DESCRIPTIVE STATISTICS

Introduction to Statistics

Statistics is that branch of mathematics which is concerned with the collection, organization, interpretation, presentation and analysis of numerical data. To a statistician, any information collected is called **data.**

Statistics is therefore a set of concepts, rules, and procedures that help us to:

- **organize** numerical information in the form of tables, graphs, and charts;
- **understand** statistical techniques underlying decisions that affect our lives and well-being; and
- **make** informed decisions basing on information generated from processed data.

Statistics plays a vital role in every field of human activity. For example it has an important role in determining the existing position of per capita income, unemployment, population growth rate, housing etc…in a country.

Statistics is a vital tool in fields like industry, commerce, trade, Physics, Chemistry, Economics, Mathematics, Biology, Botany, Psychology, and Astronomy.

Statistical methods are used in research to collect, analyse, and formulate research findings in every field at higher institutions of learning.

Raw data: This is the data which is not organized numerically. Thus when data has not been ordered in any specific way after collection, it is called raw data.

Discrete raw data

This is the data that takes only exact values. This data is normally collected by counting and usually takes integral values. For example;

the number of students in class,
the number of books each students reports to school with at the beginning of the term,
the number of cars passing a certain trading centre in 1 hour,
the number of students served in the dinning hall in five minutes.

The numbers of students doing Subsidiary mathematics in 20 randomly chosen schools are as shown below;

63 50 49 62 89 152 180 90 21 29

55 89 60 49 65 72 79 81 68 47

This is an example of discrete raw data. The data is raw because it has not been ordered in any way.

Continuous Raw data

This is the data that cannot take exact values, but can be given only within certain range or measured to a certain degree of accuracy. Continuous data can therefore take on any value; for example

the speeds of cars passing a certain trading centre,

the heights of students in class,

the time taken by each student in Subsidiary mathematics class to solve a problem.

The heights of 20 children in a school measured correct to the nearest cm are shown below;

133	136	120	138	133	131	127	141	127	143
130	131	125	144	128	134	135	137	133	129

This is an example of continuous raw data.

Types of data

Data can be defined as groups of information that represent the qualitative or quantitative attributes of a variable or set of variables. A **variable** is a quantity, which is counted or measured. Thus data can be any set of information that describes a given entity. Data in statistics can be classified into grouped data and ungrouped data.

Ungrouped Data

Any data that you first gather is ungrouped data. Ungrouped data is data in the raw. An example of ungrouped data is any list of numbers that you can think of. Below is an example of ungrouped data.

The following table shows the data on the length of time (in minutes) it took 80 students in a S.5 Subsidiary mathematics Class to complete a certain exercise.

23	24	18	14	20	24	24	26	23	21
16	15	19	20	22	14	13	20	19	27
29	22	38	28	34	32	23	19	21	31
16	28	19	18	12	27	15	21	25	16
30	17	22	29	29	18	25	20	16	11
17	12	15	24	25	21	22	17	18	15
21	20	23	18	17	15	16	26	23	22
11	16	18	20	23	19	17	15	20	10

This data is ungrouped because it is not organized in any form.

Representation of ungrouped data using frequency tables

Ungrouped data is concisely represented or illustrated by counting the number of times each value occurs and forming a **frequency distribution** table.

Frequency

This refers to the number of times an item occurs. In most cases many items occur more than once, therefore a frequency distribution table is used to remove repetitions. The frequency is easily obtained by tallying the data.

Example 1

The following data gives the number of blind students in 10 randomly chosen classes in a certain school. 0, 2, 1, 4, 2, 3, 2, 1, 4, 5. Form a frequency table for the data.

Solution

Number of students	**Tally**	**Frequency**
0	/	1
1	//	2
2	///	3
3	/	1
4	//	2
5	/	1
Total	10	10

Example 2

The following temperatures were recorded over a 20-day period in January 2012.

23°, 19°, 15°, 18°, 20°, 18°, 21°, 18°, 20°, 21°,
20°, 22°, 20°, 20°, 15°, 25°, 20°, 21°, 18°, 22°.

Represent this raw data in a suitable frequency distribution table.

Solution

Temperature (^{0}c)	**Tally**	**Frequency**
15	//	2
18	////	4
19	/	1
20	~~////~~	6
21	///	3
22	//	2
23	/	1
25	/	1
Totals		20

Example 3

The following data shows the numbers of children in 30 randomly chosen families in Kampala District.

1	2	4	0	2	3	1	4	2	3	5	2	2	3	2
2	3	1	2	3	2	0	1	1	2	0	3	2	3	3

Represent the above data using a frequency distribution table.

Solution

Number of children in family	**Tally**	**Frequency**
0	///	3
1	~~////~~	5
2	~~////~~ ~~////~~ /	11
3	~~////~~ ///	8
4	//	2
5	/	1
Total		30

Grouped Data

Grouped data is data that has been organized into groups known as classes. Grouped data has been 'classified' and thus some level of data analysis has taken place, which means that the data is no longer raw.

A data class is group of data which is related by some user defined property. For example, if you were collecting the ages of the people you met during a census, you could group them into classes as those in their teens, twenties, thirties, forties and so on. Each of those groups is called a class.

Each of those classes is of a certain width and this is referred to as the **Class width**, **Class Interval** or **Class Size**. This class interval is very important when it comes to drawing Histograms and Frequency diagrams. All the classes may have the same class size or they may have different class sizes depending on how you group your data. The class interval is always a whole number.

Below is an example of grouped data where the classes have the same class interval.

Age (years)	Frequency
0 – 9	12
10 - 19	30
20 - 29	18
30 - 39	12
40 - 49	9
50 - 59	6
60 - 69	0

Below is an example of grouped data where the classes have different class interval.

Age (years)	Frequency	Class Interval
0 - 9	15	10
10 - 19	18	10
20 - 29	17	10
30 - 49	35	20
50 - 79	20	30

General rules for grouping data:

Given a set of raw or ungrouped data, how would you group that data into suitable classes that are easy to work with and at the same time meaningful? The following is followed;

(i) Calculating Class Interval

The first step is to determine how many classes you want to have. Next, you subtract the lowest value in the data set from the highest value in the data set and then you divide by the number of classes that you want to have:

$$Class\ interval = \frac{Highest\ value\ -\ Lowest\ value}{number\ of\ classes\ you\ want}$$

(ii) Finding the class frequencies.

The number of observations falling into each class interval also known as the **frequency** is best determined by using **tallies**.

The results are then displayed in the form of a table called the frequency distribution table.

Example 1

The following table shows the data on the length of time (in minutes) it took 80 students in a S.5 Subsidiary mathematics Class to complete a certain exercise.

23	24	18	14	20	24	24	26	23	21
16	15	19	20	22	14	13	20	19	27
29	22	38	28	34	32	23	19	21	31
16	28	19	18	12	27	15	21	25	16
30	17	22	29	29	18	25	20	16	11
17	12	15	24	25	21	22	17	18	15
21	20	23	18	17	15	16	26	23	22
11	16	18	20	23	19	17	15	20	10

Group the following raw data into six classes and form a frequency distribution table.

Solution

The first step is to identify the highest and lowest number. Smallest number is 10 and the highest value is 38.

$$Class\ interval = \frac{Highest\ value\ -\ Lowest\ value}{number\ of\ classes\ you\ want} = \frac{38-10}{6} = 4.66\dot{6}$$

The Class interval should always be a whole number and yet in this case we have a decimal number. The solution to this problem is to round off to the nearest whole number.

In this example, $4.66\dot{6}$ gets rounded up to 5. So our class width/class interval will be 5; meaning that we group the above data into groups of 5 as in the table below.

Time (Minutes)	**Tally**	**Frequency**
10 – 14	~~////~~ ///	8
15 – 19	~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ///	28
20 – 24	~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ //	27
25 – 29	~~////~~ ~~////~~ //	12
30 - 34	////	4
35 - 39	/	1
Total		80

Class Limits and Class Boundaries

Class limits refer to the actual values that you see in the table. Taking an example of the table above, **10** and **14** would be the class limits of the first class. Class limits are divided into two categories: lower class limit and upper class limit. In the table above, for the first class, **10** is the lower class limit while **14** is the upper class limit. Thus 10, 15, 20, 25, 30 and 35 are called the lower class limits, and 14, 19, 24, 29, 34 and 39 are called the upper class limits. Note that the **class widths** are not given by the differences between the respective class limits.

On the other hand, class boundaries are not always observed in the frequency table. **Class boundaries** give the true class interval, and similar to class limits, are also divided into lower and upper class boundaries.

Class boundaries are related to class limits by the given relationships:

Upper class boundaries = Upper class limits + 0.5
Lower class boundaries = Lower class limits - 0.5

This is only true when the data values are whole numbers.

As a result of the above, the lower class boundary of one class is equal to the upper class boundary of the previous class.

For example;

For the class 10 – 14, the lower class boundary is 9.5 and the upper is 14.5

For the class 15 – 19, the lower class boundary is 14.5 and the upper is 19.5

NOTE: It should however be noted that the class boundaries depend on the degree of accuracy.

If the data values are rounded off to one decimal place (1d.p);

Upper class boundaries = Upper class limits + 0.05
Lower class boundaries = Lower class limits - 0.05

If the data values are rounded off to two decimal places (2d.p);

Upper class boundaries = Upper class limits + 0.005
Lower class boundaries = Lower class limits - 0.005

For example the class;

7.0 – 7.4 has a lower class boundary of 6.95 and upper class boundary of 7.45.

0.14 – 0.15 has a lower class boundary of 0.135 and upper class boundary of 0.155

The relationship between the class boundaries and the class interval is given as follows:

$$Class\ interval = upper\ class\ boundary - lower\ class\ boundary$$

For 10 – 14. The class interval = 14.5 – 9.5 = 5

For 7.0 – 7.4. The class interval = 7.45 – 6.95 = 0.5

For 0.135 – 0.155. The class interval = 0.155 – 0.135 = 0.002

Note: Class interval, width, size and length are the same.

Class Marks

The class marks are simply the midpoints of the classes. They are the averages of the lower and upper class limits and they are obtained by the formula:

$$\text{Class mark} = \frac{\text{lower class limit} + \text{upper class limit}}{2} \text{ Or } \frac{\text{upper class boundary} + \text{lower class boundary}}{2}$$

For example; for 10 – 14, the class mark is $\frac{10+14}{2} = 12$

Class limits and class boundaries play separate roles when it comes to representing statistical data diagrammatically as we shall see in the next sub – topic.

Example 2

The masses of 50 mangoes brought from Ntinda Market measured in grams (g), were noted and shown in the table below.

86	101	114	118	87	92	93	116
105	102	97	93	101	111	96	117
100	106	118	101	107	96	101	102
104	92	99	107	98	105	113	100
103	108	92	109	95	100	103	110
113	99	106	116	101	105	86	88
108	92						

Construct a frequency distribution, using equal class intervals of width 5g, and taking the lower class boundary of the first interval as 84.5g.

Solution

Since the lower class boundary of the first interval should be 84.5, then the lower class limit of the first class must be 85.

Mass (g)	Tally	Frequency	Class boundaries
85 – 89	////	4	84.5 – 89.5
90 – 94	~~////~~ /	6	89.5 – 94.5
95 – 99	~~////~~ //	7	94.5 – 99.5
100 – 104	~~////~~ ~~////~~ ///	13	99.5 – 104.4
105 – 109	~~////~~ ~~////~~	10	104.4 – 109.5
110 – 114	~~////~~	5	109.5 – 114.4
115 – 119	~~////~~	5	114.5 – 119.5

NOTE: The classes and class boundaries of Grouped data may be given in different ways. The following frequency distributions show some of the ways that data may be grouped.

(i) The frequency distribution shows the length of 30 rods measured to the nearest mm.

Length, x (mm)	Frequency	Class boundaries
27 – 31	4	$26.5 \leq x \leq 31.5$
32 – 36	11	$31.5 \leq x \leq 36.5$
37 – 41	12	$36.5 \leq x \leq 41.5$
42 – 46	3	$41.5 \leq x \leq 46.5$

The interval '27 – 31' means 26.5 mm ≤ length ≤ 31.5 mm. Thus if we are given data in the form $26.5 \leq x \leq 31.5$, then 26.5 and 31.5 are the class boundaries.

(ii) The frequency distribution shows the marks of 50 students in a Subsidiary mathematics test marked out of 20

Marks	Frequency	Class boundaries
0 -	9	0 – 3
3 -	12	3 – 6
6 -	15	6 – 9
9 -	10	9 – 12
12 -	4	12 – 15
15 -	0	15 – 18

The interval '3–' means 3 marks ≤ mark < 6 marks, so any mark including 3 marks and up to (but not including) 6 minutes come into this interval and so on. When the data is given in the form shown, the class boundaries are obtained directly as shown in the table above.

(iii) The frequency distribution table shows the masses of 40 phones made by different manufacturers.

Mass (g)	Frequency	Class boundaries
–100	8	0 – 100
–200	10	100 – 200
–300	16	200 – 300
–400	6	300 – 400

The interval '–100' means 0g < mass ≤ 100g. Similarly '–200' means 100g < mass ≤ 200g, so any mass over 100 grams up to and including 200 grams comes into this interval. The class boundaries are therefore 0, 100, 200, 300 and 400 as shown in the table above.

(iv) On a particular day, the length of stay of each car at a taxi park was recorded as shown in the table below.

Length of stay (min)	Frequency
$t < 25$	62
$25 \le t \le 50$	70
$50 \le t \le 75$	88
$75 \le t \le 100$	280
$100 \le t \le 125$	30

The interval $t < 25$ means that the time lies between 0 and 25 exclusive. Thus lengths of stay given represent the class boundaries. The class boundaries are therefore 0, 25, 50, 75, 100 and 125. These intervals can be written as '0–' and the next is '25–' and the next is '50–',etc.

Representation of data using frequency diagrams

Histogram

In statistics, a *histogram* is a graphical representation showing a visual impression of the distribution of data. A histogram consists of tabular frequencies, shown as adjacent rectangles, erected over discrete intervals, with an area equal to the frequency of the observations in the interval.

It is therefore a graphical display of data using bars of different heights where the class frequencies are plotted against class boundaries and NO gaps are left between the bars.

Example 1

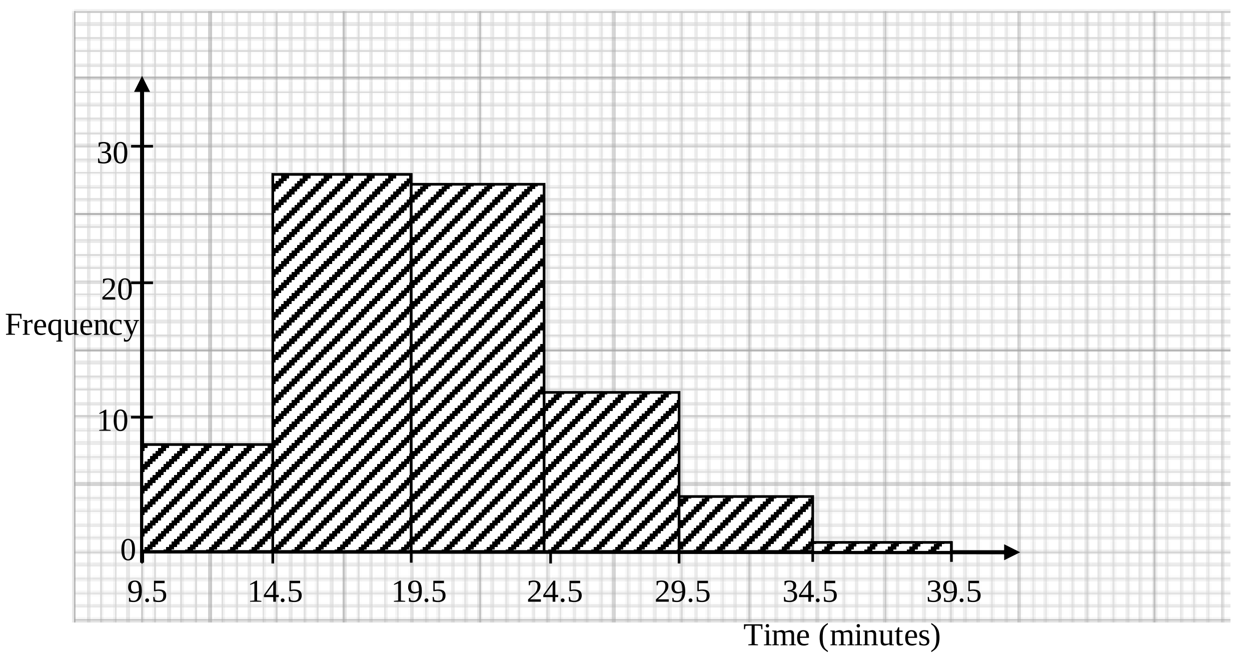
Draw a histogram for the data given below.

Time (Minutes)	Tally	Frequency
10 – 14	~~////~~ ///	8
15 – 19	~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ///	28
20 – 24	~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ //	27
25 – 29	~~////~~ ~~////~~ //	12
30 - 34	////	4
35 - 39	/	1

Solution

Time (Minutes)	Class boundaries	Frequency
10 – 14	9.5 – 14.5	8
15 – 19	14.5 – 19.5	28
20 – 24	19.5 – 24.5	27
25 – 29	24.5 – 29.5	12
30 - 34	29.5 – 34.5	4
35 - 39	34.5 – 39.5	1

We then plot frequency against class boundaries.



NOTE:

(i) A suitable **uniform scale** for each axis should be used and each axis should have a starting value.

(ii) The horizontal axis should start with the lowest class boundary

(iii) Shading of the histogram is not important but once done it should be uniform.

(iv) The **width** of each **rectangle** is the **difference** between **upper** and **lower class boundaries**, **NOT** the difference between **upper** and **lower class limits**.

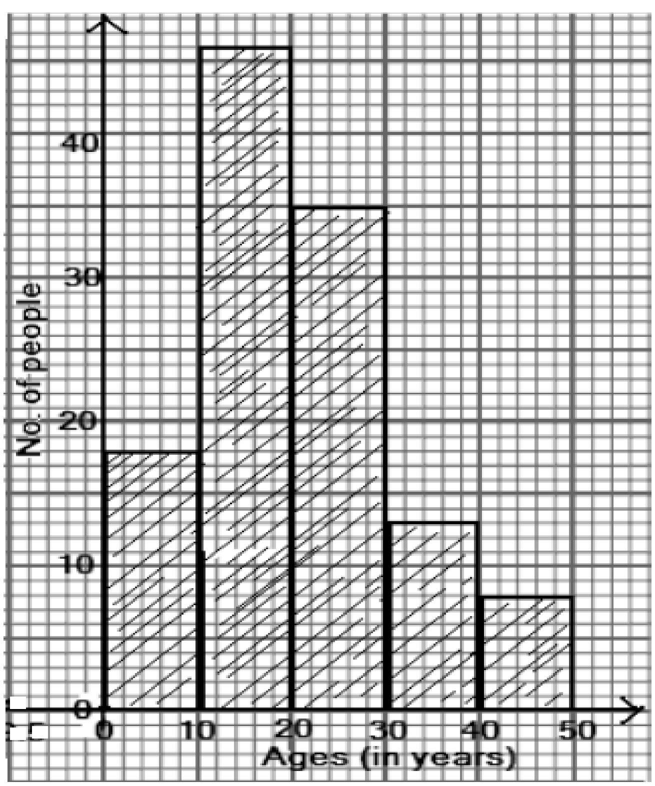
Example 2

The ages of 120 people who travelled to Fort Portal on Christmas day using two Link Buses were recorded and are shown in the frequency table. Draw a histogram to illustrate the data.

Age (years)	Frequency
– 10	18
– 20	46
– 30	35
– 40	13
- 50	8

Solution

The notation '- 10' means, '0 < age ≤ 10' and similarly '– 20' means, '10 < age ≤ 20'. Therefore the class boundaries are 0, 10, 20, 30, 40 and 50. The histogram is then drawn as shown in the diagram below;



Cumulative Frequency

Cumulative frequency is defined as a running total of frequencies. It can also be defined as the sum of all previous frequencies up to the current point. Therefore the cumulative frequency is the total frequency up to a particular **upper class boundary**.

The cumulative frequency is important when analyzing data, where the value of the cumulative frequency indicates the number of elements in the data set that lie below the current value. The cumulative frequency is also useful when representing data using diagrams like the Ogive or cumulative frequency curve.

Example 1

The set of data below shows the length of 30 rods measured to the nearest mm. Draw a cumulative frequency table for the data.

Length, x (mm)	Frequency
27 – 31	4
32 – 36	11
37 – 41	12
42 – 46	3

Solution

The cumulative frequency at a certain point is found by adding the frequency at the present point to the cumulative frequency of the previous point.

The cumulative frequency for the first data point is the same as its frequency since there is no cumulative frequency before it. The cumulative table is as shown below;

Length, x (mm)	Frequency	Cumulative frequency
27 – 31	4	4
32 – 36	11	4+11=15
37 – 41	12	15+12=27
42 – 46	3	27+3=30

Cumulative Frequency Graph (Ogive)

A cumulative frequency graph, also known as an Ogive, is a curve showing the cumulative frequency for a given set of data. The cumulative frequencies are plotted on the y-axis against the upper class boundaries and the points are joined with a *smooth curve*.

An Ogive is used to study the growth rate of data as it shows the accumulation of frequency and hence its growth rate.

Example 2

The table below shows the distribution of marks of 81 students in a S.5 End of Year Subsidiary Mathematics examination.

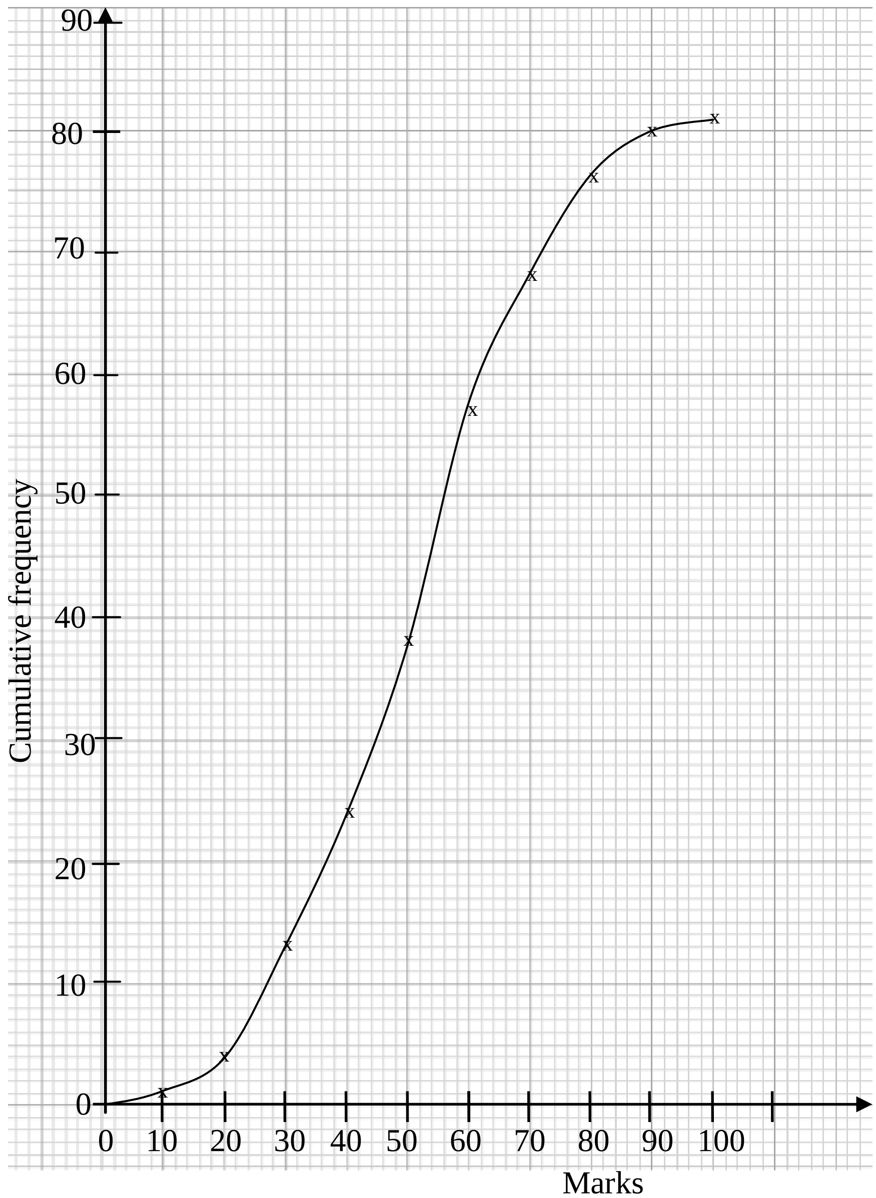
Mark	No. of candidates
$0 < x \leq 10$	1
$10 < x \leq 20$	3
$20 < x \leq 30$	9
$30 < x \leq 40$	11
$40 < x \leq 50$	14
$50 < x \leq 60$	19
$60 < x \leq 70$	11
$70 < x \leq 80$	8
$80 < x \leq 90$	4
$90 < x \leq 100$	1

Draw a cumulative frequency curve to show the data.

Solution

Mark	No. of candidates	Class boundaries	Cumulative frequency
$0 < x \leq 10$	1	0 – 10	1
$10 < x \leq 20$	3	10 – 20	4
$20 < x \leq 30$	9	20 – 30	13
$30 < x \leq 40$	11	30 – 40	24
$40 < x \leq 50$	14	40 – 50	38
$50 < x \leq 60$	19	50 – 60	57
$60 < x \leq 70$	11	60 – 70	68
$70 < x \leq 80$	8	70 – 80	76
$80 < x \leq 90$	4	80 – 90	80
$90 < x \leq 100$	1	90 – 100	81

Since the marks have been given in the form $0 < x \leq 10$, the class boundaries are 0, 20, 30, 40, 50, 60, 70, 80, 90, 100. The cumulative frequency curve is then drawn as shown below;



NOTE:

(i) A class with zero frequency is added at the beginning of the distribution so that its upper class boundary is the lower class boundary of the first class. The resulting curve starts from the origin.

(ii) The scale should be uniform and the curve should always be made to touch both axes. It should not be left hanging.

Example 3

Thirty pencils were collected from S.5 students offering Subsidiary mathematics and their lengths were measured, correct to the nearest cm. The frequency distribution is given below.

Length (cm)	3 - 5	6 - 8	9 - 11	12 - 14	15 - 17	18 – 20
Frequency	1	2	11	10	5	1

(a) Construct the cumulative frequency table and draw a cumulative frequency curve for the data.

(b) (i) Estimate from your curve the number of pencils that were less than 10cm long.

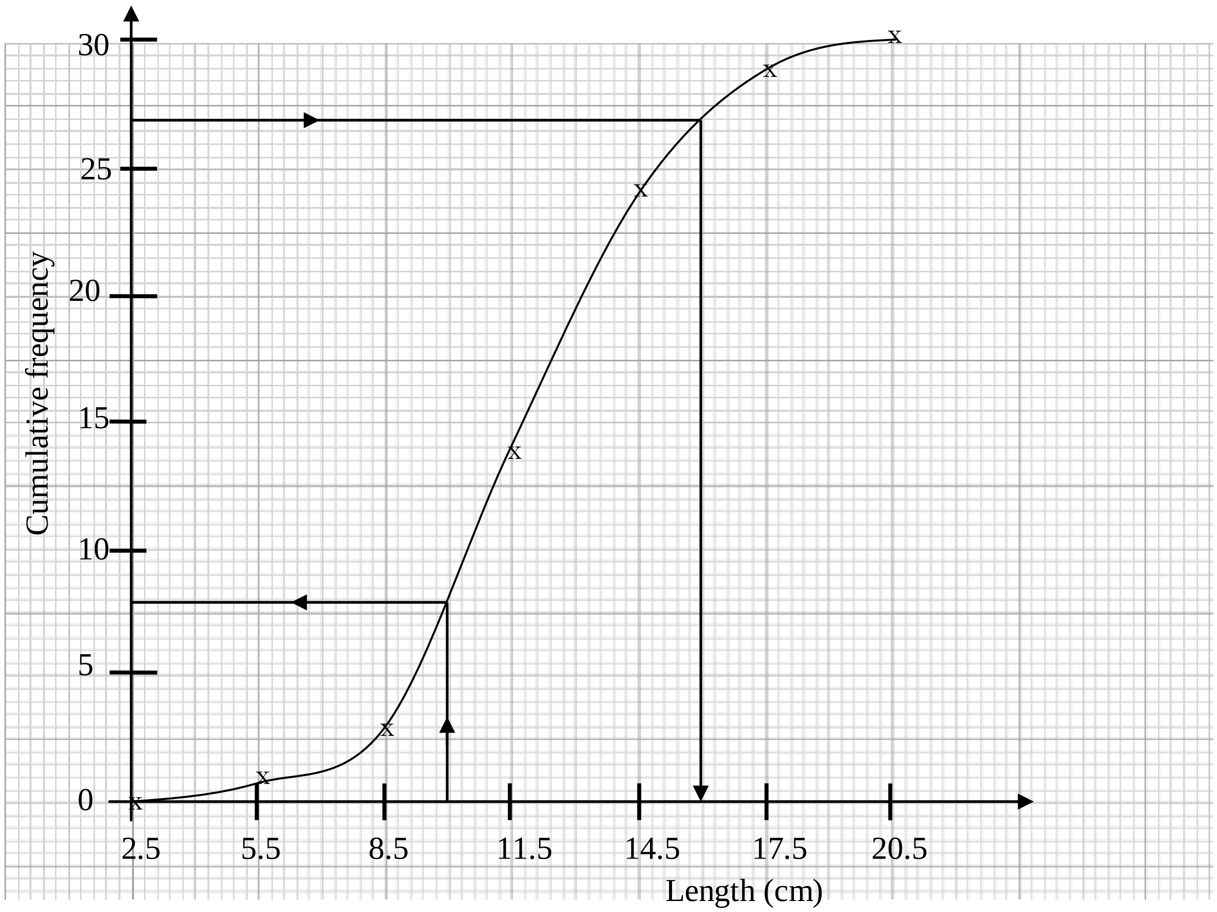
(ii) 10% of the pencils were of length x cm or more. Find x.

Solution

(a)

Length	Frequency	Class boundaries	Cumulative frequency
3 – 5	1	2.5 – 5.5	1
6 – 8	2	5.5 – 8.5	3
9 – 11	11	8.5 – 11.5	14
12 – 14	10	11.5 – 14.5	24
15 – 17	5	14.5 – 17.5	29
18 – 20	1	17.5 – 20.5	30

Cumulative frequency curve to show the length of the 30 Pencils



(b)

(i) To find how many pencils were less than 10cm long tall, find the length 10cm on the horizontal axis. Draw a vertical line to meet the curve and then draw a horizontal line to meet the cumulative frequency axis. Then read off the value.

From the graph it can be estimated that 8 pencils were less than 10 cm long.

(ii) 10% of the pencils were of length *x* cm or more = $\frac{10}{100} \times 30 = 3$ pencils. Thus (30 – 3) = 27 pencils were less than *x* cm long.

Hence find 27 on the cumulative frequency axis and draw a horizontal line to meet the Ogive. Then draw a vertical line to meet the length axis and read off the value.

From the graph, 27 pencils were less than 16 cm long.

Therefore 10% of the pencils were of length 16cm or more, so the value of *x* is 16.

Measures of central tendency

The measures of central tendency are the **mean**, **median** and **mode**. These are Values about which the distribution of a set of data is considered to be roughly balanced.

The Mean, $\bar{x}$

The mean is the ordinary **arithmetical average.** It is the value obtained when then the total sum of the values of the members in a list, set or distribution is divided by the total number or frequency. The mean can be obtained for grouped or ungrouped data.

$$Mean, \bar{x} = \frac{sum\ of\ \ values\ in\ set}{number\ of\ \ values\ in\ set}$$

Given a set of n values $x_1, x_2, x_3, ..., x_n,\ then \quad \bar{x} = \frac{x_1 + x_2 + x_3 + ... + x_n}{n} = \frac{\sum x}{n}$

NOTE: the symbol $\sum$ is a Greek letter which means 'the sum of' and it is read 'sigma'.

Example 1

Find the mean of the set of numbers 3, 4, -1, 22, 14, 0, 9, 18, 7, 0, 1

Solution

The first step is to count how many numbers there are in the set, which we shall call **n**

n = 10, $\sum$x = 3 + 4 + -1 +22 +14 + 0 +9 + 18 + 7 +0 +1 = 77

Therefore $\bar{x} = \frac{\sum x}{n} = \frac{77}{10} = 7.7$

Example 2

On a certain day, nine students received, respectively, 1, 3, 2, 0, 1, 5, 2, 1and 3 pieces of mail. Find the mean.

Solution

Mean, $\bar{x} = \frac{\sum x}{n} = \frac{1+3+2+0+1+5+2+1+3}{9} = \frac{18}{9} = 2$

Mean can also be found for a frequency distribution. If data is in the form of a frequency distribution, the mean is calculated using the formula:

Mean, $\bar{x} = \dfrac{\sum fx}{\sum f}$ where $\sum fx$ is 'the sum of the products' i.e. $\sum$ (number × frequency) and

$\sum$ f means 'the sum of the frequencies'.

Example 3

The following frequency distribution shows the temperatures that were recorded over a 20-day period in January 2012. Calculate the mean daily temperature over this period.

Temperature (^{0}c)	15	18	19	20	21	22	23	25
Frequency	2	4	1	6	3	2	1	1

Solution

x	f	fx
15	2	30
18	4	72
19	1	19
20	6	120
21	3	63
22	2	44
23	1	23
25	1	25
	$\sum f = 20$	$\sum fx = 396$

Mean daily temperature, $\bar{x} = \dfrac{\sum fx}{\sum f} = \dfrac{396}{20} = 19.8^0 c$

Example 4

The marks obtained by 100 students in a test were as follows: Find the mean mark.

Mark (x)	0	1	2	3	4
Frequency (f)	4	19	25	29	23

Solution

Mark (x)	Freq. (f)	fx
0	4	0
1	19	19
2	25	50
3	29	87
4	23	92
Total	100	248

Mean mark, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{248}{100} = 2.48$

Mean for grouped data.

When the data have been grouped into intervals we do not know the actual values, so we can estimate the mean. We take the mid – point of an interval to represent that interval. Thus for **grouped data**, each class can be represented approximately by its mid-point (class mark).

Example 5

The results of 24 students in a Subsidiary mathematics test are given in the table.

Mark	Freq. (f)	Mid-point(x)	fx
85 – 99	4	92	368
70 – 84	7	77	539
55 – 69	8	62	496
40 - 54	5	47	235
	$\sum f$ =24		$\sum fx$ = 1,638

The Mean mark, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{1,638}{24} = 68.25$

Example 6

The length of 40 bean pods were measured to the nearest cm and grouped as shown. Estimate the mean length, giving the answer to one decimal place.

Length (cm)	$3\leq x<9$	$9\leq x<13$	$13\leq x<19$	$19\leq x<23$	$23\leq x<29$	$29\leq x<33$
Frequency	2	4	7	14	8	5

Solution

Length (cm)	Mid-point(x)	f	fx
$3\leq x<9$	6	2	12
$9\leq x<13$	11	4	44
$13\leq x<19$	16	7	112
$19\leq x<23$	21	14	294
$23\leq x<29$	26	8	208
$29\leq x<33$	31	5	155
		$\sum f=40$	$\sum fx=825$

The Mean length, $\bar{x}=\frac{\sum fx}{\sum f}=\frac{825}{40}=20.6(1d.p)$

Method of assumed mean

When the number of members in a list is large, the above method of finding the mean is quite difficult and tiring. Assumed mean, like the name suggests, is a guess or an assumption of the mean. Assumed mean is most commonly denoted by the letter **A**. It doesn't need to be correct or even close to the actual mean and choice of the assumed mean is at your discretion except for where the question explicitly asks you to use a certain assumed mean value.

From the data we can guess the expected value of the mean. This expected value is called **the assumed or working mean**. The assumed mean is approximately in the middle of the data. In a frequency distribution table, the modal value provides the best assumed value. How ever the assumed mean does not need to be one of the values given.

Example 7

The table below shows the profit made by a trader in 100 days.

Profit in '000 Ush.	115	125	135	145	155	165
No. of days	8	18	30	26	12	6

Calculate the mean profit using the method of assumed mean.

Solution

To calculate the mean of the above data using an assumed mean, the following steps should be followed.

1. Choose an appropriate assumed mean (A) from the range of the values given if the question has not specified the value.
2. Find by how much each of the values (x) differs from this assumed mean (A). These differences obtained are called deviations (d). Thus, d = x – A.
3. Multiply each frequency (f) by its corresponding value of deviation (d) to obtain the product (fd).
4. Find the sum of the products, $\sum fd$, and then divide this sum by the sum of the frequencies to obtain the mean of the deviations, $\frac{\sum fd}{\sum f}$
5. Find the mean by using the formula: $\bar{x} = A + \frac{\sum fd}{\sum f}$

The table below is a summary of the calculations where the assumed mean, A is 135 (the modal value).

Profit (x)	Deviations d= x - A	Freq. f	Products (fd)
115	-20	8	-160
125	-10	18	-180
135	0	30	0
145	10	26	260
155	20	12	240
165	30	6	180
		Σf = 100	Σ(fd) = 340

The actual mean = assumed mean + mean deviations

$$\bar{x} = A + \frac{\sum fd}{\sum f}$$

$$= 135 + \frac{340}{100} = 135 + 3.4 = 138.40$$

The mean profit is Sh. 138.40 × 1000 = Sh. 138,400

Example 8

The student body of a certain school was polled to find out what their hobbies were. The number of hobbies each student had was then recorded and the data obtained was grouped into classes shown in the table below.

Number of hobbies	Frequency
0 - 4	45
5 - 9	58
10 - 14	27
15 - 19	30
20 - 24	19
25 - 29	11
30 - 34	8
35 - 40	2

Using an assumed mean of 17, find the mean for the number of hobbies of the students in the school.

Solution

We have been given the assumed mean **A** as **17** and we know the formula for finding mean from the assumed mean as $\bar{x} = A + \frac{\sum fd}{\sum f}$

So we can solve the rest of this problem using a table where by we find each remaining component of the formula and then substitute at the end:

Number of hobbies	x	Frequency f	d (x – A)	fd
0 - 4	2	45	-15	-675
5 - 9	7	58	-10	-580
10 - 14	12	27	-5	-135
15 - 19	17	30	0	0
20 - 24	22	19	5	95
25 - 29	27	11	10	110
30 - 34	32	8	15	120
35 - 40	37	2	20	40
		$\sum f = 200$		$\sum fd = -1025$

Mean number of hobbies, $\bar{x} = A + \frac{\sum fd}{\sum f}$

$$= 17 + \frac{-1025}{200} = 17 - 5.125 = 11.875$$

Median

The median is defined as the number in the middle of a given set of numbers arranged in order of increasing magnitude. When given a set of numbers, the median is the number positioned in the exact middle of the list when you arrange the numbers from the lowest to the highest. The median is also a measure of average. The median is important because it describes the behavior of the entire set of numbers.

Example 1

Find the median in the set of numbers given below

15, 16, 15, 7, 21, 18, 19, 20, 11

Solution

From the definition of median, we should be able to tell that the first step is to rearrange the given set of numbers in order of increasing magnitude, i.e. from the lowest to the highest

7, 11, 15, 15, 16, 18, 19, 20, 21

Then we inspect the set to find that number which lies in the exact middle.

$$\cancel{7}, \cancel{11}, \cancel{15}, \cancel{15}, 16, \cancel{18}, \cancel{19}, \cancel{20}, \cancel{21}$$

Median = 16

Example 2

Find the median of the given data: 13, 0, 5, 8, -8, -5, 10, 7, 1, 0, 0, 4, 6, 16

Solution

We start off by rearranging the data in order from the smallest to the largest.

$$-8, -5, 0, 0, 0, 1, 4, 5, 6, 7, 8, 10, 13, 16$$

We inspect the data to find the number that lies in the exact middle.

$$\cancel{-8}, \cancel{-5}, \cancel{0}, \cancel{0}, \cancel{0}, \cancel{1}, 4, 5, \cancel{6}, \cancel{7}, \cancel{8}, \cancel{10}, \cancel{13}, \cancel{16}$$

We can see from the above that we end up with two numbers (**4** and **5**) in the middle. We can solve for the median by finding the mean of these two numbers as follows:

$$Median = \frac{4+5}{2} = 4.5$$

Therefore the middle position in an array of N data items is the position numbered $\frac{N+1}{2}$.

If N is odd, there is a data item at the middle and we take this item as the median. If N is even, we take the average of the two middle data items as median.

For **grouped data**, the median is given by the following formula:

$$Median = L_1 + \left(\frac{\frac{N}{2} - c.f_b}{f_m} \right) \times c$$

Where

L_1 = lower class boundary of the median class;

$c.f_b$ = cumulative frequency of the class before the median class;

f_m = frequency of the median class;

c = class width of the median class;

N = total frequency.

Example 3

The table below shows the weights of 40 poles in kg. Find the median weight.

Weight (kg)	frequency	Cumulative frequency	Class boundaries
118 – 126	3	3	117.5 – 126.5
127 – 135	5	8	126.5 – 135.5
136 – 144	9	17	135.5 – 144.5
145 – 153	12	29	144.5 – 153.5
154 – 162	5	34	153.5 – 162.5
163 – 171	4	38	163.5 – 171.5
172 - 180	2	40	171.5 – 180.5

We find the median class first. Since $\frac{N}{2} = \frac{40}{2} = 20$, we look for where 20 is cumulated in the cumulative frequency column. We note that this lies in the cumulative of 29. And so '145 – 153' is the median class.

Thus $L_1 = 144.5$; $c.f_b = 17$; $f_m = 12$; $c = 9$

Hence $Median = L_1 + \left(\frac{\frac{N}{2} - c.f_b}{f_m} \right) \times c \;=\; 144.5 + \left(\frac{20-17}{12} \right) \times 9$

$= \ 144.5 + 2.25 = 146.75kg$

Example 4

The age of people in Mukono town after the 2002 census were as follows

Age (years)	Number in thousands
0 - <5	4.4
5 - <15	8.1
15 - <30	10.5
30 - <50	14.6
50 - <70	9.8
70 - <90	4.7

Determine the median and mean.

Solution

Age (years)	x	f	fx	Cumulative frequency
0 - <5	2.5	4.4	11	4.4
5 - <15	10	8.1	81	12.5
15 - <30	22.5	10.5	236.25	23
30 - <50	40	14.6	584	37.6
50 - <70	60	9.8	588	47.4
70 - <90	80	4.7	376	52.1
		$\sum f = 52.1$	$\sum fx = 1876.25$	

NOTE:

(i) The ages are given as class boundaries already.

(ii) The class width of the median class does not necessary need to be equal to the width of other classes.

The $\frac{N}{2}^{th}$ $position = \frac{52.1}{2} = 26.02$. This lies in the cumulative frequency of 37.6. Hence the median class is 30 - < 50 and thus $L_1 = 30$; $c.f_b = 23$; $f_m = 14.6$; $c = 20$

Therefore $Median = L_1 + \left(\frac{\frac{N}{2} - c.f_b}{f_m} \right) \times c = 30 + \left(\frac{26.05 - 23}{14.6} \right) \times 20$

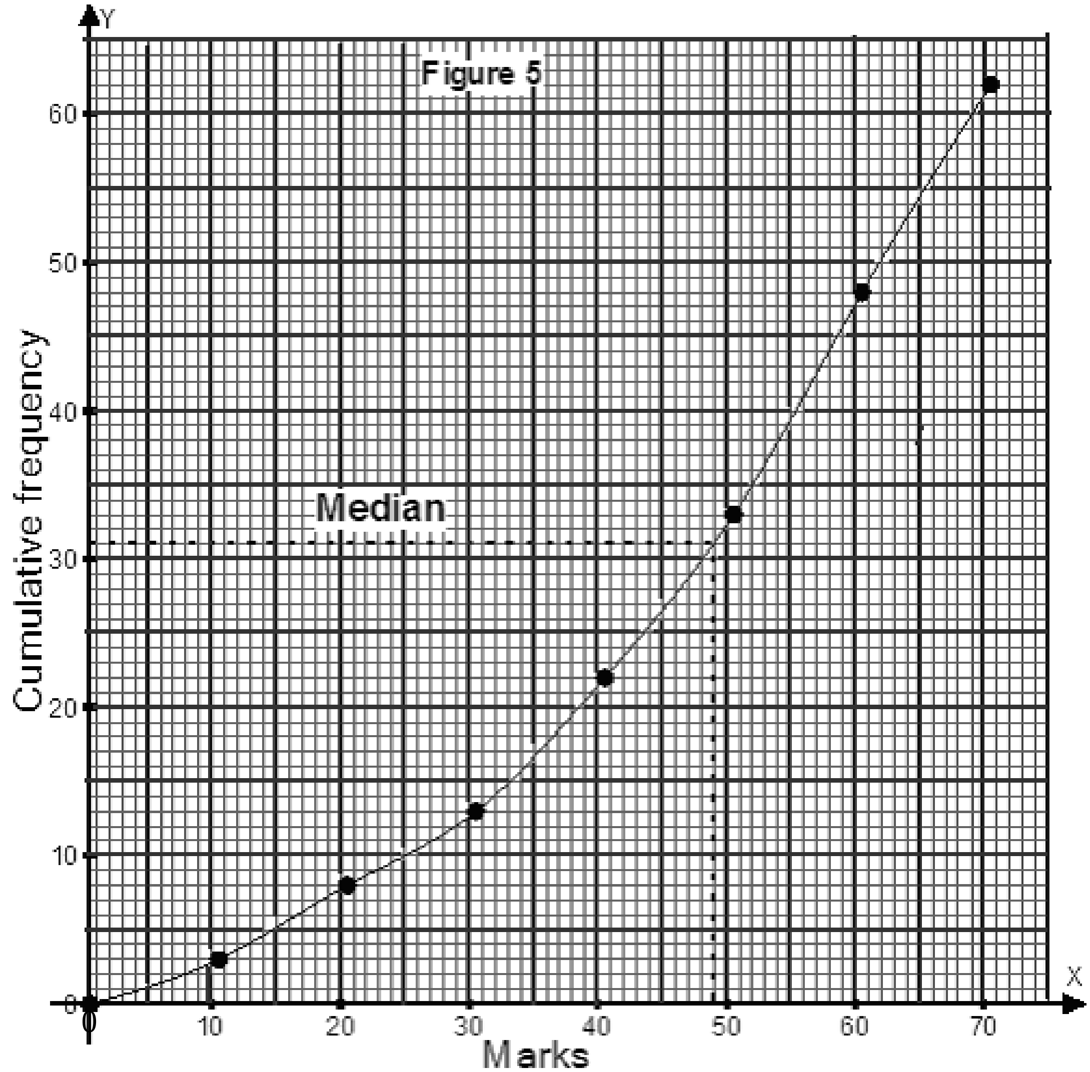
$= 30 + 4.178 = 34.178$ years

$$Mean = \frac{\sum fx}{\sum f} = \frac{1876.25}{52.1} = 36.012 years$$

Median from an Ogive:

A cumulative frequency curve/graph shows the median at the $\frac{N}{2}^{th}$ position of the cumulative frequency.

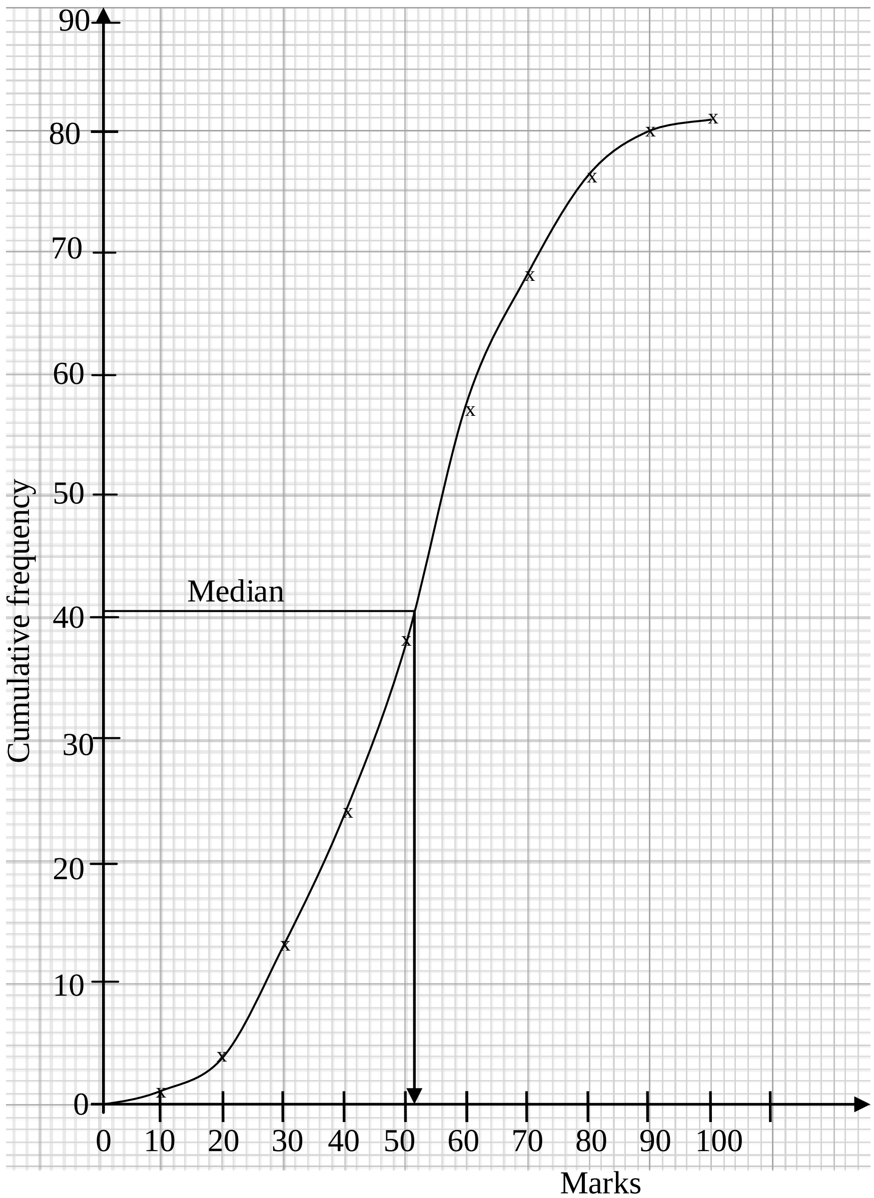
For instance, in figure 5, N = 62 and the therefore the $\frac{N}{2}^{th}$ value, that is the 31st data item gives the median which is obtained by drawing a horizontal line, from the cumulative frequency axis to the curve, and then drawing the line down wards to the horizontal axis. From the Ogive, the median mark is 49.



Example 5

The cumulative frequency graph below shows the distribution of marks of 81 students in a S.5 End of Year Subsidiary Mathematics examination.

Use it to estimate the median.



For the median, we find the $\frac{1}{2}(81)^{th}$ value, i.e. the 40.5th value. This is shown on the graph.

From the curve, an estimate of the median is 52 marks.

The mode

Another measure which is sometimes used to describe the "middle" of a set of data is the mode. The mode is defined as the element that appears most frequently in a given set of elements.

Using the definition of frequency given above, mode can also be defined as the element with the highest frequency in a given data set.

For a given data set, there can be more than one mode. As long as those elements all have the same frequency and that frequency is the highest, they are all the modal elements of the data set.

Example 1

Find the mode of the following numbers: 5, 4, 10, 3, 3, 4, 7, 4, 6, 5.

Solution

The mode is 4. (There are more 4's than any other number).

Example 2

Find the Mode of the following data set.

3, 12, 15, 3, 15, 8, 20, 19, 3, 15, 12, 19, 9

Solution

Mode = 3 and 15

Mode for Grouped Data

For **grouped data**, the mode is estimated using the following formula:

$$Mode = L_1 + \left(\frac{d_1}{d_1 + d_2} \right) \times c$$

Where, L_1 = lower class boundary of the modal class (i.e. the class containing the mode);

d_1 = the difference between the frequency of the modal class and the frequency of the class before it;

d_2 = the difference between the frequency of the modal class and the frequency of the class after it;

c = class width of the modal class.

Example 2

Find the mode of the following data.

Class	Frequency
20 – 22	3
23 – 25	6
26 – 28	12
29 – 31	9
32 - 34	2

Solution

The modal class is the class with the highest frequency. Thus from the table, the modal class for this data is '26 – 28'.

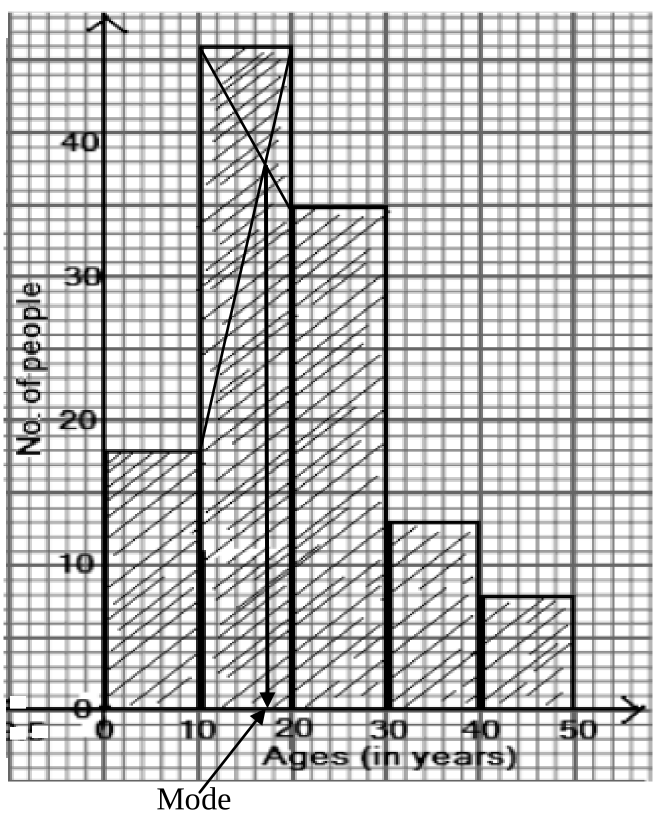
So, $L_1 = 25.5$; $d_1 = (12 - 6) = 6$; $d_2 = (12 - 9) = 3$; $c = 3$

$$Mode = L_1 + \left(\frac{d_1}{d_1 + d_2}\right) \times c \;=\; 25.5 + \left(\frac{6}{6+3}\right) \times 3$$

$$= 25.5 + 2 = 27.5$$

Estimating the mode using the histogram

When a histogram is drawn, the mode is obtained by joining the ends of the highest rectangle to the opposite ends of the rectangles next to it as shown in the figure below.



The point of intersection of the two diagonals is read off and gives an estimate of the mode.

From the graph, the estimate of mode is 18 years.

Measures of variation (dispersion)

Dispersion measures how the various elements behave with regards to some sort of central tendency, usually the mean. Measures of dispersion include range, interquartile range, variance and standard deviation.

Averages, being measures of **central location** only, do **not** indicate how the values in the data are **spread**. Thus the range, variance and standard deviation gives information on how the values in the data are spread.

Range

The **range** in a set of data is the **difference** between the **highest value** and the **lowest value** in the set and this is the simplest measure of **dispersion** (or spread).

Thus Range = highest value – lowest value

Example 1

Find the range of the data set below

3, 12, 15, 3, 15, 8, 20, 19, 3, 15, 12, 19, 9

Solution

Range = 20 – 3 = 17

Quartiles and Percentiles

Quartiles

The term quartile is derived from the word quarter which means one fourth of something. Thus quartiles are values that divide the data into four equal parts. When you arrange a data set in increasing order from the lowest to the highest, then you divide this data into groups of four, you end up with quartiles.

There are three quartiles that are studied in statistics. They are denoted by Q_1, Q_2 and Q_3. Q1 divides the data into $\frac{1}{4}$ and is called the lower quartile. Q_2 divides the data into $\frac{1}{2}$ and is called the median while Q_3 divides the data into $\frac{3}{4}$ and is called the upper quartile. In other words, for Q_1, 25% of the values are below it, Q_2 takes 50% and Q_3 takes 75%.

For **ungrouped data**, the position of Q_1 is given by $\frac{1}{4}$ (n+1)th value, while that of Q_3 by $\frac{3}{4}$ (n+1)th value. In each case n is the number of observations.

Example 2

Find the lower and upper quartiles of the following set of numbers:

(a) 3, 12, 4, 6, 8, 5, 4.

(b) 10, 12, 13, 15, 19, 19, 24, 26, 26

Solution

(a) Arranging the numbers in order of size, we have; 3, 4, 4, 5, 6, 8, 12.

Position of Q_1: $\frac{1}{4}$ (7+1)th = 2^{nd} value.

Therefore, the lower quartile is 4.

Position of Q_3: $\frac{3}{4}$ (7+1)th = 6^{th} value.

Therefore, the upper quartile is 8.

(b) The numbers are arranged in order

$$\begin{array}{cccccccccccc} 10 & 12 & \uparrow & 13 & 15 & 19 & 19 & 24 & \uparrow & 26 & 26 \\ & & Q_1 & & & & & & Q_3 & & \end{array}$$

Position of Q_1: $\frac{1}{4}$ (9+1)th = 2.5^{th} value.

Therefore, the lower quartile $Q_1 = \frac{1}{2}(12+13) = 12.5$

Position of Q_3: $\frac{3}{4}$ (9+1)th = 7.5^{th} value.

Therefore, the upper quartile $Q_3 = \frac{1}{2}(24+26) = 25$

For **grouped data**, the following formulae are used:

$$Q_1 = L_1 + \left(\frac{\frac{N}{4} - c.f_b}{f_m} \right) \times c \quad \text{and} \quad Q_3 = L_1 + \left(\frac{\frac{3N}{4} - c.f_b}{f_m} \right) \times c$$

Where

L_1 = lower class boundary of the quartile class;

$c.f_b$ = cumulative frequency of the class before the quartile class;

f_m = frequency of that quartile class;

c = class width of the quartile class;

N = total frequency.

Example 3

The table below shows the distribution of marks gained by a group of students in a Subsidiary mathematics test marked out of 50.

Estimate the lower and upper quartiles.

Marks	Frequency	Cumulative Frequency
1 – 10	15	15
11 – 20	20	35
21 – 30	32	67
31 – 40	26	93
41 - 50	7	100

Position of Q_1 is $\frac{1}{4}$ (100) = 25th observation. Using the cumulative frequency column, the 25th observation lies within the cumulative of 35. Thus the lower quartile class is '11 – 20'

So, $L_1 = 10.5$, $c.f_b = 15$, $f_m = 20$, $c = 20.5 – 10.5 = 10$, N = 100

Therefore $Q_1 = L_1 + \left(\frac{\frac{N}{4} - c.f_b}{f_m}\right) \times c \;=\; 10.5 + \left(\frac{\frac{100}{4} - 15}{20}\right) \times 10$

$$= 10.5 + 5 = 15.5 \text{ marks}$$

Position of Q3: $\frac{3}{4}$ (100) = 75th observation (mark). Using the cumulative frequency column, the 75th observation lies within the cumulative of 93 and is located in the '31 – 40' class.

So, $L_1 = 30.5$, $c.f_b = 67$, $f_m = 26$, $c = 40.5 – 30.5 = 10$, N = 100

Therefore $Q_3 = L_1 + \left(\dfrac{\frac{3N}{4} - c.f_b}{f_m} \right) \times c = 30.5 + \left(\dfrac{75-67}{26} \right) \times 10$

$= 30.5 + 3.0769 = 33.58$(2d.p)

Interquartile range

The Interquartile range is the difference between the upper quartile and the lower quartile. It is a useful measure of spread of a distribution. It is the range of the middle 50% of the values.

The Interquartile range = upper quartile – lower quartile

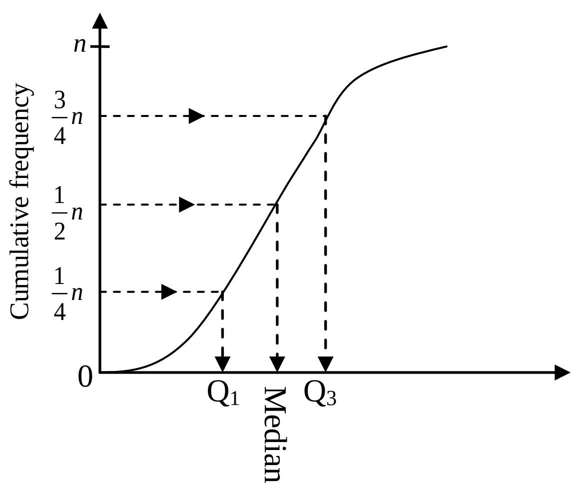
$= Q_3 - Q_1$

Semi – Interquartile range

We define semi-interquartile range as: $\frac{1}{2}(Q_3 - Q_1)$

Estimating the quartiles using the cumulative frequency graph

The values of the quartiles can be read off the cumulative frequency graph as shown:



Thus on the cumulative frequency curve (Ogive), the lower quartile, Q_1corresponds to the $\frac{1}{4}n^{\text{th}}$ reading, Median to the $\frac{1}{2}n^{\text{th}}$ reading and the upper quartile, Q_3 to the $\frac{3}{4}n^{\text{th}}$ reading.

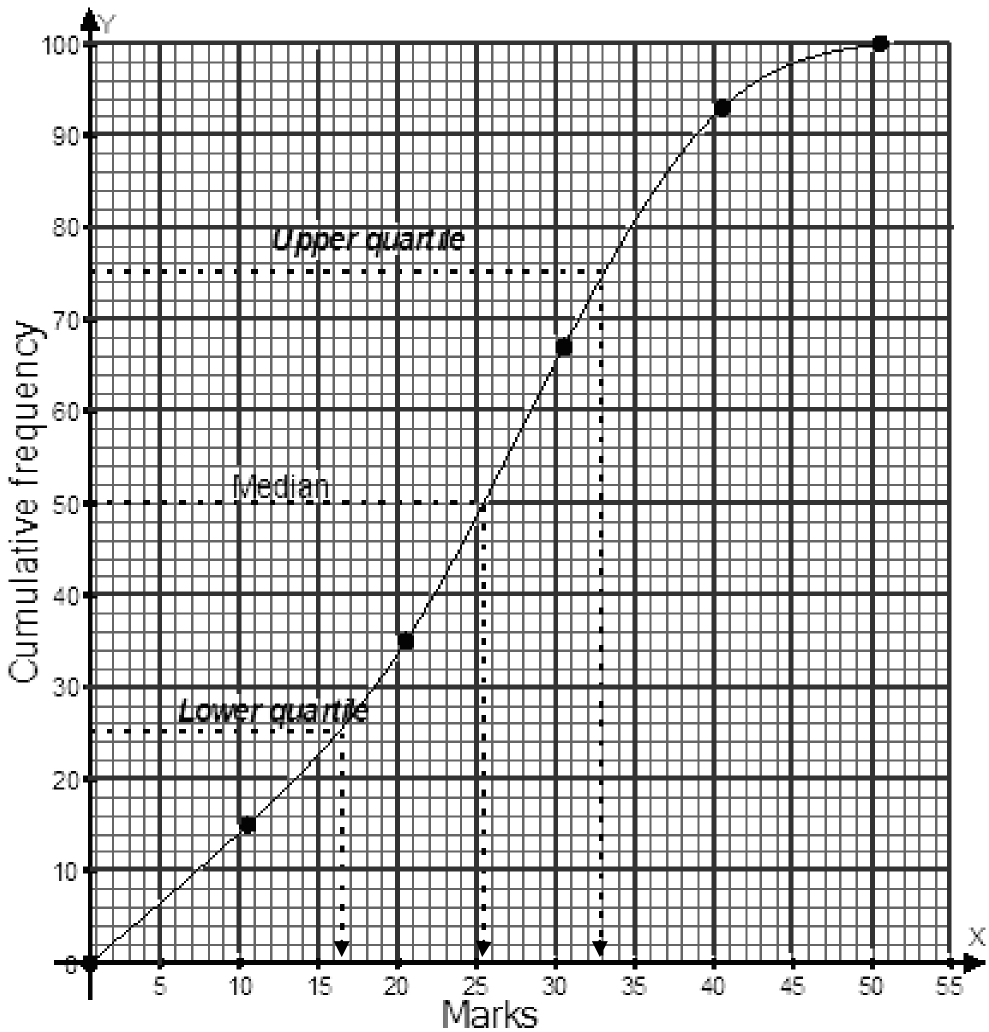
Example 4

Plot an Ogive for the above data and use it to estimate the semi-interquartile range.

Marks	Frequency	Class boundaries	Cumulative frequency
1 – 10	15	10.5	15
11 – 20	20	20.5	35
21 – 30	32	30.5	67
31 – 40	26	40.5	93
41 - 50	7	50.5	100

Solution

Cumulative frequency curve



From the Ogive;

Lower quartile, Q_1= 16.5 marks and upper quartile, Q_3 = 33 marks

The interquartile range = $Q_3 - Q_1$

= 33 – 16.5 = 16.5 marks

Percentiles

A percentile is a certain percentage of a set of data. Percentiles are used to observe how many of a given set of data fall within a certain percentage range; for example; a thirtieth percentile indicates data that lies in the 30% mark of the entire data set.

Calculating Percentiles

Let designate a percentile as p_m where **m** represents the percentile we are finding, for example for the tenth percentile, **m** would be 10. Given that the total number of elements in the data set is **N,** then $p_m = \frac{m}{100} \times N$

On the cumulative frequency curve (Ogive), the m^{th} percentile corresponds to the $\left(\frac{m}{100} \times N\right)^{th}$ reading or value. For example given a total of 100 observations, the 10th percentile corresponds to $\left(\frac{90}{100} \times 100\right) = 90^{th}$ value. Thus from the Ogive, the value at the 25th percentile is the lower quartile, and that at the 75th percentile is the upper quartile.

Example 5

Survey results of the heights (cm) of 400 students in a certain school are shown in this frequency table.

Height (cm)	Cumulative frequency
$100 \le x < 110$	27
$110 \le x < 120$	85
$120 \le x < 130$	215
$130 \le x < 140$	320
$140 \le x < 150$	370
$150 \le x < 160$	395
$160 \le x < 170$	400

(a) Draw a cumulative frequency curve.

(b) Find an estimate of the median.

(c) Find the upper and lower quartiles and determine the interquartile range.

(d) Determine the

(i) 10th percentile

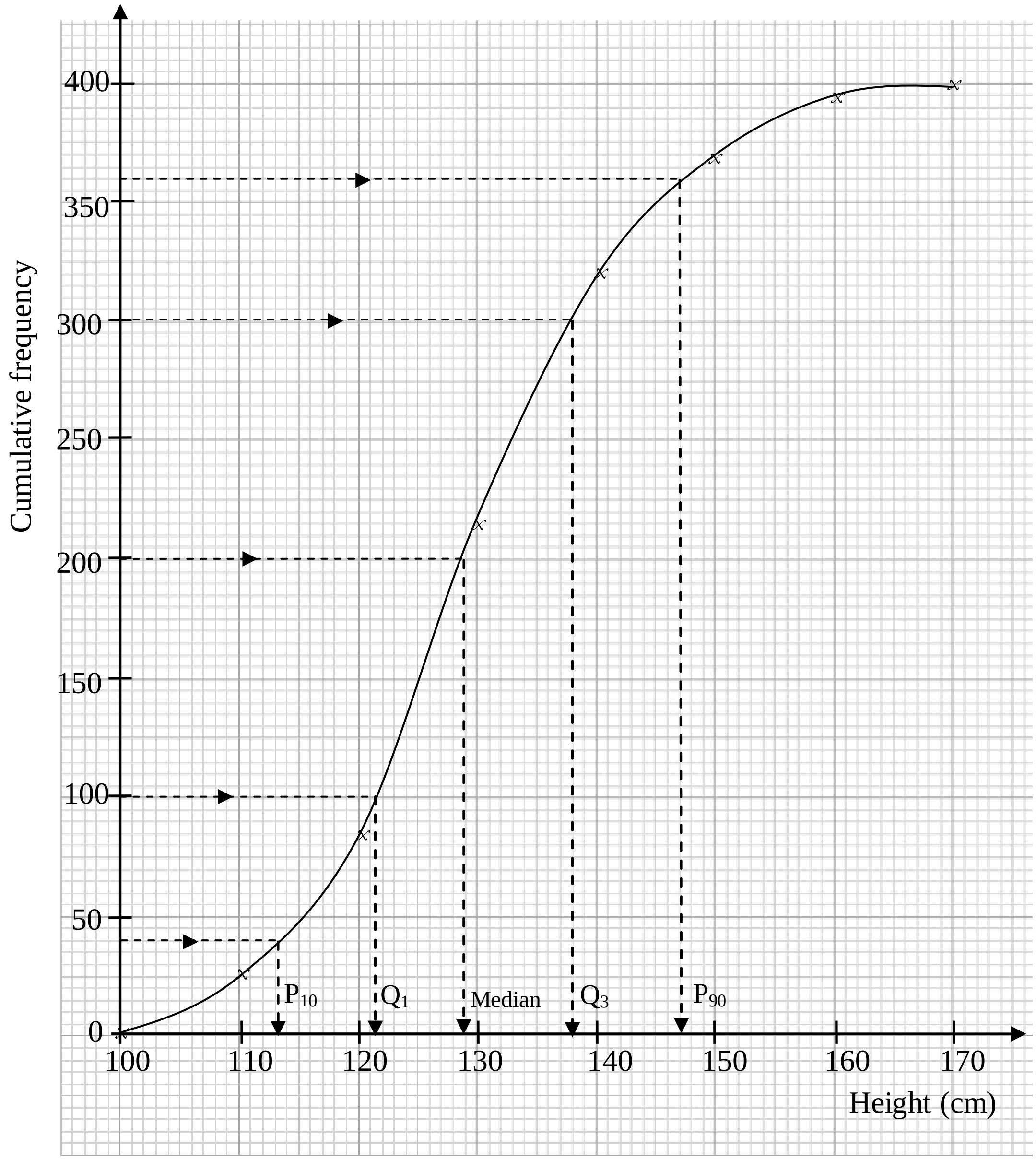
(ii) 90th percentile

(iii) The 10 to 90 percentile range.

Solution

(a)

Cumulative frequency curve showing the heights of 400 students



From the Ogive;

(b) An estimate of the median is 129cm.

(c) Upper quartile, Q_3 = 138 and lower quartile, Q_1 = 121.5cm.

Hence interquartile range = Q3 – Q1

= 138 – 121.5 = 16.5cm

(d) 90th percentile, P_{90} corresponds to $\frac{90}{100}(400) = 360^{th}\,value$

10^{th} percentile, P_{10} corresponds to $\frac{10}{100}(400) = 40^{th}\, value$

Thus;

(i) $P_{90} = 147$cm

(ii) $P_{10} = 113$cm

(iii) The 10 to 90 percentile range $= P_{90} - P_{10}$

$= 147 - 113 = 34$cm

Exercise

1. Find the mean of each of the following sets of numbers.

   (i) 1769, 1771, 1772, 1775, 1778, 1781, 1784

   (ii) 0.85, 0.88, 0.89, 0.93, 0.94, 0.96

2. The mean of 10 numbers is 8. If an eleventh number, c is now included in the results, the mean becomes 9. What is the value of c.

3. The mean of n numbers is 5. If the number 13 is now included with the n numbers, the new mean is 6. Find the value of n.

4. If the mean of the following frequency distribution is 3.66, find the value of a.

x	1	2	3	4	5	6
f	3	9	a	11	8	7

5. The marks below were obtained by students of S5 in a certain school in the end of year mathematics examination.

Marks	20 – 29	30 – 39	40 – 49	50 – 59	60 – 69	70 – 79	80 – 89
No of students	3	8	18	24	20	12	5

(i) Find the average mark for the class

(ii) Draw an ogive to represent the marks and use it to estimate the median and semi-interquartile range

(iii) Using the ogive estimate the probability that a student selected at random scored a mark between 32.5 and 49.5

6. In an experiment, 50 people were asked to guess the weight of a mobile phone in grams. The guesses were as follows:

47	39	21	30	42	35	44	36	19	52
23	32	66	29	5	40	33	11	44	22
27	58	38	37	48	63	23	40	53	24
47	22	44	33	13	59	33	49	57	30
17	45	38	33	25	40	51	56	28	64

Construct a frequency table using intervals 0 – 9, 10 – 19, 20 – 29, etc.

Hence draw a cumulative frequency curve and estimate:

(a) the median weight,

(b) the inter-quartile range,

7. In a competition, 30 children had to pick up as many money notes of 50,000 as possible in one minute using their hands. The results were as follows:

3	17	8	11	26	23	18	28	33	38
12	38	22	50	5	35	39	30	31	43
27	34	9	25	39	14	27	16	33	49

Construct a frequency table using intervals of width 10, starting with 1 – 10.
From the frequency table, estimate the
(i) mean,
(ii) median of the distribution.

8. The mean weight of 8 boys is 55 kg and the mean weight of a group of girls is 52 kg. The mean weight of all the children is 53.2 kg. How many girls are there?

9. A group of 50 people were asked how many books they had read in the previous year; the results are shown in the frequency table below. Calculate the mean number of books read per person.

No. of books	0	1	2	3	4	5	6	7	8
Frequency	5	5	6	9	11	7	4	2	1

10. The heights in cm of S5 students in a certain school were recorded as shown in the table below:

Height (cm)	Frequency (f)
149 – 152	5
153 – 156	17
157 – 160	20
161 – 164	25
165 – 168	15
169 – 172	6
173 - 176	2

(a) Estimate the mean height of the student.
(b) Plot a cumulative frequency curve (ogive) and use it to estimate the median height.

11. The grouped frequency distribution shown below gives the speed of service of the EOP 50 performers in men's professional tennis in a given year.

Service speed (m.p.l)	90 – 94	95 –99	100 – 104	105 –109	110 –114	115 –119	120 – 124	125– 129
Frequency	2	7	9	14	9	4	3	2

(a) Calculate the mean and modal speed of the performers.

(b) Draw a cumulative curve for the data. Use your curve to estimate the

(i) Median and semi – interquartile range of the speeds.

(ii) 39^{th} to 89^{th} percentile speed range.

12. The table below shows the marks obtained, out of 50, by S.5 students in a Subsidiary mathematics test.

Mark (x)	No. of students (f)
1 – 10	9
11 – 20	10
21 – 30	11
31 – 40	8
41 - 50	7

Using an assumed mean, calculate the mean mark to the nearest whole number.

13. The heights of 50 army recruits were measured and tabulated as shown below.

Height (cm)	No. of recruits
151 – 155	8
156 – 160	16
161 – 165	14
166 – 170	10
171 - 175	2

Using 160 cm as an assumed mean, calculate the mean height.

14. The times taken by a group of students to solve a mathematical problem are given below:

Time(min)	5 - 9	10 - 14	15 - 19	20 - 24	25 - 29	30 – 34
No. of students	5	14	30	17	11	3

(a) Draw a histogram for the data. Use it to estimate the modal time for solving a problem.

(b) Calculate the mean time of solving a problem.

15. The speeds of public service vehicles during a police check are shown in the table below.

Speed (km/h)	31 – 40	41 – 50	51 – 60	61 – 70	71 – 80	81 – 90	91 - 100
No. of vehicles	5	10	15	30	55	70	15

(a) Draw a cumulative frequency curve for this data.

(b) Use your graph to estimate:

(i) the median speed.

(ii) the lower and upper quartiles.

(iii) the percentage of vehicles traveling between 64km/h and 72 km/h.

16. The table below shows the ages of students in a training college.

Age	Frequency
$18 \leq x < 20$	7
$20 \leq x < 22$	10
$22 \leq x < 24$	33
$24 \leq x < 26$	21
$26 \leq x < 28$	14
$28 \leq x < 30$	13
$30 \leq x < 32$	10
$32 \leq x < 34$	4

Calculate the median age.

17. The table below shows the time (in seconds) taken to solve a certain problem

Time(s)	30 – 49	50 – 69	70 – 89	90 – 109	110 – 119
No. of students	10	30	25	20	15

Draw a histogram for the data and use it to estimate the mode.

Variance and standard deviation

Variance

Variance is a measure of how spread out a data set is. Variance is used to indicate how spread out the elements in the set is from the mean of the population in the set. Unlike range and quartiles, the variance combines all the values in a data set to produce a measure of spread. It is symbolized by $\mathbf{S^2}$.

The variance of a set of n numbers, $x_1, x_2, ..., x_n$, with mean $\overline{x}$ is given by S^2 where

$$S^2 = \frac{\sum x^2}{n} - \overline{x}^2$$

For data in form of a frequency distribution, if $x_1, x_2, ..., x_n$ occur with frequencies $f_1, f_2, ..., f_n$ then the variance S of the distribution with mean $\overline{x}$ is given by:

$$S^2 = \frac{\sum fx^2}{\sum f} - \overline{x}^2$$

NOTE: For grouped data, x represents the mid – point of an interval.

Standard deviation

Standard deviation is the measure of spread most commonly used in statistical practice when the mean is used to calculate central tendency. It measures spread around the mean. It is symbolized by **S**. Standard deviation is the square root of variance.

So standard deviation, S = $\sqrt{Variance}$

The standard deviation of n numbers is given by S where

$$S = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2}$$

For a frequency distribution, the standard deviation is given by:

$$S = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}$$

Standard deviation is a measure of how precise the mean of a population or sample is. It is used to indicate trends in the elements in a given data set with respect to the mean, i.e. the spread of these elements from the mean. Because of its close links with the mean, standard deviation can be greatly affected if the mean gives a poor measure of central tendency.

Example 1

A hen lays eight eggs. Each egg was weighed in grams and recorded as follows:

60, 56, 61, 68, 51, 53, 69, 54.

Calculate the standard deviation

Solution

First calculate the mean;

Mean, $\bar{x} = \frac{\sum x}{n} = \frac{472}{8} = 59$

x	60	56	61	68	51	53	69	54	
x^2	3600	3136	3721	4624	2601	2809	4761	2916	$\sum x^2$ =28168

$$S = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2} = \sqrt{\frac{28168}{8} - 59^2}$$

$$= \sqrt{3521 - 3481}$$

$$= \sqrt{40} = 6.32\text{g (2d.p)}$$

Example 2

The length of 32 leaves were measured in mm. Find the mean length, variance and the standard deviation.

Length (mm)	20 - 22	23 - 25	26 - 28	29 - 31	32 - 34
Frequency	3	6	12	9	2

Solution

Length (mm)	Mid – point, x	x^2	f	fx	fx^2
20 – 22	21	441	3	63	1323
23 – 25	24	576	6	144	3456
26 – 28	27	729	12	324	8748
29 – 31	30	900	9	270	8100
32 – 34	33	1089	2	66	2178
			$\sum f = 32$	$\sum fx = 867$	$\sum fx^2 = 23805$

Hence $$Mean, \bar{x} = \frac{\sum fx}{\sum f} = \frac{867}{32} = 27.1 \text{ (1d.p)}$$

$$Variance, S^2 = \frac{\sum fx^2}{\sum f} - \bar{x}^2 = \frac{23805}{32} - \left(\frac{867}{32}\right)^2 = 9.835$$

and $$Standard\ deviation, S = \sqrt{9.835} = 3.14 \text{ (2d.p)}$$

Analysing data using variance and standard deviation

The variance and standard deviation is used to quickly analyze data. Data extracted by people is not perfect but one can interpret the results in order to make informed decisions. For instance,

(i) I could say that the average weight for a baby is 3kg. Based on this number, any person having a baby would expect it to weigh approximately this much. However, based on standard deviation the average baby could actually never weigh close to 3kg. After all, the average of 1 and 5 is also 3.

(ii) A survey made in Uganda shows that the average Ugandan woman spends 9 hours a day on domestic tasks, such as preparing food and clothing, fetching water and firewood, and caring for the elderly, the sick as well as orphans. But there are some women who do not even spend more than 2 hours a day on domestic tasks.

Thus the averages, being measures of central location only, do not indicate how the values in the data are spread and may not give us enough information to analyse the data, hence the need to find the standard deviation.

Standard deviation is useful when comparing the spread of two separate data sets that have approximately the same mean. The data set with the smaller standard deviation has a narrower spread of measurements around the mean and therefore usually has comparatively fewer high or low values. An item selected at random from a data set whose standard deviation is low has a better chance of being close to the mean than an item from a data set whose standard deviation is higher.

Generally, the more widely spread the values are, the larger the standard deviation is. For example, imagine that we have to separate two different sets of exam results from a class of 30 students the first exam has marks ranging from 31% to 98%, the other ranges from 82% to 93%. Given these ranges, the standard deviation would be larger for the results of the first exam. Thus if the standard deviation is a large number, the mean might not represent the data very well.

Example 1

Two machines, A and B, are used to pack biscuits. A sample of 10 packets was taken from each machine and the mass of each packet, measured in grams, was noted. Find the standard deviation of the masses of the packets taken in the sample from each machine. Analyse the data and comment on which machine is more reliable.

Machine A (mass in g)	196, 198, 198, 199, 200, 200, 201, 201, 202, 205
Machine B (mass in g)	192, 194, 195, 198, 200, 201, 203, 204, 206, 207

Solution

For Machine A, mean $\bar{x} = \frac{\sum x}{n} = \frac{2000}{10} = 200$

For Machine B, mean $\bar{x} = \frac{\sum x}{n} = \frac{2000}{10} = 200$

The standard deviation for machine A, S_A

x	196	198	198	199	200	200	201	201	202	205	
x^2	38416	39204	39204	39601	40000	40000	40401	40401	40804	42025	$\sum x^2 = 400056$

$$S_A = \sqrt{\frac{\sum x^2}{n} - \overline{x}^2} = \sqrt{\frac{400056}{10} - 200^2}$$

$$= \sqrt{40005.6 - 40000} = \sqrt{5.6}$$

$= 2.37$ (2d.p)

The standard deviation for machine B, S_B

x	192	194	195	198	200	201	203	204	206	207	
x^2	36864	37636	38025	39204	40000	40401	41209	41616	42436	42849	$\sum x^2 = 400240$

$$S_B = \sqrt{\frac{\sum x^2}{n} - \overline{x}^2} = \sqrt{\frac{400240}{10} - 200^2}$$

$$= \sqrt{40024 - 40000} = \sqrt{24}$$

$= 4.90$ (2d.p)

The standard deviation for machine A is 2.37g and the standard deviation for machine B is 4.90g. Since the standard deviation for machine A is smaller than that of machine B, then machine A is more reliable.

Example 2

Thirteen students from each of the two classes, S.5A and S.5B did a certain mathematics test and there scores are as shown below;

Test scores from S.5A Class
65, 66, 67, 69, 70, 70, 70, 71, 71, 72, 73, 74, 76

Test scores from S.5B Class
50, 53, 54, 55, 55, 58, 59, 59, 75, 95, 98, 100, 100

Analyse the test scores and comment on which class is more reliable.

Solution

First calculate the mean mark for each class

For S.5A class, the mean $\overline{x} = \frac{\sum x}{n} = \frac{914}{13} = 70.3$

For S.5B class, the mean $\bar{x} = \frac{\sum x}{n} = \frac{911}{13} = 70.1$

The standard deviation for S.5A class

x	65	66	67	69	70	70	70	71	71	72	73	74	76	
x^2	4225	4356	4489	4761	4900	4900	4900	5041	5041	5184	5329	5476	5776	$\sum x^2$ =64378

$$S = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2} = \sqrt{\frac{64378}{13} - 70.3^2}$$

$$= \sqrt{4952.15 - 4942.09} = \sqrt{10.06}$$

$$= 3.17 \text{ (2d.p)}$$

The standard deviation for S.5B class

x	50	53	54	55	55	58	59	59	75	95	98	100	100	
x^2	2500	2809	2916	3025	3025	3364	3481	3481	5625	9025	9604	10000	10000	$\sum x^2$ =68855

$$S = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2} = \sqrt{\frac{68855}{13} - 70.1^2}$$

$$= \sqrt{5296.54 - 4914.01} = \sqrt{382.53}$$

$$= 19.56 \text{ (2d.p)}$$

Notice that both classes have the same mean (about 70) but the distribution of scores is very different. A mark selected at random from a data set whose standard deviation is low has a better chance of being close to the mean than a mark from a data set whose standard deviation is higher. Since the standard deviation for S.5A class is low, it is a more reliable class.

Exercise

1. Find the mean, median and standard deviation of the following values

   2,4,6,3,1,2,1,1,5,4,2,5,6,8,15,7

2. Find the mean and standard deviation of the following sets of numbers

(i) 11, 14, 17, 23, 29

(ii) 200, 203, 206, 207, 209

3. The mean of the numbers 3, 6, 7, a, 14, is 8. Find the standard deviation of the set of numbers.

4. A certain factory produces ball bearings. A sample of the bearings from the factory produced the following results

Diameter of bearing mm	***Frequency***
81 – 93	4
94 – 96	6
97 – 99	34
100 – 102	40
103 – 105	13
106 – 108	3

Determine the mean and variance of the diameter of the sample bearings.

5. 100 students were tested to determine their intelligence quotient (IQ), and the results were as shown below. All IQs are given to the nearest integer)

IQ	45 –	55 –	65 –	75 –	85 –	95 –	105 –	115 –
Number of pupils	1	1	2	6	21	29	24	16

(a) Calculate the mean and standard deviation
(b) Draw a cumulative frequency curve and use it to estimate
  (i) median
  (ii) semi interquartile range

6. The table shows the times taken on 30 consecutive days for Kalita Bus to complete one journey on a particular route. Find the mean time for the journey and the standard deviation.

Time (minutes)	Frequency
60 – 63	1
64 – 67	3
68 – 71	12
72 – 75	10
76 – 79	4

7. The table below shows the distribution of marks obtained by a class of 30 students of a certain school;

Marks	15 - 19	20 - 24	25 - 29	30 - 34	35 - 39
Frequency	10	6	5	4	5

Calculate;

(i) the mean and variance

(ii) Median and modal mark

(iii) The percentage of students who scored less than 33 marks.

8. A box contains 100 tokens which differ in mass, but are otherwise identical. 20 of them have a mass of 4.8g each, 35 a mass of 5.2 g each, 25 a mass of 5.7g each, 15 a mass of 6.5 g each and the remaining 5 have a mass of 8.0g each. Calculate the expected value and variance of the mass of a token chosen at random from the box.

9.

MOVING AVERAGES

Moving averages is one of the methods used in business and other areas of life for forecasting numerical data.

Moving average is obtained by finding average changes over successive periods of time. The time periods can be seconds, minutes, days, weeks, months, years or decades.

Moving averages help us to study changes in the magnitude of quantities such as population figures, rainfall figures, growth figures etc over periods of time. The pattern of change in figures is called a *trend* and moving averages help us to form a *trend line* which can be extrapolated to predict future values.

It is important to have an idea about the trend that is underlying movement of the data in consideration because this can be used to see if there are any **variations**. For example there could be particular points during a year when the sales are lower than previous time.

Types of variations (trends) of data

(i) Secular trend: Is formed when the direction of the data keeps going upwards or downwards over a long period of time. For example the high jump record keeps increasing over a long period, giving an upwards moving secular trend in the data. Also the winning time in marathon keeps decreasing over a longtime period, thereby giving a downwards moving secular trend in the data.

(ii) Seasonal variation: This occurs when the data follows a pattern during corresponding months in successive years. For example heating bills. Electricity and heating bills fluctuate with seasons; higher bills in the colder weather and lower bills in the warm weather. Also we would expect that sales of umbrellas would increase during the rainy season and then decrease during the sunny season.

(iii) Cyclical variation: This occurs when long periods of time follow the trend line. For example several years of prosperity in economy followed by several years of recession forming a pattern over time.

(iv) Random variation: This occurs when unpredictable events like war or a "crash" in the stock markets happens. The trend of data cannot therefore be predicted easily and cannot be "smoothed out" using moving averages since the variations are irregular.

Calculation of moving averages

Moving averages can be calculated to smooth out the seasonal variations so that *trends* can be spotted. It can be calculated using 2, 3, 4, 5, 6, 7 or more items in the data known as *data points*.

Moving average = $\dfrac{moving\ \ total}{number\ of\ \ data\ points}$

The moving total is obtained by adding the given data items over successive periods of time. The moving totals and averages are even or odd depending on whether the data points considered are even or odd respectively.

The moving totals and averages are written at the mid – point of the data from which it is calculated.

Example 1

The monthly rainfall in millimeters at a weather station for eight consecutive months was recorded as shown below.

Jan	Feb	Mar	Apr	May	June	July	Aug
7	18	25	20	12	8	6	4

Calculate the 4 – monthly moving averages from January to August.

Solution

Month	**Rainfall (mm)**	**4 – point moving totals**	**4 – point moving averages**
Jan	7		
Feb	18		
		7+18+25+20 = 70	$\frac{70}{4} = 17.5mm$
Mar	25		
		18+25+20+12= 75	$\frac{75}{4} = 18.5mm$
Apr	20		
		25+20+12+8 = 65	$\frac{65}{4} = 16.25mm$
May	12		
		20+12+8+6 = 46	$\frac{46}{4} = 11.5mm$
June	8		
		12+8+6+4 = 30	$\frac{30}{4} = 7.5mm$
July	6		
Aug	4		

Hence the 4 – monthly moving average rainfall is 17.5mm, 18.75mm, 16.25mm, 11.5mm and 7.5 mm.

NB: The first moving average is computed from Jan, Feb, Mar and Apr, giving the first moving average mid – way between Feb and Mar and so on.

Example 2

Grace a student in OLAN obtained the following marks in her weekly tests in subsidiary mathematics during the last two months.

Week	1	2	3	4	5	6	7	8
Marks in %	38	52	65	44	58	56	70	72

Calculate her 5 – monthly moving averages and hence comment on her progress in subsidiary mathematics.

Solution

Week	Marks in %	5 – point moving totals	5 – point moving averages
1	38		
2	52		
3	65	38+52+65+44+58=257	51.4
4	44	52+65+44+58+56=275	55
5	58	65+44+58+56+70=293	58.6
6	56	44+58+56+70+72=300	60
7	70		
8	72		

From the results of moving averages it can be concluded that Grace is following an increasing performance trend in the scores of subsidiary mathematics. She is therefore improving over the two months.

N.B: The first moving average is obtained by averaging out the data for week 1, 2, 3, 4 and 5 giving the first moving average mid – way at week 5 and so on.

Graphs of moving averages

Many sets of data display **trends** which depend upon the time of year or the particular month or even the time of day etc. When the data is plotted on a line graph, the graph obtained is called a **time series graph** and this graph in most cases shows wildly fluctuating trends.

In attempting to glean meaningful information from such graphs, we really need to isolate the different seasons, each of which exerts its own seasonal influence. One way of doing this is to use what are termed **moving averages**, which are designed to *level* out the large fluctuations which can occur in a set of data that varies over time.

Moving Averages, when graphed, allow us to see any trends in data that are cyclical. By calculating the average of 2 or more items in the data, any peaks and troughs are smoothed out. The graph of moving average therefore presents a steady trend of change of data.

The positioning of the moving average is very important. The moving average must be plotted at the *mid – point* of the data from which it is calculated. The points on the graph are then joined one after the other successively using straight lines to give a trend line. The line graph of the moving averages is therefore called a *trend line*. The trend line can be used to forecast future values by extrapolating it.

When the plotted points of moving averages do not give a straight line on joining, then we simply draw a best fitting trend line through the plotted moving averages.

Example 1

The table below shows the number of properties let by an Estate Agency over the past three years.

	1st quarter	**2nd quarter**	**3rd quarter**	**4th quarter**
2009	328	317	335	151
2010	355	343	361	379
2011	381	371	388	406

(i) Plot these data on the graph.

(ii) Calculate the 4 – point moving averages

(iii) Plot these averages on the same graph and draw the trend line.

(iv) Use your graph to estimate how many properties are likely to be let in the 1st quarter of 2012.

(v) Why do we use moving averages?

Solution

(ii)

Year	Quarter	No. of property let	4 – point moving totals	4 –point moving averages
	1st	328		
2009	2nd	317		
			1331	332.75
	3rd	335		
			1358	339.5
	4th	351		
			1384	346
	1st	355		
			1410	352.5
2010	2nd	343		
			1438	359.2
	3rd	361		
			1464	366
	4th	379		
			1492	373
	1st	381		
			1519	379.75
2011	2nd	371		
			1546	386.5
	3rd	388		
	4th	406		

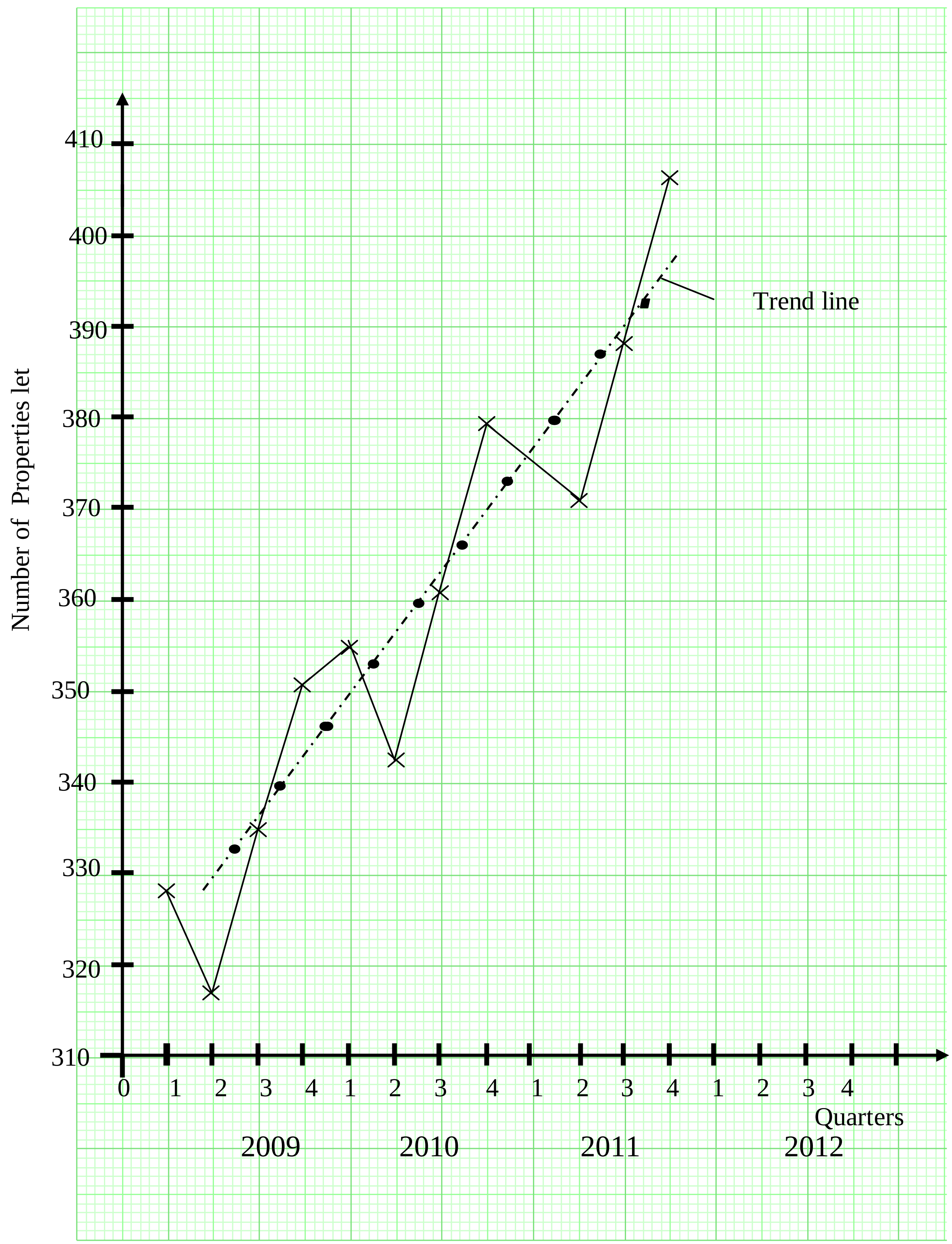
From the graph the next moving average = 393.

(vi) Let the estimate of the number of properties likely to be let in the 1st quarter of 2012 be x.

Working backwards;

$$\frac{371+388+406+x}{4}=393$$
$$1165+x=1572$$
$$x=407$$

(v) We use moving averages to "smooth" data so that we can try to predict future values.



N.B: The first moving average is calculated from quarters 1, 2, 3 and 4 of 2009, giving the first moving average mid – way between quarters 2 and 3 of 2009.This is where it should be plotted and so on.

Advantages of using moving averages

- It reduces the effects of random variation which is due to the cumulative effects of all the short term unpredictable influences.
- It gives room different statisticians presented with same time series to come up with one approach to explain given data.
- They can be used to predict the future trend of any quantity but the estimate should not be too far ahead.

Example 2

The following table shows the daily sales of a certain book at *Muhumuza and Sons* bookshop for two consecutive weeks.

1st week	10	9	6	11	14
2nd week	9	8	6	13	12

(a) Workout the 5 – day moving averages.

(b) On the same graph paper draw both;

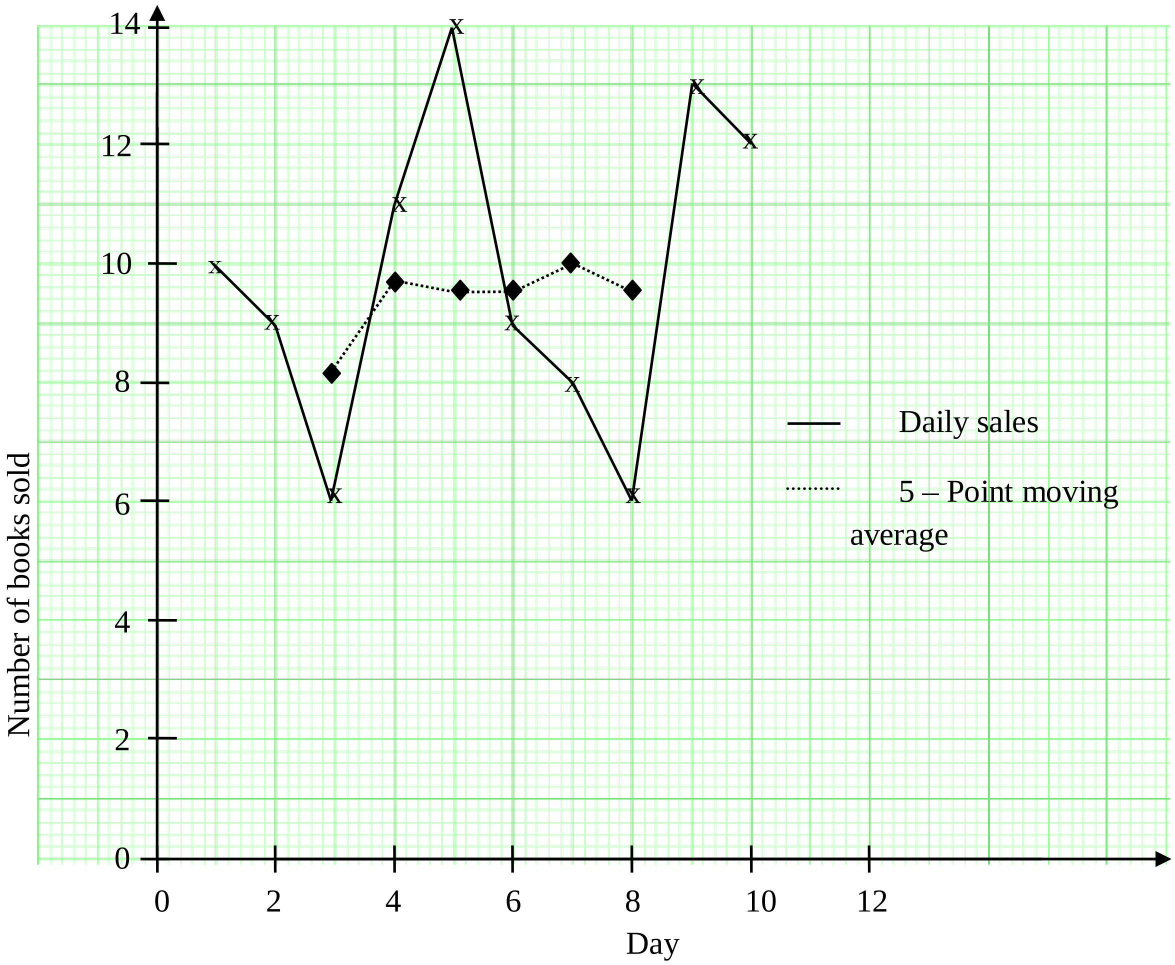
(i) The graph of daily sales, and

(ii) The graph of 5 – day moving averages.

Solution

(a)

	Day	**No. of books sold**	**5 –point moving totals**	**5 – point moving averages**
	1	10		
	2	9		
Week 1	3	6	41	8.2
	4	11	49	9.8
	5	14	48	9.6
	6	9	48	9.6
Week 2	7	8	50	10
	8	6	48	9.6
	9	13		
	10	12		



Example 3

The table shows the number of units of electricity used each quarter by a householder in Mukono over a period of 3 years.

Year	2010				2011				2012			
Quarter	1	2	3	4	1	2	3	4	1	2	3	4
Units used	680	810	470	740	640	850	420	750	970	880	490	760

a) Plot these values on a line graph.

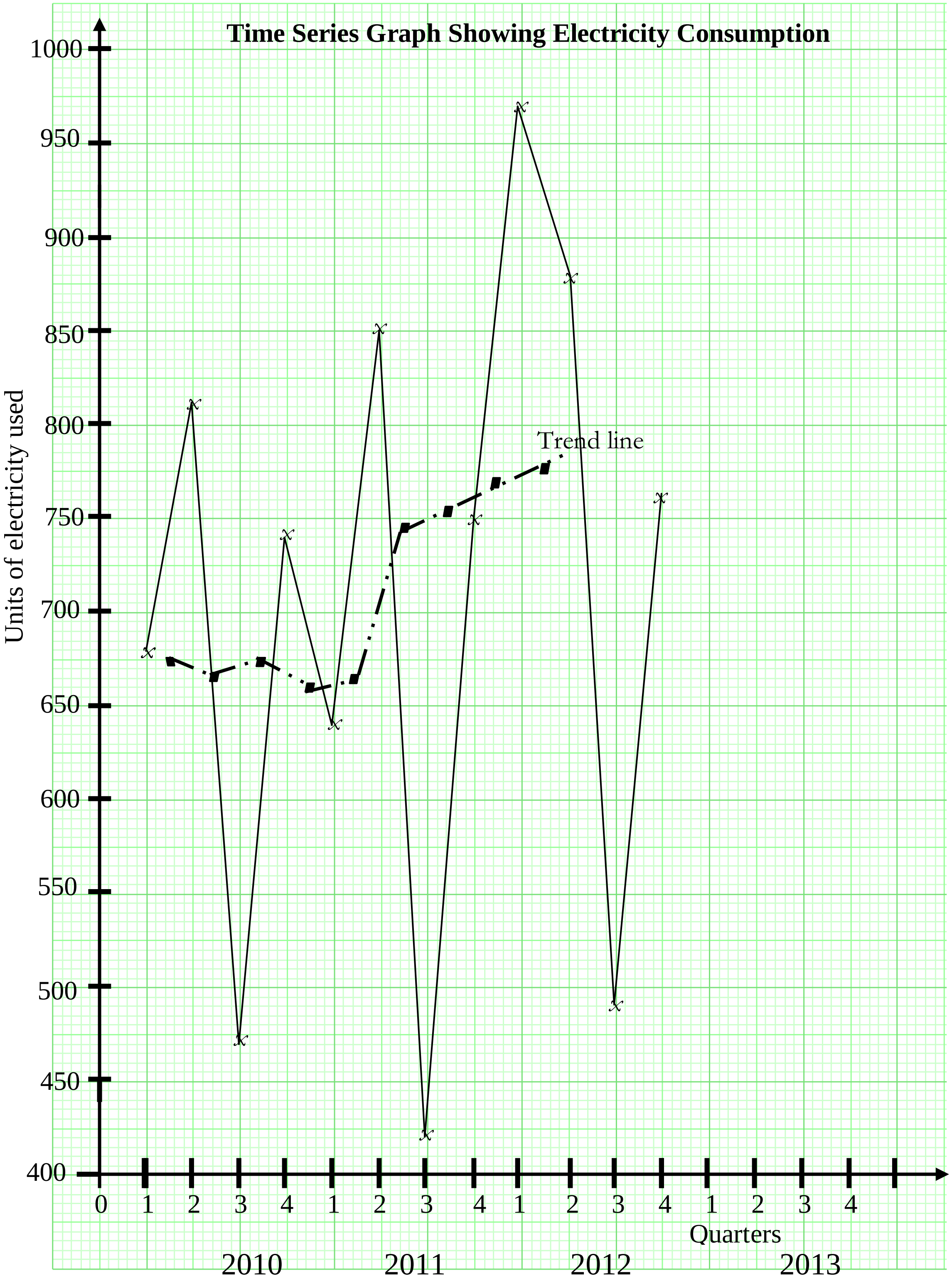
b) Calculate a 4-point moving average.

c) Plot the moving averages on your line graph. Draw in the trend line.

d) Comment on the trend in the units of electricity used.

Solution

(a)



Time Series Graph Showing Electricity Consumption
Units of electricity used
1000
950
900
850
800
750
700
650
600
550
500
450
400
Trend line
0
1
2
3
4
1
2
3
4
1
2
3
4
1
2
3
4
Quarters
2010
2011
2012
2013

(b) The first 4-point moving average is calculated as the mean of the 4 values from 2010. The mean of 680, 810, 470 and 740 is 675.

The second 4-point moving average is calculated as the mean of the Q_2, Q_3 and Q_4 values from 2010 and the Q_1 value from 2011. The mean of 810, 470, 740 and 640 is 665.

The 3rd 4-point moving average is the mean of Q_3 and Q_4 from 2010 and Q_1 and Q_2 from 2011. This moving average is 675.

Units used	680	810	470	740	640	850	420	750	970	880	490	760

Moving average 675 665 675 662.5 665 747.5 755 772.5 775

c) The moving averages have been added to the graph in (a). Each moving average is plotted above the middle of the interval from which it was calculated. The trend line is drawn through the middle of the points.

d) The trend line shows that electricity usage is increasing over time. There is a seasonal pattern to the usage, with Quarter 3 seeing the least electricity used.

Exercise

1. A young baby was weighed by a nurse every week –end. Following is the record of her weights.

No. of week end	1	2	3	4	5	6	7	8	9	10	11	12
Weight (kg)	3.5	3.7	4.0	4.25	4.6	4.9	5.18	5.4	5.63	5.8	6.0	6.18

Calculate the 8 – weekly mowing averages for her weights during these three months.

2. The collections in thousands of shillings from the sales of tickets during the 10 days of league matches at Namboole stadium were as given in the following table.

Day	1	2	3	4	5	6	7	8	9	10
Collection	250	180	180	190	260	200	280	250	320	360

Find the 7 – day moving average and comment about your answer.

3. The following data gives the typical price (in pounds) for a textbook of subsidiary mathematics.

Year	2003	2004	2005	2006	2007	2008	2009	2010	2011

Price	63	51	49	80	92	57	62	60	61

Determine and plot the three year moving averages for these data.

4. The number of bicycles sold by a shop for the first 10 weeks of the period February to April 2012 is as in the table below.

Week	1	2	3	4	5	6	7	8	9	10
No. sold	7	4	6	4	5	3	4	1	5	2

(i) Draw a graph showing the weekly sales.

(ii) Calculate the 4 – weekly moving averages of sales.

(iii) On the same diagram as for part (i) draw the graph of the moving average. Which of these graphs gives you a clear idea of the trend of sales and why.

(iv) Comment on the trend of sales for this period.

5. Karungi has the following information about the oil bills for her home over the period from 2011 to 2012.

Year	1st quarter	2nd quarter	3rd quarter	4th quarter
2011	222	120	58	172
2012	234	128	64	182

(a) Plot these quarterly bills as a time series.

(b) Work out the four-point moving averages.

(c) On the same axes as the time series, plot the moving averages and draw the trend line.

(d) Make three comments about the variations in Karungi's oil bills from 2011 to 2012.

6. The table below shows the sales of a particular commodity over the past three years by Owino market vendors.

Year	2010				2011				2012			
Quarter	1	2	3	4	1	2	3	4	1	2	3	4
Sales	189	244	365	262	190	266	359	250	201	259	401	265

(a) Plot the values of sales on a line graph.

(b) Calculate a 4-point moving average.

(c) Plot the moving averages on your line graph. Draw in the trend line.

7. The table below shows the average termly marks scored in mathematics tests by a certain student from his Senior One in 2009 to his Senior Four in 2012.

YEAR	**TERMLY MARKS (%)**		
	1st	**2nd**	**3rd**
2009	36	50	54
2010	40	45	60
2011	39	46	70
2012	49	50	……

(i) Calculate the four – point moving averages for the data.

(ii) Plot the termly marks and the four – point moving averages on the same graph.

(iii) Comment on the trend of the students' performance in mathematics during the period. What was likely to be his mark in mathematics in the final examination?

8. The table below shows Fiona's quarterly electricity bills in (£) over a three year period.

Year	**1st quarter**	**2nd quarter**	**3rd quarter**	**4th quarter**
2010	147	105	122	142
2011	118	78	89	115
2012	89	49	55	88

(i) Plot these data on graph paper using suitable axes and scales.

(ii) Calculate the quarterly moving averages.

(iii) Plot these averages on the graph obtained in (i) and draw the trend line.

(iv) Use the trend line to estimate Fiona's electricity bill for the first quarter in 2013 showing clearly where your reading is taken.

9. The table below shows the amount of copper in thousands of tones exported by Uganda in the period 1990 – 5

Year	1990	1991	1992	1993	1994	1995
Quantity	14	16	18	17	19	18

On the same coordinate axes draw a line graph representing this data and a graph of three - point moving averages.

10. The sales of a computer company are given for the period of five years in the table below.

	First half	Second half
2007	230	810
2008	241	852
2009	259	902
2010	272	934
2011	288	966

(i) Draw a graph of these sales and on it superimpose the 2 – point moving average.

(ii) Estimate from your graph the sales for the first half of 2012.

INDEX NUMBERS

Index numbers are statistical measures designed to show changes in a variable or group of related variables with respect to time, geographical location or other characteristics such as income, profession etc. They provide a measure of the relative change in some variable or group of variables at a specified date when compared with some fixed period in the past.

Index numbers may be classified in terms of the variables that they are intended to measure. In business, different groups of variables in the measurement of which index number techniques are commonly used are; (i) price, (ii) quantity, (iii) value etc. The simplest example of an index number is a price index (price relative).

- ***Price Index (Price relative)*** *– The Price Index shows how the price of something changes over a period of time. Price index numbers measure the relative changes of a commodity or commodities between two periods.*

Terminologies used in price index.

In general the present level of prices is compared with the level of prices in the past. The present period is called the ***current period*** and some period in the past is called the ***base period***. The period can be days, weeks, months, years or decades. For example;

Base year is the year to compare all the other prices with, or the year on which the price changes are based.

Current year is the year for which the index is to be calculated or the year being compared with the base year.

Simple price index

A simple price index number measures the relative change in price of just one variable with respect to the base price.

If we denote the price in the base period as p_o and the price in the current period as p_1, then

Simple price index, $P = \frac{p_1}{p_0} \times 100$

Example 1

In January 2010, the price of a kilogram of sugar was 4200=. In January 2012, the price was 6300=. Taking 2010 as the base year, find the price index.

Solution

$$\text{Price index} = \frac{p_{2012}}{p_{2010}} \times 100$$

$$= \frac{6300}{4200} \times 100 = 150$$

So the price index or price relative = 150.

This indicates that the price of a kilogram of sugar increased by 50% between 2010 and 2012.

Example 2

One litre of petrol costs 2500= in 2005 and 3000= in 2009. Taking 2005 as the base year, find the price relative in 2009.

Solution

$$\text{Price relative} = \frac{p_{2009}}{p_{2005}} \times 100$$

$$= \frac{3000}{2500} \times 100 = 120$$

This indicates that the price of petrol went up by 20% between 2005 and 2009.

Example 3

The table below shows how the price of one house has changed over the years 2007 to 2009.

Year	2007	2008	2009
Price (£)	62,000	66,960	73,780

Taking 2007 as the base year, calculate the simple price indices for 2008 and 2009.

Solution

For 2008, the price index is = $\frac{66960}{62000} \times 100 = 108$

The index number 108 shows that the price has increased by 8 % since 2007

For 2009, the index number is = $\frac{73780}{62000} \times 100 = 119$

The index number 119 shows that the price has increased by 19 % since the base period.

The base year is 2007. We give this year an index number of 100.

This is what the table looks like now.

Year	2007	2008	2009
Price index	100	108	119

Example 4

In 2005, the price index of a commodity using 2001 as the base year was 112. In 2011, the index using 2005 as base year was 85. What would have been the index in 2011, using 2001 as base year?

Solution

Taking 2001 as the base year, then;

$$\frac{p_{2005}}{p_{2001}} \times 100 = 112 \Rightarrow \frac{p_{2005}}{p_{2001}} = 1.12 \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Taking 2005 as the base year, then;

$$\frac{p_{2011}}{p_{2005}} \times 100 = 85 \Rightarrow \frac{p_{2011}}{p_{2005}} = 0.85 \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Multiplying equations (i) and (ii) gives;

$$\frac{p_{2005}}{p_{2001}} \times \frac{p_{2011}}{p_{2005}} = 1.12 \times 0.85$$

$$\frac{p_{2011}}{p_{2001}} = 0.952$$

Thus the price index in 2011 taking 2001 as the base year would have been;

$$\frac{p_{2011}}{p_{2001}} \times 100 = 0.952 \times 100 = 95.2$$

Example 5

The 2011 price index for a pair of shoes was 120 taking 2007 as base year. Calculate the 2007 index, referred to 2011 as base.

Solution

Price index taking 2007 as base year, $\frac{p_{2011}}{p_{2007}} \times 100 = 120 \Rightarrow \frac{p_{2011}}{p_{2007}} = 1.2$

Therefore referred to 2011 as base year $\frac{p_{2007}}{p_{2011}} = \frac{1}{\frac{p_{2011}}{p_{2007}}} = \frac{1}{1.2}$

The 2007 price index, referred to 2011 as base; $\frac{p_{2007}}{p_{2011}} \times 100 = \frac{1}{1.2} \times 100 = 83.3$

Simple aggregate price index

The simple aggregate price index is obtained by calculating the total price of a group of items as a ratio of the total price of the same group of items in the base year. It consists of expressing the aggregate price of all commodities in the current year as a percentage of the aggregate price in the base year.

Simple aggregate price index, P = $\frac{\sum p_1}{\sum p_0} \times 100$

Where $\sum p_1$ is the sum of the prices for the current year and $\sum p_0$ is the sum of the prices for the base year.

Example 1

The table below shows the price of items in Uganda shillings of rice, sugar, eggs and groundnuts in 2008 and 2010.

Item	**2008 Price**	**2010 Price**
Rice (1kg)	2500	3200
Sugar (1kg)	3000	3500
Eggs (1 dozen)	4000	4500
Groundnuts (1kg)	2000	2600

Taking 2008 as the base year, find the simple aggregate price index for the items.

Solution

Item	**2008 Price**	**2010 Price**
Rice (1kg)	2500	3200
Sugar (1kg)	3000	3500
Eggs (1 dozen)	4000	4500
Groundnuts (1kg)	2000	2600
Total	**11500**	**13800**

Simple aggregate price index = $\frac{\sum p_1}{\sum p_0} \times 100$

$$= \frac{13800}{11500} \times 100 = 120$$

This implies that the prices of these group of items increased by 20% between 2008 and 2010.

Example 2

Below is a table showing the prices of items A, B, C, D, E and F in US dollars in 1995 and 2005. Using 1995 as the base year, calculate the aggregate price index for these items.

Item	1995 Price	2005 Price
A	$ 7.7	$8.9
B	18.5	18.4
C	8.8	10.1
D	14.6	15.6
E	15.8	17.0
F	44.0	46.2

Solution

Item	1995 Price	2005 Price
A	$ 7.7	$8.9
B	18.5	18.4
C	8.8	10.1
D	14.6	15.6
E	15.8	17.0
F	44.0	46.2
Total	$**109.4**	$**116.2**

Simple aggregate price index = $\frac{\sum p_1}{\sum p_0} \times 100$

$$= \frac{116.2}{109.4} \times 100 = 106.2$$

Weighted Index Numbers

An index number can include a number of different items. The index number must take into account the proportions of the different items. These are called weightings. The final index number is called a weighted index number. For example;

The table shows the indices for the basic costs of a factory for two years.

Year	2008 (Base Year)	2009	Weighting
Production	100	105	210
Services	100	103	40
Rent	100	109	12

N.B: 2008 =100 means that 2008 is taken as the base year.

The weighting reflects the proportion of money spent on each item. The increase of 5 % on production costs in 1999 would have a greater effect on the total costs from the 9 % increase on rent. This is because more money is spent on production than the rent.

We use the weightings to find the weighted index for costs.

Weighted Index for Costs = $\frac{\sum(weight \times index)}{\sum weights}$

$$= \frac{(210 \times 15) + (40 \times 103) + (12 \times 109)}{210 + 40 + 12} = 105$$

This index is sometimes called the weighted mean index, or the weighted average.

Weighted Price Index

This puts into consideration the relative importance of the commodities. In this case the weights of different commodities together with their prices or Price relatives are given.

Weighted aggregate price index numbers is the simple aggregative type in which weights are assigned to the various items included in the index.

In general weighted aggregate price index, $\bar{P} = \frac{\sum W(\frac{p_1}{p_0} \times 100)}{\sum W}$ where W is the weight.

But $\frac{p_1}{p_0} \times 100$ is the Price index or Price relative. Let $\frac{p_1}{p_0} \times 100 =$ A

Then weighted price index, $\bar{P} = \frac{\sum WA}{\sum W}$

The weighted price index is some times referred to as the composite index.

Example 1

1. The table below lists seven categories of house holds expenditure and records the value of the price index in 2008 taking 2006 as base year. The weights assigned to the seven categories are also given.

Expenditure category	2008 Price index	Weight
Food	106.7	163
Medical care	113.4	50
Clothing	109.4	78
Foot wear	106.6	36
Electricity	118.3	160
Drinks	101.6	55
Fuel	105.2	72

(a) Calculate the weighted aggregate price index for these seven categories of expenditure.
(b) A household's weekly expenditure on these seven categories in 2006 was approximately £115. Estimate the weekly expenditure on these categories for a similar household in 2008.

Solution

(a)

Expenditure category	A	W	WA
Food	106.7	163	17392.1
Medical care	113.4	50	5670
Clothing	109.4	78	8533.2
Foot wear	106.6	36	3837.6
Electricity	118.3	160	18928
Drinks	101.6	55	5588
Fuel	105.2	72	7574.4
		$\sum$ W=614	$\sum$ WA=67523.3

Weighted aggregate price index = $\frac{\sum WA}{\sum W} = \frac{67523.3}{614} = 110.0$ (1d.p)

(b) Weekly expenditure in 2008 $\approx$ 110% of expenditure in 2006

$$= \frac{110}{100} \times 115 = £126.5$$

Example 2

Calculate a weighted price index for the following figures for 2009 based on 2007.

Item	2007 price (£)	2009 Price(£)	Weight
Food	55	60	4
Housing	48	52	2
transport	16	20	1

Solution

Item	p_0	p_1	W	$A=(\frac{p_1}{p_0} \times 100)$	WA
Food	55	60	4	109.09	436.36
Housing	48	52	2	108.33	216.66
transport	16	20	1	125	125
			$\sum$W=7		$\sum$WA=778.02

Weighted price index = $\frac{\sum WA}{\sum W} = \frac{778.02}{7} = 111.1$ (1d.p)

Example 3

Calculate to the nearest integer, the weighted price index for the following table of price relatives and weights.

	Price relative	weight
Food	118	40
Rent	102	8
Clothing	114	12
Fuel	120	10
Others	110	30

Solution

	A	W	WA
Food	118	40	4720
Rent	102	8	816
Clothing	114	12	1368

Fuel	120	10	1200
Others	110	30	3300
		$\sum$ W=100	$\sum$ WA=11404

$$\text{Weighted price index} = \frac{\sum WA}{\sum W} = \frac{11404}{100} = 114.04$$

The weighted price index or weighted aggregate price index = 114 (to the nearest integer)

N.B: When calculating the Price indices, care must be taken on the units of currency used. The prices of the commodities in both the current and base year must be in the same currency.

Example 4

A mechanical product consists of five raw materials A, B, C, D and E in the proportions of 16, 46, 11, 11 and 16% respectively. The following table shows the price indices of the raw materials.

Raw materials	Price in year 2009($)	Price in the year 2010($)	Price index in 2010 based on 2009
A	2.50	3.10	124
B	z	23.85	150
C	12.00	18.00	150
D	17.50	21.35	y
E	2.50	x	108

(a) Find the values of x, y and z

(b) Calculate the aggregate weighted price index representing the cost of the mechanical product in the year 2010 using the year 2009 as the base year.

(c) If the total monthly cost of the raw materials in the year 2009 is $1.3 million, find the total monthly cost of the raw materials in the year 2010.

(d) If the cost of each raw material rises by 40% from the year 2010 to 2012, find the weighted price index representing the cost of the mechanical product in the year 2012 based on the year 2009.

Solution

(a) For raw material E,

$$\frac{x}{2.50} \times 100 = 108$$

$$x = \frac{2.50 \times 108}{100} = 2.70$$

For raw material D, $\frac{21.35}{17.50} \times 100 = y$

$y = 122$

For raw material B, $\frac{23.85}{z} \times 100 = 150$

$z = \frac{23.85 \times 100}{100} = 15.90$

(b)

Raw materials	A	W	WA
A	124	16	1984
B	150	46	6900
C	150	11	1650
D	122	11	1342
E	108	16	1728
		$\sum$ W=100	$\sum$ WA=13604

Weighted aggregate price index in 2010, $\bar{P} = \frac{\sum WA}{\sum W} = \frac{13604}{100} = 136$

(c) Let the total monthly cost in the year 2010 = P

$$\frac{P}{1.3} \times 100 = 136$$

$$P = \frac{136 \times 1.3}{100} = \$1.77 million$$

(d) Since the cost of each raw material rises by 40% from the year 2010 to 2012, this implies;

$$\frac{P_{2012}}{P_{2010}} \times 100 = 140 \ldots\ldots\ldots\ldots\ldots(1) \quad \text{but also} \quad \frac{P_{2010}}{P_{2009}} \times 100 = 136 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From equations (1) and (2)

The weighted price index representing the cost of the mechanical product in the year 2012 based on the year 2009, $\frac{P_{2012}}{P_{2009}} \times 100 = 136 \times \frac{140}{100} = 190.4$

Value index

A **value index** measures changes in both the price and quantities involved. Value is the product of price and quantity. A simple ratio is equal to the value of the current year divided by the value of base year. If the ratio is multiplied by 100 we get the value index number.

A value index, such as the index of department store sales, needs the original base-year prices, the original base-year quantities, the present-year prices, and the present year quantities for its construction.

Its formula is: Value index V = $\frac{\sum p_1q_1}{\sum p_0q_0}\times 100$ where p_1q_1 is the product of price and quantity for the current period and p_0q_0 is the product of price and quantity for the base period.

Example 1

The prices and quantities for various items sold at Select Garments for May 2007 and May 2012 are as follows;

Item	2007 price (Ush.)	Quantity sold	2012 price (Ush.)	Quantity sold
Ties (each)	3,000	1000	6,000	900
Suits (each)	90,000	100	120,000	120
Shoes (pair)	30,000	500	24,000	500

What is the value index for May 2012 using May 2007 as the base period?

Solution

Item	p_0	q_0	p_1	q_1	p_0q_0	p_1q_1
Ties (each)	3,000	1000	6,000	900	3,000,000	5,400,000
Suits (each)	90,000	100	120,000	120	9,000,000	14,400,000
Shoes (pair)	30,000	500	24,000	500	15,000,000	12,000,000
					$\sum p_0q_0$ =27,000,000	**$\sum p_1q_1$ =31,800,000**

Value index = $\frac{\sum p_1q_1}{\sum p_0q_0}\times 100$

$$= \frac{31,800,000}{27,000,000}\times 100 = 117.8 \text{ (1d.p)}$$

Example 2

The figures given below show the prices and quantities of maize, wheat and beans a certain family used for the year 2007 and 2010.

	2007		**2010**	
	Quantity	Price	Quantity	Price
Maize	20	650	25	700
Wheat	10	1500	8	1600
beans	5	150	8	200

Calculate the value index for 2010 for the family taking 2007 as the base year.

Solution

	2007		2010			
	q_0	p_0	q_1	p_1	p_0q_0	p_1q_1
Maize	20	650	25	700	13000	14000
Wheat	10	1500	8	1600	15000	16000
beans	5	150	8	200	750	1000
					$\sum p_0q_0 = 28750$	$\sum p_1q_1 = 31000$

$$\text{Value index} = \frac{\sum p_1q_1}{\sum p_0q_0}\times 100$$

$$= \frac{31900}{28750}\times 100 = 111$$

This shows an increase of 11% in the money spent for the items between 2007 and 2009.

Exercise

1. In 2011, the price index of a commodity was 135 when 2009 was taken as base year. The value of the commodity in 2011 was £54 and in 2010 was £46. Find
   (a) The value of the commodity in 2009
   (b) The price index of the commodity in 2010 when 2009 was taken as base year.
2. The price of New Vision newspaper in 2010 was Ush.1500. The index number for the same price of this newspaper in 2000 was 160 based on 1990. In 2010 it was 75, based on 2000. Calculate
   (a) The price index in 2010, based on 1990.
   (b) The prices of the newspaper in 1990 and 2000.
3. The price relative of a commodity in 2001 using 2000 as base year was 105. The price relative of the same commodity in 2002, using 2001 as base year was 95. Given that the cost of the commodity in 2000 was 50,000=, find its cost in 2002.
4. A certain family in a Kampala suburb spent the following amounts per month on the items shown in the years 2008 and 2009.

Item	2008 amount in Uganda sh.	2009 amount in Uganda sh.
Housing	80,000	100,000
Clothing	20,000	20,000
Electricity	40,000	50,000
Water	10,000	12,000
Food	140,000	160,000
Transport	50,000	60,000

Medical fees	30,000	35,000
Miscellaneous	30,000	40,000

Using 2008 as the base year, calculate the simple aggregate expenditure index for 2009.

5. The prices of unit values of four commodities A, B, C and D in the years 2004 and 2006 were as below;

Commodities	2004 Price	2006 Price	Weights
A	4,000	5,000	7
B	9,000	11,000	2
C	6,000	7,000	3
D	6,000	8,000	6

(a) Taking 2004 as the base year, calculate the

(i) Simple price index for 2006.

(ii) Weighted aggregate price index.

6. The table above shows the price indices and the percentage of usage of four major ingredients used to produce a certain kind of food.

Ingredient	**Price index for the year 2010 based on the year 2009**	**Percentage of usage (%)**
W	121	11
X	x	21
Y	139	13
Z	113	55

(a) Calculate

(i) The price of ingredient Y in the year 2009 if the price in the year 2010 is £99,

(ii) The price index of ingredient Z in the year 2010 based on the year 2007 if the price index in the year 2009 based on the year 2007 is 140.

(b) The weighted aggregate index representing the cost of production of the food in the year 2010 based on the year 2009 is 116. Calculate

(i) the value of x,

(ii) The cost of one unit of the food in the year 2009 if one unit of the food costs £14.50 in the year 2010.

7. The table above shows the prices of four raw materials, A, B, C and D, used to produce a technology product in the years 2007 and 2009, and their corresponding weightings.

Raw material	**Price per Unit in dollars**		**Weights**
	2007	**2009**	

A	57.30	42.98	8
B	72.10	36.05	8
C	84.20	17.88	2
D	90.10	166.69	5

(a) Calculate the composite index representing the prices of the four raw materials in the year 2009 based on the year 2007.

(b) If the prices of the raw materials, *A*, *B*, *C* and *D* changed by +14%, +12%, +54% and +75% respectively in the year 2012 based on the year 2009, find the composite index representing the prices of the four raw materials in the year 2012 based on the year 2009.

8. The table below shows the prices and quantities of four cereals A, B, C and D used to produce a food product in years 2008 and 2010

Cereal	**2008**		**2010**	
	Quantity (kg)	**Price (sh.)**	**Quantity (kg)**	**Price (sh.)**
A	25	7500	40	16000
B	30	12000	50	15000
C	10	8000	25	10000
D	20	12000	15	12000

(a) Calculate the simple price indices for the cereals based on 2008.

(b) Calculate the value index for the production of the food product in 2010 based on 2008.

9. The table below shows the change in the cost of a representative basket of goods and services. The weighting of each item comes from the survey that was carried out to find the cost of living index. The indices are worked out for 2012 taking 2009 as the base and weights given. Calculate the weighted index for the 14 groups.

Group	Weighting	Index
1. Food	154	111.3
2. Catering	49	118
3. Alcoholic Drink	83	114.7
4. Tobacco	36	106.4
5. Housing	175	138
6. Fuel and Light	54	109
7. Household Goods	71	110
8. Household Services	41	113
9. Clothing and Footwear	73	111
10. Personal Goods and Saving	37	115
11. Motoring Expenditure	128	120
12. Travel Costs	23	126

13. Leisure Goods	47	107
14. Leisure Services	29	117

Correlation

Correlation is the method used to determine relationships between two or more variables being investigated. A set of **paired** observations from **two** random variables is called a ***bivariate distribution***. If, in a bivariate distribution, a **change** in **one** variable is **matched** by a **similar proportional change** in the **other** variable, then the technique used to measure the degree of association is called **correlation**.

Before quantifying the interdependence of two variables it is often useful to plot the paired observations on a **scatter diagram**. By inspection, if a straight line can be drawn to fit the data reasonably well, then there exists a **linear correlation** between the variables: This is achieved by drawing scatter diagrams.

Scatter diagrams

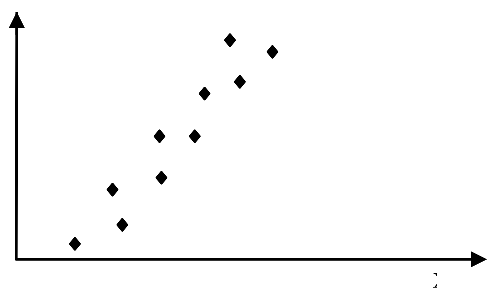
A scatter diagram (also known as a scatter plot) is a graphic representation of the relationship between two variables. It helps us visualize the apparent relationship between two variables that are plotted in pairs. Scatter diagram is a tool for analyzing relationships between two variables. One variable is plotted on the horizontal axis and the other is plotted on the vertical axis. The pattern of their intersecting points can graphically show relationship patterns.

If x and y are the two variables under consideration, then plotting the values of y against the values of x gives a scatter diagram. Scatter diagrams are used in research to investigate relationships between two variables such as cause-and- effect relationships. In scatter diagrams, the relationship or correlation between two variables may be either positive, negative, or zero.

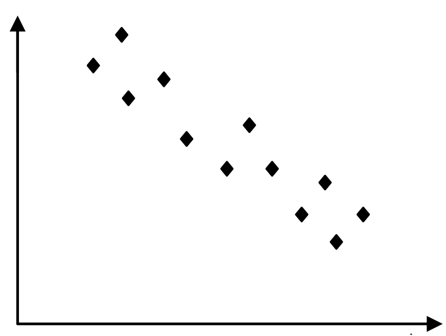
Types of correlation

The type of correlation explains the relationship between the two variables say, x and y being investigated. When a scatter diagram is plotted, the following correlation types can be found.

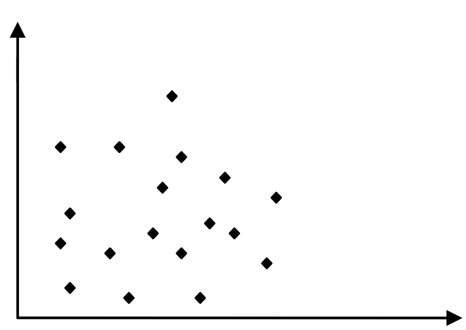
(a) **Positive correlation**: If one variable y tends to increase as the other x increases, then there is positive correlation.



(b) **Negative correlation:** If one variable y tends to decrease as the other x increases, then there is negative correlation.



(c) **Zero or No correlation:** If the data points are scattered in a shapeless pattern. You can conclude that the two variables are not correlated over the ranges for which the data was collected. Thus if there is no relationship between x and y, then there is no correlation.



Constructing a Scatter Diagram

Axes

The axes are constructed at right angles to each other. The scales for each axis (variable) are from the lowest to highest value or score. Thus, it is not necessary to have a zero for scatter diagram scales.

The next step is to select suitable scale divisions or intervals for labeling. The intervals selected are the same for each variable, if each variable is measured in the same unit. The spacing of the scale divisions on each axis needs to be large enough to accommodate the symbols.

Plotting Data

Each entry that is recorded by a symbol in the proper coordinates (or cell) always represents two numerical values, one measured on the X-axis and the other on the Y-axis. These are then plotted to give a scatter diagram.

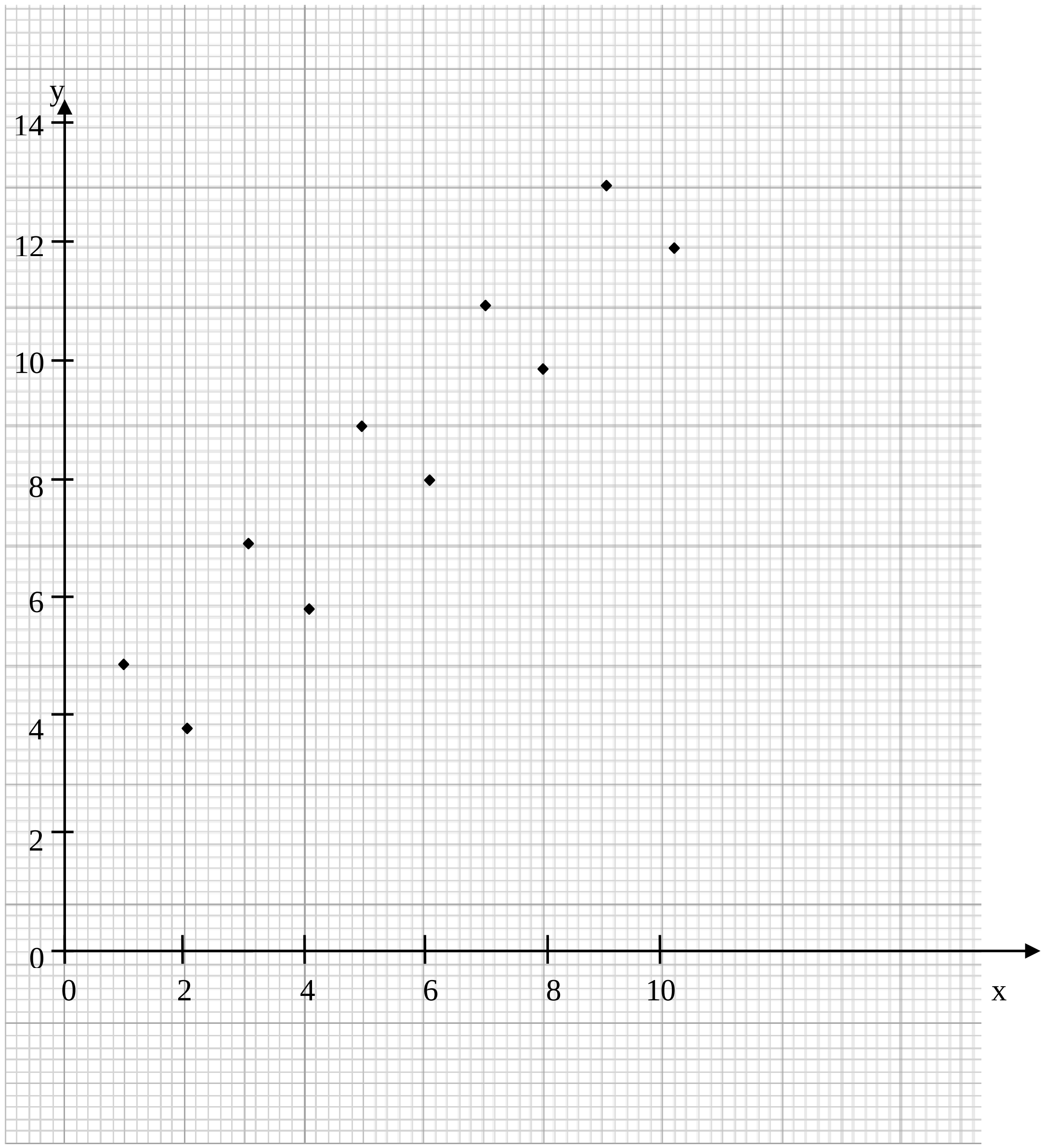
Example 1

Draw a scatter diagram for the following data.

x	1	2	3	4	5	6	7	8	9	10
y	5	4	7	6	9	8	11	10	13	12

Solution

Scatter Graph



Comment: There is a positive correlation between x and y.

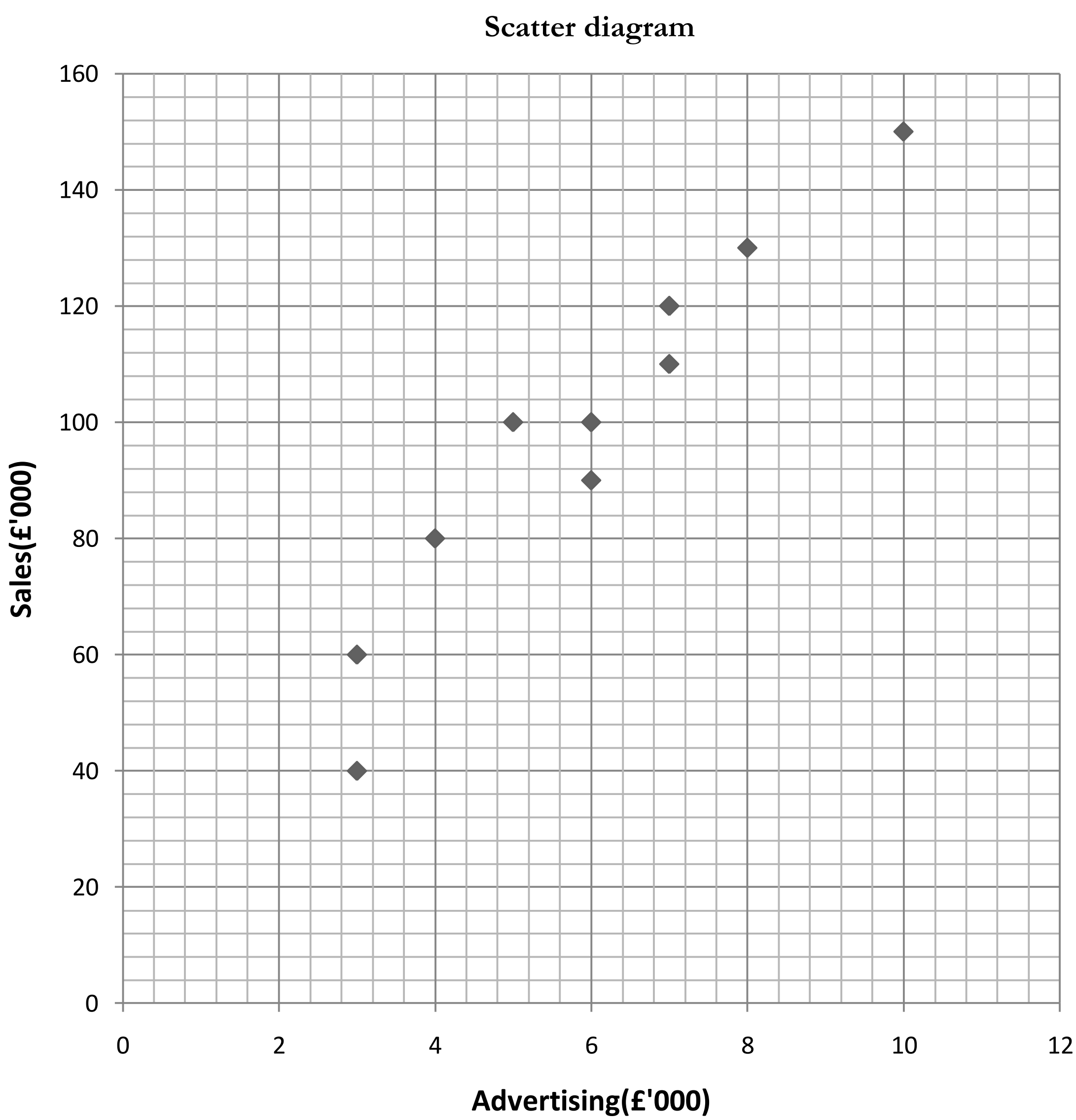
Example 2

The local bakery would like to be able to predict the demand, (Sales), for their bread from the amount they spend, (Advertising), on advertising it. The table below shows the data.

Advertising(£'000)	3	3	5	4	6	7	7	8	6	7	10
Sales(£'000)	40	60	100	80	90	110	120	130	100	120	150

Plot a scatter diagram and comment on the correlation.

Solution

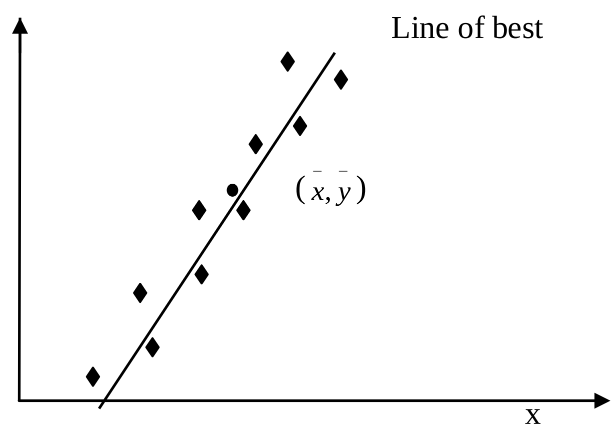


Comment: There is a positive correlation between the sales and advertising.

Drawing a line of best fit

The line of best fit is a straight line. If your scatter graph suggests that there may be a linear relation between the two variables, a line of best fit can reasonably be drawn on the scatter graph. This is the line that passes closest to each of the plotted point and a mean point $(\bar{x}, \bar{y})$.

This line can be drawn by first calculating the co-ordinates of the point $(\bar{x}, \bar{y})$ where $\bar{x} = \frac{\sum x}{n}$ and $\bar{y} = \frac{\sum y}{n}$. Then draw a line of best fit, ensuring that it passes through $(\bar{x}, \bar{y})$



The line of best fit is drawn such that it passes through the points or divides the points equally and is suitable if there is very little scatter. The line of best fit is used to estimate other data values.

Example 3

The table below shows how x varies with y.

x	10	20	30	40
y	312	509	682	865

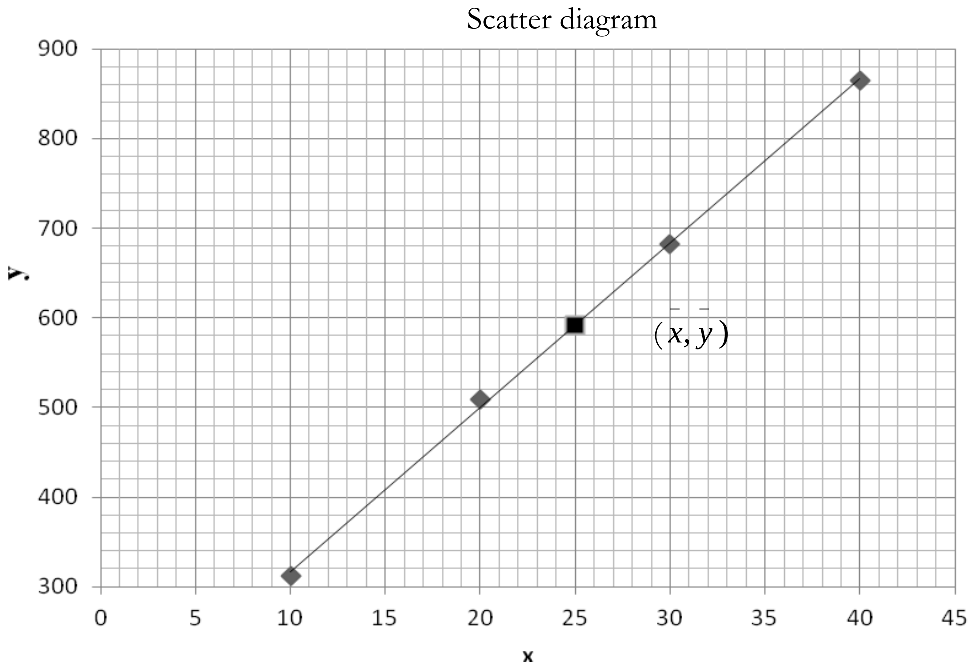
Plot these data on a scatter diagram. State the type of correlation which the data displays.

Draw on your diagram, the line of best fit.

Solution

$$\bar{x} = \frac{\sum x}{n} = \frac{10+20+30+40}{4} = 25 \quad \text{and} \quad \bar{y} = \frac{312+509+682+865}{4} = 592$$

Mean point, $(\bar{x}, \bar{y})$ = (25,592)



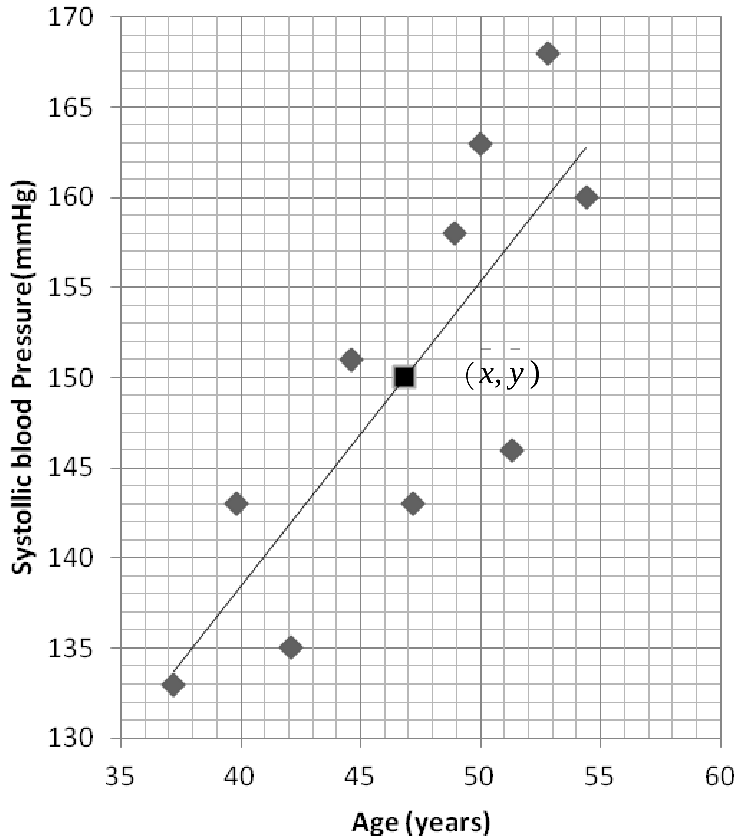
The data displays a positive correlation.

Example 4

A company doctor is investigating the possible effect of stress upon the health of the company's management employees. She suspects that employees under stress will suffer from high systolic blood pressure (mmHg). She takes a random sample of ten employees, aged between 35 and 55 years, and records their age and blood pressure:

Management employee	Age (years) (x)	Systolic blood pressure (y)
A	37.2	133
B	39.8	143
C	42.1	135
D	44.6	151
E	47.2	143
F	48.9	158
G	50.0	163
H	51.3	146
I	52.8	168
J	54.4	160

Draw a scatter diagram to show the relation ship between the age and the systolic blood pressure and best line of fit. Comment on the relationship. A scatter graph showing the data is given below:



The mean age of the employees is: $\bar{x} = \dfrac{37.2 + 39.8 + \ldots + 54.4}{10} = 46.83$

The mean systolic blood pressure is: $\bar{y} = \dfrac{133 + 143 + \ldots + 160}{10} = 150$

The scatter graph shows a relationship between blood pressure and age. Blood pressure appears to increase with age (i.e. older people tend to have higher blood pressure than young people). We say that the two variables are *positively correlated.*

Example 5

The table below shows the widths and length of seven textbook covers.

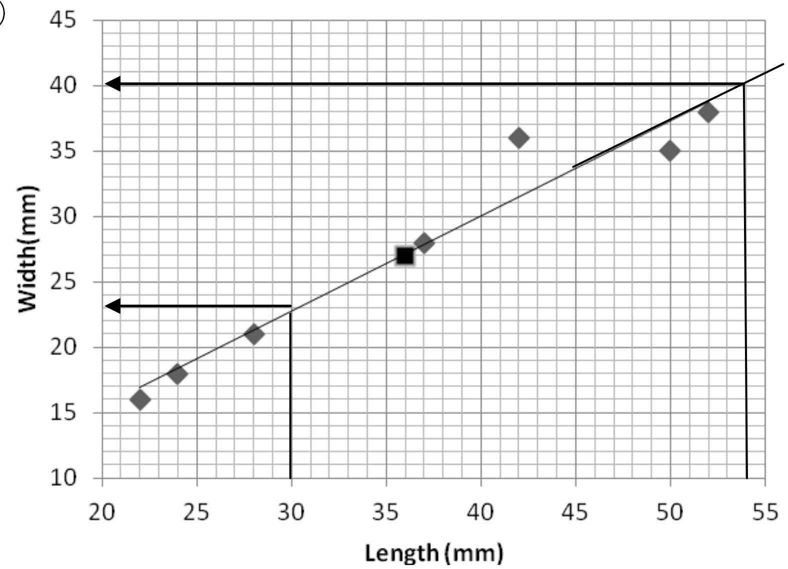
Length (mm)	42	28	50	24	37	52	22
Width (mm)	30	21	35	18	28	38	16

(a) Draw a scatter diagram and comment on the relationship between length and width.

(b) Draw the best line of fit and use it to estimate the width of length

(i) 30mm (ii) 54mm

(a)



There is a positive correlation between the length and width of the textbooks.

(b) The points lie in an almost straight line to plot the line of best fit you need to find the mean of the lengths and widths.

Mean of lengths, $\bar{x} = \dfrac{42+28+50+24+37+52+22}{7} = \dfrac{255}{7} = 36.4$

Mean of widths, $\bar{y} = \dfrac{30+21+35+18+28+38+16}{7} = \dfrac{186}{7} = 26.6$ $\quad (\bar{x}, \bar{y})$ = (36.4, 26.6)

Draw the line through the mean point and through the middle of the rest of the points this is the line of best fit.

(i) Draw a vertical line from 30mm on the length axis to the line of best fit then a horizontal line to the width axis. Read off this value, an estimate for the width is 23mm.

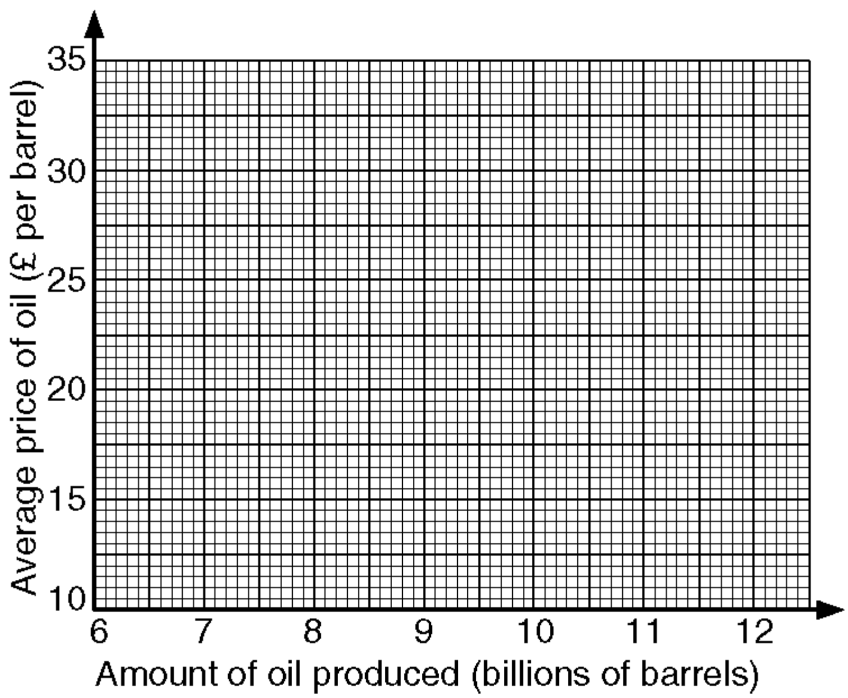
(ii) Another estimate for the width corresponding to 54mm is 40mm. This estimate is not as good as the estimate before as it is outside the range of lengths in the table.

Exercise

1. Information about oil was recorded each year for 12 years. The table shows the amount of oil produced (in billions of barrels) and the average price of oil (in £ per barrel).

Amount of oil produced (billions of barrels)	Average price of oil (£ per barrel)
7.0	34
11.4	13
10.8	19
11.3	12
9.6	23
8.2	3
7.7	30
10.9	12.5
8.0	28.5
9.9	13.5
9.2	26.5
9.4	15.5

(a) Draw a scatter graph to show the information in the table



(b) Describe the correlation between the average price of oil and the amount of oil produced.

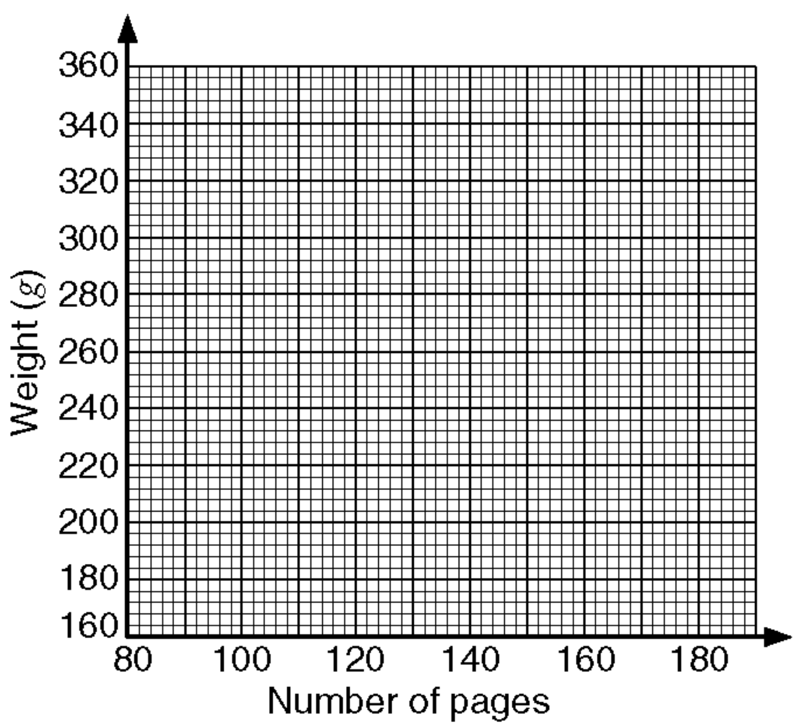
(c) Use your graph to estimate the average price of 9.5 billion barrels of oil

2. The table lists the weights of twelve books and the number of pages in each one.

Number of pages	Weight (g)
80	160
155	330
100	200
125	260
145	320
90	180
140	290
160	330
135	260
100	180
115	230
165	350

(a) Draw a scatter graph to show the information in the table.

(b) Describe the correlation between the number of pages in these books and their weights.



(c) Draw a line of best fit across the point you have plotted.

(d) Use your line of best fit to estimate

(i) the weight of a book with 130 pages
(i) the number of pages in a book which weighs 250 g.

3. The table lists the ages and prices of five cars.

Age in years	3	4	1	5	4
Price in £	3000	2500	3750	1500	2250

(a) Plot this information on the scatter graph.

(b) Describe the correlation between the ages of these five cars and their prices.

4. The table shows the temperature and the relative humidity at one place at regular intervals during one day:

Temperature °F, x	65	68	68	70	72	74	78	81	79	78	77	75
Relative humidity %, y	52	52	53	45	42	33	32	28	30	31	32	32

(a) Draw a scatter diagram and comment on the correlation.

5. In a field, the growth of the lettuces (in cm) is measured every week, together with the rainfall (in mm). The data collected are as follows:

x=Rainfall	0.5	1	1.5	2
y=growth	8	5	4	2

(a) Draw a Scatter diagram

(b) Draw the best line of fit and use it to predict what the growth would be if the rainfall were 2.5mm.

6. Draw a scatter diagram for the following data.

x	3	7	9	11	14	14	15	21	22	23	26
y	5	12	5	12	10	17	23	16	10	20	35

(a) Comment on the correlation.

(b) Draw the best line of fit.

7. Four identical money boxes contain coins of sh.100. The table below show the number of coins in each box and the combined weight of the coins and the box.

No. Of coins in box, x	10	20	30	40
Combined weights of coins and box, y	312	509	682	865

(a) Plot these data on a scatter diagram, labelling the axes clearly. State which correlation the data displays.

(b) State the coordinates of one of the points through which the line of best fit must pass.

(c) Draw on your diagram the line of best fit.

(d) Estimate from your best line of fit;

(i) The weight of an empty box

(ii) The mean weight of a single coin.

Rank correlation coefficient

The *rank correlation coefficient* tells you the strength of linear relationship between two random variables. It is used to find the type of correlation between two sets of data.

Instead of using the actual values of the variables, the ranks in which the data are ordered are obtained. We rank the values in order of size, using the numbers 1, 2, 3... n. A correlation coefficient based on the *ranks* can then be determined. The most popular of the methods used for determining a coefficient for rank correlation is **Spearman's coefficient of rank correlation**.

Spearman's coefficient of rank correlation (r_s)

$r_s = 1 - \frac{6\sum d^2}{n(n^2-1)}$ where d is the **difference** in the values of the **ranks** between pairs.

Example 1

Calculate **Spearman's rank correlation coefficient** for the following data:

x	25	27	27	28	29	31	32	33	34	34
y	45	49	51	54	52	60	60	62	63	64

Method of ranking

The sets of values for **x** and **y** are arranged in ascending order of magnitude and the ranks are assigned from the lowest to the highest. Where two or more *equal* values occur, the rank assigned to **each** is the **average** of the **positions occupied** by the **tied values**. Then we have:

x	25	27	27	28	29	31	32	33	34	34	
y	45	49	51	54	52	60	60	62	63	64	
Rank x	1	2.5	2.5	4	5	6	7	8	9.5	9.5	
Rank y	1	2	3	5	4	6.5	6.5	8	9	10	
Rank x – rank y (d)	0	0.5	-0.5	-1	1	-0.5	0.5	0	0.5	-0.5	
d^2	0	0.25	0.25	1	1	0.25	0.25	0	0.25	0.25	$\sum d^2 = 3.5$

Note: (i) The 2nd and 3rd places represent the same values of x (27), so we assign the average rank 2.5 to both these places.

(ii) The 7th and 8th places represent the same values of y (60), so we assign the average rank 6.5 to both of these places.

$$r_s = 1 - \frac{6\sum d^2}{n(n^2-1)}$$

$$r_s = 1 - \frac{6(3.5)}{10(100-1)} = 1 - \frac{21}{990} = \frac{323}{330} = 0.98(2d.p)$$

The value of r_s can lie anywhere between -1 and +1.
If *r* is positive, this indicates a positive correlation between the variables. If *r* is negative, it indicates a negative correlation. The further *r* is from 0, the stronger the association between the two random variables. In particular:

Correlation coefficient	**interpretation**
1	Perfect positive correlation
$0.7 \leq r_s < 1$	Strong positive correlation
$0.4 \leq r_s < 0.7$	Fairly positive correlation
$0 < r_s \; 0.4$	Weak positive correlation
0	No correlation

$0 > r_s$ -0.4	Weak negative correlation
$-0.4 \geq r_s > -0.7$	Fairly negative correlation
$-0.7 \geq r_s < -1$	Strong negative correlation
-1	Perfect negative correlation

Example 2

The marks of 10 students in English and Kiswahili tests marked out of 20 are as follows;

English ,x	12	8	16	11	7	10	13	17	12	9
Kiswahili , y	6	5	7	7	4	9	8	13	10	11

Calculate the correlation coefficient. Comment on your answer.

English ,x	12	8	16	11	7	10	13	17	12	9	
Kiswahili , y	6	5	7	7	4	9	8	13	10	11	
Rank x	6.5	2	9	5	1	4	8	10	6.5	3	
Rank y	3	2	4.5	4.5	1	7	6	10	8	9	
d	3.5	0	4.5	0.5	0	-3	2	0	-1.5	-6	
d^2	12.25	0	2.25	0.25	0	9	4	0	2.25	36	$\sum d^2$ **=84**

$r_s = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6(84)}{10(100-1)} = 0.49(2d.p)$ .This indicates a fairly positive correlation between the marks of English and Kiswahili.

Example 3

Two competitors rank the eight photographs in a competition as follows:

Photograph	A	B	C	D	E	F	G	H
1st Competitor	2	5	3	6	1	4	7	8
2nd Competitor	4	3	2	6	1	8	5	7

Calculate the coefficient of rank correlation for the data and comment.

In this example, the data has already been ranked.

Rank x	2	5	3	6	1	4	7	8	
Rank y	4	3	2	6	1	8	5	7	
d	-2	-2	1	0	0	-4	2	1	
d^2	4	4	1	0	0	16	4	1	$\sum d^2$ **= 30**

$$r_s = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6(30)}{8(64-1)} = 0.64(2d.p).$$

The coefficient of rank correlation is 0.64 indicating a fair positive correlation between the competitors.

Exercise

1. The table below shows the marks awarded to six children in a competition. Calculate a coefficient of rank correlation for the data.

Child	A	B	C	D	E	F
Judge 1	6.8	7.3	8.1	9.8	7.1	9.2
Judge 2	7.8	9.4	7.9	9.6	8.9	6.9

2. Seven army recruits were given two separate aptitude tests. Their orders of merit in each test were as follows;

Order of merit	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}	7^{th}
1^{st} test	G	F	A	D	B	C	E
2^{nd} test	D	F	E	B	G	C	A

Find the coefficient of rank correlation between the two orders and comment briefly on the correlation obtained.

3. The following table shows the marks of eight pupils in biology and chemistry. Rank the results and find the value of the coefficient of rank correlation.

Biology, x	65	65	70	75	75	80	85	85
Chemistry, y	50	55	58	55	65	58	61	65

4. Two adjudicators at a music competition award marks ten pianists as follows;

				Pianist						
	A	B	C	D	E	F	G	H	I	J
Adjudicator 1	78	66	73	73	84	66	89	84	67	77
Adjudicator 2	81	68	81	75	80	67	85	83	66	78

Calculate a coefficient of rank correlation for these data.

5. These are the marks obtained by 8 students in subsidiary mathematics and economics. Calculate the coefficient of rank correlation.

Subsidiary mathematics (x)	67	42	85	51	39	97	81	70
Economics (y)	70	59	71	38	55	62	80	76

6. A company doctor is investigating the possible effect of stress upon the health of the company's management employees. She suspects that employees under stress will suffer from high systolic blood pressure. She takes a random sample of ten employees, aged between 35 and 55 years, and records their age and blood pressure:

Management employee	Age (*x*)	Systolic blood pressure (*y*)
A	37.2	133
B	39.8	143
C	42.1	135
D	44.6	151
E	47.2	143
F	48.9	158
G	50.0	163
H	51.3	146
I	52.8	168
J	54.4	160

Calculate a coefficient of rank correlation between the age and systolic blood pressure and comment on the value obtained.

Probability theory

Probability theory is the branch of mathematics concerned with prediction, uncertainty. It was developed from the theory of games of chance and gambling. It prays a very important role in astronomy, physics, chemistry, engineering, economics, business, social science, psychology and research.

Experimental probability

Probabilities can be estimated from experiments. An **experiment** is a situation involving chance or probability that leads to results called outcomes. An **outcome** is the result of a single trial of an experiment while an **event** is one or more possible outcomes of a random experiment.

Examples of outcomes and events

(a) One toss of a coin results in the outcomes head or tail $\{H, T\}$. One event of this experiment is obtaining a head.

(b) One throw of an ordinary die results in the outcomes $\{1, 2, 3, 4, 5, 6\}$. An example of an event of this experiment is obtaining a number greater than 4.

(c) Two tosses of a coin result in the outcomes $\{HH, HT, TH, TT\}$. An example of an event of this experiment obtaining two heads.

(d) A spinner has 4 equal sectors colored yellow, blue, green and red. The possible outcomes are landing on yellow, blue, green or red. One event of this experiment is landing on blue.

Events are denoted by Capital letters. Using example (b) we can let A be an event "obtaining a number greater than 4", then A= (5, 6).

Experimental probability is therefore an estimate of the probability of an event. It is called *experimental* because you perform activities or experiments to find the number of times a certain event actually happened after repeating the experiment a certain number of times.

$$\text{Experimental probability} \approx \frac{\textit{number of times a certain event happens}}{\textit{total number of trials}}$$

Suppose you have a bag filled with marbles of different colors. You pull out 12 marbles without replacing them, and 5 of those marbles are red. What is the experimental probability of getting another red marble the next time you pull a marble from the bag?

$$P(\text{red}) \approx \frac{\textit{number of red marbles pulled from bag}}{\textit{total number of marbles pulled from bag}} = \frac{5}{12}$$

The probability of the next marble being red is $\frac{5}{12}$.

Probability Defined

Every random event has a different number of possible outcomes. The probability of an event is a value that describes the chance or likelihood that the event will happen or the chance or likelihood that the event will end with a particular outcome.

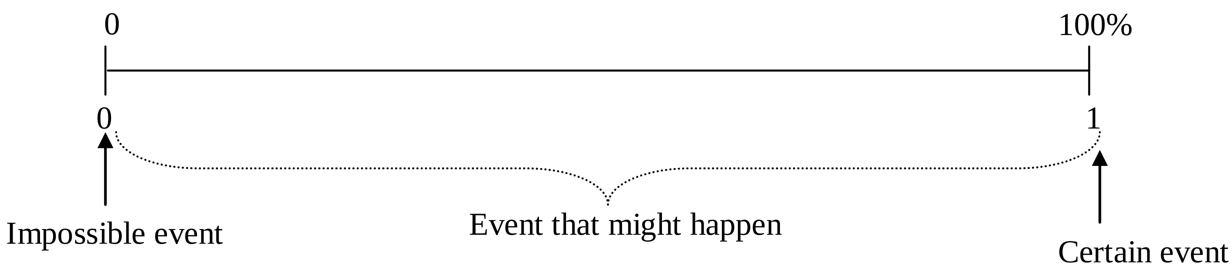
We use the notation P(A) to denote the probability that event "A" will happen.

Certain, Impossible and Uncertain Events

The probability of an event that is certain to happen is 100% or 1, while the probability of an event that is impossible is simply 0% or 0. Any other event that has some possibility of happening will therefore have a probability that lies somewhere between 0 and 1 (or between 0% and 100%).

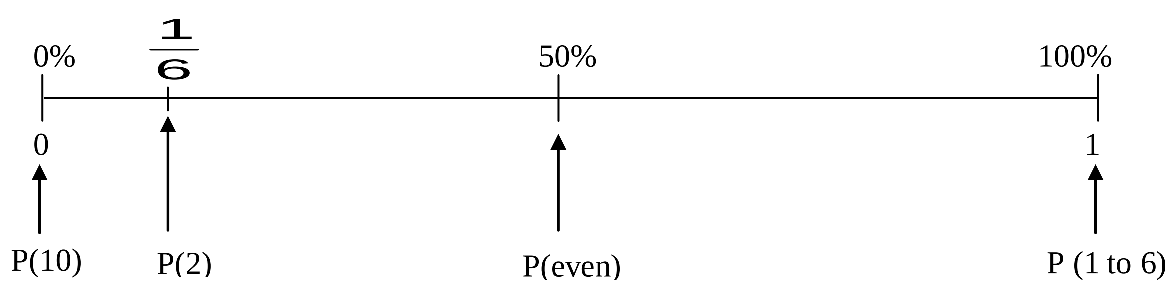
In other words, the probability of any event can be expressed as a fraction lying between 0 and 1.

Since an uncertain event has a probability that lies between 0 and 1, the higher the chance of the outcome, the closer the probability of that outcome will lie to 1.



For example, suppose a dice is rolled; the outcomes are (1, 2, 3, 4, 5, 6)

- P(dice will land on a number from 1 to 6) = certain or "1" or 100%
- P(dice will land on 10) = impossible or "0" or 0%
- P(dice will land on an even number) = $\frac{3}{6}$ = 50%
- P(dice will land on 2) = $\frac{1}{6}$



Different Ways of Describing the Probability of an Event

Hopefully you have seen from the section that there are three main ways in which we can choose to describe the probability of an event:

1. As a fraction − i.e. a value that lies between 0 and 1.
2. As a decimal − i.e. the result of computing the fraction;
3. As a percentage – i.e a value that lies between 0% and 100%

For Example:

The probability of a fair coin landing on heads is: P(heads) $= \frac{1}{2}$

We could also choose to write this as:
P(heads) = 0.5 or P(heads) = 50%

Terminologies used in probability theory

(a) Sample Space (S)

This is the set of all possible outcomes of an experiment. It is some times termed as the possibility space. Each possible outcome is called a sample point.

- For example, if an unbiased coin is tossed then the two possible outcomes are 'head' and 'tail'. The set of all possible outcomes is therefore $\{H, T\}$. This is called the sample space of the experiment
- The sample space of flipping two coins -{ HH, HT, TH, TT }.
- The following table illustrates a sample space for the sum obtained when rolling two dice.

	1	(1, 1)	(1,2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
Number on First die	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)
		1	2	3	4	5	6
		Number on second die					

Table of sums

First Die	Second Die					
	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

The sample space S has 36 sample points, each of which is equally likely to occur. The values in the columns represent the sums on the two dice.

If the sample space has a countable number of sample points then we denote the number of points in S by n(S). The probability of an event A which is a subset of S denoted P (A) is defined as

$$P(A) = \frac{n(A)}{n(S)}$$

(b) Intersection of events

Given A and B are two events in a sample space S, the intersection of A and B, denoted $A \cap B$ is the event containing all sample points that are both in A and B. Sometimes we use A and B for intersection.

(c) Union of events

The union of events A and B, denoted $A \cup B$ is the event containing all sample points in either A or B or both. Sometimes we use A or B for union.

(d) Compliment of events

If A is an event of a sample space S, the compliment of A denoted $\bar{A}$ or A^c is the event containing all sample points that are not in A. Sometimes we use not A for compliment.

(e) Mutually exclusive events(Disjoint Events)

Two events A and B are said to be mutually exclusive if their intersection is empty having no sample points in common (i.e. $A \cap B = \varnothing$). In this case P ($A \cap B$) =0

(f) Independence of Events

Two events are independent if one event does not affect the probability of the other event. Thus Events are *independent* when the outcome of one event has no effect on the outcome of a second event. For example throwing a die and flipping a coin are **independent events.**

For two independent events A and B, P ($A \cap B$) = P (A) ×P (B).

Example 1

The numbers 1 to 20 are each written on a card. The 20 cards are mixed together. One card is chosen at random from the pack. Find the probability that the number on the card is:

(a) Even
(b) A factor of 24
(c) Prime

Solution
We will use 'P(x)' to mean 'the probability of x'. Let S be the sample space such that S = {1, 2, 3, 4, 5, 6, 7, 8… 20}

(a) P(even) = $\dfrac{\text{number of even numbers}}{\text{total number of numbers in the pack}} = \dfrac{10}{20} = \dfrac{1}{2}$

(b) P(a factor of 24) = $\dfrac{\text{number of factors of } 24}{\text{total number of numbers in the pack}}$

The factors of 24 are: $F_{24} = \{1,2,3,4,6,8,12,24\} \quad \cap(F_{24}) = 8$

Therefore P(a factor of 24) = $\dfrac{8}{20} = \dfrac{2}{5}$

(c) Prime numbers in the pack = {2,3,5,7,11,13,17,19}

$$P(\text{Prime}) = \frac{\text{number of prime numbers in the pack}}{\text{total number of numbers in the pack}} = \frac{8}{20} = \frac{2}{5}$$

Example 2

One letter is selected at random from the word "SUBSIDIARYMATHEMATICS". Find the probability of selecting

(a) an A

(b) an E

The word "SUBSIDIARYMATHEMATICS" has a sample space of 21 letters

(a) P(selecting A)= $\dfrac{\text{number of } A^{\prime s}}{\text{total number letters}} = \dfrac{3}{21} = \dfrac{1}{7}$

(b) P(selecting E) = $\dfrac{\text{number of } E^{\prime s}}{\text{total number letters}} = \dfrac{1}{21}$

Example 3

The two sides of a coin are known as 'head' and 'tail'. Two coins are tossed at the same time. Illustrate the possible outcomes on a sample space diagram and find the probability of obtaining

(a) Two heads

(b) A head and a tail.

The sample space S for the outcomes when the two coins are tossed is as shown.

Second coin		
H	HH	TH
T	HT	TT
	H	T
	Fist coin	

S= {HH, TH, HT, TT} $\cap$(S) = 4

(a) Let A be the event ' two heads are obtained'

A= {HH} → $\cap$(A) =1

Therefore P(A) = $\frac{n(A)}{n(S)} = \frac{1}{4}$. The probability that two heads are obtained is $\frac{1}{4}$

(b) Let E be the event ' a head and a tail is obtained'

E = {HT, TH} → $\cap$(E) = 2

Therefore P(E) = $\frac{n(E)}{n(S)} = \frac{2}{4} = \frac{1}{2}$

Example 4

A black die and a white die are thrown at the same time. Display all the possible outcomes. Find the probability of obtaining:

(a) A total of 5,

(b) A total of 11,

(c) A 'two' on the black die and a 'six' on the white die.

Z	1	(1, 1)	(1,2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)

	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)
		1	2	3	4	5	6
		Number on white die					

There are 36 possible outcomes, shown above hence $n(S) = 36$

(a) Let A be an event ' obtaining a total of 5 on the two dice'

A = {(1,4), (4,1), (2,3), (3,2)} $\Rightarrow$ $n(A) = 4$

P(A) = $\frac{n(A)}{n(S)} = \frac{4}{36} = \frac{1}{9}$

(b) Let B be an event ' obtaining a total of 11 on the two dice

B = {(6,5), (5,6)} $\Rightarrow$ $n(B) = 2$

P(B) = $\frac{n(B)}{n(S)} = \frac{2}{36} = \frac{1}{18}$

(c) Let C be an event ' obtaining a 'a two' on the black die and a 'six' on the white die'

C={(2,6)} $\Rightarrow$ $n(C) = 1$

P (C) = $\frac{n(C)}{n(S)} = \frac{1}{36}$

Example 5

A fair coin is tossed and a fair die is rolled. Find the probability of obtaining a 'head' and a 'six'.

Solution

When a coin is tossed once, the sample space is: S = {H, T} where H denotes a 'head' and T a 'tail'. So P(H) = P(T) = ½.

Similarly the sample space when a die is tossed once is: S = {1, 2, 3, 4, 5, 6} So $P(six) = \frac{1}{6}$

The two events are independent. Therefore, P (head and six) = $\frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$

Exercise

1. An ordinary die is thrown. Find the probability that the number obtained

   (a) is s multiple of 3, (b) is less than 7, (c) is a factor of 6

2. A bag contains 6 red balls and 4 green balls

   (a) Find the probability of selecting at random

      (i) a red ball (ii) a green ball

   (b) One red ball is removed from the bag. Find the new probability of selecting at random:

      (i) a red ball (ii) a green ball

3. From a set of cards numbered 1 to 20 a card is drawn at random. Find the probability that the number (a) is divisible by 4, (b) is greater than 15 ,(c) is divisible by 4 and greater than 15

4. One letter is selected at random from the word 'UNNECESSARY'. Find the probability of selecting:
   (a) an R (b) an E (c) an O

5. Two ordinary dice are thrown. Find the probability that (a) the sum on the two dice is 3, (b) the sum on the two dice exceeds 9, (c) the two dice show the number.

6. An ordinary die and a fair coin are thrown together. Show the possible outcomes on a sample space diagram and find the probability that (a) a head and a 2 are obtained, (b) a tail and a 7 are obtained, (c) a head and an even number are obtained.

7. The letters of the word 'INDEPENDENCE' are written on individual cards and the cards are put into a box. A card is selected at random from the box. Find the probability of obtaining : (a) a card with letter E (b) a card with letter N

8. Two tetrahedral dice (4 sided), each with faces labeled 1, 2, 3 and 4 are thrown. The score is the sum of the two numbers on which the dice land. Find the possibility space and the probability of each element of the space.

9. One ball is selected at random from a bag containing 12 balls of which x are white.

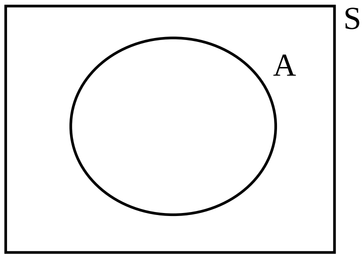
(a) What is the probability of selecting a white ball? (b) When a further 6 white balls are added the probability of selecting a white ball is doubled. Find x.

10. A bead is drawn from a container containing 10 red, 15 black, 5 green and 10 yellow beads. Find the probability that the bead is

    (a) black, (b) not green, (c) red or black, (d) not blue.

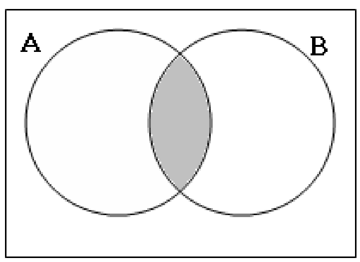
Venn diagrams

Probabilities can be illustrated on a Venn diagram. The rectangle represents the entire sample space, and the circle represents the event A, as shown below.

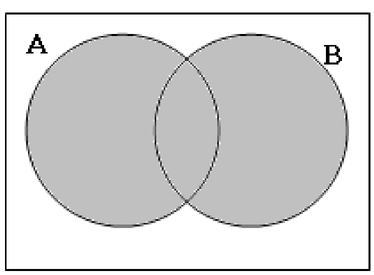


Combinations of probabilities can be shown on a Venn diagram, as follows;

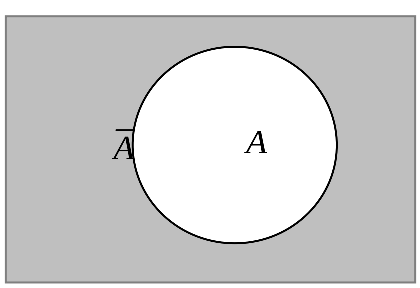
(i) A and B, $A \cap B$. This is the overlap of the regions corresponding to A and B as shown by the shaded region in the figure below.



(ii) A $\cup$ B is the region of points in either the A region or the B region (or both). Note that the word or is inclusive. 'A or B' means 'A or B or both', as shown by the shaded region below.



(iii) $\bar{A}$ is the region of points that are not in A, as shown by the shaded region below.



Probability Laws and Notations in relation to set theory

Let a random experiment have sample space S. Any assignment of probabilities to events must satisfy the following basic laws of probability.

1. Probabilities are real numbers on the interval from 0 to 1 i.e $0 \leq P(A) \leq 1$ for any event A and $P(S) = 1$

2. **Complementary law:**

   $P(A) + P(\bar{A}) = 1 \quad \Rightarrow \quad P(\bar{A}) = 1 - P(A)$

3. **Additive law:**

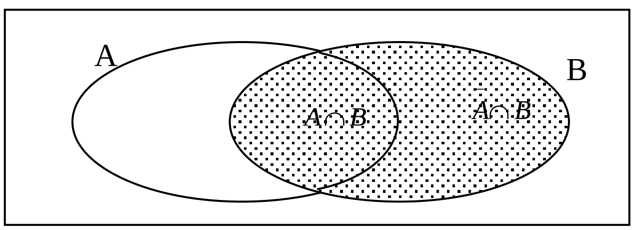
   For any two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

   If A and B are mutually exclusive $P(A \cap B) = 0$ and thus $P(A \cup B) = P(A) + P(B)$

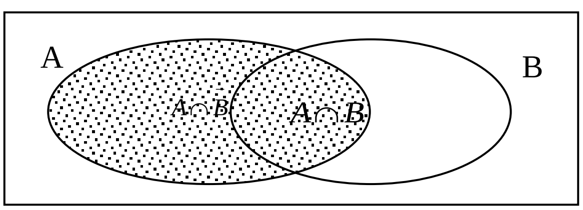
4. **Law of total probability:**

   For two events A and B where A, $\bar{A}$ and B, $\bar{B}$ are complementary then

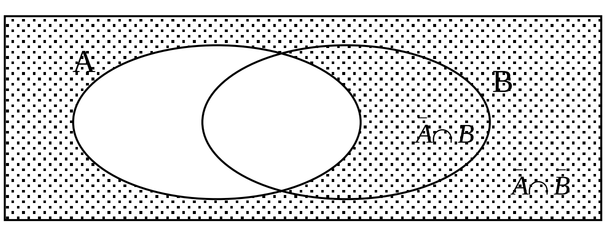
   (i) $P(B) = P(A \cap B) + P(\bar{A} \cap B)$



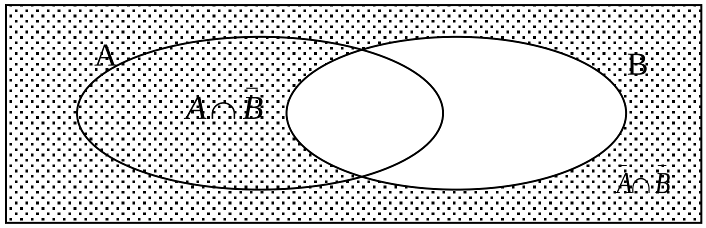
   (ii) $P(A) = P(A \cap B) + P(A \cap \bar{B})$



   (iii) $P(\bar{A}) = P(\bar{A} \cap B) + P(\bar{A} \cap \bar{B})$



(iv) $P(\bar{B}) = P(A \cap \bar{B}) + P(\bar{A} \cap \bar{B})$



5. **De - morgan's laws**

   For any two events A and B

   (i) $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$

   (ii) $P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B})$

6. **Multiplicative law for independent events** – the **and r**ule.

   If events A and B are independent, $P(A \cap B) = P(A) \times P(B)$

Contingency table

A contingency table provides a way of portraying data that can facilitate calculating probabilities. We use a contingency table to represent the probabilities of two events, A and B, which may or may not be independent. The contingency table might look like this:

Event	A	$\bar{A}$	
B	$A \cap B$	$\bar{A} \cap B$	$P(B)$
$\bar{B}$	$A \cap \bar{B}$	$\bar{A} \cap \bar{B}$	$P(\bar{B})$
	$P(A)$	$P(\bar{A})$	1

Some important relationships are visible in the contingency table. In particular:

1. $P(A) = P(A \cap B) + P(A \cap \bar{B})$.
2. $P(B) = P(A \cap B) + P(\bar{A} \cap B)$.
3. $P(\bar{A}) = P(\bar{A} \cap B) + P(\bar{A} \cap \bar{B})$.
4. $P(\bar{B}) = P(A \cap \bar{B}) + P(\bar{A} \cap \bar{B})$.
5. $P(A) + P(\bar{A}) = 1$
6. $P(B) + P(\bar{B}) = 1$

These relationships can easily be obtained by constructing a contingency table.

Example 1

Events A and B are such that P(A) $=\frac{19}{30}$, P(B) $=\frac{2}{5}$ and $P(A\cup B)=\frac{4}{5}$. Find $P(A\cap B)$.

From $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$$\frac{4}{5}=\frac{19}{30}+\frac{2}{5}-P(A\cap B)$$

$$\Rightarrow \quad P(A\cap B)=\frac{19}{30}+\frac{2}{5}-\frac{24}{30}=\frac{7}{30}$$

Example 2

Events A and B are such that P(A) = 0.5, P(B) =0.7 and P $(A\cap B)$ = 0.3. Find

(i) $P(A\cup B)$ (ii) $P(A\cap \bar{B})$

(i) $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$P(A\cup B)$ = 0.5 + 0.7 – 0.3

= 0.9

(ii) From P(A) = P$(A\cap B)$ + P$(A\cap \bar{B})$

$$\Rightarrow \quad P(A\cap \bar{B})=P(A)-P(A\cap B)$$

= 0.5 – 0.3 = 0.2

Example 3

Events A and B are such that P(A) = $\frac{1}{3}$ and P$(A\cap B)$ = $\frac{1}{12}$. If A and B are independent events, find **(a)** P(B), (b) $P(A\cup B)$

(a) Since A and B are independent events $P(A\cap B)=P(A)\times P(B)$

$$\frac{1}{12}=\frac{1}{3}P(B)$$

$$P(B)=\frac{1}{4}$$

(b) From $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$$\Rightarrow P(A\cup B)=\frac{1}{3}+\frac{1}{4}-\frac{1}{12}=\frac{1}{2}$$

Example 4

Given that A and B are mutually exclusive events such that P(A) = 0.5, $P(A\cup B)$ = 0.9, find

(i) $P(\bar{A}\cap\bar{B})$ (ii) $P(\bar{A}\cup B)$

Solution

(i) $P(\bar{A}\cap\bar{B}) = P(\overline{A\cup B})$ But $P(\overline{A\cup B})$ = 1 - $P(A\cup B)$

= 1 – 0.9 = 0.1

(ii) $P(\bar{A}\cup B) = P(\bar{A}) + P(B) - P(\bar{A}\cap B)$ But $P(\bar{A})$ = 1 – P(A) = 1 – 0.5 = 0.5

Since A and B are mutually exclusive $P(A\cup B)$ = P (A) + P (B) and $P(A\cap B)$ =0

P (B) = 0.9 – 0.5 = 0.4

Also $P(\bar{A}\cap B) = P(B) - P(A\cap B)$ = 0.4 – 0 = 0.4

$\Rightarrow$ $P(\bar{A}\cup B)$ = 0.5 + 0.4 – 0.4 = 0.5

Exercise

1. For events A and B it is known that P(A) $=\frac{2}{3}$, $P(A\cup B)=\frac{3}{4}$ and $P(A\cap B)=\frac{5}{12}$. Find P(B)
2. In a group of 30 students all study at least one of the subjects physics and biology. 20 attend the physics class and 21 attend the biology class. Find the probability that a student chosen at random studies both physics and biology.
3. For events A and B it is known that P(A) = P(B) and $P(A\cap B)$ = 0.1 and $P(A\cup B)$ =0.7. Find P(A).
4. The probability that a boy in S.5 class is in the football team is 0.4 and the probability that he is in the volleyball team is 0.5. If the probability that a boy in the class is in both teams is 0.2, find the probability that a boy chosen at random is in the football or volleyball team.
5. Given that $P(\bar{A})=\frac{2}{3}$, $P(B)=\frac{1}{2}$ and $P(A\cap B)=\frac{1}{12}$, find $P(A\cup B)$.
6. If events A and B are such that they are independent and P(A) = 0.3, P(B) = 0.5, find

   (a) $P(A\cap B)$, (b) $P(A\cup B)$. Are events A and B mutually exclusive?

7. Events A and B are such that P(A) = 0.4 and P(B) = 0.25. If A and B are independent events, find (a) $P(A\cap B)$, (b) $P(A\cap\bar{B})$, (c) $P(\bar{A}\cap\bar{B})$

8. The probability of two independent events A and B occurring together is $\frac{1}{8}$. The probability that either or both events occur is $\frac{5}{8}$. Find the

(i) P (A)
(ii) P (B)

9. If two events A and B are independent and 3 $P(A\cup B)$ = 5P(B) = 4P(A).Find

(i) P(A)

(ii) $P(\bar{A}\cap\bar{B})$

10. The events A, B and C are mutually exclusive such that P(A)= 0.3, P(B) =0.2 and P(C)= 0.3. Find (i) $P(A\cup C)$ (ii) $P(A\cup B)^C$

11. The probability that a student X can solve a certain problem is $\frac{2}{5}$ and that student Y can solve it is $\frac{1}{2}$. Find the probability that the problem will be solved if both X and Y try to solve it independently

Probability Situations

The OR situation.

This deals with the probability of either one or the other or even both events occurring. If A and B are two events, the probability that either A or B or even both occur is denoted by $P(A\cup B)$ where $P(A\cup B)=P(A)+P(B)-P(A\cap B)$. If the two events are mutually exclusive, the probability that A or B will occur equals the sum of their probabilities. i.e. P (A or B) = $P(A\cup B)$ =P(A) + P(B). **'Or Law**' in probability means that the **probabilities** are **added**. We must take care to use it only when the two events cannot occur at the same time.

The AND situation

This deals with the probability of two events A and B occurring together. **'And Law'** in probability means that the **probabilities** are **multiplied**. P (A and B) = $P(A\cap B)$. For independent events, P (A and B) = P (A) × P (B)

Example 1

One ball is selected at random from a bag containing 5 red balls, 2 yellow balls and 4 white balls. Find the probability of selecting a red ball or white ball.

Solution

The two events are exclusive.

P (red ball or white ball) = P (red) + P (white)

$$= \frac{5}{11} + \frac{4}{11} = \frac{9}{11} .$$

Example 2

John, Paul and mark compete in a 100m race. The probability that John wins is 0.3, the probability that Paul wins is 0.2 and the probability that Mark wins is 0.4. Find the probability that

(a) John or Mark wins

(b) Neither John nor Paul wins.

Solution

Since only one person can win, the events are mutually exclusive.

(a) P(John or Mark wins) = P(John wins) + P(Mark Wins)

= 0.3+ 0.4

= 0.7

(b) P(neither John nor Paul wins) = $P(\textit{John or Paul wins})^C$

= 1- P(John or Paul wins)

= 1- (0.3+0.2)

= 0.5

Example 3

A coin and a die are thrown together. Find the probability of obtaining

(a) a head

(b) a number greater than 4

(c) a head and a number greater than 4

(d) a head or a number greater than 4

Solution

The sample space S when a coin is thrown is S= {H,T}

Let A be the event ‘ a head is obtained’, so n(A)=1 and n(S) =2

(a) $P(A) = \frac{1}{2}$.

The sample space when a die is thrown is S={1, 2, 3, 4, 5, 6}

Let B be the event ‘a number greater than 4 is obtained’, so B={5, 6} n(B) = 2 and n(S) = 6

(b) $P(B) = \frac{2}{6} = \frac{1}{3}$.

(c) P(head and a number greater 4) = $P(A \text{ and } B)$

Since A and B are independent = P(A) × P(B)

$$= \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}.$$

(d) P(a head or a number greater than 4) = P(A or B) = $P(A \cup B)$.

Since A and B are not mutually exclusive $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

$$= \frac{1}{2} + \frac{1}{13} - \frac{1}{6} = \frac{8}{12} = \frac{2}{3}$$

Example 4

In a group of 40 students, 14 are girls. Of the girls 8 wear sweaters and 4 out of the 26 boys wear sweaters. What is the probability that a student chosen at random from the group is a girl or someone who wears a sweater?

Solution

Let G be the event ‘the person chosen is a girl’ and S be the event ‘the person chosen wears a sweater’.

$P(W) = \frac{14}{40} = \frac{7}{20}$, $P(G) = \frac{12}{40} = \frac{6}{20}$, P(W and G) = $P(W \cap G) = \frac{8}{40} = \frac{4}{20}$

$P(W \text{ or } G) = P(W \cup G) = P(W) + P(G) - P(W \cap G)$

$$= \frac{7}{20} + \frac{6}{20} - \frac{4}{20} = \frac{8}{12} = \frac{9}{20}.$$

EXERCISE

1. An ordinary die is thrown. Find the probability that the number obtained is

   (a) even, (b) prime, (c) even and prime) (d) even or prime

2. A and B are independent events and P(A) = 0.3, P(B) = 0.75. Find the probability that

   (a) Both A and B occur

   (b) A or B occurs

3. The probability that a student in a S.5 class is left – handed is. $\frac{1}{6}$. From the class of 15 girls and 5 boys a student is chosen at random. Assuming that 'left – handedness' is independent of the sex of a student, find the probability that a student chosen is a boy or is left – handed.

4. The probability of a student in S.5 getting an A in Subsidiary mathematics is 0.24 and that of getting a B in Economics is 0.28. What is the probability that a randomly selected student from this class will get an A or a B?

5. A fair die is thrown twice. Find the probability of obtaining a 4 on the first throw and an odd number on the second throw.

6. A bag contains 10 red balls, 5 blue balls and 7 green balls. Find the probability of selecting at random

   (a) a red ball, (b) a green ball, (c) a blue or a red ball, (d) a red or a green ball.

7. A box contains red, green, blue and yellow counters. The table shows the probability of getting each colour.

Colour	Red	Green	Blue	Yellow
Probability	0.4	0.25	0.25	0.1

   A counter is taken from the box at random. What is the probability of getting a red or blue counter?

8. Aggie and Rose both try to score a goal in netball. The probability that Aggie will score a goal on the first try is 0.65. The probability that Rose will score a goal on the first try is 0.8.

   (a) Work out the probability that Aggie and Rose will both score a goal on their first tries.

   (b) Work out the probability that neither Aggie nor Rose will score a goal on their first tries.

9. Samantha takes examinations in maths and English. The probability that she passes maths is 0.7. The probability that she passes English is 0.8. The results in each subject are independent of each other. Calculate the probability that

   (i) Samantha passes both subjects;

   (ii) Samantha passes maths and fails English.

The conditional Probability

This is the probability that an event will occur given that another event has already occurred.

For example: A certain school has male and female teachers who teach Arts and Science subjects. Suppose that one teacher is selected at random and it is known that the teacher is a female, what is the probability that the teacher teaches a science subject? So this is the probability of selecting a teacher who teaches a science subject given that the teacher is a female.

If A and B are two events, where P (A) ≠ 0 and P(B) ≠ 0, then the probability of A, given that B has occurred is denoted P(A/B). This is read this as 'the probability of A, given B'

$$P\ (A/B) = \frac{P(A \cap B)}{P(B)}$$

It follows that the probability of B, given A has occurred is denoted P (B/A) and is given by;

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \quad \text{Since } P(B \cap A) = P(A \cap B)$$

These results can be written as $P(A \cap B) = P\ (A/B) \times P\ (B) = P\ (B/A) \times P\ (A)$

Note:

(i) If A and B are mutually exclusive events then, since $P(A\cap B)=0$ and P (A) $\neq 0$ and P(B) $\neq 0$, it follows that P(A/B)= 0 and P(B/A)=0.

(ii) If events A and B are independent, $P(A\cap B)$ = P (A) ×P (B). Thus P(A/B) = P(A) and P(B/A) = P(B)

Example 1

Given that P (H) = $\frac{13}{52}$ and $P(H\cap G)=\frac{3}{52}$. Find P (G/H).

Solution

$$P(G/H)=\frac{P(H\cap G)}{P(H)}$$

$$=\frac{\frac{3}{52}}{\frac{13}{52}}=\frac{3}{13}$$

Example 2

The probability of selecting a student in a certain high school who does Geography is 0.2. The probability selecting a student who does Subsidiary mathematics and Geography is 0.03. Find the probability of selecting a student who does subsidiary mathematics given that he/she does Geography.

Solution

Let G be the event ‘a student selected does Geography’

Let S be the event ‘a student selected does Subsidiary mathematics’

$\Rightarrow P(G)=0.2, P(S \text{ and G}) = \text{P(S}\cap\text{G)=0.03}$

Thus $$P(S/G)=\frac{P(S\cap G)}{P(S)}$$

$$=\frac{0.03}{0.2}=0.15$$

Example 3

`If A and B are events such that $P(B)=\frac{1}{6}, P(A\cap B)=\frac{1}{12}$ and $\text{P(B/A)}=\frac{1}{3}$. Calculate

(i) P(A), (ii) P(A/B) , (iii) P(A/ $\bar{B}$).

Solution

(i) Using $P(B/A) = \frac{P(A \cap B)}{P(A)} \Rightarrow$ P(A) $= \frac{P(A \cap B)}{P(B/A)}$

$$= \frac{\frac{1}{12}}{\frac{1}{3}} = \frac{1}{4}$$

(ii) $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{2}$

$P(A/\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})}$ But P(A $\cap \bar{B}) = P(A) - P(A \cap B)$

$$= \frac{1}{4} - \frac{1}{12} = \frac{1}{6}$$

(iii) Also P($\bar{B}) = 1 - P(B)$

$$= 1 - \frac{1}{6} = \frac{5}{6}$$

$\Rightarrow$ P(A/$\bar{B}$) = $\frac{\frac{1}{6}}{\frac{5}{6}} = \frac{1}{5}$

Example 4

A card is picked at random from a pack of 20 cards numbered 1, 2, 3, …, 20. Given that the card shows an even number, find the probability that it is a multiple of 4.

Solution

The sample space, S= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20} $\Rightarrow$ n(S) = 20

Let E be the event 'the card picked shows an even number'

Let M be the event 'the card picked shows a multiple of 4'

E= {2, 4, 6, 8, 10, 12, 14, 16, 18, 20} n(E) = 10 $\Rightarrow P(E) = \frac{n(E)}{n(S)} = \frac{10}{20} = \frac{1}{2}$

M = {4, 8, 12, 16, 20} n(M) =5 $\Rightarrow P(M) = \frac{n(M)}{n(S)} = \frac{5}{20} = \frac{1}{4}$

M $\cap$E) = {4, 8, 12, 16, 20} n (M $\cap$E) = 5 $\Rightarrow P(M \cap E) = \frac{n(E \cap M)}{n(S)} = \frac{5}{20} = \frac{1}{4}$

Hence P(M/E) $= \frac{P(M \cap E)}{P(E)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

Exercise

1. If P(A/B) = $\frac{2}{5}$, P(B) = $\frac{1}{4}$, P(A)= $\frac{1}{3}$, find

   (a) P(B/A), (b) $P(A\cap B)$

2. A number is picked at random from the digits 1, 2, …, 9. Given that the number is a multiple of 3, find the probability that the number is

   (a) even (b) a multiple of 4

3. X and Y are two events such that P(X) = $\frac{2}{5}$, P(X/Y) = $\frac{1}{2}$ and P(Y/X) $\frac{2}{3}$=. Find

   (a) P(X $\cap$ Y), (b) P(Y), (c) P(X $\cup$ Y).

4. The two events A and B are such that P(A) = 0.6, P(B) =0.2, P(A/B) = 0.1. Calculate the probabilities that

   (i) both of the events occur (ii) B occurs, given that A has occurred.

5. Given P(A) =$\frac{3}{5}$, P(B) =$\frac{1}{2}$ and P($A\cap B$) =$\frac{1}{3}$

   Find:

   (i) P($A\cap \bar{B}$)

   (ii) P($A/\bar{B}$)

   (iii) P($\bar{A}\cap B$)

Probability tree diagrams

A tree diagram is a diagram that is shaped like a tree, with the branches of the diagram representing all of the different possible outcomes for the event taking place. It may be drawn to represent probabilities when only two possible have to be considered. For example tossing a coin where the outcome at each toss can only be one of the two possibilities; Heads or Tails.

Tree diagrams are especially useful when working with two or more events that are happening at the same time and when the number of possible outcomes of an event is not immediately obvious. They are used to:

(i) provide us with a way to *visually* represent all of the possible outcomes of an event so as to generate a sample space

(ii) Calculating probabilities when two events are combined.

Example 1

A husband and wife are considering having three children.

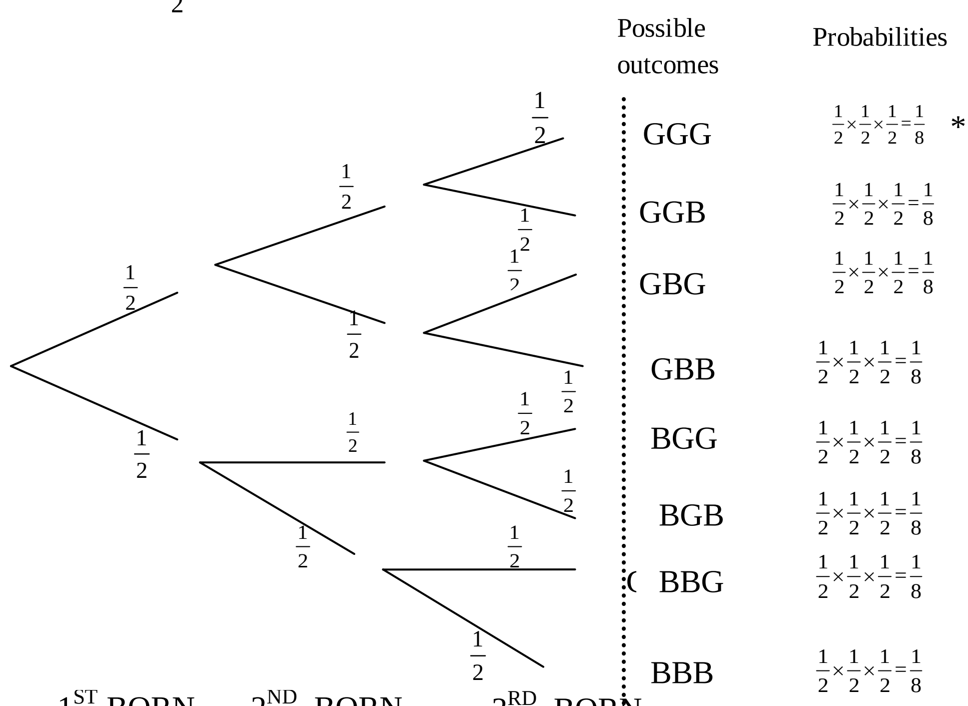
(a) Draw a tree diagram to show all of the possible combinations of boys and girls that can exist with three children and write down the sample space. (Possible outcomes)

(b) What is the probability that all the three of the children will be girls?

(c) What is the probability that 2 of the children will be boys 1 will be a girl? (in any order)

Solution

At each birth, there are two possible outcomes – Girl (G) or Boy (B). Each of these outcomes has $\frac{1}{2}$ chance of happening. This event can be represented on a tree diagram as:



(a) The sample space, S = {BBB, GGB, GBG, GBB, BGG, BGB, BBG, BBB}

NOTE:

(i) *We fill the possible outcome for each event at the end of every branch.*

(ii) *We include a branch for every possible outcome of an event.*

(iii) *We write the probability of each possible outcome on the branches of the "tree".*

RULES OF OBTAINING PROBABILITIES:

(i) When the required outcome is given by a path along the branches of a probability tree, multiply the probabilities along that path.

(ii) When the required outcome is given by more than one path in a probability tree, add the probabilities resulting from each path. This is possible when we are calculating the combination of different ways in which the two events can occur according to a set of conditions.

(b) The first path marked * involves having all the three children as girls. The probability of this event is obtained by simply multiplying the fractions on the three branches along the path.

Therefore, P (all the three children are girls) = P (GGG) = $\frac{1}{8}$.

(c) The paths marked # involve having 2 of the children as boys and 1 as a girl. To find the probability that 2 of the children will be boys and one will be a girl, we add the probabilities resulting from each path.

Therefore, P (two boys and one girl) = P (GBB) + P (BGB) + P (BBG)

$$= \frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$$

Example 2

A bag contains 5 red balls and 3 green balls. A ball is drawn at random and then replaced. Another ball is drawn. What is the probability that both balls are green?

Solution

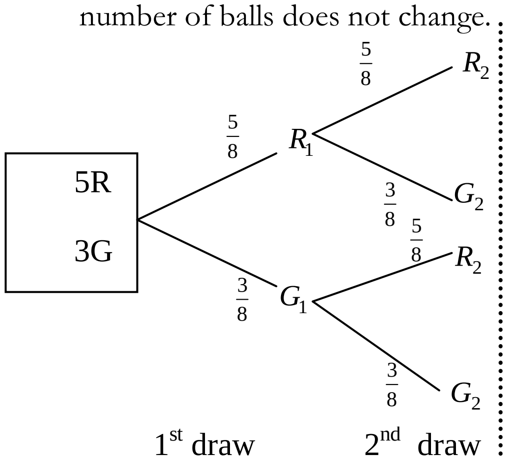
Let R_1 be the event 'a red ball is drawn first'

R_2 be the event 'a red ball is drawn second'

G_1 be the event 'a green ball is drawn first'

G_2 be the event 'a green ball is drawn second'

The tree diagram is as shown below. Since the ball is replaced after the first draw the total number of balls does not change.



$P(R_1 \cap R_2) = \frac{5}{8}\times\frac{5}{8}=\frac{25}{64}$

$P(R_1 \cap G_2) = \frac{5}{8}\times\frac{3}{8}=\frac{15}{64}$

$P(G_1 \cap R_2) = \frac{3}{8}\times\frac{5}{8}=\frac{15}{64}$

$P(G_1 \cap G_2) = \frac{3}{8}\times\frac{3}{8}=\frac{9}{64}$

P (both balls are green) = $P(G_1 \cap G_2) = \frac{3}{8}\times\frac{3}{8}=\frac{9}{64}$

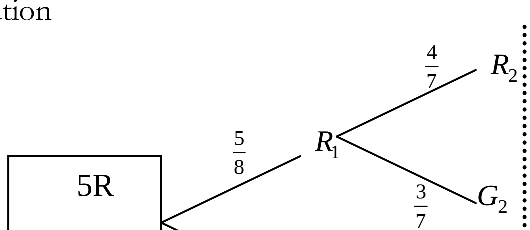
Example 3

A bag contains 5 red balls and 3 green balls. A ball is selected at random and not replaced. A second ball is then selected. Find the probability of selecting;

(a) two green balls

(b) one red ball and one green ball.

Solution



$P(R_1 \cap R_2) = \frac{5}{8}\times\frac{4}{7}=\frac{10}{28}$

$P(R_1 \cap G_2) = \frac{5}{8}\times\frac{3}{7}=\frac{15}{56}$

(a) P (two green balls) = $P(G_1 \cap G_2) = \frac{3}{8} \times \frac{2}{7} = \frac{3}{28}$

(b) P (one red, one green ball) = $P(R_1 \cap G_2) + P(G_1 \cap R_2)$

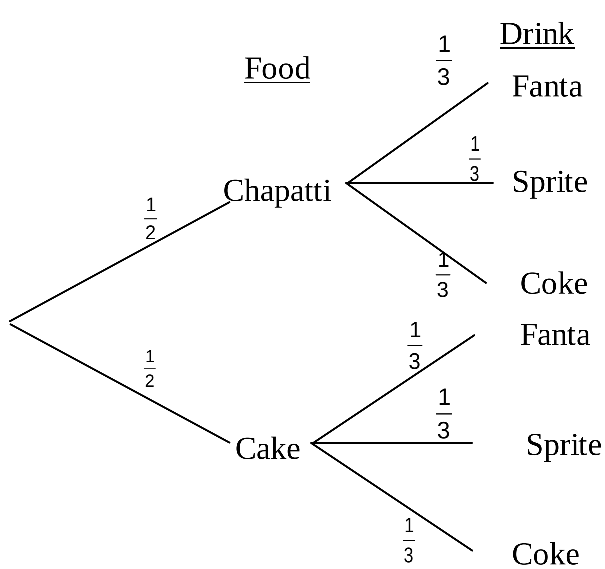
$$= \frac{15}{56} + \frac{15}{56} = \frac{15}{28}$$

Example 4

A school canteen sells Chapatti and Cakes, and Fanta, Sprite and Coke cool drinks.

(a) If a student wants to buy something to eat and drink from the canteen, write down all of the possible combinations that the student could choose.

(b) The tree diagram below shows all of the possible combinations of food and drink at this canteen.



(i) What is the probability that a student will choose a Chapatti and a Fanta?

(ii) What is the probability that the student will end up with some sort of food and a Coke?

Solution

(a) The possible combinations that a student can choose include;

{(Chapatti and Fanta), (Chapatti and Sprite), (Chapatti and Coke), (Cake and Fanta), (Cake and Sprite), (Cake and Coke)}

(b) (i) P (Chapatti and a Fanta) = $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

(ii) P (Some sort of food and a Coke) = P (Chapatti and Coke) + P (Cake and Coke)

$$= \left(\frac{1}{2} \times \frac{1}{3}\right) + \left(\frac{1}{2} \times \frac{1}{3}\right)$$

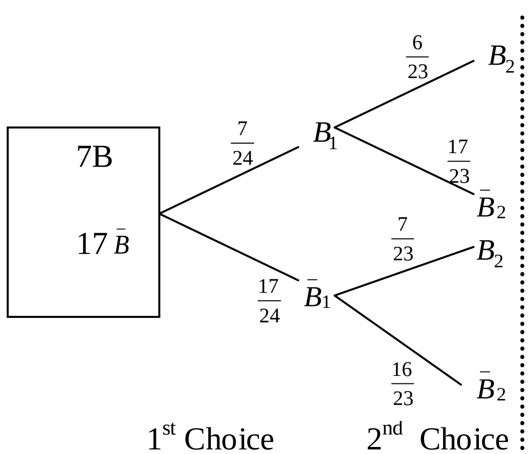
$$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Example 5

In a class of 24 girls, 7 have black hair. If 2 girls are chosen at random from the class, find the probability that (a) they both have black hair, (b) neither has black hair.

Solution

Let B be the event 'the girl chosen has black hair', then P (B) = $\frac{7}{24}$ and $P(\bar{B}) = \frac{17}{24}$. In this case $\bar{B}$ means that 'the girl chosen is not having black hair'



$$P(B_1 \cap B_2) = \frac{7}{24} \times \frac{6}{23} = \frac{7}{92}$$

$$P(B_1 \cap \bar{B}_2) = \frac{7}{24} \times \frac{17}{23} = \frac{119}{552}$$

$$P(\bar{B}_1 \cap B_2) = \frac{17}{24} \times \frac{7}{23} = \frac{119}{552}$$

$$P(\bar{B}_1 \cap \bar{B}_2) = \frac{17}{24} \times \frac{16}{23} = \frac{34}{69}$$

(a) P (both have black hair) = $P(B_1 \cap B_2) = \frac{7}{24} \times \frac{6}{23} = \frac{7}{92}$

(b) P (neither has black hair) = $P(\bar{B}_1 \cap \bar{B}_2) = \frac{17}{24} \times \frac{16}{23} = \frac{34}{69}$

Exercise

1. By drawing a tree diagram, find the possible outcomes when a coin is tossed three times.

2. The probability that a biased die falls showing a 6 is 0.2. This biased die is thrown twice.

   (a) Draw a tree diagram showing the possible outcomes and the corresponding probabilities, considering the event 'a six is thrown'.

   (b) Find the probability that exactly one six will be obtained.

3. A box contains ten green and six white marbles. A marble is chosen at random, its colour noted and it is not replaced. This is repeated once more. What is the probability that the marble chosen are of the same colour? $Ans.\left(\frac{1}{2}\right)$

4. A bag contains 8 white counters and 3 black counters. Two counters are drawn, one after the other. Find the probability of drawing one white and one black counter;

   (a) If the first counter is replaced. $Ans.\left(\frac{48}{121}\right)$

   (b) If the first counter is not replaced. $Ans.\left(\frac{24}{55}\right)$

5. A bag contains 3 black and 5 white balls. 2 balls are drawn at random one at a time without replacement. Find

   (i) The probability that the second ball is white. $Ans.\left(\frac{5}{8}\right)$

   (ii) The probability that the first ball is white given that the second is white. $Ans.\left(\frac{4}{7}\right)$

6. A box contains 3 red, 2 green and 5 blue crayons. Two crayons are randomly selected from the box with out replacement. Find the probability that:

   (i) The crayons are of the same colour. $Ans.\left(\frac{14}{25}\right)$

   (ii) At least one red crayon is selected. $Ans.\left(\frac{8}{15}\right)$

7. (a) A bag X contains 5 white balls and 3 black balls and bag Y contains 2 white balls and 3 black balls. A ball is drawn at random from each bag. Find the probability that

   (i) Both balls are white

   (ii) One ball is white and another black.

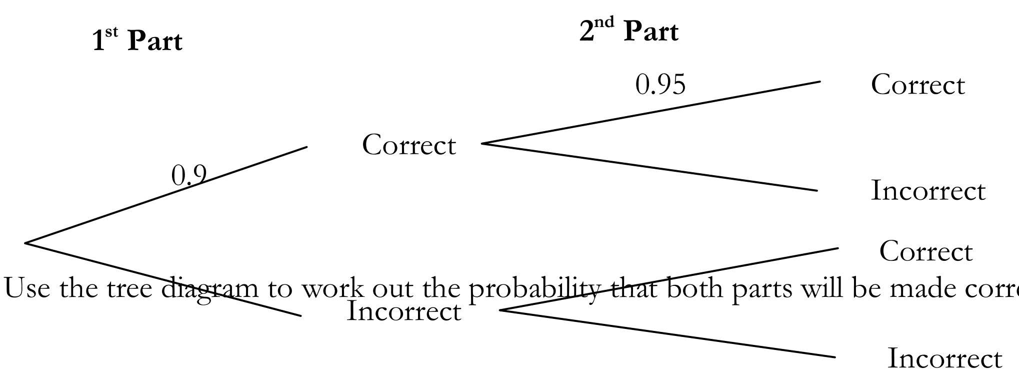
   (b) A ball is drawn at random from bag X and then put into bag Y. Then a ball is drawn at random from bag Y. Find the probability that the balls drawn from bags X and Y are of different colours.

8. A box contains 3 blue marbles and some green marbles all of identical size. Two marbles are drawn at random without replacement. If the probability of drawing two blue marbles is $\frac{3}{28}$, how many green marbles are in the box?

9. A box A contains 3 red balls and 4 black balls. A box B contains 3 red balls and 2 black balls. One box is selected at random and then from that box, one ball is selected at random. Find:
   (i) the probability that the ball is red
   (ii) the probability that the ball came from A, given that it is red.

10. A machine makes two parts which fit together to make a tool. The probability that the first part will be made correctly is 0.9. The probability that the second part will be made correctly is 0.95.

(a) Complete the tree diagram below giving the missing probabilities.



(b) Use the tree diagram to work out the probability that both parts will be made correctly.

Permutations and Combinations

Permutations

Permutations refer to all possible ways of arranging objects (different arrangements) that can be made out of a given set of objects, by taking some or all of them at a time.
Informally, a permutation of a set of objects is an arrangement of these objects into a particular order. For example there are six permutations of the set $\{1, 2, 3\}$, namely (1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2) and (3, 2, 1)

Factorial notation n!

The continued product of the first n natural numbers (i.e. the continued product of n consecutive integers beginning with one and ending with n) is denoted by the symbol n! and is read as factorial n.

Thus $n! = 1 \times 2 \times 3 \ldots \times n$.

$7! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7.$

$5! = 1 \times 2 \times 3 \times 4 \times 5.$

$3! = 1 \times 2 \times 3.$

Arrangements of objects in a row

Result1

The number of ways of arranging n unlike objects in a row or line is n!

Example 1

How many ways can the letters A, B, C, D be arranged.

Solution

The number of ways of arranging the 4 letters is $4! = 1 \times 2 \times 3 \times 4 = 24$.

When the letters A, B, C, D are considered;

The first letter can be chosen in 4 ways (either A or B or C or D),

the second letter can be chosen in 3 ways,

the third letter can be chosen in 2 ways,

the fourth letter can be chosen in only 1 way.

Therefore the number of ways of arranging the 4 letters is (4) (3) (2) (1) = 24

Example 2

Find the number of possible arrangements for five students to sit on a desk.

Solution

Number of possible arrangements of five students = $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$

Considering the five students, the 1st seat can be occupied in 5 different ways (any of the five students can occupy the 1st seat), the 2nd seat can be occupied in 4 different ways, the 3rd seat can be occupied in 3 different ways, the 4th seat can be occupied in 2 different ways and the 5th seat can be occupied in only one way. This is as shown below;

5	4	3	2	1
Seat 1	Seat 2	Seat 3	Seat 4	Seat 5

Thus there are $5 \times 4 \times 3 \times 2 \times 1 = 120$ possible arrangements.

Result 2

The number of ways of arranging in a row n objects, of which p are alike is $\frac{n!}{p!}$.

Example 3

How many ways can the letters A, A, A, D be arranged?

Solution

The number of ways of arranging 4 objects of which 3 are alike = $\frac{4!}{3!} = \frac{1\times2\times3\times4}{1\times2\times3} = 4$

Example 4

How many permutations can be made out of the letters of the word 'GOOD' taken all together?

Solution

In the given word "GOOD", there are 4 letters and the two O's are alike.

The number of permutations of 4 objects of which 2 are alike = $\frac{4!}{2!} = \frac{1\times2\times3\times4}{1\times2\times} = 12$

Result 3

The number of ways of arranging in a row n objects of which p of one type are alike, q of a second type are alike, r of the third type are alike, and so on is $\frac{n!}{p!q!r!\ldots}$.

Example 5

In how many ways can the letters of the word "STASTISTICS" be arranged.

Solution

Consider the word 'STATISTICS', there are 10 letters and S occurs 3 times

T occurs 3 times

I occurs 2 times

Therefore the number of ways $= \frac{10!}{3!3!2!} = 50400$.

Example 6

How many permutations can be made out of the letters of the word "MISSISSIPPI" taken all together?

Solution

In the given word "MISSISSIPPI" there are eleven letters, and 'S' occurs 4 times,

I occurs 4 times

P occurs 2 times

The number of permutations of eleven letters taken all at a time

$$= \frac{11!}{4! \times 4! \times 2!} = 34650 .$$

Permutation notation ^nP_r

The number of permutations of n different objects taken r at a time is usually denoted by the symbol ^nP_r where $^nP_r = \frac{n!}{(n-r)!}$

Example 7

How many permutations are there of 3 letters chosen from eight unlike letters of the word "RELATION".

Solution

The number of permutations of 8 different objects taken 3 at a time is given by;

$$^8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 336 \text{ ways}$$

Example 8

A number of four different digits is formed by using the digits 1, 2, 3, 4, 5, 6, 7 in all possible ways. Find how many such numbers can be formed.

Solution

There are 7 digits given. The number of ways of 7 objects taking 4 at a time is given by;

$$^7P_4 = \frac{7!}{(7-4)!} = \frac{7!}{3!} = 840 \text{ ways}$$

Thus the different 4 – digit numbers that can be formed are 840.

Arranging Objects in a Circular form or ring

The number of ways of arranging n unlike objects in a ring or circular form is (n – 1)! In circular permutation, we fix the position of one object and then arrange the remaining (n – 1) out of n objects in all possible ways. This can only be done in (n – 1)! ways.

For example, consider 4 people A, B, C and D who are to be seated at a round table.

The number of different arrangements of 4 people around the table is 3!

Example 9

A, B, C, D, E are four different coloured beads which are threaded on a ring. In how many ways can these threads be arranged?

Solution

There are five beads to be arranged in a circular form.

Number of ways = $(5 - 1)! = 4! = 24$

Example 10

In how many ways can 7 gentlemen and 7 ladies sit at a round table so that no two ladies are together?

Solution

The number of ways in which 7 gentlemen can be seated around a round table = $(7 - 1)! = 6!$

Since no two ladies are to be together, the ladies can be put in, one between every two gentlemen, thus there are 7 places for ladies and they are to use all of them and this can be done in 7! ways.

Hence the number of ways in which both the gentlemen as well as the ladies can be seated around a round table so that no two ladies are together = $6! \times 7! = 3{,}628{,}800$

Condition permutation

This involves getting the number of ways of arranging objects following a given number of restrictions.

Example 1

(a) How many different arrangements can be made by using all the letters in the word 'MATHEMATICS'?

(b) How many of them begin with C?

(c) How many of them begin with T?

Solution

(a) In the given word 'MATHEMATICS', there are eleven letters. 'M' occurs twice, 'A' occurs twice and 'T' occurs twice.

The number of permutations of these eleven letters taken all at a time = $\frac{11!}{2!2!2!} = 4989600$

(b) To find how many of these arrangements begin with 'C', then C takes the first position as shown;

1	2	3	4	5	6	7	8	9	10	11
C										

We have now to arrange the remaining 10 letters.

This can be accomplished in $\dfrac{10!}{2! \times 2! \times 2!} = 253,600$ ways

(c) To find how many of them begin with 'T', then T takes the first position. We are now left with 10 letters. The letter 'T' no longer occurs twice for one of the two T's has occupied the first position. We have now to arrange the remaining 10 letters. This can be done in $\dfrac{10!}{2! \times 2!} = 907,200$ ways

Example 2

In how many different ways can the letters of the word 'CONSTITUTION' be arranged? How many of these will have the letter 'N' both at the beginning and at the end?

Solution

The word 'CONSTITUTION' consists of 12 letters having O, N, I occurring two times each and T occurring three times.

The required number of different ways = $\dfrac{12!}{2! \times 2! \times 2! \times 3!} = 9,979,200$ ways

For having N both at the beginning and at the end, let us fix one N each at the beginning and at the end. Then we are left with arranging 10 letters and N no longer occurs twice.

The number of arrangements would therefore be = $\dfrac{10!}{2! \times 2! \times 3!} = 151,200$ ways

Example 3

How many arrangements of the letters of the word COMRADE can be made if the vowels are never separated.

Solution

There are 7 different letters in the word 'COMRADE' of which 3 are vowels and 4 consonants.

If vowels are never separated, we regard the vowels (o, a, e) as one letter, then we have to arrange 5 letters (o, a, e), c, m, r, d taken altogether.

We get 5! arrangements in each of which the vowels are not separated. And since in each of these arrangements keeping all the other letters in their own places, the three vowels can be arranged among themselves in 3! ways.

The total number of required arrangements = $5! \times 3! = 720$

Example 4

In how many ways can the letters of the word "ARRANGE" be arranged? How many of these arrangements are there in which

(i) The two R's come together?

(ii) The two R's do not come together?

(iii) The two R's and the two A's come together?

Solution

There are 7 letters in the word "ARRANGE" out of which 2 are A, 2 are R, 1 is N, 1 is G and 1 is E. The letters of the word may be arranged in $\frac{7!}{2! \times 2!} = 1260$ ways

(i) For the two R's to come together, let us combine these into one letter. In such a situation we have 6 letters out of which 2 are A, 1 is R, 1 is N, 1 is G and 1 is E.

Required number of arrangements = $\frac{6!}{2!} = 360$

(ii) The number of ways for two R's not to come together = total number of arrangements – number of ways when two R's come together.

Required number of arrangements = 1260 – 360 = 900

(iii) Treating both the R as well as the A as one individual letter, then there shall be 5 different letters which may be arranged in 5! = 120 ways.

Example 5

(a) How many distinguishable ways are there to rearrange the letters in the word COMBINATORICS?

(b) How many distinguishable arrangements are possible with the restriction that all vowels ("A", "I", "O") are always grouped together to form a contiguous block?

(a) There are 13 letters in the word COMBINATORICS, including two C's, two O's and two I's. So, the total number of arrangements is $\frac{13!}{2! \times 2! \times 2!}$

(b) If all five vowels are consecutive, they form a single block. Then first we need to count permutations of the consonants and one block of vowels. Given eight consonants with two C's), we have $\frac{9!}{2!}$. But every arrangement of consonants and the block of vowels can be combined with any permutation of vowels inside the block. For five vowels including two O's and two I's we have $\frac{5!}{2! \times 2!}$ possible permutations inside the block. Then by the product rule we get the answer: $\frac{9! \times 5!}{2! \times 2! \times 2!}$

Example 6

Students A, B, C, D, E, F, G, H, I, and J must sit in ten chairs lined up in a row. Answer the following questions based on the restrictions given below.

(a) How many ways can the students sit if the two students on the ends of the row have to be vowel-named students?
(b) How many ways can the students sit if no two students with vowel names can sit adjacent to each other?

Solution

(a) There are three vowels, thus for 2 students on the ends of the row to be vowel – named, this can be done in ${}^3p_2$ ways. Following those two choices, we can arrange the rest of the 8 students left in 8! ways. Thus, the total number of ways the students can sit is (${}^3p_2$) (8!).
(b) Place all seven consonants like so (C designates an arbitrary consonant):

___ C ___ C ___ C ___ C ___ C ___ C ___ C ___

Now, the empty slots (___) represent possible locations for the vowels. There are 8 places which can be filled with 3 vowels and this can be done in ${}^8p_3 =$ (8)(7)(6) ways to place the vowels. The 7 consonants can be ordered in 7! ways. Thus, there are (8)(7)(6)(7!) ways the students can sit without any vowel-named students sitting next to each other.

Example 7

How many permutations can be made out of the letters of the word INDEPENDENCE? In how many of them the vowels occur together?

Solution

The word INDEPENDENCE has 12 letters of which N occurs 3 times, D occurs 2 times and E occurs 4 times.

The number of permutations = $\frac{12!}{3! \times 2! \times 4!}$

For vowels to occur together, then we consider the 5 vowels as one letter so that we have 8 letters N, D, P, N, D, N, C, (I, E, E, E, E) to arrange of which N occurs 3 times and D occurs 2 times.

Hence the number of arrangements of the letters is $\frac{8!}{3! \times 2!}$

Again, the vowels being 5 in number of which E occurs 4 times can be arranged among themselves in $\frac{5!}{4!}$

Thus number of permutations in which vowels occur together $= \dfrac{8!}{3! \times 2!} \times \dfrac{5!}{4!}$

Exercise

1. If the letters of the word "WOMAN" be permutated, find the numbers of words that can be formed. *Ans.*120
2. Find the number of words that can be formed by considering all possible permutations of the letters of the word "FATHER". *Ans.*720
3. How many different six digit numbers can be formed from the digits of 5, 4, 8, 4, 5, 4. *Ans.*60
4. How many arrangements can be made out of the letters in the word "TERRITORY"? *Ans.*30240
5. In how many ways can 5 people sit at a round table conference? *Ans.*24
6. Nine children play a party game and hold hands in a circle. In how many different ways can this be done?
7. A circular ring has ten different beads. In how many ways can the beads be arranged along the ring?
8. In how many ways can 5 gentlemen and 3 ladies sit at a round table so that no two ladies are together? *Ans.*1440
9. In how many ways can 7 people sit at a round table? How many of these possible arrangements can a husband and wife sit together?
10. In how many ways can the letters of the word "LAUGHTER" be arranged so that the vowels may never be separated? *Ans.*4320
11. Determine all the possible arrangements of the letters KAMPALA. How many of these end with (i) A (ii) K? Hence find the probability that an arrangement ends with A or K.
12. In how many ways can the letters in the word SISTER be arranged if the vowels are not to come together?
13. In how many different ways can the letters of the word MISCELLANEOUS be arranged if the E's cannot come together?
14. In how many different ways can the letters in the word ARRANGEMENTS be arranged? Find the probability that an arrangement chosen at random begins with the letters EE.
$Ans.\left(\dfrac{12!}{(2!)^4}, \dfrac{1}{66}\right)$
15. If the letters of the word PROBABILITY are arranged at random, find the probability that the two I's are separated. $Ans.\dfrac{9}{11}$

16. In how many ways can the letters of the word FACETIOUS be arranged in a line. What is the probability that an arrangement begins with F and ends with S? *Ans.*$(9!\ ,\ \frac{1}{72})$

17. If the letters of the word ABSTEMIOUS are arranged at random, find the probability that the vowels and consonants appear alternately. *Ans.*$\frac{1}{126}$

18. A class has 8 girls and 4 boys. If the class contains 6 sets of identical twins, where each child is indistinguishable from their twin, how many different ways can the class line up to go to assembly? *Ans.*$\frac{12!}{(2!)^6}$

Combinations

Combination is the number of ways of selecting a group of objects from a given set of objects. The different groups or selections or collections that be formed out of a given set of objects by taking some or all of them at a time (without regard to order of their arrangements) are called their combinations. With combinations, the order does not matter; for example a subject combination.

If a student has eight subjects at 'O' level and he/she is to select the different combinations at 'A' level, then he/she may chose Arts or Sciences and create the different combinations of 3 subjects say; HEG(History, Economics and Geography), PCB(Physics, Chemistry and Biology), etc. In combinations PCB and BCP mean the same thing since the order in which the elements are chosen does not matter.

Example 1

List all the three letter combinations that can be formed from the five letters A,B, C, D, E and hence state the number of combinations.

Solution

ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE and CDE

Thus 10 three letter combinations can be formed.

Example 2

List all the possible three subject combinations that can be formed from the following four Arts subjects; History(H), Economics(E), Geography(G), Literature(L).

Solution

HEG, HEL, HGL, LEG

There are four possible combinations.

Combination Notation $\binom{n}{r}$ or ^nC_r

The number of combinations of n different objects taken r at a time is usually denoted by the symbol $\binom{n}{r}$ or ^nC_r read as '*the combinations of r objects from n objects*' and is given by $^nC_r = \frac{n!}{(n-r)!r!}$

Example 1

A committee of four people is formed from nine people. In how many ways can this be done?

Solution

The number of combinations of 4 people from 9 people is given by

$$^9C_4 = \frac{9!}{(9-4)!4!} = \frac{9!}{5!4!} = 126 \text{ committees}$$

Example 2

In how many ways can a hand of 4 cards be selected from an ordinary pack of 52 playing cards?

Solution

$$^{52}C_4 = \frac{52!}{(52-4)!4!} = \frac{52!}{48!4!} = 270725 \text{ ways}$$

Example 3

Four letters are chosen at random from the word RANDOMLY

(a) How many possible combinations can be formed?

(b) How many of these can be formed if all the four letters chosen are consonants and hence find the probability that all four letters chosen are consonants.

Solution

(a) Possible combinations = $^8C_4 = \frac{8!}{(8-4)!4!} = \frac{8!}{4!4!} = 70$

(b) There are six consonants, so if all the four letters are consonants

Possible combinations = $^6C_4 = \frac{6!}{(6-4)!4!} = \frac{6!}{2!4!} = 15$

Let S be the sample space n(S) = 70

Let E be the event 'four consonants are chosen'. n(E) = 15

$$P(E) = \frac{n(E)}{n(S)} = \frac{15}{70} = \frac{3}{14}$$

Example 4

A committee of three men and one woman is obtained from five men and three women. In how many ways can the members be chosen?

Solution

Three men can be selected from five men in $^5C_3$ ways and one woman can be selected from three women in $^3C_1$ ways.

Thus number of ways of selecting members = $^5C_3 \times {}^3C_1$

$$= \frac{5!}{2!3!} \times \frac{3!}{2!1!} = \frac{5!}{2!2!} = 30$$

Therefore 30 such committees can be formed.

Example 5

Three letters are selected at random from the word BIOLOGY. Find the number of selections which can be made;

(i) Without the letter O

(ii) Contains both of the letters O.

Solution

(i) Number of selections without the letter O = number of ways to choose 3 letters from B, I, L, G, Y

$= \ {}^{5}C_3 = 10.$

(ii) Number of selections with 2 letters O for example $\boxed{\text{O,}\quad}$, $\boxed{\text{O,}\quad}$ and so on = number of ways to choose 1 letter from B, I, L, G, Y

$= \ {}^{5}C_1 = 5.$

Example 6

From 6 boys and 4 girls, 5 are to be selected for admission for LAW course at the University. In how many ways can this be done if there must be exactly 2 girls?

Solution

Since there must be exactly 2 girls, then there should be 3 boys in order to make a group of 5 students.

The number of ways of choosing 3 boys out of 6 is ${}^{6}C_3$ and the number of ways of choosing 2 girls out of 4 is ${}^{4}C_2$

Hence the number of ways = ${}^{6}C_3 \times {}^{4}C_2$ = 120 ways.

Example 7

There are 3 boys and 4 girls at a birth day party. In how many ways can a team of 3 students be formed so as to include at least one boy?

Solution

Since the team is to include at least one boy, then possible number of boys is 1, 2 or all 3. The total number of students should be 3; hence the following teams can be formed;

Number of teams with 1 boy and 2 girls = ${}^{3}C_1 \times {}^{4}C_2 = 18$

Number of teams with 2 boys and 1 girl = ${}^{3}C_2 \times {}^{4}C_1 = 12$

Number of teams with 3 boys and no girls = ${}^{3}C_3 \times {}^{4}C_0 = 1$

Total is 18 + 12 + 1 = 31. Thus 31 such teams can be formed.

Example 8

The question paper of Subsidiary mathematics consists of 10 questions divided into two sections A and B of 5 questions each. In how ways can a student select 6 questions taking at least two questions from each section.

Solution

The question paper may be answered in the following ways

(i) 2 questions from section A and 4 questions from section B

(ii) 3 questions from section A and 3 questions from section B

(iii) 4 questions from section A and 2 questions from section B

For alternative (i) number of choices = ${}^5C_2 \times {}^5C_4 = 50$

For alternative (ii) number of choices = ${}^5C_3 \times {}^5C_3 = 100$

For alternative (iii) number of choices = ${}^5C_4 \times {}^5C_2 = 50$

Total number of choices = 50 + 100 + 50 = 200 ways.

Exercise

1. A committee of five students to comprise the school council is to be selected from eight male students and five female students. Find how many possible committees can be obtained. *Ans.*1287
2. In how many can a committee of 3 ladies and 4 gentlemen be appointed from a meeting consisting of 8 ladies and 7 gentlemen? *Ans.*1960
3. A committee of four people is to be selected from six women and three men. In how many ways can this committee be chosen, if there has to be at least a lady on the committee?
4. A games coach has to choose a team of six out of ten of the best athletes and one of the six has to be made a captain. How possible teams can the coach obtain?
5. From twenty athletes selectors have to allocate four to compete in the first race and another four to compete in the second race. How many possible choices are there?
6. From 5 gentlemen and 6 ladies a committee of 5 is to be formed. In how many ways can this be done if;

   (i) The committee is to include at least one lady. *Ans.*461

(ii) There is no restriction about its formation. *Ans.*231

(iii) Not more than 3 gentlemen? *Ans.*431

PROBABILITY DISTRIBUTIONS

Random Variable

A random variable is a numeric outcome that results from an experiment that can take on different values whose (numerical) value is determined by the outcome of a random experiment.

Example: toss two fair coins

- Let H denote a head, T a tail

There are four possible outcomes:
{HH, HT, TH, TT}

Now consider a variable X, defined as "the number of heads that are observed in the throw of two fair coins". The situation is as follows

Possible outcomes	Number of heads
TT	0
TH	1
HT	1
HH	2

The variable, X, "number of heads", is a random variable and has 3 possible outcomes: 0, 1 or 2. Random Variables are denoted by upper case letters (X) and individual outcomes for random variables are denoted by lower case letters (x)
Random variables (r.v) may be discrete or continuous:

Discrete Random Variable

A discrete random variable is one which may take on only a countable number of distinct values such as 0, 1, 2, 3, 4, ... Discrete random variables are usually (but not necessarily) counts. If a random variable can take only a finite number of distinct values, then it must be discrete. For example if a husband and wife are considering having three children, the number of boys could be 0, 1, 2 or 3. The variable being considered is 'the number of boys produced' and it can be denoted by X. It can only take exact values, 0, 1, 2 and 3 and so is called a **discrete** variable.

More examples of discrete random variables include the number of sweaters sold at school, the number of traffic police reporting for duty, the Sunday morning attendance at a church, the number of patients in a doctor's surgery, the number of defective light bulbs in a box of ten., the number of heads obtained if we toss a coin twice.

Probability Distribution

A ***probability distribution*** is a table, formula, or graph that describes the values of a random variable and the probability associated with these values. It is also sometimes called the probability function or the probability mass function.

Discrete Probability Notation

An upper-case letter will represent the ***name*** of the random variable, usually **X**. Its lower-case counterpart, x, will represent the ***value*** of the random variable.

The probability that the random variable **X** will equal x is:

$P(X = x)$ Or more simply $p(x)$

The probability distribution of a discrete random variable X is sometimes called the ***probability density function*** (p.d.f) of X and is responsible for allocating probabilities.

Properties of a p.d.f of a discrete random variable X

(i) $\sum_{all\ x} P(X = x) = 1$

(ii) $0 \leq P(X = x) \leq 1$ for all values of X. i.e. the probability of X taking the value of x lies between 0 and 1

Note: The probability density function can be expressed as a formula, i.e. $P(X = x) = f(x)$

Example 1

An ordinary die is thrown once. Write the probability distribution of the score on the die.

Solution

The sample space S = {1, 2, 3, 4, 5, 6}

Let X be a random variable 'the score obtained when a die is thrown'

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

NOTE: $\sum_{all\ x} P(X = x) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$, confirming that X is a random variable.

Example 2

The probability mass function of a discrete random variable X is defined as $p(x) = ax$ for $x = 1,2,3,4$. Find the value of a hence represent the distribution on a vertical line graph.

Solution

The probability distribution of X is

x	1	2	3	4
$P(X = x)$	a	2a	3a	4a

Since X is a random variable, $\sum_{all\ x} P(X = x) = 1$

So $$a + 2a + 3a + 4a = 1$$

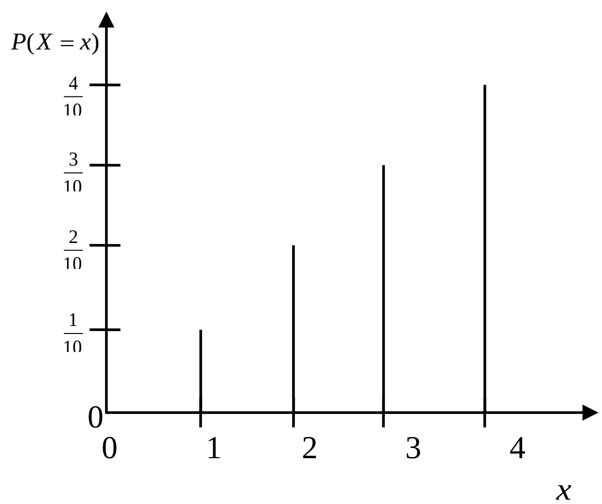
$$10a = 1$$

$$a = \frac{1}{10}$$

Hence the probability distribution is

x	1	2	3	4
$P(X = x)$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

and it can be represented by a vertical line graph:



Example 3

A random variable t has the following distribution

P(t = 1) = 0.1, P(t = 2) = 0.2, P(t =3) =0.3, P(t =4) =0.4.

Show that the distribution above is a probability distribution and determine

(i) P(t <3)

(ii) P (1≤ t <4)

(iii) P(t ≥2)

Solution

The probability density function for the random variable T is as shown below;

t	1	2	3	4
$P(T=t)$	0.1	0.2	0.3	0.4

$$\sum_{all\ t} P(T=t) = 0.1 + 0.2 + 0.3 + 0.4 = 1$$

Since $\sum_{all\ t} P(T=t) = 1$, then the distribution above is a probability distribution.

(i) P(t <3) = P(T=2) + P(T=1)

= 0.2 + 0.1 = 0.3

(ii) P (1≤ t <4) = P(T=1) + P(T=2) + P(T=3)

= 0.1 + 0.2 + 0.3 = 0.6

(iii) P(t ≥2) = P(T=2) + P(T=3) + P(T=4)

= 0.2 + 0.3 + 0.4 = 0.9

Example 4

The probability density function of the discrete r.v. is given by:

$P(X=x) = c\left(\frac{1}{4}\right)^x \quad for\ x = 0,1,2,3,....$ Find the value of the constant, c.

Solution

$P(X=x) = c\left(\frac{1}{4}\right)^x \quad for\ x = 0,1,2,3,....$ implies that x goes up to infinity

Since X is a random variable $\sum_{all\ x} P(X=x) = 1$

Substituting for x in the function gives: $P(X=0) = c\left(\frac{1}{4}\right)^0 = c$

$$P(X=1) = c\left(\frac{1}{4}\right)^1$$

$$P(X=2) = c\left(\frac{1}{4}\right)^2$$

$P(X=3) = c\left(\frac{1}{4}\right)^3$ and so on.

So $\sum_{all\ x} P(X=x) = c + c\left(\frac{1}{4}\right) + c\left(\frac{1}{4}\right)^2 + c\left(\frac{1}{4}\right)^3 + ...$

$$= c\left[1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \ldots\right]$$

This represents the sum to infinity of a geometric progression with the first term $a = 1$ and common ratio $r = \frac{1}{4}$

Thus the sum to infinity of a G.P is given by: $\frac{a}{1-r}$

$$\Rightarrow c\left(\frac{1}{1-\frac{1}{4}}\right) = 1$$

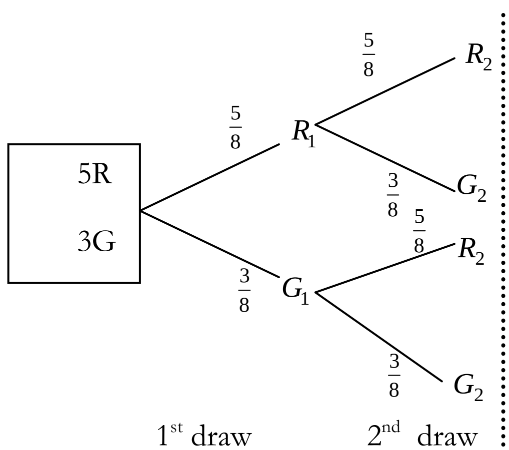
Hence
$$4c = 3$$
$$c = \frac{3}{4}$$

Example 5

A drawer contains 5 red socks and 3 green socks. A sock is taken at random from the drawer, its colour is noted and it is then replaced. Another sock is taken. If X is the random variable 'the number of red socks taken', find the probability distribution for X.

Solution

We first construct a tree diagram to obtain the probabilities



$$P(R_1 \cap R_2) = \frac{5}{8} \times \frac{5}{8} = \frac{25}{64}$$

$$P(R_1 \cap G_2) = \frac{5}{8} \times \frac{3}{8} = \frac{15}{64}$$

$$P(G_1 \cap R_2) = \frac{3}{8} \times \frac{5}{8} = \frac{15}{64}$$

$$P(G_1 \cap G_2) = \frac{3}{8} \times \frac{3}{8} = \frac{9}{64}$$

If X is the random variable 'the number of red socks taken', then X can take on values 0, 1 or 2. The probability distribution is formed as follows:

x	0	1	2

$P(X=x)$	$\frac{9}{64}$	$\frac{15}{32}$	$\frac{25}{64}$

Expectation of a discrete random variable, E(X)

The expectation or mean of a given random variable X (or expected value), written E(X), is given by:

$$E(X)=\sum_{all\ x} xP(X=x)$$

The symbol μ is usually used for the expectation so that $\mu = E(X)$

Example 6

A random variable X has probability density function (p.d.f) as shown. Find the expectation E(X).

x	1	2	3	4	5
$P(X=x)$	0.1	0.2	0.4	0.2	0.1

Solution

x	1	2	3	4	5	
$P(X=x)$	0.1	0.2	0.4	0.2	0.1	
$xP(X=x)$	0.1	0.4	1.2	0.8	0.5	$\sum xP(X=x)$ =3

$$E(X)=\sum_{all\ x} xP(X=x) = 0.1+0.4+1.2+0.8+0.5 = 3$$

Example 7

A random variable X has the probability – density function

$$f(x)=\begin{cases}\dfrac{2^x}{k} : x = 1, 2, 3, 4. \\ 0; else\ where\end{cases}$$

Find:

(i) the value of k
(ii) expectation of X
(iii) P(X≤3/X>1)

Solution

The probability distribution of X is

x	1	2	3	4
$P(X=x)$	$\frac{2}{k}$	$\frac{4}{k}$	$\frac{8}{k}$	$\frac{16}{k}$

(i) Since X is a random variable, $\sum_{all\ x} P(X=x)=1$

So $\frac{2}{k}+\frac{4}{k}+\frac{8}{k}+\frac{16}{k}=1$

$$\frac{30}{k}=1$$

$$k=30$$

Hence the probability distribution is

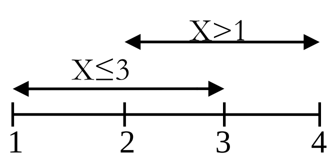
x	1	2	3	4
$P(X=x)$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{8}{15}$

(ii) The expectation of X, $E(X)=\sum_{all\ x} xP(X=x)$

x	1	2	3	4	
$P(X=x)$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{8}{15}$	
$xP(X=x)$	$\frac{1}{15}$	$\frac{4}{15}$	$\frac{12}{15}$	$\frac{32}{15}$	$\sum_{all\ x} xP(X=x)=\frac{49}{15}$

$\therefore$ The expectation, $E(X)=\frac{49}{15}$

(iii) $P(X\leq 3/X>1)=\frac{P(X\leq 3\cap X>1)}{P(X>1)}$



$$\Rightarrow P(X\leq 3\cap X>1)=P(X=2)+P(X=3)$$

$$=\frac{2}{15}+\frac{4}{15}=\frac{6}{15}$$

$$P(X>1)=P(X=2)+P(X=3)+P(X=4)$$

$$=\frac{2}{15}+\frac{4}{15}+\frac{8}{15}=\frac{14}{15}$$

$$\therefore P(X \le 3 / X > 1) = \frac{\frac{6}{15}}{\frac{14}{15}} = \frac{6}{14} = \frac{3}{7}$$

Example 8

A variable X has a probability mass function shown below

x	1	2	5	10
f(x)	0.5	a	0.12	b

Given that x can take on only the values given and E(X) = 2.5, find the values of a and b.

Solution

x	1	2	5	10	
$P(X=x)$	0.5	a	0.12	b	$\sum P(X=x) = 0.62 + a + b$
$xP(X=x)$	0.5	2a	0.6	10b	$\sum xP(X=x) = 1.1 + 2a + 10b$

Since X is a random variable, $\sum_{all\ x} P(X=x) = 1$

$$\Rightarrow 0.62 + a + b = 1$$

$$a + b = 0.38 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Since E(X) = 2.5, $\sum_{all\ x} xP(X=x) = 2.5$

$$\Rightarrow 1.1 + 2a + 10b = 2.5$$

$$2a + 10b = 1.4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

Solving equations (i) and (ii) simultaneously gives: $a = 0.3 \quad and \quad b = 0.08$

Properties of the expectation, E(X)

The definition of expectation can be extended to any function of the random variable. In general, the following results hold when X is a discrete random variable.

1. $E(a) = a, \;\; where\ a\ is\ any$ constant.
2. $E(aX) = aE(X), \;\; where\ a\ is\ any$ constant.
3. $E(aX + b) = aE(X) + b, \;\; where\ a$ and b are any constants.

Example 9

The r.v. X has p.d.f. $P(X=x)$ as shown in the table:

x	1	2	3	4	5	6
$P(X=x)$	0.0625	0.1875	0.25	0.25	0.1875	0.0625

Calculate (a) $E(X)$, (b) $E(4)$, (c) $E(5X)$, (d) $E(3X-5)$, (e) $E(2X+2)$

Solution

x	1	2	3	4	5	6	
$P(X=x)$	0.0625	0.1875	0.25	0.25	0.1875	0.0625	
$xP(X=x)$	0.0625	0.375	0.75	1	0.9375	0.375	$\sum xP(X=x)=3.5$

(a) $E(X)=\sum_{all\ x} xP(X=x)=3.5$

(b) Since $E(a)=a,$ *where a is any* constant.

$\Rightarrow E(4)=4$

(c) Since $E(aX)=aE(X),$ *where a is any* constant.

$\Rightarrow E(5X)=5E(X)=5\times 3.5=17.5$

(d) Since $E(aX+b)=aE(X)+b,$ *where a* and *b* are *any* constants.

$\Rightarrow E(3X-5)=3E(X)-5$

$= 3(3.5)-5=5.5$

(e) $E(2X+2)=2E(X)+2$

$= 2(3.5)+2=9$

Variance of a discrete random variable, $Var(X)$

For a discrete random variable X, with $E(X)=\mu$, the variance of X, written $Var(X)$ is given by:

$Var(X)=E(X^2)-\mu^2$ where $E(X^2)=\sum_{all\ x} x^2P(X=x)$

$\Rightarrow Var(X)=E(X^2)-[E(X)]^2$

$[E(X)]^2$ is written as $E^2(X)$; so we have: $Var(X)=E(X^2)-E^2(X)$

Standard deviation of a discrete random variable

The standard deviation of a discrete random variable X = $\sqrt{Var(X)}$.

Example 10

The r.v. X has probability distribution as shown in the table:

x	0	2	5	6
$P(X=x)$	0.11	0.35	0.46	0.08

Find (a) $E(X)$, (b) $E(X^2)$, (c) $Var(X)$, (d) *the* standard *deviation of* X

Solution

x	0	2	5	6	
$P(X=x)$	0.11	0.35	0.46	0.08	
$xP(X=x)$	0	0.7	2.3	0.48	$\sum xP(X=x)=3.48$
$x^2P(X=x)$	0	1.4	11.5	2.88	$\sum x^2P(X=x)=15.78$

(a) $E(X)=\sum_{all\ x} xP(X=x)=3.48$

(b) $E(X^2)=\sum_{all\ x} x^2P(X=x)=15.78$

(c) $Var(X)=E(X^2)-E^2(X)$

$$\begin{aligned}&=15.78-3.48^2\\&=15.78-12.1104\\&=3.6696\\&=3.67\ (2d.p)\end{aligned}$$

(d) Standard *deviation of* $X=\sqrt{Var(X)}$.

$$=\sqrt{3.67}=1.916\ (3d.p)$$

Properties of the Variance, $Var(X)$

For the discrete random variable X and constants a and b,

1. $Var(a)=0$
2. $Var(aX)=a^2Var(X)$
3. $Var(aX+b)=a^2Var(X)$

Example 11

The discrete random variable X has probability distribution as shown in the table.

x	10	20	30
$P(X=x)$	0.1	0.6	0.3

Find (a) $Var(X)$, (b) $Var(20)$, (c) $Var(3X)$, (d) $Var(2X+3)$, (e) standard deviation of X

Solution

x	10	20	30	
$P(X=x)$	0.1	0.6	0.3	
$xP(X=x)$	1	12	9	$\sum xP(X=x)=22$
$x^2P(X=x)$	10	240	270	$\sum xP(X=x)=520$

(a) $Var(X)=E(X^2)-[E(X)]^2$

$$E(X)=\sum_{all\ x} xP(X=x)=22$$

$$E(X^2)=\sum_{all\ x} x^2P(X=x)=520$$

$$\Rightarrow Var(X)=520-22^2$$
$$=36$$

(b) Since $Var(a)=0$

$\Rightarrow Var(20)=0$

(c) Since $Var(aX)=a^2Var(X)$

$$\Rightarrow Var(3X)=3^2Var(X)$$
$$=9\times 36=324$$

(d) Since $Var(aX+b)=a^2Var(X)$

$$\Rightarrow Var(2X+3)=2^2Var(X)$$
$$=4\times 36=144$$

(e) Standard *deviation of* $X=\sqrt{Var(X)}.$

$$=\sqrt{36}=6$$

Mode

The mode of a discrete probability distribution is the value x at which its probability mass function takes its maximum value. A **mode** of X is a number m such that P(X = m) is largest. This is the most likely value of X or values of X if it has several values with the same largest probability.

Example 12

A random variable has the following probability distribution

x	0	1	2	3	4	5	6
$P(X=x)$	0.03	0.04	0.06	0.12	0.4	0.15	0.2

Determine the mode

Solution

The mode is 4, since its probability is the greatest. Thus X = 2 is the mode.

Example 13

The probability distribution of X is as follows:

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{16}$

Find the mode.

Solution

The mode is 3 and 4 since the probability $P(X=3)=\frac{1}{4}$ and $P(X=4)=\frac{1}{4}$ is the greatest.

The Cumulative Distribution Function (Distribution function)

The cumulative distribution function (c.d.f.) of a discrete random variable X is the function $F(x)$ which tells you the probability that X is less than or equal to x. The cumulative distribution function is a cumulative sum of the probabilities up to a particular value.

So if X has probability density function $P(X=x)$, the c.d.f. is denoted by $F(x)$ and is mathematically described as:

$$F(x) = P(X \le x)$$

$$= \sum_{x=x_1}^{x} P(X=x) \qquad x = x_1,\ x_2,\ x_3,\ x_n$$

The cumulative distribution function is sometimes known as just the distribution function and its table called the **distribution table**.

Example 14

The probability distribution for the random variable X is shown in the table:

x	1	2	3	4	5
$P(X=x)$	0.05	0.25	0.3	0.15	0.25

Construct the cumulative distribution table.

Solution

We know that $F(x)=P(X\le x)=\sum_{x=1}^{x}P(X=x)$ $\quad x=1,\ 2,\ldots,5$

So
$$F(1)=P(X\le 1)=0.05$$
$$F(2)=P(X\le 2)=0.05+0.25=0.3$$
$$F(3)=P(X\le 3)=0.05+0.25+0.3=0.6$$

and so on
$$F(4)=P(X\le 4)=0.75$$
$$F(5)=P(X\le 5)=1$$

NOTE: $F(5)=1$, thus always the last value of $F(x)=1$

The cumulative distribution table is:

x	1	2	3	4	5
$F(x)$	0.05	0.3	0.6	0.75	1

Example 15

For the discrete random variable X the cumulative distribution function $F(x)$ is as shown:

x	0	1	2	3	4
$F(x)$	0.2	0.32	0.67	0.9	1

Construct the probability distribution of X, and find $P(X>2)$.

Solution

$$P(X=0)=F(0)=0.2$$
$$P(X=1)=F(1)-F(0)$$
$$=0.32-0.2=0.12$$
$$P(X=2)=F(1)-F(0)$$
$$=0.67-0.32=0.35$$
$$P(X=3)=F(3)-F(2)$$
$$=0.9-0.67=0.23$$

$$P(X=4)=F(4)-F(3)$$
$$=1-0.9=0.1$$

The probability distribution table is:

x	0	1	2	3	4
$P(X=x)$	0.2	0.12	0.35	0.23	0.1

$$P(X>2)=P(X=3)+P(X=4)$$
$$=0.23+0.1$$
$$=0.33$$

Median

The median of a discrete random variable is the "middle" value. It is the value of X for which $P(X \le x)$ is greater than or equal to 0.5 *and* $P(X \ge x)$ is greater than or equal to 0.5. Thus the **median** of X is the least number m such that $P(X \le m) \ge 0.5$ and $P(X \ge m) \ge 0.5$.

Example 16

In a recent little league softball game, each player went to bat 4 times. The number of hits made by each player is described by the following probability distribution.

Number of hits, x	0	1	2	3	4
Probability, P(x)	0.10	0.20	0.30	0.25	0.15

Construct a cumulative distribution and find the median of the probability distribution.

Solution

The cumulative distribution table is:

Number of hits, x	0	1	2	3	4
Probability, P(x)	0.10	0.20	0.30	0.25	0.15
$F(x)$	0.10	0.30	0.60	0.85	1

The median is 2; because $P(X \le 2)$ is equal to 0.60, and $P(X \ge 2)$ is equal to 0.70. The computations are shown below.

$$P(X \le 2) = P(x=0) + P(x=1) + P(x=2) = 0.10 + 0.20 + 0.30 = 0.60$$
$$P(X \ge 2) = P(x=2) + P(x=3) + P(x=4) = 0.30 + 0.25 + 0.15 = 0.70$$

Example 17

The probability distribution of a random variable X is given by:

x	0	1	2	3
$P(X=x)$	0.008	0.096	0.384	0.512

Find the median and mode.

Solution

The median is 3, since P(X ≤ 3) = 1 ≥ 0.5 and P(X ≥ 3) = 0.512 ≥ 0.5. Further, 3 is the least value of X with this property.

The mode is also 3, since its probability is the greatest.

Exercise

1. The probability distribution of X is a shown:

x	2	3	4	5	6	7	8
$P(X=x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

   Show that X is a random variable.

2. The probability density function of a discrete random variable X is given by $P(X=x)=kx$ for $x=12,13,14.$. Find the value of the constant k.

3. The p.d.f. of a discrete random variable X is given by $P(X=x)=kx^2$, for $x=0,1,2,3,4.$ Given that k is a constant, find the value of k. $Ans.\left(\frac{1}{30}\right)$

4. The discrete random variable Y has p.d.f. shown

y	-1	0	1	2	3
$P(Y=y)$	0.1	0.2	0.35	0.15	c

   Find (a) the value of c, (b) $P(-1\le Y<2)$, (c) $P(Y>1)$, (d) $P(-1\le Y\le 2)$, (e) the mode.

5. A drawer contains 8 brown socks and 4 blue socks. A sock is taken from the drawer at random, its colour is noted and it is then replaced. This procedure is performed twice more. If X is the r.v. ‘the number of brown socks taken’, find the probability distribution for X.

6. The probability distribution of a r.v. X is as shown in the table:

x	1	2	3	4	5
$P(X=x)$	0.15	0.25	d	0.1	0.2

   Find (a) the value of d, (b) $E(X)$, (c) $Var(X)$, (d) the standard deviation of X

7. A bag contains 5 black beads and 6 red beads. Two beads are picked, one at a time, and not replaced. Let X be the r.v. ‘the number of red beads picked’. Find $E(X)$.

8. The discrete r.v. X has p.d.f. $P(X=0)=0.05, P(X=1)=0.45, P(X=2)=0.5$.

   Find (a) $\mu=E(X)$, (b) $E(X^2)$, (c) $E(2X+1)$, (d) $Var(X)$, (e) $Var(3X-3)$

9. The probability distribution for the r.v. X is shown in the table.

x	0	1	2	3	4	5	6
$P(X=x)$	0.03	0.04	0.06	0.12	0.4	0.15	0.2

Construct the cumulative distribution table and find the median.

10. For a discrete r.v. R the cumulative distribution function $F(r)$ is as shown in the table:

r	1	2	3	4
$F(r)$	0.13	0.54	0.75	1

Find (a) the probability distribution of R, (b) $P(R>2)$, (c) $P(R\leq 3)$ (e) $Var(R)$, (d) $Var(4X)$

11. A discrete random variable X has the following probability distribution

x	1	2	3	4
P(X=x)	$^3/_{16}$	½	¼	$^1/_{16}$

Find the mean and variance of X.

12. A discrete random variable X has a probability function

$$P(X=x)=\begin{cases}\dfrac{x}{k}, & x=1,2\dots n\\ 0 & \text{elsewhere.}\end{cases} \qquad \text{where k is a constant}$$

Given that the expectation of X is 3, find

(i) the value of n and the constant k

(ii) the median and variance of X

(iii) $P(X = 2/X \geq 2)$

13. A regular customer at a boutique observes the number of customers, X, in the boutique when she enters has the following probability distribution.

Number of customers, x	0	1	2	3	4
Probability p(x)	0.15	0.34	0.27	0.14	0.10

(i) Find the mean and standard deviation of X

(ii) The cumulative distribution of X

(iii) The mode and median of X

14. A six – faced die is shaped in such a way that it produces a score, X, for which the probability distribution is given in the following table.

x	1	2	3	4	5	6
$P(X=x)$	k	$\frac{k}{2}$	$\frac{k}{3}$	$\frac{k}{4}$	$\frac{k}{5}$	$\frac{k}{6}$

Show that the constant $k = \frac{20}{49}$. Find

(i) the mean and variance of X.

(ii) $P(X \geq 2)$ and $P(X < 2)$

(iii) the cumulative distribution of the variable

(iv) the median of X

15. The r.v. X has p.d.f. $P(X = x) = c\left(\frac{4}{5}\right)^x$ *for* $x = 0, 1, 2, 3...$ Find the value of the constant, c.

16. The discrete random variable R has p.d.f. given by $P(R = r) = c(3 - r)$ *for* $\mathrm{r} = 0, 1, 2, 3.$

(i) Find the value of the constant c

(ii) the expectation of R

(iii) $Var(R)$

(iv) $Var(4R - 3)$

BINOMIAL PROBABILTY DISTRIBUTION

Concept of a binomial distribution

One discrete probability distribution that has many commonly encountered applications is the ***binomial probability distribution.*** The binomial distribution describes the probabilities of discrete events that have only two possible outcomes (often described as success or failure). The application of this distribution is quite varied. In statistical quality control, items are often inspected and classified as either conforming or non-conforming. Airlines book passengers who ultimately show up for the flight or do not show up for the flight. In a similar vein, a variety of organizations accept some sort of reservation (hotel bookings, restaurant reservations, automobile service appointments, etc.), and in each case the customers may either show up or not show up. These two outcomes are what are referred to as success and failure. Which we choose to label as success doesn't really matter in the analysis, however, at times the judicious assignment of the success or failure label can make computations a little more streamlined

This probability distribution results from experiments which have only two mutually exclusive outcomes. These outcomes are referred to as success and failure for example

1. Tossing a coin to get a head or a tail
2. Passing and failing exams
3. Getting a baby boy or baby girl

Properties of a binomial distribution

A binomial experiment has the following four properties:

1. The experiment consists of a sequence of n identical trials.
2. Two outcomes are possible on each trial. We refer to one as a *success* and the other as a *failure*.
3. The probability of a success, denoted by p, does not change from trial to trial. Consequently, the probability of failure, denoted by $1 - p$, does not change from trial to trial.

[Note: the probability of failure is often denoted by the letter q, where $q = (1 - p)$] thus $p+q = 1$

4. The trials are independent.

Binomial notation

If the probability that an experiment results in a successful outcome is p and the probability that the outcome is a failure is q, where $q = 1 - p$, and if X is the r.v. 'the number of successful outcomes in n independent trials', then the binomial probability distribution of X is given by:

$$P(X = x) = {}^{n}C_{x} p^{x} q^{n-x} \qquad x = 0, 1, 2, \ldots, n$$

Where x is the number of successes and ${}^{n}C_{x} = \frac{n!}{(n-x)!x!}$

(Your calculator may have a function key to calculate these numbers directly, but if not, you can use the factorial function key which will certainly be on any scientific calculator.)

If X is distributed in this way, we write $X \sim Bin(n, p, x)$ where n is the number of independent trials and p is the probability of a successful outcome in one trial and x the number of successes.

n and p are called the parameters of the distribution.

So we read the statement $X \sim Bin(n, p, x)$ thus: X follows a binomial distribution with parameters n and p.

Expectation (Mean) for a Binomial Distribution

For a binomial distribution with n trials, where the probability of success in each trial is p, the variance is:

Variance and Standard Deviation for a Binomial Distribution

For a binomial distribution with n trials, where the probability of success in each trial is p, the variance is: $Var(X) = npq \qquad where \;\; q = 1 - p$

and Standard *deviation*: $\sigma = \sqrt{Var(X)} = \sqrt{npq}$

How to determine the binomial probabilities

The probabilities of a binomial distribution can be obtained by calculating them using a formula or reading the values from binomial tables.

Example 1

The random variable X is binomially distributed with n=4 and p=0.4. Find

(i) $P(X = 2)$

(ii) $P(X \geq 1)$

(iii) $P(X < 3)$

Solution

Given that $X \sim Bin(n, p, x)$ with $n = 4, p = 0.4, q = 0.6$

and so $P(X = x) = {}^{n}C_{x} p^{x} q^{n-x} = {}^{4}C_{x}(0.4)^{x}(0.6)^{4-x} \qquad x = 0, 1, 2, 3, 4.$

(i) $P(X = 2) = {}^{4}C_{2}(0.4)^{2}(0.6)^{2}$ =0.3456 $(4d.p)$

(ii) $P(x \geq 1) = P(x = 1) + P(x = 2) + P(x = 3) + P(x = 4)$

$$= {}^4C_1(0.4)^1(0.6)^3 + {}^4C_2(0.4)^2(0.6)^2 + {}^4C_3(0.4)^3(0.6)^1 + {}^4C_4(0.4)^4(0.6)^0$$
$$= 0.3456 + 0.3456 + 0.1536 + 0.0256 = 0.8704 \quad (4d.p)$$

(iii) $P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$
$$= {}^4C_0(0.4)^0(0.6)^4 + {}^4C_1(0.4)^1(0.6)^3 + {}^4C_2(0.4)^2(0.6)^2$$
$$= 0.1296 + 0.3456 + 0.3456 + = 0.8208 \quad (4d.p)$$

Example 2

A coin is tossed 3 times and the number of heads showing up recorded

(a) Obtain a binomial probability distribution for the number of heads showing and hence
(b) Find
(i) Probability that only one head shows up
(ii) Probability that a least one head shows up
(iii) Probability that at most two heads show up

Solution

Let X be the r.v. 'the number of heads showing up'.

Then $p = P(Head) = \frac{1}{2}$ *and* $q = P(Tail) = \frac{1}{2}$

Thus $X \sim Bin(n, p, x)$ *where* $p = 0.5$ *and* $n = 3$

and so $P(X = x) = {}^nC_x p^x q^{n-x} = {}^3C_x(0.5)^x(0.5)^{4-x} \qquad x = 0, 1, 2, 3.$

$\Rightarrow P(X = 0) = {}^3C_0(0.5)^0(0.5)^3 = 0.1250 \quad (4d.p)$

$\Rightarrow P(X = 1) = {}^3C_1(0.5)^1(0.5)^2 = 0.3750 \quad (4d.p)$

$\Rightarrow P(X = 2) = {}^3C_2(0.5)^2(0.5)^1 = 0.3750 \quad (4d.p)$

$\Rightarrow P(X = 3) = {}^3C_3(0.5)^3(0.5)^0 = 0.1250 \quad (4d.p)$

(a) The probability distribution is as shown:

x	0	1	2	3
$P(X = x)$	0.1250	0.3750	0.3750	0.1250

(b)

(i) $P(X = 1) = 0.3750$

(ii) $P(X \geq 1) = P(X = 1) + P(X = 2) + P(X = 3)$
$$= 0.3750 + 0.3750 + 0.1250 = 0.8750 \ (4d.p)$$

(iii) $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$
$$= 0.1250 + 0.3750 + 0.3750 = 0.8750 \ (4d.p)$$

Example 3

The probability of winning a game is $\frac{4}{5}$. If ten games are played, what is the

(i) Mean number of successes
(ii) Variance and standard deviation
(iii) Probability of at least 8 successes in the ten games

Solution

Let 'a win' be 'success'. Then $p=\frac{4}{5}=0.8$ and $q=0.2$

Let X be the r.v. "the number of successes"

Then $X \sim Bin(x,n,p)$ $where\ p=0.8\ \ and\ \ n=10$

(i) Mean number of successes, $E(X)=np=10\times 0.8=8$

(ii) $Variance,\quad Var(X)=npq=10\times 0.8\times 0.2=1.6$

Standard deviation$=\sqrt{\text{Variance}}$

$=\sqrt{1.6}=1.265\ \ (3d.p)$

(iii) $P(X\geq 8)=P(X=8)+P(X=9)+P(X=10)$

$P(X=8)={}^{10}C_8(0.8)^8(0.2)^2\ =0.3020$

$P(X=9)={}^{10}C_9(0.8)^9(0.2)^1\ =0.2684$

$P(X=10)={}^{10}C_{10}(0.8)^{10}(0.2)^0\ =0.1074$

$P(X\geq 8)=0.3020+0.2684+0.1074=0.6778$

Example 4

The random variable X is such that $X \sim Bin(n,p)$ and $E(X)=2$, $Var(X)=\frac{24}{13}$.

Find the values of n and p.

Solution

$E(X)=n\times p\qquad \Rightarrow np=2\text{.........................}(i)$

$Var(X)=npq\qquad \Rightarrow npq=\frac{24}{13}\text{..................}(ii)$

Substituting equation (i) in (ii) gives: $2q=\frac{24}{13}$

$26q=24$

$q=\frac{24}{26}=\frac{12}{13}$

But $\quad p=1-q=1-\frac{12}{13}$

$$p = \frac{1}{13}$$

From equation (i) $np = 2$

$$\Rightarrow n \times \frac{1}{13} = 2$$

$$n = 26$$

Binomial tables

Occasionally we can avoid the binomial formula. Because probabilities of binomial variables are so common in statistics, tables are used to alleviate having to continually use the formula.

The common mathematical tables have values of n ranging from 1 to 20, and the values of p that are included are 0.01, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40, 0.45, 0.50. So, if the value of n is less than or equal to 20 and the value of p is one of the 11 values listed above, you should use the tables.

The binomial tables show the probability of x successes in n independent trials each with probability of success p. Each entry in the table is calculated by the formula ${}^nC_x p^x q^{n-x}$ and the calculations are done for you, and all you have to be able to do is read a table. You should only resort to the formula if you can not use the tables.

Example 5

A drug is used to treat patients of a certain disease. The probability that a patient cures when treated with this drug is 0.1. If 5 patients are given the drug, find the probability that 3 of them get cured.

Solution

Let X be the r.v. 'the number of patients (from the 5) who get cured'.

Then $X \sim Bin(n, p)$ with $n = 5$ and $p = 0.1$ *and* $q = 0.9$

Using the formula:

$$P(X = 3) = {}^5C_3(0.1)^3(0.9)^2 = 0.0081$$

We are required to find $P(X = 3)$ so we can use the table.

First identify the proper table for the value of n i.e. the binomial distribution table of individual terms. For the above example $n = 5$ and an extract of the table looks like this:

n	x	0.01	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
5	0	.9510	.7738	.5905	.4437	.3277	.2373	.1681	.1160	.0778	.0503	.0312
	1	.0480	.2036	.3280	.3915	.4096	.3955	.3602	.3124	.2592	.2059	.1562
	2	.0010	.0214	.0729	.1382	.2048	.2637	.3087	.3364	.3456	.3369	.3125
	3	.0000	.0011	.0081	.0244	.0512	.0879	.1323	.1811	.2304	.2757	.3125
	4	.0000	.0000	.0004	.0022	.0064	.0146	.0284	.0488	.0768	.1128	.1562
	5	.0000	.0000	.0000	.0001	.0003	.0010	.0024	.0053	.0102	.0185	.0312

Once you have identified the proper table for your value of *n*, go down to the row that corresponds to the correct value of x.

n	x	0.01	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
5	0	.9510	.7738	.5905	.4437	.3277	.2373	.1681	.1160	.0778	.0503	.0312
	1	.0480	.2036	.3280	.3915	.4096	.3955	.3602	.3124	.2592	.2059	.1562
	2	.0010	.0214	.0729	.1382	.2048	.2637	.3087	.3364	.3456	.3369	.3125
	3	**.0000**	**.0011**	**.0081**	**.0244**	**.0512**	**.0879**	**.1323**	**.1811**	**.2304**	**.2757**	**.3125**
	4	.0000	.0000	.0004	.0022	.0064	.0146	.0284	.0488	.0768	.1128	.1562
	5	.0000	.0000	.0000	.0001	.0003	.0010	.0024	.0053	.0102	.0185	.0312

Then go across to the column that corresponds to the value of *p*. Where this column crosses the row for x, you will find the probability.

n	x	0.01	0.05	**0.10**	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
5	0	.9510	.7738	**.5905**	.4437	.3277	.2373	.1681	.1160	.0778	.0503	.0312
	1	.0480	.2036	**.3280**	.3915	.4096	.3955	.3602	.3124	.2592	.2059	.1562
	2	.0010	.0214	**.0729**	.1382	.2048	.2637	.3087	.3364	.3456	.3369	.3125
	3	**.0000**	**.0011**	.0081	**.0244**	**.0512**	**.0879**	**.1323**	**.1811**	**.2304**	**.2757**	**.3125**
	4	.0000	.0000	**.0004**	.0022	.0064	.0146	.0284	.0488	.0768	.1128	.1562
	5	.0000	.0000	**.0000**	.0001	.0003	.0010	.0024	.0053	.0102	.0185	.0312

The tables tell us the probability is 0.0081. Hence $P(X=3)=0.0081\ \ (Tab)$

Example 6

A random variable $x \sim Bin(8,0.4)$
Find

(a) $P(X=2)$

(b) $P(X=0)$

(c) $P(X>6)$

Solution

$n=8,\ p=0.4,\ q=1-p=0.6$

(a) $P(X=2)=0.2090\ \ (Tab)$

(b) $P(X=0)=0.0168\ \ (Tab)$

(c) $P(X>6)=P(X=7)+P(X=8)$

$=0.0079+0.0007=0.0086\ \ (Tab)$

Cumulative Binomial probability tables

These tables give the cumulative probabilities $F(r)=P(X\leq r)$ for the possible values of r. Thus, a cumulative probability is just the probability that x has a value of *r* or less, where *r* is some possible outcome of the experiment.

We will use the symbol B (n, *p*, r) to denote cumulative binomial probabilities, which are formally defined as:

$$Bin(n,p,r)=P(X\le r)=P(X=0)+(PX=1)+P(X=2)+...+P(X=r)=\sum_{x=0}^{r}b(n,p,x)$$

This may look rather complicated at first glance, but you just have to remember that B (n, p, r) is simply the sum of binomial probabilities for the values of $x = 0$ through $x = r$.

Example 7

If X is a r.v. such that $X \sim Bin(5,0.3)$. Use the cumulative binomial probability tables to find the following:

(a) $P(X \le 4)$ (b) $P(X=2)$ (c) $P(X<3)$ (d) $P(X>1)$ (e) $P(X \ge 3)$

Solution

Since $X \sim Bin(5,0.3)$ then $n=5$ *and* $p=0.3$

Thus first identify the proper binomial cumulative table for the value of ***n*** and then go down to the row that corresponds to the correct value of ***x*** and column corresponding to the value of ***p***. For the above example $n=5$ *and* $p=0.3$. An extract of the table looks like this:

		p						
n	***x***	0.05	0.10	0.20	0.25	**0.30**	0.40	0.50
5	**0**	0.7738	0.5905	0.3277	0.2373	**0.1681**	0.0778	0.0313
	1	0.9774	0.9185	0.7373	0.6328	**0.5282**	0.3370	0.1875
	2	0.9988	0.9914	0.9421	0.8965	**0.8369**	0.6826	0.5000
	3	1.0000	0.9995	0.9933	0.9844	**0.9692**	0.9130	0.8125
	4	1.0000	1.0000	0.9997	0.9990	**0.9976**	0.9898	0.9688

(a) $P(X \le 4)=0.9976$ $(directly\ from\ table)$

(b) $P(X=2)=P(X\le 2)-P(X\le 1)$

$=0.8369-0.5282$

$=0.3087$

(c) $P(X<3)=P(X\le 2)=0.8369\ (Tab)$

(d) $P(X>1)=1-P(X\le 1)$

$=1-0.5282$

$=0.4718$

(e) $P(X\ge 3)=1-P(X\le 2)$

$=1-0.8369$

$=0.1631$

Example 8

It is known that 25% of all fish caught in a certain vicinity are infected with a particular parasite. What is the probability that between 5 and 10 inclusive of 20 randomly selected fish from this population will be infected with the parasite?

Solution

Let X be the r.v. 'the number of infected fish (from the 20) which are caught'.

Then $X \sim Bin(n, p)$ with $n = 20$ and $p = 0.25$ *and* $q = 0.75$

Now, we are asked to find the probability that the experiment will result in either 5 or 6 or 7 or 8 or 9 or 10 infected fish (the word **inclusive** means "include the boundary values 5 and 10 of the interval mentioned"). We could proceed and simply calculate these six simple probabilities and sum them up to get the required answer:

$$P(5 \le X \le 10) = P(X = 5) + P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$$

but this is beginning to look like quite a lot of work.

and so, we can calculate the desired answer to the present problem using the formula:

$$P(5 \le X \le 10) = P(X \le 10) - P(X \le 4)$$

An extract of the cumulative binomial probability table for n=20 is shown below.

		p						
n	***x***	0.05	0.1	0.2	0.25	0.3	0.4	0.5
	0	0.3585	0.1216	0.0115	0.0032	0.0008	0.0000	0.0000
20	**1**	0.7358	0.3917	0.0692	0.0243	0.0076	0.0005	0.0000
	2	0.9245	0.6769	0.2061	0.0913	0.0355	0.0036	0.0002
	3	0.9841	0.8670	0.4114	0.2252	0.1071	0.0160	0.0013
	4	0.9974	0.9568	0.6296	**0.4148**	0.2375	0.0510	0.0059
	5	0.9997	0.9887	0.8042	0.6172	0.4164	0.1256	0.0207
	6	1.0000	0.9976	0.9133	0.7858	0.6080	0.2500	0.0577
	7	1.0000	0.9996	0.9679	0.8982	0.7723	0.4159	0.1316
	8	1.0000	0.9999	0.9900	0.9591	0.8867	0.5956	0.2517
	9	1.0000	1.0000	0.9974	0.9861	0.9520	0.7553	0.4119
	10	1.0000	1.0000	0.9994	**0.9961**	0.9829	0.8725	0.5881
	11	1.0000	1.0000	0.9999	0.9991	0.9949	0.9435	0.7483
	12	1.0000	1.0000	1.0000	0.9998	0.9987	0.9790	0.8684
	13	1.0000	1.0000	1.0000	1.0000	0.9997	0.9935	0.9423
	14	1.0000	1.0000	1.0000	1.0000	1.0000	0.9984	0.9793
	15	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9941
	16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9987
	17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998
	18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Finally, we need cumulative probabilities for $x \le 10$ and for $x \le 4$, so we read the cumulative binomial tables from the rows in the n = 20 sections that are labeled x = 10 and x = 4, respectively, getting: $P(X \le 10) = 0.9961$ and $P(X \le 4) = 0.4148$

$$\Rightarrow P(5 \le X \le 10) = 0.9961 - 0.4148 = 0.5813$$

Example 9

A doctor found that 40% of children treated with a particular drug survive, 10 children were known to have the disease.
Find the probability that

(i) At least 7 children will survive after being treated
(ii) Between 3 to 5 children inclusive will survive.

Solution

Let X be the r.v. 'the number of children that will survive after treatment'.

Then $X \sim Bin(n, p)$ with $n = 10$ and $p = \frac{40}{100} = 0.4$ *and* $q = 0.6$

(i) $P(X \ge 7) = P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$

$$P(X = 7) = 0.0425$$

$$P(X = 8) = 0.0106$$

$$P(X = 9) = 0.0016$$

$$P(X = 10) = 0.0001$$

$$\Rightarrow P(X \ge 7) = 0.0548$$

Or using the cumulative binomial tables

$$P(X \le 7) = 1 - P(X \le 6)$$
$$= 1 - 0.9452 = 0.9877 \quad (Tab)$$

(ii) $P(3 \le X \le 5) = P(X = 3) + P(X = 4) + P(X = 5)$

$$= 0.2150 + 0.2508 + 0.2007 = 0.6665$$

Or using the cumulative binomial tables

$$P(3 \le X \le 5) = P(X \le 5) - P(X \le 2)$$
$$= 08338 - 0.1673 = 0.6665$$

Mode of a binomial probability distribution

The mode is the most likely value of X to occur and therefore has the highest probability. It can be found by working out the probabilities for the values of X near the mean and then the value of X near the mean with the highest probability is the mode.

Example 10

The germination of bean seeds is not easy. From experience, Mpanga the expert bean grower knows that on average the probability of a bean seed germinating is 0.4. Six seeds are planted. Determine the most likely number of germinating seeds.

Solution

Let X be the r.v. 'the number of germinating bean seeds'.

Then $X \sim Bin(n, p)$ with $n = 6$ and $p = = 0.4$ *and* $q = 0.6$

We first find the mean, $E(X) = np = 6 \times 0.4 = 2.4$

Hence the values of X near the mean are 2 and 3. Their probabilities are:

$$P(X = 2) = {}^6C_2(0.4)^2(0.6)^4 = 0.3110$$

$$P(X = 3) = {}^6C_3(0.4)^3(0.6)^3 = 0.2765$$

The most likely number of germinating seeds (mode) is 2 since it has the highest probability.

EXERCISE

1. A coin is biased so that a head is twice as likely to occur as a tail. If the coin is tossed 15 times, determine the
   (i) Expected number of heads *Ans.*(10)
   (ii) Probability of getting at most 2 tails. *Ans.*(0.0793)
2. A pair of dice tossed 10 times, determine the probability that the sum of 7 appears.
   (a) Exactly 4 times *Ans.*(0.0543)
   (b) Between 4 to 8 inclusive times *Ans.*(0.0697)
3. A family plans to have 3 children
   (i) Write down the possible sample space and construct its probability distribution given that x is the number of boys in the family.
   (ii) Find the expected number of boys.
4. The probability that Bob wins a game is $\frac{2}{3}$. He plays 8 games. What is the probability that he wins
   (i) At least 7 games *Ans.*(0.1951)
   (ii) Exactly 5 games *Ans.*(0.2731)
5. A multiple choice test has 10 questions of which each has 4 alternative answers but one is correct. If a student attempts the question through guess work, find the probability that
   (i) He gets half of the questions correct *Ans.*(0.0584)
   (ii) He gets more than 7 correct answers. *Ans.*(0.0004)

6. Calculate the probability of a drawing pin landing on its back 3 times in 5 trials if the probability of getting a drawing pin to land on its back on any toss is 0.4 *Ans.*(0.2304)
7. It is known that 60% of candidates will achieve in an examination. If 5 people sit the examination find the probability:

   (a) Exactly 3 candidates will achieve.

   (b) At least 3 candidates will achieve.
8. Suppose 8 flies are selected at random from a population in which 42% of all flies have red eyes. What is the probability that 3 of these flies will have red eyes? *Ans.*(0.2723)
9. It is thought that 25% of all fish caught in a certain vicinity are infected with a particular parasite.
   (a) What is the probability that 6 of 20 randomly selected fish from this population will be infected with the parasite? *Ans.*(0.1686)
   (b) What is the probability that there will be six or seven infected fish in the sample of twenty fish selected randomly. *Ans.*(0.2810)
10. If x is binomially distributed with mean 0.45 and n = 10, find the most likely value of x.
11. In Iganga, 40% of the inhabitants have a particular eye disorder. If 12 are waiting to see the doctor, what is the most likely number of them to have the eye disorder?
12. In a certain city, rain falls on average one day out of three. If three dates are selected at random, what is the probability that rain will fall on at least one of the three dates?
13. In darts game Peter finds that the probability of getting the bull's eye is 0.25. In 10 trials, find the probability that
    (i) at least two attempts are successful
    (ii) between two and five inclusive attempts are successful
    (iii) at most three are on target.
14. In a given community, the probability of finding a person with secondary level of education is 2 in 5. If a random sample of 6 persons is taken, find the probability that
    (i) Exactly 3,
    (ii) at most 4 persons, have had secondary education.
15. The probability that a student is awarded a distinction in the subsidiary mathematics examination is 0.25. Find the probability that in a group of 10 students more than five pass the examination with a distinction.
16. The random variable X is distributed binomially with mean 2 and variance 1.6. Find
    (a) the mode of X
    (b) $P(X \le 6)$
17. Use cumulative binomial probability tables to find the following:
    (a) $X \sim Bin(6, 0.2)$, find

        (i) $P(X \le 3)$ (ii) $P(X \ge 4)$ (iii) $P(X = 5)$
    (b) $X \sim Bin(6, 0.2)$, find

(i) $P(X=6)$ (ii) $P(X \le 3)$ (iii) $P(X<5)$ (iv) $P(X \ge 3)$

18. The probability that it will be a fine day is 0.4. Determine the most likely number of fine days in a week and standard deviation.

CONTINUOUS RANDOM VARIABLES

A continuous random variable (r.v.) is the one that takes on an uncountable number of values. This means you can never list all possible outcomes even if you had an infinite amount of time. For example, X could be the continuous random variable 'the time taken, in minutes to finish eating supper', H could be a continuous random variable 'the height, in cm, of 50 students in senior five of a particular school'.

CONTINUOUS PROBABILITY DENSITY

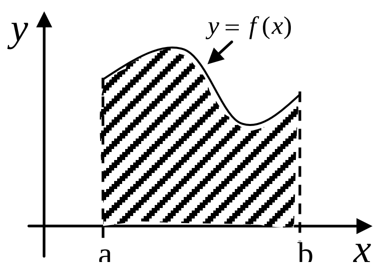
Continuous probability density functions are probability functions where by the random variable X take on any value in some interval of values. Because a continuous r.v. can take an infinite number of values, the probability of it taking any one is always measured over an interval.

The probability density function is written $f(x)$ where $f(x) \geq 0$ throughout the range of values for which x is valid. This probability density function (p.d.f) can be represented by a curve, and the probabilities are given by the area under the curve.

For example the function $f(x) = \begin{cases} kx, & 0 \leq x \leq 2 \\ 0, & elsewhere \end{cases}$ has values of x in the range of 0 to 2. This kind of function is a continuous probability distribution function.

Properties of a continuous p.d.f.

Consider the curve below which shows a p.d.f. $f(x)$ in which the r.v. X is valid over the range $a \leq x \leq b$

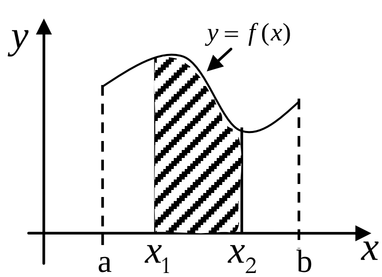


The following properties can then be established:

(i) $\int_{all\ x} f(x)dx = 1 \qquad \Rightarrow \int_a^b f(x)dx = 1$

Thus the total area under the curve $y = f(x)$ between $x = a$ and $x = b$ is 1.

(ii) $P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f(x)dx$ is the area under the curve between x_1 and x_2 where $x_2 > x_1$



Note: For continuous p.d.f. one requires the knowledge of integration.

Example 1

Given $f(x) = \begin{cases} kx, & 0 \le x \le 2 \\ 0, & elsewhere \end{cases}$

(a) Find the value of k.
(b) Find $P(0 \le X \le 1)$

Solution

(a) Since X is a random variable the total probability is 1,

So $$\int_0^2 (kx)dx = 1$$

$$\left[\frac{kx^2}{2}\right]_0^2 = 1$$

$$\frac{4k}{2} - 0 = 1$$

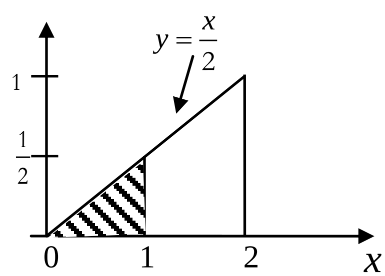
$$4k = 2$$

$$k = \frac{1}{2}$$

Therefore $f(x) = \dfrac{x}{2}, 0 \le x \le 2$

(b) $P(0 \le X \le 1) = \displaystyle\int_1^2 \frac{x}{2}\, dx$

$$= \frac{1}{2}\left[\left(\frac{x^2}{2}\right)\right]_0^1 = \frac{1}{4}$$



Similarly

$P(0 \le X \le 1) = \textit{Shaded area under the graph}$

$$= \frac{1}{2}\left(1 \times \frac{1}{2}\right) = \frac{1}{4}$$

Example 2

Suppose that the error in the reaction temperature in ${}^0C$ for a controlled laboratory experiment is a continuous random variable **X** having the probability density function:

$$f(x) = \frac{x^2}{3} \quad , \quad -1 < x < 2$$
$$= 0 \qquad otherwise$$

(a) Show that $\int_{all\ x} f(x)\,dx = 1$

(b) Find $P(0 < X \leq 1)$

(c) find $P(0 < x < 3)$

Solution

(a) $\int_{-1}^{2} \frac{x^2}{3} \quad dx = \left[\frac{x^3}{9}\right]_{-1}^{2} = \frac{1}{9}[8-(-1)^3] = \frac{8+1}{9} = 1$

(b) $P(0 < X \leq 1) = \int_{0}^{1} \frac{x^2}{3} dx = \left[\frac{x^3}{9}\right]_{0}^{1} = \frac{1}{9}$

(c) $P(0 < x < 3) = \int_{0}^{2} \frac{x^2}{3} dx = \left[\frac{x^3}{9}\right]_{0}^{3} = \frac{(2)^3}{9} = \frac{8}{9}$

Example 3

The random variable X has a p.d.f. given by

$$f(x) = \begin{cases} ax, & 0 \leq x \leq 1 \\ a, & 1 \leq x \leq 2 \\ 3a - ax, & 2 \leq x \leq 3 \\ 0, & elsewhere \end{cases}$$

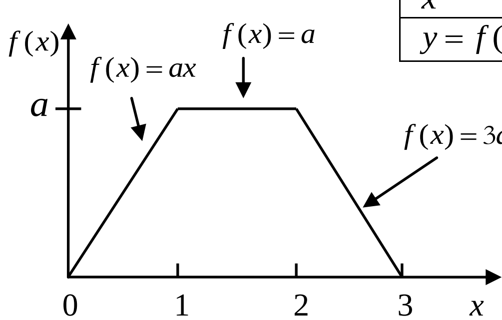
(i) Sketch the graph of $y = f(x)$ and find the value of the constant a

(ii) Find $P(1.5 \leq x \leq 3)$

Solution

(i) ***The sketch of*** $y = f(x)$

x	0	1	2	3
$y = f(x)$	0	a	a	0



The area under the graph = area of the trapezium

$$Area = \frac{1}{2} \times a(3+1)$$
$$= 2a$$

But the total area is 1,

Thus $\quad 2a = 1$

$$a = \frac{1}{2}$$

Therefore $f(x) = \begin{cases} \frac{x}{2} & 0 \le x < 1 \\ \frac{1}{2} & 1 \le x < 2 \\ \frac{3}{2} - \frac{x}{2} & 2 \le x < 3 \\ 0 & \text{otherwise} \end{cases}$

Similarly a be obtained using integration by considering the fact that $\int_{all\ x} f(x)dx = 1$. Thus:

$$\int_0^1 ax\ dx + \int_1^2 a\ dx + \int_2^3 (3a - ax)\ dx = 1 \quad \Rightarrow \quad \left[\frac{ax^2}{2}\right]_0^1 + [ax]_1^2 + \left[3ax - \frac{ax^2}{2}\right]_2^3 = 1$$

$$\frac{a}{2} + (2a - a) + \left(a - \frac{9a}{2}\right) - \left(6a - \frac{4a}{2}\right) = 1$$

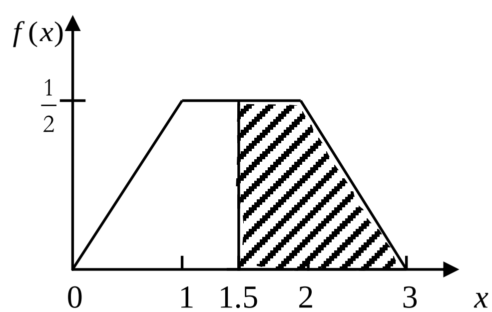
$$\frac{a}{2} + a + \left(\frac{18a - 9a}{2}\right) - \left(\frac{12a - 4a}{2}\right) = 1$$

$$\frac{3a}{2} + \frac{9a}{2} - \frac{8a}{2} = \frac{3a + 9a - 8a}{2} = 1$$

$$\frac{4a}{2} = 1$$

$$a = \frac{1}{2}$$

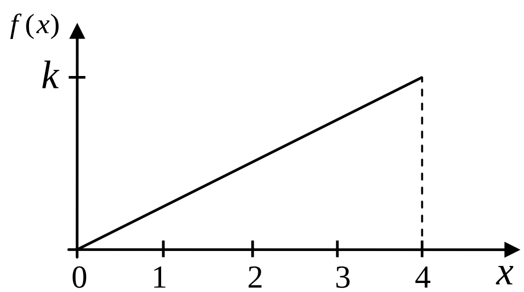
(ii) $P(1.5 \le x \le 3) = \int_{1.5}^{2} \frac{x}{2}\ dx + \int_2^3 \left(\frac{3}{2} - \frac{x}{2}\right) dx$



$$
\begin{aligned}
&= \left[\frac{x^2}{2}\right]_{1.5}^{2} + \left[\frac{3x}{2} - \frac{x^2}{4}\right]_{2}^{3} \\
&= \left(1 - \frac{9}{16}\right) + \left[\left(\frac{9}{2} - \frac{9}{4}\right) - (3-1)\right] \\
&= \frac{7}{16} + \left(\frac{9}{4} - 2\right) \\
&= \frac{11}{16}
\end{aligned}
$$

Example 4

A probability density function is defined by the diagram given below



where k is a constant

(a). Find the value of k and hence obtain the expression for $f(x)$

(b). Determine the mean and variance

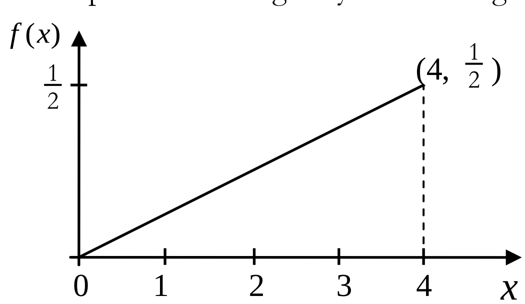
Solution

(a) The value of k can be got by considering the fact that the total area under the graph = 1

$$\Rightarrow \frac{1}{2} \times 4 \times k = 1$$

$$k = \frac{1}{2}$$

The expression for is got by determining the equation of the line



Gradient of line $= \dfrac{\frac{1}{2} - 0}{4 - 0} = \dfrac{1}{8}$

Equation of line is got from $\dfrac{y-0}{x-0} = \dfrac{1}{8}$

$$\Rightarrow y = \frac{1}{8}x$$

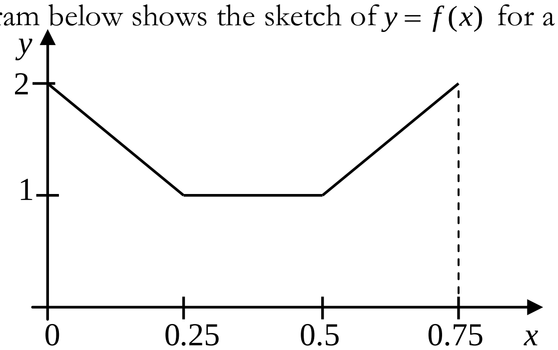
Thus $f(x)=\begin{cases}\frac{1}{8}x & 0\le x\le 4\\ 0, & otherwise\end{cases}$

(b) $Mean, \mu = \int_{all\ x} xf(x)\,dx = \int_0^4 \frac{x^2}{8}\,dx = \left[\frac{x^3}{24}\right]_0^4 = \frac{64}{24} = 2.67$

$Variance = \int_{all\ x} x^2 f(x)\,dx - \mu^2 = \int_0^4 \frac{x^3}{8} dx - (2.67)^2 = \left[\frac{x^4}{32}\right]_0^4 - 7.13 = \frac{256}{32} - 7.13 = 0.87$

Example 5

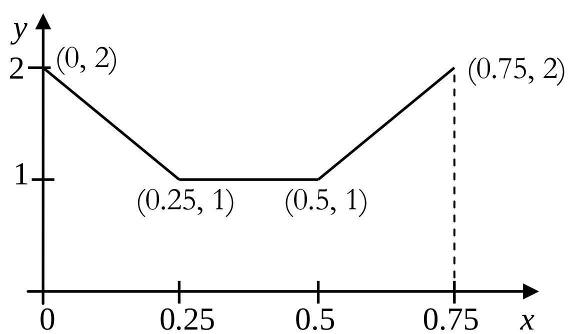
The diagram below shows the sketch of $y = f(x)$ for a certain random variable X



Use the sketch to obtain the expression for the probability density function $f(x)$

Solution

The probability density function is got by obtaining the equations of the lines in the different intervals given.



For $0\le x\le 0.25$ Gradient of line = $\frac{2-1}{0-0.25} = -4$

Equation of line is got from $\frac{y-2}{x-0} = -4$

$$\Rightarrow y = 2-4x$$

For $0.25 \le x \le 0.5$ Equation of line is $y = 1$

For $0.5 \le x \le 0.75$ Gradient of line = $\dfrac{2-1}{0.75-0.5} = 4$

Equation of line is got from $\dfrac{y-1}{x-0.5} = 4$

$$\Rightarrow y = 4x - 1$$

Thus $f(x) = \begin{cases} 2-4x & 0 \le x < 0.25 \\ 1 & 0.25 \le x < 0.5 \\ 4x-1 & 0.5 \le x < 0.75 \\ 0 & otherwise \end{cases}$

Example 6

The continuous r.v. X has probability density function $f(x)$ where

$$f(x) = \begin{cases} \tfrac{1}{8}(x+2)^2 & -2 \le x < 0 \\ \tfrac{1}{2} & 0 \le x < \tfrac{4}{3} \\ 0 & otherwise \end{cases}$$

(a) Find $P(-1 \le X \le 1)$

(b) Find $P(X > 1)$

Solution

(a) $P(-1 \le X \le 1) = \int_{-1}^{0} \tfrac{1}{8}(x+2)^2 dx + \int_{0}^{1} \tfrac{1}{2}\, dx$

$$= \tfrac{1}{24}\left[(x+2)^3\right]_{-1}^{0} + \left[\frac{x}{2}\right]_0^1$$

$$= \tfrac{1}{24}(8-1) + (\tfrac{1}{2} - 0)$$

$$= \frac{7}{24} + \frac{1}{2} = \frac{19}{24}$$

(b) $P(X > 1) = \int_{1}^{\frac{4}{3}} \frac{1}{2} dx$

$$= \left[\frac{x}{2}\right]_1^{\frac{4}{3}} = \left(\frac{4}{6} - \frac{1}{2}\right)$$

$$= \frac{1}{6}$$

Mean (Expectation) of a continuous random variable

If X is a continuous random variable with probability density function $f(x)$, then the expectation of X denoted $E(X)$ or μ is defined as:

$$\mu = E(X) = \int_{all\ x} xf(x)\ dx$$

Variance and Standard deviation of a continuous random variable

If X is a continuous random variable with probability density function $f(x)$, then the Variance of X denoted $Var(X)$ is defined as:

$$Var(X) = \int_{all\ x} x^2 f(x)\ dx - \mu^2 \qquad \text{where } \mu = \int_{all\ x} xf(x)\ dx$$

The standard deviation of X is often denoted as σ, and given by $\sigma = \sqrt{Var(X)}.$

Example 5

Given that $f(x) = \begin{cases} \dfrac{x}{2} & 0 \le x < 2 \\ 0, & elsewhere \end{cases}$

Find the mean value and standard deviation of the function

Solution

The mean of X, $\mu = \int_{all\ x} xf(x)\ dx$

$$= \int_0^2 \frac{x^2}{2}\ dx = \frac{1}{2}\left[\frac{x3}{3}\right]_0^2$$

$$= \tfrac{1}{2} \times \tfrac{8}{3} = \frac{8}{6} = \frac{4}{3}$$

The variance of X, $Var(X) = \int_{all\ x} x^2 f(x)\ dx - \mu^2$

$$= \int_0^2 \frac{x^3}{2}\ dx - \left(\frac{4}{3}\right)^2 = \frac{1}{2}\left[\frac{x^4}{4}\right]_0^2 - \frac{16}{9}$$

$$= \left(\tfrac{1}{2} \times \tfrac{16}{4}\right) - \frac{16}{9}$$

$$= 2 - \tfrac{16}{9} = \frac{2}{9}$$

The variance is $\frac{2}{9}$ and thus the standard deviation, $\sigma = \sqrt{Var(X)} = \sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{3}$

Example 6

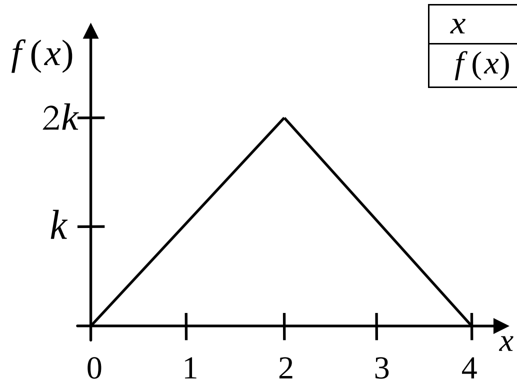
A random variable has a p.d.f. given by

$$f(x)=\begin{cases} kx, & 0\le x\le 2\\ k(4-x), & 2\le x\le 4\\ 0, & elsewhere\end{cases}$$

(a) Sketch the graph of $f(x)$
(b) Find the value of k
(c) Determine $E(X)$ and $Var(X)$.

Solution

(a) Sketch graph of $f(x)$



x	0	2	4
$f(x)$	0	2k	0

Area under graph = 1

$$\tfrac{1}{2}bh=1$$
$$\tfrac{1}{2}\times 4\times 2k=1$$
$$4k=1$$
$$k=\tfrac{1}{4}$$

(b) Alternatively

Since $\int\limits_{all\ x} f(x)\ dx=1$

$$\int_0^2(kx)dx+\int_2^4(4k-kx)dx=1$$
$$\left[\frac{kx^2}{2}\right]_0^2+\left[4kx-\frac{kx^2}{2}\right]_2^4=1$$
$$\frac{4k}{2}+\left(16k-\frac{16k}{2}\right)-\left(8k-\frac{4k}{2}\right)=1$$
$$2k+16k-8k-8k+2k=1$$
$$4k=1$$
$$k=\frac{1}{4}$$

(c) $$Mean,\mu=\int\limits_{all\ x} xf(x)\ dx=\int_0^2(kx^2)dx+\int_2^4(4kx-kx^2)dx=\left[\frac{kx3}{3}\right]_0^2+\left[\frac{4kx^2}{3}-\frac{kx^3}{3}\right]_2^4$$
$$=\frac{8k}{3}+\left(\frac{64k}{2}-\frac{64k}{3}\right)-\left(\frac{16k}{2}-\frac{8k}{3}\right)=\frac{8k}{3}+32k-\frac{64}{3}k-8k+\frac{8k}{3}=\frac{24k}{3}=8k$$
$$But\ k=\frac{1}{4}\qquad\Rightarrow\mu=8\times\frac{1}{4}=2$$

$$Var(X) = \int_{all\ x} x^2 f(x)\, dx - \mu^2$$

(d)

$$\int_{all\ x} x^2 f(x)\, dx = \int_0^2 (kx^3) dx + \int_2^4 (4kx^2 - kx^3) dx = \left[\frac{4x^4}{4}\right]_0^2 + \left[\frac{4kx^3}{3} - \frac{kx^4}{4}\right]_2^4$$

$$= \frac{16k}{4} + \left(\frac{256k}{3} - \frac{256k}{4}\right) - \left(\frac{32k}{3} - \frac{16k}{4}\right) = 4k + \frac{256k}{3} - \frac{256k}{4} - \frac{32k}{3} + 4k$$

$$= 8k + \frac{224k}{3} - 64k = \frac{224k}{3} - 56k$$

$$Since\ k = \frac{1}{4}, \Rightarrow \int_{all\ x} x^2 f(x)\, dx = \left(\frac{224}{3} \times \frac{1}{4}\right) - \left(56 \times \frac{1}{4}\right) = \frac{224}{12} - \frac{56}{4} = \frac{14}{3}$$

$$Hence\ \ Var(X) = \frac{14}{3} - 2^2 = \frac{14}{3} - 4 = \frac{2}{3}$$

Example 7

The probability density function for a continuous random variable X is

$$f(x) = \begin{cases} a + bx^2, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

where a, b are some constants. Find

(a) The values of a and b if $E(X) = \dfrac{3}{5}$

(b) $Var(X)$.

Solution

(a) Using ;

$$\int_0^1 f(x) dx = 1 \iff \int_0^1 \left(a + bx^2\right) dx = 1 \iff ax + \frac{b}{3}x^3 \Big|_0^1 = 1$$

$$\iff a + \frac{b}{3} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

And

$$E(X) = \int_0^1 xf(x) dx = \int_0^1 x\left(a + bx^2\right) dx = \frac{a}{2}x^2 + \frac{b}{4}x^4 \Big|_0^1 = \frac{a}{2} + \frac{b}{4} = \frac{3}{5} \ldots\ldots\ldots\ldots(ii)$$

Solve for the two equations, we have

$$a = \frac{3}{5},\ b = \frac{6}{5}$$

(b) $f(x) = \frac{3}{5} + \frac{6}{5}x^2,\ 0 \le x \le 1$

$0,$ otherwise.

Thus,

$$Var(X) = \int_{all\ x} x^2 f(x)dx - [E(X)]^2 = \int_{all\ x} x^2 f(x)dx - \left(\frac{3}{5}\right)^2$$

$$= \int_0^1 x^2 f(x)dx - \frac{9}{25} = \int_0^1 x^2\left(\frac{3}{5} + \frac{6}{5}x^2\right)dx - \frac{9}{25}$$

$$= \frac{1}{5}x^3 + \frac{6}{25}x^5 |_0^1 - \frac{9}{25} = \frac{1}{5} + \frac{6}{25} - \frac{9}{25} = \frac{2}{25}$$

Exercise

1. Verify that $f(x) = \begin{cases} 2x & (0 \le x \le 1) \\ 0 & elsewhere \end{cases}$ is a legitimate probability density function and find $P\left[-\frac{1}{2} < X < \frac{1}{2}\right]$. $Ans.(\frac{1}{4})$

2. The lead concentration in gasoline ranges from 0.2 to 0.6 grams per liter. The density of the random variable is given by $f(x) = \begin{cases} kx - 1 & for\ (0.2 \le x \le 0.6) \\ 0 & elsewhere \end{cases}$

   (a) Find the value of k

   (b) What is the probability that a liter of gas will have between 0.3 and 0.5 grams of lead?

3. The random variable X has a p.d.f given by

$$f(x) = \begin{cases} kx, & 0 \le x \le 1 \\ \frac{1}{3}k(4 - x), & 1 \le x \le 4 \\ 0, & elsewhere \end{cases}$$

   (i) Sketch the graph of $f(x)$ and find the value of k. $Ans.(k = \frac{1}{2})$

   (ii) Determine the mean and $Var(X)$ $Ans.(E(X) = \frac{5}{3}, Var(X) = \frac{13}{18})$

4. Verify that the function defined by

$$f(x) = \begin{cases} \frac{x}{6}, & 0 \le x \le 2 \\ \frac{1}{2} - \frac{x}{12}, & 2 \le x \le 6 \\ 0, & elsewhere \end{cases}$$

is a probability density function.

Find

(i) $P(0.5 \le x \le 1.5)$ *Ans.*$(\frac{5}{3})$

(ii) $E(X) and\ Var(X)$ *Ans.*$(E(X) = \frac{8}{3}, Var(X) = 1.6)$

5. Given that the p.d.f. of a certain continuous random variable is defined as:

$$f(x) = \begin{cases} kx, & 0 \le x \le 1 \\ k(2-x), & 1 \le x \le 2 \\ 0, & elsewhere \end{cases}$$

(i) Determine the value of k

(ii) Determine $E(X)$ and the Standard deviation.

6. A random variable X has the p.d.f given by

$$f(x) = \begin{cases} kx, & 0 \le x \le 1 \\ k(4-x^2), & 1 \le x \le 2 \\ 0, & elsewhere \end{cases}$$

(i) Find the constant k *Ans.*$(k = \frac{6}{13})$

(ii) Determine $E(X)$ and $Var(X)$ *Ans.*$(E(X) = 1.2, Var(X) = 0.12)$

7. A continuous random variable X has a p.d.f

$$f(x) = \begin{cases} kx(3-x), & 0 \le x \le 2 \\ k(4-x), & 2 \le x \le 4 \\ 0, & elsewhere \end{cases}$$

Find

(i) The value of k *Ans.*$(\frac{3}{16})$

(ii) The mean *Ans.*(1.75)

(iii) $P(1 \le x \le 3)$ *Ans.*$(\frac{11}{16})$

(iv) Sketch the function $f(x)$

8. A discrete random variable x has a probability density function

$$f(x) = \begin{cases} \frac{2}{3a}(x+a), & -a \le x < 0 \\ \frac{1}{3a}(2a-x), & 0 \le x \le 2a \\ 0, & elsewhere \end{cases}$$

$$where\ a\ is\ a\ constant.$$

Determine

(i) The value of a *Ans.*$(a = 1)$

(ii) The variance

(iii) $P(x \le 1.5/x > 0)$ *Ans.*$(\frac{15}{16})$

9. A continuous random variable X has probability density function $f(x)$ defined by

$$f(x) = \begin{cases} k(x+2), & -1 \le x \le 0 \\ 2k(1-x), & 0 \le x \le 1 \\ 0, \ elsewhere \end{cases}$$

(i) Sketch the function
(ii) Find k and the mean of x *Ans.*$(k = \frac{2}{5}, E(X) = \frac{-2}{5})$
(iii) Find the $P\left(0 < x < \frac{1}{2} / x > 0\right)$ *Ans.*$(\frac{3}{4})$

10. The continuous random variable R has probability density function given by

$$f(r) = \begin{cases} \frac{r}{6}, & 0 \le x \le 3 \\ \frac{1}{2}(4-r), & 3 \le x \le k \\ 0, & elsewhere \end{cases}$$

(a) Sketch $f(r)$ and show that $k=4$
(b) Find
   (i) $P(1 \le x \le 2)$ *Ans.*$\left(\frac{1}{4}\right)$
   (ii) Standard deviation

11. The random variable X has a p.d.f. given by

$$f(x) = \begin{cases} \frac{x}{3}, & 0 \le x \le 2 \\ \frac{-2x}{3} + 2, & 2 \le x \le 3 \\ 0, & elsewhere \end{cases}$$

(a) Sketch the graph of $f(x)$
(b) $Evaluate\ P(1 \le x \le 2.5)$
(c) Find the mean

12. The probability density function of a random variable X is given by

$$f(x) = \begin{cases} k(x+2), & -1 \le x \le 0 \\ 2k, & 0 \le x \le 1 \\ \frac{k}{2}(5-x), & 1 \le x \le 3 \\ 0, & elsewhere \end{cases}$$

(a) Sketch the function $f(x)$
(b) Find the
   (i) The value of k

(ii) Mean of x

13. A continuous random variable X takes values in the intervals shown. The probability density function of X is defined by

$$f(x) = \begin{cases} kx, & 0 \le x \le 1 \\ k, & 1 \le x \le 2 \\ 0, & elsewhere \end{cases} \qquad \text{where } k \text{ is a constant}$$

(i) Show that $k = \frac{1}{2}$
(ii) The mean and variance of x

14. A random variable X has a p.d.f. given by

$$f(x) = \begin{cases} ax - bx^2, & 0 \le x \le 2 \\ 0, & elsewhere \end{cases}$$

where $a \text{ and } b \text{ are constants}.$ Given that $E(X) = 1.$

(i) Find the values of a and b
(ii) Find the variance
(iii) Sketch the function
(iv) Find $P(1 \le X \le 2)$

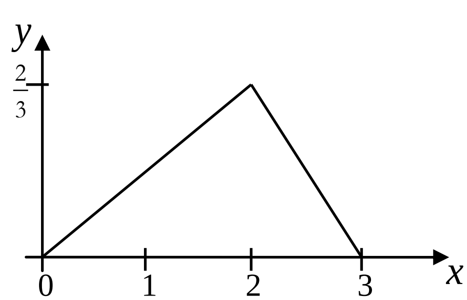
15. $A\ random\ variable\ x\ has\ a\ p.d.f\ given\ by$

$$f(x) = \begin{cases} ax^2 + bx & 0 \le x \le 2 \\ 0 & elsewhere \end{cases}$$

$where\ a\ and\ b\ are\ constants. Given\ that E(x) = 1.25$

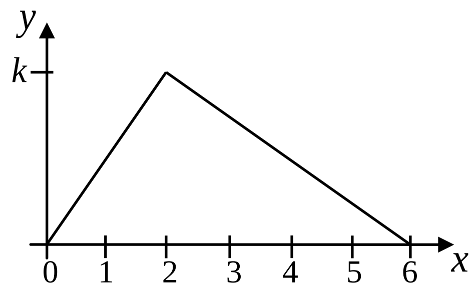
(i) Find the value of a and b *Ans.* $\left(a = \frac{3}{4}, b = \frac{-3}{16}\right)$
(ii) Find the variance *Ans.*(0.24)
(iii) Sketch the function
(iv) Find $P(1 \le x \le 2)$ *Ans.*(0.69)

16. The diagram below shows the probability density function for a continuous random variable X



(a) Obtain the expression for $f(x)$
(b) Determine the mean and variance

17. A p.d.f. is defined by the diagram given below



where k is a constant

(a) Find the value of k and hence obtain the expression for $f(x)$ Ans.$\left(k = \frac{1}{3}\right)$

$$f(x) = \begin{cases} \frac{x}{6} & 0 \le x < 2 \\ \frac{-x}{12} + \frac{1}{2} & 2 \le x < 6 \\ 0, & otherwise \end{cases}$$

(b) Determine the mean and variance

18. A random variable x takes on the values of the interval
$0 < x < 2$ $and has\ a\ p.d.f\ given\ by$

$$f(x) = \begin{cases} a, & 0 \le x \le 1\frac{1}{2} \\ \frac{a}{2}(2 - x), & 1\frac{1}{2} < x \le 2 \\ 0, & elsewhere \end{cases}$$

$Find$

(i) The value of a
(ii) $P(x < 1.6)$

19. The probability density function for a continuous random variable X is

$$f(x) = \begin{cases} \frac{x+2}{18}, -2 < x < 4 \\ 0, \text{otherwise} \end{cases}$$

Find (a) $P(|X| < 1)$ (b) $P(X^2 < 9)$ (c) $E(X)$ and $Var(X)$

20. The continuous random variable, X, has probability density function defined by

$$f(x) = \begin{cases} kx, & 0 \le x \le 8 \\ 8k, & 8 \le x \le 9 \\ 0, & otherwise \end{cases}$$

where k is a constant.

(i) Sketch the graph of $f(x)$

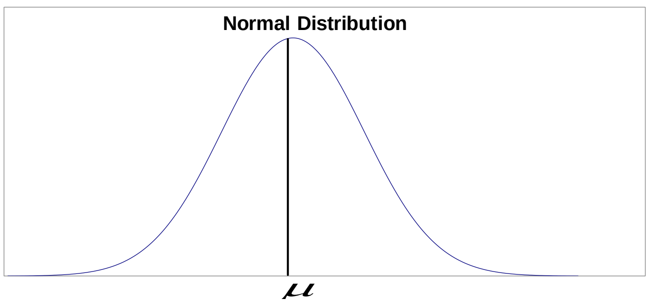
(ii) Show that $k = 0.025$.

(iii) Calculate the probability that an observed value of X exceeds 6.

NORMAL DISTRIBUTION

Concept of normal distribution

Normal distribution is probability distribution of a continuous random variable and is the most commonly used in statistics. A variable with a normal distribution displays what we call a bell-shaped distribution. It is called "normal" because it is a good model for random error around a particular value. For instance, a person's height will not be exactly equal to the average height for all people in the world. However, we would expect a lot of people to be fairly close to the average, and less people to be much taller or shorter. Additionally, we might expect that there will be as many people below average as there are people above average. For this reason, we might use a bell-shaped distribution to signify more people in the middle and fewer in the extremes.



This famous bell – shaped curve is the graph of a normal distribution. Many characteristics of manufactured products and human characteristics such as height, mass, intelligence, and various kinds of abilities approximate a normal distribution.

The Normal Distribution has 2 parameters, the mean μ and the standard deviation σ. The mean represents the central point of the distribution, while the standard deviation describes the width of the distribution. The higher the standard deviation the wider the Normal curve will be.

Properties of a Normal Distribution

1. It is a continuous distribution.
2. Every normal distribution has a mean, μ, and a standard deviation, σ.
3. The curve is bell – shaped and symmetric about the mean μ.
4. The total area under the curve and above the horizontal axis is equal to 1.
5. Almost all the population lies within 3 standard deviations of the mean.

Calculating probabilities using the Normal Distribution

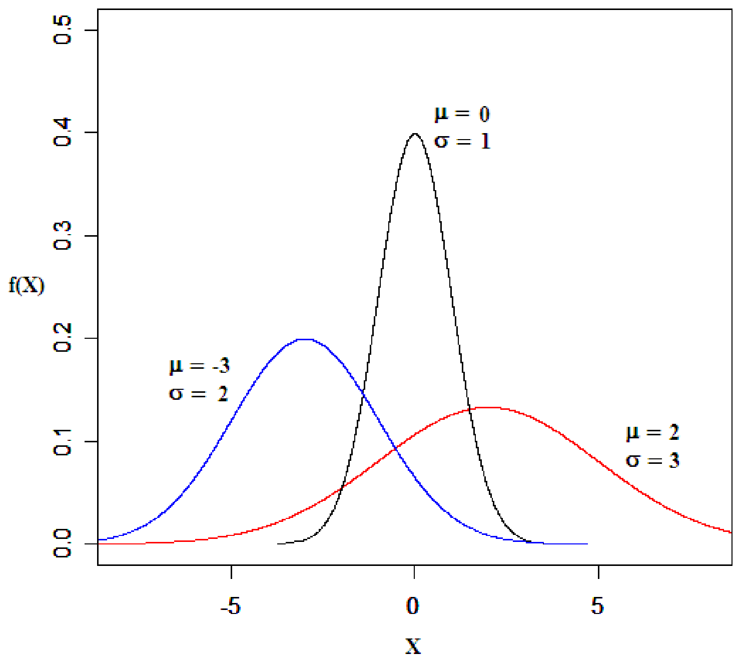
For a continuous random variable X, the formula for the density curve of the normal distribution is:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$ (this determines the bell shape)

While the formula is too complicated to integrate, by observation you can see that the formula is entirely determined given values of μ and σ. For this reason, the mean and standard deviation completely define a normal distribution.

If X is distributed in this way we write $X \sim N(\mu, \sigma^2)$

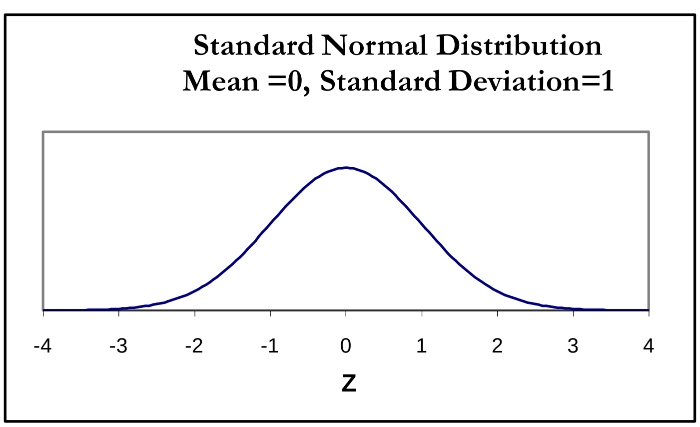
Here is a plot of normal distributions with different values of μ and σ:



In order to obtain the probabilities under the normal curve, there are Normal Probability Tables available, with this calculation done for one particular Normal Distribution called the *Standard Normal Distribution*.

The Standard Normal Distribution

The **standard normal distribution** is the normal distribution with mean 0 and standard deviation 1. It is often denoted Z to differentiate it from a regular random variable X or Y. So $Z \sim N(0,1)$



To calculate probabilities for a Normal Distribution, with a different mean and standard deviation, we must first convert the values (X) from that distribution into standard values (Z) using the following formula: $Z = \frac{X - \mu}{\sigma}$ where μ = Mean and σ =Standard Deviation. The process of obtaining the standard normal variable Z is known as standardization.

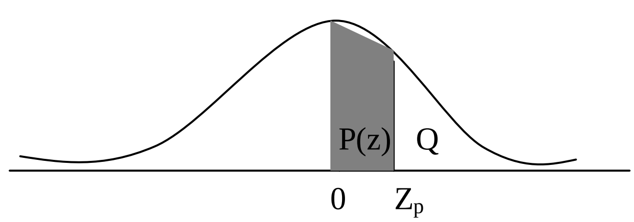
Standard Normal tables

These tables are used find areas below certain values of Z of a normal distribution. The standard normal tables give cumulative probabilities. The organization of the table is as follows:

- The left column gives a value z to one decimal place.
- The top row gives the 2nd decimal place for z.
- The numbers in the body of the table give the area *below* the z-score given by the corresponding numbers in the row/column.

i.e. the numbers in the body of the table are $P(Z < z)$, where Z is the standard normal random variable.

Note that the table only gives the area *below* particular z-values, but not the area *above* or *between* values. To find these areas, you must use the fact that the area under the entire curve is 1.



Example 1

Consider a normal distribution with mean 70 and standard deviation 2. What is $P(X < 68)$?

Solution

To find areas under this special distribution, statisticians used numerical integration methods to find areas below certain values of Z, (these are called z-scores). These areas have been compiled into a table (Table A-2, back cover of the book). The organization of the table is as follows:

- The left column gives a value z to one decimal place.
- The top row gives the 2nd decimal place for z.
- The numbers in the body of the table give the area *below* the z-score given by the corresponding numbers in the row/column.

i.e. the numbers in the body of the table are $P(Z < z)$, where Z is the standard normal random variable.

Note that the table only gives the area *below* particular z-values, but not the area *above* or *between* values. To find these areas, you must use the fact that the area under the entire curve is 1.

MECHANICS

Mechanics is a branch of the physical sciences concerned with the state of rest or motion of bodies that are subjected to the action of forces. It is therefore concerned with much that we meet in every day life – the motion of engines, the flight of aero – planes, the stresses in bridges and frameworks and many similar problems. Such problems, as we shall constantly note have to be simplified and somewhat idealized to make them capable of solution by simple mathematics, but this simplification is a necessary preliminary to any more detailed attack.

MOTION IN A STRAIGHT LINE

A body is said to be in motion if its position changes with time. The position of the object can be specified with respect to some assigned origin called a reference position.

Terms used to describe motion in a straight line

Distance

This is the length between any two specified points.

Speed

This is the rate of change of distance.

$$\Rightarrow \text{Speed} = \frac{\text{Distance covered}}{\text{Time taken}}$$

Thus: Distance travelled = speed × time

The speed expresses the rate of motion without specifying the direction. Speed is therefore a quantity having magnitude only, and is defined completely when we know this magnitude. Any quantity having magnitude and no direction is called a **scalar quantity**, so that speed is a scalar.

Uniform speed may be defined as such that the body moves through equal length of its path in equal times. Thus a body is said to be moving with constant/uniform speed if its speed remains unchanged such that the body travels the same distance in equal time intervals.

For example a car moving at constant speed of $50kmh^{-1}$ implies that; the car will travel 50km in each hour. The car would therefore travel 100km in 2 hours, 150km in 3 hours etc.

Displacement

This is the distance measured in a specified direction. Displacement is measured from a given initial position in the direction given. Displacement is a quantity specified by both magnitude and direction. Any quantity having both magnitude and direction is called a **vector quantity**, so that displacement is a vector quantity.

For example; if we say that a car travels a distance of 100m, the expression "100m" is a scalar quantity. But if the car happens to be moving along a straight line and we mention the direction of travel say 100m due east, we are now dealing with a vector quantity; and this is called the displacement of the car.

When the direction of motion of a body remains unchanged, then the distance travelled is equal to the displacement. If the direction of motion changes part way through the motion, then the distance travelled is not equal to the displacement. In formulae S is used to denote both distance and displacement.

Suppose a body moves 15km due east and then 10km due west;

Distance moved = 15km + 10km = 25km

Displacement from initial position = 15km E + 10km W = 5km E

Velocity

This is the rate of change of displacement. It is a vector quantity. The velocity of a body is a measure of the speed at which it is travelling in a particular direction. If a body has **constant or uniform velocity**, then both the speed and the direction of motion of the body remain unchanged.

In most cases speed is referred to even if it is velocity. Thus the velocity of a car may be stated as $80kmh^{-1}$ due South and the speed of this car is then $80kmh^{-1}$. The letter v is used to denote both speed and velocity but the difference between them need to be remembered.

Thus;

Distance travelled = speed × time

$$S = v \times t$$

Similarly;

Distance travelled in a particular direction = velocity × time taken

$$S = v \times t$$

Acceleration

This is the rate of change of velocity. It is a vector quantity.

$$\Rightarrow Acceleration, a = \frac{\text{change in velocity}}{Time\ \ taken} = \frac{dv}{dt}$$

Uniform or constant acceleration is when equal changes in velocity occur in equal time intervals. Thus if a body moves so that the changes of velocity in any equal times are the same in direction and equal in magnitude, the acceleration is said to be uniform.

Consider a car travelling in a straight line. If initially its velocity is $5ms^{-1}$ and 3 seconds later its velocity is $11ms^{-1}$, the car is said to be accelerating.

For the above example, the car would be accelerating with acceleration $a = \dfrac{11-6}{3} = 2ms^{-2}$.

Average speed

This is the ratio of the total distance covered to the total time taken.

$$\Rightarrow \text{Average speed} = \frac{\text{total distance covered}}{\text{total time taken}}$$

Average velocity

This is the ratio of the total distance covered in a particular direction to the total time taken.

$$\Rightarrow \text{Average velocity} = \frac{\text{total distance covered in a particular direction}}{\text{total time taken}}$$

Example 1

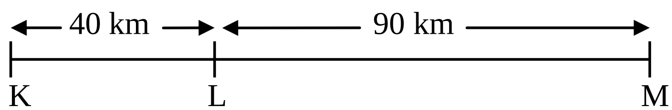
K, L and M are three points in that order in a straight road with KL = 40km and LM = 90km. A woman travels from K to L at 10 kmh^{-1} and then from L to M at 15 kmh^{-1}. Calculate:

(a) the time taken to travel from K to L

(b) the time taken to travel from L to M

(c) the average speed of the woman for the journey from K to M.

Solution



(a) Using $S = v \times t$ for K to L gives:

$$40 = 10t$$
$$t = 4h$$

$\therefore$ The time taken to travel from K to L is 4 hours.

(b) Using $S = v \times t$ for L to M gives:

$$90 = 15t$$
$$t = 6h$$

$\therefore$ The time taken to travel from L to M is 6 hours.

(c) Using $\text{Average speed} = \frac{\text{total distance travelled from K to M}}{\text{total time taken}}$

$$v = \frac{40+90}{4+6}$$
$$v = 13kmh^{-1}$$

The average speed for the whole journey is 13 kmh^{-1}.

Example 2

A body travelling at a constant speed covers a distance of 200m in 8 seconds. Find the speed of the body.

Solution

Using $\text{Speed} = \frac{\text{Distance covered}}{\text{Time taken}}$

$$v = \frac{200}{8} = 25ms^{-1}$$

Example 3

A, B and C are points on a straight line. At time t=0, a body passes through a point A and is moving with constant velocity of $4ms^{-1}$

(a) Find how far the body is from A when t= 3s.
(b) What is the value of t when the body is at a point B, 24m from A?
(c) If the body rests for 5s at B and it then moves with constant velocity of $6ms^{-1}$ to point C which is at a distance 30m from B, find the average velocity of the body's journey from A to C.

Solution

(a) Using $S = v \times t$

Distance from A = 4×3 = 12m

The body is 12m from A when t=3s.

(b) Using $S = v \times t$ for A to B

$$24 = 4t$$

$$t = 6s$$

The body takes 6s to move from A to B

(c) Using $S = v \times t$ for B to C

$$30 = 6t$$

$$t = 5s$$

Thus the body takes 5s to move from B to C

$$\Rightarrow \text{Average velocity} = \frac{\text{total distance covered from A to C}}{\text{total time taken}}$$

$$= \frac{24+30}{6+5+5} = \frac{54}{16} = 3.375ms^{-1}$$

The average velocity for the journey is 3.375ms-1

Change of Units (kmh^{-1} to ms^{-1})

The speed or velocity in kmh^{-1} can be changed to ms^{-1} by considering the fact that a body which is travelling at a certain number of kilometers per hour is of course travelling a number of metres per second.

1km = 1000m
1 hour = (60×60) s = 3600s

Example 1

Express a speed of $36kmh^{-1}$ in ms^{-1}.

Solution

$$36kmh^{-1} = \frac{36 \times 1000}{3600} = 10ms^{-1}$$

A speed of 36kmh^{-1} is equivalent to a speed of 10ms^{-1}.

Example 2

Express a speed of 81kmh^{-1} to ms^{-1}.

Solution

$$81kmh^{-1} = \frac{81 \times 1000}{3600} = 22.5ms^{-1}$$

In calculation, the units of the quantities involved must be consistent. If the speed is in kmh^{-1}, then the time must be in hours and the distance in km. However if the speed is in ms^{-1}, then the time must be in seconds and distance in metres.

Example 3

Find the distance travelled in 3 minutes by a body moving with a constant speed of 15kmh^{-1}. Find also the time taken by this body to travel 200m at the same speed.

Solution

$v = 15kmh^{-1}$ t=3 minutes = $\frac{3}{60} = \frac{1}{20}h$

$S = vt$ gives S = $15 \times \frac{1}{20} = \frac{3}{4}km$ Or $\frac{3}{4} \times 1000 = 750m$

To find the time taken to travel 200m

$S = 200m$ $v = 15kmh^{-1} = \frac{15 \times 1000}{3600} = \frac{25}{6}ms^{-1}$

Using $S = vt$ gives $200 = \frac{25}{6} \times t$ where t is measured in seconds

$$t = 48s$$

Exercise

1. Find the distance travelled in 5 seconds by a body moving with constant speed of 3.2 ms^{-1}.
2. If an athlete runs a 1500 metre race in 3 minutes 33 seconds, find his average speed for the race.
3. A body travelling at a constant speed covers a distance of 3km in 2 minutes. Find the speed of the body in ms^{-1}.
4. Express a speed of 35kmh^{-1} in ms^{-1}.

5. A, B and C are three points lying in that order on a straight road with AB= 5km and BC=4km. A man runs from A to B at 20kmh^{-1} and the walks from B to C at 8kmh^{-1}. Find;

   (a) the total time taken to travel from A to C.

   (b) the average speed of the man for the journey from A to C.

6. A car is driven from Town A to Town B, 40km away at an average speed of 60kmh^{-1}. The car is at B for 10 minutes and is then driven back to A.

(a) Find the average speed for the journey B to A if the average speed for the complete journey is 60kmh-1.

(b) What is the average velocity of the car for the complete journey?

GRAPHICAL REPRESENTATION

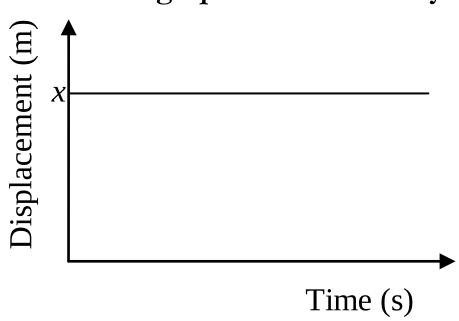
Linear motion can be represented graphically. The most important graphs are the s – t (distance or displacement – time graphs and the v – t (velocity – time) graphs.

Displacement – time graph

This is a graph that shows the displacement, S of the body at any time t. The displacement time graph shows the displacement of an object as a function of time. Positive displacements indicate the object's position is in the positive direction from its starting point, while negative displacements indicate the object's position is opposite the positive direction.

The gradient of a displacement – time graph represents velocity. Thus $\frac{dS}{dt}$ is a measure of the velocity in ms^{-1}.

Displacement – time graph for a stationery body

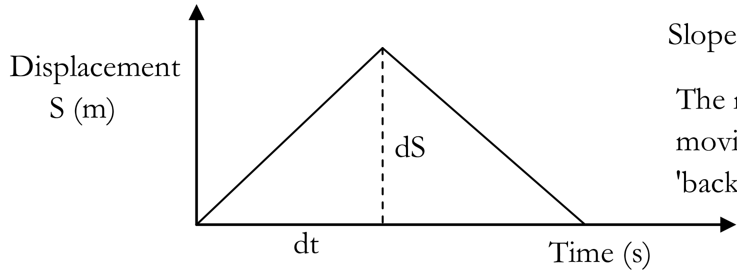


Stationary - object stays at position 'x' all of the time!

The Gradient of the graph = 0 ms^{-1} so velocity = 0 ms^{-1}

Displacement – time graph for a body moving with uniform velocity

A body moving with uniform velocity covers equal displacements in equal times. The displacement – time graph is a straight line like OA. The steeper the line the bigger the gradient and hence the greater the velocity.

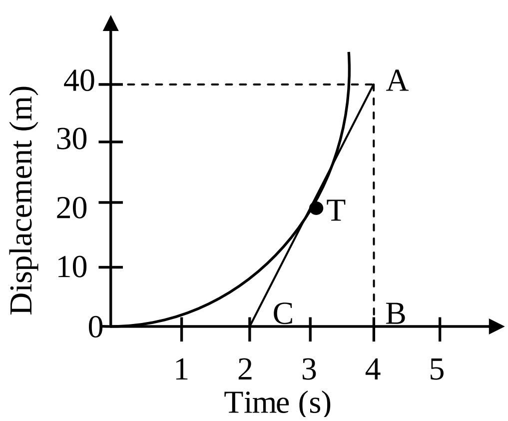


Slope or Gradient = $\frac{dS}{dt}$ = Velocity

The negative gradient indicates the body is moving in the opposite direction (moving 'backwards') - so it finishes up where it started!

Displacement – time graph for a body moving with non –uniform velocity

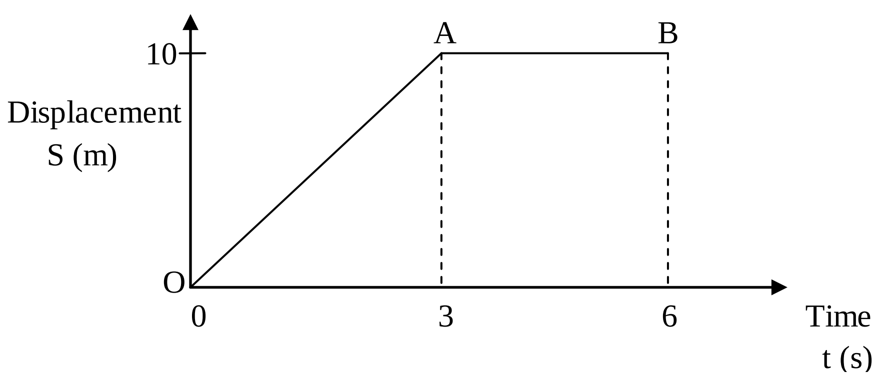
When the velocity of the body is changing, the slope of the displacement – time graph varies, and at any point equals to the slope of the tangent. For example in the figure shown below, the slope of the tangent at T is $\frac{AB}{BC} = \frac{40}{2} = 20$. The velocity at the instant corresponding to T is therefore 20ms^{-1}.



Example 1

The graph below shows the displacement time graph for a moving object. Describe the motion in regions OA and AB.

What is the velocity of the object during the first 3 seconds?



Solution

From O to A, the object is moving with uniform velocity and from A to B, the object is stationery i.e. having zero speed.

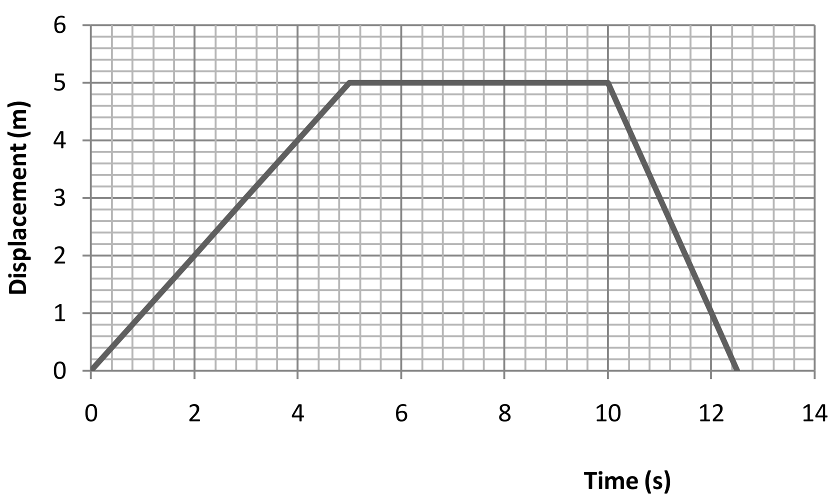
The velocity of object during the first 3s = Gradient of AB

$$\frac{dS}{dt} = \frac{10}{3} = 3.3ms^{-1}$$

Example 2

Mr. Olal is jogging away from his house at a uniform velocity and stops after 5 seconds when he is 5m away. He then decides to take a short five-second rest in the playground. After his five second rest, he runs back to the house in $2\frac{1}{2}$ seconds. Draw the displacement-time graph for his motion and describe Olal's motion.

Solution

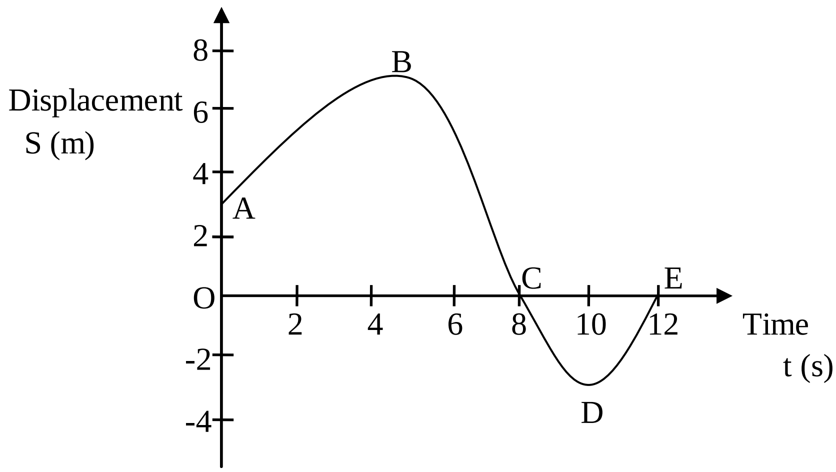


From the graph, Mr. Olal's displacement begins at zero meters at time zero. Then, as time progresses, his displacement increases at a rate of 1 ms^{-1}, so that after one second, Olal is one meter away from his starting point. After two seconds, he is two meters away, and so forth, until he reaches his maximum displacement of five meters from his starting point at a time of five seconds. He then remains at that position for 5 seconds while he takes a rest. Following his rest, at time t=10 seconds, Olal races back to the house at a speed of 2 ms^{-1}, so the graph ends when he returns to her starting point at the house, a total distance traveled of 10m, and a total displacement of zero meters.

As we look at the displacement – time graph, notice that at the beginning, when Olal is moving in a positive direction, the graph has a positive slope. When the graph is flat (has a zero slope) he is not moving and when the graph has a negative slope, Olal is moving in the negative direction that is back to his house.

Example 3

The facts below can be seen from the displacement – time graph below;



1. The starting or initial position A of the body is 3m from a reference position O. The initial velocity is given by the gradient at A and is positive. The body is travelling away from O in the positive direction.

2. From A to B the body travels further away from O but its velocity is decreasing, as the gradient is decreasing.

3. At B, 7m from O, the velocity is zero. The body is momentarily at rest. The time is 5s from the start.

4. From B to C the gradient is negative. Hence the velocity is negative that is the body is travelling back towards O and arrives there after 8s from the start. However the distance travelled up to this time is 3+7 = 10m but the displacement from O is zero.

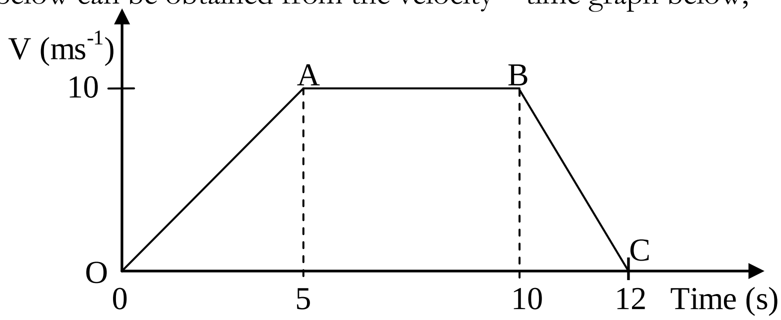
5. The body reaches position D (3m from O in the opposite direction to B) after 10s and is momentarily at rest.

6. From D to E, the body is again travelling in the positive direction and reaches O again after 12s from the start.

Velocity – time graph

The velocity – time graph is the v – t graph relating velocity to time. The gradient of a velocity – time graph represents acceleration.

The facts below can be obtained from the velocity – time graph below;



- The gradient of this graph $\left(\frac{dV}{dt}\right)$ measures the acceleration in ms^{-2}.
- The body starts at t=0, from rest (with zero velocity). From O to A the velocity increases until it reaches 10ms^{-1} at time 5s. Since OA is a straight line, the acceleration is constant or uniform and is equal to gradient of OA = $\frac{10-0}{5-0} = \frac{10}{5} = 2ms^{-2}$. At A acceleration stops.
- From A to B the body travels with constant velocity (10ms^{-1}) and thus acceleration is zero.
- From B to C the velocity decreases steadily and the body comes to rest again at t= 12s. From B to C the acceleration is negative also known as retardation. The body has a uniform retardation equal to the gradient of BC $= -\frac{10-0}{12-10} = -\frac{10}{2} = -5ms^{-2}$.

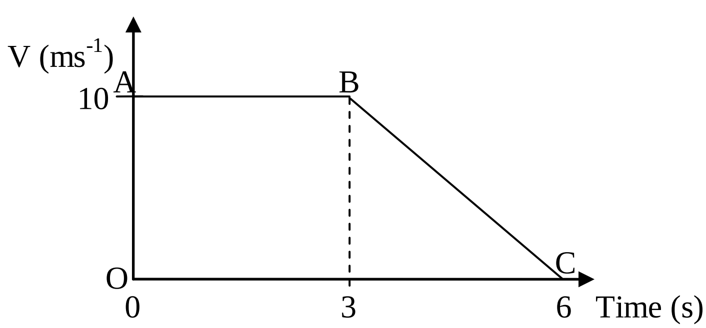
Area under the velocity – time graph

The area under a velocity – time graph is numerically equal to the distance covered by the body and represents strictly the displacement.

If velocity is measured in ms^{-1} and time in s, the area will give the displacement in m; if velocity is in kmh^{-1} and time in h, the area represents the distance measured in km. Thus care must be taken to work in consistent units, not for example ms^{-1} and h.

Example 1

The graph below represents the velocity – time graph of an object for which it is in motion. Calculate the total distance travelled and the average speed during this period.



Solution

Total distance travelled = area under velocity – time graph

$$= \text{area of trapezium OABC}$$

$$= \frac{1}{2} OA(AB + OC)$$

$$= \frac{1}{2} \times 10(3 + 6)$$

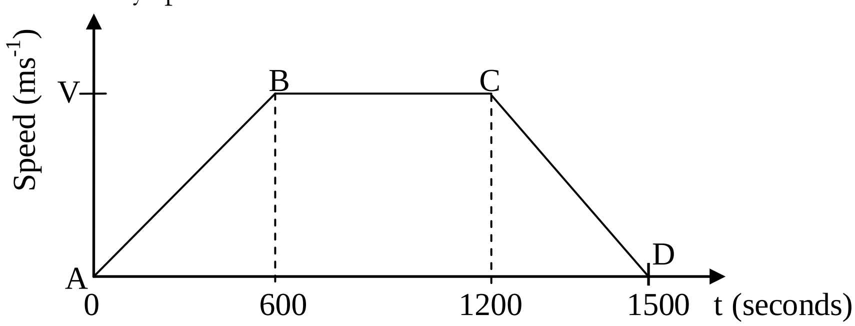
$$= 5(9) = 45m$$

Example 2

A car accelerates from rest to reach a certain speed in 600 seconds. It then continues at this speed for another 600 seconds and decelerates to rest in further 300 seconds. The total distance covered is 17500m. Find the steady speed reached.

Solution

Let the steady speed reached be V



The total distance covered = area under the velocity time graph (area of trapezium ABCD)

$$17500 = \frac{1}{2} \times V(AD + BC)$$

$$17500 = \frac{1}{2} \times V(1500 + 600)$$

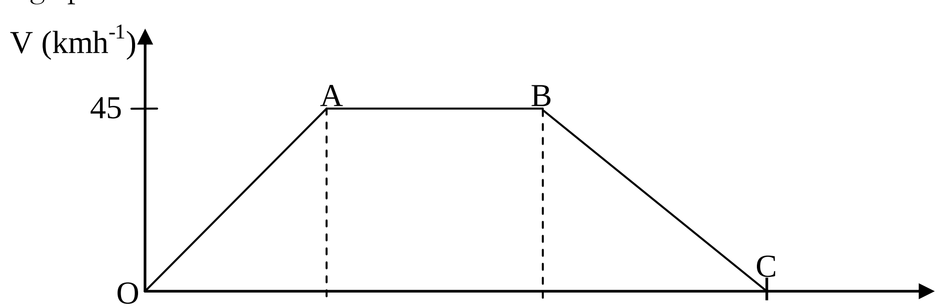
$$35000 = 2100V$$

$$16.7ms^{-1} = V$$

The steady speed reached by the car is $16.7ms^{-1}$.

Example 3

The graph below shows the motion of a car.



0 5 10 15 t (minutes)

Use the graph to find (i) the acceleration
(ii) the retardation, both in ms^{-2}.
(iii) the distance travelled in km.

Solution

Since the acceleration and deceleration are required in ms^{-2}, then the velocity should be in ms^{-1} and time in seconds.

$45kmh^{-1} = \frac{45\times1000}{3600} = 12.5ms^{-1}$ and 5 minutes = $5\times60=300s$

(i) The acceleration = Gradient of velocity time graph along OA

$$= \frac{dV}{dt} = \frac{22.5-0}{300-0} = 0.075ms^{-2}$$

(ii) The retardation = Gradient of velocity time graph along BC

$$= \frac{dV}{dt} = \frac{22.5-0}{(10-15)60} = \frac{22.5}{-300} = -0.075ms^{-2}$$

(iii)Since the distance is required in km, the velocity should be in kmh^{-1} and the time in h

Distance travelled = area under velocity – time graph (area of trapezium OABC)

$$= \frac{1}{2}\times45(AB+OC)$$

$$= \frac{1}{2}\times45\left(\frac{5}{60}+\frac{15}{60}\right)$$

$$= 7.5km$$

Example 4

A train starts from station P and accelerates uniformly for 2 minutes reaching a speed of $48kmh^{-1}$. It continues at this speed for 5 minutes and then is retarded uniformly for a further 3 minutes to come to rest at station Q. Find

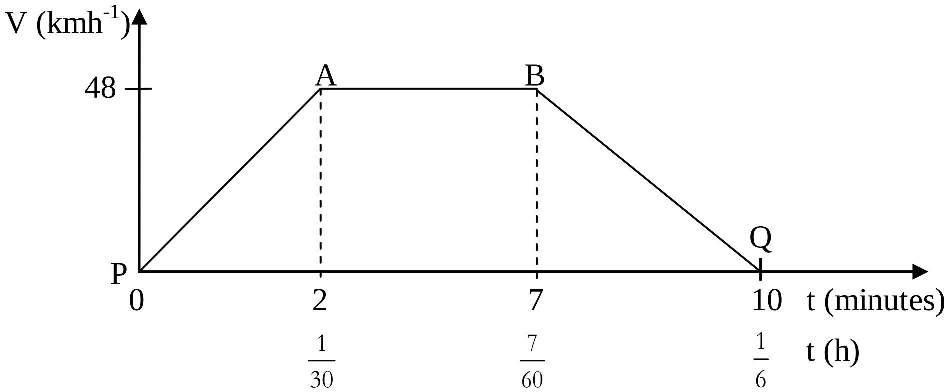
(i) the distance PQ in km.

(ii) the average speed of the train.

(iii) the acceleration in ms-2

(iv) the time taken to cover half the distance between P and Q.

Solution



(i) The area under the graph represents the distance travelled

The graph is a trapezium of area = $\frac{1}{2}(sum$ of parrallel sides$)\times$height

$$=\frac{1}{2}(AB+PQ)\times 48$$

$$=\frac{1}{2}\times\left(\frac{5}{60}+\frac{1}{6}\right)\times 48 = 6km$$

So the distance PQ = 6km

(ii) The average speed = $\frac{total \text{ distance travelled}}{\text{time taken}}$

$$= \ 6\Big/\tfrac{1}{6} = 36kmh^{-1}$$

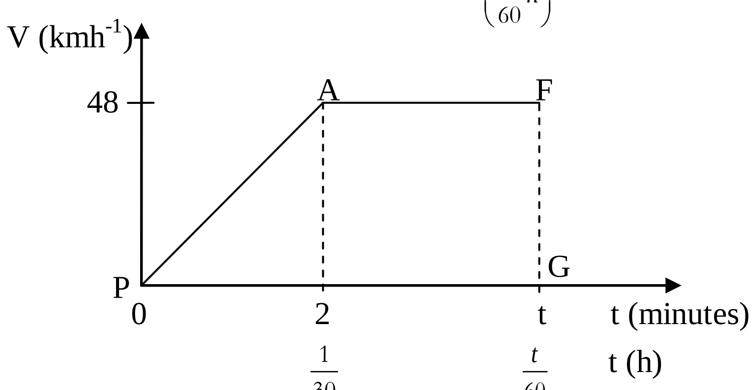
(iii) To find the acceleration over section OA, we use metres (m) and seconds (s)

$$48kmh^{-1} = \frac{48\times 1000}{3600}ms^{-1} = \frac{40}{3}ms^{-1}$$

Acceleration = Gradient of velocity time graph

$$\text{Gradient of OA} = \frac{40}{3}\Big/120 = 0.11ms^{-2}$$

(iv) Let half the distance be covered in t min $\left(\frac{t}{60}h\right)$



The area OAFG represents 3km (half the distance covered)

$$\therefore 3 = \frac{1}{2}\left(AF + OG\right)\times 48 = \frac{1}{2}\left(\frac{t}{60} - \frac{1}{30} + \frac{t}{60}\right)\times 48$$

$\Rightarrow 7.5 = 2t - 2$ Or $\Rightarrow 7.5 = 2t - 2$ *Or* $t = 4.75$ minutes.

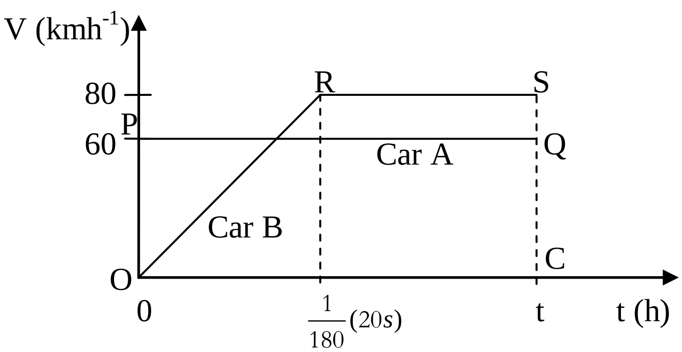
Example 5

Car A is travelling at a steady speed of 60kmh^{-1} and passes a stationery car B. The driver of car B immediately accelerates reaching 80kmh^{-1} in 20s and continues at this speed until he overtakes car A. Find the distance travelled by car B when this happens.

Solution

Since the speed has been given in kmh^{-1}, the time must be converted to hours.

$$20s = \frac{20}{3600} = \frac{1}{180}\ hours$$



Since the two cars A and B travel the same distance, the area ORSC = the area OPQC

Let t (in hours) be the time taken for car B to overtake car A

Then area ORSC = the area OPQC

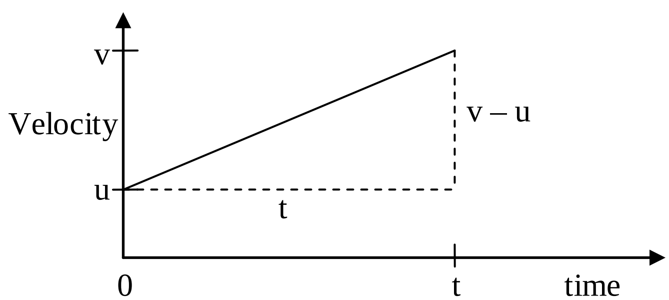
$$\frac{1}{2}\times 80\left(t + t - \frac{1}{180}\right) = 60\times t$$

$$40\left(\frac{180t+180t-1}{180}\right)=60t$$
$$40(360t-1)=10800t$$
$$360t-1=270t$$
$$90t=1$$
$$t=\frac{1}{90}h$$

Hence the distance travelled by B = $v\times t=60\times\frac{1}{90}=0.67km.$

Equations of linear motion

A basic type of motion is that of a body travelling in a straight line with constant or uniform acceleration. For this motion we can derive useful equations known as the equations of motion. Consider the motion of a body which accelerates uniformly from an initial velocity **u** to a final velocity **v** in time **t**. The velocity – time graph can be represented as shown below;



The acceleration of the body is defined as the rate of change of velocity i.e. $a=\frac{v-u}{t}$

1. Since the **gradient** of a **velocity/time graph** gives **acceleration**, we have:

   $$a=\frac{v-u}{t}$$

   $\Rightarrow at=v-u$ (By cross multiplication)

   Or $\boxed{v=u+at}$ ………………………….. (Equation 1)

2. The area under the velocity – time graph represents the displacement.

   Hence $S=\frac{1}{2}(u+v)\times t$ (Area of Trapezium)

   Substituting for v in this equation, $S=\frac{1}{2}(u+u+at)\times t$ gives

   $$\boxed{S=ut+\frac{1}{2}at^2}$$

…………………………….. (Equation 2)

3. From the first two equations; $S = \frac{1}{2}(u+v)\times t = \frac{1}{2}(u+v)(\frac{v-u}{a})$ Or $2as = v^2 - u^2$

$\Rightarrow$ $\boxed{v^2 = u^2 + 2as}$ ……………………………… (Equation 3)

The three equations for uniformly accelerated motion in a straight line involving uniform acceleration can be used to solve problems instead of using a velocity – time graph or in addition to doing so.

Important points to note:

- A body starting at rest has initially velocity zero.
- A body brought to rest has its final velocity as zero.
- A stationery body has velocity zero.
- A deceleration implies negative acceleration.

Example 1

A body starts from rest and moves in a straight line, accelerating uniformly at $5ms^{-2}$. Find its velocity after 6 seconds.

Solution

Using $v = u + at$

We have: u=0; a = 5; t = 6

$\therefore$ v = 0+5(6)

$\therefore$ v = $30ms^{-1}$

This means that the velocity after 6 seconds is $30ms^{-1}$.

Example 2

A particle starts with velocity $3ms^{-1}$ and accelerates at $0.5\ ms^{-2}$. What is its velocity after

(i) 3s (ii) 10s? How far has it travelled in these times?

Solution

(i) Using $v = u + at$

We have: u=$3ms^{-1}$; a = $0.5ms^{-2}$; t = 3s; v=?

$\therefore$ v = 3+0.5(3)

$\therefore$ v = $4.5ms^{-1}$

This means that the velocity after 3 seconds is $4.5ms^{-1}$.

(ii) Using $v = u + at$

We have: $u=3ms^{-1}$; $a = 0.5ms^{-2}$; $t = 10s$; $v=?$

$\therefore v = 3+0.5(10)$

$\therefore v = 8ms^{-1}$

This means that the velocity after 10 seconds is 8 ms^{-1}.

The distance travelled in 3s; $S = ut + \frac{1}{2}at^2$ where $u=3ms^{-1}$; $a= 0.5ms^{-2}$; $t=3s$

$$\Rightarrow S = 3\times 3 + \frac{1}{2}\times 0.5\times 3^2$$

$$= 9+2.25$$

$$S = 11.25m$$

The distance travelled in 10s; $S = ut + \frac{1}{2}at^2$ where $u=3ms^{-1}$; $a= 0.5ms^{-2}$; $t=10s$

$$\Rightarrow S = 3\times 10 + \frac{1}{2}\times 0.5\times 10^2$$

$$= 30+25$$

$$S = 55m$$

Example 3

A body moves along a straight line from A to B with uniform acceleration $\frac{2}{3}ms^{-2}$. The time taken is 12s and the velocity at B is $25ms^{-1}$. Find;

(i) the velocity at A

(ii) the distance AB

Solution

(i) Using $v = u + at$

We have: $v=25ms^{-1}$; $a = \frac{2}{3}ms^{-2}$; $t = 12s$; $u=?$

$\therefore 25 = u+\frac{2}{3}(12)$

$\therefore u = 17ms^{-1}$

Therefore the velocity at A is 17 ms^{-1}.

(ii) Using $S = ut + \frac{1}{2}at^2$ where $u=17ms^{-1}$; $a= \frac{2}{3}ms^{-2}$; $t=12s$

$$\Rightarrow S = 17\times 12 + \frac{1}{2}\times \frac{2}{3}\times 12^2$$

$$= 204+48$$

$$S = 252m$$

The distance AB is 252m

Similarly the distance AB can be obtained by using the third equation:

$v^2 = u^2 + 2as$ where v=25ms^{-1}; u= 17ms^{-1}; a= $\frac{2}{3}ms^{-2}; S = AB?$

$$\Rightarrow 25^2 = 17^2 + 2\times\left(\frac{2}{3}\right)S$$

$$625 - 289 = \frac{4}{3}S$$

1008 = 4S (By cross - multiplication)

S = 252 (as before)

Example 4

A cyclist travelling down hill accelerates uniformly at 1.5ms^{-2}. If his initial velocity at the top of the hill is 3ms^{-1}, find how far he travels before reaching a velocity of 7ms^{-1}.

Solution

Using $v^2 = u^2 + 2aS$ where v=7ms^{-1}; u= 3ms^{-1}; a= $1.5ms^{-2}; S = ?$

$$\Rightarrow 7^2 2 = 3^2 + 2\times 1.5S$$

$$49 - 9 = 3S$$

40 = 3S

S = $\frac{40}{3}$ m

The cyclist travels a distance of $\frac{40}{3} = 13.33m$ before reaching a velocity of 7ms^{-1}.

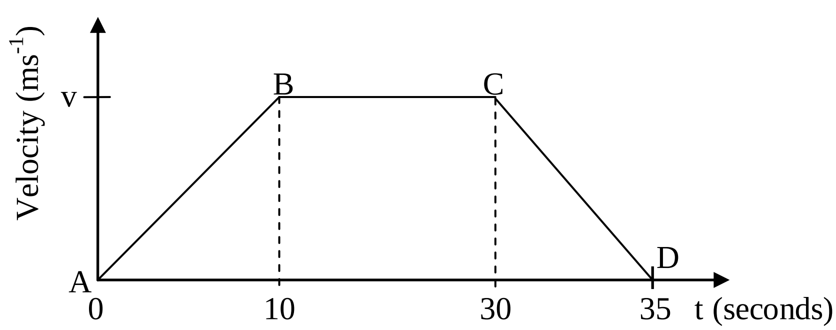
Example 5

A car starts from rest, accelerating at 1ms^{-2} for 10s. It then continues at a steady speed for a further 20s and decelerates to rest in 5s. Draw the velocity – time graph and find;

(i) the distance travelled in m.

(ii) the average speed in ms^{-1}.

Solution



Using $v = u + at$ for the acceleration ;

We have: $u = 0ms^{-1}$; $a = 1ms^{-2}$; $t = 10s$; $v = ?$

$\therefore v = 0 + 1(10)$

$\therefore v = 10ms^{-1}$

This means that steady speed reached after 10 seconds is 10 ms^{-1}.

(i) Distance travelled = area under velocity – time graph (area of trapezium ABCD)

$$= \frac{1}{2} \times v(BC + AD)$$

$$= \frac{1}{2} \times 10\left(20 + 35\right)$$

$$= 275m$$

(ii) The average speed = $\dfrac{total \text{ distance travelled}}{\text{time taken}}$

$$= \frac{275}{35} = \frac{55}{7}m$$

Example 6

A motorist travelling on a straight stretch of motorway at a steady speed of $32ms^{-1}$ sees a warning sign for road works asking motorists to reduce speed. She immediately decelerates uniformly for 24 seconds until she reaches a speed of $20ms^{-1}$. She travels at this speed for 1500m and then accelerates at $1.2ms^{-2}$ until she regains her original speed of $32ms^{-1}$.

(i) Draw a speed/time graph to illustrate this information, marking the time intervals on the horizontal axis.
(ii) Calculate how far, in metres, the motorist travels from the time she sees the warning sign until she regains her original speed.
(iii) Calculate how much less time it would have taken the motorist to complete her journey if there had been no road works.

Solution

(i) Time taken by motorist travelling at constant speed of 20ms-1 can be got from:

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}} = \frac{1500}{20} = 75 \text{ seconds}$$

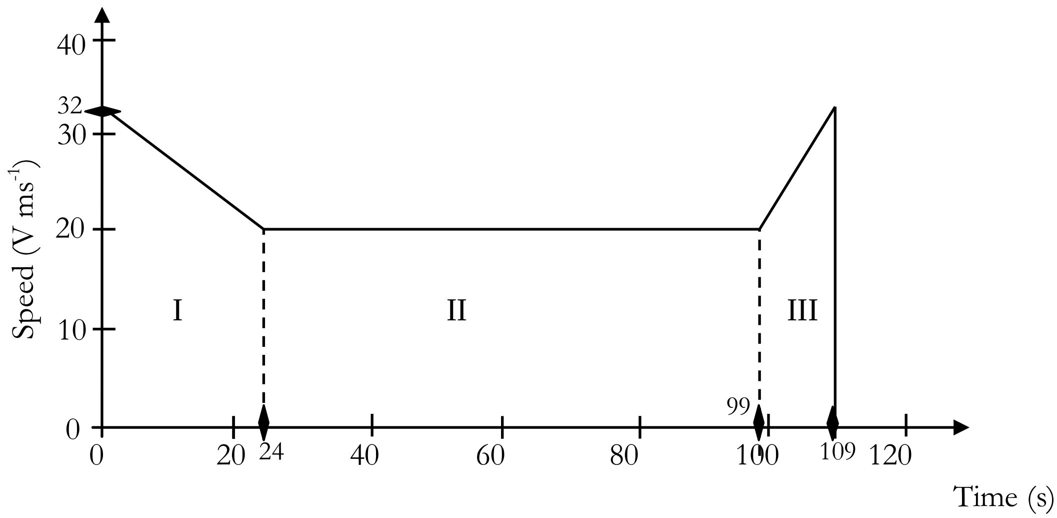
Time for the acceleration from $20ms^{-1}$ to $32ms^{-1}$ can be got using the equation:

$v = u + at$

$\Rightarrow 32 = 20 + 1.2t$

$t = 10s$

The speed – time graph is as shown below;



(ii) Distance = Total area under the speed – time graph

Area of trapezium I = $\frac{1}{2}\times 24\left(32+20\right)=624m$

Area of rectangle II = $75\times 20=1500m$

Area of trapezium III = $\frac{1}{2}\times 10\left(32+20\right)=260m$

The total area = $624+1500+260=2384m$

Hence the distance the motorist travels from the time she sees the warning sign until she regains her original speed is 2384m

(iii) With road works, time = 109s

Without road works, time = $\frac{\text{Distance}}{\text{Speed}}=\frac{2384}{32}$=74.5s

Hence if there had been no road works, it would have been less by (109 – 74.5) = 34.5 seconds for the motorist to complete her journey

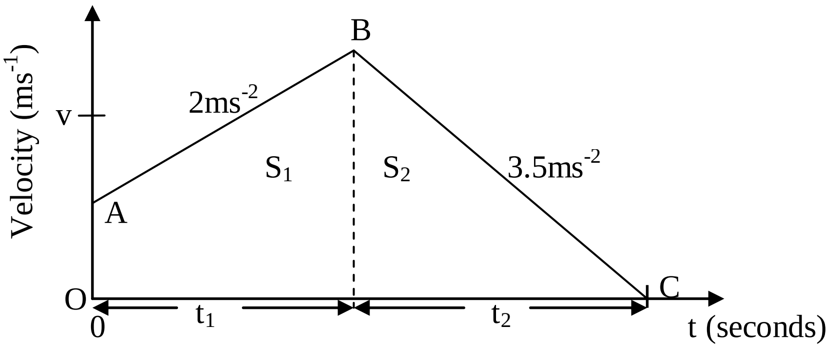
Example 7

A, B and C are three points on a straight road. A car passes A with a speed of 5ms^{-1}and travels from A to B with a constant acceleration of 2ms^{-2}. From B to C the car has a constant retardation of 3.5ms^{-2} and comes to rest at C. If the total distance from A to C is 475m, find

a. the speed of the car at B.
b. the distance of B from A.

Solution

Sketch of v – t graph



Using the equation v = u+at ; the speed at B is given by

$v = 5 + 2t_1$..............................(i)

From B to C, using the same equation

$0 = v - 3.5t_2$............................(ii)

This gives $v = 3.5t_2 = 5 + 2t_1$....................(iii)

The distance from A to B (S_1) is given by the equation $S = ut + \frac{1}{2}at^2$

Hence $S_1 = \frac{1}{2}(5 + 5 + 2t_1)t_1 = 5t_1 + t_1^2$

and the distance from B to C (S_2) is similarly given by

$$S_2 = \frac{1}{2}(3.5t_2)t_2 = 1.75t_2^2$$

Total distance is $475 = S_1 + S_2 = 5t_1 + t_1^2 + 1.75t_2^2$

Substituting for t_2 in terms of t_1, using equation (iii)

Then $475 = 5t_1 + t_1^2 + 1.75\left(\frac{5 + 2t_1}{3.5}\right)^2 = 5t_1 + t_1^2 + \frac{25 + 20t_1 + 4t_1^2}{7}$ leading to

$$3325 = 35t_1 + 7t_1^2 + 25 + 20t_1 + 4t_1^2$$

$$11t_1^2 + 55t_1 - 3300 = 0$$

$$t_1^2 + 5t_1 - 300 = 0$$

$$(t_1 - 15)(t_1 + 20) = 0$$

$$t_1 = 15s$$

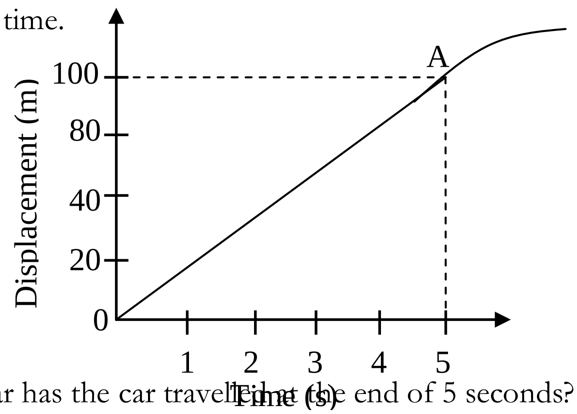
Hence the speed of the car at B = $v = 5 + 2t_1 = 5 + 2(15) = 35ms^{-1}$

and the distance of B from A is $S_1 = \frac{1}{2}(5 + 5 + 2t_1)t_1$

$$S_1 = \frac{1}{2}(5+35)\times 15$$
$$= 300\text{m}$$

Exercise

1. The manufacturer of a new car claims that it can accelerate from rest to 90kmh^{-1}in 10seconds. Find the acceleration.
2. A body moves along a straight line uniformly increasing its velocity from 2ms^{-1} to 18ms^{-1} in a time interval of 10s. Find the acceleration of the body during this time and the distance travelled.
3. The graph in the figure below represents the displacement – time graph travelled by a car plotted against time.



   a. How far has the car travelled at the end of 5 seconds?
   b. What is the velocity of the car during the first 5 seconds?
   c. What has happened to the car after A?

4. A car starts from rest, accelerates at 0.8ms^{-2} for 10s and then continues at a steady speed for a further 20s. Draw the velocity – time graph and find the total distance travelled.
5. A body decelerates at 0.8ms^{-2}, passes a certain point with a speed of 30ms^{-1}. Find its velocity after 10s, the distance covered in that time and how much further the body will go until it stops.
6. A particle travelling with acceleration of 0.75ms^{-2} passes a point O with speed 5ms^{-1}. How long will it take to cover a distance of 250m from O. What will its speed be at that time?
7. A particle starting from rest moves with constant acceleration x $\text{ms}^{-1}$for 10s, travels with constant velocity for a further 10s and then retards at 2x ms^{-2} to come to rest 300m away from its starting point. Find the value of x.
8. If a particle passes a certain point with speed 5ms^{-1} and is accelerating at 3ms^{-2}, how far will it travel in the next 2s? How long will it take (from the start) to travel 44m?
9. A particle moving along a straight line with uniform acceleration covers the first two consecutive distances of 100m and 140m in time intervals of 20 and 40 seconds respectively.
   a. Calculate the:
   (i) acceleration and initial velocity of the particle
   (ii) total time taken before it comes to rest.
   b. Sketch the motion of the particle.

10. Two cars start from the same place. One accelerates at $1ms^{-2}$ for 10s, the other accelerates at $0.8ms^{-2}$ for 20s. Both cars continue with speed then reached. How long after the start will the second car overtake the first and in what distance.
11. A bus travelling at $15ms^{-1}$ and accelerating at $0.2ms^{-2}$ passes a stationery car. The bus accelerates for 5s and then continues at a steady speed. Thirty seconds after being passed the car starts with acceleration $0.5ms^{-2}$ until it reaches a speed of $30ms^{-1}$, with which speed it continues to travel.
    (i) What is the distance between the bus and the car 0.5 minutes later? *Ans.*1917.5*m*
    (ii) When will the car overtake the bus?
12. A motorist accelerates uniformly from rest at a rate of a ms^{-2} for 10s and then travels at constant speed of $20ms^{-1}$ and slows down to rest at s constant retardation of 2a ms^{-2}. If the total distance is 550m,
    (i) Sketch the velocity – time graph of the motion of the motorist.
    (ii) Find the value of a
    (iii) Find the maximum speed attained by the motorist.
    (iv) Find the acceleration and retardation of the motorist.
13. A train travels along a straight piece of track between two stations A and B. The train starts from rest at A and accelerates at $1.25ms^{-2}$ until it reaches a speed of $20ms^{-1}$. It then travels at this steady speed for a distance of 1.56km and then decelerates at $2ms^{-2}$ to come to rest at B. Find:
    a. The distance from A to B.
    b. The total time taken fro the journey.
    c. The average speed.
14. Four points A, B, C and D lie on a straight road such that BC, CD are 448cm and 576cm respectively. A cat moving along this road covers each of these distances from A to D at 8 second intervals with a constant acceleration. Find
    (i) the constant acceleration.
    (ii) its speeds at A and D
    (iii) the distance AB.
15. A train stops at stations R and T which are 2.1km apart in a straight line. It accelerates uniformly from R at $1ms^{-2}$ for 20s and maintains a constant speed for a time before decelerating uniformly to rest at T at $2ms^{-2}$.
    (i) Sketch a velocity – time graph for the motion of the train.
    (ii) Find the time taken for which the train is travelling at a constant speed.
    (iii) Determine the average speed for the whole journey.
16. A particle moving in a straight line with a constant velocity of $6ms^{-1}$ for 3s. Then there is a constant acceleration of $-3ms^{-2}$ for 5s.
    a. Draw a velocity – time graph for its motion.
    b. Find
        (i) the distance it has travelled
        (ii) how far the particle is from the start?

(iii) the velocity after 8 seconds.

c. If the velocity is then kept constant, how long will it take to get back to the start?

17. Two stations A and B are a distance of 6x m apart along a straight track. A train starts from rest at A and accelerates uniformly to a speed Vms^{-1}, covering a distance of x m. the train then maintains this speed until it has travelled a further 3x m, it then retards uniformly to rest at B. Make a sketch of the velocity – time graph for the motion and show that if T is the time taken for the train to travel from A to B, then $T = \frac{9x}{v}$ seconds.

18. A particle starts from rest, moving with constant acceleration of $1.5ms^{-2}$, for 12 seconds. For the next 48 seconds the acceleration is $\frac{1}{8}$ ms^{-2}, and for the last ten seconds it decelerates uniformly to rest.
    (i) Sketch the velocity – time graph for the particle's motion.
    (ii) Find the distance travelled by the particle.
    (iii) Calculate the average velocity of its motion.

19. A motorist starting a car from rest accelerates uniformly to a speed of Vms^{-1}in 10 seconds. He maintains this speed for another 50s and then applies the brakes and decelerates uniformly to rest. His deceleration is numerically equal to twice his previous acceleration.
    (i) Sketch a velocity – time graph
    (ii) Calculate the time during which deceleration takes place.
    (iii) Calculate the initial acceleration.

20. Three collinear points A, B and C are such that 4AC=3AB. A particle moving with uniform acceleration passes A with a velocity of 10ms-1and reaches B ten seconds later.
    Given that AB=300m, find
    (i) the acceleration.
    (ii) the velocities at C and B.

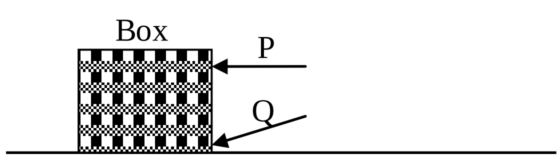
21. The table below shows the velocity of a particle during the course of its motion

t (s)	0	5	10	15	20	25	30
V (ms^{-1})	0	4	8	8	8	7	6

Plot a graph of velocity against time and use it to find
    (i) The retardation of the body during the last 10s.
    (ii) The total distance travelled by the particle
    (iii) Describe the conditions of the particle during the period 10s to 20s.

RESULTANTS AND COMPONENTS OF FORCES

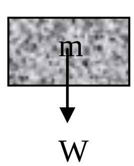
A force is a vector quantity that causes a change in the state of motion of a body. A body in motion can change its velocity or direction only if a resultant force acts on it. The unit of force is the newton (N).
A force vector can be expressed in two dimensions on the (x, y) plane. For example, imagine the surface of a table top to be an (x, y) plane. I can push a box across this table surface in several different directions not just parallel to the length or with table. I can push it across a table top or at a slanted direction relative to the edges of the table top.



Hence we can represent a force by a straight line. The length of the line represents the magnitude of the force and the arrow show the direction of the force. It will thus make some difference if I push the box at P or Q in the direction shown.

Types of force
The following types of force have particular names;

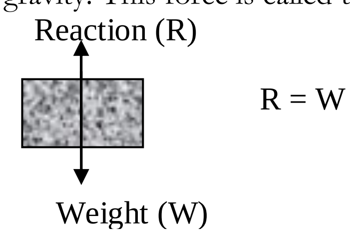
Weight
This is the gravitational attraction of the earth on a body. It always acts vertically downwards. If I hold a brick, I feel it exerting a downward force on my hand. If I let it go, the brick will fall to the ground. The force I felt on my hand is now free to pull the brick towards the ground. This force is called the weight.



The weight of the brick W= mg where m is the mass of the block and g is the acceleration due to gravity. A body of mass m kg thus has a weight of mg N. The value of g should be taken as $9.8ms^{-2}$ unless stated otherwise.
The weight of a body acts through a point called the centre of gravity.

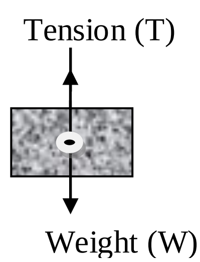
Reaction

If I place the brick of weight W on a horizontal table, it does not move and it is said to be in equilibrium. It is not realistic to assume that the gravitational attraction force acting downwards has ceased, so there must be an opposing force exactly equal to W acting vertically upwards and also through the centre of gravity. This force is called the **normal reaction**.



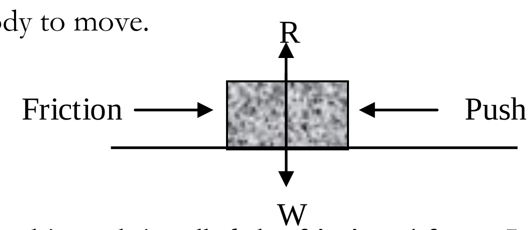
Tension

If a body, say a brick is suspended by a string or spring and the string does not break, the brick is again in equilibrium. In this case the force acting upwards which equalizes the weight passes along the string and is called the **tension** in the string.



Friction

If a body is placed on a horizontal table and it is pushed along the table, a force has to be exerted in order for the body to move.

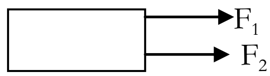


The resistance to this push is called the **frictional force**. It opposes the relative motion. In this situation there are four forces acting on the brick. If the body is pushed harder, eventually the frictional force is overcome and the brick moves.

Resultant of parallel forces

A resultant force is a single force which has the same effect as the two or more forces acting at a point. When two or more forces act on a body, the total force on the body is called the resultant force.

- When two or more parallel forces act on the body in the same direction, the resultant force is got by addition.



Resultant force $F = F_1 + F_2$

Example 1

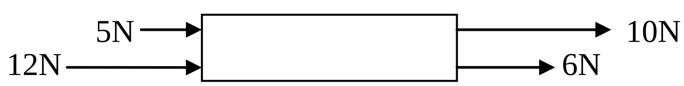
Find the resultant of the following forces.

Solution

Resultant force = 10 + 6 = 16N.

Example 2

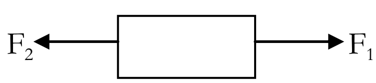
Find the resultant force acting on the body shown.



Solution

Resultant force F = 5 + 12 +10 +6 = 33N

- When two or more parallel forces act in opposite directions, the resultant force is got by subtraction.



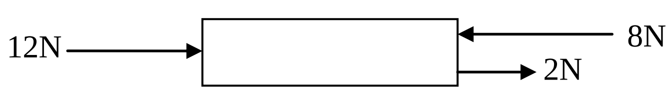
Resultant force;

$F = F_1 - F_2$ if $F_1 > F_2$

Or $F = F_2 - F_1$ if $F_2 > F_1$

Example 3

Three forces act on a body as shown below. Find the resultant force.

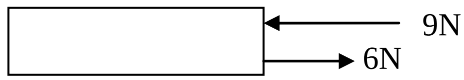


Solution

Resultant force F = (12 + 2) – (8) = 6N to the right.

Example 4

Find the resultant of the following forces acting on the body as shown.

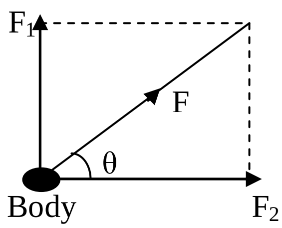


Solution

Resultant force F = (9 – 6) N = 3N to the left.

Resultant of forces at right angles

When two or more forces act on a body at right angles (perpendicular to each other), the resultant force is obtained by use of Pythagoras theorem.



Resultant force F is got from;

$$F^2 = F_1^2 + F_2^2 \qquad \text{(Pythagoras theorem)}$$

$$F = \sqrt{F_1^2 + F_2^2}$$

Direction of the resultant force

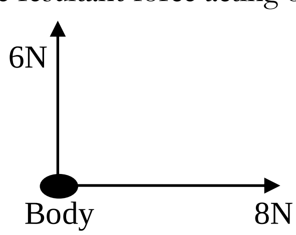
The direction or inclination of the resultant force can be obtained from $\tan\theta = \frac{F_1}{F_2}$ such that;

$\theta = \tan^{-1}\left(\frac{F_1}{F_2}\right)$ to the F_2 force.

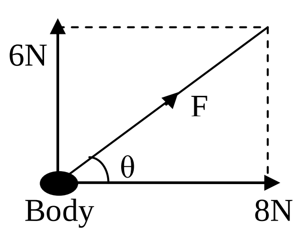
NOTE: Since force is a vector quantity, its magnitude and direction should always be stated.

Example 5

Find the resultant force acting on the body shown.



Solution



Resultant force F is got from;

$$F^2 = 6^2 + 8^2$$

$$F = \sqrt{36 + 64}$$

$$F = \sqrt{100} = 10N$$

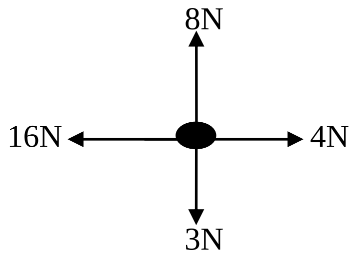
The direction of the resultant force F is obtained from;

$$\tan\theta = \frac{6}{8} = \frac{3}{4} \qquad \Rightarrow \theta = \tan^{-1}\left(\frac{3}{4}\right)$$

Therefore the resultant force is 10N making an angle with the 8N force.

Example 6

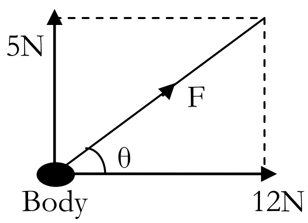
Four forces act on a body as shown below. Find the resultant force.



Solution

Resultant force horizontally = (16N – 4N) = 12N to the left.

Resultant force vertically = (8N – 3N) = 5N to the North.



Resultant force F is got from;

$$F^2 = 12^2 + 5^2$$

$$F = \sqrt{144+25}$$

$$F = \sqrt{169} = 13N$$

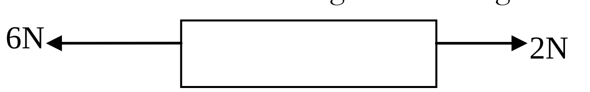
The direction of the resultant force F is obtained from;

$$\tan\theta = \frac{5}{12} \qquad \Rightarrow \theta = \tan^{-1}\left(\frac{5}{12}\right)$$

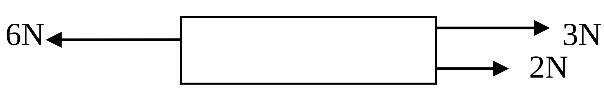
Therefore the resultant force is 13N making an angle with the 4N force.

Exercise

1. Find the resultant of the following forces acting on the body as shown below.



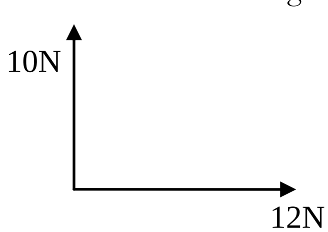
2. Forces act on a body on a smooth ground as shown. Find the resultant force.



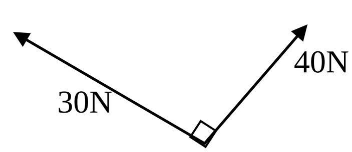
3. A horizontal force of 0.6N acts on a body. There is a resistance of 0.15N opposing the first force. What is the resultant force on the body?

4. Find the resultant of the following forces.

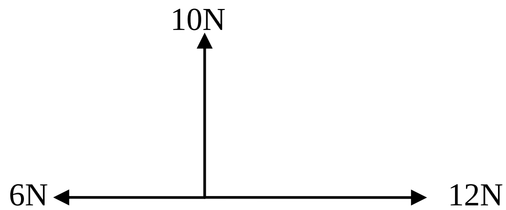
a.



b.



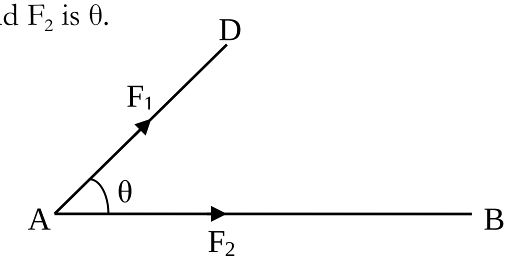
5. Two forces of 7N and 24N act away from the point A and make an angle of 90^0 with each other. Find the magnitude and direction of their resultant.
6. A body is acted on by 3 forces of 10N, 6N and 3N as shown below. Find the resultant force.



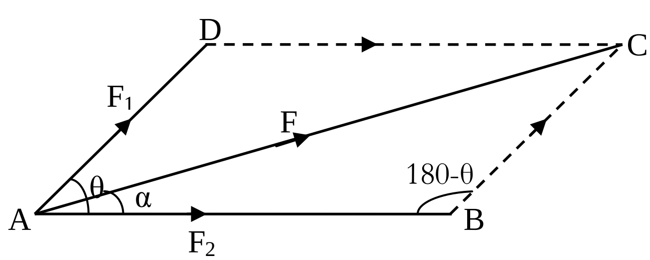
Parallelogram of forces

It two forces are acting at an angle θ to each other when θ is not equal to 90^0, the resultant force is obtained from parallelogram law.

Consider two forces F_1 and F_2 represented by the line segments AB and AD such that the angle between the forces F_1 and F_2 is θ.



The parallelogram ABCD can be completed by drawing BC and DC as shown below;



To find the resultant of the forces F_1 and F_2, we consider $\vec{AB}+\vec{AD}=\vec{AC}$. Hence the resultant of the forces F_1 and F_2 can be represented by the line segment AC which is a diagonal of the parallelogram ABCD. This is referred to as a parallelogram of forces.

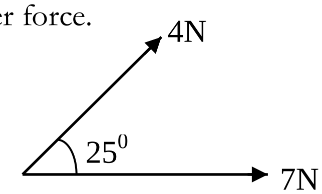
The resultant force F can be obtained using the parallelogram law;

$$F^2 = F_1^2 + F_2^2 + 2F_1F_2\cos\theta$$
$$F = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta}$$

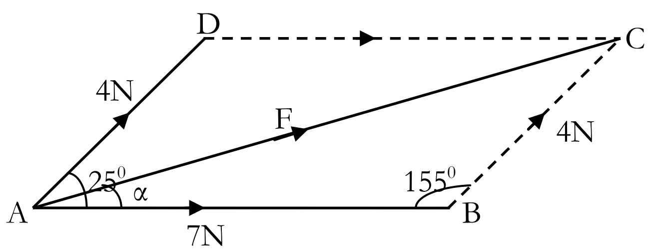
The direction of the resultant force, α which is the angle that the resultant makes with $F_{2,}$ can be got by sine rule: $\dfrac{F_1}{\sin\alpha} = \dfrac{F}{\sin(180-\theta)}$

Example 7

Find the magnitude of the resultant of the forces shown in the sketch and the angle that the resultant makes with the larger force.



Solution



By parallelogram law $F^2 = 4^2 + 7^2 + 2\times4\times7\cos 25^0$

The resultant force $F = \sqrt{16+49+56\cos 25}$

$F = 10.75N$

The direction of the resultant can be obtained by using sine rule:

$$\frac{4}{\sin\alpha} = \frac{F}{\sin 155^0}$$

$$\sin\alpha = \frac{4\sin 155^0}{10.75}$$

$$\alpha = 9.05^0$$

The resultant force is 10.75N making an angle of 9.05^0 with the larger force.

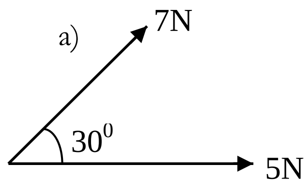
Exercise

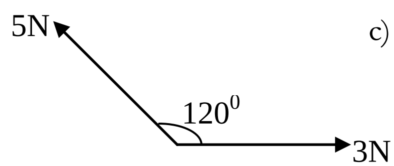
1. Two forces of 5N and 8N act away from the point A and make an angle of 40^0 with each other. Find the resultant force and the angle the resultant makes with the 8N force.

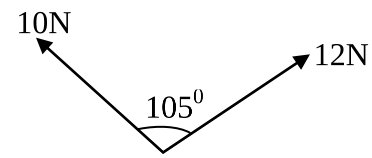
2. Find the angle between a force of 7N and a force of 4N if their resultant has magnitude of 9N.

3. Find the magnitude of the resultant of two forces of sizes 5N and 8N acting at 50^0 to each other.

4. In each of the following diagrams, two forces are shown. Find the magnitude of their resultant and the angle it makes with the larger of the two forces.

a) 

b) 

c) 

5. Forces of 3N and 2N act along OA and OB respectively, the direction of the forces being indicated by the order of the letters. If $A\hat{O}B = 150^0$ find the magnitude of the resultant of the two forces and the angle it makes with OA.

6. Find the angle between a force of 6N and a force of 5N given that their resultant has magnitude of 9N.

7. The angle between a force of P N and a force of 3N is 120^0. If the resultant of the two forces has magnitude 7N, find the value of P.

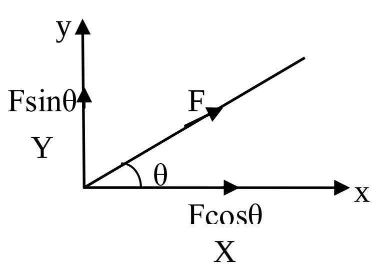
Components of a force

The components of the force **F** in any direction is a measure of the effect of the force **F** in that direction. Mathematically, the component of a force in any direction is the product of the magnitude of the force and the **cosine** or **sine** of the angle between the force and the required direction.

Resolving forces

The separation of a force into its components is called the **resolution of a force**. Forces can be resolved into **two components** at **right angles** or perpendicular to each other when a right angled triangle is constructed around the force, making it the hypotenuse.

Suppose a force F acts in the direction shown at an angle θ to the x – axis.



We use cosine if resolving the force to the required direction closes the angle and sine if resolving it results into opening the angle. Thus the force F can be resolved into two components, one along x – axis and the other along y – axis.
Let X and Y represent the horizontal and vertical components of F, along the x and y – axis.

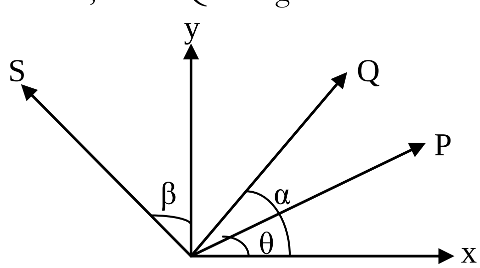
Resolving in the x – direction (horizontally): $X = F\cos\theta$
Resolving in the y – direction (vertically): $Y = F\sin\theta$

Therefore the force F is equivalent to a force $F\cos\theta$ along the x – axis and $F\sin\theta$ along the y – axis. It is important to remember that when a force F has been resolved into its components in two perpendicular directions, the force F is the resultant of these two components.

Resultant of a number of forces

When there are a number of forces acting, their components in a particular direction can be added together, due regard being given to the directions of the components.

Consider the forces S, P and Q acting in the directions shown.

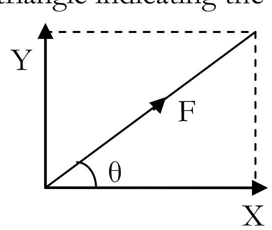


The forces can be resolved into two perpendicular components. It is however important to note that the direction of the arrow in the forces being resolved has to be considered as the direction will determine whether the components must be added or subtracted.

Let X and Y be the components of resultant of the forces along the x and y- axis respectively. Then;

Resolving in the x – direction: $X = P\cos\theta + Q\cos\alpha - S\cos\beta$
Resolving in the y – direction: $Y = P\sin\theta + Q\sin\alpha + S\sin\beta$

Thus in a force diagram, the forces S, P and Q can be replaced by their components. The forces are represented in magnitude and direction by the sides of the triangle, the arrows on the sides of the triangle indicating the directions of the forces.



The resultant force F of the forces S, P and Q can thus be obtained from;

$F^2 = X^2 + Y^2$ (Pythagoras theorem)

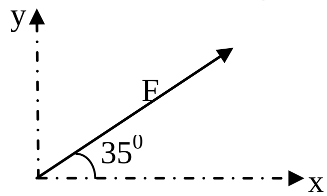
$F = \sqrt{X^2 + Y^2}$

Since force is a vector quantity, it must be specified both in magnitude and direction. The direction of the resultant force θ can be calculated from $\tan\theta = \frac{Y}{X}$ hence $\theta = \tan^{-1}\left(\frac{Y}{X}\right)$. Thus the resultant of the forces should be stated in magnitude and direction.

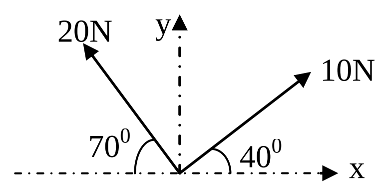
Example 8

Find the components of the given forces in the directions of the x – axis and the y – axis.

(a)



(b)



Solution

(a) Component along the x – axis = $5\times\cos35^0 = 4.10$N

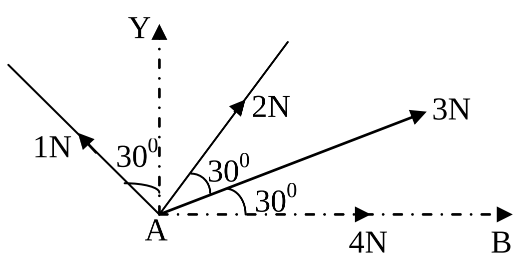
Component along the y – axis = $5\times\sin 35^0 = 2.87$N

(b) Resolving along the x – axis gives us: $10\cos 40^0 - 20\cos 70^0 = 7.66 - 6.84 = 0.82N$

Resolving along the y – axis gives us: $10\sin 40^0 - 20\sin 70^0 = 6.43 + 18.79 = 25.22N$

Example 9

Forces of 4, 3, 2 and 1N act at a point A as shown in the diagram below. Find the magnitude of their resultant and direction from AB.



Solution

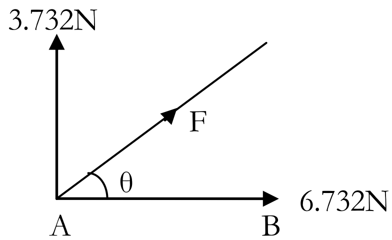
Let X and Y be the components of the resultant of the forces along AB and AY respectively;

Resolving horizontally: X $= 4 + 3\cos 30^0 + 2\cos 60^0 - 1\cos 30^0 = 6.732N$

Resolving vertically: Y = $3\sin 30^0 + 2\sin 60^0 + 1\sin 30^0 = 3.732N$

The resultant force $F = \sqrt{X^2 + Y^2} = \sqrt{(6.732^2 + 3.732^2)} = 7.7N$

If θ is the angle the resultant made by F with AB, then

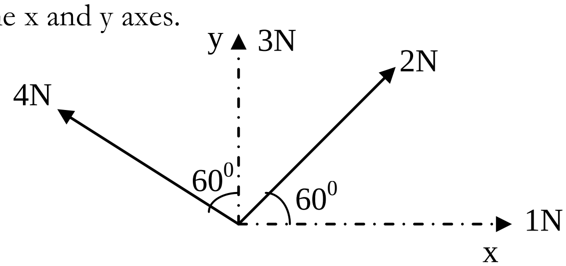


$$\tan\theta = \frac{Y}{X} = \frac{3.732}{6.732}$$
$$\Rightarrow \theta = \tan^{-1}\left(\frac{3.732}{6.732}\right) = 29.0^0$$

Therefore the resultant of the forces above is 7.7N and acts at an angle of 29.0^0 to AB.

Example 10

Find the resultant of the given forces, by finding the components of the forces in the direction of the x and y axes.



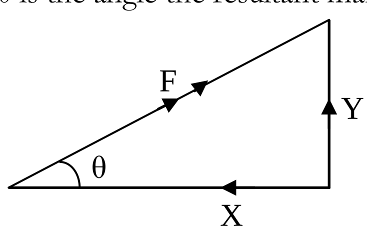
Solution

Resolving along the x – axis: $X = 1 + 2\cos 60^0 - 4\sin 60^0 = -1.464N$

Resolving along the y – axis: $Y = 3 + 2\sin 60^0 + 4\cos 60^0 = 6.732N$

The resultant force $F = \sqrt{X^2 + Y^2} = \sqrt{(1.464^2 + 6.732^2)} = 7N$

If θ is the angle the resultant makes with the horizontal, then



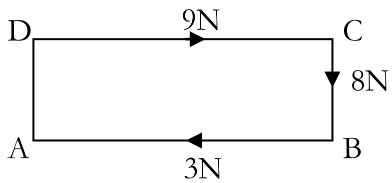
$$\tan\theta = \frac{Y}{X} = \frac{6.732}{1.462}$$
$$\Rightarrow \theta = \tan^{-1}\left(\frac{6.732}{1.464}\right) = 77.7^0$$

The resultant force is 7N and the direction of the resultant force is 77.7^0 to the horizontal.

Example 11

ABCD is a rectangle. Forces of 9N, 8N and 3N act along the lines DC, CB and BA respectively, in the directions indicated by the order of the letters. Find the magnitude of the resultant and the angle it makes with DC.

Solution

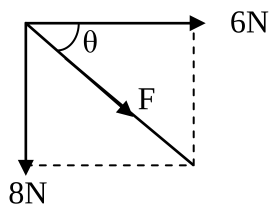


Resolving horizontally: $X = 9 - 3 = 6N$ to right

Resolving vertically: $Y = 8N$ to South

The resultant force $F = \sqrt{X^2 + Y^2} = \sqrt{(6^2 + 8^2)} = 10N$

If θ is the angle the resultant makes with DC, then



$$\tan\theta = \frac{Y}{X} = \frac{8}{6} = \frac{4}{3}$$

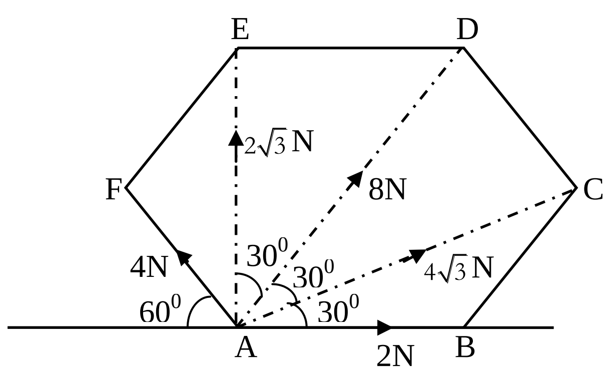
$$\Rightarrow \theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.13^0$$

The resultant is 10N making an angle 53.13^0 with DC.

Example 12

ABCDEF is a regular hexagon. Forces of $2, 4\sqrt{3}, 8, 2\sqrt{3}$ and 4N act at A in direction AB, AC, AD, AE and AF respectively. Find the magnitude of their resultant and its direction from AB.

Solution



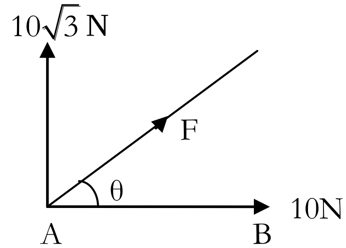
Let the resultant force be F

Resolving horizontally: $X = 2 + 4\sqrt{3}\cos 30^0 + 8\cos 60^0 - 4\cos 60^0 = 10N$

Resolving vertically: $Y = 2\sqrt{3} + 4\sqrt{3}\sin 30^0 + 8\sin 60^0 + 4\sin 60^0 = 10\sqrt{3}N$

$$F = \sqrt{X^2 + Y^2} = \sqrt{(10\sqrt{3})^2 + 10^2} = \sqrt{400} = 20N$$

If θ is the angle the resultant F, makes with the horizontal AB,



$$\tan\theta = \frac{Y}{X} = \frac{10\sqrt{3}}{10}$$
$$\Rightarrow \theta = \tan^{-1}\left(\sqrt{3}\right) = 60^0$$

The resultant force is 20N inclined at angle of 60^0 to AB.

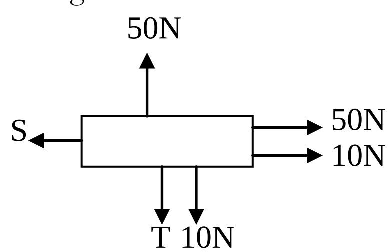
Conditions for equilibrium of any number of forces acting on a body

If the forces acting on a body are in equilibrium, the resultant force F must be zero. This is only possible when X=0 and Y=0. It follows therefore that if any number of forces is in equilibrium, the algebraic sums of their components in two directions perpendicular to each other must separately be zero. i.e. the forces in opposite directions balance.

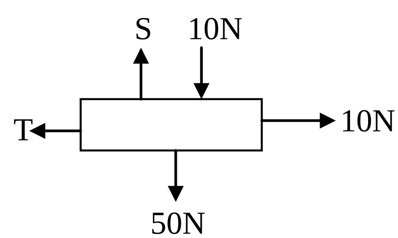
Example 13

Find the magnitude of the unknown forces S and T if the body is in equilibrium under the action of the given forces.

(a)



(b)



Solution

(a) Since the body is in equilibrium, then;
The horizontal forces balance and therefore: $S = 50 + 10 = 60N$

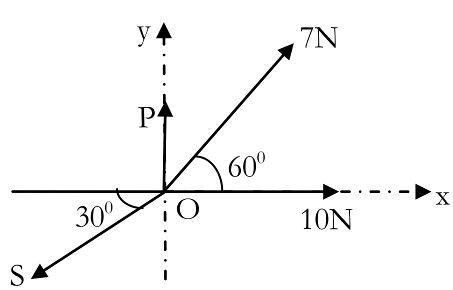
The vertical forces balance and therefore: $T + 10 = 50$
$$T = 40N$$

(b) Horizontally: $T = 10N$

Vertically: $S = 10 + 50 = 60N$

Example 14

The given forces act on a particle at O which is in equilibrium. By resolving, find P and S.



Solution

Resolving along the x – axis: $X = 10 + 7\cos 60^0 - S\cos 30^0$

Resolving along the y – axis: $Y = P + 7\sin 60^0 - S\sin 30^0$

Since the particle is in equilibrium;

$X = 0 \qquad \therefore 10 + 7\cos 60^0 - S\cos 30^0 = 0$

$$\Rightarrow S\cos 30^0 = 10 + 7\cos 60^0$$

$$\Rightarrow \frac{\sqrt{3}}{2}S = \frac{27}{2}$$

$$S = 9\sqrt{3}N$$

Similarly, $Y = 0 \qquad \therefore P + 7\sin 60^0 - S\sin 30^0 = 0$

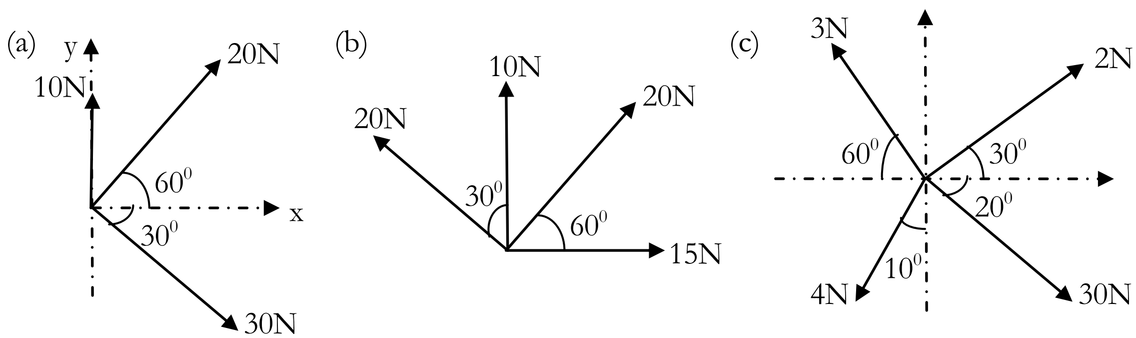
$$\Rightarrow P + 7\sin 60^0 = S\sin 30^0$$

$$\Rightarrow P + \frac{7\sqrt{3}}{2} = 9\sqrt{3} \times \frac{1}{2}$$

$$\mathrm{P} = \sqrt{3}N$$

Exercise

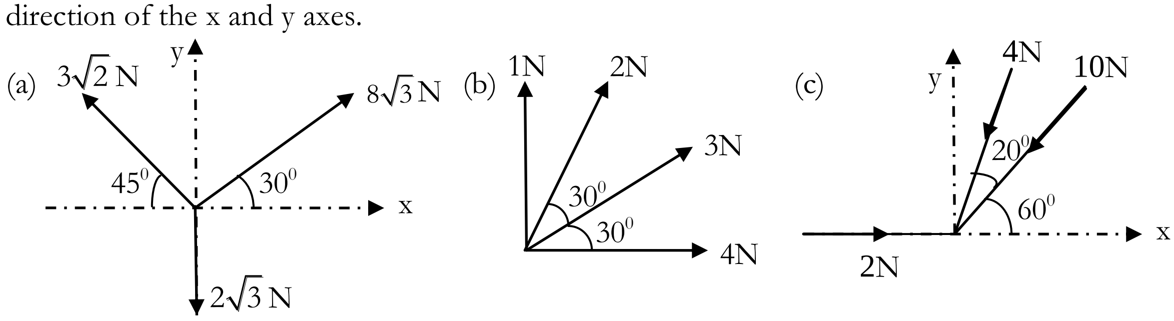
1. Find the resultant of the following forces in magnitude and direction.



2. Forces of 5N, 9N and 7N act along the sides A, BC and CA respectively of an equilateral triangle ABC in the directions indicated by the letters. Find the resultant in magnitude and direction.

3. The resultant of two forces X N and 3N is 7N. If the 3N force is reversed, the resultant is $\sqrt{19}N$. Find the value of X and the angle between the two forces.

4. Find the resultant of the given forces, by finding the components of the forces in the direction of the x and y axes.

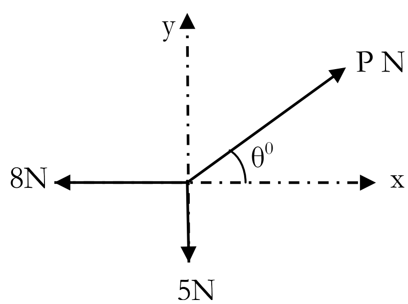


5. ABCD is a rectangle. Forces of 3N, 4N and 1N act along AB, BC and DC respectively, in the directions indicated by the order of the letters. Find the magnitude of the resultant and the angle it makes with AB.

6. ABCDEF is a regular hexagon. Forces of 3, 4, 8, 2 and 6N act in the directions AF, FE, ED, CD and AB respectively. Find the magnitude of their resultant and its inclination to AB where AB is horizontal.

7. Four forces of magnitudes 4N, 13N, 20N and 3N act along the sides $\vec{AB}, \vec{BC}, \vec{DC}$ and $\vec{DA}$ respectively of a square of sides a metres. Find the magnitude of the resultant force and direction the resultant force makes with AB.

8. The diagram below shows three forces acting on a particle in the directions shown.

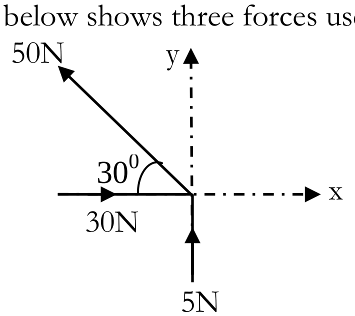


Find P and θ if,

(i) the three forces are in equilibrium.
(ii) the resultant of the three forces is 10N due North.

9. A regular hexagon PQRSTU has forces of 10N, 12N, 8N, 20N, 12N and 15N acting along PQ, QR, SR, SP, TP and UT respectively. In each case the direction of the force is given according to the order of the letters. If PQ is horizontal, find the magnitude and direction of the resultant force with respect to PQ.

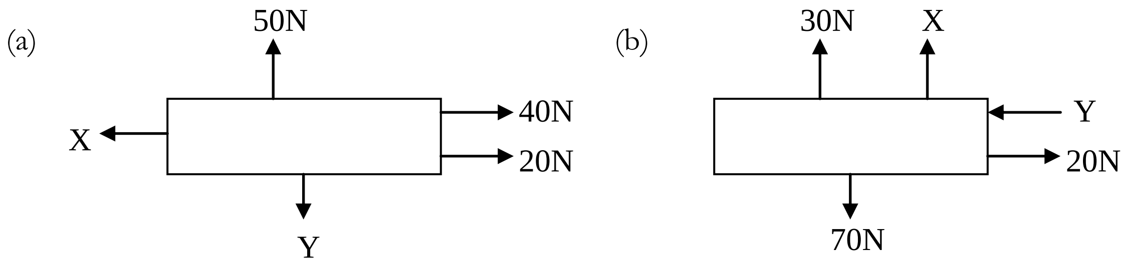
10. The diagram below shows three forces used to direct a space shuttle S towards a target area.



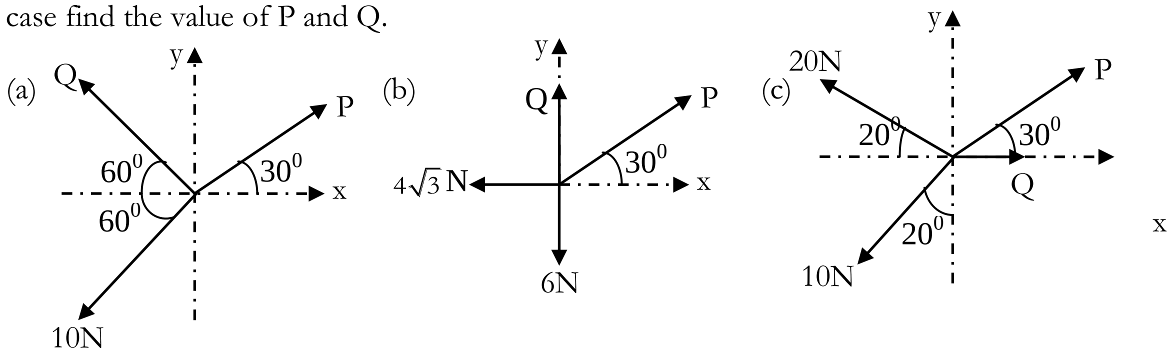
Find the magnitude of the resultant force.

11. ABCDEF is a regular hexagon. Forces act along AB, AC, AE and AF of magnitudes 3N, 2N, 1N and 3N respectively. Find the magnitude of the resultant and the direction it makes with AB.

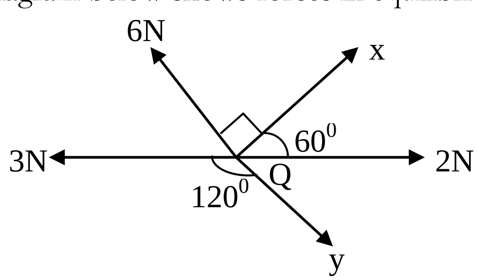
12. Find the magnitudes of the unknown forces X and Y if the particle is in equilibrium under the action of the given forces.



13. Each of the diagrams below shows a particle in equilibrium under the forces shown. In each case find the value of P and Q.



14. The diagram below shows forces in equilibrium acting at Q.



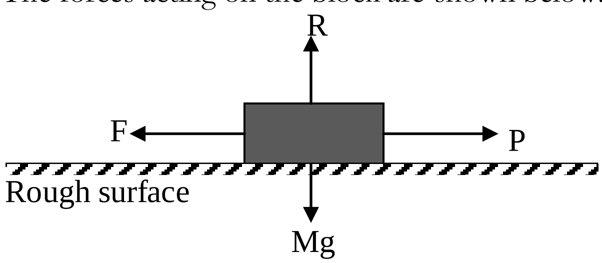
(i) Find the values of x and y.
(ii) If the 6N force is removed, calculate the magnitude and direction of the resultant force at Q.

FRICTION

Friction is the force which opposes relative motion between two bodies in contact.

Concept of friction

A block of mass m kg rests on a horizontal table and a horizontal force of P newtons is applied to the block. The forces acting on the block are shown below:



The force F which opposes the motion of the block is called the **friction force**. The friction force acts parallel to the surfaces in contact and in a direction so as to oppose the motion of one body across the other.

If the surfaces were perfectly smooth, F would be zero and then motion would take place however small the applied force P might be. However when surfaces are surfaces are rough, the block will only move if P is greater than the frictional force F.

Limiting equilibrium

Limiting equilibrium is said to occur when a body in contact with a certain surface is about to move with application of a force. The frictional force F for a particular block and surface is not constant, but increases as the applied force P increases until the force F reaches a maximum value F_{max} beyond which it cannot increase. The block is then on the point of moving and is said to be in a state of *limiting equilibrium*.

Suppose $F_{max} = P_1$. If the applied force P is increased still further to a value P_2, the frictional force cannot increase as it has already reached its maximum value and the block will therefore move and we say frictional force has been overcome.

Coefficient of friction

This is the ratio of the maximum frictional force to the normal reaction. The magnitude of the maximum frictional force is a fraction of the normal reaction R. This fraction is called the coefficient of friction μ for the two surfaces in contact.

$F_{\max} = \mu R$. For a perfectly smooth surface $\mu = 0$. The maximum frictional force only acts where there is a state of limiting equilibrium or when motion is taking place.

In many machines, where metal parts move against each other, friction is a nuisance and every effort is made to reduce it by lubrication. For example, the piston in a car cylinder is lubricated by the oil pumped up from the sump. On the other hand, without friction we could not move or even write. We rely on the friction between our shoes and the ground to push us forward. On smooth ice or a polished floor we tend to slip as this frictional force is small.

Similarly the frictional force between the tyres of a car and the road is essential to motion. In solving problems, first mark and find the normal reaction between the two surfaces. Then provided sliding is taking place or about to take place, put $F = \mu R$. Otherwise F< μR

Friction on horizontal planes

The frictional force has a maximum value μR where R is the normal reaction between the surfaces in contact and μ is the coefficient of friction between the surfaces.

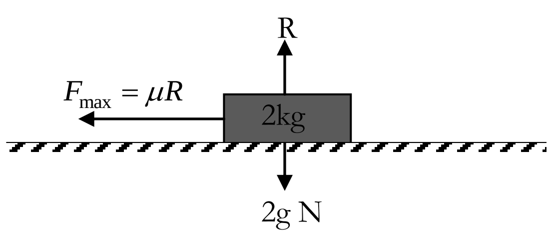
Thus $F = \mu R$

Example 1

Calculate the maximum frictional force which can act when a block of mass 2kg rests on a rough horizontal surface, the coefficient of friction between the surfaces being;

(a) 0.7 (b) 0.2

Solution



R = 2g
= 2×9.8 = 19.6 N

(a) Maximum friction force $F_{\max} = \mu R$
$= 0.7 \times 19.6 = 13.7$ N

(b) $F_{\max} = \mu R$

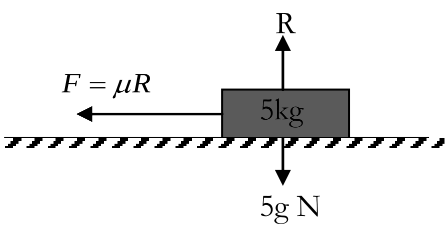
$= 0.2 \times 19.6$

$= 3.92$ N

Example 2

A block of mass 5kg rests on a rough horizontal plane, the coefficient of friction between the block and the plane being 0.6. Calculate the frictional force acting on the block if the block is about to slide.

Solution



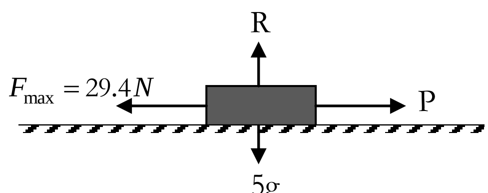
$F = \mu R$

R = 5×9.8 = 49N

$F_{\max} = 0.6 \times 49$

$= 29.4$N

This implies that if a horizontal force P is applied, the block will only move when P is greater than the maximum frictional force of 29.4N.



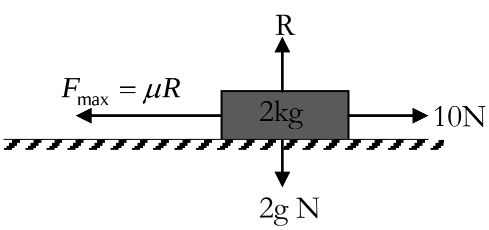
If P< $F_{\max}$ then there is no motion and the frictional force F=P.

If P> $F_{\max}$, then the block will move and the maximum value of frictional force μR will be maintained.

Example 3

If a force of 10N is just sufficient to move a mass of 2kg resting on a rough horizontal table, find the coefficient of friction.

Solution



Since the mass is at a point of limiting equilibrium,

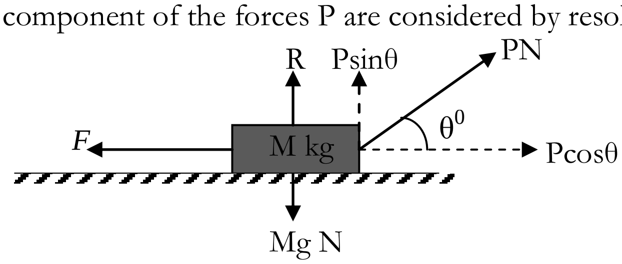
$F_{\max} = \mu R = 10N$

R = 2×9.8 = 19.6N

$10 = \mu \times 19.6$

$\mu = \frac{10}{19.6} = 0.51$

When the applied force P on the block of mass M in inclined at an angle θ above the horizontal, then the component of the forces P are considered by resolving.



In this case the normal reaction R is not equal to Mg as we must take into account the vertical component of P.

Resolving horizontally: $F = P\cos\theta$

Resolving vertically: $R + P\sin\theta = Mg$

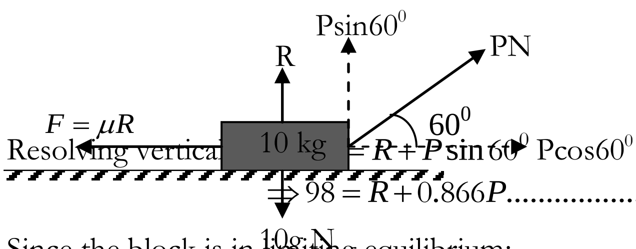
$$\Rightarrow R = Mg - P\sin\theta$$

Thus the component of P in a vertical direction decreases the magnitude of the normal reaction, and the component of P in a horizontal direction tends to move the block.

Example 4

A block of mass 10kg rests on a horizontal floor, coefficient of friction 0.4. What force is required just to make the block move when pulling at an angle of 60^0 to the horizontal?

Solution



Resolving vertically: $= R + P\sin 60^0$

$$\Rightarrow 98 = R + 0.866P \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Since the block is in limiting equilibrium;

Resolving horizontally: $P\cos 60^0 = F = \mu R$

$$\Rightarrow P \times 0.5 = 0.4R$$

$$\therefore R = 1.25P \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

Substituting (ii) in (i) gives

$98 = 1.25P + 0.866P$

$98 = 2.116P$

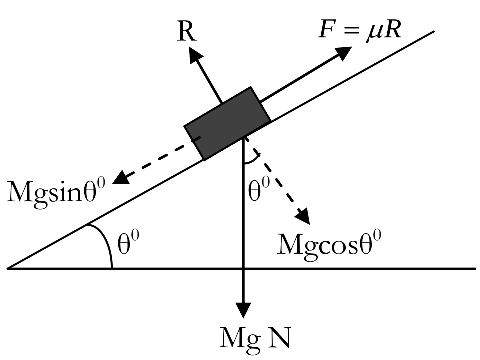
$\therefore P = 46.3N$

For motion to take place the applied force must exceed 46.3 N.

Friction on inclined planes

When a body is on an inclined plane, then it is usually important to resolve the forces on it parallel to and at right angles to the surface of the plane. In this case it is necessary to decide in which direction the friction is going to act as illustrated by the next examples.

Suppose a body of mass M kg rests on a plane which is inclined at an angle θ to the horizontal.



The vertical force Mg can be resolved into two components, parallel to and perpendicular to the surface of the plane. The plane exerts a normal reaction R on the body.

Resolving perpendicular to the plane: $R = Mg\cos\theta$

Resolving parallel to the plane: $F = Mg\sin\theta$

The component Mgsinθ acting down the plane will cause motion unless the frictional force F acting up the plane balances it.

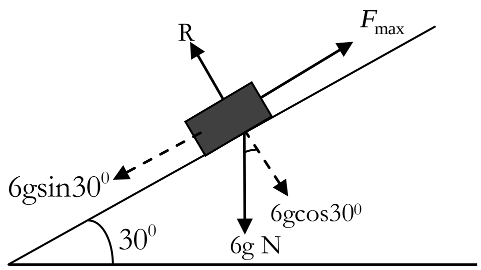
The maximum frictional force $F_{\max} = \mu R = \mu Mg\cos\theta$

At the point of limiting equilibrium: $\mu Mg\cos\theta = Mg\sin\theta$

Example 5

A mass of 6kg rests in equilibrium on a rough plane inclined at 300 to the horizontal. Find the coefficient of friction the mass and the plane.

Solution



The frictional force F acts upwards since the mass is on the point of moving down the plane.

Resolving perpendicular to the plane gives: $R = 6g\cos 30^0 = 6g \times \frac{\sqrt{3}}{2} = 3g\sqrt{3}$

Resolving parallel to the surface of the plane gives: $F_{\max} = 6g\sin 30^0$

$$\mu R = 6g\sin 30^0$$

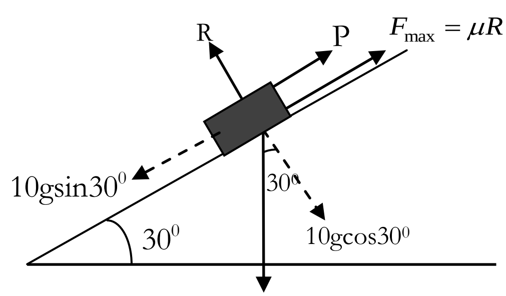
$$\mu \times 3g\sqrt{3} = 6g \times \frac{1}{2}$$

$$\therefore \mu = \frac{1}{\sqrt{3}}$$

Example 6

A mass of 10kg is placed on a plane inclined at an angle of 30^0 to the horizontal. What force parallel to the plane is required to hold the mass at rest?($\mu = 0.4$).

Solution



As the block is just being held at rest, it is on the verge of slipping down. Hence $F_{\max} = (\mu R)$ acts upwards.

Resolving perpendicular to the planes gives: $R = 10g\cos 30^0 = 10g \times \dfrac{\sqrt{3}}{2}$

$$\Rightarrow R = 5g\sqrt{3}$$

Resolving parallel to the surface of the plane gives: $P + \mu R = 10g\sin 30^0$

$$P + 0.4(5g\sqrt{3}) = 10g(0.5)$$

$$\Rightarrow P = 5g - 2\sqrt{3}g$$

$$P = 15N$$

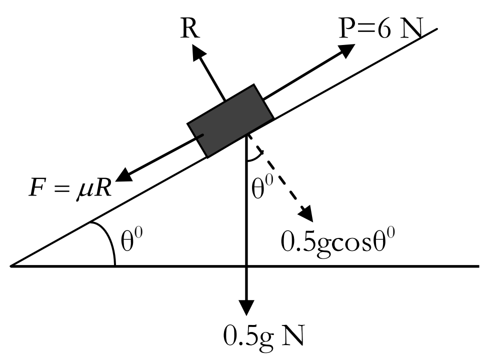
Example 7

A mass of 0.5 kg rests on a rough plane. The coefficient of friction between the mass and the plane is $\frac{1}{\sqrt{2}}$, and the plane is inclined at an angle θ to the horizontal such that $\sin\theta = \frac{1}{3}$.

Find the limiting frictional force if the mass is to move up the plane.

If a 6N force is applied up the plane along a line of greatest slope, will motion take place?

Solution



If the mass is to move up the plane, then $F_{\max} = (\mu R)$ acts downwards.

Since $\sin\theta = \frac{1}{3}$, then $\cos\theta = \frac{2\sqrt{2}}{3}$

Resolving perpendicular to the plane: $R = 0.5g\cos\theta$

$$R = 0.5g \times \frac{2\sqrt{2}}{3} = \frac{g\sqrt{2}}{3}$$

The limiting friction force $F_{\max} = \mu R$

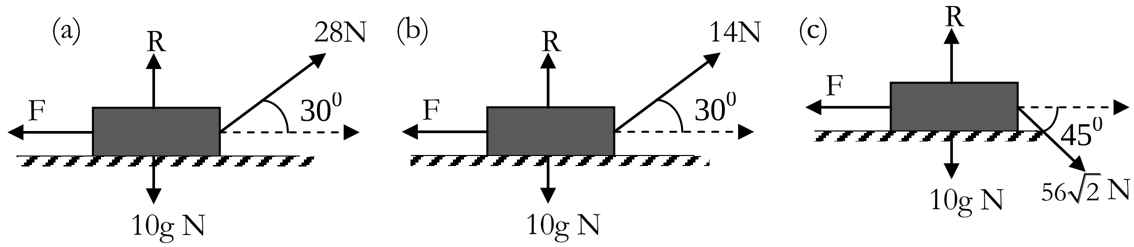
$$= \mu \times \frac{g\sqrt{2}}{3} = \frac{1}{\sqrt{2}} \times \frac{g\sqrt{2}}{3} = \frac{g}{3}N = 3.27N$$

The 6N force is greater than the limiting frictional force acting down the plane. Thus friction is overcome and motion takes place up the plane.

Exercise

1. A mass of 4 kg rests on a rough table ($\mu = 0.4$). Find the least force sufficient to make it move.
2. A block of mass 10kg rests on a horizontal floor, coefficient of friction 0.4. What force is required to make the block move when pulling horizontally?
3. When a horizontal force of 28N is applied to a body of mass 5 kg which is resting on a rough horizontal plane, the body is found to be in limiting equilibrium. Find the coefficient of friction between the body and the plane.
4. A 10kg trunk lies on a horizontal rough floor. The coefficient of friction between the trunk and the floor is $\frac{\sqrt{3}}{4}$. Calculate the magnitude of the force P which is necessary to move the trunk horizontally if P is applied;

   (a) Horizontally (b) at 30^0 above the horizontal
5. A body of mass 10kg is initially at rest on a rough horizontal plane. The coefficient of friction between the body and the plane is $\frac{1}{7}$. Find the value of the maximum frictional forces in each of these diagrams.



6. When a horizontal force of 0.245N is applied to a body of mass 250g which is resting on a rough horizontal plane, the body is found to be in limiting equilibrium. Find the coefficient of friction between the body and the plane.
7. A block of mass 10kg is placed on an inclined plane of angle 30^0 to the horizontal where $\mu = 0.5$. Find the least force parallel to the plane required to keep the blocker rest.
8. A body of mass 500g is placed on a rough plane which is inclined at 40^0 to the horizontal. If the coefficient of friction between the body and the plane is 0.6, find the frictional force acting and state whether motion will occur.
9. A mass of 3kg rests on a rough plane inclined at 60^0 to the horizontal and the coefficient of friction between the mass and the plane is $\frac{\sqrt{3}}{5}$. Find the force P, acting parallel to the plane, which must be applied to the mass in order to just prevent motion down the plane.
10. A body of mass 5kg lies on a rough plane which is inclined at 35^o to the horizontal. When a force of 20N is applied to the body, parallel and up the plane, the body is found to be on the point of moving down the plane. Find μ; the coefficient of friction between the body and the plane.

11. A block of mass 1 kg is placed on an inclined plane of angle 60^0 and is just held there at rest by a horizontal force P. If the coefficient of friction is 0.4, find P.

12. A horizontal force of 1N is just sufficient to prevent a brick of mass 600g sliding down a rough plane which is inclined at $\sin^{-1}\left(\frac{5}{13}\right)$ to the horizontal. Find the coefficient of friction between the brick and the plane.

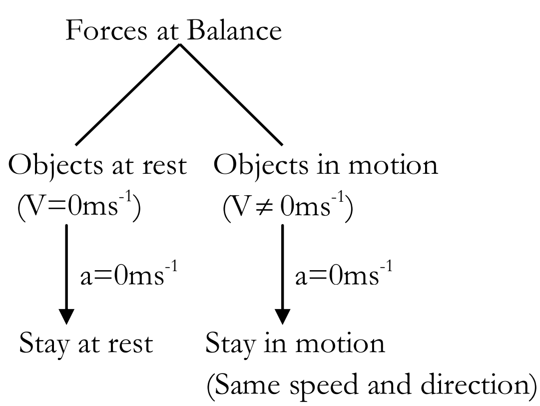
NEWTON'S LAWS OF MOTION

Newton's laws of motion are three physical laws that form the basis for classical mechanics. They describe the relationship between the forces acting on a body and its motion due to those forces. They are used in the theory of motion of bodies and the common bodies include vehicles, lifts, pulleys, connected bodies like trucks and trains etc.

Newton's first law

It states that: A body will remain at rest or will continue to move with constant velocity, unless external forces cause to change this state. This law therefore emphasizes that a change in the state of motion of a body is caused by a force.

Thus there are two parts in the statement of this law– one that predicts the behavior of bodies at rest and the other that predicts behavior of bodies in motion. The two parts are summarized in the following diagram



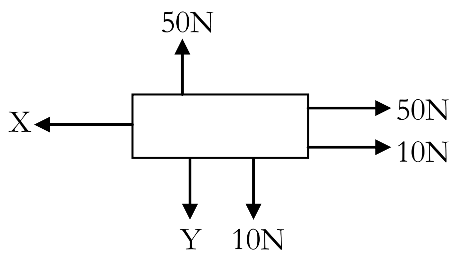
A body at rest

If forces at on a body and it does not move, the forces must balance. Hence, if a body is not moving, then the resultant force in any direction must be zero. Thus any forces acting on a body at rest cancel out.

Example 1

A body is at rest when subjected to the forces shown in the diagrams. Find X and Y.

a.



b.

Solution

a. Since the body is at rest;
The horizontal forces must balance: $X = 50 + 10$
$X = 60N$

The vertical forces must balance: $Y + 10 = 50$
$Y = 40N$

b. Horizontal forces must balance: X = 20N

Vertical forces must balance: Y = 98N

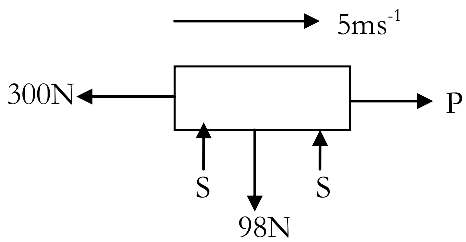
A body in motion

If a body is moving with constant velocity, there can be no resultant force acting on it. This is because a body can only change its velocity if a resultant force acts upon it. So if a body is travelling with constant speed then either no force whatever is acting on it or no resultant force acts on the body.

It follows that if a body is not at rest or moving with uniform velocity, it has an acceleration (retardation) and some force is acting on it. So acceleration is linked with any external force or forces acting. Newton's second law leads to the precise relation between force and acceleration.

Example 2

A body moves horizontally at a constant speed of 5ms-1 subject to the forces shown. Find the values of P and S.



Solution

There is no vertical motion: $\therefore S+S=2000$

$$S=1000N$$

The horizontal velocity is constant: P = 300N

Newton's Second law

If a force acts on a body and produces a certain acceleration, then the force is proportional to the product of the mass of the body and the acceleration. Also the acceleration takes place in the direction of the force. Thus Newton's second law leads to the precise relation between force and acceleration. It can be summarized by the equation $F=ma$, which is often referred to as the equation of motion.

Example 3

A body of mass 10 kg is acted upon by a force of 5N. Find the acceleration.

Solution

Using F = ma

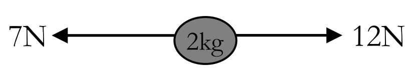
$$5=10a$$

$$\Rightarrow a=0.5ms^{-2}$$

Example 4

A body of mass 2kg rests on a smooth horizontal surface. Horizontal forces of 12N and 7N start to act on the particle in opposite directions. Find the acceleration of the body.

Solution



The resultant force F = 12 – 7 =5N

From $F = ma$

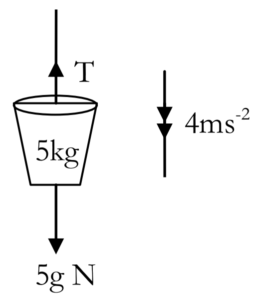
$5 = 2a$

$a = 2.5ms^{-2}$

Example 5

A bucket of water of mass 5kg is lowered vertically by a rope. Find the force in the rope (Tension) when the bucket is lowered with an acceleration of $4ms^{-2}$.

Solution



The resultant vertical force on the bucket F = (5g – T) N acting downwards.

Using $F = ma$

$5g - T = 5 \times 4$

$T = 5g - 20$

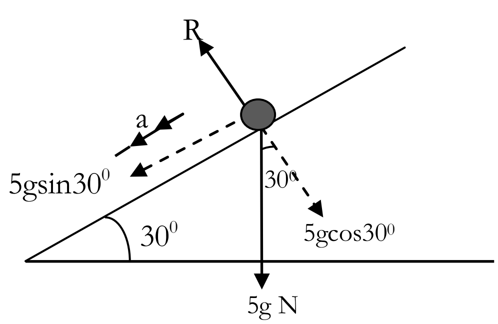
$T = (5 \times 9.7) - 20 = 29N$

The force (tension) in the rope is 29N

Example 6

A particle of mass 5kg slides down a smooth plane inclined at 30^0 to the horizontal. Find the acceleration of the particle and the reaction between the particle and the plane.

Solution



Since the acceleration is down the plane, the resultant force is also down the plane.

Resultant force down the plane F= $5g\sin 30^0$

Using $F = ma$

$5g \sin 30^0 = 5a$

$5 \times 9.8 \times 0.5 = 5a$

$a = 4.9ms^{-2}$

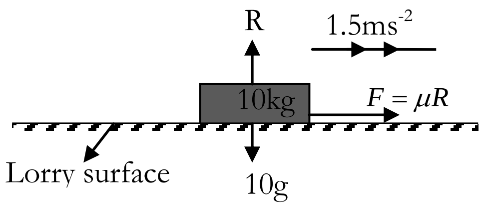
Resolving perpendicular to the plane: R = $5g\cos 30^0$

R = 5×9.8×0.866 = 42.43N

Example 7

A parcel of mass 10kg rests on a lorry which is moving along a straight flat road. When the lorry is accelerating at $1.5ms^{-2}$ the parcel is just on the point of sliding backwards. What is the coefficient of friction between the parcel and the lorry?

Solution



As the parcel is on the point of sliding backwards, the friction force acts forwards in the direction of the acceleration and is the only force acting on the parcel.

Using $F = ma$

$F = 10 \times 1.5 = 15N$

But $F = \mu R$

Since the parcel is vertically in equilibrium, R = 10g N

Therefore $\mu R = 15$

$\mu \times 10g = 15$

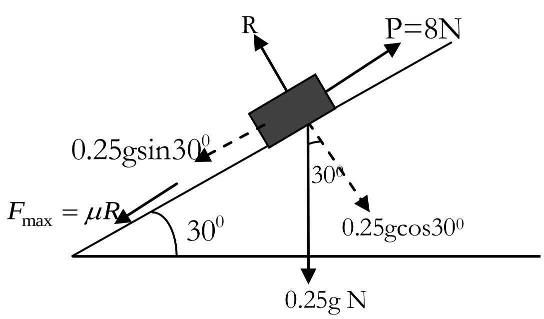
$$\mu = \frac{15}{10 \times 9.8} = 0.15$$

The coefficient of friction between the parcel and the lorry is 0.15

Example 8

A body of mass 0.25 kg rests on a rough plane, inclined at 30^0 to the horizontal. The coefficient of friction between the body and the plane is 0.2. Find the acceleration of the body when a force of 8N is applied up the plane.

Solution



As the body is being pulled upwards by a force P, then the friction force $F_{\max} = (\mu R)$ acts downwards.

Resolving perpendicular to the plane: $R = 0.25g\cos 30^0 = 2.12N$

$F_{\max} = \mu R$

$\Rightarrow F_{\max} = 0.2 \times 2.12 = 0.42N$

Resolving parallel to the plane: The component of the weight acting down the plane $= 0.25g\sin 30^0 = 1.23N$

Therefore the forces acting down the plane are 0.42N and 1.23N.

The resultant force F acting up the plane F = 8 – (0.42 + 1.23) = 6.35N

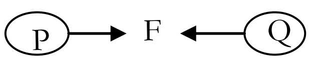
Using $F = ma$

$6.35 = 0.25a$

$a = 25.4ms^{-2}$

Newton's Third law

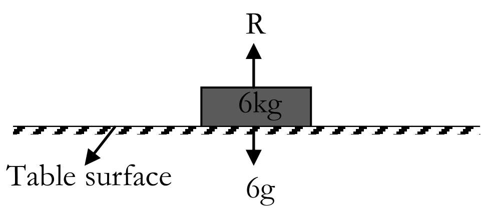
This law states that: Action and Reaction are equal and opposite. This means that if two bodies P and Q are in contact and exert forces on each other, the forces are equal in magnitude and opposite in direction.



If body P exerts a force F in the direction shown, body Q exerts an equal for force F but in the opposite direction as shown.

For example;

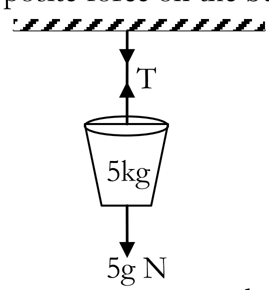
1. Suppose a suit case with a mass of 6kg rests on a horizontal table. The case exerts a force on the table and the table **'reacts'** by exerting an equal and opposite force on the suit case.



Since the suit case is a rest, the reaction force

R = 6g which is equal to the weight of the case.

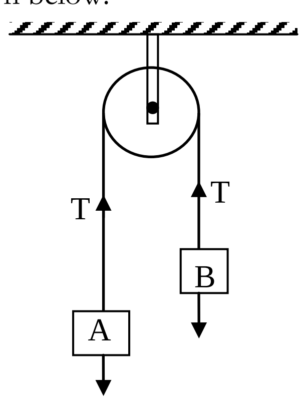
2. Suppose a bucket of mass 5kg hangs on a vertical rope attached to a ceiling. The rope exerts an equal and opposite force on the bucket.



Tension force = weight of the bucket

T = 3g N

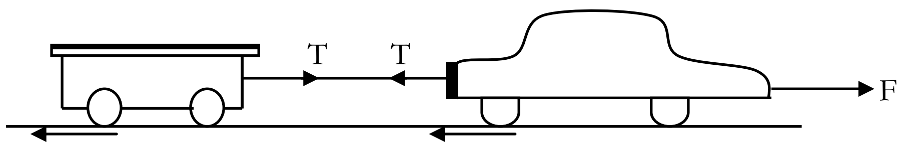
3. Suppose two masses are suspended by a string over a frictionless (smooth) pulley. The string transmits a tension force T which pulls upwards when considering A but pulls B upwards when considering B as shown below.



Example 9

A car of mass 800kg is pulling a caravan of mass 500kg by means of a light inextensible string at a steady speed. The resistance to motion of the car and the caravan is 5N per kg of mass for each vehicle. What is the tractive force of the car? If the car and the caravan accelerate at $0.2ms^{-2}$, what is the new tractive force?

Solution



Resistance = (5×500) N   Resistance = (5×800 N)

By Newton's third law if the pull of the car on the caravan is T, the pull of the caravan on the car is also T (reversed in direction). The resistance to the motion of the caravan is (5×500) N and that of the car (5×800) N

For the caravan, since the speed is steady: T = 5×500 = 2500N

For the car, similarly: F = T + (5×800) = 6500N

The tractive force is thus 6500N

The weights and road reactions, being perpendicular to the line of acceleration do not enter into the solution.

When the vehicles accelerate, we have for the caravan:
Using F = ma; $T-(5\times500)=500\times0.2$

$$T=100+2500=2600N$$

For the car: $F-(T+5\times800)=800\times0.2$

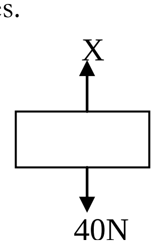
$$F=2600+4000+160=6760N$$

The new tractive force is 6760N. The 260N increase in the tractive force is the extra force necessary to accelerate the vehicles.

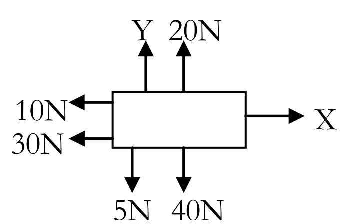
Exercise

1. Find the magnitudes of the unknown forces X and Y if a body is at rest under the action of certain forces.

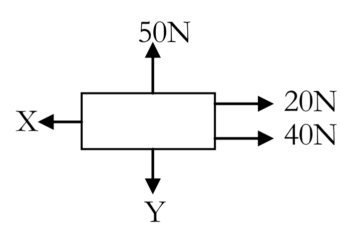
(a)



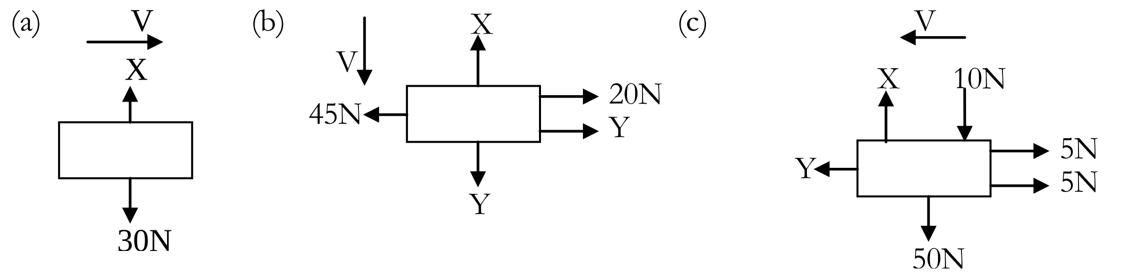
(b)



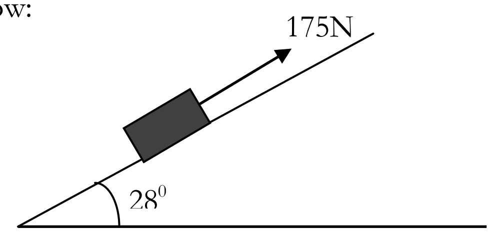
(c)



2. In each of the following situations a body is shown moving with constant velocity V under the action of certain forces. Find the magnitudes of the unknown forces X and Y.



3. If a force of 20N acts on a mass of 2kg what is the acceleration produced?
4. A mass of 1.5kg has an acceleration of $0.8ms^{-2}$. What force is acting on it?
5. A mass of 5kg is dragged across a rough surface (frictional force equal to 3N opposing the motion) by a horizontal force of 20N. What is the acceleration produced?
6. A horizontal force of 0.6N acts on a body of mass 0.3kg. There is a resistance of 0.15N opposing the first force. What acceleration will be produced? $Ans(1.5ms^{-2})$
7. Two forces F_1 and F_2 act on a body of mass 2.5kg. F_1 has magnitude 5N and in the direction $N30^0E$ and F_2 has magnitude 8N in direction E. Find the acceleration of the body. $Ans(4.54ms^{-2})$
8. A body of mass 500g experiences a resultant force of 3N. Find:
   (a) the acceleration produced.
   (b) the distance travelled by the body whilst increasing its speed from $1ms^{-1}$ to $7ms^{-1}$.
9. Two forces, 20 and 10N act on a body of mass 0.5kg at right angles to each other. What is the acceleration of the mass in magnitude and direction?
10. A body of mass 10kg slides down a smooth slope whose inclination to the horizontal is 30^0. What is its acceleration?
11. A pack of bricks of mass 100kg is lifted up the side of a house. Find the force in the lifting rope when the bricks are lifted with an acceleration of $0.25ms^{-2}$. $Ans(1005N)$
12. A particle of mass 10kg is pulled up a smooth slope inclined at 600 to the horizontal by a string parallel to the slope. If the acceleration of the particle is $0.98ms^{-2}$, find the tension in the string.
13. A boat of mass 200kg stands on a rough ramp inclined at 28° to the horizontal. A light inextensible rope attached to the rope prevents it from sliding down the ramp as shown in the figure below:



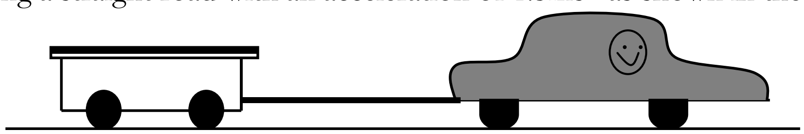
When the tension in the rope is 175N the boat is on the point of sliding down the ramp.
   (i) Copy the diagram overleaf and include on it **all** the forces acting on the boat.
   (ii) Calculate the coefficient of friction between the boat and the ramp, giving your answer correct to two decimal places.

(iii) The rope is removed and the boat moves down the ramp from rest. The coefficient of friction between the boat and the ramp remains the same. Calculate the acceleration of the boat down the ramp.

14. A body of mass 20kg is initially at rest on a rough slope inclined at 20^o to the horizontal. The body is released from rest, and moves down the slope. After 10 seconds it is moving with a velocity of $8ms^{-1}$. Find:
   (i) the acceleration of the body.
   (ii) the frictional force opposing the motion of the body, giving your answer correct to 2 decimal places,
   (iii) the coefficient of friction between the body and the rough surface, giving your answer correct to 2 decimal places.
15. A particle of mass 8kg is pulled along a smooth horizontal surface by a string inclined at 30^0 to the horizontal. If the tension in the string is 10N, find the acceleration of the particle.

16. A mass of 2kg is at rest on a rough horizontal table. A force of 20N is applied to the mass, the force making an angle of 300 with the table. Frictional resistance is equal to 5N. What is the acceleration of the mass?

17. A car of mass 1000kg pulls a trailer of mass 800kg by means of a light inextensible horizontal tow-bar along a straight road with an acceleration of $1.5ms^{-2}$ as shown in the figure below:



The car's engine exerts a tractive force of 4000N and the resistance to motion of the car is 0.8N per kg. Find:

   (i) the resistance to motion of the trailer; $Ans.(500N)$
   (ii) the tension in the tow-bar. $Ans.(1700N)$

At the instant when the speed of the car and the trailer is $10ms^{-1}$ the tow-bar breaks. Find
   (iii) the time taken by the trailer in coming to rest; $Ans.(16s)$
   (iv) the distance it travels in this time. $Ans.(80m)$

18. A car of mass 1200kg is towing a trailer of mass 600kg by means of a light horizontal tow-bar. The car and trailer are travelling along a straight horizontal road at a constant speed of $12ms^{-1}$.The resistance to the motion of the car is 1.2 N per kg of mass. The resistance to the motion of the trailer is 0.8 N per kg of mass. Find:

   (i) the tractive force of the car's engine, $Ans.(1920N)$
   (ii) the tension in the tow-bar. $Ans.(480N)$

After travelling for a short time at 12 ms^{-1} the car accelerates at 0.3 ms^{-2}. Find
   (iii) the new tractive force of the engine, $Ans.(2460N)$
   (iv) the speed of the car and trailer after accelerating for a period of 25 seconds. $Ans.(19.5ms^{-1})$

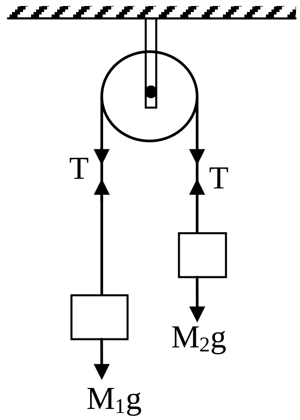
At the end of this period the tow-bar breaks. Find, giving your answer to the nearest metre,

   (v) the additional distance travelled by the trailer before it comes to rest. $Ans.(0.8m)$

MOTION OF CONNECTED BODIES

If two or more bodies are in contact, actually touching or connected by a string, rope or tie rod, they will have an effect on each other and are known as **connected bodies**. In these problems, the **strings** connecting two bodies (particles) are considered to be **light** and **inextensible**: An inextensible string is the one which is inelastic such that its length does not alter under tension.

Consider two bodies of unequal mass M_1 and M_2 connected by a light string passing over a fixed pulley as shown in the diagram below;



Since the string is *light*, its **weight** can be **ignored**.

Also, since it is *inextensible*, when the system is released both bodies have the **same speed** and **acceleration** along the line of the string while the string is kept *taut*.

By Newton's Third Law, the **tension** in the **string** acting on **both bodies** is **equal in magnitude** and **opposite in direction**.

Note also that a *smooth* surface offers **no resistance** to the motion of a body across it.

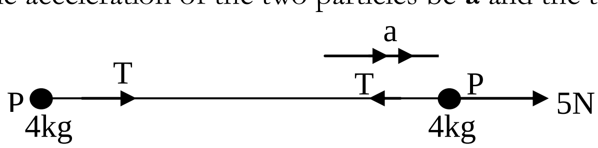
To analyse the motion of the system of connected particles, the forces acting on each body must be considered separately and the equation $F = ma$ applied to each particle in turn.

Example 1

Two particles, **P** and **R** are at rest on a smooth horizontal surface. **P** has mass **3kg** and **R** has mass **4kg**. **P** is connected to **R** by a light inextensible string, which is taut. A force of **5N** is exerted on particle **R** in the direction **PR**.
Find the acceleration of the two particles and the tension in the string.

Solution

Let the acceleration of the two particles be **a** and the tension in the string be **T**



Using $F = ma$:

For particle P: $T = 3a$ .... (i)

For particle R: $5 - T = 4a$ .... (ii)

Adding (i) and (ii) gives $5 = 7a$

$$a = \frac{5}{7} ms^{-2}$$

Substituting the value of a in (i) gives $T = \frac{15}{7}N$

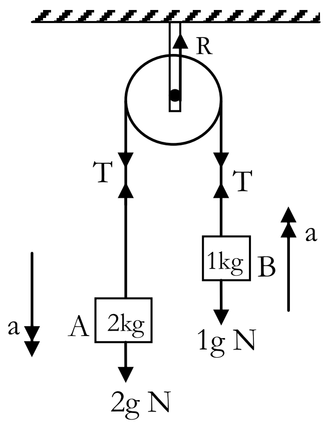
$\therefore$ The acceleration of the particles is $\frac{5}{7}ms^{-2}$ and the tension in the string is $T = \frac{15}{7}N$

This problem illustrates the important point that each mass must be considered separately with all the forces acting on that mass taken into account.

Example 2

A light inelastic string is placed over a smooth pulley. To the ends of the string are attached masses of 2kg (A) and 1kg (B) and both parts of the string are vertical. With what acceleration does the system move? What is the reaction at the axle of the pulley?

Solution



The system is as shown aside. Let the acceleration of the 2kg mass to be a ms^{-2} downwards and hence the 1kg mass will have the same acceleration upwards. Now consider each mass and the pulley separately.

The string transmits a tension T and the reaction at the axle of the pulley is R.

For the mass A, since the acceleration is downwards: $2g - T = 2a$......................(i)

For the pulley, since it has no acceleration vertically: $R = 2T$...............................(ii)

For the mass B, since the acceleration is upwards: $T - g = 2$................................(iii)

From (i) and (ii) by addition, $g = 3a$

$$a = \frac{g}{3} = \frac{9.8}{3} = 3.27ms^{-2}$$

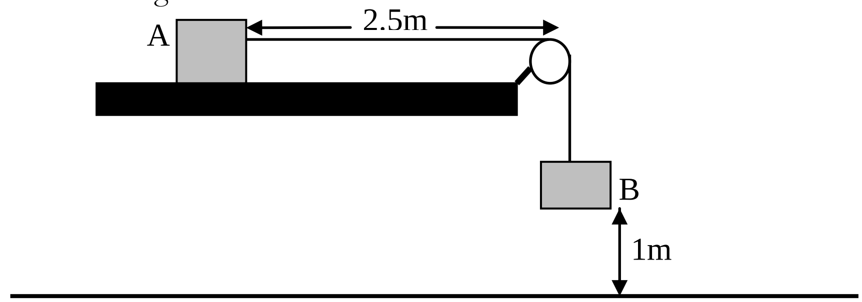
From (iii) $T = a + g = 3.27 + 9.8 = 13.1N$

Finally from (ii) $R = 2 \times 13.1 = 26.2N$

$\therefore$ The system moves with acceleration of 3.27ms^{-2} and the reaction at the axle of the pulley is 26.2N

Example 3

Two packages A and B of masses 3kg and 2kg respectively are connected by a light inextensible string which passes over a smooth pulley fixed at the edge of a smooth horizontal platform. Package A is held on the platform 2.5m from the pulley. Package B hangs freely 1m above the ground as shown in the figure below:



Package A is released from rest. Calculate:

(i) the acceleration of the packages;

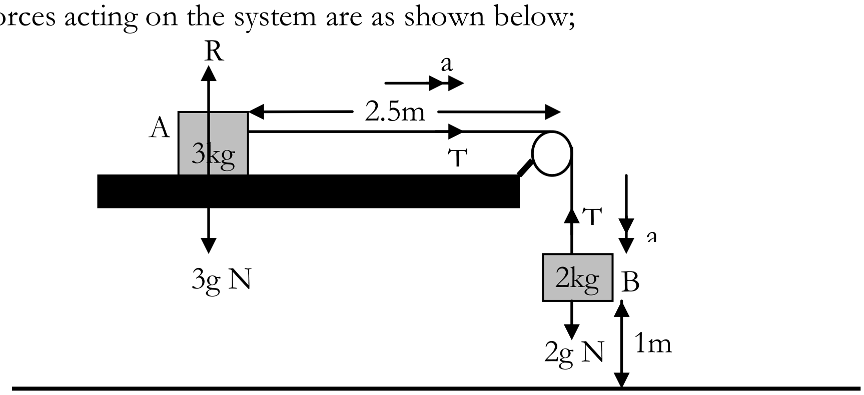
(ii) the tension in the string;

(iii) the speed with which B hits the ground, giving your answer correct to 2 decimal places;

(iv) the time which elapses between B hitting the ground and A reaching the pulley, giving your answer correct to 2 decimal places. (Throughout this question take g = $10ms^{-2}$)

Solution

The forces acting on the system are as shown below;



Using $F = ma$;

For the 2kg mass: $2g - T = 2a$........................(i)

For the 3kg mass: $T = 3a$.................................(ii)

(i) Adding equations (i) and (ii) gives $2g = 5a$

$$\Rightarrow 20 = 5a$$

$$\Rightarrow a = 4ms^{-2}$$

(ii) From (ii) $T = 3\times 4 = 12N$

(iii) Using $v^2 = u^2 + 2as$

$$\Rightarrow v^2 = 0 + 2(4)(1)$$

$$\Rightarrow v = 2.83ms^{-1}$$

(iv) When B hits the ground, A is travelling at a steady speed of 2.83ms^{-1} and still has a distance of 1.5m to go before reaching the edge.

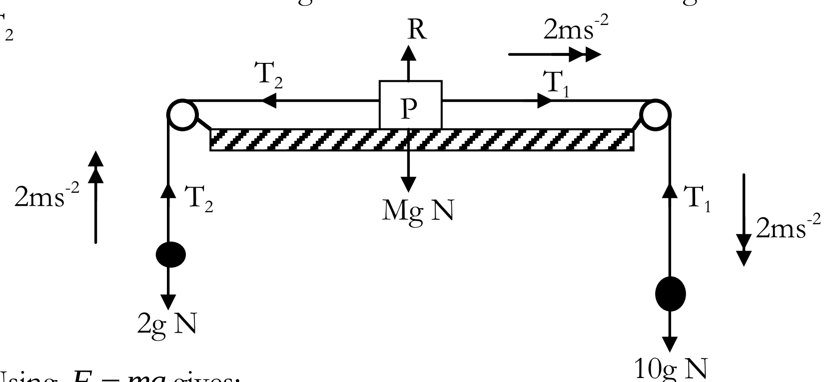
$$\Rightarrow \text{ Time that elapses } = \frac{\text{Distance}}{\text{Speed}} = \frac{1.5}{2.83} = 0.53s$$

Example 4

A body P rests on a smooth horizontal table. Two bodies of mass 2kg and 10kg, hanging freely are attached to P by strings which pass over smooth pulleys at the edges of the table. The two strings are taut. When the system is released from rest, it accelerates at 2ms^{-2}. Find the mass of P.

Solution

Let the mass of P be M kg. The tensions in the two strings will be different; let them be T_1 and T_2



Using $F = ma$ gives:

For 2kg mass: $T_2 - 2g = 2\times 2$

$$T_2 - 2g = 4 ....(i)$$

For P: $T_1 - T_2 = 2M ....(ii)$

For 10kg mass: $10g - T_1 = 10\times 2$

$$10g - T_1 = 20 ....(iii)$$

Adding equations (i), (ii) and (iii) gives

$$8g = 2M + 24$$

$$(8\times 9.8) - 24 = 2M$$

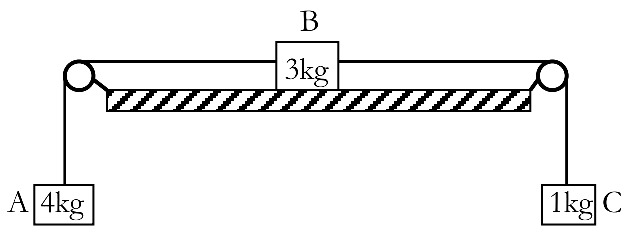
$$54.4 = 2M$$

$$\Rightarrow M = 27.2kg$$

The mass of body P is 27.2kg.

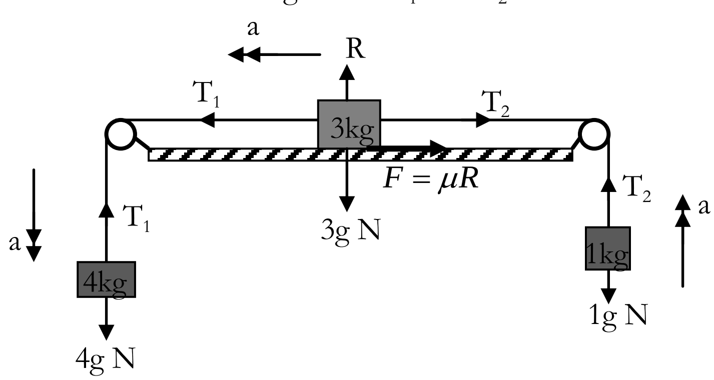
Example 5

Three masses A, B, C are connected by light inextensible strings as shown in the figure below where B is held on a rough horizontal plane (coefficient of friction 0.6). The pulleys are smooth. When B is released, what will be the acceleration of the masses?



Solution

The diagram below shows the forces acting on each mass. Let the acceleration of the masses be a and the tensions in the strings to be T_1 and T_2



Using $F = ma$ for the masses separately;

For the 4kg mass: $4g - T_1 = 4a$..........................(i)

For the 3kg mass: $T_1 - (T_2 + F) = 3a$.................(ii)

$F = \mu R$ (As B moves and acts against the motion.) $R = 3g \Rightarrow F = 0.6 \times 3g$

For the 1kg mass: $T_2 - g = a$............................(iii)

From (i) $T_1 = 4g - 4a$

From (iii) $T_2 = g + a$

Substituting in (ii) gives $4g - 4a - (g + a + 0.6 \times 3g) = 3a$

$$4g - 4a - g - a - 1.8g = 3a$$

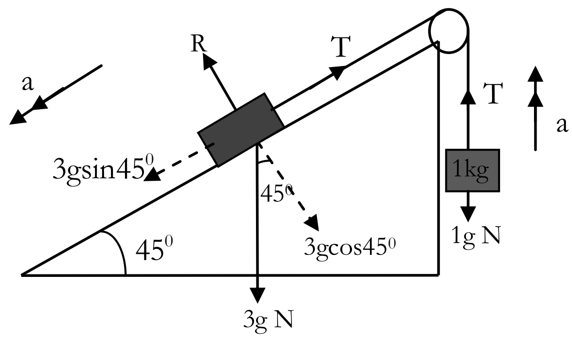
$$1.2g = 8a$$

$$a = 1.47ms^{-2}$$

The acceleration of the masses will be $1.47ms^{-2}$.

Example 6

Two bodies of masses 3kg and 1kg shown are connected by a light inextensible string which passes over a smooth pulley. Calculate the tension T, the normal reaction R and the acceleration a.



If the 1kg mass moves upwards, the 3kg mass must move down the surface of the plane

Solution

Using $F = ma$ in a vertical direction for the 1kg mass gives: $T - g = a$..............................(i)

For the 3kg mass down the plane: $3g\sin 45^0 - T = 3a$ .... (ii)

From (i) $T = g + a$

$$\Rightarrow 3g \times \frac{\sqrt{2}}{2} - (g + a) = 3a$$

$$\Rightarrow \frac{3g\sqrt{2}}{2} - g = 4a$$

$$a = 2.747ms^{-2}$$

Substitute in (i) $T = g + 2.747$

$T = 12.547N$

Resolving perpendicular to the plane: $R = 3g\cos 45^0$

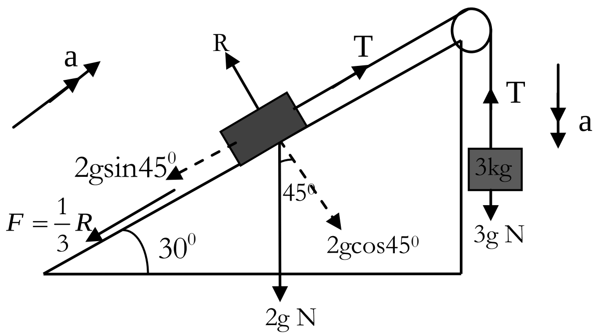
$$R = 20.79N$$

The tension in the string is 12.5N, the normal reaction is 20.79N and the acceleration of the particles is 2.747 ms^{-2}.

Example 7

A particle of mass 2kg rests on the surface of a rough plane which is inclined at 30^0 to the horizontal. It is connected by a light inelastic string passing over a light smooth pulley at the top of the plane, to a particle of mass 3kg which is hanging freely. If the coefficient of friction between the 2kg mass and the plane is $\frac{1}{3}$ find the acceleration of the system when it is released from rest and find the tension in the string.

Solution



Resolving perpendicular to the plane gives: $R = 2g\cos 30^0 = 2g \times \frac{\sqrt{3}}{2} = g\sqrt{3}$

There is an acceleration up the plane. Using $F = ma$;

For the 2kg mass: $T - \frac{1}{3}R - 2g\sin 30^0 = 2a$

$$T - \frac{1}{3}\left(g\sqrt{3}\right) - 2g \times \frac{1}{2} = 2a$$

$$T - \frac{1}{3}g\left(\sqrt{3}+3\right) = 2a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

For the 3kg mass: $3g - T = 3a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$

Adding equations (i) and (ii) gives $3g - \frac{1}{3}g\left(\sqrt{3}+3\right) = 5a$

$$9g - g\left(\sqrt{3}+3\right) = 15a$$

$$6g - \sqrt{3}g = 15a$$

$$a = \frac{1}{15}\left(6-\sqrt{3}\right)g$$

Substituting for a in (ii) gives $T = 3g - 3a$

$$T = 3g - 3 \times \frac{1}{15}\left(6-\sqrt{3}\right)g$$

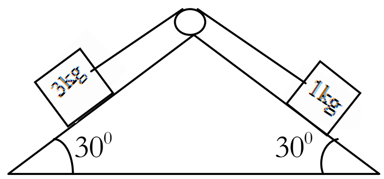
$$T = 3g - \frac{1}{5}\left(6-\sqrt{3}\right)g$$

$$T = \frac{1}{5}\left(9+\sqrt{3}\right)g$$

Therefore the acceleration of the system is $\frac{1}{15}\left(6-\sqrt{3}\right)g$ and the tension in the string is $T = \frac{1}{5}\left(9+\sqrt{3}\right)g$

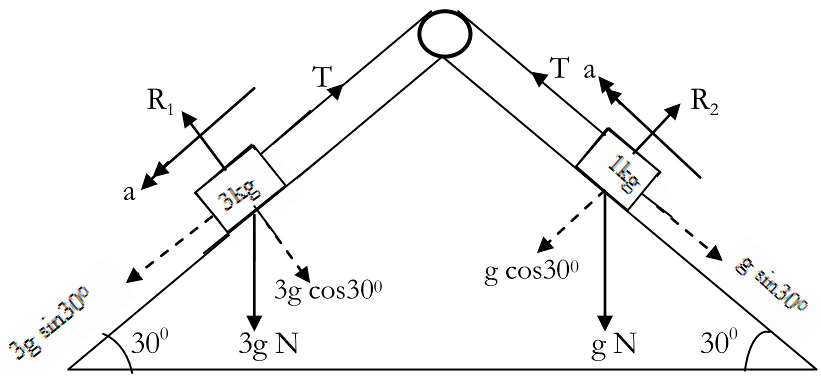
Example 8

The figure below shows a wedge with a smooth surface and two masses connected by a light string over a smooth pulley. The strings run parallel to the wedge surface. Find the acceleration of either mass and the tension in the string.



Solution

Clearly the 3kg mass will slide downwards. Consider each mass separately, marking the forces acting on it then we have:



Resolving in the direction of the acceleration of each mass and using, $F = ma$ then;

For the 3kg mass: $3g \sin 30^0 - T = 3a$............................(i)

For the 1kg mass: $T - g \sin 30^0 = a$................................(ii)

Adding equations (i) and (ii) gives $2g \sin 30^0 = 4a$

$$2 \times 9.8 \times \frac{1}{2} = 4a$$

$$9.8 = 4a$$

$$a = 2.45 ms^{-2}$$

From (ii) $T = a + g \sin 30^0$

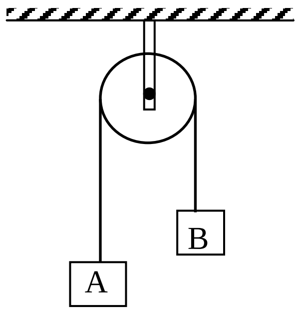
$$T = 2.45 + 4.9$$

$$T = 7.35N$$

The acceleration of either mass is 2.45 ms^{-2} and the tension in the string is 7.35N

Exercise

1. Two packages, A and B, of masses 15kg and 9kg respectively are connected by a light inextensible string passing over a smooth fixed pulley as shown in the figure below:



The packages are released from rest with the string taut. Find:

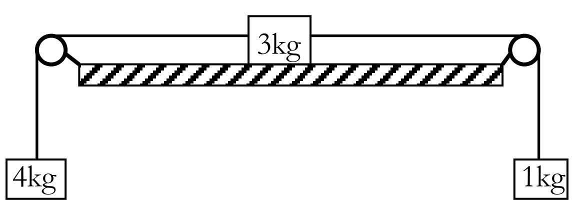
(i) the acceleration of the packages;
(ii) the tension in the string;
(iii) the force exerted by the string by the pulley.

2. Two particles of mass 5kg and 3kg are connected by a light inelastic string passing over a smooth fixed pulley. Find the acceleration of the particles and the tension in the string when the system is moving freely.

3. A body of mass 3kg at rest on a smooth horizontal table is connected by a light string, which passes over a smooth pulley at the edge of the table to another body of mass 2kg hanging freely. Find the acceleration of the particles.

4. A mass of 3kg rests on a smooth horizontal table connected by a light string passing over a smooth pulley at the edge of the table to another mass of 2kg hanging vertically. When the system is released from rest, with what acceleration do the masses move and what is the tension in the string.

5. The figure below shows three masses connected by two pieces of light inextensible strings as shown. The horizontal surface is smooth and the pulleys are free from friction. Find the acceleration with which the masses move and the tensions in the strings.



6. A body of mass 10kg lies on a smooth inclined plane. A light string attached to this body passes over a smooth pulley at the top of the plane and supports a mass of 2kg hanging freely. If the inclination of the plane is θ to the horizontal where $\sin\theta = \frac{1}{14}$. Find the acceleration of the masses.

7. A particle A of mass 10kg lies on a rough horizontal table. It is connected by a light inextensible string which passes over a smooth pulley at the end of the table, to a particle B of mass 8kg, which hangs freely above the ground. The system is released from rest with A at a distance of

4m from the edge of the table. The coefficient of friction between the particle and the table is 0.5.

(a) Find the acceleration of the particles.

(b) If B comes to rest on the ground after 1.5s, calculate

(i) The distance of A from the edge of the table.

(ii) The subsequent deceleration of A.

8. A particle of mass 10kg lies on a rough horizontal table and is connected by a light inextensible string passing over a fixed smooth light pulley at the edge of the table to a particle of mass 8kg hanging freely. The coefficient of friction between the 10kg mass and the table is $\frac{1}{4}$. The system is released from rest with the 1okg mass a distance of 1.5m from the edge of the table. Find:

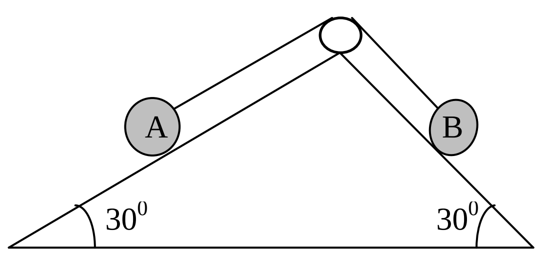
(a) the acceleration of the system

(b) the resultant force on the edge of the table

(c) the speed of the 10kg mass as it reaches the pulley.

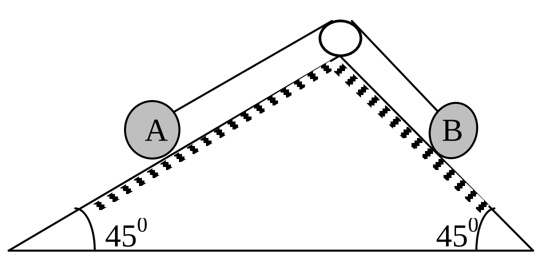
9. Two particles of masses 2kg and 6kg are attached one on each end of a long light inextensible string which passes over a fixed smooth pulley. The system is released from rest and the heavier particle hits the ground after 2 seconds. Find the height of this particle above the ground when it was released, and the speed at which it hits the ground.

10.



Two particles A and B rest on the smooth inclined faces of a fixed wedge. The particles are connected by a light inextensible string that passes over a fixed smooth pulley at the vertex of the wedge as shown in the diagram. If A and B are each of mass 4kg, find the acceleration of the masses and the tension in the string. Find also the force exerted by the string.

11.



A and B are two particles connected by a light string that passes over a smooth pulley at the top of a wedge as shown in the diagram. The mass of A is 1kg and that of B is 2kg. Contact between each particle and the wedge is rough with a coefficient of friction of $\frac{1}{3}$. When the system is allowed to move, find the acceleration and the tension in the string.

Pulley systems

In problems concerned with connected bodies and pulley systems or bodies in contact where each body is free to move, the acceleration of different parts of the system will not necessarily have the same magnitude. However a relationship between the accelerations can be found by considering the physical properties of the system. As before, Newton's laws must be applied to each body of the system.

As we will see in the following examples, there are two basic steps in solving these problems:

(i) Write down all the equations and
(ii) Relate the acceleration of the various masses by noting that the length of the string(s) does not change a fact that we call "conservation of string".

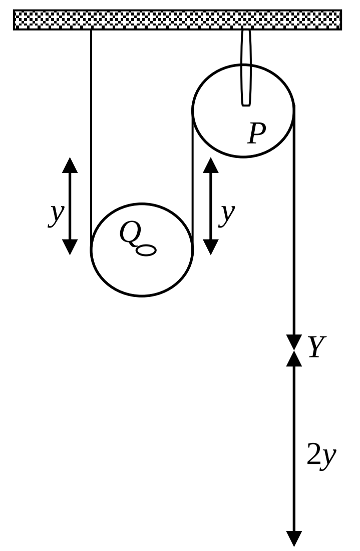
Examples of pulley systems and related accelerations

1. In this pulley system shown, pulley P is fixed and pulley Q may be raised by pulling down the end Y of the string. All the parts of the string not in contact with the pulleys are vertical.

   For Q to move upwards a distance y, a length $2y$ of string must pass over the pulley P.

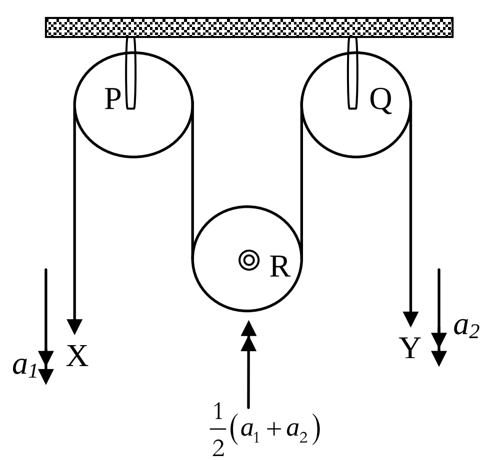
   The distance between the pulley P and the end Y of the string is therefore increased by $2y$.

   Hence using the conservation of string principle, if Q has an upward acceleration of a, then the end Y of the string will have a downward acceleration of 2a.



2. In this pulley system shown, P and Q are fixed pulleys and R is a moveable pulley. When the ends, X and Y, of the string move down distances x and y respectively, the length of the string between P and Q is shortened by $(x+y)$. Pulley R will therefore move up a distance $\frac{1}{2}(x+y)$.

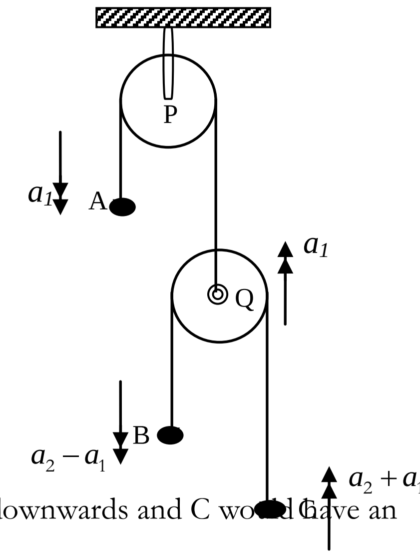
   Using the principle of conservation of string, if the downward accelerations of X and Y are a_1 and a_2, the upward acceleration of R will be $\frac{1}{2}(a_1+a_2)$



3. In this pulley system, pulley P is fixed and Q is moveable.

If A moves down with an acceleration a_1, Q moves up with an acceleration a_1.

Suppose the accelerations of B and C relative to pulley Q, are a_2 downwards and a_2 upwards respectively. Since Q has an acceleration a_1 upwards then B has an actual acceleration $(a_2 - a_1)$ downwards and C has an actual acceleration $(a_2 + a_1)$ upwards.

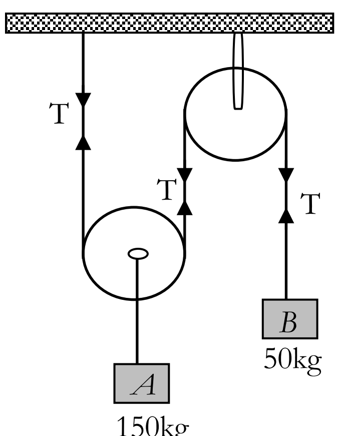


N.B: There are two different strings involved in this pulley system and their tensions are therefore not the same unless stated otherwise.

How ever if pulley Q were stationery, B would accelerate downwards and C would have an equal acceleration upwards.

Note: (In problems involving pulley systems, if we are wrong about the direction we have chosen for the various accelerations, the answers we obtain will be negative).

Example 9



A system of frictionless pulleys carries two masses hung by inextensible strings as shown below. Find the tension in the strings and the acceleration of the masses A and B.

Solution

Let a be the upward acceleration of A, then the upward acceleration of B will be $2a$

Using $F = ma$ gives:

For A: $2T - 150g = 150 \times a$

$\Rightarrow 2T - 150g = 150a$..................................(i)

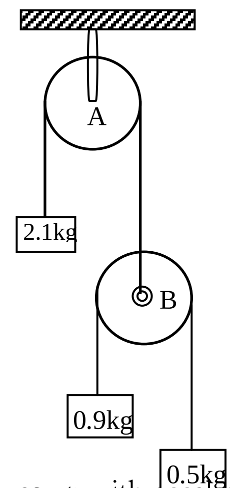
For B: $50g - T = 50 \times 2a$

$\Rightarrow 50g - T = 100a$.......................................(ii)

Solving (i) and (ii) gives: $a = 1.4ms^{-2}$ *and* $T = 630N$

The tension in the string is $630N$, the acceleration of A is $1.4ms^{-2}$ and the acceleration of B is $2.8ms^{-2}$.

Example 10



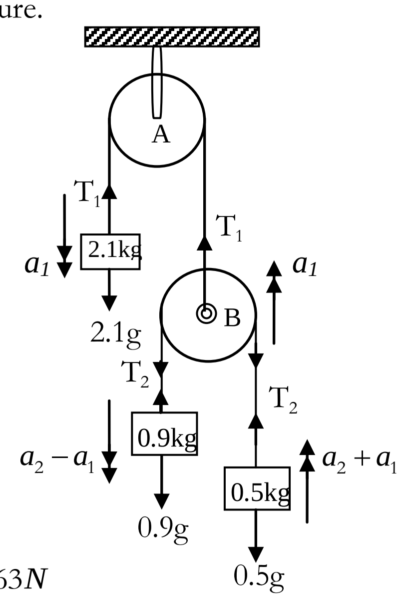
The figure shows a pulley system in which both pulleys A and B are weightless and smooth but A is fixed, where as B is on one end of a light inextensible string passing over A, on the other end of which is a weight of mass 2.1 kg. Over pulley B is another light inextensible string with masses of 0.9kg and 0.5kg on the ends. If the system is free to move, find the acceleration of B and the tensions in the strings.

Solution

Let the 2.1 kg mass moves down with acceleration a_1, pulley B moves up with acceleration a_1.

Suppose the accelerations of 0.9kg mass and 0.5kg mass relative to pulley B, are a_2 downwards and a_2 upwards respectively. Then the 0.9kg mass has an actual acceleration $(a_2 - a_1)$ downwards and 0.5kg mass has an actual acceleration $(a_2 + a_1)$ upwards.

Let the tensions in the strings be T_1 *and* T_2 as shown in the figure.



Applying $F = ma$ to each part of the system, we have;

For 2.1 kg mass: $2.1g - T_1 = 2.1a_1$

$$20.58 - T_1 = 2.1a_1 .... (i)$$

For pulley B: $T_1 - 2T_2 = 0$.....................(ii) $(B \text{ is weightless})$

For 0.9 kg mass: $0.9g - T_2 = 0.9(a_2 - a_1)$

$$8.82 - T_2 = 0.9a_2 - 0.9a_1 .... (iii)$$

For 0.5 kg mass: $T_2 - 0.5g = 0.5(a_2 + a_1)$

$$T_2 - 4.9 = 0.5a_2 + 0.5a_1 .... (iii)$$

From equation (ii) $T_1 = 2T_2$...(iv)

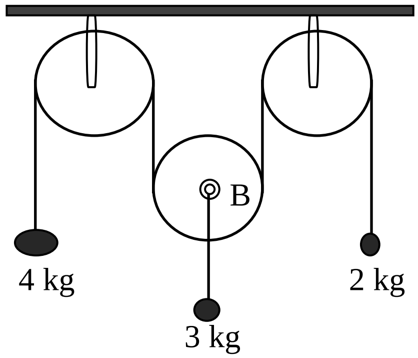
Substituting (iv) in (i) and solving the equations gives:

$a_1 = 2.357\ ms^{-2},\ a_2 = 3.473\ ms^{-2},\ T_1 = 7.815N$ *and* $T_2 = 15.63N$

The acceleration of the pulley B is $2.357\ ms^{-2}$ and the tensions in the strings are $7.815N$ *and* $15.63N$.

Example 11

Masses of 4kg and 2kg are connected by a light inextensible string passing over two fixed smooth pulleys and under a light smooth movable pulley B, which carries a mass of 3kg as shown in the diagram below. Find the acceleration of the 4kg mass and the tension in the string.



Solution

4. If the accelerations of the 4kg and the 3kg masses are a_1 and a_2 upwards respectively, then the acceleration of the pulley B (3kg mass) will be $\frac{1}{2}(a_1 + a_2)$ in the opposite direction. Let the tension in the string be T.

Using $F = ma$ to each part of the system we have

For the 4kg mass: $T - 4g = 4a_1$

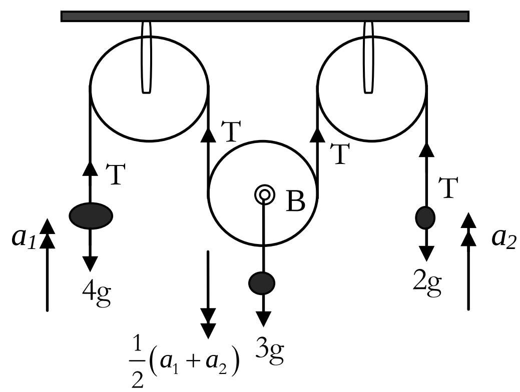
$$\Rightarrow T - 4a_1 = 39.2 \ldots\ldots\ldots\ldots\ldots (i)$$

For the 3kg mass: $3g - 2T = 3 \times \frac{1}{2}(a_1 + a_2)$

$$\Rightarrow -4T - 3a_1 - 3a_2 = -58.8 \ldots\ldots\ldots\ldots\ldots\ldots (iii)$$

For the 2kg mass: $T - 2g = 2a_2$

$$\Rightarrow T - 2a_2 = 19.6 \ldots\ldots\ldots\ldots\ldots\ldots (ii)$$



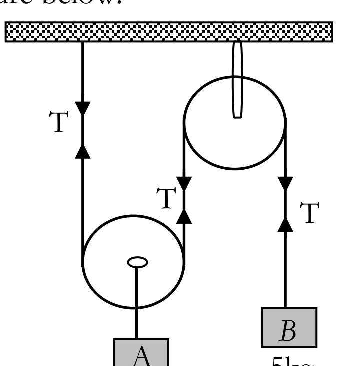
Solving equations (i), (ii) and (iii) simultaneously gives

$T = 18.816N,\ a_1 = 5.096ms^{-2}$ *and* $a_2 = 0.392ms^{-2}$

The acceleration of the 4kg mass is 5.096 ms^{-2} and the tension in the string is 18.816N

Exercise

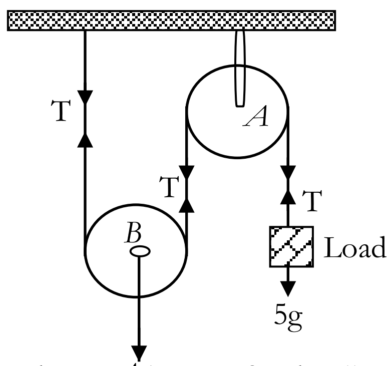
1. Determine the tension in the strings and acceleration of block A if blocks A and B are weighing 2kg and 5kg connected by a string and a frictionless and weightless pulley as shown in the figure below.



acceleration of $A = 3.56ms^{-2}$ *upwards*

Tension $= 13.36N$

2.

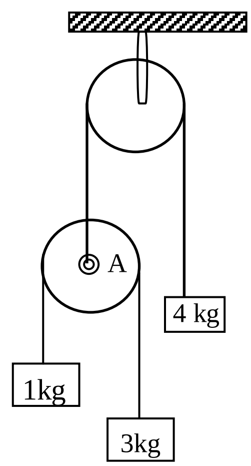


The figure shows a pulley system. A is a fixed smooth pulley and pulley B, also smooth has a mass of 4kg. A load of mass 5kg is attached to the free end of the rope. Determine the acceleration of the load and the tension in the rope when the load is released.

acceleration of Load $= 4.9\ ms^{-2}$

Tension in the string $= 24.6N$

3. The diagram shows a fixed pulley carrying a string which has a mass of 4 kg attached at one end and a light pulley A attached at the other. Another string passes over pulley A and carries a mass of 3kg at one end and a mass of 1kg at the other end.
Find:
   (i) the acceleration of pulley A
   (ii) the acceleration of the 1kg, 3kg and 4kg masses
   (iii) the tensions in the strings.



$(i)\ 1.4\ ms^{-2}$

$(ii)\ 7ms^{-2}, 4.2ms^{-2}, 1.4ms^{-2}$

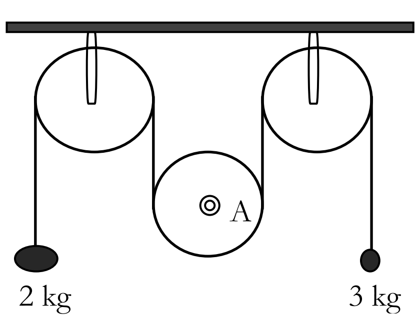
$(iii)\ 33.6N, 16.8N$

4. A fixed pulley carries a string which has a load of mass 7kg attached to one end and a light pulley attached to the other end. This light pulley carries another string which has a load of mass 4kg at one end, and another load of mass 2kg at the other end. Find the acceleration of the 4kg mass and the tensions in the strings.

Acceleration of 4kg mass is 2.38 ms^{-2}

Tensions in the strings $(59.33N, 29.66N)$

5. In the pulley system shown in the diagram below, A is a movable pulley of mass 6kg. Find the acceleration of the movable pulley and the tension in the string.



Acceleration of movable pulley is 1.09 ms^{-2}

Tensions in the string $(26.13N)$

6. A string, with one end fixed, passes under a movable pulley of mass 8kg, and over a fixed pulley; the string carries a 5kg mass at its other end. Find the acceleration of the 5kg mass and the tension in the string.

Acceleration of 5kg mass is 1.4 ms^{-2}

Tensions in the string $(42N)$

7.

WORK, ENERGY AND POWER

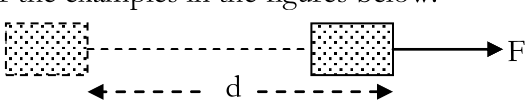
Work

When a force acts on a body and causes it to move, we say the force does work on the body. The measurement of this work will involve the distance through which the body is moved by the force and the size of the force in the direction in which the body moves.

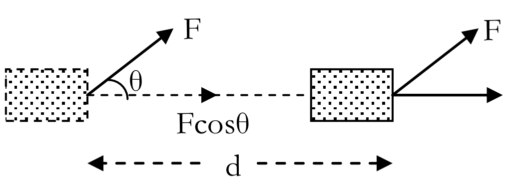
Work done by a constant force

When the force is constant, the work done is defined as the product of the force and the distance moved in the direction of the force.

Consider the examples in the figures below:

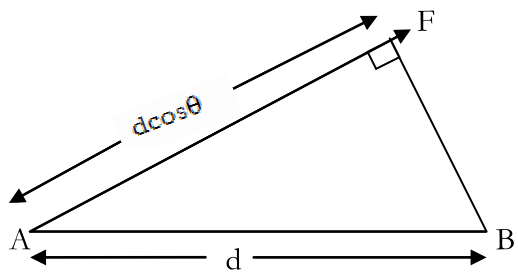


1. If the force F moves the body through a distance d as shown above in the direction of the force, the work done = $F \times d$



2. In the figure above F acts at angle θ to the direction in which the body moves. The component of F along the line of motion is Fcosθ. So the work done by the force F is $F\cos\theta \times d$. If the body moves in the same direction as the force the angle θ is 0 and so work done $= F \times d$

   This result can be obtained by considering the distance moved in the direction of the force. Consider the example in the figure below a force F acting an angle θ moves a body from point A to point B.



   The distance moved in the direction of the force is given by $d\cos\theta$

   So the work done by the force is $F \times d\cos\theta$ the same result as is before.

   The other component of the force (Fsinθ) does no work as its point of application does not move in the direction of the force.

The unit of work is **joule** abbreviated **J** (with force, **F**, in newtons **N** and distance, **d**, in metres **m**)

Larger uints for work are;

Kilojoule kJ (**1kJ=1000=10^3 J**)

Megajoule MJ (**1MJ=1000000=10^6** J)

Example 1

   How much work is done when a force of 5000N moves its point of application 600mm in the direction of the force?

Solution

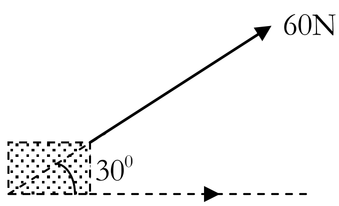
   Force, F=5000N and distance, d =600mm = 600×10^{-3}m

$$\begin{aligned} \textit{Work done} &= F \times d \\ &= (5000)\times(600\times10^{-3}) \\ &= 3000J \end{aligned}$$

Example 2

   A box is pulled horizontally through 4m by a force of 60N at an angle of 30^0 to the horizontal. What is the work done?

Solution



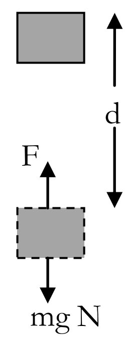
The component of the force in the direction of motion is: $60\cos 30^0 = 60 \times \frac{\sqrt{3}}{2} = 30\sqrt{3}N$

Hence $Work\ done = F \times d$

$$= 30\sqrt{3} \times 4$$
$$= 120\sqrt{3}J$$

Work done against gravity

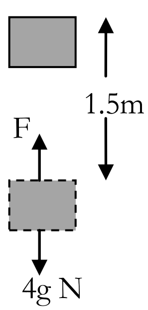
In order to raise a mass of m kg vertically at a constant speed, a force of mg N must be applied vertically to the mass. In raising the mass a distance d metres, the work done against gravity will be mgd joules.



Example 3

I lift a mass of 4kg at a steady speed through a vertical distance of 1.5m. What is the work done by me?

Solution



The force I exert (F N) will equal the weight of the mass (4g N) and the distance lifted is 1.5m in the direction of the force.

Hence the $Work\ done = F \times d$

$$= (4g) \times (1.5) = 6 \times 9.8$$
$$= 58.8J$$

Example 4

Find the work done in raising 100g of water through a vertical distance of 3m.

Solution

The force is the weight of the water, so F = 100g

$Work\ done = F \times d$

$$= (100g)\times(3) = (100\times 9.8)\times 3$$
$$= 2940J$$

Work done against friction

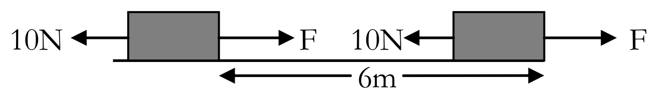
In order to move a body at constant speed against friction (resistance), a force equal in magnitude to the force of friction (resistance forces) acting on the body has to be applied to the body.

Example 5

A brick is pulled a distance of 6m across a horizontal surface against a friction force of 10N. If the brick moves at a steady speed, find the work done against friction.

Solution

Let the pulley force be F



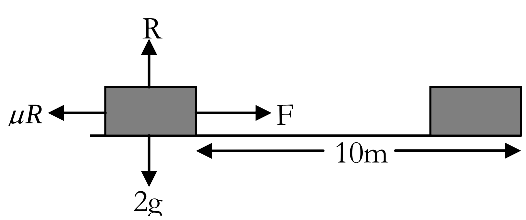
$\Rightarrow F = 10N$

$$Work\ done\ against\ \ friction = F\times d$$
$$= (10)\times(6)$$
$$= 60J$$

Example 6

A particle of mass 2kg is pulled by a horizontal force across a horizontal rough surface coefficient of friction 0.5. If the body moves a distance of 10m with a constant velocity and the only resistive force is due to friction, find the work done against friction.

Solution



The friction force is μR hence $F = \mu R$ but $R = 2g\ N$

$F = 0.5\times 2g = 9.8N$

$$Work\ done\ against\ \ friction = F\times d$$
$$= (9.8)\times(10)$$
$$= 98J$$

When a body is pulled at a steady speed up the surface of a rough inclined plane, work is done both against gravity and against the frictional force which is acting on the body due to the contact with the rough surface of the plane.

Example 7

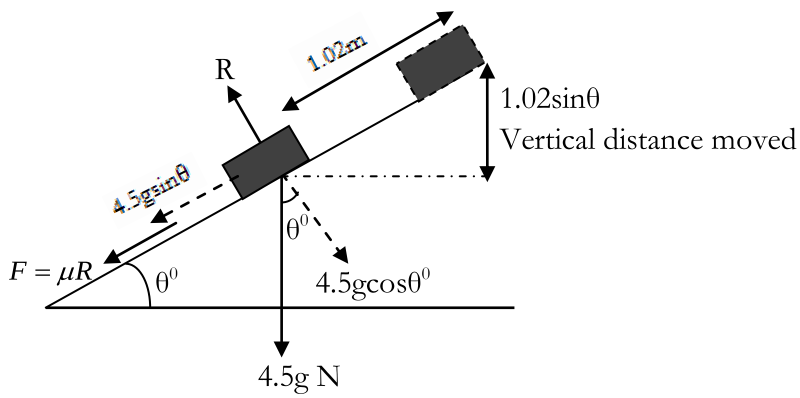
A particle of mass 4.5 kg is pulled up a rough plane inclined at an angle θ to the horizontal where sinθ = 0.2 and travels up a distance of 1.02m. If the coefficient of friction between the body and the surface of the plane is 0.6, find:

(a) the work done against gravity

(b) the work done against friction

(c) the total work done

Solution



(a) $\text{Work done against gravity} = \text{Force} \times \text{vertical distance moved}$

$$= (4.5g) \times (1.02\sin\theta)$$
$$= 4.5 \times 9.8 \times 1.02 \times 0.2 = 8.9964J$$

(b) Resolving perpendicular to surface of the plane: $R = 4.5g\cos\theta$

Given sinθ = 0.2, cosθ = 0.98 since $\left(\cos\theta = \sqrt{1-\sin^2\theta}\right)$

$$\Rightarrow F = \mu R = 0.6 \times 4.5g \times 0.98$$
$$= 25.9308N$$

$$\text{Work done against friction} = F \times d$$
$$= (25.9308) \times (1.02)$$
$$= 26.45J \ (2d.p)$$

(c) Total work done = work done against gravity + work done against friction

= 8.9964 + 26.45 = 35.4464 J

Energy

A body which has the capacity to do work is said to possess energy. The unit of energy is the same as that for work, joules J.

Kinetic energy (K.E)

Kinetic energy may be described as energy due to motion. The kinetic energy of a body may be also defined as the amount of work it can do before being brought to rest. For example when a hammer is used to knock in a nail, work is done on the nail by the hammer and hence the hammer must have possessed energy.

A body of mass m kg travelling at v ms^{-1} horizontally in a straight line has kinetic energy given by:

$$K.E = \frac{1}{2}mv^2$$

Suppose a body of mass m kg is travelling at u ms^{-1} horizontally in a straight line. A force F N now acts on it in the direction of motion and then the force ceases to act when the velocity reached is v ms^{-1}. So the initial K.E of the body $= \frac{1}{2}mu^2$ and the final K.E energy $= \frac{1}{2}mv^2$

Thus the work done by the force F = Final K.E – Initial K.E = Increase in K.E

The unit of K.E is therefore the same as the unit of work, J.

In using the formula $K.E = \frac{1}{2}mv^2$, standard units must be used – kg for mass, ms^{-1} for velocity.

The work done by the force in raising the velocity of the body from u to v is converted into the increase in K.E of the body. Conversely, some or all of the K.E possessed by a body can be converted into work. Hence loss of K.E = work done against a force.

If more than one force acts on a body then the total work done = gain in K.E of the body. The total work done is the algebraic sum of all the separate amounts of work done by or against the separate forces.

Thus work done by forces acting on a body = change of K.E in the body.

Example 8

Find the kinetic energy of a body of mass 0.2kg moving with a speed of $10\sqrt{2}\ ms^{-1}$.

Solution

$$\begin{aligned} K.E &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}(0.2)\times\left(10\sqrt{2}\right)^2 \\ &= 20J \end{aligned}$$

Example 9

A car of mass 1000kg travelling at $30ms^{-1}$ has its speed reduced to $10ms^{-1}$ by a constant breaking force over a distance of 75m. Find;

(a) the car's initial kinetic energy

(b) the car's final kinetic energy

(c) the breaking force.

Solution

(a) Initial K.E $= \frac{1}{2}mu^2$

$$= \frac{1}{2}(1000)\times(30)^2$$

$$= 450000J$$

(b) Final K.E $= \frac{1}{2}mv^2$

$$= \frac{1}{2}(1000)\times(10)^2$$

$$= 50000J$$

(c) Work done = Change in K.E

Change in K.E = 450000 – 50000 = 40000J

$\Rightarrow F\times d = 40000$

$F\times 75 = 40000$

$F = 5333.33N$

Example 10

A body of mass 2kg is brought from rest to a speed of 3ms^{-1} over a distance of 1.2m. What force was acting?

Solution

Initial K.E $= \frac{1}{2}mu^2$

$= \frac{1}{2}(2)\times(0)^2$ Since it is brought from rest u = 0ms^{-1}

$= 0J$

Final K.E $= \frac{1}{2}mv^2$

$= \frac{1}{2}(2)\times(3)^2$

$= 9J$

But work done by force = gain in K.E

Gain in K.E = 9 – 0 = 9J

$\Rightarrow F\times d = 40000$

$F\times 1.2 = 9$

$$F = \frac{9}{1.2} = 7.5N$$

Example 11

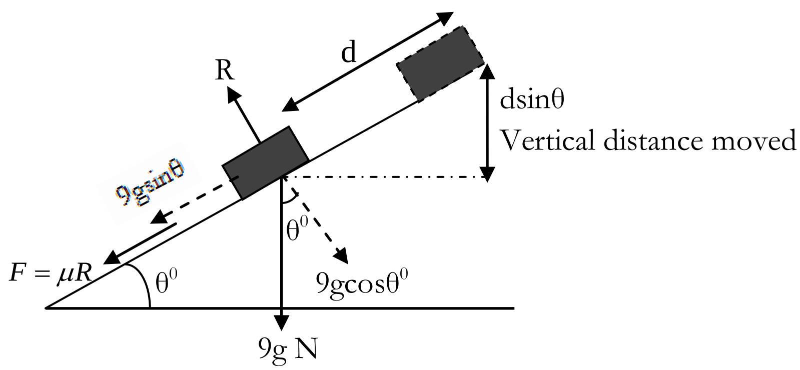
A particle of mass 9kg is projected up an incline of angle θ where sinθ = 0.2 with speed 2ms^{-1}. How far will it travel up the incline;

(i) If the surface is smooth

(ii) If the surface is rough and μ =0.4?

Solution

Let the distance moved up the incline be d



(i) When the surface is smooth, work is done only against gravity.

$$Work\ done\ against\ \text{gravity} = Force\ \times \text{vertical distance moved}$$
$$= (4.5g) \times (\sin\theta d)$$
$$= 4.5 \times 9.8 \times 0.2d = 17.64d\ \ J$$

But the work done against gravity = loss of K.E

$$Initial\ K.E = \frac{1}{2}mu^2$$
$$= \frac{1}{2}(9) \times (2)^2$$
$$= 18J$$

Final K.E = 0

Loss of K.E = Initial K.E – Final K.E

$$\Rightarrow 17.64 \times d = 18$$
$$d = 1.02m\ \ (2d.p)$$

(ii) When the surface is rough, work is done against gravity and also against friction

Total work done = work done against gravity + work done against friction

Resolving perpendicular to the surface: $R = 9g\cos\theta$

Given sinθ = 0.2, cosθ = 0.98 since $\left(\cos\theta = \sqrt{1-\sin^2\theta}\right)$

$$and\ \ friction\ \ force\ F = \mu R = 0.4 \times 4.5g \times 0.98$$
$$= 0.4 \times 9g \times 0.98N$$

$$Work\ done\ against\ \ friction = F \times d$$

$$= (0.4 \times 9 \times 9.8 \times 0.98) \times d$$
$$= 34.5744d$$

Total work done = 17.64d + 34.5774d = (17.64 + 34.5744) d

But total work done = loss of K.E

$$\Rightarrow (17.64 + 34.5744)d = 18$$
$$52.2144d = 18$$
$$d = \frac{18}{52.2144} = 0.345\ (2d.p)$$

Potential Energy (P.E)

The potential energy of a body is that energy it possesses as a reson of its position. P.E is therefore the ability to do work because of the position of the body, in the sense that if released, the body will move to a lower position and its P.E will be converted into K.E.

Formula for Potential Energy

A body is at rest on the earth's surface. It is then raised a vertical distance h above the surface. The work required to do this is: the force required × the distance h

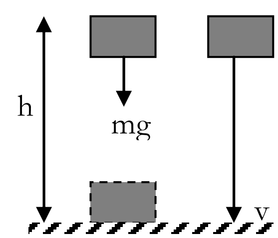
Since the force required is its weight, and weight, W = mg, the work required is mgh.

Potential energy is thus given by: $P.E = mgh$ where h is the height above the earth's surface.

The P.E of a body has no absolute value, but is relative to some point of reference (reference level), say the surface of the earth or some other level above which the body is raised and to which it can fall.

Suppose a body of mass m kg is raised through a height of h m from a floor. The work done against gravity = mgh and hence the body now possesses P.E = mgh J

Thus change in P.E = work done against gravity



If the body now falls it will acquire a velocity v on reaching its original level, where $v^2 = 2gh$. Its K.E is now $\frac{1}{2}mv^2 = \frac{1}{2}m \times 2gh = mgh$. Hence all the P.E has been converted into K.E.

Hence when a body falls through a height h the work done against gravity mgh is converted into K.E. Similarly if a body of mass m kg is raised through a vertical height (h m) by whatever path (provided smooth), the work done against gravity = mgh and is equal to the change in P.E of the body and is equal to the gain in K.E.

Example 12

What is the potential energy of a 10kg mass if it is raised 100m above the surface of the earth?

Solution

$$P.E = mgh$$
$$= 10 \times 9.8 \times 100 = 9800 J$$

Example 13

Find the change in the potential energy of a body of mass 50kg when

(i) ascending a vertical distance of 5m.

(ii) Descending a vertical distance of 5m.

Solution

(i) Change in P.E = work done against gravity

$$= mgh$$
$$= 50 \times 9.8 \times 5 = 2450\ J$$

(ii) When descending, the body is losing P.E, i.e. losing its potential energy to do work

Loss in P.E = mgh

= 50 × 9.8 × 5 = 2450 J

The change in P.E is 2450 J

The principle of conservation of Energy

It states that 'the total mechanical energy of a system remains constant provided that it is not acted on by any external force other than gravity'.

Suppose we have a situation involving a moving body in which there is no energy loss i.e. the body is not affected by any external force say friction other than gravity, the total energy possessed by the body will then be the total of its kinetic energy and its potential energy.

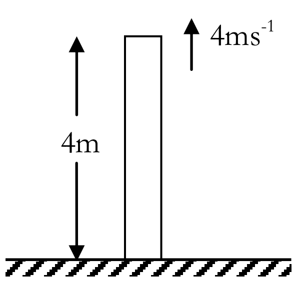
The principle of conservation of energy will then appear in the form: $K.E + P.E =$ constant

The total energy (K.E + P.E) is called its mechanical energy.

Example 14

A body of mass 5kg is thrown so that it just clears the top of a wall 4m high when its speed is $4ms^{-1}$. What is its total mechanical energy as it passes over the wall?

Solution



K.E of stone $= \frac{1}{2}mv^2 = \frac{1}{2}(5)(4)^2 = 40J$

P.E of stone $= mgh = (5)(9.8)(4) = 196J$

Total mechanical energy = $K.E + P.E = 236\ J$

Example 15

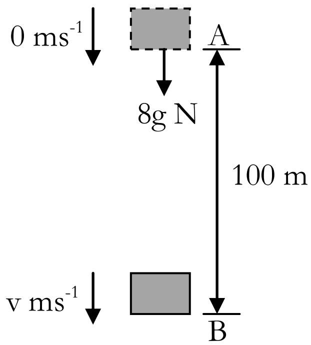
A body of mass 8kg is released from rest at point A and falls freely under gravity up to a point B which is 100m vertically below A. Find its speed at point B.

Solution

Let v ms^{-1} be the speed of the body at point B

Let point B be the reference level from where the P.E is measured.

To find the speed of the body at B, we calculate the total mechanical energy at A and the total mechanical energy at B and then use the principle of conservation of energy.



At A: $K.E = \frac{1}{2}mv^2 = \frac{1}{2}(8)\times(0)^2 = 0J$

$$P.E = mgh = (8)(9.8)(100) = 7840\ J$$

Total energy at A = 0 + 7840 = 7840 J

At B: $K.E = \frac{1}{2}mv^2 = \frac{1}{2}(8)\times(v)^2 = 4v^2$

$$P.E = mgh = (8)(9.8)(0) = 0\ J$$

Total energy at B = $4v^2$ +0 = $4v^2$

By the principle of conservation of energy:

Total energy at A = Total energy at B

$$7840 = 4v^2$$

$$v^2 = 1960$$

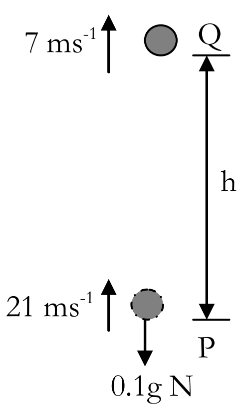
$$v = 44.27\ ms^{-1}$$

Example 16

The point Q is vertically below P. A particle of mass 0.1 kg is projected from P vertically upwards with speed 21 ms^{-1} and passes through point Q with speed 7 ms^{-1}. Find the distance from P to Q.

Solution

We shall choose to measure the P.E from the level P. Let the distance from P to Q be h m.



At P: $K.E = \frac{1}{2}mv^2 = \frac{1}{2}(0.1)\times(21)^2 = 22.05J$

$$P.E = mgh = (0.1)(9.8)(0) = 0\ J$$

Total energy at P = 22.05 + 0 = 22.05 J

At Q: $K.E = \frac{1}{2}mv^2 = \frac{1}{2}(0.1)\times(7)^2 = 2.45\ J$

$$P.E = mgh = (0.1)(9.8)(h) = 0.98h\ J$$

Total energy at Q = 2.45 + 0.98h

By the principle of conservation of energy:

Total energy at P = Total energy at Q

$$22.05 = 2.45 + 0.98h$$

$$19.6 = 0.98h$$

$$h = 20\ m$$

Power

Power is a measure of the rate at which work is done, or the rate at which energy is being transferred.

$$Power = \frac{work\ done}{time\ taken}$$

If 1 joule of work is done in 1 second, the rate of working is 1 watt (W). Thus the unit for power is the watt W.

Larger units of power are the kilowatt kW (1kW=1000 =10^3 W) and the megawatt MW (1MW = 1000000 = 10^6 W)

If work is being done by a machine or body moving at speed v against a constant force or resistance, F, then the power P is given by $P = F \times v$

Example 17

A constant force of 2kN pulls a crate of soda along a level floor for a distance of 10m in 50s. What is the power used?

Solution

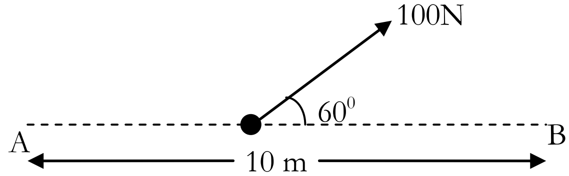
$$work\ done = force \times \text{distance}$$

$$2000 \times 10 = 20000\ J$$

$$Power = \frac{work\ done}{time\ taken} = \frac{20000}{50} = 400W$$

Example 18

A particle is acted upon by a constant force of 100N as shown in the diagram below. If the particle moves from A to B in 5 seconds, find the work done by the force and the power required.



Solution

The component of the force in the direction of motion = 100cos60^0 = 50N

$$\therefore work\ done = force \times \text{distance}$$
$$= 50 \times 10 = 500\ J$$

$$Power = \frac{work\ done}{time\ taken} = \frac{500}{5} = 100W$$

The force does 100 J of work and its rate of working is 100 watts

Example 19

A man of mass 76kg runs up a flight of stairs to a vertical height of 10m in 4s. Find his rate of working.

Solution

Work done against gravity = mgd

$$= 76 \times 9.8 \times 10 = 7448 \text{ J}$$

Hence the power developed = $\dfrac{work\ done}{time\ taken} = \dfrac{7448}{4} = 1862\ W$

Example 20

Find the power generated by a car engine when the car travels at a constant speed of 72kmhr^{-1} on a horizontal ground against resistance forces of 2500N.

Solution

$$72kmhr^{-1} = \frac{72 \times 1000}{3600} = 20ms^{-1}$$

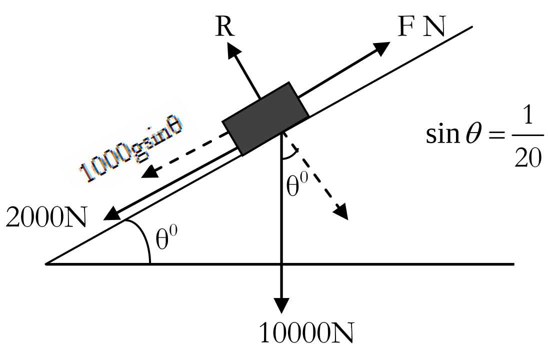
$$\begin{aligned} Power &= F \times v \\ &= 2500 \times 20 = 5000W \\ &= 50kW \end{aligned}$$

Example 21

A car of mass 1000kg travels up a hill inclined at angle θ to the horizontal, where $\sin\theta = \dfrac{1}{20}$.

The non gravitational resistance to motion is 2000N and the power output from the engine is 60kW. Find the acceleration of the car when it is travelling at 10 ms^{-1}. Take g = 10ms^{-2}.

Solution



Using $Power = F \times v$

$$60000 = F \times 10$$

$$F = 6000N$$

Let the acceleration of the car be a ms^{-2}

Using $F = ma$

$$6000 - 2000 - 1000g\sin\theta = 1000a$$

$$4000 - 10000 \times \frac{1}{20} = 1000a$$

$$3500 = 1000a$$

$$a = 3.5\ ms^{-2}$$

Exercise

1. Find the work done when a force of 100N moves a body horizontally through a distance of 200cm.
2. A block is pulled horizontally through 4m at a steady speed by a force of 20N, inclined at an angle of 60^0 to the line of motion. Find the work done.
3. A mass of 20kg is pulled across a rough horizontal floor (coefficient of friction 0.4) through 2m at a steady speed by a horizontal force. Find the work done.
4. How much work is done when a force of 2kN moves its point of application 2.25m in the direction of the force?

5. The work done in raising 100 bricks each of mass 1kg from the ground to the first floor of a building is 3000J. Find the height of the first floor from the ground.

6. Find the work done against gravity when an object of mass of 3.5 kg is raised through a vertical distance of 10m.

7. A certain metal block is pulled a distance of 8m across a horizontal surface against resistances totalling 15N. If the block moves at a steady velocity, find the work done against the resistances.

8. A horizontal force pulls a body of 5kg a distance of 15m across a rough horizontal floor, coefficient of friction 0.45. The body moves with constant velocity and the only resisting force is that due to friction. Find the work done against friction.

9. A rough surface is inclined at to the horizontal. A body of mass 5000g lies on the surface and is pulled at a uniform speed a distance of 0.75m up the surface by a force acting along a line of greatest slope. The coefficient of friction between the body and the surface is 0.4. Find;

   (a) the work done against gravity

   (b) the work done against friction

10. Find the kinetic energy of a body of mass 0.25 kg moving with a speed of $11\sqrt{3}$ ms^{-1}.

11. A bullet of mass 20g moving at 200 ms^{-1} has the same kinetic energy as a gun of mass 1kg. Find the speed at which the gun is moving.

12. A body of mass 5kg, initially moving with speed 2 ms-1, increases its kinetic energy by 30 J. Find the final speed of the body.

13. Find the potential energy gained by:

    (a) a body of mass 5kg raised through a vertical distance of 10m.

    (b) a man of mass 60kg ascending a vertical distance of 5m

14. Find the potential energy lost by:

    (a) a body of mass 20kg falling through a vertical distance of 2m.

    (b) a man of mass 90 kg descending a vertical distance of 5m.

15. A car of mass 1000kg increases its speed from 5 ms^{-1} to 6ms^{-1}. Find its gain in kinetic energy.

16. A truck of mass 500kg decreases its speed from 50 ms^{-1} to 10 ms^{-1} in an attempt to negotiate a corner. Find the loss in kinetic energy during this instance.

17. If a mass of 10kg at rest acquires a velocity of 2 ms^{-1} after being pulled through 1.5m what force is acting in the direction of motion?

18. The velocity of a body of mass 0.5 kg is reduced from 3 to 1.5 ms^{-1} in a distance of 1.5m. What force is acting on the body?

19. A ball of mass 250g is projected up a smooth incline of inclination $\sin^{-1}\left(\frac{1}{25}\right)$ with a velocity of 5 ms^{-1}. How far will it travel before coming to rest?

20. A particle of mass 1.5kg is projected up a plane inclined at 30^0 to the horizontal with an initial speed of 1 ms^{-1}. How far will it travel up the incline if (i) the surface is smooth, (ii) the coefficient of friction is 0.5.

21. A particle of mass 4kg is projected up a rough plane inclined at 45^0 to the horizontal with a speed of 4 ms^{-1}. The coefficient of friction is 0.4. How far will the particle have travelled when its speed is halved?

22. A point A is 5m vertically above the point B. A body of mass 0.25 kg is projected from A vertically downwards with speed 2 ms^{-1}. Find the speed of the body when it reaches B.

23. A body of mass 20 kg is released from rest and falls freely under gravity. Find the distance it has fallen when its speed is 21 ms^{-1}.

24. A body of mass 3kg is projected vertically downwards from a point A with speed 4 ms^{-1}. The body passes through a point B, 5m below A. Find the speed of the body at B.

25. A smooth slope is inclined at $\tan^{-1}\left(\frac{3}{4}\right)$ to the horizontal. A body of mass 4kg is released from rest at the top of the slope and reaches the bottom with speed 7 ms^{-1}. Find the length of the slope.

26. A ball is kicked from ground level with speed 15ms^{-1} and just clears a wall of height 2m. The speed of the ball as it passes over the wall is v ms^{-1}.

    (a) Assume that the ball has mass m kg and use the conservation of mechanical energy principal to write an equation.

    (b) Solve the equation to find v

27. Find the power developed when a load of 50kg is lifted vertically through 10m in 5 seconds.

28. A boy of mass 44kg runs up a flight of stairs of vertical height 4m in 5s. What power is sustaining?

29. If a car travels at a steady speed of 15ms^{-1} against resistances of 200N what power is being exerted by the engine?

30. A cyclist is travelling at a steady speed of 4ms^{-1}. The total mass of the cyclist and machine is 80kg. All resistances amount to 1.2N per kg of mass. At what rate is he working?

31. The power of the engine of a car is 7kW. What would be the maximum speed of the car on the level against resistance of 250N?

32. A car of mass 800kg travels up a hill inclined at angle θ to the horizontal, where $\sin\theta = \left(\frac{1}{20}\right)$.

    The non gravitational resistance to motion is 1600N and the power output from the engine is 45kW. Find the acceleration of the car when it is travelling at 15ms^{-1}. (Take g =10 ms^{-2})