12.0 EXPANSION AND FACTORISATION (senior two work)

		CONTENT	COMMENTS
		Objectives: To • understand key terms involving algebraic expressions. • expand algebraic expressions. • describe quadratic expressions. • derive and use the three important quadratic identities. • factorize algebraic expressions. • factorize quadratic expressions.	
		Key words. Algebraic expressions, terms(nomials), coefficients, expansion, like terms, collection of like terms, difference of two squares('DOTS'), perfect squares, factorization, quadratic expression.	
	12.1	**Nomial (a term)** is an expression with one or more numbers and /or variables within it. For example, $2y$ can be a term, 4 can be a term, $-3x^2$ can be a term. An **algebraic expression** is an expression built up from constants and variables connected by the algebraic operations + and/or -. For example, $2x^2+3x-y+1$ is an algebraic expression with four terms namely, $2x^2$, $3x$, $-y$ and 1 as the first, second, third and fourth terms respectively. Notice that each term includes the operational sign before it. The positive sign is usually not shown when naming a term. An algebraic expression with: (i) one term is called a **monomial**. e.g., $-3x^2$ (ii) two terms is called a **binomial**. e.g., 2y + 3 and $2x^2+3x$, (iii) many terms is called a **polynomial**. e.g., $2x^2+3x-y+1$ A **coefficient** of a term is the non-variable (number) part of a term. For example, in the expression $2x^2-3x+4$, the coefficient of the; - x^2term is 2. x term is -3.	
	12.2	**Expansion of algebraic expressions.** This involves opening of brackets to form an expression that occupies more space. **Expansion of monomial multipliers.** For example, consider expanding $a(b+c)$ This can be illustrated geometrically by finding the area of a rectangle with length, $(b+c)$ and side width, a as shown in Figure 1.	

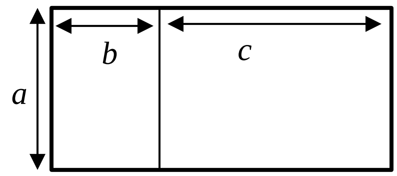


Figure 1

Area $= a(b+c) = a \times b + a \times c$

The multiplication sign can be 'hidden' to give.

Area $= a(b+c) = ab + ac$

This simply means that each term inside the bracket is multiplied by the monomial outside the bracket. …………….F 12.1

Note: This applies to monomial multiplying any polynomial.

Example 1.

Find the expansion of:

(a) $2(x+3)$
(b) $3x(x-5)$
(c) $-7(y^2+2y)$
(d) $-3a(x-y)$
(e) $2x - x(1-x^2)$

(a) $2(x+3) = 2 \times x + 2 \times 3$
$= 2x + 6$

(b) $3x(x-5) = 3x \times x + 3x \times -5$
$= 3x^2 - 15x$

(c) $-7(y^2+2y) = -7 \times y^2 + \ -7 \times 3y$
$= -7y^2 - 21y$

(d) $-3a(x-y) = -3a \times x + \ -3a \times -y$
$= -3ax + 3ay$

(e) $2x - x(1-x^2) = 2x + \ -x \times 1 + \ -x \times -x^2$
$= 2x - x + x^3$
$= x + x^3$(*after collecting 2x and -x as like terms*).

Learners:
1. should recall and use the sign rules of multiplication.

2.should check for errors in opening brackets.

Such errors have gone on to senior levels of mathematics.

Expansion of binomial multipliers.

Product of two binomials.

For example, consider expanding $(a+b)(e+f)$

This can be geometrically illustrated by finding the area of a rectangle with length, $(b+c)$and width, $(e+f)$ as shown in Figure 2.

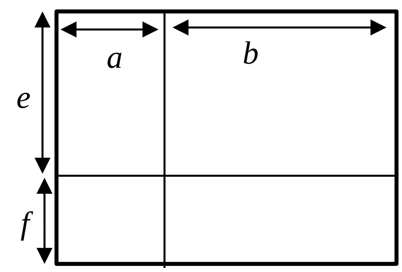


Figure 2

$$\text{Area} = (a+b)(e+f) = a \times (e+f) + b \times (e+f)$$

The multiplication sign can be 'hidden' to give.

$$\text{Area} = (a+b)(e+f) = a(e+f) + b(e+f)$$
$$= ae + af + be + bf$$

This simply means that each term inside the first bracket is multiplied by each term in the second bracket. ………………F 12.2

Note: This applies to binomial multiplying any polynomial.

Example 2.

Find the expansion of:

(a) $(2+y)(x+1)$
(b) $(3x+2)(2y+5)$
(c) $(2x+4)(1-2y)$
(d) $(2x-y)(x-y)$
(e) $(x+\frac{1}{2})(x+2)$
(f) $(x+2)^2$

(a) $(2+y)(x+1) = 2(x+1) + y(x+1)$
$= 2x + 2 + yx + y$

(b) $(3x+2)(2y+5) = 3x(2y+5) + 2(2y+5)$
$= 6xy + 15x + 4y + 10$

(c) $(2x+4)(1-2y) = 2x(1-2y) + 4(1-2y)$
$= 2x - 4xy + 4 - 8y$

(d) $(2x-y)(x-y) = 2x(x-y) + -y(x-y)$
$= 2x^2 - 2xy - xy + y^2$
$= 2x^2 - 3xy + y^2$

(e) $(x+\frac{1}{2})(x+2) = x(x+2) + \frac{1}{2}(x+2)$
$= x^2 + 2x + \frac{1}{2}x + 1$

		$= x^2 + (2 + \frac{1}{2})x + 1$ $= x^2 + \frac{5}{2}x + 1$ (*after collecting terms in x*). (f) $(x+2)^2 = (x+2)(x+2)$ $= x(x+2) + 2(x+2)$ $= x^2 + 2x + 2x + 2^2$ $= x^2 + 4x + 4$ (*after collecting terms in x*)	
	12.3	**Quadratic expressions.** A quadratic expression is one where the highest power is 2. For example, $x^2 + 2x + 1$, $x^2 + 2xy + y^2$, $x^2 - 4$, $3x^2 + 2x$ are quadratic expressions. **Quadratic identities.** By expansion, three important quadratic identities can be derived. <u>Square of a sum</u>. $(A+B)^2 = (A+B)(A+B)$ $= A(A+B) + B(A+B)$ $= A^2 + AB + BA + B^2$ $= A^2 + 2AB + B^2$ (*since AB = BA*) $\therefore \boldsymbol{(A+B)^2 = A^2 + 2AB + B^2}$ **...............F 12.3** <u>Square of a difference</u>. $(A-B)^2 = (A-B)(A-B)$ $= A(A-B) + -B(A-B)$ $= A^2 - AB - BA + B^2$ $= A^2 - 2AB + B^2$(*since AB = BA*) $\therefore \boldsymbol{(A-B)^2 = A^2 - 2AB + B^2}$ **...............F 12.4** <u>Difference of two squares(‘DOTS’).</u> $(A+B)(A-B) = A(A-B) + B(A-B)$ $= A^2 - AB + BA - B^2$ $= A^2 - B^2$ $\therefore \boldsymbol{(A+B)(A-B) = A^2 - B^2}$ **................F 12.5**	Learners should note that $(A+B)^2 \neq A^2 + B^2$ and $(A-B)^2 \neq A^2 - B^2$ Such errors have been carried on to senior level of mathematics.

Example 3

Use the identities $(A \pm B)^2 = A^2 \pm 2AB + B^2$ *to evaluate: -*

(a) 999^2
(b) 502^2
(c) $2.4^2 + (2.6)(4.8) + 2.6^2$
(d) $10.2^2 - (20.4)(0.2) + 0.2^2$

(a) For 999^2, the nearest convenient figure is 1,000.

$$\begin{aligned} 999^2 &= (1{,}000 - 1)^2 \\ &= 1{,}000^2 - 2 \times 1{,}000 \times 1 + 1^2 \\ &= 1{,}000{,}000 - 2{,}000 + 1 \\ &= 998{,}001 \end{aligned}$$

(b) For 502^2, the nearest convenient figure is 500.

$$\begin{aligned} 502^2 &= (500 + 2)^2 \\ &= 500^2 + 2 \times 500 \times 2 + 2^2 \\ &= 250{,}000 + 2{,}000 + 4 \\ &= 252{,}004 \end{aligned}$$

(c) $2.4^2 + (2.6)(4.8) + 2.6^2$

$$\begin{aligned} &= 2.4^2 + 2(2.6)(2.4) + 2.6^2 \\ &= (2.4 + 2.6)^2 \\ &= 5.0^2 \\ &= 25 \end{aligned}$$

(d) $10.2^2 - (20.4)(0.2) + 0.2^2$

$$\begin{aligned} &= 10.2^2 - 2(10.2)(0.2) + 0.2^2 \\ &= (10.2 - 0.2)^2 \\ &= 10.0^2 \\ &= 100 \end{aligned}$$

Example 4.

Use the identity $(A + B)(A - B) = A^2 - B^2$ *to evaluate:*

(a) 43×37
(b) $7.1^2 - 2.9^2$

(a) For 43×37 the convenient figure is 40 with a difference of 3 from each.

$$\begin{aligned} 43 \times 37 &= (40 + 3)(40 - 3) \\ &= 40^2 - 3^2 \\ &= 1600 - 9 \\ &= 1591 \end{aligned}$$

(b)
$$\begin{aligned} 7.1^2 - 3.9^2 &= (7.1 + 2.9)(7.1 - 2.9) \\ &= 10 \times 4.2 \\ &= 42 \end{aligned}$$

		Example 5. *Given that* $x^2 - y^2 = 15$ *and* $x + y = 5$, *find the: -* *(a) value of* $x - y$ *(b) values of x and y.* (a) Notice that $x^2 - y^2 = 15$ is a 'DOTS' So, $x^2 - y^2 = (x + y)(x - y)$…………………..(1) Substituting $x^2 - y^2 = 15$ and $x + y = 5$ into equation (1) $15 = 5 \times (x - y)$ $\frac{15}{5} = \frac{5(x-y)}{5}$ $3 = (x - y)$ $x - y = 3$ ………………….(2) (b) Let $x + y = 5$ ………………….(3) Equation (2) + equation (3) $\begin{array}{r} x-y=3 \\ +(x+y)=5 \\ \hline 2x=8 \end{array}$ $x = 4$ Substituting x = 4 into equation (3) $4 + y = 5$ $y = 1$	
	12.4	**Factorization.** The process of finding the factors of an algebraic expression. The expansion of $2(x + 3) = 2x + 6$ Therefore, 2 and (x + 3) are the factors of $2x + 6$. Notice that factorization is the opposite of expansion. **Factorization by common factors.** Some algebraic expressions have terms with common factors. So, the simplest method of factorization is by looking out for common factors in the terms. ***Example 6.*** *Factorize completely:* *(a)* $3x^2 + 6x$ *(b)* $12yx^2 - 3yx$ *(c)* $3(x + 1) + 6x(x + 1)$ (a) $3x^2 + 6x$ (*common factors are 3 and x*) $= 3x(x + 2)$	

(b) $12yx^2 - 3yx$ (*common factors are 3, y and x*)
$= 3yx(4x - 1)$

(c) $3(x+1) + 6x(x+1)$ (*common factors are 3 and (x+1)*)
$= (x+1)(3+6x)$
$= 3(x+1)(1+2x)$

Example 7.
Evaluate the following by factorization: -
(a) $3.2^2 + 6.8 \times 3.2$
(b) $55 \times 61 + 55 \times 56 - 55 \times 17$

(a) $3.2^2 + 6.8 \times 3.2$
$= 3.2(3.2 + 6.8)$
$= 3.2 \times 10$
$= 32$

(b) $55 \times 61 + 55 \times 56 - 55 \times 17$
$= 55(61 + 56 - 17)$
$= 55(61 + 56 - 17)$
$= 55 \times 100$
$= 5500$

Factorization by grouping.

The terms of the expression can be creatively grouped to have common factors in the groups.

Example 8.
Factorize completely: -
(a) $x^2 + xy + 2x + 2y$
(b) $6a^2 - 2ab - b + 3a$

(a) $x^2 + xy + 2x + 2y$
$= (x^2 + xy) + (2x + 2y)$
$= x(x + y) + 2(x + y)$
$= (x + y)(x + 2)$

(b) $6a^2 - 2ab - b + 3a$
$= (6a^2 + 3a) + (-2ab - b)$
$= 3a(2a + 1) - b(2a + 1)$
$= (2a + 1)(3a - b)$

Factorization by using ‘DOTS’.

Here, it is important to: -
(i) first look out for common factors.

(ii) then rewrite the expression so that the squares are clearly visible.

Example 9.
Factorize completely; -

(a) $x^2 - y^2$
(b) $t^2 - 121$
(c) $m^2 - 9n^2$
(d) $81x^2 - 16y^2$
(e) $(a + b)^2 - 4b^2$
(f) $4u^2 - 64v^2$
(g) $32x^4 - 2a^4y^4$

(a) $x^2 - y^2$
$= (x + y)(x - y)$

(b) $t^2 - 121$
$= t^2 - (11)^2$
$= (t + 11)(t - 11)$

(c) $m^2 - 9n^2$
$= m^2 - (3n)^2$
$= (m + 3n)(m - 3n)$

(d) $81x^2 - 16y^2$
$= (9x)^2 - (4y)^2$
$= (9x + 4y)(9x - 4y)$

(e) $(a + b)^2 - 4b^2$
$= (a + b)^2 - (2b)^2$
$= (a + b + 2b)(a + b - 2b)$
$= (a + 3b)(a - b)$

(f) $4u^2 - 64v^2$
$= (2u)^2 - (8v)^2$
$= (2u + 8v)(2u - 8v)$
$= 2(u + 4v)2(u - 4v)$
$= 4(u + 4v)(u - 4v)$

(g) $32x^4 - 2a^4y^4$
$= 2(16x^4 - a^4y^4)$
$= 2((4x^2)^2 - (a^2y^2)^2)$
$= 2(4x^2 + a^2y^2)(4x^2 - a^2y^2)$
$= 2(4x^2 + a^2y^2)\,((2x)^2 - (ay)^2)$
$= 2(4x^2 + a^2y^2)(2x + ay)(2x - ay)$

Factorization of quadratic expressions.

Considering the expansion

$$(Ax + B)(Cx + D) = Ax(Cx + D) + B(Cx + D)$$
$$= ACx^2 + ADx + BCx + BD$$
$$= ACx^2 + (AD + BC)x + BD$$

The result is a quadratic expression such that the coefficient of x,(AD + BC) is the sum of two numbers whose product is ACBD(or the product of the coefficient of the x^2 term and the constant term).

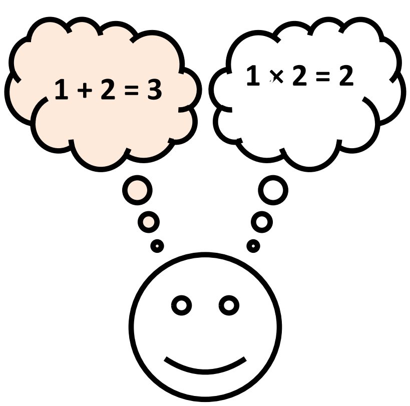
This observation provides for a method of factorization of a quadratic expression following the steps outlined below.

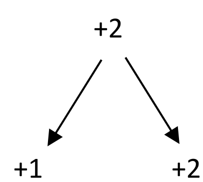
Step 1. the **coefficient** of the term in x^2(quadratic term) and the number term are identified.
Step2. the product of these numbers is obtained.
Step 3. two numbers with a sum equal to the coefficient of the term in x and the product equal to the product obtained in step 2.are identified by trial and error.
Step 4. the x term is split using the numbers identified in step 3.
Step 5. the terms are paired (grouped)up and factorized.

Example 10.
Factorize $x^2 + 3x + 2$

The coefficient of the first term is 1. The constant term is 2.
Their product is 1 × 2 = 2.
The coefficient of the middle term is 3.

Which two numbers would add to 3 and have product of 2?

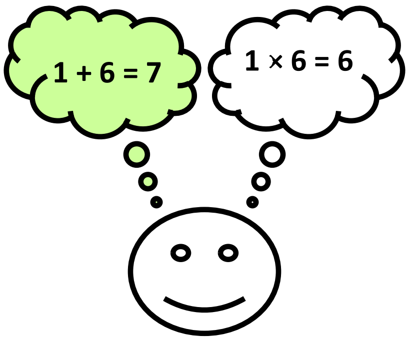


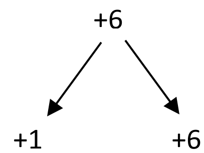


$$x^2 + 3x + 2$$
$$= x^2 + (x + 2x) + 2 \quad \textit{(breaking the middle term.)}$$
$$= (x^2 + x) + (2x + 2) \quad \textit{(grouping)}$$
$$= x(x + 1) + 2(x + 1) \quad \textit{(making common factor visible)}$$
$$= (x + 1)(x + 2) \quad \textit{(factorization).}$$

Example 11.
Factorize $2x^2 + 7x + 3$

The coefficient of the first term is 2. The constant term is 3.
Their product is 2 × 3 = 6
The coefficient of the second term is 7.
Which two numbers would add to 7 and have product of 6?

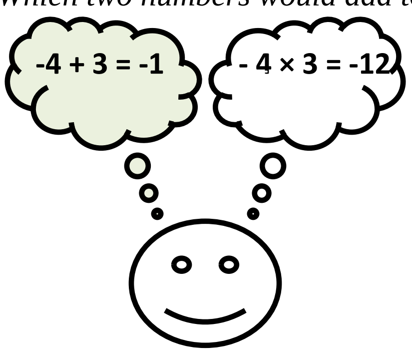


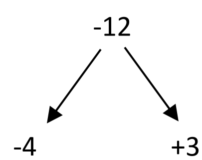


$$
\begin{aligned}
& 2x^2 + 7x + 3 \\
&= 2x^2 + (x + 6x) + 3 \quad \textit{(breaking the middle term.)} \\
&= (2x^2 + x) + (6x + 3) \quad \textit{(grouping)} \\
&= x(2x + 1) + 3(2x + 1) \quad \textit{(making common factor visible)} \\
&= (2x + 1)(x + 3) \quad \textit{(factorization).}
\end{aligned}
$$

Example 12.
Factorize $6x^2 - x - 2$

The coefficient of the first term is 6. The constant term is -2.
Their product is 6 × -2 = -12
The coefficient of the second term is -1.
Which two numbers would add to -1 and have product of -12?





$$
\begin{aligned}
& 6x^2 - x - 2 \\
&= 6x^2 + (-4x + 3x) - 2 \quad \textit{(breaking the middle term.)} \\
&= (6x^2 - 4) + (3x - 2) \quad \textit{(grouping)} \\
&= 2x(3x - 2) + (3x - 2) \quad \textit{(making common factor visible)} \\
&= (3x - 2)(2x + 1) \quad \textit{(factorizing).}
\end{aligned}
$$

	12.5	*Exercise.* 1. *Expand each of the following expressions:* *(a)* $ax(ax-2)$ *(b)* $4\left(1-\frac{3x}{2}\right)^2$ 2. *Factorize completely:* *(a)* $2axy^2 - 4a^2x^2y$ *(b) 3ab – 2 + 3a – 2b* 3. *Factorize completely:* *(a)* $9-4x^2$ *(b)* $2-18y^2$ *(c)* $a^2x^2-4x^2$ *(d)* $2x^4-2$ 4. *Without using tables or calculator, evaluate:* *(a)* 202^2 *(b)* 199^2 *(c)* $1.9^2+(0.2)(1.9)+(0.1)^2$ *(d)* $53^2-(6)(53)+3^2$ *(e)* $5(2\times 14^2 - 2\times 4^2)$ 5. *Factorize completely:* *(a)* $y^2+5y-14$ *(b)* $4b^2+8b+3$ *(c)* $3x^2-4x+1$ *(d)* $14x^2-10x-4$	