CHAPTER THREE

MACLAURIN'S EXPANSION

3.1 Maclaurin's theorem

Suppose that $f(x)$ can be expanded as an infinite series in ascending powers of x and that $f(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots$

Putting $x = 0$, gives $f(0) = A_0$

Differentiating gives, $f'(x) = A_1 + 2A_2x + 3A_3x^2 + \cdots$

Putting $x = 0$ gives, $f'(0) = A_1$

Differentiating again gives, $f''(x) = 2A_2 + 3(2)A_3x + \cdots$

Putting $x = 0$ gives, $f''(0) = 2A_2 \quad \therefore A_2 = \frac{f''(0)}{2!}$

Differentiating again and putting $x = 0$ gives

$f'''(0) = 3(2)A_3. \quad \therefore A_3 = \frac{f'''(0)}{3!}$

$\therefore \boldsymbol{f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots}$

This result is known as Maclaurin's theorem.

The expansion is valid provided that the infinite series is convergent.

Example 1

Use the Maclaurin's theorem to find the first four non-zero terms in the expansion of e^x. Hence find $e^{0.3}$ correct to four decimal places.

Solution

Let $f(x) = e^x$; $f(0) = e^{(0)} = 1$

$\Rightarrow f'(x) = e^x; \quad f'(0) = e^{(0)} = 1$

$\Rightarrow f''(x) = e^x; \; f''(0) = e^{(0)} = 1$

$\Rightarrow f'''(x) = e^x; \; f'''(0) = e^{(0)} = 1$

Using $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$

$\Rightarrow e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\Rightarrow e^{(0.3)} \approx 1 + (0.3) + \frac{(0.3)^2}{2!} + \frac{(0.3)^3}{3!}$

≈ 1.3495

Example 2

Using Maclaurin's theorem expand $\log_e(1 + x)$ up to the term in x^4

Solution

Let $f(x) = \log_e(1 + x)$; $f(0) = \log_e(1 + 0) = 0$

$\Rightarrow f'(x) = \frac{1}{1+x}; \; f'(0) = \frac{1}{1+0} = 1$

$\Rightarrow f''(x) = -\frac{1}{(1+x)^2}; \; f''(0) = -\frac{1}{(1+0)^2} = -1$

$\Longrightarrow f'''(x) = \frac{2}{(1+x)^3};\quad f'''(0) = \frac{2}{(1+0)^3} = 2$

$\Longrightarrow f''''(x) = -\frac{6}{(1+x)^4};\ f''''(0) = -\frac{6}{(1+0)^4} = -6$

$\therefore \log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$

The expansion is valid provided$-1 < x \leq 1$.

Example 3

Use Maclaurin's theorem to find the first three non-zero terms in the expansion of $sinx$. Hence find sin $(0.1)^{rad}$ correct to seven decimal places.

Solution

$let\ f(x) = sinx;\quad f(0) = sin0 = 0$

$f'(x) = cosx;\quad f'(0) = \cos 0 = 1$

$f''(x) = -sinx;\ \ f''(0) = -\sin 0 = 0$

$f'''(x) = -cosx;\ \ f'''(0) = -\cos 0 = -1$

$f''''(x) = sinx;\quad f''''(0) = \sin 0 = 0$

$f'''''(x) = cosx;\ \ f'''''(0) = \cos 0 = 1$

By Maclaurin's theorem;

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f''''(0) + \frac{x^5}{5!}f'''''(0) + \cdots$$

$$= 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \cdots$$

$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!}$ is the required expansion

Thus $\sin(0.1\ rad) \approx \frac{(0.1)}{1!} - \frac{(0.1)^3}{3!} + \frac{(0.1)^5}{5!}$

≈ 0.0998334 correct to seven decimal places

Example 4

Use Maclaurin's theorem to expand $\tan^{-1} x$ by giving the first two non-zero terms of the expansion.

Solution

Let $f(x) = \tan^{-1} x\ ;\ \ f(0) = 0$

$f'(x) = \frac{1}{1+x^2};\quad f'(0) = 1$

$f''(x) = -\frac{2x}{(1+x^2)^2};\ \ f''(0) = 0$

$f'''(x) = -\frac{2}{(1+x^2)^2} + \frac{8x^2}{(1+x^2)^3};\ \ f'''(0) = -2$

$\therefore \tan^{-1} x = x - \frac{x^3}{3} + \cdots$

Example 5

Expand $secxtanx$ as far as the term in x^3

Solution

Let $f(x) = secxtanx$; $f(0) = 0$
$f'(x) = sec^3x + secxtan^2x$; $f'(0) = 1$
$f''(x) = 3sec^3xtanx + secxtan^3x + 2tanxsec^3x$
$\quad = 5sec^3xtanx + secxtan^3x$; $f''(0) = 0$
$f'''(x) = 15sec^3xtan^2x + 5sec^5x + secxtan^4x + 3tan^2xsec^3x$; $f'''(0) = 5$
$\therefore secxtanx = x + \frac{5x^3}{3!} + \cdots$

Example 6

Find the derivative of e^x from first principles

Solution

Let $y = e^x$

As x increases by Δx, then y will increase by Δy

$\Rightarrow y + \Delta y = e^{(x+\Delta x)}$

$\Rightarrow \Delta y = e^x e^{\Delta x} - e^x$

$\Rightarrow \Delta y = e^x\left(e^{\Delta x} - 1\right)$

$\Rightarrow \Delta y = e^x\left(e^{\Delta x} - 1\right)$

Using $e^{\Delta x} = 1 + \Delta x + \frac{(\Delta x)^2}{2!} + \frac{(\Delta x)^3}{3!} + \cdots$

$\Rightarrow \Delta y = e^x\left(1 + \Delta x + \frac{(\Delta x)^2}{2!} + \frac{(\Delta x)^3}{3!} + \cdots - 1\right)$

$\Rightarrow \Delta y = e^x\left(\Delta x + \frac{(\Delta x)^2}{2!} + \frac{(\Delta x)^3}{3!} + \cdots\right)$

Dividing through by Δx

$\Rightarrow \frac{\Delta y}{\Delta x} = e^x\left(1 + \frac{(\Delta x)}{2!} + \frac{(\Delta x)^2}{3!} + \cdots\right)$

As $\Delta x \to 0, \frac{\Delta y}{\Delta x} \to \frac{dy}{dx}$

$\therefore \frac{dy}{dx} = e^x$

Exercise 3.1

1. Find the first three non-zero terms in the Maclaurin's expansion of;
   (i) $\cos x$ (ii) $\tan x$ (iii) $cosecx$
2. Use Maclaurin's theorem to expand $e^{-\frac{x}{2}}$ by giving the first four terms of the expansion. Hence find the derivative of $e^{-\frac{x}{2}}$ from first principles.
3. Find the first three terms in the Maclaurin's expansion of $e^{-2x}\sin 2x$ in ascending powers x. $Ans\left\{2x - 4x^2 + \frac{8}{3}x^3\right\}$
4. Use Maclaurin's theorem to show that the expansion of 5^{1+sin^2x} as far as a power series up to the term in x^2 is $5 + 5x^2 \ln 5 + \cdots$

5. Prove that $xcotx = 1 - \frac{x^2}{3} - \frac{x^4}{45} + \cdots$
6. Prove that $e^{tanx} = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \cdots$
7. Prove that $\log_e(secx + tanx) = x + \frac{x^3}{6} + \cdots$
8. Show that $\sqrt{(1-x^2)}\sin^{-1}x = x - \frac{x^3}{3} - \frac{2x^5}{6} + \cdots$
9. Show that $\frac{\sin^{-1}x}{\sqrt{(1-x^2)}} = x + \frac{2}{3}x^3 + \frac{8}{15}x^5 + \cdots$
10. Given that $y = \ln(x^2 + 2x + 3)$, show that $\frac{d^3y}{dx^3}(x^2 + 2x + 3) + (4x + 4)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$, hence, find Maclaurin's expression of y showing the first four non-zero terms and approximate $ln3.21$ correct to four decimal places.
11. Use Maclaurin's theorem to expand $\tan^{-1} 2x$ as far as the term in x^2.
12. Use Maclaurin's theorem to expand $\frac{1}{\sqrt{1+x}}$ up to the term in x^3.
13. Find Maclaurin's expansion of $y = ln\frac{(2-x)^2}{(1+x)^2}$, showing the first three non zero terms.