ than\ 8\ letters) = \frac{450}{500} = \frac{9}{10}$$

d) n(a word with at least 4 letters) = 60 +70 + 80 + 90 = 350

$$\therefore P(a\ word\ \ selected\ \ has\ at\ least\ 4\ letters) = \frac{350}{500} = \frac{7}{10}$$

e) n(a word with 9 letters) = 0

$$\therefore P(a\ word\ \ selected\ \ has\ 9\ letters) = \frac{0}{500} = 0$$

Example

Two dice are simultaneously thrown and the sum of their scores observed. Find the probability of getting:

a) a sum which is an even number

b) a sum which exceeds 10

c) a sum which is divisible by 2 or 3

Solution

The possibility space for the two dice tossed and the sum on their uppermost faces can be summarized in a table as below.

		Die 2					
		1	***2***	***3***	***4***	***5***	***6***
Die 1	***1***	*2*	*3*	*4*	*5*	*6*	*7*
	2	*3*	*4*	*5*	*6*	*7*	*8*
	3	*4*	*5*	*6*	*7*	*8*	*9*
	4	*5*	*6*	*7*	*8*	*9*	*10*
	5	*6*	*7*	*8*	*9*	*10*	*11*
	6	*7*	*8*	*9*	*10*	*11*	*12*

*n(**S**) = 6 x 6 = 36*

*a) Let **E** be the event that even sum obtained*

Therefore E = {2, 4, 4, 4, 6, 6, 6, 6, 6, 8, 8, 8, 8, 8, 10, 10, 10, 12}.

*n (**E**) = 18*

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{18}{36} = \frac{1}{2}$$

b) *Let **T** be the event that sum exceeds 10*
*Therefore **T** = {11, 11, 12}. n (**T**) = 3*

$$\therefore P(T) = \frac{n(T)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

c) *Let **D** be the event that sum is divisible by 2 or 3.*
*Therefore **D** = {2, 3, 3, 4, 4, 4, 6, 6, 6, 6, 6, 8, 8, 8, 8, 8, 9, 9, 9, 9, 10, 10, 10, 12}. n (**D**) = 24*

$$\therefore P(D) = \frac{n(D)}{n(S)} = \frac{24}{36} = \frac{2}{3}$$

Example

A die and a coin are tossed. Find the probability of getting:

a) a tail and even number

b) a tail and a number not less than 4

c) a head and a triangular number

Solution

Coin	***Die***						
		1	***2***	***3***	***4***	***5***	***6***
	H	*(H, 1)*	*(H, 2)*	*(H, 3)*	*(H, 4)*	*(H, 5)*	*(H, 6)*
	T	*(T, 1)*	*(T, 2)*	*(T, 2)*	*(T, 3)*	*(T, 4)*	*(T, 5)*

*n(**S**) = 2 x 6 = 12*

a) *Let **A** be the event of obtaining a tail and even number*
***A** = {(T, 2), (T, 4), (T, 6)} and n (**A**) = 3*

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{3}{12} = \frac{1}{4}$$

b) *Let **B** be the event of obtaining a tail and a number not less than 4.*
***B** = {(T, 4), (T, 5), (T, 6)} and n (**B**) = 3*

$$\therefore P(B) = \frac{n(B)}{n(S)} = \frac{3}{12} = \frac{1}{4}$$

c) *Let **C** be the event of obtaining a head and a triangular number.*
***C** = {(H, 1), (H, 3), (H, 6)} and n (**C**) = 3*

$$\therefore P(C) = \frac{n(C)}{n(S)} = \frac{3}{12} = \frac{1}{4}$$

10.4 Tree Diagram

A tree diagram can be used to generate a sample space of an experiment

10.4.1 Independent event

Events **A** and **B** are said to be independent events if their joint occurrence is equal to the product of their individual probabilities i.e.

$$P(A \cap B) = P(A) \bullet P(B)$$

Note:

If two or more events are independent, then the occurrence of one is not affected by the occurrence of the other.

Example

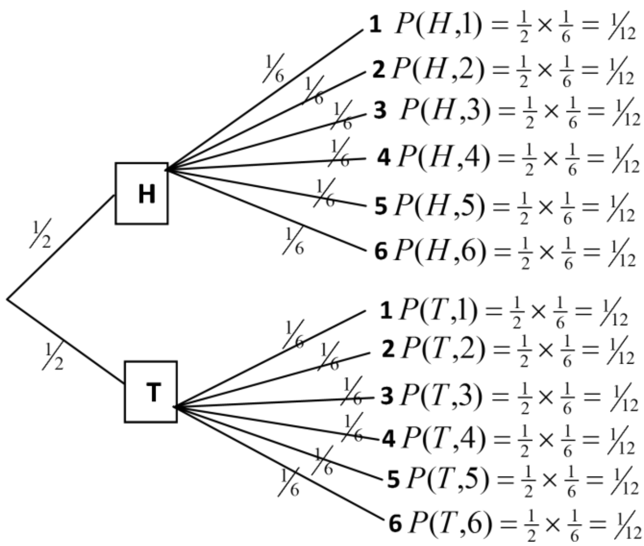
Given that, a coin and a fair die are tossed once.

a) Use a tree diagram to generate a sample space.

b) What is the probability of obtaining a head and an even number?

Solution

a) *Let **H** and **T** represent head and tail of a coin respectively. For a coin, a head and a tail are equally likely events.*
$\therefore P(H) = P(T) = \frac{1}{2}$. *For a die, all the six faces are equally likely to show up.* $\therefore P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$



*b) Let **E** stand for even number.*

$$
\begin{aligned}
P(H \cap E) &= P(H,2) + P(4) + P(6) \\
&= 1/12 + 1/12 + 1/12 \\
&= \underline{\underline{1/4}}
\end{aligned}
$$

Example

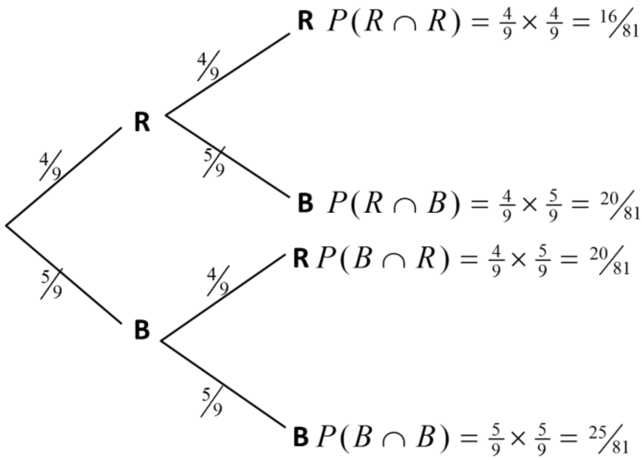
A bag contains 4 red and 5 blue beads. Two beads are taken out of it. What is the probability that they are both blue?

a) if the first bead is put back

b) if the first bead is not replaced

Solution

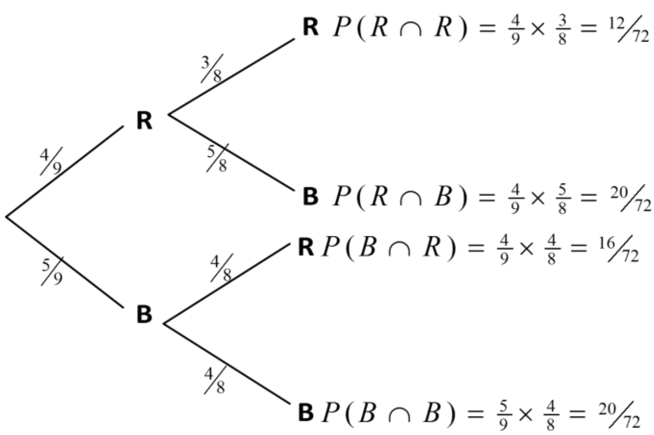
a) *Let* ***R*** *stands for red bead and* ***B*** *for blue bead*



From the tree diagram, $P(B \cap B) = \frac{5}{9} \times \frac{5}{9} = {}^{25}\!/_{81}$

b) *When the bead is not replaced, the number of beads in the second group of branches will drop to 8. If the first bead removed was blue, then there will be 4 blue and 4 red beads left. However, if the first removed was red, there will be 3 red and 5 blue beads left.*

The probabilities in the second group of branches will therefore be different. The tree diagram will look like the one below.



$\therefore P(B \cap B) = \frac{5}{9} \times \frac{4}{8} = {}^{20}\!/_{72} = {}^{5}\!/_{18}$

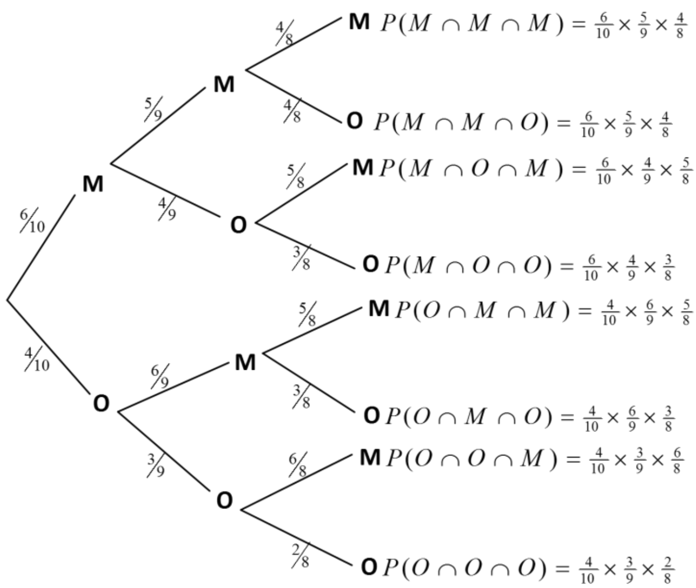
Example

A basket contains 6 mangoes and 4 oranges. Three fruits are removed from it without replacement. Use tree diagram to work out the following probabilities

a) P(three mangoes are removed)
b) P(a mango and two oranges are removed)

Solution

*Let **M** stands for mangoes and **O** for oranges*



a) *From the diagram three mangoes is **M**, **M**, **M***

$$\Rightarrow P(M \cap M \cap M) = \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{120}{720} \quad = \frac{1}{6}$$

b) *A mango and two oranges is (**M**n**O**n**O**), (**O**n**M**n**O**) and (**O**n**O**n**M**)*

$$\Rightarrow P(\text{a mango and two oranges}) = P(M \cap O \cap O) + P(O \cap M \cap O) + P(O \cap O \cap M)$$

$$= \left(\frac{6}{10} \times \frac{4}{9} \times \frac{3}{8}\right) + \left(\frac{4}{10} \times \frac{6}{9} \times \frac{3}{8}\right) + \left(\frac{4}{10} \times \frac{3}{9} \times \frac{6}{8}\right)$$

$$= \frac{3}{10}$$

10.5 Venn diagram

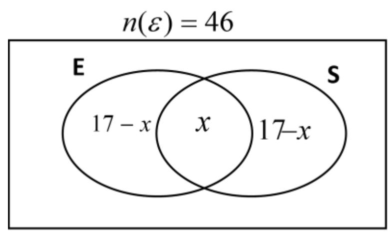
Venn diagrams can be used to solve probability problems.

Example

In a class, 30 pupils like English and 17 like science. All 46 students say they like at least one of these subjects. What is the probability that a pupil chosen at random like exactly one subject?

Solution

*Let **E** and **S** stand for English and Science respectively*



$$30 - x + x + 17 - x = 46$$
$$\therefore x = 46 - 45 = 1$$

Therefore, the number of pupil who like science only = 16, both = 1 and English only = 29

$$n(exactly\ one\ subject) = 29 + 16 = 45$$
$$\therefore P(exactly\ one\ subject) = {}^{45}\!/_{46}$$

Example

A group of 60 tourists visited three tourist sites in Uganda. The tourist sites visited were Bujagali falls (**B**), Mount Elgon (**E**), and Bwindi forest reserve (**R**). each of the 60 tourists visited at least one of the sites as follows. 38 visited Bujagali falls, 35 visited mount Elgon, 31 visited Bwindi forest reserve, 19 visited both mount Elgon and Bwindi forest, 21 visited both Bujagali and Bwindi forest, and 20 visited both Bujagali falls and mount Elgon. Using a Venn diagram, determine;

a) the number of tourist who visited all the three sites

b) the number of tourist who visited only mount Elgon

c) the probability that a tourist picked at random from the group will have visited only one site

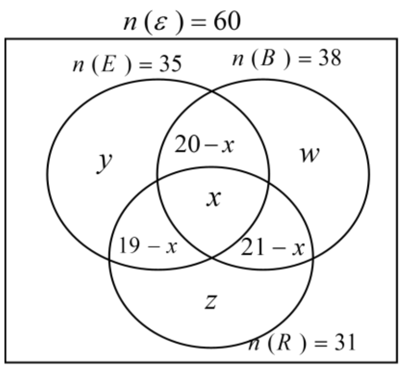
Solution

$n(\varepsilon) = 60,\ n(B) = 38,\ n(E) = 35,\ (R) = 31$

$n(B \cap E) = 20,\ n(E \cap R) = 19,\ n(B \cap R) = 21$

$Let\ n(E \cap B \cap R) = x \quad and\ y, w\ and\ z\ be\ the\ number\ of\ tourist\ who\ visited\ only$

$\text{Elgon},\ Bujagali\ and\ Bwindi\ \ forest\ respectively$



<u>For Elgon alone</u>

$$y + 20 - x + x + 19 - x = 35$$
$$y = 35 - 20 - 19 + x$$
$$\therefore\ y = x - 4 \text{.......................................}(1)$$

<u>For Bujagali alone</u>

$$w + 20 - x + x + 21 - x = 38$$
$$w = 38 - 20 - 21 + x$$
$$\therefore\ w = x - 3 \text{.......................................}(2)$$

<u>For Bwindi forest alone</u>

$$z + 19 - x + x + 21 - x = 31$$
$$z = 31 - 19 - 21 + x$$
$$\therefore\ z = x - 9 \text{.......................................}(3)$$

$$But \quad w + z + x + 20 - x + 21 - x + 19 - x = 60$$
$$\Rightarrow \quad x - 3 + x - 9 + x - 4 + 60 - 2x = 60$$
$$\Rightarrow \quad 3x - 2x = 60 - 44$$
$$\therefore\ \ x = 16$$

a) $n(B \cap E \cap R) = 16$

b) $Only\ mt\ Elgon = x - 4 = 16 - 4 = 12$

c) $Number\ who\ visited\ only\ one\ site\ = w + y + x$

$$= 12 + (16 - 3) + (16 - 9)$$
$$= 32$$

$$P(visited\ exactly\ one\ site) = \frac{32}{60} = \underline{\underline{\frac{8}{15}}}$$

10.6 Miscellaneous exercise

1. An integer between 10 and 30 (inclusive) is chosen at random. What is the probability that the chosen integer is:
   i. prime
   ii. divisible by 2, 3, or 5.
   iii. a triangle number
   iv. a factor of 240

2. A box contains 4 red and 6 blue pens. A pen is picked at random from the box and not replaced. Another pen is then picked from the box. What is the probability that:
   i. the first pen was red?
   ii. both pens were red?
   iii. the pens were of different colors?

3. Two dice are tossed and positive difference between the numbers on both of their upper most faces recorded. Find the probability of getting a positive difference which:
   i. is one
   ii. is divisible by 2
   iii. is at least 2

4. Faces of regular octahedron are marked with integers 1 to 8 respectively and those of an unbiased die are marked 1, 2, 3, 4, 5, and 6. The octahedron and a die are simultaneously tossed and the scores for the experiment are recorded as shown in the table below.

	Octahedron								
		1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Die	**1**	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(1, 7)	(1, 8)
	2								
	3								
	4					(4, 5)			
	5								
	6			(6, 3)					(6, 8)

a) Copy and complete the table above.

b) Determine the probability of getting:

i. an even number on the topmost face of the die and a prime number on top most face of the octahedron.

ii. more than five on the top most face of the octahedron but less than 3 on the top most face of a die.

iii. at least a five on the top most face of octahedron but at most a four on the top most face of the die.

iv. an even number appearing on the top most face of the die and the octahedron.

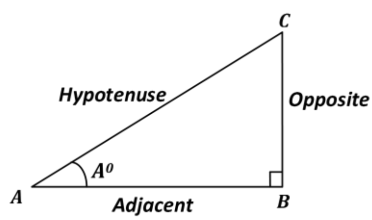
11 TRIGONOMETRY (SINE, COSINE, AND TANGENT)

11.1 Introduction:

Trigonometry is the branch of mathematics concerned with the relationships between the lengths of the sides of triangles and their angles. Under this topic, we shall look at what sine, cosine, and tangent stand for.

11.2 Right-angled Triangles:

These are triangles whose one of the angles is 90°. Now consider the right-angled triangle ABC shown below. Angle A is an acute angle.



- The longest side AC of the triangle is called the ***hypotenuse***.
- The side BC opposite to angle Ao is called the ***opposite***.
- The side AB adjacent to angle Ao is called the ***adjacent***.

11.3 Definition of sine, cosine, and tangent:

a) The ratio $\frac{side\ BC}{side\ AC}$ is called the sine of angle A

b) The ratio $\frac{side\ AB}{side\ AC}$ is called the cosine of angle A

c) The ratio $\frac{side\ BC}{side\ AB}$ is called the tangent of angle A

Sine, cosine, and tangent are abbreviated as sin, cos, and tan respectively. Therefore, sine, cosine, and tangent of angle A^o are denoted as $\sin A^o$, $\cos A^o$, and $\tan A^o$ respectively and are defined respectively as:

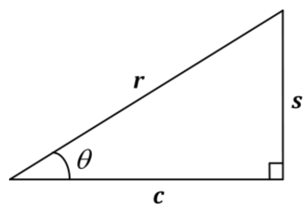
$$SinA^0 = \frac{BC}{AC} = \frac{opposite}{hypotenuse}$$

$$CosA^0 = \frac{AB}{AC} = \frac{adjacent}{hypotenuse}$$

$$TanA^0 = \frac{BC}{AC} = \frac{opposite}{adjacent}$$

11.4 Relationship between sine, cosine, and tangent:

Consider a right-angled triangle below with opposite side ***S*** units, adjacent ***C*** units and hypotenuse ***r***, units.



By definition:

$$\sin\theta = \frac{s}{r} \quad \Rightarrow s = r\sin\theta$$

$$\cos\theta = \frac{c}{r} \quad \Rightarrow c = r\cos\theta$$

$$\tan\theta = \frac{s}{c} \quad but\ s = r\sin\theta\ and\ c = r\cos\theta$$

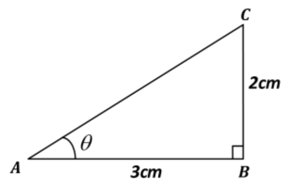
$$\therefore \tan\theta = \frac{r\sin\theta}{r\cos\theta} = \frac{\sin\theta}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

This is the relationship that connects sine, cosine, and tangent.

Example

Consider a right-angled triangle below whose adjacent is 3cm and opposite 2cm.



Find:

a)

b)

c)

Solution

a) $$\tan\theta = \frac{BC}{AB} = \frac{2}{3} = \underline{\underline{0.67}}$$

b) $$\sin\theta = \frac{BC}{AC} \quad but\ AC = unknown$$

Using Pythagoras theorem,

$$AC^2 = BC^2 + AB^2$$

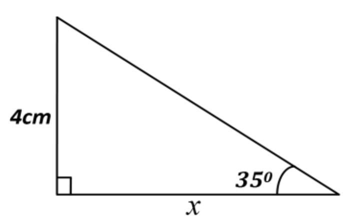
$$\therefore AC = \sqrt{2^2 + 3^2} = \sqrt{13} = 3.61cm$$

$$\Rightarrow \sin\theta = \frac{2}{3.61} = \underline{\underline{0.55}}$$

c) $$\cos\theta = \frac{AB}{AC} = \frac{3}{3.61} = \underline{\underline{0.83}}$$

Example

Find the length indicated as x in the figure below



Solution

The two indicated sides, 4cm and x are the opposite and adjacent respectively. Therefore from:

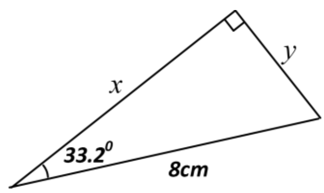
$$\tan 35^0 = \frac{4}{x}, \quad \textit{by cross – multiplication}$$

$$\Rightarrow x = \frac{4}{\tan 35^0} = \frac{4}{0.7002}, \quad \textit{where } \tan 35^0 = 0.7002 \textit{ from your calculator}$$

$$\therefore x = \underline{\underline{5.71cm}}(3.s.f)$$

Example

Find the values of x and y in the figure below correct to two significant figures.



Solution:

X is adjacent to angle 33.2^o and y is opposite angle 33.2^o. 8cm is the longest side i.e. hypotenuse.

From:

$$\sin\theta = \frac{opposite}{hypotenuse} \quad \Rightarrow \sin 33.2^0 = \frac{y}{8}$$

$$\therefore y = 8 \times \sin 33.2^0 = 8 \times 0.5476$$
$$= \underline{\underline{4.4cm}}(2.s.f)$$

Also from:

$$\cos\theta = \frac{adjacent}{hypotenuse} \quad \Rightarrow \cos 33.2^0 = \frac{x}{8}$$

$$\therefore x = 8 \times \cos 33.2^0 = 8 \times 0.8368$$
$$= \underline{\underline{6.7cm}}(2.s.f)$$

Example

Given that $\sin\theta = \frac{3}{5}$ and $\tan\theta = \frac{3}{4}$. Find $\cos\theta$ without using a calculator.

Solution

***Method 1*:**

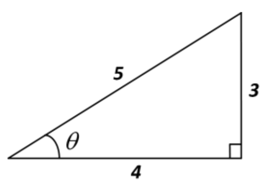
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \Rightarrow \quad \cos\theta = \frac{\sin\theta}{\tan\theta}$$

$$\therefore \cos\theta = \frac{3/5}{3/4} = \frac{3}{5} \times \frac{4}{3} \quad = \quad \underline{\underline{4/5}}$$

Method 2:

$$\sin\theta = \frac{3}{5} = \frac{oppossite}{hypotenuse} \qquad \tan\theta = \frac{3}{4} = \frac{opposite}{adjacent}$$

$$opposite = 3$$
$$\Rightarrow hypotenuse = 5$$
$$adjacent = 4$$



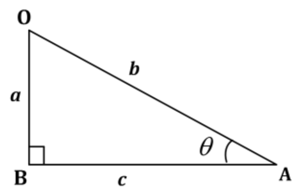
$$\therefore \underline{\underline{\cos\theta = 4/5}}$$

Example

$$\cos\theta = \tfrac{12}{13}$$

Solution:

First sketch the right angled triangle as below



$\cos\theta = \frac{c}{b} = \frac{12}{13} \Rightarrow c = 12, b = 13, a = unknown$

By Pythagoras theorem,

$b^2 = a^2 + c^2$

$\Rightarrow a = \sqrt{b^2 - c^2} = \sqrt{13^2 - 12^2} = \sqrt{25}$

$\therefore a = 5$

Hence;

$\sin\theta = \frac{a}{b} = \underline{\underline{\frac{5}{13}}}$

$\tan\theta = \frac{a}{c} = \underline{\underline{\frac{5}{12}}}$

Example

Given that $\sin\Phi = \frac{3}{5}$, where Φ is an acute angle.

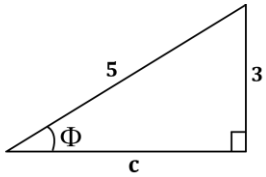
Find;

a) $\cos\Phi$

b) $\sin\Phi + \cos\Phi$

Solution

$\sin\Phi = \frac{3}{5}$



By Pythagoras theorem;

$$5^2 = c^2 + 3^2 \Rightarrow c^2 = 25 - 9 = 16$$
$$\therefore c = \sqrt{16} = 4$$

a) $\cos\Phi = c/5 = 4/5 = \underline{\underline{0.8}}$

b) $\sin\Phi + \cos\Phi = 3/5 + 4/5 = 7/5 = \underline{\underline{1.4}}$

11.5 Inverse Trigonometry

11.5.1 Introduction:

Consider the angle 30^o.

* $\sin 30^0 = 0.5$. In other words, the angle whose sine is 0.5 is 30^o i.e. 30^o is the arcsine of 0.5 abbreviated as $\sin^{-1} 0.5$ i.e. $30^o = \sin^{-1}(0.5)$.

Similarly:

* $\cos 30^0 = 0.866025403$.

$\Rightarrow 30^0 = \cos^{-1}(0.866025403)$

* $\tan 30^0 = 0.577350269$

$\Rightarrow 30^0 = \tan^{-1}(0.577350269)$

Generally therefore if;

$\sin A^0 = a$, then $A^0 = \sin^{-1} a$
$\cos B^0 = b$, then $B^0 = \cos^{-1} b$
$\tan C^0 = c$, then $C^0 = \tan^{-1} c$

Arcsine, arccosine, and arctangent are what are known as ***inverse trigonometry***.

Example

Find the angles whose cosines are given below;

i. 0.5
ii. -0.5
iii. 1
iv. 0.2588

Solution

i. *Let the angle be A i.e.*

CosA = 0.5

$\Rightarrow A = \cos^{-1}(0.5)$

$\therefore \underline{\underline{A = 60^0}}$

ii. *Let the angle be B i.e.*

CosB = -0.5

$\Rightarrow B = \cos^{-1}(-0.5)$

$\therefore \underline{\underline{B = 120^0}}$

iii. *Let the angle be i.e.*

$\Rightarrow \theta = \cos^{-1}(1)$

$\therefore \underline{\underline{\theta = 90^0}}$

iv. *Let the angle be* Ω *i.e.*

$\Rightarrow \Omega = \cos^{-1}(0.2588)$

$\therefore \underline{\underline{\Omega = 75^0}}$

Example

Obtain the angles whose tangents are given below

i. 0.5051

ii. 0.48

Solution

i. *Let the angle be D i.e.*

tanD = 0.5051

$\Rightarrow D = \tan^{-1}(0.5051)$

$\therefore \underline{\underline{D = 26.8^0}}$

ii. *Let the angle be C i.e.*

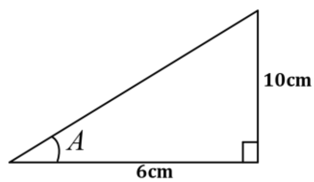
tanC = 0.48

$\Rightarrow C = \tan^{-1} 0.48$

$\therefore \underline{\underline{C = 25.64^0}}$

Example

Calculate angle **A** of the triangle below



Solution

$\tan A = \frac{10}{6} = 1.667$

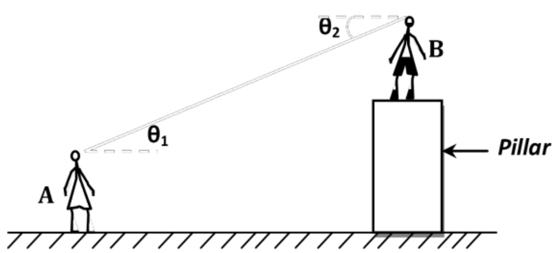
$\Rightarrow A = \tan^{-1}(1.667)$

$\therefore A = 59^0$

11.6 Application of trigonometry in real life situations

11.6.1 Angle of elevation and depression:

Consider two children **A** and **B**, with child **A** standing on the ground and child **B** standing on the pillar.

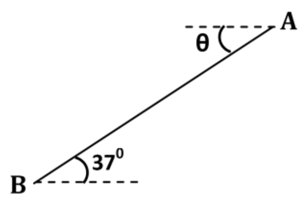


As child **A** looks up at child **B,** her line of sight is elevated at an angle θ_1 above the horizontal. This angle is what is called ***angle of elevation***. Similarly as child, **B** looks down at child **A**, his line of sight is depressed at an angle θ_2 below the horizontal. This angle is what is known as ***angle of depression***. From the property of alternate angles, $\theta_1 = \theta_2$. Therefore, angle of depression is always equal to the angle of elevation.

Example

If the angle of elevation of **A** from **B** is 37^o, what is the angle of depression of **B** from **A**?

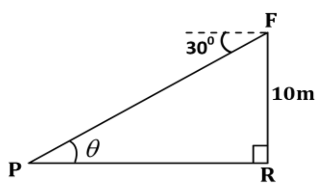
Solution



Angle of depression of B from A is $\theta = 37^0$ *by the property of alternate angles.*

Example

From the figure below,



Find:

i. The value of θ
ii. The distance PR
iii. The distance PF

Solution

i. *From the property of alternate angles* $\theta = 30^0$

ii. *From:* $\tan\theta = \dfrac{10}{PR} \therefore PR = \dfrac{10}{\tan 30^0} = \dfrac{10}{0.5774} = \underline{\underline{17.3m}}$

iii. *From :*

$$\sin\theta = \frac{10}{PF}$$

$$\Rightarrow PF = \frac{10}{\sin 30^0} = {}^{10}\!/_{0.5}$$

$$\therefore \underline{\underline{PF = 20m}}$$

Or , from:

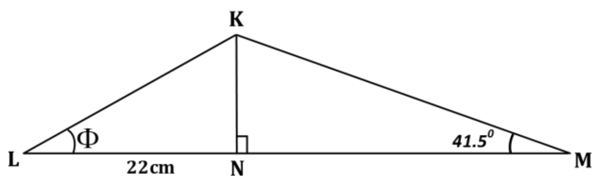
$$\cos\theta = \frac{PR}{PF}$$

$$\Rightarrow PF = \frac{PR}{\cos 30^0} = \frac{17.3205}{0.866}$$

$$\therefore \underline{\underline{PF = 20m}}$$

Example

From the figure below



Find;

a) the length KN

b)

c) the length LK

Solution

a. *From triangle KNM*

$$\tan 41.5^0 = \frac{KN}{33} \Rightarrow KN = 33\tan 41.5^0$$

$$\therefore KN = 33 \times 0.8847$$

$$= \underline{\underline{29.2cm}}$$

b) *From triangle KNL*

$$\tan\phi = \frac{KN}{22} = {}^{29.2}\!/_{22} = 1.327$$

$$\Rightarrow \phi = \tan^{-1}(1.327)$$

$$\therefore \underline{\underline{\phi = 53^0}}$$

c) *From triangle KLN*

$$\cos\phi = \frac{22}{LK} \quad \Rightarrow LK = \frac{22}{\cos 53^0}$$
$$\therefore LK = \frac{22}{0.602} = \underline{\underline{36.6cm}}$$

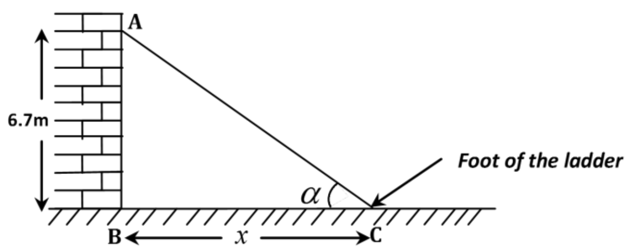
Example

A ladder makes an angle of 22^o with the wall when it reaches a window 6.7m up.

a) How far is the foot of the ladder away from the building?

b) Calculate the length of the ladder.

Solution



a) *Let x be the distance of the foot of the ladder from the building*

From:

$$\tan 22^0 = \frac{x}{AB} \quad \Rightarrow x = AB\tan 22^0$$
$$\therefore x = 6.7 \times 0.404 = \underline{\underline{2.7m}}$$

OR: using angle α;

$$\alpha = 90 - 22 = 68^0$$
$$\tan\alpha = \frac{AB}{x} \quad \Rightarrow x = \frac{6.7}{\tan 68^0}$$
$$\therefore x = \frac{6.7}{2.475} = \underline{\underline{2.7m}}$$

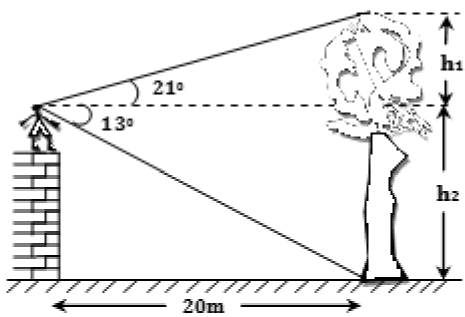
b) *Let the length of the ladder be AC = y*

$$\cos\alpha = \frac{x}{AC}$$
$$\Rightarrow AC = \frac{x}{\cos\alpha} = \frac{2.7}{\cos 68^0} = 7.2m$$
$$\therefore \underline{\underline{y = 7.2m}}$$

Example

From the roof of a house, a boy can see a coconut tree, which is 20m away from the house. He measures the angle of elevation of the top of the tree as 21^o and the angle of depression of the bottom of the tree as 13^o. Find the height of the coconut tree.

Solution



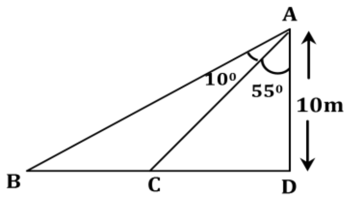
The height of the tree = $h_1 + h_2$

For h_1:	***For h_2:***
$\tan 21^0 = \frac{h_1}{20}$	$\tan 13^0 = \frac{h_2}{20}$
$\Rightarrow h_1 = 20\tan 21^0$	$\Rightarrow h_2 = 20\tan 13^0$
$\therefore \underline{\underline{h_1 = 7.68m}}$	$\therefore \underline{\underline{h_2 = 4.62m}}$

$$\text{Therefore the height of the tree} = 7.68 + 4.62 = \underline{\underline{12.3m}}$$

Example

Given the diagram below.



Using the diagram above, calculate the distance BC.

Solution

$$BC = BD - CD$$

From triangle ABD:

$$\tan 65^0 = \frac{BD}{12}$$
$$\Rightarrow BD = 12\tan 65^0$$
$$\therefore BD = 25.73m$$

From triangle ACD:

$$\tan 55^0 = \frac{CD}{12}$$
$$\Rightarrow CD = 12\tan 55^0$$
$$\therefore CD = 17.14m$$

$$\therefore BC = 25.73 - 17.14 = \underline{\underline{8.59m}}$$

Example

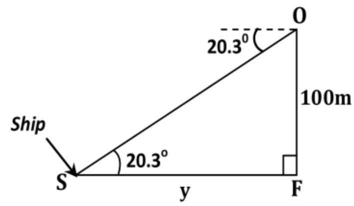
The angle of depression of the ship from the vertical cliff 100m high is 20.3^o

a) How far is the ship from the base of the cliff?

b) If there is a rock in between the ship and the vertical cliff, 70m from the ship, calculate the angle of depression of the rock from the ship.

Solution

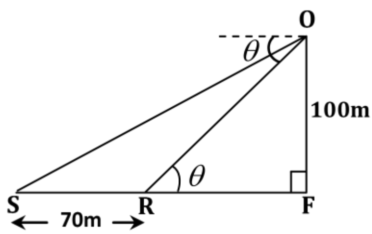
a) *Let y be the distance of the ship from the cliff*



$$\tan 20.3^0 = \frac{100}{y} \quad \Rightarrow y = \frac{100}{\tan 20.3^0}$$

$$\therefore \underline{\underline{y = 270.34m}}$$

b)



$$SF = y = 270.34m$$

$$RF = SF - SR = 270.34 - 70 = 200.34m$$

$$\therefore \tan\theta = \frac{100}{RF} = \frac{100}{200.34} = 0.4992$$

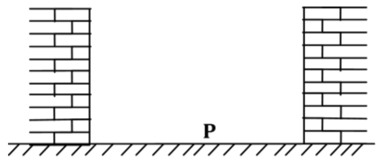
$$\Rightarrow \theta = \tan^{-1}(0.4992)$$

$$\therefore \underline{\underline{\theta = 26.5^0}}$$

Therefore, the angle of depression of the rock from O is 26.5^o

Example

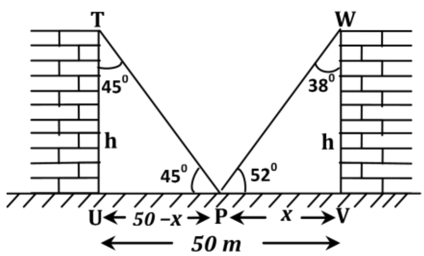
The diagram below shows two houses of the same height. The distance between them is 50m.



If from **P** the angle of elevation of one house is 45^o and that of the other is 52^o, calculate the height of the two buildings.

Solution

Let the height of the buildings be h.



From triangle PVW:

$$\tan 38^0 = \frac{PV}{h}$$

$$\therefore PV = h\tan 38^0, \ but \ PV = x$$

$$\Rightarrow x = h\tan 38^0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

From triangle PUT:

$$\frac{PU}{h} = \tan 45^0$$

$$\therefore PU = h\tan 45^0, \ but \ PU = 50 - x$$

$$\Rightarrow 50 - x = h\tan 38^0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Solving equations (1) and equation (2) simultaneously

$$50 - h\tan 38^0 = h\tan 45^0 \quad \Rightarrow 50 = h\tan 38^0 + h\tan 45^0$$
$$50 = h(\tan 38^0 + \tan 45^0) = h(0.7813 + 1)$$
$$50 = h(1.7813)$$
$$\therefore h = \frac{50}{1.7813} = \underline{\underline{28.07m}}$$

Exercise

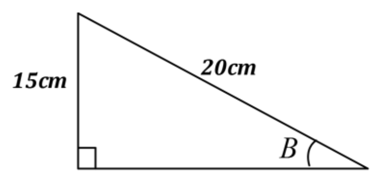
1. Given that $\tan\theta = \frac{15}{8}$ and that θ is an acute angle; find without using table or calculator:

   a) $\sin\theta$
   b) $\cos\theta$

2. If $5\sin A = 3\cos A$; find $\tan A$ without evaluating the angle.

3. Without using table or calculator, find $\cos Q - \sin Q$, given that $\tan Q = \frac{5}{12}$

4. Calculate angle B of the triangle below.



5. A woman 1.55m tall is standing on the top of a cliff 30m high looking down at a boat that is on a lake. If the boat is 80m from the base of the cliff, find the angle of depression of the boat from the woman. (*Ans:21.5^0*)

6. A girl is looking up at the top of a building. She measures the angle of elevation of the top of the building as 40^o. She walks 20m towards the building and finds that the angle of elevation of the top of the building is now 55^o. The girl is 1.5m tall.

   a) How far was she from the building when she started? (*Ans: 48.5m*)

   b) How tall is the building? (*Ans: 42.2m*)

7. A flag mast 6m long stands on top of a church tower. From a point on level ground, the angles of elevation of the top and bottom of the flag mast are 40^o and 30^o respectively.

   Find:

   a) the distance of the observer from the church tower (*Ans: 22.93m*)

   b) the height of the church tower (*Ans: 13.24m*)

   c) the shortest distance between the observer and the top of the flag mast (*Ans: 29.93m*)

11.7 General Angles

In this chapter, we shall consider trigonometric ration for any angle. Angles are generally measured in anti-clockwise direction beginning from ***positive x-axis.*** Before we go further, let us first understand the following terms:

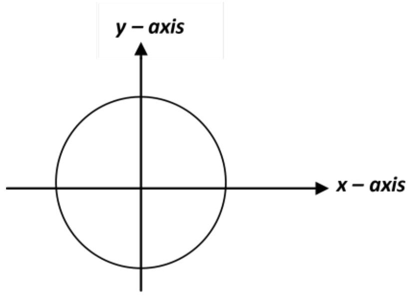
a) Clockwise direction (movement)
This is a direction or movement that follows that of the hand of the clock

b) Anti-clockwise direction (movement)
This is movement in the direction opposite to that of the hand of the clock.

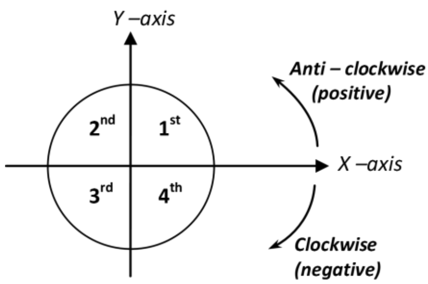
c) Acute angles
These are angles, which are greater than 0^o but less than 90^o.

d) Obtuse angles
These are angles greater than 90^o but les 180^o

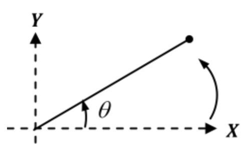
e) Reflect angles
These are angles greater than 180^0 but less than 360^o.
Now consider a circle of radius r, center (0, 0) as shown below



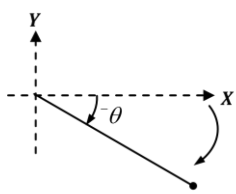
The cycle above is divided up into four equal parts by the x axis and y axis known as quadrants. These four quadrants are numbered 1 to 4 starting from the positive x-axis going to the anti-clockwise direction and are known as 1^{st}, 2^{nd}, 3^{rd}, and 4^{th} quadrant respectively.



Conventionally, we consider anti-clockwise to be positive and clockwise to be negative. Therefore, angles measured from the x-axis to the anti-clockwise direction are positive. i.e.



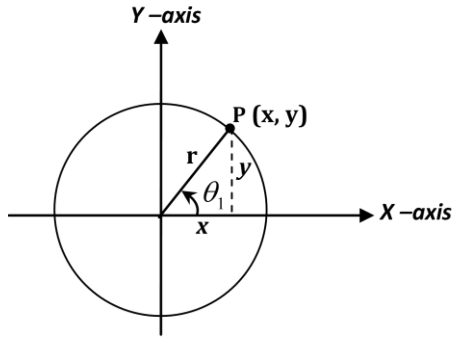
Whereas angles measured from the positive x-axis to the clockwise direction are negative. Ie



Let us now look at which trigonometrical ratios are positive and negative in each of the quadrant. Consider a circle of radius r, centre (0, 0) and let the point P(x, y) be any point on the cycle.

11.7.1 For 1st quadrant:

$_1$ in the first quadrant, x and y are positive.

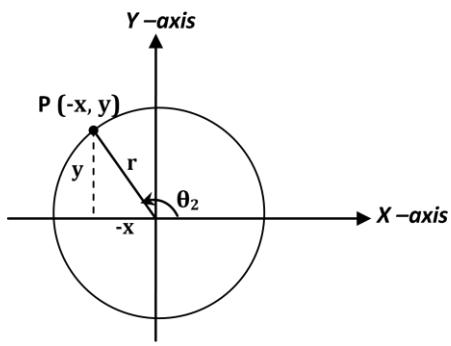


$$\sin\theta_1 = \frac{y}{r}, \cos\theta_1 = \frac{x}{r}, \tan\theta_1 = \frac{y}{x}$$

In the first quadrant therefore, all the three trigonometrical rations are positive.

11.7.2 For 2nd quadrant:

$_2$ in the second quadrant, x is negative and y is positive



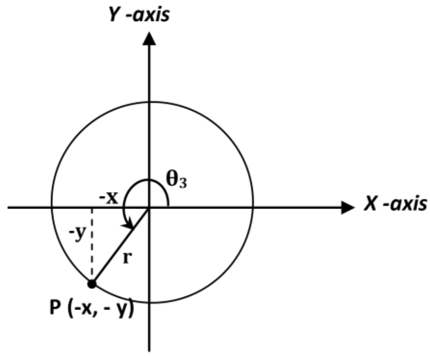
$$\sin\theta_2 = \frac{y}{r}, \cos\theta_2 = \frac{-x}{r}, \tan\theta_2 = \frac{-y}{x}$$

$_2$, which is positive; co $_2$ $_2$

are negative.

11.7.3 For 3rd quadrant:

$_3$ in the third quadrant, x and y are both negative but **r** is always positive.

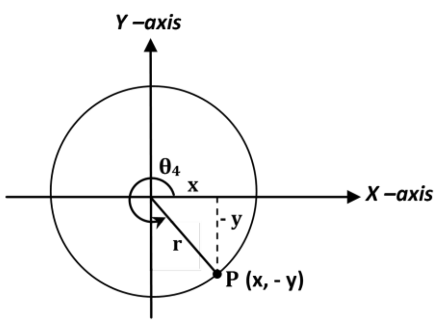


$$\sin\theta_3 = \frac{-y}{r}, \cos\theta_3 = \frac{-x}{r}, \tan\theta_2 = \frac{-y}{-x} = \frac{y}{x}$$

$_3$ $_3$ $_3$ are negative.

11.7.4 From the 4th quadrant:

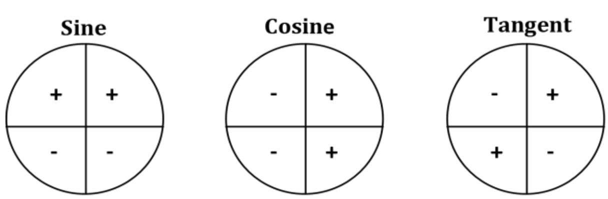
$_4$ in this quadrant x is positive, y is negative and **r** is always positive.



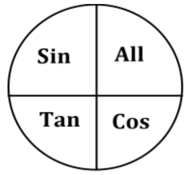
$$\sin\theta_4 = \frac{-y}{r}, \cos\theta_4 = \frac{x}{r}, \tan\theta_4 = \frac{-y}{x}$$

$_4$ $_4$ $_4$ are negative.

From the above results:



Generally, these results can be summarized by writing which rations are positive in each quadrant.



These results explain why angles, which are equally inclined to the positive or negative x-axis, have trigonometrical ratios of the same magnitude.

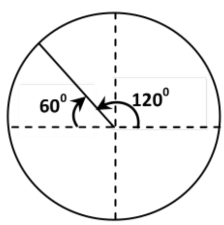
Example

Express the following trigonometrical ratios in terms of their respective acute or obtuse angles.

1. a)	Sin120^o	2. a)	sin200^o	3. a)	sin330^o
b)	Cos120^o	b)	cos 200^o	b)	cos330^o
c)	tan120^o	c)	tan200^o	c)	tan330^o

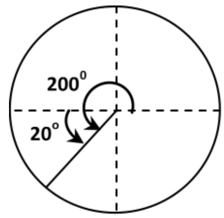
Solution

1) 120^o is in the second quadrant, where sine is positive and tangent and cosine are negative



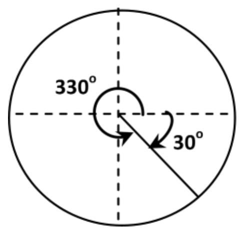
$$a), \sin 120^0 = \sin 60^0$$
$$b), \cos 120^0 = -\cos 60^0$$
$$c), \tan 120^0 = -\tan 60^0$$

2) *200^o is in the third quadrant, where sine and cosine is negative and only tangent, which is positive.*



$$a), \sin 200^0 = -\sin 20^0$$
$$b), \cos 200^0 = -\cos 20^0$$
$$c), \tan 200^0 = \tan 20^0$$

3) *330^o is in the fourth quadrant where cosine is positive and sine and tangent are negative.*



$$a), \sin 330^0 = -\sin 30^0$$
$$b), \cos 330^0 = \cos 30^0$$
$$c), \tan 330^0 = -\tan 30^0$$

$90^o < \theta < 360^o$

a. For 90^o ^{o}:

$$\sin\theta = \sin(180^0 - \theta)$$
$$\cos\theta = -\cos(180^0 - \theta)$$
$$\tan\theta = -\tan(180^0 - \theta)$$

b. For 180^o ^{o}:

$$\sin\theta = -\sin(\theta - 180^0)$$
$$\cos\theta = -\cos(\theta - 180^0)$$
$$\tan\theta = \tan(\theta - 180^0)$$

c. For 270^o ^{o}:

$$\sin\theta = -\sin(360^0 - \theta)$$
$$\cos\theta = \cos(360^0 - \theta)$$
$$\tan\theta = -\tan(360^0 - \theta)$$

Example

Express the following sines interm of sines of acute angles.

a) $\text{Sin}115^o$

b) $\text{Sin}205^o$

c) $\text{Sin}340^o$

Solution

a) 115^o lies in between $90^o$and 180^o

$$\therefore \sin 115^0 = \sin(180^0 - 115^0)$$
$$\Rightarrow \sin 115^0 = \sin 65^0$$

b) 205^o lies between 180^o and 270^o

$$\therefore \sin 205^0 = -\sin(205^0 - 180^0)$$
$$\Rightarrow \sin 205^0 = -\sin 25^0$$

c) 340^o lies between 270^o and 360^o

$$\therefore \sin 340^0 = -\sin(180^0 - 340^0)$$
$$\Rightarrow \sin 340^0 = -\sin 20^0$$

Example

Given that $\sin\theta = \frac{3}{5}$ and 90^o ^{o}.

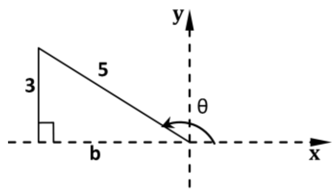
Find:

a)

b)

Solution

a) *is an obtuse angle and is in the second quadrant*



Using Pythagoras theorem:

$$5^2 = b^2 + 3^2$$
$$\Rightarrow b^2 = 25 - 9 = 16$$
$$\therefore b = \sqrt{16} = 4$$

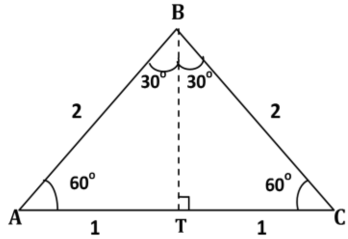
*But **b** is negative because it is in the negative x-axis. Therefore **b** = –4*

$$\therefore \cos\theta = \frac{b}{5} = \underline{\underline{-4/5}}$$

b) $\sin\theta + \cos\theta = 3/4 + -4/5 = \frac{3-4}{5} = \underline{\underline{-1/5}}$

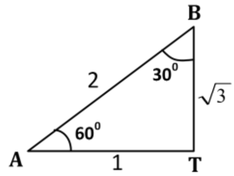
11.8 Trigonometrical Ratios of 300, 450 and450

For trigonometrical ratios of 30 and 60, consider an equilateral triangle with sides given as 2units as shown below.



11.8.1 For 60^o:

From triangle ATB:



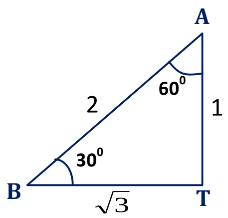
Using Pythagoras theorem;

$$AB^2 = AT^2 + TB^2$$
$$\Rightarrow 2^2 = 1^2 + TB^2$$
$$\therefore TB = \sqrt{3}$$

$$\sin 60^0 = \frac{\sqrt{3}}{2}$$
$$\cos 60^0 = \frac{1}{2}$$
$$\tan 60^0 = \sqrt{3}$$

11.8.2 For 30o:

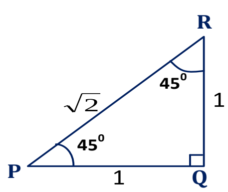
From triangle BTA:



$$\cos 30^0 = \frac{\sqrt{3}}{2}$$
$$\sin 30^0 = \frac{1}{2}$$
$$\tan 30^0 = \frac{1}{\sqrt{4}}$$

11.8.3 For 45o:

Consider a right-angled triangle shown below. PQ and QR are of equal length, say 1unit. The length PR is obtained from Pythagoras theorem.



Using Pythagoras theorem;

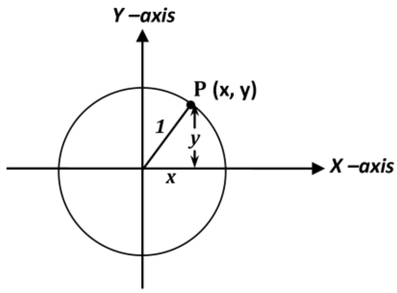
$$PR^2 = PQ^2 + QR^2$$
$$PR^2 = 1^2 + 1^2$$
$$\Rightarrow PR = \sqrt{2}$$

$$\boxed{\begin{aligned} \cos 45^0 &= \frac{1}{\sqrt{2}} \\ \sin 45^0 &= \frac{1}{\sqrt{2}} \\ \tan 45^0 &= 1 \end{aligned}}$$

11.9 Graphs of sine and cosine

11.9.1 Graph of sinθ:

Consider the unit circle below



Point P is any point on the unit circle with coordinates (x, y) where y

If we are to

i.
horizontal axis.

ii. ^{o} to 360^{o}.

iii.

iv. nding values of

sine curve because the curve is in the form of a wave. It is also known as *sine wave*.

Example

a) of 30^{o}

b) By using the graph you have drawn;

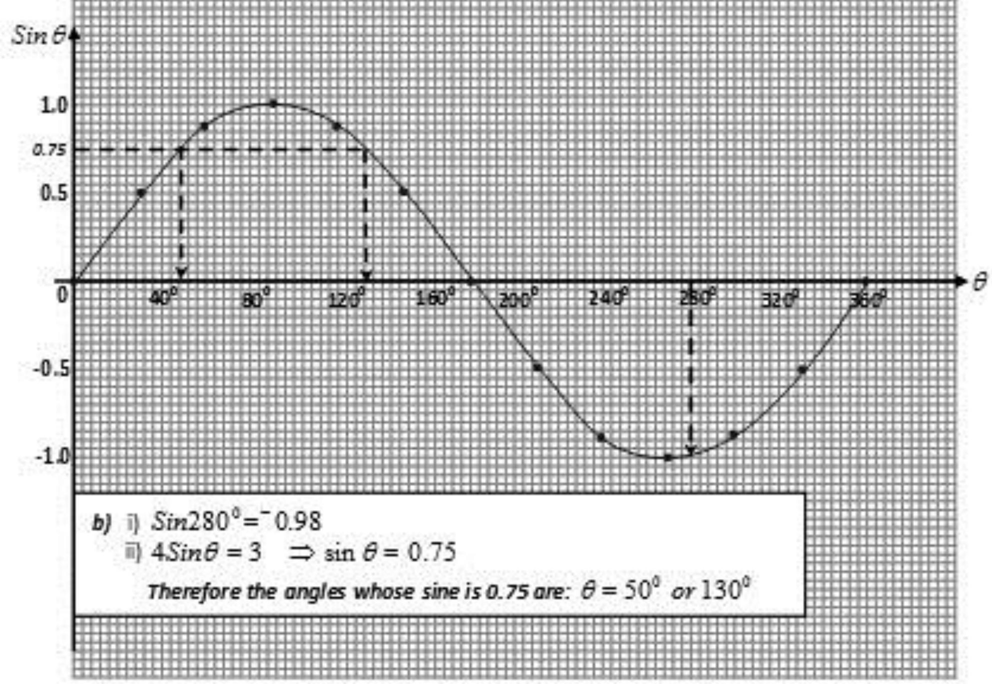
i. Find sin280^{o}

ii.

Solution

a) Table of values of y = sin

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\sin\theta$	0.0	0.50	0.87	1.0	0.87	0.50	0.0	-0.5	-0.87	-1.0	-0.87	-0.50	0.0



11.9.2 Graph of cosθ:

to t

graph obtained is called cosine graph or cosine curve.

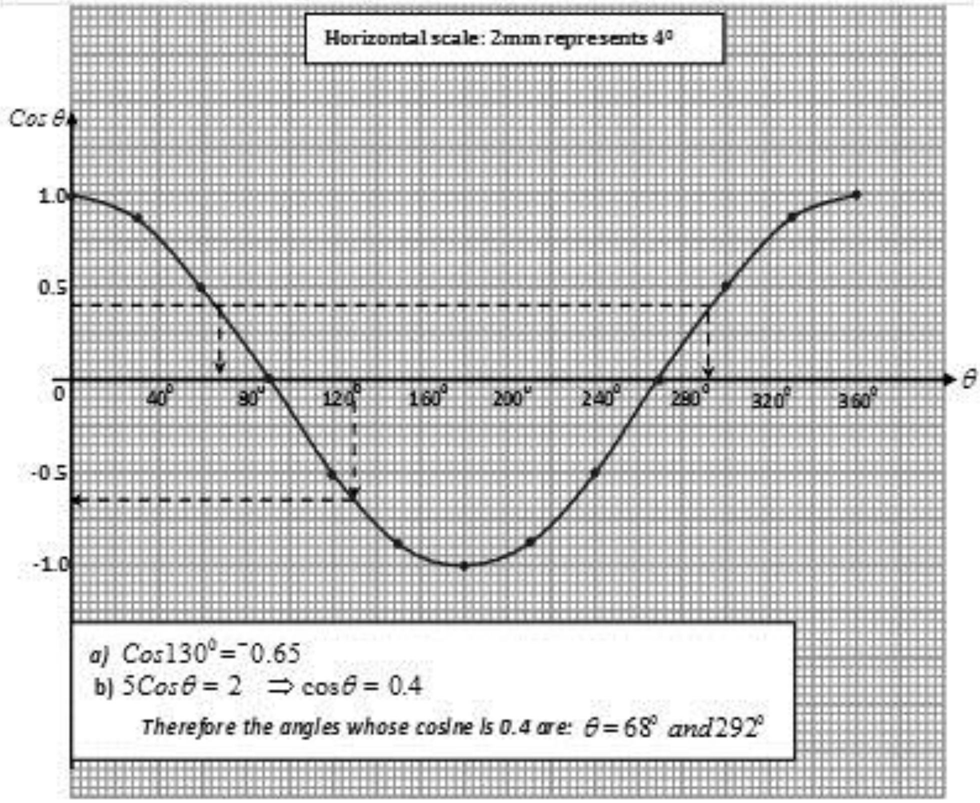
Example

i) Find $\cos 130^{o}$

ii)

Solution

θ°	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
Cosθ	1.0	0.87	0.5	0.0	-0.5	-0.87	-1.0	-0.87	-0.5	0.0	0.5	0.87	0.0



Observation:

that:

a) $^{-}1$ and $^{+}1$.

b) The sine curve and the cosine curve have the same shape. The cosine curve can be obtained by translating the sine curve through 90^{o} to the right.

c) The curves have peaks (high points) and troughs (low points)

Remarks:

places and approximate them to one decimal place when graphing.

Example

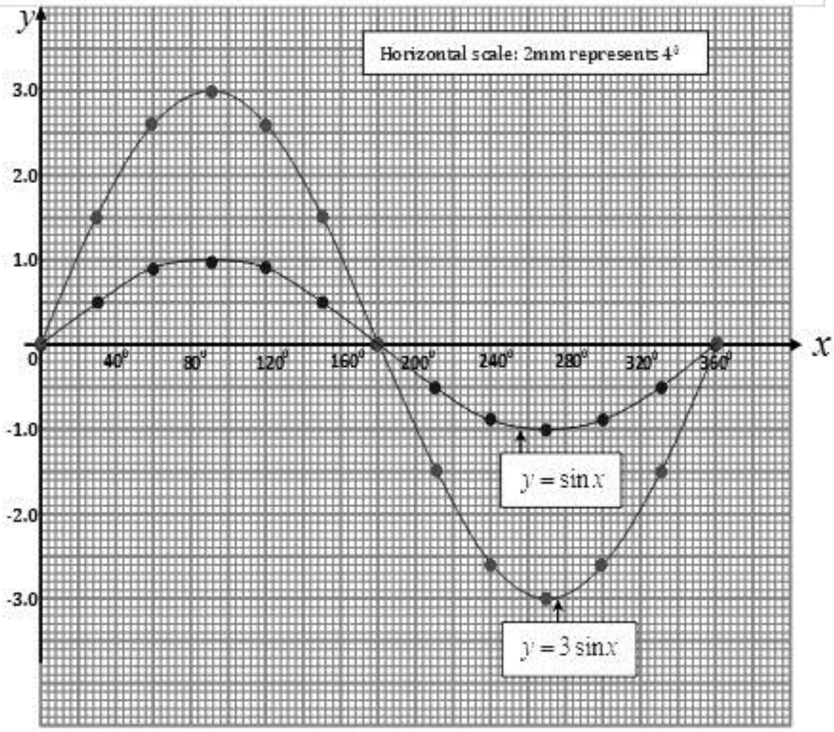
On the same axes, draw the graph of y = *sin**x*** and y = 3*sin**x*** for x such that 0^o ^{o}. Use your graph to answer the following questions:

a) State the value of *x* at which each graph has reached highest and lowest point.

b) At what values of *x* do the two graphs coincide? What do these values represent?

Solution

θ°	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
Y = sinx	0.0	0.5	0.87	1.0	0.87	0.5	0.0	-0.5	-0.87	-1.0	-0.87	-0.5	0.0
Y = 3sinx	0.0	1.5	2.6	3.0	2.6	1.5	0.0	-1.5	-2.6	-3.0	-2.6	-1.5	0.0



a) ***For y = sinx:***

For highest point; the value of x = 90^o
For lowest point; the value of x = 270^o

For y = 3sinx:
For highest point; the value of x = 90^o
For lowest point; the value of x = 270^o.

b) *At $x = 0^o$, $x = 180^o$ and $x = 360^o$*

These values represent roots of the equation sinx = 3sinx.

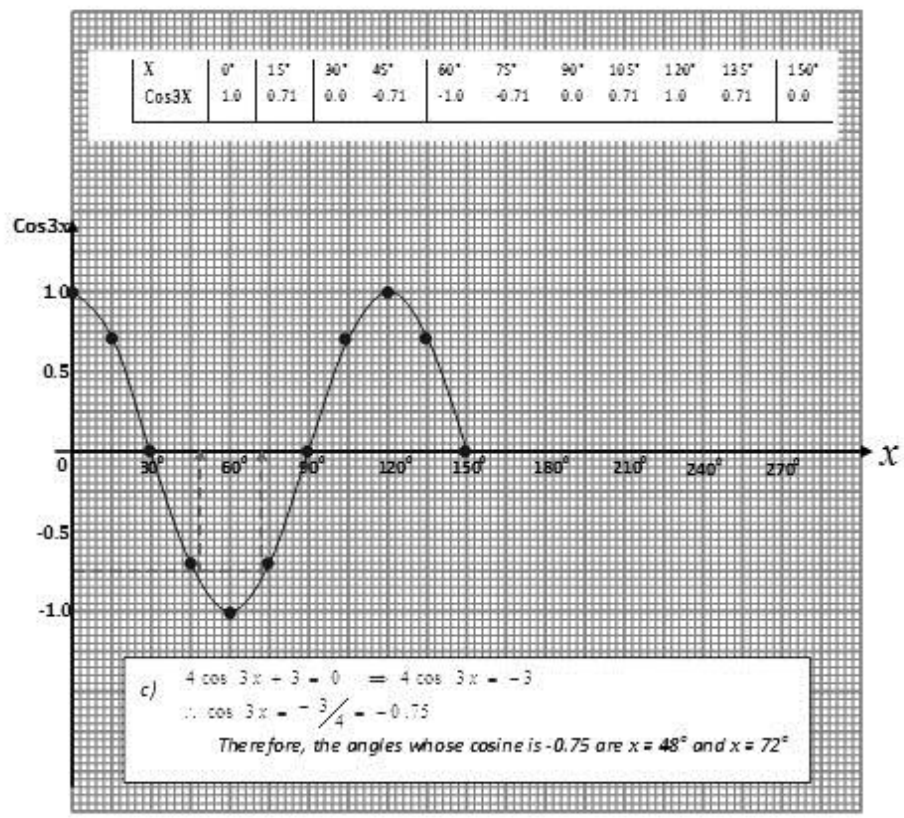
Example

a) Draw the table showing the values of *cos3*x for 0^o x ^{o} using the values of x at intervals of 15^o.

b) Use the table in a) above to draw a graph of *cos3x* with a horizontal scale of 1cm for 15^o and vertical scale of 2cm for 0.5units.

c) From the graph you have drawn, determined the values of x if 4*cos*3x + 3 = 0

Solution



X	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°
Cos3X	1.0	0.71	0.0	-0.71	-1.0	-0.71	0.0	0.71	1.0	0.71	0.0

c) $4\cos 3x + 3 = 0 \Rightarrow 4\cos 3x = -3$

$\therefore \cos 3x = -{}^{3}/_{4} = -0.75$

Therefore, the angles whose cosine is -0.75 are x = 48° and x = 72°

Exercise:

1. ^{o} ^{o} to;

   a) Find the following;
      (i) $\sin 25^{o}$
      (ii) $\sin 118^{o}$
      (iii) $\sin 350^{o}$

   b) Solve the following:
      (i)
      (ii)
      (iii) 7 = 0

2. Use the graph of y = cos *x* for 0^{o} ^{o} to;

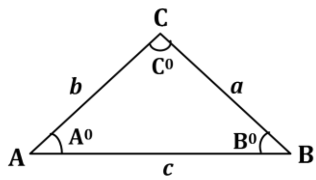
   a) Find the following:
      (i) $\cos 15^{o}$
      (ii) $\cos 245^{o}$
      (iii) Cos300

   b) Solve the following:
      (i) $3\cos x = -1$
      (ii) $13\cos(\frac{1}{2}x) - 3 = 0$

12 THE SINE AND COSINE RULE

12.1 Introduction:

Consider the triangle below:

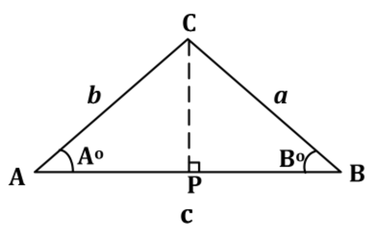


The vertices of a triangle are always labeled with capital letters, for instance **A**, **B**, and **C**, and the same symbols are used to represent the sizes of the angles at the vertices. The corresponding lower case letters, a, b, c are then used to represent the length of the sides opposite the ***a''*** is used to represent the length of the side BC,

you can in the triangle above.

12.2 The sine rule:

Consider the triangle below with vertices **A, B, C**.



In the figure above, CP is perpendicular to AB. Now considering triangle APC:

$$\frac{CP}{b} = \sin A$$

$$CP = b\sin A \text{...(1)}$$

Also considering triangle BPC:

$$\frac{CP}{a} = \sin B$$

$$CP = a\sin B \text{...(2)}$$

From equation (1) and (2;

$$a\sin B = b\sin A$$

$$\Rightarrow \frac{a}{\sin A} = \frac{b}{\sin B}$$

Applying the same argument to the line from A perpendicular to BC, we could obtain;

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Putting these expressions together, we obtain:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

This expression is what is known as the sine rule or the sine formula.

Use of the sine rule:

We use the sine rule when given:

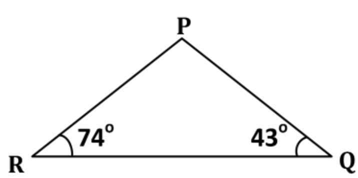
i. two sides and an angle opposite to one of the given sides, or

ii. two sides and any two angle, or

iii. two angles and a side

Example

Solve triangle PQR in which QR = 5.12cm, angle Q = 43^{o} and angle R = 74^{o} and hence calculate its area.



Solution

a) *To solve a triangle means to find the sides and angles of the triangle, which are not known.*

Angle P =?, QR =? PR =? Angle Q = 43^o, angle R = 74^o, QR = 5.12cm.

For the angle P:

From sum of interior angles of a triangle = 180^o

$$Q + P + R = 180$$
$$43^0 + 74^0 + P = 180^0$$
$$\Rightarrow P = 180 - 117^0$$
$$\therefore \underline{\underline{P = 63^0}}$$

For the length PR:

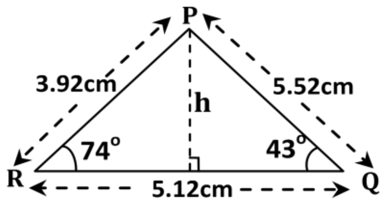
Using sine rule i.e. $\frac{QP}{\sin R} = \frac{QR}{\sin P}$

$$\Rightarrow \frac{QP}{\sin 74^0} = \frac{5.12}{\sin 63^0}$$
$$\Rightarrow QP = \frac{5.12 \times \sin 74^0}{\sin 63^0}$$
$$\therefore \underline{\underline{QP = 5.52cm}}$$

For the length PR:

$$\frac{PR}{\sin Q} = \frac{QR}{\sin P}$$
$$\frac{PR}{\sin 43^0} = \frac{5.12}{\sin 63^0}$$
$$\Rightarrow PR = \frac{5.12 \times \sin 74^0}{\sin 63^0}$$
$$\therefore \underline{\underline{PR = 3.92cm}}$$

b) *Area of the triangle*



$Area = \frac{1}{2}bh, \quad h = ?, b = 5.12cm$

From triangle RPT:

$$\frac{h}{3.92} = \sin 74^0$$

$$\Rightarrow h = 3.92\sin 74^0$$

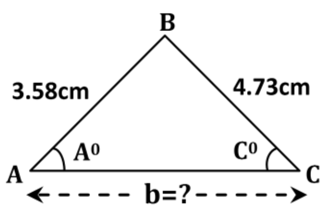
$$\therefore h = 3.77cm$$

$$Area\ of\ triangle\ QPR = \tfrac{1}{2}\times 5.12\times 3.77$$
$$= \underline{\underline{9.65cm^2}}$$

Example

In triangle ABC, **a** = 4.73cm, **c** = 3.58cm, and **C** = 42.2^0. Calculate the size of angle **A** and length **b**.

Solution



By the sin rule;

$$\frac{a}{\sin A} = \frac{c}{\sin C}, but, a = 4.73cm, C = 42.2^0, c = 3.58cm$$

$$\therefore \frac{4.73}{\sin A} = \frac{3.58}{\sin 42.2^0}$$

$$\therefore \sin A = \frac{4.73 \times \sin 42.2^0}{3.58} = 0.8875$$

$$\Rightarrow A = \sin^{-1}(0.8875)$$

$$\therefore \underline{\underline{A = 62.6^0}}$$

From:

$$A + B + C = 180^0$$

$$62.6 + 42.2 + B = 180^0$$

$$\Rightarrow B = 180^0 - 104.8^0$$

$$\therefore B = 75.2^0$$

From:

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

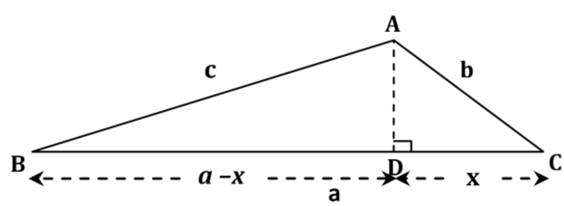
$$\therefore \frac{b}{\sin 75.2^0} = \frac{3.58}{\sin 42.2^0}$$

$$\Rightarrow b = \frac{3.58 \times \sin 75.2^0}{\sin 42.2^0}$$

$$\therefore \underline{\underline{b = 5.15cm}}$$

12.3 The cosine rule:

Consider triangle ABC below in which angle C is acute angle.



AD is perpendicular to BC. Considering triangle ABC and by Pythagoras theorem;

$$AD^2 = AB^2 - BD^2$$
$$\therefore AD^2 = c^2 - (a - x)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Considering triangle ACD:

$$AD^2 = AC^2 - DC^2$$
$$\therefore AD^2 = b^2 - x^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From equation (1) and (2)

$$c^2 - (a - x)^2 = b^2 - x^2$$
$$c^2 - (a^2 - 2ax + x^2) = b^2 - x^2$$
$$c^2 - a^2 + 2ax - x^2 = b^2 - x^2$$
$$\therefore c^2 = a^2 + b^2 - 2ax \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

From triangle ACD:

$$\frac{x}{b} = \cos C$$
$$\therefore x = b\cos C$$

Substituting for x in equation (3)

$$c^2 = a^2 + b^2 - 2bc\cos C$$

It can similarly be shown that:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$

The expression;

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2bc \cos C$$

is known as the cosine rule or formula

12.4 Use of the cosine formula

The cosine formula is used when given;

i) two sides and the included angle, or

ii) all the three sides

To find the angle of a triangle given the lengths of the three sides, we need to re-arrange the cosine rule. Thus from:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

We obtain the expression below

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Similarly,

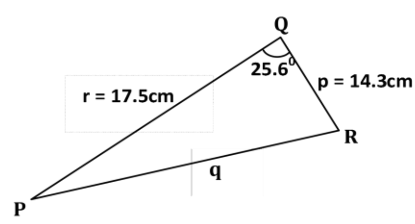
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Example

In triangle PQR, p = 14.3cm, r = 17.5cm and Q = 255.6°. Calculate the length of side PR.

Solution



PR* = *q

$$q^2 = r^2 + p^2 - 2rp\cos Q$$

$$q^2 = 17.5^2 + 14.3^2 - 2\times 17.5\times 14.3\cos 25.6^0 = 59.373$$

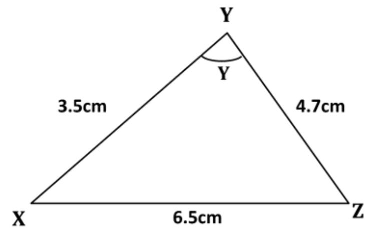
$$q = \sqrt{59.373}$$

$$\therefore \underline{\underline{q = 7.71cm}}$$

Example

In triangle XYZ, XY = 3.5cm, YZ = 4.7cm and ZX = 6.5cm. Calculate the size of angle Y.

Solution



$From: y^2 = z^2 + x^2 - 2zx\cos Y, \quad y = 6.5cm, x = 4.7cm \text{ and } z = 3.5cm$

$6.5^2 = 3.5^2 + 4.7^2 - 2 \times 3.5 \times 4.7 \cos Y$

$42.25 = 34.34 - 32.9\cos Y$

$\Rightarrow \cos Y = \dfrac{34.34 - 42.25}{32.9} = -0.2404$

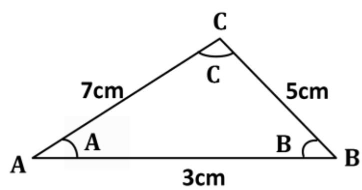
$\therefore Y = \cos^{-1}(-0.2404) \quad \Leftrightarrow \underline{\underline{Y = 103.9^0}}$

Example

Find the angles of a triangle with sides of lengths 3cm, 5cm, and 7cm.

Solution

Let the vertices of the triangle be ABC



A =? a = 5cm, B =? b =7cm, C =? c = 3cm *i.*

$$CosA = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\therefore \cos A = \frac{7^2 + 3^2 - 5^2}{2 \times 7 \times 3} = 0.7857$$

$$\Rightarrow A = \cos^{-1}(0.7857)$$

$$\therefore \underline{\underline{A = 38.2^0}}$$

ii.

$$CosB = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\therefore \cos B = \frac{5^2 + 3^2 - 7^2}{2 \times 5 \times 3} = -0.5$$

$$\Rightarrow B = \cos^{-1}(-0.5)$$

$$\therefore \underline{\underline{B = 120^0}}$$

iii.

$$CosC = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\therefore \cos C = \frac{5^2 + 7^2 - 3^2}{2 \times 5 \times 7} = 0.9286$$

$$\Rightarrow C = \cos^{-1}(0.9286)$$

$$\therefore \underline{\underline{c = 21.8^0}}$$

You can now check if the sum of these angles is 180^o i.e.
$A + B + C = 38.2^0 + 21.8^0 + 120^0 = 180^0$

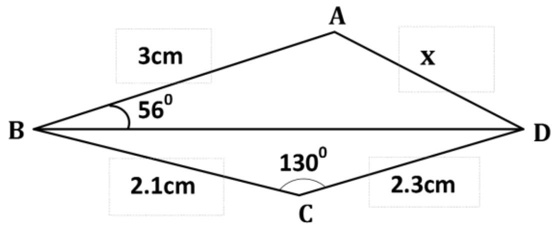
12.5 Application of the sine and cosine formula

The concept of sine and cosine formula shall highly be applied in solving problems under bearing.

Exercise

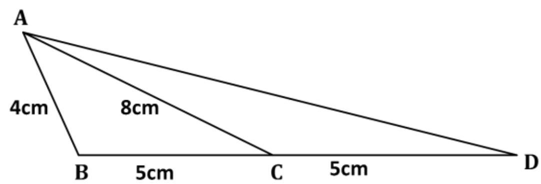
1. Solve the triangles below, giving your answers correct to 3.s.f.
   a) a = 3cm, b = 8cm, c = 7cm
   b) x = 6cm, y = 14.5cm, Z = 95^o
   c) p = 4.1cm, Q = 115.2^o, R = 38.9^o
   d) a = 3.49cm, b = 4.2cm, c = 6.93cm.

2. Given the diagram below



Use it to find x.

3. In the figure below, AB = 4cm, BC = 5cm, AC = 8cm and CD = 5cm

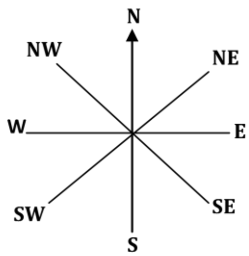


Find the length AD.

13 BEARING

13.1 Introduction:

Directions are described using north(N), south(S),east(E), and west(W) and north-east(NE), south-east(SE), south-west(SW) and north-west(NW).

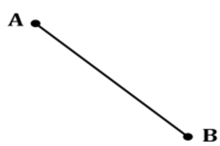


13.2 Ways of giving bearing:

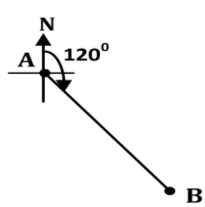
There are two ways of giving bearing. These include the following; true bearing and campus bearing.

13.2.1 True bearing

Here bearing is given as the amount of angle turned clockwise from facing true north. For instance, suppose that you are required to give the bearing of point **B** from **A** below.

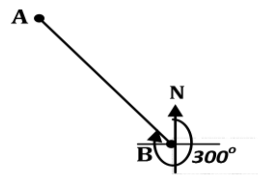


In this case, you will stand at A while facing north, and then you turn clockwise until you are facing point B directly. You then measure the angle you have turned. This gives you the bearing of B from A. Suppose the angle measured is 120^o as shown.



The bearing of point **B** from point **A** is therefore 120^o.

Similarly, the bearing of point A from point B can be obtained in the same manner.



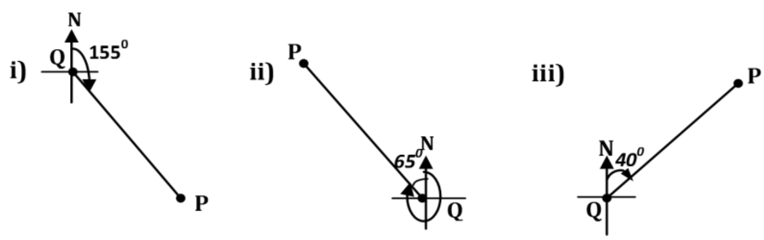
The bearing of point A from point B is therefore 300^o

Note:

For precision, three-figure bearing is normally used. E.g. 060^o, 075^o, 090^o, 180^o, 270^o, etc.

Example

a) State the bearing of P from Q and
b) Also, state the bearing of Q from P in each of these diagrams.



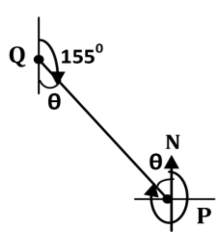
Solution

a) *i) The bearing of* P *from* Q $= 155^0$

ii) The bearing of P *from* Q $= 360^0 - 65^0 = 295^0$

iii) The bearing of P *from* Q $= 040^0$

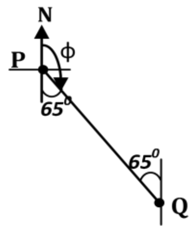
b) *i)*



$$\theta = 180^0 - 55^0$$
$$= 125^0$$

Therefore, the bearing of Q *from* P $= 360^0 - 125^0 = 235^0$

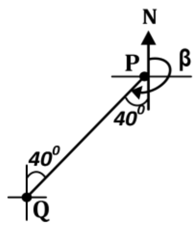
ii)



$$\phi = 180^0 - 65^0$$
$$= 115^0$$

The bearing of Q from P $= \phi = 115^0$

iii)

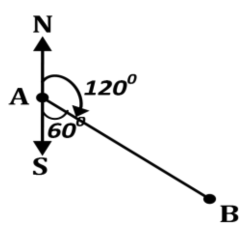


$$\beta = 180^0 + 40^0$$
$$= 220^0$$

Therefore the bearing of Q *from* P = $= 220^0$

13.2.2 Compass bearing

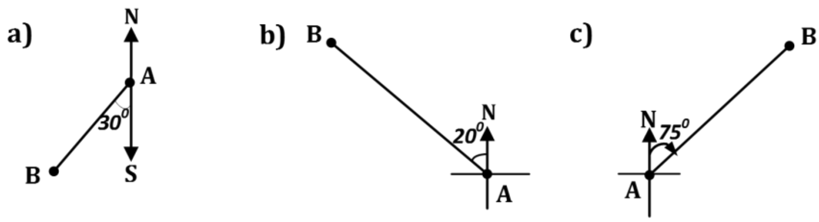
Bearing can be given by measuring the angle east or west from north or south. Consider the diagram below.



Therefore, the bearing of B from A is S60^o E. this bearing is known as compass bearing.

Example

By use of compass bearing, state the bearing of B from A from the following figures.



Solution

a) $S30^0W$

b) $N20^0W$

c) $N75^0E$

Example

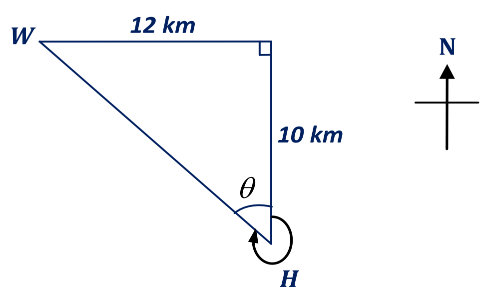
Mr. Okello walked 10km north from his home and then 12km west to the market.

a)

b)

Solution

a) *First, sketch the diagram. Let H and M stand for Okello's house and Market respectively.*



This is a right-angled triangle. The distance between the market and Okello's house is ***WH***.

$\therefore HW = \sqrt{10^2 + 12^2} = \underline{\underline{15.6\,km}}$

b) *The bearing of M from H is equal to* $360^0 - \theta$

But $\theta = ?$

$$From:\ \tan\theta = \frac{12}{10} = \frac{6}{5}$$

$$\Rightarrow\ \theta = \tan^{-1}\left(\frac{6}{5}\right) = \underline{\underline{50.2^0}}$$

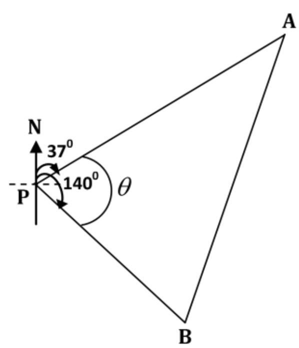
Therefore the bearing of M from H $= 360^0 - 50.2^0 = \underline{\underline{309.8^0}}$

Example

Two boats, **A** and **B** leave a port at 07:00h. Boat **A** travels at 25km/h on a bearing of 037^o, boat **B** travels at 15km/h on a bearing of 140^o. After 3 hours, how far is **A** from **B**?

Solution

Sketch:



$\theta = 140^0 - 37^0 = 103^0$

For boat A:

The distance ***A*** *has travelled in 3 hours is equal to PA*

$$From: Distance\ =\ speed \times time, \quad Speed = 25km/h, and\ time = 3hrs$$

$$\therefore PA\ =\ 25 \times 3 = \underline{\underline{75km}}$$

For boat B:

*The distance **B** has travelled in 3 hours is equal to PB*

$Speed = 15km/h, and\ time = 3hrs$

$\therefore PB = 15 \times 3$

$= \underline{\underline{45km}}$

*The distance of **A** from **B** after 3 hours is equal to AB. Using the cosine rule:*

$AB^2 = PB^2 + PA^2 - 2 \times PA \times PB \times \cos\theta$

$AB^2 = 45^2 + 75^2 - 2 \times 75 \times 45 \cos 103^0 = 9168.42$

$\therefore AB = \sqrt{9168.42} = \underline{\underline{95.75km}}$

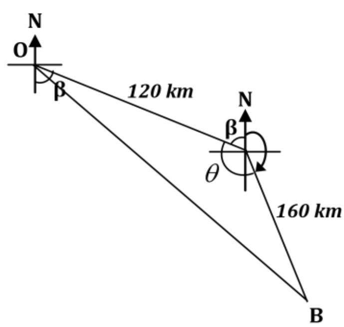
Example

An aeroplane flies 120km in the direction 113^0, then turns and flies 160km in the direction 156^o. Find its distance from the starting point.

Solution

Sketch

*Let **O** be the starting point and **B** the ending point.*



Required to find distance OB.

$\beta = 180^0 - 113^0 = 67^0$

$And\ \theta + 156^0 + \beta = 360^0$

$\therefore \theta = 360^0 - (156^0 + 67^0) = 137^0$

Using the cosine rule:

$$OB^2 = 120^2 + 160^2 - 2\times 120\times 160\times \cos 137^0$$

$$OB^2 = 68083.98$$

$$\therefore OB = \sqrt{68083.98} = \underline{\underline{260.9\ km}}$$

Example

Two planes start from an airport at the same time. One plane flies west at 400km per hour while the other flies at 500km per hour on a bearing of 040^0. What is the distance between the two planes after 15minnutes?

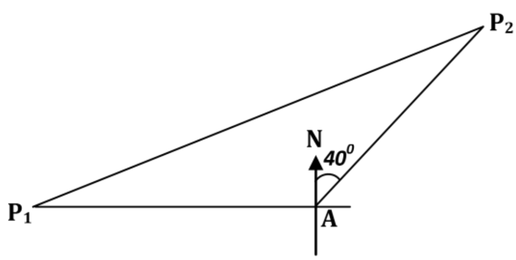
Solution

Let: **A** *be the air port*

P_1 *be position of plane to the west and* **U_1** *its speed.*

P_2 *be position of plane on a bearing of 040^o and* **U_2** *its speed.*

Sketch



Here the speed is given in km/h and time is given in minutes. This implies that, we have to convert 15minutes into hours

$$60\ \text{minutes} = 1\ hour$$

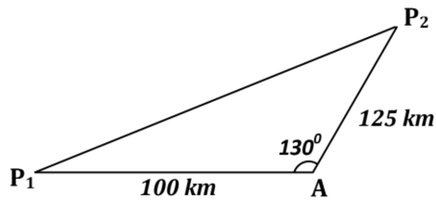
$$1\text{minute} = 1hour$$

$$\therefore 15\,\text{minutes} = \frac{1}{60}\times 15 = {}^1\!/_4\,hour. \quad So\ time\ T = 0.25hour$$

$$Distance\ AP_2 = U_2 \times T = 500\times 0.25 = 125km$$

$$Distance\ AP_1 = U_1 \times T = 400\times 0.25 = 100km$$

The distance between the two planes after 15 minutes is P_1P_2



Using cosine rule:

$$P_1P_2^{\ 2} = 100^2 + 125^2 - 2\times100\times125\times\cos130^0$$

$$P_1P_2^{\ 2} = 41694.69$$

$$\therefore P_1P_2 = \sqrt{41694.69} = \underline{\underline{204.2\ km}}$$

Example

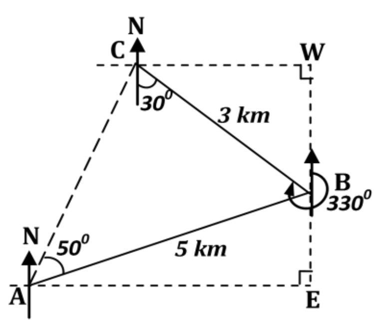
A boat sails from a port A in the direction 050^0 a distance of 5km to B. from B it sailed on a bearing of 330^0 a distance of 3km to C.

a) the eastward distance of B from A

b) the westward distance of C from B

Solution

Sketch



a) *Using triangle ABE, AE represents the eastward distance of B from A.*

$$< BAE = 40^0$$

$$Cos40^0 = \frac{AE}{5}, \quad \Rightarrow AE = 5\cos 40^0 = 3.83$$

$$\therefore Eastward\ distance\ of\ B\ from\ A = 3.83km$$

b) *Using triangle BCW, CW represents the Westward distance of C from W.*

$$< CBW = 30^0$$

$$Sin30^0 = \frac{CW}{3}, \quad \Rightarrow CW = 3\cos 40^0 = 1.5$$

$$\therefore Westward\ distance\ of\ C\ from\ B = 1.5km$$

13.2.3 Obtaining the bearing and distance of a point by use of scale drawing

To obtain the bearing or distance of one point from another, you can follow the steps below.

* First, make a rough sketch of the interpreted information.
* Choose a scale for yourself in case the scale to use is not given.
* Convert all the distances given in kilometers to centimeters using your scale.
* Draw the diagram of the sketch to scale using the distances in centimeters.
* Measure the distance of a point you have been asked to obtain using your ruler and then convert the distance you have obtain in centimeters to kilometers using your scale.
* Use your protractor to obtain the bearing of certain point.

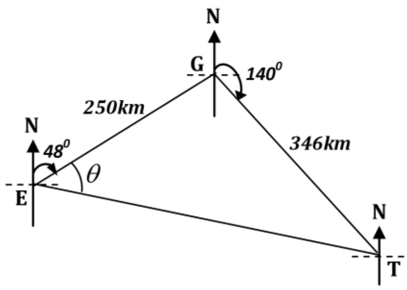
Example

An aeroplane flew from Entebbe to Gulu a distance of 250km and then to Tororo a distance of 346km. the bearing of Gulu from Entebbe was 048^o and the bearing of Tororo from Gulu was 140^o.

By scale drawing, and sing a scale of 1cm to represent 50km, find the distance and direction of Tororo from Entebbe.

Solution

Sketch



$$\underline{Scale\ }:$$

$$1cm = 50\,km$$

$$\therefore 1km = \frac{1}{50}km$$

$$From\ Entebbe\ to\ Gulu:$$

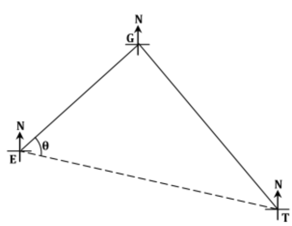
$$250\,km = \frac{1}{50} \times 250 = 5.0\,cm$$

$$Gulu\ to\ Tororo:$$

$$346km = \frac{1}{50} \times 346 = 6.9\,cm$$

<u>Steps to follow:</u>

- *Beginning E, measure 48^o clockwise using a protractor and draw a line along this direction from E. Measure a length of 5.0cm using a ruler along this line from E to G and do the same from G to T.*



- *Measure the length ET using a ruler.*

$$ET = 8.3cm$$
$$\Rightarrow ET = 8.3 \times 50 = 415km$$

Therefore, the distance from Tororo to Entebbe is 415km.

- *Also, measure angle using a protractor.*

$$\theta = 57^0$$
$$\therefore \textit{the bearing of Tororo from Entebbe} = 48^0 + 57^0 = 105^0$$

Example

Four towns XYZW are situated such that X is 20km in a direction N65^{o}E from Y, Z is 24km in the direction S48^{o}E from X while W is 27km in a direction S39^{o}W from Z.

a) By mean of scale drawing, find the respective locations of the towns.

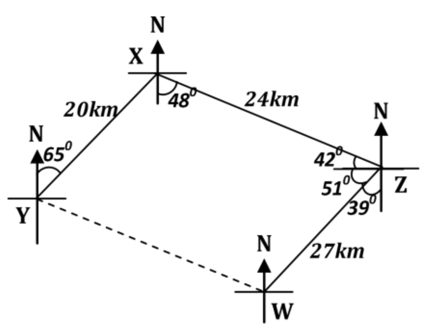
b) Using your drawing, find the distance and bearing of W from Y.

Solution

a) In this case, the scale is not given; more so, the distances between these towns are not so large. This therefore implies that you have to choose the scale to use by yourself.

Let 1cm = 5km

Sketch



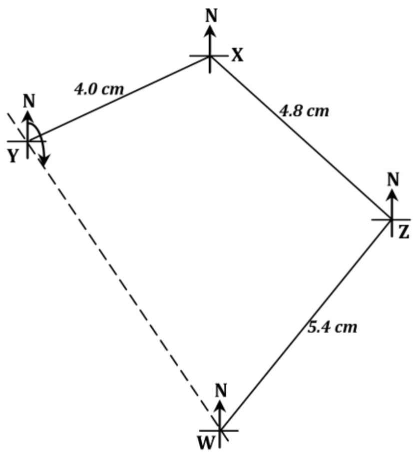
$Scale$:

$XY = 20\,km = {}^{20}\!/_{5} = 4\,cm$

$XZ = 24km = {}^{24}\!/_{5} = 4.8\,cm$

$ZW = 27km = {}^{27}\!/_{5} = 5.4cm$

Scale drawing



b) $The\ distance\ of\ W\ \ from\ Y = 6.8cm = 6.8 \times 5 = 34km$

$The\ bearing\ of\ W\ \ from\ Y = 147^0\ OR\ \ S33^0\ E$

Example

Tree points A, B, and C are 10km, 15km and 25km from an observation point O, on bearings 100^o, 240^0, 240^o, and 000^o from O respectively.

a) Find by scale drawing, the:

i. bearing of C from A

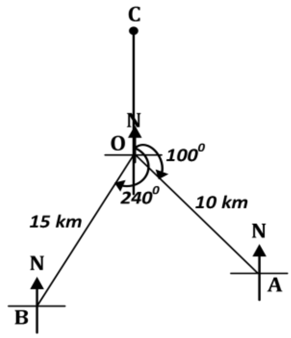
ii. bearing of C from B

iii. distances AB and BC.

b) If a cyclist is to steadily ride his bicycle from O to C via B at a speed of 12.5km/h. determine how long he would take to travel to C.

Solution

a) *Sketch*



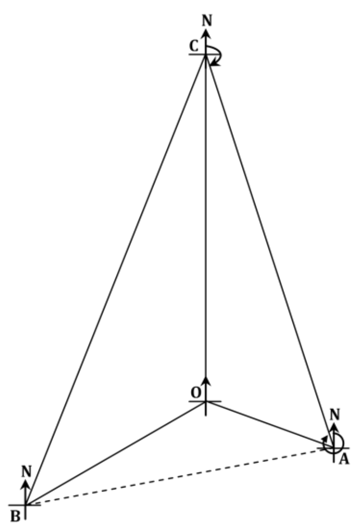
$Scale:$

$Let\ 1cm = 3\ km$

$10\,km = 10/3 = 3.3\,cm$

$15km = 15/3 = 5.0\,cm$

$25km = 25/3 = 8.3cm$



i. *The bearing of C from A is 340°*

ii. *The bearing of C from B is 022°*

iii. $Distance\ AB = 7.8cm = 7.8 \times 3 = 23.4km$

$Distance\ BC = 11.7cm = 11.7 \times 3 = 35.1km$

b) $\quad Total\ distance = OB + BC = 15km + 35.1km = 50.1km$

$$Time\ taken = \frac{total distance}{speed}, \quad Speed = 12.5km$$

$$= \frac{50.1}{12.5} = 4.008$$

$$= \underline{\underline{4hrs}}$$

13.3 Miscellaneous exercise

1. A, B and C are points on the same level. Points B and C are 100km and 150km respectively from point A. the bearing of B from A is 225^o and that of C from A is 140^o.

   a) Represent this information on a sketch diagram.

   b) From the sketch, find:
      i. the distance of B from C.
      ii. the bearing of B from C

2. Four boats P, Q R, and S are anchored on a bay such that boat Q is 180meters on a bearing of 075^0 from P, boat R is 240 meters on a bearing of 165^o from Q, boat S is 185 meters to the south of P and due west of R.

   a) Draw a sketch diagram to show the positions of P, Q, R, and S.

   b) Without using a scale diagram, calculate:

      i. the distance PR to 3 significant figures.
      ii. the bearing of P from R

3. A helicopter flies 540km from station A to station B on bearing 060^o. From station B, it travels 465km to C on a bearing of 150^o. From C it heads for station D 360km away on a bearing 265^o.

   a) Draw to scale a diagram showing the route of the helicopter. (Use the scale: 1cm to represent 50km)

   b) From your diagram, determine the distance and bearing of station A from station D.

c) Determine how long it would take the helicopter travelling at a speed of 400km/h to travel direct from station A to station C.

4. In a sports field, four points A, B, C and D are such that B is due south of A and due west of D. AB = 10.8m, BD = 18.8m, CD = 16.6m, <BAD = 60^0, <CDB = 40^0 and <BCD = 80^0.

   A vertical pole erected at D has at D has a height of 4.8cm.

   a) Draw a sketch of the relative positions of the points on the sports field.

   b) Using a scale of 1cm to represent 2cm, draw an accurate diagram to show the relative position of the points and the pole and hence, find:

      i. distances BC and AD
      ii. bearing of B from C
      iii. angle of elevation of the top of the pole from B.

   c) If an athlete runs from point A through points B, C and D and

5. Three points A, B, and c are on the same horizontal level and are such that B is 150km from A on a bearing of 060^o. The bearing of C from A is 125^o and the bearing of C from B is 60^o.

   a) By scale drawing using 1cm to represent 25km,find the distance of C from:
      i. A
      ii. B

   b) An aeroplane flies from A on a bearing of 340^0 at 300km/h. After 40minutes of flying; the pilot changes the course at point D and flies directly to C at the same speed. Include in your diagram in (a) above the route of the plane. Hence find:

      i. The time (in hours), the plane takes to travel from A to reach C.

      ii. The bearing of D from C

6. A helicopter flies from Moroto due south for 300km. It then flies on a bearing of 255^{o} for 350km. from there; it flies on a bearing of 020^{o} for 400km.
    a) Draw an accurate diagram showing the journey of the helicopter using a scale of 1cm to represent 50km.
    b) From your diagram, find the distance and bearing of Moroto from the final position of the helicopter.
    c) Given that the helicopter flies at a steady speed of 200km/h, find how long the whole journey took.

14 MATRICES OF TRANSFORMATIONS

14.1 Definition:

Transformation means a change of position or size or shape or all.

14.2 Common terms used:

Below are some of the most frequently used terms under transformation:

1. Object

   This is the initial figure (shape) formed before transformation has taken place.

2. Image

   This is the figure (shape) obtained when an object has undergone transformation.

3. Congruent (identical)

   This is when two figures have the same size and shape. Therefore, if the shape and size of an object is the same as that of its image, we then say that the object and the image are congruent or identical.

4. Invariant

   This term is used to describe a situation when there is no change in position, size, and shape after transformation. Therefore, if the position, size, and shape of an object are not changed when it has been subjected to transformation, then the position, size, and shape of the image is the same as that of the object. In this case, we say that the object and the image are invariant.

5. Mirror line

   This is a line of symmetry from where reflection of object takes place.

6. Scale factor

   This is a scalar quantity which when operated on an object, can either increase or decrease its size.

Ways by which an object can be transformed

An object can undergo transformation by the following ways:

* Translation
* Transformation by matrix multiplication
* Reflection
* Rotation
* Enlargement
* Combined transformation

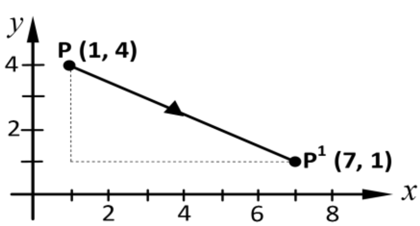
14.3 Translation

14.3.1 Definition:

Translation is the displacement of an object in a specified direction without turning. In other words, it is a movement, which has length and direction. It is described using coordinates or a column matrix.

Example

Suppose point P (1, 4) has been displaced (translated) to its image point P^1 (7, 1). To describe the translation, we need to compare the coordinates of P and P^1.



NB:

Movement to the right and movement upwards are defined as positive while movement to the left and movement downwards are defined as negative.

Now let us consider the displacement of P along the x   and y axis;

Along the x   axis, P has moved 7   1 = 6 spaces (units)

Along the y   axis, P has moved 1   4 = -3 spaces (units).

This can shortly be written as $\begin{pmatrix} 6 \\ -3 \end{pmatrix}$ which is a 2×1 matrix (column matrix) and it is known as matrix of translation denoted by T.

$$P^1 - P = T$$

$$I.e. \quad \begin{pmatrix} 7 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$$

Generally therefore, the Image point (**I**), Object point (**O**) and the matrix of Translation (**T**) are related by the expression below.

$$\text{Image point} = \textit{Translation matrix} \quad + \quad \textit{Object point}$$

$$\boxed{I = T + O}$$

This formula can be used to obtain image point given the object point and matrix of translation.

Example

Given translation $T = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$. Find the image of triangle ABC with vertices A (0, 2), B (-3, 4) and C (2, 6) under **T**.

Solution

Let the image of ABC be $A^1B^1C^1$

$$From: \text{Image} = Translation + Object$$

$$A^1 = \begin{pmatrix} -2 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \end{pmatrix}$$

$$B^1 = \begin{pmatrix} -2 \\ 5 \end{pmatrix} + \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 9 \end{pmatrix}$$

$$C^1 = \begin{pmatrix} -2 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 11 \end{pmatrix}$$

$$\therefore \; A(0, 2) \longrightarrow A^1(-2, 7)$$

$$B(-3, 4) \longrightarrow B^1(-5, 9)$$

$$C(2, 6) \longrightarrow C^1(0, 11)$$

Example

Given two matrices of translations as

$$T = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \ and \ K = \begin{pmatrix} -4 \\ 5 \end{pmatrix}. A = (0, 0), B = (0, 3) \ and \ C = (3, 3)$$

Find the image of triangle ABC under;

a) **T**

b) 2**T**

c) **K**

d) **K + T**

Solution

a) *Let the image of ABC be $A^1B^1C^1$*

$$From: I = T + O, T = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$A^1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$B^1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

$$C^1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

$$\therefore A(0,0) \longrightarrow A^1(2,3)$$
$$B(0,3) \longrightarrow B^1(2,6)$$
$$C(3,3) \longrightarrow C^1(5,6)$$

b)

$$2T = 2\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$A^1 = \begin{pmatrix} 4 \\ 6 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$B^1 = \begin{pmatrix} 4 \\ 6 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$$

$$C^1 = \begin{pmatrix} 4 \\ 6 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \end{pmatrix}$$

$$\therefore\ A(0,0) \longrightarrow A^1(4,6)$$

$$B(0,3) \longrightarrow B^1(4,9)$$

$$C(3,3) \longrightarrow C^1(7,9)$$

c)

$$k = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$$

$$A^1 = \begin{pmatrix} -4 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$$

$$B^1 = \begin{pmatrix} -4 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \end{pmatrix}$$

$$C^1 = \begin{pmatrix} -4 \\ 5 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \end{pmatrix}$$

$$\therefore\ A(0,0) \longrightarrow A^1(-4,5)$$

$$B(0,3) \longrightarrow B^1(-4,8)$$

$$C(3,3) \longrightarrow C^1(-1,8)$$

d)

$$K + T = \begin{pmatrix} -4 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$$

$$A^1 = \begin{pmatrix} -2 \\ 8 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$$

$$B^1 = \begin{pmatrix} -2 \\ 8 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 11 \end{pmatrix}$$

$$C^1 = \begin{pmatrix} -2 \\ 8 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 11 \end{pmatrix}$$

$$\therefore\ A(0,0) \longrightarrow A^1(-2,8)$$

$$B(0,3) \longrightarrow B^1(-2,11)$$

$$C(3,3) \longrightarrow C^1(1,11)$$

Example

The image of PQR is P^1 (0, 0), Q^1 (-2, 4) and R^1 (3, 4). Find the coordinates of PQR and hence sketch PQR and P^1 Q^1 R^1 on the same diagram. Take the translation vectors as $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

Solution

$$From: I = T + O, \Rightarrow O = I - T, \; and \; T = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$\Rightarrow P = P^1 - T$$

$$\therefore \; P = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

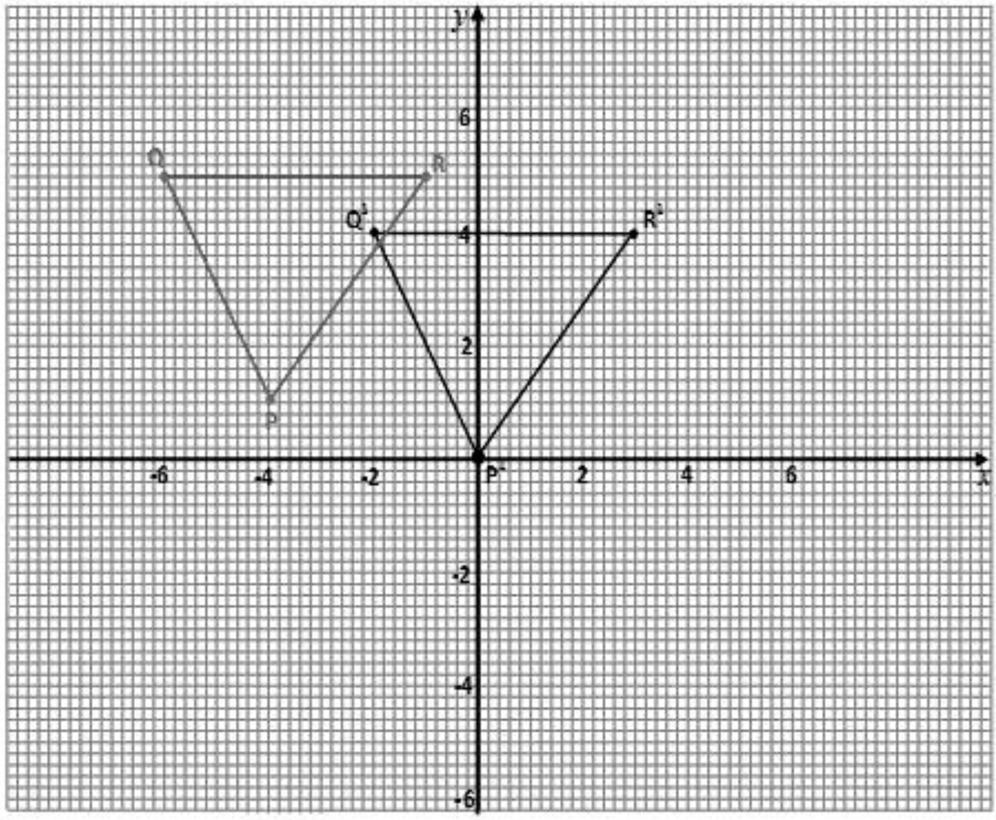
$$\Rightarrow Q = Q^1 - T, \quad Q^1 = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$\therefore \; Q = \begin{pmatrix} -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -6 \\ 5 \end{pmatrix}$$

$$\Rightarrow R = R^1 - T, \quad R^1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\therefore \; R = \begin{pmatrix} 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

$$\therefore P(-4,1) \longrightarrow P^1(0,0)$$
$$Q(-6,5) \longrightarrow Q^1(-2,4)$$
$$R(-1,5) \longrightarrow R^1(3,4)$$



14.4 Matrix of Transformation

Consider point P (x, y). Its position vector is $\begin{pmatrix} x \\ y \end{pmatrix}$. Let it be pre-multiplied i.e. multiplied from the left by the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *i.e*

$$MP = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

We say that the matrix **M** has transformed the point P (x, y) to a new point P^1 (x^1, y^1).

$$\Rightarrow \quad x^1 = ax + by$$
$$y^1 = cx + dy$$

The matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is known as the matrix of transformation.

Generally therefore given the object point (P) and the matrix of transformation (M), the image point (P^1) is calculated from the relation given below.

$$\text{Image point} = \text{Martrix of transformation} \times \text{Object point}$$

$$\boxed{P^1 = MP}$$

NB:

- In order to obtain the image point, the object point must be multiplied from the left by the matrix of transformation.
- If the matrix of transformation M is an identity matrix, i.e. $M = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then the new point is left unchanged i.e. invariant.

Example

The point P (4, 5) is transformed by matrix $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Find the image of P.

Solution

Let the image of P be P^1

$$P^1 = MP$$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 4+10 \\ 12+20 \end{pmatrix} = \begin{pmatrix} 14 \\ 32 \end{pmatrix}$$

$$\therefore P^1 = \begin{pmatrix} 14 \\ 32 \end{pmatrix}$$

$$\Rightarrow P\ (4, 5) \rightarrow P\ (14, 32)$$

Example

a) P = (6, 12) is the image of the point K under the transformation $T = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$

Find the coordinates of K.

b) Find the matrix of transformation which transformed W (1, 1) onto W^1 (1, 1) and Y (3, 2) onto Y^1 (4, 3).

Solution

a) $P = TK$

$Let\ K = (a \quad b)$

$$\Rightarrow \begin{pmatrix} 6 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a+4b \\ a+3b \end{pmatrix}$$

$\therefore\ 6 = 2a + 4b$......................................(1)

$12 = a + 3b$......................................(2)

$Eqn(2) - eqn(1)$

$$a + 3b = 12$$

$$^{-}\underline{a + 2b = 3}$$

$$b = 9\ and\ a = 12 - 3b \ \Rightarrow a = 12 - 27 = -15$$

$\therefore\ \underline{\underline{K = (-15, 9)}}$

b) *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be the matrix that transformed W and Y to* W^1 *and* Y^1 *respectively.*

$$W^1 = MW, \quad W^1(1,1), \quad W\,(1,1)$$

$$\Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix}$$

$\therefore\ 1 = a + b$......................................(1)

$1 = c + d$......................................(2)

$Also \quad Y^1 = MY, \quad Y^1(4,3), \quad Y(3,2)$

$$\Rightarrow \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3a+2b \\ 3c+2d \end{pmatrix}$$

$\therefore \; 4 = 3a + 2b$.....................................(3)

$3 = 3c + 2d$.....................................(4)

Solving eqn(1) *and eqn*(3)

$2(a + b = 1)$

$\underline{3a + 2b = 4}$

$2a + 2b = 2$

$^-\underline{3a + 2b = 4}$

$-a = -2 \Rightarrow a = 2$

$b = 1 - a \Rightarrow b = -1$

Also solving eqn(2) *and eqn*(4)

$2(c + d = 1)$

$\underline{3c + 2d = 3}$

$2c + 2d = 2$

$^-\underline{3c + 2d = 3}$

$-c = -1 \Rightarrow c = 1$

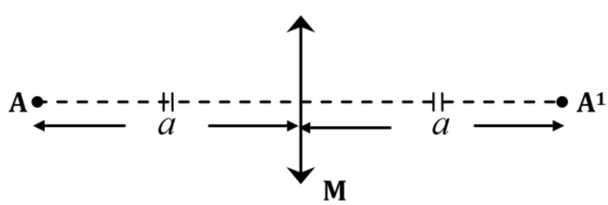
$d = 1 - c \Rightarrow d = 2$

$$\therefore M = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

14.5 Reflection:

For an object to be reflected, there must be a mirror line. A mirror line as stated earlier is a line from where reflection of object takes place.

If point **A** is to reflected along the mirror line M, then its image $\mathbf{A^1}$ will be formed to the left of the mirror line and **A** and $\mathbf{A^1}$ are of equal distance from the mirror line M.



Below are the general properties of reflection:

- Points, which are on the mirror line, are their own images i.e. they are invariant.
- The distance of the image from the mirror line is equal to the distance of the object from the mirror line.

- The mirror line M bisects the angle between the object and its image.

- In reflection, an object and its image are oppositely congruent i.e. lengths and angle remain same but direction is reversed.

a) Finding the image of an object by scale drawing

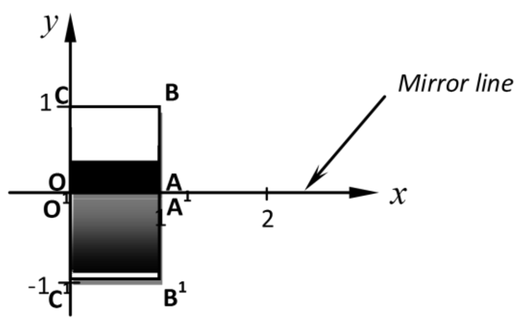
Here we shall use the properties of reflection to find the image of an object.

Example

Find the image of square OABC with O (0, 0), A (1, 0), B (1, 1) and C (0, 1) after reflection along the x axis.

Solution

X-axis is the mirror line



- *Points O (0, 0) and A (1, 0) which are on the mirror line are their own images i.e. they are invariant.*
- *Points B (1, 1) and C (0, 1) which are not on the mirror line are displaced 1 unit below the x –axis (mirror line).*

$$\Rightarrow O^1 = (0, 0) \quad A^1 = (1, 0)$$
$$B^1 = (1, 1) \quad C^1 = (0, -1)$$
$$OABC \xrightarrow{x-axis} O^1A^1B^1C^1$$

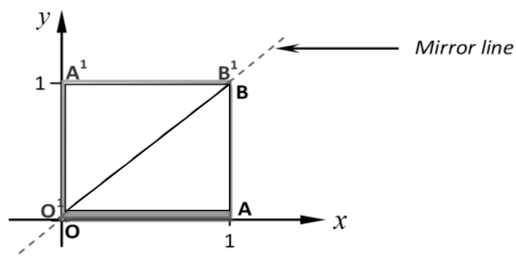
Example

a) Find the image of triangle OAB with O (0, 0), A (1, 0) and B (1, 1) after reflection along the line $y = x$.

b) Find the image A B C after a reflection along the line $y = -x$

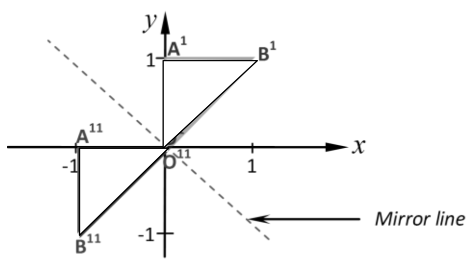
Solution

a)



$$A^1 = (0, 1)$$
$$B^1 = (1, 1)$$
$$O^1 = (0, 0)$$

b)

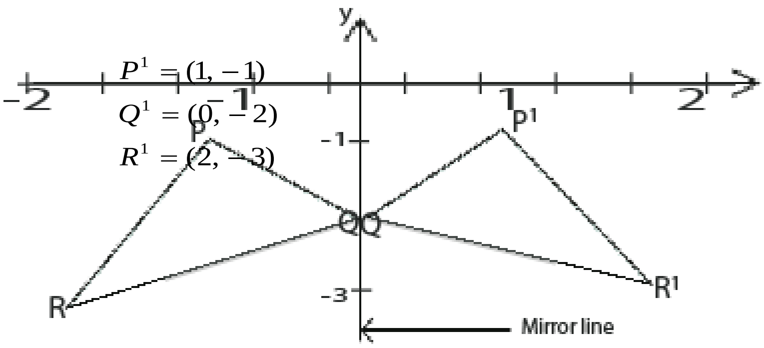


$$A^{11} = (-1, 0)$$
$$B^{11} = (-1, -1)$$
$$O^{11} = (0, 0)$$

Example

Find the image of triangle PQR with P (-1, -1), Q (0, -2) and R (-2, -3) after a reflection along the y axis.

Solution



$P^1 = (1, -1)$

$Q^1 = (0, -2)$

$R^1 = (2, -3)$

General case:

If point P (x, y) has been reflected along the following mirror lines;

i. $y-axis$

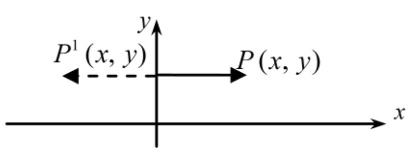
ii. $x-axis$

iii. $y = x$

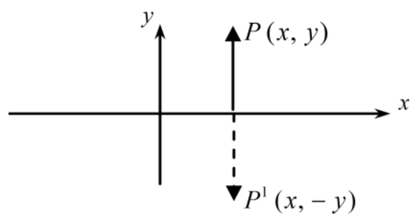
iv. $y = -x$

Then, point P (x, y) would have its image as follows:

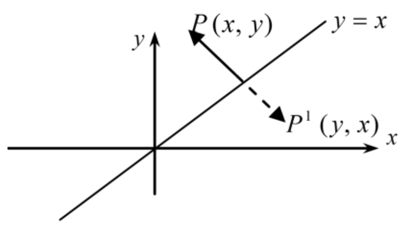
i. $P\,(x,\, y) \xrightarrow{y-axis} P^1(-x,\, y)$



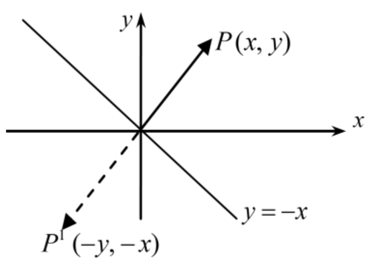
ii. $P\,(x,\, y) \xrightarrow{x-axis} P^1(x, -y)$



iii. $P\,(x,\,y) \xrightarrow{y=x} P^1(y,\,x)$



iv. $P\,(x,\,y) \xrightarrow{y=-x} P^1(-y,\,-x)$

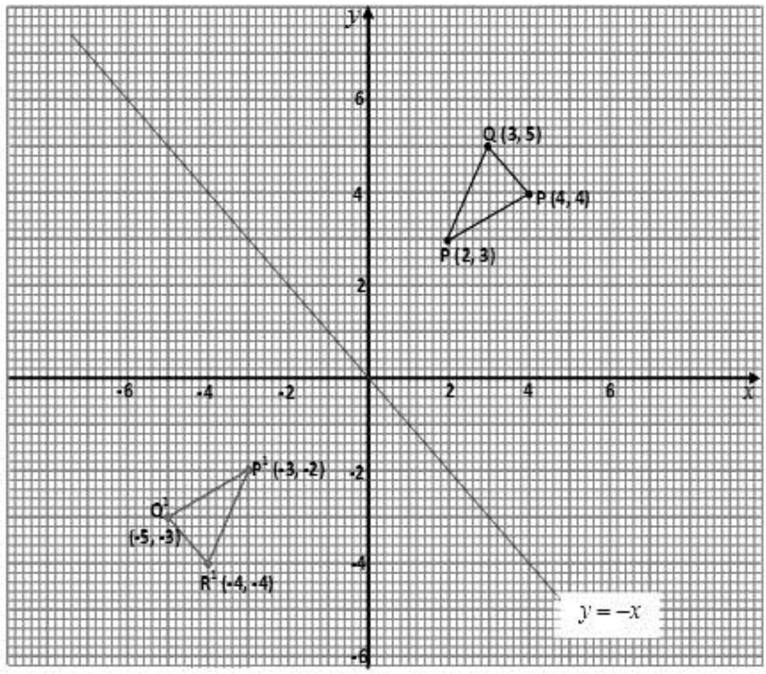


Example

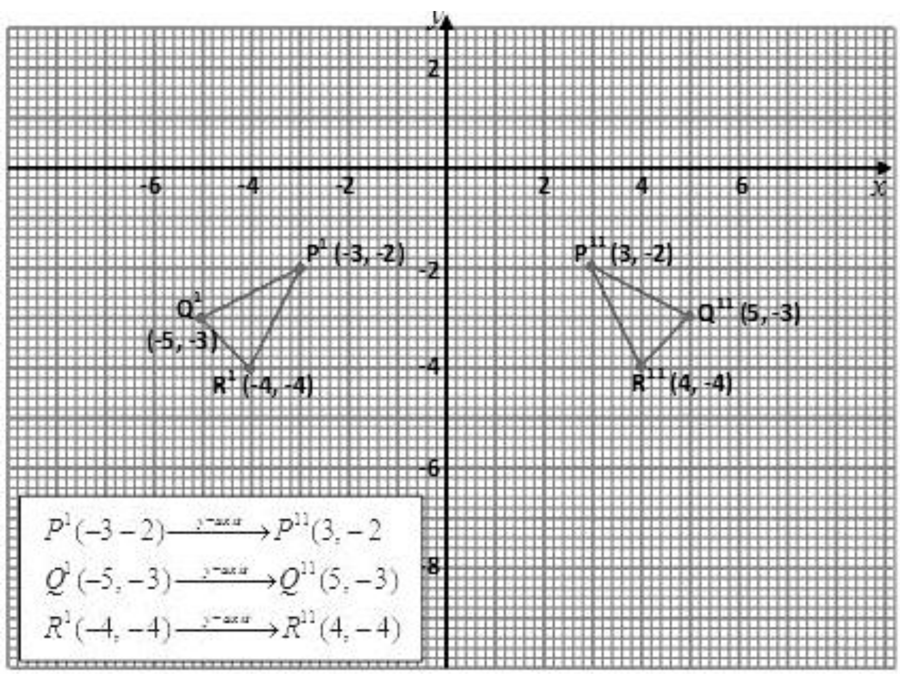
a. Find the image of triangle PQR with P (2, 3), Q (3, 5) and R (4, 4) after a reflection along the line $y = -x$

b. Also find the image of $P^1\ Q^1\ R^1$ when reflected along the y axis.

a) **Graphical solution**



b) **Graphical solution**



b) Finding the image of an object by use of calculation

Here we need to know the matrix of reflection along the given mirror line. Thereafter, we multiply the object from the left by the matrix of reflection to obtain the image.

14.6 Matrix of reflection along x –axis

Consider two points P (1, 0) and Q (0, 1) being reflected along x axis.

$$Fom: \quad P(x, y) \xrightarrow{x-axis} P^1(x, -y)$$
$$\Rightarrow \quad P(1, 0) \xrightarrow{x-axis} P^1(1, 0)$$
$$Q(0, 1) \xrightarrow{x-axis} Q^1 \ (0, -1)$$

$$Let \ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} be \ the \ matrix \ of \ reflection$$

$$\therefore P^1 = MP \ also \quad Q^1 = MQ$$

$$\begin{array}{cccc} & P^1 \quad Q^1 & & P \quad Q \\ \Rightarrow & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \therefore \begin{matrix} a = 1 & b = 0 \\ c = 0 & d = -1 \end{matrix}$$

$$\therefore M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Therefore the matrix of reflection along the x axis is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Activity:

Show that the matrix of reflection along;

i. $y - axis \ is \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

ii. $y = x \ is \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

iii. $y = -x \ is \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Example

Find the image of OABC with O (0, 0), A (1, 0), B (1, 1) and C (0, 1) being reflected along the x axis.

Solution

Method 1:

The matrix of reflection along the x –axis is $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\Rightarrow O^1A^1B^1C^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{array}{c} \begin{array}{cccc} O & A & B & C \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} O^1 & A^1 & B^1 & C^1 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix} \end{array}$$

$\therefore O^1 = (0, 0)$

$A^1 = (1, 0)$

$B^1 = (1, -1)$

$C^1 = (0, -1)$

Method 2:

$$From: \quad P\,(x, y) \xrightarrow{x-axis} P^1(x, -y)$$
$$O(0, 0) \xrightarrow{x-axis} O^1\,(0, 0)$$
$$A(1, 0) \xrightarrow{x-axis} A^1\,(1, 0)$$
$$B(1, 1) \xrightarrow{x-axis} B^1\,(1, -1)$$
$$C\,(0, 1) \xrightarrow{x-axis} C^1\,(0, -1)$$

Example

a) Find A and B the images of A and B respectively under the reflection in the x   axis with A (1, 3) and B (3, 7).

b) Find:

   i) The equation of AB

   ii) The equation of A^1 B^1

Solution

a) *The matrix of reflection along x –axis* $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$A^1 = MA \text{ and } B^1 = MB. \quad A = (1, 3), B = (3, 7)$

$A^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$\therefore A(1, 3) \longrightarrow A^1\,(1, -3)$

$B^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$

$\therefore B(3, 7) \longrightarrow B^1(3, -7)$

b) i) ***For AB:***

$A = (1, 3), B = (3, 7)$

$\text{Gradient of } AB = \frac{7-3}{3-1} = \frac{4}{2} = 2$

$\text{From } y = mx + c, m = 2 \text{ and using point } A(1,3), x = 1, y = 3$

$\Rightarrow \quad 3 = 2(1) + c \quad \therefore c = 1$

$\therefore \underline{\underline{y = 2x + 1}}$

ii) ***For A^1 B^1:***

$A^1 = (1, -3), B^1 = (3, -7)$

$\text{Gradient of } A^1B^1 = \frac{-7 - {}^-3}{3-1} = \frac{-4}{2} = -2$

$\text{From } y = mx + c, m = -2 \text{ and using point } A(1, -3), x = 1, y = -3$

$\Rightarrow \quad -3 = -2(1) + c \quad \therefore c = -1$

$\therefore \underline{\underline{y + 2x + 1 = 0}}$

14.7 Rotation

This involves change in position of points when they are turned about a fixed point known as centre of rotation. Centre of rotation is a single fixed point under rotation. Nevertheless, every other point under rotation moves along an arc of a circle with this centre.

When a point changes when it is turned about the centre of rotation, the line from the point through the centre of rotation turns through an angle known as angle of rotation. Angle of rotation therefore is the angle through which a given point has been shifted from its initial position when turned about the centre of rotation.

14.7.1 General properties of rotation:

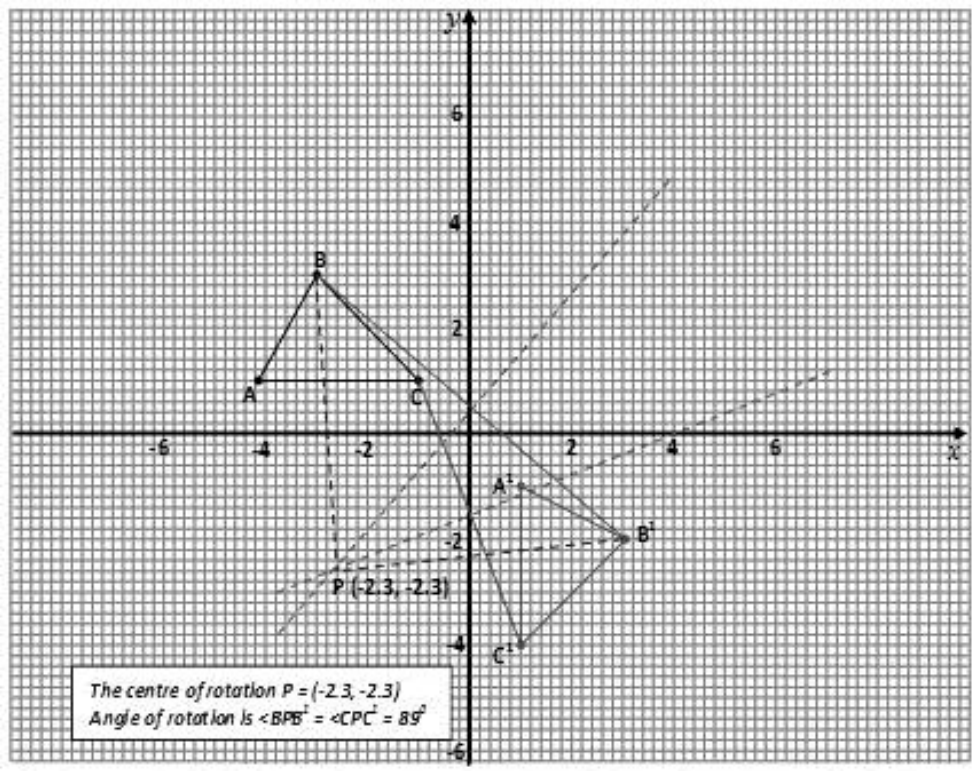
- The image is directly congruent to the object
- The distance of an image point from the centre of rotation is equal to the distance of the corresponding object point from the centre of rotation.
- Each line on the object through the centre of rotation turns through an angle equal to the angle of rotation.
- The centre of rotation is the intersection of the perpendicular bisectors of any two lines from the object to the corresponding image. For instance, given triangle ABC and its image A^1 B^1 C^1, the centre of rotation is the intersection of the perpendicular bisectors of any two of the line segments AA^1, BB^1, and CC^1.

a) *Obtaining the centre and angle of rotation*

When given an object and its image, the centre and angle of rotation can be obtained. The following example will illustrate this concept. Consider triangle ABC with vertices A (-4, 1), B (-3, 3) and C (-1, 1) being rotated on triangle $A^1B^1C^1$ with vertices A^1 (1, -1), B^1 (3, -2) and C^1 (1, -4). Determine the centre and angle of rotation.

In finding the centre and angle of rotation, the following steps should be followed:

- Plot the image $A^1B^1C^1$ and the object ABC on the same graph paper.
- Join BB^1 and then construct its perpendicular bisector.
- Join CC^1 and also construct its perpendicular bisector.
- The point P where the perpendicular bisectors meet gives the centre of rotation.
- The angle of rotation is $<BPB^1$ or $<CPC^1$. It is measured using a protractor.



b) *Finding the image of an object by scale drawing given the angle and centre of rotation*

We can find the image of an object if we know the object point, angle of and centre of rotation.

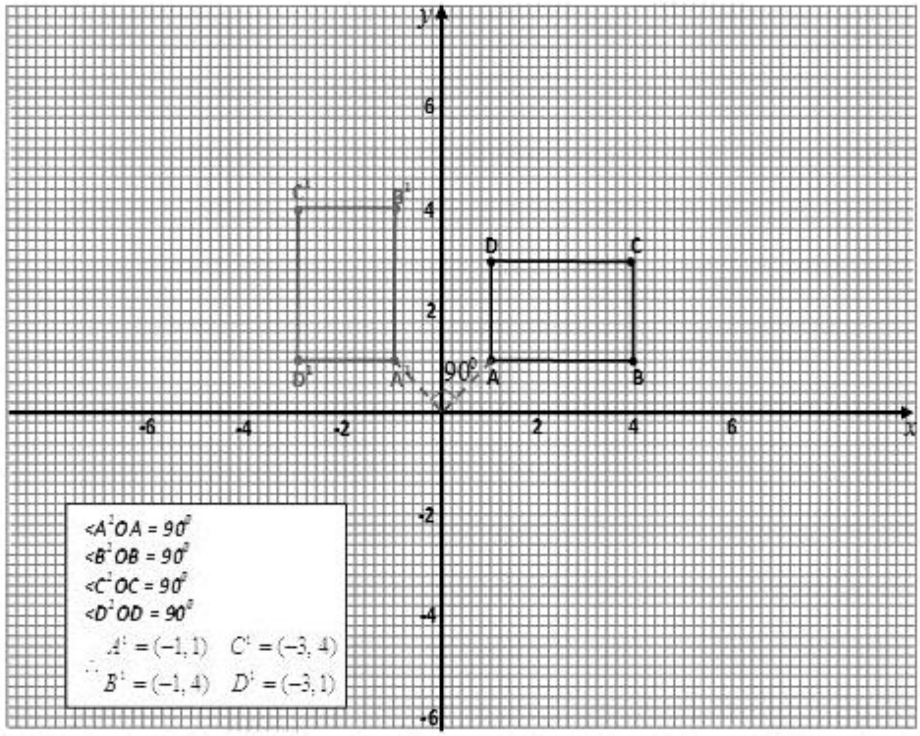
NB:

- Rotation is defined as positive when it is anti-clockwise and negative when it is clockwise.
-

  negative rotation of through an angle of (360^o
  same centre.

Example

Use graph paper to obtain the image of rectangle ABCD with vertices A (1, 1), B (4, 1), C (4, 3) and D (1, 3) when rotated through 90^o anti clockwise with O (0, 0) as the centre of rotation.



c) *Finding the image of an object by calculation*

Here we need to know the matrix of rotation. The object is then multiplied from the left by the matrix of rotation and the result obtained gives the image point. That is to say, if M is the matrix of rotation and point P is the object point, then the image point P^1 is calculate from the expression below.

$$P^1 = MP$$

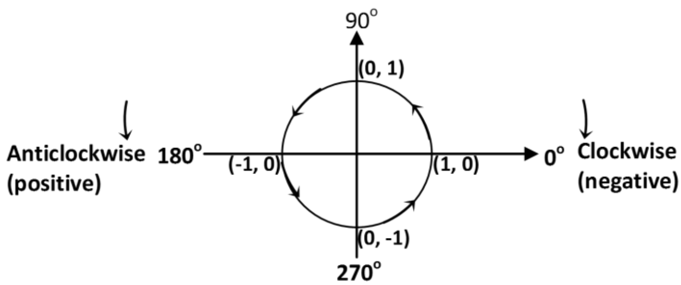
Generally, the matrix of rotation M is given the expression below:

$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Where θ is the angle of rotation

14.7.2 Common terms associated with rotation

Consider a unit circle i.e. a circle of radius 1 and centre (0, 0) as shown below.



Clockwise rotation is negative and anticlockwise rotation is positive. Below are some common terms associated with rotation.

1 *Quarter turn:*
This is the same as turning through 90^0 anticlockwise. I.e. quarter turn = 90^0.

2 *Half turn:*
This is the same as turning through180^o anticlockwise. I.e. Half turn = 180^o.

3 *Three quarter turn:*
This is the same as turning through 270^o anticlockwise. I.e. three quarter turn = 270^o.

4 *Negative quarter turn:*
This is the same as positive three quarter turn. I.e. negative quarter turn = 90^o.

Example

Find the matrix of rotation through:

a) Quarter turn

b) Half turn

c) Three quarter turn

d) Negative quarter turn

e) 30^o

Solution

$$Matrices\ of\ rotation\ M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

a) *Quarter turn: θ =90°*

$$\therefore M = \begin{pmatrix} \cos 90^0 & -\sin 90^0 \\ \sin 90^0 & \cos 90^0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

b) *Half turn: θ =180°*

$$\therefore M = \begin{pmatrix} \cos 180^0 & -\sin 180^0 \\ \sin 180^0 & \cos 180^0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

c) *Three–quarter turn = negative quarter turn: θ = 270°*

$$\therefore M = \begin{pmatrix} \cos 270^0 & -\sin 270^0 \\ \sin 270^0 & \cos 270^0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

d) *Negative Quarter turn: θ = -90°*

$$\therefore M = \begin{pmatrix} \cos -90^0 & -\sin -90^0 \\ \sin -90^0 & \cos -90^0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

e) *For 30°:*

$$\therefore M = \begin{pmatrix} \cos 30^0 & -\sin 30^0 \\ \sin 30^0 & \cos 30^0 \end{pmatrix},\quad but\ \sin 30^0 = \tfrac{1}{2} \quad and\ \cos 30 = \tfrac{\sqrt{3}}{2}$$

$$= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{pmatrix}$$

Example

Find the image of A (3, 4) after a rotation through an angle of 30°.

Solution

The matrix of rotation for 30° is

$$M = \begin{pmatrix} \cos 30^0 & -\sin 30^0 \\ \sin 30^0 & \cos 30^0 \end{pmatrix} = \begin{pmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{pmatrix}$$

$$A^1 = MA = \begin{pmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0.598 \\ 4.964 \end{pmatrix}$$

$$\therefore A^1 = (0.598, 4.964)$$

Example

Find the image of P (3, 00, Q (4, 2) and R (-3, 0) after being rotated about the origin through $\frac{3}{4}$ turn. Hence, sketch the object and its image.

Solution

$\frac{3}{4}$ turn is the same as turning through 270^o. The matrix of rotation is:

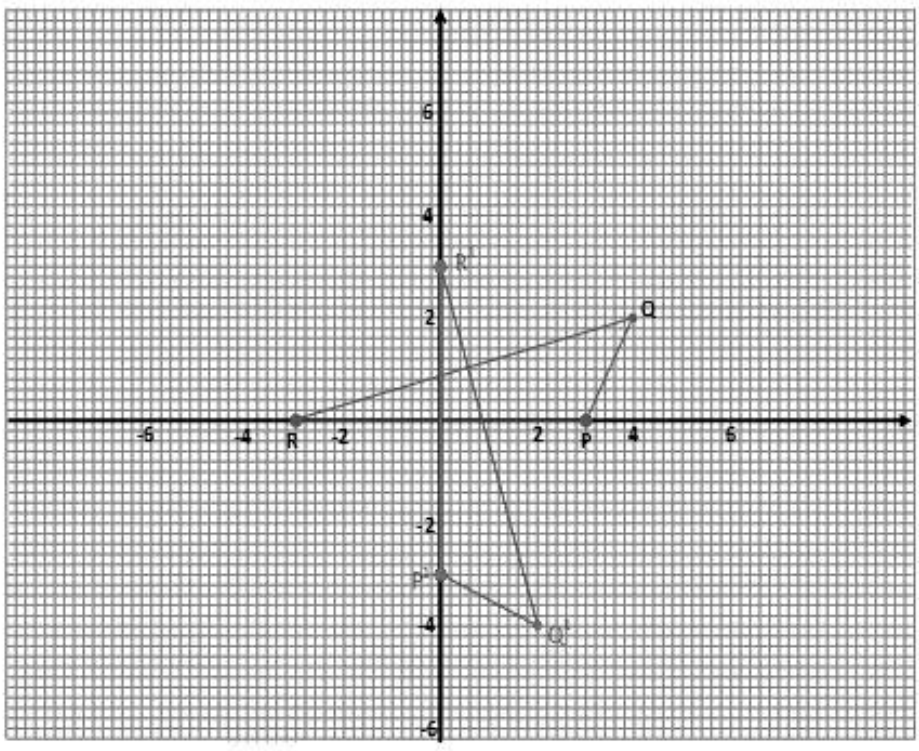
$$\therefore M = \begin{pmatrix} \cos 270^0 & -\sin 270^0 \\ \sin 270^0 & \cos 270^0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Rightarrow P^1Q^1R^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{array}{c} \begin{array}{ccc} P & Q & R \end{array} \\ \begin{pmatrix} 3 & 4 & -3 \\ 0 & 2 & 0 \end{pmatrix} \end{array} = \begin{pmatrix} 0 & 2 & 0 \\ -3 & -4 & 3 \end{pmatrix}$$

$P(3, 0) \longrightarrow P^1(0, 3)$

$Q(4, 2) \longrightarrow Q^1(2, -4)$

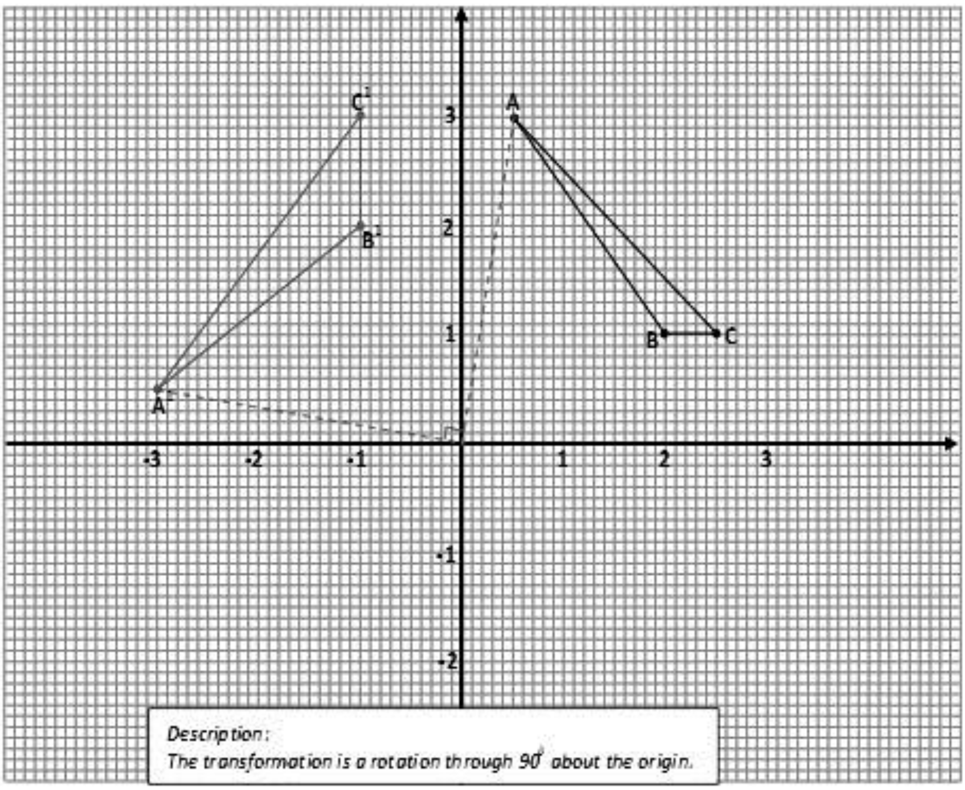
$R(-3, 0) \longrightarrow R^1(0, 3)$



Example

A triangle with vertices A (1, 3), B (2, 1) and C (3, 1) is mapped onto another triangle with vertices A^1 (-3, 1), B (-1, 2) and C (-1, 3). Describe this transformation and find its matrix.

Solution



$$\text{Let the matrix of transformation } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$From: P^1 = MP$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} A^1 & B^1 & C^1 \end{array} \\ \begin{pmatrix} a+3b & 2a+b & 3a+b \\ c+3d & 2c+d & 3c+d \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} A^1 & B^1 & C^1 \end{array} \\ \begin{pmatrix} -3 & -1 & -1 \\ 1 & 2 & 3 \end{pmatrix} \end{array}$$

$$\Rightarrow \begin{array}{lll} a+3b = -3 \ldots\ldots\ldots\ldots(1) & 2a+b = -1 \ldots\ldots\ldots\ldots(3) & 3a+b = -1 \ldots\ldots\ldots\ldots(5) \\ c+3d = 1 \ldots\ldots\ldots\ldots(2) & 2c+d = 2 \ldots\ldots\ldots\ldots(4) & 3c+d = 3 \ldots\ldots\ldots\ldots(6) \end{array}$$

$Equation(5) - eqn(3)$	$Equation(6) - eqn(4)$
$3a + b = -1$	$3c + d = 3$
$^{-}\underline{2a + b = -1}$	$^{-}\underline{2c + d = 2}$
$a = 0,$	$c = 1,$
$3a + b = -1 \quad \Rightarrow b = -1$	$3c + d = 3 \quad \Rightarrow d = 0$

$$\therefore M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

17.3 Enlargement:

We say that an object has been enlarged when its size has been increased. An enlargement is a transformation, which results in an image, such that:

* All its lengths and the corresponding lengths on the object bear a constant ration known as scale factor.
* Its angles are equal to the corresponding angles on the object. In order words the object its image are similar.

To describe an enlargement, we need to know its centre of enlargement and its scale factor. A scale factor is a factor by which the size of a given object changes.

17.3.1 General properties of enlargement

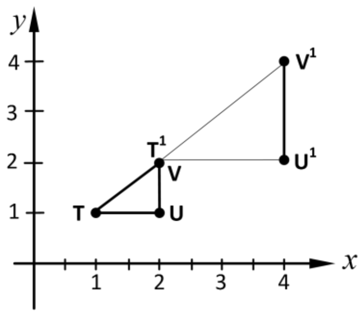
- An object and point, its image and centre of enlargement are collinear.
- For any point A on an object, $\bar{OA^1} = k\bar{OA}$, where k is a scale factor. For instance if k = 3, then this mean that $\bar{OA^1}$ is 3 times $\bar{OA}$ $i.e.\ \bar{OA^1} = 3\bar{OA}.$
- The centre of enlargement is the only point that remains fixed irrespective of the scale factor.
- If the linear scale factor is k, the area scale factor is k^2 and if the enlargement results in the formation of solid object, then the volume scale factor is k^3.

17.3.2 Obtaining the matrix of enlargement

Consider triangle TUV with vertices T (1, 1), U (2, 1) and V (2, 2) being enlarged with scale factor 2 and centre of enlargement O (0, 0). This means that every point on the triangle increases by a factor of 2, i.e.

$$T^1 = 2\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix},\quad U^1 = 2\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix},\quad V^1 = 2\begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}4\\4\end{pmatrix}$$

The size of triangles TUV and $T^1U^1V^1$ can be compared by drawing them on the same graph.



17.3.3 Matrix of enlargement:

Let the matrix of enlargement $M = \begin{pmatrix}a & b\\c & d\end{pmatrix}$

$$From \quad P^1 = MP$$

$$\Rightarrow \begin{array}{c} \begin{array}{ccc} T^1 & U^1 & V^1 \end{array} \\ \begin{pmatrix}2 & 4 & 4\\2 & 2 & 4\end{pmatrix} \end{array} = \begin{pmatrix}a & b\\c & d\end{pmatrix} \begin{array}{c} \begin{array}{ccc} T & U & V \end{array} \\ \begin{pmatrix}1 & 2 & 2\\1 & 1 & 2\end{pmatrix} \end{array}$$

$$\begin{pmatrix}2 & 4 & 4\\2 & 2 & 4\end{pmatrix} = \begin{pmatrix}a+b & 2a+b & 2a+2b\\c+d & 2c+d & 2c+2d\end{pmatrix}$$

$$a+b = 2 \text{............}(1) \qquad c+d = 2 \text{............}(4)$$
$$2a+b = 4 \text{............}(2) \qquad 2c+d = 2 \text{............}(5)$$
$$2a+2b = 4 \text{..........}(3) \qquad 2c+2d = 4 \text{..........}(6)$$

Equation(2) − eqn(1)	Equation(5) − eqn(4)
$2a + b = 4$	$2c + d = 2$
$^{-}\underline{a + b = 2}$	$^{-}\underline{c + d = 2}$
$a = 2,$	$c = 0,$
$b = 0$	$d = 2$

$\therefore$ *The matrix of enlargement M, scale factor 2 centre (0, 0) is* $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Generally, the matrix of enlargement is given by:

$$M = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

Where k is the scale factor

Note:

a) If k > 0, then the object and its image lie on the same side of the centre of enlargement.

b) If k < 0, then the object and its image are on opposite sides of the centre of enlargement.

c) If $k = \pm 1$, then the object and its image are congruent

d) If 0 < k < 1, then the image is smaller than the object.

Example

The vertices of quadrilateral ABCD have coordinates A (2, 3), B (-3, 4), C (-5, -1) and D (4, -5).

Find the images of the vertices of the quadrilateral under enlargement with centre (0,) and :

a) With the matrices:

i. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

ii. $\begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$

b) With matrix $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

In both a) and b) show the images and the object on the same graph paper.

Solution

a) $\text{Image} = \text{matrix of enlargement} \times \text{object}$

i.
$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 2 & -3 & -5 & 4 \\ 3 & 4 & -1 & -5 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} A^1 & B^1 & C^1 & D^1 \end{array} \\ \begin{pmatrix} 4 & -6 & -10 & 8 \\ 6 & 8 & -2 & -10 \end{pmatrix} \end{array}$$

$A(2, 3) \longrightarrow A^1(4, 6)$

$B(-3, 4) \longrightarrow B^1(-6, 8)$

$C(-5, -1) \longrightarrow C^1(-10, -2)$

$D(4, -5) \longrightarrow D^1(8, -10)$

ii.
$$\begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 2 & -3 & -5 & 4 \\ 3 & 4 & -1 & -5 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} A^{11} & B^{11} & C^{11} & D^{11} \end{array} \\ \begin{pmatrix} 1 & -1.5 & -2.5 & 2 \\ 1.5 & 2 & -0.5 & -2.5 \end{pmatrix} \end{array}$$

$A(2, 3) \longrightarrow A^{11}(1, 1.5)$

$B(-3, 4) \longrightarrow B^{11}(-1.5, 2)$

$C(-5, -1) \longrightarrow C^{11}(-2.5, -0.5)$

$D(4, -5) \longrightarrow D^{11}(2, -2.5)$

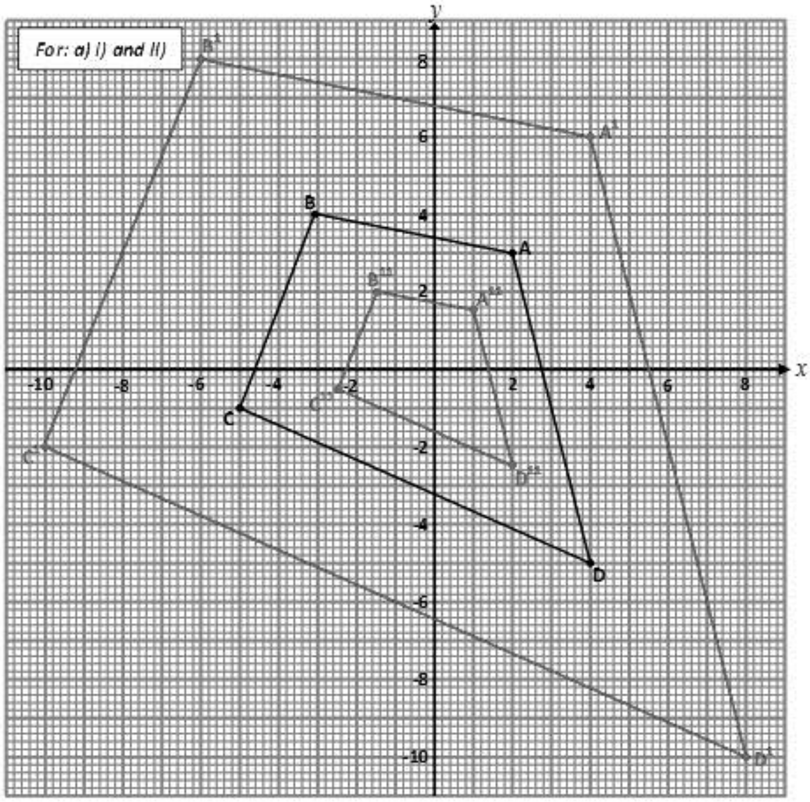
b)
$$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 2 & -3 & -5 & 4 \\ 3 & 4 & -1 & -5 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} A^{111} & B^{111} & C^{111} & D^{111} \end{array} \\ \begin{pmatrix} -4 & 6 & 10 & -8 \\ -6 & -8 & 2 & 10 \end{pmatrix} \end{array}$$

$A(2, 3) \longrightarrow A^{111}(-4, -6)$

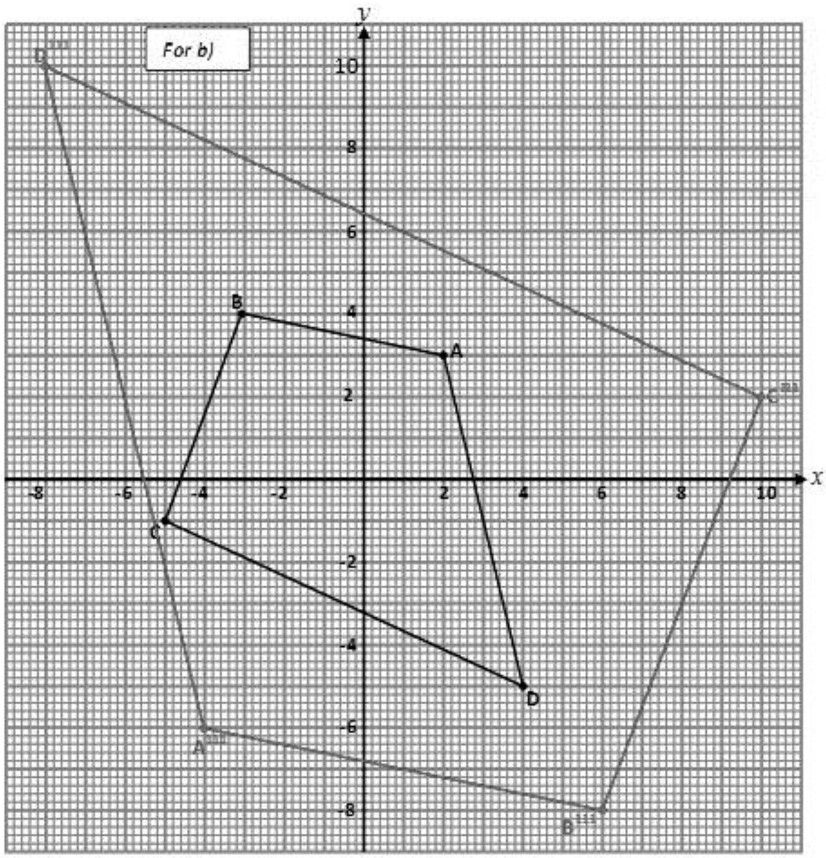
$B(-3, 4) \longrightarrow B^{111}(6, -8)$

$C(-5, -1) \longrightarrow C^{111}(10, 2)$

$D(4, -5) \longrightarrow D^{111}(-8, 10)$



For: a) i) and ii)
y
x
A
B
C
D
8
6
4
2
-2
-4
-6
-8
-10
-10
-8
-6
-4
2
4
6
8



17.3.4 Centre of enlargement (C.E)

The centre of enlargement can be calculated from the expression below

$$C.E = \frac{1}{k-1}(kO - I)$$

Where : k – *scale factor*

O – *object position*

I – *image position*

Example

The image of point A (5, 2) under an enlargement scale factor 3 is A^1 (1, 6). Determine the coordinates of the centre of enlargement.

Solution

$$From: \quad C.E = \frac{1}{k-1}(kO-I), \quad k=-3, \quad O = A = \begin{pmatrix}5\\2\end{pmatrix}, \quad I = A^1 = \begin{pmatrix}1\\6\end{pmatrix}$$

$$\therefore C.E = \frac{1}{^-3-1}\left(^-3\begin{pmatrix}5\\2\end{pmatrix}-\begin{pmatrix}1\\6\end{pmatrix}\right) = \frac{1}{-4}\begin{pmatrix}^-15-1\\^-6-6\end{pmatrix} = \frac{-1}{4}\begin{pmatrix}-16\\-12\end{pmatrix} = \begin{pmatrix}{}^-16/{}^-12\\{}^-12/{}^-4\end{pmatrix} = \begin{pmatrix}4\\3\end{pmatrix}$$

$$\Rightarrow \quad C.E = (4, 3)$$

17.4 Inverse transformation:

If M is the matrix of transformation that maps an object P onto an image point P^1, then the transformation which maps P^1 back onto P is called inverse of M written as M^{-1}, i.e.

$$If \ P^1 = MP, then:$$

$$P = M^{-1}P^1$$

This expression is useful in obtaining the object point given the image point and the transformation matrix.

Example

Under the enlargement $\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}$ the image of triangle ABC has vertices $A^1(4, 2)$, $B^1(4, 4)$ and $C^1(8, 4)$. Find the coordinates of the vertices of the object triangle ABC.

Solution

$$Let \ M = \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}$$

$$\det M = 4, \quad adjoin \ of \ M = \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}$$

$$\therefore M^{-1} = \frac{1}{4}\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & 0\\0 & \frac{1}{2}\end{pmatrix}$$

$$From \quad P = M^{-1}P^1$$

$$\begin{array}{cccc} & \begin{array}{ccc}A^1 & B^1 & C^1\end{array} & & \begin{array}{ccc}A & B & C\end{array}\\ \Rightarrow \begin{pmatrix}\frac{1}{2} & 0\\0 & \frac{1}{2}\end{pmatrix} & \begin{pmatrix}4 & 4 & 8\\2 & 4 & 4\end{pmatrix} & = & \begin{pmatrix}2 & 2 & 4\\1 & 2 & 2\end{pmatrix}\end{array}$$

Therefore, the coordinates of the vertices of triangle ABC are A (2, 1), B (2, 2), and C (4, 2).

17.5 Combined transformation:

This is when the object is performed with more than one matrices of translation. Here, you will be required to be in position to:

i. Obtain the image of an object under combined transformation.
ii. State a single transformation that would map the object onto the image obtained after combined transformation.
iii. Obtain a single matrix, which is equivalent to the combined matrices of transformation.

Example

Triangle ABC with vertices A (1, 1), B (1, 4), and C (3, 4) is given a positive quarter turn about the origin to give triangle $A^1B^1C^1$. This is then followed by a reflection along the x axis giving the image of $A^1B^1C^1$ as $A^{11}B^{11}C^{11}$.

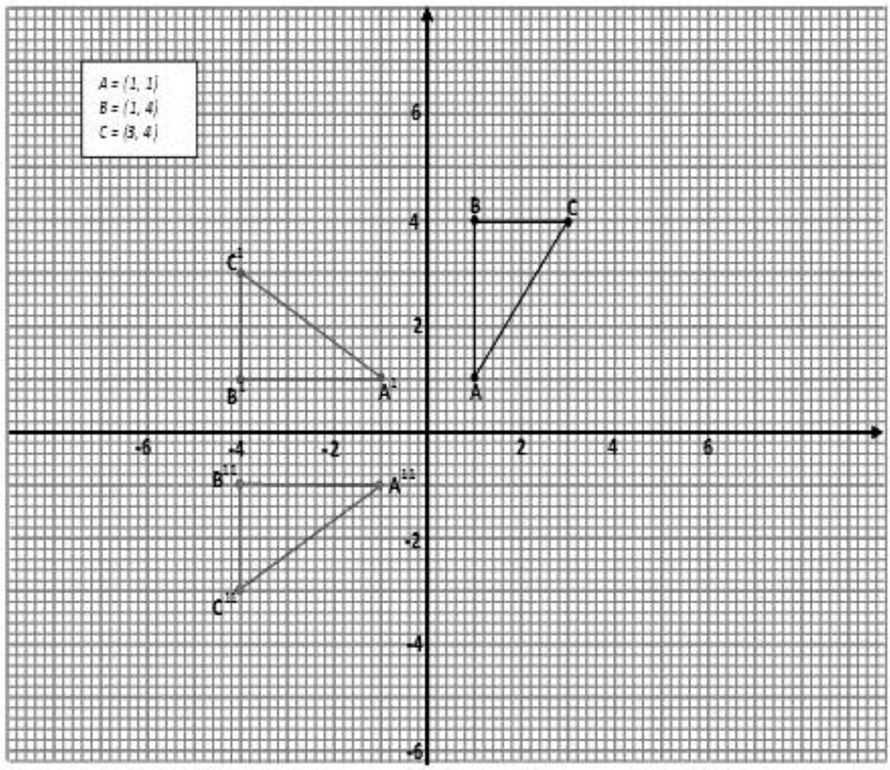
a) State the coordinates of triangles:

i. $A^1B^1C^1$
ii. $A^{11}B^{11}C^{11}$

b) i) What single transformation maps triangle ABC onto triangle $A^{11}B^{11}C^{11}$?

ii) What is the matrix of this transformation?

Solution



a)
$$A^1 = (-1, 1) \qquad A^{11} = (-1, -1)$$
$$B^1 = (-4, 1) \quad and \quad B^{11} = (-4, -1)$$
$$C^1 = (-4, 3) \qquad C^{11} = (-4, -3)$$

b) i) *A single transformation that would maps triangle ABC directly onto triangle $A^{11}B^{11}C^{11}$ is the reflection along the line* $y = -x$

ii) ***Method 1***

*Let **Q** be the matrix for quarter turn and **X** be the matrix for reflection along x –axis.*

$\underline{For\ Q}$:

$$From\ Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \theta = 90^0 \text{ for positive quarter turn}$$

$$= \begin{pmatrix} \cos 90^0 & -\sin 90^0 \\ \sin 90^0 & \cos 90^0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\underline{For\ X:}$

For the reflection along $x-axis$:

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The matrix of this transformation $M = XQ$ but <u>not</u> $M = QX$. *This is because* **Q** *was performed first on triangle ABC followed by* ***X,*** *i.e.*

$$A^{11}B^{11}C^{11} = XQABC$$

$$\Rightarrow M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \therefore \underline{\underline{M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}}$$

Which is the same as the matrix of reflection along the line $y = -x$

<u>*Method 2*</u>

Let this matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\Rightarrow A^{11}B^{11}C^{11} = MABC$$

$$\begin{matrix} & \begin{matrix} A & B & C \end{matrix} & & \begin{matrix} A^{11} & B^{11} & C^{11} \end{matrix} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \begin{pmatrix} 1 & 1 & 3 \\ 1 & 4 & 4 \end{pmatrix} & = & \begin{pmatrix} -1 & -4 & -4 \\ -1 & -1 & -3 \end{pmatrix} \end{matrix}$$

$$\begin{pmatrix} a+b & a+4b & 3a+4b \\ c+d & c+4d & 3c+4d \end{pmatrix} = \begin{pmatrix} -1 & -4 & -4 \\ -1 & -1 & -3 \end{pmatrix}$$

$$a+b = -1 \text{..........(1)} \qquad c+d = -1 \text{..........(4)}$$

$$\Rightarrow a+4b = -4 \text{........(2)} \qquad c+4d = -1 \text{.........(5)}$$

$$3a+4b = -4 \text{.......(3)} \qquad 3c+4d = -3 \text{.........(6)}$$

$Equation(2) - eqn(1)$

$$a+4b = -4$$
$$^{-}\underline{a+b=-1}$$
$$3b = -3, \quad \Rightarrow b = -1$$
$$a+b = -1 \Rightarrow a = 0$$

$Equation(5) - eqn(4)$

$$c+4d = -1$$
$$^{-}\underline{c+d=-1}$$
$$3d = 0, \quad \Rightarrow d = 0$$
$$c+d = -1 \Rightarrow c = -1$$

$$\therefore \underline{\underline{M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}}$$

General case:

- If **X** and **Q** represent transformations, then **XQ** means perform **Q** first followed by **X**.
- Similarly if **X**, **Q** and **R** are transformations, then **QRX** means **X** is performed first then **R** and finally **Q** in that order.

NB:

Make sure the order mention above is always followed.

- $\mathbf{M}^2$ is the same as **MM**, i.e. **M** followed by **M**.
- $(\mathbf{QM})^{-1} = \mathbf{M}^{-1}\mathbf{Q}^{-1}$
- Remember that: $\mathbf{XQ} \neq \mathbf{QX}$

17.6 Relationship between the area of an object and its image Area scale factor:

This is defined as the ratio of the area of the image to the area of its object, i.e.

$$Area\ scale\ \ factor\ = \frac{Area\ of\ \ image}{Area\ of\ \ object}$$

17.7 Area of an image:

Under any transformation with matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$Area\ of\ \ image = magitude\ of\ \det M \times Area\ of\ \ the\ object$$
$$I.e.$$
$$Area\ of\ \ image = |ad - bc| \times area\ of\ \ the\ object. \quad Where,\ ad - bc = \det M$$

$$\Rightarrow Area\ sacle\ \ factor\ = \frac{\det M \times Area\ of\ \ object}{Area\ of\ \ object} = \det M$$
$$\therefore Area\ scale\ \ factor\ is\ the\ same\ as\ \text{determin}ant\ of\ \ the\ operator.$$

NB:

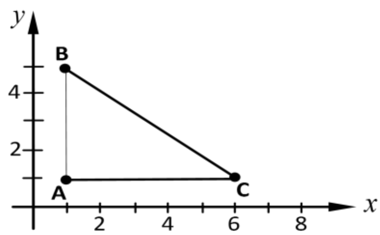
If the $\det M$ is negative, then we have to ignore the negative sign.

Example

Triangle ABC with coordinates A (1, 1), B (1, 5) and C (6, 1) undergoes a transformation represented by matrix $\begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix}$.

Find the area of the image.

Solution



$$Area\ of\ triangle\ ABC = \tfrac{1}{2}bh, \qquad b = 6 - 1 = 5\,units,\ h = 5 - 1 = 4\,units$$
$$= \tfrac{1}{2} \times 5 \times 4 = 10\,sq\ units$$
$$M = \begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix} \quad \det M = 3 \times 2 - 3 \times 1 = 3$$
$$\therefore Area\ of\ triangle\ A^1B^1C^1 = \det M \times Area\ of\ triangle\ ABC$$
$$= 3 \times 10$$
$$= \underline{\underline{30\ sq\ units}}$$

Summary:

1. Under the transformation of translation, reflection, and rotation, the size is always preserved meaning that the object and its image are identical (congruent). The three transformations above are therefore known as ***isometrics.***

2. Non isometric transformations on the other hand are transformation for which the object changes position, the size, and sometimes the shape. Enlargement is an example of this transformation.

17.8 Miscellaneous exercise

1. Triangle ABC has vertices A (-4, 1), B (-1, 1) and C (-3, 4). T is the transformation with matrix $T = \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}$

   a) Find the image of ABC under **T**.

   b) Sketch triangle ABC and its image $A^1 B^1 C^1$ after this transformation.

2. Point P (a, b) has been transformed by the transformation with matrix $\begin{pmatrix} 1 & 0 \\ 4 & 5 \end{pmatrix}$. The image of P (a, b) is P^1 (2, 9).

   Find the value of ***a*** and ***b***.

3. A triangle with coordinates A (2, 3), B (6, 3) and C (4, 6) is given a transformation represented by matrices $M = \begin{pmatrix} 0 & -3 \\ -1 & 2 \end{pmatrix}$ *and* $N = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$ to form $A^1 B^1 C^1$ and $A^{11} B^{11} C^{11}$ respectively.

   a) Find the coordinates of $A^1 B^1 C^1$ and $A^{11} B^{11} C^{11}$.

   b) Find a single matrix that maps ABC onto $A^{11} B^{11} C^{11}$.

   c) Find a single matrix that maps $A^{11} B^{11} C^{11}$ back to ABC.

   d) Find the area of triangle $A^{11} B^{11} C^{11}$.

4. Find the image of triangle A (1, 1) B (5, 1) and C (5, 3) after being reflected in the;

   i. X axis

   ii. Y axis.

5. Under a rotation X = (5, 3) is mapped onto X^1 = (-2, 5) and Y = (4, 6) is mapped onto Y^1 = (-5, 4). Find by a diagram the centre and angle of rotation as accurately as possible.

6. After a rotation, the image of P (3,0) and Q (4, 2) are P^1(-3, 0) and Q^1(-5, 1) respectively. Find the centre and angle of rotation.

7. ABC has vertices A (-5, 2), B (-1, 2) and C (3, 4). The image of triangle ABC under a rotation is the triangle $A^1B^1C^1$ with A^1 (2, 9), B^1 (2, 5) and C^1 (4, 1). Find the centre and angle of rotation.

8. a) Find the image of B (-4, 5) under a rotation about the origin of:

   i. -270^o

   ii. 45^o

   iii. 37.4^o

   b) Find the transformation matrix for a rotation of:

   i. $^+70^0$ about the origin

   ii. 38^o clockwise

9. The points A (-2, 1), B (-2, 4), C (1, 4), and D (1, 1) are vertices of a square ABCD. The images of A, B, C and D under a reflection in the line $x-y=0$ are A^1, B^1, C^1 and D^1 are then mapped onto the points A^{11}, B^{11}, C^{11} and D^{11} respectively by a positive quarter turn about the origin.

   a) Draw square ABCD and its images $A^1B^1C^1D^1$ and $A^{11}B^{11}C^{11}D^{11}$.

   b) State the coordinates of the vertices of:

   i. $A^1B^1C^1D^1$

   ii. $A^{11}B^{11}C^{11}D^{11}$

10. A triangle XYZ has vertices X (1, 0), Y (3, 0) and Z (3, 4). The triangle is given a positive quarter turn about O (0, 0) to be mapped onto triangle $X^1Y^1Z^1$. The image $X^1Y^1Z^1$ is then reflected along the line $x+y=0$ to be mapped onto triangle $X^{11}Y^{11}Z^{11}$.

    a) Plot and draw on a graph triangle X Y Z and its images $X^1Y^1Z^1$ and $X^{11}Y^{11}Z^{11}$ respectively.

    b) Using your graph, state the coordinates of the vertices of triangle $X^1Y^1Z^1$ and $X^{11}Y^{11}Z^{11}$.

11. a) PQRS is a square which has been transformed into image ABCD with vertices A (0, 0), B (6, 0), C (6, 6) and D (0, 6) by an enlargement centre (0, 0) and matrix $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

    i. Find the coordinates of PQRS

    ii. Sketch PQRS and its image on the same diagram.

    b) PQRS is transformed by an enlargement centre (0, 0) and matrix $T = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$. Sketch the image of PQRS on the same diagram as in 11 a) ii) above.

12. Unit square OABC, with O = (0, 0), A = (1, 0), B = (1, 1) and C = (0, 1) is transformed by a positive quarter turn about the origin onto OXYZ.

    a) Find the coordinates of the vertices of OXYZ

    b) OXYZ is enlarged with matrix $\begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}$ centre (0, 0). Find the area of the image of OXYZ.

13. The points P (0, 2), Q (1, 4), and R (2, 2) are vertices of triangle PQR. The images of P, Q, and R under a reflection in the line $x - y = 0$ are P^1, $Q^{1,}$ and P^1 respectively. The points P^1, Q^1 and R^1 are then mapped onto the points P^{11} Q^{11} and R^{11} respectively under an enlargement with scale factor -2 and centre of enlargement O (0, 0).

    a) Write down the matrix for the:

    i. Reflection

    ii. Enlargement.

    b) Determine the coordinates of the points:

    i. P^1Q^1 and R^1

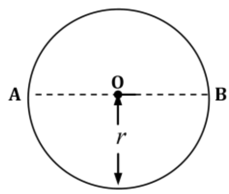
    ii. $P^{11}Q^{11}$ and R^{11}

    c) Find a single matrix of transformation that would map triangle PQR onto P 11Q11R11.

18 THE CIRCLE

18.1 Definition

A circle is a set of points, which are at the same distance from a fixed point.

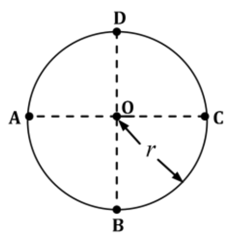


The fixed point **O** is known as the centre of the circle while the constant distance, **r** is known as the radius. The line **AB** from one point of the circle to the other point of the circle through the centre of the circle is known as the diameter and it is twice the radius, i.e.

$$Diameter, d = 2r$$

18.2 Circumference of a circle

This is the total distance round the circle



The distance A to B to C to D and back to A is equal to the circumference of the circle of radius r above. The circumference, C of the circle is given by:

$$C = 2\pi r$$

$$where\ \ r\ is\ radius\ \ of\ \ the\ \ circle$$

$$Or\ :$$

$$C = \pi d$$

$$where\ \ d\ is\ diameter\ \ of\ \ the\ \ circle\ = 2r$$

Example

Calculate the circumference of the circle whose radius is 7cm.

Solution

$$C = 2\pi r, \quad r = 7cm, \pi = \tfrac{22}{7}$$

$$\Rightarrow C = 2 \times \frac{22}{\cancel{7}} \times \cancel{7}$$

$$= \underline{\underline{44cm}}$$

Example

What is the diameter of a circle whose circumference is 300cm?

Solution

$$C = \pi d, \quad \pi = \tfrac{22}{7}$$

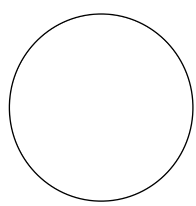
$$\Rightarrow d = \frac{C}{\pi}, \quad C = 300cm,$$

$$= 300 \div \frac{22}{7}$$

$$= \underline{\underline{95.45cm}}$$

18.3 Area of a circle

Consider the circle below of radius r.



The area of a circle is the same as the area of the shaded part and is given by:

$$Area, A = \pi r^2$$

$$: Or$$

$$Area = \frac{\pi d^2}{4} \quad where \;\; r = \tfrac{d}{2}$$

Example

Calculate the area of the circle with diameter 0.5m.

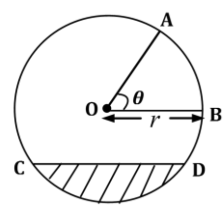
Solution

$$Area\ of\ a\ circle = \frac{\pi d^2}{4}$$
$$= \frac{\frac{22}{7} \pm \times (0.5)^2}{4}$$
$$= \underline{\underline{0.196m^2}}$$

18.4 Chord, arc, and sector

18.4.1 Chord:

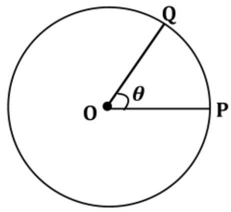
A chord is any straight line joining any two points on the circumference of the circle. Consider a circle of radius r and centre O as shown below.



- The length CD is known as the chord
- The shaded part is known as the minor segment
- The length AB is known as an arc.
- The part OAB is known as the sector.
-

18.4.2 Length of an arc:

Consider a circle of radius r, and entre O and that an arc PQ subtends an



The circumference of the circle subtends an angle of 360^o at the centre of the circle, Hence:

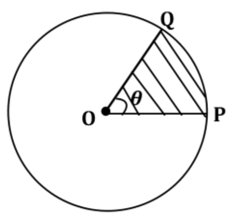
$$\frac{The\ length\ of\ arc\ PQ}{Circumferance\ of\ the\ circle} = \frac{\theta}{360^0}, \quad but\ circumferance\ of\ circle = 2\pi r$$

$$\Rightarrow \quad \frac{Length\ of\ arc\ PQ}{2\pi r} = \frac{\theta}{360^0}$$

$$\boxed{\therefore Length\ of\ an\ arc\ PQ = \frac{\theta}{360^0} \times 2\pi r}$$

18.4.3 Area of the sector:

The area of the sector of a circle can be obtained in a similar way to the length of an arc of a circle.



subtends an angle 360^o at O, thus:

$$\frac{Area\ of\ sector\ OPQ}{Area\ of\ the\ circle} = \frac{\theta}{360^0}, \quad but\ area\ of\ circle = \pi r^2$$

$$\Rightarrow \quad \frac{Area\ of\ sector\ OPQ}{\pi r^2} = \frac{\theta}{360^0}$$

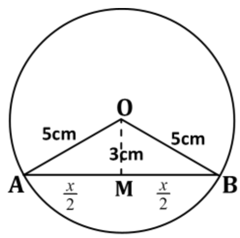
$$\therefore Area\ of\ \sec tor\ OPQ = \frac{\theta}{360^0} \times \pi r^2$$

Example

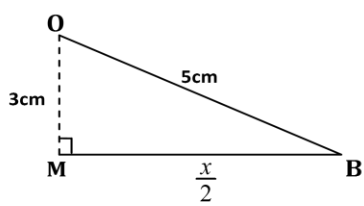
In a circle of radius 5cm, calculate the length of the chord, which is 3cm from the centre.

Solution

Let x be the length of the chord



Extracting triangle OMB



By using Pythagoras theorm

$$OB^2 = OM^2 + MB^2$$

$$5^2 = 3^2 + \left(\tfrac{x}{2}\right)^2$$

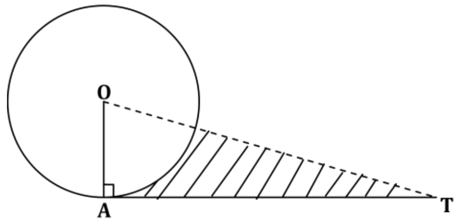
$$\Rightarrow \quad \tfrac{x^2}{4} = 25 - 9 = 16$$

$$\therefore x = \sqrt{16 \times 4} = 8cm$$

So, the length of the chord is $8cm$

Example

TA is a tangent to the circle, centre O, and radius 6cm.



Given that, OT = 12cm. Calculate:

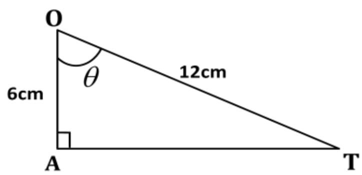
a) Length AT

b) Angle AOT

c) Area of the shaded part

Solution

a) Extracting triangle OAT



By using Pythagoras theorm

$$OA^2 + AT^2 = OT^2$$

$$6^2 + AT^2 = 12^2$$

$$\Rightarrow \quad AT^2 = 144 - 36 = 108$$

$$\therefore AT = \sqrt{108} = \underline{\underline{10.39cm}}$$

b) Let angle AOT be

$$Cos\theta = {}^{6}\!/_{12} = \tfrac{1}{2}$$

$$\therefore \theta = \cos^{-1}(\tfrac{1}{2}) = 60^0$$

$$\therefore \underline{\underline{< AOT = 60^0}}$$

c) $Area\ of\ the\ shaded\ part = Area\ of\ triangle\ AOT - Area\ of\ monor\ sector\ AOB$

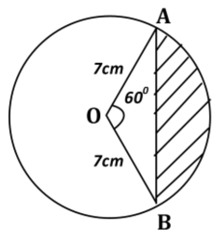
$$Area\ of\ triangle\ AOT \quad = \quad \tfrac{1}{2} \times AT \times AO = \tfrac{1}{2} \times 10.39 \times 6 = 31.18cm^2$$

$$Area\ of\ \mathrm{minor}\ sector\ AOB \quad = \quad \frac{\theta}{360^0} \times \pi r^2 = \frac{60^0}{360^0} \times 3.41 \times 6^2 = 18.85cm^2$$

$$\therefore Area\ of\ the\ shaded\ part \quad = \quad 31.18 - 18.85 = \underline{\underline{12.33cm^2}}$$

Example

The diagram below shows a circle with an arc, which subtends an angle of 60^o at the centre of the circle of radius 7cm.



a) Find the area of the circle.

b) Find the area of the minor sector AOB.

c) Find the area of the major sector AOB.

d) Find the length of the minor arc AB.

e) Find the length of the major arc AB.

f) Calculate the area of the shaded segment.

Solution

a) $Area\ of\ a\ circle = \pi r^2, \quad r = 7cm$

$$= \frac{22}{7} \times 7^2$$
$$= \underline{\underline{154cm^2}}$$

b) $Area\ of\ \text{minor sector}\ AOB\ = \frac{\theta}{360^0} \times \pi r^2, \quad \theta = 60^0$

$$= \frac{60}{360} \times \frac{22}{7} \times 7^2$$
$$= \underline{\underline{25.67cm^2}}$$

c) $Area\ of\ major\ \text{sector}\ AOB\ = \frac{300}{360^0} \times \frac{22}{7} \times 7^2, \quad \text{since}\ \theta = 360^0 - 60^0 = 300^0$

$= \underline{\underline{128.33cm^2}}$

d) $Legth\ of\ the\ arc\ =\ \frac{\theta}{360^0}\times 2\pi r$

$For\ \text{minor}\ arc,\ \ \theta = 60^0$

$\therefore the\ length\ of\ \text{minor}\ arc\ AB = \frac{60}{360}\times 2\times\frac{22}{7}\times 7 = \underline{\underline{7.33cm}}$

e) $For\ major\ arc, \theta = 300^0$

$\therefore the\ length\ of\ major\ arc\ AB = \frac{300}{360}\times 2\times\frac{22}{7}\times 7 = \underline{\underline{36.67cm}}$

f) $Area\ of\ the\ shaded\ segment = Area\ of\ \text{minor sector}\ AOB - Area\ of\ triangle\ OAB$

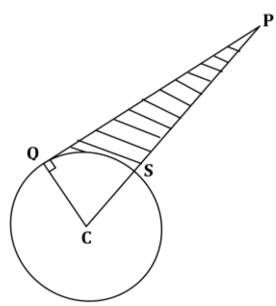
$$= 25.67 \quad - \quad \frac{1}{2}r^2\sin 60^0$$

$$= 25.67 \quad - \quad \frac{1}{2}\times 7^2\times\frac{\sqrt{3}}{2}$$

$$= \underline{\underline{4.45cm^2}}$$

Example

In the figure below, C is a centre of a circle of radius 20m, PQ is a tangent to the circle and angle CPQ = 40^o,

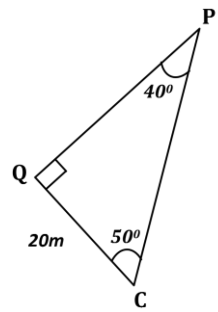


Calculate:

i. the length QP
ii. the perimeter of the shaded region
iii. the area of the shaded region

Solution

i. *Extracting triangle QCP*



$$\frac{CP}{QP} = \tan 40^0$$

$$\therefore QP = \frac{CQ}{\tan 40^0} = \frac{20}{\tan 40^0} = \underline{\underline{23.84m}}$$

ii. $$The\ length\ of\ the\ \text{minor}\ arc\ QS \quad = \quad \frac{\theta}{360^0}\times 2\pi r, \quad \theta = 90^0 - 40^0 = 50^0$$

$$\therefore \overrightarrow{QS} \quad = \quad \frac{50}{360}\times 2\times\frac{22}{7} \quad = \quad 17.46m$$

$$The\ length\ \vec{PS} = \vec{PC} - \vec{CS}, \quad but\ \vec{PC} = \frac{QP}{\cos 40^0}$$

$$\therefore \overrightarrow{PS} \quad = \quad \frac{23.84}{\cos 40^0} - 20 \quad = \quad 11.12m$$

$$\therefore Perimeter of the shaded region = \vec{PQ} + \vec{QS} + \vec{PS} = 23.84 + 17.46 + 11.2 = \underline{\underline{52.4m}}$$

iii. *Area of the shaded part = Area of triangle QCP – Area of minor sector QSC*

$$But\ area\ of\ \text{minor sec}tor\ QSC \quad = \quad \frac{\theta}{360^0}\times\pi r^2, \quad \theta = 50^0, and r = 20m$$

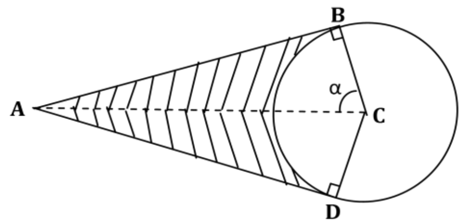
$$= \quad \frac{50}{360}\times\frac{22}{7}\times 20^2 \quad = \quad 174.6om$$

$$Area\ of\ triangle\ PQC \quad = \quad \frac{1}{2}bh \quad = \quad \frac{1}{2}\times 20\times 23.84 \quad = 2\ \ 38.4m^2$$

$$\therefore Area\ of\ shaded\ part = 238.4 - 174.6 = \underline{\underline{63.8m}}$$

Example

In the diagram below, PQ and PR are tangents to a circle of radius 15cm and centre C.



If AC = 20cm, calculate:

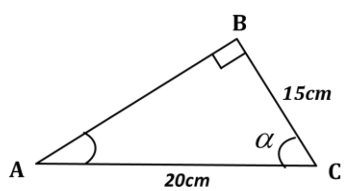
i.

ii. the length AB

iii. the area of the shaded part

Solution

i. Extracting triangle ABC



Since AB is the tangent to the circle and BC is the radius,

$< ABC = 90^0$

$$\Rightarrow \frac{BC}{AC} = \cos\alpha \quad \Rightarrow \frac{15}{20} = \cos\alpha$$

$$\therefore \alpha = \cos^{-1}\left(\frac{15}{20}\right) = \underline{\underline{41.4^0}}$$

ii. $\frac{AB}{BC} = \tan\alpha \quad \Rightarrow AB = BC\tan 41.4^0 \quad \therefore AB = 15 \times 0.88 = \underline{\underline{13.2cm}}$

iii. Area of shaded part = Area of ABCD – Area of minor sector BCD

$$Area\ of\ triangle\ ABC = \frac{1}{2}bh = \frac{1}{2}\times 150\times 13.2 = 99cm^2$$

$$\therefore Total\ area\ of\ quadrilateral = 2\times 99 = 198cm^2$$

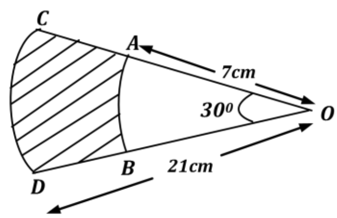
$$And\ area\ of\ \text{minor sector}\ BDC = \frac{\theta}{360}\times \pi r^2,\quad but\ \theta = 2\alpha = 2\times 41.4^0 = 82.8^0$$

$$= \frac{82.8}{360}\times\frac{22}{7}\times 15^2 = 162.6cm^2$$

$$\therefore Area\ of\ shaded\ part = 198.0 - 162.6 = \underline{\underline{35.4cm^2}}$$

Example

In the figure below, O is a centre of two arcs AB and CD with a central angle of 30^o.



Calculate:

a) the perimeter of the shaded part and

b) the area of the shaded part

Solution

a) $$The\ length\ of\ moinor\ arc\ AB = 2\pi r\times\frac{\theta}{360},\quad \theta = 30^0, r = 7cm$$

$$= 2\times\frac{22}{7}\times 7\times\frac{30}{360},\quad = 3.67cm$$

$$The\ length\ of\ moinor\ arc\ CD = 2\pi r\times\frac{30}{360},\quad , r = 21cm$$

$$= 2\times\frac{22}{7}\times 21\times\frac{30}{360},\quad = 11cm$$

$$\therefore Perimeter\ of\ shaded\ region = length\ of\ arc\ AB + length\ of\ arc\ CD + \vec{AC} + \vec{BD}$$

$$= 3.67 + 11 + (21-7) + (21-7)$$

$$= \underline{\underline{42.7cm}}$$

b) $\quad Area\ of\ shaded\ region = area\ of\ \text{sector}\ CDO - area\ of\ \text{sector}\ ABO$

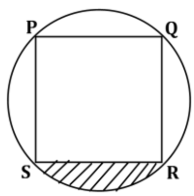
$$= \pi r_1^2 \times \frac{30}{360} - \pi r_2^2 \times \frac{30}{360}, \quad r_1 = 21cm,\ r_2 = 7cm$$

$$= \frac{22}{7} \times 21^2 \times \frac{30}{360} - \frac{22}{7} \times 7^2 \times \frac{30}{360}$$

$$= \underline{\underline{102.6cm^2}}$$

Example

The figure below shows a square PQRS inscribed in a circle of radius 21cm.



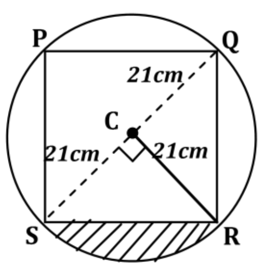
Calculate:

a) the length of the side of the square

b) the area of the shaded region

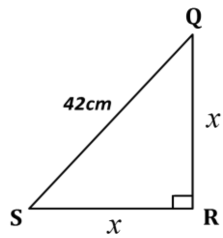
Solution

a)



$$SQ = 2 \times 21 = 42cm$$

Let x be the length of the side of the square. Extracting triangle SQR:



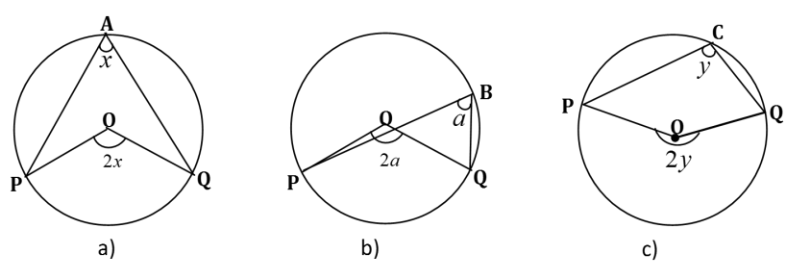
Using Pythagoras theorem:

$$42^2 = x^2 + x^2$$
$$1764 = 2x^2$$
$$\therefore\ x = \sqrt{\frac{1764}{2}} = \underline{\underline{29.7cm}}$$

b) $$\textit{Area of shaded part} = \textit{area of minor sector } CSR - \textit{area of triangle } CSR$$
$$= \pi r^2 \times \frac{90}{360} - \frac{1}{2} \times \overrightarrow{CR} \times \overrightarrow{SC}$$
$$= \frac{22}{7} \times 21^2 \times \frac{90}{360} - \frac{1}{2} \times 21 \times 21$$
$$= \underline{\underline{126cm^2}}$$

18.5 Angle properties of a circle

1. The angle subtended by the arc of a circle at the centre is twice the angle it subtends at the circumference. The following diagrams illustrate this property.

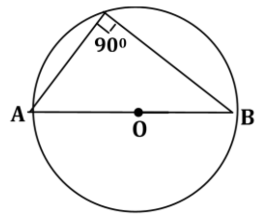


$< PAQ = \frac{1}{2} < POQ$ $\quad$ $< PBQ = \frac{1}{2} < POQ$ $\quad$ $< PCQ = \frac{1}{2} obtuse < POQ$

In cases a) and b), minor arc **PQ** subtends angles x and a respectively on the circumference. Therefore the angles subtended by the respective arcs at the centre are $2x$ and $2a$.

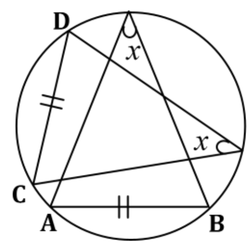
In case c), the major arc **PQ** subtends angle y on the circumference and hence the corresponding angle subtended at the centre by the same arc **PQ** is $2y$.

2. The angle subtended by the diameter at the circumference of the circle is 90^o.



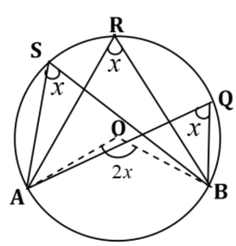
AB is the diameter

3. Equal chords subtend equal angles at the circumference.

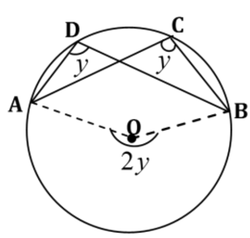


i.e. CD = AB

4. An arc of a circle subtends equal angles at the circumference.



a)



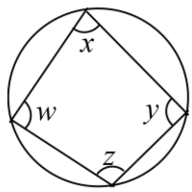
b)

In figure a) above, the minor arc AB subtends $<AQB$, $<ARB$, and $<ASB$ at the circumference and $<AOB$ at the centre of the circle. Therefore $<AQB = <ARB = <ASB = x$ and $<AOB = 2x$

In figure b) above, the major arc AB subtends $<ACB$ and $<ADB$ at the circumference and $<AOB$ at the centre of the circle. Therefore, $<ACB = <ADB = y$ and obtuse $<AOB = 2y$

18.6 Cyclic Quadrilaterals

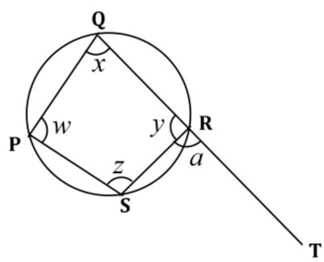
This is a quadrilateral whose all of its four vertices lie on the circumference of a cycle, figure below.



Angle x is opposite $<z$ and $<y$ is opposite $<w$.

18.7 Angle properties of a cyclic quadrilateral

Consider cyclic quadrilateral shown below.



The cyclic quadrilateral has the following angle properties.

1. Opposite angles are supplementary i.e. add up to 180^0

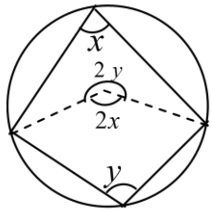
$$w + y = 180^0$$

$$Also : x + z = 180^0$$

$$\Rightarrow \quad w + y = x + z = 180^0$$

Proof:

Consider two angles x and y subtended at the circumference by the minor arc AB and major arc AB respectively.



$\therefore\ 2y + 2x = 360^0$

$\Rightarrow\ y + x = 180^0$

2. The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

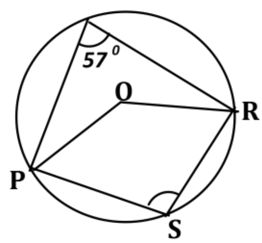
$$\Rightarrow\ w = a$$

Furthermore, sum of exterior angles add up to 180^o

$$\Rightarrow\ y + a = 180^0$$

Example

In the diagram below, O is the centre of the circle.



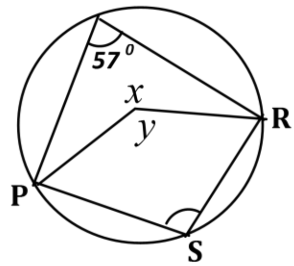
Find the following:

a) <POR

b) The reflect angle POR

c) <PSR

Solution

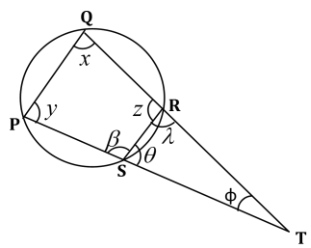


a) $< POR = x = 2\times 57 = 114^0$

b) *The reflect angle* $POR = y = 360^0 - x = 360^0 - 114^0 = 246^0$

c) $< PSR = \frac{1}{2}\times reflect < POR = \frac{1}{2}\times 246 = 123^0$

Example



o o.

Find:

i. $< \theta$

ii. $< z$

iii. $< y$

iv. $< \beta$

v. $< x$

Solution

i. *Considering triangle SRT*

$\theta + \lambda + \phi = 180^0, but \phi = 20^0, \lambda = 40^0$

$\Rightarrow \theta + 40^0 + 20^0 = 180^0$

$\therefore \theta = 180^0 - 60^0 = \underline{\underline{120^0}}$

ii. *From sum of interior angle and exterior angle = 180^o.*

$\Rightarrow \; z + \lambda = 180^0$

$\therefore \; z = 180^0 - 40^0 = \underline{\underline{140^0}}$

iii. *From the property of the cyclic quadrilateral i.e. sum of opposite angles are supplementary.*

$\Rightarrow \; y + z = 180^0$

$\therefore \; y = 180^0 - 140^0 = \underline{\underline{40^0}}$

iv. *For* $< \beta$

$\beta + \theta = 180^0$

$\therefore \; \beta = 180^0 - 120^0 = \underline{\underline{160^0}}$

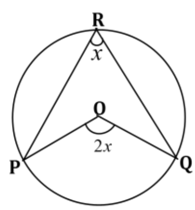
v. *For* $< x$

$x + \beta = 180^0$

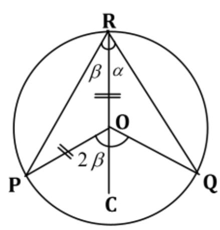
$\therefore \; x = 180^0 - 160^0 = \underline{\underline{120^0}}$

18.8 Angle at the centre of a circle

Consider the angle x being subtended by an arc PQ on the circumference of the circle. The angle therefore subtended at the centre of the circle by an arc PQ is $2x$ as shown. See figure below.

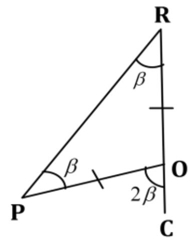


If **RO** is joined and extended to C as shown below



$\beta + \alpha = x$

OR is the radius of the circle and similarly **OP**. therefore, **OR** = **PO** implying that triangle PRO is an isosceles triangle. Now consider triangle PRO.



$$\therefore < OPR = < ORP = \beta$$

$$Also \quad < POC = < OPR + < ORP = \beta + \beta = 2\beta$$

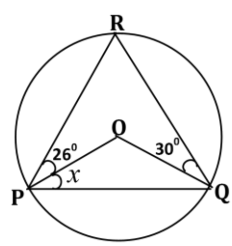
$$In\ the\ same\ way, \ < QOC = 2\alpha$$

$$< POC + < QOC = 2\beta + 2\alpha = 2(\beta + \alpha) = 2x$$

$$I.e. < POQ = 2 < PRQ$$

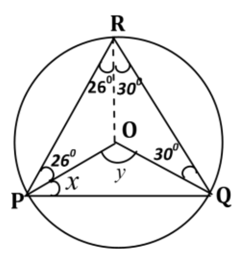
Example

In the figure below, O is the centre of the circle. Angle ABO = 26^o and angle OCA = 30^o.



Calculate the size of angle marked x.

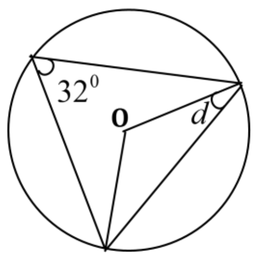
Solution



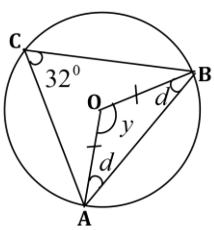
$$< BAC = 26^0 + 30^0 = 56^0$$
$$\therefore \text{ Angle } y = 2 \times 56^0 = 112^0$$
$$< OAC =< OCA = x$$
$$\Rightarrow x + x + y = 180^0$$
$$\Rightarrow 2x = 180^0 - 112^0$$
$$\therefore x = \frac{68^0}{2} = \underline{\underline{34^0}}$$

Example

Find the angle marked **d** of the circle below, given that O is the centre of the circle.



Solution

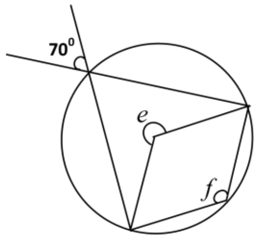


Since AO = OB = radius of the circle, <OAB = <OBA = d

$$\Rightarrow d + d + y = 180^0, \text{ but } y = 2 \times 32^0 = 64^0 \quad \therefore d = {}^{116}\!/_{2} = \underline{\underline{58^0}}$$
$$\Rightarrow 2d = 180^0 - 64^0 = 116^0$$

Example

Consider the diagram below; O is the centre of the circle.

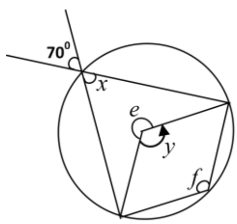


Calculate the size of angles marked:

i. e

ii. f

Solution



i. *Angle x is vertically opposite 70^o, therefore* $x = 70^0$

$Angle\ y = 2x = 2 \times 70^0 = 140^0$

$\therefore e + y = 360^0$

$e = 360^0 - 140^0 = \underline{\underline{220^0}}$

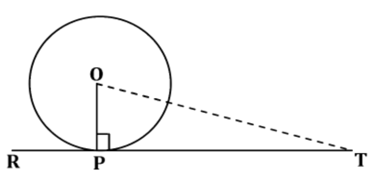
ii. *Also angle* $e = 2f$

$\Rightarrow f = \frac{e}{2} = \frac{220^0}{2} = \underline{\underline{110^0}}$

18.9 Tangent to the circle

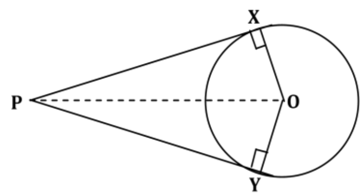
Definition:

A tangent is a line, which just touches the circle at only one point and makes an angle of 90^0 with the radius of the circle.



RT is a tangent to the circle at point P.

For any given point, we can draw two tangents to the circle. The diagram below shows two tangents PX and PY drawn from P to the circle centre O.

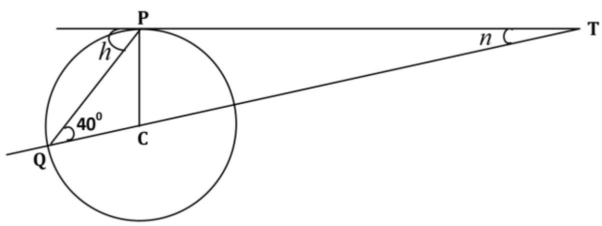


The lengths of tangents to the circle from the same external point are equal.

$$\therefore \overrightarrow{PX} = \overrightarrow{PY}$$

Example

In the diagram below TP is a tangent to the circle with centre C and <PQC = 40^o.

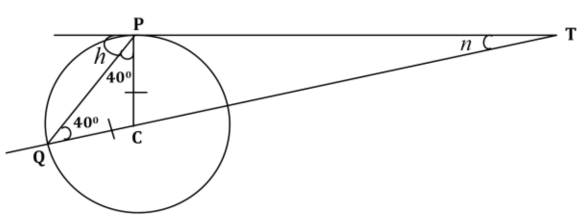


Find ***h*** and ***n***.

Solution

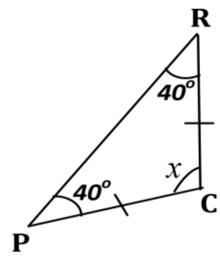
Since QC = CP = radius, this implies that, <Q = <P = 40^o

Since P is a tangent to the circle, <CPT = 90^o



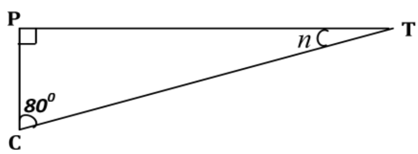
$$\Rightarrow\ h + 40^0 = 90^0$$
$$\therefore\ h = 90^0 - 40^0 = \underline{\underline{50^0}}$$

Extracting triangle PQC:



$$x + 40^0 + 40^0 = 180^0$$
$$\therefore x = 180^0 - 80^0 = 100^0$$
$$\therefore < PCT = 180^0 - 100^0 = 80^0 \quad (\textit{angle on a straight line})$$

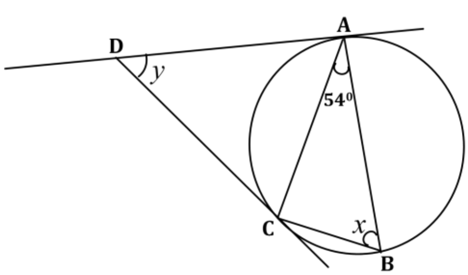
Considering triangle PCT:



$$n + 80^0 + 90^0 = 180^0$$
$$\therefore\ n = 180^0 - 170^0 = \underline{\underline{10^0}}$$

Example

In the diagram below, AB is the diameter of the circle and DA and DC are tangents to the circle at A and C respectively.



Given that angle CAB = 54°, find the values of x *and* y.

Solution

Since AB is a diameter, the angle ACB is right angle.

$$So \quad x + 54^0 + 90^0 = 180^0$$
$$\therefore \; x = 180^0 - 146^0 = \underline{\underline{36^0}}$$

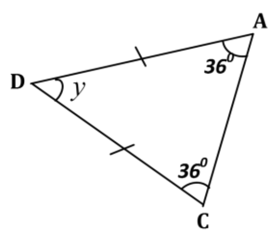
Since DA is a tangent and AB the diameter, angle DAB is right angle.

$$\Rightarrow \quad < DAC + 54^0 = 90^0$$
$$\therefore \; < DAC = 90^0 - 54^0 = \underline{\underline{36^0}}$$

Since DA and DC are tangents from the same point to the circle, the <DAC and <DCA are equal.

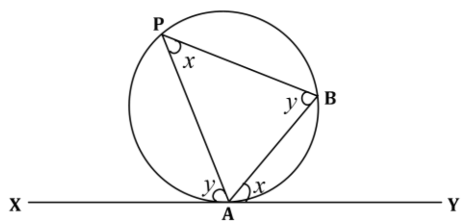
$$\Rightarrow \quad < DAC = < DCA = 36^0$$

Considering triangle ADC



$$So \quad y + 36^0 + 36^0 = 180^0$$
$$\therefore \; y = 180^0 - 72^0 = \underline{\underline{108^0}}$$

18.10 Alternate – segment theorem



If XY is a tangent to the circle at A, the angle between the tangent and the chord is equal to the angle the chord subtends in the alternate segment i.e.

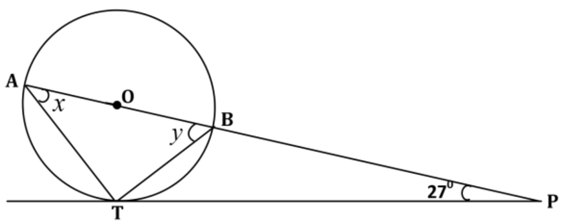
$$< BAY = < BPA$$

$$Also:$$

$$< PAX = < PBA$$

Example

In the diagram below, O is the centre of the circle and PT is a tangent to the circle at T. the angle TPB = 27^o.



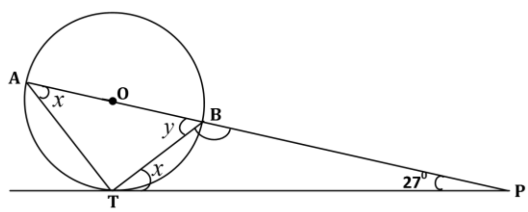
Find angles marked:

i. x

ii. y

Solution

Since AB is the diameter of the circle, <ATB = 90^o



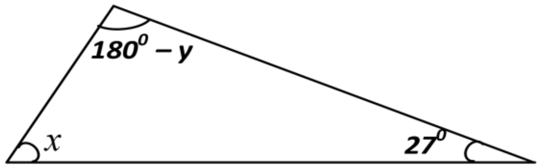
Considering triangle ABT:

$$x + y + 9^0 = 180^0$$

$$\therefore x + y = 90^0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

By the alternate segment theorem, $< PTB = < TAB = x.$

Considering triangle BTO:



$$x + 180^0 - y + 27^0 = 180^0$$

$$\therefore x - y = -27^0 \text{..}(2)$$

Solving (1) and (2) simultaneously:

$Eqn(2) + eqn(1)$

$$x + y = 90^0$$

$$^{-}\ \underline{x - y = -27^0}$$

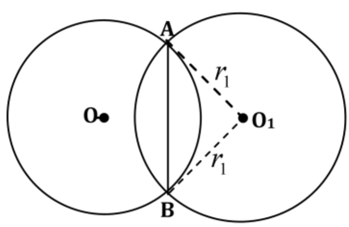
$$2x = 63^0 \ \therefore x = {}^{63}\!/_{2} \quad \underline{\underline{= 31.5^0}}$$

$$And\ \ y = 90^0 - x \Rightarrow y = 90^0 - 31.5^0 = \underline{\underline{58.5^0}}$$

18.11 Intersection of circles

When two circles intersect, they share a chord known as common chord. If we know at least one of the angles subtended at the centre of one of the circles by the chord and the radius of the same circle, we can find the length o f the chord.

Consider two circles centre O and O_1 intersecting at A and B.



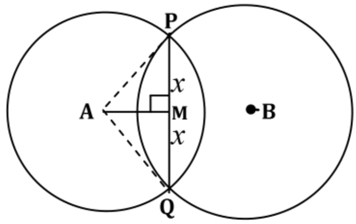
The length AB is therefore a common chord to the two circles. The following examples will illustrate how to calculate the length of the common chord.

Example

Two circles, centre A and B intersect at P and Q. circle center AA has a radius 6.5cm, and the angle subtended by PQ at A is 100^o. Calculate the length of PQ.

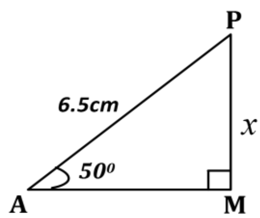
Solution

Let $PM = x = MQ$



$< PAQ = 100^0 \Rightarrow < PAM = \frac{100}{2} = 50^0$

Considering triangle APM:



$$\sin 50^0 = \frac{PM}{AP} = \frac{x}{6.5}$$

$$\Rightarrow \; x = 6.5 \times \sin 50^0 = 4.98cm$$

$$\therefore PQ = 2 \times 4.98 = \underline{\underline{9.96cm}}$$

Example

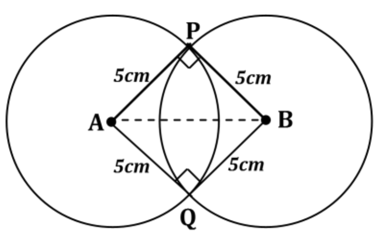
Two equal circles of radius 5cm intersect at right angles.

i. Find the distance between the two centers of the circles.

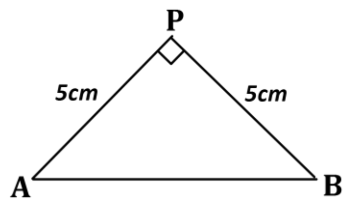
ii. Calculate the area of the common region of the circles.

Solution

i.



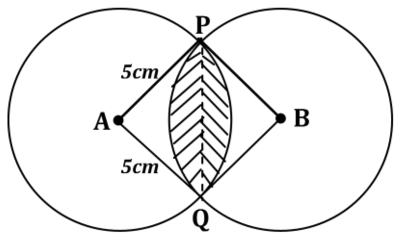
Taking triangle ABP



$$AB^2 = 5^2 + 5^2$$

$$\therefore AB = \sqrt{25+25} \quad = 5\sqrt{2} \quad = \underline{\underline{7.07cm}}$$

ii. *Area of the common region of the circle is the shaded part.*



$$Area\ of\ triangle\ APQ = \tfrac{1}{2} \times 5 \times 5 = 12.5cm$$

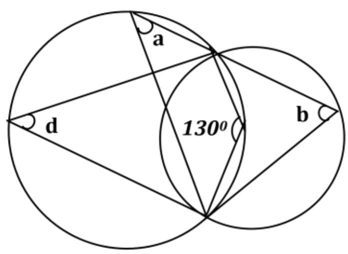
$$Area of\ \sec tor\ APQ = \tfrac{1}{2} \times 3.14 \times 5^2 = 19.625cm^2$$

$$Area\ of\ half\ shaded\ region = 19.625 - 12.5 = 7.125cm^2$$

$$\therefore Area\ of\ common\ region\ of\ the\ circle = 2 \times 7.125 = \underline{\underline{14.25cm^2}}$$

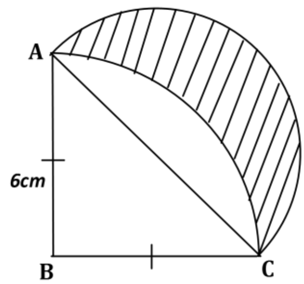
18.12 Miscellaneous Exercise

1. In the diagram below, O is the centre of the circle ABC. Angle AOC = 140°.



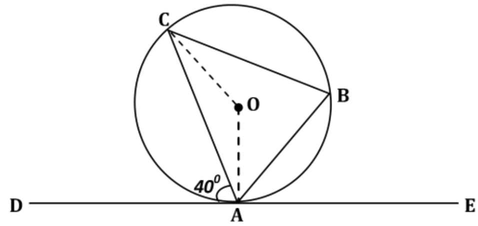
Write down the values of **a**, ***b***, and ***d***.

2. In the diagram below, ABC is an isosceles right angled triangle.



The shaded area is bounded by two circular arcs. The outer arc is a semi circle with AC as diameter and the inner arc is a quarter of a circle with centre B. Find the area of the shaded region.

3. In the diagram below, DAE is a tangent to the circle centre O at A. angle CAD = 40^o.

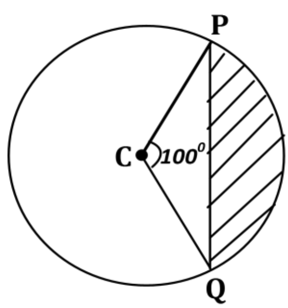


Find:

i. angle OCA

ii. angle ABC.

4. In the figure below, C is the centre of the circle of radius 21m.

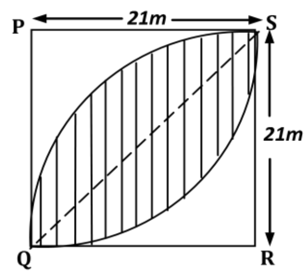


Calculate:

i. length PQ

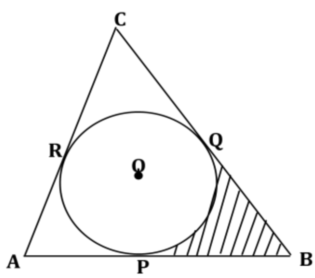
ii. the perimeter and area of the shaded part

5. In the figure below, PQRS is a square of side 21m, PQS and RQS are quadrants.



$\frac{22}{7}$, calculate the area of the shaded part.

6. In the diagram below, ABC is an isosceles triangle in which $\overrightarrow{AC} = \overrightarrow{AB} = 8cm \text{ and } \overrightarrow{BC} = 10cm.$ The circle PQR with centre O touches the sides of the triangle at points, P Q and R.



Given that, the points C, O and P are in the same straight line such that $\overrightarrow{PO} = 3\overrightarrow{OC}$.

Calculate:

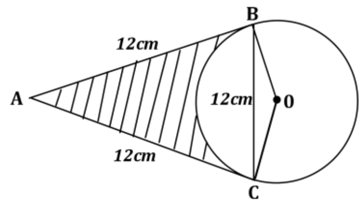
i. the radius of the circle

ii. the area of triangle ABC

iii. the area of the circle

iv. the area of the shaded portion

7. In the figure below, AB and AC are tangents to the circle at points B and C respectively. O is the centre of the circle. Given that $\overrightarrow{AB} = \overrightarrow{BC} = 12cm.$



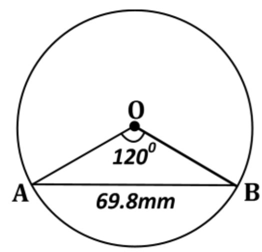
Determine:

i. the obtuse angle BOD

ii. the radius of the circle

iii. the area of minor sector BOC and hence the area of the shaded region

8. In the figure below, AB is a chord of the circle whose centre is O. angle AOB is 120^o and AB = 69.8mm of the circle.

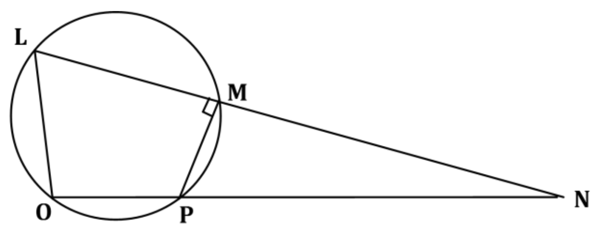


Find the radius of the circle.

(Give your answer correct to 3 significant figures)

9. In the figure below

$\overrightarrow{OL} = 4.5cm, \overrightarrow{PM} = 3cm, \overrightarrow{NM} = 4cm \text{ and } \overrightarrow{LN} = 7.5cm.$



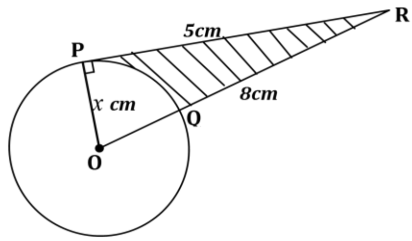
Find:

i. lengths ON and OP

ii. the radius of the circle

iii. area of OLMP

10. In the figure below, O is the centre of the circle. PR is the tangent and OR intersects the circle at Q.



Given that $\overrightarrow{RQ} = 8cm$, $\overrightarrow{PR} = 5cm$, and $\overrightarrow{OP} = x\,cm$.

a) i. Express the length OR in terms of x.

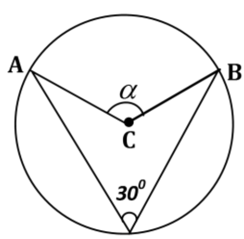
ii. Find x.

b) Calculate:

i. the area of the shaded region

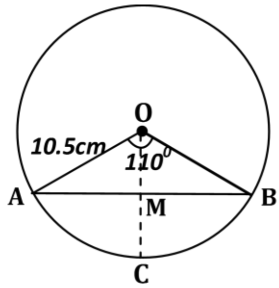
ii. Angle subtended by arc PQ at the centre.

11. In the figure below, C is the centre of the circle.



Calculate the length of the chord AB and the angle marked α

12. In the figure below, OACB is a sector of a circle centre O and radius 10.5cm. Angle AOB = 110^{o}.



Calculate:

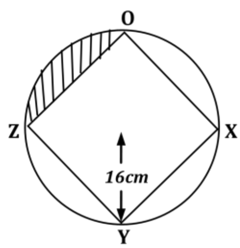
a) the length of CM

b) the length of arc BC

c) area of the minor segment cut off by the chord.

d) the perimeter of the minor segment ACB

13. In the diagram below OXYZ is a square drawn inside a circle of radius 16cm as shown in the diagram.



Calculate the perimeter and area of the shaded part.

19 AREAS AND VOLUMES OF SOLIDS

The solids under consideration include the following:

- ❖ Prisms
- ❖ Pyramid
- ❖ Cone
- ❖ Sphere
- ❖ Pipe

19.1 Surface Area of Solids

Definition:

Surface area of a solid is the sum of the areas of all the surfaces of the solid.

19.2 Surface area of a prism:

A prism is a solid, which has uniform cross-section. This includes:

- ∗ Rectangular prism i.e. cubes and cuboids.
- ∗ Triangular prism
- ∗ Circular prisms e.g. cylinders, etc.

The surface area of a prism is found as follows:

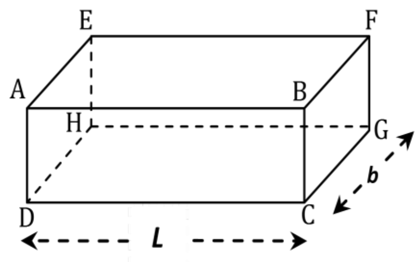
i) Find the area of cross section and multiply it by 2.

ii) Find the area of each rectangular side face and add up these areas. For the case of a cylinder, find the area of the curved surface.

iii) Add up the results to get the surface area of the prism.

19.3 Surface area of a cuboid

A cuboid is a solid with six faces. Pairs of opposite faces are identical and equal in size.



Faces ABCD and EFGH, AEHD and BFGC, AEFB and DHGC are pairs of identical faces.

- $Area\ of\ \ face\ ABCD = area\ of\ \ face\ EFGH = lh$
  $\therefore Area\ of\ \ faces\ ABCD\ and\ EFGH = lh + lh = 2lh$

- $Area\ of\ \ face\ AEHD = area\ of\ \ face\ BFGC = bh$
  $\therefore Area\ of\ \ faces\ AEHD\ and\ BFGC = bh + bh = 2bh$

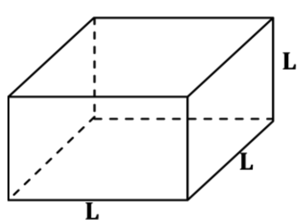
- $Area\ of\ \ face\ AEFB = area\ of\ \ face\ DHGC = lb$
  $\therefore Area\ of\ \ faces\ AEFB\ and\ DHGC = lb + lb = 2lb$

$$\boxed{\begin{aligned}\therefore Total\ \ surface\ \ area\ \ of\ \ the\ \ cuboid\ &= 2lh + 2bh + 2lb\\ &= 2(lh + bh + lb)\end{aligned}}$$

A cuboid is also referred to as rectangular block or simply a box.

19.4 Surface area of a cube

A cube is a solid with six identical square faces.



To find the surface area of the cube, we find the area of one face and multiply it by 6, i.e.

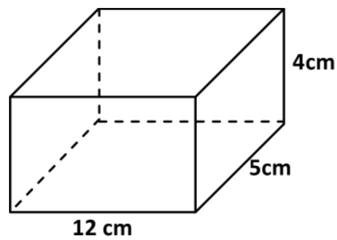
$Total\ surface\ area = 6 \times l \times l$

$$\boxed{\begin{array}{l} Surface\ area\ of\ a\ cube = 6l^2 \\ where\quad l\ is\ the\ lengh\quad of\ side \end{array}}$$

Example

Calculate the surface area of a cuboid measuring 12 cm × 5cm × 4cm.

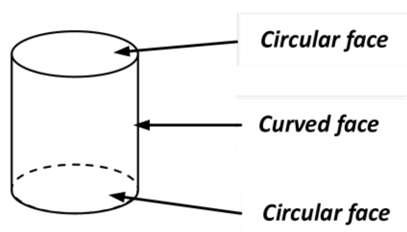
Solution



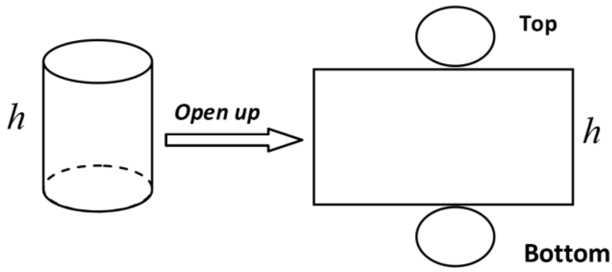
$$\begin{aligned} Surface\ area\ of\ the\ cuboid &= 2(lh + bh + lb) \\ &= 2(12 \times 4 + 5 \times 4 + 12 \times 5) \\ &= \underline{\underline{256cm^2}} \end{aligned}$$

19.5 Surface area of a cylinder

A cylinder has three surfaces: Two are circular and one is curved.



The figure below shows the shape obtained when a hollow cylinder, radius r, height h is opened up and laid out flat.



The curve surface becomes a rectangle measuring $2\pi r$ *by h unts*.

$$There\ for: Area\ of\ top = \pi r^2$$
$$Area\ of\ bottom = \pi r^2$$
$$Area\ of\ curved\ surface\ (rectangle) = 2\pi rh$$

Thus:
$Total\ surface\ area\ of\ a\ closed\ cylinder = \pi r^2 + \pi r^2 + 2\pi rh = 2\pi r(r+h)$

Note:

1. If the cylinder is hollow and has one open end, then there are only two surfaces, i.e. the curved surface and the bottom surface. In this case, $Total\ surface\ area\ of\ the\ cylinder = 2\pi rh + \pi r^2$

2. However, if the cylinder is open ended, then there is only one surface. In this case, $Surface\ area\ of\ the\ cylinder = 2\pi rh$

Example

A closed cylindrical container has a diameter of 3.2cm and height 4.9cm. Find the area of the material used to make the cylinder. Express your answer to 4sf. (Take π *as* 3.142).

Solution

$$Surface\ area\ a\ closed\ cylinder = 2\pi rh + 2\pi r^2, \quad r = 3.2/2 = 1.6cm$$
$$= 2 \times 3.13 \times 1.6(1.6 + 4.9)$$
$$= \underline{\underline{65.35cm^2\,(4.s.f)}}$$

Example

A very thin sheet of metal is used to make a cylinder of radius 5cm and height 14cm. Using $\pi = 3.142$, find the total area of the sheet that is needed to make:

a) A closed cylinder

b) A cylinder that is open on one end.

Solution

a) $r = 5cm, h = 14cm, \pi = 3.142$

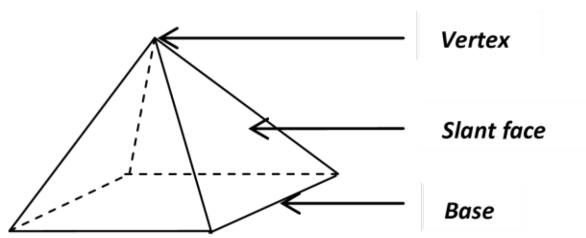
$$\text{For a closed cylinder; surface area} = 2\pi r(r+h)$$
$$= 2 \times 3.142 \times 5(5+14)$$
$$= \underline{\underline{596.98cm^2}}$$

b) For a cylinder open on one end:

$$\text{Surface area} = 2\pi rh + \pi r^2$$
$$= 2 \times 3.142 \times 5 \times 14 + 3.142 \times 5^2$$
$$= \underline{\underline{518.43cm^2}}$$

19.6 Surface area of a pyramid

Below is the structure of the pyramid.

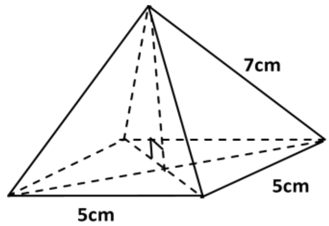


The surface area of a pyramid is obtained as the sum of the areas of the slant faces and the base. Each slanting face is an isosceles triangle.

The following examples illustrate how to obtain the surface areas of a pyramid.

Example

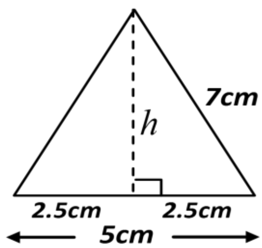
Find the surface area of the pyramid shown below.



Solution

$Area\ of\ the\ base = 5\times 5 = 25cm^2$

Each slanting face is an isosceles triangle of height h



From Pythagoras theorem:

$$5^2 = h^2 + 2.5^2$$
$$\therefore h^2 = 49 - 6.25 = 42.75$$
$$\Rightarrow h = \sqrt{42.75} = 6.538cm$$

$So\ now\ area\ of\ each\ slanting\ face = \frac{1}{2}bh = \frac{1}{2}\times 5\times 6.538 = 16.345cm^2$

$But\ there\ are\ four\ slanting\ faces.$

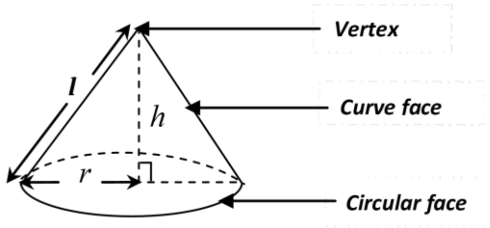
$\therefore Total\ area\ of\ the\ slantig\ faces = 4\times 16.345 = 65.38cm^2$

$\Rightarrow Total\ surface\ area\ of\ the\ pyramid = 25 + 65.38$

$= \underline{\underline{90.4cm^2}}$

19.7 Surface area of a cone

A closed cone has two surfaces; the curved surface and a circular face



If **r** is the radius of the circular face and **l** is the length of the slant edge, the:

$$Area\ of\ a\ curved\ surface = \pi rl$$

$$Area\ of\ circular\ face = \pi r^2$$

$$\therefore Total\ surface\ area\ of\ a\ closed\ cone = \pi r^2 + \pi rl$$

$$\boxed{Surface\ area\ of\ a\ closed\ cone = \pi r(r + l)}$$

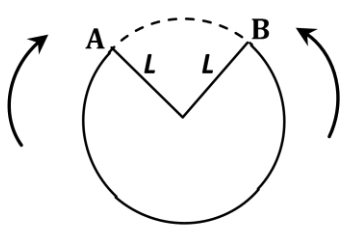
The height **h** of the cone can be obtained by applying Pythagoras theorem. Considering the figure above:

$$l^2 = h^2 + r^2 \Rightarrow h^2 = l^2 - r^2$$

$$\therefore h = \sqrt{l^2 - r^2}$$

19.8 Formation of a cone

A cone can be formed from any section of a circle. Consider a circle of radius *L* shown below.



If a section AB is cut out of the circle and folded in the direction of the arrow, then a cone whose circumference of the base equal to the length of arc AB is formed. Its slanting edge is equal to the radius of the circle.

Note:

If the cone is open, then it has only one surface, which is the curved surface. In this case, its area is simply given by:

$$Surface\ area\ of\ a\ open\ cone = \pi rl$$

Example

A section of a circle of radius 10cm having an angle of 100^o is bent to form a cone.

a) Find the length of the arc of the section

b) Determine the surface area of the cone.

Solution

a) $Length\ of\ the\ arc\ subtending\ an\ angle\ \theta\ at\ the\ centre = 2\pi l \times \frac{\theta}{360}$

$But\ \theta = 100^0,\ l = radius = 10cm$

$\Rightarrow Length\ of\ the\ arc = 2 \times 3.14 \times \frac{100}{360} = \underline{\underline{17.45cm}}$

b) $Surface\ area\ of\ a\ cone = \pi r(l + r)$

$where\ l - length\ of\ the\ slanting\ edge\ and\ r - radius\ of\ the\ base$

$Slanting\ edge\ of\ the\ cone = radius\ of\ the\ circle\ from\ which\ it\ is\ formed.$

$Length\ of\ the\ \mathrm{sector} = circumferance\ of\ the\ base\ of\ the\ cone$

$\Rightarrow 2\pi r = 17.45$

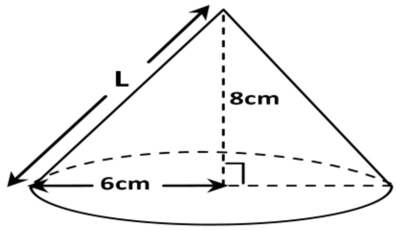
$\therefore r = \frac{17.45}{2 \times 3.14} = 2.78cm \quad and\ l = 10cm$

$\therefore Surface\ area\ of\ the\ cone = \pi r(l + r) = 3.14 \times 2.78(2.78 + 10) = \underline{\underline{111.5cm^2}}$

Example

A cone of base radius 6cm and height 8cm is slit and laid out flat into a section of circle. What angle does the section subtend at the centre?

Solution



By Pythagoras theorm,

$L = \sqrt{6^2 + 8^2} = \sqrt{100} = 10cm$

The slanting edge L of the cone = radius of the sector of the circle

$\therefore$ *radius of the sector* $= 10cm$

The circumferance of the base of the cone $= 2\pi r$ *and* $r = 6cm$

$$= 2 \times 3.14 \times 6$$

$$= 37.68cm$$

Let θ *be the angle of the sector, then the circumfereance of the sector is given by*

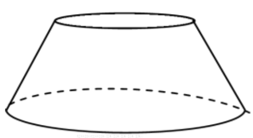
$$2\pi L \times \frac{\theta}{360}$$

$$\Rightarrow 37.68 = 2\pi L \times \frac{\theta}{360}, \quad L = 10cm$$

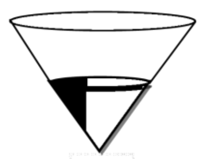
$$\therefore \theta = \frac{37.68 \times 360}{2 \times 3.14 \times 10} = \underline{\underline{216^0}}$$

19.9 The Frustum

A frustum is obtained by chopping part of a cone. It is a figure in the shape of a bucket or a lampshade as depicted below.



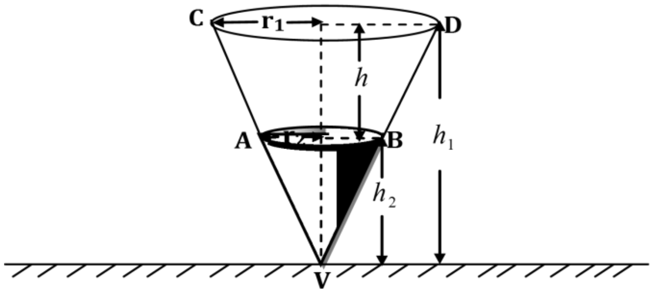
a)



b)

The shaded part of figure b) shows portion of the cone, which has been cut off.

To find the area of the frustum, we apply properties of enlargement to the cone by considering VAB as VCD.



We could also consider VCD as the image of VAB under enlargement. The linear scale factor, which maps VAB onto VCD, is:

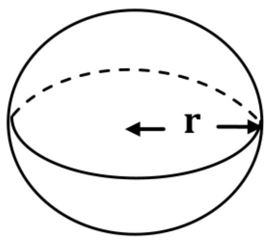
$$\frac{h_1}{h_2} = \frac{r_1}{r_2}, \quad but\ h_1 = h_2 + h \quad \Rightarrow h_2 = h_1 - h$$

$$\therefore \frac{h_1}{h_1 - h} = \frac{r_1}{r_2} \quad \Rightarrow h_1 r_2 = r_1(h_1 - h)$$

$$\Rightarrow h_1 = \frac{r_1 h}{r_1 - r_2}$$

19.10 Surface area of a sphere

The figure below represents a solid sphere of radius **r** units.



The surface area of a sphere is given by below.

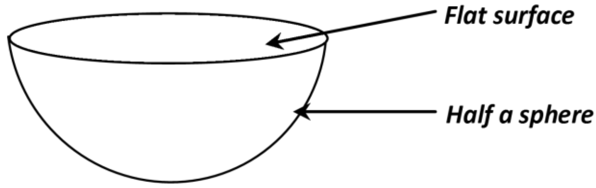
$$\boxed{Surface\ area\ of\ a\ sphere = 4\pi r^2}$$

NB:

The proof for this formula is beyond the scope of this course.

19.11 Surface area of a hemisphere

A hemisphere is half of a sphere.

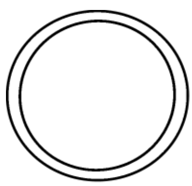


$$Area\ of\ a\ hemisphere = area\ of\ the\ flat\ surface + area\ of\ half\ of\ the\ sphere$$
$$= \pi r^2 + \frac{4\pi r^2}{2}$$
$$= \pi r^2 + 2\pi r^2 = 3\pi r^2$$

$$\boxed{Surface\ area\ of\ hemisphere = 3\pi r^2}$$

19.12 Area of a ring

A ring is a circular object with a hole at its centre. Below is a structure of a ring.



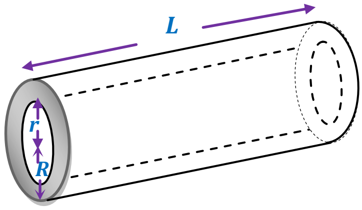
If **R** is the radius of the larger circle and **r** is the radius of the smaller circle, then area of the ring is given by:

$$Area\ of\ the\ ring = area\ of\ the\ larger\ circle + area\ of\ the\ smaller\ circle$$
$$= \pi R^2 - \pi r^2$$
$$= \pi (R^2 - r^2), \quad but\ R^2 - r^2 = (R + r)(R - r)$$

$$\boxed{\therefore\ Area\ of\ the\ ring = \pi (R + r)(R - r)}$$

19.13 Surface area of a pipe

Consider a pipe of length L with outer radius R and internal radius as shown below.



The hollow pipe has a uniform cross section, which is a ring.

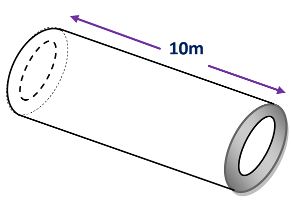
$$\begin{array}{lcl} \textit{Total surface area} & = \textit{area of two rings} & + \textit{curved surface area} \\ \textit{of the pipe} & \textit{at both ends} & \textit{of the pipe} \end{array}$$

$$\begin{array}{ll} \textit{Area of the ring at one end} & = \pi(R^2 - r^2) \\ \therefore \textit{Area of the ring at both ends} & = 2\pi(R^2 - r^2) \\ \textit{Area of the curved surface} & = 2\pi Rl \end{array}$$

$$\boxed{\therefore \textit{Surface area of the pipe} = 2\pi(R^2 - r^2) + 2\pi Rl}$$

Example

The figure below shows a cylindrical water main, which is 10cm long. The pipe has an inner radius of 30cm and outer radius of 37cm.



Calculate the total surface area of the pipe.

Solution

$$\begin{aligned}
&\textit{Area of the ring at one end} &&= \pi(R^2 - r^2), \quad R = 37cm, r = 30cm \\
& &&= \tfrac{22}{7}(37^2 - 30^2) = 1474cm^2 \\
&\therefore \textit{Area of the ring at both ends} &&= 2 \times 1474 = 2948cm^2 \\
&\textit{Area of the curved surface} &&= 2\pi Rl, \quad R = 37cm, l = 10cm \\
& &&= 2 \times \tfrac{22}{7} \times 37 \times 10 = 2325.7cm^2
\end{aligned}$$

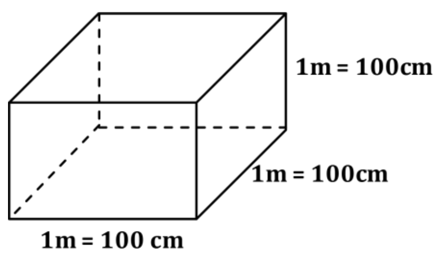
$$\textit{So total surface area of the pipe} = 2948 + 2325.7 = \underline{\underline{5273.7cm^2}}$$

19.14 VOLUME OF SOLIDS

19.14.1 Definition:

Volume is the amount of space occupied by an object.

A unit cube is used as the basic unit of volume. The SI unit of volume is the cubic meter (m^3). Consider a unit cube below i.e. a cube of side 1m.



$$\textit{The volume of the cube} = 1m \times 1m \times 1m = 100cm \times 100cm \times 100cm$$

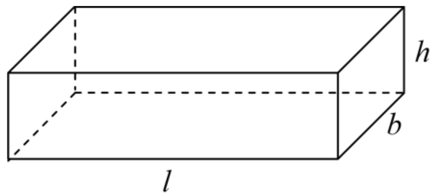
$$\Rightarrow \quad 1m^3 = 1{,}000{,}000cm^3$$

Since $1m^3$ is large for ordinary use, volumes are often measured using cm^3.

$$\boxed{\begin{aligned} 1m^3 &= 1{,}000{,}000\ cm^3 = 1.0 \times 10^6\ cm^3 \\ \therefore 1cm^3 &= \frac{1}{1{,}000{,}000} = 1.0 \times 10^{-6}\ m^3 \end{aligned}}$$

19.14.2 Volume of a cuboid

Consider a cuboid of length l, breadth b, and height h, as shown below.



$$Volume\ of\ a\ cuboid = l \times b \times h$$

19.14.3 Volume of a cuboid

A cube is just a special cuboid with *length = breadth = height*

$$Volume\ of\ a\ cube = l \times l \times l = l^3$$

Example

A rectangular tank has 70cm^3 of water. If the tank is 5cm long and the height of water is 4cm, what is the width of the tank?

Solution

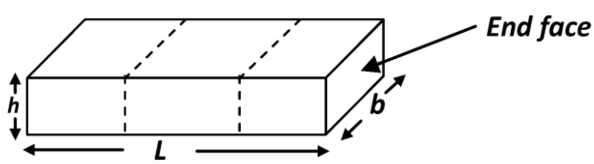
$$The\ volume\ of\ water = l \times b \times h \quad but\ V = 70m^3,\ l = 5m,\ h = 4m,\ b = ?$$

$$\Rightarrow \quad 70 = 5 \times 4 \times b$$

$$\therefore b = {}^{70}\!/_{20} = \underline{\underline{3.5m}}$$

19.14.4 Uniform cross – section

Consider the cuboid shown below, the shaded part is known as the end face.



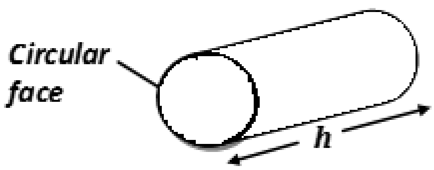
If the cuboid is sliced along the dotted lines, each slice will be parallel and identical to the end face. Such faces are known as cross section of the solid.

$$A\ cuboid\ has\ a\ uniform\ cross\ section\ of\ area = bh. \quad But\ volume = lbh$$
$$\therefore Volume = l \times area\ of\ cross\ section$$

There are many solids, which have uniform cross sections that are not rectangular; their volumes are calculated in the same way.

19.14.5 Volume of a cylinder

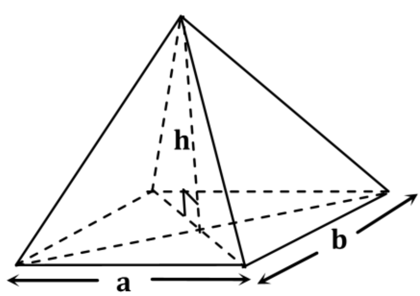
A cylinder is a solid whose uniform cross section is a circular surface.



$$Area\ of\ cross\ section = \pi r^2. \quad where\ r\ is\ the\ radius\ of\ the\ circular\ face$$
$$Volume\ of\ the\ cylinder = area\ of\ circular\ face \times height$$

$$\boxed{\therefore Volume\ of\ a\ cylinder = \pi r^2 h}$$

19.14.6 Volume of a pyramid

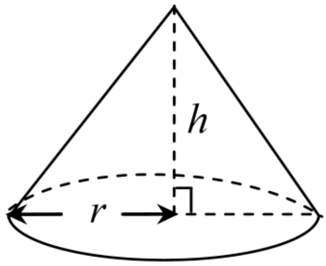


20 $$Volume\ of\ a\ pyramid = \tfrac{1}{3} \times area\ of\ base \times height$$
$$But\ area\ of\ the\ base\ A = a \times b$$

$$\boxed{\therefore Volume\ of\ a\ pyraid = \tfrac{1}{3} abh}$$

20.1.1 Volume of a cone

A cone may be considered as a pyramid with a circular base.



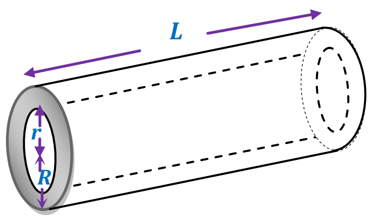
$$Volume\ of\ a\ cone = \tfrac{1}{3} \times area\ of\ base \times height$$

$$But\ area\ of\ the\ base\ A = \pi r^2$$

$$\boxed{\therefore Volume\ of\ a\ cone = \tfrac{1}{3}\pi r^2 h}$$

20.1.2 Volume of a pipe

If **R** is the outer radius and **r** is the internal radius of the pipe of length **L**.



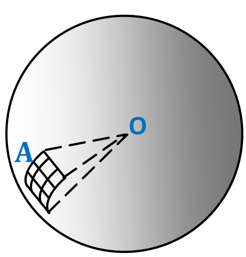
$$Volume\ of\ a\ the\ pipe = area\ of\ cross\ section \times length\ of\ the\ pipe$$

$$But\ area\ of\ cross\ section = \pi(R^2 - r^2)$$

$$\boxed{\therefore Volume\ of\ a\ pipe = \pi(R^2 - r^2)l}$$

20.1.3 Volume of a sphere

Let **A** represents a small square area on the surface of a sphere of radius r, centre O.



If **A** is very small, then it is almost flat. Therefore, the solid formed by joining the vertices of **A** to the centre **O** is a small pyramid of height equal to **r**.

$$\text{Volume of a small pyraid} = \tfrac{1}{3}Ar$$

If there are **n,** such small pyramids in the sphere with base areas, $\mathbf{A_1}$, $\mathbf{A_2}$, $\mathbf{A_3}$ $\mathbf{A_n}$.

$$\text{Their volumes are; } \tfrac{1}{3}A_1r, \tfrac{1}{3}A_2r, \tfrac{1}{3}A_3r.......\tfrac{1}{3}A_nr$$
$$\text{Total volumes} = \tfrac{1}{3}A_1r + \tfrac{1}{3}A_2r + \tfrac{1}{3}A_2r...... + \tfrac{1}{3}A_nr = \tfrac{1}{3}r(A_1 + A_2 + A_3 + ... + A_n)$$

For the whole surface of the sphere, the sum of all their base area is $4\pi r^2$ i.e.

$$A_1 + A_2 + A_3 + ... + A_n = 4\pi r^2$$

$$\text{Hence total volume } V \text{ of a sphere} = \tfrac{1}{3}r \times 4\pi r^2$$

$$\boxed{\therefore \text{Volume of a sphere} = \tfrac{4}{3}\pi r^3}$$

20.1.4 Volume of a hemisphere

Since a hemisphere is half of a sphere, its volume is equal to half of the volume of a sphere, i.e.

$$\boxed{\text{Volume of a hemisphere} = \tfrac{2}{3}\pi r^3}$$

Example

A solid hemisphere of radius 5.8cm has density of 10.5g/cm^3.

Calculate:

a) Volume of the solid

b) Mass in kg of the solid

Solution

a) *For volume of solid:*

$$Volume\ of\ \ a\ hemisphere = \tfrac{2}{3}\pi r^3, \quad r = 5.8cm$$
$$= \tfrac{2}{3} \times 3.142 \times 5.8^3$$
$$= \underline{\underline{408.7cm^3 (4sf)}}$$

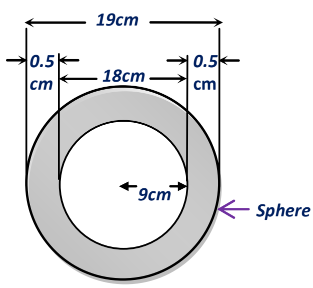
b) *For mass of solid:*

$$From: Density = \frac{Mass}{Volume}$$
$$\Rightarrow Mass \ = Density \times Volume$$
$$= 10.5 \times 408.7g$$
$$= \frac{4291.35}{1000}$$
$$= \underline{\underline{4.291kg (4sf)}}$$

Example

A hollow sphere has an internal diameter of 18cm and thickness 0.5cm. Find the volume of the material used in making the sphere.

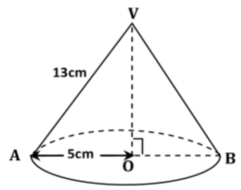
Solution



$$Internal\ \ diameter\ = 18cm. \Rightarrow Internal\ \ radius = {}^{18}\!/_{2} = 9cm$$
$$External\ \ diameter\ = 18 + 0.5 + 0.5 = 19cm. \Rightarrow External\ \ diameter\ = {}^{19}\!/_{2} = 9.5cm$$
$$Volume\ of\ \ the\ shaded\ \ part = volume\ of\ \ material\ used\ to\ make\ the\ sphere$$
$$= Volume\ of\ \ the\ whole\ sphere - Volume\ of\ \ unshaded\ \ part$$
$$= \frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^2$$
$$= \frac{4}{3} \times 3.142 (9.5^3 - 9^3)$$
$$= \underline{\underline{537.81cm^3}}$$

Example

The figure below shows a right circular cone AVB. The radius of the base is 5cm and the slanting edge 13cm.



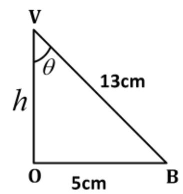
Calculate:

a) Angle VAB
b) Volume of the cone
c) Total surface area of the cone. (Take $\pi = 3.142$)

Solution

*Let **h** be the height of the cone and* **θ** *be angle OVB.*

Considering triangle VOB:



$$\sin\theta = \frac{5}{13} \quad \Rightarrow \theta = \sin^{-1}(\tfrac{5}{13}) = 22.6^0$$

$$But\ angle\ AVB = 2\theta = 2\times 22.6 = \underline{\underline{45.2^0}}$$

a) *Using Pythagoras theorem;*

$$h^2 + 5^2 = 13^2$$

$$\therefore h = \sqrt{169-25} = \sqrt{144} = 12cm$$

$$Volume\ of\ the\ cone = \tfrac{1}{3}\pi r^2 h, \quad r = 5cm$$

$$= \tfrac{1}{3}\times 3.142\times 5^2\times 12$$

$$= \underline{\underline{3145.2cm^2}}$$

b) $Total\ surface\ area\ of\ cone = Area\ of\ curved\ surface + Area\ of\ circular\ base$

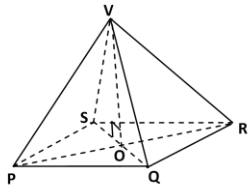
$$Areaofcircularface = \pi\ r^2 = 3.142\times 5^2 = 78.55cm^2$$

$$Area\ of\ curved\ surface = \pi rl = 3.142\times 5\times 13 = 204.23cm^2$$

$$\therefore Total\ surface\ area\ of\ cone = 204.23 + 78.55 = \underline{\underline{282.78cm^2}}$$

Example

The figure below shows a pyramid with a rectangular base PQRS. Given that ***PQ*** = 12m, ***QR*** = 9m and ***VO*** = 10m.



Calculate:

a) The length:
   i. PR
   ii. VR

b) The surface area of the pyramid

c) The volume of the pyramid

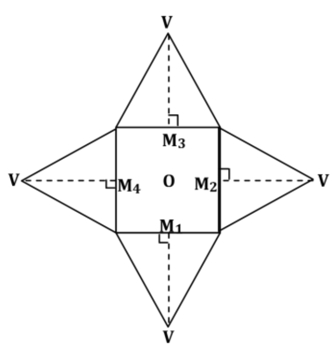
Solution

a) i) $PR = \sqrt{PQ^2 + QR^2} = \sqrt{12^2 + 9^2} = \underline{\underline{15m}}$

ii) *VO is perpendicular to the base PQRS*

$$VR = \sqrt{VO^2 + QR^2}, \quad but\ OR = \tfrac{1}{2}PR = 7.5m$$
$$= \sqrt{10^2 + 7.5^2}$$
$$= \underline{\underline{12.5m}}$$

*b) The figure below shows a net obtained by opening the pyramid at the vertex V. M_1, M_2, M_3 and M_4 are the midpoints of **PQ**, **QR**, **SR**, and **SP** respectively.*



$$But\ \bar{VS} = \bar{VR} = \bar{VQ} = \bar{VP} \quad \Rightarrow \bar{M_1V} = \bar{M_3V}\ \ and\ \bar{M_2V} = \bar{M_4V}$$

By Pythagoras theorem:

$$\bar{M_1V} = \sqrt{(VQ^2 - M_1Q^2)} = \sqrt{12.5^2 - 6^2} = 10.97m \quad \Rightarrow \bar{M_3V} = 10.97m$$

Also:

$$\bar{M_2V} = \sqrt{(VQ^2 - M_2Q^2)} = \sqrt{12.5^2 - 4.5^2} = 11.66m \Rightarrow \bar{M_4V} = 11.66m$$

$$\begin{aligned} Surface\ area\ of\ the\ pyramid &= \tfrac{1}{2}PQ\times \bar{M_1V} + \tfrac{1}{2}QR\times \bar{M_2V} + \tfrac{1}{2}SR\times \bar{M_3V} + \tfrac{1}{2}SP\times \bar{M_4V} + PQ\times QR \\ &= \tfrac{1}{2}(12\times 10.97 + 9\times 11.66 + 12\times 10.97 + 9\times 11.66) + 12\times 9 \\ &= \underline{\underline{344.31m^2}} \end{aligned}$$

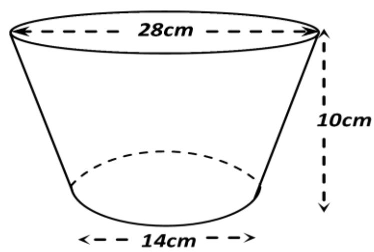
c) $Volume\ of\ the\ pyramid = \tfrac{1}{3}\times area\ of\ the\ base \times h$

$But\ area\ of\ the\ base = a\times b, \quad a = 12m, b = 9m$

$\therefore Volume\ of\ the\ pyramid = \tfrac{1}{3}\times 12\times 9\times 10 = \underline{\underline{360m^3}}$

Example

The figure below shows a bucket with a top diameter 28cma and bottom diameter 14cm. the bucket is 10cm deep.

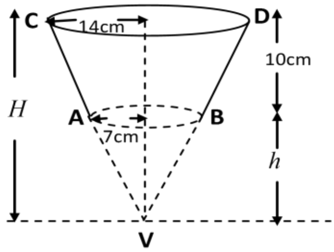


Calculate:

a) The capacity of the bucket in liters

b) The area of the plastic sheet required to make 200 such buckets, taking 5% extra for overlapping and wastage.

Solution



a) *The linear scale factor of the enlargement mapping the cone VAB to VCD is given by*

$$\frac{10+h}{h} = \frac{14}{7} = 2 \quad \Rightarrow 10+h = 2h \quad \therefore h = 10cm$$

$$\therefore Volume\ of\ cone\ VCD = \tfrac{1}{3}\pi r^2 H, \quad but\ H = 10+h = 20cm$$

$$= \tfrac{1}{3} \times \tfrac{22}{7} \times 14^2 \times 20$$

$$= 4106.7cm^3$$

$$\frac{10+h}{h} = \frac{14}{7} = 2 \quad \Rightarrow 10+h = 2h \quad \therefore h = 10cm$$

$$\therefore Volume\ of\ cone\ VCD = \tfrac{1}{3}\pi r^2 H, \quad but\ H = 10+h = 20cm$$

$$= \tfrac{1}{3} \times \tfrac{22}{7} \times 14^2 \times 20$$

$$= 4106.7cm^3$$

b)

$$Area\ of\ curve\ surface\ of\ cone\ VCD = \pi rl,\ r = 14cm,\ l = 20cm$$

$$= \tfrac{22}{7} \times 14 \times 20 = 880cm^2$$

$$Area\ of\ curve\ surface\ of\ cone\ VAB = \pi rl,\ r = 7cm,\ l = 10cm$$

$$= \tfrac{22}{7} \times 7 \times 10 = 220cm^2$$

$$\therefore Area\ of\ curve\ surface\ of\ the\ bucket = 880 - 220 = 660cm^2$$

$$\Rightarrow Total\ area\ of\ the\ bucket = 660 + \tfrac{22}{7} \times 7^2 = 814cm^2$$

$$Total\ area\ of\ the\ plastic\ material\ required$$

$$to\ make\ an\ open\ bucket = 814 \times \frac{105}{100} = \underline{\underline{854.7cm^2}}$$

20.1.5 Miscellaneous exercise:

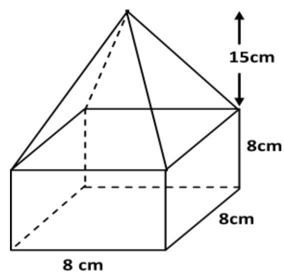
1. A solid cylinder has a radius of 18cm and height 15cm. a conical hole of radius r is drilled in the cylinder on one of the end faces. The conical hole is 12cm deep. If the material removed from the hole is 9% of the volume of the cylinder.

   Find:

   a) The surface area of the hole

   b) The radius of the spherical ball made out of the material.

2. The diagram below shows solid which comprises of a cube surmounted with a pyramid.

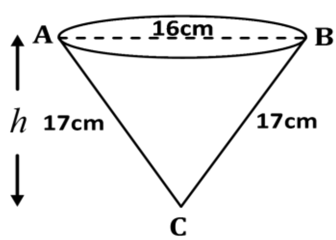


   Calculate:

   a) The surface area of the resulting solid.

   b) The volume of the solid formed.

3. The figure below shows a right circular cone ABC of vertical height h and slant side AB =BC = 17cm, and base diameter AB = 16cm.

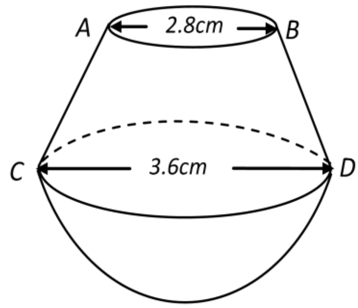


   Find:

   i. h

   ii. The capacity of the cone. Use $\pi = 3.142$

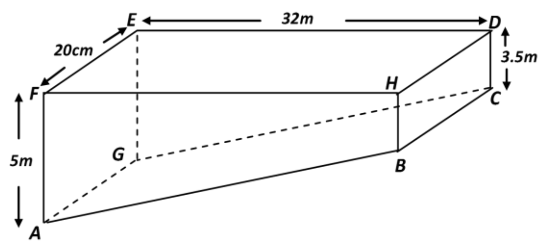
4. The diagram below represents a tank for storing water consisting of a frustum of a cone fastened to a hemisphere centre. AB = 2.8m and CD = 3.6m. The perpendicular height between AB and CD is 2.1cm.



   a) Calculate the volume of water in the tank when it is full, giving your answer to the nearest m^3.

   b) The cost of running water includes a fixed charge of shs. 150 plus shs. 50 per thousand liters used per month. If a family uses one full such tank of water per month, calculate the bill for this family in a month.

5. The diagram below shows a swimming pool 20m wide and 32m long. The pool is 3.5m deep at the shallow end and 5m deep at the deeper end.

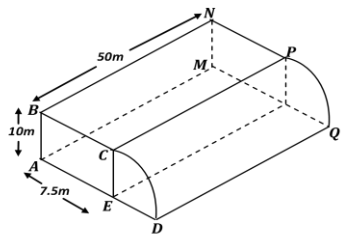


   Calculate:

   a) Volume and

   b) Surface area of the swimming pool.

6. The diagram below shows a shed with uniform cross section. ABCD consists of a rectangle ABCE and a quadrant of a circle ECD with E as the centre.



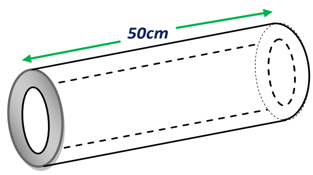
Calculate:

a) The area of cross section ABCD

b) The volume of the shed

c) Area of BCDQPN.

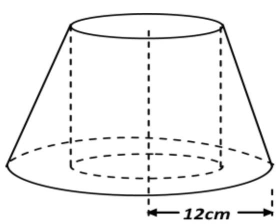
7. The figure below shows a hollow pipe of external diameter 16mm, internal diameter 10mm and length 50cm.



a) Calculate the surface area (in cm^2) of the pipe correct to two decimal places.

b) What would be the surface area of a similar pipe of length 150 cm, external diameter 48mm and internal diameter 30mm?

8. The figure below shows a right circular cone whose original height was 20cm, below part of it was cut-off. The radius of the base is 12cm and that of the top is 8cm. a circular hole of 8cm was drilled through the centre of the solid as shown in the diagram below.



Calculate the volume of the remaining solid. (Use $\pi = 3.142$).

21 LINES AND PLANES IN 3-DIMENSIONS

21.1 Introduction:

Some objects have dimensions of length, width, and height, which are all at right angle to one another. Measurement on such objects can therefore be taken in three dimensions and such objects are known as three dimensional objects.

Example of such objects includes the following:

- A box
- A cone
- A cylinder
- A pyramid

21.2 Some common term used:

21.2.1 *Lines*

A line is a set of points, which is straight and extends indefinitely in two directions, i.e.

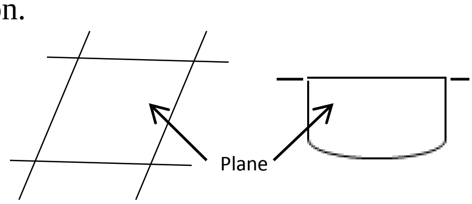
A line segment on the other hand is part of a line with two definite ends, i.e.

21.2.2 *Collinear points*

These are points lying on a single straight line. Non collinear points on the other hand are any three or more points that do not lie on a straight line.

21.3 A plane

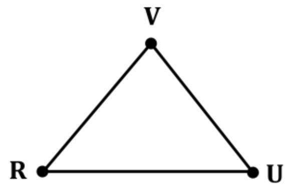
A plane is a set of points in a flat surface and extends indefinitely in all directions. However, when bounded by straight lines or curves it is called a region.



21.3.1 Determination of a plane

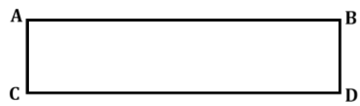
A plane is uniquely determined by:

a) Any three non collinear points i.e.



The plane RUV is formed by points **R**, **U,** and V.

b) Two parallel lines, e.g.

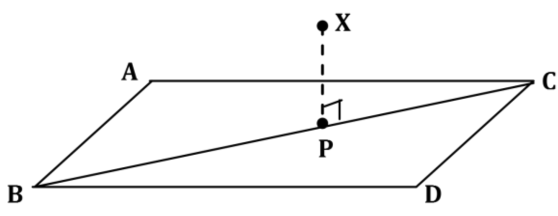


The plane ABCD is determined by the lines AB and CD.

21.4 Projection of the point and the line

21.4.1 Projection of the point onto the plane or line

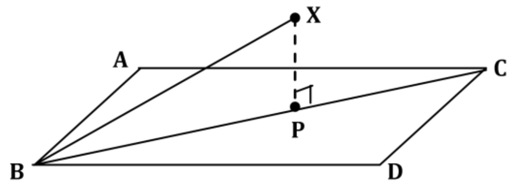
The projection of a point onto a plane or a line is the foot of the perpendicular from the point to the plane or line, i.e.



From the diagram above, P is the projection of point X onto the plane ABCD or to line AC.

21.4.2 Projection of a line onto the plane

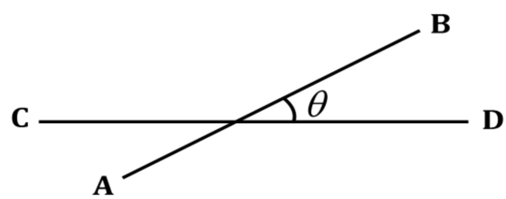
Consider the line AX below.



The projection of the line AX onto the plane ABCD is the line AP.

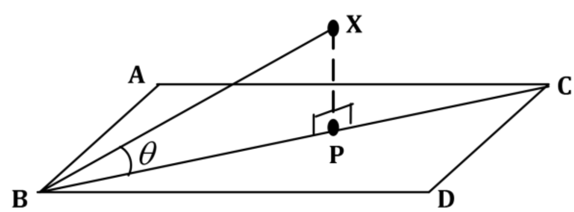
21.4.3 Angle between two lines

Angle between two lines is defined as the acute angle formed at their point of intersection. Consider lines CD and AB below.



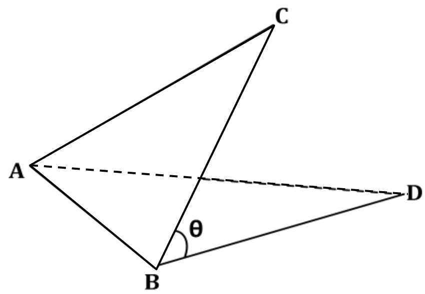
21.4.4 Angle between a line and a plane

Angle between a line and a plane is defined as the angle between the line and its projection onto the plane. Consider the line AX and its projection AP onto the plane ABCD below.



21.4.5 Angle between two planes

The angle between two planes is the angle between any two lines, one in each plane, which meet on and at right angles to the line of intersection of the planes. Consider planes ABC and ABD intersecting at AB as shown below.



The angle between the planes ABC and ABD is the same as the angle

21.5 Calculating distances and angles

In three-dimensional geometry, unknown lengths and angles can in most cases be determined by solving right angled triangle. It is therefore advisable to sketch the triangle separately from the solid.

Example

A rectangular based pyramid with vertex V is such that each of the edges VA, VB, VC, VD is 26cm long. The dimensions of the base are AB =CD =16cm and AD =BC =12cm.

Calculate:

a) The height of the pyramid

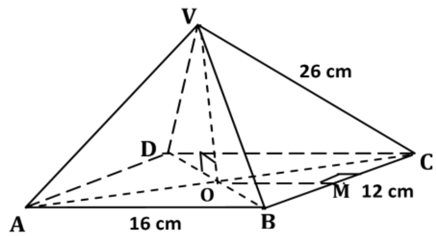
b) The angle between the edges AD and VC

c) The angle between the base and the face VBC

d) The angle between the base and slant edge.

Solution

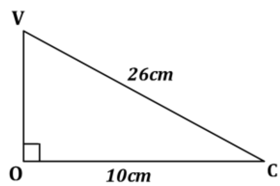
a)



In triangle ABC:

$$AC^2 = AB^2 + BC^2, \text{ by Pythagoras theorm}$$
$$= 16^2 + 12^2 = 400$$
$$\therefore AC = \sqrt{400} = 20cm$$
$$OC = \frac{1}{2}AC = \frac{20}{2} = 10cm$$

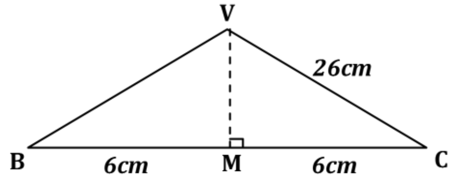
Considering triangle VOC



$$\text{By Pythagoras theorem, the height VO is calculated as follows:}$$
$$VO^2 = VC^2 - OC^2 = 26^2 - 10^2 = 576$$
$$\therefore VO = \sqrt{576} = 24cm$$

Therefore, the height VO of the pyramid is 24cm.

b) *AD and VC are skew lines (lines which are not parallel and do not meet). We therefore translate AD to BC to form the required angle VCB.*

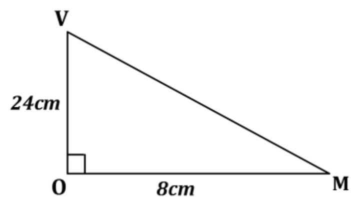


M is the midpoint of BC. From triangle VMC:

$$\cos < VCB = \frac{6}{26} = 0.2308$$
$$\therefore < VCB = \cos^{-1}(0.2308) \;=\; \underline{\underline{76.66^0}}$$

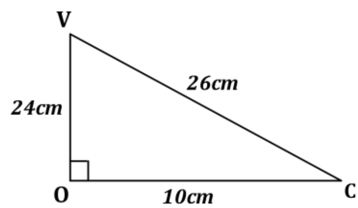
c) *BC is the line of intersection between the two planes and M is the midpoint of BC. VM and OM are lines in the plane, which are both perpendicular to BC. Thus angle VMO is the angle between the base and face VBC.*

Considering triangle VMO:



$$\tan < VMO = \frac{24}{8} = 3$$
$$\therefore < VMO = \tan^{-1}(3) \;=\; \underline{\underline{71.57^0}}$$

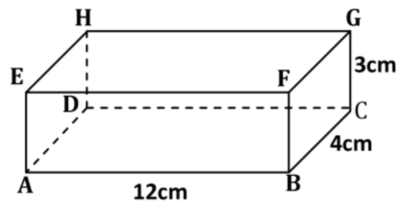
d) *Since VO is perpendicular to the base, VCO is one of the angles between the base and a slant edge. Considering triangle VCO:*



$$\tan < VCO = \frac{VO}{CO} = \frac{24}{10} = 2.4$$
$$\therefore < VCO = \tan^{-1}(2.4) \;=\; \underline{\underline{67.89^0}}$$

Example

ABCDEFGH is a cuboid with dimensions as shown in the figure below



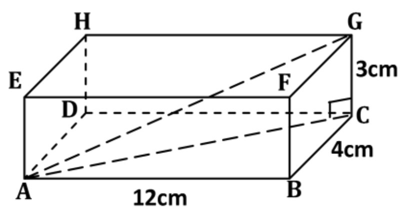
Calculate:

a) The length of AG

b) The angle that AG makes with plane BCGF

c) The shortest distance between line BF and plane ACG

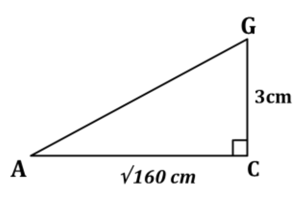
Solution



a) *The length AG can be calculated by considering triangle AGC. But we need the length AC first. This can be calculated from triangle ABC as follows:*

$$AC^2 = AB^2 + BC^2, \text{ by Pythagoras theorem}$$
$$= 12^2 + 4^2 = 160$$
$$\therefore AC = \sqrt{160} \text{ cm}$$

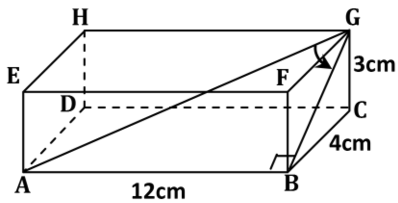
Now considering triangle AGC:



$$AG^2 = AC^2 + CG^2 = \left(\sqrt{160}\right)^2 + 3^2 = 169$$

$$\therefore AG = \sqrt{169} = \underline{\underline{13cm}}$$

b) *The angle that AG makes with plane BCGF is <AGB since BG is the projection of AG onto plane BCGF.*

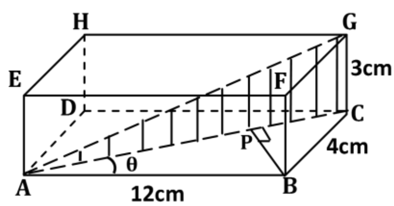


From triangle ABG:

$$\tan < ABG = \frac{AB}{GB} = \frac{12}{5} = 2.4$$

$$\therefore < ABG = \tan^{-1}(2.4) = \underline{\underline{67.38^0}}$$

c) *The shortest distance between a line and a plane is the distance between a point on the line and its projection onto the plane. For the above case, consider the diagram below:*



BP is the shortest distance between line BF and plane ACG.

$$\frac{PB}{AB} = \sin\theta$$

$$\Rightarrow PB = AB\sin\theta,\ \ AB = 12cm$$

From triangle ABC:

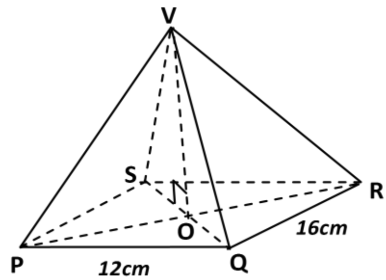
$$\tan\theta = \frac{BC}{AB} = \frac{4}{12} = {}^1\!/_3$$

$$\therefore \theta = \tan^{-1}\left({}^1\!/_3\right) = 18.43^0$$

$$\therefore PB = 12 \times \sin 18.43^0 = \underline{\underline{3.79cm}}$$

21.6 Miscellaneous Exercise

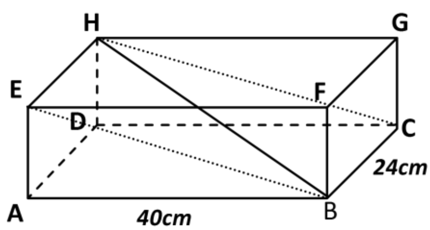
1. The figure below shows a right pyramid standing on a horizontal rectangular base PQRS. Given that PQ =12cm, QR = 16cm and V is 24cm vertically above the horizontal base PQRS



Fine:

i. The length of VQ
ii. The angle between VQ and the horizontal base
iii. The angle between the planes VPQ and VSR.

2. The diagram below shows a cuboid 40cm by 24cm by 18cm.



Calculate:

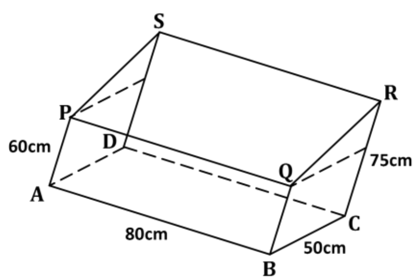
i. The length of the diagonal HB
ii. The angle between this diagonal and the base ABCD
iii. The angle between planes EBCH and ABCD

3. VEFGH is a right pyramid with a rectangular base EFGH and vertex V. O is the centre of the base and M is the point on OV such that $\bar{OM} = \frac{1}{3}\bar{OV}$. It is given that EF = 8cm, FG = 6cm, VE = VG = 15cm.

   a) Find;
      i. Length EO
      ii. The vertical height OV of the pyramid

   b) Find the angle between the opposite slant faces;
      i. VEH and VFG
      ii. VEF and VHG

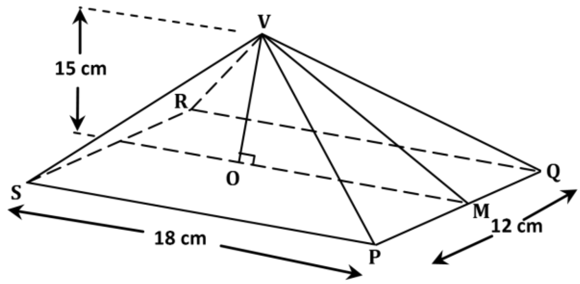
4. The figure below shows a cage in which base ABCD and roof PQRS are both rectangular. AP, BQ, CR, and DS are perpendicular to the base.



Calculate:

a) The length QR
b) The angle QRC
c) The angle between planes ABCD and PQRS
d) The inclination of PR to the horizontal.

5. The figure below shows a right pyramid on a rectangular base PQRS.



M is the midpoint of PQ. O is the centre of PQRS. Given that PQ = 12cm, QR = 18cm and VO = 15cm.

Calculate:

a) The length of VM and VQ
b) The angle between VP and the base
c) The angle between VPQ and the base.