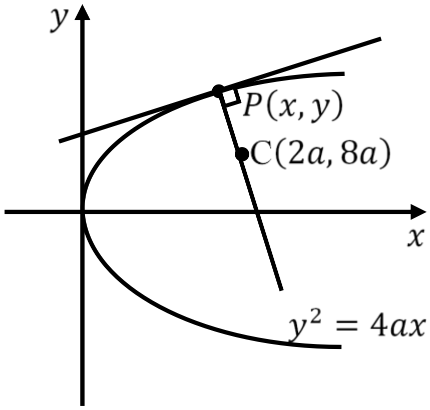
PROPOSED MARKING GUIDE
DIOCESE OF KIGEZI
PURE MATHEMATICS 2024

NO	SOLUTION	MKS	COMMENT
1	$\sin 4x \cos x = \cos 3x \sin 2x$ ${}^{1}/_{2}(\sin 5x + \sin 3x) = {}^{1}/_{2}(\sin 5x - \sin x)$ $\sin 3x = -\sin x$ $\sin 3x + \sin x = 0$ $2\sin 2x \cos x = 0$ When $\sin 2x = 0$ $2x = \sin^{-1}(0)$ $2x = 0^0, 180^0, 360^0$ $x = 0^0, 90^0, 180^0$ When $\cos x = 0$ $x = \cos^{-1}(0)$ $x = 90^0$ $\therefore x = \{0^0, 90^0, 180^0\}$ **ALT**: $\sin 4x \cos x = \cos 3x \sin 2x$ $2\sin 2x \cos 2x \cos x = \cos 3x \sin 2x$ $2\sin 2x \cos 2x \cos x - \cos 3x \sin 2x = 0$ $\sin 2x\,(2\cos 2x \cos x - \cos 3x) = 0$ $\sin 2x\,(\cos 3x + \cos x = \cos 3x) = 0$ $\sin 2x \cos x = 0$ $\sin 2x = 0$ or $\cos x = 0$ For $\sin 2x = 0$		

	$2x = \sin^{-1}(0^0)$ $2x = 0^0, 180^0, 360^0$ $x = 0^0, 90^0, 180^0$ For $\cos x = 0$ $x = \cos^{-1}(0)$ $x = 90^0$ $\therefore x = \{0^0, 90^0, 180^0\}$		
		05	
2	For A.P; $u_1 = a$ $u_2 = a + d$ $u_4 = a + 3d$ Form a G.P; $\frac{a+d}{a} = \frac{a+3d}{a+d}\ (= r)$ $(a+d)^2 = a^2 + 3ad$ $a^2 + 2ad + d^2 = a^2 + 3ad$ $ad - d^2 = 0$ $d(a - d) = 0$ $d = 0$ or $d = a$ $\therefore d = a$ ⇨$r = \frac{a+a}{a} = \frac{2a}{a} = 2$		
		05	

3



From $y^2 = 4ax$

$2y\frac{dy}{dx} = 4a$

$y\frac{dy}{dx} = 2a$

$\frac{dy}{dx} = \frac{2a}{y}$

Gradient of the normal, $m = -\frac{y}{2a}$

Equation;

$\frac{y-8a}{x-2a} = -\frac{y}{2a}$

$2ay - 16a^2 = -xy + 2ay$

$xy = 16a^2$

$x = \frac{16a^2}{y}$

Since point P(x, y) lies on the parabola, $y^2 = 4ax$, then it must satisfy it.

⇨ $(y)^2 = 4a\left(\frac{16a^2}{y}\right)$

$y^3 = 64a^3$

$y = \sqrt[3]{(4a)^3} = 4a$

When $y = 4a, x = \frac{16a^2}{4a} = 4a$

	$\therefore P(4a, 4a)$		
		05	
4	Let $y = \ln(1 + x^x)$ $u = x^x, y = \ln(1 + u)$ For $u = x^x$ $\ln u = x \ln x$ $\frac{1}{u}\frac{du}{dx} = x \cdot \frac{1}{x} + \ln x \cdot 1$ $\frac{du}{dx} = (1 + \ln x)x^x$ For $y = \ln(1 + u)$ $\frac{dy}{du} = \frac{1}{1+u}$ $\frac{dy}{du} = \frac{1}{1+x^x}$ Using $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ $= \frac{1}{1+x^x} \cdot (1 + \ln x)x^x$ $= \frac{1+\ln x}{(1+x^x)x^{-x}}$ $= \frac{1+\ln x}{1+x^{-x}}$ $\therefore \frac{d}{dx}[\ln(1 + x^x)] = \frac{1+\ln x}{1+x^{-x}}$		
		05	
5	$\frac{50}{(2+i)^2} = ai + b$ $\frac{50}{4+4i-1} = ai + b$ $\frac{50}{3+4i} = ai + b$ $\frac{50(3-4i)}{(3+4i)(3-4i)} = ai + b$		

	$\frac{50(3-4i)}{3^2+4^2} = ai + b$ $\frac{50(3-4i)}{25} = ai + b$ $6 - 8i = ai + b$ $\therefore a = -8, b = 6$		
		05	
6	Let $u = 4 - x^2$ $du = -2x\,dx$ $dx = -\frac{du}{2x}$ <table><tr><th>x</th><th>u</th></tr><tr><td>0</td><td>4</td></tr><tr><td>$1/2$</td><td>$15/4$</td></tr></table>$\Rightarrow \int_4^{15/4} \frac{4x}{u} \cdot -\frac{du}{2x}$ $= -2\int_4^{15/4} \frac{1}{u}\,du$ $= -2\Big[\ln u\Big]_4^{15/4}$ $= -2\left(\ln\left(\frac{15}{4}\right) - \ln 4\right)$ $= -2\ln\left(\frac{15}{16}\right)$ $= 2\ln\left(\frac{16}{15}\right)$ or 0.1291 (4 dps) **ALT**: $\int_0^{1/2} \frac{4x}{4-x^2}\,dx = \int_0^{1/2} \frac{4x}{(2+x)(2-x)}\,dx$ Let $\frac{4x}{(2+x)(2-x)} \equiv \frac{A}{2+x} + \frac{B}{2-x}$ $4x \equiv A(2-x) + B(2+x)$ Putting $x = 2; 8 = 4B \quad \therefore B = 2$ Putting $x = -2; -8 = 4A \quad \therefore A = -2$ $\int_0^{1/2} \frac{4x}{4-x^2}\,dx = -2\int_0^{1/2} \frac{1}{2+x}\,dx + 2\int_0^{1/2} \frac{1}{2-x}\,dx$		

	$= -2\left[\ln(2+x)\right]_0^{1/2} - 2\left[\ln(2-x)\right]_0^{1/2}$ $= -2\left(\left[\ln(4-x^2)\right]_0^{1/2}\right)$ $= -2\left(\ln\left(\frac{15}{4}\right) - \ln(4)\right)$ $= -2\ln\left(\frac{15}{16}\right)$ $= 2\ln\left(\frac{16}{15}\right)$ or 0.1291(4dps)		
		05	
7	Let θ be the required acute angle. Let $\boldsymbol{n}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \boldsymbol{n}_2 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = \lvert\boldsymbol{n}_1\rvert\lvert\boldsymbol{n}_2\rvert \cos x$ $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \sqrt{1^2 + (-1)^2} \cdot \sqrt{1^2 + 1^2 + 4^2} \cos x$ $1 + 0 - 4 = \sqrt{2}\sqrt{18}\ \cos x$ $\sqrt{36}\cos x = -3$ $\cos x = -\frac{1}{2}$ $x = \cos^{-1}\left(\frac{1}{2}\right)$ $x = 120^0$ $\therefore$ The acute angle, $\theta = 180^0 = 120^0 = 60^0$		
		05	

8	$\frac{dx}{dt} = 3t^2 - 2, \frac{dy}{dt} = 2t - 4 = 2(t-2)$ Using $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$ $\frac{dy}{dx} = 2(t-2) \times \frac{1}{3t^2-2} = 0$ $2(t-2) = 0$ $t = 2$ ⇨ $x = 2^3 - 2(2) - 2 = 8 - 6 = 2$ $y = 2^2 - 4(2) + 1 = 4 - 7 = -3$ $\therefore$ $(2, -3)$ is the stationary point.		
		05	
9	a) Let $t = \tan(\theta/_2)$ ⇨ $3\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right) = 4$ $6t - 4 + 4t^2 = 4 + 4t^2$ $6t = 8$ $t = \frac{4}{3}$ or $t = \infty$ When $t = \frac{4}{3}$; $\tan\left(\frac{\theta}{2}\right) = \frac{4}{3}$ $\frac{\theta}{2} = \tan^{-1}\left(\frac{4}{3}\right)$ $\frac{\theta}{2} = 53.13^0$ $\theta = 106.26^0$ When $t = \infty$ $\tan\left(\frac{\theta}{2}\right) = \infty$		

	$\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} = \infty$ $\cos\left(\frac{\theta}{2}\right) = 0$ $\frac{\theta}{2} = \cos^{-1}(0)$ $\frac{\theta}{2} = 90^0$ $\theta = 180^0$ $\therefore \theta = 106.26^0, 180^0$ **ALT**: Let $3\sin\theta - 4\cos\theta \equiv R\sin(\theta - \alpha)$ $3\sin\theta - 4\cos\theta \equiv R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$ $3\sin\theta - 4\cos\theta \equiv (R\cos\alpha)\sin\theta - (R\sin\alpha)\cos\theta$ Comparing coefficients of; $\sin\theta\,; R\cos\alpha = 3$ ………………….(i) $\cos\theta\,; R\sin\alpha = 4$ …………………..(ii) $(R\cos\alpha)^2 + (R\sin\alpha)^2 = 3^2 + 4^2$ $R^2(cos^2\alpha + sin^2\alpha) = 25$ $R^2 = 25$ $R = 5$ (ii)÷(i); $\tan\alpha = \frac{4}{3}$ $\alpha = \tan^{-1}\left(\frac{4}{3}\right) = 53.13^0$ $5\sin(\theta - 53.13^0) = 4$ $\theta - 53.13^0 = \sin^{-1}\left(\frac{4}{5}\right)$ $\theta - 53.13^0 = 53.13^0, 126.87^0$ $\theta = 106.26^0, 180^0$			

	ALT: $3\sin\theta = 4 + 4\cos\theta$ Squaring both sides; $9sin^2\theta = 16 + 32\cos\theta + 16cos^2\theta$ $9(1 - cos^2\theta) = 16 + 32\cos\theta + cos^2\theta$ $25cos^2\theta + 32\cos\theta + 7 = 0$ b) L.H.S $= \frac{2\cos 2A \sin A}{2\sin 4A \cos A}$ $= \frac{\cos 2A \sin A}{2\sin 2A \cos 2A \cos A}$ $= \frac{\sin A}{4\sin A \cos A \cos A}$ $= \frac{1}{4cos^2 A}$ $= \frac{1}{4}sec^2 A$		
		12	
10	a) For $n = 1$; L.H.S $= 2^3 = 8$, R.H.S$= 2(1)^2(2)^2 = 8$ It holds. For $n = 2$; L.H.S $= 2^3 + 4^3 = 8 + 64 = 72$, R.H.S $= 2(2)^2(3)^2 = 72$ It holds. Assume that the result holds for $n = k$ $2^3 + 4^3 + \cdots + (2k)^3 = 2k^2(k+1)^2$ For $n = k + 1$; $2^3 + 4^3 + \cdots + (2k)^3 + (2k+2)^3 = 2k^2(k+1)^2 + (2k+2)^3$ R.H.S $= 2k^2(k+1)^2 + (2k+2)^3$ $= 2k^2(k+1)^2 + 8(k+1)^3$ $= 2(k+1)^2[k^2 + 4k + 4]$		

$= 2(k+1)^2(k+2)^2$

But $k = n-1$

R.H.S $= 2n^2(n-1+2)^2$

$= 2n^2(n+1)^2$

It holds for $n = k+1$.

b) $\frac{1+x}{\sqrt{1-2x}} = (1+x)(1-2x)^{-1/2}$

For $(1-2x)^{-1/2}$

$= 1 - \frac{1}{2}(-2x) + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)(-2x)^2}{2!} + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{3}\right)(-2x)^3}{3!} + \cdots$

$= 1 + x + \frac{3}{2}x^2 + \frac{5}{3}x^3 + \cdots$

$\frac{1+x}{\sqrt{1-2x}} = (1+x)\left(1 + x + \frac{3}{2}x^2 + \frac{5}{3}x^3 + \cdots\right)$

$= 1 + x + \frac{3}{2}x^2 + \frac{5}{3}x^3 + \cdots$

$x + x^2 + \frac{3}{2}x^3 + \cdots$

$= 1 + 2x + \frac{5}{2}x^2 + \frac{19}{6}x^3 + \cdots$

Putting $x = \frac{1}{8}$;

$\frac{1+\frac{1}{8}}{\sqrt{1-2\left(\frac{1}{8}\right)}} \approx 1 + 2\left(\frac{1}{8}\right) + \frac{5}{2}\left(\frac{1}{8}\right)^2 + \frac{19}{6}\left(\frac{1}{8}\right)^3$

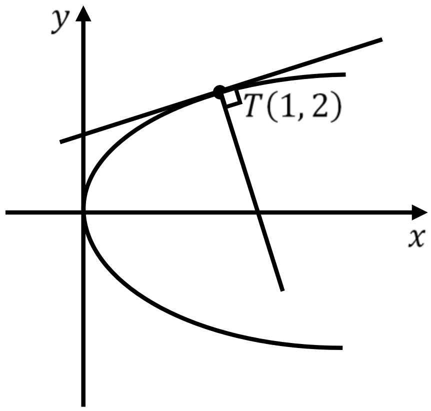
$\frac{\frac{9}{8}}{\frac{\sqrt{3}}{2}} \approx \frac{3979}{3072}$

$\frac{9}{4\sqrt{3}} \approx \frac{3979}{3072}$

$\sqrt{3} \approx \frac{9\times3072}{4\times3979}$

$\sqrt{3} \approx 1.737119879$

$= 1.737$(3dps)

<table>
<tr>
<td>11</td>
<td>a)


From $y^2 = 4x$
$2y\frac{dy}{dx} = 4$
$\frac{dy}{dx} = \frac{2}{y}$
At point (1,2); $\frac{dy}{dx} = \frac{2}{2} = 1$
Equation of the normal;
$\frac{y-2}{x-1} = -1$
$y - 2 = -x + 1$
$y = -x + 3$
Deducing;
$(3 - x)^2 = 4x$
$9 - 6x + x^2 = 4x$
$x^2 - 10x + 9 = 0$
$(x - 1)(x - 9) = 0$
$x = 1$ or $x = 9$
When $x = 1, y = -1 + 3 = 2$
When $x = 9, y = -9 + 3 = -6$
∴ The normal meets the parabola again at point $(9, -6)$</td>
<td>12</td>
<td></td>
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	b) Using $v = \pi \int_a^b x^2\, dy$ $v = \pi \int_{-6}^{2} (x_1^2 - x_2^2)\, dy$ $v = \pi \int_{-6}^{2} \left[(3-y)^2 - \left(\frac{y^2}{4}\right)^2\right] dy$ $v = \pi \int_{-6}^{2} \left[9 - 6y + y^2 - \frac{y^4}{16}\right] dy$ $v = \pi \left[9y - 3y^2 + \frac{y^3}{3} - \frac{y^5}{80}\right]_{-6}^{2}$ $v = \pi \left(\left(18 - 12 + \frac{8}{3} - \frac{32}{80}\right) - \left(-54 - 108 - \frac{216}{3} + \frac{7776}{80}\right)\right)$ $v = \pi \left(\frac{124}{15} + \frac{684}{5}\right)$ $v = \frac{2176\pi}{15}$ cubic units or 455.9238 cubic units		
		12	
12	a) At the point of intersection, $\begin{pmatrix} 8 \\ -3 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $8 + 3\lambda = 3 + \mu$ $3\lambda - \mu = -5$ …………………(i) $-3 - \lambda = 1 + 2\mu$ $-\lambda - 2\mu = 4$ …………………(ii) $7 + 2\lambda = 6 + 3\mu$ $2\lambda - 3\mu = -1$ ………………(iii) 2(i)−(ii); $7\lambda = -14$ $\lambda = -2$ From (i); $3(-2) - \mu = -5$		

	$\mu = -1$ $\begin{pmatrix}8\\-3\\7\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\2\end{pmatrix} = \begin{pmatrix}3\\1\\6\end{pmatrix} + \mu\begin{pmatrix}1\\2\\3\end{pmatrix}$ $\begin{pmatrix}8\\-3\\7\end{pmatrix} - 2\begin{pmatrix}3\\-1\\2\end{pmatrix} = \begin{pmatrix}3\\1\\6\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix}$ $\begin{pmatrix}2\\-1\\3\end{pmatrix} = \begin{pmatrix}2\\-1\\3\end{pmatrix}$ $2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ is the point of intersection. b) $\boldsymbol{r} = \begin{pmatrix}2\\-1\\3\end{pmatrix} + \mu\begin{pmatrix}3\\-1\\2\end{pmatrix} + \nu\begin{pmatrix}1\\2\\3\end{pmatrix}$		
		12	
13	a) $dt = sec^2\theta\, d\theta$ $dt = (1 + t^2)d\theta$ $d\theta = \frac{dt}{1+t^2}$ $\Rightarrow \int \frac{1}{4+5\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{dt}{1+t^2}$ $= \int \frac{1+t^2}{4(1+t^2)+5(1-t^2)} \cdot \frac{dt}{1+t^2}$ $= \int \frac{dt}{9-t^2}$ $= \int \frac{dt}{(3+t)(3-t)}$ Let $\frac{1}{(3+t)(3-t)} \equiv \frac{A}{3+t} + \frac{B}{3-t}$ $1 \equiv A(3-t) + B(3+t)$ Putting $t = 3; 1 = 6B \quad \therefore t = \frac{1}{6}$ Putting $t = -3; 1 = 6A \quad \therefore A = \frac{1}{2}$		

	$\int \frac{dt}{9-t^2} = \frac{1}{6}\int \frac{1}{3+t}dt + \frac{1}{6}\int \frac{1}{3-t}dt$ $= \frac{1}{6}\ln(3+t) - \frac{1}{6}\ln(3-t) + c$ $= \frac{1}{6}\ln\left(\frac{3+t}{3-t}\right) + c$ $= \frac{1}{6}\ln\left(\frac{3+\tan\theta}{3-\tan\theta}\right) + c$ b) $\int_0^1 \tan^{-1}x\,dx = \int_0^1 1\cdot\tan^{-1}x\,dx$ Let $u = \tan^{-1}x$, $\frac{dv}{dx} = 1$ $\frac{du}{dx} = \frac{1}{1+x^2}$, $v = x$ $\int_0^1 \tan^{-1}x\,dx = [x\tan^{-1}x]_0^1 - \int_0^1 \frac{x}{1+x^2}dx$ $= [x\tan^{-1}x]_0^1 - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1$ $= (\tan^{-1}1 - 0) - \left(\frac{1}{2}\ln 2 - \frac{1}{2}\ln 1\right)$ $= \frac{\pi}{4} - \frac{1}{2}\ln 2$ $= \frac{1}{4}(\pi - 2\ln 2)$ $= \frac{1}{4}(\pi - \ln 4)$		
		12	
14	a) Let $f(x) = \ln(1+x)$, $f(0) = 0$ $f'(x) = \frac{1}{1+x}$, $f'(0) = 1$ $f''(x) = -(1+x)^{-2}$, $f''(0) = -1$ $f'''(x) = 2(1+x)^{-3}$, $f''(0) = 2$ Using $f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f''(0)}{3!} + \cdots$ $\therefore \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$		

Validity;

b)

(i) No. of arrangements $= \frac{8!}{3!2!2!} = 1680$

(ii) No. of arrangements $= \frac{6!}{2!2!} = 180$

(iii) When O's are together;

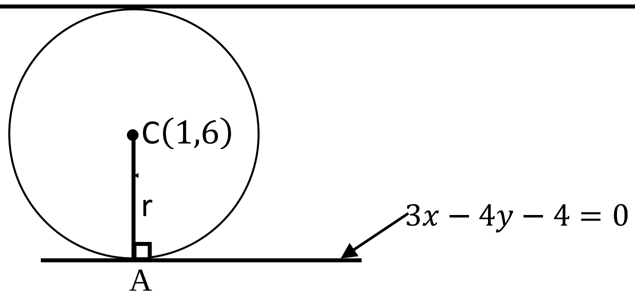
No. of ways $= 1! \times \frac{7!}{3!2!} = 420$

$\therefore$ when O's are separated $= 1680 - 420$

$= 1260$

12

15



a) $r = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}}$

$= \frac{|3(1)-4(6)-4|}{\sqrt{3^2+(-4)^2}}$

$= \frac{25}{5}$

$= 5$

Using $(x-a)^2 + (y-b)^2 = r^2$

$(x-1)^2 + (y-6)^2 = 5^2$

$x^2 - 2x + 1 + y^2 - 12y + 36 = 25$

$\therefore x^2 + y^2 - 2x - 12y + 12 = 0$ is the equation of the circle

b) Point of contact;
From $x^2 + y^2 - 2x - 12y + 12 = 0$

	$x^2 + \left(\frac{3}{4}x - 1\right)^2 - 2x - 12\left(\frac{3}{4}x - 1\right) + 12 = 0$ $x^2 + \frac{(3x-4)^2}{16} - 2x - 9x + 12 + 12 = 0$ $16x^2 + 9x^2 - 24x + 16 - 176x + 384 = 0$ $25x^2 - 200x + 400 = 0$ $x^2 - 8x + 16 = 0$ $(x-4)^2 = 0$ $x = 4$ When $x = 4, y = \frac{3}{4}(4) - 1 = 2$ $\therefore A(4,2)$ is the point of contact.		
		12	
16	a) $\frac{d}{dx}(\sin x\, y) = \tan 3x$ $\int \frac{d}{dx}(\sin x\, y)\, dx = \int \tan 3x\, dx$ $y \sin x = -\frac{1}{3}\ln(\cos 3x) + c$ b) $\frac{dP}{dt} \propto P$ $\frac{dP}{dt} = kP$ Separating variables; $\int \frac{dP}{P} = \int kdt$ $\ln P = kt + c$ When $t = o, P = P_0$ $\ln P_0 = 0 + c$ $c = \ln P_0$ $\ln P = kt + \ln P_0$		

	$kt = \ln\left(\frac{P}{P_0}\right)$ When $t = 10$ days, $P = 2P_0$ $10k = \ln\left(\frac{2P_0}{P_0}\right)$ $k = \frac{1}{10}\ln 2$ $k = 0.0693$ $\frac{1}{10}\ln(2)t = \ln\left(\frac{P}{P_0}\right)$ When $t = 20$ days, $P = ?$ $\frac{1}{10}\ln(2) \times 20 = \ln\left(\frac{P}{P_0}\right)$ $\ln\left(\frac{P}{P_0}\right) = 2\ln 2$ $\ln\left(\frac{P}{P_0}\right) = \ln 4$ $\frac{P}{P_0} = 4$ $P = 4P_0$ Percentage increase $= \frac{increase\ in\ price}{original\ price} \times 100$ $= \frac{4P_0 - P_0}{P_0} \times 100$ $= 300\%$		
		12	

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Note*:*

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Now some help;

- How do you state the validity in maclaurin's theorem?
- How do you state the degree of accuracy in No. 10