TOPIC 8: ERROR ANALYSIS

An error is an inaccuracy which cannot be avoided in a measurement or a calculation. In math an error is not a mistake because a mistake can be avoided if one is careful. Errors are made during measurement of rainfall, atmospheric pressure, weights, and calculations using estimated values.

TYPES OF ERRORS

1. **Random errors.**
   These occur due to human failure or due to machine failure. They cannot be treated numerically. Example; a student being given 54% instead of 34%
2. **Rounding errors.**
   Some numbers are normally corrected to a given number of decimal places or significant figures.

Example: Round off:

a) 3.896234 to 4 decimal places
   = 3.8962
b) 12.4872 to 2 decimal places
   =12.49
c) 0.00652673 to 3 significant figures
   = 0.00653
d) 543216 to 3 significant figures
   =543216
e) 546321 to 2 significant figures
   = 550000

NOTE: Rounding off should be done once, i.e. it should be done in a single step

3. Truncation errors.
   Occurs when an infinite process or value is terminated at some point.

Example: Truncate:

1. 0.66666667 to 4 significant figures
   $= 0.6666$
2. 6.00513 to 2 decimal places
   $= 6.00$

   Note: Truncations can also be used to write an expansion e.g The expansion of (1+x)^n to a given number of terms.

<u>Common terms used in errors</u>

1. **Approximations:**
   An approximation is a value which is close to an exact value. E.g if the exact value is 4.321, then rounding off the value is an approximation.
2. **Error :**
   An error is the difference between the exact value and the approximate value. It can be positive or negative.
   Suppose X represents the exact value and x the corresponding approximate value then the error in x denoted by $\Delta x = X - x$. ie Error = exact – estimate
3. **Absolute error**: If Δx is the error in the estimate x, then the absolute error in x is $|\Delta x|$, disregarding the sign. $|\Delta x| = |X - x|$

4. **Relative error**:
   If Δx is the error in x then the relative error in x is
   Relative error $= \frac{absolute\ error}{exact\ or\ approximate\ value}$

   $$= \left|\frac{\Delta x}{x}\right| \text{ or } \left|\frac{\Delta x}{X}\right| \text{ since } X \approx x$$
5. **Percentage error/ percentage relative error**:
   Percentage error $= \left|\frac{error}{estimate} \times 100\%\right|$
   The percentage sign (%) can be neglected.

6. **The triangular inequality**: It is useful in deducing maximum errors. It is given by

   $$|x + y| \leq |x| + |y|$$

Note:

1. **The maximum error made in rounding off a number is** given **by Error=$\frac{1}{2}\times \mathbf{10^{-n}}$, where n is the number of decimal places to which the number is rounded off. This also gives the error when the number is rounded off to a given number of significant figures.**

Example: Write down the maximum possible error in the following numbers correct to a given number of decimal places.

a) 2.31
   Error $= 0.005$
b) 23.1
   Error $= 0.05$
c) 0.0420
   Error $= 0.00005$

7. **Limits of accuracy:**
   These are maximum (upper) and minimum (lower) bounds of given figures.
   Example: if $x = 5.53$ and the number is rounded off. Find the maximum and minimum values of x and state the interval in which the exact value of x lies.
   Soln
   Maximum possible error in x, $\Delta x = 0.5 \times 10^{-2} = 0.005$
   Minimum value $= x - \Delta x$
   $=5.53 - 0.005$
   $=5.525$
   Maximum value $=x + \Delta x$
   $=5.53+0.005$
   $=5.535$
   Interval/range=$[5.525,\ \ 5.535]$ or $5.525 \le x \le 5.535$
   NOTE:
   1. **Always use closed brackets. Do not use open brackets**
   2. **If the maximum and minimum values are known then we can also say that**

Maximum absolute error, e $= \frac{maximum\ value - minimum\ value}{2}$

This implies that Error bound $= \pm e$

Maximum and minimum values of expressions

If x and y are approximate values with errors Δx and Δy respectively then we can find the maximum and minimum values of

a) $(x + y)$

$$(x+y)_{\max} = x_{\max} + y_{\max}$$

$$(x+y)_{\min} = x_{\min} + y_{\min}$$

b) $(x-y)$

$$(x-y)_{\max} = x_{\max} - y_{\min}$$

$$(x-y)_{\min} = x_{\min} - y_{\max}$$

c) $\frac{x}{y}$

$$\left(\frac{x}{y}\right)_{max} = \frac{x_{\max}}{y_{\min}}$$

$$\left(\frac{x}{y}\right)_{min} = \frac{x_{min}}{y_{max}}$$

Note: Special cases for quotients

i) $\left(\frac{x}{x-y}\right)_{\max} = \frac{(x)_{\max}}{(x-y)_{\min}}, \left(\frac{x}{x-y}\right)_{\min} = \frac{(x)_{\min}}{(x-y)_{\max}}$ when $x > y$ (both numerator and denominator are positive)

ii) $\left(\frac{x-y}{x+y}\right)_{\max} = \frac{(x-y)_{\max}}{(x+y)_{\max}}, \left(\frac{x-y}{x+y}\right)_{\min} = \frac{(x-y)_{\min}}{(x+y)_{\min}}$ when $y > x$ (the numerator is negative)

iii) $\left(\frac{y}{x-y}\right)_{\max} = \frac{(y)_{\min}}{(x-y)_{\min}}, \left(\frac{y}{x-y}\right)_{\min} = \frac{(y)_{\max}}{(x-y)_{\max}}$ when $y > x$ (the denominator is negative

d) (xy)

$$(xy)_{max} = x_{max}\, y_{max}$$

$$(xy)_{min} = x_{min}\, y_{min}$$

Example 1:

If x=5.53 and $y = 6.81$ and both numbers are rounded off.

a) State the maximum possible errors in x *and* y.

b) Find the:

I. Maximum value of $(x + y)$

II. Interval within which the exact value of $\frac{x}{y}$ lies.

Soln

a) $\Delta x = 0.005$ $\Delta y = 0.005$

b) I) $(x+y)_{max} = x_{max} + y_{max}$

$= (5.53+0.005) + (6.81+0.005)$

$=12.35$

ii) $$\left(\frac{x}{y}\right)_{max} = \frac{x_{max}}{y_{min}} = \frac{5.535}{6.805} = 0.813$$

$$\left(\frac{x}{y}\right)_{min} = \frac{x_{min}}{y_{max}} = \frac{5.525}{6.815}$$ hence the interval/range= $[0.811, 0.813]$

Example 2:

Given that $p = \frac{15.36 \times 27.1 - 1.672}{2.36 \times 1.043}$ the numbers are rounded off. Find:

i) The error in the calculation

ii) The value of the expression with error bounds

iii) The range within which the exact value lies

Soln

i) Error in 15.36=0.005

Error in 27.1=0.05

Error in 1.672=0.0005

Error in 2.36=0.005

Error in 1.043=0.0005

$$p_{\max} = \frac{15.365 \times 27.15 - 1.6715}{2.355 \times 1.0425}$$

$$=169.2356179$$

$$p_{\min} = \frac{15.355 \times 27.05 - 1.6725}{2{,}365 \times 1.0435}$$

$$=167.6259255$$

$$\text{Error in P} = \frac{p_{\max} - p_{\min}}{2}$$

$$= \frac{169.2356179 - 167.6259255}{2}$$

$$=0.8048462$$

ii) Working value $= \dfrac{15.36 \times 27.1 - 1.673}{2.36 \times 1.043}$

$$=168.42875$$

Value with error bounds $= 168.42875 \pm 0.804846$

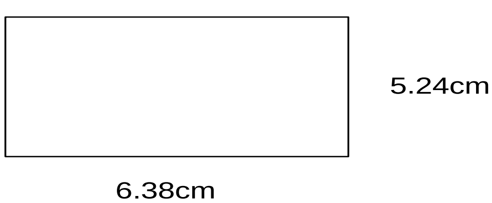
iii) Range=$[167.6259255, 169.2356179]$

Example 3:

The sides of a rectangle are measured as 5.24cm and 6.38cm. Calculate the;

i) Least value of the perimeter
ii) Limits within which the exact value of the area lies, hence determine the absolute error.

Soln



i) Perimeter = 2(l+w)

Least value=2(6.375+5.235) =23.22cm

ii) Upper limit of area $=5.245\times 6.385$

$=33.489325\text{cm}^2$

Lower limit of area $=5.235\times 6.375$ $=33.373125\text{cm}^2$

Absolute error $=\dfrac{\max-\min}{2}=\dfrac{33.489325-33.373125}{2}=0.0581$

Example 4:

The numbers $x=27.23, y=12.18$ and z=5.12 are calculated with percentage errors of 4, 3 and 2 respectively. Find the minimum value of $xy-\dfrac{y}{z}$, correct to two decimal places.

Soln

Percentage error in $x=\dfrac{\Delta x}{x}\times 100$

$$4=\frac{\Delta x}{27.23}\times 100$$

$$\Delta x=1.0892$$

$$\Delta y=\frac{3\times 12.18}{100}$$
$$\Delta y=0.3654$$

$$\Delta z=\frac{2\times 5.12}{100}=0.1024$$

Then

$$\left(xy-\frac{y}{z}\right)_{\min}=(xy)_{\min}-\frac{y_{\max}}{z_{\min}}$$

$$= (27.23 - 1.0892)(12.18 - 0.3654) - \frac{(12.18 + 0.3654)}{(5.12 - 0.1024)} = 306.34$$

Deriving formula for error propagation

Suppose x and y are approximations of X and Y respectively. Let Δx and Δy be the corresponding errors in x and y respectively, then;

a) Error in $(x+y)$

Exact value $= X + Y$

$= (x+\Delta x) + (y+\Delta y)$

Approximate value $= x + y$

Error in $x+y$ $= (x+\Delta x)+(y+\Delta y)-(x+y)$

$= \Delta x + \Delta y$

Absolute error $= |\Delta x + \Delta y|$

Since $|\Delta x + \Delta y| \le |\Delta x| + |\Delta y|$, Therefore the maximum absolute error in $(x+y)$ is $|\Delta x| + |\Delta y|$

b) Error in $x-y$

Exact value $= X - Y$

$= (x+\Delta x) - (y+\Delta y)$

Estimate value $= x - y$

Error in $x-y$ $= (x+\Delta x)-(y+\Delta y)-(x-y)$

$= \Delta x - \Delta y$

Absolute error $= |\Delta x - \Delta y|$

Since $|\Delta x - \Delta y| \le |\Delta x| + |\Delta y|$, then the maximum absolute error in $x-y$ is $|\Delta x| + |\Delta y|$

Activity: Show that the maximum absolute error in $x+y$ is $\frac{|\Delta x| + \Delta y}{|x+y|}$

c) Error in xy

Exact value $= XY$

$= (x+\Delta x)(y+\Delta y)$

Estimate value $= xy$

Error in xy $\quad = (x+\Delta x)(y+\Delta y) - xy$

$= x\Delta y + y\Delta x + \Delta x\Delta y$. For small $\Delta x, \Delta y, \Delta x\Delta y \approx 0$ (assumption)

Error in xy $\quad = x\Delta y + y\Delta x$

Absolute error in xy $\quad = |x\Delta y + y\Delta x|$. Since $|x\Delta y + y\Delta x| \le |x\Delta y| + |y\Delta x|$

Hence the maximum absolute error in xy is $|x\Delta y| + |y\Delta x|$

Note: From error in $xy = x\Delta y + y\Delta x$

Absolute error in xy $\quad = |x\Delta y + y\Delta x|$

Relative error $\quad = \left|\dfrac{x\Delta y + y\Delta x}{xy}\right|$. Since $\left|\dfrac{x\Delta y + y\Delta x}{xy}\right| \le \left|\dfrac{\Delta x}{x}\right| + \left|\dfrac{\Delta y}{y}\right|$, the maximum absolute relative error in xy is $\left|\dfrac{\Delta x}{x}\right| + \left|\dfrac{\Delta y}{y}\right|$

Example 5: If x and y are approximations to X and Y with errors of Δx and Δy respectively, Show that;

i) The maximum absolute error in $\frac{x}{y}$ is given by $\dfrac{|y||\Delta x| + |x||\Delta y|}{y^2}$

ii) The maximum possible relative error in $\frac{x}{y}$ is given by $\left|\dfrac{\Delta x}{x}\right| + \left|\dfrac{\Delta y}{y}\right|$

Soln

Exact value $\quad = \dfrac{X}{Y}$

$$= \frac{x+\Delta x}{y+\Delta y}$$

Estimate value $\quad = \dfrac{x}{y}$

Error in $\frac{x}{y}$ $\quad = \dfrac{x+\Delta x}{y+\Delta y} - \dfrac{x}{y}$

$$= \left(\frac{x+\Delta x}{y}\right)\left(1 + \frac{\Delta y}{y}\right)^{-1} - \frac{x}{y}$$

But from $\left(1+\frac{\Delta y}{y}\right)^{-1}=1-\frac{\Delta y}{y}+\frac{\Delta y^2}{y^2}+\ldots\ldots\ldots\ldots$

Assumption: For small $\Delta y, \Delta y^2 \approx 0$ and neglecting higher powers since they are very small too.

$$\left(1+\frac{\Delta y}{y}\right)^{-1}=1-\frac{\Delta y}{y}$$

Error in $\frac{x}{y}$ $=\left(\frac{x+\Delta x}{y}\right)\left(1-\frac{\Delta y}{y}\right)-\frac{x}{y}$

$=\frac{-x\Delta y}{y^2}+\frac{\Delta x}{y}-\frac{\Delta x\Delta y}{y^2}$ For small $\Delta x, \Delta y, \Delta x\Delta y \approx 0$

Error in $\frac{x}{y}=\frac{\Delta x}{y}-\frac{x\Delta y}{y^2}$

Absolute error in $\frac{x}{y}=\left|\frac{y\Delta x-x\Delta y}{y^2}\right|$ since $\left|\frac{y\Delta x-x\Delta y}{y^2}\right|\leq\frac{|y\Delta x|+|x\Delta y|}{y^2}$. Therefore the maximum absolute error in $\frac{x}{y}$ is $\frac{|y||\Delta x|+|x||\Delta y|}{y^2}$

Relative error $=\left|\frac{y\Delta x-x\Delta y}{y^2}\right|\div\frac{x}{y}=\left|\frac{y\Delta x-x\Delta y}{xy}\right|=\left|\frac{\Delta x}{x}-\frac{\Delta y}{y}\right|$

Since $\left|\frac{\Delta x}{x}-\frac{\Delta y}{y}\right|\leq\left|\frac{\Delta x}{x}\right|+\left|\frac{\Delta y}{y}\right|$. Hence the maximum relative error in $\frac{x}{y}$ is $\left|\frac{\Delta x}{x}\right|+\left|\frac{\Delta y}{y}\right|$

Note: Never use. As $\Delta x \to 0, \Delta y \to 0, \Delta x\Delta y \to 0$. When dealing with errors we are dealing with numbers. Instead we use $\Delta x\Delta y \approx 0$

Errors in functions f(x)

These include $y = \cos x, y = \sin x, y = 2^x$ and other trigonometric and exponential functions. Use of calculus can be used.

Consider $\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$ for small changes and since errors are small changes then

$\frac{\Delta y}{\Delta x} \approx f^{'}(x)\ \Delta y \approx \Delta x\ f^{1}(x)$ but Δy represents the error in f(x)

Example:

Error in x^n

Let $f(x) = x^n \Rightarrow f^{1}(x) = n x^{n-1}$

Error in f(x) = $n\Delta x\, x^{n-1}$

Alternatively: Suppose x is an approximate value of X and Δx is the error in x

Exact value $= f(X)$

$= f(x + \Delta x)$

Estimate $= f(x)$

Error in $f(x)$ $= f(x + \Delta x) - f(x)$

Using Taylor's expansion

$f(a+h) = f(a) + h\, f^{'}(a) + \frac{h^2 f^{''}(a)}{2!} + \ldots\ldots$ Where h is very small compared to a

$$f(x+\Delta x) = f(x) + \Delta x\, f^{'}(x) + \frac{(\Delta x)^2 f^{''}(x)}{2!} + \ldots\ldots$$

For $forsmall \Delta x, \Delta x^2 \approx 0$

$$f(x+\Delta x) = f(x) + \Delta x f(x)$$

Error in $f(x) = f(x) + \Delta x\, f^{1}(x) - f(x)$

$$= \Delta x \; f^1(x)$$

The absolute error in $f(x) = \left|\Delta x \; f^1(x)\right|$

Example 6:

Find the errors in the following functions.

i) $\sin x$ ii) $\cos x$. Given that $x = 30^0$ and is rounded off.

Soln

i) Let $f(x) = \sin x, x = 30^0, \Delta x = 0.5^0 = \frac{0.5\pi}{180}\, in radians$

Error in $f(x) = \Delta x f^1(x)$

$$= \left| \; \Delta x \cos x \; \right|$$
$$= \left|\frac{0.5}{180}\cos 30\right|$$
$$= 0.0075575$$

ii) Let $f(x) = \cos x$

$$Error = \left| \; \Delta x f^1(x) \; \right|$$
$$= \left|-\Delta x \sin x\right|$$
$$= \left|-\frac{0.5\pi}{180}\sin 30\right|$$
$$= 0.0043633$$

Note: For angles in degrees the error must be changed to radians.

Example 2:

If $y = \sin\theta$, find the interval within which y lies given that $\theta = 60^0$.

Soln

$$\Delta y = \left|\Delta\theta . y^1(\theta)\right|$$

$$\Delta\theta = 0.5^0 = 0.5\frac{\pi}{180}\,radians$$

$$\Delta y = \left|\frac{0.5\pi}{180}\cos 60\right| = 0.004363$$

$$y_{\max} = y + \Delta y = \sin 60 + 0.004363 = 0.87039$$

$$y_{\min} = y - \Delta y = \sin 60 - 0.004363 = 0.8617$$

Interval $[0.8617, 0.87039]$

WORKED EXAMPLE:

Derive an expression for the maximum absolute relative error in x^2y with an estimate of x and y hence find the maximum percentage error in x^2y if $x = 3.14, y = 2.888$ and are rounded off.

Soln

Error in x^2y $= (x+\Delta x)^2(y+\Delta y) - x^2y$

$$= (x^2 + 2x\Delta x + \Delta x^2)(y+\Delta y) - x^2y$$

For small $\Delta x, \Delta x^2 \approx 0$

Error in x^2y $= (x^2 + 2x\Delta x)(y+\Delta y) - x^2y$

$$= x^2\Delta y + 2xy\Delta x + \Delta x\Delta y \text{ but } \Delta x\Delta y \approx 0$$

$$= x^2\Delta y + 2xy\Delta x$$

Absolute relative error in $x^2y = \left|\frac{x^2\Delta y + 2xy\Delta x}{x^2y}\right| = \left|\frac{2\Delta x}{x} + \frac{\Delta y}{y}\right|$

Since $\left|\frac{2\Delta x}{x}+\frac{\Delta y}{y}\right| \leq 2\left|\frac{\Delta x}{x}\right|+\left|\frac{\Delta y}{y}\right|$.

The maximum absolute relative error in x^2y is $2\left|\frac{\Delta x}{x}\right|+\left|\frac{\Delta y}{y}\right|$

The percentage error in x^2y $= 2\left|\frac{\Delta x}{x}\right|+\left|\frac{\Delta y}{y}\right|\times 100\%$

$$=\left\{2\left(\frac{0.005}{3.14}\right)+\left(\frac{0.0005}{2.888}\right)\right\}\times 100$$

$=0.34\%$

ASSIGNMENT 8.1.11

1. A value of P=673.16 was obtained in a certain experiment. Given that the relative error in the measurement of this value is 0.01%, find the limits within which the value of P is expected lie.

2. The relative error obtained in determining the value of T=873.16 is 0.02%, find
   (i) The error in the measurement of this value
   (ii) The value within which T lies

3. A student measured the length and the breadth of a rectangular sheet of iron as 3.6m and 2.3m respectively.
(i) Write down the maximum possible error in each measurement
(ii) Find the limits within which the area of the sheet lies.

4. Given that $Z=|\boldsymbol{x}||\boldsymbol{y}|\boldsymbol{sin\theta}$

(a) Derive an expression for the maximum possible relative error in Z is given that Δx, Δy *and* $\Delta\theta$ are small numbers compared to x, y and θ respectively

(b) Find the maximum percentage relative error in Z, given that $x = 5.5cm$, $y = 16.8cm$ and $\theta = 45°$ and are rounded off.

5. Find the range of values within which the exact value of $2.6954\left(4.6006 - \frac{1.6175}{0.82}\right)$ lies if the numbers are rounded off to the given number of decimal places.

6. a) Given that x = 4.00 and y = 2.0, find the maximum error in $\frac{x+y}{x-y}$, correct to 4 decimal places.

b) Given that $y = 5^x$ and x is measured with a value of 2.45, determine the absolute error in y hence 0r otherwise determine the interval within which y lies. **(Hint: Use error in a function)**

7. (a)Round off to three significant figures;

(i) 6.9449 (ii) 10.459 (iii) 12436 (v) 0.01004

b)) Numbers X and Y were estimated with maximum possible errors of ΔX and ΔY respectively. Show that the maximum possible relative error in the estimation of $X\sqrt{Y}$ is given $\left|\frac{\Delta X}{X}\right| + \frac{1}{2}\left|\frac{\Delta Y}{Y}\right|$

c) Given that A=7.4, B=80.03 and C=14.801 are rounded off with corresponding percentage errors of 0.5, 0.5 and 0.005. Calculate the relative error in; i) $\frac{AB}{C}$

ii) A-C

8. (a) Given that a and b are estimated with corresponding errors of Δa and Δb. Show that the relative error in the product ab is $\left|\frac{\Delta a}{a}\right| + \left|\frac{\Delta b}{b}\right|$.

(b) The values $p = 4.7$, $q = 80.00$ and $r = 15.900$ are rounded off with corresponding percentage errors of 0.5, 0.05 and 0.05. Find the relative error in $\left(\frac{q}{r} - p\right)$.

9. a) Two sides of a triangle PQR are p and q such that $\angle PRQ = \alpha$.

i) Find the maximum possible error in the area of this triangle

ii) hence find the percentage error made in the area if $p = 4.5cm$,

$q = 8.4cm$ and $\alpha = 30^0$

(b) Find the range within which $\frac{3.679}{2} - \frac{7.0}{5.48}$ lies.

10. If $y = 5^{2x}$, find the absolute error in y when x=0.21. **(Hint: Use error in a function/calculus)**

11. An error of 2.5% is made in measuring the area of a circle. Determine the corresponding percentage error in its radius.

12. Evaluate with error bounds sin30^0. **(Hint: Use error in a function/calculus can be used)**

13. If $x = 4.95$ and $y = 2.013$ are each rounded off to a given number of decimal places, calculate the maximum and minimum values of

i) $\frac{y-x}{x+y}$ ii) $\frac{y^2}{y-x}$ **(hint: special case for quotients)**