LINEAR INTERPOLATION

We shall assume that $A(x_0, y_0)$ and $B(x_2, y_2)$ are two points on a curve. Suppose the equation of curve is either unknown or transcendental (*i.e., if given a value of x, the value of y cannot* [*easily*] *be calculated*), then one method is to assume that the points $A(x_0, y_0)$ and $B(x_2, y_2)$ are connected by a straight line, which we use to estimate the value of *y* given *x*, or vice–versa. This method is hence called **linear interpolation.**

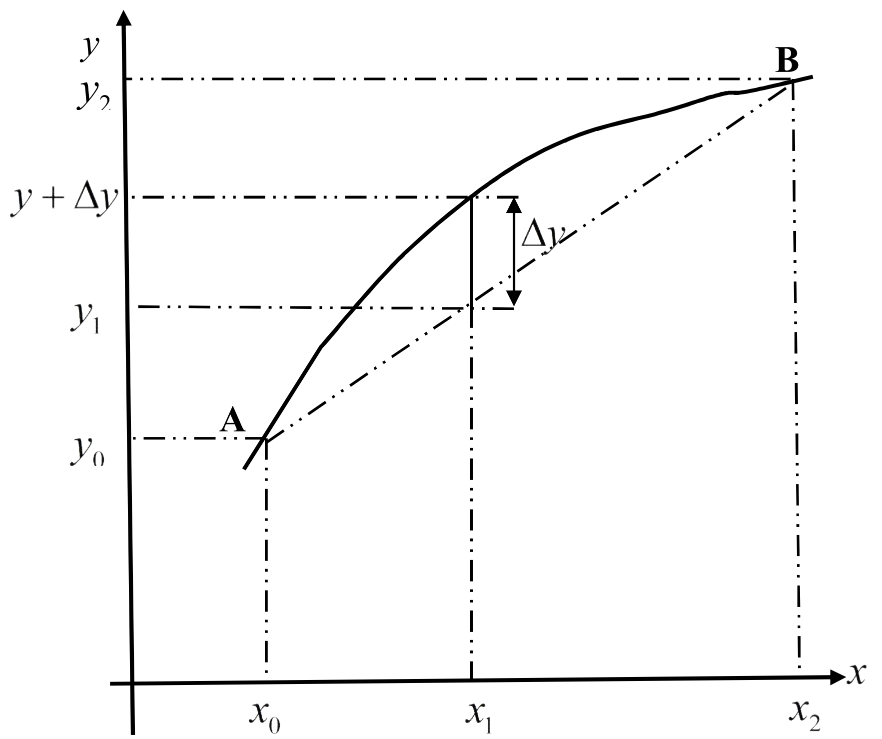


Fig. 1

Using the line AB, x_1 corresponds to y_1. Notice that this method comes with an error, Δy in the value y_1 obtained using linear interpolation.

Since the points (x_0, y_0), (x_1, y_1) and (x_2, y_2) are now assumed to lie on a straight line, then we can use method of;

(a) gradient of a line.

(b) similarity.

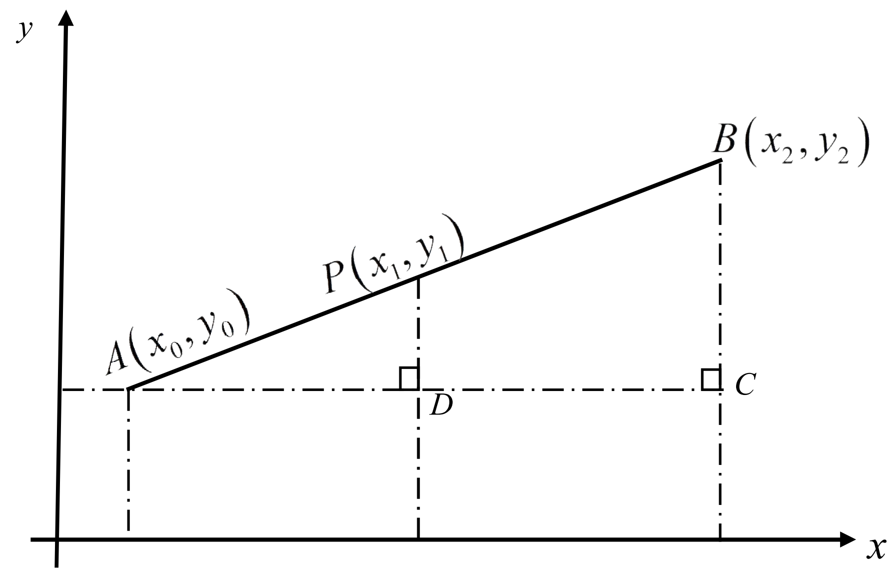


Fig. 2

(a) **<u>Gradient Method</u>**: Gradient of line AB is $\dfrac{y_2 - y_0}{x_2 - x_0}$ (using ΔABC)

or $\dfrac{y_1 - y_0}{x_1 - x_0}$ (using ΔAPD)

Since the gradient of a line is constant, we have

$$\boxed{\frac{y_2 - y_0}{x_2 - x_0} = \frac{y_1 - y_0}{x_1 - x_0}}$$

(b) **<u>Similarity</u>**: The triangles *APD* and *ABC* are similar, hence;

$$\frac{PD}{BC} = \frac{AD}{AC} \text{ i.e. } \boxed{\frac{y_1 - y_0}{y_2 - y_0} = \frac{x_1 - x_0}{x_2 - x_0}}$$

Error reduction using Linear interpolation

Refer to **Fig.1**, the error, using linear interpolation can be reduced using tabular points which are close to each other as illustrated in **Fig.3**.

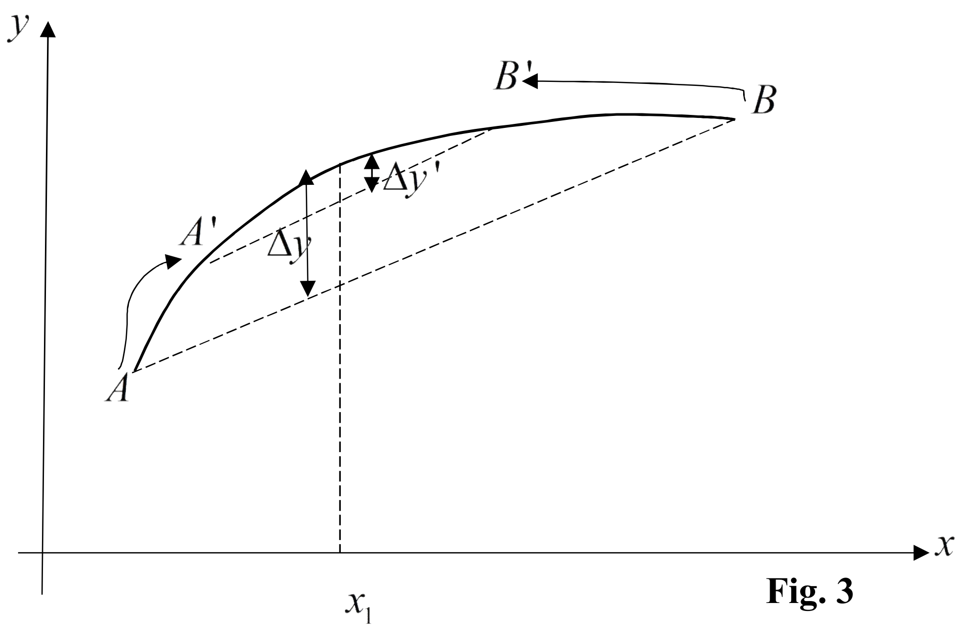


Fig. 3

The points A and B are moved towards each other to the new positions A' and B' respectively, thereby reducing the error from Δy and $\Delta y'$.

Example 1: Consider the function $y = x^2$, using the points

(a) $A(1,1)$ **and** $B(2,4)$**;**

(b) $A(1,1)$ **and** $B(1.5, 2.25)$**, use linear interpolation to find the value of** y **when** $x = 1.2$ **and in each case compute the error in your answer.**

Solutions:

(a)

x	1	1.2	2
y	1	y_1	4

$$\Rightarrow \frac{y_1 - 1}{4 - 1} = \frac{1.2 - 1}{2 - 1}$$

$$\Rightarrow y_1 \approx 1.6 \left(\text{approximate value}\right)$$

Using $y = x^2$; when $x = 1.2$, $y = 1.2^2 = 1.44$ (exact value)

Thus Error $= |1.6 - 1.44| = 0.16$.

(b)

x	1	1.2	1.5
y	1	y_1	2.25

$$\Rightarrow \frac{y_1 - 1}{2.25 - 1} = \frac{1.2 - 1}{1.5 - 1}$$

$\Rightarrow y_1 \approx 1.5$ (new approximate value)

Now Error $= |1.5 - 1.44| = 0.06$ which is a smaller error compared to 0.16 in part (a)

In example 1, the equation of the curve is given, but in most cases the equation is not given.

Example 2:

x	0.1	0.2	0.3
y	0.5	0.8	1.6

Use linear interpolation to find;

(i) y **when** $x = 0.16$

(ii) x **when** $y = 1.0$

Solutions:

(i)

0.1	0.16	0.2
0.5	y	0.8

Now;

$$\frac{y - 0.5}{0.8 - 0.5} = \frac{0.16 - 0.1}{0.2 - 0.1}$$

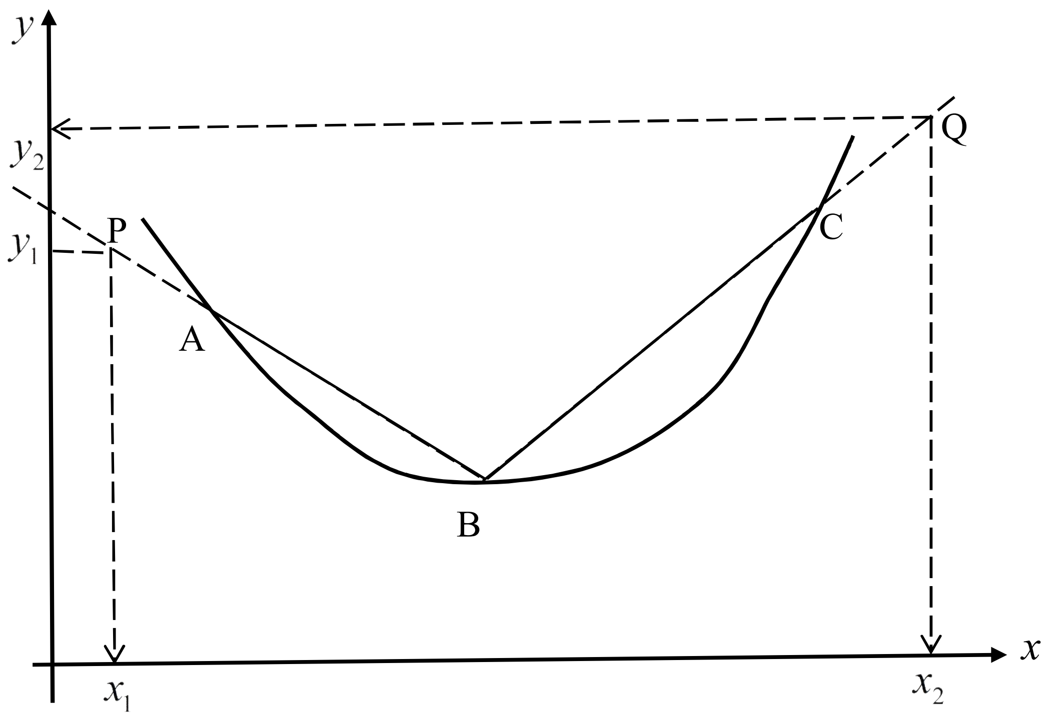
$$\Rightarrow y \approx 0.68$$

(ii)

0.2	x	0.3
0.8	1.0	1.6

$$\Rightarrow \frac{x - 0.2}{0.3 - 0.2} = \frac{1.0 - 0.8}{1.6 - 0.8} \quad \Rightarrow \quad x \approx 0.225$$

LINEAR EXTRAPOLATION



A, B and C are the tabled values. At point P, to estimate y_1 given x_1 or x_1 given y_1 the line $\overrightarrow{BA}$ is produced beyond point A.

At point Q, to estimate y_2 given x_2 or x_2 given y_2, the line $\overrightarrow{BC}$ is produced beyond point C, hence this method is called **Linear Extrapolation**.

We use the methods of gradients and (or) similarity to find the required coordinates.

Example 3: **Study the table below:**

$\theta°$	30	45	60	75
$\cos\theta$	0.8660	0.7071	0.5000	0.2588

Using linear interpolation/extrapolation, estimate;

(a) $\cos 54°$

(b) $\cos^{-1}(0.3420)$

(c) $\cos 80°$

(d) $\cos^{-1}(0.9659)$

Solutions:

(a) $\cos 54° \Rightarrow \theta = 54° \left(\text{given}\right)$ hence we need to get $\cos\theta$.

$54°$ lies between $45°$ and $60°$; i.e.

θ	45	54	60
$\cos\theta$	0.7071	y	0.5000

; $y = \cos 54°$

$$\Rightarrow \frac{y - 0.7071}{0.5 - 0.7071} = \frac{54 - 45}{60 - 45} \text{ (Using interpolation)}$$

$$\therefore \quad \cos 54° \approx 0.5828 \text{ (4dpls)}.$$

(b) Let $\theta = \cos^{-1}\left(0.3420\right) \quad \Rightarrow \quad \cos\theta = 0.3420$, so it is θ that is required.

θ	60	θ	75
$\cos\theta$	0.5000	0.3420	0.2588

$$\Rightarrow \quad \frac{\theta - 60}{75 - 60} = \frac{0.3420 - 0.5000}{0.2588 - 0.5000}$$

$$\theta \approx 69.83°$$

(c)

θ	60	75	80
$\cos\theta$	0.5000	0.2588	y

; $y = \cos 80°$

Notice that $80°$ is outside the given values of θ, so we shall use extrapolation.

$$\Rightarrow \frac{y - 0.2588}{0.2588 - 0.5000} = \frac{80 - 75}{75 - 60}$$

$$\Rightarrow \cos 80° \approx 0.1784$$

(d) Let $\theta = \cos^{-1} 0.9659 \quad \Rightarrow \quad \cos\theta = 0.9659\left(\text{given}\right)$; so we need θ

θ	θ	30	45
$\cos\theta$	0.6959	0.8660	0.7071

0.9659 is to the left of the tabled value 0.8660, so we shall use extrapolation

”

$$\Rightarrow \frac{\theta-30}{30-45}=\frac{0.9659-0.8660}{0.8660-0.7071}$$

$$\Rightarrow \quad \theta \approx 20.57°$$

<u>Example 4</u>: A 'Safe Boda' operator charges x shillings for a distance *dkm,* from the city up to the customer's destination. Some of his charges for known destinations are given in the table below.

d	5	8	10	12
x	3000	5000	8000	10000

Estimate (a) d **when** $x = 16000$

(b) x **when** $d = 2km$

<u>Solution</u>

(a)

d	10	12	d
x	8000	10000	16000

$$\Rightarrow \frac{d-12}{12-10}=\frac{16000-10000}{10000-8000}$$

$$\Rightarrow \quad d = 18km$$

(b)

d	2	5	8
x	x	3000	5000

$$\Rightarrow \frac{x-3000}{3000-5000}=\frac{2-5}{5-8}$$

$$\Rightarrow \quad x = shs.1000$$