# TOPIC 6: SERIES

## Sequences

A list of numbers written in a definite order, with a simple rule by which the terms are obtained is called a sequence

**For example**

1,3,5,7………………………., the simple rule is "add two'' to get the next term.

## Example 1

Write down the next two terms in each of the following sequences.

(a) 1, 2, 4, 8, 16…………
(b) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \dots \dots \dots \dots \dots$
(c) $1, 2, 6, 24, 120 \dots \dots \dots \dots$
(d) $4, 2, 0, -2 \dots \dots \dots \dots \dots$

**Solution**

(a) 1, 2, 4, 8, 16, 32, 64,…………….
We obtain the next terms by doubling the pervious term.
(b) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}$,…………………

$$\text{To obtain the next term use } \frac{n}{n+1},$$

$$\text{where } n = 1,2,3,4, \dots \dots \dots \dots \dots . \text{ n represents the number of terms.}$$

(c) 1, 2, 6, 24, 120, 720, 5040, . To get the next term multiply previous term by n+1, where n=1,2,3,4,…………..
(d) 4, 2, 0, -2, -4, -6 ……………………….
To get the next term subtract two from the previous term.

## Series

The expression obtained by writing the terms of a sequence as a sum is called a series.

Consider the sequence

1, 3, 5, 7, 9, 11, 13, 15, 17, 19, the sum 1+3+5+7+9+11+13+15+19 is called a series. This series has a **finite** number of terms, that is ten terms, it is called a **finite** series.

However, if the series does not end then it is called an **infinite** series.

**Arithmetical progressions (A.P)**

A series, where the next term may be obtained by adding a certain number to the pervious term, is called an **Arithmetical progression** (A.P).

This number added to the pervious term to get the next term is called the ***common difference***, denoted by d

**Example 2**

Find the fourth, ninth, twentieth and $n^{th}$ terms of the A.P with first term 4 and common difference 3.

**Solution**

Common difference = d = 3

First term = 4

Therefore, second term = 4 + 3 = 7

Third term = 7 + 3 = 10

= (4+3) + 3 =10

= 4 + (3x2)

Fourth term =10 + 3

= 4 + (3x2) +3

= 4 + (3x3)

Note that, $4^{th}$ term = 4 + 3 x (4-1)

$3^{rd}$ term = 4 + 3 x (3-1)

Also $2^{nd}$ term = 4 + 3 x (2-1)

$1^{st}$ term = 4 + 3 x (1-1)

Therefore, $9^{th}$ term = 4 + 3 x (9-1)

= 4 + (3x8) = 28

$20^{th}$ term = 4 + 3 (20-1)

= 4 + (3 x 19)

= 61

$N^{th}$ term = 4 + 3(n-1)

= 4 + 3n - 3

= 3n + 1

Note that any term can be obtained using the expression for the nth term.

For example, to get the first term put n=1, to get third, ninth and $20^{th}$ put n = 3, 9 and 20 respectively.

**Examples 3**

Find the sum of the first eighteen terms of the A.P 4 + 10 + …………

**Solution**

This first term = 4 and common difference = 6.

The eighteenth term = 4 + 6(18-1)

= 4 + (6x17)

= 106

Let $S_{18}$ = sum of the first eighteen terms

$S_{18}$ = 4 + 10 + ……….. + 106

Also $S_{18}=106 + 100 + \ldots\ldots\ldots + 4$

$2S_{18}=110 + 110 + \ldots\ldots\ldots + 110$

$= 110 \times 18$

$\therefore S_{18} = \frac{110 X 18}{2} = 990.$

**Exercise 1**

1. Write down the next two terms in each of the following sequences:
   (a) 2, 5, 8, 11, ……..
   (b) $1^3, 2^3, 3^3, 4^3$……………
   (c) $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \ldots \ldots ..$
   (d) $1, \frac{2}{3}, \frac{3}{9}, \frac{4}{27} \ldots \ldots \ldots \ldots \ldots \ldots$
   (e) 1, 4, 9, 16, 25, ……
   (f) 1, -1, 1, -1, ………..

2. Which of the following series are A.Ps? Write down the common difference of those that are.

   $(a) - 2, -5, -8, -11$

   $(b)\ 1 + 1.1 + 1.2 + 1.3,$

   (c) $1^2+2^2+3^2+4^2$,

   (d) $\frac{1}{2} + \frac{5}{6} + \frac{7}{6} + \frac{3}{2}$

   $(e)\ 1 + \frac{1}{2} + \frac{1}{3} +$

   $(f)\ 1\frac{1}{8} + 2\frac{1}{4} + 3\frac{3}{8} + 4\frac{1}{2},$

   $(g)\ n + 2n + 3n$

   $(h)\ 1 + 1.1 + 1.11 + 1.111,$

3. Write down the terms indicated in each of the following A. P. s

   (a) $7 + 6\frac{1}{2} + \cdots \ldots \ldots \ldots .40th, nth$

   $(b)\ 3 + 7 + \cdots \ldots \ldots \ldots \ldots, 90th, (n+1)th$

   $(c)\ 8 + 5 + \cdots \ldots \ldots .14th, 30th$

   $(d)\ 50 + 48 + \cdots \ldots .100th, nth$

   (e) $\frac{1}{4} + \frac{7}{8} + \cdots \ldots \ldots \ldots .10th, nth$

$4.$ Find the number of terms in the following A. Ps

(a) $2+4+6+\cdots\ldots\ldots 46,$ $(b)\ 50+47+44+\cdots..+14,$

$(c)\ 2.7+3.2+\cdots\ldots..+17.7$ $(d)\ 2-9\ldots\ldots\ldots\ldots.-130,$

$(e)\ x+2x+\cdots\ldots\ldots.+nx,$ $(f)\ 2+4+\cdots\ldots\ldots 4n.$

$5.$ Find the sums of the following A. Ps

(a) $1+3+5+\cdots\ldots\ldots.+101$ $(b)\ 71+67+63\ldots\ldots\ldots-53$

(c) $-10-7-4+--+50,$ $(d)\ x+3x+5x+\ ---+21x$

$(e)\ \ a+(a+1)+\ ----+(a+n-1),$

$(f)\ \ a+(a+d)+\ --+\{a+(n-1)d\}$

$6.$ The second term of an A. P is $15,$ and the fifth is $21.$ Find the common difference, the first term and the sum of the first ten terms.

7. The fourth term of an A.P is 18, and the common difference is -5. Find the first term and the sum of the first sixteen terms

8. Find the difference between the sums of the first ten terms of the A.P.s whose first terms are 12 and 8, and whose common differences are respectively 2 and 3

9. Find the sum of the odd numbers between 1000 and 2000

10. An A.P has thirteen terms whose sum is 143. The third term is 5. Find the first term and the sum of the first thirty terms,

**Geometrical progressions**

A series, where the ratio of a term to the pervious one is a constant, is called a **geometrical progression** (G.P)

The constant ratio is called the **common ratio**, denoted by r.

For example, in the series

$3 + 6 + 12 + 24 +$ --------, the common ratio is 2.

For this G.P, the 2nd term $= 3 \times 2^1 = 3 \times 2^{2-1}$

3rd term $= 3 \times 2^2 = 3 \times 2^{3-1}$

4th term $= 3 \times 2^{4-1}$

10th term $= 3 \times 2^{10-1}$

Therefore, nth term $= 3 \times 2^{n-1}$

**Example 4**

Find the fourth, twelveth, twenty-first and $n^{th}$ terms of the G.P, which begins

$5 + 10 +$ ------

**Solution**

First term = 5

Common ratio, $r = \frac{10}{5} = 2$

Fourth term = $5 \times 2^{4-1}$

$= 5 \times 2^3 = 40$

Twelveth term = $5 \times 2^{12-1}$

$5 \times 2^{11} = 10{,}240$

Twenty first term $= 5 \times 2^{21-1} = 5 \times 2^{20} = 5{,}242{,}880$

nth term $= 5 \times 2^{n-1}$

**Example 5**

Find the sum of the first ten terms of the geometrical progression

$1 + 3 + 9 +$ ----------

**Solution**

Let $S_{10}$ be the sum of the first ten terms of the G.P

The $10^{th}$ term = $1 \times 3^{10-1}$

$= 3^9$

$S_{10} = 1 + 3 + 9 + \text{------------} + 3^9$

$3S_{10} = 3 + 9 + \ldots\ldots\ldots\ldots\ldots + 3^9 + 3^{10}$

$3S_{10} - S_{10} = 3^{10} - 1$

$2S_{10} = 3^{10} - 1$

$\therefore\ S_{10} = \frac{1}{2}(3^{10} - 1)$

$= 29{,}524$

**Exercise 2**

1. Find the sums of the following G.P.S as far as the terms indicated. Simplify, but do not evaluate, your answers.

(a) $15 + 5 + 1\frac{2}{3} + \ldots$, 20th term, (b) $1 - 2 + 4 - \ldots$ 5oth term

(c) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ 13th term, (d) $1 - \frac{1}{3} + \frac{1}{9} \ldots$, nth term

2. The first term of a G. P is 10 and the fith is 80. Find the common ratio,

the first term and thesum of the first six terms.

3. The third term of a geometrical progressio is 2, and the fifth is 18.

Find two possible values of the common ratio, and the second term in each case.

4. the three numbers, $n - 2, n, n + 3$ are comsecutive terms of a G. P.

Find n and the term after $n + 3$,

5. Find the ratio of the sum of the first 10 terms of the series
$\log x + \log x^2 + \log x^4 + \log x^8 + \cdots$ to the first term.

**Formulae for the sums of A.Ps and G.Ps**

(a) Consider an A.P with first term, a, and common difference, d.
(b) Let the last term (nth term) be $\ell$

$\ell = \mathrm{a} + (\mathrm{n} - 10\mathrm{d}.$

$S_n = \mathrm{a} + (\mathrm{a} + \mathrm{d}) + (\mathrm{a} + 2\mathrm{d}) + -- + (\ell - \mathrm{d}) + \ell$

$S_n = \ell + (\ell - \mathrm{d}) + (\ell - 2\mathrm{d}) + - + (\mathrm{a} + \mathrm{d}) + \mathrm{a}$

Adding

$2S_n = (\mathrm{a} + \ell) + (\mathrm{a} + \ell) + (\mathrm{a} + \ell) + - \mp (\mathrm{a} + \ell) + (\mathrm{a} + \ell)$

$= \mathrm{n}(\mathrm{a} + \ell)$

$\therefore \quad S_n = \frac{\mathrm{n}}{2}(\mathrm{a} + \ell)$

substituting $\ell = \mathrm{a} + (\mathrm{n} - 1)\mathrm{d}$, we get,

$$S_n = \frac{\mathrm{n}}{2}[\mathrm{a} + \mathrm{a} + (\mathrm{n} - 1)\mathrm{d}]$$

$$\therefore S_n = \frac{\mathrm{n}}{2}\{2\mathrm{a} + (\mathrm{n} - 1)\mathrm{d}\}$$

(b)Consider a G. P, with first term, a, and common ratio r.

The nth term $= \mathrm{a}r^{n-1}$

Therefore,

$$S_n = \mathrm{a} + \mathrm{ar} + \mathrm{a}r^2 + -- + \mathrm{a}r^{n-2} + \mathrm{a}r^{n-1}$$

$$\mathrm{r}S_n = \quad \mathrm{ar} + \mathrm{a}r^2 + ------ + \mathrm{a}r^{n-1} + \mathrm{a}r^n$$

subtracting,

$$\mathrm{r}\,S_n - S_n = \mathrm{a}r^n - \mathrm{a}$$

$$(\mathrm{r} - 1)S_n = \mathrm{a}(r^n - 1)$$

$$\therefore S_n = \frac{\mathrm{a}(r^n - 1)}{(\mathrm{r} - 1)} --- (1)$$

If we multipy numerator and denominator of equation (1) by $-1$ we get

$$S_n = \frac{a(1 - r^n)}{1 - r} - - - (2)$$

formula (1)is converient if $r > 1$ and formula 2 is converient if $r < 1$.

**Example 6**

In an A. P, the sum of the first five terms is 30, and the third term is equal to the sum of the first two. Write down the first five terms of the progresssion.

**Solution**

Let a and d be the first term and common difference respectively.

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$S_5 = \frac{5}{2}[2a + (5 - 1)d]$$

$S_5$ = 5a +10d

∴ 5a + 10d = 30

∴ a + 2d = 6................................(1)

Also 3rd term = 1st +2rd term

a+2d = a + a + d

∴ a = d……………….(2)

Subtitute for a in (1)

$d + 2d = 6$

∴ $3d = 6$

∴ d = 2

and a = 2

Therefore, the first five terms of the A.P are 2, 4, 6, 8, 10.

**Example 7**

In a G.P, the sum of the second and third terms is 9, and the seventh term is eight times the fourth. Find the first term, the common ratio, and the fifth term.

**Solution**

Let the first term and common ratio be a and r respectively,

$2^{nd}$ term + $3^{rd}$ term = 9

$\therefore$ $ar + ar^2 = 9$……………….(1)

$7^{th}$ term = 8 x $4^{th}$ term

$ar^6 = 8ar^3$ …………………..(2)

From equation (2)

$ar^6 - 8ar^3 = 0$

$ar3(r3 - 8) = 0$

$\therefore \; r = 2, \quad since\ ar3 \# 0$

$substitute\ for\ r\ in\ (1)$

$2a + 4a = 9$

$6a = 9$

$\therefore \quad a = \frac{9}{6} = \frac{3}{2}$

$Therefore\ 5th\ term = ar4$

$= \frac{3}{2} x\ 24$

$= 24$

**Example 8**

The second , fourth and eighth terms of an A. P are geometricial prgression, and the sum of the third and fifth terms is 20. Find the first four terms of the progression.

**Solution**

Let the first term and common difference of the A. P be a and d respectively

The second term = a + d

$4^{th}$ term = a + 3d

$8^{th}$ term = a+7d

The three form a G.P

$\therefore \frac{(a+3d)}{(a+d)} = \frac{a+7d}{a+3d}$

$(a + 3d)^2 = (a + d)(a+7d)$

$a^2 + 6ad + 9d^2 = a^2 + 7ad + ad + 7d^2$

$\therefore\ 2d^2 - 2ad = 0$

$\therefore\ 2d(d-a) = 0$

But d #0, therefore

a = d……………………….(1)

also, $3^{rd}$ term + $5^{th}$ term =20

a+ 2d + a + 4d = 20

2a + 6d = 20

$\therefore$ a + 3d = 10……………………(2)

Substitute for a in (2)

d + 3d = 10

4d = 10

$d = 2\frac{1}{2}$ $and$ $a = 2\frac{1}{2}$

Therefore the first four terms are $2\frac{1}{2}, 5, 7\frac{1}{2}, 10$.

**Arithmetic and geometric means**.

If three numbers a, b, and c are in arithmetical progression, b is called the **arithmetic mean** of a and c.

The common difference = b-a or c-b

Therefore,

$$b - a = c - b$$

$$\therefore\ 2b = a + c$$

$$b = \frac{(a+c)}{2}$$

This is the average of a and c.

If, however, a, b and c are in geometrical progression, b is called the geometric mean.

The common ratio = $\frac{b}{a}$ $or$ $\frac{c}{b}$

$$\therefore\ \frac{b}{a} = \frac{c}{b}$$

$$b^2 = ac$$

$$\therefore\ b = \sqrt{ac}$$

Therefore the geometric mean of a and c is $\sqrt{ac}$.

**The harmonic mean.**

The reciprocal of the harmonic means of two numbers is the arithmetic means of their reciprocals.

Therefore, if h is **harmonic** mean of a and b, $\frac{1}{h}\ (\frac{\frac{1}{a}+\frac{1}{b}}{2})$

$$\therefore h = \frac{2ab}{a+b}$$

**Exercise 3**

1. The fifth term of an A.P is 17 and the third term is 11. Find the sum of the first seven terms.
2. The fourth of a G.P is -6 and the seventh term is 48. Write down the first three terms of the progression.
3. The sum of the second and fourth terms of an A.P is 15, and the sum of the fifth and sixth terms is 25. Find the first term and the common difference.
4. The fourth term of an A.P is 15, and the sum of the first five terms is 55. Find the first term and common difference, and write down the first five terms.
5. Find how many terms of the G.P 1+3+9+………… are required to make a total of more than a million.
6. The sum of n terms of a certain series is $4^n$-1 for all values of n. Find the first three terms and the nth term, and show that that the series is a geometrical progression.
7. (a) Find (i) the arithmetical mean, (ii) the geometric mean 4 and 64.
   (b) Find the harmonic mean of 10 and 30

**Proof by induction**

In proof by induction, it is shown that if the result holds for some particular value of n, say k, then it also holds for n = k+1.
It is then verified that the result does hold for some value of n, usually 1 or 2.

**Example 9**

Prove by induction that

$1^2 + 2^2 + \ldots\ldots + n^2 = \frac{1}{6}\, n(n+1)(2n+1)$

**Solution**

$1^2 + 2^2 + \ldots\ldots + n^2 = \frac{1}{6}\, n(n+1)(2n+1)$

$Suppose\ the\ result\ holds\ for\ a\ particular\ value\ of\ n, say\ k, that\ is,$

$1^2 + 2^2 + \ldots.. + k^2 = \frac{1}{6}\ k(k+1)(2k+1)$

Adding the next term of the series, $(k+1)^2$, to both sides, we obtain

$1^2 + 2^2 + \ldots\ldots + k^2 + (k+1)^2 = \frac{1}{6} k(k+1)(2k+1) + (k+1)^2$

$= \frac{1}{6}(k+1)[k\ (2k+1) + 6\ (k+1)]$

$= \frac{1}{6}(k+1)\ (2k^2 + k + 6k + 6)$

$= \frac{1}{6}(k+1)(2k^2 + 7k + 6)$

$= \frac{1}{6}(k+1)(2k^2 + 4k + 3k + 6)$

$= \frac{1}{6}(k+1)((2k\ (k+2) + 3(k+2)\ )$

$= \frac{1}{6}(k+1)(k+2)(2k+3)$

$= \frac{1}{6}(k+1)[(k+1)+1][2(k+1)+1]$

This is the formula with n = k+1. Therefore, if the result holds for n = k, then it also holds for n = k+1

If n = 1

L.H.S = $1^2$ =1

$\mathrm{R.H.S} = \frac{1}{6}\ \mathrm{x}\ 2\ \mathrm{x}\ 3 = 1$

L.H.S = R.H.S, therefore the result is true for n = 1

If n = 2,

L.H. S = $1^2 + 2^2 = 5$

$\mathrm{R.H.S} = \frac{1}{6}\ \mathrm{x}\ 2\ \mathrm{x}\ 3\ \mathrm{x}\ 5 = 5$

L.H.S = R.H.S, therefore the result is true for n=2

Therefore, since the result is true for n =1, n = 2, ……………., n = k and n = k+1, it follows by induction, that the result is true for all positive integral values of n.

**Example 10**

Prove by induction that $3^{2n}$-1 is divisible by 8 for all positive values of n

**Solution**

Let f(n) = $3^{2n}-1$

If n = 1,

$f(1) = 3^2 - 1 = 9 - 1$

$= 8 = 8 \times 1$

This is a multiple of 8. Therefore, result is divisible by 8

If n = 2,

$f(2) = 3^4 - 1 = 81 - 1$

$= 80 \quad = 8 \times 10$

A multiple of 8. Therefore, result is divisible by 8

Suppose $3^{2n}-1$ is divisible by 8 for n = k,

Let f(k) = 8p, a multiple of 8 i.e. $3^{2k}-1 = 8p$

$f(k+1) - f(k) = 3^{2(k+1)} - 1 - (3^{2k} - 1)$

$= 3^{2k+2} - 1 - 3^{2k} + 1$

$= 3^{2k+2} - 3^{2k}$

$= 3^{2k}(3^2 - 1)$

$f(k+1) - f(k) = 8(3^{2k})$

But f(k) = 8p

$\therefore f(k+1) = 8p + 8(3^{2k})$

$= 8(p+3^{2k})$

This is a multiple of 8.

Therefore, f (k+1) is divisible by 8.

Since $3^{2n}-1$ is divisible by 8 for n = 1, 2, ……, k and k+1, by induction it is divisible by 8 for all positive integral values of n.

**Exercise 4**

Prove the following results by induction:

1. $1^3+2^3+\ldots\ldots n^3=\frac{1}{4}n^2(n+1)^2$
2. $3+8+---+(n^2-1)=\frac{1}{6}n\,(n-1)(2n+5)$
3. $1 \times 3 + 2 \times 4 + \ldots\ldots + n\,(n+2) = \frac{1}{6}n\,(n+1)(2n+7)$
4. $1^2 + 3^2 + 5^2 + \ldots\ldots + (2n-1)^2 = \frac{1}{3}n(4n^2-1)$
5. Prove by induction that $6^n-1$ is divisible by 5 for all positive integral values of n.

**The $\sum$ *notation***

The notation is used to write series in short form.

For example $1^3 + 2^3 + 3^3 + \ldots.. + m^3 = \sum m^3$

**Example 11**

Write in full: (i) $\sum_1^n m^m$, (ii) $\sum_3^6 \frac{(-1)^m}{m}$

**Solution**

(i) $\sum_1^n m^m = 1^1 + 2^2 + 3^3 + \ldots\ldots\ldots\ldots + n^n$

$= 1 + 2^2 + 3^3 + \ldots\ldots\ldots\ldots + n^n$

(ii) $\sum_3^6 \frac{(-1)^m}{m} = \frac{(-1)^3}{3} + \frac{(-1)^4}{4} + \frac{(-1)^5}{5} + \frac{(-1)^6}{6}$

$= -\frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6}$

**Further series**

Certain series can be summed by means of the results:

(i) $1 + 2 + \ldots. + n = \frac{1}{2}n(n+1)$

(ii) $1^2 + 2^2 + \ldots. + n^2 = \frac{1}{6}n(n+1)(2n+1)$

(iii) $1^3 + 2^3 + \ldots.. + n^3 = \frac{1}{4}n^2(n+1)^2$

**Example 12**

Find the sum of the series:

$1^3 + 3^3 + 5^3 + \ldots. + (2n+1)^3$

**Solution**

$1^3 + 3^3 + 5^3 + \ldots. + (2n+1)^3$

$= 1^2 + 2^3 + 3^3 + 4^3 + 5^3 + \ldots. + (2n+1)^3 - 2^3 - 4^3 - \ldots. - (2n)^3$

$= \frac{1}{4}(2n+1)^2\ (2n+1+1)^2 - 2^3(1^3 + 2^3 + - - +n^3)$

$= \frac{1}{4}\ (2n+1)^2\ \big(2(n+1)\big)^2(n+1)\big)2 - 8\ \text{x}\frac{1}{4}n^2(n+1)^2$

$= (2n+1)^2(n+1)^2 - 2n^2(n+1)^2$

$= (n+1)^2[4n^2 + 4n + 1 - 2n^2]$

$= (n+1)^2\ (2n^2+4n+1).$

**Example 13**

Find the sum of n terms of the series 1x 2+2x3+3x4+4x5 +……

**Solution**

The m $^{th}$ term = m (m+1)

$= m^2 + m$

Therefore, the required sum = $1^2 + 1$

$+ 2^2 + 2$

$+ 3^3 + 3$

$+ 4^2 + 4$

+--------

$+ n^2 + n$

$= (1^2 + 2^2 + 3^3 + 4^2 + \text{--------} + n^2) + (1 + 2 + 3 + 4 + \ldots\ldots + n)$

$= \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)$

$= \frac{1}{6}n(n+1)[(2n+1)+3]$

$= \frac{1}{6}n(n+1)(2n+4)$

$= \frac{1}{3}n(n+1)(n+2)$

**Infinite geometrical progressions**

Consider a G.P whose first term is 1 and common ratio $\frac{1}{3}$, that is,

$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + --- + \frac{1}{3n-1}$

For r < 1, use sn=a $\left(\frac{1-rn}{1-r}\right)$

Therefore

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + --+ \frac{1}{3n-1} = 1\ \frac{\left[1 - \left(\frac{1}{3}\right)n\right]}{1 - \frac{1}{3}}$$

$$= \frac{\left[1 - \left(\frac{1}{3}\right)n\right]}{\frac{2}{3}}$$

$$= \frac{2}{3} - \frac{1}{2}\left(\frac{1}{3n-1}\right)$$

As n tends to infinity,

$\frac{1}{3n-1}$ *tends to zero, therefore, sum of the first n terms tends to* $\frac{3}{2}$.

$$\therefore 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + --+ \frac{1}{3n-1} = \frac{3}{2}$$

The limit $\frac{3}{2}$ is called its sum *to infinity*.

In general, for any geometrical progression whose common ratio r lies between -1 and +1, that is, $|r|<1$,

As n tends to $\infty$, $r^n$ tends to 0.

Therefore, the sum to infinity of the series

$$a + ar + ar^2 + --+ar^{n-1} = \frac{a}{1-r}$$

**Example 14**

Express the following recurring decimals as rational numbers:

(a) $0.\dot{1}\dot{2}$ and (b) $0.\dot{5}$

**Solution**

(a) $0.\dot{1}\dot{2}$=0.12121212---------

$=\frac{12}{100} + \frac{12}{10000} + \frac{12}{1000000}$+-----

This is a G.P, with a = $\frac{12}{100}$ *and* $r = \frac{1}{100}$

Therefore, $s_\infty = \frac{12}{100}\left[\frac{1-\left(\frac{1}{100}\right)n}{1-\frac{1}{100}}\right]$

As n $\rightarrow \infty$, $\left(\frac{1}{100}\right)^n$ tends to 0

$$s_\infty = \frac{12}{100} \text{ x } \frac{100}{99}$$

$= \frac{4}{33}$

$\therefore\ 0.\dot{1}\dot{2} = \frac{4}{33}$

(b) $0.\dot{5} = 0.5555$-------

$= \frac{5}{10} + \frac{5}{100} + \frac{5}{1000} + \frac{5}{10000} + ----$

This is a G.P, with a = $\frac{5}{10}$ $and\ r = \frac{1}{10}$.

Therefore, $S_n = \frac{5}{10}\left[\frac{1-\left(\frac{1}{10}\right)n}{1-\frac{1}{10}}\right]$

$S\infty = \frac{5}{10}\left(\frac{1}{\frac{9}{10}}\right)$

$= \frac{5}{10} \text{ x } \frac{10}{9}$

$\therefore\ 0.\dot{5} = \frac{5}{9}$

**Exercise 5**

1. Using the results

   (i) $1 + 2 + 3 + \ldots\ldots + n = \frac{1}{2}n\,(n+1)$,

   (ii) $1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$,

   (iii) $1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \frac{1}{4}n^2(n+1)^2$

   Find the sums of the following series

(a) $1^2 + 2^2 + 3^2 + \ldots. + (n-1)^2$

(b) $3 + 5 + 7 + \ldots.. + (2n+1)$

(c) $2 + 5 + 10 + \ldots. + (n^2+1)$

(d) 1x3 + 2x4 + 3x5 + 4x6 +….to n terms

(e) $2 + 10 + 30 + \ldots\ldots + (n^3+n)$

2. (a) If the sum to infinity of a G.P is three times the first term, what is the common ratio?

(b) The sum to infinity of a G.P is 4 and the second term is 1. Find the first, third and fourth terms.

(c)The second term of a G.P is 24 and its sum to infinity is 100. Find the two possible values of the common ratio and the corresponding first terms.

3. How many terms of the arithmetical progression $2 + 3\frac{1}{4} + 4\frac{1}{2} + ---$ are needed to make a total of 204?

4. The sum of n terms of a certain series is $3n^2 + 10n$ for all values of n. Find the $n^{th}$ term and show that the series is an A.P.

5. Find the sum of the series $2 + 6 + \dots + (n^2 - n)$.

6. The eighth term of an A.P is twice the third term, and the sum of the first eight terms is 39. Find the first three terms of the progression and show that its sum to n terms is $\frac{3}{8}n\,(n+5)$.