INTEGRATION

Integration is the reverse of differentiation.
It is denoted by $\int(\ldots)\,dx$ to mean integration with respect to x.

When integrating, we increase the power by 1 and divide by the new power.

Definite integrals have limits whereas indefinite integrals have no limits.

Examples:

Integrate the following with respect to x:

a): $\int x\,dx$.

b): $\int y^{1/2}\,dy$.

c): $\int (2x^2-2x+4)\,dx$

(d): $\int_1^2 (4x-x^2)\,dx$.

(e): $\int_1^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$

Solution.

a): $\int x\,dx = \frac{1}{2}x^2 + c$.

b): $\int y^{\frac{1}{2}}\,dy = \frac{y^{3/2}}{3/2} + c$

$= \frac{2}{3}y^{3/2} + c$.

c) $\int (2x^2-2x+4)\,dx = \frac{2}{3}x^3 - \frac{2}{2}x^2 - \frac{4x}{1} + c$

$= \frac{2}{3}x^3 - x^2 - 4x + c$.

d): $\int_1^2 (4x-x^2)\,dx = \left[\frac{4x^2}{2} - \frac{x^3}{3}\right]_1^2$.

$= \left[2x^2 - \frac{1}{3}x^3\right]_1^2$.

$= \left[2(2)^2 - \frac{1}{3}(2)^3\right] - \left[2(1)^2 - \frac{1}{3}(1)^3\right]$

$= 16/3 - 5/3$.

$= \cancel{8/}\ 11/$

e): $\int_1^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx = \int_1^9 \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right) dx$.

$$= \left[\left(\frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2}\right)\right]_1^9.$$

$$= \left(\frac{9^{3/2}}{3/2} + \frac{9^{1/2}}{1}\right) - \left(\frac{1^{3/2}}{3/2} + \frac{1^{1/2}}{1/2}\right).$$

$$\int_1^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx = \cancel{25.3333}\ \ 64/3.$$

Find;

(a): $\int \frac{8x^5 - 3x}{x^3} dx$:

b): $\int (2(\sqrt{x} - 3)(1 - \sqrt{x}) dx$.

solution.

a): $\int \frac{8x^5 - 3x}{x^3} dx = \int 8x^2 - 3x^{-2} \, dx$.

$$= \frac{8x^3}{3} - \frac{3(x^{-1})}{-1} + c.$$

$$= \frac{8}{3}x^3 + \frac{3}{x} + c.$$

b): $\int (2(\sqrt{x} - 3))(1 - \sqrt{x}) \, dx = \int (2\sqrt{x} - 2x - 3 + 3\sqrt{x}) dx$.

$$= \int (5\sqrt{x} - 2x - 3) dx.$$

$$= \int (5x^{1/2} - 2x - 3) dx.$$

$$= \frac{5x^{3/2}}{3/2} - \frac{2x^2}{2} - 3x + c.$$

$$= \frac{10}{3}x^{3/2} - x^2 - 3x + c:$$

CamScanner

Qn: A curve passes ~~f~~ through point $A(2,0)$ and its gradient function is $\cancel{\frac{3x^2-1}{x^2}}$ $3x^2 - \frac{1}{x^2}$. Find the eqn of the curve:

solution:

Gradient function means $\frac{dy}{dx}$:

$$\frac{dy}{dx} = 3x^2 - \frac{1}{x^2}.$$

Separating variables gives;

$$dy = 3x^2 - \frac{1}{x^2}\, dx.$$

Integrating gives ;

$$\int dy = \int 3x^2 - \frac{1}{x^2}\, dx$$

$$y = \frac{3x^3}{3} - \frac{x^{-1}}{-1}.$$

$$y = x^3 + \frac{1}{x} + c.$$

at point $A(2,0)$;

when $x = 2$, $y = 0$.

$$c = 0 - \left[(2^3) + \frac{1}{2}\right].$$

$$c = \frac{-17}{2}.$$

$\therefore$ $y = x^3 + \frac{1}{x} - \frac{17}{2}$ as eqn of curve:

Qn: Given that $\frac{dv}{dt} = (t+1)(3t-7)$ and $V = 36$ wh $t = 5$;

Show that : $V = (t+1)^2(t-4)$:

soln.

from; $\frac{dv}{dt} = (t+1)(3t-7)$.

$dv = (t+1)(3t-7)\,dt$.

$\int dv = \int (t+1)(3t-7)\,dt$.

$V = \int (3t^2 - 7t + 3t + -7)\,dt$.

$V = \int (3t^2 - 4t + -7)\,dt$.

$V = \frac{3t^3}{3} - \frac{4t^2}{2} + -7t + c$.

$V = t^3 - 2t^2 + -7t + c$.

when $V = 36$, $t = 5$.

$36 = 5^3 - 2(5^2) + -7(5) + c$.

$c = 36 - 125 + \overset{50}{\cancel{125}} + 35$.

$C = -4 \,.= -4$

$\therefore V = t^3 - 2t^2 + -7t + -4$.

$V = t^3 - 2t^2 + -7t - 4$.

By inspection;

$t = -1$ is a root.

Using long division;

$t(t^2-5t+7)+1$

$$
\begin{array}{r|l}
 & t^2 - 3t - 4 \\
\hline
(t+1) & t^3 - 2t^2 - 7t - 4 \\
 & -(t^3 + t^2) \\
\hline
 & -3t^2 - 7t - 4 \\
 & -(-3t^2 - 3t) \\
\hline
 & -4t - 4 \\
 & -(-4t - 4) \\
\hline
 & - \quad -
\end{array}
$$

$t(t^2-2t+7)-4$

$(t-t)($

$(t+1)(t^2-3t-4)$

factorise $t^2 - 3t - 4$.

$t^2 - 4t + t - 4$.

$t(t-4) + 1(t-4)$.

$(t+1)(t-4)$.

$\therefore$ $(t+1)^2(t-4)$ as required.

METHODS OF INTEGRATION:

1: Recognizing the Presence of a derivative:

OR: By Inspection:

Examples:

a): Find; $\int x(2x^2+3)^5 dx$.

solution:

Mtd I: By Inspection:

We increase the power of the function by one and then differentiate;

$$\therefore \frac{d(2x^2+3)^6}{dx} = 6(2x^2+3)^5(4x).$$

$$\frac{d(2x^2+3)^6}{dx} = 24x(2x^2+3)^5.$$

$$d(2x^2+3)^6 = 24x(2x^2+3)^5 dx.$$

$$\int d(2x^2+3)^6 = 24\int x(2x^2+3)^5 dx.$$

$$\frac{1}{24}(2x^2+3)^6 = \int x(2x^2+3)^5 dx.$$

$$\therefore \int x(2x^2+3)^5 dx = \frac{1}{24}(2x+3)^6 + C$$

MTD II: Recognizing the presence of a derivative:

Let $u = 2x^2+3$.

$$\frac{du}{dx} = 4x.$$

$$dx = \frac{du}{4x}.$$

$$\int x(2x^2+3)^5 dx = \int x(u)^5 \cdot \frac{du}{4x}$$

$$\int x(2x^2+3)^5 dx = \frac{1}{4}\int u^5 du.$$

$$\int x(2x^2+3)^5 dx = \frac{1}{4}\left[\frac{u^6}{6}\right] + C.$$

$$\int x(2x^2+3)^5 dx = \frac{1}{24}u^6 + C = \frac{1}{24}(2x^2+3)^6 + C.$$

b): find $\int (5x+7)^5 \, dx$.

MTD 1:

Let $u = 5x+7$.

$$\frac{du}{dx} = 5.$$

$$dx = \frac{du}{5}$$

$$\int (5x+7)^3 = \int u^5 \frac{du}{5}$$

$$= \frac{1}{5}\int u^5 du.$$

$$= \tfrac{1}{5}\left[\frac{u^6}{6}\right] + c.$$

$$= \frac{1}{30}u^6 + c$$

$$\int (5x+7) dx = \underline{\frac{1}{30}(5x+7)^6 + c.}$$

MTD II:

$$\int \frac{d(5x+7)^4}{dx} = 4(5x+7)^3(5).$$

$$\frac{d(5x+7)^4}{dx} = \cancel{4}\ 20(5x+7)^3.$$

$$d(5x+7)^4 = 20(5x+7)^3 dx.$$

$$\frac{1}{20}d(5x+7)^4 = (5x+7)^3 dx.$$

$$\int \frac{1}{20} d(5x+7)^4 = \int (5x+7)^3 dx.$$

$$\therefore \int (5x+7)^3 dx = \underline{\frac{1}{20}(5x+7)^4 + c}:$$

c): find; $\int \frac{x^2-1}{\sqrt{x^3-3x}} dx$.

soln;

$$\int \frac{x^2-1}{\sqrt{x^3-3x}} dx = \int \frac{x^2-1}{(x^3-3x)^{1/2}} dx. = \int (x^2-1)(x^3-3x)^{-1/2} dx$$

MTD 1:

$$\frac{d(x^3-3x)^{1/2}}{dx} = \tfrac{1}{2}(x^3-3x)^{-1/2}(3x^2-3)$$

$$\frac{d(x^3-3x)^{1/2}}{dx} = \frac{1}{2}(x^3-3x)^{-1/2}(3)(x^2-1).$$

$$\frac{2}{3}\frac{d(x^3-3x)^{1/2}}{dx} = (x^2-1)(x^3-3x)^{-1/2}$$

$$\int \frac{2}{3} d(x^3-3x)^{1/2} = \int (x^2-1)(x^3-3x)^{-1/2}\,dx$$

$$\int (x^2-1)(x^3-3x)^{-1/2}\,dx = \frac{2}{3}(x^3-3x)^{1/2} + c$$

MTD II:

Let $u = x^3 - 3x$.

$$\frac{du}{dx} = (3x^2-3)$$

$$dx = \frac{du}{3(x^2-1)}.$$

$$\int (x^2-1)(x^3-3x)^{-1/2}\,dx = \int (x^2-1)(u)^{-1/2}\cdot\frac{du}{3(x^2-1)}$$

$$= \frac{1}{3}\int u^{-1/2}\,du.$$

$$= \frac{1}{3}\left[\frac{u^{1/2}}{1/2}\right] + c.$$

$$\int \frac{x^2-1}{\sqrt{x^3-3x}}\,dx = \frac{2}{3}(x^3-3x)^{1/2} + c.$$

Exercise:

Integrate the following w.r.t. x.

(a): $4(1-x)^{1/2}$.

(b): $\dfrac{x}{(x^2+1)^2}$

(c): $\dfrac{3}{(2x+1)^3} + \sqrt{1-2x}$

(d): $\dfrac{x-1}{(2x^2-4x+1)^{1/2}}$.

(e): $\dfrac{2x}{(4x^2-7)^2}$.

a: Integration of Trigonometric functions:

Consider:

$$\int \cos mx \, dx = \frac{1}{m}\sin mx + c.$$

$$\int \sin mx \, dx = -\frac{1}{m}\cos mx + c.$$

E.g:

Find: i). $\int \cos 7x \, dx = \frac{1}{7}\sin 7x + c.$

ii). $\int \sin 7x \, dx = -\frac{1}{7}\cos 7x + c.$

Integration of a product of two cosines, two sines or a sine and acsine

In this case, you first express the product as the sum or a difference of trigonometrical functions by using the factor formulae ie:

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}.$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}.$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}.$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}.$$

Examples.

1): find: $\int 2\cos 5x \cos 3x \, dx.$

soln.

from; $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}.$

$$\frac{1}{2}(\cos A + \cos B) = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}.$$

$\therefore \frac{A+B}{2} = 5x.$ $\qquad \frac{A-B}{2} = 3x.$

$A+B = 10x$ —— ①. $\qquad A-B = 6x$ —— ②.

① + ②

$2A = 16x.$

$A = 8x \quad , B = 2x.$

$$\therefore \quad = \int (\cos 8x + \cos 2x)\, dx$$

$$= \int \cos 8x \, dx + \int \cos 2x \, dx$$

$$= \frac{1}{8}\sin 8x + \frac{1}{2}\sin 2x + c.$$

$\frac{d\sin x}{dx} = \cos x$

$$\therefore \int 2\cos 5x \cos 3x \, dx = \frac{1}{8}\sin 8x + \frac{1}{2}\sin 2x + c;$$

2: Evaluate: $\int \cos 3x \sin 5x \, dx$.

soln

From; $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$;

$$\frac{1}{2}(\sin A + \sin B) = \sin\frac{A+B}{2}\cos\frac{A-B}{2}.$$

$\therefore \frac{A+B}{2} = 5$ $\qquad \frac{A-B}{2} = 3$

$A+B = 10$ — (1) $\qquad A-B = 6$ — (2).

$$\int \cos 3x \sin 5x \, dx = \int \frac{1}{2}(\sin 8x + \sin 2x)\, dx.$$

$$= \frac{1}{2}\int (\sin 8x + \sin 2x)\, dx.$$

$$\int \qquad = \frac{1}{2}\left(-\frac{1}{8}\cos 8x - \frac{1}{2}\cos 2x\right) + c.$$

$$\int \cos 3x \sin 5x \, dx = -\frac{1}{16}\cos 8x - \frac{1}{4}\cos 2x + c.$$

3: Evaluate: $\int \sin x \sin 3x \, dx$.

using; $\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$.

$$-\frac{1}{2}(\cos A - \cos B) = \sin\frac{A+B}{2}\sin\frac{A-B}{2}.$$

$A = 4$ and $B = 2$.

$A+B=6$
$A-B=2$

$$\int \sin x \sin 3x \, dx = \int -\frac{1}{2}(\cos 4x - \cos 2x)\, dx.$$

$$= -\frac{1}{2}\left(\frac{1}{4}\sin 4x - \frac{1}{2}\sin 2x\right) + c.$$

$$= \frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x + c.$$

41. Show that $\int_0^{\pi/3} 2\sin 3x \cos x \, dx = 9/8$:

soln

from: ~~cosA - cos~~ $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$.

$\frac{A+B}{2} = 3$ $\qquad$ $\frac{A-B}{2} = x$

$A+B = 6$ —— (1) $\qquad$ $A-B = 2$ —— (2).

(1) + (2) :

$A = 4x$ and $B = 2$:

$$\therefore \int_0^{\pi/3} 2\sin 3x \cos x \, dx = \int_0^{\pi/3} (\sin 4x + \sin 2x)\, dx.$$

$$= \left[-\frac{1}{4}\cos 4x - \frac{1}{2}\cos 2x\right]_0^{\pi/3}$$

$$= \left[-\frac{1}{4}\cos\frac{4\pi}{3} - \frac{1}{2}\cos\frac{2\pi}{3}\right] - \left[-\frac{1}{4}\cos 0 - \frac{1}{2}\cos 0\right]$$

$$\int_0^{\pi/3} 2\sin 3x \cos x \, dx = \left(-\frac{1}{4}\cos\frac{4\pi}{3} - \frac{1}{2}\cos\frac{2\pi}{3}\right) - \left(-\frac{1}{4}(1) - \frac{1}{2}\right).$$

$$= \left(-\frac{1}{4}\cos\frac{4\pi}{3} - \frac{1}{2}\cos\frac{2\pi}{3}\right) + 3/4.$$

$$= \left(+\frac{1}{8} + \frac{1}{4}\right) + 3/4$$

$$\int_0^{\pi/3} 2\sin 3x \cos x \, dx = 9/8 \ \#.$$

Integrands Involving even powers of $\sin x$ and $\cos x$:

Here, we use an identity of trigonometry involving double angles : ie

$$\cos 2x = 1 - 2\sin^2 x.$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x).$$

(OR) :

$$\cos 2x = 2\cos^2 x - 1.$$

$$\cos^2 x = \frac{1}{2}(\cos 2x + 1):$$

Examples:

1): Evaluate : $\int \cos^4 x \, dx.$

$$\int \cos^4 x \, dx = \int (\cos^2 x)^2 \, dx.$$

$$= \int \left[\frac{1}{2}(\cos 2x + 1)\right]^2 dx.$$

$$= \int \frac{1}{4}(\cos 2x + 1)^2 dx.$$

$$= \frac{1}{4}\int (\cos^2 2x + 2\cos 2x + 1) dx.$$

$$= \frac{1}{4}\left(\int \cos^2 2x \, dx + \frac{2}{2}(\sin 2x) + x\right).$$

$$= \frac{1}{4}\left(\int \cos^2 2x \, dx + \sin 2x + x\right).$$

But ; from; $\cos 2x = 2\cos^2 x - 1.$

$$\cos x = 2\cos^2 \frac{x}{2} - 1.$$

$$\cos 4x = 2\cos^2 2x - 1.$$

$$\cos^2 2x = \frac{1}{2}(\cos 4x + 1).$$

$$= \frac{1}{4}\left(\int \left(\frac{1}{2}(\cos 4x + 1) dx\right) + \sin 2x + x\right)$$

$$= \frac{1}{4}\left(\frac{1}{2}\left(\frac{1}{4}\sin 4x + x\right) + \sin 2x + x\right) + C.$$

$$= \frac{1}{4}\left(\frac{1}{8}\sin 4x + \frac{1}{2}x + \sin 2x + x\right) + C.$$

$$\therefore \int \cos^4 x\,dx = \frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + \frac{3}{8}x + c$$

2. Evaluate: $\int \sin^4 x\,dx$.

$$\int \sin^4 x\,dx = \int (\sin^2 x)^2\,dx.$$

$$= \int \left[\frac{1}{2}(1-\cos 2x)\right]^2 dx.$$

$$= \int \frac{1}{4}\left(1 + -2\cos 2x + \cos^2 2x\right) dx.$$

$$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx.$$

$\cos 4x = 2\cos^2 2x - 1$; $\cos^2 2x = \frac{1}{2}(\cos 4x + 1)$:

$$= \frac{1}{4}\left[\left(x - \sin 2x + \int \cos^2 2x\,dx\right)\right].$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \int \cos^2 2x\,dx.$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{4}\int \frac{1}{2}(\cos 4x + 1)\,dx.$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}\left(\frac{1}{4}\sin 4x + x\right) + c.$$

$$= \frac{1}{32}\sin 4x + 3/8\, x - 1/4 \sin 2x + c.$$

$$= \frac{1}{32}\sin 4x - \frac{1}{4}\sin 2x + 3/8\, x + c :$$

3): Show that;

$$\int_0^{\pi/2} (3\sin^2 x + 2\cos^2 x)\,dx = \frac{5\pi}{4}$$

$$= \int_0^{\pi/2} 3\left(\frac{1}{2}(1-\cos 2x)\right) + 2\left(\frac{1}{2}(\cos 2x + 1)\right) dx.$$

$$= \int_0^{\pi/2} 3/2(1-\cos 2x) + (\cos 2x + 1)\,dx.$$

$$= \frac{3}{2}\left[\left(x + \frac{1}{2}\sin 2x\right)\right]_0^{\pi/2} + \left[\frac{1}{2}\sin 2x + x\right]_0^{\pi/2}.$$

$$= 3/2\left[\left(\pi/2 + 1/2 \sin\pi\right) - \left(0 + \tfrac{1}{2}\sin 0\right)\right] + \left[\left(\tfrac{1}{2}\sin\pi + \pi/2\right) - \left(\tfrac{1}{2}\sin\ldots\right.\right.$$

$$= 3/2\left(\pi/2 - 0\right) + \left(\pi/2\right)$$

$$= \frac{3\pi}{4} + \pi/2.$$

$$= \frac{5\pi}{4} \text{ as required}$$

4: Evaluate $\int \cos^4 2x \, dx$.

soln.

$$\int \cos^4 2x \, dx = \int (\cos^2 2x)^2 \, dx.$$

$$= \int \left(\tfrac{1}{2}(\cos 4x + 1)\right)^2 dx.$$

$$= \frac{1}{4}\int \left(\cos^2 4x + 2\cos 4x + 1\right) dx.$$

$$= \frac{1}{4}\left[\int \cos^2 4x \, dx + \frac{1}{2}\sin 4x + x\right].$$

$$= \frac{1}{4}\left(\frac{1}{2}\sin 4x + x\right) + \frac{1}{4}\int \cos^2 4x \, dx.$$

$$= \frac{1}{8}\sin 4x + \frac{1}{4}x + \frac{1}{4}\int \frac{1}{2}(\cos 8x + 1)\, dx.$$

$$= \frac{1}{8}\sin 4x + \frac{1}{4}x + \frac{1}{64}\sin 8x + \frac{1}{8}x + C.$$

$$= \frac{1}{64}\sin 8x + \frac{1}{8}\sin 4x + 3/8\, x + C.$$

$\cos 2x = 2\cos^2 x - 1.$

$\cos 4x = 2\cos^2 2x - 1.$

$\cos 8x = 2\cos^2 4x - 1.$

$\cos 16x = 2\cos^2 8x - 1.$

$\frac{16}{\times 4}$

$\cos^2 4$

$\cos 8x = 2\cos^2 4x - 1$

Integrands Involving odd Powers of $\cos x$ and $\sin x$.

Here, we use the identity ; $\cos^2 x + \sin^2 x = 1$.

Example :

a) Evaluate : $\int \sin^3 x \, dx$;

soln

$$\int \sin^3 x \, dx = \int (\sin^2 x) \sin x \, dx .$$

$$= \int (1 - \cos^2 x) \sin x \, dx .$$

$$= \int \sin x - \cos^2 x \sin x \, dx .$$

$$= \int \sin x \, dx - \int \sin x \cos^2 x \, dx .$$

$$= -\cos x - \int \sin x \cos^2 x \, dx .$$

$$= \cancel{-\cos x - \int \sin x \left(\tfrac{1}{2}(1 + \cos 2x)\right) dx} .$$

$$= \cancel{-\cos x - \int \tfrac{1}{2} \sin x + 1}$$

but $\int \sin x \cos^2 x \, dx = ?$

$$\int \cos^2 x \sin x \, dx = ?$$

$$\therefore \frac{d \cos^3 x}{dx} = -3 \sin x \cos^2 x .$$

$$d \cos^3 x = -3 \sin x \cos^2 x$$

$$\int d \cos^3 x = -3 \int \sin x \cos^2 x \, dx .$$

$$-\frac{1}{3} \int d \cos^3 x = \int \sin x \cos^2 x \, dx .$$

$$\therefore \int \sin x \cos^2 x \, dx = -\frac{1}{3} \cos^3 x + c .$$

$$\therefore \int \sin^3 x \, dx = -\cos x - \left(-\tfrac{1}{3} \cos^3 x\right) + c .$$

$$\int \sin^3 x \, dx = \frac{1}{3} \cos^3 x - \cos x + C$$

b); Evaluate: $\int \cos^3 3x \, dx$.

soln.:

$$\int \cos^3 3x \, dx = \int \cos 3x \cdot (\cos^2 3x) \, dx.$$

$$= \int \cos 3x \, (1 - \sin^2 3x) \, dx.$$

$$= \int (\cos 3x - \cos 3x \sin^2 3x) \, dx.$$

$$\int \cos^3 3x \, dx = \int \cos 3x \, dx - \int \cos 3x \sin^2 3x \, dx.$$

$$= +\frac{1}{3} \sin 3x - \int \cos 3x \sin^2 3x \, dx.$$

$\frac{d}{dx}$

but; $\int \sin^2 3x \cos 3x \, dx = ?$

$$\because \frac{d \, \cancel{\cos} (\sin^3 3x)}{dx} = (3) 3 \sin^2 3x \cos 3x$$

$$d \sin^3 3x = 9 \sin^2 3x \cos 3x \, dx.$$

$$\int d(\sin^3 3x) = 9 \int \sin^2 3x \cos 3 dx,$$

$$\frac{1}{9} \int d(\sin^3 3x) = \int \sin^2 3x \cos 3x \, dx,$$

$$\therefore \int \sin^2 3x \cos 3x \, dx = \frac{1}{9} \sin^3 3x + c.$$

$$\int \cos^3 3x \, dx = \frac{1}{3} \sin 3x + \frac{1}{9} \sin^3 3x + c.$$

$$\int \cos^3 3x \, dx = \frac{1}{3} \sin 3x - \frac{1}{9} \sin^3 3x + c.$$

Exercise:

1: Evaluate;

(i). $\int \sin^5 2x\,dx$.

(ii). $\int \cos^5 2x\,dx$

(iii). $\int \tan x \sec^2 x\,dx$.

(iv). $\int \cos^2 3x\,dx$.

(v). $\int \sin^4 5x\,dx$.

Solution.

iii) $\int \tan x \sec^2 x\,dx$.

let $u = \tan x$

$\frac{du}{dx} = \sec^2 x$.

$dx = \frac{du}{\sec^2 x}$.

$\therefore \int \cancel{t} u \sec^2 x \cdot \frac{du}{\sec^2 x}$.

$\int u\,du$

$\frac{u^2}{2} + c$

$\frac{1}{2}(\tan^2 x) + c$.

ii). $\int \cos^5 2x\,dx. = \int (\cos^2 2x)^2 (\cos^2 2x) \cos 2x\,dx$.

$= \int (1-\sin^2 2x)(1-\sin^2 2x)\cos 2x\,dx$.

$= \int (1-2\sin^2 2x + \sin^4 2x)\cos 2x\,dx$.

$= \int \cos 2x - 2\sin^2 2x \cos 2x + \cos 2x \sin^4 2x\,dx$.

$= \int \cos 2x\,dx - 2\int \sin^2 2x \cos 2x\,dx + \int \cos 2x \sin^4 2x\,dx$

$= -2\sin 2x - 2\int \sin^2 2x \cos 2x\,dx + \int \cos 2x \sin^4 2x\,dx$

from:

$\frac{d(\sin^3 2x)}{dx} = 3\sin^2 2x \cdot (2) \cdot \cos 2x$.

$\frac{d\sin^3 2x}{dx} = 6\sin^2 2x \cos 2x$.

$\int d\sin^3 2x = \int 6\sin^2 2x \cos 2x\,dx$.

$\int \sin^2 2x \cos 2x\,dx = \frac{1}{6}\sin^3 2x + c$.

$= -2\sin 2x - \frac{1}{3}\sin^3 2x + \int \cos 2x \sin^4 2x\,dx$

d

$= -2\sin 2x - \frac{1}{3}\sin^3 2x + \frac{1}{10}\sin^5 2x + c$.

$\sin^5 2x = 5\sin^4 2x\,(2)\cos 2x$

Integration of functions using Sine and tan substitution:

When integrating the integrand involving $\frac{1}{a^2+b^2x^2}$, we use the tan substitution and w/n integrating the integrand involving $\frac{1}{\sqrt{a^2-b^2x^2}}$, we use sine substitution:

1): Integrands involving use of sine substitution:

a): Evaluate: $\int \frac{2}{\sqrt{16-x^2}}\,dx$.

Soln.:

Let $16-x^2 = 16-16\sin^2 u$
$= 16(1-\sin^2 u)$
$16-x^2 = 16\cos^2 u$ ——(i)

but; $x^2 = 16\sin^2 u$.
$x = 4\sin u$.
$dx = 4\cos u\,du$ ——(ii)

but. $\sin u = \frac{1}{4}x$
$u = \sin^{-1}(\frac{1}{4}x)$.

$\therefore = \int \frac{2}{\sqrt{16\cos^2 u}} \cdot 4\cos u\,du$

$= \int \frac{2}{4\cos u} \cdot 4\cos u\,du$.

$= 2\int du$.

$= 2u + c$.

$= 2\sin^{-1}\left(\frac{1}{4}x\right) + c$;

2: Evaluate; $\int \frac{1}{\sqrt{2-3x^2}}\,dx$.

let $2-3x^2 = 2-2\sin^2 u = 2\cos^2 u$;

but; $3x^2 = 2\sin^2 u$.
$\sqrt{3}x = \sqrt{2}\sin u$.
$dx = \frac{\sqrt{2}}{\sqrt{3}}\cos u\,du$.

; $\sin u = \frac{\sqrt{3}x}{\sqrt{2}}$.
$u = \sin^{-1}\left(\frac{\sqrt{3}}{\sqrt{2}}x\right)$.

$= \int \frac{1}{\sqrt{2\cos^2 u}} \cdot \frac{\sqrt{2}}{\sqrt{3}}\cos u\,du$.

$= \int \frac{1}{\sqrt{2}\cos u} \cdot \frac{\sqrt{2}\cos u\,du}{\sqrt{3}}$

$= \frac{1}{\sqrt{3}} u + c$.

$= \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{\sqrt{3}}{\sqrt{2}} x\right) + c$:

3: Evaluate: $\int \frac{x}{\sqrt{16 - x^4}} \, dx$.

soln.

let; $16 - x^4 = 16 - 16\sin^2 u = 16\cos^2 u$.

but; $x^4 = 16\sin^2 u$.

$x^2 = 4\sin u$. , $u = \sin^{-1}\left(\frac{x^2}{4}\right)$

$\frac{d(x)^2}{du} = 4\cos u$.

$2x\,dx = 4\cos u\, du$

$dx = \frac{2\cos u\, du}{x}$.

$\therefore \int \frac{x}{\sqrt{16\cos^2 u}} \cdot 2\cos u\, du$.

$\int \frac{x}{4\cos u} \cdot \frac{2\cos u\, du}{x}$.

$\frac{1}{2} u + c$.

$\frac{1}{2} \sin^{-1}\left(\frac{1}{4} x^2\right) + c$.

4: Evaluate: $\int \frac{1}{(1 - 4x^2)^{3/2}} \, dx$.

let $1 - 4x^2 = 1 - \sin^2 u = \cos^2 u$.

$4x^2 = \sin^2 u$.

$2x = \sin u$.

$2\,dx = \cos u\, du$.

$dx = \frac{1}{2} \cos u\, du$.

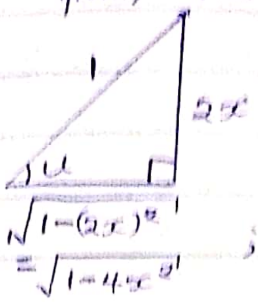
$= \int \frac{1}{(\sqrt{\cos^2 u})^3} \cdot \frac{1}{2} \cos u\, du$.

$= \int \frac{1}{(\cos u)^3} \cdot \frac{1}{2} \cos u\, du$.

$$= \frac{1}{2}\int \frac{1}{\cos^2 u}\,du = \frac{1}{2}\int \frac{1}{\cos^2 u}\,du \;;\; \text{but } \frac{1}{\cos^2 u} = \sec^2 u.$$

$$= \frac{1}{2}\int \sec^2 u\,du.$$

from; $\sin u = 2x$;



$$= \frac{1}{2}\tan u + C \;;$$

$$= \frac{1}{2}\left(\frac{2x}{\sqrt{1-4x^2}}\right) + C.$$

; $\tan u = \dfrac{2x}{\sqrt{1-4x^2}}$

$$= \frac{x}{\sqrt{1-4x^2}} + C.$$

5): Evaluate : $\displaystyle\int \frac{1}{\sqrt{3-2x-x^2}}\,dx.$

Solution

By completing squares ;

$$= (3-2x-x^2)^2 = -1(x^2+2x-3).$$

$$= -1(x^2+2x-3)$$

$$= -1\left(x^2+2x+(1)^2-(1)^2-3\right).$$

$$= -1\left((x+1)^2-4\right).$$

$$= 4-(x+1)^2.$$

$$\therefore \int \frac{1}{\sqrt{4-(x+1)^2}}\,dx = \int$$

Let ; $4-(x+1)^2 = 4-4\sin^2 u = 4\cos^2 u.$

$$(x+1)^2 = 4\sin^2 u.$$

$$x+1 = 2\sin u. \quad ; \quad u = \sin^{-1}\left(\frac{x+1}{2}\right).$$

$$dx = 2\cos u\,du$$

$$dx = 2\cos u\,du.$$

$$\therefore \int \frac{1}{\sqrt{4\cos^2 u}}\cdot 2\cos u\,du = \int \frac{1}{2\cos u}\cdot 2\cos u\,du$$

$$= \int du = u + c.$$

$$= \sin^{-1}\left(\tfrac{1}{2}(x+1)\right) + c.$$

Exercise.

i): Evaluate ; $\int \frac{9x^2}{\sqrt{1-x^6}}\,dx$.

ii) Evaluate : $\int \frac{x^2}{(4-x^2)^{5/2}}\,dx$

iii) Evaluate : $\int \frac{1}{\sqrt{1+8x-4x^2}}\,dx$

iv) Evaluate : $\int \frac{2x+3}{\sqrt{4-x^2}}$.

$4-x^2 = 4-4\sin^2 u = 4\cos^2 u$.

$x^2 = 4\sin^2 u$.

$x = 2\sin u$.

$\frac{dx}{du} = 2\cos u$.

$\int \frac{2x+3}{\cancel{2}\cos u} \cdot \cancel{2}\cos u\,du$.

$\int (2x+3)\,du$.

$\int 2x\,du + \int 3\,du$.

$4\int \sin u\,du + \int 3\,du$.

$4\cos u + 3u + C$.

C Integrands Involving tan substitution:

Examples.

1): Evaluate : $\int \frac{1}{9+x^2}\,dx$.

soln.

$$\int \frac{1}{9+x^2}\,dx.$$

let $9+x^2 = \cancel{9+9\sec^2}\ 9+9\tan^2 u = 9(1+\tan^2 u)$

$= 9\sec^2 u$.

$$\int \frac{1}{9\sec^2 u} \cdot dx.$$

but $x^2 = 9\tan^2 u$.

$x = 3\tan u$; $u = \tan^{-1}(\frac{1}{3}x)$.

$dx = 3\sec^2 u\,du$.

$$= \int \frac{1}{9\sec^2 u} \cdot 3\sec^2 u\,du.$$

$$= \frac{1}{3}\int du = \frac{1}{3}u + C.$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{1}{3}x\right) + C.$$

2); Evaluate : $\int \frac{1}{5+9x^2}dx$.

soln

let $5+9x^2 = 5+5\tan^2 u = 5(1+\tan^2 u) = 5\sec^2 u$.

$= \int \frac{1}{5\sec^2 u}dx$.

but ; $9x^2 = 5\tan^2 u$.

$3x = \sqrt{5}\tan u$; ($u = \tan^{-1}\left(\frac{3}{\sqrt{5}}x\right)$).

$3dx = \sqrt{5}\sec^2 u\, du$.

$dx = \frac{\sqrt{5}}{3}\sec^2 u\, du$.

$= \int \frac{1}{5\sec^2 u} \cdot \frac{\sqrt{5}}{3}\sec^2 u\, du$.

$= \frac{\sqrt{5}}{15}\int du = \frac{\sqrt{5}}{15}u + c$.

$= \frac{\sqrt{5}}{15}\tan^{-1}\left(\frac{3}{\sqrt{5}}x\right) + c$

3); Evaluate : $\int \frac{1}{3x^2+6x+5}dx$.

soln

By completing squares:

$3x^2+6x+5 = \cancel{x^2 + \frac{6}{3}x + \frac{7}{3}}$.

$= 3\left(x^2+2x+\frac{5}{3}\right)$.

$= 3\left(x^2+2x+(1)^2+\frac{5}{3}-(1)^2\right)$.

$= 3\left((x+1)^2+\frac{2}{3}\right) = 3(x+1)^2+2$.

$\therefore \int \frac{1}{\frac{2}{3}+3(x+1)^2}dx$.

let $\frac{2}{3}+3(x+1)^2 = \frac{2}{3}+\frac{2}{3}\overset{\tan^2 u}{\cancel{(x+1)^2}} = \frac{2}{3}\sec^2 u$

$\cancel{\therefore \frac{2}{3}(x+1)^2 = (x+1)^2}$.

$3(x+1)^2 = 2\tan^2 u$

$\sqrt{3}(x+1) = \sqrt{2}\tan u \quad ; \quad u = \tan^{-1}\left(\frac{\sqrt{3}}{\sqrt{2}}(x+1)\right)$

$\sqrt{3}\,dx = \frac{\sqrt{2}}{\sqrt{2}}\sec^2 u\,du \quad ; \quad dx = \frac{\sqrt{2}}{\sqrt{3}}\sec^2 u\,du$

$= \int \frac{1}{2\sec^2 u} \cdot \frac{\sqrt{2}}{\sqrt{3}}\sec^2 u\,du$

$= \frac{\sqrt{2}}{2\sqrt{3}}\int du = \frac{\sqrt{2}}{2\sqrt{3}}u + C$

$= \underline{\frac{\sqrt{2}}{2\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}(x+1)}{\sqrt{2}}\right) + C}$

$(x+1) = \frac{\sqrt{2}}{\sqrt{3}}\tan u$

Exercise:

Evaluate the following;

i) $\int \frac{1}{1+16x^2}\,dx$ (ii) $\int \frac{1}{x^2-2x+5}\,dx$

iii) $\int \frac{1}{4x^2+9}\,dx$ (iv) Show that; $\int_{\frac{\sqrt{3}}{2}}^{3/2} \frac{dx}{9+4x^2}\,dx = \frac{\sqrt{3}}{6}\left(\frac{\pi}{12}\right)$

$3(x+1)^2 = 2\tan^2 u$.

$\sqrt{3}(x+1) = \sqrt{2}\tan u$; $u = \tan^{-1}\left(\frac{\sqrt{3}}{\sqrt{2}}(x+1)\right)$.

$\sqrt{3}\,dx = \frac{\sqrt{2}}{\sqrt{3}}\sec^2 u\,du$; $dx = \frac{\sqrt{2}}{\sqrt{3}}\sec^2 u\,du$.

$$= \int \frac{1}{2\sec^2 u} \cdot \frac{\sqrt{2}}{\sqrt{3}}\sec^2 u\,du.$$

$$= \frac{\sqrt{2}}{2\sqrt{3}}\int du = \frac{\sqrt{2}}{2\sqrt{3}}u + C.$$

$$= \frac{\sqrt{2}}{2\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}(x+1)}{\sqrt{2}}\right) + C.$$

$(x+1) = \frac{\sqrt{2}}{\sqrt{3}}\tan u$

Exercise:

Evaluate the following;

i) $\int \frac{1}{1+16x^2}\,dx$ (ii) $\int \frac{1}{x^2-2x+5}\,dx$.

iii) $\int \frac{1}{4x^2+9}\,dx$ (iv) show that ; $\int_{\frac{\sqrt{3}}{2}}^{3/2} \frac{dx}{3+4x^2}\,dx = \frac{\sqrt{3}}{6}($

Integration of exponential and Logarithmic functions:

a): $\frac{d(e^{ax})}{dx} = ae^{ax}$.

$d(e^{ax}) = ae^{ax}\,dx$.

$\int \frac{1}{a} d(e^{ax}) = \int \frac{a}{a} e^{ax}\,dx$.

$\frac{1}{a} e^{ax} = \int e^{ax}$.

$\therefore \int e^{ax} = \frac{1}{a} e^{ax} + c$.

b): $\frac{d(a^x)}{dx}$.

let $y = a^x$.

$\ln y = x \ln a$.

$\ln y = x \ln a$.

$\frac{1}{y}\frac{dy}{dx} = \ln a$.

$\frac{dy}{dx} = y \ln a$.

$\frac{dy}{dx} = a^x \ln a$.

$\frac{1}{\ln a} dy = a^x\,dx$.

$\frac{1}{\ln a}\int dy = \int a^x\,dx$

$\int a^x\,dx = \frac{y}{\ln a} + c$.

$\int a^x\,dx = \frac{a^x}{\ln a} + c$.

c): $\frac{d}{dx}(\ln(ax+b))$:

let $y = \ln(ax+b)$.

$\cancel{\frac{dy\,y}{d}} = \quad \frac{dy}{dx} = \frac{a}{ax+b}$.

$$\int dy = \int \frac{a}{ax+b}\,dx .$$

$$\int d(\ln(ax+b)) = \int \frac{a}{ax+b}\,dx .$$

$$\ln(ax+b) = \int \frac{a}{ax+b}\,dx .$$

$$\therefore \int \frac{f'(x)}{f(x)} = \ln f(x) + c .$$

<u>Examples :</u>

i): Find ;(i): $\int xe^{3x^2}\,dx$.

(ii): $\int \sec x \tan x\, e^{\sec x}\,.dx$.

(iii): $\int \frac{e^{\cot x}}{\sin^2 x}\,dx$.

(iv): $\int \frac{1}{3x+4}\,dx$.

(5): $\int \frac{x}{1-5x^2}\,dx$.

(6): $\int \frac{x+1}{3+4x^2}\,dx$.

(7): $\int x\, 2^{3x^2}\,dx$.

Solution.

(i) $\int x e^{3x^2} dx.$

$$\frac{d(e^{3x^2})}{dx} = \cancel{6e^{3x^2}} \; 6x e^{3x^2}.$$

$$\frac{1}{6} d e^{3x^2} = x e^{3x^2} dx.$$

$$\frac{1}{6}\int d e^{3x^2} = \int x e^{3x^2} dx.$$

$$\int x e^{3x^2} dx = \frac{1}{6} e^{3x^2} + c.$$

(OR):

let $u = 3x^2.$

$$\frac{du}{dx} = 6x.$$

$$dx = \frac{du}{6x}.$$

$$\int x e^{u} \cdot \frac{du}{6x} = \frac{1}{6}\int e^{u} du.$$

$$= \frac{1}{6}\left[e^{u}\right] + c.$$

$$= \frac{1}{6} e^{3x^2} + c.$$

(ii) $\int \sec x \tan x \, e^{\sec x} dx.$

let $u = \sec x.$

$$\frac{du}{dx} = \sec x \tan x.$$

$$= \int \sec x \tan x \, e^{u} \cdot \frac{du}{\sec x \tan x}.$$

$$= \int e^{u} du = e^{u} + c.$$

$$= e^{\sec x} + c.$$

(iii) $\frac{d(e^{\cot x})}{dx} = -\operatorname{cosec}^2 x \, e^{\cot x}$.

$$\frac{-1}{\operatorname{cosec}^2 x} \, d e^{\cot x} = e^{\cot x} dx.$$

$$-d e^{\cot x} = \operatorname{cosec}^2 x \, e^{\cot x} dx.$$

$$-\int d e^{\cot x} = \int \frac{e^{\cot x}}{\sin^2 x} dx \quad ; \quad \operatorname{cosec}^2 x = \frac{1}{\sin^2 x}.$$

$$\int \frac{e^{\cot x}}{\sin^2 x} dx = \underline{-e^{\cot x} + c}$$

(iv) let $u = 3x + 4$.

$$\frac{du}{dx} = 3.$$

$$dx = \frac{du}{3}.$$

$$\int \frac{1}{3x+4} \cdot \frac{du}{3}.$$

$$\frac{1}{3} \int \frac{1}{u} du = \frac{1}{3} \ln u + c.$$

$$= \underline{\frac{1}{3} \ln(3x+4) + c}.$$

(v): $\int x^2 \, 2^{3x^2} dx$.

let $y = \cancel{3x^2} \cdot 2^{3x^2}$

$$\ln y = \ln 2^{3x^2}.$$

$$\ln y = 3x^2 \ln 2.$$

$$\frac{1}{y} \frac{dy}{dx} = 6x \ln 2.$$

$$\frac{dy}{dx} = y \, 6x \ln 2.$$

$$\frac{dy}{dx} = 6x \ln 2 \cdot 2^{3x^2}.$$

(vii) $\int x \cdot 2^{3x^2} dx$.

let $y = 2^{3x^2}$.

$\ln y = \ln 2^{3x^2}$.

$\ln y = 3x^2 \ln 2$.

$\frac{1}{y}\frac{dy}{dx} = 6x \ln 2$.

$\frac{dy}{dx} = 6x \ln 2 \, (2^{3x^2})$.

$dy = 6x \ln 2 \, (2^{3x^2}) \, dx$.

~~$d(2x$~~

$d(2^{3x^2}) = 6x \ln 2 \, (2^{3x^2}) \, dx$.

$\int d(2^{3x^2}) = 6 \ln 2 \int x \cdot 2^{3x^2} dx$.

$$\int x 2^{3x^2} dx = \frac{1}{6 \ln 2} (2^{3x^2}) + c.$$

(viii) $\int \frac{x+1}{3+4x^2} dx = \int \frac{x}{3+4x^2} dx + \int \frac{1}{3+4x^2} dx$.

Note:

i) $\int \cot x \, dx = \ln(\sin x) + c$.

ii) $\int \tan x \, dx = \cancel{\ln(\operatorname{cosec} x) + c} \; \ln(\sec x) + c$

$= \int \frac{\sin x}{\cos x} dx$.

$= -\ln(\cos x) + c$.

$= \ln(\sec x) + c$.

~~TAN SUBSTITUTION~~ :

PARTIAL FRACTIONS :

For this case, we shall look at three types of partial fractions depending on the nature of the denominator.

These are denominators with ;

ie -Linear factors ie $(x+1)$, $(2x+1)$, $(x-1)$, - - -

- Repeated factors ie $(x+1)^2$, $(x-1)^3$, $(x+2)^3$, - - -
- Quadratic factors ie (x^2+1) , x^2 , (ax^2+bx+c) , - - -

LINEAR FACTORS :

Each linear factor in the denominator is in the form $ax+b$ and has corresponding partial fraction of the term $\frac{A}{ax+b}$ where; A, a, b are constants:

Examples:

i) Express $\frac{x-1}{(x+1)(x-2)}$ in partial fractions and hence integrate with respect to x :

solution:

$$\frac{x-1}{(x+1)(x-2)} \equiv \frac{A}{x+1} + \frac{B}{x-2}$$

$$\frac{x-1}{(x+1)(x-2)} = \frac{A(x-2)+B(x+1)}{(x+1)(x-2)}$$

$$\therefore x-1 = A(x-2)+B(x+1)$$

Equating coefficients ;

x ; $1 = A+B$ - - - (i)

x^0 ; $-1 = -2A+B$ - - - (ii)

(i) − (ii)

$2 = 3A$. Sub A in (i) .

$A = 2/3$. $B = 1/3$.

$$\therefore \frac{x-1}{(x+1)(x-2)} = \frac{2}{3(x+1)} + \frac{1}{3(x-2)}$$

Mtd 2:

Sub $x=2$ gives ;

$2-1 = A(2-2) + B(2+1)$.

$1 = 3B$.

$B = 1/3$

Sub $x=-1$ gives;

$-2 = A(-1-3) + 0$.

$-2 = -3A$.

$A = 2/3$

$$\int \frac{x-1}{(x+1)(x-2)}\,dx = \int \frac{2}{3(x+1)}\,dx + \int \frac{1}{3(x-2)}\,dx.$$

$$= \frac{2}{3}\int \frac{1}{(x+1)}\,dx + \frac{1}{3}\int \frac{1}{x-2}\,dx.$$

$$= \frac{2}{3}\ln(x+1) + \frac{1}{3}\ln(x-2) + C.$$

2: Express $\frac{1}{x(x-3)(x+3)}$ in partial fractions and hence find $\int f(x)\,dx$.

solution:

$$\frac{1}{x(x-3)(x+3)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+3}.$$

$$\frac{1}{x(x-3)(x+3)} = \frac{A(x-3)(x+3) + B(x)(x+3) + C(x+3)(x)}{(x)(x-3)(x+3)}.$$

$$1 = A(x-3)(x+3) + Bx(x+3) + C(x-3)(x).$$

Sub $x=0$ gives;

$1 = -9A$.

$A = -\frac{1}{9}$.

Sub $x=3$ gives;

$1 = 18B$.

$B = 1/18$.

Sub $x=-3$ gives ;

$1 = -18C$.

$C = -\frac{1}{18}$.

$$\therefore \frac{1}{x(x-3)(x+3)} = \frac{-1}{9(x)} + \frac{1}{18(x-3)} + \frac{1}{18(x+3)}.$$

$$\int \frac{1}{x(x-3)(x+3)}\,dx = -\frac{1}{9}\int \frac{1}{x}\,dx + \frac{1}{18}\int \frac{1}{(x-3)}\,dx + \frac{1}{18}\int \frac{1}{x+3}\,dx$$

$$= -\frac{1}{9}\ln x + \frac{1}{18}\ln(x-3) + \frac{1}{18}\ln(x+3) + c$$

3: Express $\frac{4x-9}{x^2-5x+6}$ as a partial fraction and hence integrate w.r.t. x:

solution.

$$\frac{4x-9}{x^2-5x+6} = \frac{4x-9}{x^2-2x-3x+6} = \frac{4x-9}{x(x-2)-3(x-2)}$$

$$\frac{4x-9}{x^2-5x+6} = \frac{4x-9}{(x-3)(x-2)}.$$

Sum: -5
Product $= 6$.
factors $(-2,-3)$.

$$\therefore \frac{4x-9}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}.$$

$$4x-9 = A(x-2) + B(x-3);$$

Sub $x=2$ gives;

$$-1 = 0 - B$$

$$B = 1$$

Sub $x=3$ gives;

$$3 = A$$

$$\therefore A = 3$$

$$\therefore \frac{4x-9}{x^2-5x+6} = \frac{3}{x-3} + \frac{1}{x-2}.$$

$$\int \frac{4x-9}{x^2-5x+6}\,dx = \int \frac{3}{x-3}\,dx + \int \frac{1}{x-2}\,dx.$$

$$= 3\ln(x-3) + \ln(x-2) + c.$$

b): Denominator with repeated factors:

~~Equa~~ Each repeated factor $(ax+b)^n$ in the denominator has a corresponding partial fraction of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_n}{(ax+b)^n}$$ where

a, b, A_i are constants;

Examples:

1): Express $\frac{4x-9}{(x-3)^2}$ in partial fraction and find

$\int f(x)\,dx$:

solution:

$$\frac{4x-9}{(x-3)^2} = \frac{A}{(x-3)} + \frac{B}{(x-3)^2}.$$

$$4x-9 = A(x-3) + B.$$

Sub $x=3$ gives;

$3 = B.$

$B = 3.$

Sub $x=0$ gives;

$-9 = -3A + B.$

$-9 = -3A + 3.$

$A = 4.$

$$\therefore \frac{4x-9}{(x-3)^2} = \frac{4}{x-3} + \frac{3}{(x-3)^2},$$

$$\int \frac{4x-9}{(x-3)^2}\,dx = \int \frac{4}{x-3}\,dx + \int \frac{3}{(x-3)^2}\,dx.$$

$$= 4\ln(x-3) + \frac{-3}{(x-3)} + C.$$

$$= 4\ln(x-3) - \frac{3}{(x-3)} + C;$$

2): Express $\frac{x-3-2x^2}{x^2(x-1)}$ in partial fractions and hence intergrate w.r.t. x.

Solution:

$$\frac{x-3-2x^2}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)}.$$

$$x-3-2x^2 = A(x(x-1)) + B((x-1)) + C(x^2).$$

Sub; $x=0$ gives;

$-3 = -B.$

$B = 3$

Sub $x=1$ gives;

$C = -4$

Sub $x=2$ gives;

2-3-8

2-11 = -9

1-3-2.

1-5

-4

$-9 = 2A + B + 4C.$

$-9 = 2A + 3 - 16.$

$-9 = 2A - 13.$

$2A = 4.$

$A = 2$

$$\therefore \frac{x-3-2x^2}{x^2(x-1)} = \frac{2}{x} + \frac{3}{x^2} + \frac{-4}{(x-1)}.$$

$$= \frac{2}{x} + \frac{3}{x^2} - \frac{4}{(x-1)}.$$

$$\therefore \int \frac{x-3-2x^2}{x^2(x-1)}\,dx = \int \frac{2}{x}\,dx + \int \frac{3}{x^2}\,dx - \int \frac{4}{(x-1)}\,dx.$$

$$= 2\ln x - \frac{3}{x} - 4\ln(x-1) + C.$$

3): Express $\frac{10+6x-3x^2}{(2x-1)(x+3)^2}$ in partial fractions. and hence integrate :

$\int \frac{3}{(x-3)^2}\,dx$

$3\int \frac{1}{(x-3)^2}$

$u = x-3$

$\frac{du}{dx} = 1.$

$\int \frac{1}{u^2}\,du.$

$u^{-2}\,du.$

$\cancel{2u}$

$\frac{u^{-1}}{-1}.$

$= \frac{-3}{u}.$

c): Denominator with a quadratic factor:

Each quadratic factor of the form (ax^2+bx+c) in the denominator which can not be factorised has a corresponding partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$ where a, b, c, A and B are constants;

examples:

1): Express $f(x) = \frac{2x-1}{(x-1)(x^2+1)}$ in partial fractions hence integrate:

Solution:

$$\frac{2x-1}{(x-1)(x^2+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+1)}.$$

$$2x-1 = A(x^2+1)+(Bx+C)(x-1);$$

~~Sub x=1~~; $2x-1 = Ax^2+A+Bx^2-Bx+Cx-C.$

~~1 = 2A~~ $2x-1 = Ax^2+Bx^2-Bx+Cx+A-C$

~~A =~~ Equating coefficients;

x^2 ; $0 = A+B$ --- (i)

x' ; $2 = -B+C$ --- (ii)

x^0 ; $-1 = A-C$ --- (iii)

from (iii);

$C=(1+A)$ --- ⊛.

Sub ⊛ in (ii)

$2 = -B+1+A.$

$-C=-1-A$

$C=1+A$:

$1 = A-B$ --- ⊛⊛.

(i) + ⊛⊛;

$1 = 2A.$

$A = 1/2$

from; $A-C = -1.$

$-C = +\cancel{1/2} -1-1/2.$

$C = 3/2$

from; $A+B = 0.$

$B = -1/2$

$\therefore \dfrac{2x-1}{(x-1)(x^2+1)} = \dfrac{1}{2(x-1)} - \dfrac{x}{2(x^2+1)} + \dfrac{3}{2(x^2+1)}.$

Hence;

$$\int \frac{2x-1}{(x-1)(x^2+1)}\,dx = \frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x^2+1) + \frac{3}{2}\tan^{-1}(x$$

$$= \underline{\frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x^2+1) + \frac{3}{2}\tan^{-1}(x) + c}$$

2): Express the following in partial fractions;

(i) $\dfrac{5x^2-10x+11}{(x-3)(x^2+4)}$.

(ii) $\dfrac{5x^2-2x-1}{(x+1)(x^2+1)}$.

(iii) show that $\displaystyle\int \frac{4x^2+4x+25}{x(4x^2+25)}\,dx = \ln x + \frac{2}{5}\tan^{-1}\left(\frac{2x}{5}\right) + c$

(iv) $f(x) = \displaystyle\int_1^3 \frac{2x^2-x+14}{(4x^2-1)(x+3)}\,dx$.

Improper Fractions:

Examples are $\frac{x}{x}$, $\frac{x^2}{x}$, $\frac{x^2+2}{(x+1)(x+2)}$, $\frac{x^3+2x+1}{x^2+1}$.

Improper fractions are those whose index of the numerator is equal to or greater than that of the denominator; Therefore, they are first changed to proper fractions by long division before applying any method of integration:

Examples:

1): Express $\frac{3x^2+5x+3}{x(x-3)}$ in partial fractions and hence integrate:

Solution.

$$\frac{3x^2+5x+3}{x^2-3x} =$$

$$\begin{array}{r|l} & 3 \\ \hline x^2-3x & 3x^2+5x+3 \\ & -(3x^2-9x \\ \hline & 14x+3 \end{array}$$

$$\therefore \frac{3x^2+5x+3}{(x^2-3x)} = 3 + \frac{14x+3}{x(x-3)}.$$

$$\therefore \frac{14x+3}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3}.$$

$$14x+3 = A(x-3) + Bx.$$

$$14x+3 = Ax + Bx - 3A.$$

Equating coefficients;

$$x^1;\ 14 = A+B \text{ --- (i)}.$$

$$x^0;\ 3 = -3A.$$

$$A = \underline{-1}.$$

$$\therefore B = 14+1.$$

$$B = \underline{15}.$$

$$\therefore \frac{3x^2+5x+3}{x(x-3)} = \underline{3 + \frac{-1}{x} + \frac{15}{x-3}}.$$

$$\int \frac{3x^2+5x+3}{x(x-3)}\,dx = \int 3\,dx - \int \frac{1}{x}\,dx + \int \frac{15}{x-3}\,dx$$

$$= 3x - \ln x + 15\ln(x-3) + c$$

2: Express $\frac{5x^2-71}{(x+5)(x-4)}$ as a partial fraction, and then integrate.

soln:

$$\frac{5x^2-71}{(x+5)(x-4)} = \frac{5x^2-71}{x^2+x-20}.$$

$$= \begin{array}{r|l} & 5 \\ \hline x^2+x-20 & 5x^2-71 \\ & -(5x^2+5x-100) \\ \hline & -5x+29 \end{array}$$

$$\therefore \frac{5x^2-71}{(x+5)(x-4)} = 5 + \frac{-5x+29}{(x+5)(x-4)}$$

$$\frac{-5x+29}{(x+5)(x-4)} = \frac{A}{(x+5)} + \frac{B}{(x-4)}.$$

$$-5x+29 = A(x-4) + B(x+5).$$

when $x = 4$;

$9 = 9B.$

$B = 1$

when $x = -5$;

$54 = -9A.$

$A = -6$

$$\frac{5x^2-71}{(x+5)(x-4)} = \cancel{-6}\ 5 - \frac{6}{x+5} + \frac{1}{x-4}.$$

$$\int \frac{5x^2-71}{(x+5)(x-4)}\,dx = \int 5\,dx - 6\int \frac{1}{x+5}\,dx + \int \frac{1}{x-4}\,dx.$$

$$= 5x - 6\ln(x+5) + \ln(x-4) + C.$$

Expressing the following in partial fractions and hence integrate

3(a): $f(x) = \dfrac{x^3 + 2x^2 - 10x - 9}{x^2 - 9}$

(b): $f(x) = \dfrac{3x^4 + 7x^3 + 8x^2 + 53x - 186}{(x+4)(x^2+9)}$, hence find $\int_0^2 f(x)$.

(c): $f(x) = \dfrac{2x^3 + 3x^2 - x - 4}{x^2(x+1)}$

(d): $f(x) = \dfrac{x^2 + 7x - 14}{(x+5)(x-3)}$.

e): Evaluate; $\displaystyle\int_0^1 \frac{3-2x}{1+x}\,dx$.

Exercise:

1): Express each of the following in partial ~~p~~ fractions and hence integrate:

Linear:

(a): $\dfrac{(3x-1)}{(x+3)(x-2)}$. (b): $\dfrac{5x+6}{(x+4)(x-3)}$. (c) $\dfrac{2x+1}{(x+2)(x+1)}$.

Quadratic:

(d) $\dfrac{5x^2 - 3x + 1}{(x^2+1)(x-2)}$ (e): $\dfrac{9x+7}{(2x^2+3)(x+2)}$.

f): $\dfrac{6x+7}{(x^2+2)(x+3)}$.

Repeated:

(g): $\dfrac{2x+3}{(x+2)^2}$ (h): $\dfrac{7+5x-6x^2}{(2x+1)^2(x+2)}$.

i): $\dfrac{x^2 - 3x - 9}{x^3 - 6x^2 + 9x}$.

Denominator with 3 distinct factors:

a): $\dfrac{2}{(x+3)(x+2)(x+1)}$ (b): $\dfrac{x^2 - 9x + 3}{(x+1)(x-1)(x-2)}$

(c): $\dfrac{2x^2 + 11 + 3}{\ldots}$

Integration by Parts:

We can extract the mtd of integration by parts from differentiating the product of a function; $y = uv$, then;

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

$$\frac{d(vu)}{dx} - v\frac{du}{dx} = u\frac{dv}{dx}.$$

$$\int \frac{d(vu)}{dx}dx - \int v\frac{du}{dx}dx = \int u\frac{dv}{dx}dx.$$

$$\int d(uv) - \int v\,du = \int u\,\frac{dv}{\cancel{dx}}.$$

$$uv - \int v\,du = \int u\,dv.$$

$\therefore \int u\,dv = uv - \int v\,du$; as the mtd of integration by parts.

The function chosen to be u should be easily differentiated whereas the other function v should be easily integrated.

The above expression of integration by parts can be summarized by using a technique known as the basic technique of integration by parts as summarized below:

Sign:	Differentiated	Integrated:
		v
$+$	$\longrightarrow u$	$\triangleright v_1$
$-$	$\longrightarrow u_1$	$\triangleright v_2$
$+$	$\longrightarrow u_2$	$\triangleright v_3$.
$-$	u_3	

Note: The signs change as $+, -, +, -, +, -$.
The u function is differentiated until a zero value is obtained or else, we continue with differentiation.

Integration by parts is applied in the following ways:

i): Product of 2 polynomials:

Example: Find; $\int x(x+5)^5\,dx$:

soln.

Mtd: 1:

Let $u = x$. Let; $\frac{dv}{dx} = (x+5)^5$.

$\frac{du}{dx} = 1$.

$du = dx$. $v = \frac{1}{6}(x+5)^6 + c$.

from; $I = uv - \int v\,du$.

$I = x\left(\frac{1}{6}(x+5)^6\right) - \int \frac{1}{6}(x+5)^6\,du$.

$I = \frac{x}{6}(x+5)^6 - \frac{1}{42}(x+5)^7 + c$.

$I = \frac{1}{6}\left(x(x+5)^6 - \frac{(x+5)^7}{7}\right) + c$.

$I = \frac{1}{6}(x+5)^6\left(x - \frac{(x+5)}{7}\right) + c$.

Mtd 2.

Sign:	Differentiated	Integrated:
+	x	$(x+5)^5$
−	1	$\frac{1}{6}(x+5)^6$
+	0	$\frac{1}{42}(x+5)^7$.

$= x/6\,(x+5)^6 - 1/42\,(x+5)^7$

$= 1/6\,(x+5)^6\left(x - \frac{(x+5)}{7}\right) + c.$

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2) $\int (x+3)(x-4)^5 \, dx$.

$u = x+3$ $\qquad$ $\frac{dv}{dx} = (x-4)^5$.

$\frac{du}{dx} = 1$. $\qquad$ $v = \frac{1}{6}(x-4)^6$.

$I = uv - \int v \, du$.

$I = \frac{1}{6}(x+3)(x-4)^6 - \int \frac{1}{6}(x-4)^6 \, dx$.

$I = \frac{1}{6}(x+3)(x-4)^6 - \frac{1}{42}(x-4)^7 + c$.

$I = \frac{1}{6}(x-4)^6 \left[(x+3) - \frac{1}{7}(x-4) \right] + c$.

OR:

Sign:	Differentiate	Integrate:
$+$	$x+3$	$(x-4)^5$
$-$	1	$\frac{1}{6}(x-4)^6$
$+$	0	$\frac{1}{42}(x-4)^7$.

$I = \frac{1}{6}(x-4)^6 (x+3) - \frac{1}{42}(x-4)^7 + c$.

3). Find: $\int \frac{(3x-4)}{(x-2)^2} \, dx$.

$I = \int (3x-4)(x-2)^{-2} \, dx$.

let $u = 3x-4$. $\qquad$ $\frac{dv}{dx} = (x-2)^{-2}$.

$\frac{du}{dx} = 3$. $\qquad$ $v = \frac{1}{-1}(x-2)^{-1}$.

$dx = \frac{du}{3}$.

$du = 3dx$. $\qquad$ $v = \frac{-1}{(x-2)}$.

$I = uv - \int v \, du$.

$I = (3x-4)\left(\frac{-1}{x-2}\right) - \int \frac{-1}{x-2} \, dx$.

$I = \frac{-(3x-4)}{(x-2)} - \int \frac{-1}{x-2} \cdot \cancel{\frac{du}{3}} \, 3dx$.

$I = \frac{4-3x}{x-2} + \int \frac{3}{(x-2)} \, dx$.

$I = \frac{(4-3x)}{(x-2)} + 3\ln(x-2) + c$.

2. Polynomial and trigonometric function:

Examples:

i) $\int x \sin x \, dx$:

let $u = x$. $\qquad \frac{dv}{dx} = \sin x$

$du = dx$. $\qquad v = -\cos x$.

$I = uv - \int v \, du$.

$I = x(-\cos x) - \int -\cos x \, dx$.

$I = -x\cos x + \int \cos x \, dx$.

$I = -x\cos x + \sin x + c$;

OR:

sign		Diff.		Integrated:
+	$\longrightarrow$	x	$\searrow$	$\sin x$
−	$\longrightarrow$	1	$\searrow$	$-\cos x$
+		0		$-\sin x$.

$I = -x\cos x + \sin x + c$.

ii) $\int x^2 \cos x \, dx$.

$u = x^2$. $\qquad \frac{dv}{dx} = \cos x$.

$\frac{du}{dx} = 2x$. $\qquad v = \sin x$.

$du = 2x \, dx$.

$I = uv - \int v \, du$.

$\frac{d \sin x}{dx} = \int \cos x$.

$I = x^2 \sin x - \int \sin x \cdot 2x \, dx$

$I = x^2 \sin x - \int 2x \sin x \, dx$.

~~I~~ let $u = 2x$ $\qquad \frac{dv}{dx} = \sin x$.

$du = 2dx$. $\qquad v = -\cos x$.

$I = x^2 \sin x - \left(-2x\cos x - \int -2\cos x \, dx\right)$.

$I = x^2 \sin x - \left(-2x\cos x + 2\sin x\right) + c$.

$I = x^2 \sin x + 2x\cos x - 2\sin x + c$.

OR:

Sign:	Diff:	Integrated:
		$\cos x$
+	x^2	$\sin x$
-	$2x$	$-\cos x$
+	2	$-\sin x$.
-	0	

$$I = x^2\sin x + 2x\cos x - 2\sin x + C.$$

$$I = (x^2-2)\sin x + 2x\cos x + C;$$

3): $\int x^2 \sin^2 x \, dx$:

let $u = x^2$. $\quad \frac{dv}{dx} = \sin^2 x$.

$du = 2x\,dx$. $\quad \frac{dv}{dx} = \frac{1}{2}(1-\cos 2x)$.

Sign:	Diff:	Integrated:
+	x^2	$\frac{1}{2}(1-\cos 2x)$.
-	$2x$	$\frac{1}{2}x - \frac{1}{4}\sin 2x$.
+	2	$\frac{1}{4}x^2 + \frac{1}{8}\cos 2x$.
-	0	$\frac{1}{12}x^3 + \frac{1}{16}\sin 2x$

$$I = x^2\left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right) - 2x\left(\frac{1}{4}x^2 + \frac{1}{8}\cos 2x\right) + 2\left(\frac{1}{12}x^3 - \frac{1}{16}\sin 2x\right.$$

$\frac{d\cos x}{dx} = -\sin x$

$\frac{d\sin x}{dx} = \cos x$

$$I = \frac{1}{2}x^3 - \frac{x^2}{4}\sin 2x - \frac{1}{2}x^3 - \frac{x}{4}\cos 2x + \frac{1}{6}x^3 + \frac{1}{8}\sin 2x$$

$$I = \frac{1}{6}x^3 - \frac{1}{4}x^2\sin 2x - \frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + C$$

$\frac{1}{2} - \frac{1}{2}\cos x.$

8: Polynomial and Inverse Trigonometric function:

Examples.

(i) $\int \sin^{-1}(x)\,dx$:

let $u = \sin^{-1}(x)$

$x = \sin u.$

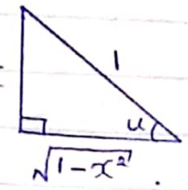
$dx = \cos u\,du.$

$du = \frac{dx}{\cos u}$:

$du = \frac{dx}{\sqrt{1-x^2}}.$

$\frac{dv}{dx} = 1.$

$v = x.$



$\cos u = \frac{\sqrt{1-x^2}}{1}.$

$\therefore \quad I = uv - \int v\,du.$

$I = x\sin^{-1}(x) - \int x \cdot \frac{dx}{\sqrt{1-x^2}}.$

For $\int \frac{x}{\sqrt{1-x^2}}\,dx$

; $1-x^2 = 1-\sin^2 u = \cos^2 u.$

$x^2 = \sin^2 u.$

$x = \sin u,$

$dx = \cos u\,du.$

+c.

$\int \frac{x}{\sqrt{\cos^2 u}} \cdot \cos u\,du = \int \frac{x}{\cos u} \cdot \cos u\,du = \int x\,du.$

+c.

but $x = \sin u.$

$= \int \sin u\,du$

$= -\cos u + c.$

$= -\sqrt{1-x^2} + c$

$I = \int \sin^{-1}(x)\,dx = x\sin^{-1}(x) + \sqrt{1-x^2} + c.$

2. $\int x\tan^{-1}(x)\,dx$.

$\frac{dv}{dx} = x$.

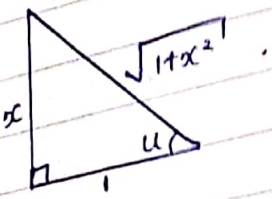
$v = \cancel{x\,1} \cdot \frac{x^2}{2}$.

let $u = \tan^{-1}(x)$

$\tan u = x$.

$\sec^2 u\,du = dx$;

$(1+\tan^2 u)du = dx$.



$du = \frac{dx}{(1+\tan^2 u)}$.

$du = \frac{dx}{1+x^2}$,

$I = uv - \int v\,du$.

$I = \frac{x^2}{2}\tan^{-1}(x) - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$.

$I = \frac{x^2}{2}\tan^{-1}(x) - \frac{1}{2}\int\left(1 - \frac{1}{x^2+1}\right)dx$.

$I = \frac{x^2}{2}\tan^{-1}(x) - \frac{1}{2}x + \frac{1}{2}\tan^{-1}(x) + c$,

$I = \frac{1}{2}\tan^{-1}(x)\left(x^2+1\right) - \frac{1}{2}x + c$.

3: show that;

$\int_0^1 x\sin^{-1}(x)\,dx = \frac{\pi}{8}$.

let $u = \sin^{-1}(x)$

$\frac{dv}{dx} = x$.

$\cancel{su}\ \sin u = x$,

$v = \frac{x^2}{2}$,

$\cos u\,du = dx$.

~~$\cos u = dx$~~

$\cos u = \sqrt{1-x^2}$

$du = \frac{dx}{\sqrt{1-x^2}}$

$I = \frac{x^2}{2}\sin^{-1}(x) - \int \frac{x^2}{2}\left(\frac{dx}{\sqrt{1-x^2}}\right)$.

$I = \frac{x^2}{2}\sin^{-1}(x) - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx$.

let $1-x^2 = 1-\sin^2 u = \cos^2 u$.

$x^2 = \sin^2 u$

$x = \sin u$:

$dx = \cos u \, du$:

$$= \int_0^1 \frac{\sin^2 u}{\cos u} \cdot \cos u \, du = \int_0^1 \sin^2 u \, du.$$

$$= \int_0^1 \frac{1}{2}(1-\cos 2u) \, du:$$

$$= \frac{1}{2}u - \frac{1}{4}\sin 2u + c \Big|_0^1$$

$$= \frac{1}{2}(\sin^{-1}(x)) - \frac{1}{4}\cancel{\sin} \times 2 \times x\sqrt{1-x^2}.$$

$$I_1 = \frac{1}{2}\sin^{-1}(x) - \frac{1}{2}x\sqrt{1-x^2}$$

$$I = \frac{x^2}{2}\sin^{-1}(x) - \frac{1}{4}\sin^{-1}(x) + \frac{1}{4}x\sqrt{1-x^2}.$$

$$I = \frac{1^2}{2}\sin^{-1}(1) - \frac{1}{4}\sin^{-1}(1) + \frac{1}{4}\times 1 \times \sqrt{1-1^2}.$$

$I = \pi/8$ as required.

Polynomial and Logarithmic function:

examples:

i) $\int \ln x^2 \, dx = \int 2\ln x \, dx$

$u = \ln x$ $\qquad \frac{dv}{dx} = 2$

$\frac{du}{dx} = \frac{1}{x}$ $\qquad v = 2x$

$I = \ln x(2x) - \int 2x \cdot \frac{1}{x} \cdot dx$

$I = 2x\ln x - 2x + c$

$I = 2x(\ln x - 1) + c$

ii) $\int \ln x \, dx$

let $u = \ln x$ $\qquad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{x}$ $\qquad v = x$

$du = \frac{1}{x} \cdot dx$

$I = x\ln x - \int x \cdot \frac{1}{x} \, dx$

$I = x\ln x - x + c$

$I = x(\ln x - 1) + c$

iii) find $\int x\ln(x^2-1) \, dx$:

$\frac{2x}{x^2-1}$

$u = \ln(x^2-1)$ $\qquad \frac{dv}{dx} = x$

$u = \frac{\ln x^2}{\ln 1}$ $\qquad v = \frac{x^2}{2}$

$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$u = \ln x^2 - \ln 1$

$u = 2\ln x - \ln 1$

$\frac{du}{dx} = 2\left(\frac{1}{x}\right) - 0$

$\frac{du}{dx} = 2/x$

$u = \ln(x^2-1)$

$\frac{du}{dx} = \frac{2x}{x^2-1}$

$$I = \ln(x^2-1)\cdot\frac{x^2}{2} - \int \frac{x^2}{2}\cdot\frac{2x}{x^2-1}\,dx.$$

$$I = \frac{1}{2}x^2\ln(x^2-1) - \int \frac{x^3\,dx}{x^2-1}.$$

$$\cancel{I = \frac{x^2}{x}\ln(x^2-1) - \frac{1}{2}x^3 + c};$$

$$I = \frac{1}{2}x^2\ln(x^2-1) - \int \frac{x^2}{2}\cdot\frac{2x}{x^2-1}\,dx.$$

$$I = \frac{1}{2}x^2\ln(x^2-1) - \int \frac{x^3}{x^2-1}\,dx;$$

From; $\int \frac{x^3}{x^2-1}\,dx = \int \frac{x^3}{x^2-1}\,dx.$

using long division;

$$\begin{array}{r|l} & x \\ x^2-1 & x^3 \\ & -(x^3 - x) \\ \hline & x. \end{array}$$

$$\int \frac{x^3}{x^2-1}\,dx = \int\left(x + \frac{x}{x^2-1}\right)dx = \int x\,dx + \int \frac{x}{x^2-1}\,dx$$

$$\int \frac{x^3}{x^2-1}\,dx = \frac{1}{2}x^2 + \frac{1}{2}\ln(x^2-1) + c.$$

$u = x^2-1$

$\frac{du}{dx} = 2x.$

$dx = \frac{du}{2x}.$

$\int \frac{\cancel{x}}{u}\cdot\frac{du}{2\cancel{x}}$

$\frac{1}{2}$

$$\therefore I = \frac{1}{2}x^2\ln(x^2-1) - \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2-1) + c;$$

$$I = \frac{1}{2}x^2\left(\ln(x^2-1) - 1\right) - \frac{1}{2}\ln(x^2-1) + c;$$

(iv) Find; $\int x^{-3}\ln x\,dx.$

6: Exponential and trigonometric functions:

Examples;

(i) Find; $\int e^x \sin x \, dx$.

$I = \int e^x \sin x \, dx$:

$u = e^x$. $\frac{dv}{dx} = \sin x$.

$\frac{du}{dx} = e^x$.

$v = -\cos x$.

$\frac{dv}{dx} = \cos x$.

$I = -e^x \cos x - \int -\cos x \, e^x \, dx$.

$v = \sin x$.

$u = e^x$, $\frac{du}{dx} = e^x$.

$I = -e^x \cos x + \int e^x \cos x \, dx$.

$I = -e^x \cos x + \left[e^x \sin x - \int e^x \sin x \, dx \right]$

$I = -e^x \cos x + e^x \sin x - I$.

$2I = -e^x \cos x + e^x \sin x$.

$I = \frac{1}{2}(-e^x \cos x + e^x \sin x) + c$

ii) Find; $\int e^{-x} \sin x \, dx$.

let $I = \int e^{-x} \sin x \, dx$.

$u = e^{-x}$

$\frac{du}{dx} = -e^{-x}$. $\frac{dv}{dx} = \sin x$.

$v = -\cos x$.

$I = -e^{-x} \cos x + \int -e^{-x} \cos x \, dx$.

$\int -e^{-x} \cos x \, dx =$

$u = -e^{-x}$ $\frac{dv}{dx} = \cos x$.

$\frac{du}{dx} = e^{-x}$

$v = \sin x$.

$I = -e^{-x} \cos x + \left[-e^{-x} \sin x - \int e^{-x} \sin x \, dx \right]$.

$I = -e^{-x} \cos x + \left[-e^{-x} \sin x - I \right]$.

$$2I = -e^{-x}\cos x - e^{-x}\sin x.$$

$$I = \frac{1}{2}\left(-e^{-x}\cos x - e^{-x}\sin x\right) + c;$$

$$I = -\frac{1}{2}e^{-x}\left(\cancel{1}\cos x + \sin x\right) + c:$$

iii) Find : $\int e^{2\theta}\cos 3\theta \, d\theta.$

Let $I = \int e^{2\theta}\cos 3\theta \, d\theta.$

$u = e^{2\theta}.$

$\frac{du}{d\theta} = 2e^{2\theta}$ $\qquad \frac{dv}{d\theta} = \cos 3\theta.$

$v = \cancel{\cos}\frac{1}{3}\sin 3\theta.$

$$I = \frac{1}{3}e^{2\theta}\sin 3\theta - \int \frac{1}{3}\sin 3\theta . 2e^{2\theta} \, d\theta.$$

$$I = \frac{1}{3}e^{2\theta}\sin 3\theta - \int \frac{2}{3}e^{2\theta}\sin 3\theta \, d\theta.$$

$$I = \frac{1}{3}e^{2\theta}\sin 3\theta - \frac{2}{3}\int e^{2\theta}\sin 3\theta \, d\theta.$$

For; $\int e^{2\theta}\sin 3\theta \, d\theta.$

$u = e^{2\theta}.$ $\qquad \frac{dv}{d\theta} = \sin 3\theta.$

$\frac{du}{d\theta} = 2e^{2\theta}$

$\therefore v = -\frac{1}{3}\cos 3\theta.$

$$I = \frac{1}{3}e^{2\theta}\sin 3\theta - \frac{2}{3}\left[\frac{1}{3}\overset{\cos 3\theta}{\cancel{e^{2\theta}}}(-2e^{2\theta}) + \int \frac{1}{3}\cos 3\theta . 2e^{2\theta} \, d\theta\right]$$

$$I = \frac{1}{3}e^{2\theta}\sin 3\theta - \frac{2}{3}\left[-\frac{1}{3}e^{2\theta}\cos 3\theta + \frac{1}{3}\int \frac{2}{3}e^{2\theta}\cos 3\theta \, d\theta\right].$$

$$I = \frac{1}{3}e^{2\theta}\sin\theta + \frac{2}{9}e^{2\theta}\cos 3\theta - \frac{4}{9}\int e^{2\theta}\cos 3\theta \, d\theta$$

$$I + \frac{4}{9}\int e^{2\theta}\cos 3\theta \, d\theta = \frac{1}{3}e^{2\theta}\sin\theta + \frac{4}{9}e^{2\theta}\cos 3\theta.$$

$$I + 4/9\, I = \frac{1}{3}e^{2\theta}\sin 3\theta + \frac{2}{9}e^{2\theta}\cos 3\theta.$$

$$I = \frac{9}{13}\left(\frac{1}{3}e^{2\theta}\sin 3\theta + \frac{2}{9}e^{2\theta}\cos 3\theta\right) + c.$$

$$I = \frac{1}{13} e^{2\theta}(3\sin 3\theta + 2\cos 3\theta) + c.$$

Trigonometric Functions:

Examples:

i) Find; $\int \tan x \, dx = \ln \sec x + c.$

$$\int \operatorname{cosec} x \, dx = -\ln(\operatorname{cosec} x + \cot x) + c.$$

$$\frac{d}{dx}\left(\ln(\operatorname{cosec} x + \cot x)\right) = \frac{-\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x}{\operatorname{cosec} x + \cot x}.$$

$$= \frac{-\operatorname{cosec} x(\cot x + \operatorname{cosec} x)}{\operatorname{cosec} x + \cot x}.$$

$$= -\operatorname{cosec} x.$$

Exercise;

1: Find;

(i) $\int \sec^3 x \, dx.$

(ii) $\int \operatorname{cosec}^3 x \, dx.$

2: Show that $\int_{\pi/6}^{\pi/3} \tan x \sec^3 x \, dx = \frac{8}{27}(9 - \sqrt{3}).$