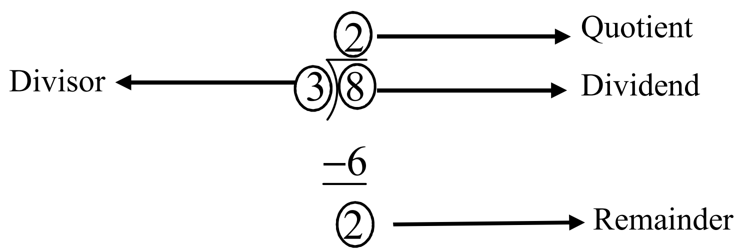
POLYNOMIALS

By the end of this discussion, you should have acquired knowledge about the following concepts:

- ✓ **Long division**
- ✓ **The remainder theorem**
- ✓ **The factor theorem**

Background:

Recall the concept of long division of numbers in early algebra; say one wishes to divide **8** by **3**. Note that **3** is referred to as the **divisor**, and the result of the operation is the **quotient**. In many cases, we also have a **remainder**, as illustrated below;



Note that $8 = 3 \times 2 + 2$, which we can translate to;

$$\boxed{\textbf{Dividend} \quad = \quad (\textbf{Quotient}) \times (\textbf{Divisor}) + \textbf{Remainder}}$$

Note also, that the Remainder is strictly **less than** the Divisor.

Long Division of Polynomials

Number concepts like long division can also be transformed to expressions or polynomials. Suppose we have a polynomial $P(x)$ that we need to divide by another (divisor), $D(x)$. This can be done, but we require that $\mathbf{deg}\left[D(x)\right] \leq \mathbf{deg}\left[P(x)\right]$. With this we have that

$$\begin{array}{r|l} & Q(x) \\ \hline D(x) & P(x) \\ & \cdots\cdots\cdots \\ & \cdots\cdots\cdots \\ \hline & R(x) \end{array}$$

Hence we can now write that;

$\boxed{P(x) = D(x) \times Q(x) + R(x)}$;This is the remainder theorem.

Where $Q(x)$ is the quotient. The remainder is denoted $R(x)$ because in many instances, (*generally when the divisor is non-linear*) the remainder is another function, whose nature depends on $D(x)$.

Case 1: Linear divisor

Suppose our divisor is $D(x) = (x-a)$, then

$$p(x) = (x-a) \times Q(x) + R$$

substituting $x = a$ gives

$$p(a) = R$$

Example 1: Find the remainder when the polynomial $27x^3 + 9x^2 - 6x + 10$ is divided by;

(i) $x-2$

(ii) $3x-1$

Solutions:

(i) Using long division, we have

$$
\begin{array}{r|l}
 & 27x^2 + 63x + 120 \\
\hline
x-2 & 27x^3 + 9x^2 - 6x + 10 \\
 & -\left(27x^3 - 54x^2\right) \downarrow \\
\hline
 & \qquad 63x^2 - 6x. \\
 & \quad -\left(63x^2 - 126x\right) \downarrow \\
\hline
 & \qquad\qquad 120x + 10 \\
 & \qquad\qquad -120x - 240 \\
\hline
 & \qquad\qquad\qquad 250
\end{array}
$$

Hence we have the remainder as $R = 250$ and the $Q(x) = 27x^2 + 63x + 120$

Or we could just substitute into the theorem statement above, and we'd have;

$$p(x) = D(x) \times Q(x) + R$$

$$27x^3 + 9x^2 - 6x + 10 = (x-2) \times Q(x) + R$$

Substituting $x = 2$ gives;

$$27(2)^3 + 9(2)^2 - 6(2) + 10 = (2-2) \times Q(2) + R$$

$$\Rightarrow \quad R = 250 \text{ as obtained before.}$$

(ii) The same can be done by taking $x = \dfrac{1}{3}$

Case 2: Non–linear divisors

The remainder in the above example turned out to be a constant. This is the case for linear divisors. However, the situation is different when the divisor is non–linear, and we can summarise the expected results in a table.

Nature of divisor	Nature of remainder	
	General	**Other**
1. Linear	Constant	None
2. Quadratic	Linear	Constant
3. Cubic	Quadratic	Linear, Constant
4. Quartic	Cubic	Quadratic, Linear, Constant

In brief, the degree of the remainder is 1 less than that of the divisor.

Example 2: Find the quotient and remainder when $3x^3+5x^2+4x+3$ is divided by x^2+2x-1.

Solution:

This requires long division, and we have;

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad\quad\;\; 3x-1 \\ \hline x^2+2x-1 & \;\; 3x^3+5x^2+4x+3 \\ & -\left(3x^3+6x^2-3x\right) \downarrow \\ \cline{2-2} & \quad\quad\quad\;\; -x^2+7x+3 \\ & \quad\quad\quad -\left(x^2-2x+1\right) \\ \cline{2-2} & \quad\quad\quad\quad\quad\quad 9x+2 \end{array}$$

Thus we have $Q(x)=3x-1$ as the quotient, and remainder is $R(x)=9x+2$

N.B: The remainder theorem can only work for non–linear divisors iff the divisor is factorable into linear factors.

Example 3: **Find the remainder when $3x^4+x^3-11x^2-4x+4$ is divided by x^2-x-2.**

Solution:

Suppose we have the remainder as $R(x)=mx+c$, then we have;

$$3x^4+x^3-11x^2-4x+4=\left(x^2-x-2\right)\times Q(x)+(mx+c)$$
$$=(x-2)(x+1)\times Q(x)+mx+c$$

Substituting $x=2$ and $x=-1$ in turn gives us

$$2m+c=8 \quad \cdots\cdots(i)$$
$$-m+c=-1 \quad \cdots\cdots(ii)$$

Solving equations (i) and (ii) simultaneously gives us $m=3; c=2$.

Thus the remainder is $R(x)=3x+2$.

Example 4: **A polynomial $f(x)$ is such that $f(4)=7$ and $f(-3)=-14$. Find the remainder when $f(x)$ is divided by x^2-x-12.**

Solution:

From the remainder theorem, $f(x)=D(x)\times Q(x)+R(x)$.

$$\therefore f(x)=\left(x^2-x-12\right)\times Q(x)+ax+b \qquad ; R(x)=ax+b$$
$$f(x)=(x-4)(x+3)\times Q(x)+ax+b$$

Substituting $x=4$ and $x=-3$ in turn gives us;

$$f(4)=4a+b \quad (=7) \quad \cdots\cdots(i)$$
$$f(-3)=-3a+b \quad (=-14) \quad \cdots(ii)$$

Solving (i) and (ii) simultaneously gives $a = 3$ and $b = -5$, thus the remainder can be stated as $\underline{R(x) = 3x - 5}$.

The Factor Theorem

Note that a number is only a factor of another if upon division the remainder is **0**(zero), otherwise it is not a factor.

Similarly, an expression is considered to be a factor of another iff the remainder after division is **0**. The remainder theorem was earlier stated as;

$$f(x) = D(x) \times Q(x) + R(x)$$

Now that $R(x) = 0$, the theorem takes new shape;

$$f(x) = D(x) \times Q(x)$$

This is the **factor theorem**, and as we can see, the divisor and quotient in this case are all factors of $f(x)$.

(*For linear divisors, x–a is a factor of $f(x)$ if* $f(a) = 0$)

Example 5: $x - 3$ **and** $2x + 1$ **are both factors of the expression** $f(x) = 2x^4 - 5x^3 + px^2 + 25x + q$**. Determine the values of *a* and *b* and hence solve the equation** $f(x) = 0$**.**

Example 6: **The expression** $3x^3 + 2x^2 - bx + a$ **is divisible by** $x - 1$ **but has a remainder 10 when divided by** $x + 1$**. Find the values of *a* and *b*.**