CHAPTER TWO
EXPONENTIAL AND LOGARITHMIC FUNCTIONS
2.1 Exponential functions

Exponential functions are categorized into two,

(a) <u>Differentiating exponential functions of the form $y = e^x$.</u>

When differentiating functions of this form we simply multiply the function by the derivative of its power. I.e.

$\frac{d}{dx}(e^x) = e^x$, $\frac{d}{dx}(e^{2x}) = 2e^{2x}$, $\frac{d}{dx}\left(e^{sinx}\right) = cosxe^{sinx}$

Example 1

Find $\frac{dy}{dx}$ of the following functions

(a) $y = e^{3x}$ (b) $y = 5e^{\left(\frac{1}{x}\right)}$

Solution

(a) Using the chain rule,

$\frac{dy}{dx} = e^{3x}(3x)' = 3e^{3x}$

(b) Using the chain rule,

$\frac{dy}{dx} = 5e^{\left(\frac{1}{x}\right)}\left(\frac{1}{x}\right)' = 5e^{\left(\frac{1}{x}\right)}\left(-\frac{1}{x^2}\right)$

$\therefore \frac{dy}{dx} = -\frac{5}{x^2}e^{\left(\frac{1}{x}\right)}$

Example 2

Find $\frac{dy}{dx}$ for each of the following,

(a) $y = (1 - e^x)^4$ (b) $y = \frac{2}{3+e^{3x}}$

Solution

(a) Let $u = 1 - e^x$; $\frac{du}{dx} = -e^x$

$\Longrightarrow y = u^4$; $\frac{dy}{du} = 4u^3$

Using the chain rule;

$\Longrightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\Longrightarrow \frac{dy}{dx} = 4u^3\ (-e^x)$

$\therefore \frac{dy}{dx} = -4e^x(1 - e^x)^3$

(b) $y = 2(3 + e^{3x})^{-1}$

Let $u = 3 + e^{3x}$; $\frac{du}{dx} = 3e^{3x}$

$\Longrightarrow y = 2u^{-1}$; $\frac{dy}{du} = -2u^{-2}$

Using the chain rule;

$\Longrightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\Longrightarrow \frac{dy}{dx} = (-2u^{-2}) \times (3e^{3x})$$

$$\therefore \frac{dy}{dx} = -\frac{6e^{3x}}{(3+e^{3x})^2}$$

Example 3

Find $\frac{dy}{dx}$ for each of the following.

(a) $y = x^2e^x$ (b) $y = \frac{e^{3x}}{x}$

Solution

(a) Using the product rule;

$$\Longrightarrow \frac{dy}{dx} = x^2(e^x)' + e^x(x^2)'$$

$$= x^2e^x + 2xe^x$$

$$\therefore \frac{dy}{dx} = xe^x(x+2)$$

(b) Using the quotient rule;

$$\Longrightarrow \frac{dy}{dx} = \frac{x\left(e^{3x}\right)' - e^{3x}(x)'}{x^2}$$

$$= \frac{3xe^{3x} - e^{3x}}{x^2}$$

$$= \frac{e^{3x}(3x-1)}{x^2}$$

We can now integrate exponential functions of this form i.e. $\int e^x\, dx$

When integrating exponential functions of the form e^x, we multiply the function by the inverse of the derivative its power.

Example 4

Find each of the following integrals

(a) $\int e^{ax}\, dx$ (b) $\int 2e^{-x}\, dx$ (c) $\int (1-e^{-3x})^2\, dx$ (d) $\int 5xe^{x^2}\, dx$

Solution

(a) $\int e^{ax}\, dx = \frac{1}{\frac{d}{dx}(ax)}(e^{ax}) + c = \frac{1}{a}e^{ax} + c$

(b) $\int 2e^{-x}\, dx = -2e^{-x} + c$

(c) $\int (1-e^{-3x})^2\, dx = \int (1 - 2e^{-3x} + e^{-6x})\, dx = x + \frac{3}{2}e^{-3x} - \frac{1}{6}e^{-6x} + c$

(d) Let $u = x^2$; $\frac{du}{dx} = 2x \Longrightarrow xdx = \frac{du}{2}$

$$\Longrightarrow \int 5xe^{x^2}\, dx = \int 5e^u \frac{du}{2} = \frac{5}{2}e^{x^2} + c$$

Exercise 2.1

1. Differentiate the following with respect to x.

(a) $e^{\sqrt{x}}$ (b) e^{cosx} (c) $sinx\, e^{x}$ (d) xe^{4x} (e) xe^{sinx} (f) $(1+x)e^{x}$ (g) e^{a-bx}

2. Integrate the following with respect to x.

   (a) e^{4x} (b) e^{ax+b} (c) e^{1-x} (d) $x^2e^{(1+x^3)}$ (e) $\frac{1}{\sqrt{x}}e^{\sqrt{x}}$ (f) sec^2xe^{tanx}

3. Evaluate the following;

   (a) $\int_1^2 e^{2x}\, dx$ (b) $\int_2^3 xe^{x^2}\, dx$

4. The distance s metres travelled by a particle in time t seconds is given by $s = te^{-\frac{1}{2}t^2}$. Show that the velocity in m/s is given by $e^{-\frac{1}{2}t^2}(1-t^2)$.
5. If $y = e^{2x}\cos x$, show that $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$.
6. Differentiate $e^{ax}(cosbx + sinbx)$ with respect to x
7. If $y = \frac{e^x}{1+x^2}$, show that $(x^2+1)\frac{dy}{dx} = y(x-1)^2$
8. Given that $y = \sin(e^x - 1)$, show that $\frac{d^2y}{dx^2} - \frac{dy}{dx} + ye^{2x} = 0$
9. Find the minimum value of $\frac{e^x}{x}$. $Ans(e)$
10. If $e^x \sin x = \frac{d}{dx}\{e^x(A\sin x + B\cos x)\}$, find the values of A and B. Hence find $\int e^x \sin x\, dx$. $Ans\left(\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}(sinx - cosx)e^x\right)$
11. Differentiate $(x-1)e^x$ and hence find $\int xe^x\, dx$. $Ans(xe^x; (x-1)e^x)$
12. Find the values of A and B if $\frac{d}{dx}\{(x^2 + Ax + B)e^x\} = x^2e^x$. Hence find $\int x^2e^x\, dx$. $Ans(-2, 2;\ (x^2 - 2x + 2)e^x)$
13. Find $f'(x)$ for each of the following;

    (a) $f(x) = (1+e^x)^2$ (b) $f(x) = \frac{1}{3-e^{4x^2}}$ (c) $f(x) = \sqrt{1-2e^{4x}}$

14. Find each of these integrals;

    (a) $\int e^x(3+e^x)^2\, dx$ (b) $\int \frac{4e^{-2x}}{(1+e^{-2x})^2}\, dx$

15. If $y = (x - 0.5)e^{2x}$ find $\frac{dy}{dx}$
16. Find $\frac{dy}{dx}$ if $y = e^{tan^2(x+1)}$
17. If $y = -e^x \cos 2x$ show that $\frac{d^2y}{dx^2} = 5e^x \sin(2x + \alpha)$ where $\alpha = \tan^{-1}\left(\frac{3}{4}\right)$
18. Find the equation of the tangent to the curve $y = e^x$ at the point given by $x = a$. Deduce the equation of the tangent to the curve which passes through the point $(1, 0)$
19. Find the maximum and minimum values of the function $(1+2x)e^{-x^2}$
20. If $y = e^{-x}\cos x$, determine the three values of x between 0 and 3π for which $\frac{dy}{dx} = 0$. Show that the corresponding values of y foyrm a geometric progression.

21. Given that $y = e^{2x}\sin 3x$, prove that $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$

22. Given that $y = e^{tanx}$ show that $\frac{d^2y}{dx^2} - (2tanx + sec^2x)\frac{dy}{dx} = 0$

23. If $y = Ae^{-x}\cos(x + \alpha)$ where A and α are constants show that

    (i) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$

    (ii) $\frac{d^4y}{dx^4} + 4y = 0$

24. Given that $e^x = \tan 2y$ show that $\frac{d^2y}{dx^2} = \frac{e^x - e^{3x}}{2(1+e^{2x})^2}$

2.2 Differentiation of natural logarithms

Logarithms to the base e are called **natural logarithms.** The notation lnx is used as the standard abbreviation for $\log_e x$. The function lnx is the inverse function of e^x.

The following formulae connecting logarithms have already been proved.

1. $\log_a x + \log_a y = \log_a xy$
2. $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$
3. $\log_a x^n = n\log_a x$
4. $\log_b x = \frac{\log_a x}{\log_a b}$.

Notice that

$lne^x = xlne = x(1)$

Therefore $\ln(e^x) = x$ and $e^{lnx} = x$.

When differentiating natural logarithm we simply find the derivative of the function and divide by itself i.e. $\frac{d}{dx}(lnx) = \frac{1}{x}$, $\frac{d}{dx}(ln\sin x) = \frac{\cos x}{\sin x} = \cot x$, $\frac{d}{dx}\left(ln(x^2 - 4)\right) = \frac{2x}{x^2-4}$

Example 5

Find $\frac{dy}{dx}$ for each of the following

(a) $y = \ln 3x$ (b) $y = \ln(x^2 - 1)$

Solution

(a) Using the chain rule, we have

$\frac{dy}{dx} = \frac{1}{3x} \times (3x)'$

$\therefore \frac{dy}{dx} = \frac{1}{x}$

(b) Using the chain rule, we have

$\frac{dy}{dx} = \frac{1}{x^2-1} \times (x^2 - 1)'$

$\therefore \frac{dy}{dx} = \frac{2x}{x^2-1}$

Example 6

Find $\frac{dy}{dx}$ for each of the following

(a) $\ln \frac{x}{\sqrt{(x^2+1)}}$ (b) $\ln sin^2 x$ (c) $\ln\left\{x\sqrt{(x+1)}\right\}$ (d) $\ln 2cos^3 x$

Solution

(a) Let $y = \ln \frac{x}{\sqrt{(x^2+1)}} = \ln x - \frac{1}{2}\ln(x^2+1)$

$\frac{dy}{dx} = \frac{1}{x} - \frac{2x}{2(x^2+1)}$

$= \frac{x^2+1-x^2}{x(x^2+1)}$

$\therefore \frac{dy}{dx} = \frac{1}{x(x^2+1)}$

(b) Let $y = \ln sin^2 x = 2\ln \sin x$

$\therefore \frac{dy}{dx} = \frac{2\cos x}{\sin x} = 2\cot x$

(c) Let $y = \ln\left\{x\sqrt{(x+1)}\right\} = \ln x + \frac{1}{2}\ln(x+1)$

$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2(x+1)}$

$= \frac{2x+2+x}{2x(x+1)}$

$\therefore \frac{dy}{dx} = \frac{3x+2}{2x(x+1)}$

(d) Let $y = \ln 2cos^3 x = \ln 2 + 3\ln\cos x$

$\frac{dy}{dx} = 0 - \frac{3sinx}{cosx}$

$\therefore \frac{dy}{dx} = -3\tan x$

Application of natural logarithms to differentiation

Complicated products and quotients are often best differentiated by taking logarithms before differentiation.

Example 7

Differentiate the following with respect to x.

(a) $\frac{(1+2x)\sqrt{(1+x)}}{(1-x)}$ (b) $y = \left(x + \frac{1}{x}\right)^2$ (c) $x^{\sin x}$ (d) x^x (e) $y = \log_{10}\left(\frac{e^x}{\cos 3x}\right)$

Solution

(a) Let $y = \frac{(1+2x)\sqrt{(1+x)}}{(1-x)}$

Taking natural logarithms on both sides

$$\ln y = \ln\left[\frac{(1+2x)\sqrt{(1+x)}}{(1-x)}\right]$$

$$= \ln(1+2x) + \frac{1}{2}\ln(1+x) - \ln(1-x)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{1+2x} + \frac{1}{2(1+x)} + \frac{1}{(1-x)}$$

$$= \frac{4(1+x)(1-x)+(1+2x)(1-x)+2(1+x)(1+2x)}{2(1+x)(1+2x)(1-x)}$$

$$= \frac{7+7x-2x^2}{2(1+x)(1+2x)(1-x)}$$

$$\frac{dy}{dx} = \frac{7+7x-2x^2}{2(1+x)(1+2x)(1-x)} \times \frac{(1+2x)\sqrt{(1+x)}}{(1-x)}$$

$$\therefore \frac{dy}{dx} = \frac{7+7x-2x^2}{2(1-x)^2\sqrt{(1+x)}}$$

(b) Taking natural logarithms on both sides

$$lny = ln\left(x + \frac{1}{x}\right)^2 = 2\ln\left(\frac{\mathrm{x}^2+1}{\mathrm{x}}\right) = 2[\ln(\mathrm{x}^2+1) - \ln \mathrm{x}]$$

$$\frac{1}{y}\frac{dy}{dx} = 2\left[\frac{2x}{x^2+1} - \frac{1}{x}\right]$$

$$= \frac{4x^2-2x^2-2}{x(x^2+1)}$$

$$= \frac{2(x^2-1)}{x(x^2+1)}$$

$$\frac{dy}{dx} = \frac{2(x^2-1)}{x(x^2+1)} \times \left(\frac{x^2+1}{x}\right)^2$$

$$\therefore \frac{dy}{dx} = \frac{2(x^4-1)}{x^2}$$

(c) Let $y = x^{\sin x}$

$$lny = lnx^{\sin x} = sinxlnx$$

$$\frac{1}{y}\frac{dy}{dx} = cosxlnx + \frac{sinx}{x}$$

$$\therefore \frac{dy}{dx} = \left(cosxlnx + \frac{sinx}{x}\right)x^{\sin x}$$

(d) $let\ y = x^x$

$lny = lnx^x = xlnx$

$\frac{1}{y}\frac{dy}{dx} = lnx + 1$

$\therefore \frac{dy}{dx} = (lnx + 1\ )x^x$

(e) $y = \log_{10}\left(\frac{e^x}{\cos 3x}\right)$

We can change the base from 10 to e.

$y = \frac{\log_e\left(\frac{e^x}{\cos 3x}\right)}{\log_e 10} = \frac{\log_e e^x - \log_e \cos 3x}{\log_e 10} = \frac{x - \log_e \cos 3x}{\log_e 10}$

$\frac{dy}{dx} = \frac{1+\frac{3\sin 3x}{\cos 3x}}{\log_e 10}$

$\therefore \frac{dy}{dx} = \frac{1+3\tan 3x}{\log_e 10}$

Example 8

Find the equation of the tangent and normal to the curve $y = ln\left(\frac{x-1}{x+1}\right)$ at the point P where $x = 3$

Solution

$\Longrightarrow \frac{dy}{dx} = \frac{1}{\left(\frac{x-1}{x+1}\right)}\left[\frac{((x+1))-(x-1)}{(x+1)^2}\right]$

$= \left(\frac{x+1}{x-1}\right)\left[\frac{2}{(x+1)^2}\right]$

$= \frac{2}{x^2-1}$

When $x = 3$; $\quad y = \ln\left(\frac{3-1}{3+1}\right) = \ln\left(\frac{1}{2}\right) = -\ln 2$

At $P(3, -\ln 2)$; $\frac{dy}{dx} = \frac{2}{(3)^2-1} = \frac{1}{4}$

The tangent is of the form $y = \frac{1}{4}x + c$

Using $P(3, -\ln 2)$, we have $-\ln 2 = \frac{3}{4} + c \quad \therefore c = -\ln 2 - \frac{3}{4}$

The equation of the tangent line is $y = \frac{1}{4}x - \ln 2 - \frac{3}{4}$ or $x - 4y - 4\ln 2 + 3$

When the gradient of the tangent at P is $\frac{1}{4}$, the gradient of the normal at P is -4.
Therefore, the normal is of the form $y = -4x + A$. Using $P(3, -\ln 2)$ gives

$-\ln 2 = -4(3) + A; \quad \therefore A = 12 - \ln 2$

The equation of the normal is $y = -4x + 12 - \ln 2$

Example 9

Find and classify the stationary points on the curve $y = x^2e^x$.

Solution

At stationary point $\frac{dy}{dx} = 0$. Using the product rule, we have

$\frac{dy}{dx} = x^2e^x + 2xe^x$

When $\frac{dy}{dx} = 0$

$\Longrightarrow x^2e^x + 2xe^x = 0$

$\Longrightarrow xe^x(x + 2) = 0$

Solving gives $x = 0 \ and \ x = -2$

When $x = 0, y = 0$; when $x = -2, y = 4e^{-2}$

The points are $(0, 0) and \ (-2, 4e^{-2})$ are stationary points.

To determine their nature, we consider $\frac{d^2y}{dx^2}$.

Now $\frac{d^2y}{dx^2} = x^2e^x + 4xe^x + 2e^x$

$\quad\quad = e^x(x^2 + 4x + 2)$

When $x = 0$; $\left.\frac{d^2y}{dx^2}\right|_{x=0} = 2 > 0$ therefore $(0, 0)$ is a minimum.

When $x = -2$; $\left.\frac{d^2y}{dx^2}\right|_{x=-2} = -2e^{-2} < 0$ therefore $(-2, 4e^{-2})$ is a maximum.

(b) Differentiating exponential functions of the form $y = a^x$.

We use the natural logarithms to do this differentiation i.e.

$lny = lna^x$

$\quad = xlna$

$\frac{1}{y}\frac{dy}{dx} = lna$

$\therefore \frac{dy}{dx} = a^x lna$

Example 10

Differentiate the following with respect to x

(a) 3^x (b) 3^{cosx} (c) a^{4x} (d) 4^{1-x}

Solution

(a) Let $y = 3^x$

$\Longrightarrow lny = ln3^x = xln3$

$\frac{1}{y}\frac{dy}{dx} = \ln 3$

$\therefore \frac{dy}{dx} = 3^x ln3$

(b) Let $y = 3^{cosx}$

$\Longrightarrow lny = ln3^{cosx} = cosxln3$

$\frac{1}{y}\frac{dy}{dx} = -sinxln3$

$\therefore \frac{dy}{dx} = -3^{cosx}.sinx.ln3$

(c) Let $y = a^{4x}$

$\Longrightarrow lny = lna^{4x} = 4xlna$

$\frac{1}{y}\frac{dy}{dx} = 4lna$

$\therefore \frac{dy}{dx} = 4a^{4x}lna$

(d) Let $y = 4^{1-x}$

$\Longrightarrow lny = ln4^{1-x} = (1-x)ln4$

$\frac{1}{y}\frac{dy}{dx} = -ln4$

$\therefore \frac{dy}{dx} = -4^{(1-x)}ln4$

Integration of exponential functions

(a) When integrating exponential function of the form $\int e^x\,dx$, we simply multiply the function by the inverse of the derivative of its power. i.e. $\int e^x\,dx = e^x + c$, $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$

Example 11

Integrate the following with respect to x.

(a) $\int e^{5x}\,dx$ (b) $\int x^2e^{x^3}\,dx$ (c) $\int e^{\sin x}\cos x\,dx$ (d) $\int_1^2 e^{2x}\,dx$

Solution

(a) $\int e^{5x}\,dx = \frac{1}{5}e^{5x} + c$

(b) Let $u = x^3; \frac{du}{dx} = 3x^2; \frac{du}{3} = x^2dx$

$$\Rightarrow \int x^2 e^{x^3}\,dx = \int e^u \frac{du}{3}$$
$$= \frac{e^u}{3} + c$$
$$= \frac{e^{x^3}}{3} + c$$

(c) Let $u = sinx$; $\frac{du}{dx} = cosx$; $du = cosxdx$

$$\Rightarrow \int e^{\sin x} \cos x\,dx = \int e^u\,du$$
$$= e^u + c$$
$$= e^{sinx} + c$$

(d) $\int_1^2 e^{2x}\,dx = \left[\frac{1}{2}e^{2x}\right]_1^2$

$$= \frac{1}{2}\left\{\left(e^{2(2)}\right) - \left(e^{2(1)}\right)\right\}$$
$$= 23.6046$$

(b) When integrating exponential function of the form $\int a^x\,dx$, we proceed as follows

From $\frac{dy}{dx} = a^x lna$

$$\Rightarrow \frac{dy}{lna} = a^x dx$$
$$\Rightarrow \int \frac{dy}{lna} = \int a^x\,dx$$
$$\Rightarrow \frac{y}{lna} = \int a^x\,dx$$
$$\therefore \int a^x\,dx = \frac{a^x}{lna} + c$$

Example 12

Integrate the following with respect to x.

(a) $\int 3^{2x}\,dx$ (b) $\int 2^{tanx} sec^2x\,dx$ (c) $\int x\,3^{x^2}\,dx$

Solution

(a) Let $u = 2x$; $\frac{du}{dx} = 2$; $\frac{du}{2} = dx$

$$\Rightarrow \int 3^{2x}\,dx = \int 3^u \frac{du}{2}$$
$$= \frac{1}{2}.\frac{3^u}{ln3} + c$$
$$= \frac{3^{2x}}{ln9} + c$$

(b) Let $u = tanx$; $\frac{du}{dx} = sec^2x$; $du = sec^2xdx$

$$\Rightarrow \int 2^{tanx} sec^2x\,dx = \int 2^u\,du$$
$$= \frac{2^u}{ln2} + c$$

$$= \frac{2^{tanx}}{ln2} + c$$

(c) Let $u = x^2$; $\frac{du}{dx} = 2x$; $\frac{du}{2} = xdx$

$\Longrightarrow \int x\, 3^{x^2}\, dx = \int 3^u \frac{du}{2}$

$$= \frac{3^{x^2}}{ln9} + c$$

Integrals of the form $\int \frac{f'(x)}{f(x)} dx$

An integral of the form $\int \frac{f'(x)}{f(x)} dx$ may be reduced to the form $\int \frac{1}{u} du$, by the substitution $u = f(x)$ as seen below.

If $u = f(x)$; $du = f'(x)dx$

$\Longrightarrow \int \frac{f'(x)}{f(x)} dx = \int \frac{1}{u} du$

$= \ln(u) + c$

$= lnf(x) + c$; let $c = lnk$

$\therefore \int \frac{f'(x)}{f(x)} dx = ln\{kf(x)\}$

Example 13

integrate the following with respect to x.

(a) $\int \frac{1}{x} dx$ (b) $\int \frac{x}{x^2+1} dx$ (c) $\int_1^2 \frac{2}{1-3x} dx$ (d) $\int \frac{1}{xlnx} dx$

Solution

(a) $\int \frac{1}{x} dx = lnx + c$

(b)) $\int \frac{x}{x^2+1} dx = \frac{1}{2}\ln(x^2+1) + c = \ln\left\{k\sqrt{(x^2+1)}\right\}$ where $c = lnk$

(c) $\int_1^2 \frac{2}{1-3x} dx = -\frac{2}{3}[|1-3x|]_1^2$

$= -\frac{2}{3}(ln|-5| - ln|-2|)$

$= -\frac{2}{3} ln\left(\frac{5}{2}\right)$

Note: when given limits, we use the magnitude symbols since natural logarithm is undefined for negative values.

(d) $\int \frac{1}{xlnx} dx = \int \frac{\left(\frac{1}{x}\right)}{lnx} dx$

$= \ln(lnx) + c$

Example 14

Find the following integrals with respect to x.

(a) $\int tanx\, dx$ (b) $\int secx\, dx$ (c) $\int cot(1-3x)\, dx$

Solution

(a) $\int tanx\, dx = \int \frac{\sin x}{\cos x} dx$

Let $u = \cos x$; $du = -sinxdx$; $-du = sinxdx$

$\Rightarrow \int tanx\, dx = -\int \frac{1}{u} du$

$= -lncosx + c = \ln(cosx)^{-1} + c = lnsecx + c$

(b) $\int secx\, dx = \int secx \left[\frac{secx+tanx}{secx+tanx}\right] dx$

$= \int \frac{sec^2x+secxtanx}{tanx+secx} dx$

$= \ln(tanx + secx) + c$

(c) $\int cot(1-3x)\, dx = \int \frac{\cos(1-3x)}{\sin(1-3x)} dx$

Let $u = \sin(1-3x)$; $du = -3\cos(1-3x)\, dx$

$\Rightarrow \int cot(1-3x)\, dx = -\int \frac{1}{u} . \frac{du}{3}$

$= -\frac{1}{3} lnu + c$

$= -\frac{1}{3} lnsin(1-3x) + c$

Exercise 2.2

1. Differentiate the following with respect to x.

   (a) $y = \log_e(x^2+1)$ (b) $\log_e \sqrt{\cos x}$ (c) $y = (\log_e x)^2$ (d) $y = (\log_e \sin x)^2$
2. Differentiate with respect to x.

   (a) $y = (2-3lnx)^3$ (b) $y = \frac{1}{\sqrt{1+lnx}}$ (c) $y = xlnx$ (d) $y = \frac{ln2x}{x^3}$
3. Differentiate the following with respect to x.

   (a) $\sqrt{\frac{x-1}{x+1}}$ (b) $\sqrt{\frac{x^2-1}{x^2+1}}$ (c) $\frac{(x-1)(x-2)}{(x-3)}$ (d) $\sqrt{\frac{(x-1)(x-2)}{(x-3)}}$ (e) $(\log_e x)^x$
4. Find the maximum value of $\frac{\log_e x}{x}$ $Ans: (\frac{1}{e})$
5. Find the maximum value of $\frac{(x+1)^2(x+2)}{(x+3)^3}$ $Ans(\frac{2}{27})$
6. Differentiate $\log_e\left\{x + \sqrt{(x^2+1)}\right\}$ $Ans\left\{\frac{1}{\sqrt{x^2+1}}\right\}$
7. Given that $y = \frac{\ln(1+x)}{x^2}$, show that $x^2\frac{dy}{dx} + 2xy = \frac{1}{1+x}$
8. Given that $y = ln\left(\frac{1+x}{1-x}\right)$, show that $(1-x^2)\frac{dy}{dx} = 2$
9. Find the equation of the tangent to the curve $y = x + e^{2x}$ at the point where $x = 0$. $Ans(y = 3x + 1)$
10. Find the equation of the normal to the curve $y = \ln(1+x)$ at the point where $x = 2$. $Ans(3y - x = 3\ln 3 - 2)$
11. Find the equation of the tangent and the normal to the curve $y = e^x \ln x$ at the point where $x = 1$. $Ans(y = e(x-1);\ \ ey + x = 1)$

12. Given that $y = x^2e^{-x}$, show that $\frac{dy}{dx} = x(2-x)e^{-x}$. Hence find the coordinates of the two points on the curve $y = x^2e^{-x}$ where the gradient is zero.
    $Ans\left\{(0,0);\ \left(2,\frac{4}{e^2}\right)\right\}$
13. Given that $y = \frac{\ln x}{x^2}$ for $x > 0$, show that $\frac{dy}{dx} = \frac{1-\ln x}{x^2}$. Hence find the coordinates of the points on the curve $y = \frac{\ln x}{x^2}$ where the gradient is zero. $Ans\left\{\left(e,\frac{1}{e}\right)\right\}$
14. Given that $y = \frac{e^x}{x^2-3}$, show that $\frac{dy}{dx} = \frac{e^x(x+1)(x-3)}{(x^2-3)^2}$
15. Find the area between the curve $y = e^{2x}$ and the x-axis from $x = 0$ to $x = 3$.
    $Ans\left(\frac{e^6-1}{2}\right)$
16. Find the area between the curve $y = \frac{2}{x+3}$ and the x-axis from $x = 2$ to $x = 7$.
    $Ans(2ln2)$
17. The line $y = \frac{1}{3}$ meets the curve $y = \frac{1}{x+1}$ at the point P.
    (a) Find the coordinates of P.
    (b) Calculate the area bounded by the line, the curve and the y-axis
    $Ans\left(\left(2,\frac{1}{3}\right),\left(ln3-\frac{2}{3}\right)\right)$
18. The line $y = x + 1$ meets the curve $y = \frac{8}{5-x}$ at the points P and Q.
    (a) Find the coordinates of P and Q
    (b) Show that the area enclosed between the curve and the line between P and Q is $6 - 8ln2$. $Ans\{(1,2),(3,4)\}$
19. The region bounded by the curve $y = e^x + 1$, the x-axis, the line $x = 0$ and the line $x = 2$ is rotated through 360^0 about the x-axis. Calculate the volume of the solid generated. $Ans\left\{\frac{\pi}{2}(e^4 + 4e^2 - 1)\right\}$
20. The region R is bounded by the curve $y = 3 + \frac{2}{x+1}$, the x-axis, the y-axis and the line $x = 4$.
    (a) Show that the area of R is $12 + 2ln5$.

    R is rotated through 360^0 about the x-axis.

    (b) Show that the volume of the solid generated is $\frac{4\pi}{5}[49 + 15ln5]$
21. Find the equation of the tangent and the normal to the curve
    $e^{x+y} = 1 + x^2 + -y^2$ at the point $(3,-3)$. $Ans\{y = -x;\ \ y = x - 6\}$
22. Find the equations of the tangent and the normal to the curve
    $\ln(x^2 - y + 1) = 8x - y^2$ at the point $(2,4)$. $Ans\{7y - 4x = 20, 4y + 7x = 30\}$

23. Show that the tangent to the curve $e^y + x^2 = 2e^2$, at the point $(e, 2)$ passes through the point $(0, 4)$
24. Given that $xlnx + 2y = 3$, show that $\frac{dy}{dx} = \frac{y(2y-3)}{x(2y+x)}$
25. Find the equation of the tangent and the normal to the curve $x = e^t + t,\ y = e^{3t} - 2t$, at the point where $t = 0$. $Ans\{2y - x = 1, y + 2x = 3\}$
26. Find the equation of the tangent and the normal to the curve $x = 2 + lnt,\ y = t^3$, at the point $(2, 1)$. $Ans\{y = 3x - 5,\ 3y + x = 5\}$
27. Show that $\frac{d}{dx}[\ln(\sec x + \tan x)] = \sec x$
28. Workout the following integrals;

(a) $\int \frac{\cos x}{1+\sin x} dx$ (b) $\int \frac{sec^2 x}{1+\tan x} dx$

29. Given $y = \ln(x^2 - 4x + 5)$, find an expression for $\frac{dy}{dx}$. Hence find $\int_3^4 \frac{x-2}{x^2-4x+5} dx$.
30. Differentiate the following with respect to x.

    (a) 3^{sinx} (b) xe^{sinx} (c) a^{tanx} (d) $\cot^{-1}(\ln x)$ (e) $x10^{\sin x}$

31. Integrate the following with respect to x.

    (a) e^{ax+b} (b) $(1 + x)e^{x^2+2x}$

32. Find the following integrals with respect to x;

    (a) $\int 5^{2x} dx$ (b) $\int 10^x dx$ (c) $\int \frac{3^{cotx}}{sin^2 x} dx$ (d) $\int 3^{\sqrt{2x-1}} dx$ (d) $\int \frac{1+\ln x}{x \ln x} dx$

33. Find the following integrals with respect to x.

    (a) $\int cosecx\, dx$ (b) $\int cot\frac{x}{2} dx$ (c) $\int \frac{2x-3}{3x^2+9x+4} dx$ (d) $\int \frac{1-\tan x}{1+\tan x} dx$

    (e) $\int \frac{1-\sin 2x}{x-sin^2 x} dx$ (f) $\int \frac{sin^{-1}x}{\sqrt{1-x^2}} dx$ (g) $\int \frac{tan^{-1}x}{1+x^2} dx$