UGANDA ADVANCED CERTIFICATE OF EDUCATION

PRE-UNEB TEST1 2024

PURE MATHEMATICS

PAPER 1

3 HOURS

INSTRUCTIONS.

Attempt all questions from section A and any five from section B

SECTION A

1. Express $5sin^2x - 3cosxsinx + cos^2x$ in the form of $a + bcos(2x - B)$
2. Given that points P (3, 4, 6) and Q (5, 7, 4). Find the coordinates of C such that it divides the line PQ in the ratio of 3:4 (a) internally (b) externally
3. Given that A (-3, 0) and B (3, 0) are fixed points. Show that the locus of P (x, y) which moves such that $PB = 2PA$ is a circle and find its radius and centre.
4. Solve the inequality $\frac{x-2}{x+1} \leq \frac{x+1}{x+3}$
5. Evaluate $\int \frac{dx}{1+cosx}$ b) $\int 3x^2\, e^x dx$
6. The sum of the height and radius of a right circular cone is 9cm. show that the maximum volume of the cone is $36\pi cm^2$
7. Prove by induction that $8^n - 7n + 6$ is divisible by 7 for all $n \geq 1$.
8. Solve the differential equation $\frac{dy}{dx} + 2ycotx = cosec^2x$

SECTION B

Attempt any five questions from this section

9. Express $f(x) = \frac{x^3-x^2-3x+5}{(x-1)(x^2-1)}$ as a partial fraction and hence show $\int_0^2 f(x)\,dx = \frac{1}{2}In27$ (12 marks)
10. The parametric equations $x = \frac{1+t}{1-t}$ and $y = \frac{2t^2}{1-t}$ represent a curve.

Kadowebenon@gmail.com

a). Find the Cartesian equation of the curve

b). Determine the turning points of the curve and their nature.

c). State the asymptotes and intercepts of the curve

d). Hence sketch the curve. (12 marks)

11. (a) Find the equation of the plane passing through the points A (2, 5, 6) B (1, 7, 4) and C (1, 9, 13) (04 marks)

b) Find the angle between the plane above and the line $\frac{x-4}{3} = \frac{2-y}{5} = \frac{z-6}{2}$ (03 marks)

c) Show that the lines $\boldsymbol{r} = (-\boldsymbol{2i} + \boldsymbol{5j} - \boldsymbol{11k}) + \mu(\boldsymbol{3i} + \boldsymbol{j} + \boldsymbol{3k})\boldsymbol{and}\,\boldsymbol{r} = (\boldsymbol{8i} + \boldsymbol{9j}) + \boldsymbol{t}(\boldsymbol{4i} + \boldsymbol{2j} + \boldsymbol{5k})$ intersect hence find the position vector of their point of intersection. (05 marks)

12. (a) Solve the differential equation; $x^2\frac{dy}{dx} = x^2 + y^2 + xy$

(4 marks)

(b) The rate of cooling of a body in air is said to be proportional to the difference between the temperature, θ of the body and temperature θ_0 of air. If the temperature of air is kept constant at 20^oC and the body cools from 100^0C to 60^0C in 20 minutes. In what time will the body cool to 30^oC (08marks)

13. Prove that the chord P(ap^2, 2ap) and Q(aq^2, 2aq) on the parabola $y^2 = 4ax$ has the equation $(p+q)y = 2x + 2apq$ (05 marks)

b) Show that the if the chord above makes a right angle at the origin show that $pq = -4$

c) Show that the locus of the midpoint of PQ is $y^2 = 2a(x - 4a)$ (07 marks)

14. Given that x and y are real values such that $xz + yz^- = 7i + 2$, where $z = 2 + i$, find the modulus of $x + yi$ (04 marks)

b) Using Demovres theorem, prove that $16sin^5\theta = sin5\theta - 5sin3\theta + 10sin\theta$ (04 marks)

c) Find the locus of Z which moves such that $|z - 3i| = 3|z + 2|$ (04)

15. a) Determine the maximum point of the expression $6sinx - 3cosx$ (03marks)

b) Prove that $\frac{cos11^0+sin11^0}{cos11^0-sin11^0} = tan56^0$ (03marks)

c) In a triangle ABC, prove that $cos2A + cos2B + cos2C = -1 - 4cosAcosBcosC$ (06 marks)

16. a) The second, fourth and eighth terms of an AP are in GP. If the sum of the third and the fifth term is 20. Find the first four terms of the progression. (05 marks)

b) Expand $\sqrt{\frac{1+5x}{1-5x}}$ as far as the term including x^3 . taking the first three terms, evaluate $\sqrt{14}$ to 3sfs (07 marks)

kadowebenon@gmail.com

PRE-UNEB PURE MATHS GUIDE 2024

NUMBER	SOLUTIONS
1.	Let $5sin^2x - 3cosxsinx + cos^2x = A$ $$A = \frac{5}{2}(1 - cos2x) + \frac{1}{2}(1 + cos2x) - \frac{3}{2}sin2x$$ $$= 3 - (2cos2x + 1.5sin2x)$$ $$3 - (2cos2x + 1.5sin2x) \equiv 3 - Rcos(2x - B)$$ $$= 3 - (Rcos2xcosB + Rsin2xsinB)$$ By comparison, $$2 = RcosB \ \ and\ 1.5 = RsinB$$ $$R = \sqrt{2^2 + 1.5^2} = 2.5$$ $$B = \tan^{-1}\frac{1.5}{2}$$ $$B = 36.87$$ $$A = 3 - 2.5\cos(2x - 36.87)$$
2.	a) $\boldsymbol{OC} = \frac{4\begin{pmatrix}3\\4\\6\end{pmatrix}+3\begin{pmatrix}5\\7\\4\end{pmatrix}}{4+3}$ $$= \frac{1}{7}\begin{pmatrix}27\\37\\36\end{pmatrix}$$ $$The\ coordinates\ of\ C\ \left(\frac{27}{7}, \frac{37}{7}, \frac{36}{7}\right)$$ b) $\boldsymbol{OC} = \frac{-4\begin{pmatrix}3\\4\\6\end{pmatrix}+3\begin{pmatrix}5\\7\\4\end{pmatrix}}{-4+3}$ $$= \begin{pmatrix}-3\\-5\\12\end{pmatrix}$$ $The\ coordinates\ of\ C$ (-3, -5, 12)
3.	$$\sqrt{(x-3)^2 + (y-0)^2} = 2\sqrt{(x+3)^2 + (y-0)^2}$$ $$x^2 - 6x + 9 + y^2 = 4(x^2 + 6x + 9 + y^2)$$ $x^2 + y^2 + 10x + 27 = 0$ is a circle By comparison with the general equation of the circle $$2g = 10,$$ $$g = 5$$ $$f = 0$$ The centre is (-5, 0)

$$Radius = \sqrt{25-3}$$
$$Radius = \sqrt{22}units$$

4.

$$\frac{x-2}{x+1} - \frac{x+1}{x+3} \leq 0$$
$$\frac{-x-7}{(x+1)(x+3)} \leq 0$$

For critical values,

$$x = -7, x = -3\ x = -1$$

Table of signs

	$x < -7$	$-7 < x < -3$	$-3 < x < -1$	$x > -1$
$-x-7$	+	-	-	-
$x+1$	-	-	-	+
$x+3$	-	-	+	+
Net sign	+	-	+	-

$The\ required\ range\ is -7 \leq x \leq -3\ and\ x \geq -1$

5.

a) $\int \frac{1}{1+cosx} dx = \int \frac{dx}{1+2cos^2\frac{x}{2}-1}$

$$= \int \frac{dx}{2cos^2\frac{1}{2}}$$
$$= \frac{1}{2}\int sec^2\frac{x}{2}dx$$
$$= tan\frac{x}{2} + c$$
$$accept\ the\ t-substitution$$

b)

$$\int 3x^2e^x\,dx$$

Sign	Differentiation	Integration
+	$3x^2$	e^x
-	$6x$	e^x
+	6	e^x
-	0	e^x

$$\int 3x^2e^x\,dx = 3x^2e^x - 6xe^x + 6e^x + A$$

6.

$$h + r = 9$$
$$h = 9 - r$$

	$$v = \frac{1}{3}\pi r^2 h$$ $$v = \frac{1}{3}\pi r^2(9-r) = \frac{1}{3}\pi(9r^2 - r^3)$$ $$\frac{dv}{dr} = \frac{1}{3}\pi(18r - 3r^2) = 0$$ $$3r(6-r) = 0$$ $$r = 0 \; and \; r = 6cm$$ $$r = 6cm$$ $$h = 9 - 6$$ $$h = 3cm$$ $$v_{max} = \frac{1}{3}\pi(6)^2(3) = 36\pi$$
7.	$$let \; a_n = 8^n - 7n + 6$$ Trying n=1 $$a_1 = 8^2 - 7(1) + 6$$ $$= 7$$ It is true for n=1 Trying n=2 $$a_2 = 8^2 - 7(2) + 6$$ $$= 56$$ It is true for n=2 $$a_2 - a_1 = 56 - 7$$ $$= 49$$ Suppose it is true for n=k $$a_k = 8^k - 7k + 6$$ Trying n=k+1 $$a_{k+1} = 8^{k+1} - 7k - 7 + 6$$ $$a_{k+1} - a_k = (8^{k+1} - 7k - 7 + 6) - (8^k - 7k + 6)$$ $$= 7(8^k - 1)$$ Since it holds for n=1, n=2, n=k and n=k+1, then $8^n - 7n + 6$ is divisible by 7 for all $n \geq 1$
8.	$$\frac{dy}{dx} + 2ycotx = cosec^2x$$ $$R = e^{2\int cotxdx} = cos^2x$$ $$cos^2x\frac{dy}{dx} + 2ycotxcos^2x = cot^2x$$

	$$\frac{d}{dx}(ycot^2x) = cot^2x$$ $$\int d(ycos^2x) = \int cot^2x\,dx$$ $$ycos^2x = \int (cosec^2x - 1)\,dx$$ $$ycos^2x = -cotx - x + A$$
	SECTION B (60MARKS)
9.	$$Let\ \frac{x^3 - x^2 - 3x + 5}{(x-1)(x-1)(x+1)} \equiv A + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x+1)}$$ $x^3 - x^2 - 3x + 5$ $$= A(x-1)^2(x+1) + B(x^2-1) + C(x+1) + D(X-1)^2$$ Putting x=1, $1 - 1 - 3 + 5 = C(1+1)$, $$C = 1$$ $$Putting\ x = -1$$ $$-1 - 1 + 3 + 5 = D(-1-1)^2$$ $$D = \frac{3}{2}$$ $$Putting\ x = 0$$ $$5 = A - B + 1 + 1.5$$ $$5 = 2A - 2B \ldots\ldots\ldots\ldots\ldots\ldots (1)$$ $$Putting\ x = 2$$ $$8 - 4 - 6 + 5 = 3A + 3B + 3 + 1.5$$ $$-1 = 2A + 2B \ldots\ldots\ldots\ldots\ldots\ldots..(2)$$ By solving $A = 1\ and\ B = \frac{-3}{2}$ $$\frac{x^3 - x^2 - 3x + 5}{(x-1)(x-1)(x+1)} = 1 - \frac{3}{2(x-1)} + \frac{1}{(x-1)^2} + \frac{3}{2(x+1)}$$ $Hence\ \int_0^2 f(x)\,dx$ $$= \int_0^2 1\,dx - \frac{3}{2}\int_0^2 \frac{1}{x-1}dx + \int_0^2 \frac{1}{(x-1)^2}dx + \frac{3}{2}\int_0^2 \frac{1}{x+1}dx$$ $$= \left\| x - 1.5In(x-1) - \frac{1}{x-1} + 1.5In(x+1)\right\|_0^2$$ $$= 2 - 1 + 1.5In3 - (0 - 0 + 1 + 0)$$ $$= \frac{1}{2}In27$$

10.

$$x - xt = 1 + t$$
$$x - 1 = t(x + 1)$$
$$t = \frac{x - 1}{x + 1}$$

Putting t into y $y = 2(\frac{x-1}{x+1})\ ^{2}x\left(\frac{x+1}{x+1-x+1}\right)$

$$y = \frac{x^2 - 2x + 1}{x + 1}$$

b.

$$\frac{dy}{dx} = \frac{(x + 1)(2x - 2) - (x^2 - 2x + 1)(1)}{(x + 1)^2} = 0$$
$$x^2 + 2x - 3 = 0$$
$$(x + 3)(x - 1) = 0$$
$$x = -3 \ or \ x = 1$$
$$when \ x = -3, y = \frac{(-3 - 1)^2}{-3 + 1} = -8 \ \ (-3, -8)$$
$$when \ x = 1, y = \frac{(1 - 1)^2}{1 + 1} = 0 \ (1, 0)$$

	L	$x = -3$	R	L	$x = 1$	R
$sign \ on \ \frac{dy}{dx}$	+	0	-	-	0	+
		maxima			minima	

c.

Vertical asymptotes

$$As \ y \to \pm\infty$$
$$(x + 1)^2 \to 0$$
$$x = -1 \ is \ the \ equation \ of \ the \ vertical \ asymptote$$

Slanting asymptote

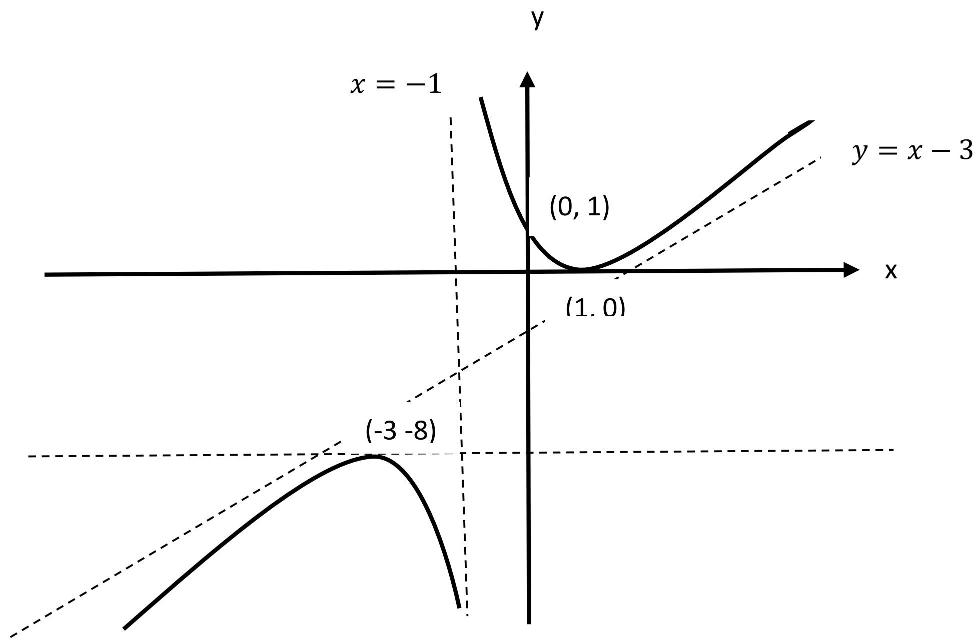
By long division $y = x - 3 + \frac{4}{x+1}$

$$As \ x \to \pm\infty$$
$$y \to x - 3$$
$$y = x - 3 \ is \ the \ equation \ of \ the \ horizontal \ asymptote$$

d.

Intercepts

$$When \ y = 0$$
$$(x - 1)^2 = 0$$
$$x = 1 \ (1, 0)$$
$$When \ x = 0, y = 1 \ (0, 1)$$

	
11.	$\boldsymbol{AB} = \begin{pmatrix}1\\7\\4\end{pmatrix} - \begin{pmatrix}2\\5\\6\end{pmatrix} = \begin{pmatrix}-1\\2\\-2\end{pmatrix}$ $\boldsymbol{AC} = \begin{pmatrix}1\\9\\13\end{pmatrix} - \begin{pmatrix}2\\5\\6\end{pmatrix} = \begin{pmatrix}-1\\4\\7\end{pmatrix}$ $\boldsymbol{n} = \begin{vmatrix}\boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k}\\-1 & 2 & -2\\-1 & 4 & 7\end{vmatrix}$ $= 22\boldsymbol{i} + 9\boldsymbol{j} - 2\boldsymbol{k}$ $\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}22\\9\\-2\end{pmatrix} = \begin{pmatrix}22\\9\\-2\end{pmatrix}.\begin{pmatrix}2\\5\\6\end{pmatrix}$ $22x + 9y - 2z = 77$
b.	$\cos(90-\theta) = \frac{\begin{pmatrix}22\\9\\-2\end{pmatrix}.\begin{pmatrix}3\\-5\\2\end{pmatrix}}{\sqrt{(22^2+9^2+(-2)^2)(9+25+4)}}$ $sin\theta = \frac{17}{\sqrt{569x38}}$ $\theta = \sin^{-1}\frac{17}{\sqrt{569x38}}$ $\theta = 6.64$

Kadowebenon@gmail.com

c.	$$\begin{pmatrix}-2+3\mu\\5+\mu\\-11+3\mu\end{pmatrix}=\begin{pmatrix}8+4t\\9+2t\\5t\end{pmatrix}$$ $$3\mu-4t=10, \dots\dots\dots(1)$$ $$\mu-2t=4, \dots\dots\dots..(2)$$ $$3\mu-5t=11 \dots\dots\dots..(3)$$ $$eqn(1)-eqn(2)$$ $$\mu=2$$ $$from\; eqn(2)$$ $$t=-1$$ Putting the unknowns into eqn (3) $$3\mu-5t=3(2)-5(-1)$$ $$=11$$ Since $LHS = RHS$ then the lines intersect $$\boldsymbol{r}=\begin{pmatrix}-2+3(2)\\5+2\\-11+3(2)\end{pmatrix}$$ Point of intersection is (4, 7, -5)
12.	$$x^2\left(v+x\frac{dv}{dx}\right)=x^2+(vx)^2+vx^2$$ $$x\frac{dv}{dx}=(1+v^2)$$ $$\int\frac{dv}{(1+v^2)}=\int\frac{dx}{x}$$ $$\tan^{-1}v=Inx+c$$ $$\frac{y}{x}=\tan(Inx+c)$$ $$y=xtan(Inx+c)$$
b.	$$\frac{-d\theta}{dt}\propto(\theta-20)$$ $$\int\frac{d\theta}{(\theta-20)}=-k\int 1\,dt$$ $$In(\theta-20)=-kt+c$$ $$t=0,\theta=100^0C$$ $$In(100-20)=c$$ $$c=In80$$ $$In\left(\frac{\theta-20}{80}\right)=-kt$$ $$t=20\;minutes,\theta=60$$

	$$In\frac{40}{80} = -20k$$ $$k = \frac{In2}{20}$$ $$In\left(\frac{\theta - 20}{80}\right) = \frac{-tIn2}{20}$$ $$In\frac{10}{80} = -t\frac{In2}{20},$$ $$t = 60\ minutes$$
13.	$$\frac{y-2aq}{x-aq^2} = \frac{2ap-2aq}{ap^2-aq^2}$$ $$\frac{y-2aq}{x-aq^2} = \frac{1}{p+q}$$ $$y(p+q) = x + aq^2 + 2apq$$ $$y(p+q) = 2x + 2apq$$
b.	$$Gradient\ of\ OP = \frac{2ap-0}{ap^2-0} = \frac{2}{p}$$ $$Gradient\ of\ OQ = \frac{2aq-0}{aq^2-0} = \frac{2}{q}$$ $$Gradient\ of\ OPx\ Gradient\ of\ OQ = -1$$ $$\frac{2}{p}x\frac{2}{q} = -1$$ $$pq = -4$$
c.	$$Y = \frac{2ap+2aq}{2} = a(p+q)$$ $$p + q = \frac{Y}{a} \dots\dots\dots\dots\dots(1)$$ $$X = \frac{a(p^2+q^2)}{2}$$ $$2X = a((p+q)^2 - 2pq)$$ $$2X = a\left(\frac{Y}{a}\right)^2 - 2a(-4)$$ $$Y^2 = 2a(X - 4a)$$
14.	$$x(2+i) + y(2-i) = 7i + 2$$ $$2x + xi + 2y - yi = 7i + 2$$ $$(x-y)i + (2x+y) = 7i + 2$$ Equating real and imaginary terms $$x - y = 7 \dots\dots\dots..(1)$$ $$2x + y = 2 \dots\dots\dots.(2)$$

	$eqn(1) + eqn(2)$ $x = 3 \; and \; y = -4$ $x + yi = 3 - 4i$ $\|3 - 4i\| = \sqrt{3^2 + (-4)^2} = 5units$
b.	$From \; (2isin\theta)^5 = \left(z + \frac{1}{z}\right)^5$ $32isin^5\theta = z^5 + 5z^4\frac{1}{-z} + 10z^3\left(\frac{-1}{z}\right)^2 + 10z^2\left(\frac{-1}{z}\right)^3 + 5z^1\left(\frac{-1}{z}\right)^4 + \left(\frac{-1}{z}\right)^5$ $32isin^5\theta = \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$ $32isin^5\theta = (2isin5\theta) - 5(2isin3\theta) + 10(2isin\theta)$ $16sin^5\theta = sin5\theta - 5sin3\theta + 10sin\theta$
c.	$\|x + (y-3)i\| = 3\|(x+2) + yi\|$ $\sqrt{x^2 + (y-3)^2} = 3\sqrt{(x+2)^2 + y^2}$ $x^2 + y^2 - 6y + 9 = 9(x^2 + 4x + 4 + y^2)$ $8y^2 + 8x^2 + 42x + 27 = 0$ The locus is a circle
15.	$Let \; 6sinx - 3cosx \equiv Rsin(x - B)$ $6sinx - 3cosx \equiv RsinxcosB - RcosxsinB$ Equating $6 = RcosB$ $2 = RsinB$ $R = \sqrt{6^2 + 3^2} = \sqrt{45}$ $B = \tan^{-1}\frac{3}{6} = 26.57$ $6sinx - 3cosx = \sqrt{45}\sin(x - 26.57)$ $(6sinx - 3cosx)max = \sqrt{45}$ Occurs when $x - 26.57 = \sin^{-1} 1$ $x = 90 + 26.57 = 116.57$ $\max point \; (116.57, \sqrt{45})$
b.	$RHS = \tan(45 + 11)$ $= \frac{tan45 + tan11}{1 - tan11tan45}$ $= \left(\frac{sin11 + cos11}{cos11}\right)x\left(\frac{cos11}{cos11 - sin11}\right)$

	$$=\frac{sin11+cos11}{cos11-sin11}$$
c.	$$LHS=2\cos(A+B)\cos(A-B)+2cos^2C-1$$ But $cosC=\cos\big(180-(A+B)\big)$ $$=-\cos(A+B)$$ $$LHS=2\cos(A+B)\cos(A-B)+2cos^2(A+B)-1$$ $$=2\cos(A+B)\{\cos(A+B)+\cos(A-B)\}-1$$ $$=2\cos(A+B)\,(2cosAcosB)-1$$ $$=-1-4cosAcosBcosC$$ $$=RHS$$
16.	$$a+d,a+3d,a+7d$$ $$\frac{a+3d}{a+d}=\frac{a+7d}{a+3d}$$ $$8ad-6ad=2d^2$$ $$2ad-2d^2=0$$ $$2d(a-d)=0$$ Either $d=0\ or\ d=a$ $a=d$ $$The\ progression\ is\ 2d,4d,8d$$ $$r=\frac{4d}{2d}$$ $$r=2$$ $$2d,4d,8d,16d,32d$$ $$8d+32d=20$$ $$40d=20$$ $$d=0.5\ ,a=0.5$$ $$The\ first\ four\ terms\ are\ 1,2,4,8$$
b.	$$\sqrt{\frac{1+5x}{1-5x}}=\sqrt{\frac{(1+5x)^2}{1-25x^2}}$$ $$=(1+5x)(1-\frac{1}{2}(-25x^2)$$ $$=1+5x+\frac{25}{2}x^2+\frac{125}{2}x^3$$
	$$using\ x=\frac{1}{9}$$

	$$\sqrt{\frac{1+\frac{5}{9}}{1-\frac{5}{9}}} \approx 1 + 5\left(\frac{1}{9}\right) + \frac{25}{2}\left(\frac{1}{9}\right)^2$$ $$\sqrt{14} \approx \frac{2x277}{2x81}$$ $$\sqrt{14} \approx 3.42$$