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# PROBABILITY:

**Introduction:** **Probability** is a measure of the expectation that an event will occur or a statement is true. Probabilities are given a value between 0 (will not occur) and 1 (will occur). The higher the probability of an event, the more certain we are that the event will occur.

When dealing with **experiments** that are **random** and **well-defined** in a purely theoretical setting (like tossing a fair coin), probabilities describe the statistical number of outcomes considered divided by the number of all outcomes (tossing a fair coin twice will yield HH with probability 1/4, because the four outcomes HH, HT, TH and TT are possible). When it comes to practical application, however, the word **probability** does not have a singular direct **definition.**

The probability of an **event** A is written as P(A), p(A) or Pr(A). This mathematical definition of probability can extend to infinite sample spaces, and even uncountable sample spaces, using the concept of a measure.

The **opposite or complement** of an event A is the event [not A] (that is, the event of A not occurring); its probability is given by P(not A) = 1 - P(A). As an example, the chance of not rolling a six on a six-sided die is 1 – (chance of rolling a six) = $1 - {}^{1}/_{6} = {}^{5}/_{6}$

If both events A and B occur on a single performance of an experiment, this is called the intersection or **joint probability** of A and B, denoted as $P(A \cap B)$

**Notes:**

Probability is a value that represents the occurrence of an event when compared with the total number of trials. If P(A) represents the probability of A then P(A) lies in the interval from 0 to 1.

$\therefore \quad 0 \angle P(A) \leq 1$

$$P(A) = \frac{\text{Number of trials for event A}}{\text{Total number of trials in samples}}$$

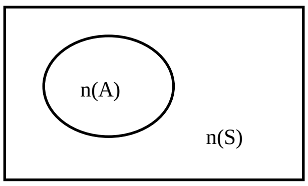
$$= \frac{n(A)}{n(S)}$$

Where A is an event with S possibilities.

If P(A) = 1, we are absolutely sure that event A will happen.

If P(A) = 0, then event A can never happen.

From $P(A) = \dfrac{n(A)}{n(S)}$




n(A) is greater or equal to zero.

n(S) is greater or equal to n(A)

Since A is a subset S.

$0 \leq n(A) \leq n(S)$ divide by n(S)

$$\frac{0}{n(S)} \leq \frac{n(A)}{n(S)} \leq \frac{n(S)}{n(S)}$$

$$0 \leq \frac{n(A)}{n(S)} \leq 1 \quad \text{but} \quad \frac{n(S)}{n(S)} = 1$$

$$0 \leq P(A) \leq 1$$

There are mainly two types of probabilities.

**Empirical or experimental probability**: It arises out of carrying out experiments practically.

**Theoretical probability**: Where values for probability are obtained from daily experience.

**Example**: Probability of a head or a tail when a coin is tossed is $\frac{1}{2}$.

**COMPLEMENT (EXHAUSTIVE) PROBABILITY**

The complement of event A is A or A implying A does not occur.

$$P = (\bar{A}) = \frac{n(\bar{A})}{n(S)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

But $n(A) + n(\bar{A}) = n(S)$

$$\therefore\ n(\bar{A}) = n(S) - n(A) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Substitute (ii) into (i)

$$P(\bar{A}) = \frac{n(S) - n(A)}{n(S)}$$

$$= \frac{n(S)}{n(S)} - \frac{n(A)}{n(S)}$$

$$P(\bar{A}) = 1 - \frac{n(A)}{n(S)} \quad \text{but} \quad \frac{n(A)}{n(S)} = P(A)$$

$$P(\bar{A}) = 1 - P(A)$$

$$\mathbf{P(A) + P(\bar{A}) = 1}$$

**LAWS OF PROBABILITY**

They are mainly classified into two parts.

**Additional law**

It mainly includes:

Mutually exclusive events.

Non – mutually exclusive events.

Exhaustive events.

**(ii) Multiplication law.**

Independent events.

Dependent events or conditional probability.

Baye’s theorem.

(i) **ADDITIONAL LAW**

(a) **MUTUALLY EXCLUSIVE EVENTS**

They are events that cannot occur at the same time thus has no intersection. If the events are A and B then $P(A \cap B) = 0$

In probability OR is represented by U (union) while AND is represented by $\cap$ (intersection) also OR can be represented with + (addition sign) and can be represented with X (multiplication sign)

$$P(A \text{ or } B) = P(A + B)$$
$$= P(AUB)$$
$$= P(A) + P(B)$$

It can be extended to three events A, B and C

P(A or B or C) = P(A) + P(B) + P(C) for events $A_1, A_2, A_3, \ldots\ldots\ldots$, And then

$P(A, \text{ or } A_2 \text{ or } A_3 \text{ or } \ldots\ldots \text{ or } A_n) = P(A_1) + P(A_2) + P(A_3) + \ldots\ldots + P(A_n)$

(b) **NON-MUTUALLY EXCLUSIVE EVENTS**

They are events that occur at the same time. If the two events are A and B then

$P(A \cap B) \neq 0$ then probability that A occurs plus the probability that B occurs
Subtract probability that both A and B occur P(A or B) = P(AUB)

$$P(A + B) = P(A) + P(B) - P(A \cap B)$$
$$\therefore \quad P(AUB) = P(A) + P(B) - P(A \cap B)$$

If three events A, B and C are non-mutually exclusive then

**$P(AUBUC) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$**

(c) **EXHAUSTIVE EVENTS**

They are those events whose probability sum up to one. If A and B are mutually exhaustive, then

$$P(\text{ A or B }) = P(AUB)$$
$$= P(A) + P(B)$$
$$= 1$$
$$\therefore P(AUB) = 1 \quad \text{or} \quad P(A) + P(B) = 1$$

If three events A, B and C are exhaustive events then

$$P(\text{A or B or C}) = P(A) + P(B) + P(C)$$
$$= 1$$

In general for events $A_1$, $A_2$, $A_3$, ……$A_n$ , If they are exhaustive events then,

$$P(A_1 \text{ or } A_2 \text{ and } A_3 \text{ and } \ldots\text{and } A_n) = P(A_1 \cap A_2 \cap A_3 \cap \ldots\ldots \cap A_n)$$
$$= P(A_1) \text{ x } P(A_2) \text{ x2}(A_3 \text{ x } \ldots\text{x } P(A_n)$$

**MULTIPLICATION LAW**

**CONDITIONAL PROBABILTY (DEPENDENT EVENTS)**

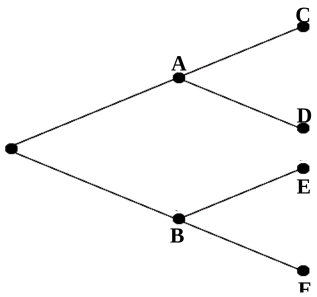
If A and B and are two events, where $P(A) \neq 0$ and $P(B) \neq 0$, then the probability of A, given that B has already occurred is written as P(A/B) and is probability of A given B.

$$\text{A and B}) = P(B) \text{ x } P(\text{A given}B)$$
$$P(A \cap B) = P(B) \text{ x } P(A \cap B)$$
$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

**TREE DIAGRAM**

### Definition of 'Tree Diagram'

A diagram used in strategic decision making, valuation or probability calculations. The diagram starts at a single node, with branches emanating to additional nodes, which represent mutually exclusive decisions or events. In the diagram below, the analysis will begin at the first blank node. A decision or event will then lead to node A or B. From these secondary nodes, additional decisions or events will occur leading to the third level of nodes, until a final conclusion is reached.




**How to Use a Probability Tree for Probability Questions**

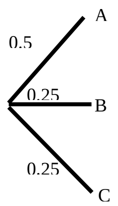
Sometimes, you'll be faced with a probability question that just doesn't have a simple solution. Drawing a **probability tree** (or a **tree diagram**) is a way for you to visually see all of the possible choices, and to avoid making mathematical errors. Below is a step-by-step process of using a decision tree.

**Example "An airplane manufacturer has three factories A B and C which produce 50%,**

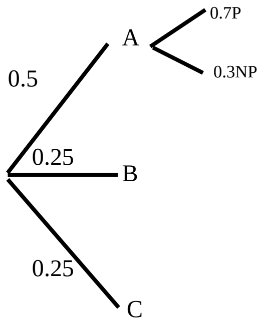
**25%, and 25%, respectively, of a particular airplane. Seventy percent of the airplanes produced in factory A are passenger airplanes, 25% of those produced in factory B are passenger airplanes, and 25% of the airplanes produced in factory C are passenger airplanes. If an airplane produced by the manufacturer is selected at random, calculate the probability the airplane will be a passenger plane."**

**Step 1:Draw lines to represent the first set of options in the question** (in our case, 3 factories). Label them (our question list A B and C so that is what we'll use here).

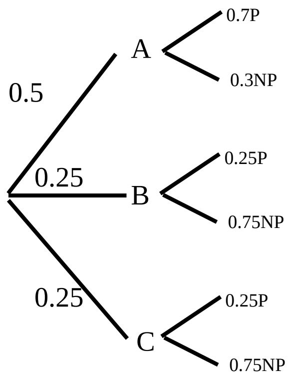
**Step 2: Convert the percentages to decimals**, and place those on the appropriate branch in the diagram. For our example, 50% = 0.5, and 25% = 0.25.



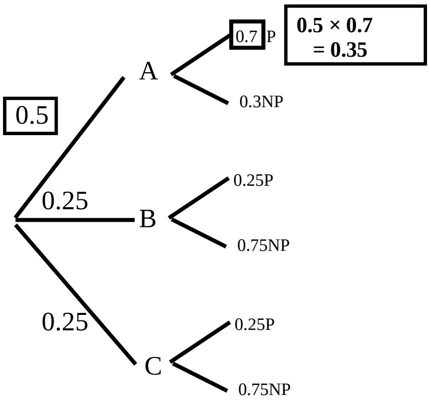

**Step 3: Draw the next set of branches.** In our case, we were told that 70% of factory A's output was passenger. Converting to decimals, we have 0.7 P ("P" is just my own shorthand here for "Passenger") and 0.3 NP ("NP" = "Not Passenger").



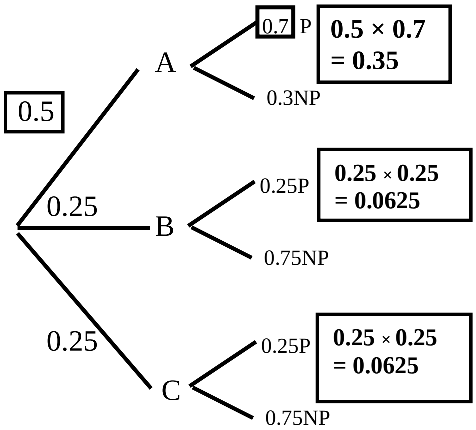

**Step 4:Repeat step 3 for as many branches as you are given.**




**Step 5:** Multiply the probabilities of the first branch that produces the desired result together. In our case, we want to know about the production of passenger places, so we choose the first branch that leads to P.




Step 6: **Multiply the remaining branches that produce the desired** result. In our example there are two more branches that can lead to P.




**Step 6: Add up all of the probabilities you calculated in steps and 6.**

In our example, we had:

0.35 + 0.0625 + 0.0625 = 0**.475**

**Note:**

It solves problems that are mutually exclusive events and exclusive. The sum of probabilities from the same point on the tree diagram adds up to one. It solves two cases of problems.

(i) Picking with replacement, here the sample size in the second stage does not change since it is replaced.

(ii) Picking without replacement, the sample size in the second stage reduces by one since it is not replaced.

**Example 1:** Given the $P(A) = 0.3$, $P(B) = 0.4$ and $P(A \cap B) = 0.1$

Find: (i) $P(AUB)$

(ii) $P(AUB)^1$

(iii) $P(A^1UB)$

**Solution:**

(i) It's a non-mutually excusive event.

$$P(AUB) = P(A) + P(B) - P(A \cap B)$$
$$P(AUB) = 0.3 + 0.4 - 0.1$$
$$= 0.6$$

(ii) Are exhaustive events.

$$P(AUB)^1 + P(AUB) = 1$$

$$P(AUB)^1 = 1 - P(AUB)$$
$$= 1 - 0.6$$
$$= 0.4$$

(iii) Intersection = 0.3

$$P(A^1 \cap B) = 0.3$$
$$P(A^1UB) = P(A^1) + P(B) - P(A^1 \cap B)$$
$$= 0.7 + 0.4 - 0.3$$
$$= 0.8$$

**Example 2: The probability that two events occur together is $\frac{2}{15}$**

**The probability that either or both events occur is $\frac{3}{5}$. Find the individual probabilities of the two events.**

**Solution:**

Let the two events be A and B.

$$P(A \cap B) = P(A)\,.P(B)$$
$$P(A).P(B) = \frac{2}{15}$$
$$P(A) = \frac{2}{15P(B)}$$
$$P(AUB) = P(A) + P(B) - P(A \cap B)$$
$$\therefore \quad P(A) + P(B) - P(A \cap B) = \frac{3}{5}$$
$$P(A \cap B) = \frac{2}{15}$$
$$P(A) + P(B) = \frac{3}{5} + \frac{2}{15}$$
$$= \frac{11}{15}$$

Substitute (i) into (ii)

$$\frac{2}{15P(B)} + P(B) = \frac{11}{15}$$
$$2 + 15P(B)^2 = 11P(B)$$
$$15P(B)^2 - 11P(B) + 2 = 0$$
$$P(B) = \frac{2}{5} \text{ or } \frac{1}{3}$$

When $P`(B) = \frac{2}{5}$;

$$P(A) = \frac{2}{15}.\frac{5}{2}$$
$$= \frac{5}{15}$$
$$= \frac{1}{3}$$

When $P(B) = \frac{1}{3}$;

$$P(A) = \frac{2}{15}.\frac{3}{1}$$
$$= \frac{2}{5}$$

**Example 3: A die is tossed three times, what is the probability of getting.**
**exactly one 2**
**At least one 2.**

**Solution:**

P (exactly one 2) = P(2). P(2).P(2) + $P(\bar{2})$ . P(2) .$P(\bar{2})$ + $P(\bar{2})$ . $P(\bar{2})$ .P(.2)

But, $P(2) = \frac{1}{6}$, $P(\bar{2}) = \frac{5}{6}$

P(exactly one 2 ) = $\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$

= **0.3472**

(b)P(At least one 2 ) = 1 – P(no six )

= 1 - $P(\bar{2})$ , $P(\bar{2})$ ,$P(\bar{2})$

= $1 - \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}$

= $1 - \frac{75}{216}$

= $\frac{91}{216}$

**Example 4: If the two events A and B such that P(A) = $\frac{1}{3}$, P(B) - $\frac{1}{2}$**

**and P(A/B) = $\frac{P(A \cap B)}{P(B)}$. Find**

**(a) $P(A \cap B)$**
**(b) $P(A \cup B)$**
**(c) $P(B/\bar{A})$**

**Solution:**

(i) $P(A \cap B)$ = P(B).P(A/B)

= $\frac{1}{2} \times \frac{1}{4}$

= $\frac{1}{8}$

(ii) $P(A \cup B)$ = $P(A) + P(B) - P(A \cap B)$

= $\frac{1}{3} + \frac{1}{2} - \frac{1}{8}$

= $\frac{8 + 12 - 3}{24}$ = $\frac{17}{24}$

(iii) $P(B/\bar{A})$ = $\frac{P(B \cap \bar{A})}{P(A)}$

$P(\bar{A}) + P(A)$ = 1

$P(\bar{A})$ = 1 – P(A)

= $1 - \frac{1}{3}$ = $\frac{2}{3}$

$P(AUB) + P(AUB)^1$ = 1

$P(AUB)^1$ = 1 – P(AUB)

= $1 - \frac{17}{24}$ = $\frac{7}{24}$

But, $P(B \cap \overline{A}) = \frac{3}{8}$

$$P(B/\overline{A}) = \frac{3/8}{2/3}$$

$$= \frac{3}{8} \times \frac{3}{2} = \frac{9}{16}$$

**Example 5:**

**Given that A and B are two events such that P(A) = 0.5, P(B) = 0.7 and P(A∪B) = 0.8 Find.**

**(i) P(A∩B)**

**(ii)** $P(A \cap \overline{B})$

**Solution:**

We have

P(A) = 0.5 , P(B) = 0.7 and P(A∪B) = 0.8

(i) $P(A\cap B) = P(A)+P(B) - P(A\cup B)$

$= 0.5 + 0.7 - 0.8$

$\therefore P(A\cap B) = 0.4$

(ii) $P(A\cap \overline{B}) = P(A) - P(A\cap B)$

$= 0.5 - 0.4$

$= 0.1$

$\therefore P(A\cap \overline{B}) = 0.1$

**Example 7:**

The probability that a student X can solve a certain problem is $^2/_3$ and that student y can solve it is ½ . Find the probability that the problem will be solve if both X and Y try to solve it independently.

**Solution:**

The problem shows two independent events

Defining events

$X_1$ 'student x can solve the problem'

$Y_1$ : 'student y can solve the problem'

we are given

$$P(X_1) = \frac{2}{5}$$

$$P(Y_1) = \frac{1}{2}$$

we are asked $P(X_1 \cup Y)$

Now $P(X_1 \cup Y_1) = P(X_1) + P(Y_1) - P(X_1 \cap Y_1)$

Now for independent events ,

$P(X_1 \cap Y_1) = P(X_1) \times P(Y_1)$

$\therefore P(X_1 \cap Y_1) = \frac{2}{5} + \frac{1}{2} - \frac{2}{5} \times \frac{1}{2}$

$\therefore P(X_1 \cap Y_1) = 0.7$ #

**Example 8 :**

**The probability that I have to wait at the traffic lights on my way to school is $\frac{1}{4}$.**

**Find the probability that, on two consecutive mornings, I have to wait on at least one morning.**

**Solution:**

. Let $W_1$ be the event "waiting on traffic lights on first day"

$W_2$ be the event "waiting on the traffic lights on $2^{nd}$ day"

Now , $P(W_1) = 1/4$ , and

$P(W_2) = 1/4$ .

$W_1$ and $W_2$ are independent events.

We are asked to find $P(W_1 \cup W_2)$.

$$\therefore\ P(W_1 \cup W_2) = P(W_1) + P(W_2) - P(W_1 \cap W_2)$$

$$= \frac{1}{4} + \frac{1}{4} - \frac{1}{4}.\frac{1}{4}$$

$$= \frac{1}{2} - \frac{1}{16}$$

$$\therefore\ P(W_1 \cup W_2) = \frac{7}{16} \qquad \#$$

## EXERCISE 2

1.A stack of 20 cards contains 4 cards of each of 5 different colours, namely white, black, green, red and blue. A part from the colour the cards are indistinguishable. A set of 3 cards is draw at random from the stack. Find the chances that the 3 cards are:

(i)all white

(ii)all of one colour

(iii) of different colours

2. Box A contains 3 white and 3 black balls and box B contains 4 white and 3 black balls. One ball is transferred from A to B. One ball is then drawn from B and is found to be white. What is the probability that the transferred ball was white?

3.A sample poll of 200 Voters revealed the following information concerning three candidates A, B and C of a certain party who were running for three different offices. 28 in favour of both A and B, 122 in favour of B or C but not A, 98 in favour of A or B but not C, 64 in favour of C but not A or B, 42 in favour of B but not C, 14 in favour of A and C but not B.

Find the probability of voters who were in favour of :

(a) all the three candidates

(b) A irrespective of B or C

(c) B irrespective of A or C

(d) C irrespective of A or B

(e) A and B but not C

(f) only one of the candidates.

4.Three cards are drawn from a deck of 52 cards. Find the probability that:

(a) two are jacks and one is a King

(b) all cards are of one suit.

(c) all cards are of different suits.

(d) at least two aces are drawn.

5.. If 10% of the rivets produced by a machine are defective, what is the probability that out of 5 rivets chosen at random:

(a) none will be defective

(b) one will be defective

(c) at least two will be defective?

6.. Two marbles are drawn in succession from the box containing 10 red, 30 white, 20 blue 15 orange marbles. Replacement being made after each drawing. Find the probability that:

(a)both are white.

(b)the first is red and the second is white.

(c)neither is orange

(d)they are either red or white or both (red and white)
(e)the second is not blue.
(f)the first is orange.
(g)at least one is blue.
(h)at most one is red.
(i)the first is white but the second is not.
(j)only one is red.

7..A factory has three machines 1,2 and 3 producing a particular type of item. One item is drawn at random from the factory is production. Let B denote the event that the chosen item is defective and let $A_K$ denote the event that the item was produced on machine K, where K = 1,2 or 3. Suppose that the machines 1,2 and 3 produce respectively 35%, 45% and 20% of the total production of items and that $P(B/A_1) = 0.02$, $P(B/A_2) = 0.01$

$P(B/A_3) = 0.03$, Given that an item chosen at random is defective. Find which machine was the most likely to have produced it.

8..Three events A, B and c are defined in the sample space. The events A and C are mutually exclusive. The events A and B are independent.

Given that $P(A) = \frac{1}{3}$, $P(C) = \frac{1}{5}$, $P(AUB) = \frac{2}{3}$ Find:

(a)P(AUC)
(b)P(B)
(c)$P(A \cap B)$

Given also that $P(BUC) = \frac{3}{5}$, determine whether or not B are independent.

**ANSWERS**

**1. (i)** $\frac{1}{285}$ **(ii)** $\frac{1}{57}$ **(iii)** $\frac{32}{57}$

**2.** $\frac{5}{9}$

**3. (a)** $\frac{1}{25}$ **(b)** $\frac{39}{100}$ **(c)** $\frac{43}{100}$ **(d)** $\frac{51}{100}$ **(e)** $\frac{1}{10}$ **(f)** $\frac{71)}{100}$

**4. (a)** $\frac{6}{5525}$ **(b)** $\frac{22}{425}$ **(c)** $\frac{169}{425}$ **(d)** $\frac{73}{5525}$

**5.( (a) 0.59049 (b) 0.32805 (c) 0.08866 )**

**6.(a) 4/25 (b) 4/75 (c) 16/25 (d) 64/225**
**(e) 11/15 (f) 1/5 (g) 104/225 (h) 221/225**
**(i) 6/25(j) 52/225 )**

**7.(Machine C probability = 0.4 )**

**8.(a)** $\frac{8}{15}$ **(b)** $\frac{1}{2}$ **(c)** $\frac{1}{6}$ **Independent**