SELF STUDY NOTES FOR SENIOR FIVE AND SIX.

UNIT: DIFFERENTIAL EQUATIONS

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Chapter 1

DIFFERENTIAL EQUATIONS

1.1 Introduction

In the Science and engineering, mathematical models are developed to aid the understanding of of physical phenomena. These models often yield an equation that contains some derivatives of an unknown function. Such an equation is called a differential equation A differential equation is an equation which relates a variable , y, to its derivatives, $\frac{dy}{dx}$ or sometimes $\frac{d^2y}{dx^2}$.
The behavior of many real-life systems in both nature and technology can be modelled by differential equations. Finding ways to produce a function which satisfies the relationship in the differential equation has resulted in many scientific advances in recent times. The laws of physics are generally written down as differential equations. Therefore, all of science and engineering use differential equations to some degree.

1.1.1 Classification of differential equations

There are many types of differential equations, and we classify them into different categories based on their properties. We broadly classify them into two categories;

a. **Ordinary differential equations or (ODEs)**. Are equations where the derivatives are taken with respect to only one variable. That is, there is only one independent variable.

b. **Partial differential equations or (PDEs)**. Are equations that depend on partial derivatives of several variables. That is, there are several independent variables.

In our study, we shall only consider the former, while the later will be considered at an advanced level.
We also distinguish how the dependent variables appear in the equation. In particular, we say an equation is **linear** if the dependent variables and their derivatives appear linearly. Otherwise, the equation is called **nonlinear**. For example, an ordinary differential equation is linear if it can be put into the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$$

the functions $a_0, a_1, \ldots, a_n$ are called the coefficients. for example

$$2x\frac{d^2y}{dx} + \cos x\frac{dy}{dx} + xy = x$$

is a linear equation while

$$2y\Big(\frac{d^2y}{dx}\Big)^2 + \cos x\frac{dy}{dx} + xy = x$$

is nonlinear.
A linear equation may further be called **homogeneous**, if all terms depend on dependent variables. Otherwise, the equation is called **nonhomogeneous** or **inhomogeneous** In particular **nonhomogeneous equations** take the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$$

for example

$$\frac{d^2y}{dx} + \cos x\frac{dy}{dx} + xy = 0$$

is homogeneous while

$$\frac{d^2y}{dx} + \cos x\frac{dy}{dx} + xy = x^2$$

is non homogeneous.

Definition 1.1.1. The order of a differential equation is determined by the highest differential coefficient present.

For example equations $\frac{dy}{dx} = 3x^2$ and $\frac{dy}{dx} = -\frac{x}{y}$ are first order differential equations while $\frac{d^2y}{dx^2} = 3x^2$ is a second order differential equation.
The solution of a differential equation defines some property common to a family of curves.
We have two types of solution to a differential equation namely;

- **General solution** involves one or more arbitrary constants, and is the equation of any member of the family,
- **Particular solution** is the equation of one member of the family.

problem 1.1. *Given the differential equation* $\frac{dy}{dx} = 5$,

i. *obtain the general solution,*

ii. *determine its particular solution if* $y = 7$ *when* $x = 1$.

solution. i. *We obtain the general solution* $y = 5x + c$, *where* c *is an arbitrary constant by integrating* $\frac{dy}{dx} = 5$,

ii. *When* $y = 7$, $x = 1$ *implies that*

$$7 = 5 \times 1 + c$$
$$7 = 5 + c \text{ and on solving for } c\text{, we obtain that } c = 2.$$

so the particular solution becomes $y = 5x + 2$.

1.2 Forming simple differential equations (DEs)

The simplest differential equation

$$\frac{dy}{dx} = f(x)$$

has a solution

$$y = \int f(x)dx,$$

$\frac{dy}{dx} = f(x)$ is a first order differential equation because it only contains the first derivative $\frac{dy}{dx}$

Example 1.2.1. A body falling through a fluid experiences resistance, causing it to lose speed at a rate proportional to its speed at that instant. Write down a differential equation satisfied by the speed v at time t.

solution.

$$\frac{dv}{dt} \propto -v \implies \frac{dv}{dt} = -kv \textit{ where } k \textit{ is a constant of proportionality.}$$

Note: Because the speed is decreasing, it makes sense to put a negative in the DE

Example 1.2.2. Water is leaking from a tank. The rate at which the depth of water is decreasing is proportional to the square root of the current depth,. Write down a differential equation satisfied by the depth h at time t.

solution.

$$\frac{dh}{dt} \propto -\sqrt{h} \implies \frac{dh}{dt} = -k\sqrt{h} \textit{ where } k \textit{ is a constant of proportionality.}$$

Example 1.2.3. A rectangular tank has a base of $5m^2$. Water is flowing into the tank at a constant rate of 10 litres per second ($= 0.01m^3s^{-1}$). At the same time, water is leaking out at a rate proportional to $\sqrt{h}$, where h is the depth of the water at time t. When the depth is 1 metre, the water is leaking out at 2 litres per second. Find the differential equation describing the rate of change of h with time t.

solution. *If water comes in at a rate* $0.01m^3s^{-1}$ *and the base area is* $5m^2$*, the water coming in is increasing the depth at a rate of* $0.002ms^{-1}$*,*
so the DE is

$$\frac{dh}{dt} = 0.002t - k\sqrt{h}.$$

If water is leaking out at 2 *litres per second (* $0.002m^3s^{-1}$*) then this is decreasing the depth at a rate of*

$$\frac{0.002}{5} = 0.0004ms^{-1}.$$

when $h = 1$*,* $k\sqrt{h} = 0.0004 \implies k = 0.0004$*, giving*

$$\frac{dh}{dt} = 0.002t - 0.0004\sqrt{h}.$$

Exercises for section (1.2)

1 The rate at which body temperature, T, falls is proportional to the difference between the body temperature T and the temperature T_0 of the surroundings. Find a differential equation relating body temperature, T, and time t.

2 A certain substance is formed in a chemical reaction. The mass of the substance formed t seconds after the start of the reaction is x grams. At any time the rate of formation of the substance is proportional to $(30 - x)$. Find a differential equation relating x and t.

3 In studying the spread of COVID-19, a scientist thinks that the rate of infection is proportional to the product of the number of people infected and the number of people uninfected. If N is the number of infected at time t and P is the total number of people in the population, form a differential equation to summarize the scientist's theory.

4 The rate of increase of a population is proportional to its size at the time. Write down a differential equation to describe this situation. Its also known that when the population was 2 million, the rate of increase was $140,000$ per days. Find the constant of proportionality.

Answers for section (1.2) ***exercises.***

1 $\frac{dT}{dt} = -k(T - T_0)$

2 $\frac{dx}{dt} = k(30 - x)$

3 $\frac{dN}{dt} = kN(P - N)$

4 $\frac{dN}{dt} = kN;\ k = 0.07$

1.3 Solving First order differential equations with separable variable

From section (1.1), we saw that when asked to solve the differential equation $\frac{dy}{dx} = 2x$, we get that

$$\frac{dy}{dx} = 2x \implies \int 1dy = \int 2xdx \implies y = x^2 + c.$$

If we need to solve a differential equation such as

$$\frac{dy}{dx} = \frac{f(x)}{g(y)},$$

we can write

$$g(y)\frac{dy}{dx} = f(x),$$

giving

$$\int g(y)\frac{dy}{dx}dx = \int f(x)dx.$$

This has solution

$$\int g(y)dy = \int f(x)dx.$$

Definition 1.3.1. A differential equation which can be rearranged to the form

$$g(y)\frac{dy}{dx} = f(x)$$

is said to have separable variables.

Example 1.3.1. Solve $\frac{dy}{dx} = 2y$.

solution.

$$\begin{aligned}
\frac{dy}{dx} &= 2y \\
\frac{1}{y}\frac{dy}{dx} &= 2 && \boxed{\textit{separate the variables first}} \\
\int \frac{1}{y}dy &= \int 2dx && \boxed{\textit{Integrate both sides}} \\
\ln y &= 2x + c \quad \textit{c is the constant of integration} && \boxed{\textit{on simplifying, we obtain}} \\
y &= e^{2x+c} = e^{2x}.e^{c} && \boxed{\textit{Let } A = e^{c}\textit{, we shall therefore obtain that}} \\
y &= Ae^{2x}.
\end{aligned}$$

Example 1.3.2. Solve $\frac{dy}{dx} = 2xy$.

solution.

$$\begin{aligned}
\frac{dy}{dx} &= 2xy \\
\frac{1}{y}\frac{dy}{dx} &= 2x \\
\int \frac{1}{y}dy &= \int 2xdx \\
\ln y &= x^2 + c \quad \textit{c is the constant of integration} \\
y &= e^{x^2+c} = e^{x^2}.e^c \ \textit{let } B = e^c \textit{ and we shall therefore obtain} \\
y &= Be^{x^2}.
\end{aligned}$$

Example 1.3.3. Solve $\frac{dy}{dx} = \cos x$

solution.

$$\begin{aligned}
\frac{dy}{dx} &= \cos x \\
\int 1dy &= \int \cos x dx \\
y &= \sin x + c \quad \textit{c is the constant of integration.}
\end{aligned}$$

Example 1.3.4. Solve $\frac{dy}{dx} = \cos^2 y$

solution.

$$\begin{aligned}
\frac{dy}{dx} &= \cos^2 y \\
\frac{1}{\cos^2 y}\frac{dy}{dx} &= 1. \quad \textit{Since } \frac{1}{\cos^2 y} = \sec^2 y, \textit{ we shall obtain that} \\
\sec^2 y\frac{dy}{dx} &= 1 \\
\int sec^2 ydy &= \int 1dx \quad \textit{Integrate both sides} \\
\tan y &= x + c \quad \textit{c is the constant of integration} \\
y &= \tan^{-1}(x + c).
\end{aligned}$$

Example 1.3.5. Solve $\frac{dy}{dx} = \frac{3x^2 \sin^2 y}{x^3+2}$

solution.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{3x^2 \sin^2 y}{x^3 + 2} \\
\frac{1}{\sin^2 y}\frac{dy}{dx} &= \frac{3x^2}{x^3 + 2}. \quad \textit{Since } \frac{1}{\sin^2 y} = cosec^2 y, \textit{ we shall obtain that} \\
cosec^2 y\frac{dy}{dx} &= \frac{3x^2}{x^3 + 2} \\
\int cosec^2 ydy &= \int \frac{3x^2}{x^3 + 2}dx \quad \textit{Integrate both sides} \\
-\cot y &= \ln(x^3 + 2) + k \quad \textit{k is the constant of integration} \\
\cot y &= -\ln(x^3 + 2) - k \quad \textit{let } c = -k \textit{ then we shall therefore obtain that} \\
\cot y &= -\ln(x^3 + 2) + c.
\end{aligned}$$

Example 1.3.6. Solve $\frac{dy}{dx} = y^2 \ln x$

solution.

$$\begin{aligned}
\frac{dy}{dx} &= y^2 \ln x \\
\frac{1}{y^2}\frac{dy}{dx} &= \ln x \qquad \textit{separate the variables} \\
\int \frac{1}{y^2} dy &= \int \ln x dx \qquad \textit{we will to integrate this by parts} \\
-\frac{1}{y} &= x \ln x - \int x.\frac{1}{x} dx \\
&= x \ln x - x + k \qquad k \textit{ is the constant of integration} \\
\frac{1}{y} &= -x \ln x + x - k \qquad \textit{let } c = -k \textit{ then we shall therefore obtain that} \\
\frac{1}{y} &= -x \ln x + x + c.
\end{aligned}$$

Exercises for section (1.3)

1. Find the general solutions to the following differential equations

   a. $\frac{dy}{dx} = \frac{x}{y}$
   b. $\frac{dy}{dx} = \frac{x^3}{y}$
   c. $\frac{dy}{dx} = \frac{x^2+x+1}{y^2}$
   d. $\frac{dy}{dx} = y(x+3)$
   e. $\frac{dy}{dx} = \frac{2}{\cos y}$
   f. $\frac{dy}{dx} = \frac{1}{xy}$
   g. $\frac{dy}{dx} = \frac{\cos x}{\cos y}$
   h. $\frac{dy}{dx} = \frac{\cos^2 y}{x}$
   i. $2\frac{dy}{dx} = 4x^3(y^2-1)$
   j. $2\frac{dy}{dx} = \frac{y^2-1}{x^2+x}$
   k. $2y\frac{dy}{dx} = 4x^3$
   l. $x^2\frac{dy}{dx} = 2y^3$
   m. $\frac{dy}{dx} = y \cos x$
   n. $\frac{dy}{dx} = 2xe^y$
   o. $\frac{dy}{dx} = 2x \sec y$
   p. $\frac{dy}{dx} = \frac{y^2}{x(x+2)}$
   q. $\frac{dy}{dx} = \frac{x(1+y^4)}{y^3}$
   r. $\frac{dy}{dx} = (\frac{x+3}{x^2+x})y^2$
   s. $\frac{dy}{dx} = xe^x \sec y$

2. Solve $(\frac{-2}{y^3})\frac{dy}{dx} = \frac{e^x}{1+e^{2x}}$ use the substitution $u = e^x$.

3. Solve $24y\frac{dy}{dx} = \frac{27}{9+x^2}$ use the substitution $x = 3\tan\theta$.

Answers for section (1.3) ***exercises.***

1. a. $y^2 = x^2 + c$
   b. $2y^2 = x^4$
   c. $2y^3 = 2x^3 + 3x^2 + 6x + c$
   d. $y = Ae^{\frac{1}{2}x^2+3x}$
   e. $\sin y = 2x + c$
   f. $\frac{1}{2}y^2 = \ln x + c$
   g. $\sin y = \sin x + c$
   h. $\tan y = \ln x + c$
   i. $\ln|\frac{y-1}{y+1}| = x^4 + c$
   j. $\frac{y-1}{y+1} = A\frac{x}{x+1}$
   k. $y^2 = x^4 + c$
   l. $\frac{1}{4y^2} = \frac{1}{x} + c$
   m. $\ln|y| = \sin x + c$
   n. $e^{-y} = -x^2 + c$
   o. $\sin y = x^2 + c$
   p. $\frac{-1}{y} = \frac{1}{2}\ln|\frac{x}{x+2}| + c$
   q. $\frac{1}{4}\ln(1+y^4) = \frac{1}{2}x^2 + c$
   r. $\frac{1}{y} = \ln|A\frac{(x+1)^2}{x^3}|$
   s. $\sin y = xe^x - e^x + c$

2. $\frac{1}{y^2} = \tan^{-1}(e^x) + c.$

3. $\frac{1}{2}y^2 = 9\tan^{-1}(\frac{x}{3}) + c.$

1.4 First order exact equations

Recall that

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

the derivative of a product. When the R.H.S. of this formula occurs in a differential equation, its integral is uv. For example, consider the differential equation

$$x\frac{dy}{dx} + y = e^x \tag{1.4.1}$$

We recognize that the L.H.S of equation (1.4.1) is the derivative of the product xy i.e

$$\frac{d}{dx}(xy) = x\frac{dy}{dx} + y.$$

Thus integrating both sides of equation (1.4.1) with respect to x gives

$$xy = e^x + A$$

where A is constant of integration.

Definition 1.4.1. A differential equation in said to exact if part of it is the exact derivative of a product.

Example 1.4.1. Solve $2xy\frac{dy}{dx} + y^2 = e^{2x}$

solution. *The derivative of product xy^2 is given by the L.H.S of $2xy\frac{dy}{dx} + y^2 = e^{2x}$ i.e*

$$\frac{d}{dx}(xy^2) = 2xy\frac{dy}{dx} + y^2$$

and so its an exact differential equation.
so on integration both sides, we obtain

$$\begin{aligned} xy^2 &= \int e^{2x}dx \\ &= \frac{1}{2}e^{2x} + c, \text{ where } c \text{ is the constant of integration} \end{aligned}$$

$\therefore xy^2 = \frac{1}{2}e^{2x} + c.$

Example 1.4.2. Find the general solution of the differential equation

$$x^2\cos y\frac{dy}{dx} + 2x\sin y = \frac{1}{x^2}$$

solution. *The L.H.S. is the derivative w.r.t. x of $x^2\sin y$. Hence integrating both sides w.r.t. x gives*

$$\begin{aligned} x^2\sin y &= \frac{-1}{x} + c, \text{ where } c \text{ is the constant of integration} \\ x^3\sin y &= cx - 1. \end{aligned}$$

$\therefore x^3\sin y = cx - 1$ *is the general solution of the given differential equation.*

Exercises for section (1.4)

1. Solve the following exact differential equations

a. $x^2\frac{dy}{dx}+2xy=1$
b. $\frac{t^2}{x}\frac{dx}{dt}+2t\ln x=3\cos t$
c. $e^y+xe^y\frac{dy}{dx}=2$
d. $y^2+2xy\frac{dy}{dx}=\frac{1}{x^2}$
e. $xy^2+x^2y\frac{dy}{dx}=\sec^2(2x)$
f. $\ln y+\frac{x}{y}\frac{dy}{dx}=\sec x\tan x$
g. $(1-2x)e^y\frac{dy}{dx}-2e^y=\sec^2 x$

Answers for section* (1.4) *exercises.

1. a. $x^2y=x+c$
   b. $t^2\ln x=3\sin t+c$
   c. $xe^y=2x+c$
   d. $y^2=\frac{c}{x}-\frac{1}{x^2}$
   e. $y^2=\frac{1}{x^2}(\tan(2x+c))$
   f. $x\ln y=\sec x+c$
   g. $(1-2x)e^y=\tan x+c$

1.5 Solving differential equations using an Integrating factor

There are some differential equations which may not be exact as they stand, but which may be made so by multiplying each side by an integrating factor. Below is the procedure one follows in order to solve a D.E. using this approach.

1. Get the equation into the form
$$\frac{dy}{dx}+P(x)y=Q(x)$$
2. Compute the integrating factor
$$v(x)=e^{\int P(x)dx}$$
3. Multiply through by $v(x)$ the integrating factor and integrate. The L.H.S. of this equation becomes $v(x)y(x)$ and the equation is now
$$v(x)y(x)=\int v(x)Q(x)dx.$$
4. Divide by $v(x)$ and you are done
$$y(x)=\frac{1}{v(x)}\int v(x)Q(x)dx.$$

Below is an illustration how to use apply the technique
The equation
$$\frac{dy}{dx}+P(x)y=Q(x)$$
is solved by multiplying both sides by the integrating factor $e^{\int pdx}$
Multiplying $\frac{dy}{dx}+Py=Q$ by $e^{\int pdx}$, we get
$$e^{\int pdx}\frac{dy}{dx}+Pe^{\int pdx}y=Qe^{\int pdx}$$
Since the left-hand side side is the differential of $ye^{\int pdx}$, we therefore have
$$\frac{d}{dx}(ye^{\int pdx})=Qye^{\int pdx}$$

which gives

$$ye^{\int pdx} = \int Qye^{\int pdx}dx$$

The right-hand side is often integrated by parts.

Example 1.5.1. Solve $\frac{dy}{dx} + 3y = x$

solution. *The integrating factor is* $e^{\int 3dx}$*, which is* e^{3x}*.*
Multiplying both sides by e^{3x}*, we obtain*

$$e^{3x}\frac{dy}{dx} + 3e^{3x}y = xe^{3x}$$

$$\implies \frac{d}{dx}(ye^{3x}) = xe^{3x}$$

Integrating both sides, we obtain

$$\begin{aligned} ye^{3x} &= \int xe^{3x}dx \\ &= \frac{1}{3}e^{3x} \times x - \int \frac{1}{3}e^{3x}dx \text{ which gives} \\ ye^{3x} &= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c \end{aligned}$$

Dividing both sides by e^{3x} *gives*

$$y = \frac{1}{3}x - \frac{1}{9} + ce^{-3x}$$

Example 1.5.2. Solve $x\frac{dy}{dx} + 3y = \frac{e^x}{x^2}$

solution. *First the equation must be written in the standard form* $\frac{dy}{dx} + P(x)y = Q(x)$*. i.e.*

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{e^x}{x^3}$$

$\implies p(x) = \frac{3}{x}$.
The integrating factor is $v(x)$ *where* $v(x) = e^{\int p(x)dx}$

$$\int p(x)dx = \int \frac{3}{x}dx = 3\ln x = \ln x^3$$
$$\implies v(x) = e^{\ln x^3} = x^3$$

multiplying the standard form of the given equation by $v(x)$*, we get*

$$x^3\frac{dy}{dx} + 3x^2y = e^x$$

The L.H.S is a derivative w.r.t x *for the product* x^3y*. hence integrating both sides, we shall obtain*

$$\begin{aligned} x^3y &= \int e^x dx \\ &= e^x + c \end{aligned}$$

$\therefore x^3y = e^x + c$ *is the general solution of the given differential equation.*

Example 1.5.3. Solve $\frac{dy}{dx} + y\cot x = \cos x$

solution. *The integrating factor is*

$$e^{\int \cot x dx} = e^{\ln \sin x} = \sin x$$

.
Multiplying each side of the given equation by $\sin x$ *we obtain*

$$sinx\frac{dy}{dx} + y\cos x = \cos x \sin x$$

$$y \sin x = \frac{1}{2}\sin^2 x + c$$

is the general solution.

Exercises for section (1.5)

1. Find the integrating factors required to make the following differential equation into exact equations, and solve them.

   a. $xy\frac{dy}{dx} + y^2 = 3x$
   b. $x\frac{dy}{dx} + 2y = e^{x^2}$
   c. $xe^y\frac{dy}{dx}2e^y = x$
   d. $2x^2y\frac{dy}{dx} + xy^2 = 1$
   e. $r\sec^2\theta + 2\tan\theta\frac{dr}{d\theta} = \frac{2}{r}$

Answers for section (1.5) ***exercises.***

1. a. $2x,\ x^2y^2 = 2x^3 + c$
   b. $x,\ x^2y = \frac{1}{2}e^{x^2} + c$
   c. $x\ x^2e^y = \frac{1}{3}x^3 + c$
   d. $\frac{1}{x}\ xy^2 = \ln kx$
   e. $r,\ r^2\tan\theta = 2\theta + c$

1.6 Finding particular solutions to differential equations

In the previous sections, we have been able to find the general solutions of given differential equations: a family of solutions. If you have to have a condition which gives a point on the solution curve, then the particular value of the constant of integration can be determined. such a condition is called initial condition or sometimes its called boundary condition. In the sequel, we shall obtain the particular solutions for given differential equations.

Example 1.6.1. Solve the differential equation $\frac{dy}{dx} + 3y = e^{2x}$, given that $y = \frac{6}{5}$ when $x = 0$.

solution. *The integrating factor is* $e^{\int 3dx} = e^{3x}$. *Multiplying each side of the given equation by* e^{3x},

$$e^{3x}\frac{dy}{dx} + 3e^{3x}y = e^{5x} \quad \textit{integrate both sides}$$

$$e^{3x}y = \frac{1}{5}e^{5x} + c$$

Therefore the general solution is

$$e^{3x}y = \frac{1}{5}e^{5x} + c$$

But when $y = \frac{6}{5}\ x = 0$,

$$\frac{6}{5} = \frac{1}{5} + c, \implies c = 1$$

. *Therefore the particular solution is*

$$y = \frac{1}{5}e^{2x} + e^{-3x}$$

.

Example 1.6.2. Find the solution of the differential equation $(1+x)\frac{dy}{dx} = 1 - \sin^2 y$, for which $y = \frac{\pi}{4}$ when $x = 0$.

solution.

$$\begin{aligned}
\frac{1}{1-\sin^2 y}\frac{dy}{dx} &= \frac{1}{1+x} && \boxed{\text{separate variables}} \\
\int \frac{1}{1-\sin^2 y}\frac{dy}{dx}dx &= \int \frac{1}{1+x}dx && \boxed{\text{integrate w.r.t } x \text{ both sides}} \\
\int \frac{1}{\cos^2 y}dy &= \int \frac{1}{1+x}dx && \boxed{\text{use the pythagorean identity } \sin^2 y + \cos^2 y = 1} \\
\int \sec^2 y dy &= \int \frac{1}{1+x}dx \\
\tan y &= ln|1+x| + c
\end{aligned}$$

when $x = 0$, $y = \frac{\pi}{4}$. *Thus*

$$\tan\frac{\pi}{4} = \ln 1 + c \qquad \boxed{\text{Find the constan } c}$$

$1 = 0 + c \implies c = 1$. *So* $\tan y = \ln|1+x| + 1$ *and we shall therefore have on further simplification*

$$y = \tan^{-1}(\ln|1+x| + 1).$$

Example 1.6.3. Find the curve which satisfies $\frac{dy}{dx} = \frac{4xy}{x^2+3}$ and passes through $(0, 9)$.

solution.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{4x}{x^2+3} \\
\frac{1}{y}\frac{dy}{dx} &= \frac{4xy}{x^2+3} && \boxed{\text{separate variables}} \\
\int \frac{1}{y}dy &= \int \frac{4xy}{x^2+3}dx \\
\ln y &= 2\ln(x^2+3)^2 + c = \ln A(x^2+3)^2 \\
y &= A(x^2+3)^2
\end{aligned}$$

we are given that when $x = 0$, $y = 9$ *we shall therefore get that*

$$9 = 9A$$

and so $A = 1$.
The particular solution is

$$y = (x^2+3)^2$$

.

Exercises for section (1.6)

1. Find the particular solutions of the following differential equations which satisfy the given conditions.

a. $(x+1)\frac{dy}{dx} - 3y = (x+1)^4$, $y = 16$ when $x = 1$

b. $\frac{dy}{d\theta} + u\cot\theta = 2\cos\theta$, $u = 3$ when $\theta = \frac{\pi}{2}$,

c. $(x+y)\frac{dy}{dx} = x - y$, $y = -2$ when $x = 3$,

d. $\frac{dy}{dx} = y\cos x$, $x = \frac{\pi}{2}$ when $y = 1$,

e. $\frac{dy}{dx} = \frac{2}{\cos y}$, $x = 0$ when $y = \frac{\pi}{2}$

f. $(x^2 - y^2)\frac{dy}{dx} = xy$, $y = 2$ when $x = 4$,

g. $x - 1 + \frac{dx}{dt} = e^{-t}t^{-2}$, $x = 1$ when $t = 1$

2. Solve the differential equation: $(x+2)\frac{dy}{dx} = (2x^2 + 4x + 1)(y - 3)$ given that $x = 0$,$y = 7$.

3. On a certain curve for which $\frac{dy}{dx} = x + \frac{a}{x^2}$, the point $(2, 1)$ is a point of inflection. Find the value of a and the equation of the curve.

4. Find the general solution of $\frac{dy}{dx} + 2xy = x$. What is the particular solution given by $y = \frac{-1}{2}$ when $x = 0$?

5. A curve passes through the point $(1, 2)$ and its gradient function is $\frac{2x+1}{2y}$ Find the equation of the curve.

6. A curve is such that $\frac{dy}{dx} = \sqrt{\frac{y}{x+1}}$ and the point $(3, 9)$ lies on the curve . Find the equation of the curve.

7. A curve is such that $x^3\frac{dy}{dx} = \sec y$ and the point $(1, \frac{\pi}{6})$ lies on the curve. Find the equation of the curve.

8. A curve is such that $e^y\frac{dy}{dx} - 2\sec^3 x = 10$ and the point $(\frac{\pi}{4}, 0)$ lies on the curve. Find the equation of the curve.

9. A curve is such that $\sqrt{xy}\frac{dy}{dx} = 1$ and the point $(4, 9)$ lies on the curve. Find the equation of the curve.

10. A curve is such that $\frac{dy}{dx} = \frac{2e^{-y}x}{x^2+1}$ and the point $(0, 0)$ lies on the curve, Find the equation of the curve.

11. A curve is such that $x(x+1)\frac{dy}{dx} = y$ and point $(1, 3)$ lies on the curve. Find the equation of the curve.

12. A curve is such that $e^{-y}\frac{dy}{dx} = -e^{3x}$ and the point $(0, 0)$ lies on the curve. Find the equation of the curve.

13. Solve the differential equation, $\frac{dy}{dx} = 3x\sin^2 y$ given that $y = -\frac{\pi}{4}$ when $x = \frac{1}{\sqrt{3}}$.

14. Find the general solution of the DE: $2(x^2+1)\frac{dy}{dx} = x(4 - y^2)$ given $y = 1$ when $x = 0$.

1.7 Second order linear differential equations with constant coefficients

Second order means that the second derivative will appear, but not derivatives of higher order, linear means that none of the terms containing y will be squared, or cubed or raised to any power but one and with constant coefficients means that the coefficients will be constant. In particular we shall be interested in equations of the standard form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0, \tag{1.7.1}$$

where a, b, c are real numbers with $a \neq 0$. The nature of solutions of equation (1.7.1) depends upon the relative magnitudes of the constants a, b, and c. To solve equation (1.7.1), we shall consider the auxiliary quadratic equation

$$am^2 + bm + c = 0 \tag{1.7.2}$$

where $m = \frac{d}{dx}$. The general solution of equation (1.7.2) is given by

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

three cases arise from the above equation;

(a) if $b^2 > 4ac$, then it has two real, distinct roots,

(b) if $b^2 = 4ac$, it has identical, real roots,

(c) if $b^2 < 4ac$, it has a pair of conjugate complex roots.

Each case gives rise to a distinct type of solution to the original differential equation (1.7.1) and we shall consider each in turn

1.7.1 Type I: with real, distinct roots

Suppose that the roots of the equation (1.7.2) are α and β, the it can therefore be written as

$$m^2 - (\alpha + \beta)m + \alpha\beta = 0$$

and the corresponding differential equation is of the form

$$\frac{d^2y}{dx^2} - (\alpha + \beta)\frac{dy}{dx} + \alpha\beta y = 0$$

and its solution is of the form

$$y = Ae^{\alpha x} + Be^{\beta x}$$

Example 1.7.1. Solve the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 0.$$

Hence find the values of the arbitrary constants such that when $x = 0$, $y = 5$ and $\frac{dy}{dx} = 23$.

solution. *The auxiliary equation is*

$$m^2 + 2m - 15 = 0$$

we shall therefore have

$$(m + 5)(m - 3) = 0$$

hence

$$m = -5 \ orm = 3$$

Since the roots are real and distinct, its general solution is

$$y = Ae^-5x + Be^{3x}$$

When $x = 0$,$y = 5$, *so*

$$A + B = 5$$

Also, when $x = 0$, $\frac{dy}{dx} = 23$
now

$$\frac{dy}{dx} = -5Ae^-5x + 3Be^{3x}$$

it follows that

$$23 = -5A + 3B$$

solving this pair of simultaneous equations for A *and* B *we shall get*

$$A + B = 5$$
$$-5A + 3B = 23$$

we shall obtain $A = -1$ *and* $B = 6$. *Hence the solution which fits the given conditions is*

$$y = -e^{-5x} + 6e^{3x}$$

.

Exercises

1. Find the general solution to the following differential equations.

   a. $\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 20y = 0$ b. $15\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + y = 0$ c. $2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 3y = 0$

1.7.2 Type II: With identical roots

For this type, the auxiliary equation takes the form

$$m^2 - 2pm + p^2 = 0$$

And on factorizing, we obtain

$$(m - p)^2$$

so that the solution of the auxiliary equation becomes $m = p$. The corresponding differential equation is

$$\frac{d^2y}{dx^2} - 2p\frac{dy}{dx} + p^2y = 0.$$

The general solution of this type takes the form

$$y = (Ax + B)e^{px}$$

Example 1.7.2. Find the general solution of the differential equation

$$4\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 9y = 0.$$

solution. *The auxiliary equation takes the form*

$$4m^2 - 12m + 9y = 0$$

and it therefore follows that

$$(2m-3)^2 = 0$$

$$2m = 3$$

$$m = \frac{3}{2}$$

The auxiliary equation has real identical roots and so the general solution takes the form

$$y = (Ax + B)e^{\frac{3}{2}x}$$

Exercises

1. Find the general solution to the following differential equations.

   a. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$ b. $100\frac{d^2y}{dx^2} - 60\frac{dy}{dx} + 9y = 0$ c. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

1.7.3 Type III: With complex roots

Suppose now that the roots of the auxiliary equation (1.7.2), are $p = \pm iq$, where p and q are real numbers. Then the sum of the roots is

$$(p + iq) + (p - iq) = 2p$$

and the product of the roots is

$$(p + iq)(p - iq) = p^2 + q^2$$

And so the auxiliary equation can be written as

$$m^2 - 2pm + (p^2 + q^2) = 0$$

and the corresponding differential equation is

$$\frac{d^2y}{dx^2} - 2p\frac{dy}{dx} + (p^2 + q^2)y = 0.$$

The general solution of this type takes the form

$$y = (A\cos qx + B\sin qx)e^{px}$$

Example 1.7.1. Solve the differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = 0$$

Given the initial conditions, $y = 0$ and $\frac{dy}{dx} = 1$, when $x = 0$.

solution. *The auxiliary equation takes the form*

$$m^2 + 6m + 10 = 0$$

solving by the formula,

$$m = \frac{-6 \pm \sqrt{6^2 - (4 \times 1 \times 10)}}{2 \times 1}$$

we shall obtain

$$m = \frac{-6 \pm \sqrt{36 - 40}}{2}$$

$$= \frac{-6 \pm \sqrt{(-4)}}{2}$$

$$= \frac{1}{2}(-6 \pm 2i)$$

$$\therefore m = -3 \pm i$$

Since the roots are complex, its general solution takes the form

$$y = (A\cos x + B\sin x)e^{-3x}$$

The initial conditions are $y = 0$ *and* $\frac{dy}{dx} = 1$*, when* $x = 0$*, so*

$$0 = (A\cos o + B\sin 0)$$

$$\therefore A = 0$$

Hence

$$y = Be^{-3x}\sin x$$

and

$$\frac{dy}{dx} = Be^{-3x}\cos x - 3Be^{-3x}\sin x$$

putting $x = 0$*, and* $\frac{dy}{dx} = 1$*, gives*

$$1 = B\cos 0$$

$$\therefore B = 1$$

Hence the solution which fits the initial conditions is

$$y = e^{-3x}\sin x.$$

Exercises

1. Find the general solution to the following differential equations.

   a. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 50y = 0$ b. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 34y = 0$ c. $36\frac{d^2y}{dx^2}y = 0$

1.8 Application of differential equations

Example 1.8.1. The rate at which a body loses temperature at any instant is proportional to the amount by which the temperature of the body at that instant, exceeds the temperature of its surroundings. A container of hot liquid is placed in a room of temperature 18°C and in 6 minutes the liquid cools from 82°C to 60°C. How long does it take for the liquid to cool from 26°C to 20°C?

solution.

$$\begin{aligned}\frac{dT}{dt} &\propto -(T-18)\\ \frac{dT}{dt} &= -K(T-18)\\ \int \frac{dT}{T-18} &= -k\int dt\\ \ln(T-18) &= -Kt+c\\ \textit{Thus } T &= 18+Ae^{-Kt}\end{aligned}$$

when 82°C, $t=0$ *and*

$$82 = 18 + Ae^{-k.0} \qquad \textit{thus } A = 64$$

$$\therefore T = 18 + 64e^{-kt}$$

when $T = 60$°C, $t = 6$

$$\begin{aligned}60 &= 18 + 64e^{-6K}\\ 42 &= 64e^{-6k} \textit{ from which we shall obtain the value of } K \textit{ as}\\ K &= -\frac{1}{6}\ln\left(\frac{21}{32}\right) \textit{ from which we obtain the expression of } T \textit{ in terms of } t \textit{ only as}\end{aligned}$$

$\therefore T = 18 + 64e^{\frac{1}{6}\ln(\frac{21}{32})t}$

when $T = 26$°C,

$$\begin{aligned}26 &= 18 + 64e^{\frac{1}{6}\ln(\frac{21}{32})t} \quad \textit{solving for } t \textit{ we obtain}\\ t &= 29.62\textit{minutes.}\end{aligned}$$

when $T = 20$°C,

$$\begin{aligned}20 &= 18 + 64e^{\frac{1}{6}\ln(\frac{21}{32})t} \quad \textit{solving for } t \textit{ we obtain}\\ t &= 49.37\textit{minutes.}\end{aligned}$$

Time to cool is given by

$$49.37 - 29.62 = 19.75$$

It will therefore take 19.75*minutes for the liquid to cool from* 26°C *to* 20°C.

Example 1.8.2. A sample of a radioactive radium losses mass at a rate which is proportional to the amount present. If M is the mass after time t years,

i. Form a differential equation connecting M and t.

ii. If initially the mass of radium is M_0, deduce that $M = M_0e^{-kt}$.

iii. Given that its initial mass halved in 6000years, find the value of the constant k, hence determine the number of years it takes for $5g$ of radium to reduce to $3.6g$.

solution. *i.*

$$\frac{dM}{dt} \propto M \implies \frac{dM}{dt} = -kM$$

ii.

$$\begin{aligned}\frac{dM}{dt} &= -kM\\ \frac{1}{M}\frac{dM}{dt} &= -k \qquad \textit{separate the variables}\\ \int \frac{dM}{M} &= -k\int dt \qquad \textit{integrate both sides}\\ \ln M &= -kt + c\end{aligned}$$

when $t = 0$, $M = M_0$, *thus* $\ln M_0 = c$. *so*

$$\ln M - \ln M_0 = -kt$$

or

$$\ln\left(\frac{M}{M_0}\right) = -kt$$

Thus

$$\begin{aligned}\frac{M}{M_0} &= e^{-kt}\\ \therefore M &= M_0 e^{-kt}.\end{aligned}$$

iii. *when* $t = 1600 years$, $M = \frac{1}{2}M_0$

$$\frac{1}{2}M_0 = M_0 e^{-1600k}$$

so

$$\begin{aligned}1600k &= \ln 2\\ k &= \frac{\ln 2}{1600} \approx 0.00043322.\end{aligned}$$

when $M_0 = 5g$, $M = 3.6g$

$$\begin{aligned}t &= \frac{1}{k}\ln\left(\frac{M_0}{M}\right)\\ &= \frac{1600}{\ln 2}\ln\left(\frac{5}{3.6}\right)\\ &= 758.289901331859.\end{aligned}$$

It will take approximately 758.29 *years for* $5g$ *of radium to reduce to* $3.6g$.

Example 1.8.3. A mathematical epidemiologist studying the spread of COVID-19 made an assumption that the spread of the virus is directly proportional to the number of infected persons at the moment. If N is the number of cases after time t in days

(a) form a differential equation connecting N and t

(b) If initially on the first day of testing, one person tested positive, deduce that $N = e^{-kt}$ where k is constant of proportionality.

(c) If after 29 days there were 81 new cases who tested positive. Determine how many people will be infected after

   (i) 10 days

(ii) 20 days

(iii) 40 days

(iv) 60 days

solution. *(a)*

$$\frac{dN}{dt} \propto N \implies \frac{dN}{dt} = kN$$

(b)

$$\begin{aligned}\frac{dN}{dt} &= kN \textit{ separate the variables and integrate both sides}\\ \int \frac{1}{N} dN &= k \int dt\\ \ln N &= kt + c_1 \textit{ where } c_1 \textit{ is constant of integration}\\ \textit{or } N &= e^{kt+c_1} = e^{kt} \times e^{c_1} = ce^{kt}\end{aligned}$$

Initially $t = 0,\ N = 1$

$$\implies c = 1$$

Hence

$$N = e^{kt}$$

(c) When $t = 29,\ N = 81$ *we shall therefore have that*

$$81 = e^{29k}$$

$$\implies k = \frac{1}{29} \ln 81$$

Thus

$$N = e^{(\frac{1}{29} \ln 81)t}$$

(i) Then after 10 days , the number of affected people will be

$$\begin{aligned}N &= e^{(\frac{1}{29} \ln 81)\times 10}\\ &= 4.550933\end{aligned}$$

Approximately 5 people will be infected in 10 days time.

(ii) Then after 20 days , the number of affected people will be

$$\begin{aligned}N &= e^{(\frac{1}{29} \ln 81)\times 20}\\ &= 20.7107852\end{aligned}$$

Approximately 21 people will be infected in 20 days time.

(iii) Then after 20 days , the number of affected people will be

$$\begin{aligned}N &= e^{(\frac{1}{29} \ln 81)\times 40}\\ &= 428.93662419\end{aligned}$$

Approximately 429 people will be infected in 40 days time.

(iv) Then after 60 days , the number of affected people will be

$$N = e^{(\frac{1}{29}\ln 81)\times 60}$$
$$= 8883.6142942$$

Approximately 8884 people will be infected in 60 days time.

Exercises for section (1.8)

1. a. Solve the differential equation $\frac{dy}{dx} = \frac{x^2+y^2}{x^2}$
   b. The temperature of a cooling body is known to decrease at a rate proportional to the temperature difference of the body and the surrounding. A police man found a dead body of a man lying on the road side at 6:00 am. The temperature of the body and the surrounding were 33°C and 18°C respectively. At $7:00am$, the temperature of the body and surrounding were found to be 28°C and 18°C respectively. If the normal temperature of the human being is known to be 37°C.
      i. Write down a differential equation to represent the above information
      ii. Estimate the time when the man was killed.

2. The size of a swamp of desert locust, n, which fluctuates during the year, has been modelled by an entomologist with the differential equation

$$\frac{dn}{dt} = 0.2n(0.2 - \cos t),$$

   where t is the number of weeks from the start of the observations. There are $4,000$ locusts in the swamp initially.

   a. Solve the differential equation to find n in terms of t.
   b. Find how many locust are there after 3 weeks.
   c. Show that the number of locust s reaches a minimum after a proximately 9.6 days , and find the number locusts at that time.
   d. how many locusts are there after 2π weeks.

3. In a certain city, the rate at which buildings are collapsing is proportional to those that have collapsed. If initially B_0 is the number that have already collapsed.

   a. Show that $B = B_0e^{kt}$, where k is a constant and B_0 is the number of buildings that have already collapsed.
   b. If the number of collapsed buildings doubled the initial number in 10 years, find the value of k.
   c. Determine the number of buildings that would have collapsed after 30 years in terms of the initial number.

4. A water tank of uniform cross-sectional area $2cm^2$ has a tap at the back. When the tap is opened water flows out at a rate proportional to the depth of the water in the tank.

   i. Show that $\frac{dh}{dt} = -\lambda h$.
   ii. If the depth of water is $1cm$ when the tap opened. Find the time it will take until the depth is $50cm$, assume $\lambda = \frac{1}{50}$.

5. A disease spreads at a rate proportional to the product of the number of people, n, infected and the number of people not yet infected. The population has size p. Initially 5% of the population is infected

   a. Write down a differential equation which is satisfied by n and the time t. After 3 days, its found that 10% of the population is infected.
   b. Solve this differential equation to find the number of people infected after t days
   c. How long will it take for half the population to be infected?
   d. Explain why every one in the population will eventually be infected?

6. A mathematical model for a regular 6km morning jog for an athlete makes an assumption that the speed of the athlete is proportional to the distance that he has to still cover so as to reach the end.

   a. i. If he starts off at a speed of $8kmh^{-1}$, and after t hours he has travelled xkm, explain why $\frac{dx}{dt} = k(6-x)$, where k is a constant.
      ii. Show that $k = \frac{4}{3}$.
   b. By solving the differential equation $\frac{dt}{dx} = \frac{3}{4(6-x)}$, Find the time taken for the athlete to complete $4km$ of the jog and state the predicted speed at the time.

1.9 Unit summary questions

Exercises for section (1.9)

1. Find the general solution to the following differential equations.

   a. $\frac{dy}{dx} = \frac{x^2}{y^2}$
   b. $\frac{dy}{dx} = 2x\cos^2 y$
   c. $\frac{dy}{dx} = e^{3x}\sec y$
   d. $x\tan y\frac{dy}{dx} = 1$
   e. $\frac{dy}{dx} = e^{3x-y}$
   f. $\frac{dy}{dx} = \frac{xy}{x^2+1}$
   g. $(x^2+6)\frac{dy}{dx} = 8xy$
   h. $\sin 3\theta\frac{d\theta}{dx} = (x+2)\cos 3\theta$

2. Find the particular solutions to the following differential equations, with given initial conditions

   a. $\frac{dy}{dx} = \frac{3x^2+x}{4y^3}$; $x = 2, y = 1$
   b. $\frac{dy}{dx} = 2y^2\sin x$; $x = \pi, y = 1$
   c. $\frac{dy}{dx} = e^{3x-y}$; $x = 0, y = 0$
   d. $\frac{dv}{dt} = e^{-t}\sqrt{v}$; $t = 0, v = 9$
   e. $\frac{dy}{dx} = xe^{x-y}$; $x = 0, y = 0$
   f. $\ln y\frac{dy}{dx} = \frac{x-1}{x^2+x}$; $x = 1, y = 1$

3. A curve is such that $\frac{dy}{dx} = \frac{y}{\sqrt{x+1}}$ and the point $(1,1)$ lies on the curve. Find the equation of the curve.

4. A curve is such that $\frac{dy}{dx} = xe^x\sec y$ and the point $(0, \frac{\pi}{6})$ lies on the curve. Find the equation of the curve.

5. The variables x and y are related by the differential equation

$$\frac{dy}{dx} = \frac{4ye^{2x}}{5+e^{2x}}$$

   Given that $y = 36$ when $x = 0$, find an expression for y in terms of x.

6. Find the general solution of the following differential equations

a. $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = 0$
c. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 29y = 0$
e. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 20y = 0$
b. $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$
d. $16\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + y = 0$
f. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 0$

7. Solve the following differential equations

   a. $\frac{d^2y}{dx^2} - 4y = 0$
   b. $\frac{d^2y}{dx^2} + 4y = 0$
   c. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$

   In each case find the solution for which $y = 0$ and $\frac{dy}{dx} = 2$, when $x = 0$.

8. The spread of a rumour in a large group of people is thought to reach new people at a rate proportional to the product of the number of people who have heard the rumour and the number who have not heard it. In a population of 2000 people, initially 50 people have heard the rumour. Two hours later, 300 people have heard it.

   a. Write down a differential equation relating the number of people, N, who have heard the rumour with the time t.
   b. Solve the differential equation to find an expression for N in terms of t.
   c. Calculate how long it takes for the majority of the people to have heard the rumour.

9. The temperature of a hot body decrease at a rate proportional to the difference between its temperature and the air temperature around it. A body is heated to 90°C and placed in air which is at 20°C.

   a. Write down a differential equation relating the temperature T of the body and the time t.
      After 4 minutes the body has cooled to 60°C.
   b. Solve the differential equation to find an expression for the temperature T in terms of t.
   c. Calculate the temperature after 10 minutes
   d. Calculate how long it takes for the body to get down to within 5°C of the air temperature.