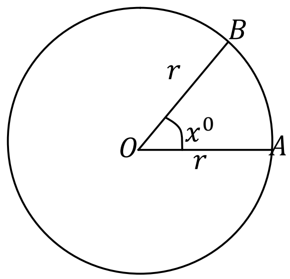
# CHAPTER ONE
# TRIGONOMETRY (CALCULAS)
## 1.1. Radians

So far we have given solutions to trigonometric equations in degrees. However, using the fact that $180^0 = \pi$ radians.

Consider the circle with, centre O, radius r and an arc AB subtending an angle of $x^0$ at O.




(a) The length of an arc of a given circle is proportional to the angle it subtends at the centre. But an angle of $360^0$ is subtended by an arc of length $2\pi r$, therefore an angle of $x^0$ is subtended by an arc of length $\frac{x}{360} \times 2\pi r$

$\therefore$**The length of the arc AB is** $\boldsymbol{\frac{x}{180} \times \pi r}$**.**

(b) The area of a sector of a given circle is proportional to the angle at the centre. But a sector containing an angle of $360^0$ is the whole circle, which has an area of $\pi r^2$, therefore a sector containing an angle of $x^0$ has an area of $\frac{x}{360} \times \pi r^2$.

$\therefore$ **The area of the sector OAB is** $\boldsymbol{\frac{x}{360} \times \pi r^2}$**.**

**Example 1**

Convert the following to degrees

(a) $\frac{\pi}{2}$ (b) $\frac{2\pi}{3}$

Solution

(a) $\frac{\pi}{2} = \frac{180^0}{2} = 90^0$

(b) $\frac{2\pi}{3} = \frac{2 \times 180^0}{3} = 120^0$

**Example 2**

Convert to radians, leaving $\pi$ in your answer:

(a) $300^0$ (b) $450^0$

Solution

(a) $300^0 = \frac{\pi}{180} \times 300 = \frac{5\pi}{3}$

(b) $450^0 = \frac{\pi}{180} \times 450 = \frac{5\pi}{2}$

- We can give the solutions of trigonometric equations in radians.

**Example 3**

Solve the equation $2\sin\theta - \cos\theta = 0$, for $-\pi \le \theta \le \pi$

Solution

Rearranging gives

$2\sin\theta = \cos\theta$

$\Longrightarrow \frac{2\sin\theta}{\cos\theta} = 1$

$\Longrightarrow \tan\theta = \frac{1}{2}$

$\Longrightarrow \theta = \tan^{-1}\left(\frac{1}{2}\right)$

$\Longrightarrow \theta = 0.46$ rad.

The other solution in the required range is $\theta = -\pi + 0.46 = -2.68$ rad.

The solution are $\theta = 0.46$ rad and $-2.68$ rad for the range $-\pi \le \theta \le \pi$

**Example 4**

Solve the equation $2\cos\theta = \sec\theta$, for $-2\pi \le \theta \le 2\pi$. Give your answer in radians.

Solution

Since $\sec\theta = \frac{1}{\cos\theta}$, we have

$2\cos\theta = \frac{1}{\cos\theta}$

$\Longrightarrow 2\cos^2\theta = 1$

$\Longrightarrow \cos\theta = \pm\frac{1}{\sqrt{2}}$

$\Longrightarrow \theta = \pm\frac{\pi}{4}$

The other solutions are $\theta = \pm\left(2\pi - \frac{\pi}{4}\right) = \pm\frac{7}{4}\pi$

$\therefore$ The solutions are $\theta = \pm\frac{\pi}{4}, \pm\frac{7}{4}\pi$ for the range $-2\pi \le \theta \le 2\pi$

**Exercise 1.1**

1. Convert the following to degrees

   (a) $\frac{\pi}{6}$ (b) $\frac{3\pi}{5}$ (c) $4\pi$ (d) $\frac{7\pi}{3}$

2. Convert the following to radians leaving $\pi$ in your answer

   (a) $150^0$ (b) $15^0$ (c) $270^0$

3. What is the length of an arc which subtends an angle of 0.8 rad at the centre of a circle of radius 10 cm? Ans(8cm)
4. The area of a sector of a circle, diameter 7 cm, is 18.375 cm2. What is the length of the sector?
5. In the triangle XYZ, $x = 29, y = 21$ and $z = 20$. Calculate;
   (a) the area of the triangle,
   (b) the length of the perpendicular from Z to XY.
6. Solve each of the following equations for $0 \le \theta \le 2\pi$ rad, giving your answers in radians.

(a) $\cos\theta = 0.4$ (b) $\sin\theta = 0.7$ (c) $\tan\theta = 4$

Ans$\{$(a)$1.16, 5.12$ (b)$0.78, 2.37$ (c)$1.33, 4.47\}$

7. Solve each of the following equations for $-\pi \le \theta \le \pi$ giving your answers in radians correct to two decimal places.

(a) $\sin(\theta - 0.1) = 0.2$ (b) $\cos(\theta + 0.2) = 0.6$ (c) $\tan(\theta - 2) = 3$

Ans$\{$(a)$0.30, 3.04$ (b) $-1.13, 0.73$ (c) $-1.39, 1.75\}$

8. Solve each of the following equations for $0 \le \theta \le 2\pi$, giving your answer in radians in terms of $\pi$

(a) $\operatorname{cosec}2\theta = 2$ (b) $\tan\left(2\theta - \frac{\pi}{6}\right) = 1$ (c) $\cos 3\theta + \cos\theta = 0$

Ans $\left\{(a)\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}\ (b)\frac{5\pi}{24}, \frac{17\pi}{24}, \frac{29\pi}{24}, \frac{41\pi}{24}\ (c)\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{2}\right\}$

## 1.2. Derivatives of Trigonometric Functions

### Small angles

- When $\theta$ is a very small angle then
  (i) $\sin\theta \approx \theta$
  (ii) $\cos\theta \approx 1$
  (iii) $\tan\theta \approx \frac{\sin\theta}{\cos\theta} \approx \frac{\theta}{1} \approx \theta$
- The above approximations are required when proving the derivatives of trigonometric functions from first principles.
- We can use the formula $f'(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$ to differentiate from first principles or use the approach of increments.
- Here we look at the derivative of trigonometric functions from first principles

**(1)** $y = \sin x$ then $\frac{dy}{dx} = \cos x$

Proof

As x increases by $\delta x$ then y will increase by $\delta y$

$\Rightarrow y + \delta y = \sin(x + \delta x)$

$\Rightarrow \delta y = \sin(x + \delta x) - \sin x$

Applying the factors formula

$\Longrightarrow \delta y = 2\cos\left(\frac{x+\delta x+x}{2}\right)\sin\left(\frac{x+\delta x-x}{2}\right)$

$= 2\cos\left(\frac{2x+\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)$

$= 2\cos\left(x+\frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)$

$\Longrightarrow \frac{\delta y}{\delta x} = \frac{2\cos\left(x+\frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}$

since $\delta x$ is very small, then $\sin\left(\frac{\delta x}{2}\right) \to \frac{\delta x}{2}$

$\Longrightarrow \frac{\delta y}{\delta x} = \frac{2\cos\left(x+\frac{\delta x}{2}\right)\times\frac{\delta x}{2}}{\delta x}$

As $\delta x \to 0, \frac{\delta y}{\delta x} \to \frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \cos x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \blacksquare$

**(2)** $y = \tan x$ then $\frac{dy}{dx} = \sec^2 x$

**Proof**

Let $y = f(x)$

$\Longrightarrow f'(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$

$= \lim_{h\to 0}\left[\frac{\tan(x+h)-\tan x}{h}\right]$

$= \lim_{h\to 0}\left[\frac{\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}}{h}\right]$

$= \lim_{h\to 0}\left[\frac{\sin(x+h)\cos x-\cos(x+h)\sin x}{h\cos(x+h)\cos x}\right]$

$= \lim_{h\to 0}\left[\frac{\sin(x+h-x)}{h\cos(x+h)\cos x}\right]$

$= \lim_{h\to 0}\left[\frac{\sin(h)}{h\cos(x+h)\cos x}\right]$

Since h is very small then $\sin h \to h$

$= \lim_{h\to 0}\left[\frac{h}{h\cos(x+h)\cos x}\right]$

$= \frac{1}{\cos x\cos x}$

$= \frac{1}{\cos^2 x}$

$= \sec^2 x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \blacksquare$

**(3)** If $y = \sec x$ then $\frac{dy}{dx} = \sec x\tan x$

Proof

As x increases by $\delta x$ then y will increase by $\delta y$

$$\Rightarrow y + \delta y = \sec(x + \delta x)$$
$$\Rightarrow \delta y = \sec(x + \delta x) - \sec x$$
$$\Rightarrow \delta y = \frac{1}{\cos(x + \delta x)} - \frac{1}{\cos x}$$
$$\Rightarrow \delta y = \frac{\cos x - \cos(x + \delta x)}{\cos(x + \delta x)\cos x}$$
$$\Rightarrow \delta y = \frac{-[\cos(x + \delta x) - \cos x]}{\cos(x + \delta x)\cos x}$$
$$\Rightarrow \delta y = \frac{-\left[-2\sin\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)\right]}{\cos(x + \delta x)\cos x}$$
$$\text{Since } \delta x \text{ is very small, } \sin\left(\frac{\delta x}{2}\right) \to \frac{\delta x}{2}$$
$$\Rightarrow \frac{\delta y}{\delta x} = \frac{2\sin\left(x + \frac{\delta x}{2}\right).\frac{\delta x}{2}}{\delta x\cos(x + \delta x)\cos x}$$
$$\text{As } \delta x \to 0, \frac{\delta y}{\delta x} \to \frac{dy}{dx}$$
$$\therefore \frac{dy}{dx} = \frac{\sin x}{\cos x \cos x} = \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) = \sec x \tan x \ldots\ldots\ldots\ldots. \blacksquare$$

**Differentiation of $\sin nx$ and $\cos nx$**

To differentiate functions of the form $\sin nx$ *and* $\cos nx$, we use the chain rule.

For example, if $y = \sin 4x$ , then let $u = 4x$ gives

$$y = sinu \quad and \quad u = 4x$$
$$\Rightarrow \frac{dy}{du} = cosu \quad and \quad \frac{du}{dx} = 4$$

By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = cosu \times 4$$
$$\therefore \frac{dy}{dx} = 4cos4x$$

In practice, we write $\frac{dy}{dx} = cos4x \times (4x)' = 4cos4x$

**Example 5**

Find $\frac{dy}{dx}$ for each of the following functions.

(a) $y = \cos 3x$ (b) $y = \sin(x^2 + 2)$ (c) $y = \cos\sqrt{x}$

Solution

(a) When $y = \cos 3x$ ; $\frac{dy}{dx} = -\sin 3x \times (3x)' = -3\sin 3x$

(b) When $y = \sin(x^2 + 2)$ ; $\frac{dy}{dx} = \cos(x^2 + 2) \times (x^2 + 2)' = 2x\cos(x^2 + 2)$

(c) When $y = \cos\sqrt{x}$ ; $\frac{dy}{dx} = -\sin\sqrt{x} \times \left(\sqrt{x}\right)' = -\frac{1}{2}x^{-\frac{1}{2}}\sin\sqrt{x} = -\frac{1}{2\sqrt{x}}\sin\sqrt{x}$

So since $\frac{d}{dx}(\sin x) = cosx$ and $\frac{d}{dx}(\cos x) = -sinx$
We also have the following integrals:
$\int sinx\, dx = -cosx + c$
$\int cosx\, dx = sinx + c$
**Note** when integrating $sinax$ and $cosax$ we proceed as follows,
$\int cosax\, dx = \frac{1}{a}\sin ax + c$and $\int sinax\, dx = -\frac{1}{a}\cos ax + c$
Proof
Let $u = ax, \frac{du}{dx} = a,\ dx = \frac{du}{a}$
$\int cosax\, dx = \int cosu \frac{du}{a} = \frac{1}{a}\int cosu\, du = \frac{1}{a}\sin u + c = \frac{1}{a}\sin ax + c$

**Example 6**

Find each of the following integrals
(a) $\int sin5x\, dx$ (b) $\int x^2\cos(x^3-2)\, dx$ (c) $\int \sin x\cos x\, dx$
Solution
(a) $\int sin5x\, dx = -\frac{1}{5}\cos 5x + c$
(b) Let $u = x^3 - 2, \frac{du}{dx} = 3x^2, \Rightarrow x^2dx = \frac{du}{3}$

$$\Rightarrow \int x^2\cos(x^3-2)\, dx = \int \cos u\frac{du}{3} = \frac{1}{3}\sin u + c = \frac{1}{3}\sin(x^3-2) + c$$

(c) $\int \sin x\cos x\, dx = \int \frac{1}{2}(sin2x)dx = \frac{1}{2}\left(-\frac{1}{2}cos2x\right) + c = -\frac{1}{2}cos2x + c$

**Differentiation of $sin^n x$ and $cos^n x$**

To differentiate functions of the form $sin^n x$ and $cos^n x$, we also use the chain rule. For example , if $y = cos^2x$, then this can be written as $y = (cosx)^2$. Letting $u = cosx$ gives $y = u^2$ and $u = cosx$
$\therefore \frac{dy}{du} = 2u$ and $\frac{du}{dx} = -sinx$
By chain rule;

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 2u(-sinx)$$
$$\therefore \frac{dy}{dx} = -2cosxsinx$$

**Example 7**

Find$\frac{dy}{dx}$ for each of the following functions.
(a) $y = sin^4x$ (b) $y = (sinx + cosx)^5$ (c) $y = cos^3 2x$

Solution

(a) When $y = sin^4x$ ; $\frac{dy}{dx} = 4sin^3x \times (sinx)' = 4sin^3xcosx$

(b) When $y = (sinx + cosx)^5$ ; $\frac{dy}{dx} = 5(sinx + cosx)^4 \times (sinx + cosx)'$

$$= 5(sinx + cosx)^4(cosx - sinx)$$

(c) When $y = cos^3 2x = (cos2x)^3$ ; $\frac{dy}{dx} = 3(cos2x)^2 \times (cos2x)'$

$$= 3(cos2x)^2(-2sin2x)$$

$$\therefore \frac{dy}{dx} = -6cos^2 2xsin2x$$

**Example 8**

Find $\frac{dy}{dx}$ for each of the following functions

(a) $y = xsinx$ (b) $y = sin^2xcos2x$ (c) $y = \frac{sinx}{cosx}$

Solution

(a) $y = xsinx$

Using the product rule, we have

$$\frac{dy}{dx} = x(sinx)' + sinx(x)'$$

$$= xcosx + sinx$$

(b) $y = sin^2xcos2x$

Using the product, we have;

$$\frac{dy}{dx} = sin^2x(-2sin2x) + cos2x(2sinxcosx)$$

$$= -2sin^2xsin2x + cos2xsin2x$$

$$\therefore \frac{dy}{dx} = sin2x(cos2x - 2sin^2x)$$

(c) $y = \frac{sinx}{cosx}$

Using the quotient rule, we have;

$$\frac{dy}{dx} = \frac{cosx(sinx)' - sinx(cosx)'}{cos^2x}$$

$$= \frac{cosx(cosx) - sinx(-sinx)}{cos^2x}$$

$$= \frac{cos^2x + sin^2x}{cos^2x}$$

$$= \frac{1}{cos^2x}$$

$$\therefore \frac{dy}{dx} = sec^2x$$

**Example 9**

Find each of the following integrals.

(a) $\int \cos x\, sin^2x\, dx$ (b) $\int \frac{\cos x}{\sqrt{2+\sin x}} dx$

Solution

(a) In this case $\cos x$ is a derivative of $\sin x$

Let $u = sinx,\ \frac{du}{dx} = cosx,\ du = \cos x\, dx$

$\Rightarrow \int \cos x\, sin^2x\, dx = \int u^2\, du = \frac{1}{3}u^3 + c = \frac{sin^3x}{3} + c$

(b) We notice that $2 + sinx$ is a function and $\cos x$ is the derivative

$\Rightarrow$Let$u = 2 + sinx,\ \frac{du}{dx} = cosx, du = cosxdx$

$\Rightarrow \int \frac{\cos x}{\sqrt{2+\sin x}} dx = \int \frac{1}{\sqrt{u}} du = 2\sqrt{2 + sinx} + c$

**Exercise 1.2**

1. Differentiate the following with respect to $x$ from first principles

   (a) $y = cos^2x$ (b) $y = \sqrt{sinx}$ (c) $y = \sin x^2$

2. Find $\frac{dy}{dx}$for each of the following

   (a) $y = sin3x$ (b) $y = 8sin\frac{1}{2}x$ (c) $y = \cos\,(x + 3)$ (d) $y = 8\sin\left(\frac{3x-\pi}{2}\right)$

3. Differentiate each of the following with respect to $x$

   (a) $y = sinx^2$ (b) $y = 3sin(2x^3 + 3)$ (c) $y = \sin(x^3 - 3x^2)$ (d) $y = \cos\left(\frac{1}{x}\right)$

4. Find $f'(x)$ for each of the following 2x

   (a) $f(x) = sin^2x$ (b) $f(x) = \frac{1}{cos^2x}$ (c) $f(x) = cos^6\left(\frac{1}{2}x\right)$ (d) $f(x) = 2\sqrt{cos4x}$

5. Find $\frac{dy}{dx}$ for each of the following

   (a) $y = (1 + sinx)^2$ (b) $y = \frac{1}{1+cosx}$ (c) $y = \sqrt{1 - 6sinx}$ (d) $y = (1 + sin^2x)^3$

   (e)$y = -\frac{4}{\sqrt{1-sin6x}}$ (f) $y = (sinx + cos2x)^3$ (g) $y = -\frac{3}{1+cos3x}$

6. Differentiate each of the following with respect to $x$.

   (a) $y = xsinx$ (b) $y = x^2cosx$ (c) $y = xsin^5x$ (d) $y = 3x^2cos^42x$ (e)$y = \frac{1+sin2x}{cos2x}$ (f) $y = \frac{x}{1+cos^2x}$ (g) $y = \frac{1+sinx}{1+cosx}$

7. Show that $\frac{d}{dx}\left\{\frac{cosx+sinx}{cosx-sinx}\right\} = \frac{2}{1-sin2x}$

8. Given that $y = Asin3x + Bcos3x$, where A and B are constants, show that $\frac{d^2y}{dx^2} + 9y = 0$

9. Given that $y = sinx + 3cosx$, show that $cosx\frac{dy}{dx} + ysinx = 1$

10. Given that $y = cos4x$, show that $\frac{d^2y}{dx^2} = -16y$

11. Given that $y = xsin2x$, show that $y\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^2 = y^3 sinx$
12. Find each of the following integrals
    (a) $\int 2cos2x\, dx$ (b) $\int \cos(2x-1)\, dx$ (c) $\int x\cos(x^2)\, dx$
    (d) $\int 2(x-2)\cos(x^2-4x)\, dx$ (e) $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx$ (f) $\int \sin\left(\frac{5x-\pi}{4}\right) dx$
13. Find each of these integrals
    (a) $\int 4\cos x\, sin^3x\, dx$ (b) $\int sinxcos^2x\, dx$ (c) $\int \frac{sinx}{(1+cosx)^2} dx$ (d) $\int \frac{x-sinx}{\sqrt{x^3+2\cos x}} dx$
    (e) $\int (1-cosx)(x-sinx)^2\, dx$ (f) $\int 2sin4x\sqrt{6+cos4x}\, dx$

**Differentiation of $\boldsymbol{tanx}$, $\boldsymbol{cosecx}$, $\boldsymbol{secx}$ and $\boldsymbol{cotx}$**

If $y = tanx, then\ \frac{dy}{dx} = sec^2x$

If $y = cosecx, then \frac{dy}{dx} = -cosecxcotx$

If $y = secx, then \frac{dy}{dx} = secxtanx$

If $y = cotx, then \frac{dy}{dx} = -cosec^2x$

**Example 10**:

Find $\frac{dy}{dx}$ for each of these functions.

(a) $y = tan3x$ (b) $y = \sec(2x^2-1)$ (c) $y = 4cosec^2x$

Solution

(a) When $y = tan3x$; $\frac{dy}{dx} = sec^2 3x(3x)' = 3sec^2 3x$

(b) When $y = \sec(2x^2-1)$ ; $\frac{dy}{dx} = sec(2x^2-1)tan(2x^2-1) \times (2x^2-1)'$

$$= 4xsec(2x^2-1)tan(2x^2-1)$$

(c) Let $u = cosecx, y = 4u^2$

$$\frac{dy}{du} = 8u \ \ and \ \ \frac{du}{dx} = -cotxcosecx$$

Using the chain rule,

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (8cosecx) \times (-cotxcosecx) = -8cotxcosec^2x$$

**Example 11**

Find $\frac{dy}{dx}$ for each of the following functions

(a) $y = 3xcotx$ (b) $y = \frac{x}{cosec2x}$

Solution

(a) Using the product rule, we have

$$\frac{dy}{dx} = 3x(cotx)' + cotx(3x)'$$

$$= -3xcosec^2x + 3cotx$$
$$= 3(cotx - cosec^2x)$$

(b) Using the quotient rule,

$$\frac{dy}{dx} = \frac{cosec2x(x)' - x(cosec2x)'}{cosec^2 2x}$$
$$= \frac{cosec2x - x(-2cosec2xcot2x)}{cosec^2 2x}$$
$$= \frac{cosec2x(1 + 2xcot2x)}{cosec^2 2x}$$
$$= \frac{1 + 2cot2x}{cosec2x}$$

**Note the following**

(i) $\int secxtanxdx = secx + c$

(ii) $\int sec^2x\, dx = tanx + c$

(iii) $\int cotxcosecx\, dx = -cosecx + c$

(iv) $\int cosec^2x\, dx = -cotx + c$

Therefore $\int secaxtanaxdx = \frac{1}{a}secax + c$

**Example 12**

Find each of these integrals,

(a) $\int 2sec3xtan3x\, dx$ (b) $\int xsec^2(1 - x^2)\, dx$ (c) $\int \frac{cosec^2\sqrt{x}}{\sqrt{x}} dx$

Solution

(a) $\int 2sec3xtan3x\, dx = \frac{2}{3}\sec 3x + c$

(b) $\int xsec^2(1 - x^2)\, dx = -\frac{1}{2}\tan(1 - x^2) + c$

(c) $\int \frac{cosec^2\sqrt{x}}{\sqrt{x}} dx = -2\cot\sqrt{x} + c$

**Applications**

**Example 13**

Find the equation of the tangent to the curve $y = x + tanx$ at the point where $x = \frac{\pi}{4}$.

Solution

We need $\frac{dy}{dx}$ when $x = \frac{\pi}{4}$, since $y = x + tanx$, we have

$$\frac{dy}{dx} = 1 + sec^2x$$

When $x = \frac{\pi}{4}, \frac{dy}{dx} = 1 + sec^2\left(\frac{\pi}{4}\right) = 1 + \frac{1}{cos^2\left(\frac{\pi}{4}\right)} = 1 + \frac{1}{\frac{1}{2}} = 3$

The gradient of the tangent line is 3. Therefore the equation of the tangent is of the form $y = 3x + c$

When $x = \frac{\pi}{4}, y = \frac{\pi}{4} + 1$

$\therefore$ The tangent passes through the point $\left(\frac{\pi}{4}, \frac{\pi}{4}+1\right)$

$$\Rightarrow \frac{\pi}{4}+1 = 3\left(\frac{\pi}{4}\right)+c, \quad c = 1-\frac{\pi}{2}$$

The equation of the tangent is $y = 3x+1-\frac{\pi}{2}$ or $2y-6x = 2-\pi$

**Example 14**

A curve is given by the equation $y = 2sin^3t \; and \; x = 2cos^3t$. Find the equation of the normal to the curve at the point where $t = \frac{\pi}{6}$.

Solution

We first need to find $\frac{dy}{dx}$ by differentiation parametrically

When $y = 2sin^3t$, $\frac{dy}{dt} = 6sin^2t\cos t$

When $x = 2cos^3t$, $\frac{dx}{dt} = 6cos^2t(-sint) = -6cos^2tsint$

By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx}$$

$$= \frac{6sin^2t\cos t}{-6cos^2tsint} = -tanx$$

When $t = \frac{\pi}{6}$, $\frac{dy}{dx} = -tan\left(\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

Therefore the gradient of the normal is $\sqrt{3}$. The normal has equation of the form

$$y = \sqrt{3}x + c$$

When $t = \frac{\pi}{6}$, $x = 2cos^3\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{4}$ and $y = 2sin^3\left(\frac{\pi}{6}\right) = \frac{1}{4}$

The normal passes through the point $\left(\frac{3\sqrt{3}}{4}, \frac{1}{4}\right)$

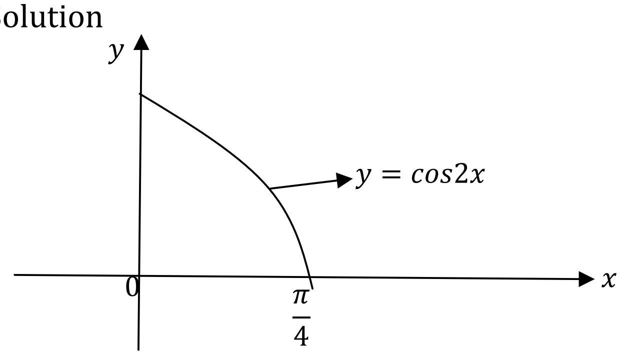
$\Rightarrow \frac{1}{4} = \sqrt{3}\left(\frac{3\sqrt{3}}{4}\right)+c, \quad c = -2$

$\therefore$ the equation of the normal is $y = \sqrt{3}x - 2$

**Example 15**

Find the area enclosed between the curve $y = \cos 2x$, the x-axis and the y-axis

Solution



$A = \int_0^{\frac{\pi}{4}} cos2x\, dx$

$= \left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}}$

$= \left(\frac{\sin 2\left(\frac{\pi}{4}\right)}{2}\right) - \left(\frac{\sin 2(0)}{2}\right)$

$= \frac{1}{2}$

**Example 16**

A curve is given by the equations $y = 2sin^3t$ and $x = 2cos^3t$. Find the equation of the normal to the curve at the point where $t = \frac{\pi}{6}$

Solution

We first find $\frac{dy}{dx}$ by differentiating parametrically.

When $y = 2sin^3t$, $\frac{dy}{dt} = 6sin^2t \cos t$

When $x = 2cos^3t$, $\frac{dx}{dt} = 6cos^2t(-\sin t) = -6cos^2t \sin t$

By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$= -\frac{6sin^2t \cos t}{6cos^2t \sin t} = -\tan t$

When $t = \frac{\pi}{6}, \frac{dy}{dx} = -\tan\left(\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

Therefore the gradient of the normal is $\sqrt{3}$.

The normal has the equation of the form $y = \sqrt{3}x + c$

When $t = \frac{\pi}{6}$, $x = 2cos^3\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{4}$ and $y = 2sin^3\left(\frac{\pi}{6}\right) = \frac{1}{4}$

The normal passes through the point $\left(\frac{3\sqrt{3}}{4}, \frac{1}{4}\right)$

$\Rightarrow \frac{1}{4} = \sqrt{3}\left(\frac{3\sqrt{3}}{4}\right) + c, \quad c = -2$

$\therefore$ The equation of the normal is $y = x\sqrt{3} - 2$

**Differentiation of inverse trigonometric function**

The inverse trigonometric functions can be differentiated as follows,

1. Given that $y = \sin^{-1} x$

   $\Rightarrow siny = x$

$\Longrightarrow cosy\frac{dy}{dx} = 1$

$\Longrightarrow \frac{dy}{dx} = \frac{1}{cosy}$

$\Longrightarrow \frac{dy}{dx} = \frac{1}{\sqrt{cos^2y}}$

$\Longrightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-sin^2y}}$

$\Longrightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

2. Given that $y = \tan^{-1} x$

   $\Longrightarrow \tan y = x$

   $\Longrightarrow sec^2y\frac{dy}{dx} = 1$

   $\Longrightarrow \frac{dy}{dx} = \frac{1}{sec^2y}$

   $\Longrightarrow \frac{dy}{dx} = \frac{1}{1+tan^2y}$

   $\Longrightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$

3. Given that $y = \sec^{-1} x$

   $\Longrightarrow secy = x$

   $\Longrightarrow secytany\frac{dy}{dx} = 1$

   $\Longrightarrow \frac{dy}{dx} = \frac{1}{xtany}$

   $\Longrightarrow \frac{dy}{dx} = \frac{1}{x\sqrt{tan^2y}}$

   $\Longrightarrow \frac{dy}{dx} = \frac{1}{x\sqrt{sec^2y-1}}$

   $\Longrightarrow \frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$

## Exercise 1.3

1. Find $f'(x)$ for each of the following

   (a) $f(x) = tan^2x$ (b) $f(x) = sec^3x$ (c) $f(x) = cot^4x$ (d) $f(x) = -tan^25x$

2. Find $\frac{dy}{dx}$ for each of the following

   (a) $y = (1 + tanx)^2$ (b) $y = \frac{3}{\sqrt{1-cot4x}}$ (c) $y = \frac{1}{(1+sec^2x)}$

3. Differentiate each of the following

   (a) $xtanx$ (b) $x^3cosecx$ (c) $3x^2sec^4x$ (d) $\frac{1}{1+tanx}$ (e) $\frac{x^2}{sec2x}$ (f) $\frac{1+cosecx}{x}$ (g) $\frac{1}{secx+tanx}$

4. Show that $\frac{d}{dx}\left(\frac{tanx}{1+secx}\right) = \frac{1}{1+cosx}$

5. Show that $\frac{d}{dx}\left(\frac{1+cotx}{1-cotx}\right) = \frac{2}{sec2x-1}$

6. Given that $y = secx + 2tanx$, show that $cosx\frac{dy}{dx} + 3tanx = 2y$
7. Given that $y = \frac{x}{1+tanx}$, show that $(1 + tanx)\frac{dy}{dx} + ysec^2x = 1$
8. Given that $y = tan^2x$, show that $\left(\frac{dy}{dx}\right)^2 = 4y(1 + y)^2$
9. Find the equation of the tangent to the curve $y = x + sinx$ at the point where $x = \frac{\pi}{3}$
10. Find the equation of the tangent and the normal to the curve$y = xcosx$ at the point where $x = \pi$.
11. The normals to the curve $y = cos2x$ at the points $A\left(\frac{\pi}{4}, 0\right)$ and $B\left(\frac{3\pi}{4}, 0\right)$meet at the point C. Find the coordinates of the point C, and the area of the triangle ABC. $Ans\left(\left(\frac{\pi}{2}, \frac{\pi}{8}\right), \frac{\pi^2}{32}\right)$.
12. Find the equation of the tangent and the normal to the curve $y = \frac{1}{1+2sinx}$ at the point where $x = \frac{\pi}{6}$
13. Find the coordinates of the points on the curve $y = sinx(2cosx + 1)$, in the range $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, where the gradient is $-\frac{1}{2}$. $Ans\left(\left(-\frac{\pi}{3}, -\sqrt{3}\right), \left(\frac{\pi}{3}, \sqrt{3}\right)\right)$
14. Find the equation of the tangent and the normal to the curve ,$xtany = 6 - x^2$, at the point $\left(2, \frac{\pi}{4}\right)$. $Ans\ (4y + 5x = \pi + 10,\ 20y - 16x = 5\pi - 32)$
15. Given that $y = sec2x$, show that $\frac{d^2y}{dx^2} = 4y(2y^2 - 1)$
16. Find each of the following integrals,
    (a) $\int 2sec^2x\,dx$ (b) $\int 3sec6xtan6x\,dx$ (c) $\int 8xsecx^2\tan x^2\,dx$ (d) $\int 4cosec^28x\,dx$
17. Find each of the following integrals,
    (a) $\int 5sec^2xtan^4x\,dx$ (b) $\int sec^2x(3 + tanx)^3\,dx$ (c) $\int \frac{cosecx\ cotx}{(1+cosecx)^2}dx$
    (d) $\int cosec^25x\sqrt{2 + \cot 5x}\,dx$ (e) $\int sec^5x\tan x\,dx$ (f) $\int sec^2x\sqrt{\cot x}\,dx$
    (g) $\int \frac{tanx}{\sqrt{cos2x+1}}dx$ (h) $\int cosxcosec^3xdx$ (i) $\int x^2\sqrt{x^3 + 1}dx$ (j) $\int \frac{(x-2)}{(x+2)^3(x-6)^3}dx$
    (k) $\int x(x^2 + 5)^6\,dx$ (l) $\int \frac{x}{\sqrt{2x^2-5}}dx$
18. Differentiate the following with respect to x
    (a) $\cos^{-1}x$ (b) $\cot^{-1}x$ (c) $cosec^{-1}x$
19. Find $\frac{dy}{dx}$ in the simplest form,
    (a) $cos^{-1}\left[\frac{1-x^2}{1+x^2}\right]$ (b) $tan^{-1}\left[\frac{1-\sqrt{x}}{1+\sqrt{x}}\right]$ (c) $sec^{-1}\left(\frac{x}{1-x^2}\right)$ (d) $\sin^{-1}\left[\frac{3+5cosx}{5+3cosx}\right]$
20. Given that $y = x - arctanx$, show that $\frac{d^2y}{dx^2} - 2x\left[1 - \frac{dy}{dx}\right]^2 = 0$