$\left(x + \sqrt{4 + (\ldots}\right.$

Second derivatives

Suppose y has been given as a function of x, then the first derivative of y with respect to x is denoted by $\frac{dy}{dx}$. The result of differentiating $\frac{dy}{dx}$ with respect to x is called second derivative. Proceeding further we have third derivative, however, according to our coverage, we shall end on second derivative.
Now, the second derivative of y with respect to x is

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Suppose that the displacement s of the moving object is given as function of time t, then the first derivative of s with respect to t will give us velocity. I.e $v = \frac{ds}{dt}$ and the second derivative of s with respect to time, t gives us acceleration. I.e $a = \frac{d^2s}{dt^2}$

(see the application of differentiation for details).

If y is a function of x given by $y = f(x)$;
Then the first derivative of $f(x)$ with respect to x is denoted by $f'(x)$ and the second derivative by $f''(x)$

Note: throughout this section, try to avoid use of **Product** and **Quotient** rules.

Examples

Find $\frac{d^2y}{dx^2}$ for each of the following

1. $y = x^5$
2. $y = x^2(x+3)$
2. $y = x^3(4 - x^2)$
3. $y = \sqrt{x}(5 + x)$
4. $y = 6\sqrt{x}(x^3 - 2x + 1)$
5. $y = 6x^{\frac{1}{3}}(2x - 5)$
6. $y = 2x^{\frac{1}{4}}(x^2 - 2)$
7. $y = (x+3)(x-4)$
8. $y = (x+4)^2$
9. $y = (x+5)(2x-1)$
10. $y = 2(x-3)^2$

Find $f''(x)$ for each of the following

11. $f(x) = 2x^2(x-1)^2$
12. $f(x) = \frac{\sqrt{x}+1}{\sqrt{x}}$

Solution

1. $\dfrac{dy}{dx} = 5x^4$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(5x^4\right) = 20x^3$$

2. $y = x^3(4 - x^2)$

$$y = 4x^3 - x^5$$

$$\frac{dy}{dx} = 12x^2 - 5x^4$$

$$\frac{d^2y}{dx^2} = 24x - 20x^3$$

3. $y = \sqrt{x}(5 + x)$

$$y = 5x^{\frac{1}{2}} + x^{\frac{3}{2}}$$

$$\frac{dy}{dx} = \tfrac{5}{2}x^{-\frac{1}{2}} + \tfrac{3}{2}x^{\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = -\tfrac{5}{4}x^{-1} + \tfrac{3}{4}x^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = -\frac{5}{4x} + \frac{3}{4\sqrt{x}}$$

4. $y = 6\sqrt{x}(x^3 - 2x + 1)$

$$y = 6x^{\frac{7}{2}} - 2x^{\frac{3}{2}} + x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 21x^{\frac{5}{2}} - 3x^{\frac{1}{2}} + \tfrac{1}{2}x^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(21x^{\frac{5}{2}} - 3x^{\frac{1}{2}} + \tfrac{1}{2}x^{-\frac{1}{2}}\right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\,\frac{105}{2}x^{\frac{3}{2}} - \frac{3}{2}x^{-\frac{1}{2}} - \tfrac{1}{4}x^{-1}$$

$$\frac{d^2y}{dx^2} = \frac{105}{2}x^{\frac{3}{2}} - \frac{3}{2\sqrt{x}} - \frac{1}{4x}$$

5. $y = 6x^{\frac{1}{3}}(2x - 5)$

Turning /stationary points

A point on a curve such that it's gradient is zero i.e. $\frac{dy}{dx}=0$, is called a stationary point. At this point, the tangent to the curve is horizontal and the curve is 'flat'.

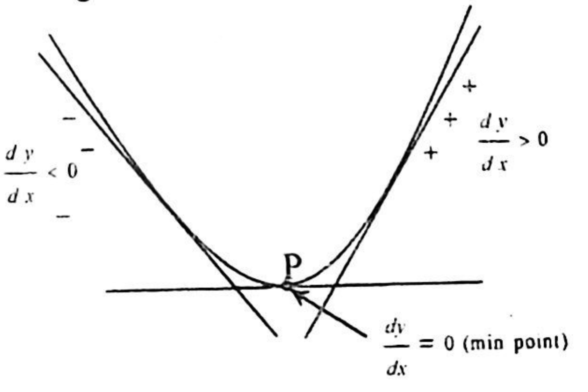
There are three types of stationary points;

- Minimum point
- Maximum point
- Point of inflexion

However, in this book, we shall only look at minimum and maximum as stipulated by the syllabus.

Minimum point

This is obtained at the lowest point of the curve (valley like). In this case, the gradient of the curve is negative to the left of the turning point and positive to the right.



In summary we have:-

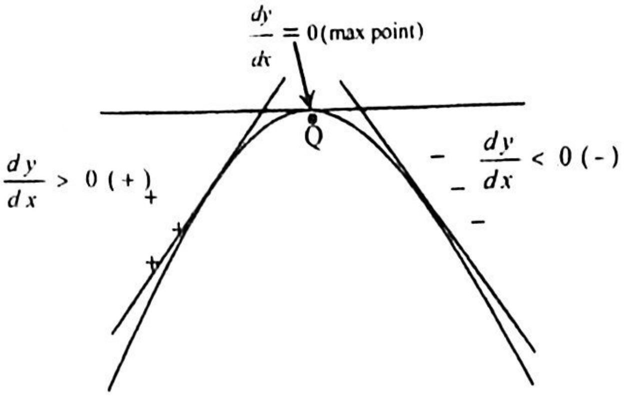
To the left of P	At point P	To the right of P
$\frac{dy}{dx}<0(-)$	$\frac{dy}{dx}=0$	$\frac{dy}{dx}>0(+)$

When using second derivative to investigate whether the turning point is minimum, in this case, considering point P, the gradient of the derived function is positive.

i.e. at point P, $\frac{d^2y}{dx^2}>0$ (positive value)

Maximum point

This is obtained at the highest point of curve (mountain like). In this case, the gradient of the curve is positive to the left and negative to the right of turning point.



In summary, we have;

To the left of Q	A point Q	To the right of Q
$\frac{dy}{dx}>0(+)$	$\frac{dy}{dx}=0$	$\frac{dy}{dx}<0(-)$

When using second derivative to investigate whether the turning point is maximum, in this case considering point Q, the gradient of the derived function is negative, i.e at point Q, $\frac{d^2y}{dx^2}<0$

(negative value)

Examples

1.Find the coordinates of the stationary points on the curve $y=x^3+3x^2+1$ and determine their nature.

Solution

$y=x^3+3x^2+1$

$\frac{dy}{dx}=3x^2+6x$

At stationary point,

$\frac{dy}{dx}=0$

$\therefore 3x^2+6x=0$

$3x(x+2)=0$

Either $3x=0,\ x=0$

Or $x+2=0,\ x=-2$

When $x=0,\ y=1$

$(x,y)=(0,1)$

When $x=-2,\ y=(-2)^3+3(-2)^2+1=5$

$(x,y)=(-2,5)$

Hence the coordinates of the stationary points are (0,1) and (-2,5).

Determining the nature of turning points;

For the point (0, 1)

x	-0.5	0	0.5
$\frac{dy}{dx}$	-2.25	0	3.75

Negative min Positive

The stationary point (0, 1) is minimum

Alternatively

Using second derivative to determine the nature of the stationary points,

$\frac{dy}{dx} = 3x^2 + 6x$

$\frac{d^2y}{dx^2} = 6x + 6$

At point (0, 1)

$\frac{d^2y}{dx^2} = 6(0) + 6 = 6$

Since $\frac{d^2y}{dx^2} > 0$, hence the stationary point (0, 1) is minimum

For the point (-2, 5)

x	-2.5	-2	-1.5
$\frac{dy}{dx}$	3.75	0	-2.25

Positive max Negative

$\therefore$ the stationary point (-2, 5) is maximum

Alternatively

Using second derivative

$\frac{dy}{dx} = 3x^2 + 6x$

$\frac{d^2y}{dx^2} = 6x + 6$

At point (-2, 5)

$\frac{d^2y}{dx^2} = 6(-2) + 6 = -6$

Since $\frac{d^2y}{dx^2} < 0$, hence the stationary point (-2, 5) is maximum

2. Find and distinguish between the nature of the turning points of the curves, $y = x^3 - x^2 - 5x + 6$

Solution

(i) $y = x^3 - x^2 - 5x + 6$

$\frac{dy}{dx} = 3x^2 - 2x - 5$

At turning points, $\frac{dy}{dx} = 0$

$\Rightarrow 3x^2 - 2x - 5 = 0$

$3x^2 + 3x - 5x - 5 = 0$

$3x(x+1) - 5(x-1) = 0$

$(x+1)(3x-5) = 0$

Either $x + 1 = 0$, $x = -1$

or $3x - 5 = 0$, $x = \frac{5}{3}$

When $x = -1$, $y = (-1)^3 - (-1)^2 - 5(-1) + 6 = 9$

when, $x = \frac{5}{3}$, $y = \left(\frac{5}{3}\right)^3 - \left(\frac{5}{3}\right)^2 - 5\left(\frac{5}{3}\right) + 6 = -\frac{13}{27}$

Hence turning point are (-1, 9) and $\left(\frac{5}{3}, -\frac{13}{27}\right)$

For (-1, 9)

x	-2	-1	0
$\frac{dy}{dx}$	11	0	-5

Positive max Negative

The stationary (-1, 9) is maximum

For $\left(\frac{5}{3}, -\frac{13}{27}\right)$

x	1	$\frac{5}{3}$	2
$\frac{dy}{dx}$	- 4	0	3

Negative min Positive

$\therefore$ the stationary point (-1, 9) is minimum

Alternatively

Using second derivative to determine the nature of turning points,

$\frac{dy}{dx} = 3x^2 - 2x - 5$

$\frac{d^2y}{dx^2} = 6x - 2$

At point (-1, 9)

$\frac{d^2y}{dx^2} = 6(-1) - 2 = -8$

Since $\frac{d^2y}{dx^2} < 0$, therefore the point (-1, 9) is maximum

At point $\left(\frac{5}{3}, -\frac{13}{27}\right)$

$\frac{d^2y}{dx^2} = 6\left(\frac{5}{3}\right) - 2 = 8$

Since $\frac{d^2y}{dx^2} > 0$, therefore the point $\left(\frac{5}{3}, -\frac{13}{27}\right)$ is minimum

Alternatively

Using second derivative for test;

CURVE SKETCHING

Whenever, we are told to sketch a graph of any kind, there are basic procedures that must be followed.
These procedures will depend on the nature of the type of graph at hand.

However, since under this topic, we are only going to look at quadratic equations, there are two approaches that will employed. These are:

A. Using the general procedure of finding intercepts and turning point by differentiation.

Steps taken

(i) Find axes intercepts
- The curve cuts the $x-$axis at points for which $y=0$
- The curve cuts the $y-$axis at points for which $x=0$

(ii) The curve passes through the origin if $(x,y)=(0,0)$

(iii) Find the turning points and their nature

Illustrative examples

1. Sketch the graph of $y=5+4x-x^2$

Solution

Steps taken

- Finding intercepts;

x - intercept: when $y=0$

$5+4x-x^2=0$

$5+5x-x-x^2=0$

$5(1+x)-x(1+x)=0$

$(5-x)(1+x)=0$

Either, $5-x=0, x=5$

Or, $1+x=0, x=-1$

Hence the curve cuts the $x-$axis at points $(-1,0)$, $(5,0)$

y – intercept: when $x=0, \Rightarrow y=5$,

Hence the curve cuts the $y-$axis at point $(0,5)$

- Finding turning point(s)

$\frac{dy}{dx}=4-2x$

At turning point, $\frac{dy}{dx}=0$

$4x-2=0,\ x=2$

When $x=2, y=5+4(2)-2^2=9$

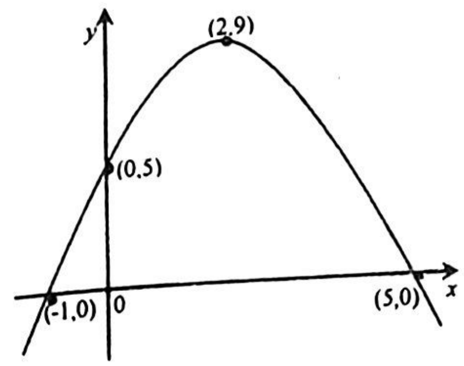
Hence turning point = $(2, 9)$

Finding the nature of turning point

$\frac{dy}{dx}=4-2x$

$\frac{d^2y}{dx^2}=-2$

Since $\frac{d^2y}{dx^2}<0$, Hence the turning point is maximum



2. Given that curve $y=x^2+2x-8$

(a) Find the intercepts

(b) Find the turning point and distinguish it

(c) Sketch the curve

Solution

Steps taken

(a) Finding intercepts;

x - intercept: when $y=0$

$x^2+2x-8=0$

$x^2+4x-2x-8=0$

$x(x+4)-x(x+4)=0$

$(x+4)(x-2)=0$

Either $x-2=0, x=2$

Or $x+4=0, x=-4$

Hence the curve cuts the $x-$axis at points $(-4,0)$, $(2,0)$

$y-$intercept: when $x=0, \Rightarrow y=-8$,

Hence the curve cuts the $y-$axis at point $(0,-10)$

(b) Finding turning point(s)

$\frac{dy}{dx}=2x+2$

At turning point, $\frac{dy}{dx}=0$

$2x+2=0,\ x=-2$

When $x=-2, y=4+2(-2)-8=-8$

Hence turning point = $(-2, -8)$

Finding the nature of turning point

$\frac{d^2y}{dx^2}=2$

Since $\frac{d^2y}{dx^2}>0$, Hence the turning point is minimum

(c)

CHAPTER 7

Integration

Definition
It is the reverse process of differentiation.
This simple definition can be used to generate the general rules of integration.
The learner by now knows the following concepts from differentiation:

(a) **Polynomial functions**

(i) $\frac{d}{dx}(x) = 1 \Rightarrow \int 1\, dx = \int x^0 dx = x + c$

(ii) $\frac{d}{dx}(x^2) = 2x \Rightarrow \int 2x\, dx = x^2 + c$ i.e.

$\int x^1\, dx = \frac{1}{2}x^2 + c$

(iii) $\frac{d}{dx}(x^3) = 3x^2 \Rightarrow \int 3x^2\, dx = x^3 + c$ I.e.

$\int x^2\, dx = \frac{1}{3}x^3 + c$

(iv) $\frac{d}{dx}(x^4) = 4x^3 \Rightarrow \int 4x^3\, dx = x^4 + c$ i.e.

$\int x^3\, dx = \frac{1}{4}x^4 + c$

In all the above cases, c is called the constant of integration.
The above examples show that, in general,

$\int x^n\, dx = \frac{1}{n+1}x^{n+1} + c$ for $n \neq -1$

For example $\int x^9\, dx = \frac{1}{10}x^{10} + c$

and $\int \frac{1}{\theta^2} d\theta = \int \theta^{-2} d\theta = \frac{\theta^{-1}}{-1} + c = c - \frac{1}{\theta}$

The general rule is thus:
Increase the power by one and divide the term by the new power.

Definite and indefinite integrals
In this context, definite integrals have no specific constants of integration.
Consider the two integrals:

$\int x^2 dx$ and $\int_0^1 x^2 dx$

The solutions may be laid down as follows

$\int x^2 dx = \frac{x^3}{3} + c$; this is indefinite integral because it does not give a definite answer as c is said to arbitrary constant.

On the other hand $\int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \left(\frac{1}{3} - 0\right) = \frac{1}{3}$;
this is called definite integral because it gives a definite answer. This is a definite integral.

Note: In the second integral above:

(i) the numbers 0 and 1 are known as the lower and upper limits respectively.
(ii) with definite integral, the constant is omitted.

Examples
1. Integrate each of the following with respect to x
(a) x^5 (b) $2x^3$ (c) $-3x^6$ (d) 10
(e) $\frac{2}{3}x^4$ (f) $-\frac{1}{2}x^3$

Solution

(a) $\int x^5 dx = \frac{x^6}{6} + c$

(b) $\int 2x^3 dx = \frac{2x^4}{4} + c = \frac{x^4}{2} + c$

(c) $\int -3x^6 dx = \frac{-3x^7}{7} + c$

(d) $\int 10 dx = \int 10x^0 dx = \frac{10x^1}{1} + c = 10x + c$

(e) $\int \frac{2}{3}x^4 dx = \frac{2x^5}{3 \times 5} + c = \frac{2x^5}{15} + c$

(f) $\int -\frac{1}{2}x^3 dx = -\frac{x^4}{2 \times 4} + c = -\frac{x^4}{8} + c$

2. Find each of the following integrals
(a) $\int x^{-3} dx$ (b) $\int \frac{1}{3}x^{-3} dx$ (c) $\int \frac{2}{x^4} dx$

(d) $\int \frac{2}{3x^4} dx$

Solution

(a) $\int x^{-3} dx = \frac{x^{-3+1}}{-3+1} + c = \frac{x^{-2}}{-2} + c = -\frac{x^{-2}}{2} + c$

(b) $\int \frac{1}{3}x^{-3} dx = \frac{x^{-2}}{-6} + c = -\frac{x^{-2}}{6} + c$

(c) $\int \frac{2}{x^4} dx = \int 2x^{-4} dx = \frac{2x^{-3}}{-3} + c = -\frac{2x^{-3}}{3} + c$

(d) $\int \frac{2}{3x^4} dx = \int \frac{2x^{-4}}{3} dx = -\frac{2x^{-3}}{9} + c$

3. Find each of the following integrals
(a) $\int 3x^{\frac{1}{2}} dx$ (b) $\int \sqrt[4]{x} dx$ (c) $\int \sqrt{x^3} dx$
(d) $\left(\int \sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$ (e) $\left(\int 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} - 5x^{\frac{1}{3}}\right) dx$

Solution

(a) $\int 3x^{\frac{1}{2}} dx = \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + c = 2x^{\frac{3}{2}} + c$

(b) $\int \sqrt[4]{x} dx = \int x^{\frac{1}{4}} dx = \frac{x^{\frac{1}{4}+1}}{\frac{1}{4}+1} + c$

$= \frac{x^{\frac{5}{4}}}{\frac{5}{4}} + c = \frac{4x^{\frac{5}{4}}}{5} + c$

(c) $\int \sqrt{x^3} dx = \int x^{\frac{3}{2}} dx = \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c$

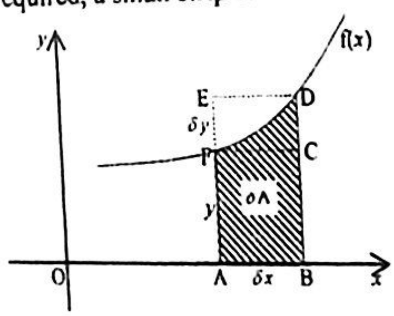
5. Find $\int y\,dx$ for each of the following
   (a) $y = x^2(x+5)$ (b) $y = \sqrt{x}(x+3)$
   (c) $y = 3\sqrt{x}(x^2 - x + 1)$ (d) $y = x(x-1)^2$
6. Find the following definite integrals
   (a) $\int_0^2 x^2 dx$ (b) $\int_0^3 4x^3 dx$ (c) $\int_2^3 \left(6x^2 - 1\right)dx$
   (d) $\int_1^2 \frac{4}{x^3}dx$ (e) $\int_{-2}^{-1} \frac{2}{x^3}dx$
7. Find the following definite integrals
   (a) $\int (\sin x + 2\cos x)dx$ (b) $\int (\sin 2x - \cos 3x)dx$
   (c) $\int (\cos 2x + \sin x)dx$ (d) $\int (\cos 4x - \sin 5x)dx$

Some applications of integration

Like differentiation, integration has wider spectrum of application:

Area under a curve

If the area under the curve $y = f(x)$ for $\alpha \le x \le \beta$ is required, a small strip can be used for the analysis.



Suppose we let the area of shaded region be δA, now, the area of the shaded strip lies between areas of rectangles ABCF and ABDE.

I.e. Area of ABCF $\le \delta A \le$ area of ABDE

$$y\delta x \le \delta A \le (y + \delta y)\delta x$$

Dividing through by δx

$$y \le \frac{\delta A}{\delta x} \le y + \delta y$$

$$\lim_{\delta x \to 0} \frac{\delta A}{\delta x} \to \frac{dA}{dx} \text{ and } \delta y \to 0$$

Hence $\frac{dA}{dx} = y$

Integrating both sides with respect to x;

$$\int \frac{dA}{dx}dx = \int y\,dx$$

$$A = \int y\,dx$$

Now for the interval $\alpha \le x \le \beta$,

$$A = \int_\alpha^\beta y\,dx \text{ or } A = \int_\alpha^\beta f(x)dx$$

Note: When finding the area under the curve, it is advisable that you sketch the curve first in order to establish the required region. The area under the curve is then got by integrating the function given with respect to x between the given x values

Area between the curve and x - axis

Example 1

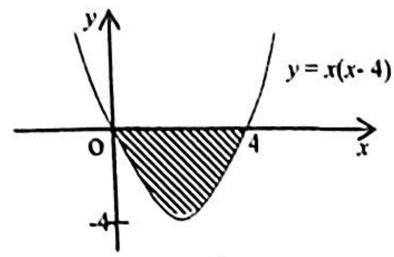
Find the area enclosed by the curve(s)

(i) $y = x(x-4)$ and the x - axis

(ii) $y = x^2 - 4$, the x – axis and line $x = 3$.

Solution

(i) By sketching the graph of $y = x(x-4)$ with the x – axis, we have;



Area required $= \int_0^4 x(x-4)dx$

$$= \int_0^4 \left(x^2 - 4x\right)dx$$

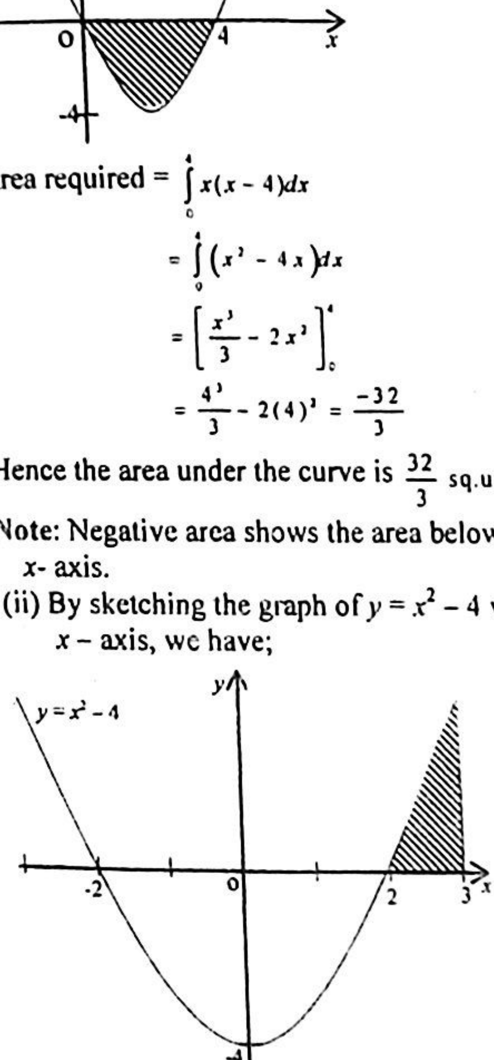
$$= \left[\frac{x^3}{3} - 2x^2\right]_0^4$$

$$= \frac{4^3}{3} - 2(4)^2 = \frac{-32}{3}$$

Hence the area under the curve is $\frac{32}{3}$ sq.units

Note: Negative area shows the area below the x- axis.

(ii) By sketching the graph of $y = x^2 - 4$ with the x – axis, we have;



Required $= \int_2^3 \left(x^2 - 4\right)dx = \frac{1}{3}\left[x^3 - 4x\right]_2^3 = \frac{7}{3}$ sq units

Example 3

Find the area enclosed between the curve $y = 2x^2 - 5x + 6$ and the x-axis

Solution

By sketching the curve, we have:

Required area $= \int_{-1}^{3} x^2 - 2x - 3\,dx$

$$= \left[\frac{1}{3}x^3 - x^2 - 3x\right]_{-1}^{3}$$

$$= \left(\frac{3^3}{3} - 3^2 - 3(3)\right) - \left(\frac{(-1)^3}{3} - (-1^2) - 3(-1)\right)$$

$$= -9 - 1.6666$$

$$= -10.6666$$

$$= 10.667 \text{ sq.units}$$

Area between two curves

Suppose we want to find the area between two intersecting functions f(x) and g(x), required is to

(i) find the points of intersection of the functions

(ii) sketch the functions f(x) and g(x)

Note: If f(x) is above g(x), then the required area

$$= \int f(x)dx - \int g(x)dx$$

Example 6

Find the area enclosed between the curves

(i) $y = x^2 - 4$ and $y = 4 - x^2$

(ii) $y = 2x^2 + 7x + 3$ and $y = 9 + 4x - x^2$

Solution

(i) Finding the points of intersection

$x^2 - 4 = 4 - x^2$

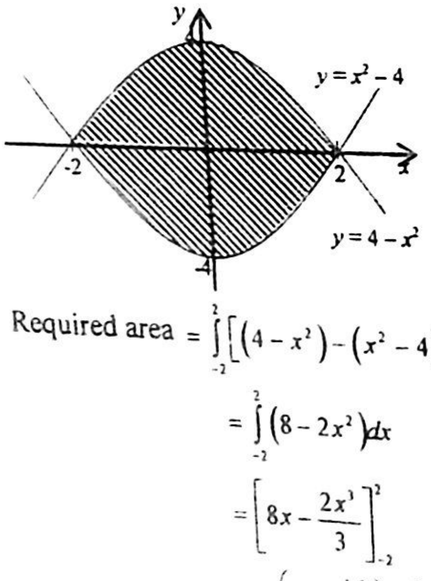
$2x^2 = 8$

$x = -2$ or $x = 2$

when $x = -2$, $y = 0$

and when $x = 2$, $y = 0$

The sketch of the functions:



Required area $= \int_{-2}^{2}\left[\left(4 - x^2\right) - \left(x^2 - 4\right)\right]dx$

$$= \int_{-2}^{2}\left(8 - 2x^2\right)dx$$

$$= \left[8x - \frac{2x^3}{3}\right]_{-2}^{2}$$

$$= \left(16 - \frac{16}{3}\right) - \left(-16 + \frac{16}{3}\right)$$

$$= \frac{32}{3} + \frac{32}{3} = \frac{64}{3} \text{ sq units}$$

(ii) Finding the points of intersection

$2x^2 + 7x + 3 = 9 + 4x - x^2$

$3x^2 + 3x - 6 = 0$

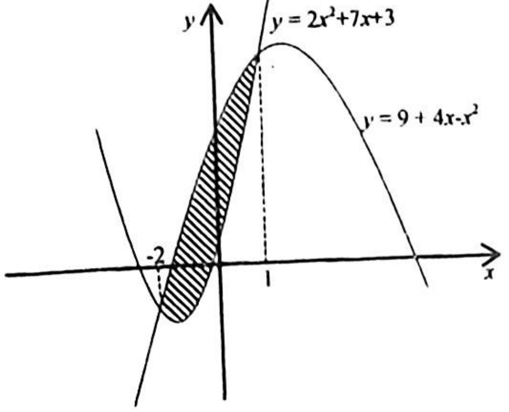
$x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0$

$x = -2$ or $x = 1$

when $x = -2$, $y = -3$

and when $x = 1$, $y = 12$



Required area $= \int_{-2}^{1}\left[\left(9 + 4x - x^2\right) - \left(2x^2 + 7x + 3\right)\right]dx$

$$= \int_{-2}^{1}\left(6 - 3x - 3x^2\right)dx$$

$$= \left[6x - \frac{3x^2}{2} - x^3\right]_{-2}^{1}$$

$$= \left(6 - \frac{3}{2} - 1\right) - \left(-12 - 6 + 8\right)$$

$$= 3.5 + 10 \text{ sq.units}$$

Example 7

Find the area enclosed between the curve $y = x^2 - x - 3$ and the line $y = 2x + 1$

Solution

Finding the points of intersection

$x^2 - x - 3 = 2x + 1$

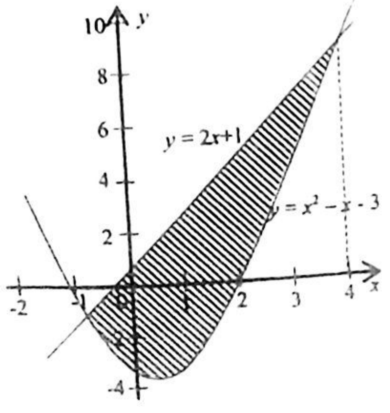
$x^2 - 3x - 4 = 0$

$(x + 1)(x - 4) = 0$

$x = -1$ or $x = 4$

when $x = -1$, $y = -1$

and when $x = 4$, $y = 9$



Required area $= \int_{-1}^{4}\left[\left(2x + 1\right) - \left(x^2 - x - 3\right)\right]dx$

Chapter 9

Differential equations (D.Es)

Definition

These are equations involving differential coefficients (derived functions) like $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots, \frac{d^ny}{dx^n}$.

The order of a differential equation

The **order** of a differential equation is the order n of the highest derivative $\frac{d^ny}{dx^n}$ contained in the differential equation. For example,

(i) $\frac{dy}{dx} + y = 4$ (1st order D.E)

(ii) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5 = 0$ (2nd order D.E)

(iii) $\frac{dy}{dx} + 5\frac{d^3y}{dx^3} = 4$ (3rd order D.E)

Solution to differential equation

This involves the elimination of all the differential coefficients in the given equation. This is normally done by integration.

Example1

Solve the differential equations

(i) $\frac{dy}{dx} - 5 = 0$ (ii) $\frac{dy}{dx} + 10 = 0$

Solution

(i) $\frac{dy}{dx} - 5 = 0$

$\int dy = \int 5\,dx$

$y = 5x + c$

(ii) $\frac{dy}{dx} + 10 = 0$

$\int dy = -\int 10dx$

$y = -10x + c$

Types of solutions of differential equation

There are two types of solutions to differential equation, namely:

- General solution,
- Particular solution.

General solution is that one that contains an arbitrary constant of integration. For example, the solutions in (i) and (ii) above are general solutions because they contain constants, (arbitrary constant), c.

The particular solution is that one that involves the elimination of arbitrary constant of integration. This is normally done by use of initial conditions given. For example, suppose that in (i) above, solve the equation given that $y = 1$ when $x = 0$.

Note: In (i), the solution is $y = 5x + c$, substituting for $x = 0$ and $y = 1$; $c = 1$.

Hence the equation becomes $y = 5x + 1$ (this is a particular equation).

Note: With general solution, there may be many solutions brought about by the value of c which varies with the order of the differential equation.

Example 2

Solve the differential equations

(a) $ydy = x^2\,dx$

(b) $ydy = xdx$, given that $y = 1$ when $x = 0$

Solution

(a) $\int ydy = \int x^2dx$

$\frac{1}{2}y^2 = \frac{1}{3}x^3 + c$ (General solution)

(b) $\int ydy = \int xdx$

$\frac{1}{2}y^2 = \frac{1}{2}x^2 + c$

Substituting for $x = 0$ and $y = 1$, $c = \frac{1}{2}$

$\therefore\ \frac{1}{2}y^2 = \frac{1}{2}x^2 + \frac{1}{2}$

$\Rightarrow y^2 = x^2 + 1$ [Particular solution].

Solving 1st order differential equations

There are three basic methods employed to solve first order differential equations. These will depend on the type of differential equation at hand.

(a) Separable Differential Equations.

(b) Differential Equations with Non-Separable Variables.

(c) Homogenous differential equations.

However, according to our coverage, we shall only look at separable differential equations

Separable Differential Equations

Suppose that the given differential equation is in the form of $f(y)\frac{dy}{dx} = g(x)$, we separate variables in such a way that $f(y)dy = g(x)dx$

Integrating both sides, we have;

$\int f(y)dy = \int g(x)dx$

Exercise 9A

1. Find the general solution to each of the following differential equations

(a) $\frac{dy}{dx}=\frac{x-1}{y}$ (b) $\frac{dy}{dx}=\frac{x^2+1}{y}$ (c) $y^2\frac{dy}{dx}=x-3$

(d) $\frac{dy}{dx}=\frac{x^2-1}{2y}$ (e) $\cos y\frac{dy}{dx}=\sin x$

2. Find the particular solution to each of the following differential equations

(a) $\frac{dy}{dx}=4-3x^2$, $y=5$ at $x=1$

(b) $\frac{dy}{dx}=\frac{x+1}{y}$, $y=3$ at $x=-2$

(c) $(y-3)\frac{dy}{dx}=x+3$, $y=4$ at $x=0$

(d) $3y^2\frac{dy}{dx}+2x=1$, $y=2$ at $x=4$

(e) $x^3\frac{dy}{dx}=2y^2$, $y=4$ at $x=2$

Application of differential equations

Differential equations are met in several fields among which we have the following:

1: Gradient of tangents.
2: Displacement, velocity and acceleration (linear motion), *see more details under chapter 8*
3. Purchase of items/products
4. Formation of products
5. Etc.

Gradient of tangents

As earlier seen, the gradient at any given point on the tangent, in terms of x and y is $\frac{dy}{dx}$

Example 1

The gradient of the tangent at any point on a curve is $\frac{x}{2y}$.Given that the curve passes through the point (2, 4), find its particular differential equation.

Solution

$\frac{dy}{dx}=\frac{x}{2y}$

$2ydy=xdx$

$2\int ydy=\int xdx$

$y^2=\frac{x^2}{2}+c$

Substituting for $x=2$ and $y=4$

$4^2=\frac{2^2}{2}+c$

$16-2=c$

$c=14$

Hence the particular equation becomes $y^2=\frac{x^2}{2}+14$

or $y=\sqrt{\frac{x^2}{2}+14}$

Example 2

A curve passes through the point (2, 3). The gradient of the curve is given by $\frac{dy}{dx}=3x^2-2x-1$

Find its particular differential equation

Solution

$dy=(3x^2-2x-1)dx$

$\int dy=\int(3x^2-2x-1)dx$

$y=x^3-x^2-x+c$

Substituting for $x=2$ and $y=3$

$3=2^3-2^2-2+c$

$3=8-4-2+c$

$c=1$

Hence the particular equation becomes

$y=x^3-x^2-x+1$

Example 3

A curve has an equation which satisfies $\frac{dy}{dx}=k(x-1)$, k is a constant. Given that the gradient of the curve at point (2, 1) is 12,

(a) find the value of k
(b) find its particular differential equation

Solution

(a) $\frac{dy}{dx}=k(x-1)$

Substituting for $x=2$

$12=k(2-1)$

$k=12$

(b) $\frac{dy}{dx}=12(x-1)$

$dy=12x-12$

$\int dy=\int(12x-12)dx$

$y=6x^2-12x+c$

Substituting for $x=2$ and $y=1$

$1=6(2^2)-24+c$

$1=24-24+c$

$c=1$

Hence the particular equation becomes

$y=6x^2-12x+1$

$$\frac{dP}{dt} \propto p^{-1}$$

$$\frac{dp}{dt} = kp^{-1}$$

$$\frac{dp}{dt} = \frac{k}{p}$$

Solving the equation;

$$\frac{dp}{dt} = \frac{k}{p}$$

$$pdv = kdt$$

$$\int pdv = \int kdt$$

$$\frac{p^2}{2} = kt + c$$

Example 6

The rate of change of price of simplified A' level mathematics by Mungawu peter varies inversely as the price (P) of the paper used in production.

(a) Form a differential equation to model the above statement

(b) Solve the differential equation.

Solution

$$\frac{dP}{dt} \propto p^{-1}$$

$$\frac{dp}{dt} = kp^{-1}$$

$$\frac{dp}{dt} = \frac{k}{p}$$

Solving the equation;

$$\frac{dp}{dt} = \frac{k}{p}$$

$$pdv = kdt$$

$$\int pdv = \int kdt$$

$$\frac{p^2}{2} = kt + c$$

Example 7

The rate of change of consumption (C) of a certain product varies directly as the time period (t)

(a) Form a differential equation to model the above statement

(b) Given that C = 50 when t = 0 and C = 100 when t = 2, solve the particular differential equation.

(c) Find C when t = 6seconds

Solution

(a) $\frac{dC}{dt} \propto t$

$$\frac{dC}{dt} = kt$$

Solving the equation;

$$\frac{dC}{dt} = ktdt$$

$$dC = ktdt$$

$$\int dC = \int ktdt$$

$$C = \frac{kt^2}{2} + A$$

(b) Substituting for C = 50 and t = 0

$$50 + 0 = A$$

$$A = 50$$

$$\Rightarrow C = \frac{kt^2}{2} + A$$

Substituting for C = 100 and t = 2

$$100 = 2k + 50$$

$$100 - 50 = 2k$$

$$50 = 2k$$

$$k = 25$$

The differential equation becomes

$$C = \frac{25t^2}{2} + 50$$

(c) When t = 6seconds;

$$C = \frac{25t^2}{2} + 50$$

$$C = \frac{25 \times 6^2}{2} + 50$$

$$C = \frac{900}{2} + 50$$

$$C = 450 + 50$$

$$C = 500$$

Formation of products

In situations where there is a change in the rate of formation of a product, two cases are considered:

- If there is an increase in the final formation of the product, then such formation is said to have positive variation
- If there is a decrease in the final formation of the product, then such formation is said to have negative variation

The above mentioned cases are applied to laws of growth and decay. For example:

(i) If the rate of growth of x is proportional to x, then we have $\frac{dx}{dt} = kx$, where k is a positive constant

(ii) If the rate of decay of x is proportional to x, then we have $\frac{dx}{dt} = -kx$, where k is still a positive constant

Example 8

At time t seconds, the rate of growth of x is proportional to $\frac{1}{x}$

$$dC = kt\,dt$$

$$\int dC = \int kt\,dt$$

$$C = \frac{kt^2}{2} + A$$

(b) Substituting for C = 50 and $t = 0$

$$50 + 0 = A$$

$$A = 50$$

$$\Rightarrow C = \frac{kt^2}{2} + A$$

Substituting for C = 100 and $t = 2$

$$100 = 2k + 50$$

$$100 - 50 = 2k$$

$$50 = 2k$$

$$k = 25$$

The differential equation becomes

$$C = \frac{25t^2}{2} + 50$$

(c) When t = 6seconds;

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$$C = \frac{25 \times 6^2}{2} + 50$$

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Example 8

At time t seconds, the rate of growth of x is proportional to $\frac{1}{x}$

(a) Write down a differential equation representing the above information

(b) If $x = 4$ at $t = 0$ and $x = 6$ at $t = 2$, find x at $t = 4$

Solution

(a) $$\frac{dx}{dt} \propto \frac{1}{x}$$

$$\frac{dx}{dt} = \frac{k}{x}$$

(b) Solving the equation;

$$x\,dx = k\,dt$$

$$\int x\,dx = \int k\,dt$$

$$\frac{x^2}{2} = kt + c$$

Substituting for $x = 4$ and $t = 0$

$$8 = c$$

Substituting for $x = 6$ and $t = 2$

$$18 = 2k + 8$$

$$18 - 8 = 2k$$

$$2k = 10$$

$$k = 5$$

The equation becomes

$$\frac{x^2}{2} = 5t + 8$$

Substituting for $t = 4$

$$\frac{x^2}{2} = 5 \times 4 + 8 \quad 28$$

$$x^2 = \cancel{2(25+8)}$$

$$x^2 = 66$$

$$x = \sqrt{66} = 8.12$$

Example 9

At time t seconds, the rate of decay of x is proportional to x^2

(a) Write down a differential equation representing the above information

(b) If $x = 4$ at $t = 0$ and $x = 2$ at $t = 2$, find x $t = 4$

Solution

(a) $$\frac{dx}{dt} \propto -x^2$$

$$\frac{dx}{dt} = -kx^2$$

(b) Solving the equation;

$$\frac{dx}{x^2} = -k\,dt$$

$$\int \frac{dx}{x^2} = -\int k\,dt$$

$$\int x^{-2}\,dx = -\int k\,dt$$

$$-x^{-1} = -kt + c$$