PROBABILITY THEORY

This is a branch of Mathematics that deals with *uncertainty* ie a measure of chance of an ***outcome*** of ***event***.

Applications

Used in predictions of outcomes.

1. Gambling

Games of chance; card game, slot machine, lottery etc.

2. Professions

Include; insurance, weather forecasting etc.

Simple Probability experiments

Leads to well-defined *possible outcomes*. *Outcome* is the result of a single trial of probability experiment.

1. Toss/flip a coin
2. Roll/throw a die
3. Draw a card from a pile
4. Pick a ball from a bag at random

These are commonly used in *games of chance*.

Sample space S

Set of all possible outcomes of expt.

Examples:

(a) tossing a coin, S= {h, t}

(b) rolling a die, S = {1, 2, 3, 4, 5, 6}

Event E

Subset of possible outcomes of a particular interest.

Examples:

(a) head on tossing a coin, E= {h}

(b) odd on rolling a die, E = {1, 3, 5}

The 3 special types of Events

1. Equally likely events
2. Mutually Exclusive events
3. Independent events

Equally likely events

Their outcomes have equal probability.

Examples: Unbiased coin and die.

Mutually Exclusive events

They cannot occur at the same time ie *disjoint*. Examples:

1. Toss a coin. Either h or t, but not both.

A = {head} and B = {tail}. Not both.

2. Roll a die. Only one of 1, 2, 3, 4, 5, 6.

A = {odd} and B = {even}. Not both.

Independent events

If one occurs the probability of the other to occur is **not** affected. 2 experiments done.

Examples:

Toss 2 coins/Roll 2 dice, or Toss a coin and Roll a die.

Outcome on one coin/die has no effect on probability on another coin/die.

Qn. Suggest any other events which are:

(a) ***not*** *mutually exclusive events*

(b) ***not*** *independent events*

See more under Probability rules/laws.

The 2 main types of Probabilities

1. Classical probabilities
2. Empirical probabilities

Classical probabilities

Assumes that certain outcomes are equally likely.

$P(E) = n(E)/n(S)$

Where, S = sample space, E = event

See Probability Rules/Laws later.

Empirical probabilities

Relies on actual experience to determine probability. Frequencies of occurrences are known.

$P(E) = classfrequency/totalfrequency$

Probability distributions

Normally shown in a probability distribution table.

Simple Example 1

Find probability distribution for:

(a) tossing a coin

(b) rolling a die

(c) tossing 2 coins

Soln:

(a) S = {h, t}

P(h) = P(t) = ½

(b) S = {1, 2, 3, 4, 5, 6}

P(1) = P(2) = P(3) = P(4) = P(5)= P(6) = 1/6

(c) S = {hh, ht, th, tt}

P(hh) = P(ht) = P(th) = P(tt) = 1/4

Alt: Tabular format is better for (c).

Coins	h	t
h	hh	th
t	ht	tt

Simple Example 2

Find the probability of obtaining:

(a) odd numbers on rolling a die
(b) one head on tossing 2 coins
(c) same numbers on rolling 2 dice

Soln:

(a) S = {1, 2, 3, 4, 5, 6}, E = {1, 3, 5}
P(E) = n(E)/n(S) = 3/6 = 1/2
Alt: P(odd)
= P(1) + P(3) + P(5)= 1/6 + 1/6 + 1/6 = 1/2
(b) S = {hh, ht, th, tt}, E = {ht, th}
P(E) = n(E)/n(S) = 2/4 = ½
(c)

Dice	1	2	3	4	5	6
1	1,1	2,1	3,1	4,1	5,1	6,1
2	1,2	2,2	3,2	4,2	5,2	6,2
3	1,3	2,3	3,3	4,3	5,3	6,3
4	1,4	2,4	3,4	4,4	5,4	6,4
5	1,5	2,5	3,5	4,5	5,5	6,5
6	1,6	2,6	3,6	4,6	5,6	6,6

P(E) = n(E)/n(S) = 6/36 = 1/6

Qn. If 2 dice are rolled find probability of getting: (a) even ordered pairs
(b) odd sums of pairs

Set Theory in Probability

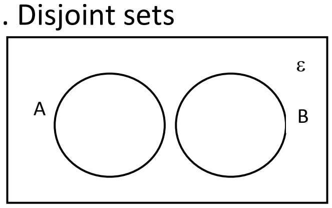
Venn diagrams are used in Probability.

1. Complement of Event

Written in 3 ways as: $\overline{A}$ or $A^{/}$ or A^{c}

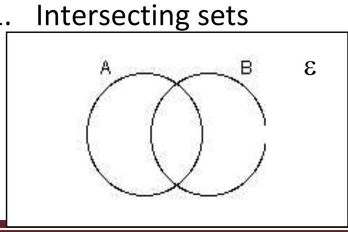
Elements are in universal set ε, but not in A.

2. Disjoint sets



$n(A \cup B) = n(A) + n(B)$

1. Intersecting sets



$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Qn. Why subtract $n(A \cap B)$?

Probability Rules/Laws

1. Condition for Probability

$0 \leq P(A) \leq 1$

Probability is positive, and cannot exceed 1.
P(E) = 0; event E cannot occur.
P(E) = 1; event E certainly occurs.

Qn. Give examples of P(A) = 0 & P(A) = 1.

2. Sum of Probabilities

$P(A_1) + P(A_2) + P(A_3) + \cdots = 1$

Simply with summation symbol as:

$$\sum_{i=1}^{n} P(A_i) = 1$$

3. Complement of Event

$P(A) + P(\overline{A}) = 1$

Simply: Outcomes of $\overline{A}$ not included A.

4. Addition rule*

(a) Mutually exclusive events.

P(AUB) = P(A or B) = P(A) + P(B)

(b) **Not** mutually exclusive events.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
P(A or B) = P (A) + P(B) - P(A and B)

5. Multiplication Rule*

(a) Independent events

$P(A \cap B) = P(A).P(B)$
P(A and B) = P (A) x P(B)

(b) **Not** independent events/Dependent
Refers to Conditional Probability

$P(A/B) = P(A \cap B)/P(B)$, where $P(B) \neq 0$

Simply: P(A/B) = Probability of A given B
Means: Probability of event A, given that event B has occurred.
For independent events:
$P(A/B) = P(A)$ and $P(B/A) = P(B)$. Why?

☺ *To verify* $0 \leq P(A) \leq 1$
Tossing 2 coins: S = {hh, ht, th, tt}
P(hh) = P(ht) = P(th) = P(tt) = ¼

😐 *To verify*

$P(A_1) + P(A_2) + P(A_3) + \cdots = 1$

Tossing 2 coins: S = {hh, ht, th, tt}

P(hh)+P(ht)+P(th)+P(tt) = ¼ +¼ +¼ +¼ = 1

😐 *To prove* $\mathrm{P}\left(\frac{\mathrm{A}}{\mathrm{B}}\right) = \mathrm{P(A)}$

for independent events.

$$\mathrm{P(A/B)} = \frac{\mathrm{P(A \cap B\,)}}{\mathrm{P(B)}} = \frac{\mathrm{P(A).P(B)}}{\mathrm{P(B)}} = \mathrm{P(A)}$$

Example 1

(a) Two coins are tossed. Find probability of getting at least one head.

(b) A died is rolled. Find the probability of getting a number less than 3 or greater than 4.

Soln:

(a) S = {hh, ht, th, tt}

Let, A = {at least 1 h} ⇒ $\overline{\mathrm{A}}$ = {no head} = {tt}

P(A) + P($\overline{\mathrm{A}}$) = 1 ⇒ P(A) = 1 - P($\overline{\mathrm{A}}$)

P($\overline{\mathrm{A}}$) = ¼ ⇒ P(A) = 1 – ¼ = ¾

Alt:

A = {hh, ht, th}. P(A) = ¼ + ¼ + ¼ = ¾

Sometimes 2nd is tedious!

(b) Let, A = {no. < 3} and B = {no. > 4}

A = {1, 2} and B = {5, 6}

A & B are mutually exclusive events.

P(A∪B) = P(A) + P(B)

P(A) = 2/6, P(B) = 2/6

P(A or B) = 2/6 + 2/6 = 4/6 =2/3

Alt:

S = {1, 2, 3, 4, 5, 6}, E = {1, 2, 5, 6}

P(E) = n(E)/n(S) = 4/6 = 2/3

Sometimes 2nd is tedious!

Example 2

On a given day, probability of driving while drug is 0.32, of having accident is 0.09, and of having accident while drug is 0.06. Find the probability of driving while drug or having accident.

Soln:

D = driving while drug & A = having accident

Not mutually exclusive events.

P(D∪A) = P (D) + P(A) - P(D∩A)

P(D or A) = P (D) + P(A) - P(D and A)

P(D or A) = 0.32 + 0.09 – 0.06 = 0.35

Example 3

A bag contains 3 red and 4 blue balls. A ball is drawn and replaced, then a second is drawn. Find the probability of getting 2 red balls.

Soln:

Let, R = red ball, B = blue ball

P(R) = 3/7 and P(B) = 4/7

P(R∩R) = P (R).P(R)

P(R and R) = P(RR) = (R) x P(R)

P(RR) = 3/7 x 4/7 =12/49

Qn. What happens to probability if a ball is drawn/picked without replacement?

See a better Tree diagram method later.

Example 4

In a factory there are 8 technicians and 5 engineers. 7 technicians and 3 engineers are female. Management is to select one staff at random. Find the probability the staff is technician or a male.

Soln:

Staff	**Female**	**Male**	**Total**
Technician	7	1	8
Engineer	3	2	5
Total	10	3	13

Let T = Technician and M = Male

P(T or M) = P (T) + P(M) - P(T and M)

=8/13 + 3/13 – 1/13 = 10/13

Tree diagrams

Branches represent mutually exclusive events.

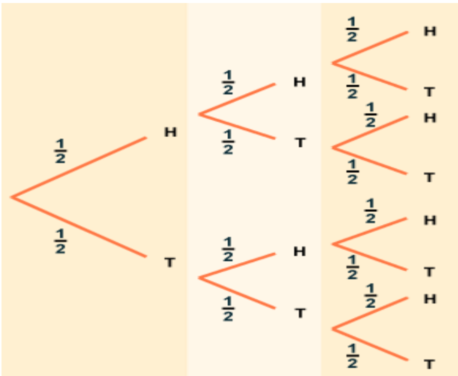
Example 5

A coin is tossed 3 times. Find the probability of getting:

(a) all heads (b) only one head

Soln:

Tree diagram can be used.



P(all heads) = P(hhh)

$= \frac{1}{2} \text{ x } \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

P(one head) = P(htt) + P(tht) + P(tth)

$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$

There are 8 mutually exclusive events.

Prob of each mutually exc event = 1/8

Total probability = 8 x 1/8 = 1

Qn.

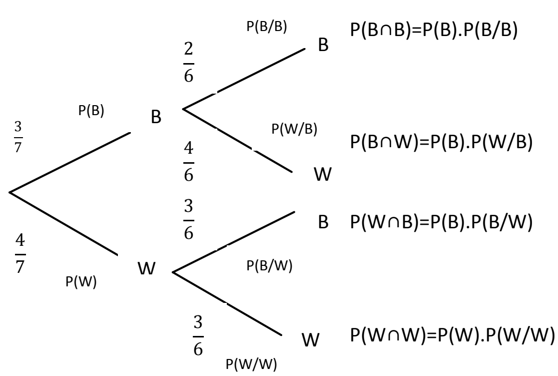
Similarly, find probabilities of other branches of the tree diagram.

See also *Binomial probability* later.

Example 6

There are 3 black and 4 white identical balls in a box. Two balls are picked from the box at random without replacement. Find the probability that the balls will be of different colours.

Soln:



P(BW or WB) = P(BW) + P(WB)

= 2/7+2/7 = 4/7

There are 4 mutually exclusive events.

P(BB) = P(B).P(B/B) = $\frac{3}{7} \times \frac{2}{6} = \frac{1}{7}$

P(BW) = P(B).P(W/B) = $\frac{3}{7} \times \frac{4}{6} = \frac{2}{7}$

P(WB) = P(B).P(B/W) = $\frac{4}{7} \times \frac{3}{6} = \frac{2}{7}$

P(WW) = P(W).P(W/W) = $\frac{4}{7} \times \frac{3}{6} = \frac{2}{7}$

Total probability = 1/7+2/7+2/7+2/7=1

Probability Relations

1. $P(A) = P(A \cap B) + P(A \cap \overline{B})$
2. $P(B) = P(A \cap B) + P(\overline{A} \cap B)$
3. $P(\overline{A}) = P(\overline{A} \cap B) + P(\overline{A} \cap \overline{B})$
4. $P(\overline{B}) = P(A \cap \overline{B}) + P(\overline{A} \cap \overline{B})$

Qn. Shade regions to prove prob relations.

Contingency table

Used to for probability relations above.

Events	A	$\overline{A}$	Add
B	$P(A \cap B)$	$P(\overline{A} \cap B)$	P(B)
$\overline{B}$	$P(A \cap \overline{B})$	$P(\overline{A} \cap \overline{B})$	$P(\overline{B})$
Add	P(A)	$P(\overline{A})$	1

1. $P(A) + P(\overline{A}) = 1$
   $P(B) + P(\overline{B}) = 1$
2. $P(A \cap B) + P(A \cap \overline{B}) = P(A)$
   $P(A \cap B) + P(\overline{A} \cap B) = P(B)$
3. $P(\overline{A} \cap B) + P(\overline{A} \cap \overline{B}) = P(\overline{A})$
   $P(A \cap \overline{B}) + P(\overline{A} \cap \overline{B}) = P(\overline{B})$

Qn. Shade to prove these equations.

de Morgan's laws

Law 1

Sets: $A' \cap B' = (A \cup B)'$

Probs: $P(A' \cap B') = P(A \cup B)'$

Law 2

Sets: $A' \cup B' = (A \cap B)'$

Probs: $P(A' \cup B') = P(A \cap B)'$

Qn. Shade to prove de Morgan's laws.

Bayes' Rule

By the English *Rev. Thomas Bayes* (1740s).

Later improved by *Pierre Simon Laplace*.

Relates to conditional probabilities.

$$P(A/B) = \frac{P(B/A).P(A)}{P(B)}$$

☹ *To prove:*

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)}$$

$\Rightarrow$ *P(B). P(A/B) = P(A). P(B/A)*

$$\Rightarrow P(A/B) = \frac{P(B/A).P(A)}{P(B)}$$

Generally:

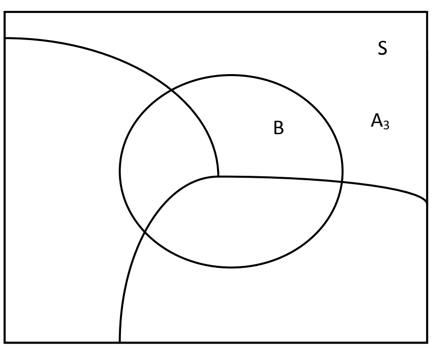
$$P(A/B) = \frac{P(B/A).P(A)}{\sum P(B/A_k).P(A_k)}$$

Where k = 1, 2, …, n

Proof for 3 events

Let A_1, A_2 and A_3 be mutually exclusive events within sample space S.

Also A_1, A_2 and A_3 events have intersections with event B (circle inside S).



P(B) = P($A_1 \cap$B) + P($A_2 \cap$B) + P($A_3 \cap$B)

$$\mathrm{P(A/B)} = \frac{\mathrm{P(A \cap B)}}{\mathrm{P(B)}} = \frac{\mathrm{P(B/A).P(A)}}{\mathrm{P(B)}}$$

$$\Rightarrow \mathrm{P(A \cap B) = P(B/A).P(A)}$$

Similarly:

$$P(A_1 \cap B) = P(B/A_1).P(A_1)$$

$$P(A_2 \cap B) = P(B/A_2).P(A_2)$$

$$P(A_3 \cap B) = P(B/A_3).P(A_3)$$

P(B) = P(B/A_1)P(A_1)
+ P(B/A_2) P(A_2)
+ P(B/A_3) P(A_3)

$$= \sum \mathrm{P(B/A_k).P(A_k)}$$

Where k = 1, 2, 3.

Example 7

Given 51% of the population are male.
12% and 8% of the male and female patients are susceptible to heart attack, respectively.
A person is selected at random and found to be susceptible to heart attack.
What is the probability that he is a male?
(The answer is 0.61)

Soln:

$Let\ M = male\ and\ \bar{M} = female$

$$P(M) = \frac{51}{100}\ and\ P(\bar{M}) = \frac{49}{100}$$

Let S = susceptible and $\bar{\mathrm{S}}$ = not susceptible

$$P(S/M) = prob\ susc\ given\ male = \frac{12}{100}$$

$$P(S/\bar{M}) = prob\ susc\ given\ female = \frac{8}{100}$$

Find:

$$P(M/S) = prob\ male\ given\ susceptible$$

$$P(M/S) = \frac{P(S/M).P(M)}{P(S/M).P(M) + P(S/\bar{M}).P(\bar{M})}$$

$$= 0.61$$

Exercise

1. A die is tossed. Find the probability of getting a number bigger 3 or less than 5.
2. Find probability distribution for:
   (a) tossing 3 coins
   (b) sums of numbers on rolling 2 dice
3. Find the probability of obtaining:
   (a) numbers $\leq$ 4 on rolling a die
   (b) at least 1 head on tossing 2 coins
   (c) both numbers > 3 on rolling 2 dice
4. A coin is tossed and a die is rolled. Find the probability of getting:
   (a) a head and an odd number
   (b) a tail and a number greater than 2
5. Find the probability of:
   (a) selecting at random a vowel from alphabetical letters
   (b) picking at random a red ball from a bag with 2 red and 3 blue balls

6. Bag A contains 3 red beads and 4 blue beads, while Bag B contains 5 red beads and three blue beads. If a bead is taken from each bag in turn, what is the probability of getting a blue followed by a red bead?
7. The probability that a fisherman catches a fish on a cloudy day is 7/10 and on a clear day is 1/5. If the probability of a cloudy day is 3/5, find the probability that the day was cloudy given that he got the fish.