UCE
Essential Mathematics

M. Forman and D. Nyakairu

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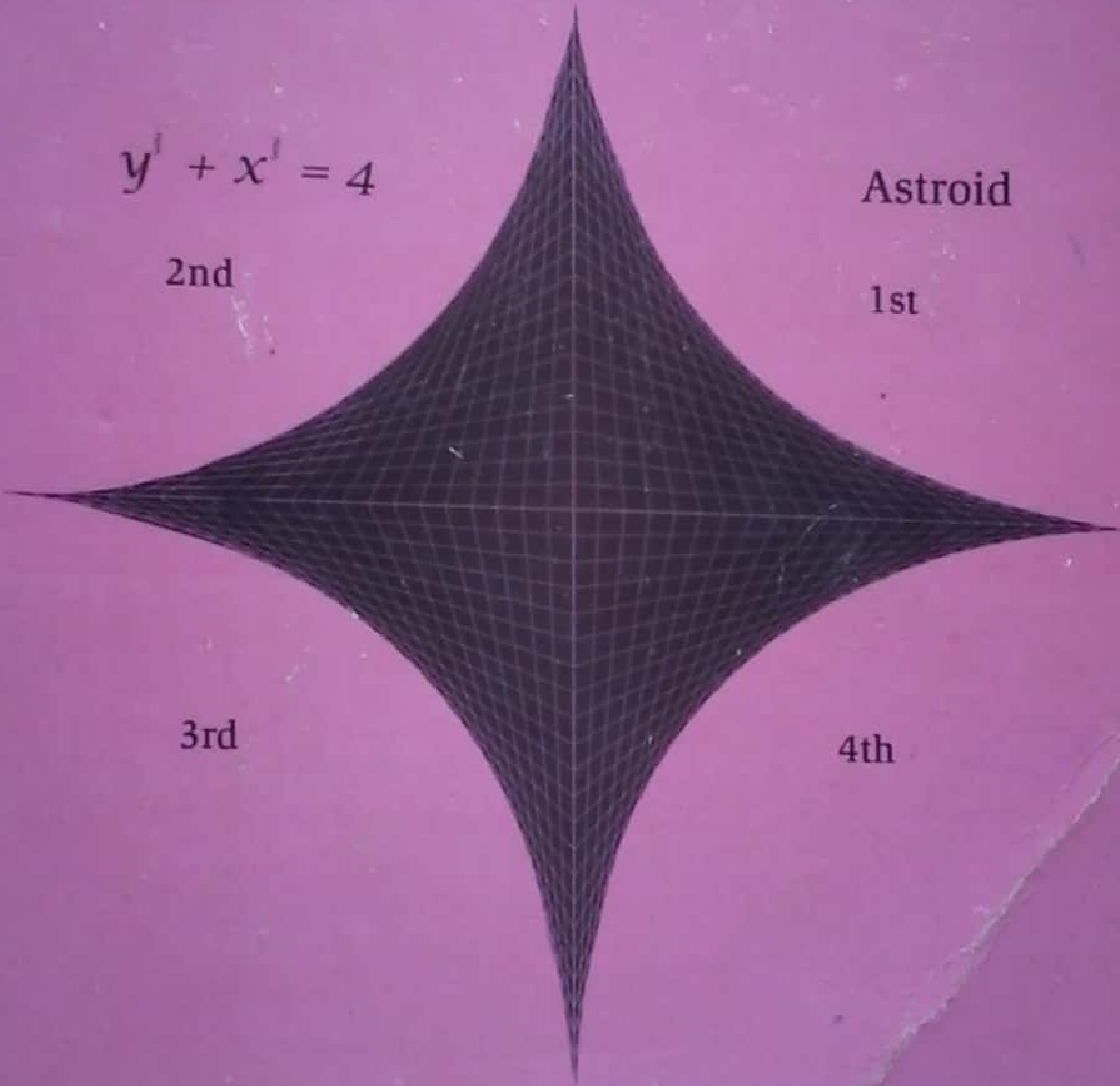

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UCE
Essential Mathematics

A Complete Course

Mont Forman
David Nyakairu

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Introduction

This text is for the use of pupils in Senior 1 to Senior 4 who will be sitting the Uganda Certificate of Education in Mathematics and for the teachers of these pupils. It may be used as a revision book in S.4 or as a supplement to course books throughout S.1 to S.4. In either case it will prove to be a valuable companion particularly as examinations approach.

Furthermore, *U.C.E. Essential Mathematics* will be found ideal for Grade III and Grade V teacher trainees.

The book is presented as a series of sixty Topics which together cover the entire U.C.E. Mathematics Syllabus as specified by the National Curriculum Development Centre. For further enrichment some Topics extend slightly beyond the syllabus. Specimen examination papers are included so that pupils may sharpen their technique before sitting the final examination. Answers are provided to enable the reader to check that correct methods have been used and accurate calculations performed.

Pupils starting this Book are assumed only to have knowledge of the concepts of the Primary Mathematics Syllabus, although even the more advanced topics in this are revised. All new terminology is highlighted in **bold** type and important instructions and ideas to be stressed are *italicised*. There are many cross references within the text and an index is provided to facilitate the finding of particular terms and topics.

Thus the reader of this Book can exploit its considerable potential by working through each Topic in turn to revise and consolidate or by selecting those topics which need strengthening.

Both teachers and pupils are strongly recommended to read the short section on 'How to use the Book' before getting down to work with it.

We feel confident that all those using this Book will improve their Mathematics and at the same time derive great enjoyment from it.

Mont Forman
David Nyakairu

Acknowledgement

The authors would like to express their appreciation to A C R Bull for the additional material he has provided.

How to use the Book

To the Teacher

When used as a supplementary text, select the Topics in the order given in the teaching syllabus for S.1 to S.4 as produced by the National Curriculum Development Centre. Each of the sixty Topics has been divided into a number of Sections, for example **1.1**, **1.2**, **2.1** and so on. Depending on the ability of the class and the difficulty of the work, the teacher may, in general, spend one or two periods on one section. Run through the material of each section and any worked examples with the class. If a section contains a cross reference, look this up when preparing the lesson. For example, Section **53.6** contains cross references to Section **41.1**. Thus, before beginning this section, it may be necessary for pupils to revise the work on mirror lines. The questions of each Exercise have been carefully graded and some questions depend upon others having been answered, for example **Ex 6c Q22**. Therefore, if not setting the class an entire Exercise, take care to select appropriate questions by working through the Exercise during lesson preparation.

When this Book is used for revision purposes, for example in S.4, select appropriate Topics in which you feel your class is weak or that consolidation is required. It is recommended that the whole of a Topic is revised and not just part of it. Lesson preparation should be carried out section by section as above. S.4 pupils should work through the Specimen Papers as preparation for the mock and final examinations.

When used as a classroom text, the teacher may prefer to remove the answers from pupils' copies as indicated.

To the Pupil

Pupils may use this Book to study on their own. Start at the beginning and follow through the sequence of Topics from **1** to **60** in order. The material becomes progressively harder as you go through the Book. Read carefully through each Section and through any worked Examples, then answer the questions of the Exercise which follows. In each Exercise, the questions become progressively more difficult to answer. It is therefore important that you attempt the questions in the order **1**, **2**, **3** and so on. After answering two or three questions, check your understanding of the work by seeing that you have the answers which are given at the back. If all is well, proceed with the rest of the Exercise. If you have difficulty, re-read the text and pay particular attention to any Examples which appear in the section. Follow up any cross references if you have not done so. For example, in Section **1.1** there is a cross reference to Section **5.1**. By referring to **5.1** you can confirm that your ideas about a set are correct. Rework the questions of the Exercise until you are correct and confident.

At the end of the course make sure you go through each of the five Specimen Examination Papers before sitting for U.C.E. You should time yourself to see that you answer the required number of questions within the allotted 2½ hours.

Pupils may use this Book to revise on their own. Use the Index to locate a term or topic you wish to revise. Having located the reference we advise you to read through the whole Topic which contains it. For example, if you wish to revise **union** you are referred to Section **5.7**. For a full understanding you should work through the whole of Topic **5** on Sets and not just Section **5.7**. Work through the Topic section by section as above. Repeat for other references. Before the final examination go through the Specimen Papers as advised above.

Note Three-figure tables are provided for your use and have been used throughout this Book. Answers are usually given correct to 3 significant figures where appropriate. However, this accuracy may not necessarily be attained with 3- or even 4-figure tables.

1 NATURAL NUMBERS

1.1 Basic Ideas

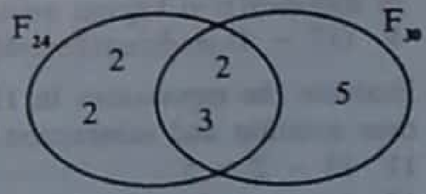
The set (see **5.1**) of **natural** or **counting** numbers is {1, 2, 3, 4, 5, ...}. Since another natural number can always be obtained by adding 1 to an existing one there are infinitely many natural numbers. The set of **whole** numbers is {0, 1, 2, 3, 4, 5, ...}. Note that the number 0 (zero) is a whole number but is not a natural or counting number. Whole numbers are represented as points on a **number line** as shown.

The set of whole numbers may be divided into two disjoint subsets: {all **odd numbers**} and {all **even** numbers}. These are {1, 3, 5, 7, 9, 11, ...} and {0, 2, 4, 6, 8, 10, ...}. All odd numbers end in 1, 3, 5, 7 or 9 and all even numbers end in 0, 2, 4, 6 or 8.
The number 24 is divisible by 1, 2, 3, 4, 6, 8, 12 and 24. These are the **factors** of 24. The factors of 30 are 1, 2, 3, 5, 6, 10, 15 and 30. The **common factors** of 24 and 30 are 1, 2, 3 and 6. The **Highest Common Factor** (HCF) of 24 and 30 is 6. (See also **Exercise 5c Q7**)
The Highest Common Factor is sometimes called the **Greatest Common Divisor** (GCD).
Multiples of 4 are 4, 8, 12, 16, 20, 24, 28, 32, 36, ... and of 6 are 6, 12, 18, 24, 30, 36, 42, 48, ...
The **common multiples** of 4 and 6 are 12, 24, 36, 48, ... The **Lowest Common Multiple** (LCM) of 4 and 6 is 12. (See also **Exercise 5c Q8**)

Tests of Divisibility:
A number is divisible by **2** if the last **digit** is even.
A number is divisible by **3** if the sum of the digits is divisible by 3.
A number is divisible by **4** if the number formed by the last two digits is divisble by 4.
A number is divisible by **5** if it ends in 5 or 0.
A number is divisible by **6** if it is divisible by 2 and by 3.
A number is divisible by **8** if the number formed by the last three digits is divisible by 8.
A number is divisible by **9** if the sum of the digits is divisible by 9.
A number is divisible by **10** if it ends in 0.

A **prime number** has just two different factors, 1 and itself. It has no other factors. Note that 1 is not a prime number (it has only one factor). A number which is not a prime may be written as a product of prime factors. This may be used to find the HCF and LCM of two or more numbers.
For example $24 = 2 \times 2 \times 2 \times 3$ and $30 = 2 \times 3 \times 5$
The figure shows the Venn diagram (see **5.2**) for the sets of prime factors of 24 and 30:

$$F_{24} = \{2, 2, 2, 3\} \text{ and } F_{30} = \{2, 3, 5\}$$



The intersection gives the HCF of 24 and 30 as $2 \times 3 = 6$.
The union gives their LCM as $2 \times 2 \times 2 \times 3 \times 5 = 120$.

Exercise 1a

1 Write down the number *five hundred and forty-six thousand two hundred and thirty-nine.*
2 Write down the number *twenty million two hundred and two thousand and twenty.*
3 Write in words only the number (i) 798,142, (ii) 100,101,001.
4 Write down the first fifteen members of the set of prime numbers.
5 List the members of {51, 53, 57, 59, 61, 63, 67, 69, 71, 73} which are prime.
6 List all the factors of (i) 16, (ii) 18, (iii) 20, (iv) 144, (v) 256.
7 Find the number divisible by 2, 3 and 5 among {306, 465, 500, 526, 1,365, 1,650, 1,780}.

8 Find the HCF of 32 and 56.
9 Find the LCM of 9 and 15.
10 (i) Find the HCF of 30 and 42. (ii) Find the LCM of 30 and 42.
(iii) Find the product of your answers for (i) and (ii), ie. HCF $\times$ LCM.
(iv) Find the product 30×42. (v) How are your answers to (iii) and (iv) related?
11 Repeat **Q10** for the numbers 72 and 126.
12 By use of prime factors, find the HCF of 210, 1,540 and 2,275.
13 Find the LCM of 210, 1,540 and 2,275.
14 A **perfect number** is one in which the sum of the factors, apart from the actual number, equals the number itself. Thus $28 = 1 + 2 + 4 + 7 + 14$, and so 28 is a perfect number. Show that 496 and 8,128 are members of the set of perfect numbers.

1.2 Operations

Of the four operations multiplication and division take preference over addition and subtraction.

So, for example $6 + 4 \times 3 = 6 + 12$ (and not 10×3) $= 18$
$12 \div 3 - 2 = 4 - 2$ (and not $12 \div 1$) $= 2$
$15 \times 4 + 8 \div 2 = 60 + 4$ (and not $15 \times 12 \div 2$) $= 64$

To avoid any doubt use brackets: $(6 + 4) \div 2 = 10 \div 2 = 5$
$(16 - 10) \times 3 = 6 \times 3 = 18$
$4 \times (5 + 3) \div 8 = 4 \times 8 \div 8 = 32 \div 8 = 4$

For a calculation without brackets such as $12 - 4 + 3 - 2$ which involves only addition and subtraction, perform the operations in order from left to right:

$$12 - 4 + 3 - 2 = 8 + 3 - 2 = 11 - 2 = 9$$

We do not advise the use of *BODMAS* because it is not universally true. It may also be applied without understanding which should always be avoided.

Exercise 1b

Evaluate the expressions in **1** to **10**.

1 $(5 + 4) \times 2$
2 $5 \times (4 + 2)$
3 $5 + (4 \times 2)$
4 $(24 - 6) \div 3$
5 $24 - (6 \div 3)$
6 $17 - (4 - 1)$
7 $(17 - 4) - 1$
8 $17 - (4 + 3) - (2 + 5)$
9 $(17 - 4) + 3 - (2 + 5)$
10 $17 - 4 + 3 - 2 + 5$

Evaluate the expressions in **11** to **20** in which multiplication and division take preference over addition and subtraction.

11 $13 - 2 \times 5$
12 $15 + 6 \div 2$
13 $6 \times 4 + 5 \times 3$
14 $27 \div 3 - 18 \div 2$
15 $16 - 4 \div 2$
16 $20 \div 4 - 4$
17 $16 \times 2 + 8 \div 2$
18 $16 \div 4 \times 2 + 7 - 2$
19 $39 \div 3 + 16 \times 2$
20 $32 \div 4 + 36 \times 2 - 16$

In **21** to **30** insert brackets so that the required answer is obtained.

21 $24 - 7 + 1$ to give 18
22 $24 - 7 + 1$ to give 16
23 $36 \times 2 + 7$ to give 79
24 $36 \times 2 + 7$ to give 324
25 $60 \div 4 \times 3$ to give 5
26 $60 \div 4 \times 3$ to give 45
27 $128 \div 8 + 8 \times 2$ to give 32
28 $128 \div 8 + 8 \times 2$ to give 16
29 $128 \div 8 + 8 \times 2$ to give 4
30 $128 \div 8 + 8 \times 2$ to give 48

2 INTEGERS

2.1 Basic Ideas

The subtraction 6 – 4 gives the natural number 2. However, 4 – 6 does not give a natural number. To provide an answer to such a subtraction **negative numbers** are used. Therefore $4 - 6 = -2$. These may be represented on the number line by extending it to the left of zero.

These numbers are called {**integers**} and are **ordered**. For example, $-3 < +2$ because -3 is to the left of $+2$ on the number line; $+4 > -3$ because $+4$ is to the right of -3.
Integers are sometimes called **directed numbers**. There are an infinite number of integers since another integer can always be obtained by adding 1 to or subtracting 1 from an existing integer.

Exercise 2a

1 Give the two integers 'next to' +1 on the number line.
2 Give the two integers 'next to' −5 on the number line.

Draw a number line from −6 to +6 in your Exercise Book. and use it to answer, inserting < or > as appropriate, in **3** to **14**.

3	+4 ... +6	**4**	−6 ... +5	**5**	+1 ... −6	**6**	−1 ... −3
7	−1 ... −2	**8**	+8 ... +6	**9**	+2 ... −2	**10**	−3 ... +3
11	0 ... +4	**12**	+4 ... 0	**13**	−4 ... 0	**14**	0 ... −4

2.2 Addition and Subtraction

The addition of two integers always gives another integer. For example

$$(+4) + (+3) = +7, \quad (-5) + (+2) = -3 \quad \text{and} \quad (-5) + (-2) = -7$$

Likewise the subtraction of one integer from another always gives an integer:

$$(+6) - (+2) = +4, \quad (+2) - (+6) = -4 \quad \text{and} \quad (-5) - (-2) = -3$$

Although it is usual to omit the positive sign in front of a positive integer it is included in the following Exercise, **1** to **10** to stress the rules concerning directed numbers.

Exercise 2b

Evaluate the following.

1 $(+3) + (+5)$
2 $(+5) - (+8)$
3 $(-5) + (-3)$
4 $(-5) - (-3)$
5 $(+5) - (-5)$
6 $(+3) - (+5)$
7 $(-5) - (-5)$
8 $(-1) + (-1) - (-2) + (-2)$
9 $(-3) - (-2) - (-1)$
10 $(+1) - (+1) - (+2) + (-2)$
11 $27 - (-33)$
12 $-49 - 42$
13 $(-82) - (-28)$
14 $32 - 39$
15 $-43 + 49$
16 $-62 - (-39)$
17 $-112 + 29 - (-81)$
18 $27 - (-39) + 32$
19 $17 - 42 - (-9)$
20 $39 - (-42) - 50$

2.3 Multiplication and Division

Multiplication of two integers always gives an integer. If the integers are both positive or both negative then their product is positive. If the signs are different their product is negative.

For example: $2 \times 3 = 6$ and $-2 \times -3 = 6$

$-2 \times 3 = -6$ and $2 \times -3 = -6$

A similar rule holds for division.

For example: $12 \div 3 = 4$ and $-12 \div -3 = 4$

$12 \div -3 = -4$ and $-12 \div 3 = -4$

Exercise 2c

In **1** to **10** evaluate the given expression.

1 -3×-4

2 -4×3

3 -3×4

4 $-12 \div 6$

5 $-15 \div -5$

6 $(-15 \times 2) + (-15 \div -3)$

7 $[(4 + (-3)) \times 2] - (-1)$

8 $(-5 \div 5) + (-5 \times 5)$

9 $[(4 \times -3) - (-2)] + 1$

10 $(-6 \times 2) - (105 \div -5)$

In **11** to **18** insert brackets so that the required number is obtained.

11 $4 \times -3 + (-2) \times -1$ to give -10

12 $4 \times -3 + (-2) \times -1$ to give 20

13 $4 \times -3 + (-2) \times -1$ to give 14

14 $4 \times -3 + (-2) \times -1$ to give -4

15 $-32 \div 8 + 8 \times -2$ to give -20

16 $-32 \div 8 + 8 \times -2$ to give 4

17 $-32 \div 8 + 8 \times -2$ to give -8

18 $-32 \div 8 + 8 \times -2$ to give 1

2.4 Applications

Negative numbers are necessary to describe certain measurements. For example, a temperature may be recorded as -5°C. The negative sign indicates that the temperature is 5°C *below* zero. Such a temperature may be recorded inside the freezing compartment of a refrigerator. (See **7.9**)

Exercise 2d

In **1** to **5** describe the required measurement using a negative number.

1 Musisi's bank balance shows an overdraft of sh200,000. How much money has he in the account?

2 Eilu reverses his car at a speed of 3m/s. What is his forward speed?

3 Achoroi throws a stone vertically upwards. Due to Earth's gravity it has a downward acceleration of 10 m/s^2. What is its upward acceleration?

4 The soldiers had to retreat 100m. How far did they advance?

5 A rocket is to be launched at 12 noon. How many minutes *after* launch time is it at 11.50am on that day?

6 Nansamba has sh6,700 in her bank account. She withdraws sh9,000. What is her balance?

7 The temperature inside a refrigerator is 8°C If this falls by 25°C, what will be the new temperature?

8 Akol has an overdraft of sh15,500 in his bank account. If he pays in sh23,000, what will be the balance?

9 The temperature inside a deep-freeze rises by 13°C. If the temperature is now -19°C, what was it before the rise.

10 A bird flies at a height of 135m above the Dead Sea, the surface of which is -395m above mean sea level. What is the height of the bird above mean sea level?

3 FRACTIONS

3.1 Basic Ideas

A **proper fraction** has the **numerator** (top) less than the **denominator** (bottom), for example $\frac{3}{4}$, where $3 < 4$. An **improper fraction** has the numerator greater than the denominator, for example $\frac{4}{3}$.
A **mixed number** is the sum of an integer and a proper fraction, for example $5\frac{1}{2}$ which is equivalent to $5 + \frac{1}{2}$. A mixed number can be changed to an improper fraction as follows.

$$5\tfrac{1}{2} = 5 + \tfrac{1}{2} = \tfrac{10}{2} + \tfrac{1}{2} = \tfrac{10+1}{2} = \tfrac{11}{2}$$

An improper fraction can be changed into a mixed number as follows.

$$\tfrac{17}{4} = \tfrac{16+1}{4} = \tfrac{16}{4} + \tfrac{1}{4} = 4 + \tfrac{1}{4} = 4\tfrac{1}{4}$$

To obtain an **equivalent fraction** to, say, $\frac{2}{3}$, multiply the numerator and denominator by the same number, for example,

$$\tfrac{2}{3} = \tfrac{2\times2}{3\times2} = \tfrac{4}{6} \quad \text{or} \quad \tfrac{2}{3} = \tfrac{2\times3}{3\times3} = \tfrac{6}{9}$$

A fraction in which the numerator and denominator have no common factor is said to be in its **lowest terms**.

Example Express $\frac{18}{24}$ in lowest terms.

The HCF of 18 and 24 is 6. We therefore cancel numerator and denominator by 6:

$$\tfrac{18}{24} = \tfrac{3\times\cancel{6}^{1}}{4\times\cancel{6}_{1}} = \tfrac{3}{4}$$

Exercise 3a

1 Copy and fill in the missing numbers: $\frac{1}{2} = \frac{\;}{10} = \frac{6}{\;} = \frac{\;}{50} = \frac{50}{\;}$

2 Change into improper fractions: $1\frac{3}{4}$, $4\frac{1}{5}$, $5\frac{5}{6}$, $7\frac{1}{7}$

3 Express as mixed numbers: $\frac{7}{4}$, $\frac{21}{5}$, $\frac{37}{7}$, $\frac{41}{9}$

4 By expressing each of the following with the same denominator arrange the fractions in ascending order: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{2}{5}$, $\frac{3}{7}$

5 Which of the following are equivalent: $\frac{2}{3}$, $\frac{4}{7}$, $\frac{4}{8}$, $\frac{5}{10}$, $\frac{6}{9}$, $\frac{8}{12}$, $\frac{9}{15}$?

6 Reduce to lowest terms: $\frac{8}{12}$, $\frac{21}{24}$, $\frac{36}{40}$, $\frac{128}{144}$, $\frac{256}{272}$

7 Express as twenty-fourths: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{3}{8}$, $\frac{7}{12}$

8 Express the first quantity as a fraction of the second and reduce the fraction to lowest terms.

(i) sh200 : sh2,000	(ii) 60cm : 1m	(iii) 100m*l* : 1 litre
(iv) 25 min : 1 hour	(v) 30° : 1 right-angle	(vi) $1\frac{1}{2}$ hours : 1 day

3.2 Addition and Subtraction

To add two (or more) fractions of equal denominator simply add the numerators.

For example: $\frac{5}{7} + \frac{1}{7} = \frac{5+1}{7} = \frac{6}{7}$

If the denominators are unequal find equivalent fractions to make the denominators equal.

For example: $\frac{3}{4} + \frac{2}{3} = \frac{9}{12} + \frac{8}{12} = \frac{9+8}{12} = \frac{17}{12} = 1\frac{5}{12}$

Mixed numbers are added as follows:

Either $2\frac{1}{3} + 1\frac{1}{2} = 3\left(\frac{1}{3} + \frac{1}{2}\right) = 3\left(\frac{2}{6} + \frac{3}{6}\right) = 3\left(\frac{2+3}{6}\right) = 3\frac{5}{6}$

or $2\frac{1}{3} + 1\frac{1}{2} = \frac{7}{3} + \frac{3}{2} = \frac{14}{6} + \frac{9}{6} = \frac{14+9}{6} = \frac{23}{6} = 3\frac{5}{6}$

Subtraction is dealt with in the same way.

1. Denominators equal: $\frac{5}{7} - \frac{1}{7} = \frac{5-1}{7} = \frac{4}{7}$
2. Denominators unequal: $\frac{3}{4} - \frac{2}{3} = \frac{9}{12} - \frac{8}{12} = \frac{9-8}{12} = \frac{1}{12}$
3. Mixed: $2\frac{1}{3} - 1\frac{1}{2} = \frac{7}{3} - \frac{3}{2} = \frac{14}{6} - \frac{9}{6} = \frac{14-9}{6} = \frac{5}{6}$

   or: $2\frac{1}{3} - 1\frac{1}{2} = 1\left(\frac{1}{3} - \frac{1}{2}\right) = 1\left(\frac{2}{6} - \frac{3}{6}\right) = 1\left(\frac{2-3}{6}\right) = \frac{6+2-3}{6} = \frac{5}{6}$

Exercise 3b

Give answers in lowest terms.

1 $\frac{3}{7} + \frac{2}{7}$ 2 $\frac{1}{6} + \frac{1}{3}$ 3 $\frac{4}{9} - \frac{2}{7}$ 4 $5\frac{1}{2} + 2\frac{3}{4}$ 5 $6\frac{1}{3} - 4\frac{1}{2}$ 6 $\frac{1}{4} - \frac{1}{5} + \frac{1}{3}$

7 $1\frac{1}{4} + 2\frac{1}{2} - 1\frac{3}{4}$ 8 $\frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$ 9 $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6}$ 10 $2\frac{1}{3} + 3\frac{1}{4} - 4\frac{3}{7}$

3.3 Multiplication

When multiplying fractions multiply the numerators together and multiply the denominators together. Change mixed numbers into improper fractions before proceeding and write any integers as improper fractions with 1 as denominator.

Examples $\frac{2}{3} \times \frac{3}{5} = \frac{2 \times \cancel{3}^{1}}{{}_{1}\cancel{3} \times 5} = \frac{2}{5}$ (note the cancellation of the '3's *before* multiplying)

$2\frac{1}{2} \times 1\frac{1}{3} = \frac{5}{{}_{1}\cancel{2}} \times \frac{\cancel{4}^{2}}{3} = \frac{10}{3} = 3\frac{1}{3}$

$2\frac{1}{2} \times 3 = \frac{5}{2} \times \frac{3}{1} = \frac{15}{2} = 7\frac{1}{2}$

3.4 Division

Since $\frac{3}{4}$ when divided into three equal parts is $\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$ it follows that $\frac{3}{4} \div 3 = \frac{1}{4}$.
Now $\frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$. Therefore $\frac{3}{4} \div 3 = \frac{3}{4} \times \frac{1}{3}$ This leads to the rule 'invert and multiply' when dividing one fraction by another.

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Exercise 3c

Give answers in lowest terms.

1 $\frac{1}{3} \times \frac{2}{7}$ 2 $\frac{1}{2} \times \frac{3}{5}$ 3 $2\frac{1}{4} \times \frac{2}{3}$ 4 $1\frac{2}{3} \times 1\frac{2}{5}$ 5 $1\frac{5}{12} \times 3$ 6 $\frac{15}{24} + \frac{3}{4}$

7 $2 \div \frac{3}{4}$ 8 $\frac{7}{12} + 3\frac{1}{4}$ 9 $2\frac{1}{2} \times 3\frac{2}{3} + 2\frac{1}{3}$ 10 $6\frac{1}{4} \times 1\frac{1}{15} + 5$

3.5 Percentages

All percentages represent fractions with a denominator of 100. Per cent means 'out of a 100'.

$$\text{Thus } 50\% = \frac{50}{100} = \frac{1}{2}$$

To find the percentage of a given quantity use fractions. For example, 75% of 1 day is

$$\frac{75}{100} \times 24\text{ h} = \frac{3}{4} \times 24 = 18\text{ h.}$$

Exercise 3d

In **1** to **10** express as fractions in lowest terms.

1 25%	2 40%	3 45%	4 15%	5 85%
6 24%	7 36%	8 87%	9 95%	10 99%

In **11** to **20** express as percentages.

11 $\frac{1}{4}$ 12 $\frac{2}{5}$ 13 $\frac{1}{8}$ 14 $\frac{3}{8}$ 15 $\frac{9}{20}$ 16 $\frac{17}{20}$ 17 $\frac{18}{25}$ 18 $\frac{9}{10}$ 19 $\frac{11}{40}$ 20 $\frac{12}{55}$

Find the value of

21 5% of sh100	22 4% of 50g	23 10% of 1 hour
24 15% of 1kg	25 40% of 1l	26 30% of 1 min
27 75% of 1 tonne	28 $2\frac{1}{2}$% of sh8,000	29 $17\frac{1}{2}$% of 80 cm^3

3.6 Farey Sequences

The set of proper fractions with denominators 3 or less is $\{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$ where the fractions are arranged in ascending order. We will denote this set by F_3. Also $F_4 = \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\}$. To obtain F_5 we need to insert in F_4 the fractions $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}$ and $\frac{4}{5}$ in the correct order.

$$\text{Since } \frac{1}{5} < \frac{1}{4} \text{ and } \frac{1}{3} < \frac{2}{5} < \frac{1}{2} \text{ we get } F_5 = \{\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}\}$$

Such sets of fractions, arranged in ascending order as above, are called **Farey seqences or fractions.** Other properties are:

If $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$ are successive terms of a Farey sequence then

$$(1)\ bc - ad = 1 \qquad \text{and} \qquad (2)\ \frac{c}{d} = \frac{a+e}{b+f}$$

Show that (1) and (2) hold for F_5.

Exercise 3e

1 What fractions must be added to F_5 to give F_6? List F_6.
2 What fractions must be added to F_6 to give F_7? List F_7.
3 Give F_8.
4 Three consecutive members of a Farey sequence are $\frac{1}{6}, \frac{2}{11}, \frac{1}{5}$ ...
Find the next number in the sequence.

4 DECIMALS

4.1 Basic Ideas

The decimal notation gives a way of writing a fraction. For example, the **decimal** (or decimal fraction) 0·875 means $\frac{875}{1000}$ or $\frac{8}{10} + \frac{7}{100} + \frac{5}{1000}$ ie. eight tenths, seven hundredths and five thousandths. Note that the **decimal point** in, for example, 79·4 separates the whole number part (seventy nine) from the fractional part (four tenths).

Changing decimals to fractions

$0{\cdot}36 = \frac{36}{100} = \frac{9}{25} \qquad 0{\cdot}036 = \frac{36}{1000} = \frac{9}{250} \qquad 3{\cdot}648 = 3 + \frac{648}{1000} = 3 + \frac{81}{125} = 3\frac{81}{125}$

Changing fractions to decimals (see also **4.6**)

$\frac{12}{25} = \frac{12\times4}{25\times4} = \frac{48}{100} = 0{\cdot}48 \qquad \frac{88}{125} = \frac{88\times8}{125\times8} = \frac{704}{1000} = 0{\cdot}704$

4.2 Addition and Subtraction

To add or subtract decimals ensure that the decimal points are aligned vertically.

Exercise 4a

1 Write as decimals: $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{100}$, $\frac{3}{1000}$, $\frac{2}{10000}$

2 Write as decimals: $\frac{1}{8}$, $\frac{7}{50}$, $\frac{3}{16}$, $\frac{1}{20}$, $\frac{3}{24}$, $\frac{1}{32}$, $\frac{1}{64}$, $\frac{3}{40}$

3 Write as fractions in lowest terms: 0·6, 0·05, 0·041, 0·26, 0·35

4 Express as fractions in their lowest terms: 0·12, 0·045, 0·008, 0·066, 0·014

In **5** to **10** evaluate:

5 5·6 + 4·4 + 3·9

6 5·6 + 4·4 − 3·9

7 5·06 − 4·4

8 5 + 0·5 + 0·05 + 0·005

9 5 − 0·5 + 0·05 − 0·005

10 127·5 − 31·002 + 42·5 − 9·998

4.3 Multiplication and Division

If 32·25 is multiplied by 10 it becomes 322·5. Multiplied by 100 it becomes 3,225·0. This leads to the rule 'when multiplying by 10^n, where n is a positive integer, move the decimal point n places to the right'.

To multiply 0·04 × 0·3 we have $0{\cdot}04 \times 0{\cdot}3 = \frac{4}{100} \times \frac{3}{10} = \frac{12}{1000} = 0{\cdot}012$

The 2 decimal places in the first number and the 1 decimal place in the second give 3 decimal places in the answer. This leads to a general rule for multiplying two decimals:
(i) Multiply the two numbers ignoring decimal points.
(ii) Count the number of decimal places in each of the numbers given and put the decimal point that number of places from the right.

Example 1 Evaluate $0{\cdot}6 \times 0{\cdot}009$

$6 \times 9 = 54$ The total number of decimal places is $1 + 3 = 4$

The answer is $0{\cdot}0054$

When a decimal is divided by 10 the point moves one place to the left, for example, $32{\cdot}3 \div 10 = 3{\cdot}23$. Generally when a decimal is divided by 10^n the point moves n places to the left.

To divide one decimal by another make the divisor a whole number by moving the decimal point to the right and adjust the other number accordingly. Then perform the division.

Example 2 Evaluate $3{\cdot}01 \div 0{\cdot}7$

The second number (divisor) is made a whole number by multiplying by 10, or moving the decimal point 1 place to the right. The first number (dividend) is treated likewise.

Hence $\frac{3{\cdot}01}{0{\cdot}7} = \frac{30{\cdot}1}{7} = 4{\cdot}3$

Exercise 4b

Evaluate the following

1	$0{\cdot}045 \times 100$	**2**	$3{\cdot}2515 \times 1{,}000$	**3**	$3{\cdot}25 \times 0{\cdot}045$
4	$3{\cdot}05 \times 4{\cdot}52$	**5**	$0{\cdot}87 \div 100$	**6**	$0{\cdot}88 \div 0{\cdot}4$
7	$0{\cdot}044 \div 0{\cdot}88$	**8**	$\frac{0{\cdot}3 \times 1{\cdot}15}{0{\cdot}5}$	**9**	$\frac{1{\cdot}31 + 0{\cdot}51}{1{\cdot}31 - 0{\cdot}51}$

10 Multiply 16 by 21 and then write down the answer to (i) $1{\cdot}6 \times 21$, (ii) $1{\cdot}6 \times 2{\cdot}1$, (iii) $0{\cdot}16 \times 0{\cdot}21$.

4.4 Decimal Places

A decimal is often given correct to a specified number of decimal places (dp). For example, 2·376354 correct to 2 dp is 2·38. Because the third decimal place, 6, is more than 5 the value is closer to 2·38 than 2·37. Correct to 3 dp gives 2·376 and to 4 dp 2·3764. To determine whether to round up or down examine the next place and if it is 5 or more increase the previous number by 1. If the next place is less than 5 no alteration is needed to the previous place.

4.5 Significant Figures

If the population of a town is given as 57,379 it is unlikely to remain at that value. The stated number is too accurate (see Topic **43**) and a more sensible figure would be 57,000. We say there are two **significant figures** (sf), the 5 and the 7. The noughts are place holders to give the 'thousands'. If the population was given to the nearest hundred we would have 57,400 as 57,379 is nearer to this than 57,300. The population given correct to 3 sf is 57,400.

Decimals can also be expressed correct to a given number of significant figures.

For example 0·06035 is 0·0604 (3 sf), 0·060 (2 sf) and 0·06 (1 sf).

In the original number, the 6 is the first significant figure, as it is the first non-zero digit. The first two zeros are only there to give place value and are not significant. The zero following the 6 is the second significant figure. The 3 and 5 are the third and fourth significant figures respectively. The rule for rounding up or down is the same as that for decimals.

Exercise 4c

1 Give the following correct to 1 dp: 17·74, 17·75, 159·46, 2·12, 0·0695
2 Give the following correct to 2 dp: 17·745, 17·752, 159·466, 0·0641, 2·299
3 Give the following correct to 3 dp: 0·72598, 1·8089, 0·002141, 0·004715
4 Give the numbers of **1** correct to 2 sf.
5 Give the numbers of **2** correct to 3 sf.
6 Give the numbers of **3** correct to 4 sf.
7 State the number of significant figures in each of the following: 0·051, 6·02, 8,900, 7,070, 0·0430
8 Express 0·0092598 correct to (i) 2 dp, (ii) 4 dp, (iii) 2 sf, (iv) 4 sf.
9 Between what limits must the following numbers lie? (i) 6,700 to the nearest hundred, (ii) 17 to the nearest unit, (iii) 0·8 to 1 dp, (iv) 634,000 to the nearest thousand, (v) 0·027 to 2 sf
10 A square is of side 10cm, correct to the nearest centimetre. What is the least possible length of the square? What is the greatest possible length? Within which limits does the perimeter lie?

4.6 Recurring Decimals

A fraction can be converted to a decimal by dividing the numerator by the denominator. For some fractions the division never ends. It is then sensible to give the answer correct to a certain number of decimal places. For $\frac{17}{24}$ the decimal equivalent is 0·708333... and the 3 is repeated. This is called a **recurring decimal**. The repeated pattern is indicated by placing a dot over the recurring digit or over the first and last digits in a group of digits that recur.

Thus $\frac{17}{24} = 0{\cdot}708\dot{3}$, $\frac{2}{7} = 0{\cdot}\dot{2}8571\dot{4}$ ($\frac{2}{7}$ correct to 3 dp is 0·286), $\frac{13}{44} = 0{\cdot}29\dot{5}\dot{4}$

and $\frac{7}{27} = 0{\cdot}\dot{2}5\dot{9}$

To convert a recurring decimal into a fraction note the method for 0·6363636...

$$\text{let } x = 0{\cdot}6363\ldots \quad \ldots\ldots(1)$$

$$\text{so } 100x = 63{\cdot}6363\ldots \quad \ldots\ldots(2)$$

Subtracting (1) from (2) gives $100x - x = 63{\cdot}6363\ldots - 0{\cdot}6363\ldots$

$$\therefore \quad 99x = 63$$

$$\therefore \quad x = \frac{63}{99} = \frac{7\times9}{11\times9} = \frac{7}{11}$$

Hence $0{\cdot}6363636\ldots = \frac{7}{11}$

Exercise 4d

1 Express as decimals correct to 3 dp: $\frac{5}{7}$, $\frac{7}{9}$, $\frac{4}{11}$, $\frac{3}{19}$, $\frac{11}{17}$

2 Express as recurring decimals: $\frac{1}{3}$, $\frac{3}{7}$, $\frac{7}{9}$, $\frac{4}{11}$, $\frac{6}{13}$

3 Show that $\frac{1}{17}$ is $0{\cdot}0\dot{5}88235294117647\dot{0}$

4 Express the following as fractions: $0{\cdot}\dot{1}\dot{5}$, $0{\cdot}\dot{1}4\dot{8}$, $0{\cdot}\dot{1}5384\dot{6}$

Mº 1168

5 SETS (1)

5.1 What is a Set?

A set is a collection of *distinct* items. Sets can have special names. For example, a set of sheep is called a *flock*; a set of cows is a *herd*. The items contained in a set are its **members** (or **elements**).

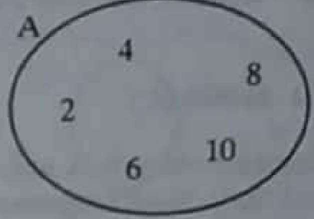
5.2 Set Notation

In mathematics, sets are represented using curly brackets, or by a **Venn diagram**. Often they are named using a capital letter of the alphabet such as A or B. Ways of representing the set, A, of numbers 2, 4, 6, 8 and 10 are as follows.

(a) A = {2, 4, 6, 8, 10}

(b) A = {all even numbers from 2 to 10 inclusive}

(c)



Notes:
(a) Members of the set are listed within curly brackets. We read this as *the set of 2, 4, 6, 8, 10*. The order of members or elements is not important. For example, {10, 4, 2, 8, 6} is the *same* set. But {10, 4, 2, 10, 6, 8, 4} is not a set as 10 and 4 are included twice.
(b) A description (or definition) of the set is given within curly brackets. The description must be *precise*. For example, A = {some even numbers} is not clear and therefore unacceptable.
(c) This is a Venn diagram. Here the members are displayed within a circle, oval or other closed figure. The name of the set (in this case, A) is placed next to the figure as shown.

In the above examples, the number 4 is a **member** of the set A. We write this as:

$4 \in A$ or $4 \in \{2, 4, 6, 8, 10\}$ or $4 \in$ {all even numbers from 2 to 10 inclusive}

where $\in$ means *is a member of* or *is an element of*.
We can write $7 \notin \{2, 4, 6, 8, 10\}$. What do you think $\notin$ means?
Also, the set A has 5 members. This is written as $n(A) = 5$. The notation $n(A)$ means *the number of members in set A* or *the number of elements in A*.

5.3 The Empty Set (∅)

This is a set which has no members. It is denoted by ∅ (or { }). It is also called a **null set**.
For example, B = {all natural numbers less than 9 which are divisible by 9}
is an empty set since none of the numbers 1, 2, 3, 4, 5, 6, 7, 8 is divisible by 9. So B = ∅. Another example of an empty set is

{boys in S1 at Tororo Girls High School}

Exercise 5a

Questions 1 to 24 of this Exercise refer to the following sets.

A = {all odd numbers between 0 and 12} (See 1.1)
B = {all natural numbers less than 16 which are divisible by 3} (See 1.1)
C = {all integers between −3·5 and +3·5} (See 2.1)
D = {all factors of 36} (See 1.1)
E = {the first five multiples of 7} (See 1.1)
F = {all countries of East Africa}
G = {all whole numbers less than zero} (See 1.1)
H = {all proper fractions between 0 and 1} (See 3.1)

In 1 to 8 list the members of the given set (a) using curly brackets, (b) in a Venn diagram.

1 A 2 B 3 C 4 D 5 E 6 F 7 G 8 H

In 9 to 16 write down whether the given statement is true or false.

9 $1 \in A$	10 $4 \notin B$	11 $-4 \in C$	12 $0 \in D$
13 $35 \in E$	14 Uganda $\notin$ F	15 $-2 \notin G$	16 $\frac{57}{129} \in H$

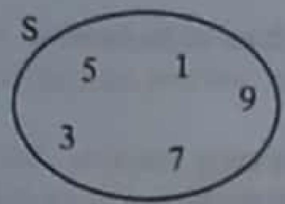
In 17 to 24 write down the number of elements.

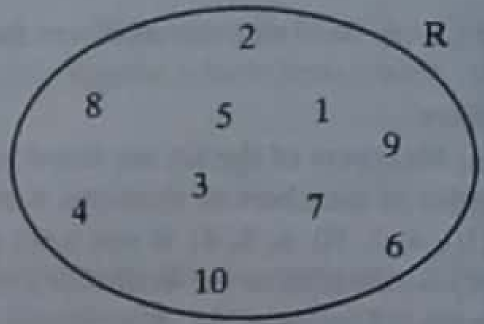
17 n(A)	18 n(B)	19 n(C)	20 n(D)
21 n(E)	22 n(F)	23 n(G)	24 n(H)

25 Write down three examples of an empty set.

5.4 Subset (⊂)

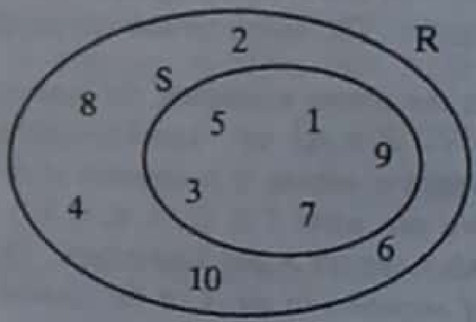
Consider the sets R and S below and their Venn diagrams (right).





R = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
S = {1, 3, 5, 7, 9}

Notice that *all* the members of set S belong to the set R. In such a case we say that S is a **subset** of R and write $S \subset R$. Also the two diagrams above are combined in one Venn diagram as shown on the right.. Note that the ring for S is completely within the ring for R.
Can we write $R \subset S$? No, because not all members of R belong to S. For example, 8 is a member of R but is not a member of S. So R is not a subset of S and we write $R \not\subset S$.



Exercise 5b

1 to 10 refer to the sets of **Exercise 5a** and the following sets.

I = {the first 15 natural numbers}
J = {all primes less than 12}
K = {all factors of 12}

Insert ⊂ or ⊄ as appropriate. In each case where the first set is a subset of the second, draw the corresponding (single) Venn diagram.

1 A ... I	2 B ... I	3 D ... I	4 J ... I	5 E .. A
6 F ... B	7 K ... D	8 G ... C	9 H ... A	10 G ... H

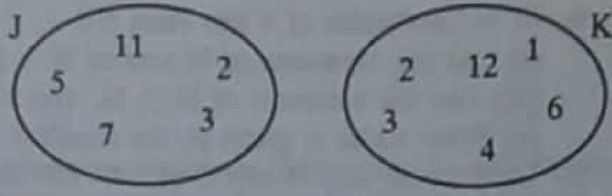
In 11 to 16 relate, if possible, each pair of sets using ⊂.

11 L = {three-sided figures} M = {right-angled triangles}
12 N = {four-legged animals on the school farm} P = {poultry on the school farm}
13 Q = {all S1 pupils at Teso College Aloet} T = {all pupils at Teso College Aloet}
14 U = {all cows in Kagoro's herd} V = {black cows in Kagoro's herd}
15 W = {pupils at Namasagali College who wear glasses}
X = {pupils at Namasagali College who wear size 8 shoes}
16 Y = {1, 8, 27, 64, ..., 512}
Z = {125, 343}

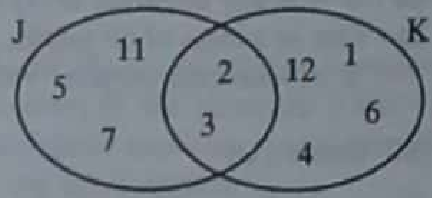
5.5 Intersection (∩)

Consider the sets J and K of **Ex 5b** and their Venn diagrams.

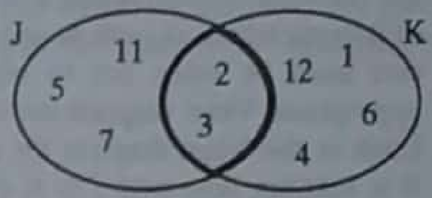
$J = \{2, 3, 5, 7, 11\}$
$K = \{1, 2, 3, 4, 6, 12\}$



Note that *some* members belong to *both* sets, namely 2 and 3. In such a case the two diagrams are combined into one Venn diagram as shown on the right. It shows the **intersection** of the two sets.



The intersection can be indicated more clearly as shown in the next diagram. This intersection is written as J ∩ K and read as *J intersection K*.



The intersection is itself a set and we write:

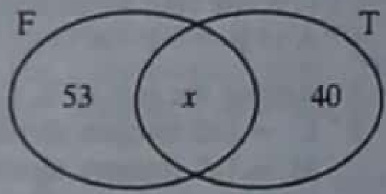
$J \cap K = \{2, 3\}$

The intersection of two sets is the set whose elements belong to both the sets.

Example A certain school has 120 pupils in S1. Each pupil takes part in at least one sport out of football and tennis. If 80 play football and 67 play tennis, how many play both of these sports?

Method 1

Draw the Venn diagram and enter the *numbers* of pupils in each region as you calculate them.



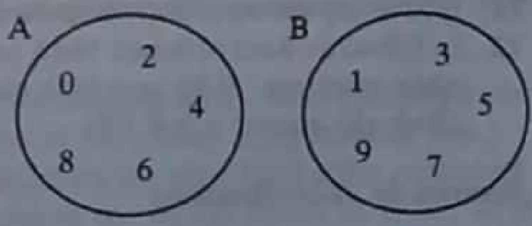
If 80 play football then 120 − 80 = 40 do not play football
If 67 play tennis then 120 − 67 = 53 do not play tennis
$x = 120 - (53 + 40) = 120 - 93 = 27$
Therefore 27 pupils play both sports.

Method 2

Number of pupils who play football + number who play tennis = 80 + 67 = 147.
But there are only 120 pupils in S1. This means that 147 − 120 = 27 must play both sports.

5.6 Disjoint Sets

Two sets are **disjoint** if none of the members of one set belong to the other, ie. they have no common element.
For example A = {all even numbers less than 10}
and B = {all odd numbers less than 10}
are disjoint sets and the Venn diagram is as shown on the right. Note that $A \cap B = \varnothing$.



Exercise 5c

1 to **6** refer to the sets of **Exercise 5a**. List the members of the two given sets. Draw a Venn diagram to illustrate the intersection and list its members. Save your diagrams for use in **Exercise 5d**.

1 A ∩ B **2** A ∩ C **3** B ∩ D **4** C ∩ D **5** B ∩ E **6** C ∩ G

7 P = {all factors of 24} Q = {all factors of 30}
(i) List all the members of P and of Q. (ii) Draw a Venn diagram to illustrate P ∩ Q.
(iii) List the members of P ∩ Q. (iv) Describe P ∩ Q in relation to 24 and 30.
(v) What is the name given to the greatest member of this set? (See **1.1**)

8 M = {multiples of 4 less than 50} N = {multiples of 6 less than 50}
(i) List the elements of M and of N. (ii) Draw a Venn diagram to illustrate M ∩ N.
(iii) List the elements of M ∩ N. (iv) Describe M ∩ N in relation to the numbers 4 and 6.
(v) What name is given to the smallest element of this set?

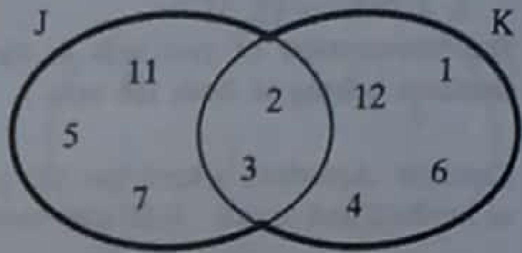
9 Look at the sets R and S of **5.4**. Do they have some common elements? What is R ∩ S?

10 If X is any set, what is (i) X ∩ X, (ii) X ∩ ∅?

11 Out of a class of 50 pupils, 43 passed Mathematics and 35 passed English. All pupils passed in at least one subject. How many pupils passed both subjects?

5.7 Union (∪)

We can *unite* two sets A and B by finding the set whose elements are either members of A, members of B or members of both A and B. This set is called *A union B*, and is written A ∪ B. Particular care must be taken not to repeat members which are common to both sets. Drawing the appropriate Venn diagram avoids this error.



Look at the Venn diagram for sets J and K of **5.5**.
It is redrawn here with J ∪ K clearly indicated.
From the diagram, J ∪ K = {1, 2, 3, 4, 5, 6, 7, 11, 12}.
For the disjoint sets A and B of **5.6**, we have

A ∪ B = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

Exercise 5d

In **1** to **6**, use your Venn diagrams for **1** to **6** of **Exercise 5c** to list the elements of the given union.

1 A ∪ B **2** A ∪ C **3** B ∪ D **4** C ∪ D **5** B ∪ E **6** C ∪ G

7 Among the sets A, B, C, D and E of **Exercise 5a** name *three* pairs of disjoint sets.

8 L = {all integers divisible by 3 between 19 and 41}
M = {all integers divisible by 4 between 19 and 41}
List the elements of L ∪ M

9 P = {all multiples of 2 less than 20} and Q = {all multiples of 3 less than 20].
Draw a Venn diagram which relates the sets P and Q. On your diagram, clearly mark the region representing natural numbers less than 20 which are divisible by 2 or by 3 or by both 2 and 3. How is this set named?

10 Look at sets R and S of **5.4**, and the corresponding Venn diagram. List the elements of R ∪ S.

11 If X is any set what is (i) X ∪ X, (ii) X ∪ ∅?

12 In Enwaku's herd of mixed black and white cattle, 45 have black markings on them, 35 have white markings. If 20 have both black and white markings, how many cattle are in the herd? (*Hint:* the answer is *not* 100)

Exercise 5e (miscellaneous)

In **1** to **8**, P and Q are any sets. Write down whether the given statement is true or false.

1 P ∩ ∅ = ∅ **2** P ∪ ∅ = P **3** Q ∩ Q = ∅ **4** Q ∪ Q = Q
5 ∅ ⊂ P **6** ∅ ∈ ∅ **7** {0} = ∅ **8** n(∅) = 0

9 If A ⊂ B, draw the Venn diagram and hence express (i) A ∩ B, (ii) A ∪ B as a single set.

10 If X ⊂ Y and Y ⊂ X, what can you say about the sets X and Y?

11 If P = {all prime numbers} and T = {all multiples of 2}, how are P and T related?

12 E = {all multiples of 8} and F = {all multiples of 4}. How are these sets related? Why?

13 If Q = {all quadrilaterals} and S = {all squares}, how are Q and S related? Draw the Venn diagram.

14 If n(A) = 31, n(B) = 28 and n(A ∩ B) = 16, find n(A ∪ B).

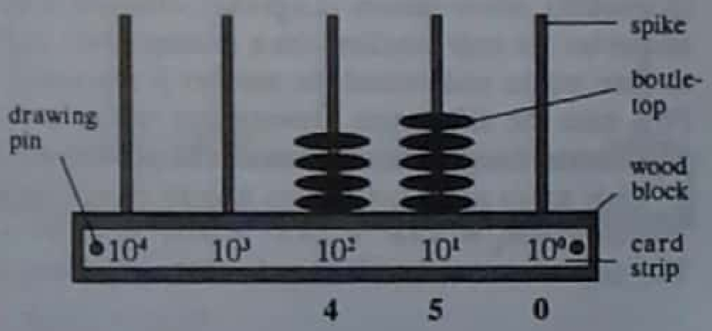
6 NUMBER BASES

6.1 Numbers and Numerals

A **number** (the full name is a **cardinal number**) gives *how many* objects there are in a set of objects. For example, {Charles, Juliet, Rita} has *three* members; your class has *forty-five* pupils. Here, *three* and *forty-five* are numbers.
A **numeral** is a symbol which represents a number. Different numerals can be used to represent the *same* number. Probably you would represent the above numbers by the numerals 3 and 45. However, the Romans used the numerals III and XLV to represent these numbers. In **6.3** you will see that, for example, the numerals 450 and 296 *can* represent the same number!
Even though there is a substantial difference between numbers and numerals, the latter are (even in this Book) usually loosely referred to as numbers, for example 'the number 3'. However, to master this Topic you need to understand the difference.

6.2 Base Ten - Denary

This is the system of numeration used in everyday life in Uganda and in many (but not all) countries of the world. It uses ten **digits**. These are 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. Each number is represented by a numeral which uses one or more of these digits, for example six (6), thirty-seven (37), four hundred and fifty (450), one eighth ($\frac{1}{8}$), twenty-nine hundredths (0·29).
Now 37 means 3 *tens* and 7 *units*, 450 means 4 *ten-tens (hundreds)* 5 *tens* and 0 *units*. You can see why the system is *based* on the number *ten* (**base ten**). It is also an example of a **place value system** because the value of each digit depends on its *place* in the numeral. For example, the 2's in 202 have different values. The first 2 represents 2 *hundreds* while the last 2 represents 2 *units*. The 0 (zero) in 202 has no value (0 *tens*) but is necessary as a **place holder**. Without it we would have 22 which represents a different number.
Numbers can be represented on an **abacus**. The *spike* abacus on the right shows the number four hundred and fifty.



[*Note:* The abacus is made by inserting lengths of stiff straight wire through small holes drilled into a block of wood. The wire must fit tightly into the holes. Long nails might also be used. Counters are placed onto the spikes. These can be bottle-tops with holes punched in them. A strip of thin card labelling spikes with appropriate place values (base ten, base eight, etc.) is attached to the front of the abacus using drawing pins.]
The abacus shown is a *base ten* abacus. A bottle-top placed on the right hand spike represents 1 unit. Two tops placed on this spike represent 2 units and so on. A top placed on spikes to the left of this would represent *ten*, a *hundred*, a *thousand* and *ten thousand* respectively. Because 9 is the highest digit, not more than 9 bottle-tops may be on any one spike. This abacus could be extended to the left to include more spikes for a *hundred thousand*, a *million* and so on; or to the right for the decimal values *tenths, hundredths,* etc.
The spikes are labelled using **index** notation:10^4, 10^3, 10^2, 10^1, 10^0 (see **14.1** and **14.3**). So the number represented above is $\mathbf{4}\times10^2 + \mathbf{5}\times10^1 + \mathbf{0}\times10^0 = \mathbf{4}\times100 + \mathbf{5}\times10 + \mathbf{0}\times1 = 400 + 50 + 0 = \mathbf{450}$. and is read as four hundred and fifty. Note that the digits **4**, **5** and **0** (marked in the diagram above) correspond to the number of bottle-tops on each spike.
Since other number bases are used in this Topic, this base ten numeral will be written as 450_{10} and read as *four five zero base ten*.

Exercise 6a

In 1 to 6, (i) write the number in words, (ii) represent it on an abacus, (iii) express it using index notation.

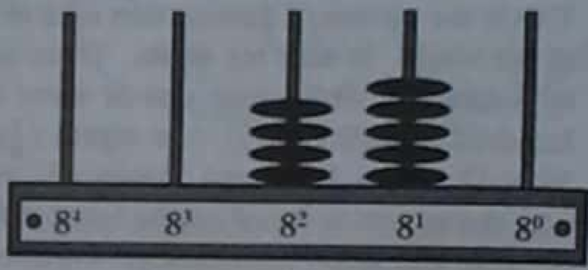
1 91_{10} 2 154_{10} 3 630_{10} 4 $1{,}027_{10}$ 5 $27{,}001_{10}$ 6 $101 \cdot 11_{10}$

7 Why do you think *ten* was chosen as number base? Why not *six* or *twelve*?
8 Name another numeration system which is in use today. List the digits used in this system.
9 Add $88{,}664{,}422_{10}$ to $12{,}345{,}679_{10}$. Write your answer in words.
10 Subtract $452{,}678{,}209_{10}$ from $533{,}478{,}217_{10}$. Write your answer in words.
11 Multiply $12{,}345{,}679_{10}$ by 9.
12 Divide $121{,}212_{10}$ by 6. Did you leave out the zero place holders?

6.3 Base Eight - Octal

The answer to Q7 of **Exercise 6a** is that a human being has *ten* digits, ie. fingers and thumbs, on his hands. Suppose we used only our *eight* fingers for counting (and not our thumbs) then probably we would have used **base eight**. We shall consider this number base in which only *eight* digits are used: 0, 1, 2, 3, 4, 5, 6 and 7.

A number is represented on the base eight abacus shown on the right.



The numeral for this number is written as 450_8 because the digits used must correspond to the number of bottle-tops on each spike. The small 8 tells us that base eight is being used. This is read as *four five zero base eight*.

What number does 450_8 represent?

Now the *language* we use for numbers, for example, *twenty-three* (English), *ishirini na tatu* (Kiswahili), *abbiri nusatu* (Luganda), *akaisarei kanuuni* (Ateso) ***refers to the base ten system***. This means we are only familiar with a number when expressed in base ten and we need to *convert* 450_8 to base ten to understand the number it represents.

First note the differences between this abacus and the one for base ten.

1. Powers of eight are used instead of powers of ten.
2. Only up to *seven* bottle-tops may be placed on any one spike.

Therefore $\mathbf{450_8} = 4\times8^2 + 5\times8^1 + 0\times8^0 = 4\times64 + 5\times8 + 0\times1 = 256 + 40 + 0 = 296_{10}$.

The number is two hundred and ninety-six. Note, of course, that $450_8 \neq 450_{10}$.

Example 1 Convert (i) 29_{10} (ii) 743_{10} to base 8.

(i) We find the number of 8's that go into 29_{10}, noting the remainder.

$29_{10} \div 8 = 3$ remainder 5 (3 r 5)

So $29_{10} = 3\times8 + 5$

$= 3\times8^1 + 5\times8^0 = 35_8$

$$\begin{array}{rl} 8)29 & \\ 8)3 & \text{r } 5 \\ 0 & \text{r } 3 \end{array}\uparrow$$

The division is usually set out as the working shown on the right. The required number is given by reading the remainders from bottom to top as the arrow indicates.

(ii) The above method is extended here.
Repeatedly divide 743_{10} by 8 until the quotient is 0, as shown.

This gives $743_{10} = 1347_8$.

$$\begin{array}{rl} 8)743 & \\ 8)92 & \text{r } 7 \\ 8)11 & \text{r } 4 \\ 8)1 & \text{r } 3 \\ 0 & \text{r } 1 \end{array}\uparrow$$

Example 2 Add 256_8 to 615_8 giving your answer in base eight.

The working (in base eight) is set out as shown on the right.

$$\begin{array}{r} 615 \\ +\,256 \\ \hline 1073 \end{array}$$

Notes $5 + 6 = 11_{10} = 13_8$ Write 3 and carry 1.

$1 + 5 + 1(\text{carried}) = 7_{10} = 7_8$ Write 7.

$6 + 2 = 8_{10} = 10_8$ Write 10. So $256_8 + 615_8 = 1073_8$

Exercise 6b

In **1** to **6** (i) show the number on a base eight abacus, (ii) find the number represented by converting to base ten.

1 10_8 **2** 27_8 **3** 77_8 **4** 100_8 **5** 777_8 **6** 1000_8

In **7** to **12** convert the base ten numbers to base eight.

7 10_{10} **8** 16_{10} **9** 37_{10} **10** 99_{10} **11** 955_{10} **12** 2000_{10}

In **13** to **15** perform the additions in base eight. Give your answer in base eight.

13 $12_8 + 14_8$ **14** $47_8 + 15_8$ **15** $763_8 + 125_8$

16 What is wrong with this numeral: 782_8?

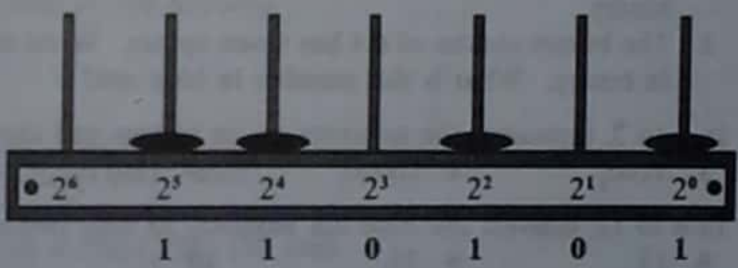
17 Try this subtraction in base eight: $1503_8 - 631_8$.

18 By first converting to base ten, work out the multiplication (i) $35_8 \times 10_8$ (ii) $472_8 \times 10_8$. Give your answers in base eight. Suggest a rule for multiplying a number in base eight by 10_8.

6.4 Base Two - Binary

In this base just two digits are used, 0 and 1. This system is of great importance in the world of computers. These represent numbers by means of many tiny electrical circuits each of which can be turned off (0) or on (1). Although a pocket calculator may display denary numbers, all the calculations are performed in the binary system.

A base two or binary abacus is shown on the right. It has seven spikes and can represent numbers up to more than a hundred. What number does the abacus represent? The numeral is 110101_2. Use index notation, shown on the abacus, to convert this to base ten.



$$\begin{aligned} 110101_2 &= 1\times2^5 + 1\times2^4 + 0\times2^3 + 1\times2^2 + 0\times2^1 + 1\times2^0 \\ &= 1\times32 + 1\times16 + 0\times8 + 1\times4 + 0\times2 + 1\times1 \\ &= 32 + 16 + 0 + 4 + 0 + 1 \\ &= 53_{10} \end{aligned}$$

So 110101_2 represents the number fifty-three.

Note: On the binary abacus, no more than one bottle-top may be placed on any one spike.

Example 1 Convert 39_{10} into binary.

We use the method of the Example of **6.3** and repeatedly divide 39_{10} by 2 as shown on the right. Reading the remainders upwards in the direction of the arrow as in the octal system gives

$$39_{10} = 100111_2$$

$$\begin{array}{rl} 2)\underline{39} & \\ 2)\underline{19} & \text{r } 1 \\ 2)\underline{9} & \text{r } 1 \\ 2)\underline{4} & \text{r } 1 \\ 2)\underline{2} & \text{r } 0 \\ 2)\underline{1} & \text{r } 0 \\ 0 & \text{r } 1 \end{array} \uparrow$$

Example 2 Add 11101_2 to 110101_2.

The working on the right gives $110101_2 + 11101_2 = 1010010_2$

$$\begin{array}{r} 110101 \\ +\ \ 11101 \\ \hline 1010010 \\ \hline \end{array}$$

We can compare the corresponding calculation in base ten.

$$\begin{aligned} 110101_2 &= 32 + 16 + 4 + 1 = 53_{10} \\ 11101_2 &= 16 + 8 + 4 + 1 = 29_{10} \\ 1010010_2 &= 64 + 16 + 2 = 82_{10} \quad \text{and} \quad 53_{10} + 29_{10} = 82_{10} \end{aligned}$$

6.5 Point Notation

In ordinary (base ten) arithmetic we use a decimal point as an extension of the place value idea in order to express fractions. For example $4{\cdot}1 = 4\frac{1}{10}$. We may use point notation in other bases. Look at the answer to part (iii) of **Exercise 6a Q6**.

$$101{\cdot}11_{10} = 1\times10^2 + 0\times10^1 + 1\times10^0 + 1\times10^{-1} + 1\times10^{-2} = 101\tfrac{11}{100}$$

Suppose this had been in binary, ie $101{\cdot}11_2$. What number would this represent?

$$\begin{aligned} 101{\cdot}11_2 &= 1\times2^2 + 0\times2^1 + 1\times2^0 + 1\times2^{-1} + 1\times2^{-2} \\ &= 1\times4 + 0\times2 + 1\times1 + 1\times\tfrac{1}{2} + 1\times\tfrac{1}{4} \\ &= 4 + 0 + 1 + \tfrac{1}{2} + \tfrac{1}{4} \\ &= 5\tfrac{3}{4} \end{aligned}$$

Here, the point cannot be called a decimal point. It is called a **bicimal** point! Any decimal number can be converted to a bicimal number and vice-versa.

Exercise 6c

1 Represent in turn the numbers from one to ten on a binary abacus. Write these numbers in binary.

2 The binary abacus of **6.4** has seven spikes. Write down the greatest number it can represent in binary. What is this number in base ten?

In **3** to **7**, represent the numbers on an abacus and then express them in base ten.

3 1100_2 **4** 10000_2 **5** 11011_2 **6** 1011010_2 **7** 10000001_2

In **8** to **12**, convert the base ten numbers to base two and show your answers on an abacus.

8 13_{10} **9** 25_{10} **10** 47_{10} **11** 104_{10} **12** 202_{10}

In **13** to **17**, perform the addition in binary. Convert the original numbers and your answer to base ten and compare the corresponding calculation in base ten.

13 $11_2 + 101_2$ **14** $1110_2 + 11011_2$ **15** $101010_2 + 10101_2$

16 $11111_2 + 11111_2$ **17** $10100101_2 + 1011011_2$

18 What number is represented by (i) the bicimal $1{\cdot}101_2$ (ii) the 'octimal' $1{\cdot}54_8$?

19 Convert $2{\cdot}25_{10}$ into (i) bicimal, (ii) octimal.

20 Convert 10_8 into base two by first converting it to base ten.

21 Convert 567_8 to base two. What is the relation between the digits of 567_8 and the digits of your answer? (*Hint:* Convert the digits 5, 6 and 7 to binary.)

22 Write down 4, 7 and 5 in binary. Hence convert 475_8 to binary using the relation discovered in **Q21**. Check that your answer is correct by converting it and 475_8 to base ten.

23 Repeat **Q22** for 423_8. (*Hint* Did you use 3-digit numerals for 2 and 3?)

24 Use the relation discovered in **Q21** to convert (i) 111110101_2 (ii) 1110011101_2 to base eight.

25 Perform the following operations in binary. Check your arithmetic by converting to denary.

(i) $11010011_2 - 1101001_2$ (ii) $10101_2 \times 1011_2$ (iii) $10001111_2 + 1101_2$

7 METRIC MEASURE

7.1 Length

The basic unit of length is the metre. Other units are derived from this:

1 metre (m) = 100 centimetres (cm)
1 centimetre = 10 millimetres (mm)
1 kilometre (km) = 1,000 metres

When converting from one unit to another in the metric system simply multiply or divide by a power of 10 (see **4.3**).

For example

$0{\cdot}0591\text{m} = 0{\cdot}0591 \times 1{,}000\text{mm} = 59{\cdot}1\text{mm}$
$985\text{cm} = 985 \div 100\text{m} = 9{\cdot}85\text{m}$
$23{,}400\text{m} = 23{,}400 \div 1{,}000\text{km} = 23{\cdot}4\text{km}$

Exercise 7a

1 Express in millimetres (i) 7cm, (ii) 2·5cm, (iii) 0·25cm.
2 Express in centimetres (i) 7mm, (ii) 7m, (iii) 0·07m.
3 Express in metres (i) 150cm, (ii) 2,500mm, (iii) 0·059km.
4 Express in kilometres (i) 2,500m, (ii) 259m, (iii) 250,000cm.

7.2 Area

The **area** of a square of side 1cm is 1 square centimetre (cm^2) or 10×10 mm^2. Hence $1\text{ cm}^2 = 100\text{mm}^2$. Now $1\text{m} = 100\text{cm}$ so $1\text{m}^2 = 100 \times 100\text{cm}^2 = 10{,}000\text{cm}^2$. For areas of land these units are too small and we use ares, hectares and square kilometres.
1 are = 100m^2, 1 hectare = $10{,}000\text{m}^2$, 1km^2 = 100 hectares

Exercise 7b

1 Express in mm^2 (i) 10cm^2, (ii) $2{\cdot}6\text{cm}^2$, (iii) $0{\cdot}2\text{m}^2$.
2 Express in cm^2 (i) 900mm^2, (ii) $8{,}000\text{mm}^2$, (iii) 79mm^2.
3 Express in cm^2 (i) 5m^2, (ii) $9{\cdot}2\text{m}^2$, (iii) 2 ares.
4 Express in hectares (i) $900{,}000\text{m}^2$, (ii) $4{\cdot}2\text{km}^2$, (iii) 150 ares.
5 Kampala City has a land area of 198km^2. What is this area in hectares?

7.3 Volume and Capacity

The basic unit of **volume** is the cubic metre (m^3). This is the volume of a cube of edge 1m.
Hence $1\text{m}^3 = 100 \times 100 \times 100 = 1{,}000{,}000\text{cm}^3$
Also $1\text{cm}^3 = 10 \times 10 \times 10 = 1{,}000\text{mm}^3$
Capacity is the volume of liquid a container can hold. The basic unit of capacity is the litre(l). Other units are 1 decilitre(dl) = $\frac{1}{10}l$ and 1 millilitre(ml) = $\frac{1}{1000}l$. The relation between the units of volume and capacity is $1\text{cm}^3 = 1ml$.

Exercise 7c

1 Express in litres (i) $100ml$, (ii) $2{,}567\text{cm}^3$, (iii) 1m^3.
2 Express in decilitres (i) $2{\cdot}5l$, (ii) $25ml$, (iii) 324cm^3.
3 Express in cm^3 (i) $10dl$, (ii) $242ml$, (iii) $0{\cdot}2\text{m}^3$.
4 Express in litres (i) $10dl$, (ii) $240dl$, (iii) $0{\cdot}5\text{m}^3$.

7.4 Mass

The basic unit of **mass** in the metric system is the gram (g).

$$\begin{aligned} 1\text{g} &= 1{,}000 \text{ milligrams (mg)} \\ 1 \text{ kilogram (kg)} &= 1{,}000\text{g} \\ 1 \text{ tonne (t)} &= 1{,}000\text{kg} \end{aligned}$$

Exercise 7d

1 Express in grams (i) 9·2kg, (ii) 0·092kg, (iii) 125,000mg.
2 Express in kilograms (i) 2,400g, (ii) 942g, (iii) 65g.
3 Express in tonnes (i) 250kg, (ii) 25,000kg, (iii) 25kg.
4 Express in kilograms (i) 2·5t, (ii) 0·25t, (iii) 0·00025t.

7.5 Mensuration

Mensuration is the study of length, area and volume. For a fuller treatment including the derivation of formulae see Topic **31**.

Rectangles *A*rea = *l*ength × *b*readth or $A = l \times b$
Perimeter = Total distance around the rectangle
$P = 2l + 2b = 2(l + b)$

Squares *A*rea = (*l*ength)² or $A = l^2$
*P*erimeter = 4 × *l*ength or $P = 4l$

Parallelograms *A*rea = *b*ase × perpendicular *h*eight or $A = b \times h$

Triangles *A*rea = $\frac{1}{2}$ *b*ase × perpendicular *h*eight or $A = \frac{1}{2}b \times h$

Trapezia If the lengths of the parallel sides are a and b and are h units apart then the area A is given by $A = \frac{1}{2}(a + b) \times h$. This is sometimes given as: 'half the sum of the parallel sides times the distance between them'.

Cubes *V*olume = (*e*dge)³ or $V = e^3$

Cuboids *V*olume = *l*ength × *b*readth × *h*eight or $V = lbh$

Prisms *V*olume = *A*rea of cross-section × *l*ength or $V = A \times l$

Pyramids *V*olume = $\frac{1}{3}$ × base *A*rea × perpendicular *h*eight or $V = \frac{1}{3}Ah$

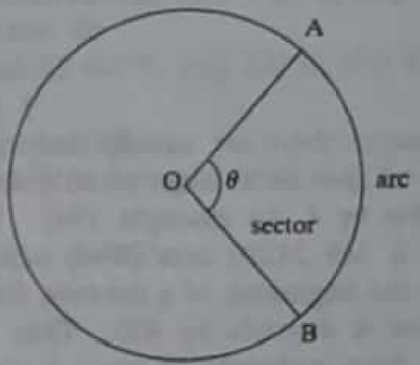
Exercise 7e

1 Calculate the outside perimeter and area of a path of width 1m which runs round a **rectangular** house 30m by 25m.
2 A **square** has a perimeter of 24m. Calculate its area.
3 Calculate the perimeter in km and area in hectares of a rectangular piece of land measuring 450m by 300m.
4 Calculate the vertical height of a **parallelogram** with area 24cm² and base 8cm.
5 Calculate the base of a **triangle** with area 24cm² and vertical height 8cm.
6 The area of a **trapezium** is 48cm². One parallel side is twice the length of the other and their distance apart is 4cm. Calculate the lengths of the parallel sides

7 Find the volume and total surface area of a **cube** of edge 6cm.
8 An empty tank in the form of a **cuboid** measures 50cm by 32cm and has a height of 28cm. If 28*l* of petrol are poured into it find the distance of the surface of the petrol below the top of the tank.
9 A water-trough in the form of a **prism** has a trapezium as uniform cross-section with parallel sides 40cm and 20cm and distance apart 25cm. Its length is 180cm. Calculate its capacity in litres.
10 Calculate, correct to 3 sf, the volume of a **pyramid** with a rectangular base measuring 19cm by 16cm and a vertical height of 11cm.

7.6 Mensuration of the Circle

The **circumference** C of a circle of diameter d is given by $C = \pi d$. If r is the radius of the circle then $d = 2r$ and the **formula** (see Topic **47**) for the circumference becomes $C = 2\pi r$.
The **area**, A, of a circle is given by $A = \pi r^2$.
The value of π is given by the non-terminating, non-recurring decimal 3·141592... (see **4.6** and **26.2**). Correct to 2 decimal places its value is 3·14. Sometimes we use $\frac{22}{7}$ as an approximate value of π.
The diagram shows an **arc** AB of a circle which **subtends** an angle $\theta°$ at the centre, O. The length of the arc is calculated as the fraction $\frac{\theta}{360}$ of the circumference $2\pi r$.

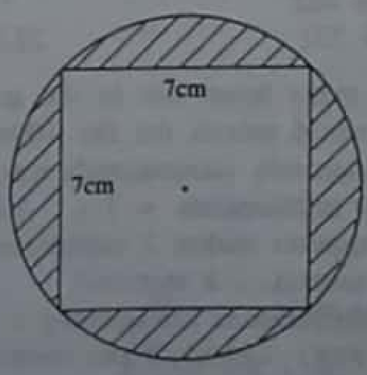


$$\text{Arc AB} = \frac{\theta}{360} \times 2\pi r$$
$$= \frac{\theta\pi r}{180}$$

The area of **sector** AOB $= \frac{\theta}{360} \times \pi r^2$

Exercise 7f Take π to be $\frac{22}{7}$ unless otherwise stated.

1 Find the circumference and area of a circle of diameter 10·5cm.
2 A circle has a circumference of 44cm. Find (i) the radius, (ii) the area of the circle.
3 Find the area of a circle whose circumference measures 220m.
4 Given that the area of a circle is 616mm² find (i) its radius, (ii) its circumference.
5 A circle has an area of 314cm². Find the circumference. (Take $\pi = 3 \cdot 14$)
6 What length of fencing is required to make a circular cattle enclosure of area 154m²?
7 Find the length of an arc of a circle of radius 12·6cm which subtends an angle of 136° at the centre. Find also the area of the corresponding sector of the circle.
8 Taking the circle of latitude at the equator to have a diameter of 12,600km, find the length of arc which subtends an angle of 1° at the centre of the Earth.
9 The diagram shows a square of side 7cm inscribed in a circle. Find the area of the shaded portion.



7.7 Money

The unit of money (currency) in Uganda is the **shilling** (sh). The face value of a coin or note is its **denomination**.

Exercise 7g

1 List in ascending order the denominations of (i) the coins, (ii) the notes in use in Uganda.
2 What salary per month is equivalent to sh414,000 per annum?
3 Convert a salary of sh6,200 per month into shillings per annum.
4 Evaluate, (i) sh980 × 5, (ii) sh1,650 × 7, (iii) sh2,700,000 × 6.
5 Henry buys 2 pairs of shorts at sh3,600 each, 3 T shirts at sh2,700 each and a pair of shoes at sh7,500. How much does he spend altogether?

7.8 Time

One day is the time taken for the Earth to rotate once on its axis. One year is the time taken by the Earth to revolve once round the Sun. Units of time are

$$\begin{aligned} 60 \text{ seconds (s)} &= 1 \text{ minute (min)} \\ 60 \text{ minutes} &= 1 \text{ hour (h)} \\ 24 \text{ hours} &= 1 \text{ day} \\ 1 \text{ year} &\approx 365\tfrac{1}{4} \text{ days} \end{aligned}$$

In practice there are usually three years of 365 days followed by one of 366 days (called a **leap year**) to give an average of $365\frac{1}{4}$ days. A leap year normally occurs when the year number is divisible by 4, for example 1992. However, the actual length of the year (called the **tropical year**) is 365·24220 days (8 sf) which is slightly less than $365\frac{1}{4}$. Because of this, years which mark the beginning of a century, for example 1900 and 2000, are not leap years unless the year number is divisible by 400. Thus 1900 was not a leap year but the year 2000 will be.
Time from midnight to noon is called ante meridian (am) and from noon to midnight post meridian (pm). To avoid confusion between am and pm, 24-hour time can be used, in which, for example, 1pm is 13.00 (read as thirteen hundred hours), 2pm is 14.00 hours and 11.30pm is 23.30 hours.

Exercise 7h

1 Copy and complete the table.

FLIGHT No	DEPARTURE TIME	ARRIVAL TIME	FLIGHT TIME
QU 320	09.00	10.45	?
ET 856	09.43	11.28	?
KQ 411	13.55	?	1h 10 min
MS 822	?	08.00	4h 40 min
SN 571	22.30	?	7h 55 min

2 How many hours are in the month of February 1992?
3 The world record for the 100m was broken by 0·04 seconds. How many milliseconds, microseconds, nanoseconds was this?
(1,000 milliseconds = 1 s 1,000 microseconds = 1 ms 1,000 nanoseconds = 1 mcs)
4 A computer makes 2 calculations in a nanosecond. How many calculations will it make in a second,... a minute?
5 The instructions for cooking a turkey are '15 min per $\frac{1}{2}$kg plus 5 min per kg for each kg over 6kg'. Calculate the cooking time for a turkey of mass 10kg.

7.9 Temperature

Heat is a form of energy while **temperature** is a measure of the 'hotness' or 'coldness' of an object. The temperatures at which water boils and freezes at sea level are taken as reference points on the **Celsius**(°C) temperature scale. These are 100°C and 0°C respectively. The lowest possible temperature is about −273°C. This is absolute zero for the **Kelvin**(°K) scale of temperature. The absolute temperature is obtained by adding 273° to the Celsius temperature.

Example Convert (i) 50°C to °K, (ii) 180°K to °C.

(i)
$$\begin{aligned} 50°\text{C} &= (50 + 273)°\text{K} \\ &= 323°\text{K} \end{aligned}$$

(ii)
$$\begin{aligned} 180°\text{K} &= (180 - 273)°\text{C} \\ &= -93°\text{C} \end{aligned}$$

Exercise 7i

1 Give the temperature on the Kelvin scale for (i) 0°C, (ii) 100°C, (iii) −23°C, (iv) 47°C, (v) −272°C.
2 Give the temperature on the Celsius scale for (i) 300°K, (ii) 400°K, (iii) 222°K, (iv) 150°K, (v) 100°K.
3 The melting point of sulphur is 444·4°C. What is this in °K?
4 Degrees **Fahrenheit**(°F) and degrees Celsius are related by the formula $9C = 5(F - 32)$. Use this formula to give the following in °C.
(i) 113°F, (ii) 212°F, (iii) 140°F, (iv) 302°F, (v) 32°F, (vi) −40°F.
5 Give your answers to 4 in °K.

8 RATE

8.1 Basic Ideas

Examples of rate are kilometres per hour (km/h), metres per second (m/s), litres per minute (l/min), shillings per hour (sh/h), grams per cm^3 (g/cm^3), shillings per kg (sh/kg). *Per* means *for each* and is given the symbol / in the above examples. The idea of rate occurs often in real life situations.

8.2 Speed

Speed is the distance travelled in unit time. If the distance is given in kilometres and the time in hours then the speed is in km/h. If the distance is in metres and the time in seconds then the speed is in m/s. Speeds at sea or in the air are usually given in knots. One knot is 1 nautical mile per hour (see **59.5**). If s is the speed, t the time and d the distance then:

$$d = s \times t, \quad s = \frac{d}{t} \quad \text{and} \quad t = \frac{d}{s}$$

$$\text{Average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

Example A car travels 48km in 45 minutes. Find its average speed in km/h.

$$45 \text{ min} = \tfrac{45}{60}\text{h} = \tfrac{3}{4}\text{h} \qquad \text{average speed} = \frac{48}{\frac{3}{4}} = \frac{48 \times 4}{3} = 64\text{km/h}$$

Exercise 8a

1 A lorry travels 63km in 1h 30min. Find its average speed in km/h.
2 How long does a cyclist take to travel 56km at a speed of 14km/h?
3 A journey took 1h 20min at an average speed of 81km/h. Find the distance travelled.
4 I travel from Soroti to Kampala (350km) at an average speed of 70km/h and return at an average speed of 50km/h. What is the difference in times for the journeys?
5 What is the average speed for the whole journey from Soroti to Kampala and back in 4 above?

8.3 Acceleration

If a car is moving along a straight road, its **acceleration** is the rate of change of its speed. If during a time of 10s the speed changes from 9m/s to 18m/s then

$$\text{average acceleration} = \frac{\text{change in speed}}{\text{time taken}} = \frac{18-9}{10} = \frac{9}{10} = 0{\cdot}9\text{m/s}^2$$

Notice the units. We have divided a speed by a time. So the units are (metres per second) per second, which is written as m/s^2.
If the acceleration is negative the car is slowing down. We say it is **decelerating** or **retarding**.
If a is the acceleration, s the change in speed, and t the time taken then

$$a = \frac{s}{t} \qquad s = at \quad \text{and} \quad t = \frac{s}{a}$$

Example A car has a speed of 8m/s at 10s and 16m/s at time 14s after the start. Find the average acceleration in m/s^2.

$$a = \frac{s}{t} = \frac{16-8}{14-10} = \frac{8}{4} = 2\text{m/s}^2$$

Exercise 8b

1 Find the acceleration of a car that increases its speed from 6m/s to 12m/s in 1·5s.
2 Find the speed of a car starting from rest which accelerates at 0·75m/s² for 20s.
3 Find the time taken for a car to increase its speed by 5m/s if the acceleration is $\frac{1}{2}$m/s².
4 The speed of a train leaving a station is shown in the table.

Time in seconds	0	2	4	6	8	10
Speed in m/s	0	3	5	6	7	8

Find the average acceleration during (i) the first 2 seconds, (ii) the first 4 seconds, (iii) the last 2 seconds, (iv) the last 4 seconds, (v) the whole 10 seconds.

8.4 Flow

Example Water flows from a tap at a rate of 50m*l* /s. How many minutes does it take to fill a 20*l* *debe*?

50m*l* /s = 50 × 60m*l* /min = $\frac{50\times60}{1000}$ = 3*l* /min

Time = $\frac{\text{volume delivered}}{\text{flow rate}}$ = $\frac{20}{3}$ = $6\frac{2}{3}$ min

Other relations are: Volume delivered = flow rate × time; Flow rate = $\frac{\text{volume delivered}}{\text{time}}$

Exercise 8c

1 A tap fills a 750m*l* bottle in 10s. Find the rate of flow in *l* /min.
2 How many hours does it take a tap to fill a water tank of capacity 900*l* if the delivery is 125m*l* /s?
3 A pipe delivers 30m*l* /s and takes 3 minutes to fill a petrol tank. What is the capacity of the tank in litres?
4 Tap A can fill a tank in 5 min and tap B fills the same tank in 7 min. How long does it take to fill the tank if both tap A and tap B are on?

8.5 Cost 'per' Unit

Prices are usually quoted as a given number of shillings per unit, as in the following Example.

Example Opolot buys 3kg of onions for sh1,650. What is the price of onions per kg?

Price per kg = (Total cost) ÷ (Number of kilograms) = 1,650 ÷ 3 = sh550 per kg

Exercise 8d

In 1 to 5 give the rate in its simplest form.
1 5kg of sugar for sh3,750 (sh per kg)
2 6kg of rice for sh3,900 (sh per kg)
3 7kg of potatoes for sh1,750 (sh per kg)
4 20 eggs for sh1,500 (sh per dozen)
5 9*l* of petrol for sh5,850 (sh per *l*)

In 6 to 10 give the cost per kg.
6 6kg of carrots for sh2,700
7 8kg of oranges for sh3,600
8 5·5kg of tangerines for sh2,200
9 3·5kg of tomatoes for sh1,750
10 7·5kg of meat for sh6,375

8.6 Density

Some substances, because they are more **dense**, are heavier than others. The **density** of a substance is the mass per unit volume. For example, iron has a density of $7 \cdot 6 g/cm^3$.

$$\text{density} = \frac{\text{mass}}{\text{volume}} \quad \text{or} \quad \text{mass} = \text{volume} \times \text{density} \quad \text{or} \quad \text{volume} = \frac{\text{mass}}{\text{density}}$$

Example If $4cm^3$ of copper (density $8 \cdot 9 g/cm^3$) is melted with $6cm^3$ of lead (density $11 g/cm^3$), what is the density of the alloy formed?

Mass of copper is $4 \times 8 \cdot 9 = 35 \cdot 6g$, mass of lead is $6 \times 11 = 66g$

Therefore mass of alloy $= 35 \cdot 6 + 66 = 101 \cdot 6g$, and volume $= 4 + 6 = 10cm^3$

Therefore density of alloy $= \frac{\text{mass}}{\text{volume}} = \frac{101 \cdot 6}{10} = 10 \cdot 16 g/cm^3$

Exercise 8e

1 What is the density of a mixture of equal quantities by volume of mercury ($13 \cdot 5 g/cm^3$) and lead? (*Hint* Suppose there are $10cm^3$ of each substance.)
2 Repeat 1 when the mixture contains an equal mass of mercury and lead. Give your answer to 3 sf.
3 What is the density of a mixture of equal volumes of lead, copper and mercury? Give your answer to 2 dp.
4 What is the density of a mixture of lead and copper if the mixture contains lead and copper in the ratio (see Topic 9) 3 : 2 by volume?
5 Repeat 4 if the ratio is '3 : 2 by mass' giving your answer to 2 dp.

8.7 Population Density

This is defined as the number of people per unit area.

So population density $= \frac{\text{population}}{\text{area}}$ and is usually given as the number of people per km^2.

Example The population density of Kampala is $3{,}900/km^2$. If the area of the city is $198km^2$, what is the population of Kampala?

Population = density × area = 3,900 × 198 = 772,000 (to the nearest 1,000)

Exercise 8f

1 If the population of Uganda is 16,600,000 and the land area is $200{,}000km^2$, calculate the population density.
2 Copy and complete the table based on information from the Statistics Department.

District	Population	Land area (km^2)	Density (persons/km^2)
Bushenyi	750,000	5,000	?
Iganga	?	4,750	200
Kabarole	738,000	?	90
Kotido	191,400	13,200	?
Mubende	?	6,250	80
Mukono	828,000	?	180
Nebbi	322,000	?	115
Pallisa	345,000	1,500	?

9 RATIO

9.1 Basic Ideas

A **ratio**, which is a type of fraction, enables us to compare two or more quantities of the same kind. If two packets of detergent contain 375g and 500g respectively then the ratio of the smaller to the larger is

$$\frac{375\text{g}}{500\text{g}} = \frac{125\times3}{125\times4} = \frac{3}{4}$$

We write the ratio as 3 : 4. Note that the quantities must be of the same unit and that this unit is omitted in the final ratio.

Exercise 9a

In **1** to **10** express each of the ratios in its simplest form.

1 sh6,000 : sh9,000
2 200m : 1km
3 250ml : 2l
4 $2\frac{1}{2}$h : 1 day
5 $\frac{2}{3}$kg : 600g
6 50m² : 1 are
7 15° : $1\frac{1}{3}$ right angles
8 $\frac{3}{4}$h : $2\frac{1}{2}$min
9 24mm : $2\frac{1}{2}$cm
10 120,000cm³ : 0·065m³

11 A rectangle has a length of 24cm and a width of 18cm. Give the ratio of (i) its length to width, (ii) its width to length.

12 Two squares are of side 18cm and 12cm respectively. Give the ratio of (i) their sides, (ii) their perimeters, (iii) their areas.

9.2 Division in a given Ratio

To divide sh12,000 between Adhiambo, Angwero and Asanda in the ratio 3 : 5 : 7 we first divide sh12,000 into 15 equal parts because 3 + 5 + 7 = 15. This gives Adhiambo $\frac{3}{15}$ of sh12,000 or sh2,400, Angwero $\frac{5}{15}$ of sh12,000 or sh4,000 and Asanda $\frac{7}{15}$ of sh12,000 or sh5,600. As a check note that 2,400 + 4,000 + 5,600 = 12,000.

Exercise 9b

In **1** to **8** divide the number or quantity in the given ratio.

1 50; 2 : 3
2 40; 1 : 7
3 sh10,000; 13 : 7
4 sh10,000; 3 : 7 : 10
5 sh3,600; 1 : 2 : 6
6 sh2,400 6 : 3 : 1
7 1m; 1 : 2 : 3 : 4
8 1·5km 1 : 2 : 3 : 4 : 5

9 A piece of string 3m long is cut into 3 lengths which are in the ratio 7 : 5 : 3. Find the length of the shortest piece

10 When a sum of money is divided in the ratio 2 : 3 : 7 the smallest share is sh1,500. What is the original sum of money?

11 A cake requires sugar, margarine and flour to be mixed in the ratio 2 : 3 : 5. If the cake contains 750g of flour, how much sugar does it contain?

12 Idro is 9, Ochung is 12 and Nsubuga is 15 years old. They decide to share a number of mangoes in the ratio of their ages. If Ochung and Nsubuga together receive 42 mangoes, how many does Idro get?

9.3 Increasing and Decreasing Ratios

To increase 24cm in the ratio 3 : 2 means $\frac{\text{new length}}{\text{old length}} = \frac{3}{2}$

So new length $= \frac{3}{2} \times$ old length $= \frac{3}{2} \times 24\text{cm} = 36\text{cm}$

To decrease 24cm in the ratio 2 : 3 proceed in the same way to give

new length $= \frac{2}{3} \times$ old length $= \frac{2}{3} \times 24 = 18\text{cm}$

Exercise 9c

1 Increase sh6,000 in the ratio 6 : 5

2 Decrease sh6,000 in the ratio 5 : 6

3 Increase 48cm^3 in the ratio 4 : 3

4 Decrease 48cm^3 in the ratio 3 : 4

5 Increase the sides of a rectangle measuring 18cm by 12cm in the ratio 5 : 2. What is the ratio of (i) old perimeter to new perimeter, (ii) old area to new area?

6 If the radius of a circle is increased in the ratio 4 : 3, what will be the ratio of (i) old circumference to new circumference, (ii) old area to new area?

9.4 Multiple Ratios

If A : B = 4 : 3 and B : C = 2 : 1 what is A : B : C? The two ratios are combined by making B's number the same in both ratios. Since the LCM (see 1.1) of 2 and 3 is 6, we multiply the first ratio by 2 and the second ratio by 3. So A : B = 4 : 3 = 8 : 6 and B : C = 2 : 1 = 6 : 3. These two ratios may now be combined to give A : B : C = 8 : 6 : 3.

Exercise 9d

1 If A : B = 3 : 4 and B : C = 1 : 2, find A : B : C.

2 If D : E = 7 : 6 and E : F = 4 : 13, find D : E : F.

3 If X : Y = 3 : 5 and Y : Z = 4 : 7, find X : Y : Z.

4 If W : X : Y = 1 : 2 : 3 and X : Y : Z = 4 : 6 : 7, find W : X : Y : Z.

5 If A : B : C = 1 : 2 : 3 and C : D : E = 4 : 3 : 2, find A : B : C : D : E.

6 If P : Q = 5 : 4 and R : Q = 2 : 3, find P : R.

9.5 Percentage Increase and Decrease

To increase a length of 24cm by 25% means

new length : old length = (100 + 25) : 100 = 125 : 100

So new length $= \frac{125}{100} \times 24 = 30\text{cm}$

To decrease the same length by 25% means

new length : old length = (100 − 25) : 100 = 75 : 100

So new length $= \frac{75}{100} \times 24 = 18\text{cm}$

Exercise 9e

1 Increase sh1,200 by 25%

2 Increase 600cm^3 by $33\frac{1}{3}\%$

3 Decrease 1l by 40%

4 Decrease 1kg by 47%

5 Increase 1t by 59%

6 Increase 360g by $2\frac{1}{2}\%$

7 Decrease sh 16,000 by $17\frac{1}{2}\%$

8 Decrease 1 day by $12\frac{1}{2}\%$ (in hours)

9 Increase sh2,000,000 by $13\frac{1}{2}\%$

10 Increase 3 days by 14% (in hours)

9.6 Representative Fraction

If 1cm on a map represents 10,000cm on the ground then the ratio 1 : 10,000 is called the **representative fraction** (R.F.). What length on this map would represent 1km?

Now 1km = 100,000cm so length on map = $\frac{100,000}{10,000}$ = 10cm

Representative fractions are expressed in the form $1 : n$, where n is a positive integer.

Exercise 9f

For 1 to 4 copy and complete the following table.

	Distance on ground	Distance on map	R.F.
1	$\frac{1}{2}$km	5cm	?
2	?km	2cm	1 : 50,000
3	1km	?cm	1 : 40,000
4	?km	1cm	1 : 500,000

5 According to the scale on a map of Uganda, 5cm represents 1km. What is the representative fraction of this map?

6 Two maps have R.F. 1 : 10,000 and 1 : 15,000 respectively. A distance is represented by 6cm on the first map. What would this be represented by on the second map?

7 The straight line distance on an atlas whose R.F. is 1 : 10,000,000 between Kampala and Nairobi is 5·1cm. What does this give for the actual distance, in km, between these two cities?

10 PROPORTION

10.1 Direct Proportion

If the cost of 1kg of potatoes is sh300, what is the cost of $2\frac{1}{2}$ kg? We know that the cost will be two and a half times sh300 because this is an example of **direct proportion**. If two quantities are in direct proportion, then an increase (or decrease) in one item causes an increase (or decrease) in the othe· in the same ratio (see Topic **9**).

Example Find the cost of 5·5m of material if 4m cost sh11,200.

This is an example of direct proportion. The length has increased in the ratio $5{\cdot}5 : 4$ or $11 : 8$, so the cost of the material increases in the ratio $11 : 8$.

Therefore cost $= \frac{11}{8} \times$ sh11,200 $=$ sh15,400

Exercise 10a

Assume direct proportion for 1 to **10**.

1 If the cost of 1kg of potatoes is sh300, what is the cost of $2\frac{1}{2}$kg?

2 If 6 equal bags of flour have a mass of 15kg, how many such bags are required for 35kg of flour?

3 A car travels 94km on 14 litres of petrol. How far does it travel on 21 litres?

4 A man earns sh4,200 in 8 hours. How many hours does it take him to earn sh7,350?

5 If 30 tins cost sh12,900, how many such tins can be bought for sh43,000?

6 How much seed is needed for a field of area 210m^2 if 8kg are needed for 120m^2?

7 If a watch loses 30 seconds in 4 days how much does it lose in a month of 30 days?

8 If 100cm^3 of a metal has a mass of 750g find the mass of 1m^3 of this metal in kg.

9 A sheet of metal 2m by 3m has a mass of 6kg. Find the mass of a similar sheet measuring 4m by 5m.

10 A piece of cloth 2m by 1·4m costs sh2,800. Find the cost of a similar piece of cloth 3m by 1·6m.

11 Which of the following could be examples of direct proportion? In each such case give an expression for the amount stated.

(i) If the cost of n kg of salt is sh500 what is the cost of m kg?

(ii) If a man has 3 wives when he is 25 years old how many does he have when he is 75 years old?

(iii) A man is paid sh$h$ for 4 hours. How much is he paid for 9 hours?

(iv) A team scored n goals in 5 matches. How many do they score in 10 matches?

(v) A man has a mass of 65kg at the age of 20. What is his mass when he is aged 50?

(vi) If A cm^3 of a metal has a mass of K kg, what is the mass of N m^3?

(vii) A woman has x children at the age of 20. How many does she have when she reaches the age of 80?

(viii) A clock gains 6 seconds in x days. How many minutes does it gain in a year?

(ix) A chilli bush 1m high gives 150 chillies. How many chillies does a bush 1·5m high yield?

(x) This year Ojangole planted 4 acres of cotton which yielded x bags. How many bags of cotton will he get next year when he plants 10 acres?

10.2 Indirect Proportion

If two quantities are in **indirect proportion** (or **inverse proportion**) then an increase in one causes a decrease in the other and vice-versa. Again we may use ratio as shown in the next Example.

Example If it takes 8 days for 9 men to weed a field of millet, how long will it take 6 men?

The men have decreased in the ratio 6 : 9. So the time will increase in the ratio 9 : 6.

Time = $\frac{9}{6} \times 8 = 12$ days

Exercise 10b

Assume indirect proportion.

1 If 2 men can paint a house in 6 days how many days would it take 3 men to paint a similar house?
2 If 15 women can thatch a roof in 24 hours how long does it take 9 women?
3 If I spend sh600 per day my money lasts 15 days. How many days will it last if I spend only sh400 per day?
4 A water tank is filled in 8 hours by 5 taps. How many taps were used if the tank was filled in 2 hours?
5 If 4 cows take 7 hours to graze a field how many hours will it take 14 cows?
6 A poultry farmer has sufficient grain to feed 40 hens for 12 days. If he sells 10 hens how long will the grain last?
7 The width of a rectangular room is halved but its area remains the same. What is the change in the length of the room.
8 A taxi travels from Kumi to Mukura in 16 minutes at an average speed of 70km/h. How many minutes would it take to cover the same distance at 80km/h?
9 If 18 boxes contain 20 pineapples in an export consignment how many boxes are needed if each box contains instead 12 pineapples?
10 If it takes 5 hens 5 days to lay 5 eggs, how long does it take for 1 hen to lay 1 egg?
11 Tap A fills a water tank in 3h, tap B fills it in 4h and tap C fills it in 6h. How many hours will it take if taps A, B and C are used together?
12 Give expressions as answers for each of the following.
   (i) If it takes x hours to fill a tank using y taps, how long will it take using z taps?
   (ii) If it takes s men t days to repair a road, how long will it take v men?
   (iii) If it takes p days for n hens to finish a certain amount of grain, how long will it take m hens?
   (iv) If it takes c days for d men to build a house, how long will it take e men?
   (v) If it takes c days for d men to build a house, how many men are needed to build this house in f days?

11 SQUARES AND ROOTS

11.1 Squares

When a number is multiplied by itself, for example 3×3, the product 9 is called a square. Using index notation, $3^2 = 9$. This is read as 'three squared equals nine'.
Also $(1\frac{1}{2})^2 = 1\frac{1}{2} \times 1\frac{1}{2} = \frac{3}{2} \times \frac{3}{2} = \frac{9}{4} = 2\frac{1}{4}$ and $1{\cdot}2^2 = 1{\cdot}2 \times 1{\cdot}2 = 1{\cdot}44$
To find the square of a number, multiply the number by itsef. If we cannot find the square mentally long multiplication or tables are used. For example, $4{\cdot}76^2 = 22{\cdot}6576$ using long multiplication (check that this is correct), or, using tables of squares, $4{\cdot}76^2 = 22{\cdot}7$ (3sf). Note that 3-figure tables give the answer correct to 3 sf, whereas long multiplication gives the exact value. When using tables, care must be taken to obtain the correct place value in the answer. This is done by using approximation.

Example Find $(0{\cdot}286)^2$ using tables.

Looking up 286 in tables of squares gives 818.
Now $0{\cdot}286 = 0{\cdot}3$, correct to 1 sf, and $0{\cdot}3^2 = 0{\cdot}09$
Therefore $0{\cdot}286^2 = 0{\cdot}0818$ (3 sf)

Exercise 11a

1 Evaluate mentally and write down the squares of the following.
(a) $\frac{3}{4}$ (b) $2\frac{1}{2}$ (c) 0·03 (d) 0·5 (e) 0·05
(f) 1·1 (g) 1,000 (h) 10^4 (i) -6 (j) $-0{\cdot}1$

2 Find the exact value of the squares of the following.
(a) 123 (b) 789 (c) 4·56 (d) 0·987 (e) 0·0909

3 Use tables of squares to find the squares of the following.
(a) 17 (b) 31 (c) 32 (d) 87 (e) 105
(f) 3·16 (g) 3·17 (h) 0·234 (i) 0·0123 (j) 0·00202

4 Evaluate the following.
(a) $3^2 + 4^2$ (b) $(3 + 4)^2$ (c) $10^2 - 8^2$ (d) $(10 - 8)^2$
(e) $3^2 - 4^2$ (f) $(3 - 4)^2$ (g) $(3^2 + 4^2)^2$ (h) $(3^2 - 4^2)^2$

5 A square is of side 6cm correct to the nearest centimetre. What is the least possible length of the square? What is its greatest possible length? Between what limits does the area lie? (See also **Q11, Ex43c**)

11.2 Square Roots

Square rooting is the reverse of squaring. For example, the square of 3 is 9, and the square root of 9 is 3. This last statement is written as $\sqrt{9} = 3$. To obtain the square root of a number, find the number which, when multiplied by itself, gives that number.

Thus $\sqrt{49} = 7$, $\sqrt{5\frac{4}{9}} = \sqrt{\frac{49}{9}} = \frac{7}{3} = 2\frac{1}{3}$ and $\sqrt{1{\cdot}44} = 1{\cdot}2$

Example 1 Find the square root of 1,764 by the factor method.

$1{,}764 = 2 \times 2 \times 3 \times 3 \times 7 \times 7$, factorising
$= (2 \times 3 \times 7) \times (2 \times 3 \times 7)$, writing factors in two *equal* groups
$= 42 \times 42$
So $\sqrt{1{,}764} = 42$

To find the square root of a number which does not factorise easily or is not an exact square, use either of the methods given below.

Example 2 Find $\sqrt{2{,}189}$ correct to 3 sf.

Method 1: Mechanical

Pair off the digits in either direction starting from the decimal point. Note that to each *pair* of digits there corresponds *one* digit in the answer, and decimal points are aligned. Take the pair of digits on the left which is 21. The largest number which, when multiplied by itself, gives a product of 21 or less is 4. Place 4 in the answer space corresponding to 21 and subtract 16 (4×4) from 21. This leaves 5. Bring down the next pair of digits (89) to give 589. Double 4 to give 8 and look for the largest digit $\boxed{}$ for which $8\boxed{} \times \boxed{}$ equals 589 or less.

```
             4 | 6 | 7 | 8
   4 ) 2 1|8 9|0 0|0 0
      -1 6
   8[6]) 5 8 9
        -5 1 6
   9 2[7]) 7 3 0 0
          -6 4 8 9
   9 3 4[8]) 8 1 1 0 0
            -7 4 7 8 4
               6 3 1 6
```

Now $8\boxed{5} \times \boxed{5} = 425$ is too small and $8\boxed{7} \times \boxed{7} = 609$ is too large. It is $8\boxed{6} \times \boxed{6} = 516$. Write 6 in the answer space corresponding to 89 and subtract 516 from 589. This leaves 73. Bring down the next pair of digits (00) to give 7300.
We then repeat this process until the calculation terminates or the required degree of accuracy is achieved. Here, double 46 gives 92 and $92\boxed{7} \times \boxed{7} = 6489$ etc.

Therefore $\sqrt{2{,}189} = 46{\cdot}78 = 46{\cdot}8$ to 3 sf

Method 2: Square root tables

Correct the number to 3 sf, if necessary. Here 2,189 = 2,190 to 3 sf. From square root tables, to 219 corresponds 148 or 468. To decide which number to use and determine the position of the decimal point, perform mentally the first steps of the mechanical method. This shows that the answer begins with a 4 and there are two digits in front of the decimal point.

```
 4 |   |
2 1|8 9|0 0
```

Therefore $\sqrt{2{,}189} = 46{\cdot}8$ to 3 sf

Example 3 Find $\sqrt{0{\cdot}000754}$ (i) to 3 sf, (ii) to 4 sf.

(i) Corresponding to 754 from square root tables we have 275 or 868. Pairing off digits from the decimal point gives the answer starting as 0·02.

So $\sqrt{0{\cdot}000754} = 0{\cdot}0275$ to 3 sf

(ii) Using the mechanical method gives the working shown.
The last calculation $\boxed{}$ gives at least 5. Thus the previous 5 rounds up to 6.

So $\sqrt{0{\cdot}000754} = 0{\cdot}02746$ to 4 sf

```
       0 | 0 | 2 | 7 | 4 | 5 | [ ]
   2 ) 0 0|0 0|0 7|5 4|0 0|0 0|0 0
          -4
       4[7]) 3 5 4
            -3 2 9
       5 4[4]) 2 5 0 0
              -2 1 7 6
       5 4 8[5]) 3 2 4 0 0
                -2 7 4 2 5
       5 4 9 0[ ]) 4 9 7 5 0 0
```

Exercise 11b

1 Evaluate mentally and write down the square roots of the following.

(a) 144 (b) $\frac{4}{9}$ (c) 0·36 (d) 0·04 (e) $2\frac{1}{4}$

(f) 0·0064 (g) 0·0001 (h) 10,000 (i) 10^6 (j) $\frac{1}{10^4}$

2 By factorising the following numbers, find their square roots.
(a) 576 (b) 2,025 (c) 11,025 (d) 5,929 (e) 254,016

3 Use the mechanical method to find the exact square roots of the following numbers.
(a) 474,721 (b) 9,102,289 (c) 3·0976 (d) 246·49 (e) 0·087616

4 Use the mechanical method to find the square roots of the following correct to 3 sf.
(a) 39 (b) 4·76 (c) 2 (d) 1,000 (e) 0·036

5 Use tables to find the square roots of the following numbers.
(a) 4·56 (b) 17·6 (c) 45·6 (d) 176 (e) 40
(f) 2 (g) 5 (h) 1·02 (i) 1,270 (j) 13,000
(k) 0·139 (l) 0·0139 (m) 0·001 (n) 0·00079 (o) 0·00002

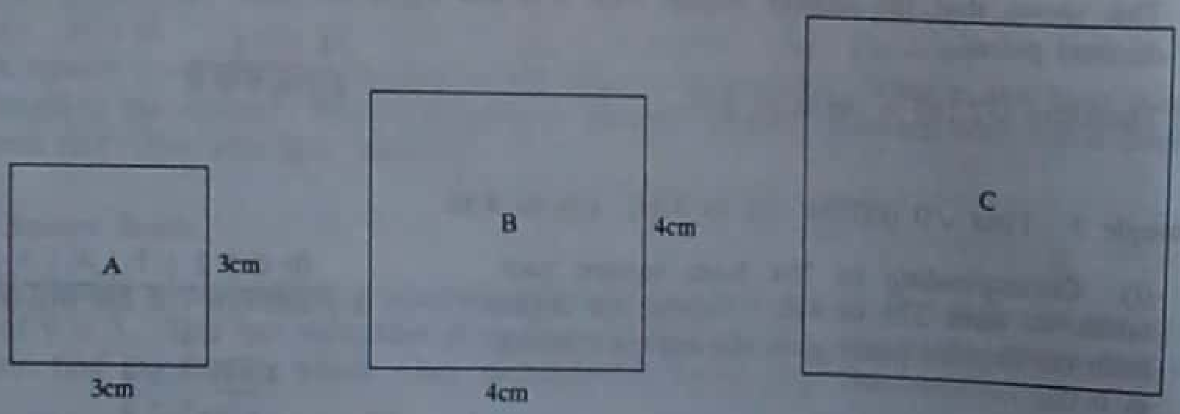
In 6 and 7 give your answers correct to 3 sf.

6 Find the area of a square having the following length of side.
(a) 1·75cm (b) 20·7m (c) 343km (d) 0·078mm

7 Find the length of side of each of the following squares for which the area is given.
(a) 33cm^2 (b) 500m^2 (c) $2{,}000\text{km}^2$ (d) $0{\cdot}4\text{mm}^2$

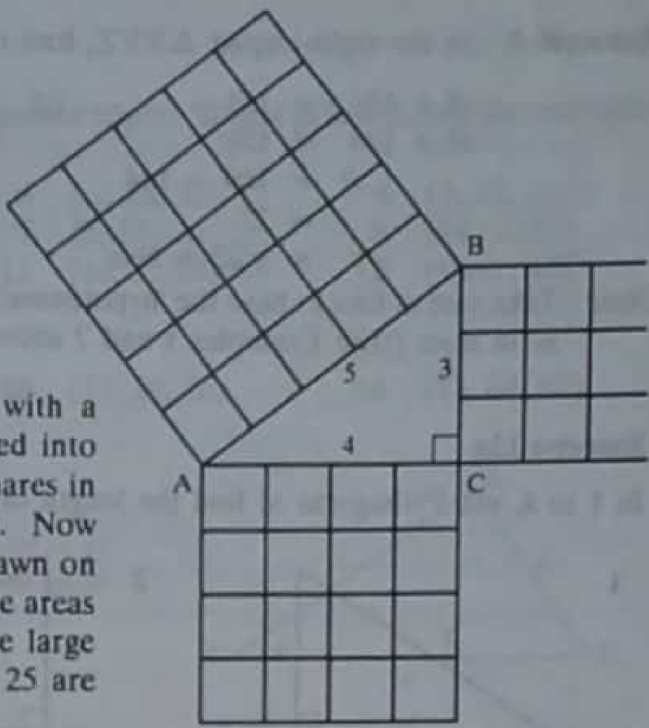
8 Evaluate the following.
(a) $\sqrt{(3^2 + 4^2)}$ (b) $\sqrt{(13^2 - 5^2)}$ (c) $\sqrt{(59^2)}$ (d) $(\sqrt{600})^2$
(e) $\sqrt{(3^2 + 4^2 + 12^2)}$ (f) $\sqrt{(17^2 - 9^2 - 8^2)}$

9 Two squares A and B have sides 3cm and 4cm as shown below. Square C has an area equal to the sum of the areas of A and B. Find the length of the side of square C.



12 PYTHAGORAS

12.1 Leading to Pythagoras



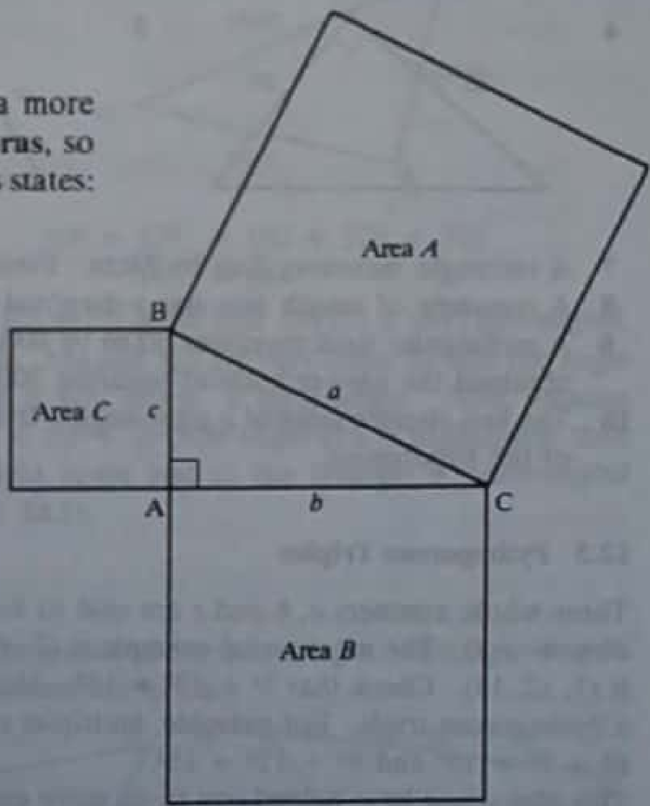
Construct accurately triangle ABC ($\triangle ABC$) with $AC = 4$cm, $\angle ACB = 90°$ and $BC = 3$cm (see 23.3). Measure the length of the **hypotenuse** AB. (The hypotenuse is the longest side of a right-angled triangle and is always opposite the right angle.) You should find that this is exactly 5cm. The figure shows the triangle you have drawn with a square on each of its sides. Each square is divided into centimetre squares. Count the number of cm. squares in the two smaller squares. There are 9 + 16 or 25. Now count the number of cm. squares in the square drawn on the hypotenuse. There are 25. Thus the sum of the areas of the two smaller squares equals the area of the large square. Note that the three numbers 9, 16 and 25 are square numbers (see 11.1 and 16.2) and that

$$3^2 + 4^2 = 5^2.$$

12.2 The Theorem of Pythagoras

The result obtained in **12.1** is a special case of a more general result known as **The Theorem of Pythagoras**, so named after the famous Greek mathematician. This states:

In any right-angled triangle, the square on the hypotenuse equals the sum of the squares on the other two sides.



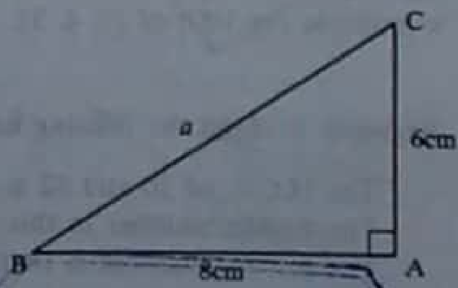
The figure on the right shows a right-angled triangle ABC with sides of length a, b, c and squares drawn on these sides. From the above statement:

$$\text{Area } A = \text{Area } B + \text{Area } C.$$

Now since Area $A = a^2$, Area $B = b^2$ and Area $C = c^2$, we have $a^2 = b^2 + c^2$.

The following examples show how this relation is used to determine one side of a right-angled triangle when the other two sides are given.

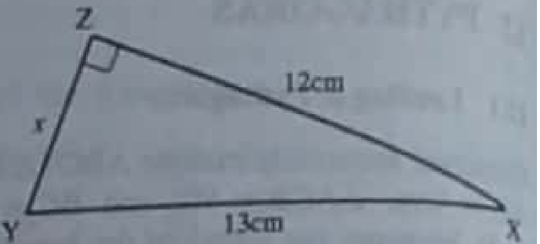
Example 1 In the right-angled triangle shown, find a.



In $\triangle ABC$, by Pythagoras

$$a^2 = 6^2 + 8^2 \quad \ldots\ldots \text{(i)}$$
$$= 36 + 64$$
$$= 100$$
$$\therefore \quad a = \sqrt{100} = 10\text{cm}$$

Example 2 In the right-angled $\triangle XYZ$, find x.



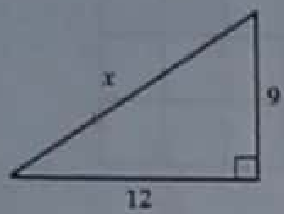
$$x^2 + 12^2 = 13^2 \quad \ldots\ldots \text{(i)}$$
$$\therefore \quad x^2 + 144 = 169$$
$$\therefore \quad x^2 = 169 - 144$$
$$= 25$$
$$\therefore \quad x = \sqrt{25} = 5\text{cm}$$

Note Take care at first to have the 'hypotenuse' measuremnt alone on one side of the 'equals' sign, as in lines (i) in Examples 1 and 2 above. For example, $x^2 = 12^2 + 13^2$ would be *wrong*.

Exercise 12a

In **1** to **6**, use Pythagoras to find the length of the unknown side. Dimensions are in cm.

1



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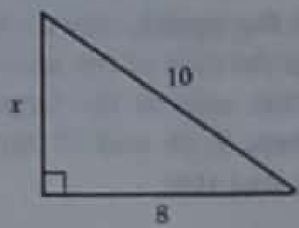

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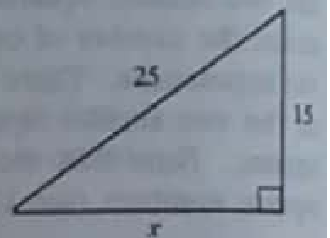

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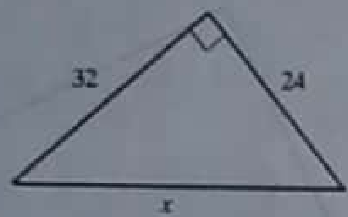

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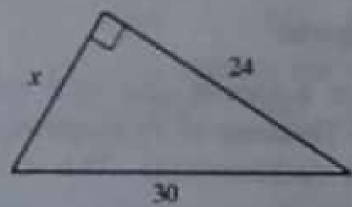

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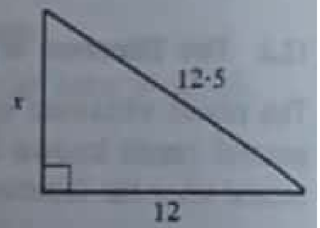


7 A rectangle measures 7cm by 24cm. Find the length of a diagonal.

8 A rectangle of length 6cm has a diagonal of length 6·5cm. Find the width.

9 A rectangular field measures 300m by 400m. Find the length of a diagonal. Could you have obtained the answer without squaring 300 and 400? (See **12.3**)

10 The two shorter sides of a right-angled triangle are of length 8cm and 15cm. Find the length of the hypotenuse.

12.3 Pythagorean Triples

Three whole numbers a, b and c are said to form a **Pythagorean triple** if they satisfy the relation $a^2 + b^2 = c^2$. The most useful example is (3, 4, 5) since $3^2 + 4^2 = 5^2$. Another common example is (5, 12, 13). Check that $5^2 + 12^2 = 13^2$. Also any multiple of a Pythagorean triple will itself be a Pythagorean triple. For example, multiples of (3, 4, 5) are (6, 8, 10) and (9, 12, 15). Check that $6^2 + 8^2 = 10^2$ and $9^2 + 12^2 = 15^2$.

This idea could have helped you to do more easily many of the questions of the previous Exercise. For example, in **Q.9**, look for the number which completes the triple (300, 400, ...). This is clearly a multiple (by 100) of (3, 4, 5). Therefore the missing number is $5 \times 100 = 500$.

Example Find the missing number in the Pythagorean triple (20, ..., 52).

The H.C.F. of 20 and 52 is 4, (see **1.1**), so (20, ..., 52) = 4 × (5, ..., 13).
The middle number in this last triple is known to be 12. (See above)
The missing number is therefore $4 \times 12 = 48$. Check that $20^2 + 48^2 = 52^2$.

Exercise 12b

For 1 to 12, find the missing number in each Pythagorean triple. Where possible, use the multiple idea to help you.

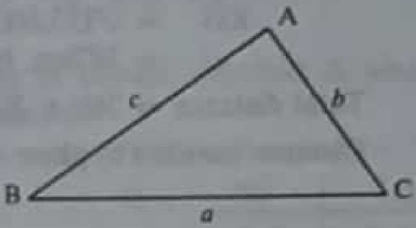
1 (3, 4, ...)	**2** (9, ..., 15)	**3** (..., 28, 35)	**4** (5, 12, ...)
5 (10, ..., 26)	**6** (..., 60, 65)	**7** (8, 15, ...)	**8** (24, ..., 51)
9 (..., 150, 170)	**10** (7, 24, ...)	**11** (14, ..., 50)	**12** (18, ..., 82)

In **13** to **16**, which are Pythagorean triples?

13 (12, 35, 37)	**14** (15, 16, 22)	**15** (17, 19, 26)	**16** (13, 84, 85)

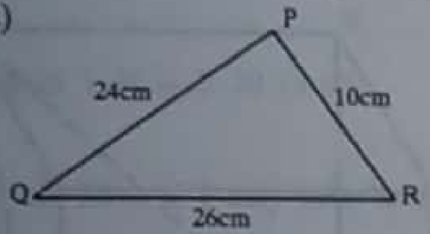
12.4 Converse of Pythagoras

This states that if $\triangle ABC$ (shown on the right) has $a^2 + b^2 = c^2$ then the triangle is right-angled at A. This idea can be used to test a triangle for a right angle.

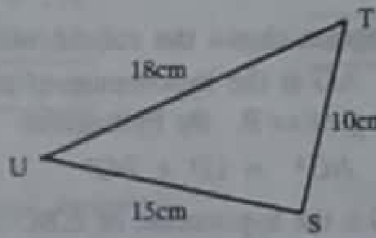


Example Test the following triangles for a right angle.

(a)



(b)



(a) $10^2 + 24^2 = 100 + 576 = 676$
$26^2 = 676$

$\therefore \quad 10^2 + 24^2 = 26^2$

Therefore the triangle is right-angled. The longest side (hypotenuse) is 26cm so the right angle is at P.

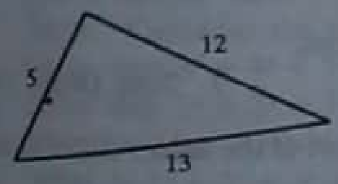
(b) $10^2 + 15^2 = 100 + 225 = 325$
$18^2 = 324$

So $10^2 + 15^2 \neq 18^2$ and $\triangle STU$ is not right-angled.
Note 324 and 325 are very close, so that the angle at S is *very nearly* a right angle. Also, because $18^2 < 10^2 + 15^2$, the angle at S is slightly less than a right angle and so the triangle is *acute-angled* (see **24.1**).

Exercise 12c

In each Question, test the triangle for a right angle. In a case where the triangle is not right-angled state whether it is acute- or obtuse-angled. Dimensions are in cm.

1



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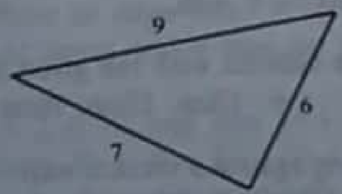

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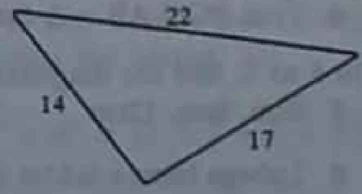

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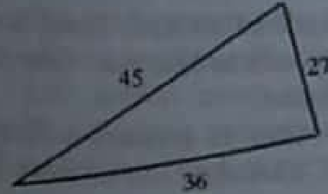

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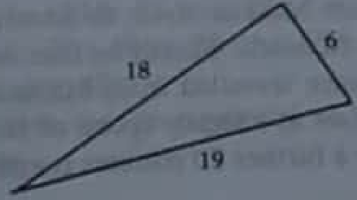

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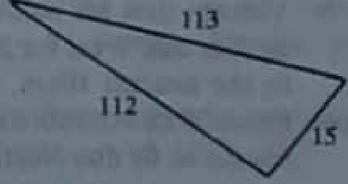
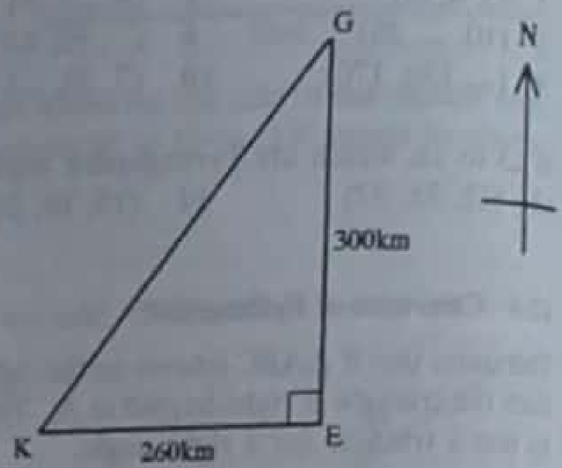


12.5 Further Applications

Example 1 A plane flies due East from Kasese to Entebbe, a distance of 260km. It then flies 300km due North to Gulu. Finally it flies directly back to Kasese. Calculate the distance travelled.

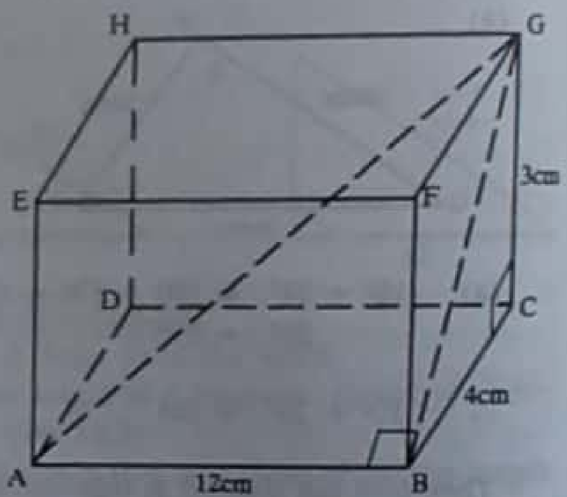


ΔKEG represents the path travelled by the plane.
The triangle is right-angled at E.
In ΔKEG, by Pythagoras

$$\begin{aligned} KG^2 &= 260^2 + 300^2 \\ &= 67{,}600 + 90{,}000 \\ &= 157{,}600 \\ \therefore \quad KG &= \sqrt{157{,}600} \\ &= 397\text{km, from square root tables} \end{aligned}$$

Total distance = 260 + 300 + 397 = 957
Distance travelled by plane = 960km (nearest 10km)

Example 2 Find the length of a diagonal of a cuboid measuring 12cm by 4cm by 3cm.



The diagram shows the cuboid with diagonal AG drawn. AG is the hypotenuse of ΔABG which is right-angled at B. By Pythagoras

$$AG^2 = 12^2 + BG^2$$

But BG is the hypotenuse of ΔBCG which is right-angled at C.
So $BG^2 = 3^2 + 4^2$ (*NB* do not calculate this yet)

$$\begin{aligned} AG^2 &= 12^2 + (3^2 + 4^2) \\ &= 144 + 9 + 16 \\ &= 169 \qquad \therefore AG = \sqrt{169} = 13 \end{aligned}$$

The length of the diagonal is 13cm.

Exercise 12d

1 to 4 refer to the right-angled triangle shown.

1. Find AC if AB = 6cm and BC = 5cm.
2. Find AB if AC = 2cm and BC = 1cm.
3. Find AC if AB = 1cm and BC = 1cm.
4. Find BC if AB = 6·8cm and AC = 9·4cm.

In 5 to 7, find the diagonals of the cuboids with the given dimensions.

5. 8cm, 9cm, 12cm
6. 12cm, 15cm, 16cm
7. 7cm, 14cm, 14cm
8. Lubega leans a ladder 6·5m long against a vertical wall. If the foot of the ladder is 2·2m from the wall, how far up the wall does it reach?
9. Vincent flies his helicopter from Masaka direct to Soroti, a distance of 300km. From Soroti he flies due West for 210km to Masindi. Finally he flies due South back to Masaka. Calculate, to the nearest 10km, the distance travelled from Masindi to Masaka.
10. From X an aircraft flies due East at a steady speed of 180km/h. After 45 minutes it changes course to fly due North. After a further 20 minutes it arrives at Y. Calculate the distance XY.

13 CUBES AND ROOTS

13.1 Cubes

If a number is multiplied together three times, for example $4 \times 4 \times 4$, the product 64 is the **cube** of that number. We write $4^3 = 64$, read as 'four cubed equals sixty-four'. Fractions and decimals are cubed in the same way.

Example Evaluate $1{\cdot}3^3$

Method 1 Long multiplication gives $1{\cdot}3^3 = 1{\cdot}3 \times 1{\cdot}3 \times 1{\cdot}3 = 1{\cdot}69 \times 1{\cdot}3 = 2{\cdot}197$

Method 2 Logarithms (see **15.4**) give two methods of working as shown. Method **B**, where the logarithm of 1·3 is simply multiplied by 3, is usually preferable.

A

No	Log
1·3	0·114
1·3	0·114
1·3	0·114
2·20	0·342

B

No	Log
1·3	0·114
	× 3
2·20	0·342

This gives $1{\cdot}3^3 = 2{\cdot}20$ (3 sf)

Note that *Method* 1 may take longer but gives the exact answer.

Exercise 13a

In **1** to **10** find the exact cube of each number.

1 2	**2** 3	**3** 5	**4** 6	**5** 7
6 8	**7** 9	**8** 10	**9** 15	**10** 2·3

In **11** to **20** find the cubes of the numbers correct to 3 sf.

11 12	**12** 1·2	**13** 120	**14** 0·12	**15** 37
16 5·8	**17** 16·5	**18** 4·46	**19** 0·893	**20** 0·142

21 Find the volume of a cube of edge 4·7cm to 3 sf.

22 A solid metal cube of edge 10cm is melted down to make some smaller cubes each of edge 3cm. How many such smaller cubes may be formed?

23 Jars of a certain brand of hair conditioner are contained in small boxes which are packed into a carton. Each box is a cube of edge 5cm. The carton is also a cube but of edge 30cm. Calculate the number of boxes which may be packed into the carton.

24 Show that $3^3 + 4^3 + 5^3 = 6^3$.

25 Show that $1^3 + 2^3 + 3^3 = (1 + 2 + 3)^2$ and that $1^3 + 2^3 + 3^3 + 4^3 = (1 + 2 + 3 + 4)^2$.
Use the same method to evaluate $1^3 + 2^3 + 3^3 + 4^3 + 5^3$.
What is $1^3 + 2^3 + 3^3 + ... + n^3$ by the same method?

26 The perfect numbers (see Ex **1a Q.14**) 28 and 496 can be given as the sum of the cubes of consecutive odd numbers, ie. $1^3 + 3^3 = 28$ and $1^3 + 3^3 + 5^3 + 7^3 = 496$. Express the next perfect number 8,128 in this way.

27 The smallest natural number that can be expressed as the sum of two cubes in *two* ways is 1,729. Find these two ways.

13.2 Cube Roots

Cube rooting is the reverse process of cubing. For example, to obtain the cube root of 125, written $\sqrt[3]{125}$, find the number which, when multiplied together three times, gives 125. We know that $5 \times 5 \times 5 = 125$. Therefore $\sqrt[3]{125} = 5$. Exact cube roots are found by use of factors, otherwise logarithms are used.

Example 1 Find the exact cube root of 5,832.

$$\begin{aligned} 5{,}832 &= 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \\ &= (2 \times 3 \times 3) \times (2 \times 3 \times 3) \times (2 \times 3 \times 3), \text{ forming three } \textit{equal} \text{ groups} \\ &= 18 \times 18 \times 18 \end{aligned}$$

Therefore $\sqrt[3]{5{,}832} = 18$

Example 2 Find $\sqrt[3]{18\cdot2}$ correct to 3 sf.

The logarithm of 18·2 is divided by 3 and the antilogarithm, 2·63, is found as shown.

$\sqrt[3]{18\cdot2} = 2\cdot63$ to 3 sf

No	Log
18·2	1·260
	÷ 3
2·63	0·420

Example 3 Find $\sqrt[3]{0\cdot628}$ to 3 sf.

The logarithm of 0·628 is $\bar{1}\cdot798$. To divide this by 3 we proceed as follows:

$\bar{1}\cdot798 = -1 + 0\cdot798 = -3 + 2\cdot798$

$$\begin{aligned} \text{So } \bar{1}\cdot798 \div 3 &= (-3 + 2\cdot798) \div 3 \\ &= -1 + 0\cdot933 \\ &= \bar{1}\cdot933 \end{aligned}$$

This is set out on the right.

So $\sqrt[3]{0\cdot628} = 0\cdot857$ to 3 sf

No	Log
0·628	$\bar{1}\cdot798$
0·877	$\bar{3}\cdot\overset{2}{7}98 \div 3$
0·857	$\bar{1}\cdot933$

Exercise 13b

In **1** to **5** find the cube roots of the numbers using factors.

1 216 **2** 512 **3** 729 **4** 3,375 **5** 74,088

In **6** to **15** find the cube roots of the numbers using logarithms.

6 6 **7** 10 **8** 100 **9** 58·2 **10** 187,000

11 0·436 **12** 0·0708 **13** 0·00496 **14** 0·000924 **15** 0·00001

16 Find the edge of a cube whose volume is 20cm³ to 3 sf.

17 Ten solid metal cubes of edge 2cm are melted down to form one large cube. Find the edge of the large cube.

18 A solid metal cube of edge 16cm is melted down to form two equal cubes. Find the edge of these cubes.

19 Use logarithms to calculate $\dfrac{6\cdot73 \times \sqrt[3]{0\cdot27}}{1\cdot82}$ to 3 sf.

20 Use logarithms to calculate $\sqrt[3]{\left(\dfrac{807}{0\cdot023}\right)}$ to 3 sf.

14 INDICES

14.1 Index Notation

The product $3 \times 3 \times 3 \times 3 \times 3$ may be written as 3^5, where 3 is called the **base** and 5 an **index** giving the number of times the base is multiplied. We read 3^5 as *three to the power five*. So $3^5 = 243$.

Similarly $2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$.

Generally $a^n = a \times a \times a \times \ldots \times a$ (n times) where n is a positive integer.

14.2 Laws of Indices

First Law What is $4^2 \times 4^3$?

$$4^2 \times 4^3 = (4 \times 4) \times (4 \times 4 \times 4) = 4 \times 4 \times 4 \times 4 \times 4 = 4^5$$

However, we could add the indices instead since the base, 4, is the same:

$$4^2 \times 4^3 = 4^{(2+3)} = 4^5$$

Generally $\boldsymbol{a^m \times a^n = a^{m+n}}$

Second Law What is $6^7 \div 6^5$?

$$6^7 \div 6^5 = \frac{6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6}{6 \times 6 \times 6 \times 6 \times 6} = 6 \times 6 = 6^2$$

This could be worked out by subtracting the indices:

$$6^7 \div 6^5 = 6^{(7-5)} = 6^2$$

Generally $\boldsymbol{a^m \div a^n = a^{m-n}}$

Third Law What is $(5^2)^3$?

$$(5^2)^3 = 5^2 \times 5^2 \times 5^2 = (5 \times 5) \times (5 \times 5) \times (5 \times 5) = 5^6$$

This is worked out by multiplying the indices:

$$(5^2)^3 = 5^{(2 \times 3)} = 5^6$$

Generally $\boldsymbol{(a^m)^n = a^{mn}}$

Other useful relations are $a^n \times b^n = (ab)^n$ and $a^n \div b^n = (a \div b)^n$

Example Evaluate $\left(\dfrac{5^5 \times 5^4 \times 2^{12}}{5^6 \times 2^7 \times 2^2}\right)^2$

$$\left(\frac{5^5 \times 5^4 \times 2^{12}}{5^6 \times 2^7 \times 2^2}\right)^2 = \left(\frac{5^9 \times 2^{12}}{5^6 \times 2^9}\right)^2 \text{ using } \textit{First Law}$$

$= (5^3 \times 2^3)^2$ using *Second Law*

$= (10^3)^2$ using $a^n \times b^n = (ab)^n$

$= 10^6$ using *Third Law*

$= 1{,}000{,}000$

Exercise 14a

In **1** to **15** evaluate the expressions.

1 2^{10} **2** 10^7 **3** 5^4 **4** $2^{15} \div 2^{12}$ **5** $5^6 \times 2^6$

6 $20^5 \div 2^5$ **7** $(-2)^6$ **8** $(-3)^5$ **9** $\left(\frac{1}{3}\right)^4$ **10** $\left(1\frac{1}{2}\right)^3$

11 $(2^3)^3$ **12** $2^{3^3} \div 2^{5^2}$ **13** $\dfrac{2^7 \times 10^8}{5^4 \times 2^{11}}$

14 $\dfrac{6^{10} \times 10^6}{2^{16} \times 3^{10} \times 5^6}$ **15** $\dfrac{2^{10} \times 3^{11} \times 5^{12}}{30^{10}}$

In **16** to **18** simplify the expression as far as possible.

16 $\dfrac{a^7b^8}{(ab^2)^3}$ **17** $\dfrac{x^{10}}{y^3} \div \left(\dfrac{x}{y}\right)^9$ **18** $\left(\dfrac{3m}{2n}\right)^7 \times \left(\dfrac{4n^2}{3m}\right)^5$

14.3 Zero and Negative Indices

What is the meaning of 4^0? We may write 0 as, for example $3 - 3$.

Therefore $4^0 = 4^{3-3} = 4^3 \div 4^3$ (from *Second Law*) $= \dfrac{4 \times 4 \times 4}{4 \times 4 \times 4} = 1$

In general $a^0 = 1$

What is the meaning of 4^{-3}? We may write -3 as, for example, $2 - 5$.

So $4^{-3} = 4^{2-5} = 4^2 \div 4^5$ (from *Second Law*) $= \dfrac{4 \times 4}{4 \times 4 \times 4 \times 4 \times 4} = \dfrac{1}{4 \times 4 \times 4} = \dfrac{1}{4^3}$ (ie. $\frac{1}{64}$)

In general $a^{-m} = \dfrac{1}{a^m}$ Similarly $\dfrac{1}{a^{-m}} = a^m$ and $\left(\dfrac{a}{b}\right)^{-m} = \left(\dfrac{b}{a}\right)^m$

Example Evaluate $6^{-3} \times \left(\frac{2}{3}\right)^{-4} \div 2^{-1}$

$$6^{-3} \times \left(\tfrac{2}{3}\right)^{-4} \div 2^{-1} = \tfrac{1}{6^3} \times \left(\tfrac{3}{2}\right)^4 \div \tfrac{1}{2} \quad \text{changing to positive indices}$$

$$= \frac{1}{2^3 \times 3^3} \times \frac{3^4}{2^4} \times 2 = \frac{3^1}{2^6} = \frac{3}{64}$$

Exercise 14b

Evaluate the expressions in **1** to **10**.

1 3^{-1}

2 2^{-3}

3 $\dfrac{1}{2^{-2}}$

4 $\left(\frac{3}{5}\right)^{-2}$

5 6×3^{-2}

6 5^0

7 $(3^{-2})^2 \div 2^{-6} \times \left(\frac{2}{3}\right)^{-5}$

8 $\left(-\frac{1}{2}\right)^{-6}$

9 $(-2)^{-10} \times \left(\frac{1}{4}\right)^{-5} \div \dfrac{1}{10^{-2}}$

10 $6^9 \times (-8)^{-7} \times \left(1\frac{1}{3}\right)^{-10}$

Simplify the expessions in **11** to **15**.

11 $a^5 \times a^2$

12 $2c^{-2} \times 3c^{-3}$

13 $8e^{-3} \div (2e)^{-4}$

14 $(3x^{2}y^{-3})^2 \div \left(\frac{x}{y}\right)^5$

15 $\left(\dfrac{h^3}{k^3}\right)^0 \times \left(\dfrac{k}{h}\right)^{-1}$

14.4 Fractional Indices

To give a meaning to a number raised to a fractional index, such as $9^{\frac{1}{2}}$, we apply the *Laws of Indices* as shown below.

What is $9^{\frac{1}{2}}$? If we multiply $9^{\frac{1}{2}}$ by itself this gives $9^{\frac{1}{2}} \times 9^{\frac{1}{2}} = 9^{\left(\frac{1}{2}+\frac{1}{2}\right)} = 9^1 = 9$.

Therefore $9^{\frac{1}{2}} = \sqrt{9}$ (which equals 3).

What is $8^{\frac{1}{3}}$? Cubing $8^{\frac{1}{3}}$ gives $(8^{\frac{1}{3}})^3 = 8^{\frac{1}{3}\times 3} = 8^1 = 8$.

Therefore $8^{\frac{1}{3}} = \sqrt[3]{8}$ (which equals 2).

Similarly $16^{\frac{1}{4}} = \sqrt[4]{16} = 2$. Note $\sqrt[4]{16}$ is read as *the fourth root of 16.*

Generally $a^{\frac{1}{n}} = \sqrt[n]{a}$, the nth root of a, (ie. the number which multiplied n times gives a).

What is $8^{\frac{2}{3}}$?

Method 1 $\quad 8^{\frac{2}{3}} = 8^{\frac{1}{3}\times 2} = (8^{\frac{1}{3}})^2 = (\sqrt[3]{8})^2 = 2^2 = 4$

Method 2 $\quad 8^{\frac{2}{3}} = 8^{2\times\frac{1}{3}} = (8^2)^{\frac{1}{3}} = \sqrt[3]{(8^2)} = \sqrt[3]{64} = 4$

The first method leads to simpler calculation.

So $8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = \sqrt[3]{(8^2)}$

Generally $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{(a^m)}$ where m and n are natural numbers.

Example $\quad$ Evaluate $\left(\frac{8}{27}\right)^{-\frac{2}{3}}$

$$\left(\tfrac{8}{27}\right)^{-\frac{2}{3}} = \left(\tfrac{27}{8}\right)^{\frac{2}{3}} = \sqrt[3]{\left(\tfrac{27}{8}\right)}^2 = \left(\tfrac{3}{2}\right)^2 = \tfrac{9}{4}$$

Exercise 14c

Evaluate the expressions in 1 to 13.

1 $16^{\frac{1}{2}}$ $\quad$ 2 $125^{\frac{1}{3}}$ $\quad$ 3 $81^{\frac{1}{4}}$ $\quad$ 4 $32^{\frac{1}{5}}$ $\quad$ 5 $(27^2)^{\frac{1}{6}}$

6 $16^{\frac{3}{4}}$ $\quad$ 7 $9^{\frac{3}{2}}$ $\quad$ 8 $4^{\frac{7}{2}}$ $\quad$ 9 $\sqrt[3]{27^4}$ $\quad$ 10 $4^{-\frac{1}{2}}$

11 $\left(3\frac{3}{8}\right)^{-\frac{2}{3}}$ $\quad$ 12 $2^{\frac{1}{3}} \times 4^{\frac{1}{3}}$ $\quad$ 13 $3^{\frac{3}{2}} \times 12^{-\frac{1}{2}}$

Simplify the expressions in 1 to 18.

14 $(xy)^{\frac{1}{2}} \times \left(\frac{x}{y}\right)^{\frac{1}{2}}$ $\quad$ 15 $(8a^2)^{\frac{1}{3}}$

16 $-16x^4 \times (4x^2)^{-\frac{3}{2}}$ $\quad$ 17 $\sqrt{(9h^4)} \div h^2$

18 $\sqrt[3]{(8p^2q^2)} \times (pq)^{\frac{1}{3}} \div \sqrt[3]{p}$

15 LOGARITHMS

15.1 Standard Form

A number written in the form $A \times 10^n$, where $1 \leq A < 10$ and n is an integer, is said to be in **standard form**. Thus 343 expressed in standard form is $3{\cdot}43 \times 10^2$.
Also $0{\cdot}236 = 2{\cdot}36 \div 10 = 2{\cdot}36 \times 10^{-1}$ which is in standard form.

Exercise 15a

Express the following in standard form.

1 42	**2** 873	**3** 0·4	**4** 0·067	**5** 5,070
6 0·0005806	**7** 39×10^2	**8** 10^5	**9** 3	**10** $0{\cdot}016 \times 10^4$

15.2 Logarithms to Base 10

A **logarithm** is a power or index. Consider the relation $1{,}000 = 10^3$. We say 3 is the logarithm of 1,000, base 10. This is written as

$$\log_{10} 1{,}000 = 3$$

Generally, if $a = 10^c$ then $\log_{10} a = c$. ***In this book all logarithms are base 10*** and we shall write $\log_{10} a$ as simply $\log a$.
The logarithm of a number is the power to which 10 must be raised to give the number. For example, $\log 10{,}000 = 4$ since 10 must be raised to the power 4 to give 10,000.
Logarithms of numbers which are not integral powers of 10 are found from tables of logarithms. These give the logarithms of numbers between 1 and 10, correct to 3 sf. For example, $\log 3{\cdot}86 = 0{\cdot}587$ (ie. $3{\cdot}86 = 10^{0{\cdot}587}$). Logarithms of other numbers are found by first expressing them in standard form.

Example 1 Find the logarithm of 47·2.

$$\begin{aligned} 47{\cdot}2 &= 4{\cdot}72 \times 10^1 && \text{using standard form} \\ &= 10^{0{\cdot}674} \times 10^1 && \text{from tables of logarithms} \\ &= 10^{1{\cdot}674} && \text{using the } \textit{First Law} \text{ of indices (see 14.2)} \end{aligned}$$

So $\log 47{\cdot}2 = 1{\cdot}674$

In the logarithm 1·674, the integer 1 is called the **characteristic** while the part following the decimal point, ·674, is the **mantissa**. The mantissa is always positive but the characteristic may be negative as the following example shows.

Example 2 Find log 0·0538

$$\begin{aligned} 0{\cdot}0538 &= 5{\cdot}38 \times 10^{-2} && \text{using standard form} \\ &= 10^{0{\cdot}731} \times 10^{-2} && \text{from tables} \\ &= 10^{-2 + 0{\cdot}731} && \text{using the } \textit{First Law} \text{ of indices} \end{aligned}$$

Although $-2 + 0{\cdot}731 = -1{\cdot}269$, we shall write it as $\bar{2}{\cdot}731$ in order to keep the mantissa ·731 positive. Only the characteristic -2 is negative as the minus sign above the 2 shows.

So $\log 0{\cdot}0538 = \bar{2}{\cdot}731$ which is read as *bar two point seven three one*.

Exercise 15b

In **1** to **10** find mentally and write down the logarithm of each number.

1 100	**2** 100,000	**3** 10	**4** 1	**5** $\frac{1}{10}$
6 $\frac{1}{100}$	**7** 10^7	**8** 10^{-4}	**9** 0·001	**10** $\sqrt{10}$

In **11** to **20** use tables to find the logarithms of each number.

11 5	**12** 8·4	**13** 4·27	**14** 26·3	**15** 278
16 576,000	**17** 0·345	**18** 0·0407	**19** 0·00598	**20** 0·000608

15.3 Antilogarithms

To find a number whose logarithm is given, we use the reverse process of finding a logarithm. For this purpose tables of **antilogarithms** are used. Alternatively a table of logarithms (as given in this book) may be used in reverse.

Example Find the number whose logarithm is $\bar{3}\cdot236$

Look up the number corresponding to the mantissa ·236 in tables of antilogarithms and then use the characteristic -3 to place the decimal point.

The antilogarithm of $\bar{3}\cdot236 = 1\cdot72 \times 10^{-3} = 0\cdot00172$

Exercise 15c

Find the antilogarithms of the following.

1 0·473	**2** 0·911	**3** 0·096	**4** 1·230	**5** 2·807
6 $\bar{1}\cdot512$	**7** $\bar{2}\cdot123$	**8** 4·600	**9** $\bar{3}\cdot928$	**10** $\bar{4}\cdot869$

15.4 Multiplication

Logarithms are used to work multiplications such as $4\cdot39 \times 87\cdot6$:

$4\cdot39 \times 87\cdot6 = 10^{0\cdot642} \times 10^{1\cdot943}$ from logarithm tables
$= 10^{0\cdot642 + 1\cdot943}$ using the *First Law* of indices
$= 10^{2\cdot585}$ using addition
$= 385$ from antilogarithm tables

Number	Logarithm
4·39	0·642
87·6	1·943
385	2·585

Long multiplication gives $4\cdot39 \times 87\cdot6 = 384\cdot564$, so the above answer is correct only to 3 sf. The above working is set out in vertical form as shown on the right.

Rule To **multiply** two numbers **add** their logarithms.

Example Use logarithms to calculate $0\cdot0782 \times 0\cdot647$.

No	Log
0·0782	$\bar{2}\cdot893$
0·647	$\bar{1}\cdot811$
0·0506	$\bar{2}\cdot704$

$0\cdot0782 \times 0\cdot647 = 0\cdot0506$ (3 sf)

Note the addition of characteristics:
$\bar{2} + \bar{1} + 1\text{(carried)} = (-2) + (-1) + 1 = -2 = \bar{2}$

15.5 Division

A division such as 184 ÷ 38·9 can be worked as follows:

$$184 \div 38{\cdot}9 = 10^{2{\cdot}265} \div 10^{1{\cdot}590} \quad \text{from logarithm tables}$$
$$= 10^{2{\cdot}265 - 1{\cdot}590} \quad \text{using the } \textit{Second Law} \text{ of indices}$$
$$= 10^{0{\cdot}675} \quad \text{using subtraction}$$
$$= 4{\cdot}73 \quad \text{from antilogarithm tables}$$

No	Log
184	2·265
38·9	1·590
4·73	0·675

This is set out in vertical form as shown.

Rule To *divide* one number by another *subtract* their logarithms.

Example Calculate 0·25 ÷ 0·877

0·25 ÷ 0·877 = 0·285 (3sf)

No	Log
0·25	$\bar{1}{\cdot}398$
0·877	$\bar{1}{\cdot}943$
0·285	$\bar{1}{\cdot}455$

Note the subtraction of characteristics:

$\bar{1} - 1(\text{borrowed}) - \bar{1} = (-1) - 1 - (-1) = -2 + 1 = -1 = \bar{1}$

Exercise 15d

Use logarithms to calculate the following to 3sf.

1 39·4 × 1·95 **2** 39·4 ÷ 1·95 **3** 681 × 8·47
4 681 ÷ 8·47 **5** 0·325 × 0·227 **6** 0·325 ÷ 0·227
7 0·325 × 0·914 **8** 0·325 ÷ 0·914 **9** 0·529 × 0·0806
10 0·529 ÷ 0·0806 **11** 5·721 × 612·9 **12** 5·721 ÷ 612·9

15.6 Reciprocals

Two numbers are **reciprocals** if their product is equal to 1. If $a \times b = 1$ then a is the reciprocal of b and vice versa. For example, the reciprocal of 5 is $\frac{1}{5}$ or 0·2, since 5 × 0·2 = 1. The reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$ or $1\frac{1}{3}$.

Reciprocal tables are used to obtain reciprocals of numbers given in decimal form. The tables give reciprocals for numbers from 1 to 10. However, reciprocals of other numbers such as 84·3 and 0·0454 can be found by first writing the number in standard form as follows.

$$84{\cdot}3 = 8{\cdot}43 \times 10^1$$

So the reciprocal of 84·3 is 0·119 (from tables) × 10^{-1} (change the sign of the index), ie. 0·0119.

$$0{\cdot}0454 = 4{\cdot}54 \times 10^{-2}$$

So the reciprocal of 0·0454 = 0·220 × 10^2 = 22·0.

Exercise 15e

In **1** to **10** *write down* the reciprocals.

1 4 **2** 100 **3** $\frac{1}{2}$ **4** 0·8 **5** $\frac{2}{3}$
6 $3\frac{1}{3}$ **7** 0·0001 **8** −10 **9** 1 **10** 0

In **11** to **15** use tables to find the reciprocals.

11 2·57 **12** 67·1 **13** 543 **14** 0·926 **15** 0·0073

In **16** to **20** calculate using reciprocal tables.

16 $\frac{1}{4{\cdot}56}$ **17** $\frac{1}{0{\cdot}237}$ **18** $\frac{10}{517}$ **19** $\frac{1000}{0{\cdot}00042}$ **20** $\frac{2}{781}$

16 NUMBER PATTERNS

16.1 Sequences

A **sequence** is a list of numbers usually forming a regular pattern.
Examples are:

$1, 2, 3, 4, 5, 6, 7, 8$
$2, 4, 6, 8, 10, 12, \ldots, 40$
$3, 2, 1, 0, -1, -2, -3, -4, \ldots$
$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$

Each number is called a **term**. The dots (. . .) indicate missing terms which may be found by studying the pattern. The first two sequences **terminate**, their last terms being 8 and 40 respectively. The other sequences do not terminate as indicated by the dots. Their **next terms** are -5 and $\frac{1}{7}$ respectively.
Note that the numbers of a sequence may be considered as members of a set. For example, $\{1, 3, 5, 7, 9, 11, \ldots\}$ shows the sequence of odd numbers.

Exercise 16a

In **1** to **18** write down the next two terms of the given sequence.

1 $1, 3, 5, 7, 9, 11, \ldots$
2 $1, 4, 7, 10, 13, 16, \ldots$
3 $6, 10, 14, 18, 22, \ldots$
4 $0, 9, 18, 27, 36, \ldots$
5 $4, 2, 0, -2, -4, -6, \ldots$
6 $9, 7, 5, 3, 1, \ldots$
7 $250, 200, 150, 100, 50, \ldots$
8 $1, -2, 3, -4, 5, -6, \ldots$
9 $1, 2, 4, 8, 16, 32, \ldots$
10 $2, 3, 5, 9, 17, 33, \ldots$
11 $1, 4, 9, 16, 25, \ldots$
12 $1, 8, 27, 64, 125, \ldots$
13 $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$
14 $6, 3, 1\frac{1}{2}, \frac{3}{4}, \ldots$
15 $1\frac{5}{6}, 1\frac{1}{2}, 1\frac{1}{6}, \frac{5}{6}, \frac{1}{2}, \ldots$
16 $-3, -\frac{1}{2}, 2, 4\frac{1}{2}, \ldots$
17 $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$
18 $1, 1, 2, 3, 5, 8, 13, \ldots$

In **19** to **26** write down the missing terms (—) in the given sequence.

19 $0, 2, 4, —, —, 10$
20 $0, 3, —, 9, —, 15$
21 $—, —, -1, -2, -3, -4, -5$
22 $\frac{1}{2}, \frac{2}{3}, —, \frac{4}{5}, \frac{5}{6}, —$
23 $6, —, 24, —, 96, 192$
24 $—, -1, -7, —, -19, -25$
25 $\frac{2}{3}, 1, 1\frac{1}{2}, —, —, 5\frac{1}{16}$
26 $1, 2, 6, 24, 120, —, 5040, —$

16.2 Number Patterns

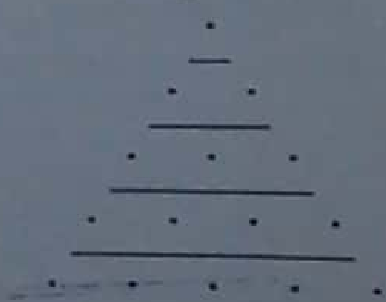
The following dot patterns represent **square, rectangle** and **triangle numbers** respectively.


$4 \times 4 = 16$


or

$3 \times 5 = 15$

(Consider these to be the same)


$1+2+3+4+5 = 15$

These patterns show that 16 is a square number, 15 is a rectangle (or **composite**) number and that 15 is also a triangle number. Note how the triangular pattern is built of rows containing 1, 2, 3, 4 and 5 dots. The sequence of square numbers is 1, 4, 9, 16, 25, 36, 49, . . . and the sequence of triangle numbers is 1, 3, 6, 10, 15, 21, 28, . . .
A rectangle number is any natural number (see 1.1) which is not prime. The sequence of rectangle numbers 1, 4, 6, 8, 9, 10, 12, 14, . . . has no regular pattern.

Exercise 16b

1 Write down the first twelve members of {square numbers}.

2 Write down the first twelve members of {triangle numbers}.

3 Write down the first twenty members of {rectangle numbers}.

4 Find the first two numbers which are both square and triangular numbers, ie. the first two members of the set {square numbers} ∩ {triangle numbers}.
The nth triangle number is given by $\frac{1}{2}n(n + 1)$. Use this to find the next member of {square numbers} ∩ {triangle numbers}.

5 The number pattern shown below is known as **Pascal's triangle**. The arrows show how each number is obtained by adding two numbers in the row above.
The next row is: 1, 6, 15, 20, 15, 6, 1

(i) Write down the next three rows.

(ii) What sequence is formed by the numbers in bold type: 1, 3, 6, . . .?

(iii) Give the next two terms of the sequences
1, 4, 10, 20, 35, 56, . . .
and
1, 5, 15, 35, 70, 126 . . .

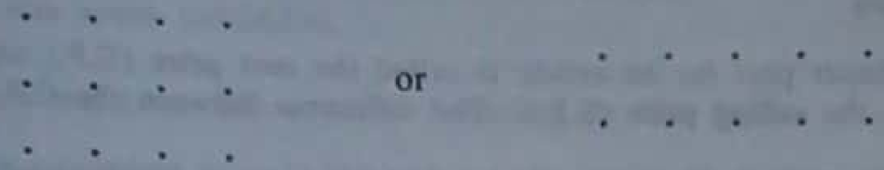
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
. . . .
. . . .

6 Consider the sequence of odd numbers {1, 3, 5, 7, 9, 11, 13, . . .}. Copy and complete the following table:

odd numbers	total
1	1
1 + 3	4
1 + 3 + 5	9
1 + 3 + 5 + 7	.
.	.
.	.
.	.
1 + 3 + 5 + . . . + 19	

What do you notice about the totals?

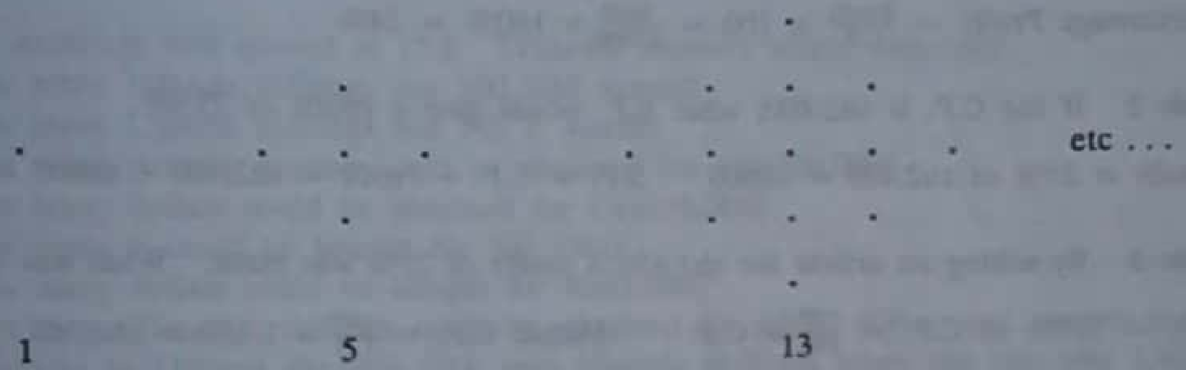
7 Some rectangle numbers can be represented in more than one way. For example, 12 can be represented in two different ways.



Note that a single row of twelve dots is not considered to be a rectangle.
In how many different ways can the following be represented as rectangle numbers?

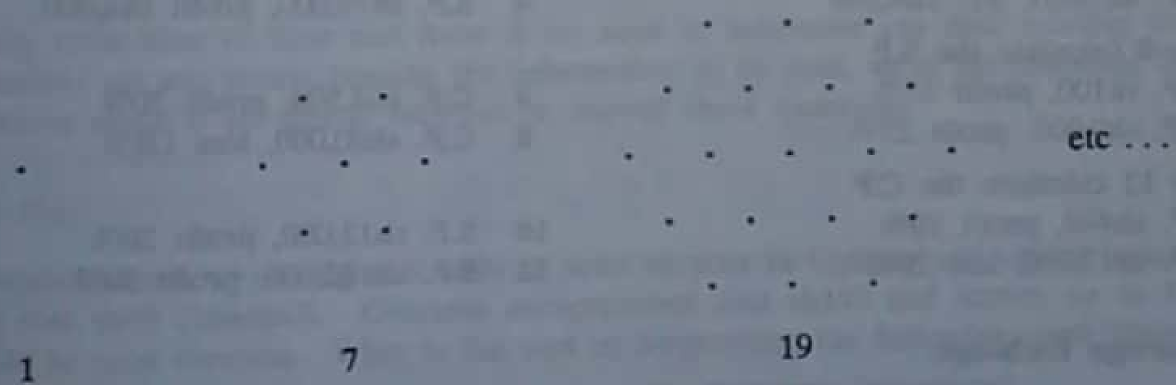
(i) 20	(ii) 24	(iii) 28	(iv) 36	(v) 45
(vi) 100	(vii) 120	(viii) 128	(ix) 192	(x) 221

8 The dot pattern for the sequence of diamond numbers 1, 5, 13, . . . is as follows:



What are the next two terms of this sequence?

9 The dot pattern for the sequence of hexagonal numbers 1, 7, 19, . . . is as follows:



What are the next two terms of this sequence?

17 COMMERCE

17.1 Profit and Loss

The price a shopkeeper pays for an article is called the **cost price** (C.P.) while the price at which he sells it is the **selling price** (S.P.). The difference between the C.P. and S.P. is the **profit** (or **loss**).

NB The percentage profit (or loss) is always calculated as a percentage of the *cost price*.

$$\text{Percentage Profit (or loss)} = \frac{\text{Profit (or loss)}}{\text{C.P.}} \times 100\%$$

Example 1 An article with C.P. sh15,000 is sold at a S.P. of sh18,600. What is the percentage profit?

Profit = S.P. − C.P. = sh(18,600 − 15,000) = sh3,600

$$\text{Percentage Profit} = \frac{\text{Profit}}{\text{C.P.}} \times 100 = \frac{3600}{15000} \times 100\% = 24\%$$

Example 2 If the C.P. is sh2,400, what S.P. would give a profit of 25%?

Profit = 25% of sh2,400 = sh600 S.P. = C.P. + Profit = sh2,400 + sh600 = sh3,000

Example 3 By selling an article for sh1,450 a profit of 25% was made. What was the C.P.?

S.P. = 125% of C.P. = $\frac{125}{100} \times$ C.P. Hence C.P. = $\frac{100}{125} \times 1{,}450$ = sh1,160

Exercise 17a

In **1** to **4** find the percentage profit or loss.

1 C.P. sh1,200, S.P. sh1,500
2 C.P. sh2,000, profit sh300
3 C.P. sh7,000, S.P. sh4,200
4 S.P. sh79,000, profit sh4,000

In **5** to **8** calculate the S.P.

5 C.P. sh100, profit 15%
6 C.P. sh4,500, profit 20%
7 C.P. sh9,000, profit 25%
8 C.P. sh90,000, loss $12\frac{1}{2}$%

In **9** to **12** calculate the C.P.

9 S.P. sh440, profit 10%
10 S.P. sh12,000, profit 20%
11 S.P. sh12,000, loss 20%
12 S.P. sh182,000, profit 30%

17.2 Foreign Exchange

The following is a simplified extract from the exchange rates given on a certain day in the National Newspapers.

Currency	Mean Rate in Ush
1 US dollar ($)	970
1 Sterling pound (£)	1,650
1 French franc (F)	160
1 Deutsche mark (DM)	530
100 Kenya shillings (Ksh)	3,100
100 Tanzania shillings (Tsh)	390

Its use is explained in the following examples.

Example 1 How many Uganda shillings could \$25 buy?

Since \$1 = Ush970, then \$25 = 25 × 970 = Ush24,250
Hence \$25 was worth Ush24,250.

Example 2 How much was \$25 worth in £ sterling on that day?

Since \$25 is equivalent to sh24,250 and sh1,650 equals £1 sterling, we have

$$\$25 = £(24{,}250 \div 1{,}650) = £14{\cdot}7 \text{ (3 sf, using logarithms)}$$

\$25 was worth about £14.

Note when giving the answer in the above Example we have **truncated** 14·7 to 14. (*Truncated* here means rounding down to the nearest whole number below the original value.)

Exercise 17b

Use the exchange rate quoted in **17.2**. Truncate answers where necessary.

1 How many Uganda shillings was 100 DM worth?
2 How many Uganda shillings was 100 F worth?
3 How many pounds sterling could be obtained for Ush100,000?
4 How many dollars could be obtained for Ush150,000?
5 How many F could be bought for 100 DM?
6 How many dollars could be bought for Ksh1,000?
7 How many Tanzania shillings could be obtained for £200? (Give your answer to 3 sf.)
8 A tourist to Uganda changed £100 into Uganda shillings when the rate was 1,400. He spent sh80,000 and changed the remainder to £ when the rate was 1,600. How many pounds sterling did he receive?

17.3 Postal Rates, Electricity Charges

These vary from time to time and there is no need to memorise any data relating to either. Any questions set will always contain the information to be used, as in the following Exercise, and common sense is all that is required to answer these questions.

Exercise 17c

1 Unsealed letters up to 10g cost sh40 to send by post in Uganda and sealed letters up to 10g cost sh60 (internal). Overseas aerogrammes cost sh140 and letters up to 10g cost sh240 to send overseas. What is the cost of despatching the following (each under 10g)?

43 unsealed letters (internal)	24 aerogrammes (international)
17 sealed letters (internal)	14 letters (international)

2 My electricity bill is made up as follows:– meter rent per month sh500; first 20 units cost sh5 per unit and the rest cost sh10 per unit; each unit consumed is subject to sh1 tax. The meter read 210429 units at the beginning of the month and then 210854 at the end. What was the cost of my electricity for this month?

17.4 Commission

When goods are sold by one person on behalf of another a **commission** may be earned. This is usually stated as a percentage of the value of the goods sold.

Example A salesman sells 550 articles for sh1,200 each and receives 5% commission on the first sh500,000 worth and $7\frac{1}{2}$% on the rest. How much commission does he earn?

550 articles sold @ sh1,200 = 550 × 1,200 = sh660,000. For the first sh500,000 he earns 5% and on the remaining sh160,000, $7\frac{1}{2}$%.

$$\text{Commission} = 500{,}000 \times \frac{5}{100} + 160{,}000 \times \frac{7.5}{100} = 25{,}000 + 12{,}000 = \text{sh}37{,}000$$

Exercise 17d

1 Fred gets $3\frac{1}{2}$% commission for the first sh500,000 of goods he sells and 4% for the rest. What commission does he earn when his sales amount to sh750,000?
2 George sells 450 articles for sh1,600 each and gets 4% commission on the first sh600,000 and 5% on the rest. How much commission does he earn?
3 Hezbon gets sh35,000 commission for selling goods worth sh700,000. What was the single percentage rate of commission?
4 If the commission earned in 3 was 4% for the first sh500,000 of sales, what was the rate of commission on the rest?
5 Juanita sells goods worth sh950,000. Her total commission earned was sh45,000. If the commission on the first sh500,000 was half that earned on the rest, what were the two rates of commission?

17.5 Discount and Bills

A discount means you pay *less*. Suppose you were given a 10% discount when paying a bill of sh9,500 you would pay sh9,500 × $\frac{10}{100}$ less or sh950. Hence you pay 9,500 − 950 = sh8,550.

Exercise 17e

1 Gertrude buys the following items:-

6 hankies	at sh250 each	2 dresses	at sh12,000 each
4 face flannels	at sh400 each	1 bath towel	at sh4,000
2 blouses	at sh3,500 each	2 hand towels	at sh1,500 each

Make out a bill, give a 10% discount and hence determine how much she paid.
2 Gladys received a bill for sh70,000 but only paid sh61,250. What percentage discount was she given?
3 Grace paid sh15,300 after being given a 15% discount. What was her original bill?

17.6 Simple Interest

If you pay money into a building society or a bank deposit account you receive **interest**. If you borrow money you will pay interest. Interest is payment for the use of money deposited or borrowed, called the **principal** and is calculated as a percentage of the principal over an agreed period of time. If you deposit sh100,000 in a bank which pays 19% interest per annum (p.a.), then you will receive sh100,000 × $\frac{19}{100}$ = sh19,000 after one year. However, if you borrow sh100,000 under the same terms you will pay the bank sh19,000. If the interest is withdrawn (or paid) at the end of the year then the principal remains the same. This is an example of **simple interest**. If sh100,000 is invested for 5 years at 19% p.a. simple interest this is sh19,000 per year or sh19,000 × 5 = sh95,000. Thus the simple interest earned over the 5 years is

$$\text{sh}100{,}000 \times \frac{19}{100} \times 5 = \text{sh}95{,}000$$

If P = Principal, R = Rate (percent), T = Time then I the interest is given by $I = \frac{PRT}{100}$
To find the Principal, Rate and Time use $P = \frac{100I}{RT}$, $R = \frac{100I}{PT}$ and $T = \frac{100I}{PR}$ respectively.

Example What sum of money invested for 5 years at $7\frac{1}{2}\%$ p.a. yields sh150,000 simple interest? What is the **amount** at the end of this period?

$$P = \frac{100I}{RT} = \text{sh } \frac{100 \times 150000}{7{\cdot}5 \times 5} = \text{sh}400{,}000$$

$$\text{Amount } (A) = P + I = \text{sh}(400{,}000 + 150{,}000) = \text{sh}550{,}000$$

Exercise 17f

1 If P = sh100,000, T = 4 years and R = 8% p.a., find I.
2 If I = sh4,500, T = 3 years and $R = 7\frac{1}{2}\%$ p.a., find P.
3 If P = sh800,000 T = 6 months and I = sh40,000, find R.
4 If P = sh10,000,000, R = 11% p.a. and I = sh2,200,000, find T.
5 If T = 9 months, I = sh150,000 and $R = 12\frac{1}{2}\%$ p.a., find P and A.

17.7 Compound Interest

With **compound interest,** the interest is added to the principal each year, or other agreed period of time. The new principal is used to calculate the interest in the next year.

Example Calculate the amount after two years when sh100,000 is invested at 10% p.a. compound interest. Find also the total interest earned.

First year: Principal = sh100,000

Interest $= \text{sh}100{,}000 \times \frac{10}{100} = \text{sh}10{,}000$

Second year: Principal = sh(100,000 + 10,000) = sh110,000

Interest $= \text{sh}110{,}000 \times \frac{10}{100} = \text{sh}11{,}000$

Amount after 2 years = sh(110,000 + 11,000) = sh121,000

Interest earned = sh(121,000 − 100,000) = sh21,000

Exercise 17g

In 1 to 4 calculate the amount obtained by making the stated investment. Find also the total interest earned.

1 Sh200,000 for 2 years at 10% p.a. compound interest
2 Sh100,000 for 2 years at 8% p.a. compound interest
3 Sh50,000 for 2 years at 6% p.a. compound interest
4 Sh100,000 for 3 years at 10% p.a. compound interest
5 Rukuba opens a deposit account which will earn 20% compound interest per annum. He pays sh100,000 into the account at the beginning of each year. What will be the value of his deposit just after the beginning of the third year?
6 Adolu borrowed sh10,000,000 from his bank at 10% compound interest per annum. After two years he repaid sh6,000,000. How much did he owe the bank by the end of the third year?

17.8 Hire Purchase

If an item is too expensive to purchase by a single payment it may be bought on Hire Purchase (H.P.) in which a **deposit** (or **down payment**) is made and the balance is repaid in **fixed monthly instalments**. Since H.P. is a form of borrowing money, interest is charged and therefore the total repaid will be more than the original price of the article.

Example A refrigerator has a marked price of sh500,000. A customer can either (a) pay cash for which he will get a 5% discount, or (b) make a 10% deposit and 12 monthly instalments of sh45,000. How much more would he pay by H.P. than by paying cash?

Cash price $= 500{,}000 - 500{,}000 \times \frac{5}{100} = 500{,}000 - 25{,}000 = \text{sh}475{,}000$

H.P. price $=$ deposit $+$ instalments

$= 500{,}000 \times \frac{10}{100} + 45{,}000 \times 12$

$= 50{,}000 + 540{,}000 = \text{sh}590{,}000$

Difference $= \text{sh}(590{,}000 - 475{,}000) = \text{sh}115{,}000$

Exercise 17h

1 The cash price of a radio cassette is sh60,000. Kasami buys it on hire purchase terms. He pays a deposit of sh10,000 and the balance in 20 monthly instalments of sh3,500. How much does he pay altogether by H.P.? How much would he have saved by paying cash, assuming he would have got a 10% discount?

2 Oluk buys a TV set on H.P. terms. He makes a down payment of sh20,000 and 15 monthly instalments of sh18,000. In all he pays sh70,000 more than the cash price. What was the cash price?

3 Limuri buys a video cassette recorder costing sh300,000 by paying a 15% deposit plus 12 equal monthly instalments. In total he pays 20% more than the cash price. How much was each monthly instalment?

4 Egau bought a car priced at sh2 million by making a 25% down payment. The balance was subject to the equivalent of 20% simple interest per annum over a 2 year period. It was agreed he would make 24 equal monthly instalments to repay the balance. How much did Egau pay each month?

MQ 1163

18 ALGEBRA

18.1 Basic Ideas

A letter such as x, which represents a number, is called an **unknown**. (In later work we call it a **variable**.) If operations are performed on an unknown an **expression** is formed, for example, $2x + 1$. Here $2x$ and 1 are **terms**. If an expression is put equal to another expression or number, then an **equation** is formed, for example $2x + 1 = 7$.

18.2 Symbolic Representation of Statements

An equation may be formed from a statement by taking the following steps:

Step 1 Introduce an unknown
Step 2 Form expressions involving the unknown
Step 3 Form an equation

Example Form an equation from the following statement: I think of a number, double it, add one and the result is seven.

Step 1 Let x be the number
Step 2 'double and add one' means $2x + 1$
Step 3 $2x + 1 = 7$

Exercise 18a

For each Question form an equation. Keep your results for use in Topic **21**.

1. I think of a number and subtract 3. I multiply the result by 4 and get 12.
2. Okello, who is x years old now, is 14 years older than Nansubuga. In 5 years' time he will be twice Nansubuga's age.
3. The measures of two angles of a triangle are $x°$ and $(x + 10)°$. The third angle is twice the sum of these.
4. Nyambura walks from home to market, a distance of s km, at 4km/h. After spending 30 minutes there she returns home at 3km/h. She takes 4 hours altogether.
5. Tumwesigye hires a car for two weeks. There is a basic charge of sh50,000 plus sh120 per kilometre. He drives d km and pays a total of sh230,000.
6. A salesman is paid sh5,000 plus a 5% commission (see **17.4**) for selling n books which are worth sh1,000 each. He is paid sh17,000 altogether.
7. A loaf of bread costs sh b. A packet of milk costs sh150 less than this and a tin of margarine costs sh300 more. Fatuma buys 2 loaves of bread, 3 packets of milk and a tin of margarine. She pays sh2,550 altogether.
8. A car is travelling at u m/s. It then **accelerates** (see **8.3**) at 2m/s² for 10s, after which its speed is 26m/s.

18.3 Simplification of Algebraic Expressions

An expression such as $2x + 3x$ may be simplified by adding the **coefficients** 2 and 3 since $2x$ and $3x$ are **like terms**.

$$\text{So} \quad 2x + 3x = (2 + 3)x = 5x$$

The expression $2x + 3x^2$ may not be simplified in this way since $2x$ and $3x^2$ are **unlike terms**.

Example Simplify $4x + 3ay + 2y^2 + x + 2ay - 5y^2$

Collect like terms:
$$4x + x = (4 + 1)x = 5x$$
$$3ay + 2ay = (3 + 2)ay = 5ay$$
$$2y^2 - 5y^2 = (2 - 5)y^2 = -3y^2$$

Therefore $4x + 3ay + 2y^2 + x + 2ay - 5y^2 = 5x + 5ay - 3y^2$

Exercise 18b

Simplify where possible.

1 $4x + 3x - 2x + x$

2 $3x + t - 5x - 3t + 2x$

3 $ax + x + 1 + 3ax - 2x - 5$

4 $5a + 4b - 3ab$

5 $p - 3pq + 2p - 3pq$

6 $u^2 + v^2 - 5uv + u^2 - v^2 + 4uv$

7 $3 - x + y + 2x^2 - 2y^2 + 6xy - 1 - x - 2y - x^2 - y^2 - 5xy$

8 $2mn - 5m^2n^2 - 5mn + m^2n^2$

9 $pq^2 + qp^2 + 3pq + 1$

10 $x^3 + 3x^2y - xy^2 + y^3 - 2x^2y - 2xy^2 - 2y^3$

18.4 Brackets

An expression may contain a **bracket**, for example $3(x + 2)$ which means $3 \times (x + 2)$. To remove the bracket use the distributive law.

$$3(x + 2) = 3 \times (x + 2) = 3 \times x + 3 \times 2 = 3x + 6$$

This process is called **expansion**. The reverse of expansion is **factorisation**.

Example 1 Expand and simplify $5(3x - 4) - 2(2x - 6)$

$$\begin{aligned} 5(3x - 4) - 2(2x - 6) &= 5 \times 3x + 5 \times (-4) - 2 \times 2x - 2 \times (-6) \\ &= 15x - 20 - 4x + 12 \\ &= 11x - 8 \end{aligned}$$

Example 2 Factorise $12p^2q^2 - 8pq^3$

The HCF (see 1.1) of the two terms is $4pq^2$ $\quad\therefore\quad 12p^2q^2 - 8pq^3 = 4pq^2(3p - 2q)$

Always check a factorisation by expansion. In this case:

$$4pq^2(3p - 2q) = 4pq^2 \times 3p - 4pq^2 \times 2q = 12p^2q^2 - 8pq^3$$

Exercise 18c

In 1 to 8 expand and simplify the expression.

1 $4(3x - 2)$

2 $3(x + 2) + 2(x + 3)$

3 $4(y - 7) + 5(2y + 7)$

4 $p(q + 4) - q(p - 2)$

5 $3mn(m + n)$

6 $2a(x^2 + y^2) - a(x^2 - y^2 + 1)$

7 $a(b - c) + b(c - a) + c(a - b)$

8 $\frac{1}{3}xy(x^2 + y^2 - 2xy)$

In 9 to 14 factorise the expression.

9 $6x + 4$

10 $9x + 9y + 3$

11 $2m^2n^2 - 2mn$

12 $4x^2 + 2x$

13 $15p^2q^2 - 10p^2q^3$

14 $\frac{3}{4}x^2y + \frac{1}{4}xy$

18.5 Substitution

An algebraic expression may be evaluated by **substitution**.

Example Evaluate $x^2 - y^2 + 2x - 2y + 1$ when $x = 2$ and $y = -3$.

Substitute the values of x and y into the expression:

$$\begin{aligned} x^2 - y^2 + 2x - 2y + 1 &= 2^2 - (-3)^2 + 2 \times 2 - 2 \times (-3) + 1 \\ &= 4 - 9 + 4 + 6 + 1 \\ &= 6 \end{aligned}$$

Exercise 18d

Find the value of each expression when $w = 0$, $x = 3$, $y = -2$ and $z = -1$.

1 $w + x + y + z$
2 $w - 2x + 3y - 4z + 9$
3 $w^2 + x^2 + y^2 + z^2$
4 $2x^2 - 3y^2 + xyz$
5 $3xy(2x - 5y + 7)$
6 $3xy - 2yz + 4wx$
7 $6xy^2z^4 - 5w^2x^2y^2$
8 $x^3 + y^3 - 3x^2y - 3xy^2 - 8$

18.6 Algebraic Operations

In arithmetic and algebra we have certain standard operations for which there are standard symbols. Examples are *addition* (+), *subtraction* (−), *multiplication* (×), *division* (÷) and *take the square root of* (√). For certain purposes we may wish to define other operations and assign symbols to them as in the following Example.

Example The operations $*$ and @ are defined as $a * b = a^2 + b^2$ and $a @ b = ab - b - 2$. Find the value of (a) $x * y$, (b) $w @ (x * y)$ when $w = 3$, $x = 1$ and $y = -2$.

(a)
$$\begin{aligned} x * y &= x^2 + y^2 && \text{according to the definition of } * \\ &= 1^2 + (-2)^2 && \text{substituting for } x \text{ and } y \\ &= 1 + 4 \\ &= 5 \end{aligned}$$

(b)
$$\begin{aligned} w @ (x * y) &= w @ 5 && \text{since } x * y = 5 \text{ from (a)} \\ &= w \times 5 - 5 - 2 && \text{according to the definition of @} \\ &= 3 \times 5 - 5 - 2 && \text{substituting for } w \\ &= 15 - 5 - 2 \\ &= 8 \end{aligned}$$

Exercise 18e

Use the information in the Example above to evaluate the expressions in **1** to **8**.

1 $w * x$
2 $x * w$
3 $x @ y$
4 $y @ x$
5 $x * x$
6 $y @ y$
7 $y @ (x * w)$
8 $(y @ x) * w$

9 An operation $*$ is defined thus: $a * b = a^2 - 2ab + 1$.
(i) Evaluate (a) $3 * 4$, (b) $4 * 3$, (c) $-3 * -4$, (d) $-4 * -3$.
(ii) What is $a * (a * a)$? Simplify your result as far as possible.

10 The operations $\oplus$ and $\otimes$ are defined to give the arithmetic mean and the geometric mean of two numbers respectively, ie. $a \oplus b = \frac{1}{2}(a + b)$ and $a \otimes b = \sqrt{(ab)}$.
Calculate (i) $2 \otimes (8 \oplus 28)$, (ii) $(2 \otimes 8) \oplus 28$.

19 ALGEBRAIC FRACTIONS

19.1 Simplification (Multiplication and Division)

The rules relating to the manipulation of algebraic fractions are like those for numerical fractions. Compare the methods in the numerical and algebraic examples which follow here and in 19.2.

Example 1 Simplify (i) $\frac{15}{18}$ (ii) $\frac{ac}{bc}$

(i) $\frac{15}{18} = \frac{5 \times 3}{6 \times 3} = \frac{5}{6}$ (ii) $\frac{ac}{bc} = \frac{a \times c}{b \times c} = \frac{a}{b}$

Example 2 Simplify (i) $\frac{3}{7} \times \frac{7}{10}$ (ii) $\frac{x}{y} \times \frac{y}{z}$

(i) $\frac{3}{7} \times \frac{7}{10} = \frac{3 \times 7}{7 \times 10} = \frac{3}{10}$ (ii) $\frac{x}{y} \times \frac{y}{z} = \frac{x \times y}{y \times z} = \frac{x}{z}$

Example 3 Simplify $\frac{a^2b^4}{a^3b^2}$

$$\frac{a^2b^4}{a^3b^2} = \frac{a \times a \times b \times b \times b \times b}{a \times a \times a \times b \times b} = \frac{b^2}{a}$$

Exercise 19a

Simplify the expression where possible.

1 $\frac{xy}{zx}$ 2 $\frac{a}{b} \times \frac{b}{a}$ 3 $\frac{p}{q} \times \frac{r}{p}$ 4 $\frac{m^2n}{mn}$

5 $\frac{xr^2}{r^2x}$ 6 $\frac{a^2b^2}{c^2}$ 7 $\frac{ab^3}{a^3b}$ 8 $\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d}$

9 $\frac{a^2b}{c^2} \times \frac{ac^3}{b^2}$ 10 $\frac{vu}{w} \times \frac{u^2w^2}{v^2} \times \frac{w^3v^3}{u^3}$ 11 $\frac{8x^2}{y} \times \frac{y^2}{4x}$ 12 $\sqrt{\left(\frac{s^3}{t^3} \times \frac{t}{s}\right)}$

13 $\frac{p^2}{q} \times \frac{q^2}{r} \times \frac{r^2}{p}$ 14 $\frac{a^5b^3}{a^7} \times \frac{a^2b^2}{b^5}$ 15 $\frac{x}{y} \div \frac{3}{4}$ 16 $\frac{x}{y} \div \frac{x}{4}$

17 $\frac{x}{y} \div \frac{x}{z}$ 18 $\frac{x}{y} \div \frac{y}{x}$ 19 $\frac{x}{y} \div \frac{x^2}{y^2}$ 20 $\frac{p^2}{q^2} \div \frac{p^3}{q^3}$

19.2 Simplification (Addition and Subtraction)

Example 1 Simplify (i) $\frac{2}{3} + \frac{2}{5}$ (ii) $\frac{x}{3} + \frac{x}{5}$ (iii) $\frac{2}{y} + \frac{2}{z}$

(i) $\frac{2}{3} + \frac{2}{5} = \frac{2 \times 5}{3 \times 5} + \frac{3 \times 2}{3 \times 5} = \frac{(2 \times 5) + (3 \times 2)}{3 \times 5} = \frac{10 + 6}{15} = \frac{16}{15}$

(ii) $\frac{x}{3} + \frac{x}{5} = \frac{x \times 5}{3 \times 5} + \frac{3 \times x}{3 \times 5} = \frac{(x \times 5) + (3 \times x)}{3 \times 5} = \frac{5x + 3x}{15} = \frac{8x}{15}$

(iii) $\frac{2}{y} + \frac{2}{z} = \frac{2z + 2y}{y \times z} = \frac{2(z + y)}{yz}$

Example 2 Express as a single fraction $\frac{2}{x} - \frac{3}{x + 1}$

$$\frac{2}{x} - \frac{3}{x + 1} = \frac{2(x + 1)}{x(x + 1)} - \frac{3x}{x(x + 1)} = \frac{2(x + 1) - 3x}{x(x + 1)} = \frac{2x + 2 - 3x}{x(x + 1)} = \frac{2 - x}{x(x + 1)}$$

Exercise 19b

In 1 to 25, express as a single fraction and simplify.

1 $\frac{x}{3} + \frac{x}{4}$

2 $\frac{x}{3} + \frac{y}{4}$

3 $\frac{3}{x} + \frac{4}{x}$

4 $\frac{3}{x} + \frac{4}{y}$

5 $\frac{x}{3} + \frac{4}{x}$

6 $\frac{x}{3} + \frac{4}{y}$

7 $\frac{2}{1} + \frac{1}{x}$

8 $3 + \frac{2}{y}$

9 $x + \frac{1}{x}$

10 $\frac{1}{1 + x} + 1$

11 $\frac{1}{1 + x} - 1$

12 $\frac{1}{x} - \frac{2}{x + 1}$

13 $\frac{3}{x - 1} - \frac{2}{x}$

14 $\frac{1}{x + 1} + \frac{1}{x + 2}$

15 $\frac{1}{x + 1} - \frac{1}{x + 2}$

16 $\frac{2}{3x + 1} - \frac{3}{2x + 1}$

17 $\frac{x + 1}{x - 1} - \frac{1}{2}$

18 $\frac{x}{3} + \frac{x}{4} - \frac{x}{2}$

19 $\frac{1}{2} - \frac{1}{x} - \frac{2}{x + 1}$

20 $\frac{2}{1 + t^2} - 1$

21 $\frac{3}{4x} - \frac{1}{2} + \frac{1}{6x}$

22 $\frac{x + 1}{x - 1} + \frac{x - 1}{x + 1}$

23 $\frac{1}{(x + 1)(x + 2)} + \frac{1}{x + 2}$

24 $\frac{1 + t}{1 - t} - \frac{1 - t}{1 + t}$

25 $\frac{1}{(x + 1)(x + 2)} + \frac{1}{(x + 2)(x + 3)} - \frac{1}{(x + 1)(x + 3)}$

In 26 to 40 simplify each expression by first factorising numerator and/or denominator.

26 $\frac{2x + 4}{2x + 6}$

27 $\frac{3}{3x + 12}$

28 $\frac{2x + 6}{3x + 9}$

29 $\frac{x^2 + 3x}{2x}$

30 $\frac{a^2 - 2a}{a^2 - a}$

31 $\frac{x^2 - 4x}{2x - 8}$

32 $\frac{x^2 + 3x + 2}{x^2 + 4x + 3}$ (See 34.4 for factorisation)

33 $\frac{x^2 + x - 2}{x^2 + 2x - 3}$

34 $\frac{x^2 - 4}{2x - 4}$

35 $\frac{2x^2 - 3x - 2}{4x^2 - 1}$

36 $\frac{a^2 + 2ab + b^2}{a^2 - b^2}$

37 $\frac{a^2 - 2ab + b^2}{a^2 - b^2}$

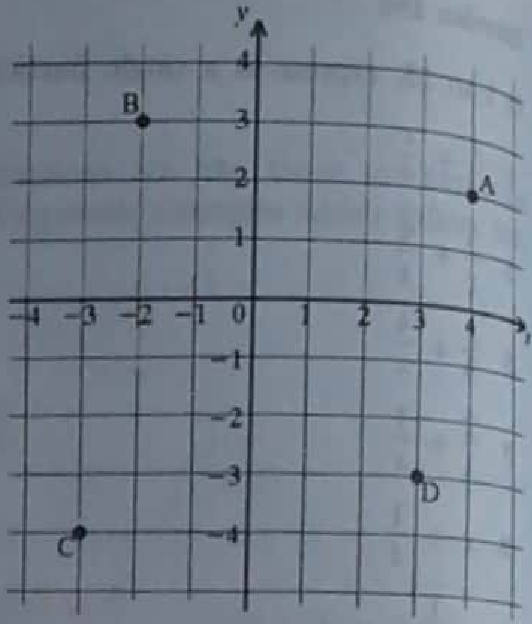
38 $\frac{4x^2 - 9}{4x^2 - 12x + 9}$

39 $\frac{(x - 1)^2 - 9}{(x - 2)^2 - 16}$

40 $\frac{(3m + 2n)(m - n) + 2mn}{(3m + 2n)(m + n) - 10mn}$

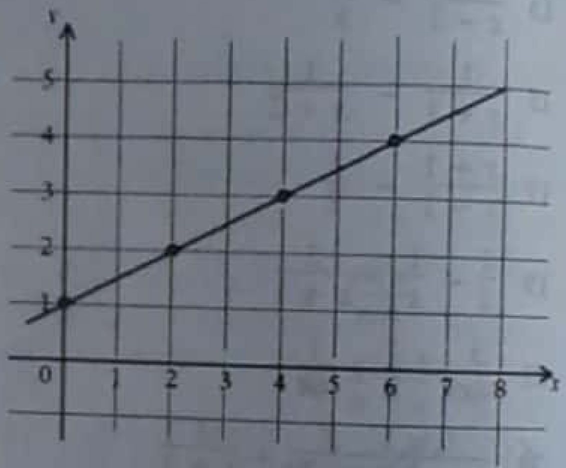
20 COORDINATES AND LINES

20.1 Points and Coordinates



In the **grid** shown the position of **point** A is described by its **coordinates** (4, 2). Its **x-coordinate** is 4 and its **y-coordinate** is 2. Similarly B is the point (−2, 3), C is (−3, −4) and D(3, −3). Note that these are **ordered** pairs. For example, $(4, 2) \neq (2, 4)$.
Copy the grid and plot the points P(1,0), Q(3, 4), R(−1, 2) and S(−3, −2). If you plot these points correctly and then join them up you should find that PQRS is a **rhombus**.

20.2 Lines and Equations



A **line** is a set of points, for example $\{(x, y) : 2y - x = 2\}$, ie. the set of all points (x, y) such that $2y - x = 2$. The equation $2y - x = 2$ is called the **equation of the line**. Ordered pairs which satisfy this equation are (0, 1), (2, 2), (4, 3), (6, 4). If we plot these as points on a grid, they are seen to lie on a straight line or, simply, a line. By joining the points with a ruler we draw the **graph** of the line $\{(x, y) : 2y - x = 2\}$. (See diagram)

The following Example shows one method of finding the equation of a line when given a number of points which lie on it. (See **20.4** and **39.2** for alternative methods.)

Example Find the equation of the line containing the points (i) (0, 2), (1, 5), (2, 8), (3, 11) (ii) (0, 4), (2, 3), (4, 2), (6, 1).

(i) The first two rows in the table opposite give the x- and y-coordinates of the four points. The equation of a line is the relation between the x- and y-coordinates of all points of it. To find this relation study the number patterns (see Topic **16**) for x and y. The x-coordinates increase by 1 each time, while y increases by 3. So multiply x by 3 to obtain $3x$ as shown. Now if 2 is added to $3x$ we obtain the y-coordinates as the last line in the table shows.
Hence $y = 3x + 2$ and this is the equation of the line.

x	0	1	2	3
y	2	5	8	11
$3x$	0	3	6	9
$3x + 2$	2	5	8	11

(ii) Here the x-coordinates increase by 2 each time hence multiply the y-coordinates by 2. Since x increases while y decreases we add $2y$ to x. In each case the result is 8 as shown in the table.
Therefore the required equation is $x + 2y = 8$.

x	0	2	4	6
y	4	3	2	1
$2y$	8	6	4	2
$x + 2y$	8	8	8	8

Exercise 20a

In **1** to **10** the equation of a line is given. Find three ordered pairs which satisfy this equation. Hence draw the graph of the line by plotting points on a grid.

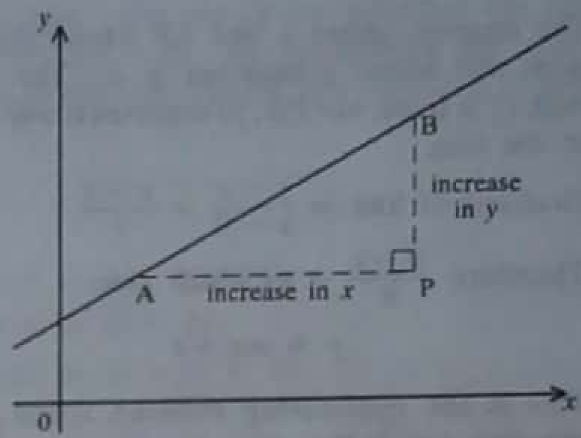
1 $y = 2x$	**2** $y = x + 1$	**3** $y = 2x - 3$	**4** $x + y = 4$	**5** $x + 2y = 5$
6 $4x + y = 8$	**7** $x + y = 0$	**8** $x = -2$	**9** $y = 4$	**10** $2y + x + 2 = 0$

In **11** to **14** find the equation of the line through the given set of points.

11 $\{(0, 1), (1, 5), (2, 9), (3, 13)\}$ **12** $\{(0, 4), (3, 3), (6, 2), (9, 1)\}$
13 $\{(-2, -1), (0, 2), (2, 5), (4, 8)\}$ **14** $\{(-4, 9), (4, 3), (8, 0), (12, -3)\}$

20.3 Gradient

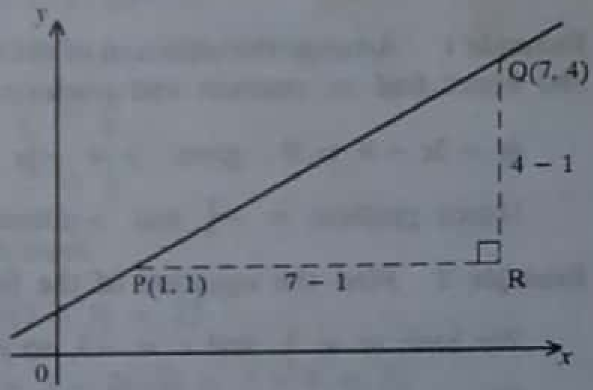
The **gradient** (or slope) of a line is found by taking any two points A and B on it. (See diagram)



$$\text{Gradient of line} = \frac{\text{increase in } y}{\text{increase in } x} = \frac{PB}{AP}$$

The gradient of a line is constant.

Example Find, by drawing graphs, the gradient of the line through the points (i) P(1, 1) and Q(7,4), (ii) L(−2, 6) and M(3, 1).



(i) $\text{Gradient} = \dfrac{\text{increase in } y}{\text{increase in } x}$

$= \dfrac{RQ}{PR}$

$= \frac{4-1}{7-1} = \frac{3}{6} = \frac{1}{2}$

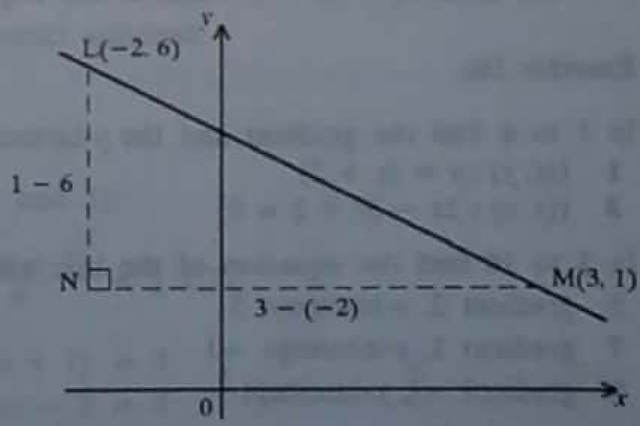
(ii) $\text{Gradient} = \dfrac{\text{increase in } y}{\text{increase in } x}$

$= \dfrac{LN}{MN}$

$= \dfrac{1-6}{3-(-2)}$

$= \frac{-5}{5}$

$= -1$



Note here that the increase in y is negative.

Exercise 20b

In **1** to **10** find the gradient of the line through the given two points.

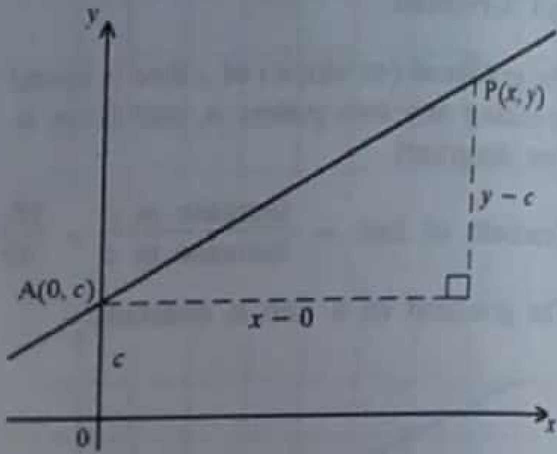
1 (2, 1), (5, 4)	**2** (1, 0), (4, 6)	**3** (1, 1), (7, 3)
4 (0, 5), (4, 1)	**5** (1, 4), (7, 1)	**6** (1, 9), (4, 0)

7 $(-1, 2), (3, 6)$ 8 $(-4, 4), (4, 2)$ 9 $(-3, -2), (4, -1)$
10 $(-1, 5), (8, -1)$

In **11** to **22** draw the graph of the line whose equation is given and find its gradient.

11 $y = 2x + 1$ **12** $y = 3x$ **13** $y = x - 1$
14 $x + y = 6$ **15** $x + 2y = 9$ **16** $2x = y - 2$
17 $3y - 2x = 10$ **18** $2y - 3x - 5 = 0$ **19** $y + 4 = 4x$
20 $5x + 3y = 15$ **21** $y = -2$ **22** $x = 3$

20.4 $\{(x, y) : y = mx + c\}$



The diagram shows a line AP whose gradient is m and whose **y-intercept** is c. The point A$(0, c)$ is fixed, but P(x, y) represents *any* point of the line.

Gradient of line $= \dfrac{y - c}{x - 0} = \dfrac{y - c}{x}$

Therefore $\dfrac{y - c}{x} = m$ which gives

$$y = mx + c$$

This is the relationship between x and y for this line and is therefore its equation.

Example 1 Arrange the equation of the line $\{(x, y) : 4y + 3x - 6 = 0\}$ in the form $y = mx + c$ and hence find its gradient and y-intercept.

$4y + 3x - 6 = 0$ gives $y = -\frac{3}{4}x + \frac{3}{2}$

Hence gradient $= -\frac{3}{4}$ and y-intercept $= \frac{3}{2}$

Example 2 Find the equation of the line with gradient $\frac{1}{2}$ and y-intercept -3.

We have $m = \frac{1}{2}$ and $c = -3$, so $y = mx + c$ gives $y = \frac{1}{2}x - 3$

This gives $2y = x - 6$ which is the required equation.

Exercise 20c

In **1** to **4** find the gradient and the y-intercept of the given line.

1 $\{(x, y) : y = 3x + 2\}$ 2 $\{(x, y) : 2y = x + 5\}$
3 $\{(x, y) : 2x - 3y + 2 = 0\}$ 4 $\{(x, y) : x + 2y + 4 = 0\}$

In **5** to **10** find the equation of the line with the given gradient and y-intercept.

5 gradient 2, y-intercept 5 **6** gradient $\frac{1}{2}$, y-intercept 3
7 gradient 1, y-intercept -1 **8** gradient -2, y-intercept 0
9 gradient $-\frac{1}{3}$, y-intercept $\frac{5}{3}$ **10** gradient $-\frac{1}{2}$, y-intercept $-2\frac{1}{2}$

In **11** to **20** find the equation of the line through the given points.

11 $(0, 1), (3, 10)$ **12** $(0, 2), (5, 7)$ **13** $(0, -1), (3, 5)$
14 $(0, 3), (4, -1)$ **15** $(-5, 1), (0, 6)$ **16** $(-4, -4), (0, -2)$
17 $(0, 0)$ $(2, 3)$ **18** $(-2, 1), (4, -2)$ **19** $(-4, -5), (2, 4)$
20 $(-1, 1), (-3, 6)$

21 SIMPLE EQUATIONS

21.1 The Linear Equation

An equation such as $3(2x + 3) = 3 - 2(x - 5)$ is called a **linear equation** in one unknown. All such equations can be expressed in the form $ax = b$, where x is the unknown and a and b are known. Note that such an equation does not contain x^2 or any higher power of x.

21.2 Method of Solution

An equation has two sides separated by the 'equals' sign. For the above equation the left side (LS) is $3(2x + 3)$ and the right side (RS) is $3 - 2(x - 5)$. By performing the *same* operations on both sides, thus keeping the two sides 'equal', we reduce the equation to the form $ax = b$ and hence solve it.

Example 1 Solve $3(2x + 3) = 3 - 2(x - 5)$

Step 1: Remove the brackets $\quad 6x + 9 = 3 - 2x + 10$

$$6x + 9 = 13 - 2x$$

Step 2: Add $2x$ to both sides $\quad 6x + 9 + 2x = 13 - 2x + 2x$

$$8x + 9 = 13$$

Step 3: Subtract 9 from both sides $\quad 8x + 9 - 9 = 13 - 9$

$$8x = 4 \quad \text{(the form } ax = b\text{)}$$

Step 4: Divide both sides by 8 $\quad \frac{8x}{8} = \frac{4}{8}$

$$x = \tfrac{1}{2}$$

Check: Substitute $x = \frac{1}{2}$ in the left and right sides.

$$LS = 3(2x + 3) = 3(2 \times \tfrac{1}{2} + 3) = 3(1 + 3) = 12$$

$$RS = 3 - 2(x - 5) = 3 - 2(\tfrac{1}{2} - 5) = 3 - 2(-\tfrac{9}{2}) = 3 + 9 = 12$$

So LS = RS and $x = \frac{1}{2}$ is the correct solution.

Example 2 Solve $\dfrac{4x + 5}{4} - \dfrac{3x + 1}{6} = \dfrac{1}{3}$

Multiply both sides by 12 (the LCM of 4, 6 and 3):

$$\frac{12(4x + 5)}{4} - \frac{12(3x + 1)}{6} = \tfrac{1}{3} \times 12$$

Simplify: $\quad 3(4x + 5) - 2(3x + 1) = 4$

Remove brackets: $\quad 12x + 15 - 6x - 2 = 4$

Collect like terms: $\quad 6x + 13 = 4$

Subtract 13: $\quad 6x = -9$

Divide by 6: $\quad x = -\frac{9}{6} = -1\frac{1}{2}$ or $-1{\cdot}5$

Check: $\quad LS = \dfrac{4(-1{\cdot}5) + 5}{4} - \dfrac{3(-1{\cdot}5) + 1}{6} = -\frac{1}{4} + \frac{7}{12} = \frac{1}{3} = RS$

Exercise 21a

Solve the following equations.

1 $x + 15 = 22$	2 $3x - 2 = 16$
3 $2x + 11 = 5$	4 $13 + 3x = 12$
5 $7 - 2x = 1$	6 $2 - 3x = 8$
7 $3x + 4 = 0$	8 $\frac{3}{4}x - 2 = 5\frac{1}{2}$
9 $x + 2 = 5 - 2x$	10 $2(x + 4) = 5x - 1$
11 $1 - 2(x + 1) = x + 2$	12 $x + 1 + 3(x + 3) = 2(x + 2)$
13 $\frac{1}{2}(x + 5) = 3x$	14 $\frac{x + 1}{2} = \frac{x + 2}{3}$
15 $\frac{2}{3}(x + 2) - \frac{1}{2}(x + 1) = 1$	16 $5(2x - 1) + 2(x + 3) = 1$
17 $\frac{x}{3} - \frac{x - 1}{4} + \frac{x + 3}{6} = 0$	18 $\frac{1}{4}(5x - 4) - \frac{3}{8}(x + 1) - \frac{5}{6}(x - 1) = 0$

21.3 Problems

Equations are used to solve problems.

Example There are two paths, one 1km longer than the other, connecting a boy's home to his school. His walking speeds along the longer and shorter paths are 4km/h and 3km/h respectively. Find the length of the shorter path if it takes the boy 10 minutes longer to walk to school by this route.

Let the length of the shorter path be x km,

then the length of the longer path is $(x + 1)$km.

Time taken by shorter path $= \frac{\text{distance}}{\text{speed}} = \frac{x}{3}$ h

Time taken by longer path $= \frac{\text{distance}}{\text{speed}} = \frac{x + 1}{4}$ h

The difference in these times $= \frac{x}{3} - \frac{x + 1}{4}$ h

But this is equal to 10 minutes or $\frac{10}{60} = \frac{1}{6}$h

$$\text{So} \quad \frac{x}{3} - \frac{x + 1}{4} = \frac{1}{6}$$

$$\therefore \quad 4x - 3(x + 1) = 2$$

$$\therefore \quad 4x - 3x - 3 = 2$$

$$\therefore \quad x = 5$$

The shorter distance is 5km.

Check: The longer distance is $5 + 1 = 6$km
Time by shorter path $= \frac{5}{3}$h $=$ 1h 40min
Time by longer path $= \frac{6}{4}$h $=$ 1h 30min
Time difference $=$ 1h 40min $-$ 1h 30min $=$ 10 minutes

Exercise 21b

For **1** to **8** refer to **1** to **8** respectively of **Exercise 18a.**

1 Find the number.
2 Find Okello's present age.
3 Find the three angles.
4 Find s.
5 Find the distance he drives.
6 How many books did he sell?
7 Find the cost of one packet of milk.
8 Find u.

9 In the market, a kilogram of onions costs sh100 more than a kilogram of tomatoes. Ochar buys 2kg of tomatoes and 4kg of onions and pays sh2,500 altogether. Find the cost of 1kg of onions.

10 Ekek uses his 10ha piece of land to grow maize, beans and potatoes. The areas for growing beans and potatoes are in the ratio 2 : 1 and he uses 4ha more for maize than potatoes. Find the area he uses for maize.

11 A motorist leaves Kampala at 9 am for Masaka, a distance of 130km. He stops for 10 minutes at Mpigi for a cup of tea and arrives in Masaka at 11 am. Before Mpigi his average speed was 60km/h and after Mpigi it was 75km/h. Find the distance of Mpigi from Kampala.

12 A cyclist leaves Fort Portal at 9.30 am and heads towards Kyenjojo at an average speed of 15km/h. At 10.30 am on the same day a motorist sets out from Fort Portal. He takes the same road as the cyclist and averages 60km/h. At what distance from Fort Portal does the motorist overtake the cyclist? Find also at what time this occurs.

13 Two taps A and B can together fill a water tank in 6 hours. If Tap A, when turned on alone, can fill the tank in 10 hours, how long would it take tap B alone to fill it? (*Hint:* Let b hours be the time Tap B alone takes to fill the tank and suppose the tank has a capacity of 100 litres.)

22 SIMULTANEOUS EQUATIONS

22.1 Equations with Two Unknowns

An equation with two unknowns such as $x + y = 7$ has many possible solutions, for example $x = 4$ and $y = 3$ or $x = 5$ and $y = 2$. Solutions can be written as ordered pairs, ie. (4, 3) and (5, 2) in this case. If a second equation involving x and y is given, for example $x - y = 3$, then we have two equations

$$x + y = 7 \quad \ldots\ldots\ldots (i)$$

$$\text{and} \quad x - y = 3 \quad \ldots\ldots\ldots (ii)$$

Equations (i) and (ii) are **simultaneous equations**. By trial it can be seen that there is only one (unique) ordered pair which satisfies both equations. This is (5, 2) since $5 + 2 = 7$ and $5 - 2 = 3$. The **solution** (5, 2) **satisfies** both equations.

22.2 Graphical Method of Solution

Consider the pair of equations

$$x + 2y = 4$$

and $\quad 4y - 3x = 12$

Using the same axes draw the graphs of the following sets of points:

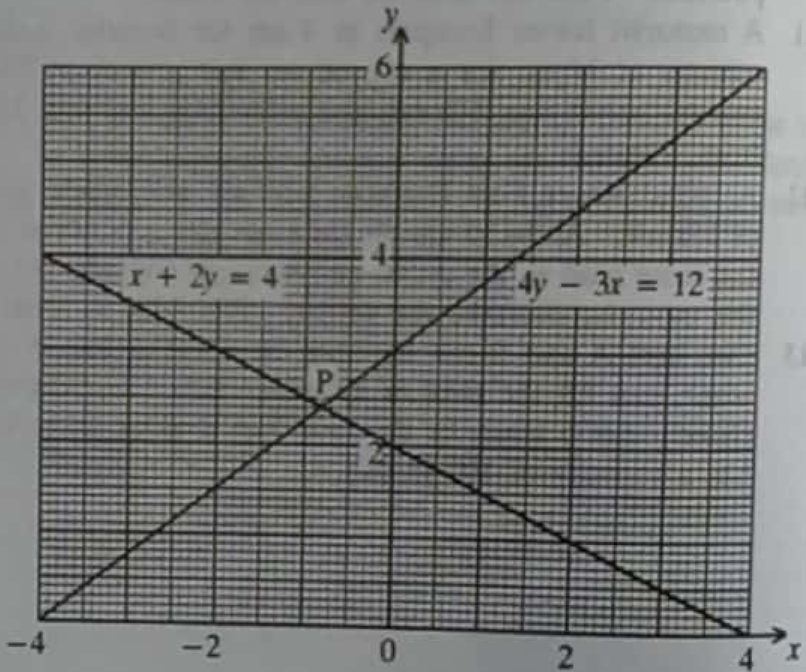
$$S = \{(x, y) : x + 2y = 4\}$$

and $\quad T = \{(x, y) : 4y - 3x = 12\}$

This gives the two lines shown (see **20.2**) which intersect at P, ie. $S \cap T = \{\text{point P}\}$.

The coordinates of P(−0·8, 2·4) give the solution to the equations.

Check that (−0·8, 2·4) satisfies both equations.



Exercise 22a

Solve graphically the following pairs of simultaneous equations.

1 $2x + y = 8$
  $x + 2y = 7$

2 $x + 2y = 4$
  $2y = x + 8$

3 $y = x - 4$
  $2x + y = 5$

4 $x - 3y = 1$
  $2y - 3x = 4$

5 $m + 3n = 2$
  $2m - 4n = 1$

6 $2b - 3a = 7$
  $3b - 6a = 13$

22.3 Solution by Substitution

Consider the equations

$$y + 5x = 1 \quad \ldots\ldots\ldots (i)$$

$$\text{and} \quad 2x - 4y = 7 \quad \ldots\ldots\ldots (ii)$$

From (i) we have $y = 1 - 5x$, in which one unknown (y) is expressed explicitly in terms of the other (x).

Substitute this value for y in (ii):

$$\begin{aligned} 2x - 4(1 - 5x) &= 7 \\ \therefore \quad 2x - 4 + 20x &= 7 \\ \therefore \quad 22x &= 11 \\ \therefore \quad x &= \tfrac{1}{2} \end{aligned}$$

Substitute $x = \frac{1}{2}$ in (i)

$$\begin{aligned} y + 5 \times \tfrac{1}{2} &= 1 \\ \therefore \quad y &= -1\tfrac{1}{2} \end{aligned}$$

Check in (ii): $\text{LS} = 2x - 4y = 2 \times \frac{1}{2} - 4 \times (-1\frac{1}{2}) = 1 + 6 = 7 = \text{RS}$

Note: If the second substitution had been made in (ii), we would have checked using (i).

Exercise 22b

Use the method of substitution to solve the following.

1 $y = 5x - 3$
$2x + 3y = 8$

2 $x = 3y - 3$
$5y - 3x = 1$

3 $3x + 7y = 1$
$y = 5 + 2x$

4 $y - 4x = 1$
$3x - 2y = 3$

5 $5x - y = 8$
$2x - 3y = 11$

6 $4x = 2 - 5y$
$5x + 4y = 1$

22.4 Solution by Elimination

By a process of adding or subtracting simultaneous equations we may **eliminate** one of the unknowns.

Example 1 Solve the equations:

$$\begin{aligned} 3x + 2y &= 7 \quad \ldots\ldots\ldots \text{(i)} \\ 5x - 2y &= 1 \quad \ldots\ldots\ldots \text{(ii)} \end{aligned}$$

Eliminate y: add (i) to (ii)

$$\begin{aligned} 3x + 2y + 5x - 2y &= 7 + 1 \\ \therefore \quad 8x &= 8 \\ \therefore \quad x &= 1 \end{aligned}$$

Substitute $x = 1$ in (i):

$$\begin{aligned} 3 \times 1 + 2y &= 7 \\ \therefore \quad y &= 2 \end{aligned}$$

Check in (ii): $\text{LS} = 5x - 2y = 5 \times 1 - 2 \times 2 = 1 = \text{RS}$

Example 2 Solve the equations:

$$\begin{aligned} 6x + 5y &= 7 \quad \ldots\ldots\ldots \text{(i)} \\ 4x + 3y &= 5 \quad \ldots\ldots\ldots \text{(ii)} \end{aligned}$$

Eliminate x: Make coefficients of x equal in both equations by multiplying (i) by 2 and (ii) by 3, since the LCM of 6 and 4 is 12.

$$\begin{aligned} \text{(i)} \times 2: \quad 12x + 10y &= 14 \quad \ldots\ldots\ldots \text{(iii)} \\ \text{(ii)} \times 3: \quad 12x + 9y &= 15 \quad \ldots\ldots\ldots \text{(iv)} \end{aligned}$$

Subtract (iv) from (iii):

$$\begin{aligned} 12x + 10y - (12x + 9y) &= 14 - 15 \\ \therefore \quad y &= -1 \end{aligned}$$

Substitute $y = -1$ in (i):

$$\begin{aligned} 6x + 5(-1) &= 7 \\ \therefore \quad x &= 2 \end{aligned}$$

Check in (ii): $\text{LS} = 4x + 3y = 4 \times 2 + 3(-1) = 8 - 3 = 5 = \text{RS}$

Rules for Elimination

1. Decide which unknown to eliminate.
2. If necessary, multiply one or both equations by suitable numbers to make the coefficients of this unknown numerically equal in the two equations.
3. If the 'equal' coefficients have *like* signs *subtract* the equations. If they have *unlike* signs then *add* the equations.

Exercise 22c

Solve the following by elimination.

1 $5x + 2y = 9$ $3x + 2y = 7$	**2** $x + 4y = 1$ $3x - 4y = 19$	**3** $2y - x = 5$ $3y + 2x = 4$
4 $3y - 4x = 13$ $7y - 2x = 1$	**5** $6x - 4y = 3$ $8x - 6y = 1$	**6** $3m + 4n = 0$ $4m + 3n + 14 = 0$

22.5 Problems

Example John bought 4 *chapatti* and 3 *mandazi* for sh900. From the same shop Charles bought 3 *chapatti* and 6 *mandazi* for sh1,050. Find the cost of a *chapatti*.

Let a *chapatti* cost sh c and a *mandazi* sh m.

Then	$4c + 3m = 900$	........(i)
and	$3c + 6m = 1{,}050$	........(ii)
Eliminate m: (i) × 2 gives	$8c + 6m = 1{,}800$	........(iii)
Subtract (ii) from (iii)	$5c = 750$	
	$\therefore \quad c = 150$	

A *chapatti* costs sh150.

Check: Substituting $c = 150$ in (i) gives $4 \times 150 + 3m = 900$ which gives $m = 100$
Substituting $c = 150$ and $m = 100$ in (ii) gives
$\text{LS} = 3c + 6m = 3 \times 150 + 6 \times 100 = 450 + 600 = 1{,}050 = \text{RS}$

Exercise 22d

1 In the market Mildred buys 4kg of tomatoes and 2kg of potatoes for sh1,900. Eunice buys 5kg of tomatoes and 3kg of potatoes for sh2,500. Find the cost of 1kg of potatoes.

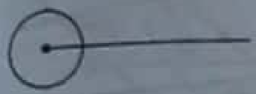
2 With sh4,000 Ali can buy 10 pencils and 5 rubbers or 4 pencils and 10 rubbers. Find the cost of 1 pencil and 1 rubber.

3 A bus travels from Moroto to Mbale via Soroti. It leaves Moroto at 8.30 am, stops in Soroti for 45 minutes and arrives in Mbale at 1.30 pm. Its average speeds between Moroto and Soroti and between Soroti and Mbale are 56km/h and 80km/h respectively. Given that a motorist makes the same journey non-stop at an average speed of 67km/h in 4 hours, find the distances from Moroto to Soroti and from Soroti to Mbale.

4 An alloy of density 7g/cm^3 (see **8.6**) is made from two metals A and B by mixing them in the ratio 3 : 2 by volume. Another alloy of density $5{\cdot}5\text{g/cm}^3$ is formed by mixing the same metals in the ratio 3 : 7 by volume. Find the densities of A and B.

23 ANGLES AND CONSTRUCTIONS

23.1 Types of Angle



Full turn or
Complete revolution 360°

Half-turn or
Straight angle 180°

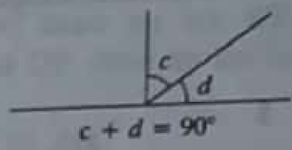
Quarter-turn or
Right-angle 90°

Acute angle less than 90°

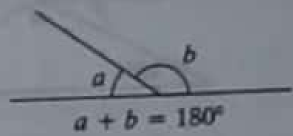
Obtuse angle greater than 90°, less than 180°

Reflex angle greater than 180°, less than 360°

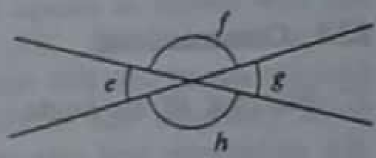
Complementary angles are angles which add up to 90°.



Supplementary angles are angles which add up to 180°.



When two lines intersect, the **vertically opposite angles** are equal, so $e = g$ and $f = h$.



Exercise 23a

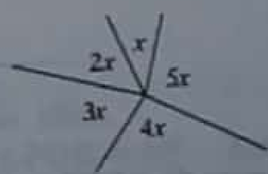
1 Of the following angles, which are acute, obtuse or reflex?
(i) 10°, (ii) 100°, (iii) 200°, (iv) 300°, (v) 50°, (vi) 150°, (vii) 250°

2 Which of these angles are equivalent to acute, obtuse or reflex angles?
(i) 410°, (ii) 690°, (iii) 1,000°, (iv) 2,000°, (v) 3,000°

3

(i) Find f, g and h if $e = 70°$
(ii) Find e, f, g and h if $f = 2e$

4 Find the angles x, $2x$, etc., in the diagram on the right.



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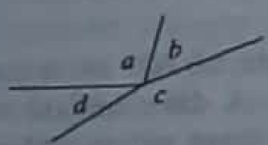


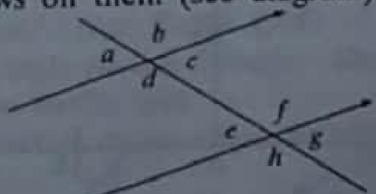
Find d if $a = 110°$, $c = 2b$ and $d = b$.

6 Find the angle between the hour and minute hands of a clock at the following times:
(i) 12 noon, (ii) 9 am, (iii) 6 pm, (iv) 4.30 pm, (v) 3.45 am

23.2 Parallel Lines

Parallel lines have the same direction. They are a fixed distance apart and therefore never meet. Two lines are shown as parallel by placing arrows on them (see diagram).

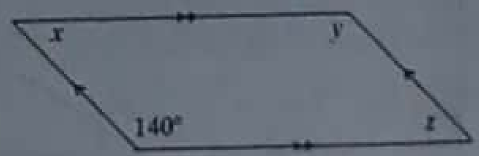
A **transversal** is a line cutting two or more parallel lines, in which case **corresponding angles** are equal. So $a = e$, $b = f$, $c = g$ and $d = h$. Also the **alternate angles** are equal, therefore $c = e$ and $d = f$.



Exercise 23b

Find the unknown angles in each diagram.

1



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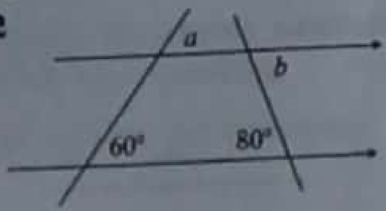

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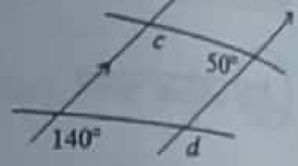

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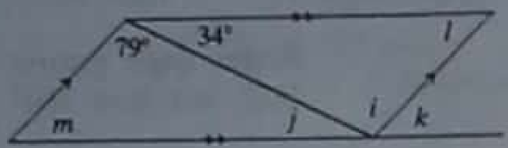

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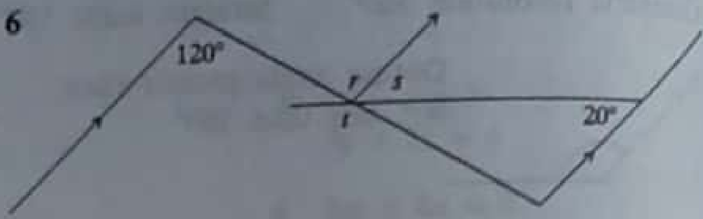

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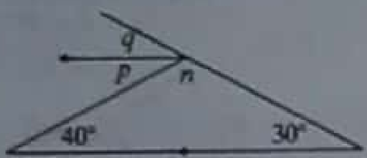

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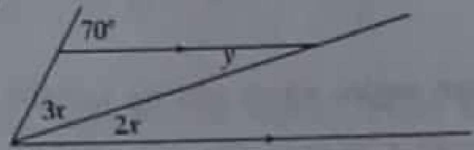

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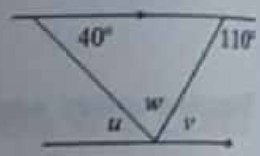
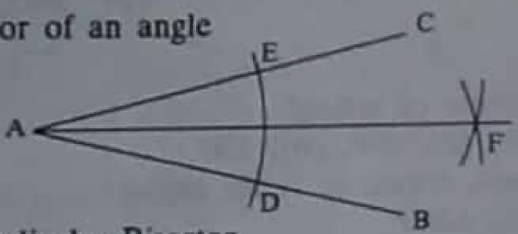


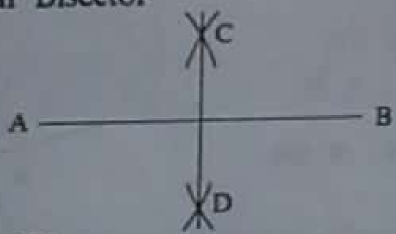
23.3 Constructions

1. Bisector of an angle



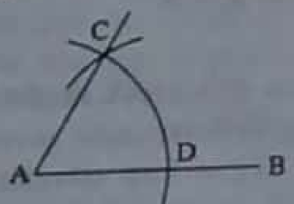
To bisect angle BAC: with centre A draw an arc to cut AC and AB at E and D. With centres E and D draw arcs to intersect at F. Join AF. Then ∠CAF = ∠BAF.

2. Perpendicular Bisector



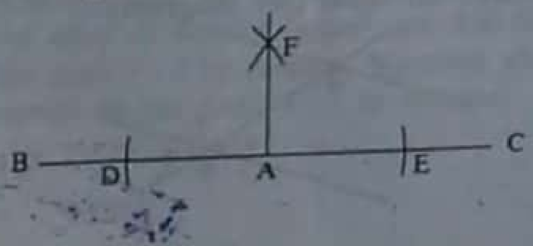
To obtain the **perpendicular bisector** of line segment AB (also called the **mediator**): draw arcs with A and B as centres to intersect at C and D. Join CD. The line CD is the perpendicular bisector of AB.

3. An angle of 60°



To construct an angle of 60° at A on the line AB: with centre A draw an arc to cut AB at D. With the same radius and centre D draw an arc to cut the first arc at C. Join AC. Then ∠BAC = 60°.

4. An angle of 30° — Construct an angle of 60°, then bisect it.
5. An angle of 90°, ie. a line perpendicular to a given line at a given point on the line.

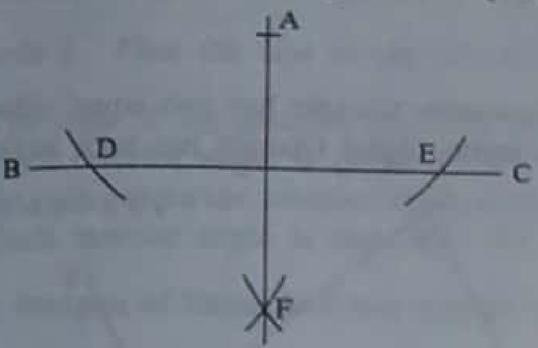


A is the given point on the line BC. With centre A draw arcs to cut BC at D and E. With centres D and E and a radius greater than before draw arcs to intersect at F. Join AF. The line AF is perpendicuar to BC.

6. An angle of 45°

Construct an angle of 90°, then bisect it.

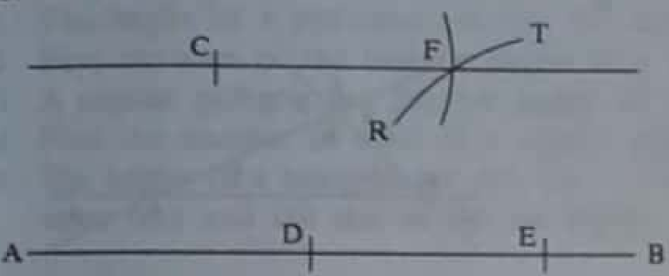
7. A perpendicular to a given line (BC) from a given point (A) not on the line.



With centre A draw two arcs to cut BC at D and E. With centres D and E draw arcs to intersect at F. Join AF.
The line AF is perpendicular to BC.

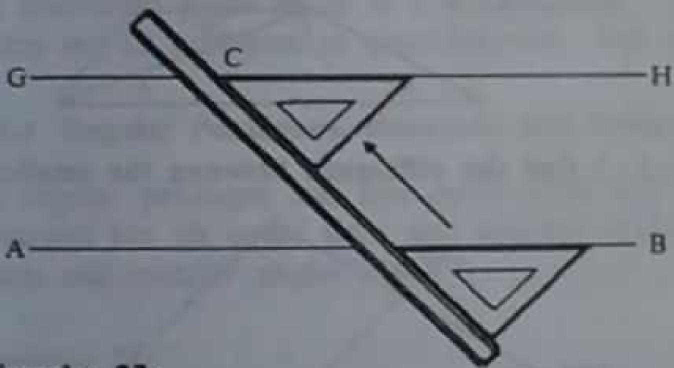
8. A line through a given point (C) parallel to a given line (AB).

Method 1



Mark any two points D and E on AB. With centre E and radius DC draw an arc RT. With centre C and radius DE draw an arc to cut RT at F. Join CF.
The line CF is parallel to AB.

Method 2



Place ruler and set square as shown with one side aligned along AB. Slide the set square along the ruler so that this side passes through C. Draw line (GH in the diagram) along side of set square to give line parallel to AB.

Exercise 23c

Use ruler and compasses.

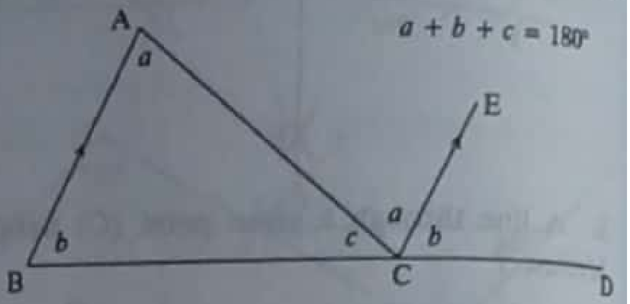
1. Construct a triangle with angles 30°, 60° and 90° and longest side 10cm.
2. Construct triangle PQR in which QR = 12cm, PQ = 8cm and ∠PQR = 60°. From P construct the perpendicular to QR meeting QR at S. Construct the bisector of ∠PRQ to meet PS at X. Measure and write down the length of QX.
3. Construct the quadrilateral ABCD such that AB = 7cm, AD = 4cm, DC = 5cm, BC = 4cm and ∠BAD = 45°. Measure and write down the lengths of BD and AC.
4. Construct triangle ABC in which AB = 5cm, BC = 9cm and ∠ABC = 60°. At the point C construct a perpendicular to BC. By construction find the points P and Q which lie on this perpendicular and are 7cm from A. Measure and write down the length of PQ.
5. Construct triangle ABC with sides 7cm, 8cm and 9cm. Construct the mediator of each side and let these meet at O. With centre O draw a circle with radius equal to OV where V is one vertex of the triangle. What do you notice? (See **Ex57a Q15**)
6. Repeat 5 but construct the angle bisectors of each angle of the triangle. Do they meet at a point? Of what circle is this the centre? (See **Ex57a Q16**)
7. Repeat 5 but construct the perpendicular from each vertex to the opposite side. Do they meet at a point?

24 POLYGONS

24.1 Triangles

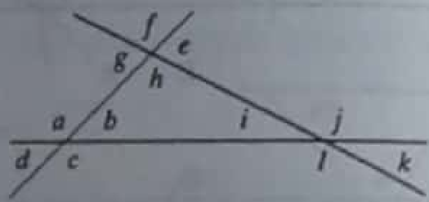
A **scalene** triangle has sides of different lengths, an **isosceles** triangle has two equal sides and an **equilateral** triangle has all three sides equal. An **acute-angled** triangle has each angle less than 90°, a **right-angled** triangle has one angle equal to 90° and an **obtuse-angled** triangle has one angle greater than 90°.

Angle sum: The sum of the interior angles of a triangle is 180°. The outline proof is shown in the diagram.



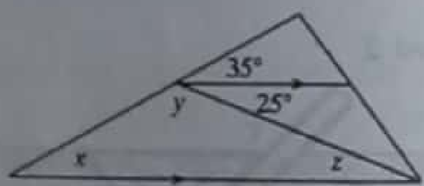
Exercise 24a

1 In the diagram $a = 150°$ and $f = 110°$. Find all the other angles.



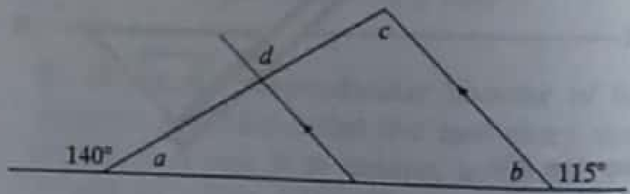
2 Find the angles of a triangle which are $3x°$, $4x°$ and $5x°$.

3 Find x, y and z in the figure.



4 If a triangle has its angles in the ratio $1 : 2 : 3$ find the difference between the smallest and largest angle.

5 Find a, b, c and d in the figure.



24.2 Angle Sum of Polygons

A **polygon** is a many-sided plane figure. Triangles, quadrilaterals, pentagons and so on., are all examples of polygons. A polygon with n sides can be divided into $(n - 2)$ triangles each with a sum of 180° or 2 rt $\angle$s. The sum of the angles of all these triangles is therefore $180(n - 2)°$ or $(2n - 4)$ rt $\angle$s.

Example 1 A decagon has ten sides. What is the sum of its interior angles?

Sum of interior angles is $(10 - 2) \times 180° = 8 \times 180° = 1{,}440°$

The exterior angles of a polygon are formed by continuing one side at each vertex as shown. If the polygon was drawn on the floor then a person starting at A and walking clockwise round the polygon would turn through each exterior angle and on returning to A would have made a complete revolution of 360°. Hence for any polygon the sum of the exterior angles is 360°.

A **regular** polygon has all its sides equal in length and all its interior angles equal. An equilateral triangle and a square are the most common examples of a regular polygon.

Example 2 Find the size of the interior angles of a regular decagon.

The sum of the exterior angles is 360°.
Each exterior angle is therefore 360° ÷ 10 = 36°.
At each vertex the exterior and interior angles add up to 180° (ie. they are supplementary).
Each interior angle is therefore 180° − 36° = 144°.

If the decagon of **Example 1** was regular then each interior angle would be 1,440 ÷ 10 = 144°.

Exercise 24b

1 The angles of a pentagon are $4x°$, $5x°$, $6x°$, $7x°$ and $8x°$. What are they?
2 Find the sum of the interior angles of a 17-sided polygon.
3 A regular polygon has interior angles of 162°. How many sides does it have?
4 Find the number of sides of a regular polygon with an exterior angle of 10°.
5 The angles of a hexagon are $4x°$, $(5x - 10)°$, $6x°$, $(7x - 40)°$, $8x$ and $(9x - 10)°$. Find the value of x and the size of the six angles.

24.3 Quadrilaterals

A four-sided plane figure is a **quadrilateral**. Squares, rectangles, parallelograms, rhombi and kites are all examples of quadrilaterals. For cyclic quadrilaterals, see **30.3**.

24.4 Regular Pentagons, Hexagons and Octagons

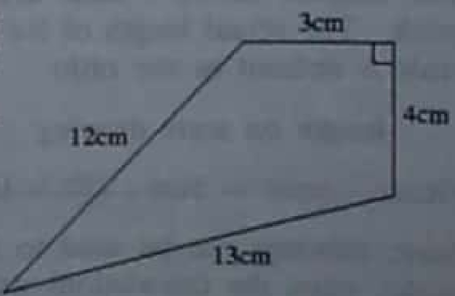
A regular pentagon has five equal sides and five interior angles each of 108°. A regular hexagon has six equal sides and interior angles of 120°. A regular octagon has eight equal sides and interior angles of 135°.

Exercise 24c

Draw diagrams to help you in **1** to **4**.
1 Two sides AB, DC of a regular pentagon ABCDE are produced to meet at H. Calculate the measure of the angle at H.
2 ABCDE is a regular pentagon. Calculate the sizes of the angles of triangles ABC and ACD.
3 ABCDEF is a regular hexagon. Calculate the angles of triangle ACD.
4 ABCDEFGH is a regular octagon. Calculate the angles of triangles ABD, ADG and AEF.

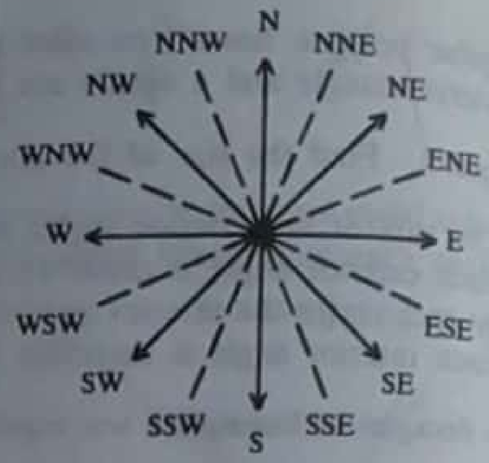
5 Draw accurately a regular pentagon, hexagon and octagon. Join the mid-points of adjacent sides in each of your drawings. What are the figures called that are so formed?

6 Find the area of the quadrilateral shown in the diagram.



25 BEARINGS

25.1 Compass points



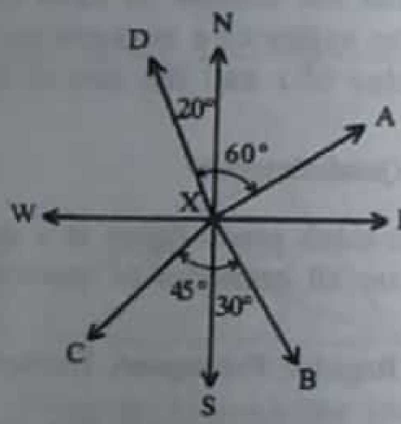
The compass has sixteen **points** as shown. The angle between directions that are next to each other is 360 ÷ 16 or $22\frac{1}{2}°$.

Exercise 25a

In **1** to **10** give the angle turned through in a clockwise direction.

1 N to SE **2** NNE to ESE **3** NE to SW **4** E to NE **5** S to E
6 SSW to S **7** SW to SSW **8** WSW to ENE **9** SSE to ENE **10** NW to WSW

25.2 Bearings



A **bearing** gives the direction of one point from another as (i) an angle measured from the North-South direction towards the East or West or (ii) a *three* digit angle measured *clockwise* from the North.

In the figure, the bearings of A, B, C and D from the point X are (i) N 60° E (read as *North sixty degrees East*), S 30° E, S 45° W (or SW) and N 20° W respectively in N-S notation, or (ii) 060°, 150°, 225° and 340° in three-digit angle notation.

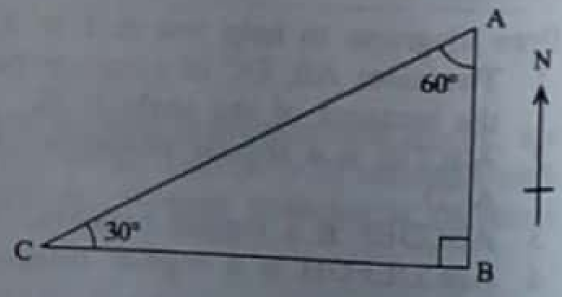
Exercise 25b

In **1** to **10** give the compass directions in (i) N-S bearing notation, (ii) three-digit angle notation.

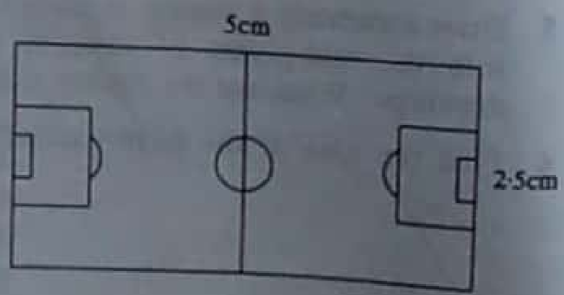
1 ENE **2** SE **3** E **4** SSW **5** WSW **6** NW **7** NNW **8** S **9** W **10** N

11 In the diagram find the bearing, in three-digit angle notation, of

(i) B from C (ii) A from C
(iii) B from A (iv) C from A
(v) C from B (vi) A from B



25.3 Scale Drawing



The diagram shows a **scale drawing** of a football pitch. The actual length of the pitch is 50m. The scale is defined as the ratio

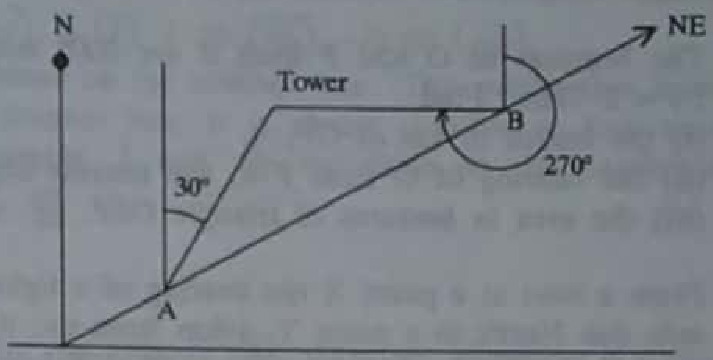
length on scale drawing : actual length

Hence scale = 5cm : (50 × 100cm) or 1 : 1000.

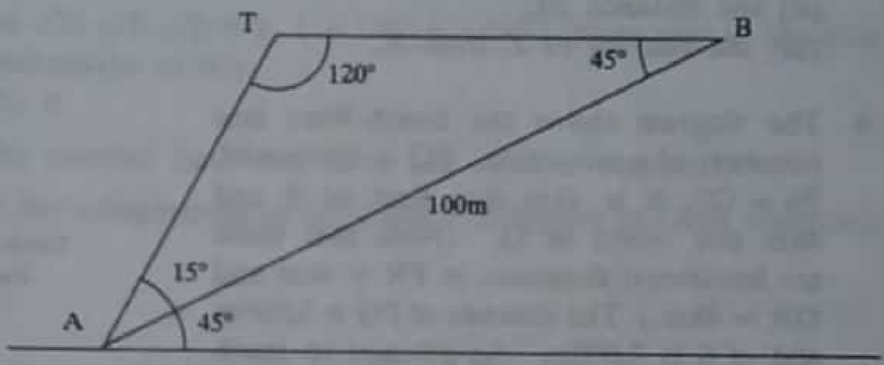
Scale drawings can be used to find distances and angles when the calculations would be difficult.

Example A straight road runs in the direction NE. From a point A on this road a water tower has a bearing of 030° and from a point B 100m along the road it has a bearing of 270°. Find the shortest distance of the tower from the road.

Step 1. Make a rough sketch of the situation.

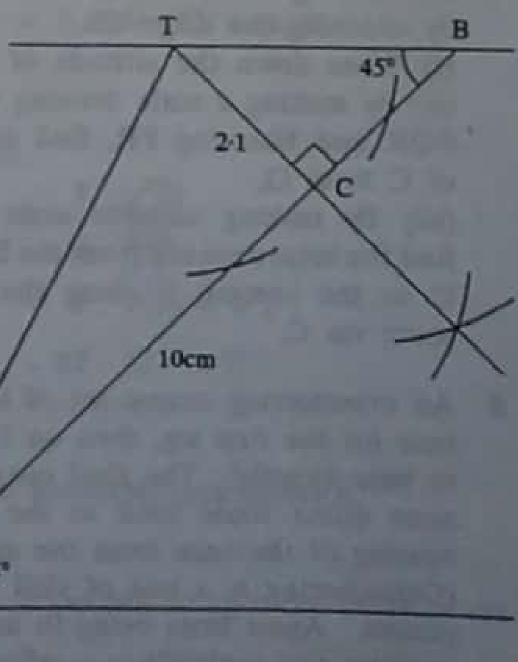


Step 2 Put all the known angles and lengths on triangle ABT.



Step 3 Choose a suitable scale. This means not too small, which would make reading lengths and angles difficult and not too large which would mean it would not fit on the paper. In this case a scale of 1cm to 10m or 1 : 1,000 is suitable.

Step 4 Draw triangle ABT accurately to scale.



The shortest distance of the tower from the road is the distance from the tower which is *perpendicular* to the road.

Construct a perpendicular (see **23.3** Constr. 7.) to AB to pass through T.
Measure TC.

It is 2·1cm.

Therefore the actual distance

$$= \frac{2{\cdot}1 \times 1000}{100} \text{ m}$$

= 21m (to the nearest metre)

NB Always use a sharp pencil when making scale drawings.

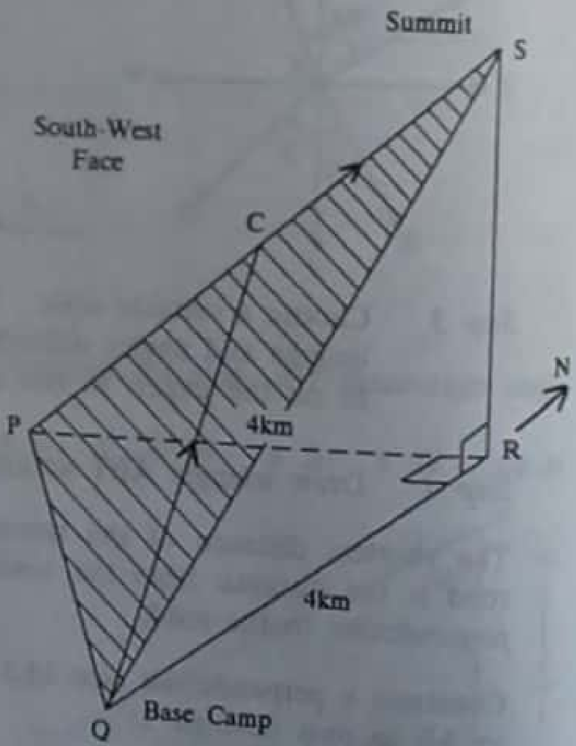
Exercise 25c

1 From a point A on a straight stretch of shoreline running EW a ship has a bearing 060°, and from B a point 500m East of A it has a bearing 300°. What is the distance of the ship from the shoreline?

2 The bearings of O and P from S are 200° and 290° respectively and SO = 3·6km and PS = 2·7km. Find
(i) the length in km of OP,
(ii) the bearing of O from P to the nearest degree,
(iii) the area in hectares of triangle OSP.

3 From a boat at a point X the bearing of a lighthouse at a point L is S 68° E. The boat sails due North to a point Y, 10km from the lighthouse. The bearing of the lighthouse from Y is S 30° E. The boat then sails NE from Y for 8km to a point Z. Find by scale drawing,
(i) the distance XY
(ii) the distance ZL,
(iii) the bearing of Z from X.

4 The diagram shows the South-West face (shaded) of a mountain. PQ is horizontal, PS = QS, S is 4km due East of P and 4km due North of Q. (Note that these are horizontal distances, ie PR = 4km and QR = 4km.) The altitude of PQ is 3,000m and of S is 7,000m. An attempt to reach the summit S from the Base Camp Q is to be made by a team of mountaineers by going from Q to C, where C is half way between P and S, establishing a camp at C and making the final assault on the summit by climbing the ridge CS.
(i) Write down the altitude of C,
(ii) By making a scale drawing of triangle PQR and bisecting PR, find the bearing of C from Q,
(iii) By making suitable scale drawings, find the total distance from the Base Camp Q to the summit S along the proposed route via C.



5 An orienteering course set of instructions reads 'Proceed on bearing 060° for 1km from base for the first leg, then on 180° for a distance of 2km for the second leg, then return to base directly'. The final instruction 'return to base directly' in the third leg means the most direct route back to the starting point. What is this distance back to and the bearing of the base from the end of the second leg?
(Orienteering is a test of skill in finding your way round a course planned by another person. Apart from being fit and healthy, all you need is a map of the area, a magnetic compass and a whistle as a safety precaution. Competitors are timed and the fastest is the winner.)

26 SURDS

26.1 Rational Numbers

A number is said to be **rational** if it can be expressed as the ratio of two integers. Examples are $\frac{3}{4}$, $\frac{5}{2}$, $3\frac{2}{3}$ $\left(\frac{11}{3}\right)$, 5 $\left(\frac{5}{1}\right)$, -3 $\left(\frac{-3}{1}\right)$, $-\frac{2}{3}$ $\left(\frac{-2}{3}\right)$, 0 $\left(\frac{0}{1}\right)$, $2{\cdot}37$ $\left(\frac{237}{100}\right)$, $-0{\cdot}03$ $\left(\frac{-3}{100}\right)$.
The members of {rational numbers} are **dense** on the number line. This means that however close two rational numbers are on the number line, it is always possible to find another rational number between them. For example, $\frac{1}{2}$ and $\frac{51}{100}$ are close on the number line. Between them is their mean $\frac{1}{2}\left(\frac{1}{2} + \frac{51}{100}\right)$, ie. $\frac{101}{200}$.

26.2 Irrational Numbers

Although the rational numbers lie dense on the number line, there are points on it which do not correspond to any rational number. An example is $\sqrt{2}$. It may be shown that $\sqrt{2}$ cannot be expressed in the form $\frac{a}{b}$, where a and b are integers. Numbers on the number line which are not rational are **irrational**.
Examples of irrational numbers are $\sqrt{2}$, $\sqrt{3}$, $\sqrt{2} + 1$, $1 - \sqrt{5}$, π, $\sqrt{50}$. (Note that in the case of π, $\frac{22}{7}$ is merely a rational approximation to it!)
Using set notation (see **55.1**, **55.2**), if

$$\mathcal{E} = \{\text{all numbers on the number line}\} \text{ and } Q = \{\text{all rational numbers}\}$$

then {all irrational numbers} is Q' the complement of Q. Here $\mathcal{E}$ is known as {**real** numbers} and is usually denoted by R.

26.3 Simplification of Surds

Irrational numbers such as $\sqrt{2}$, $\sqrt{50}$, $1 - \sqrt{3}$ are called **surds**. Some surds may be simplified, for example

$$\sqrt{50} = \sqrt{(25 \times 2)} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$$

Example Simplify $\sqrt{27} + \frac{\sqrt{6}}{\sqrt{2}}$ leaving the answer in surd form.

$$\sqrt{27} + \frac{\sqrt{6}}{\sqrt{2}} = \sqrt{(9 \times 3)} + \sqrt{\left(\frac{6}{2}\right)} = \sqrt{9} \times \sqrt{3} + \sqrt{3} = 3\sqrt{3} + \sqrt{3} = 4\sqrt{3}$$

Exercise 26a

Simplify, leaving your answer in surd form where necessary.

1 $\sqrt{8}$	2 $\sqrt{18}$	3 $\sqrt{72}$
4 $\sqrt{75}$	5 $\sqrt{2} + \sqrt{32}$	6 $\frac{\sqrt{20}}{\sqrt{5}}$
7 $243^{\frac{1}{2}} - 27^{\frac{1}{2}}$	8 $\frac{\sqrt{28}}{\sqrt{7}}$	9 $\sqrt[3]{16} + \sqrt[3]{2}$
10 $\frac{\sqrt{54} + 2\sqrt{48} + \sqrt{3}}{\sqrt{27}}$	11 $(1 + \sqrt{2})^2$	12 $(2 - \sqrt{3})^2$

26.4 Rationalisation of Denominator

This procedure is used to simplify a fraction which has an irrational denominator.

Example 1 Simplify $\frac{4}{\sqrt{2}}$

Since $\sqrt{2} \times \sqrt{2} = 2$, multiply the denominator by $\sqrt{2}$. So as not to alter the value of the fraction the numerator is also multiplied by $\sqrt{2}$.

$$\therefore \quad \frac{4}{\sqrt{2}} = \frac{4 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

Example 2 Express $\frac{1+\sqrt{5}}{3+\sqrt{5}}$ in the form $a + b\sqrt{5}$ where a and b are rational.

The denominator $3 + \sqrt{5}$ is rationalised by multiplying by its **conjugate surd** $3 - \sqrt{5}$, since the difference of two squares (see **34.3**) gives

$$(3 + \sqrt{5})(3 - \sqrt{5}) = 3^2 - (\sqrt{5})^2 = 9 - 5 = 4$$

$$\begin{aligned} \frac{1+\sqrt{5}}{3+\sqrt{5}} &= \frac{(1+\sqrt{5})(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})} \\ &= \frac{3 + 3\sqrt{5} - \sqrt{5} - 5}{9 - 5} \\ &= \frac{-2 + 2\sqrt{5}}{4} \\ &= -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5} \text{ in the required form} \end{aligned}$$

Example 3 Given that $\sqrt{2} = 1{\cdot}4142$ (5 sf), evaluate $\frac{\sqrt{2}-1}{3\sqrt{2}-4}$ to 5 sf.

$$\begin{aligned} \frac{\sqrt{2}-1}{3\sqrt{2}-4} &= \frac{(\sqrt{2}-1)(3\sqrt{2}+4)}{(3\sqrt{2}-4)(3\sqrt{2}+4)} \quad \text{rationalising the denominator} \\ &= \frac{6 + 4\sqrt{2} - 3\sqrt{2} - 4}{(3\sqrt{2})^2 - 4^2} \\ &= \frac{2+\sqrt{2}}{18-16} = \frac{2+\sqrt{2}}{2} = \frac{2 + 1{\cdot}4142}{2} = \frac{3{\cdot}4142}{2} = 1{\cdot}7071 \text{ (5 sf)} \end{aligned}$$

Exercise 26b

In **1** to **8** simplify by rationalising the denominator.

1 $\frac{3}{\sqrt{3}}$ **2** $\frac{20}{\sqrt{5}}$ **3** $\frac{2-\sqrt{2}}{\sqrt{2}}$

4 $\frac{6}{4+\sqrt{10}}$ **5** $\frac{6}{4-\sqrt{10}}$ **6** $\frac{1+\sqrt{3}}{2-\sqrt{3}}$

7 $\frac{\sqrt{3}-1}{\sqrt{3}+1}$ **8** $\frac{\sqrt{2}}{\sqrt{2}-1}$

Evaluate in **9** to **18** given $\sqrt{2} = 1{\cdot}41421$ and $\sqrt{3} = 1{\cdot}73205$ (to 6 sf).

9 $\frac{1}{\sqrt{2}}$ **10** $\frac{1}{\sqrt{3}}$ **11** $\frac{1}{\sqrt{2}+1}$

12 $\frac{\sqrt{12}-\sqrt{3}}{\sqrt{3}}$ **13** $\frac{1}{\sqrt{3}-\sqrt{2}}$ **14** $\frac{\sqrt{6}-\sqrt{2}}{\sqrt{2}}$

15 $\frac{2+\sqrt{3}}{2-\sqrt{3}}$ **16** $\frac{2-\sqrt{3}}{2+\sqrt{3}}$ **17** $\frac{1}{1+\sqrt{2}} - \frac{1}{1-\sqrt{2}}$

18 $\frac{\sqrt{3}}{1+\sqrt{3}} + \frac{\sqrt{3}}{1-\sqrt{3}}$

19 Evaluate $\frac{1}{\sqrt{20}}$ given $\sqrt{5} = 2{\cdot}23607$ (to 6 sf).

20 Given that $\frac{3-2\sqrt{3}}{3+2\sqrt{3}} - \frac{3+2\sqrt{3}}{3-2\sqrt{3}} = a + b\sqrt{3}$ find rational values of a and b.

27 FURTHER LOGARITHMS

27.1 Laws of Logarithms

These are obtained from the definition of base 10 logarithms (see **15.2**) by applying the laws of indices.

First Law $\quad \log a + \log b = \log ab$

Let $\log a = x$ and $\log b = y$
then $a = 10^x$ and $b = 10^y$ by definition of a logarithm

$$\therefore \quad ab = 10^x \times 10^y$$

$$\therefore \quad ab = 10^{x+y} \quad \text{from the } \textit{First Law} \text{ of indices}$$

$$\therefore \quad x + y = \log ab \quad \text{from the definition of a logarithm}$$

$$\therefore \quad \log a + \log b = \log ab$$

Second Law $\quad \log a - \log b = \log \frac{a}{b}$

Proceeding as above we have $\quad \frac{a}{b} = 10^x \div 10^y$

$$\therefore \quad \frac{a}{b} = 10^{x-y} \quad \text{from the } \textit{Second Law} \text{ of indices}$$

$$\therefore \quad x - y = \log \frac{a}{b} \quad \text{from the definition of a logarithm}$$

$$\therefore \quad \log a - \log b = \log \frac{a}{b}$$

Third Law $\quad \log a^n = n \log a$

Let $\log a = x$ then $a = 10^x$ by definition

$$\therefore \quad a^n = (10^x)^n \quad \text{raising both sides to the } n\text{th power}$$

$$\therefore \quad (a^n) = 10^{nx} \quad \text{from the } \textit{Third Law} \text{ of indices}$$

$$\therefore \quad \log (a^n) = nx \quad \text{from the definition of a logarithm}$$

$$\therefore \quad \log a^n = n \log a$$

Example Express $\log \frac{x^3}{\sqrt{y}}$ in terms of $\log x$ and $\log y$.

$$\log \frac{x^3}{\sqrt{y}} = \log x^3 - \log \sqrt{y} \quad \text{from the } \textit{Second Law} \text{ of logarithms}$$

$$= \log x^3 - \log y^{\frac{1}{2}}$$

$$= 3 \log x - \tfrac{1}{2} \log y \quad \text{from the } \textit{Third Law} \text{ of logarithms}$$

Exercise 27a

Express the logarithms of the following in terms of $\log x$ and $\log y$.

1 xy^2	**2** x^4y^{-3}	**3** $x \times \sqrt{y}$	**4** $x^2 \div y$	**5** $\sqrt{(xy)}$
6 $\frac{\sqrt[3]{x}}{y^3}$	**7** $10xy^{-1}$	**8** $\frac{10}{xy}$	**9** $y \times \sqrt{(10x^3)}$	**10** $\frac{\sqrt[4]{(xy^3)}}{100}$

27.2 Computation

When using logarithms for calculation purposes, the laws are applied.

Example 1 Calculate $1{\cdot}14^5 \times 1{,}750$

$$\log(1{\cdot}14^5 \times 1{,}750) = 5 \log 1{\cdot}14 + \log 1{,}750$$
$$= 5 \times 0{\cdot}057 + 3{\cdot}243$$
$$= 3{\cdot}528$$

$$\therefore \quad 1{\cdot}14^5 \times 1{,}750 = \text{antilog } 3{\cdot}528$$
$$= 3{,}370$$

No	Log
1·14	0·057
	×5
	0·285
1750	3·243
3370	3·528

The calculation is normally set out as shown on the right.

Example 2 Calculate $\dfrac{\sqrt[4]{(0{\cdot}147)} \times 2{\cdot}53}{0{\cdot}0828}$

Use the fact that $\log \dfrac{\sqrt[4]{(0{\cdot}147)} \times 2{\cdot}53}{0{\cdot}0828} = \frac{1}{4}\log 0{\cdot}147 + \log 2{\cdot}53 - \log 0{\cdot}0828$

and calculate as shown on the right.

No	Log
0·147	$\bar{1}{\cdot}167$
	$\bar{4}{\cdot}\,^{3}167 \div 4$
	$\bar{1}{\cdot}792$
2·53	$0{\cdot}403$
	$0{\cdot}195$
0·0828	$\bar{2}{\cdot}918$
18·9	$1{\cdot}277$

So $\dfrac{\sqrt[4]{(0{\cdot}147)} \times 2{\cdot}53}{0{\cdot}0828} \simeq 18{\cdot}9$

Note To divide $\bar{1}{\cdot}167$ by 4, increase the number under the bar so that it is divisible by 4, ie. write $\bar{1}{\cdot}167$ as $\bar{4} + 3{\cdot}167$. (See also **13.2 Example 3**)

Exercise 27b

Use tables to calculate the following.

1 $\dfrac{33{\cdot}7 \times 42{\cdot}9}{76{\cdot}1}$

2 $\dfrac{523}{27{\cdot}2 \times 4{\cdot}86}$

3 $\dfrac{4{\cdot}76 \times 0{\cdot}0273}{58 \times 0{\cdot}00629}$

4 $575 \times 1{\cdot}09^6$

5 $0{\cdot}324 \times \sqrt[5]{(8{\cdot}26)}$

6 $\dfrac{1}{\sqrt{(20{\cdot}5)}}$

7 $\sqrt[3]{\left(\dfrac{4{\cdot}06}{153}\right)}$

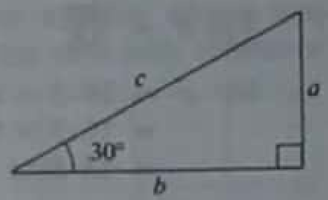
8 $\dfrac{3}{0{\cdot}63 \times \sqrt{(0{\cdot}357)}}$

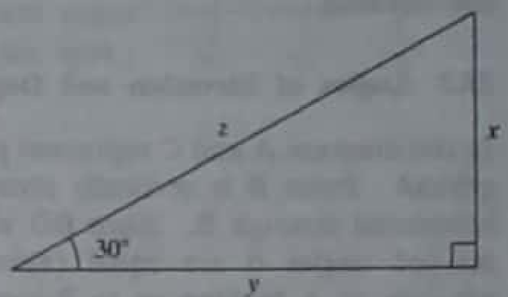
9 $\dfrac{4{\cdot}2^4 - 3{\cdot}2^4}{4{\cdot}2^2 - 3{\cdot}2^2}$

10 $\sqrt[5]{(0{\cdot}73^5 + 0{\cdot}27^5)}$

28 TRIGONOMETRY

28.1 Sine, Cosine and Tangent

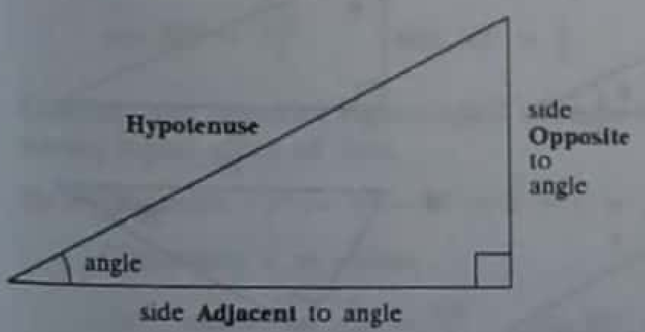




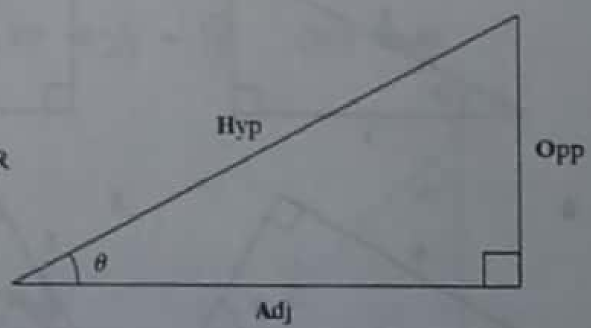
Trigonometry is the mathematical study of the triangle. Right-angled triangles are considered first.
The triangles shown have two angles equal in each and so they are **similar** (see **37.1**). Therefore the ratios of corresponding sides are equal, for example $\frac{a}{c} = \frac{x}{z}$. Choosing suitable measurements, make an accurate drawing of one of the triangles and find the value of this ratio. It is $\frac{1}{2}$ or 0·5. The value is dependent upon the 30° angle and is called the **Sine** of this angle. So Sine 30° $= \frac{1}{2} = 0{\cdot}5$. Check that this is the value given in your table of Natural Sines. This table gives sines of angles between 0° and 90°.
Other basic trigonometric ratios are:

Cosine 30° $= \frac{b}{c} = \frac{y}{z}$ and **Tangent** 30° $= \frac{a}{b} = \frac{x}{y}$

Determine the value of these ratios from your drawing and check your accuracy in Natural Cosine and Natural Tangent tables. You should find from tables that Cosine 30° = 0·866 and Tangent 30° = 0·577. For the purposes of memorising the ratios, the sides of the triangle are referred to as shown in the diagrams below.



OR

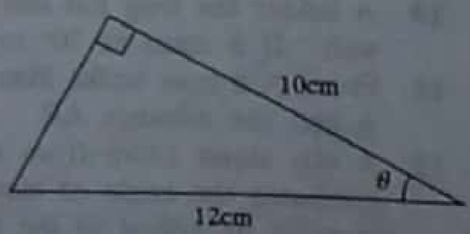


Abbreviations for Sine, Cosine and Tangent are Sin, Cos and Tan.

$$\text{Sin } \theta = \frac{\text{Opp}}{\text{Hyp}} \qquad \text{Cos } \theta = \frac{\text{Adj}}{\text{Hyp}} \qquad \text{Tan } \theta = \frac{\text{Opp}}{\text{Adj}}$$

The mnemonic (memory aid) OŜHAĈHOT̂A may be helpful.

Example 1 Find θ.

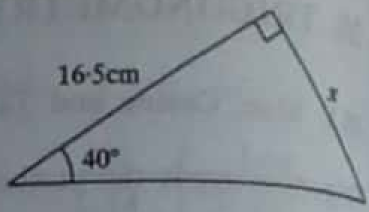


Adjacent = 10cm, Hypotenuse = 12cm

$\cos\theta = \frac{\text{Adj}}{\text{Hyp}} = \frac{10}{12} = 0{\cdot}833$

So $\theta = 33{\cdot}6°$ from tables

Example 2 In the diagram, find x.



Opposite $= x$, Adjacent $= 16\cdot5\text{cm}$

$\text{Tan}\,\theta = \dfrac{\text{Opp}}{\text{Adj}}$ $\therefore$ $\tan 40° = \dfrac{x}{16\cdot5}$

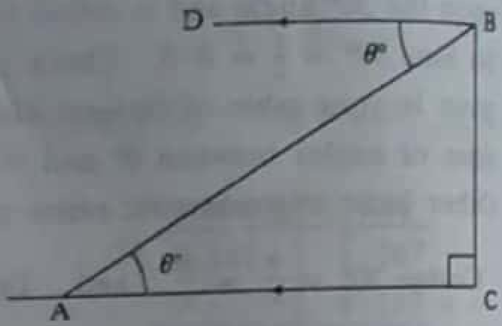
So $x = 16\cdot5 \times \tan 40°$
$= 16\cdot5 \times 0\cdot839$ from tables
$= 13\cdot8\text{cm}$ using logarithms

No	Log
16·5	1·217
0·839	$\bar{1}$·924
13·8	1·141

Note that a log tangent table could be used to shorten the working.

28.2 Angles of Elevation and Depression

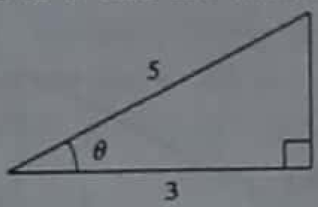
In the diagram A and C represent points on horizontal ground. Point B is vertically above C and BD is the horizontal through B. Since BD is parallel to CA the marked angles θ are equal (alternate angles). An observer at A looking up to B elevates his eyes by $\theta°$. Angle CAB is called the **angle of elevation**. An observer at B looking down at A depresses his eyes by $\theta°$. This angle ($\angle$DBA) is called the **angle of depression**.



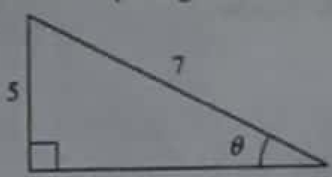
Exercise 28a

In **1** to **9** find the unknowns indicated. (Lengths are in cm)

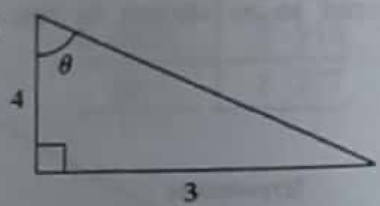
1



2



3



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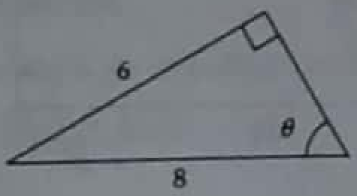

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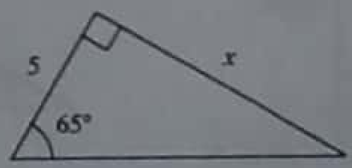
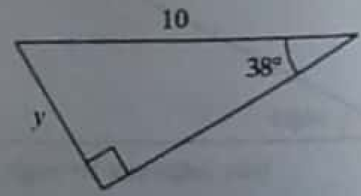
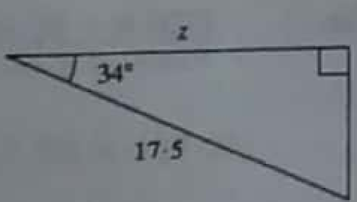


6



7



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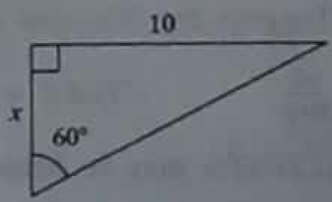
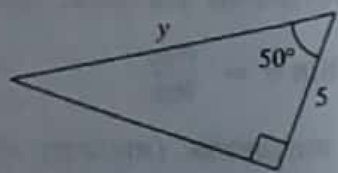


9



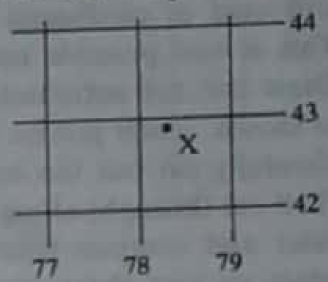
10 A ladder 6m long has one end on horizontal ground and the other end against a vertical wall. If it leans at 70° to the horizontal, how far up the wall does it reach?

11 From A a man walks 3km North then 4km East to B. Calculate the bearing of B from A and the distance AB.

12 A boy stand 150m from the foot of the Kenyatta Conference Centre in Nairobi. He estimates the angle of elevation of the top of the building to be 30°. What would this give for its height to the nearest 5m?

13 From a helicopter hovering at an altitude of 700m above horizontal ground the angle of depression of an object on the ground is 5°. What horizontal distance would the pilot have to fly so that his machine will be directly above the object?

14 Tumusiime uses a map in which grid lines are 1km apart. From a point whose map reference is 574646 he walks 9km on a bearing of 130°. Calculate his new map reference. (A map has a grid of squares drawn on it. The grid lines are normally 1km apart and are numbered. To describe the position of a place on the map we can give its **map reference** which contains six digits. The first three digits refer to the **easting** and the second three to the **northing**. The system is similar to the xy-coordinate system. For example the reference 783429 refers to the point (78·3, 42·9). In the diagram, X marks the spot.)

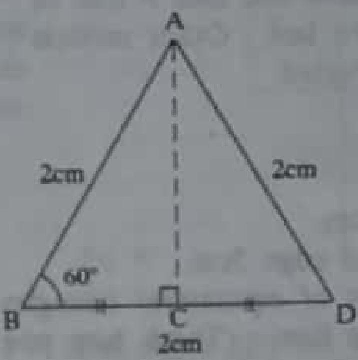


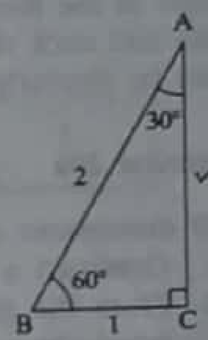
28.3 Ratios of 30°, 45° and 60°

Consider the equilateral triangle ABD of side 2cm, shown. C is the mid-point of AD, so BC = 1cm.
In the right-angled $\triangle$ABC, by Pythagoras:

$$AC^2 = AB^2 - BC^2$$
$$= 2^2 - 1^2 = 4 - 1 = 3$$

Therefore $AC = \sqrt{3}$cm





From $\triangle$ABC (shown separately) we have

$\cos 60° = \frac{1}{2}$ $\quad\sin 60° = \frac{\sqrt{3}}{2}$ $\quad\tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\cos 30° = \frac{\sqrt{3}}{2}$ $\quad\sin 30° = \frac{1}{2}$ $\quad\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ (See 26.4)

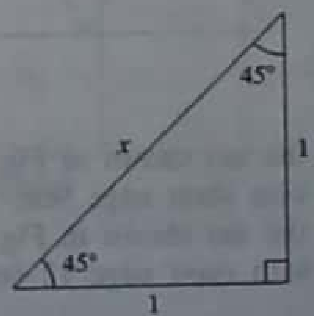
Consider the isosceles right-angled triangle shown having equal sides of 1cm.

By Pythagoras: $x^2 = 1^2 + 1^2 = 1 + 1 = 2$

Therefore $x = \sqrt{2}$cm

So $\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\tan 45° = \frac{1}{1} = 1$

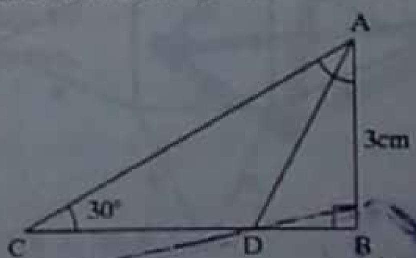


Exercise 28b

1 An isosceles triangle has its equal sides 6cm and base angles 30°. Calculate its area to the nearest mm². (Take $\sqrt{3} = 1{\cdot}732$)

2 A rhombus has side 4cm and one angle 45°. Calculate its area to the nearest mm². (Take $\sqrt{2} = 1{\cdot}414$)

3 The figure shows a right-angled $\triangle$ABC in which $\angle ACB = 30°$, AB = 3cm and AD bisects $\angle$BAC. Show that the area of $\triangle ADC = 3\sqrt{3}$cm².



29 SOLIDS

29.1 Nets of Solids

A solid such as a **cube** may may be constructed by first drawing its **net** on manilla or other stiff card or cardboard.
This is one possible net. (See Fig. 1)
Draw the net accurately. Place large tabs T as shown. Their precise shape is not important. Carefully cut out the net with scissors. Score (half cut through) along the broken lines using ruler and compass-point. Fold along scored edges so that these appear on the *outside* of the model. Join edges by gluing tabs to the *inside* of the model. Leave one face F free of tabs and stick that down last. Other models may be similarly constructed.

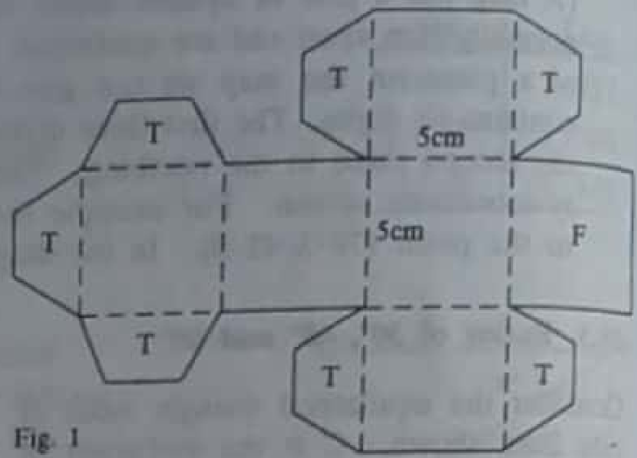


Fig. 1

Exercise 29a

All dimensions are in cm.

1 Construct a cube of edge 5cm.
2 From the given net of equilateral triangles, shown in Fig. 2 below, construct a regular **tetrahedron** of edge 8cm. (Think how you are going to place the tabs.)
3 Construct a **cuboid** from the net shown in Fig. 3.


Fig. 2

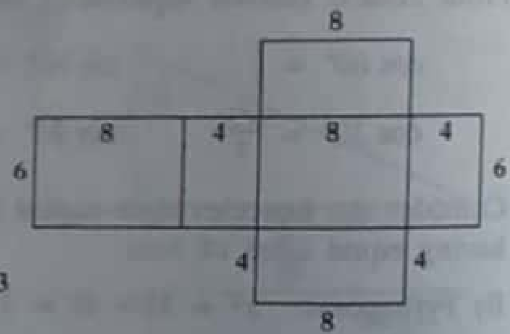


Fig. 3

4 Use the net shown in Fig. 4 below to construct a **pyramid on a square base** of side 8cm and with slant edge 9cm.
5 Use the net shown in Fig. 5 to construct a **pyramid on a rectangular base** 12cm by 9cm and with slant edge 12·5cm.

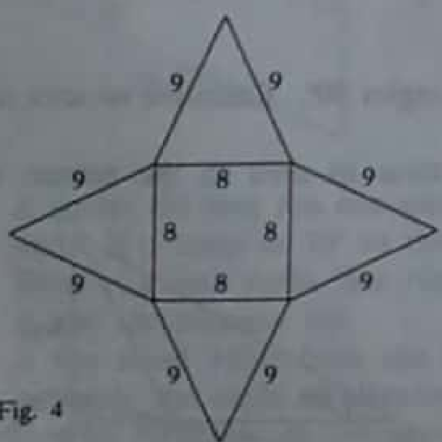


Fig. 4

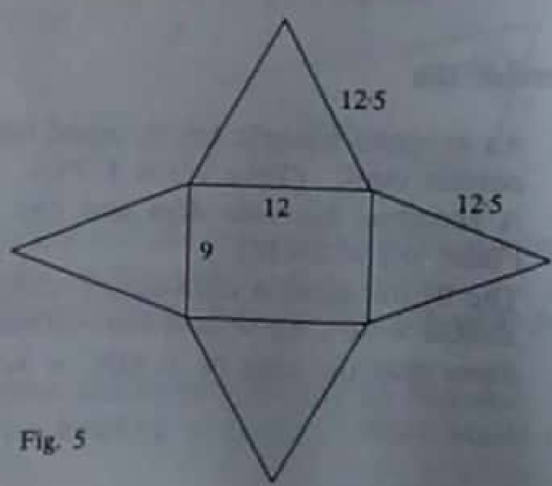


Fig. 5

6 Construct a **triangular prism** from the net given in Fig. 6.

7 From an A4 sheet of paper, cut out the rectangle and tab shown in Fig. 7. Do not score or fold the tab. Glue tab so as to form a **cylinder** of height 21cm and circumference 22cm.

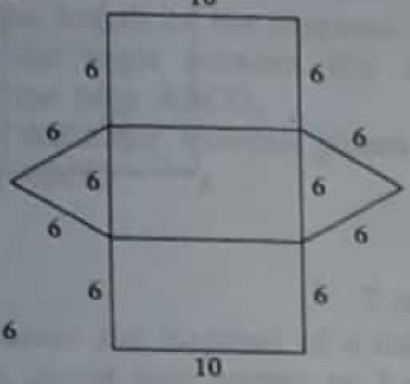


Fig. 6

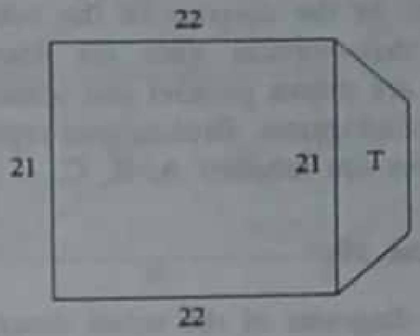


Fig. 7

8 Fig. 8 shows another possible net which could be used to form a cube. Draw as many different cube nets as you can. (There are 12 altogether.)

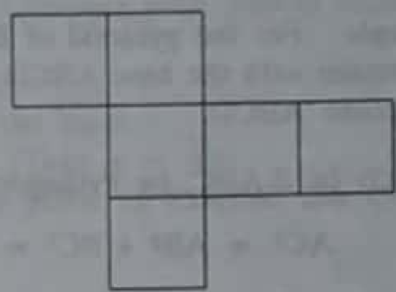


Fig. 8

9 Count the number of faces (F), vertices (V) and edges (E) for each of the models you have made and complete the table below.

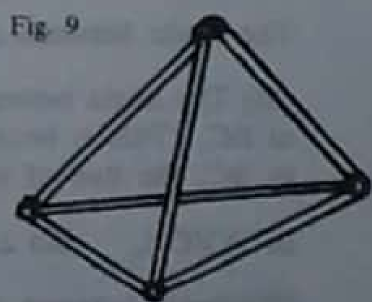
Solid	F	V	E	F + V	E + 2
Cube	6	8	12	14	14
Tetrahedron					
Cuboid					
Square-based pyramid					
Rectangular-based pyramid					
Triangular prism					

What is the relation between F + V and E + 2 for each solid?
This is called **Euler's Relation**.

29.2 Skeleton Solids

These are outline solids in which only the edges are seen. For example, a regular tetrahedron can be made from 6 matchsticks by gluing the ends together with quick drying glue. (See Fig. 9) For larger models, drinking straws cut to suitable lengths may be used. These can be either glued together or fastened together by passing cotton thread through them. For most solids extra straws, for example diagonals, should be added to obtain rigidity.

Fig. 9



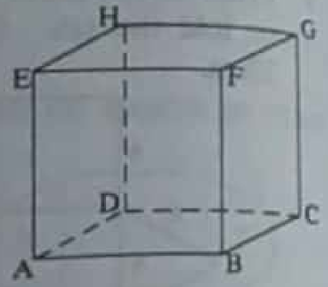
Exercise 29b

Make skeleton models of the solids described in **Exercise 29a 1 to 6**.

29.3 Drawing Solids

Diagrams of most solids are best drawn on squared paper. In the diagram of the cube, shown in Fig. 10, note that vertical lines are drawn vertical, parallel edges are drawn parallel and some square faces appear as parallelograms. Broken lines represent 'hidden' edges. Vertices are labelled A, B, C, . . .

Fig. 10



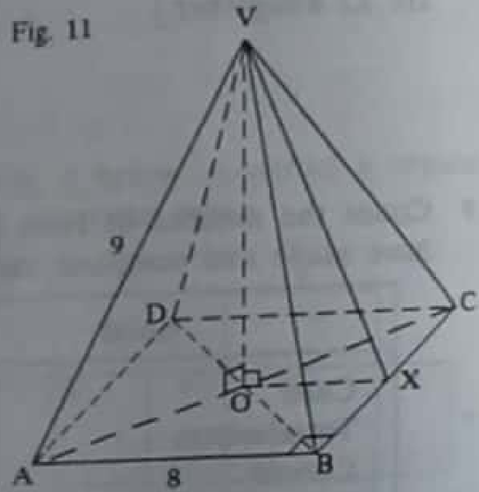
Exercise 29c

Draw diagrams of the solids described in **Exercise 29a 1 to 7**

29.4 Three-Dimensional Problems

Example For the pyramid of **Exercise 29a Q4** calculate (i) its height, (ii) the angle that edge AV makes with the base ABCD (see Fig. 11), (iii) the angle between the triangular plane VBC and plane ABCD.

Fig. 11



(i) In $\triangle ABC$, by Pythagoras,

$$AC^2 = AB^2 + BC^2 = 8^2 + 8^2 = 128$$

$$\therefore \quad AC = \sqrt{128} = 8\sqrt{2} \quad \text{(see 26.3)}$$

$$\therefore \quad AO = \tfrac{1}{2}AC = \tfrac{1}{2} \times 8\sqrt{2} = 4\sqrt{2}$$

In $\triangle AVO$, by Pythagoras,

$$VO^2 = AV^2 - AO^2$$
$$= 9^2 - (4\sqrt{2})^2 = 81 - 32 = 49$$

$$\therefore \quad VO = \sqrt{49} = 7$$

The height of the pyramid is 7cm.

(ii) Point O is the **projection** of point V on plane ABCD because VO is perpendicular to this plane. Also AO is the projection of AV on this plane. The angle between line AV and plane ABCD is the angle between the line and its projection on the plane, ie. $\angle VAO$.

In $\triangle VAO$, $\quad \sin \angle VAO = \frac{VO}{AV} = \frac{7}{9} = 0{\cdot}778 \quad \therefore \quad \angle VAO = 51{\cdot}1^\circ$

The angle between a slant edge and the base is $51{\cdot}1^\circ$.

(iii) The angle between plane VBC and plane ABCD is $\angle VXO$ where X is the mid-point of BC. This is because lines VX and OX lie in each plane and are both perpendicular to BC, the line of intersection of the planes.

In $\triangle VOX$, $\quad \tan \angle VXO = \frac{VO}{OX} = \frac{7}{4} = 1{\cdot}75 \quad \therefore \quad \angle VXO = 60{\cdot}3^\circ$

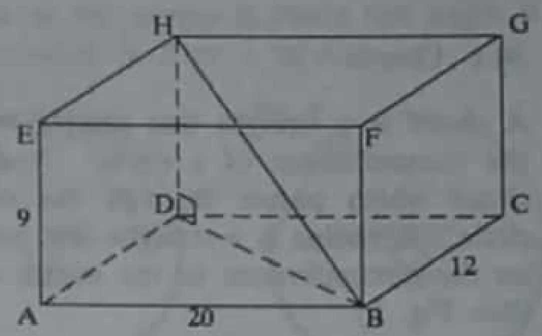
The angle between a triangular face and the base is $60{\cdot}3^\circ$.

Take the corresponding measurements from the skeleton model of **Exercise 29b Q4** and compare these with the above calculated results.

Exercise 29d

1 The diagram (Fig. 12) shows a cuboid 20cm by 12cm by 9cm. Calculate
(i) the length of the diagonal HB,
(ii) the angle between this diagonal and the base ABCD,
(iii) the angle between planes EBCH and ABCD.

Fig. 12



2 Calculate the diagonal of a cube of edge 5cm and the angle this diagonal makes with a face, giving your answer to 3 sf.

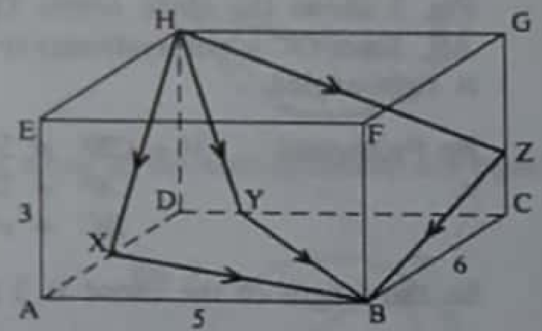
3 Calculate the height of the pyramid of **Exercise 29a Q5**. Compare your answer with the corresponding measurement from your model.
Find also (i) the angle a slant edge makes with the base,
(ii) the angle each triangular face makes with the base.

4 Calculate the radius of the cylinder of **Exercise 29a Q7**. (Take $\pi = \frac{22}{7}$)

5 Calculate (i) the volume, (ii) the total surface area of the prism of **Exercise 29a Q6**.

6 A room is a cuboid 5m by 6m by 3m (see Fig. 13). A lizard wishes to travel from H to B by the shortest possible route. Three routes HXB, HYB and HZB each traversing two faces are shown. By considering the net of the cuboid, determine the positions of X, Y and Z so that the length of each route shall be a minimum. Use Pythagoras to find these lengths. Which route is shortest?

Fig. 13



30 THE CIRCLE

30.1 Chords

A **chord** is a straight line joining two points on the circumference of a circle. A **diameter** is a chord which passes through the centre of the circle. A **radius** is a straight line joining a point on the circumference to the centre of the circle. (See Fig. 1)

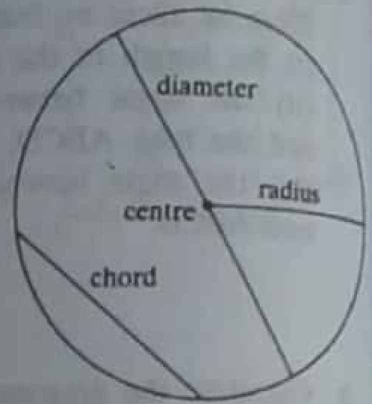



Fig. 1

Some properties of chords are:
(1) The perpendicular bisector of a chord of a circle passes through the centre.
(2) A line joining the mid-point of a chord to the centre of a circle is perpendicular to the chord.
(3) If two chords of a circle are of equal length they are the same distance from the centre.
(4) If two chords are the same distance from the centre of a circle then they are equal in length.

Example Find the length of a chord of a circle of radius 5cm which is 3cm from the centre.

Fig. 2 shows the circle centre O with chord AB. Since OC is perpendicular to AB, $\triangle OCB$ is right-angled.

By Pythagoras $3^2 + CB^2 = 5^2$

$\therefore$ $CB^2 = 25 - 9 = 16$

$\therefore$ $BC = \sqrt{16} = 4$

So the length of the chord is 2×4 or 8cm.

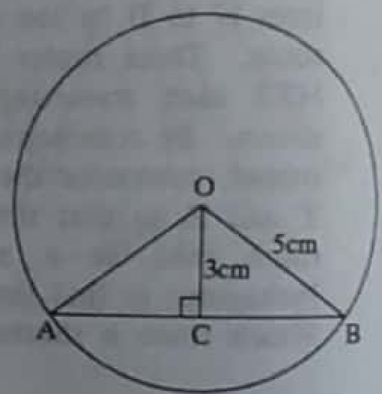


Fig. 2

Exercise 30a

For **1** to **5** refer to Fig. 3.

1 Calculate OR if PQ = 6cm and OQ = 5cm.
2 Calculate OQ if PQ = 24cm and OR = 5cm.
3 Calculate OR if PQ = 30cm and OQ = 17cm.
4 Calculate PQ if OP = 10cm and OR = 6cm.
5 Calculate OP if PQ = 10cm and OR = 5cm.

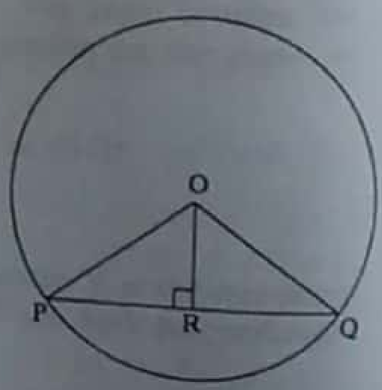


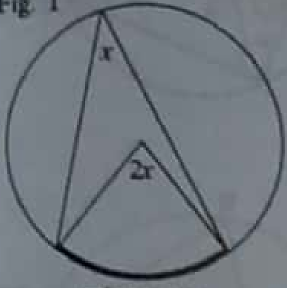
Fig. 3

6 Two parallel chords of a circle of radius 10cm are of length 12cm and 16cm. Calculate their distance apart.
7 A circle is drawn through the four vertices of a rectangle with sides 16cm and 12cm. What is the diameter of the circle?
8 A hemispherical bowl of radius 10cm has water in it that comes to 4cm below the rim of the bowl. Calculate the circular surface area of the water.

30.2 Angle Properties of a Circle

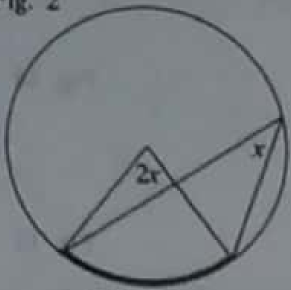
(i) The angle subtended by an **arc** (see **7.6**) of a circle at the centre is twice the angle it subtends at the circumference. This property is illustrated in Figs. 1 to 4.

Fig. 1



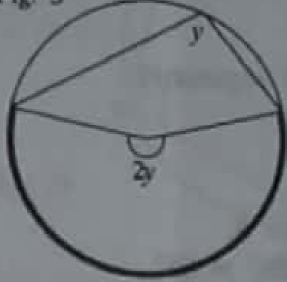
minor arc

Fig. 2



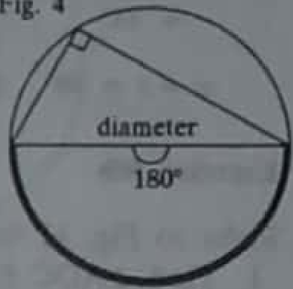
minor arc

Fig. 3



major arc

Fig. 4



semi-circle

The last diagram (Fig. 4) is a special case where the arc is a semi-circle. Since the angle at the centre is 180°, the angle at the circumference is 90°.

(2) An arc of a circle subtends equal angles at the circumference. (See Figs. 5 and 6) This property may also be stated as *angles in the same* ***segment*** (see **50.2**) *are equal.*

Fig. 5

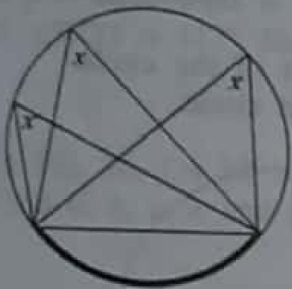


Fig. 6

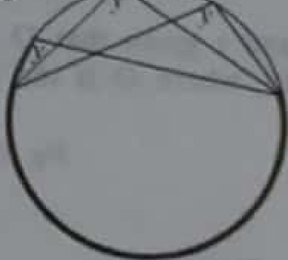


Fig. 7

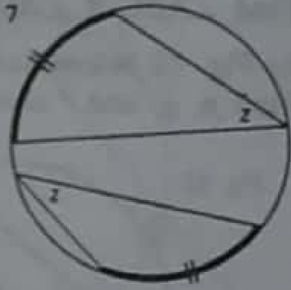


Fig. 7 shows that angles subtended by equal arcs of a circle at the circumference are equal.

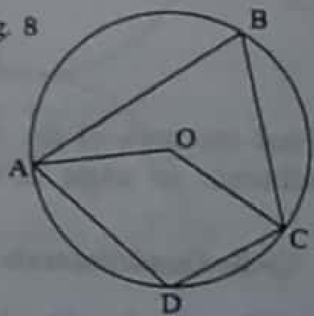
Example 1 In Fig. 8, if $\angle ABC = 70°$ calculate $\angle AOC$, reflex $\angle AOC$ and $\angle ADC$.

$\angle AOC = 2\angle ABC = 2 \times 70° = 140°$

Reflex $\angle AOC = 360° - 140° = 220°$

$\angle ADC = \frac{1}{2} \times$ reflex $\angle AOC = \frac{1}{2} \times 220 = 110°$

Fig. 8



Example 2 Find the area of (i) sector OPRQ, (ii) the minor segment formed by PQ. (See Fig. 9)

(i) Area of sector OPRQ $= \frac{120}{360} \times \frac{22}{7} \times 10{\cdot}5^2$

$= 115{\cdot}5\text{cm}^2$

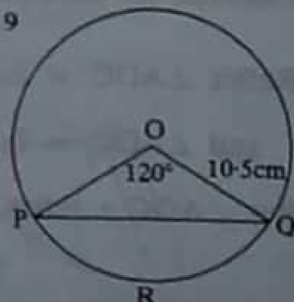
(ii) Area of the minor segment formed by PQ

= Area of (sector OPRQ − triangle OPQ)

$= 115{\cdot}5 - \frac{1}{2} \times 10{\cdot}5^2 \times \sin 120°$

$= 115{\cdot}5 - 47{\cdot}7 = 67{\cdot}8\text{cm}^2$

Fig. 9



Note See **31.2** for area of triangle formula and **44.1** for sin 120°.

Example 3 Calculate x, y, z, w and t shown in Fig. 10.

$x = 70°$, $y = 30°$, $z = 42°$ (same segments)

$t = 180 - (42 + 30 + 70)$ (angle sum of $\triangle$)
$= 38°$

$w = t = 38°$ (same segment)

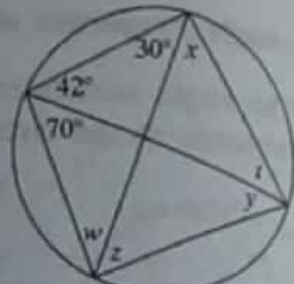


Fig. 10

Exercise 30b

Refer to Fig. 11 for 1 to 8.

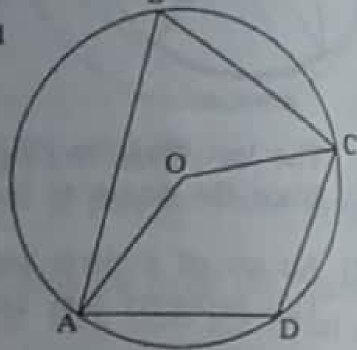


Fig. 11

1 Find $\angle AOC$ if $\angle ABC = 65°$.
2 Find $\angle ABC$ if $\angle AOC = 112°$.
3 Find reflex $\angle AOC$ if $\angle ADC = 112°$.
4 Find $\angle ADC$ if reflex $\angle AOC = 204°$.
5 Find $\angle AOC$, reflex $\angle AOC$, $\angle ADC$ if $\angle ABC = 47°$.
6 Find $\angle ADC$ if $\angle ABC = 33°$.
7 Find reflex $\angle AOC$, $\angle AOC$, $\angle ABC$ if $\angle ADC = 107°$.
8 Find $\angle ABC$ if $\angle ADC = 97°$.
9 In Fig. 12 below, calculate x, y and z given that O is the centre of the circle.
10 Find p, q, and r shown in Fig. 13, where O is the centre of the circle.

Fig. 12

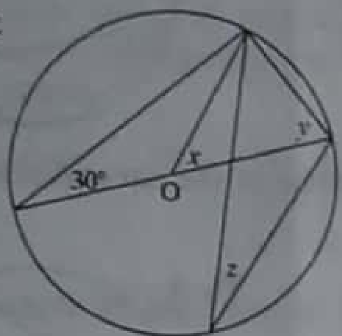
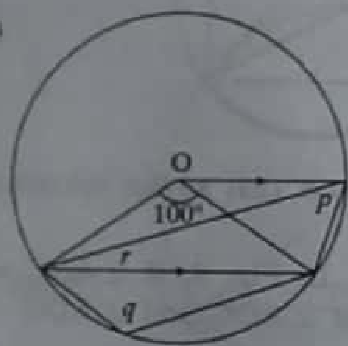


Fig. 13

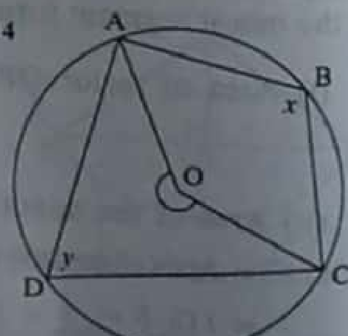


11 Find the area of the minor segment of a circle formed by a chord of length 7cm which subtends an angle of 60° at the centre of the circle.

30.3 Cyclic Quadrilaterals

A quadrilateral that has its four vertices on the circumference of a circle is called a **cyclic quadrilateral**, for example ABCD in Fig. 14.

Fig. 14



Reflex $\angle AOC = 2x$ (angle at centre)

and $\angle AOC = 2y$ (angle at centre)

But $\angle AOC + \text{reflex } \angle AOC = 360°$

$2x + 2y = 360°$

Hence $x + y = 180°$

Opposite angles of a cyclic quadrilateral are supplementary.

The next diagram (Fig. 15) shows a cyclic quadrilateral PQRS with PQ produced to T

We have $q + \angle TQR = 180°$ ($\angle$s on a straight line)
But $q + s = 180°$ (opp $\angle$s cyclic quad)

Hence $\angle TQR = s$

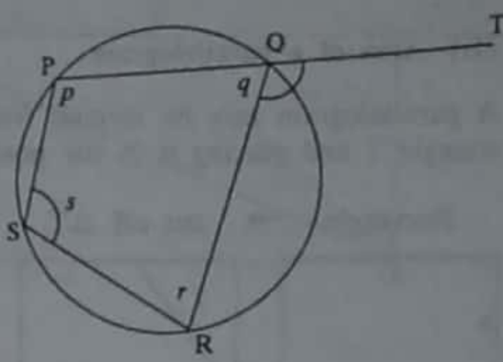


Fig. 15

When a side of a cyclic quadrilateral is produced the exterior angle so formed is equal to the opposite interior angle.

Exercise 30c

Refer to Fig. 15 for **1** and **2**.

1 If $p = 105°$, what is r?
2 If $\angle RQT = 117°$, find q and s.

3 In Fig. 16, $f = 14°$ and $e = 80°$. Find d, b, c, a and g.

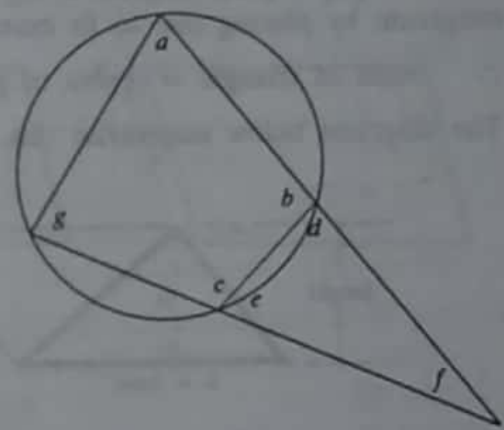


Fig. 16

4 In Fig. 17, AB is a diameter of the circle and AB is parallel to DC. Find u, w, x, y and z.
5 In Fig. 18, DC = BC and AB = DB. Find a, b, c, d and e.

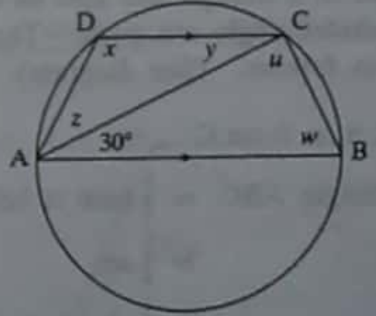


Fig. 17

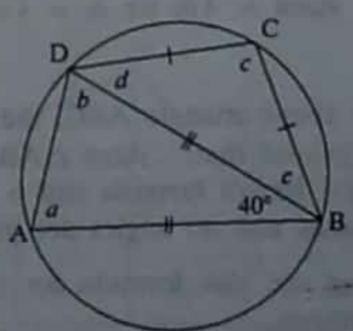
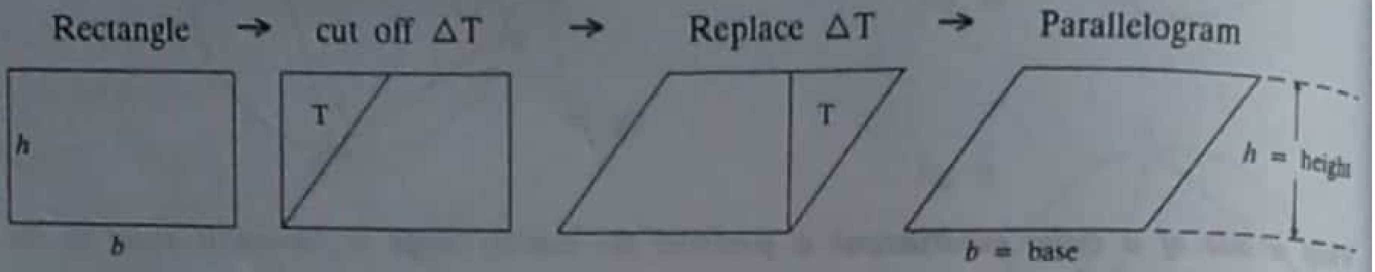


Fig. 18

31 AREA AND VOLUME

31.1 Area of a parallelogram

A parallelogram may be formed from a rectangle, such as a sheet of paper, by cutting off triangle T and placing it in the position shown.



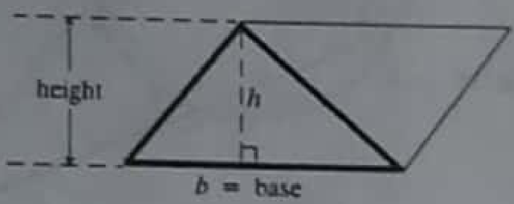
Area of parallelogram = Area of rectangle = $b \times h$ = base × height

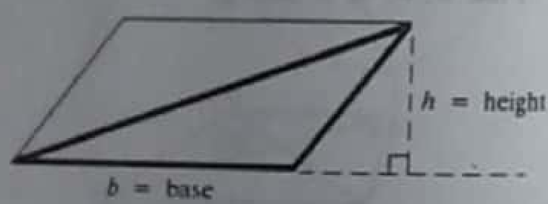
31.2 Area of a Triangle

(i) Cut a paper parallelogram along a diagonal. The two triangles formed are shown to be congruent by placing one to fit exactly on top of the other.

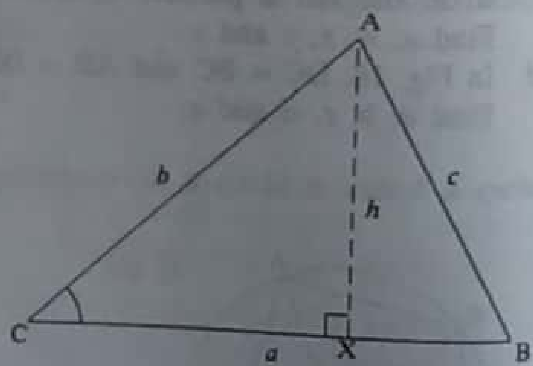
$\therefore$ Area of triangle = $\frac{1}{2}$area of parallelogram = $\frac{1}{2}$(base × height) = $\frac{1}{2}bh$

The diagrams below summarise this.





(ii) The area of a triangle can also be found if two sides and the **included angle** are given. The formula for this is derived as follows. (See diagram)



In $\triangle AXC$, $h = b \sin C$

Area of triangle ABC $= \frac{1}{2}$base × height

$= \frac{1}{2}ah$

$= \frac{1}{2}ab \sin C$

Similarly: Area $= \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B$

(iii) In the above triangle ABC, the **semi-perimeter**, s, is given by $s = \frac{1}{2}(a + b + c)$
It may be proved that: Area $\triangle ABC = \sqrt{[s(s-a)(s-b)(s-c)]}$
This is called **Hero's formula** and is mostly useful for non-right-angled scalene triangles when the three sides and no angles are given.

NB ***Do not*** use this formula for finding areas of right-angled or isosceles triangles. Use simpler methods.

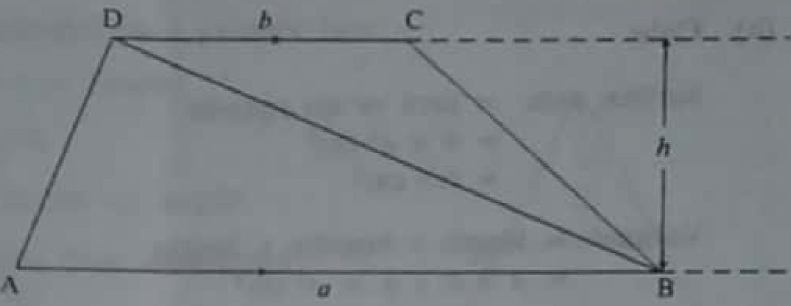
31.3 Area of a Trapezium

Area of trapezium ABCD

$= \text{area } \triangle ABD + \text{area } \triangle BCD$

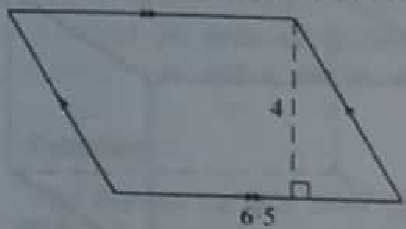
$= \frac{1}{2}ah + \frac{1}{2}bh$

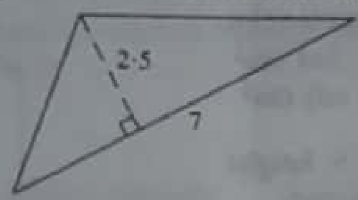
$= \frac{1}{2}h(a + b)$

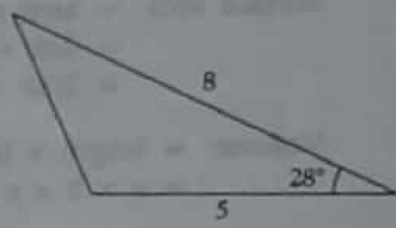


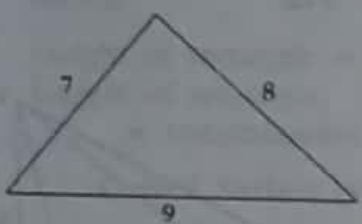
Exercise 31a

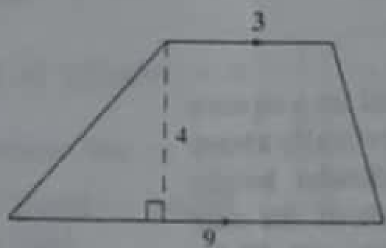
In 1 to 8 find the area of each figure where dimensions are in cm.

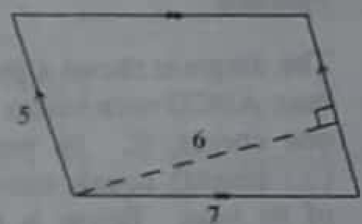
1


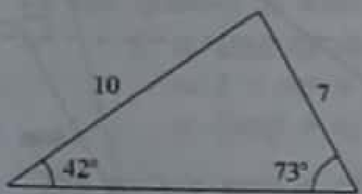
2


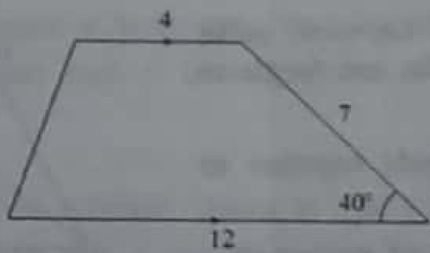
3


4


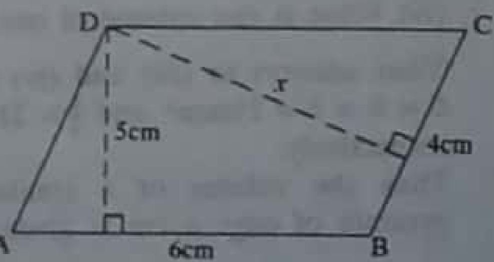
5


6


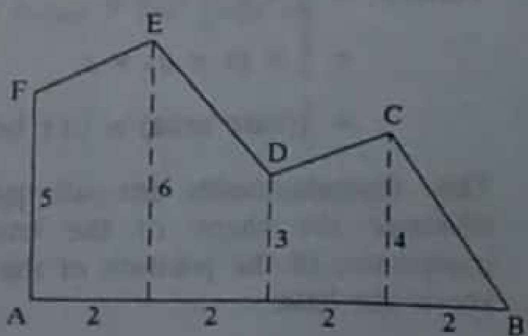
7


8


9 In the diagram, ABCD is a parallelogram. Find (i) the area of ABCD
(ii) the length x.

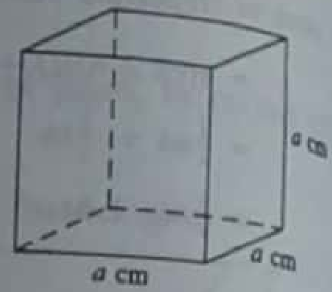


10 The polygon ABCDEF represents a scale drawing of a farm, the dimensions being in cm.
(i) By dividing the polygon into three trapezia and a triangle as shown, find the area of ABCDEF in cm².
(ii) If the scale is 1 : 20,000 find the area of the farm in hectares.



31.4 Surface Area and Volume of Solids

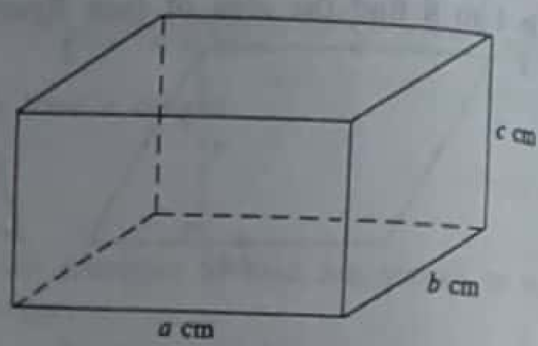
(a) Cube



Surface area = area of six squares
$= 6 \times a^2$ cm²
$= 6a^2$ cm²

Volume = length × breadth × height
$= a \times a \times a = a^3$ cm³

(b) Cuboid

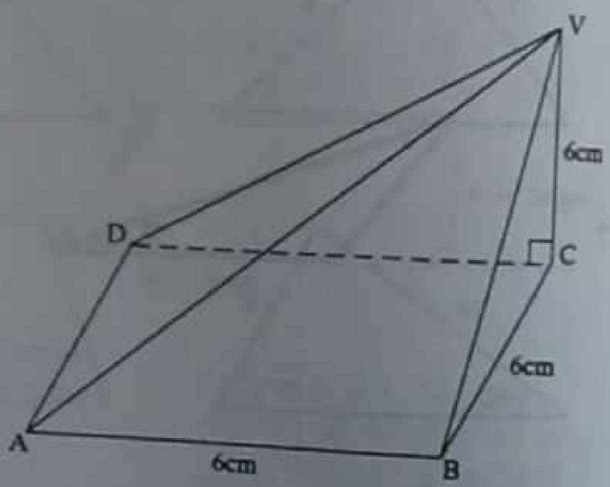


Surface area = area of six rectangles
$= 2ab + 2bc + 2ca$ cm²
$= 2(ab + bc + ca)$ cm²

Volume = length × breadth × height
$= a \times b \times c = abc$ cm³

(c) Pyramid

The diagram shows a pyramid on a square base ABCD with vertex V vertically above one corner, C. Its perpendicular height (⊥r height) equals the length of the side of its base. Below is its net(see 29.1).



(i) By drawing the nets on card, make three such pyramids. (Do not forget to put tabs.)

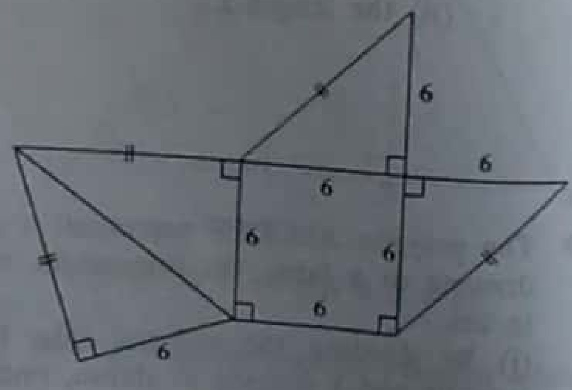
(ii) Fit the three pyramids together to form a single cube.

(iii) What is the volume of the cube?

(iv) What is the volume of one pyramid?

Your answers to (iii) and (iv) should be $6 \times 6 \times 6 = 216$cm³ and $\frac{1}{3} \times 216 = 72$cm³ respectively.

Thus the volume of a similar general pyramid of edge a cm is given by



Volume $= \frac{1}{3} \times a \times a \times a$

$= \frac{1}{3} \times (a \times a) \times a$

$= \frac{1}{3}$(base area) × (⊥r height)

This formula holds for all pyramids whatever the shape of the base and irrespective of the position of the vertex above the base.

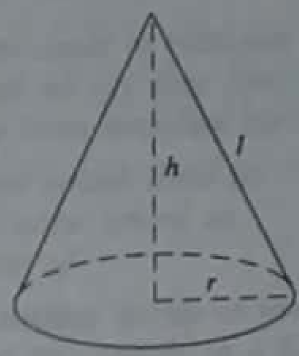
(d) Cone

A cone may be considered as a pyramid on a circular base.

$\therefore$ Volume $= \frac{1}{3} \times$ (base area) $\times$ ($\perp$r height)

$= \frac{1}{3} \times \pi r^2 \times h = \frac{1}{3}\pi r^2 h$

where r is the base radius and h is the $\perp$r height.



If l is the **slant height** (see diagram) then the curved surface area S of a cone is given by

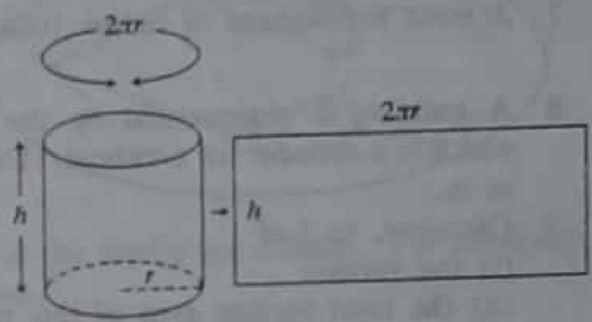
$$S = \pi r l$$

The total surface area A of a solid cone is given by

$$A = \text{area of circular base} + \text{curved surface area}$$
$$= \pi r^2 + \pi r l = \pi r(r + l)$$

(e) Cylinder

The curved surface area of a hollow cylinder can be opened out to form a rectangle as shown.



Height of rectangle = height of cylinder = h

Length of rectangle

= circumference of circular end = $2\pi r$

$\therefore$ Curved Surface Area = length $\times$ height

$= 2\pi r h$

If the cylinder is solid, or closed at both ends, then its total surface area S is given by

$$S = \text{area of two circular ends} + \text{curved surface area}$$
$$= 2 \times \pi r^2 + 2\pi r h$$
$$= 2\pi r(r + h)$$

The surface area A of a hollow cylinder, closed at one end, is given by

$$A = \text{area of one circular end} + \text{curved surface area}$$
$$= \pi r^2 + 2\pi r h$$
$$= \pi r(r + 2h)$$

The volume V of a cylinder is given by

$$V = \text{base area} \times \text{height}$$
$$= \pi r^2 h$$

(f) Sphere

The surface area S and volume V of a sphere of radius r are given by

$$S = 4\pi r^2 \quad \text{and} \quad V = \tfrac{4}{3}\pi r^3$$

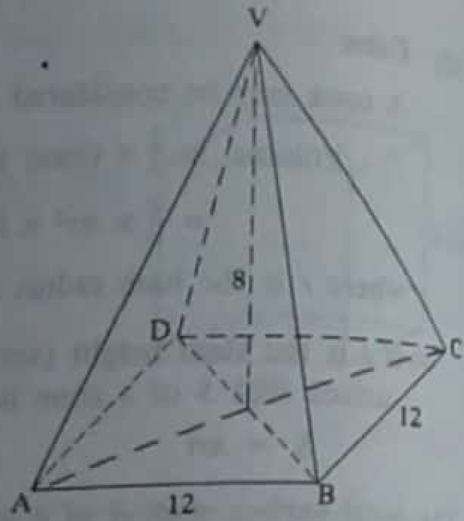
Exercise 31b

In **1** to **7** find to 3 sf (i) the surface area, (ii) the volume of the solid described. (Take $\pi = \frac{22}{7}$ or 3·14 as appropriate)

1 A cube of edge 6cm

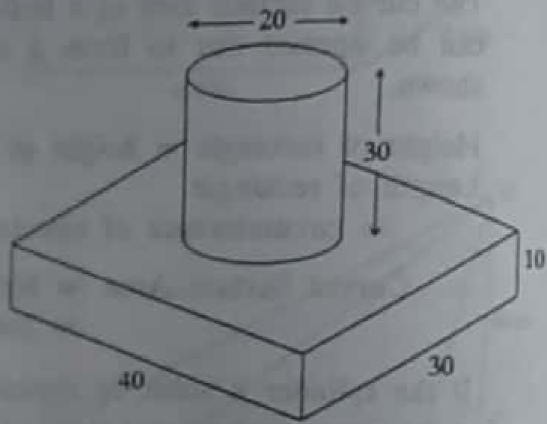
2 A cuboid 5cm by 4cm by 3cm

3 The pyramid, shown, on a square base of side 12cm and height 8cm

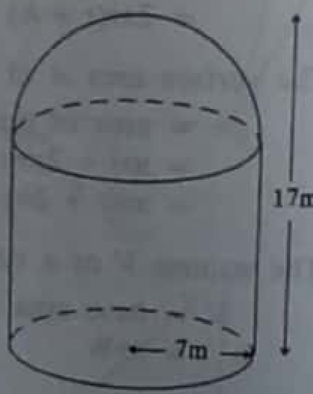


4 A solid cylinder of radius 7cm, height 10cm
5 A solid cone which has base radius 5cm and perpendicular height 12cm
6 A sphere of radius 7cm
7 A solid hemisphere of radius 10cm

8 A building is represented by the solid shown which is a cylinder on a cuboid. Dimensions are in m.
Calculate, to 3 sf
(i) the volume
(ii) the total surface area of this solid.



9 An observatory is represented by the solid shown which is a hemisphere on a cylinder. If the base radius of the cylinder is 7m and the height of the observatory is 17m, calculate to 3 sf
(i) its volume
(ii) its total surface area.



10 Cylindrical tins of diameter 10cm and height 15cm are made from thin sheet metal which costs sh2,400 per m². Ignoring wastage find the cost, to the nearest sh10, of metal used for one tin.
11 A cylindrical metal water pipe is 10m long. It has an external and internal diameter of 5·1cm and 4·7cm respectively. Find the volume of metal required to make the pipe. [*Hint:* $R^2 - r^2 = (R + r)(R - r)$]
12 The cylindrical cap of a tin of aerosol is made of plastic 0·5mm thick. Its diameter is 6cm and its height 5·5cm. By considering its surface area, find the approximate volume, in cm³, of plastic used.

32 RELATIONS AND MAPPINGS

32.1 Relations

Consider the statement: John is the uncle of Lilian. The phrase *is the uncle of* indicates that there is a connection between John and Lilian. They are *related* and we say that John and Lilian are *relations*. **A relation** may be defined on a set. Examples from everyday life are:

(i) *is taller than*
(ii) *sits next to*
(iii) *is the father of*
(iv) *has the same colour dress as*
(v) *goes to the same school as*
(vi) *is the brother of*

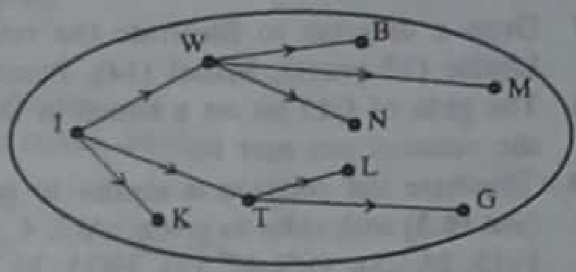
Each of the sets on which these relations are defined refers to a particular group of people. Relations (i) and (ii) could refer to the pupils of your class, (iii) and (vi) people in a family or in a village, (iv) the female population, (v) all pupils at secondary school.

A relation between members of a given set may be illustrated by means of a **papygram**.

For example, consider Isabirye's family:

Isabirye (I) has two sons Waiswa (W) and Tenywa (T) and a daughter Kawala (K). Waiswa has two sons Buyinza (B) and Muzira (M) and a daughter Naisanga (N). Tenywa has two sons Lwamusanyi (L) and Gobolo (G).

The papygram on the right illustrates the relation *is the father of* within this family.



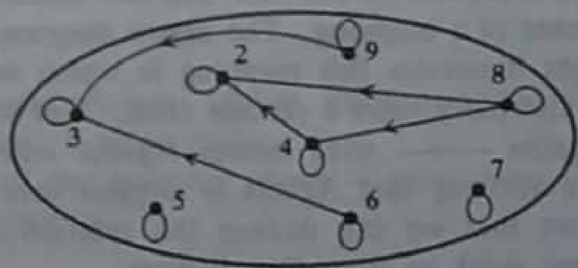
Note that arrow lines are used to show how members of the family are related. For example, W ——→— N shows that Waiswa *is the father of* Naisanga.

Examples of mathematical relations are:

(i) *is equal to*
(ii) *is greater than*
(iii) *is a multiple of*
(iv) *is the square of*
(v) *is congruent to*
(vi) *is perpendicular to*

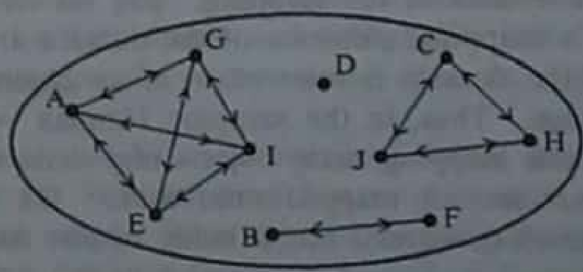
Suggest sets on which each of the above relations could be defined.

The relation *is a multiple of* is defined on the set {2, 3, 4, 5, 6, 7, 8, 9}. The papygram on the right illustrates this relation. Here, for example, 9 ——→— 3 indicates that 9 *is a multiple of* 3. Note each element has a loop because each *is a multiple of* itself. Why do 5 and 7 have no arrow lines relating them to other elements of the set?



Example At a party Agatha (A) drinks soda, Beatrice (B) tea, Christine (C) milk, Dorcas (D) passion fruit, Eunice (E) soda, Fatuma (F) tea, Gertie (G) soda, Hannah (H) milk, Irene (I) soda and Janet (J) milk. Draw a papygram to illustrate the relation *has the same drink as*.

Draw an oval shape and, inside this, mark and label dots to represent the girls in the set. Link by double-arrowed lines those having the same drink. Thus those having the same drink are grouped together as shown.



Note the double arrow, for example A ←—→ I, shows that Agatha *has the same drink as* Irene *and* Irene *has the same drink as* Agatha.

Exercise 32a

1 to 9 refer to Isabirye's family of **32.1**. In each case draw a papygram with all members of the family to illustrate the given relation.

1 *is the grandfather of* **2** *is the uncle of* **3** *is the aunt of*
4 *is the nephew of* **5** *is the niece of* **6** *is the brother of*
7 *is the sister of* **8** *is the cousin of* **9** *is the grandchild of*

In **10** to **16**, draw a papygram to illustrate the relation defined on the given set.

10 set: {2, 3, 4, 5, 6, 7, 8, 9} relation: *is a factor of*
11 set: {0, 1, 2, 3} relation: *is greater than*
12 set: {0, 1, 2, 3} relation: *is less than or equal to*
13 set: {−2, −1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9} relation: *is twice*
14 set: {−2, −1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9} relation: *is the square of*
15 set: {−2, −1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9} relation: *is the square root of*
16 set: {4, 7, 9, 6, 13, 14, 12} relation: *exceeds by more than three*

17 Draw a diagram to illustrate the relation *is older than* among the following: Louise (12 years), Mabel (14), Pamela (15), Rose (13) and Sophie (14).

18 The girls of **Q17** sit on a bench in the order L, M, P, R, S. Draw a papygram illustrating the relation *sits next to*.

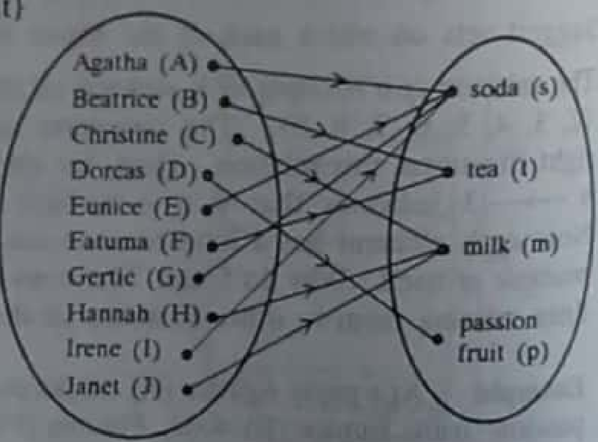
19 Illustrate the relation *is similar to* (see **37.1**) between the following right-angled triangles (see **12.3**) with sides as given: A(3, 4, 5), B(10, 6, 8), C(10, 24, 26), D(7, 24, 25), E(5, 12, 13), F(15, 20, 25), G(9, 12, 15), H(15, 36, 39), I(14, 48, 50), J(12, 16, 20), K(8, 15, 17).

32.2 Mappings

Look again at the **Example** of **32.1**. In this example there are two sets namely:

{girls} = {Agatha, Beatrice, Christine, Dorcas, Eunice, Fatuma, Gertie, Hannah, Irene, Janet}
{drinks available} = {soda, tea, milk, passion fruit}

The elements of the two sets are connected by means of a **mapping**. The **arrow diagram** on the right illustrates this mapping in which each girl at the party *takes* a certain drink. For example, Agatha ⟶ soda means Agatha *takes* soda. We also say that Agatha *is mapped onto* soda. Note that we can deduce the relation *has the same drink as* from this diagram.

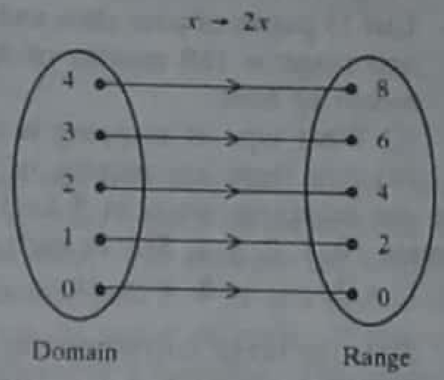


In the above example, the set on the left in the arrow diagram, ie. {girls at the party} is called the **domain** of the mapping. The set on the right, ie. {drinks available} is the **range**. Thus, in a mapping, elements of the domain are mapped onto elements of the range. An element of the domain is referred to as an **object** and the corresponding element in the range is its **image**. Thus, in the mapping Hannah → milk, *Hannah* is the object and *milk* is her image. In this mapping, many objects may share the same image. For example, *Christine*, *Hannah* and *Janet* are all mapped onto *milk*. On the other hand, each object has only one image. Therefore we call this a **many to one** mapping.

Note that *many* in this sense means *two or more*.

The next arrow diagram illustrates a mapping in which the domain is {0, 1, 2, 3, 4} and each element of the range is twice the corresponding element of the domain. In this mapping, *one* object is mapped onto *one* image, without exception. It is called a **one to one** mapping.

Notation If x represents any element in the domain, then $2x$ will represent the corresponding element in the range. We write this mapping as $x \to 2x$ (read as x is mapped onto $2x$).



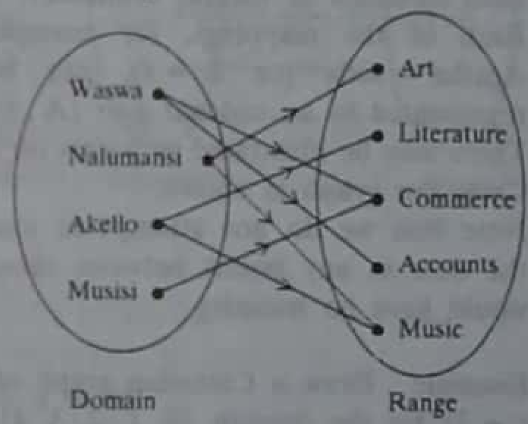
Example Optional subjects at Makerere College School are Art, Literature, Commerce, Accounts and Music. Waswa takes Commerce and Accounts, Nalumansi takes Music and Art, Akello takes Literature and Music while Musisi takes Commerce.

(i) List the elements of the domain and range for the mapping *pupils* → *subjects*.
(ii) Draw an arrow diagram to illustrate this mapping.
(iii) What type of mapping is this?

(i) Domain = {Waswa, Nalumansi, Akello, Musisi}
Range = {Art, Literature, Commerce, Accounts, Music}

(ii) Draw two ovals with pupils' names and subjects taken as shown. Connect names to subjects taken according to the given information.

(iii) *Many* objects are mapped onto the same image, for example Waswa and Musisi are both mapped onto Commerce. An object may have *many* images, for example Akello is mapped onto Literature and Music.
Therefore this is a **many to many** mapping.



Exercise 32b

1 In an S2 class at Nyakasura School, Kato, Birungi, Avinia, Mpuru, Okurut, Jongo and Nyakaisiki wear shoes of sizes 8, 7, 8, 6, 9, 7, and 8 respectively.
(i) List the elements of the domain and range of the mapping *pupils* → *shoe sizes*.
(ii) Draw an arrow diagram to illustrate this mapping.
(iii) What type of mapping is this?

2 The girls of Ex **32a Q17** are mapped onto their ages.
(i) List the elements of the domain and range.
(ii) Draw an arrow diagram to illustrate this mapping.
(iii) What type of mapping is this?

3 In a certain S3 class Achoroi plays football, Bijurenda plays tennis and volleyball, Charo plays football and volleyball and Ddungu plays football, tennis and volleyball. List the elements of the domain and range of the mapping *pupils* → *games* and draw an arrow diagram. What type of mapping is this?

4 Simba House came first, Chui second, Ndovu third and Kifaru fourth in a recent school athletics competition. Draw an arrow diagram for the mapping *house* → *position in competition*. List the members of the domain and range. What type of mapping is this?

5 The marks scored by Ali, Bob, Col, Dan, Eve and Fay in a test were 5, 4, 7, 5, 9 and 7 respectively. Illustrate the mapping *marks* → *pupils*. What type of mapping is this?

6 List 15 pupils of your class and note the month of birth of each. With domain = {pupils} and range = {all months of the year}, draw a diagram to illustrate the mapping *pupil → month of birth*.

(i) What type of mapping is this?

(ii) Are there any months of the year which have no arrows going to them?

For the mappings given in 7 to 11, list the elements of the range which correspond to the domain {0, 1, 2, 3, 4, 5}. In each case draw an arrow diagram and state the type of mapping.

7 $x \to 3x$ 8 $x \to \frac{1}{2}x$ 9 $x \to 2x + 1$ 10 $x \to 3x - 2$ 11 $x \to 5 - x$

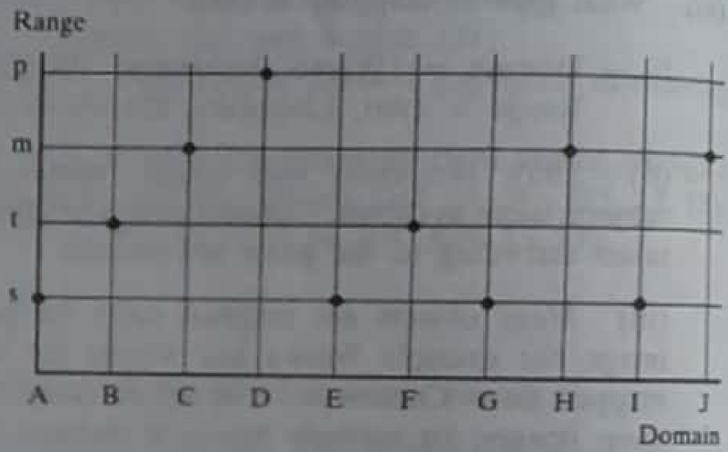
12 State the range, corresponding to the domain {−3, −2, −1, 0, 1, 2, 3}, (or {0, ±1, ±2, ±3}), for the mapping $x \to x^2$. Draw an arrow diagram. What type of mapping is this?

13 Repeat **Q12** for the mapping $x \to \pm\sqrt{x}$ and domain {0, 1, 4, 9, 16}.

14 Repeat **Q12** using the domain $\{\pm\frac{1}{3}, \pm\frac{1}{2}, \pm1, \pm2, \pm3\}$ for the mapping $x \to \frac{1}{x}$. What do you notice about the domain and range?

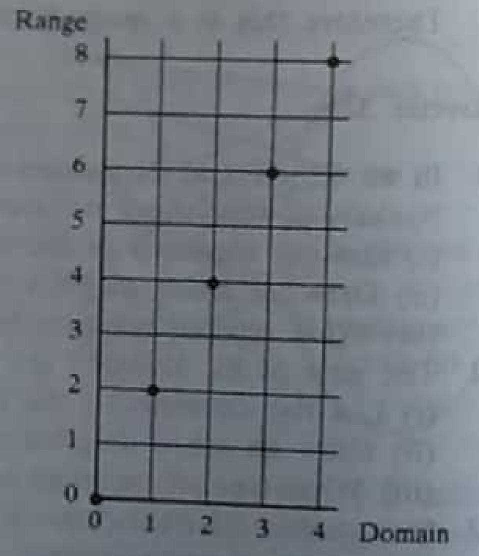
32.3 Cartesian Graphs

Look at the arrow diagram of **32.2** which maps elements of {girls at the party} onto elements of {drinks available}. Each of the mappings, for example Agatha → soda (or A → s), can be represented by an ordered pair (A , s). These may be illustrated as points on a **Cartesian graph** as shown. Note that we do not attempt to join the dots as any points between these would have no meaning.



Example Draw a Cartesian graph of the mapping $x \to 2x$ for the domain {0, 1, 2, 3, 4}.

The arrow diagram for this mapping is shown in **32.2**. We see that 0 → 0, 1 → 2, 2 → 4, 3 → 6 and 4 → 8. This gives ordered pairs (0, 0), (1, 2), (2, 4), (3, 6) and (4, 8). These points are plotted on the Cartesian graph (or, simply, **graph**) which is shown on the right. Compare this graph with that of **Ex 20a Q1**. Note that, although the points lie on a straight line, we do not join them as the domain does not contain intermediate values.



Exercise 32c

1 to 14 For each of the mappings given in **Ex 32b Q1** to **14**, (i) make a list of ordered pairs, (ii) draw a Cartesian graph.

15 For the cuboid of **Ex 29d Q1**, let edges AB, BC, CD, DA, AE, BF, CG, DH, EF, FG, GH, HE be named a, b, c, d, e, f, g, h, i, j, k, l respectively. For these edges, draw a papygram showing the relation (i) *is parallel to*, (ii) *is perpendicular to*, (iii) *has the same length as*. Illustrate the mapping *edge → length*. Name the type of mapping and draw its graph.

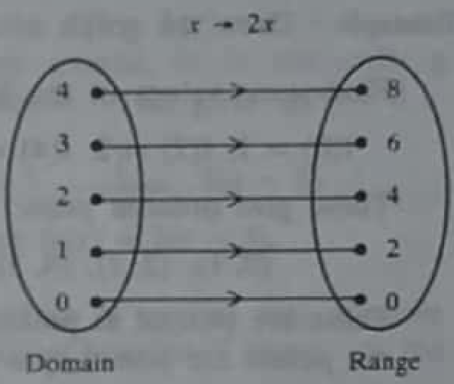
33 FUNCTIONS

33.1 Definition

A **function** is a mapping (see **32.2**) in which every object has one and only one image. So a mapping which is *one to one* or *many to one* is a function if every object in the domain has an image. Most of the mappings of **32.2** and **Ex 32b** are functions. For example the mapping {girls at the party} → {drinks available} is a many to one function and $x \to 2x$ is a one to one function. It is easy to see when a mapping is a function from its arrow diagram. If just one arrow comes from every element of the domain then the mapping is a function. The mapping in the Example of **32.2** is not a function since, for example, *two* arrows come from Waswa.

33.2 Notation

Mathematical functions such as $x \to 2x$ are given names. For example we could let f be the function $x \to 2x$. One way of writing this is $x \xrightarrow{f} 2x$. This is read as: f is the function which maps x onto $2x$. Another way of writing this is $f : x \to 2x$. A third and most important notation is $f(x) = 2x$. This is read as: f of x equals $2x$. The arrow diagram for this function is shown on the right. The domain is {0, 1, 2, 3, 4}. We see that this function maps 3 onto 6. The mapping $3 \to 6$ is written as $f(3) = 6$. Similarly, for the other elements in the domain, $f(0) = 0$, $f(1) = 2$, $f(2) = 4$ and $f(4) = 8$.



Example 1 Given the function $f(x) = 2x + 4$, find $f(3)$.

Substitute the value $x = 3$ into the function.

$f(x) = 2x + 4 \quad \therefore \quad f(3) = 2 \times 3 + 4 = 6 + 4 = 10$

Example 2 The function $g(x) = 3x + c$ and $g(4) = 7$. Find (i) the value of c, (ii) $g(2)$.

(i) $g(x) = 3x + c \quad \therefore \quad g(4) = 3 \times 4 + c = 12 + c$

But $g(4) = 7 \quad \therefore \quad 12 + c = 7 \quad \therefore \quad c = -5$

(ii) Since $c = -5$ we have $g(x) = 3x - 5 \quad \therefore \quad g(2) = 3 \times 2 - 5 = 6 - 5 = 1$

Exercise 33a

1 to 14 Study the mappings of **Ex 32b 1** to **14** and state which are functions.

In **15** to **19**, express each function using the notation $f(x) = \ldots$

15 $x \to 3x$ **16** $x \to x$ **17** $x \to 3x - 6$ **18** $x \to x^2$ **19** $x \to x^3 + 1$

20 For $f(x) = 2x - 3$, find (i) $f(4)$, (ii) $f(0)$, (iii) $f(-2)$, (iv) $f(y)$.

21 For $g(x) = \dfrac{x + 4}{x - 4}$, find (i) $g(6)$, (ii) $g(2)$, (iii) $g(0)$, (iv) $g(-4)$, (v) $g(y)$.

22 For $h(x) = x^2 + 1$, find (i) $h(0)$, (ii) $h(2)$, (iii) $h(-2)$, (iv) $h(z)$.

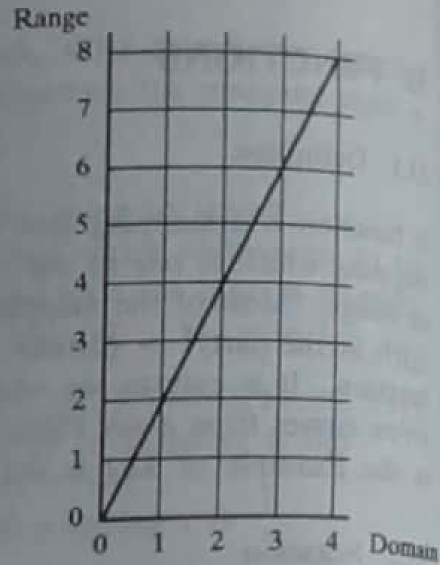
23 Obtain the range for the function $f(x) = 2^x$ when the domain is {0, 1, 2, 3, 4}.

24 Given the function $f(x) = cx + 9$ and that $f(5) = 49$, find (i) the value of c, (ii) $f(4)$.

25 It is given that $f(y) = \dfrac{ay}{y^2 - 2}$ and that $f(2) = 12$. Find (i) the value of a, (ii) $f(1)$.

33.3 Graphs of Functions

Cartesian graphs of mappings were discussed in **32.3**. Consider the graph of the mapping $x \to 2x$ in the Example of **32.3**. This is also the graph of the function $f(x) = 2x$ for the domain $\{0, 1, 2, 3, 4\}$. Note that there are no intermediate values such as 1·5, 2·4 etc. Suppose the domain included *all* values from 0 to 4. This set is written as $\{x : 0 \le x \le 4\}$. The colon (:) is read as *such that*.
In this case the graph has very many points which are represented by the straight line shown in the diagram on the right.



Example Draw the graph of the function $f(x) = \frac{1}{2}x + 1$ for the domain $\{x : 0 \le x \le 6\}$.

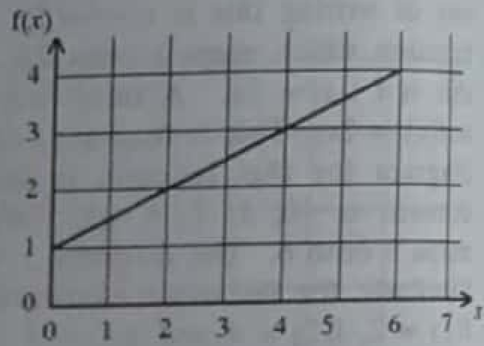
Find some values of the function:

$f(0) = 1$, $f(2) = 2$, $f(4) = 3$, $f(6) = 4$

These give ordered pairs:

$(0, 1)$, $(2, 2)$, $(4, 3)$, $(6, 4)$

These are plotted as points on a grid and the points are joined by a straight line as shown.



By letting $y = f(x)$ in the above Example, we obtain the **Cartesian equation** of the graph, ie. $y = \frac{1}{2}x + 1$. This may be rearranged as $2y - x = 2$. Compare this equation and the graph with those of **20.2**.

Exercise 33b

In **1** to **8**, draw the graph of the function for the given domain. In each case write down the Cartesian equation of the graph.

1 $f(x) = x + 1$, $\{0, 1, 2, 3, 4\}$
2 $f(x) = 2x - 1$, $\{1, 2, 3, 4, 5\}$
3 $f(x) = x$, $\{0, 1, 2, 3, 4, 5\}$
4 $f(x) = 3 - x$, $\{-1, 0, 1, 2, 3, 4\}$
5 $f(x) = 3$, $\{x : 0 \le x \le 5\}$
6 $f(x) = x + 2$, $\{x : -3 \le x \le 3\}$
7 $f(x) = 3 - x$, $\{x : -1 \le x \le 4\}$
8 $f(x) = 2 - \frac{1}{2}x$, $\{x : -2 \le x \le 6\}$

9 On the same grid, draw the graphs of $f(x) = x - 2$ and $g(x) = 4 - \frac{1}{2}x$ for the domain $\{x : 0 \le x \le 8\}$. For what value of x is $f(x) = g(x)$?
10 Repeat **Q9** for the functions $f(x) = 3x + 1$ and $g(x) = -x - 3$ and domain $\{x : -2 \le x \le 2\}$.
11 Find the range of the function $f : x \to \frac{1}{2}x^2$ for the domain $\{0, \pm1, \pm2, \pm3, \pm4, \pm5\}$. Write down the corresponding ordered pairs and plot these as points on a grid. Hence draw the graph of the function $f(x) = \frac{1}{2}x^2$ over the domain $\{x : -5 \le x \le 5\}$ by joining your points carefully with a smooth curve.
12 A function $f(x)$ is defined over the domain $\{x : 0 \le x \le 5\}$. Draw the graph of this function, given that $f(x) = 3$ when $0 \le x < 2$ and $f(x) = 5 - x$ when $2 \le x \le 5$.

33.4 Composite Functions

Two or more functions may be combined to form a **composite function**. Consider the 'double' function $f : x \rightarrow 2x$ and the 'subtract three' function $g : x \rightarrow x - 3$, that is $f(x) = 2x$ and $g(x) = x - 3$. Take a number, say 5, and first double it and then subtract 3. The result of this may be illustrated as follows:

$$5 \xrightarrow{f} 10 \xrightarrow{g} 7$$

The two stages $f(5) = 10$ and $g(10) = 7$ are combined as $g(f(5)) = 7$ since $f(5) = 10$. The outer brackets of $g(f(5))$ are removed and we write $gf(5) = 7$. Note that gf means ***function f followed by function g***. Also gf is an example of a composite function. To find this function, ie. $gf(x)$, take a general number, say a. The effect of the composite function gf on a is as shown.

$$a \xrightarrow{f} 2a \xrightarrow{g} 2a - 3$$

Therefore $gf(a) = 2a - 3$. Hence $gf(x) = 2x - 3$.
The composite function $gf(x)$ can be given a new name, say $h(x)$. Thus $h(x) = 2x - 3$.
The functions f and g above may be combined the other way around, ie. in the order g followed by f. The effect of this on the number 5 is shown below. This gives $fg(5) = 4$.

$$5 \xrightarrow{g} 2 \xrightarrow{f} 4 \qquad\qquad a \xrightarrow{g} a - 3 \xrightarrow{f} 2(a - 3)$$

The second diagram above shows that $fg(a) = 2(a - 3)$. Hence $fg(x) = 2(x - 3)$.
Note that fg is *not* the same function as gf.
A composite function may be derived from a single function which is applied two or more times. The next diagram shows the effect of applying the 'double' function f above, on the number 5.

$$5 \xrightarrow{f} 10 \xrightarrow{f} 20$$

Now ff is written as f^2, so $f^2(5) = 20$. Similarly $f^3(5) = 40$, $f^4(5) = 80$, and so on.

Example Given that $f(x) = 3x$ and $g(x) = x^2 + 3$, find (i) $gf(2)$, (ii) $fg(2)$, (iii) $gf(x)$, (iv) $fg(x)$. Find the values of x for which gf = fg.

(i) $f(2) = 3 \times 2 = 6$ and $g(6) = 6^2 + 3 = 36 + 3 = 39$ $\therefore$ $gf(2) = 39$

(ii) $g(2) = 2^2 + 3 = 4 + 3 = 7$ and $f(7) = 3 \times 7 = 21$ $\therefore$ $fg(2) = 21$

(iii) $f(a) = 3a$ and $g(3a) = (3a)^2 + 3 = 9a^2 + 3$ $\therefore$ $gf(x) = 9x^2 + 3$

(iv) $g(a) = a^2 + 3$ and $f(a^2 + 3) = 3(a^2 + 3) = 3a^2 + 9$ $\therefore$ $fg(x) = 3x^2 + 9$

If $gf(x) = fg(x)$ then $9x^2 + 3 = 3x^2 + 9$ so $6x^2 = 6$ $\therefore x^2 = 1$ $\therefore x = \pm 1$ (See **T.35**)

Exercise 33c

In each of **1** to **6**, find (i) $gf(2)$, (ii) $fg(2)$, (iii) $gf(x)$, (iv) $fg(x)$ for the given functions f and g.

1 $f(x) = 2x$, $g(x) = x + 3$
2 $f(x) = 3x$, $g(x) = x - 2$
3 $f(x) = x + 2$, $g(x) = x^2$
4 $f(x) = x + 1$, $g(x) = x^2 - 1$
5 $f(x) = 2x + 1$, $g(x) = x - 3$
6 $f(x) = x - 1$, $g(x) = 2x^2 - 3$

7 The function f is *square and add three* and the function g is *add three and square.*
(i) Find $f(5)$ and $g(5)$.
(ii) What are $f(x)$ and $g(x)$?
(iii) Are f and g the same function?

8 Given that $h(x) = x - 2$, $g(x) = 4x$ and $f(x) = x^2 - 4$, find (i) $hgf(x)$ (ii) $fgh(x)$.

9 For the functions $f(x) = 2x$ and $g(x) = x^2 + 8$, find (i) $fg(a)$ (ii) $gf(a)$. Hence find the value(s) of x for which $fg(x) = gf(x)$.

10 Repeat **Q9** for the functions $f(x) = x + 2$ and $g(x) = 2x^2 - 3$.

11 For the functions $f(x) = x$ and $g(x) = 3x + 4$, find (i) $gf(x)$ (ii) $fg(x)$. Is $gf = fg$?

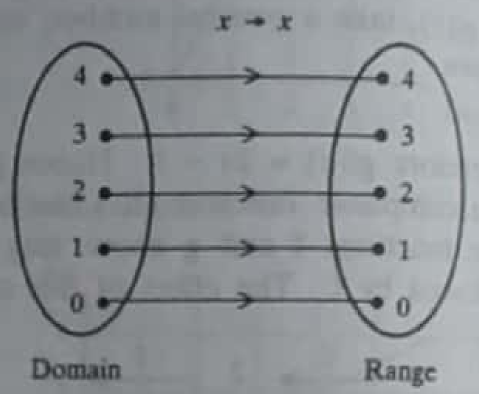
12 Repeat **Q11** for the functions $f(x) = 2x + 3$ and $g(x) = \frac{1}{2}(x - 3)$.

In **13** to **20**, find (i) $f^2(x)$, (ii) $f^3(x)$ for the given function.

13 $f(x) = x + 3$	**14** $f(x) = x - 2$	**15** $f(x) = 3x$	**16** $f(x) = 5 - x$
17 $f(x) = x^2$	**18** $f(x) = \dfrac{1}{x}$	**19** $f(x) = \dfrac{1}{1 + x}$	**20** $f(x) = \dfrac{1}{1 - x}$

33.5 The Identity Function

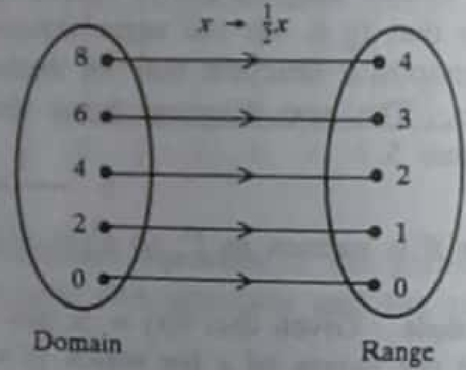
This is the function which maps every element of the domain onto itself. The diagram illustrates this function for the domain {0, 1, 2, 3, 4}. The **identity function** f is given by $f : x \to x$ or $f(x) = x$.



33.6 Inverses

Consider the function $f(x) = 2x$ with the domain {0, 1, 2, 3, 4} The arrow diagram for this function is shown on page 101. Now compare this diagram with the one shown on the right.

The range and domain have been interchanged. This is the function $x \to \frac{1}{2}x$. It has the opposite or reverse effect of $x \to 2x$ upon elements of its domain. For this reason the function $x \to \frac{1}{2}x$ is called the **inverse** of $x \to 2x$ and is denoted by f^{-1}. So $f^{-1}(x) = \frac{1}{2}x$.



Example Find the inverse of the function (i) $f(x) = x + 2$, (ii) $g(x) = 3x$, (iii) $h(x) = 3x + 2$.

(i) First draw the arrow diagram for the function $f(x) = x + 2$. (See Fig.1) Interchange the domain and range to obtain Fig.2. Finally note that this is the function $x \to x - 2$. So $f^{-1}(x) = x - 2$.

Fig. 1

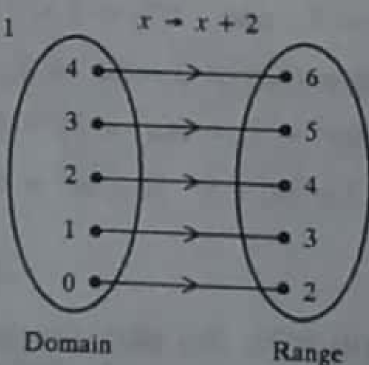
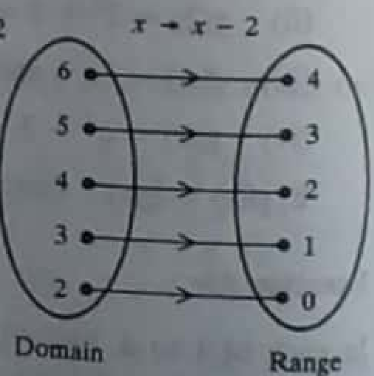


Fig. 2



(ii) Repeating the above method for $g(x) = 3x$ gives the inverse as $g^{-1}(x) = \frac{1}{3}x$.

(iii) *Method 1* We note that $h(x) = 3x + 2$ is the composite function $fg(x)$ where $f(x) = x + 2$ and $g(x) = 3x$. From the above we have $f^{-1}(x) = x - 2$ and $g^{-1}(x) = \frac{1}{3}x$. Now $h^{-1}(x)$ is given by $g^{-1}f^{-1}(x)$. Note that the inverses follow each other in the ***reverse order***. The diagram shows that $g^{-1}f^{-1}(a) = \frac{1}{3}(a - 2)$.

Hence $h^{-1}(x) = \frac{1}{3}(x - 2)$.

$$a \xrightarrow{f^{-1}} a - 2 \xrightarrow{g^{-1}} \tfrac{1}{3}(a - 2)$$

Method 2

Step 1 Use the Cartesian equation of the function $h(x) = 3x + 2$: $\quad y = 3x + 2$

Step 2 Interchange x and y in this equation: $\quad x = 3y + 2$

Step 3 Make y the subject (see **47.2**):

$$x - 2 = 3y$$
$$3y = x - 2$$
$$y = \tfrac{1}{3}(x - 2)$$

The inverse function is $h^{-1}(x) = \frac{1}{3}(x - 2)$

Note that *Method 2* could have been used for (i) and (ii).

Exercise 33d

In **1** to **8**, by drawing arrow diagrams, find the inverse of the given function.

1 $f(x) = 4x$ **2** $f(x) = x + 4$ **3** $f(x) = \frac{1}{2}x$ **4** $f(x) = x - 2$

5 $f(x) = x^2$ **6** $f(x) = \sqrt{x}$ **7** $f(x) = 5 - x$ **8** $f(x) = \dfrac{1}{x}$

In **9** to **12**, find two functions f and g which may be combined to produce the function $h(x)$. Write down the inverses of f and g and hence find $h^{-1}(x)$.

9 $h(x) = 2x + 5$ **10** $h(x) = 3x - 4$ **11** $h(x) = \frac{1}{2}(x + 7)$ **12** $h(x) = x^2 + 1$

In **13** to **16**, use *Method 2* in the Example above to find the inverse of the given function.

13 $h(x) = 2x - 3$ **14** $h(x) = 8 - x$ **15** $h(x) = \dfrac{3}{x}$ **16** $h(x) = 4 - 9x^2$

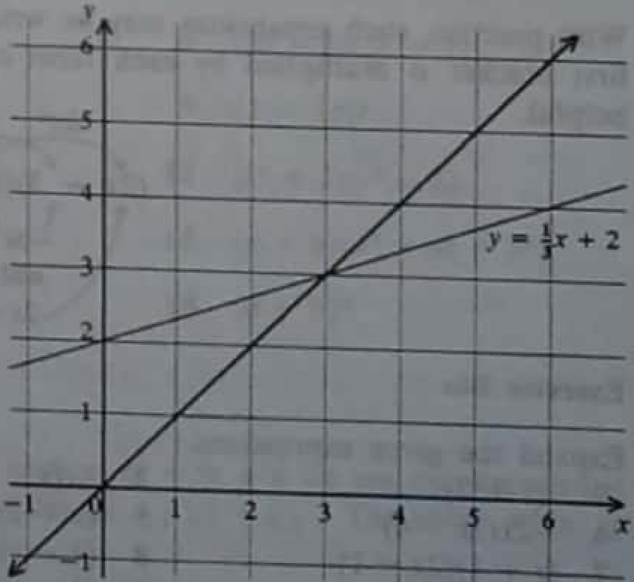
17 What is the inverse of the identity function $f(x) = x$?

18 For the function $f(x) = 2x + 1$, find $f^{-1}(x)$. Find also $ff^{-1}(x)$ and $f^{-1}f(x)$. What do you notice?

19 Repeat **Q18** for the function (i) $f(x) = 4x - 5$, (ii) $f(x) = 3 - 2x$.

20 Given that $f : x \to 1 - \dfrac{1}{x}$, find f^2 and show that f^3 is the identity function. From your results *deduce* f^{-1}. Confirm your deduction by calculating f^{-1}.

21 Consider the function $f(x) = \frac{1}{3}x + 2$.
Its graph is shown in the diagram together with that of the identity function, ie. the line $\{(x, y) : y = x\}$.
(i) Copy the diagram onto squared paper.
(ii) Taking $y = x$ as mirror line, draw the reflection of the line $y = \frac{1}{3}x + 2$ (see **41.1**).
(iii) Use the method of **20.4** or **39.2** to find the equation of this line.
(iv) Write this using the function notation $g(x) = \ldots$
(v) Find $f^{-1}(x)$. What do you notice?



22 Draw the graph of the given function on squared paper together with that of $y = x$, taking a suitable domain. In each case repeat steps (ii) to (v) of **Q.21**.
(a) $f(x) = 2x$ (b) $f(x) = \frac{1}{3}x$ (c) $f(x) = x + 2$ (d) $f(x) = 2x - 2$
(e) $f(x) = 5 - 2x$ (f) $f(x) = 5 - x$

23 Draw the graph of the function $f(x) = \frac{1}{2}x^2$ for the domain $\{x : -5 \le x \le 5\}$. Use the reflection idea of **Q.21** to draw the graph of the inverse of $f(x)$. Is the inverse a function?

34 POLYNOMIALS

34.1 Definition

A **polynomial** is a series of terms added together containing an unknown (usually x) and powers of that unknown. Examples are:

$$2x + 3, \text{ a \textbf{linear} polynomial}$$
$$3x^2 + x - 10, \text{ a \textbf{quadratic} polynomial}$$
$$x^3 - 2x^2 - 3x + 4, \text{ a \textbf{cubic} polynomial}$$

The **degree** of a polynomial is determined by the highest power it contains. The above polynomials are of the first, second and third degrees respectively. The terms of a polynomial are usually arranged in descending powers of x as above, or in ascending powers, for example $1 - x + 2x^2 - x^3$. The term not containing x, for example -10 in the quadratic polynomial above, is called the **constant term**. A polynomial is an example of an **algebraic expression**.

34.2 Expansion

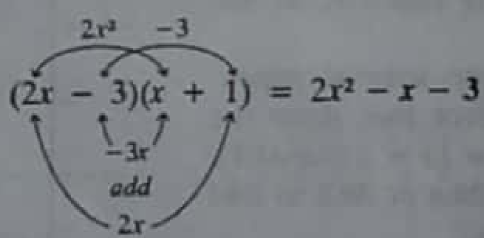
Polynomials often appear in factorised form, for example $2x(x + 3)$ or $(x + 1)(x + 2)$. Expansion of these is often necessary.

Example Expand (i) $2x(3x - 5)$ (ii) $(2x - 3)(x + 1)$

(i) $2x(3x - 5) = (2x) \times (3x) + (2x) \times (-5)$ using the distributive rule
$= 6x^2 - 10x$

(ii) $(2x - 3)(x + 1) = 2x(x + 1) - 3(x + 1)$ distributive rule
$= 2x^2 + 2x - 3x - 3$ distributive rule
$= 2x^2 - x - 3$ collecting like terms

With practice, such expansions may be written down at sight by noting that each term of the first bracket is multiplied by each term of the second. The arrow diagram below will be helpful.



Exercise 34a

Expand the given expressions.

1 $x(x + 2)$	**2** $2x(x - 1)$	**3** $3x(1 - 2x)$
4 $-2x(3x - 4)$	**5** $(x + 2)(x + 3)$	**6** $(2x + 3)(3x + 4)$
7 $(x + 1)(2x - 1)$	**8** $(1 - x)(2 + x)$	**9** $(3x - 2)(x - 3)$
10 $(x + 4)(x - 4)$	**11** $(1 - x)(1 + x)$	**12** $(x + 3)(x + 3)$
13 $(x + 2)^2$	**14** $(x - 3)^2$	**15** $(5x - 2)^2$
16 $(3 + 2x)^2$	**17** $2(x + 1)(x + 2)$	**18** $3(1 - x)(1 - 2x)$
19 $(x + 1)(x + 2)(x + 3)$	**20** $(2 - x)(2 + x)(2 + 3x)$	**21** $(x + a)(x + 2a)$
22 $(2x + a)(x - 2a)$	**23** $(x - a)(x + a)$	**24** $x(2x + 3a)(x - a)$
25 $(x + a)(x + 2a)(x + 3a)$	**26** $(a + b)^2$	**27** $(a - b)^2$
28 $(a + b)(a - b)$		

34.3 Identities

An **identity** is a relation between expressions which is always true no matter what values are attached to the unknowns.
For example, $2(x + 3a) \equiv 2x + 6a$ is an identity.
Note the sign $\equiv$ means *is identical to*. Three identities which should be memorised are:

$$(a + b)^2 \equiv a^2 + 2ab + b^2$$
$$(a - b)^2 \equiv a^2 - 2ab + b^2$$
$$(a + b)(a - b) \equiv a^2 - b^2$$

Put in several values of a and b in each of the above identities to show that the left side always equals the right side.
These identities were introduced at the end of the previous Exercise. The last is called a *difference of two squares*.

Example Expand (i) $(3p - 2q)^2$ (ii) $(3p + 2q)(3p - 2q)$

(i) Using the second identity above gives:

$$(3p - 2q)^2 = (3p)^2 - 2(3p)(2q) + (2q)^2 = 9p^2 - 12pq + 4q^2$$

(ii) Using a difference of two squares gives:

$$(3p + 2q)(3p - 2q) = (3p)^2 - (2q)^2 = 9p^2 - 4q^2$$

Exercise 34b

Expand the following.

1 $(p + q)^2$
2 $(2m - n)^2$
3 $(a + 3b)(a - 3b)$
4 $(3x + 5y)^2$
5 $(3x - 4)^2$
6 $(3x - 2)(3x + 2)$
7 $(1 - 6a)^2$
8 $(x + \frac{1}{2})^2$
9 $(1 - \frac{1}{2}x)^2$
10 $(\frac{1}{2}x - 1)(\frac{1}{2}x + 1)$
11 $(1 - 3x^2)^2$
12 $(x^2 + 2)(x^2 - 2)$
13 $(a + b)^3$
14 $(a - b)^3$
15 $(a + b)(a^2 - ab + b^2)$
16 $(a - b)(a^2 + ab + b^2)$
17 $(a + b + c)^2$
18 $(a + b)^4$

34.4 Factorisation of Quadratic Expressions

To factorise a quadratic expression (or polynomial) such as $x^2 + 5x + 6$ we use our knowledge of expansion, ie. that the factors will be of the form $(x + p)(x + q)$. Therefore work as follows:

$x^2 + 5x + 6 = (x \quad)(x \quad)$ ie. write two brackets with an x in each

Now the constants p and q must have a ***product*** of +6, the constant in the quadratic. This gives 1 and 6 or 2 and 3. The coefficient of x in the quadratic is +5, so the ***sum*** of p and q is +5. Therefore the numbers are 2 and 3. Insert these numbers in the brackets.

$$\therefore \quad x^2 + 5x + 6 = (x + 2)(x + 3)$$

The expression $x^2 + 5x + 6$ has its coefficient of x^2 equal to 1. In this case the product of the numbers (2×3) in the factors $(x + 2)(x + 3)$ equals 6, the constant. The sum $(2 + 3)$ equals 5, the coefficient of x. This is true generally in $x^2 + ax + b$ where b will be the product and a the sum of the numbers required in the factors. (See Example 1 below) However, this will not be the case if the coefficient of x^2 is not equal to 1. (See Example 2)

Example 1 Factorise $x^2 - 3x - 4$

By trial and error, the two numbers whose product is -4 (think of such numbers first) and whose sum is -3, are -4 and $+1$.

$$\therefore \quad x^2 - 3x - 4 = (x - 4)(x + 1)$$

Example 2 Factorise $5x^2 + x - 6$

Method 1: Trial and Error

Here the coefficient of x^2 is not 1, so even more care is required.

$$5x^2 + x - 6 = (5x \qquad)(x \qquad) \text{ so as to give } 5x^2$$

Two numbers whose product is -6 are: -6 and 1, 6 and -1, 3 and -2 or -3 and 2. Each of these pairs is tried either way around, for example $(5x - 6)(x + 1)$ and $(5x + 1)(x - 6)$ until the expansion

$$(5x \qquad)(x \qquad)$$

add

gives a coefficient of x equal to 1.

The correct factorisation is $5x^2 + x - 6 = (5x + 6)(x - 1)$

Method 2: Mechanical

Take the product of 5 and -6 ie. -30 and find the two factors of -30 which, on addition, give the coefficient of x, ie. $+1$. The two numbers are $+6$ and -5. Now proceed as follows:

$$\begin{aligned} 5x^2 + x - 6 &= 5x^2 + 6x - 5x - 6 \\ &= x(5x + 6) - 1(5x + 6) \quad \ldots\ldots (1) \\ &= (x - 1)(5x + 6) \end{aligned}$$

Note the expression in each bracket in (1) above is the same, ie. $(5x + 6)$.

NB Always check your answer by expansion.

Exercise 34c

Factorise the following.

1 $x^2 + 7x + 12$	2 $x^2 - 7x + 12$	3 $x^2 + x - 6$
4 $x^2 - x - 6$	5 $2x^2 + 5x + 2$	6 $3x^2 - 2x - 1$
7 $x^2 - 13x - 30$	8 $x^2 - 13x + 30$	9 $5x^2 + 27x + 10$
10 $x^2 - 14x - 72$	11 $5x^2 + 4x - 12$	12 $6x^2 - 11x - 10$
13 $x^2 + 2x + 1$	14 $x^2 - 8x + 16$	15 $4x^2 - 12x + 9$
16 $x^2 - 1$	17 $x^2 - 9$	18 $9x^2 - 4$

35 QUADRATIC EQUATIONS (1)

35.1 Definition

An equation in one unknown which can be expressed in the form

quadratic polynomial = 0

is called a **quadratic equation**.

Examples are: $x^2 + 5x - 6 = 0$, $2t^2 - 5t = 3$, $x^2 = 6x$ and $p^2 = 4$

35.2 Solution by Factorisation

This method uses the following statement:

The relation $a \times b = 0$ is satisfied only if either $a = 0$ or $b = 0$

Example 1 Solve the equation $x^2 + 5x - 6 = 0$.

Write down the equation: $x^2 + 5x - 6 = 0$
Factorise the left side: $(x - 1)(x + 6) = 0$

This is now of the form $a \times b = 0$ given above where $a = (x - 1)$ and $b = (x + 6)$.

$\therefore$ either $x - 1 = 0$ or $x + 6 = 0$

These two **linear** equations give $x = 1$ or $x = -6$

Note that there are two possible solutions or **roots**. Check that each satisfies the original equation.

Example 2 Solve the equation $2t^2 - 5t = 3$.

First write the equation in the form given in **35.1** and then proceed as in **Example 1**.

$$2t^2 - 5t - 3 = 0$$
$$\therefore \quad (t - 3)(2t + 1) = 0$$
$$\therefore \quad \text{either } t - 3 = 0 \text{ or } 2t + 1 = 0$$
$$\therefore \quad t = 3 \text{ or } -\tfrac{1}{2}$$

Exercise 35a

Solve the equations in **1** to **12**.

1 $x^2 - 5x + 6 = 0$	**2** $x^2 + 3x - 10 = 0$	**3** $x^2 - 11x - 60 = 0$
4 $x^2 + 10x - 24 = 0$	**5** $x^2 - 4 = 0$	**6** $x^2 = 6x$
7 $2t^2 + 3t - 2 = 0$	**8** $4p^2 - 4p + 1 = 0$	**9** $6a^2 + 5a = 6$
10 $(2m - 1)(2m - 3) = 8$	**11** $(s - 1)^2 = 9$	**12** $(2c + 3)^2 = 1$

13 Form a quadratic equation whose roots are 3 and -2.

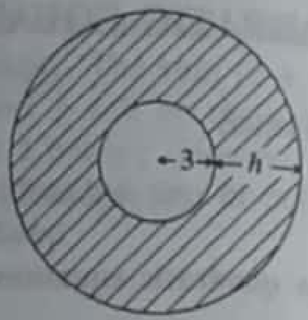
14 Form a quadratic equation whose roots are $\frac{1}{2}$ and $-\frac{2}{3}$.

15 A rectangle is such that its length is 10cm longer than its width. If the area is 96cm², find the width.
(Hint: let x cm be the width of the rectangle; form a quadratic equation and choose the appropriate solution to answer the question.)

16 Find the dimensions of a rectangle whose width is $1\frac{1}{2}$cm shorter than the length and has area $32\frac{1}{2}$cm².

17 A right-angled triangle has sides of length x cm, $(x - 2)$cm and $(x + 2)$cm. By using Pythagoras, find x and hence the lengths of the three sides.

18 The diagram shows two **concentric** circles (circles with the same centre) whose difference in radii is h cm. The radius of the inner circle is 3cm and the area (shaded) between the two circles is 286cm^2. Taking π to be $\frac{22}{7}$ show that



$$h^2 + 6h = 91$$

and hence find h.

19 A lorry travels from Mbarara to Masaka, a distance of 130km, at an average speed of v km/h. It then travels from Masaka to Kampala, a further distance of 130km at an average speed of $(v + 5)$km/h. If the journey takes 4h 10min altogether, find v.

35.3 Graphical Method of Solution

This method is particularly useful for solving two or more quadratic equations which are related, or equations for which the factorisation method fails.

Example Solve (i) $2x^2 - 3x - 3 = 0$, (ii) $2x^2 - 3x - 1 = 0$, (iii) $2x^2 - 3x + 1 = 0$.

All the above equations can be written in the form $2x^2 - 3x = y$, where y equals 3, 1 and -1 in cases (i), (ii) and (iii) respectively. We therefore draw the graph of $y = 2x^2 - 3x$.

Table of values:

x	-1	0	1	2	3	$-0{\cdot}5$	$0{\cdot}5$	$1{\cdot}5$	$2{\cdot}5$
$2x^2$	2	0	2	8	18	$0{\cdot}5$	$0{\cdot}5$	$4{\cdot}5$	$12{\cdot}5$
$-3x$	3	0	-3	-6	-9	$1{\cdot}5$	$-1{\cdot}5$	$-4{\cdot}5$	$-7{\cdot}5$
y	5	0	-1	2	9	2	-1	0	5

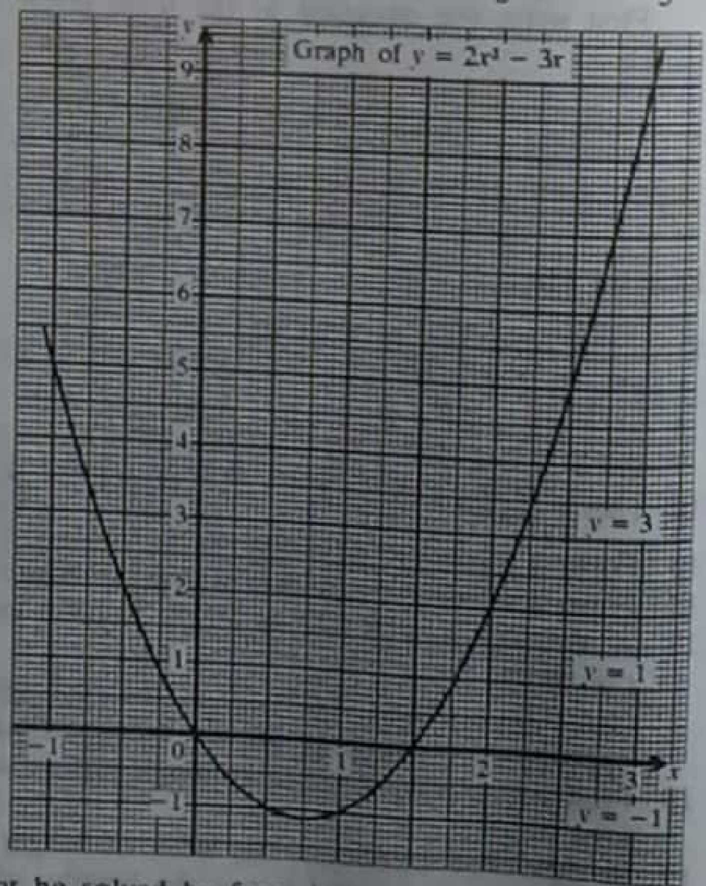
Integral values of x are taken first and the corresponding points plotted so that it can be seen which other points are needed to produce a smooth curve. In this case these are when $x = -0{\cdot}5$, $0{\cdot}5$, $1{\cdot}5$ and $2{\cdot}5$. The table of values is therefore extended as shown. These points are now plotted and the graph completed as shown.

To solve the equations, draw the lines $y = 3$, $y = 1$ and $y = -1$ as indicated and read off the values of x where the graph cuts these lines. The solutions should be correct to 1 decimal place.

(i) For $2x^2 - 3x - 3 = 0$ the solutions are $x = -0{\cdot}7$ or $2{\cdot}2$

(ii) For $2x^2 - 3x - 1 = 0$ the solutions are $x = -0{\cdot}3$ or $1{\cdot}8$

(iii) For $2x^2 - 3x + 1 = 0$ the solutions are $x = 0{\cdot}5$ or 1



Note that the first two equations could *not* be solved by factorisation.

Exercise 35b

Draw the graph of $y = 2x^2 + 3x$ for values of x from -3 to $+1$ and use your graph with suitable lines to solve the equations in **1** to **6**.

1 $2x^2 + 3x - 2 = 0$	**2** $2x^2 + 3x - 4 = 0$	**3** $2x^2 + 3x = 1$
4 $2x^2 + 3x = 0$	**5** $2x^2 + 3x + 1 = 0$	**6** $2x^2 + 3x + 2 = 0$

By drawing the appropriate graph, solve the equations in **7** to **12**.

7 $3x^2 - 5x - 4 = 0$	**8** $3x^2 - 5x + 1 = 0$	**9** $3x^2 - 5x = 1$
10 $3x^2 - 5x - 2 = 0$	**11** $3x^2 = 5x + 3$	**12** $3x^2 = 5x - 3$

13 Using a scale of 2cm for 1 unit on the x-axis and 1cm for 1 unit on the y-axis, draw an accurate graph of $y = x^2$ for values of x from -3 to $+3$.
(i) On the same axes draw the line $2y - 3x - 3 = 0$. Use the points of intersection to solve the equation $2x^2 - 3x - 3 = 0$.
(ii) Also on the same axes draw the line $2y - 3x - 1 = 0$. Hence solve the equation $2x^2 - 3x - 1 = 0$.
(iii) By drawing a suitable straight line such as in (i) and (ii) above, solve the equation $2x^2 - 3x + 1 = 0$.
Compare your graphs and solutions with those of the text Example.

In **14** to **20** solve the equations using your graph of $y = x^2$ and method of **Q13**.

14 $x^2 = x + 1$	**15** $x^2 = 3 - x$	**16** $x^2 - 2x - 2 = 0$
17 $x^2 + 2x - 1 = 0$	**18** $2x^2 + 3x = 1$	**19** $3x^2 - 5x - 4 = 0$
20 $2x^2 - 2x + 1 = 0$		

35.4 Quadratic Inequalities

These may be solved graphically as well as by the analytical method of **36.5**.

Example Solve the inequality $2x^2 - 3x < 0$.

The graph of $y = 2x^2 - 3x$ is shown on page 110. The part of the graph lying *below* the x-axis represents the inequality $2x^2 - 3x < 0$. The values of x corresponding to this lie between 0 and 1·5.
The solution is $0 < x < 1{\cdot}5$.

Exercise 35c

1 Use the graph drawn on page 110 to solve the inequalities
(i) $2x^2 - 3x < 5$ (Hint: for what values of x does the graph lie *below* the line $y = 5$?)
(ii) $2x^2 - 3x > 2$ (Hint: for what values of x does the graph lie *above* the line $y = 2$?)
(iii) $2x^2 - 3x + 1 \le 0$. (Hint: first subtract 1 from each side of the inequality)

2 Use the graph of $y = 2x^2 + 3x$ (drawn in **Exercise 35b**) to solve the inequalities
(i) $2x^2 + 3x < 0$ (ii) $2x^2 + 3x > 5$ (iii) $2x^2 + 3x \le 2$ (iv) $2x^2 + 3x + 1 \ge 0$

In **3** to **8**, solve the inequalities by drawing an appropriate graph.

3 $3x^2 - 5x < 4$	**4** $3x^2 - 5x > -1$	**5** $3x^2 - 5x \le 1$
6 $3x^2 - 5x - 2 \ge 0$	**7** $3x^2 < 5x + 3$	**8** $3x^2 \ge 5x - 3$

36 INEQUALITIES

36.1 Rules for Inequalities

The rules for manipulating or solving inequalities are the same as those for equations (see 21.2) with the following exceptions.
(i) When *multiplying* or *dividing* an inequalitiy by a ***negative*** number, ***the sign of the inequality must be reversed.***
(ii) When interchanging the LS and RS the inequality sign must be reversed, for example if $2 > x$ then $x < 2$.

Example 1 Solve the inequality $5 - 4x \geq -x + 8$

Write down the inequality:	$5 - 4x \geq -x + 8$
Add x:	$5 - 3x \geq 8$
Subtract 5:	$-3x \geq 3$
Divide by -3:	$x \leq -1$ (note change of inequality sign)

Example 2 Solve the inequality $\frac{4}{x} < 8$.

Multiplication through by x gives two possibilities.

(i) If x is positive, ie. $x > 0$, then $4 < 8x$ or $x > \frac{1}{2}$

(ii) If x is negative, ie. $x < 0$, then $4 > 8x$ or $x < \frac{1}{2}$

From (i), $x > 0$ *and* $x > \frac{1}{2}$. So $x > \frac{1}{2}$ satisfies ***both*** these conditions.

From (ii), $x < 0$ *and* $x < \frac{1}{2}$. Therefore $x < 0$

The solution is $x < 0$ or $x > \frac{1}{2}$. (See graphical representation below.)

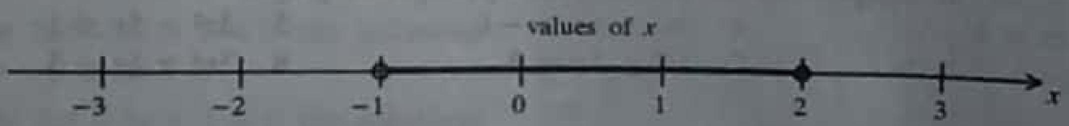
Exercise 36a

Solve the inequalities.

1 $2x + 5 \leq 11$	2 $3x - 1 \geq x + 7$	3 $5 - x < 2$
4 $2x + 3 \geq -x$	5 $2(2x + 1) < 3 - 2(x - 1)$	6 $5 - (1 + x) > 3 - (2 + 3x)$
7 $\frac{6}{x} > 2$	8 $-\frac{4}{x} \geq 8$	

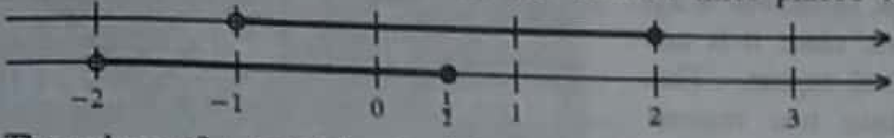
36.2 Graphical Representation

It is useful to represent solutions to inequalities graphically. Suppose x is such that $x > -1$ and $x \leq 2$. This may be written $-1 < x \leq 2$.
We may represent possible values of x on a number line.



The empty circle at -1 means that -1 is not included and the full circle at 2 means that 2 is included in the solution.

Example 1 Represent graphically all values of x for which $-1 < x \leq 2$ and $-2 < x \leq \frac{1}{2}$.

Represent the two conditions on two number lines placed together:

$-1 < x \leq 2$

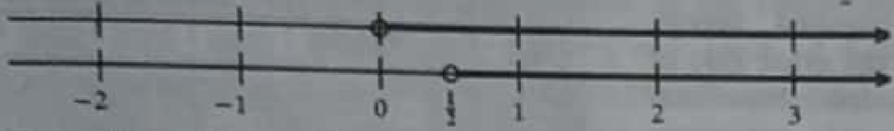
$-2 < x \leq \frac{1}{2}$

The values of x satisfying both inequalities can now be clearly seen:

This is $-1 < x \leq \frac{1}{2}$

Example 2 Represent graphically the solution to $\frac{4}{x} < 8$. (See **36.1 Ex 2**.)

In case (i), where x is positive, we have $x > 0$ and $x > \frac{1}{2}$



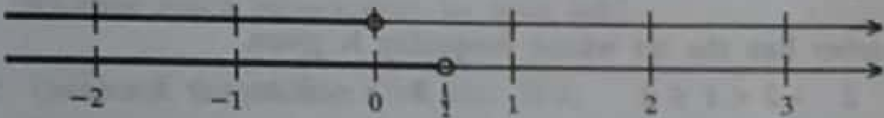
$x > 0$

$x > \frac{1}{2}$

The values of x satisfying both inequalities are represented thus:

This is $x > \frac{1}{2}$ .... (i)

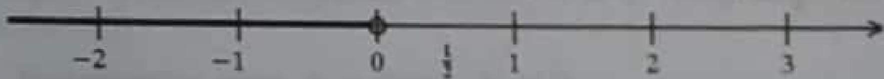
In case (ii), where x is negative, we have $x < 0$ and $x < \frac{1}{2}$



$x < 0$

$x < \frac{1}{2}$

The values of x satisfying both inequalities are represented thus:



This is $x < 0$ .... (ii)

The combined solution from (i) and (ii) is shown on one number line:

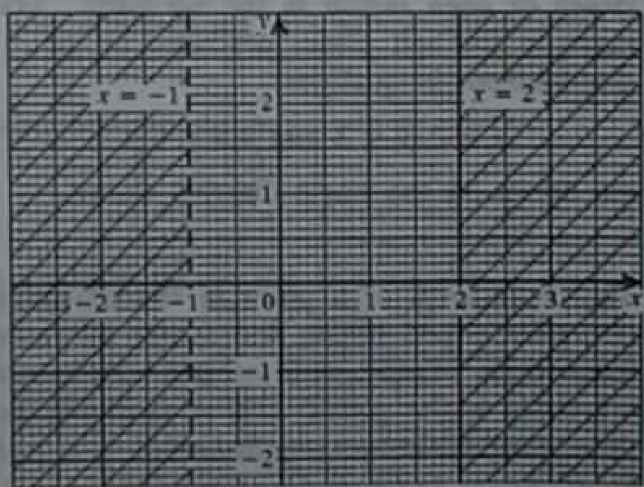
$x < 0$ or $x > \frac{1}{2}$

36.3 Inequalities in Two Unknowns

Such inequalities are represented graphically by a **region** in a plane.

Example 1 Show the region representing points (x, y) such that $-1 < x \leq 2$.

This inequality, which restricts the value of x only, is represented on a number line in **36.2**. Here it is represented as a region in the x-y plane (see diagram).



Notes

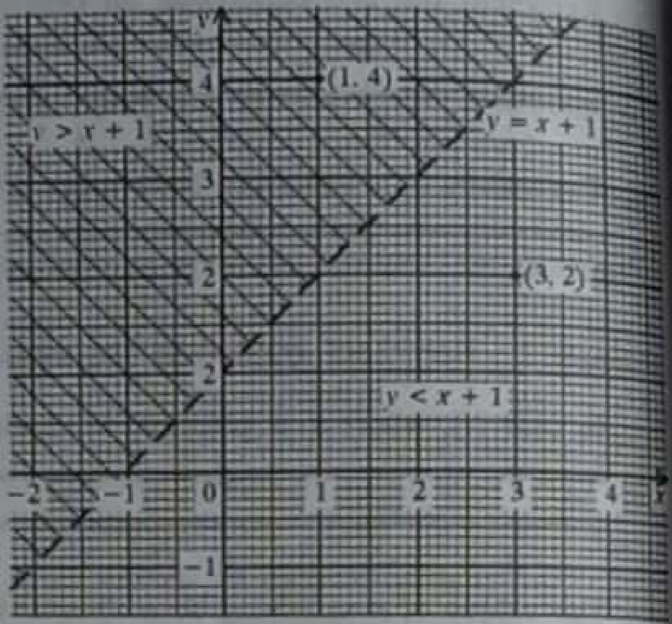
1. There is no restriction on the value of y.
2. The region representing the solution is left *unshaded*.
3. The line $x = -1$ is drawn broken as it is not in the region.

Example 2 Show the region representing $\{(x, y) : y < x + 1\}$.

First, draw the line $\{(x, y) : y = x + 1\}$. Draw it broken, as shown, since it is not to be included in the final region. This divides the x-y plane into two regions (half planes):

$$\{(x, y) : y > x + 1\}$$
$$\text{and} \quad \{(x, y) : y < x + 1\}$$

To decide which region represents $\{(x, y) : y < x + 1\}$ select a point, say (3, 2), and substitute its coordinates to see if the inequality is satisfied. It is since $2 < 3 + 1$. So (3, 2) is in the required region. Therefore the region on the other side of $\{(x, y) : y = x + 1\}$ is shaded. Note that a point such as (1, 4) is not in the region since its coordinates do not satisfy the inequality $y < x + 1$.



Exercise 36b

In 1 to 7 represent on a number line the set whose inequality is given.

1 $x < 2$

2 $-2 < x \le 1$

3 $x < 0$ or $x > 3$

4 $\frac{2}{x} < 1$

5 $\frac{2}{x} \ge 1$

6 $-\frac{1}{2} < x \le 2$ and $-1 < x \le 1$

7 $\frac{3}{1 - x} + 1 \le 0$

In 8 to 16 represent each set of points as a region by shading the unwanted half plane.

8 $\{(x, y) : x \ge 1\}$

9 $\{(x, y) : y < 3\}$

10 $\{(x, y) : -\frac{1}{2} \le x < 2\}$

11 $\{(x, y) : y \ge x + 1\}$

12 $\{(x, y) : y > 2x - 1\}$

13 $\{(x, y) : y < x\}$

14 $\{(x, y) : x + y \ge 0\}$

15 $\{(x, y) : 3x + 2y < 6\}$

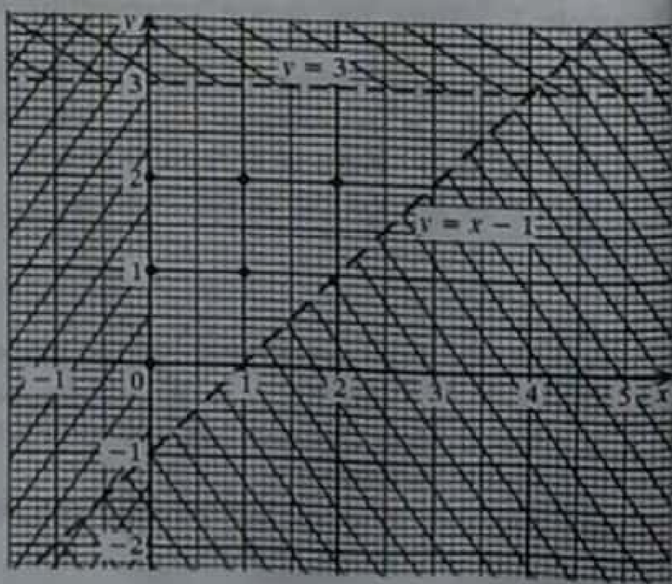
16 $\{(x, y) : x + 4y \ge 8\}$

36.4 Simultaneous Inequalities

Certain sets of points are represented as a **closed** or **open** region in the x-y plane. For example, the half planes described by the **simultaneous** inequalities

$$x \ge 0,\ y < 3 \text{ and } y > x - 1$$

are drawn on the single diagram shown.



The coordinates of a point in the unshaded triangular closed region will satisfy all three inequalities. Points with integral coordinates are called **lattice points**. Those satisfying the three inequalities are marked with dots and are (0, 0), (0, 1), (0, 2), (1, 1), (1, 2) and (2, 2).

Exercise 36c

In each of 1 to 6, show on one diagram the region described by the given inequalities.

1 $x \geq 0, y \geq 0, x \leq 3, y \leq 2$

2 $x > -1, y > 1, x + y < 4$

3 $x \geq 1, y \geq 0, y \leq x + 1$ (an open region)

4 $x \geq 0, y \geq 0, x + y < 5, x + 4y > 8$
Mark and count the number of lattice points in the region.

5 $x < 4, y < 3, x + y > 4$
Mark and name the single lattice point in the region.

6 $y < x + 2, x + 3y \geq 6, 2x + y \leq 8$
Mark and name the five lattice points in the region.

7 Find graphically the only integral solution to the simultaneous inequalities

$$3y \leq 2x + 9, \quad 2x + y < 8 \quad \text{and} \quad x + 2y > 7$$

8 A loaf of bread costs sh400 and a soda sh250. Rwerinyange has sh2,000 and buys x loaves of bread and y sodas under the following conditions. He must buy more than one loaf and the number of sodas must be at least twice the number of loaves.
(i) Show that $8x + 5y \leq 40$ and write down two other inequalities involving x and y.
(ii) Graph these inequalities.
(iii) How many loaves and sodas does Rwerinyange buy?
(iv) How much money does he have left?

36.5 Quadratic Inequalities (See also 35.4)

Example Solve the inequality $x^2 - x - 6 \leq 0$. Represent the solution set on the number line.

Factorisation gives $(x + 2)(x - 3) \leq 0$.
The 'end' or limiting values are given by $(x + 2)(x - 3) = 0$.
This gives $x + 2 = 0$ or $x - 3 = 0$, ie. $x = -2$ or $x = 3$.
Mark these values as full circles on the number line.
(Note: if we were given $(x + 2)(x - 3) < 0$, then empty circles at -2 and 3 would have been marked.)
For the product of $(x + 2)$ and $(x - 3)$ to be negative (ie. *less than* zero) then one bracket must be positive and the other negative.
So *either* $x + 2 \leq 0$ *and* $x - 3 \geq 0$ *or* $x + 2 \geq 0$ *and* $x - 3 \leq 0$
The first statement leads to $x \leq -2$ *and* $x \geq 3$ which gives no possible solutions.
The second statement leads to $x \geq -2$ *and* $x \leq 3$ ie. $-2 \leq x \leq 3$ as the solution set.
This is illustrated in the diagram below.

Exercise 36d

In 1 to 9, solve the inequality and represent its solution set on the number line.

1 $x^2 - x - 2 \leq 0$	2 $x^2 - 5x + 4 < 0$	3 $x^2 - 2x - 3 \geq 0$
4 $x^2 \geq 16$	5 $x^2 + 4x < 0$	6 $2x^2 - 9x + 4 \leq 0$
7 $(x - 2)^2 > 9$	8 $(x - 1)(x + 2) \geq 4$	9 $x \geq \frac{9}{x}$

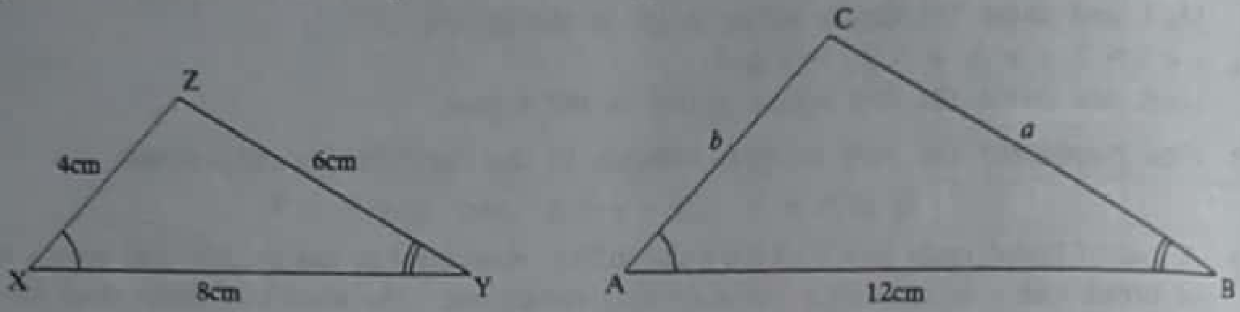
10 Draw a graph to show the region described by $4y > x^2$ and $2y < x + 4$. Mark and name the six lattice points in the region.

37 SIMILARITY AND CONGRUENCE

37.1 Similar Triangles

Two similar triangles have the same shape but different sizes, ie. one is an **enlargement** of the other (see **53.5**). If two triangles are equiangular (such as in the Example below) then they are similar. It also follows that the lengths of corresponding sides are proportional.

Example Find a and b in triangle ABC shown, given that $\angle A = \angle X$ and $\angle B = \angle Y$.



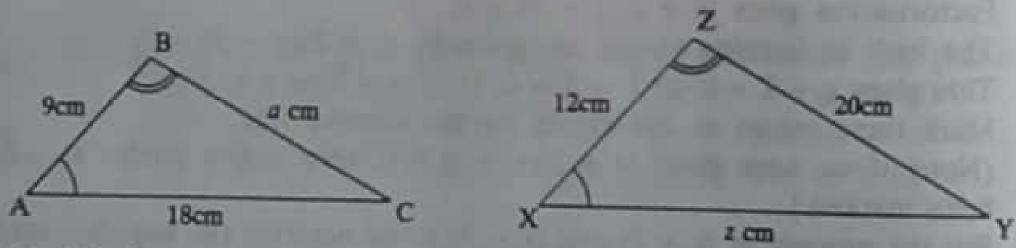
Since $\angle A = \angle X$ and $\angle B = \angle Y$, then $\angle C = \angle Z$ because the angle sum of each triangle is 180°. The triangles are equiangular and are therefore similar. The sides opposite the equal angles correspond.

So $\frac{a}{6} = \frac{b}{4} = \frac{12}{8} = \frac{3}{2}$ Hence $a = 6 \times \frac{3}{2} = 9\text{cm}$ and $b = 4 \times \frac{3}{2} = 6\text{cm}$

Triangle XYZ has been **enlarged** by a **linear scale factor** $\frac{3}{2}$ to give triangle ABC. We name triangles that are similar in the order of the equal angles, ie. $\triangle ABC$ to $\triangle XYZ$.
Then the ratios of the sides follow from this correct order: $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$

Exercise 37a

1 Find z and a.



2 When a pole of height 2 metres casts a shadow of length 3m on horizontal ground a building casts a shadow of length 37·5m. What is the height of the building?

3 In Fig. 1, given that DE is parallel to BC show that triangles ABC and ADE are similar. Hence find x.

Fig. 1

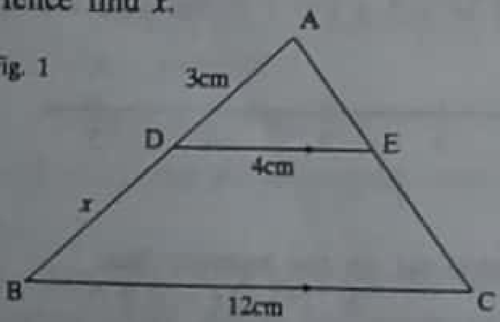
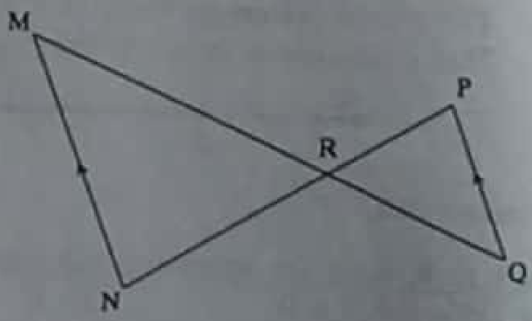


Fig. 2

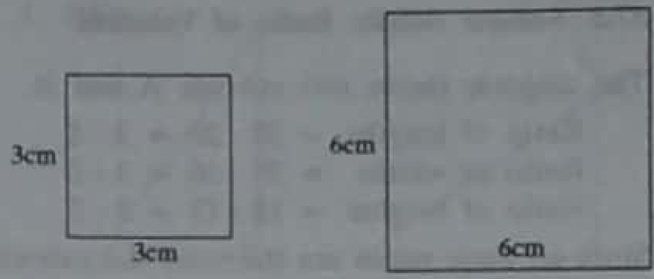


4 In Fig. 2, MN and PQ are parallel lines and PN and QM intersect at R. Given that MN = 5cm and PQ = 3cm find MR if RQ = 2·4cm.

5 Triangle ABC has a point X on AB and a point Y on AC such that XY is parallel to BC. Given that AY = 3cm, YC = 5cm and AB = 12cm calculate AX and BX.

37.2 Similar Figures: Ratio of Areas

The two squares shown are similar and the ratio of their sides is 6 : 3 or 2 : 1. Their areas are 36cm² and 9cm² respectively which means that the ratio of the areas is 36 : 9 or 4 : 1.



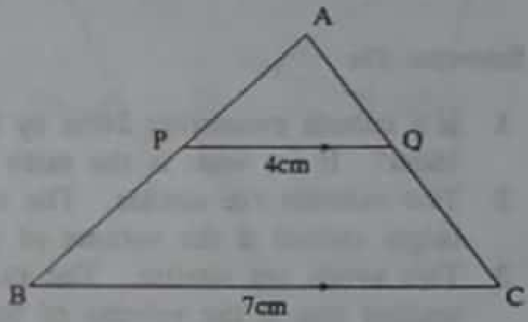
Note that the **linear scale factor** (LSF) is 2 : 1 and that the **area scale factor** (ASF) is 4 : 1. Also note that the ASF = (LSF)². This relation is true for all similar figures.

Example The points P and Q on the sides AB and AC of ΔABC are such that PQ is parallel to BC. PQ = 4cm and BC = 7cm.
Find the ratio Area of quadrilateral BPQC : Area of triangle ABC.

ΔABC is similar to ΔAPQ with LSF = $\frac{7}{4}$ $\quad \therefore$ ASF = (LSF)² = $\left(\frac{7}{4}\right)^2 = \frac{49}{16}$

Therefore $\dfrac{\text{Area of triangle ABC}}{\text{Area of triangle APQ}} = \dfrac{49}{16}$

So the area of ΔAPQ is $\frac{16}{49}$ of the area of ΔABC.



The area of quadrilateral BPQC is $\left(1 - \frac{16}{49}\right)$ or $\frac{33}{49}$ of the area of ΔABC.

$\therefore \quad \dfrac{\text{Area of quadrilateral BPQC}}{\text{Area of triangle ABC}} = \dfrac{33}{49}$

Hence the required ratio is 33 : 49.

Exercise 37b

1 Rectangle A measures 6m by 4m and rectangle B is 9m by 6m. State in simplest form (i) the ratio of their lengths, (ii) the ratio of their widths. Are these rectangles similar? Find the ratio of their areas and verify that: ratio of areas = (ratio of lengths)².

2 A piece of land has an area of 12ha. Another piece of land of the same shape has twice the perimeter. What is its area?

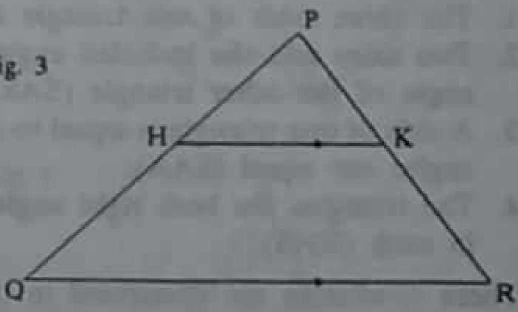
3 Two similar triangles have areas of 16cm² and 36cm² respectively. If the base of the smaller triangle is 6cm, what is the base of the larger triangle?

4 In Fig. 3, HK = 3cm, QR = 5cm and HK is parallel to QR. Find the ratio area of ΔPQR : area of ΔPHK.

5 In Fig. 3, QR = 8cm, HK = 4cm and HK is parallel to QR. Find the ratio area of ΔPQR : area of quad HQRK.

6 In Fig. 3, PH = 3cm, HQ = 4cm and HK is parallel to QR. Find the area of ΔPHK if the area of quadrilateral HQRK is 20cm².

Fig. 3



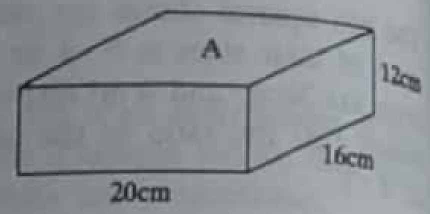
7 Erapu and Wedi have rectangular plots of land on which they will plant beans. These measure 140m by 120m and 210m by 180m respectively. Are these plots similar? An agricultural officer advises Erapu to use 160kg of fertiliser on his land. How much fertiliser should Wedi use?

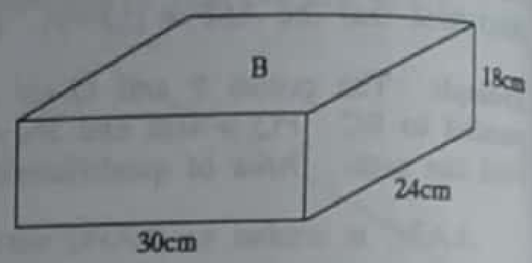
8 A farm is represented by an area of 252cm² on map A and 700cm² on map B. If the scale of map B is 1 : 30,000, what is the scale of map A?

37.3 Similar Solids: Ratio of Volumes

The diagram shows two cuboids A and B.

Ratio of lengths $= 30 : 20 = 3 : 2$
Ratio of widths $= 24 : 16 = 3 : 2$
Ratio of heights $= 18 : 12 = 3 : 2$





Since all these ratios are the same the cuboids are similar, with LSF $= \frac{3}{2}$

Volume of B $= 30 \times 24 \times 18$ cm^3
Volume of A $= 20 \times 16 \times 12$ cm^3

$$\textbf{Volume scale factor} = \frac{30 \times 24 \times 18}{20 \times 16 \times 12} = \left(\tfrac{3}{2}\right) \times \left(\tfrac{3}{2}\right) \times \left(\tfrac{3}{2}\right) = \left(\tfrac{3}{2}\right)^3$$

ie. volume of B $= \left(\frac{3}{2}\right)^3 \times$ volume of A
Note that volume scale factor (VSF) = (LSF)^3.

Exercise 37c

1 Is a cuboid measuring 24cm by 18cm by 12cm similar to one measuring 32cm by 24cm by 16cm? If so, what is the ratio of (i) their surface areas, (ii) their volumes?
2 Two cuboids are similar. The ratio of their lengths is 4 : 3. What is the volume of the larger cuboid if the volume of the smaller one is 81cm^3?
3 Two solids are similar. The ratio of their lengths is 5 : 2. What is the volume of the smaller one if the volume of the larger solid is 625cm^3?
4 A sphere has a volume of 24cm^3. What is the volume of a sphere with three times the radius?
5 Two milk containers are of the same shape. One is 80cm high and the other 64cm high. If the smaller container can hold 32 litres of milk, what is the capacity of the larger one?

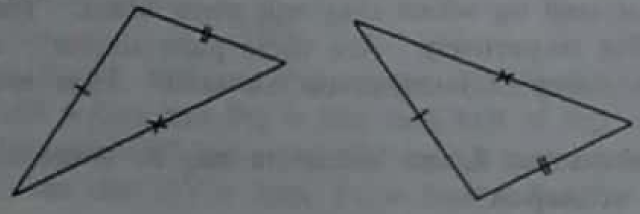
37.4 Congruent Triangles

Similar triangles have the same shape. Congruent triangles not only have the same shape, but the same size. There are four sets of conditions which ensure in each case that two triangles are congruent.

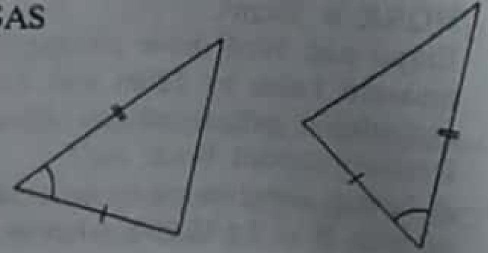
1. The three sides of one triangle are equal to the three sides of the other triangle (SSS).
2. Two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle (SAS).
3. A side of one triangle is equal to a side of the other triangle and two pairs of corresponding angles are equal (SAA).
4. The triangles are both right angled, are of equal hypotenuse and have a second side equal in each (RHS).

These conditions are illustrated in the following diagrams.

1. SSS



2. SAS



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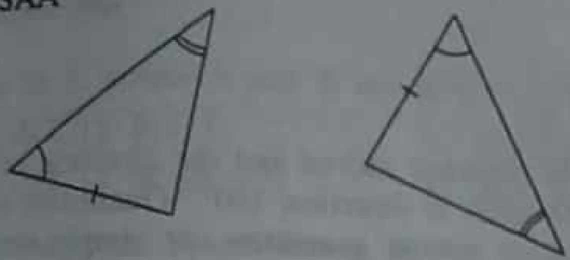
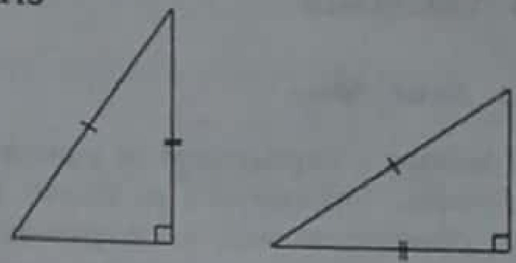
3. SAA



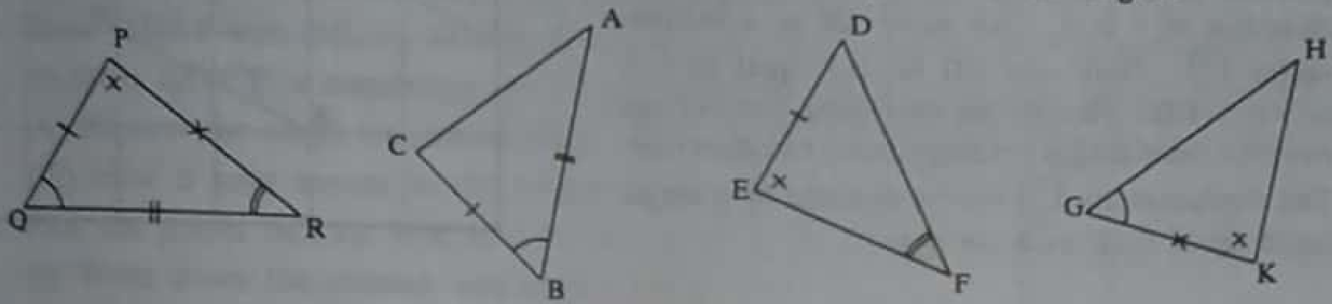
4. RHS



If two triangles are congruent then corresponding sides and angles are equal.

Exercise 37d

1 Which triangles are congruent to $\triangle PQR$? Name the equal sides and angles.

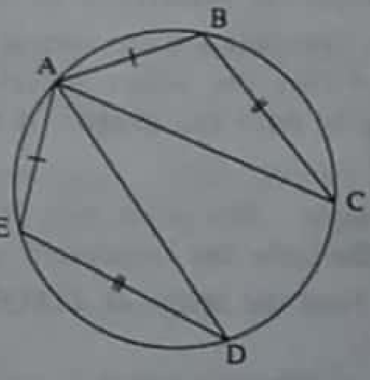


2 Which two of the following triangles are congruent?
$\triangle PQR$: $\angle R = 90°$, $PQ = 10$cm, $QR = 7$cm
$\triangle STU$: $\angle T = 90°$, $ST = 10$cm, $TU = 7$cm
$\triangle VWX$: $\angle W = 90°$, $XV = 10$cm, $VW = 7$cm

3 LMNP is a parallelogram. The sides LM, MN, NP, PL are produced to points A, B, C and D respectively, such that $LM = MA$, $MN = NB$, $NP = PC$ and $PL = LD$. Prove that $\triangle PCD$ is congruent to $\triangle MAB$.

4 In Fig. 4, $AE = AB$ and $ED = BC$. Prove that $\triangle AED$ is congruent to $\triangle ABC$.

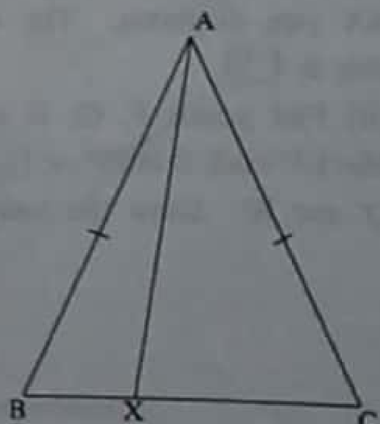
Fig. 4



5 Fig. 5 shows an isosceles triangle ABC in which $AB = AC$. The point X is on BC such that $BX \neq XC$. What is the error in the following argument?

In $\triangle$s ABX and ACX
$AB = AC$ (isosceles $\triangle$)
$\angle ABX = \angle ACX$ (isosceles $\triangle$)
AX is common
$\therefore$ $\triangle$s ABX and ACX are congruent (ASS)
Therefore $BX = XC$!

Fig. 5

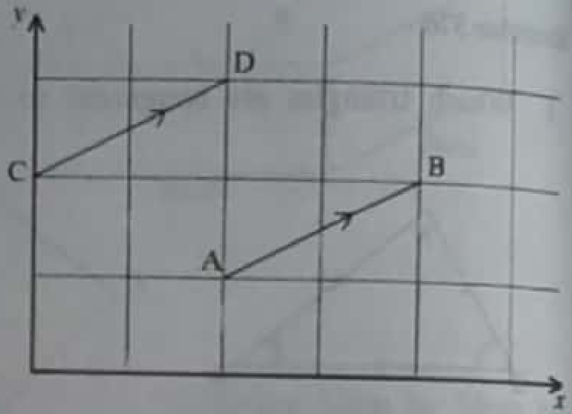


38 VECTORS

38.1 Basic Ideas

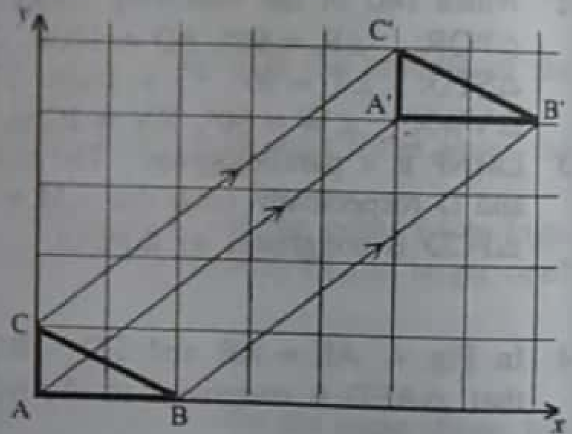
To describe a displacement or movement we give the distance moved and the direction of that movement. For example, an aircraft might fly at 400km/h in direction 150°. Quantities which have direction as well as magnitude (size) are called **vector quantities**, or simply vectors. Examples of vectors are velocity, acceleration and force. A quantity which has only magnitude is called a **scalar quantity**, or simply a scalar. Examples of scalars are mass, time, length, area, volume, a number (eg. 7).

The displacement from A to B in the diagram is denoted by **AB** (or $\underset{\sim}{AB}$). Movement in the positive direction of x is 2 and in the positive direction of y is 1. We write **AB** as a **column vector** $\binom{2}{1}$. Note that **CD** is also equal to $\binom{2}{1}$, so **AB** = **CD**. This means that these two vectors have the same length (or magnitude) and direction. The displacement $\binom{2}{1}$ can be denoted by a single small letter such as **a** (or $\underset{\sim}{a}$).



38.2 Translation

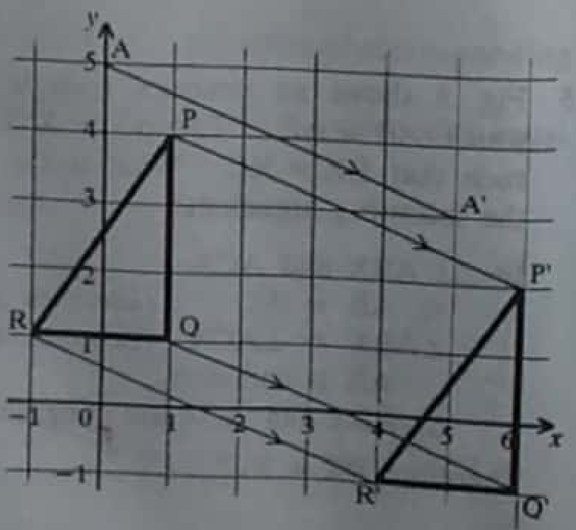
A vector is used to describe a **translation** in which an object, such as triangle ABC ($\triangle$ABC) shown in the diagram on the right, slides from one position to another. In this example any of the vectors **AA'**, **BB'** or **CC'** can be used to describe the translation as they are all equal to $\binom{5}{4}$. Translation is an example of a **transformation**. Note that the object moves in a straight line directly onto the image. It does *not* rotate.



Example The point A(0, 5) moves to A'(5, 3) under a translation.
(i) Describe the translation using a column vector
(ii) Find the image of $\triangle$PQR under this translation where P is (1, 4), Q(1, 1) and R(−1, 1).

(i) The translation is described by the vector **AA'** (see diagram). The column vector for this is $\binom{5}{-2}$.

(ii) Plot points P, Q, R and draw $\triangle$PQR. Mark P' such that **PP'** = $\binom{5}{-2}$. Similarly mark Q' and R'. Draw the image $\triangle$P'Q'R'.



Exercise 38a

In **1** to **6**, points A and B are given. Express the vector **AB** as a column vector.

1 A(2, 1), B(5, 3)	**2** A(−2, −1), B(2, 4)	**3** A(0, 4), B(3, 0)
4 A(3, 1), B(−3, −5)	**5** A(0, 0), B(−4, 3)	**6** A(0, 0), B(10, −16)

7 The point P(4, 5) moves to P'(0, 1) under a translation.
(i) Describe the translation using a column vector.
(ii) Find the image of $\triangle LMN$ under this translation where L is (2, 3), M(6, 3) and N(3, 4).

8 $\triangle ABC \rightarrow \triangle A'B'C'$ under a translation given by $\binom{4}{-6}$. Find the coordinates of A, B and C if the image coordinates are A'(2, 2), B'(1, −1) and C'(6, −1).

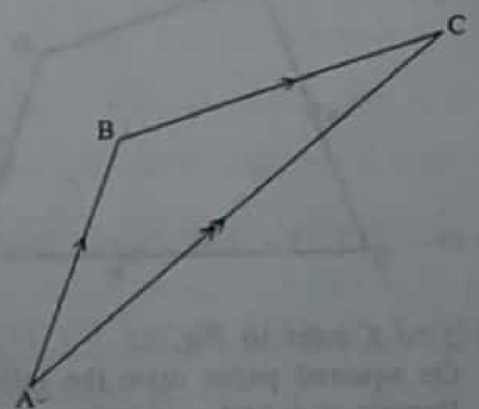
9 Draw $\triangle DEF$ with D(1, 1), E(1, 3), F(4, 4) and its image $\triangle D'E'F'$ after a translation $\binom{2}{-1}$.
(i) Give $\triangle D'E'F'$ a translation of $\binom{4}{3}$ and state the coordinates of its image $\triangle D''E''F''$.
(ii) Express the single translation which **maps** $\triangle DEF$ onto $\triangle D''E''F''$ as a column vector.
(iii) How is your answer to (ii) related to the original column vectors?

10 Plot the points A(3, 0), B(8, 3), C(0, 5), D(10, 11) on squared paper. Join AB and CD.
(i) Write down the column vectors for **AB** and **CD**.
(ii) How are these column vectors related?
(iii) What can you say about (a) the magnitudes (lengths), (b) the directions of AB and CD?

38.3 Multiplication by a Scalar

The vector $\binom{6}{3}$ is in the same direction as $\binom{2}{1}$ but is three times its length. If $\binom{2}{1}$ = **a** then $\binom{6}{3}$ = 3**a**. Multiplication by a scalar alters the magnitude but not the direction of a vector. The vector $\binom{-2}{-1}$ denoted by BA in the diagram of **38.1**, is opposite in direction to $\binom{2}{1}$. Therefore **BA** = $\binom{-2}{-1} = -1\binom{2}{1}$ = −**AB**. Multiplication by a negative scalar reverses the direction of a vector.

38.4 Addition of Vectors



From the diagram we see that the displacement AB followed by the displacement BC is the same as the single displacement AC. This is an example of vector addition and we write

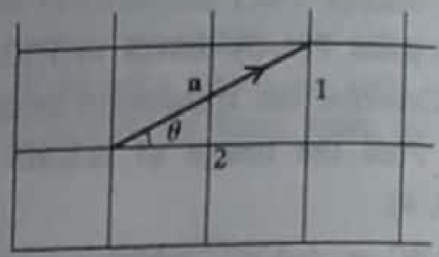
AB + BC = AC

If **AB** = $\binom{3}{2}$ and **BC** = $\binom{2}{1}$
then **AC** = $\binom{3}{2} + \binom{2}{1} = \binom{3+2}{2+1} = \binom{5}{3}$

To add two column vectors add the x-components and the y-components.

38.5 Magnitude and Direction of a Vector

The length of a vector is called its **magnitude** or **modulus**. The magnitude or modulus of a vector **a** is denoted by $|\mathbf{a}|$. The modulus of the vector **a** in the diagram below is given, using Pythagoras Theorem, by



$$|\mathbf{a}| = \sqrt{(2^2 + 1^2)} = \sqrt{(4 + 1)} = \sqrt{5}$$

Generally, if $\mathbf{a} = \binom{x}{y}$ then $|\mathbf{a}| = \sqrt{(x^2 + y^2)}$

The direction of vector **a** is given by angle θ which it makes with the positive x-direction.

Since $\mathbf{a} = \binom{2}{1}$, we have $\tan\theta = \frac{1}{2}$

This gives θ as $26{\cdot}6°$.

Example If $\mathbf{a} = \binom{2}{1}$ and $\mathbf{b} = \binom{3}{2}$ find the magnitude and direction of (i) $\mathbf{a} + \mathbf{b}$, (ii) $2\mathbf{b} - 3\mathbf{a}$.

(i) $\mathbf{a} + \mathbf{b} = \binom{2}{1} + \binom{3}{2} = \binom{5}{3}$ so $|\mathbf{a} + \mathbf{b}| = \sqrt{(5^2 + 3^2)} = \sqrt{34}$

$\tan\theta = \frac{3}{5} = 0{\cdot}6 \quad \therefore \quad \theta = 31{\cdot}0°$

(ii) $2\mathbf{b} - 3\mathbf{a} = 2\binom{3}{2} - 3\binom{2}{1} = \binom{6}{4} - \binom{6}{3} = \binom{0}{1}$

$|2\mathbf{b} - 3\mathbf{a}| = \sqrt{(0^2 + 1^2)} = \sqrt{1} = 1$

This is a **unit vector** making 90° with the positive x-direction.

38.6 Position Vector

This is the displacement vector from the origin O to a point in the x-y plane. If P has coordinates (4, 5), then the position vector of P is **OP** which, as a column vector is, $\binom{4}{5}$.

Exercise 38b

1 In Fig. 1, express **AC**, **BD**, **AD** and **DA** in terms of **a**, **b** and **c**.

Fig. 1

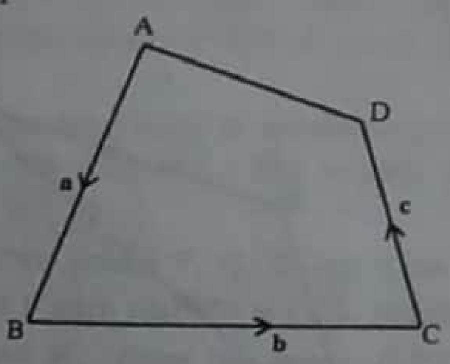
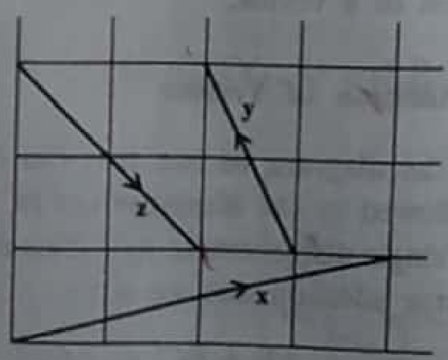


Fig. 2



For 2 to 5 refer to Fig. 2.

2 On squared paper draw the following vectors: $2\mathbf{x}$, $\mathbf{z} + \mathbf{x}$, $\mathbf{x} - \mathbf{y}$, $\mathbf{x} + \mathbf{y} + \mathbf{z}$

3 Express **x**, **y** and **z** as column vectors.

4 Find the angle, to the nearest degree, that **x**, **y** and **z** make with the positive x-direction.

5 Find, leaving your answers in surd form (see **26.3**), the modulus of **x**, **y** and **z**.

6 On squared paper mark points O(0, 0), A(3, 1) and C(2, 3).
(i) Write down the position vectors of A and C as column vectors.
(ii) The position vector of point B is the sum of those of A and C found in (i). Use this information to find the coordinates of B.
(iii) Mark B on your diagram. Draw and describe the figure OABC.

7 If $\mathbf{d} = \binom{4}{3}$ and $\mathbf{e} = \binom{-2}{1}$ express as column vectors: $\mathbf{d} + \mathbf{e}$, $2\mathbf{e}$, $2\mathbf{e} - \mathbf{d}$, $2\mathbf{e} + 3\mathbf{d}$, $\frac{1}{2}\mathbf{d} + \frac{3}{2}\mathbf{e}$.

8 The coordinates of P, Q, R and S are (1, 1), (2, 5), (5, 2) and (6, 6). Express as column vectors **OP**, **OQ**, **PQ**, **RS**, **QS** and **PR**, where O is the origin.

9 (i) Find $\mathbf{a}$ if $\binom{2}{3} + \mathbf{a} = \binom{-4}{1}$ (ii) Find x and y if $\binom{-9}{-2} - 3\binom{x}{y} = 2\binom{3}{-4}$

10 Given the vectors $\mathbf{a} = \binom{2}{7}$, $\mathbf{b} = \binom{8}{17}$ and $\mathbf{c} = \binom{17}{32}$, find the values of p and q such that $\mathbf{c} = p\mathbf{a} + q\mathbf{b}$.

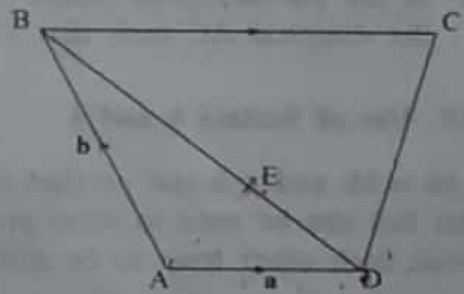
38.7 Parallel Vectors

If **AB** = k**CD** then **AB** and **CD** are parallel and the length of **AB** is k times the length of **CD**.

38.8 Collinearity

Points on the same straight line are **collinear**. If **AB** = k**BC** then **AB** is parallel to **BC**; but since both contain the point B then the points A, B and C are collinear.

Example The diagram shows a trapezium ABCD in which BC = 2AD. E is a point on BD such that BE = 2ED. AD = $\mathbf{a}$ and AB = $\mathbf{b}$. Express in terms of $\mathbf{a}$ and $\mathbf{b}$ the following vectors: **BC**, **BD**, **ED**, **AE**, **BE**, **EC**. Hence show that A, E and C are collinear.



BC = 2**AD** = 2$\mathbf{a}$

AB + **BD** = **AD** gives **BD** = **AD** − **AB** = $\mathbf{a} - \mathbf{b}$

ED = $\frac{1}{3}$**BD** = $\frac{1}{3}(\mathbf{a} - \mathbf{b})$

AE = **AD** − **ED** = $\mathbf{a} - \frac{1}{3}(\mathbf{a} - \mathbf{b}) = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$

BE = $\frac{2}{3}$**BD** = $\frac{2}{3}(\mathbf{a} - \mathbf{b})$

EC = **BC** − **BE** = $2\mathbf{a} - \frac{2}{3}(\mathbf{a} - \mathbf{b}) = \frac{4}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$

Therefore **EC** = $2(\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b})$ = 2**AE**

Hence EC is parallel to AE. But E is common to both vectors, so A, E and C are collinear.

Exercise 38c

1 Show that PRSQ of Ex **38b Q8** is a parallelogram.

2 Use vectors to find what figure is formed with the points A(1, 1), B(4, −1), C(2, −4) and D(−1, −2) as vertices.

3 If $\mathbf{a} = \binom{2}{3}$ express in the form $k\mathbf{a}$ where possible: $\binom{4}{6}$, $\binom{-1}{-1.5}$, $\binom{-1}{-2}$, $\binom{-8}{-12}$, $\binom{3}{4.5}$, $\binom{4}{11}$

4 P is the point (2, −3). If **PS** = $\binom{5}{1}$ what are the coordinates of S?

5 If **AB** = $\frac{1}{2}$**CD** what can you say about **AB** and **CD**?

6 If **AB** = $\frac{1}{3}$**BC** what can you say about A, B and C?

7 In triangle OAB, **OA** = **a** and **OB** = **b** (see Fig. 3). If C and D are the mid-points of OA and OB respectively, express **AB** and **CD** in terms of **a** and **b**. What can you say about **AB** and **CD**?

Fig. 3

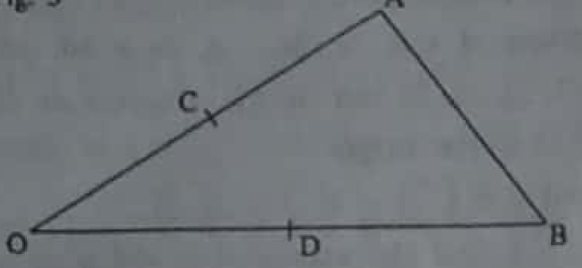
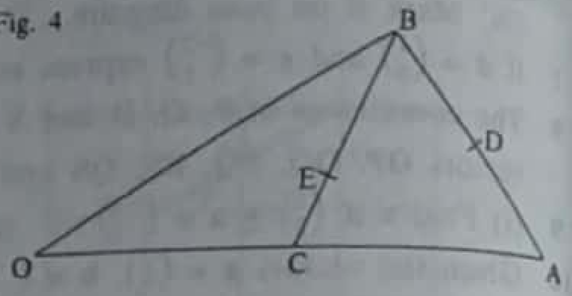


Fig. 4



8 In triangle OAB, **OA** = **a** and **OB** = **b** (see Fig. 4). C and D are the mid-points of OA and AB respectively. E is a point on BC such that **BE** = $\frac{2}{3}$**BC**. Express in terms of **a** and **b**: **BC**, **BE**, **OE**, **BA**, **BD**, **OD**.
What can you say about O, E and D?

9 OABC is a parallelogram (see Fig. 5). P, Q and R are points on OA, OB and OC respectively such that **OP** = $\frac{1}{3}$ **OA**, **OQ** = $\frac{1}{5}$**OB** and **OR** = $\frac{1}{2}$**OC**. If **OA** = **a** and **OC** = **c**, express **OP**, **AB**, **OB**, **OQ**, **OR**, **PQ** and **QR** in terms of **a** and **c** only. Hence show that PQR is a straight line.

Fig. 5

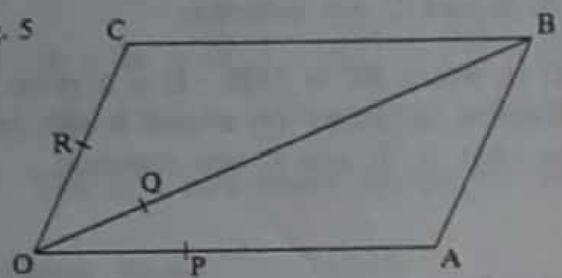
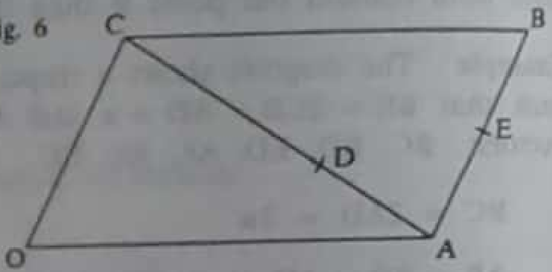


Fig. 6



10 In the parallelogram OABC (see Fig. 6), E is the mid-point of AB and D is a point on the diagonal AC such that **CD** = $\frac{2}{3}$**CA**. Show that O, D and E are collinear.

38.9 Use of Scalars h and k

If h**a** = k**b** and **a** is *not* parallel to **b** then both h and k must equal zero for this to be true. This fact can be used to solve problems in which lines intersect and the ratios in which they divide each other have to be determined.

Example In parallelogram OACB shown below, **OA** = **a**, **OB** = **b** and **AE** = $\frac{1}{3}$**AC**. In what ratio does D cut AB and OE?

Let **OD** = k**OE** and **AD** = h**AB** then **OD** = k(**a** + $\frac{1}{3}$**b**) and **AD** = h(**b** − **a**)

Now in triangle OAD, **OA** + **AD** = **OD**

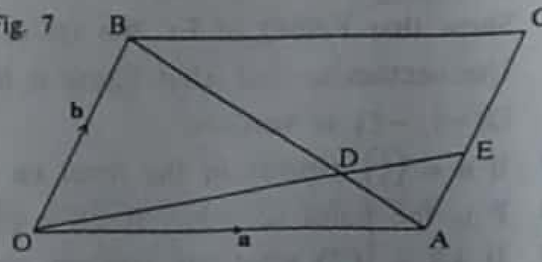
$$\mathbf{a} + h(\mathbf{b} - \mathbf{a}) = k(\mathbf{a} + \tfrac{1}{3}\mathbf{b})$$

$$\mathbf{a} + h\mathbf{b} - h\mathbf{a} = k\mathbf{a} + \tfrac{1}{3}k\mathbf{b}$$

$$\mathbf{a} - h\mathbf{a} - k\mathbf{a} = \tfrac{1}{3}k\mathbf{b} - h\mathbf{b}$$

$$(1 - h - k)\mathbf{a} = (\tfrac{1}{3}k - h)\mathbf{b}$$

Fig. 7



Both quantities in the brackets must be zero as **a** and **b** are not parallel.

Therefore $1 - h - k = 0$ and $\frac{1}{3}k - h = 0$

Simplifying these simultaneous equations we have

$$h + k = 1$$
$$\text{and} \quad k = 3h$$

Solving these equations (see **22.3**) gives $h = \frac{1}{4}$ and $k = \frac{3}{4}$

From this we deduce that D cuts AB in the ratio 1 : 3 and D cuts OE in the ratio 3 : 1.

Exercise 38d

1 to 3 refer to Fig. 7 of the text Example.

1 If $AE = \frac{3}{4}AC$, find the ratio in which D cuts OE and AB.

2 If $AE = \frac{1}{4}AC$, find the ratio in which D cuts OE?

3 If $AE = \frac{3}{5}AC$, in what ratio does D cut AB?

4 In Fig. 8:

$$OQ = 3OP \text{ and } QR = 2OS$$

Find PS and OR in terms of **a** and **b**, where $OP = \mathbf{a}$ and $OS = \mathbf{b}$. If $OX = kOR$ and $PX = hPS$, find h and k and hence find the ratio in which X divides OR and PS.

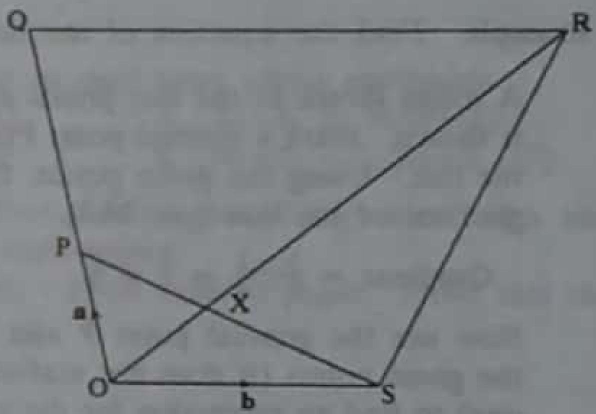
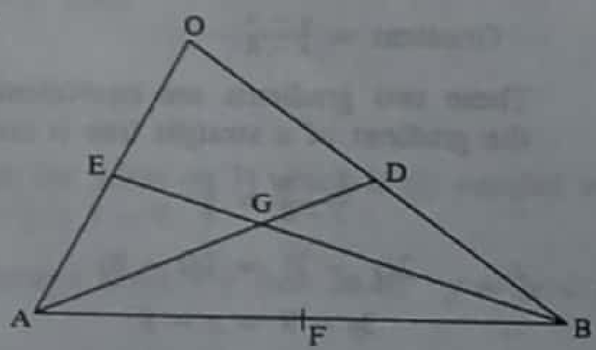


Fig. 8

5 Fig. 9 shows triangle OAB in which $OA = \mathbf{a}$ and $OB = \mathbf{b}$. D and E are the mid-points of OB and OA respectively.
(i) Find the ratio in which G divides AD and BE.
(ii) What is the position vector of G in terms of **a** and **b**?
(iii) If F is the mid-point of AB, find the position vector of F in terms of **a** and **b**.
(iv) Prove that OGF is a straight line and find the ratio in which G divides OF.
(v) What does this prove about the three medians of a triangle?

Fig. 9



39 COORDINATE GEOMETRY

39.1 Lines and Line Segments

In **20.2** we saw that a set of points which form a straight line is simply called a line. A line is understood to extend indefinitely in either direction and is named by referring to any two marked points on it such as AB in the diagram on the next page. Note that the line is drawn beyond A and B to indicate that it extends beyond these points.

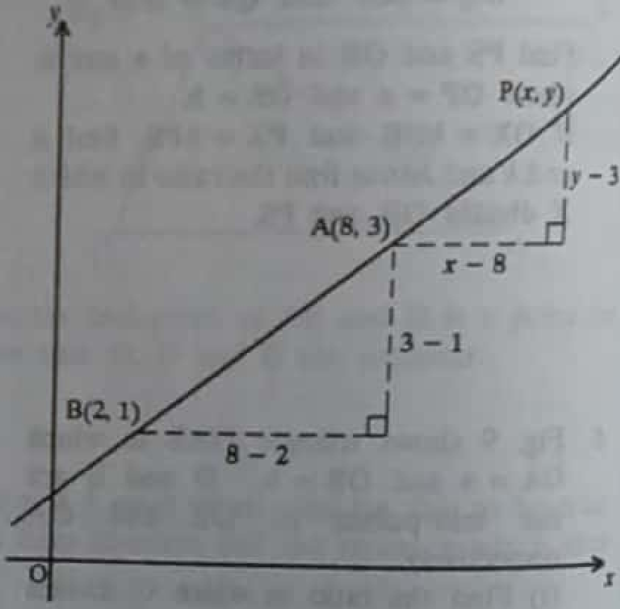
Line segment AB (denoted by $\overline{AB}$) refers to all points of line AB which lie between A and B. This includes points A and B themselves.

(*Note:* In this Book we shall denote, for example, both line AB and line segment $\overline{AB}$ by AB, unless particular attention needs to be drawn to the distinction between them.)

39.2 Equation of a Line

Two methods of finding the equation of a line where discussed in **Topic 20**. The following Example illustrates a more advanced method in which the equation of a line is determined when given two points which lie on it.

Example Find the equation of the line through the points A(8, 3) and B(2, 1).



A rough sketch of the two points and line is shown. Mark a general point P(x, y) on the line. Using the given points, find the gradient of the line (see **20.3**).

Gradient $= \frac{3-1}{8-2} = \frac{2}{6} = \frac{1}{3}$

Now use the general point P and one of the given points (it does not matter which one) to find an expression for the gradient in terms of x and y.

Gradient $= \frac{y-3}{x-8}$

These two gradients are equivalent since the gradient of a straight line is constant.

$\therefore \quad \frac{y-3}{x-8} = \frac{1}{3}$

$\therefore \quad 3(y-3) = 1(x-8)$

$\therefore \quad 3y - 9 = x - 8$

$\therefore \quad x - 3y + 1 = 0$ and this is the required equation

Exercise 39a

In **1** to **4** find the equation of the line through the given pair of points.

1 (1, 2), (5, 4) **2** (1, 7), (3, 2) **3** (−2, 1), (4, −1) **4** (−1, −3), (3, 1)

5 Find the equation of the line through the point (2, 3) with gradient 2.

6 Find the equation of the line through (−1, 1) with gradient $-\frac{1}{3}$.

7 Find the coordinates of the point of intersection of the lines $\{(x, y) : 2x + y = 1\}$ and $\{(x, y) : 3y = 4x + 8\}$. (*Hint:* Find the values of x and y which satisfy *both* equations.)

8 A line is drawn through the points A(−3, 2) and B(0, 1). Find where this line intersects the line $\{(x, y) : y = x - 1\}$.

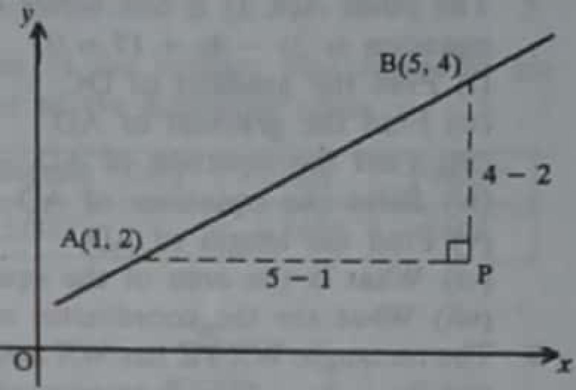
9 PQRS is a quadrilateral with P(−1, −2), Q(6, −2), R(5, 4) and S(1, 3). Calculate the coordinates of the point of intersection of its diagonals.

39.3 Length of a Line Segment

Example Find the length of the line segment AB where A is the point (1, 2) and B is (5, 4).

In the diagram, right-angled △APB is formed.
Using Pythagoras:

$$\begin{aligned} AB &= \sqrt{(AP^2 + BP^2)} \\ &= \sqrt{[(5-1)^2 + (4-2)^2]} \\ &= \sqrt{(16+4)} \\ &= \sqrt{20} \\ &= 2\sqrt{5} \quad \text{(See } \mathbf{26.3}\text{)} \end{aligned}$$



Exercise 39b

In 1 to 6 find the length of AB, leaving your answer in surd form where applicable.

1 A(2, 1), B(6, 4)
2 A(−2, 1), B(10, 6)
3 A(−6, 7), B(9, −1)
4 A(−1, −1), B(2, 2)
5 A(−3, 1), B(4, 0)
6 A(−2, −1), B(−5, −7)

7 △ABC has A(−1, −4), B(4, 1) and C(1, 2). Find the lengths of the three sides and hence, by Pythagoras, prove that the triangle is right-angled.

8 Draw △PQR which has P(−3, 0), Q(4, 6) and R(6, −3) on squared paper. Prove that the triangle is isosceles but not equilateral.

39.4 Parallel and Perpendicular Lines

Suppose two lines have gradients m and n respectively, then

(a) if $m = n$, the lines are parallel
(b) if $m \times n = -1$, the lines are perpendicular

Example Determine the equation of the line through the point (6, 3) which is (i) parallel to, (ii) perpendicular to the line whose equation is $2x - y + 1 = 0$.

Arrange the equation of the given line in the form $y = mx + c$ (see **20.4**): $y = 2x + 1$
From this we see that its gradient m equals 2

(i) The gradient of the parallel line must be equal to that of the given line, ie. 2
Thus the equation of the required line is given by $\frac{y-3}{x-6} = 2$ which gives $2x - y = 9$

(ii) If n is the gradient of the perpendicular line then $m \times n = -1$

Whem $m = 2$ this gives $2 \times n = -1$ ie. $n = -\frac{1}{2}$

Hence the equation of the perpendicular line is given by $\frac{y-3}{x-6} = -\frac{1}{2}$

This simplifies to $x + 2y = 12$

Exercise 39c

In each of 1 to 4 you are given the equations of two lines. Determine whether these lines are perpendicular, parallel or neither.

1 $6x + 4y = 1$ and $2y = 3(1 - x)$
2 $4y = 10x + 1$ and $5y + 2x = 3$
3 $3y + 2x = 5$ and $3y - 2x = 5$
4 $3x - y + 1 = 0$ and $3(x + 3y) = -4$

5 Find the equation of the line through (2, 3) parallel to the line $2x - 3y = 5$.

6 Find the equation of the line through (2, 4) perpendicular to the line $4x + 3y = 12$.

7 The point A(4, 1) is one vertex of a square ABCD. The side DC lies on the line whose equation is $3x - 4y + 17 = 0$.
(i) Find the gradient of DC.
(ii) Find the gradient of AD.
(iii) Find the equation of AD.
(iv) Solve the equations of AD and DC simultaneously to find the coordinates of D.
(v) Find the length of AD.
(vi) What is the area of the square?
(vii) What are the coordinates of B and C?

8 The rectangle WXYZ has WX and XY along the lines whose equations are $3y + 2x + 5 = 0$ and $3x - 2y - 12 = 0$ respectively. The vertex Z is (−2, 4).
(i) Find the coordinates of X.
(ii) Find the equations of WZ and ZY.
(iii) Find the coordinates of W and Y.
(iv) Calculate the area of the rectangle.

9 A **parallelogram** is defined to be a quadrilateral with both pairs of opposite sides parallel. A **rhombus** is defined to be a parallelogram with one pair of adjacent sides equal. Quadrilateral ABCD has A(2, 3), B(11, 1), C(18, 7) and D(9, 9).
(i) Show that ABCD is a rhombus. (*Hint:* First show it to be a parallelogram.)
(ii) Prove that its diagonals are perpendicular to each other.

10 PQRS is a parallelogram with P(11, −1), Q(2, 7) and R(−10, 6).
(i) Find the coordinates of S.
(ii) Prove that PQRS is a rhombus.
(iii) Show that the diagonals bisect each other at right-angles.

40 STATISTICS

40.1 Introduction

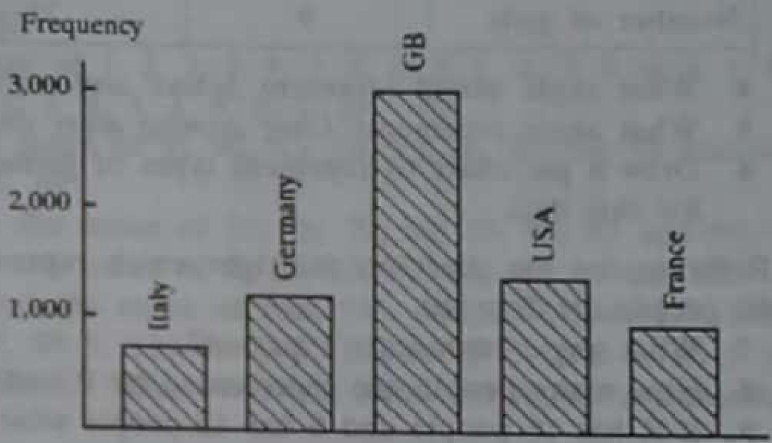
Statistics is that branch of Mathematics which is concerned with the collection, organisation and presentation of data. We look first at pictorial ways of representing data. These often give a clearer idea of a situation than a list of numerical data.

40.2 Bar Charts

The manager of a hotel made a note of the nationalities of the tourists who stayed at his establishment during the course of a year. This is shown in the following table:

Country	Italy	Germany	Great Britain (GB)	USA	France
Number of people	720	1,200	3,000	1,340	940

This information is more effectively shown on a **bar chart**. The vertical axis gives the **frequency**. The length of each bar is proportional to the frequency. The bars are labelled to show the country of origin.

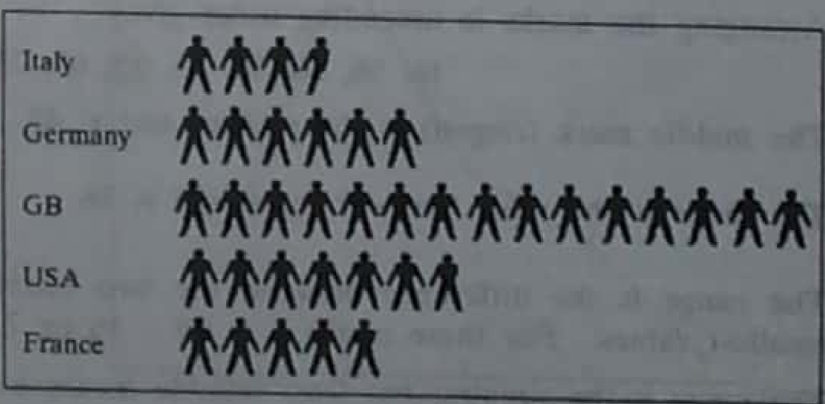


40.3 Pie Charts

In a pie chart each number is represented by the area of a sector. The frequency is proportional to the angle of the sector. Since there are 7,200 tourists in the data above then each 1° represents 7,200 ÷ 360 ie. 20 people. The sector for Great Britain, for example, will have an angle of 3,000 ÷ 20 or 150°.

40.4 Pictograms

In this method of representation each pin man in the diagram stands for 200 tourists. A part figure denotes a number less than 200.



Pictograms are sometimes called ideographs.

Exercise 40a

The number of vehicles passing a school in a certain period of time was noted as follows:

Type of vehicle	Bus	Lorry	Taxi	Car	Pick-up
Number of vehicles	30	40	60	88	22

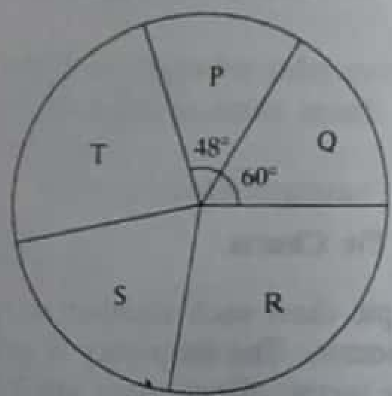
1 Illustrate the above data in a bar chart.
2 Illustrate the data above in a pie chart.
3 Using one picture vehicle to represent ten real vehicles, draw a pictogram to illustrate the above data.

To raise money for a new school bus the S4 girls put on a fashion show, details of which are shown in the table below. Answer **4** to **6**.

Type of garment	Traditional dress	Long evening dress	T-shirt & jeans	Skirt & blouse	School uniform
Number of girls	9	21	20	6	4

4 What angle would represent *School uniform* on a pie chart?
5 What angle represents *Long evening dress* on a pie chart?
6 Draw a pie chart to represent types of garment at the fashion show. Draw a pictogram for this data.

Refer to the pie chart on the right which represents 60 people for **7** to **10**.



7 What angle represents 1 person?
8 How many people are represented by P and Q?
9 If R has 16 people and S has 12 people what are their angles?
10 What is the angle of the sector T and how many people does it represent?

40.5 Mean, Median and Mode

Eleven pupils took a test and obtained the following marks:

73, 36, 45, 16, 89, 36, 36, 89, 65, 36, 73

$$\text{The } \textbf{mean} \text{ mark} = \frac{\text{sum of all marks}}{\text{number of pupils taking test}} = \frac{594}{11} = 54$$

Arranging the marks in ascending order gives

16, 36, 36, 36, 36, 45, 65, 73, 73, 89, 89

The middle mark (ringed) is the **median** and is 45.

The **mode** is the most frequent mark and is 36.

The **range** is the difference between the two extreme values, ie. between the largest and smallest values. For these marks it is 89 − 16 or 73.

The range is the simplest but least reliable measure of **spread** or **dispersion**.

40.6 Using a Working Mean

Use of such reduces the arithmetic. For example, the following eleven numbers 1,022, 986, 995, 968, 1,040, 986, 986, 1,040, 1,013, 986 and 1,022 are scattered either side of 1,000. It is therefore easier to work with +22, −14, −5, −32, +40, −14, −14, +40, +13, −14 and +22 than with the original numbers. (*Note:* 1,000, the **working mean**, has been subtracted from each of the original numbers.)
The mean of the new data is $\frac{137-93}{11} = \frac{44}{11} = 4$. The mean of the original data is 1,000 + 4 or 1,004.

Exercise 40b

1 Find the mean of 1, 2, 3, 4, 5, 6, 7, 8 and 9.
2 Find the mean of 9, 10, 11, 12, 13, 14, 15, 16, 17 and 18.
3 The mean of nine numbers is 7. If a tenth number, 8, is included, what is the mean of the ten numbers?
4 Find the median of the numbers in **Q1**.
5 Find the median of the numbers in **Q2**.
6 Find the mode, mean, median and range of 1, 2, 2, 3, 3, 3, 4, 4, 5, 6, 4, 7, 8, 9, 4, 1, 8, 2, 7, 4, 3, 6, 3, 6, 3, 5, 4, 2, 4, 1, 8, 2, 7, 3, 3, 3, 5, 3, 4, 2.
7 Use a working mean of 55 to find the mean of 51, 52, 53, 54, 55, 56, 57, 58, 59 and 60.
8 Find the mean of 1, 2, 5, 6, 9, 12, 13, 16.
9 Use your answer to **Q8** to *write down* the mean of 75, 76, 79, 80, 83, 86, 87 and 90.
10 Use your answer to **Q8** to *write down* the mean of 5, 10, 25, 30, 45, 60, 65 and 80.
11 Choose a suitable working mean to find the mean of 568, 535, 546, 559, 544, 539, 545.
12 Find the mean of 104·5, 110·5, 91·5, 89·5, 111·5, 99·5, 102·5, 89·5, 112·5, 86·5 by using a working mean of (i) 100, (ii) 100·5. Which working mean led to less arithmetic?

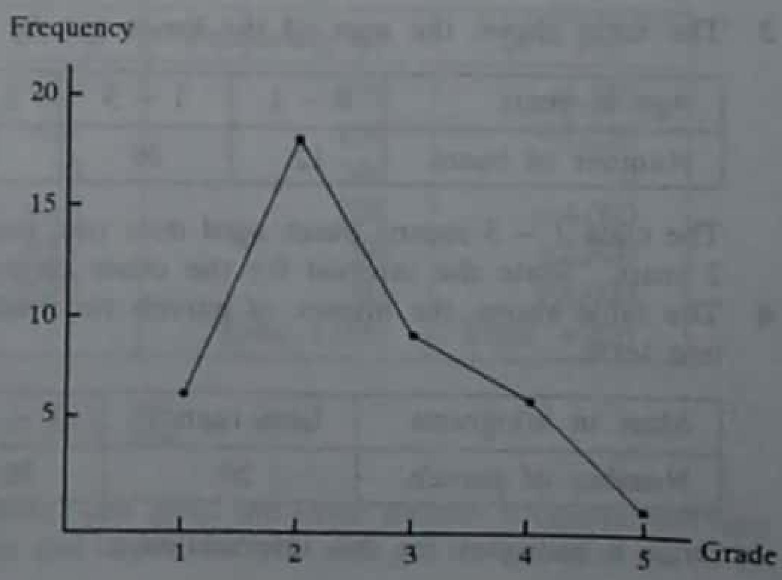
40.7 Frequency Distribution

The grades of 40 pupils who used this book to prepare for the UCE examination were

1, 2, 3, 4, 5, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3
2, 2, 4, 2, 2, 1, 2, 2, 3, 2, 4, 2, 3, 2, 2, 3, 2, 3, 2, 2

By going through this list once we get the following **tally chart** and associated **frequency polygon.**

Grade	Tally	Frequency
1	卌 /	6
2	卌 卌 卌 ///	18
3	卌 ////	9
4	卌 /	6
5	/	1



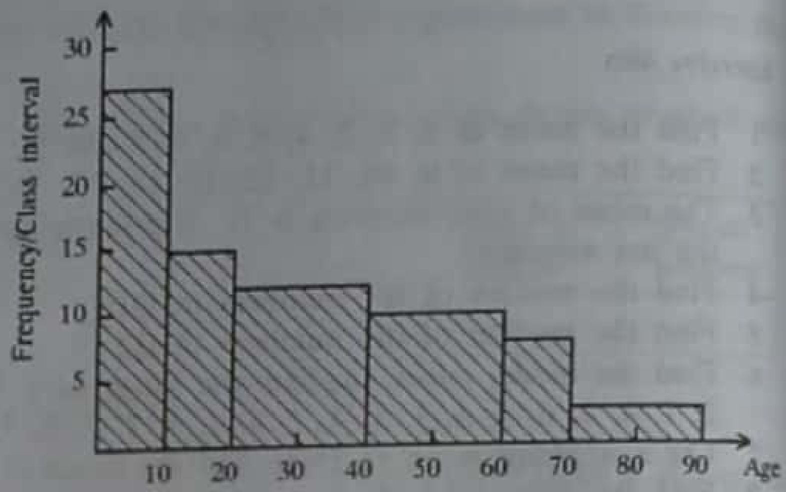
40.8 Grouped Data

When the data has a large number of different values we often place them in groups or classes. A count of the people in a village was made with the following result:

Age in years (x)	0 – 9	10 – 19	20 – 39	40 – 59	60 – 69	70 – 89
Frequency (f)	270	150	240	200	80	60

The diagram shows a **histogram** which represents these ages. The frequency is represented by the *area* of each bar and the vertical scale is the frequency divided by the **class interval**. These figures are 27, 15, 12, 10, 8 and 3 since the class intervals are 10, 10, 20, 20, 10 and 20 respectively. Note that the horizontal scale must be uniform.

The shape of the histogram gives a realistic picture of the population distribution which is **skewed** to the lower end of the age scale, perhaps indicating a high birth rate.



Exercise 40c

1 The following is a list of the number of goals scored by Chui United in 30 football matches:

6, 1, 2, 3, 2, 1, 4, 6, 1, 1, 0, 1, 2, 0, 3, 1, 4, 3, 2, 0, 2, 1, 2, 0, 3, 1, 4, 2, 2, 1

Prepare a tally chart and frequency table for this information and draw a frequency polygon.

2 Forty girls in S3 were asked the size of shoes they wear. Here are their replies:

5 9 6 6 7 5 7 6 9 4 8 6 5 8 4 7 6 8 4 9 7 6 6 5 7 5 10 5 6 7 6 8 6 8 5 5 6 4 7 8

Prepare a tally chart and frequency table for this information and draw a frequency polygon.

3 The table shows the ages of the buses run by the Simba Bus Company.

Age in years	0 – 1	1 – 3	3 – 5	5 – 9
Number of buses	12	36	20	16

The class 1 – 3 means buses aged over one year but under 3 years. The class interval is 2 years. State the interval for the other classes and draw a histogram.

4 The table shows the masses of parcels received by the boarders at a certain school over one term.

Mass in kilograms	Less than 1	1 – 2	2 – 4	4 – 8	8 – 16
Number of parcels	20	36	12	8	4

Draw a histogam for this information.

5 As part of their Soil Conservation Programme 43 boys each planted a tree on the school compound. Later the heights were measured and recorded correct to the nearest centimetre. The results were grouped to give the following table.

Height (cm)	15 – 20	21 – 23	24 – 26	27 – 32	33 – 38
Frequency	8	9	11	11	4

The class 15 – 20 contains trees from 14·5cm to 20·5cm and the class interval is 6cm. State the intervals for the other classes and draw a histogram for the information.

40.9 Mean for Grouped Data

Using the data for the ages of the village population of **40.8**, we get the following table when calculating the mean.

Age Class interval	Centre of interval Class mark x	Frequency f	xf
0 – 9	5	270	1,350
10 – 19	15	150	2,250
20 – 39	30	240	7,200
40 – 59	50	200	10,000
60 – 69	65	80	5,200
70 – 89	80	60	4,800
		Total 1,000	Total 30,800

$$\text{The mean} = \frac{\Sigma xf}{\Sigma f} = \frac{30800}{1000} = 30 \cdot 8$$

Notes 1. Since the actual ages have not been used this result is an estimate.
2. We have used the *sigma* (Σ) notation. Σ simply means *the sum of*.

40.10 Mean of Grouped Data using a Working Mean

Usually, with grouped data, we use one of the class marks (in this case 30) as the working mean. The calculation for the ages in the village becomes (a = working or **assumed** mean):

Class interval	Class mark x	$x - a$	Frequency (f)	$(x - a)f$
0 – 9	5	−25	270	−6,750
10 – 19	15	−15	150	−2,250
20 – 39	30	0	240	0
40 – 59	50	+20	200	+4,000
60 – 69	65	+35	80	+2,800
70 – 89	80	+50	60	+3,000
			Total 1,000	Total +800

$$\text{Actual mean} = 30 + \frac{800}{1000} = 30 \cdot 8$$

Use of a working mean reduces the arithmetic and gives the same answer whatever working mean is used. Take 50 as the assumed mean and check that you still get the mean to be 30·8.

Exercise 40d

1 A survey of 50 families showed that the number of children per family was distributed as follows:

Number of children per family	1	2	3	4	5
Number of families	1	4	8	18	19

Find the median and mean number of children per family. What is the mode?

2 The Fourth formers at Ololok School counted the number of cars passing their school each minute for 35 minutes and obtained the following table:

Number of cars	0	1	2	3	4	5	6
Frequency	8	7	4	4	6	3	3

This means, for example, that during 8 one-minute intervals no cars passed. Find the mode, median and mean of this distribution.

3 The times of the first 20 girls in the District Cross-Country race were noted to the nearest minute and recorded as follows:

Time (min)	12 – 14	15 – 17	18 – 20	21 – 23
Number of runners	3	5	8	4

(i) Calculate an estimate of the mean time using the method of **40.9**.
(ii) Using 16, the centre of 15 – 17, as a working mean rework the mean and compare with your answer obtained in (i).

4 The masses of 30 parcels were recorded to give the following table:

Mass in kilograms	Under 2	2 – 4	4 – 6	6 – 8
Frequency	6	11	8	5

Calculate an estimate for the mean mass (i) using 3kg, (ii) using 5kg as a working mean.

5 Forty pupils were asked the distance they lived from their school. Their answers are summarised in the following table:

Distance (km)	Less than 1	1 – 3	3 – 5	5 – 9
Frequency	4	13	17	6

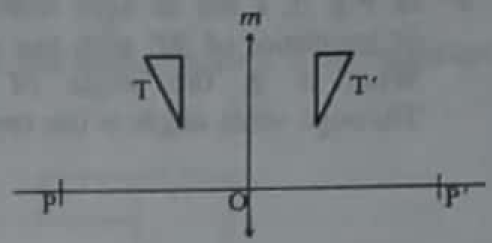
Use 2km as the working mean and calculate an estimate of the mean distance these pupils live from their school.

41 REFLECTION

41.1 Mirror Lines

The image of point P in the **mirror line** m is P′ (see Fig. 1). PP′ is perpendicular to the mirror line and OP = OP′. The **image** of the object triangle T is T′. The object and the image face opposite ways. T and T′ are **oppositely congruent**.

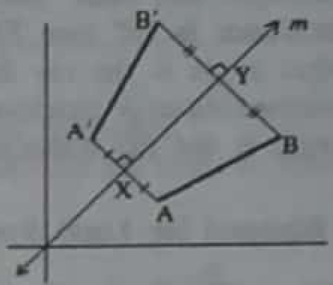
Fig. 1



Example 1 Find the image of the given line segment AB (see Fig. 2) after reflection in the mirror line m.

Construct perpendiculars to m passing through A and B. Let these meet m at X and Y respectively. Produce AX to A′ and BY to B′ where A′X = AX and B′Y = BY.
Join A′B′.

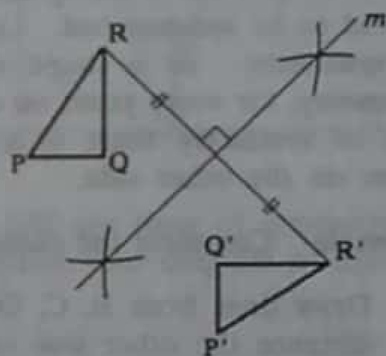
Fig. 2



Example 2 Given that P′Q′R′ is the image of PQR under reflection find the position of the mirror line (m). See Fig. 3.

Join any point on the object to its image, for example RR′. Construct the perpendicular bisector of this line to obtain the mirror line.

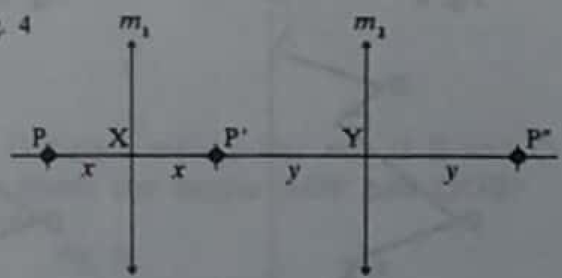
Fig. 3



41.2 Parallel Mirror Lines

The object P (see Fig. 4) is reflected in m_1 to give P′ and this is reflected in m_2 to give P″. Since PX = XP′ and P′Y = YP″, the point P moves a distance $2x + 2y$ to P″. But $2x + 2y$ is $2(x + y)$ and $(x + y)$ is the distance between the mirrors. An object reflected in two parallel mirrors moves a distance equal to twice the distance between the mirrors.

Fig. 4



Exercise 41a

In 1 to 4 find the image of (3, 4) in the given mirror line.

1 The x-axis 2 The y-axis 3 $\{(x, y) : y = x\}$ 4 $\{(x, y) : y + x = 0\}$

5 After reflection, the points (1, 2), (2, 2) and (1, 4) become (1, 0), (2, 0) and (1, −2) respectively. Find the equation of the mirror line.

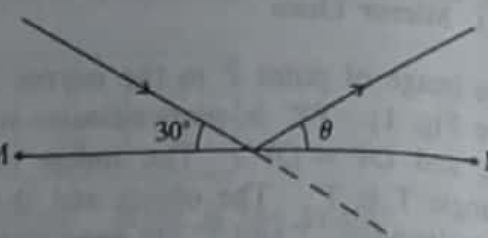
6 Find the coordinates of (0, 0), (1, 0), (1, 1) and (0, 1) after reflection in $\{(x, y) : x + y = 2\}$.

7 What is the distance moved by an object after reflection in two parallel mirrors 12 cm apart?

8 The points (1, 2), (1, 4) and (2, 4) are reflected in the line $\{(x, y) : y = x\}$ and then in the x-axis. Find the image coordinates after the double reflection.

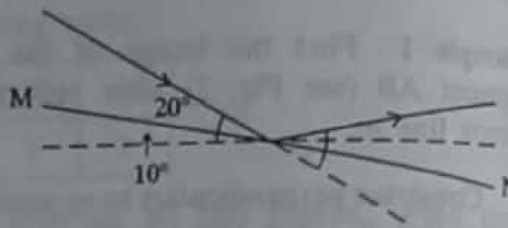
9 In Fig. 5, a ray of light makes an angle of incidence of 30° with the mirror MN. What is θ, the angle of reflection? Through what angle is the ray deflected?

Fig. 5



10 The mirror of **Q.9** is turned clockwise through 10° such that the angle of incidence is 20° (see Fig. 6). Through what angle is the ray deflected? If the mirror rotates $\phi°$ clockwise, through what angle is the ray deflected?

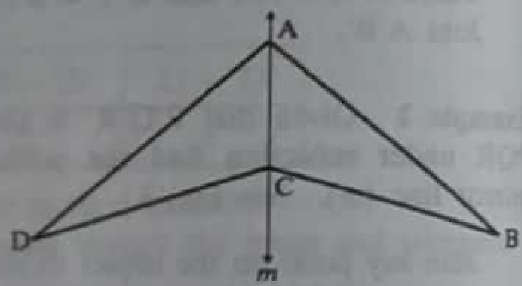
Fig. 6



41.3 Bilateral (or Line) Symmetry

If ABC, in Fig. 7, is reflected in the mirror line m to give ADC, the quadrilateral ABCD is said to be **symmetrical**. Line m is a **line of symmetry**. In a shape which has **line symmetry**, for every point on one side of the line of symmetry there is a corresponding point on the other side.

Fig. 7



Example Complete the shape ABCDEF (see Fig. 8) so that the line m is a line of symmetry.

Draw lines from B, C, D and E perpendicular to the line m and extend them the same distance the other side of m to obtain B', C', D' and E' (see Fig. 9). The points are joined to form the final complete shape (see Fig. 10)

Fig. 8

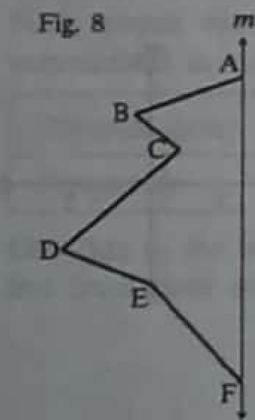


Fig. 9

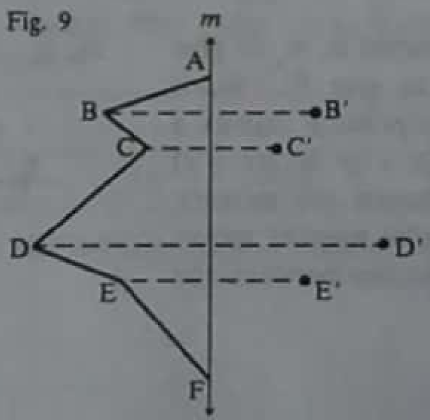
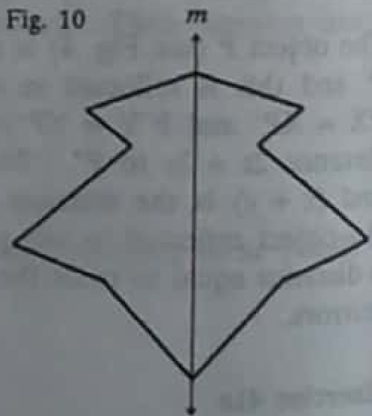


Fig. 10



Exercise 41b

In **1** to **10** use broken lines to show the line(s) of symmetry for the given shapes.

1 An isosceles triangle
2 An equilateral triangle
3 A right-angled triangle
4 A rectangle
5 A square
6 A regular pentagon
7 A regular hexagon
8 A regular octagon
9 A rhombus
10 A kite

11 Make copies of these letters and show their lines of symmetry by means of broken lines.

B D H I O T Z

12 In your exercise book copy and complete Fig. 11 so that the shape formed is symmetrical about the line MN.

Fig. 11

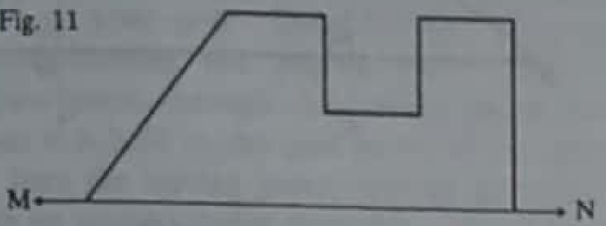
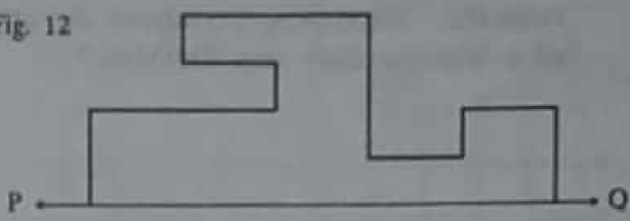


Fig. 12



13 Repeat Q.12 for the shape of Fig. 12 with PQ as the line of symmetry.

14 Copy and complete the shape shown in Fig. 13 so that OP and OQ are lines of symmetry.

Fig. 13

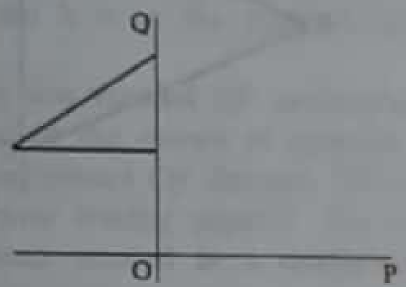
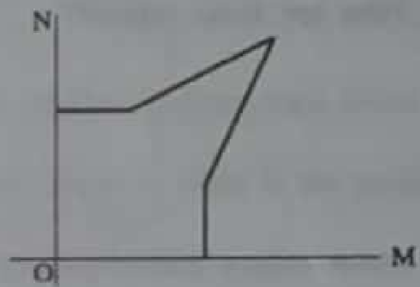


Fig. 14



15 Copy and complete the shape of Fig. 14 so that OM and ON are lines of symmetry.

16 Plot the points (1, 1), (2, 2) and (4, 1) and join them. Complete your diagram so that Ox and Oy are lines of symmetry.

41.4 Geometrical Deductions using Reflection

These are dealt with in the following Exercise.

Exercise 41c

1 Fig. 15 shows two lines AB and PQ intersecting at O, and a mirror line *m*. If B and Q are the images of P and A in *m* what can you say about the angles AOP and QOB?

Fig. 15

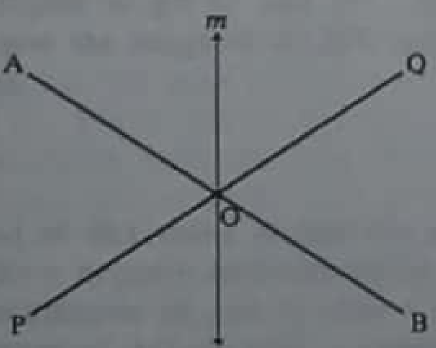
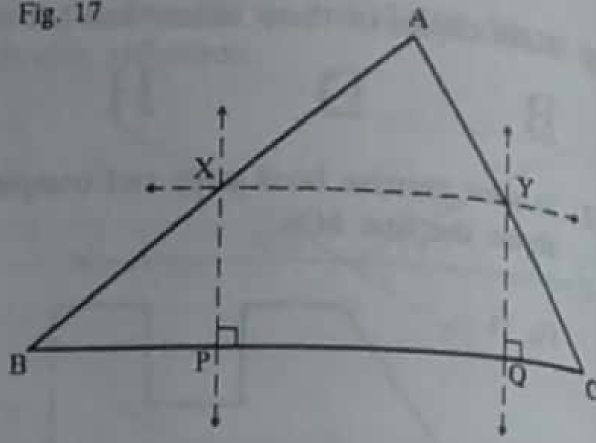


Fig. 16

2 In Fig. 16 triangle ABO is right-angled and is reflected in line *m*. If the image of B is C what type of triangle is ABC? Deduce a property about the sides and angles of $\triangle ABC$.

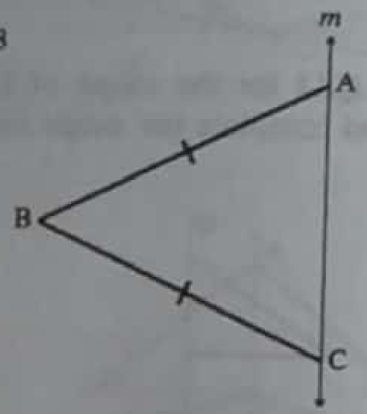
3 Draw $\triangle ABC$ (see Fig. 17) on a large piece of plain paper and cut it out. Reflect $\triangle AXY$ in XY, where X and Y are the mid-points of AB and AC respectively. Do this by folding along XY. Similarly reflect $\triangle BXP$ in XP and $\triangle CYQ$ in YQ. The images of A, B and C should now coincide. What property about the angles of a triangle does this illustrate?

Fig. 17



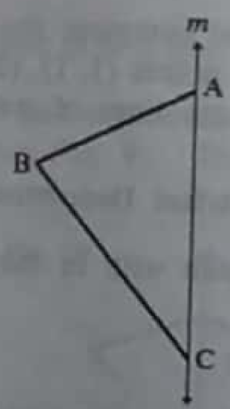
4 The triangle ABC has AB = BC and is reflected in line m so that the image of B is D (see Fig. 18). What is the name of ABCD? Draw its other line of symmetry. How are these related?

Fig. 18



5 In Fig. 19, ABC is reflected in m so that D is the image of B. What is the name of ABCD? What can you say about the angle between the diagonals AC and BD?

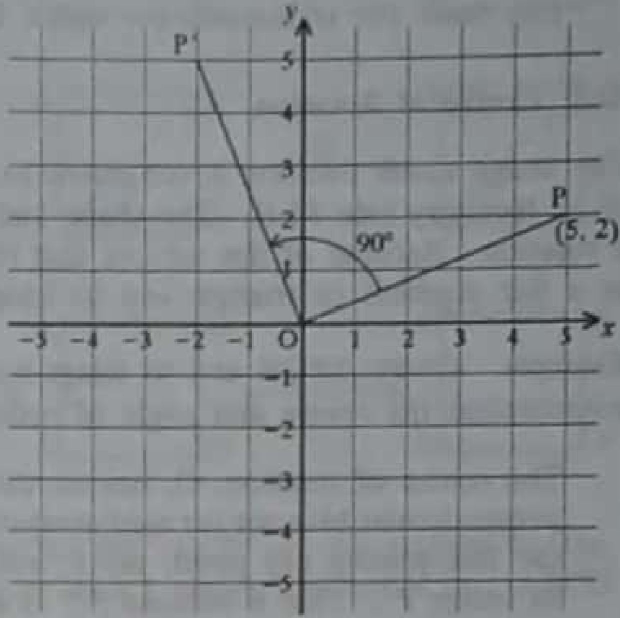
Fig. 19



42 ROTATION

42.1 Basic Ideas

Copy the x-y grid shown into your exercise book (squared paper is best). Draw OP. Place a small piece of tracing paper (or other see-through paper) on your grid. Draw OP on this paper. Without moving the paper, insert a pin or compass point through the tracing paper at O so that it is held to the grid below at that point. Now **turn** the tracing paper through an angle of 90° in an anticlockwise direction. (To help you judge 90°, mark Ox on the paper before turning it. When this line falls upon Oy on the grid you will have turned the paper through 90°.) Trace the **image** OP′ of OP onto the grid. You should find that it is in the position shown in the diagram.



In this activity you **rotated** OP **anticlockwise** through 90°, or a **positive quarter-turn** about O. Point O is called the **centre of rotation**.

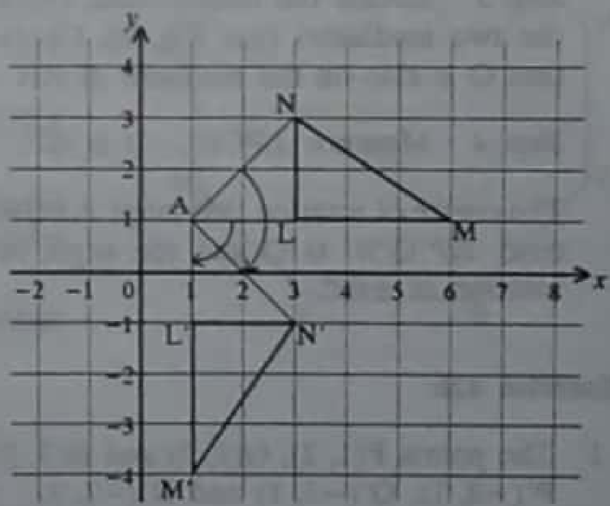
If, instead, you rotated OP through 180° (or a **positive half-turn**) about O, what is the position of P′? (Use your tracing paper!) Do you agree P moves to $(-5, -2)$?

If OP had been rotated in a **clockwise** direction then the corresponding angles would be **negative**.

Example Triangle LMN ($\triangle$LMN), shown below, is given a rotation of $-90°$ about A(1, 1). Find the image of $\triangle$LMN after this rotation.

Method 1 First trace $\triangle$LMN and point A. Use the above method to rotate through 90° *clockwise* and confirm that the image is $\triangle$L′M′N′ as shown.

Method 2 Without using tracing paper, proceed as follows. Join N to the centre of rotation (1, 1) and rotate 90° in a clockwise direction to get N′. Join L (and M) to (1, 1) and repeat to get L′ and M′. Join L′M′N′ to give the image of $\triangle$LMN under this rotation.



Exercise 42a

1 Use the grid of **42.1** above to find the image of P(5, 2) after (i) a negative quarter-turn about O, (ii) a negative half-turn about O.

2 Find the coordinates of A(3, 4) after a positive quarter-turn about the origin.

3 Find the image of A(3, 4) after a negative quarter-turn about (2, 1).

4 Draw $\triangle$LMN with L(3, 1), M(6, 1) and N(3, 3) and its image after a half-turn about the origin. State the coordinates of the vertices L′, M′ and N′ of the image triangle.

5 Rotate $\triangle$LMN of **Q4** through +90° about the origin. State the coordinates of L′, M′, N′.

6 Draw triangle T with vertices at (1, 3), (3, 3) and (1, 7).
(i) Show the image of T under a positive quarter-turn about (3, 3) and label it V.
(ii) Show the image of T under a positive quarter-turn about (1, 3) and label it W.
(iii) State the translation for which W is the image of V.

42.2 Centre of Rotation

The image under rotation is congruent to the object. For this reason, rotation is an example of an **isometry** (see **53.1**). The shape and size of the object are unaltered or **invariant** under a rotation. Because of this we can find the centre and angle of rotation when an object such as a line segment or triangle and its image are given.

Example Given $\triangle PQR$ and its image under a rotation $\triangle P'Q'R'$ as shown in Fig. 1, find by construction the centre and angle of rotation.

The centre of rotation, O, lies on the perpendicular bisector (or mediator) of the line joining any point, say P, and its image P′. This is because PP′ is a chord of a circle centre O; and we know that the centre of a circle lies on the perpendicular bisector of a chord (see **30.1**). So we proceed as follows.

Step 1 Join P to P′ and Q to Q′.

Step 2 Construct the perpendicular bisector of PP′ and QQ′. (See **23.3**)

Step 3 Locate the intersection, O, of the two mediators (see Fig. 2). Check that O is also on the mediator of RR′.

Step 4 Measure $\angle POP'$. It is 60°.

The centre of rotation that maps $\triangle PQR$ onto $\triangle P'Q'R'$ is O and the angle of rotation is +60°.

Fig. 1

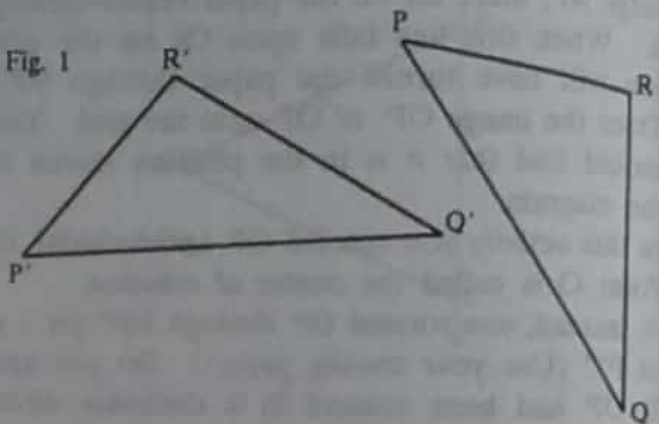
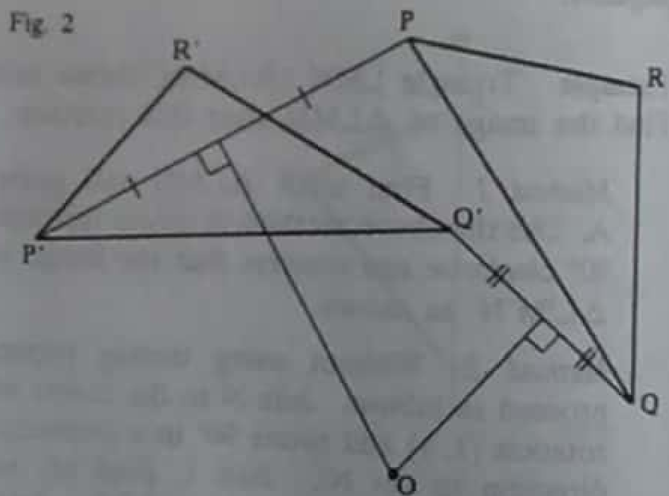


Fig. 2



Exercise 42b

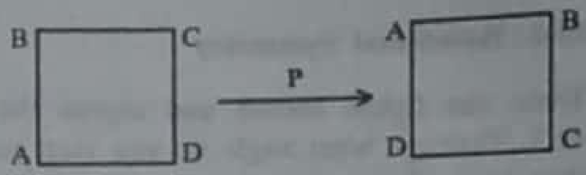
1 The points P(1, 2), Q(1, 3) and R(3, 2) are given a rotation such that the image points are P′(−2, 1), Q′(−3, 1) and R′(−2, 3). Draw triangles PQR and P′Q′R′ on a suitable grid and find the coordinates of the centre of rotation and the angle of rotation.

2 A(2, 0), B(5, 0) and C(1, 2) are mapped onto A′(−1, −1), B′(−1, 2) and C′(−3, −2) by a rotation. Find the centre and angle of this rotation.

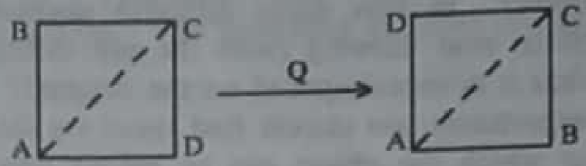
3 W(1, 1), X(3, 1), Y(3, 3) and Z(1. 3) are mapped onto W′(−2, 3), X′(−4, 3), Y′(−4, 1) and Z′(−2, 1) by a rotation. Find the centre and angle of this rotation.

4 $\triangle ABC$ has A(0, 0), B(6, 0) and C(6, 5). Draw this triangle and its image, $\triangle A'B'C'$ under a rotation centre (2, 1) through +90°. Now draw $\triangle A''B''C''$ the image of $\triangle A'B'C'$ under a rotation of +90° about the point (−2, 3). Find the centre and angle of the rotation which maps $\triangle ABC$ onto $\triangle A''B''C''$.

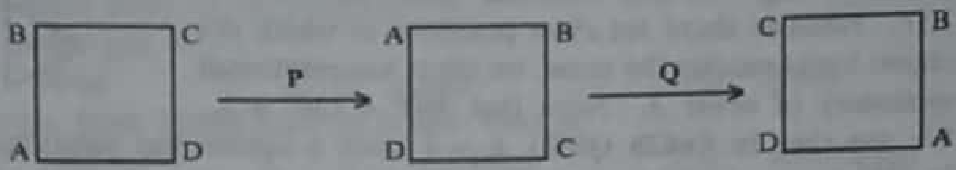
5 The operation P applied to a square of card ABCD (lettered as shown) means 'rotate through 90° clockwise about the centre'. This is shown in the diagram on the right.



The operation Q means 'turn over about the diagonal which runs from bottom left to top right' as illustrated in the diagram on the right.

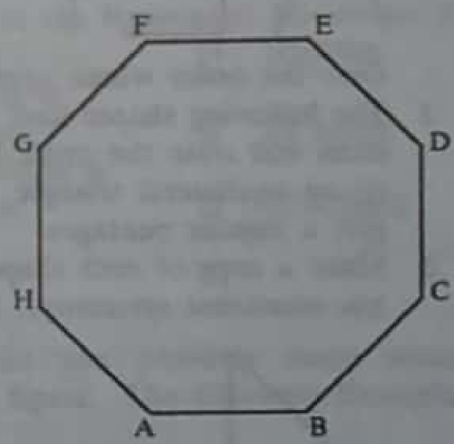


The next diagram shows the effect on the square of QP which means P followed by Q.



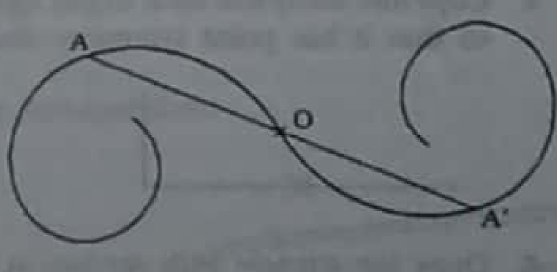
(i) Draw diagrams as above to show the effect of (a) PQ, (b) P^2 on the original square.
(ii) What is the lowest natural number, k, for which $P^k = I$, where I indicates that ABCD is unchanged?
(iii) Show that $P^2Q = QP^2$.
(iv) The result of a third operation, R, is as shown. Express R in terms of P and Q.

6 ABCDEFGH is a regular octagon (see 24.4). The rotation R maps A onto B, B onto C and so on.
(i) What is the angle of rotation for R and R^2?
(ii) What are the images of B, D and the line segment BD under R^2? Write down two relations involving BD and DF that follow from this.
(iii) Write down the image of BD under R^4.
(iv) Using your previous answers, prove that BDFH is a square.
(v) The rotation given by R^n, where n is a natural number, maps AB onto itself. Find the smallest possible value of n.
(vi) If $P = R^3$, find the least value of m for which BE maps onto itself under P^m.

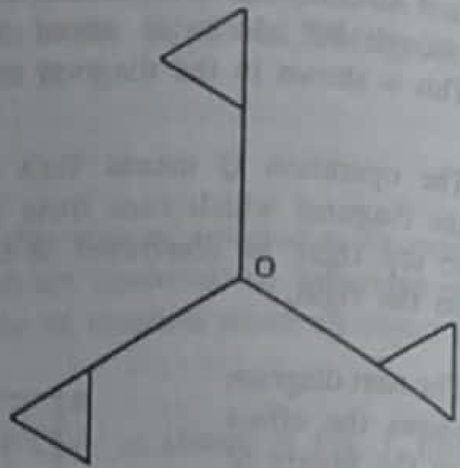


42.3 Point Symmetry

Draw the curve shown on tracing paper. Before moving the paper, place a pin or compass point through O. Now rotate your diagram so that it coincides with that of the book. You should find that a half turn about O brings the two diagrams on top of each other. The curve and your copy have **point symmetry** about O because for every point A on the curve there is a corresponding point A' such that AOA' is a straight line and AO = A'O.



42.4 Rotational Symmetry



Trace the figure shown and repeat the activity of 42.3. Through what angle do you turn your paper so that your figure coincides again with that of the book? In how many different positions can you place your drawing (with the pin through O) such that it is superimposed on the original? From your experiment you should find there are *three* positions in which the object can be placed upon itself and that the angle turned between these positions is 120°. Because there are *three* positions in which the object looks exactly the same, we say it has **rotational symmetry** of order 3. Note that $360° \div 120° = 3$.
We see that in **Ex42b Q5(ii)**, $k = 4$ since a square has rotational symmetry of order 4. In **Q6(v)**, $n = 8$ which indicates that the regular octagon has rotational symmetry of order 8. A shape which has rotational symmetry of order n, where n is even, also has point (or **half-turn**) symmetry.

Exercise 42c

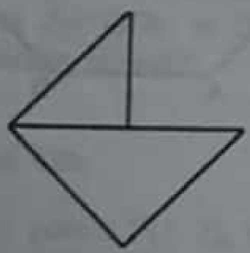
1 Which of the letters below have rotational symmetry?

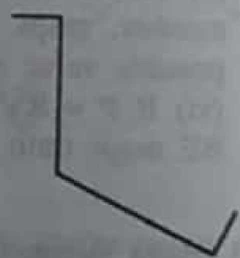
Give the order where appropriate. Which have point symmetry?

2 The following shapes have rotational symmetry. Draw each shape, mark the centre with a cross and state the order of symmetry.

(i) an equilateral triangle (ii) a rectangle
(iii) a regular pentagon (iv) a circle

3 Make a copy of each shape below and complete it by adding one or two lines so that it has rotational symmetry. State the order of rotational symmetry for each shape.





4 Copy and complete each of the following shapes so that it has point symmetry about the cross.

5 Draw the triangle with vertices at $(-1, 1)$, $(-3, 1)$ and $(-3, 2)$. Draw another triangle so that the shape made by both triangles together has point symmetry about the origin.

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43 ESTIMATION AND ERRORS

43.1 Estimation

It is useful to be able to make a reasonable estimate of a quantity without making accurate measurements or using complicated formulae and calculations. The following Exercise checks that you are able to do this.

Exercise 43a

Choose the most suitable estimate from A, B, C or D.

1 The length of your ball point pen is
A 5cm B 10cm C 15cm D 20cm

2 The distance across Uganda from North-East to South-West is
A 100km B 1,000km C 10,000km D 100,000km

3 The area of a classroom floor is
A $1m^2$ B $10m^2$ C $100m^2$ D $1,000m^2$

4 The volume of a standard football is
A $500cm^3$ B $1,000cm^3$ C $2,000cm^3$ D $6,000cm^3$

5 The mass of a saloon car is
A 1t B 10t C 100t D 1,000t

6 The weight in Newtons (N) of an average man is
A 6N B 60N C 600N D 6,000N

7 The area of a school football pitch is
A 0·5ha B 1ha C 2ha D 5ha

8 The height above sea level of the top of Mt. Margherita in the Ruwenzori Mountains is
A 50m B 500m C 5,000m D 50,000m

9 The time taken for a bus to travel from Arua to Kampala is
A 4 hours B 10 hours C 20 hours D 24 hours

10 The speed of a rocket which takes 3 days to reach the Moon is
A 50km/h B 150km/h C 500km/h D 5,000km/h

43.2 Rough Estimates and Rounding

To answer some of the questions in the previous Exercise you probably made some approximate calculations involving **rounding** to one significant figure. The following Example and Exercise give further practice at this.

Example Find an approximation for $\dfrac{32800 \times 0{\cdot}00476}{0{\cdot}0793 \times 980}$

$$\frac{32800 \times 0{\cdot}00476}{0{\cdot}0793 \times 980} \simeq \frac{30000 \times 0{\cdot}005}{0{\cdot}08 \times 1000} \quad \text{rounding to 1 sf}$$

$$= \frac{3 \times 10^4 \times 5 \times 10^{-3}}{8 \times 10^{-2} \times 10^3} \quad \text{using standard form}$$

$$= \frac{15 \times 10}{8 \times 10}$$

$$\simeq 2$$

Exercise 43b

In 1 to 4 give a one significant figure estimate.

1 $\dfrac{0{\cdot}527 \times 673000}{2270 \times 0{\cdot}00896}$

2 $\dfrac{0{\cdot}000391 \times 5{\cdot}27}{58{\cdot}2 \times 0{\cdot}0808}$

3 $\dfrac{4}{0{\cdot}00732} \times \dfrac{693}{0{\cdot}722}$

4 $\sqrt{\left(\dfrac{429 \times 0{\cdot}886}{0{\cdot}00279 \times 0{\cdot}0321}\right)}$

5 A meat canning factory makes a net profit of 2·5% of the price of every tin of meat it sells. Tins are packed in cartons of 24 and on average the factory sells 450 cartons per day. If the price of one tin is sh965, estimate to 1 sf the annual net profit.

6 Assuming that the Earth's orbit around the Sun is circular and of radius 148,600,000km, form an expression for the speed of the Earth relative to the Sun, in km per second. Estimate this speed to 1 sf.

43.3 Errors

Suppose a length is measured with a 30cm ruler and is given as 8·3cm. This figure is an estimate and not the exact length. It involves an **error** which may be due to the placing and reading of the ruler or inaccuracies in the ruler itself. The figure may be correct to the nearest mm, which means that the actual length lies between 8·25cm and 8·35cm, called the **limits of accuracy**. The **possible error**, or **absolute error**, is 0·05cm and we write the measurement as 8·3 ± 0·05cm. The accuracy of a measurement is found by calculating the **percentage error** as follows.

$$\text{Percentage error} = \frac{\text{possible error}}{\text{measurement}} \times 100\%$$

$$= \frac{0{\cdot}05}{8{\cdot}3} \times 100 \simeq 0{\cdot}6\%$$

Exercise 43c

In 1 to 8 express the measurement in the form $a \pm x$, where a is the measurement and x is the possible error. In each case find the percentage error.

1 5cm to the nearest cm

2 10km to the nearest km

3 5·2cm to the nearest mm

4 24cm to the nearest 2cm

5 500kg to the nearest 10kg

6 75km/h to the nearest 5km/h

7 4ha to the nearest $100m^2$

8 20 litres to the nearest $100cm^3$

9 By calculating percentage errors, find which is more serious:
(i) an error of 25m in giving the height of Mt. Elgon as 4,300m, or
(ii) an error of 5mm in giving a length as 10cm.

10 In a calculation, 3·8 is (i) rounded to 4, (ii) **truncated** to 3 (see **17.2**). Find the absolute and percentage errors involved in each case.

11 The area of a square whose side is given as 10 ± 1cm is found.
(i) What is the percentage error in the length?
(ii) What are the limits of accuracy for the area?
(iii) What is the percentage error in the area?

12 Find the percentage error in the volume of a cube as obtained from its edge given as 10 ± 1cm.

13 A sign near Nakuru gives the distance to Lagos along the proposed Trans African Highway (which passes through Uganda) as 5,749km.
(i) What is the possible error in this distance?
(ii) What is the percentage error?
(iii) On a map this distance is represented by 10cm. Give the possible error in this length with the above percentage error.
(iv) Is it possible to measure this length with such accuracy?

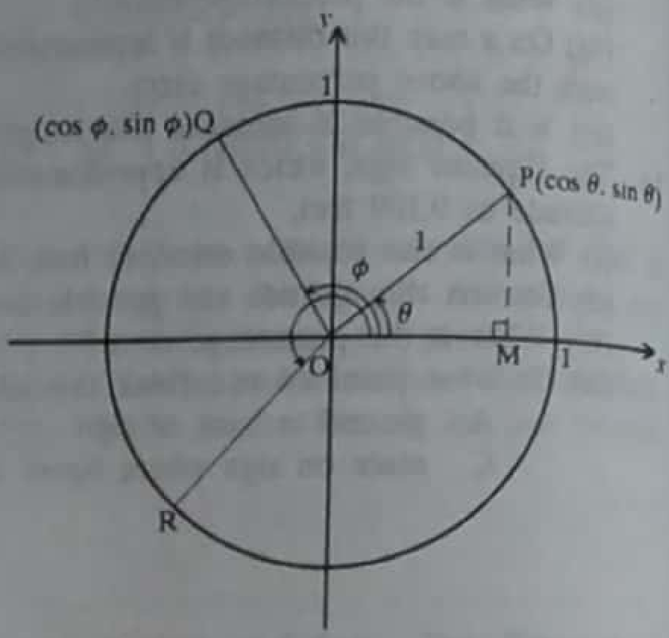
14 The Equator sign, which is approximately 3m high, near Timboroa in Kenya, states the altitude as 9,109 feet.
(i) What is the possible error, in feet, in this altitude?
(ii) Convert this altitude and possible error to metres, given that 1 foot equals 0·3048m.
(iii) What is the percentage error?
(iv) To what point do you think this altitude refers?

A ground at base of sign
B top of sign
C place on sign where figure appears
D none of these

44 TRIGONOMETRICAL GRAPHS

44.1 Ratios Redefined

We shall extend our definitions of Sine, Cosine and Tangent (see **28.1**) so as to obtain ratios of angles greater than 90°. The diagram shows a **unit circle** with radius OP making an angle θ with the positive direction of the x-axis.



In $\triangle$OPM, $\quad OM = 1 \times \cos\theta = \cos\theta$
and $\quad PM = 1 \times \sin\theta = \sin\theta$

Hence the coordinates of P are $(\cos\theta, \sin\theta)$ and this is used to ***define*** cosine and sine.

Consider the obtuse angle ϕ (see diagram). Its cosine and sine are defined as the x- and y-coordinates of Q.

Also from $\triangle$OPM, $\tan\theta = \frac{PM}{OM} = \frac{\sin\theta}{\cos\theta}$ and this defines tangent.

Similarly, $\tan\phi = \frac{\sin\phi}{\cos\phi}$

Coordinates of points such as R give the sine, cosine and tangent of reflex angles.

On graph paper, draw the unit circle with 5cm representing 1 unit together with x- and y-axes.

Draw OP with $\theta = 60°$ and OQ with $\phi = 120°$.

Write down the coordinates of P and Q.

This should give:

$$\cos 60° \simeq 0\cdot5, \quad \sin 60° \simeq 0\cdot87, \quad \cos 120° \simeq -0\cdot5, \quad \sin 120° \simeq 0\cdot87$$

$$\tan 60° = \frac{\sin 60°}{\cos 60°} \simeq \frac{0\cdot87}{0\cdot5} \simeq 1\cdot7 \quad \text{and} \quad \tan 120° = \frac{\sin 120°}{\cos 120°} \simeq \frac{0\cdot87}{-0\cdot5} \simeq -1\cdot7$$

Exercise 44a

Use your unit circle drawn as described above to find the sine, cosine and tangent of the angles in **1** to **16**.

1 20°	**2** 40°	**3** 60°	**4** 80°	**5** 100°	**6** 120°
7 140°	**8** 160°	**9** 200°	**10** 220°	**11** 240°	**12** 260°
13 280°	**14** 300°	**15** 320°	**16** 340°		

17 Give the sin of the angles 0°, 90°, 180°, 270°, 360°.
18 Give the cos of the angles 0°, 90°, 180°, 270°, 360°.
19 Give the tan of the angles 0°, 180°, 360°.
20 Can you give the tan of 90° or 270°?

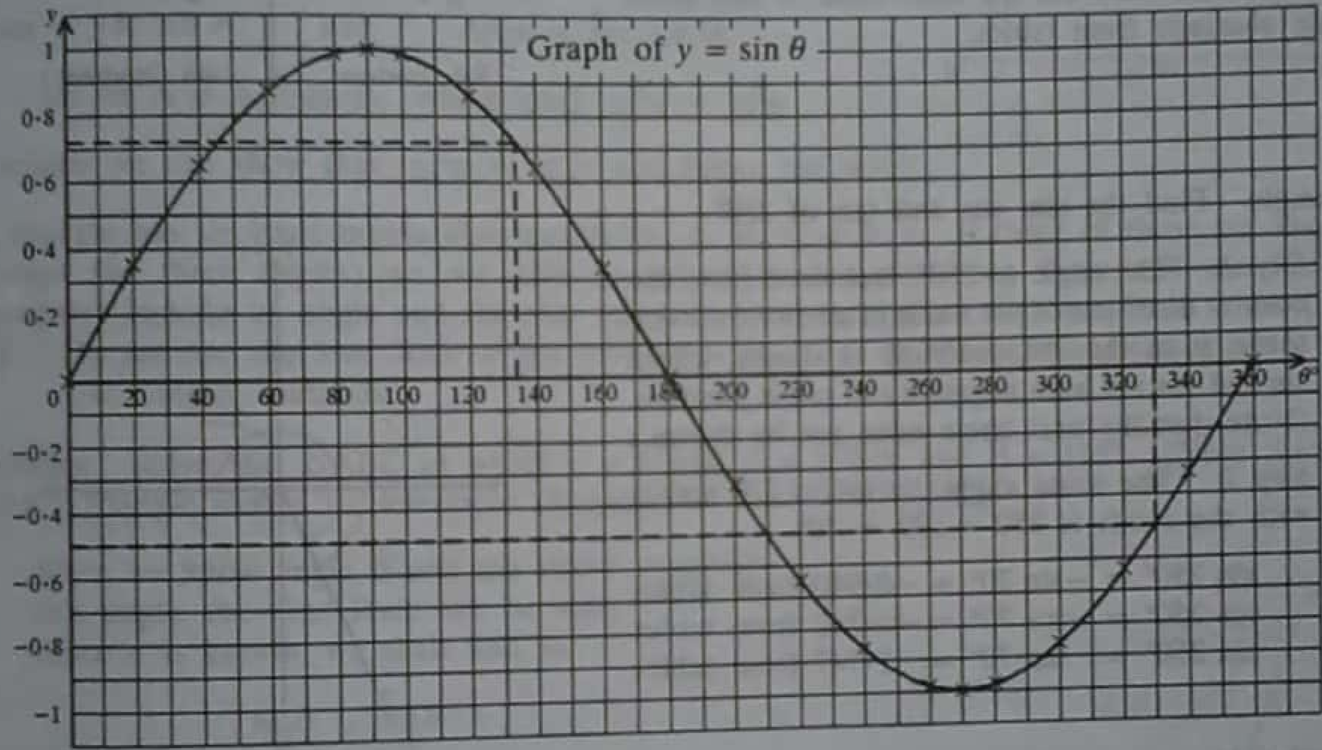
44.2 Graphs of Trigonometrical Ratios

The following table summarises the results for the sine of an angle from the previous Exercise.

$\theta°$	0	20	40	60	80	90	100	120	140	160	180
$\sin\theta$	0	0·34	0·64	0·87	0·98	1	0·98	0·87	0·64	0·34	0

$\theta°$	200	220	240	260	270	280	300	320	340	360
$\sin\theta$	−0·34	−0·64	−0·87	−0·98	−1	−0·98	−0·87	−0·64	−0·34	0

The graph of $y = \sin\theta$ is drawn on squared paper as shown below (or on graph paper) by plotting values of $\sin\theta$ against θ and joining the points with a smooth curve.



The graph may be used to read off sines, for example $\sin 134° = 0{\cdot}72$ and $\sin 330° = -0{\cdot}5$.

Exercise 44b

From the graph of $y = \sin\theta$, read off the sine of each angle in **1** to **6**.

1 46° **2** 150° **3** 210° **4** 314° **5** 24° **6** 156°

7 From your results of **Exercise 44a**, make out a table of values for $\cos\theta$ for $0° \le \theta \le 360°$. Draw the corresponding graph of $y = \cos\theta$.

In **8** to **13** use the graph of $y = \cos\theta$ to find the cosine of each angle.

8 50° **9** 130° **10** 230° **11** 310° **12** 108° **13** 252°

14 Draw the graph of $y = \tan\theta$ for $0° \le \theta \le 360°$. Note that the lines given by $\theta = 90°$ and $\theta = 270°$ are **assymptotes** (see **49.2**).

In **15** to **20** use the graph of $y = \tan\theta$ to find the tangent of each angle.

15 45° **16** 135° **17** 225° **18** 315° **19** 145° **20** 215°

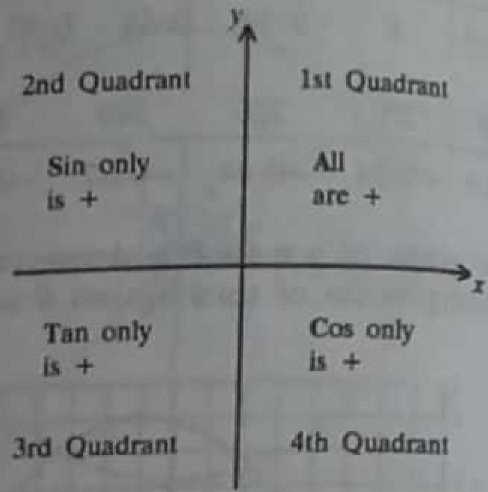
44.3 Ratios from Tables

Study the answers to the previous Exercise. You will probably notice that there is a relationship between the ratios of certain angles. For example: $\sin 120° = \sin 60° = 0{\cdot}87$, $\cos 120° = -\cos 60° = -0{\cdot}5$, $\tan 135° = -\tan 45° = -1$ and so on.

To find a ratio of an angle using tables, follow two steps:

Step 1: Decide whether the ratio will be positive or negative using the *quadrant* diagram shown. The mnemonic CAST may help you remember this.

Step 2: Find the acute angle the radius makes with the x-axis and use the same ratio of this acute angle obtained from tables.

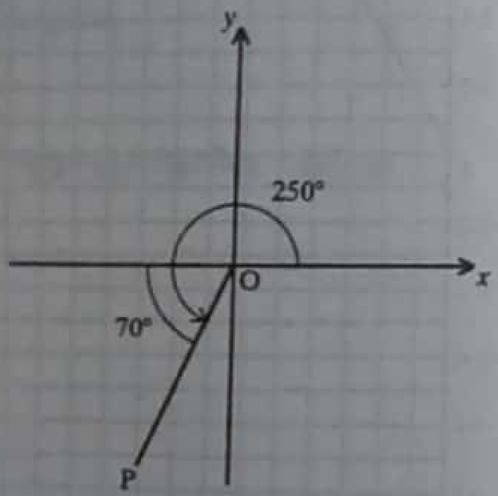


Example Find the sin, cos and tan of 250°.

Step 1: The angle of 250°, measured from the *positive* direction of the x-axis in an *anticlockwise* sense, is in the 3rd quadrant as shown, where sin and cos are both negative and tan is positive. (Note that the unit circle need not be drawn.)

Step 2: The acute angle the radius OP makes with the x-axis is $250 - 180 = 70°$.

$\therefore$ $\sin 250° = -\sin 70° = -0{\cdot}940$ from tables
$\cos 250° = -\cos 70° = -0{\cdot}342$ from tables
$\tan 250° = \tan 70° = 2{\cdot}747$ from tables



Exercise 44c

In **1** to **6** use tables to find the sin, cos and tan of the given angle.

1 115° **2** 235° **3** 295° **4** 350° **5** 400° **6** 600°

In **7** to **12** find the exact value of the sin, cos and tan of the given angle, leaving your answer in surd form where necessary. For example, $\sin 120° = \sin 60° = \frac{1}{2}\sqrt{3}$ (see **28.3**).

7 135° **8** 150° **9** 210° **10** 225° **11** 300° **12** 330°

13 Use the formula Area $= \frac{1}{2}ab \sin C$ (see **31.2**) to find the area of triangle ABC where AC = 7cm, BC = 8cm and ∠ACB = 150°.

14 Find (to 3 sf) the area of △PQR where PQ = 10cm, QR = 7cm and ∠PQR = 98°.

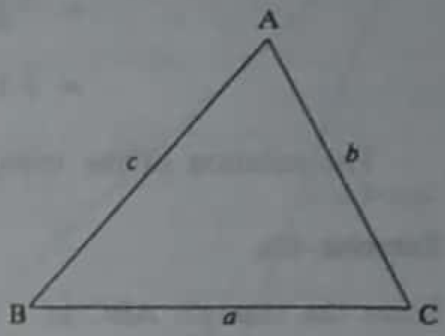
15 △ABC has AC = 8cm, BC = 6cm and an area of 20cm². Find the two possible values of ∠ACB.

45 SINE AND COSINE RULES

45.1 The Sine Rule

This Rule (and the Cosine Rule) enables us to calculate the sides and angles of any triangle when given sufficient information. The results are more accurate than would be the case using scale drawing.

The Sine Rule states that for any $\triangle ABC$ (see diagram) where $AB = c$, $BC = a$ and $CA = b$:



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

We derive this by considering the area of $\triangle ABC$ (See **31.2**)

Area $= \frac{1}{2}bc \sin A$ and Area $= \frac{1}{2}ac \sin B$

$\therefore \quad \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B$ which gives $\frac{a}{\sin A} = \frac{b}{\sin B}$

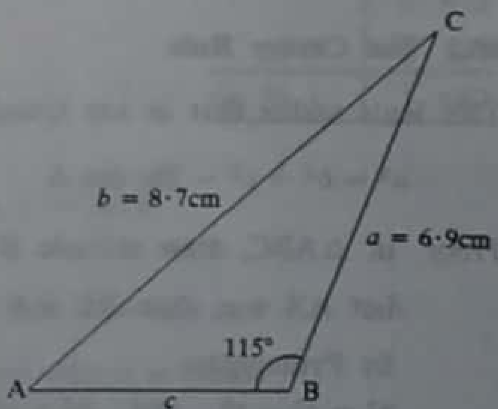
Similarly we can show that $\frac{b}{\sin B} = \frac{c}{\sin C}$ and hence the result.

A triangle has six basic measurable quantities: A, B, C, a, b, c. If three of these are known (except the three angles) we are able to determine the remaining three. If the known quantities include an angle and the side opposite that angle, then the Sine Rule is used.

NB Do ***not*** use the Sine Rule if the triangle is isosceles or if a right-angle is given. Use simpler methods.

Example Solve the $\triangle ABC$ in which $a = 6{\cdot}9$cm, $b = 8{\cdot}7$cm and $B = 115°$. (Note that to *solve* a triangle means to determine all unknown sides and angles.)

Since we know angle B and the side b opposite this angle, the Sine Rule can be used. Since a is known, we shall find A:



$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\therefore \quad \sin A = \frac{a \sin B}{b}$$

$$= \frac{6{\cdot}9 \sin 115°}{8{\cdot}7}$$

$$= \frac{6{\cdot}9 \sin 65°}{8{\cdot}7} \quad \text{(see } \mathbf{44.3}\text{)}$$

$$\therefore \quad A = 45{\cdot}9°$$

No	Log
6·9	0·839
sin 65	$\bar{1}$·957
	0·796
8·7	0·940
sin 45·9	$\bar{1}$·856

Notes

1. A logarithms of sines table has been used here. Alternatively the sine is obtained from a table of Natural Sines (as found in this Book) and its logarithm then found.
2. Sin $(180 - 45{\cdot}9)°$ ie. $\sin 134{\cdot}1°$ also has its logarithm as $\bar{1}{\cdot}856$, but A cannot be $134{\cdot}1°$ since $A + B$ must be less than $180°$.

$C = 180 - (115 + 45{\cdot}9) = 180 - 160{\cdot}9 = 19{\cdot}1°$

Using the Sine Rule again:

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\therefore \quad c = \frac{b \sin C}{\sin B}$$

$$= \frac{8{\cdot}7 \sin 19{\cdot}1°}{\sin 115°}$$

$$= 3{\cdot}15\text{cm}$$

No	Log
8·7	0·940
sin 19·1	$\bar{1}$·515
	0·455
sin 115	$\bar{1}$·957
3·15	0·498

The solution of the triangle is $A = 45{\cdot}9°$, $C = 19{\cdot}1°$, $c = 3{\cdot}15$cm (3sf)

Exercise 45a

Solve the triangle ABC in **1** to **6**.

1 $A = 60°$, $B = 50°$, $b = 5$cm

2 $a = 7$cm, $A = 38°$, $C = 75°$

3 $a = 12$cm, $A = 130°$, $b = 7$cm

4 $b = 8$cm, $c = 10$cm, $C = 98°$

5 $A = 37°$, $B = 29°$, $c = 10$cm

6 $a = 9$cm, $c = 4{\cdot}5$cm, $C = 30°$

7 In $\triangle$PQR, PQ = 10cm, QR = 9cm and $\angle$QPR = 60°. Solve this triangle giving the two possible solutions. (Hint: see *Note* 2 in the Example of **45.1**)

8 Two observers A and B are situated 500m apart on a straight piece of shoreline running north-south. The bearings of a ship S from A and B are 120° and 080° respectively. Calculate the distance of S from B.

9 A motorist drives towards Mombasa along the Nairobi-Mombasa road (assumed straight and running SE) at an average speed of 75km/h. At Athi River he notes the bearing of Kibo (on Mt. Kilimanjaro) to be 168°. Two hours later at Kibwezi, he observes the bearing of Kibo to be 222°. Calculate the distance of Kibwezi from Kibo.

45.2 The Cosine Rule

This Rule states that in any triangle ABC, with the usual notation:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Proof In $\triangle$ABC, draw altitude $BX = h$ as shown.

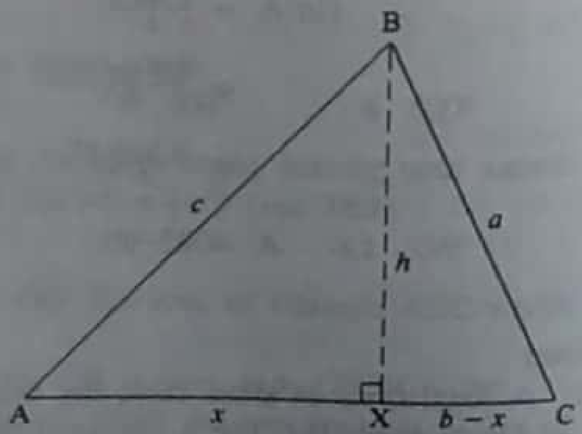
Let $AX = x$, then $XC = b - x$.

By Pythagoras:

$h^2 = c^2 - x^2$ and $h^2 = a^2 - (b - x)^2$

$\therefore \quad a^2 - (b - x)^2 = c^2 - x^2$

$$\begin{aligned} \therefore \quad a^2 &= (b - x)^2 + c^2 - x^2 \\ &= b^2 - 2bx + x^2 + c^2 - x^2 \\ &= b^2 + c^2 - 2bx \\ &= b^2 + c^2 - 2bc \cos A \quad \text{since } x = c \cos A \end{aligned}$$



Two other versions of this may be found useful:

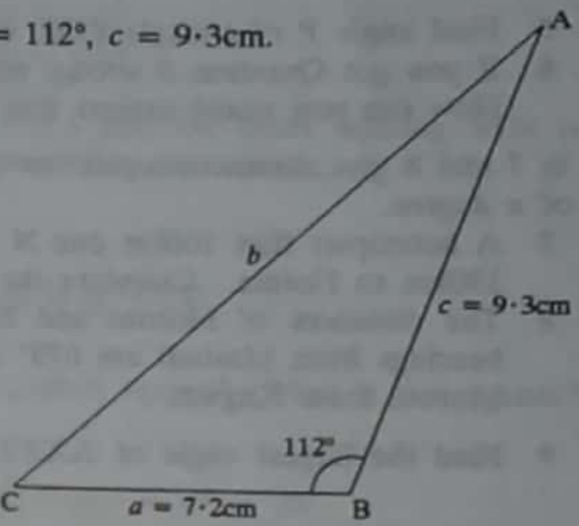
$$b^2 = a^2 + c^2 - 2ac \cos B \qquad \text{and} \qquad c^2 = a^2 + b^2 - 2ab \cos C$$

No 1168

Notes

1. The Cosine Rule is less easy to apply and less accurate than the Sine Rule. Therefore use it only as a first step for solving triangles in which the Sine Rule cannot be used.
2. Use only on scalene, non-right-angled triangles.

Example 1 Solve the $\triangle ABC$ in which $a = 7{\cdot}2$cm, $B = 112°$, $c = 9{\cdot}3$cm.



The Sine Rule may not be used since we do not know an angle and the side opposite to it.
Therefore we use the Cosine Rule:

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos B \\ &= 7{\cdot}2^2 + 9{\cdot}3^2 - 2 \times 7{\cdot}2 \times 9{\cdot}3 \cos 112° \\ &= 51{\cdot}8 + 86{\cdot}5 - 14{\cdot}4 \times 9{\cdot}3(-\cos 68°) \\ &= 138{\cdot}3 + 14{\cdot}4 \times 9{\cdot}3 \times \cos 68° \\ &= 138{\cdot}3 + 50{\cdot}1 \\ &= 188{\cdot}4 \end{aligned}$$

$$\therefore \quad b = \sqrt{188{\cdot}4} = 13{\cdot}7\text{cm}$$

No	Log
14·4	1·158
9·3	0·968
cos 68	$\bar{1}$·574
50·1	1·700
7·2	0·857
sin 68	$\bar{1}$·967
	0·824
13·7	0·137
sin 29·1	$\bar{1}$·687

Now use the Sine Rule:

$$\begin{aligned} \sin A &= \frac{a \sin B}{b} \\ &= \frac{7{\cdot}2 \sin 112°}{13{\cdot}7} \\ &= \frac{7{\cdot}2 \sin 68°}{13{\cdot}7} \end{aligned}$$

$$\therefore \quad A = 29{\cdot}1°$$

$C = 180 - (112 + 29{\cdot}1) = 180 - 141{\cdot}1 = 38{\cdot}9°$

The solution of the triangle is $A = 29{\cdot}1°$, $b = 13{\cdot}7$cm, $C = 38{\cdot}9°$

Example 2 Find the largest angle of $\triangle ABC$ where $a = 12$cm, $b = 8$cm and $c = 7$cm.

The largest angle is A because it is opposite the longest side $a = 12$cm.
Make cos A the subject of the Cosine Rule (see **47.2**):

$$\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{8^2 + 7^2 - 12^2}{2 \times 8 \times 7} = -\frac{31}{112} = -0{\cdot}277 \end{aligned}$$

$$\therefore \quad A = 180 - 73{\cdot}9 = 106{\cdot}1°$$

No	Log
31	1·491
112	2·049
0·277	$\bar{1}$·442

The largest angle of $\triangle ABC$ is $106{\cdot}1°$

(Note: $\cos 106{\cdot}1° = -\cos 73{\cdot}9° = -0{\cdot}277$, from tables)

Exercise 45b

Solve the triangle ABC in 1 to 4.

1 $A = 70°$, $b = 6\text{cm}$, $c = 8\text{cm}$

2 $A = 110°$, $b = 6\text{cm}$, $c = 8\text{cm}$

3 $a = 7\text{cm}$, $b = 8\text{cm}$, $c = 9\text{cm}$

4 $a = 10\text{cm}$, $b = 6\text{cm}$, $c = 7\text{cm}$

5 Find angle P of triangle PQR where $PQ = 9\text{cm}$, $QR = 12\text{cm}$ and $\angle PQR = 30°$.

6 If you got Question 5 wrong, repeat it by finding angle R first.
How can you guard against this error in future?

In 7 and 8 give distances correct to the nearest km and bearings correct to the nearest tenth of a degree.

7 A helicopter flies 100km due N from Entebbe, after which it flies on a bearing 294° for 130km to Hoima. Calculate the distance and bearing of Hoima from Entebbe.

8 The distances of Moroto and Kitgum from Masindi are 340km and 220km and their bearings from Masindi are 073° and 036° respectively. Find the distance and bearing of Moroto from Kitgum.

9 Find the largest angle of ΔXYZ where $XY = 8\text{cm}$, $YZ = 16\text{cm}$ and $\angle XYZ = 60°$.

46 QUADRATIC EQUATIONS (2)

46.1 Completing the Square

We can find the missing terms in: $x^2 + 6x + \ldots = (x + \ldots)^2$
by comparison with the identity: $x^2 + 2ax + a^2 = (x + a)^2$

Since $2a = 6$, the missing term in the bracket is $\frac{1}{2} \times 6 = 3$ and the other missing term is $3^2 = 9$. The process of adding 9 to $x^2 + 6x$ is called **completing the square.**

Note that $9 = (\frac{1}{2} \times 6)^2 = (\frac{1}{2}$ coefficient of $x)^2$.

The expression $x^2 + 6x + 9$ is a **perfect square** because it is $(x + 3)^2$.

Example What has to be added to $x^2 + 10x$ to form a perfect square? What is this square?

The coefficient of x in $x^2 + 10x$ is 10
Number to be added is $(\frac{1}{2}$ coefficient of $x)^2 = (\frac{1}{2} \times 10)^2 = 5^2 = 25$
The perfect square is $x^2 + 10x + 25 = (x + 5)^2$

NB The number to be added namely $(\frac{1}{2}$ coefficient of $x)^2$ will only work if the coefficient of x^2 is 1.

Exercise 46a

Find the number to be added to form a perfect square. State this square.

1 $x^2 + 2x$	**2** $x^2 + 4x$	**3** $x^2 + 8x$	**4** $x^2 + 12x$
5 $x^2 - 2x$	**6** $x^2 - 4x$	**7** $x^2 - 8x$	**8** $x^2 - 12x$
9 $x^2 + x$	**10** $x^2 - 3x$	**11** $x^2 + \frac{1}{2}x$	**12** $x^2 - \frac{3}{2}x$

46.2 Solution by Completing the Square

This method may be used to solve all quadratic equations, but especially those for which the factorisation method (see **35.2**) fails.

Example 1 Solve $x^2 - 2x - 15 = 0$ by completing the square.

Write down the equation: $x^2 - 2x - 15 = 0$
Add 15 to each side: $x^2 - 2x = 15$
Complete the square on the LS by adding 1 to each side: $x^2 - 2x + 1 = 15 + 1$
$\therefore \quad (x - 1)^2 = 16$
Take the square root of each side: $x - 1 = \pm 4$
Add 1 to each side: $x = 1 \pm 4$

The solution is $x = 1 + 4 = 5$ or $x = 1 - 4 = -3$

Note This quadratic equation is better and more quickly solved by factorisation.

Example 2 Solve the equation $2x^2 + 6x + 1 = 0$.

This cannot be solved by factorisation (try it!). To complete the square, the coefficient of x^2 must be 1. Divide throughout by 2 and rearrange the equation as follows:

$$x^2 + 3x = -\tfrac{1}{2}$$

Add $\left(\frac{3}{2}\right)^2$ or $\frac{9}{4}$ to each side: $x^2 + 3x + \left(\frac{3}{2}\right)^2 = -\frac{1}{2} + \frac{9}{4}$

$$\therefore \quad \left(x + \tfrac{3}{2}\right)^2 = \tfrac{7}{4}$$

$$\therefore \quad x + \tfrac{3}{2} = \pm\sqrt{\tfrac{7}{4}}$$

$\therefore \quad x = -\frac{3}{2} \pm \sqrt{\frac{7}{4}} = -1{\cdot}5 \pm \sqrt{1{\cdot}75} = -1{\cdot}5 \pm 1{\cdot}32$ from tables

The solution is $x = -1{\cdot}5 + 1{\cdot}32 = -0{\cdot}18$ or $x = -1{\cdot}5 - 1{\cdot}32 = -2{\cdot}82$

Note Values of x are correct only to 2 decimal places.

Exercise 46b

Solve by completing the square in 1 to 12.

1 $x^2 - 2x - 8 = 0$	**2** $x^2 + 2x - 8 = 0$	**3** $x^2 - x - 6 = 0$
4 $2x^2 + x - 6 = 0$	**5** $x^2 - 4x + 2 = 0$	**6** $x^2 + 2x - 1 = 0$
7 $x^2 - 8x = 7$	**8** $x^2 - 5x + 2 = 0$	**9** $x^2 - x - 1 = 0$
10 $2x^2 - 8x - 3 = 0$	**11** $2x^2 - 5x + 2 = 0$	**12** $3x^2 - 2x - 6 = 0$

13 Solve $3x^2 + x - 2 = 0$ by (i) completing the square, (ii) factorisation. Which method is easier?

14 Try to solve the equation $x^2 - 2x + 2 = 0$. Explain why you are unable to solve it.

15 Show that the **roots** (solutions) of the equation $2x^2 - 4x - 7 = 0$ are $x = \frac{1}{2}(2 \pm 3\sqrt{2})$.

16 Solve the equation $4x^2 + 12x + 9 = 0$ by completing the square.

46.3 Solution by Formula

By using the method of completing the square, the solutions of the equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

This is the formula for solving quadratic equations.

Example Solve the equation $x^2 - 3x - 1 = 0$ using the formula.

Substitute $a = 1$, $b = -3$, $c = -1$ into the formula.

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

$$= \frac{-(-3) \pm \sqrt{[(-3)^2 - 4 \times 1 \times (-1)]}}{2 \times 1} = \frac{3 \pm \sqrt{(9 + 4)}}{2} = \frac{3 \pm \sqrt{13}}{2} = \frac{3 \pm 3{\cdot}61}{2}$$

$\therefore \quad x = \frac{3 + 3{\cdot}61}{2} = \frac{6{\cdot}61}{2} = 3{\cdot}31$ or $x = \frac{3 - 3{\cdot}61}{2} = \frac{-0{\cdot}61}{2} = -0{\cdot}31$

Exercise 46c

Solve the equations in **1** to **9** using the formula.

1 $x^2 + 4x + 3 = 0$	**2** $x^2 - 2x - 3 = 0$	**3** $x^2 + 3x - 3 = 0$
4 $x^2 - 5x + 1 = 0$	**5** $2x^2 - x - 10 = 0$	**6** $2x^2 - 10x + 5 = 0$
7 $5x^2 + 10x + 2 = 0$	**8** $3x^2 + 11x + 2 = 0$	**9** $4x^2 + 7x - 1 = 0$

10 Use the formula to solve the equation $2x^2 - 4x + 3 = 0$. Explain why you are unable to solve it.

11 Use the formula to solve the equation $4x^2 - 20x + 25 = 0$.

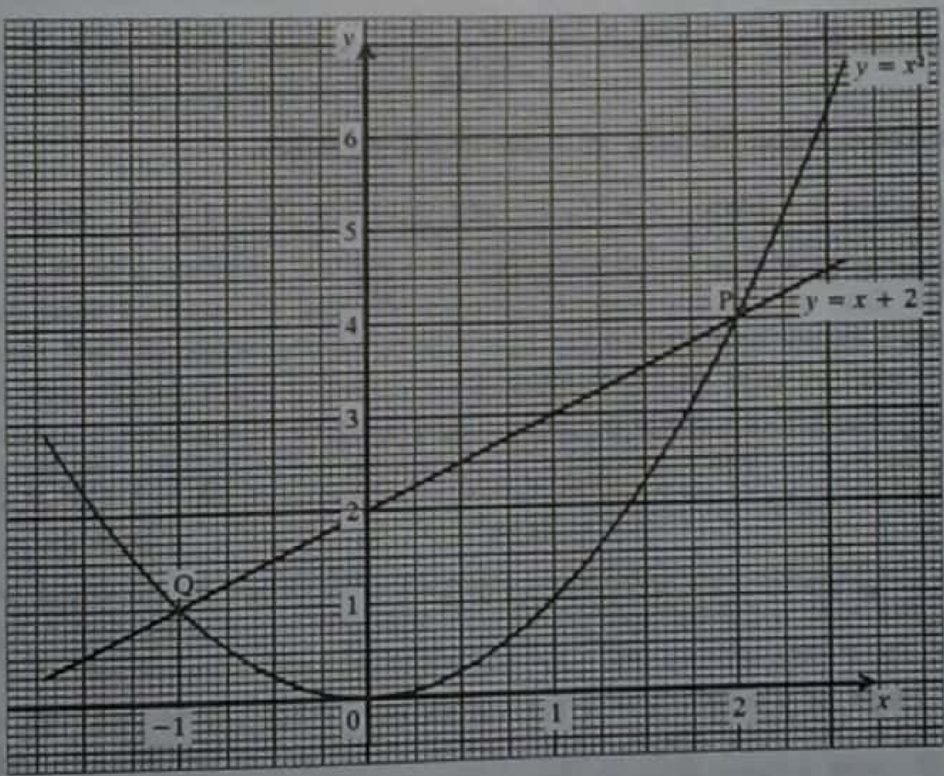
46.4 Simultaneous Equations (one Linear)

We shall consider methods of solving simultaneous equations in two unknowns in which one is linear and the other is of the second degree (see **34.1**).

1. *Graphical Solution*

Example Solve graphically the simultaneous equations $y = x^2$ and $y = x + 2$.

On the same axes, draw graphs of $\{(x, y) : y = x^2\}$ and $\{(x, y) : y = x + 2\}$ as shown.



Determine the coordinates of P and Q, the points of intersection of the two graphs. These are (2, 4) and (−1, 1) respectively.

Hence the solutions to the equations $y = x^2$ and $y = x + 2$ are

$$x = 2, y = 4 \quad \text{or} \quad x = -1, y = 1$$

Note There are two pairs of solutions.

2. *Analytical Solution*

Substitute for x or y (whichever is easier) from the linear equation into the second degree equation. This will then reduce to a quadratic equation.

Example 1 Solve the simultaneous equations $y = x^2$ and $y = x + 2$.

Substitute $y = x + 2$ into $y = x^2$:	$x^2 = x + 2$
Rearrange the equation:	$x^2 - x - 2 = 0$
Factorise the LS:	$(x - 2)(x + 1) = 0$
This gives:	$x = 2$ or -1

Substituting these values in turn into the linear equation gives $y = 4$ or 1 respectively.

The solution is $(2, 4)$ or $(-1, 1)$ (See also *Graphical Solution* on the previous page)

Example 2 Solve for x and y the equations $x^2 - 2xy - 3 = 0$ and $x + 2y - 1 = 0$.

Substitute $2y = 1 - x$ into the second degree equation:

$$x^2 - x(1 - x) - 3 = 0$$
$$\therefore \quad 2x^2 - x - 3 = 0$$
$$\therefore \quad (2x - 3)(x + 1) = 0$$
$$\therefore \quad x = \tfrac{3}{2} \text{ or } -1$$

Substitute these values in turn into the linear equation:

$\frac{3}{2} + 2y - 1 = 0$ gives $y = -\frac{1}{4}$ and $-1 + 2y - 1 = 0$ gives $y = 1$

The solution is $(1{\cdot}5, -0{\cdot}25)$ or $(-1, 1)$

Exercise 46d

Solve analytically the simultaneous equations in **1** to **6**.

1 $y = x^2 + 1$
$y = 3x - 1$

2 $y^2 = x^2 + 5$
$y = 2x + 1$

3 $x^2 + y^2 = 25$
$4x + 3y = 25$

4 $x^2 - 2xy + y^2 = 4$
$y = 2x - 3$

5 $x^2 + 4y^2 = 4$
$y = x - 1$

6 $y^2 + 3xy + 2y = 6$
$y = 3x - 2$

7 Draw the graph of $y = x^2 - x + 1$ for values of x from -1 to $+4$. By drawing another graph on the same axes, obtain approximate solutions to the simultaneous equations

$$y = x^2 - x + 1 \quad \text{and} \quad 3x - y - 1 = 0$$

Solve the equations analytically and compare your results.

8 Solve the simultaneous equations $4y = x^2$ and $y = x - 1$. From your analysis, what do you deduce about the parabola $4y = x^2$ and the line $y = x - 1$?

47 FORMULAE

47.1 Constructing Formulae

The following formulae have been derived and used in other Topics:

$A = \frac{1}{2}bh$ speed $= \dfrac{\text{distance}}{\text{time}}$ $F + V = E + 2$ $C = 2\pi r$ $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

Formulae such as these are **constructed** for a particular purpose.

Example Given that electricity costs sh5 per unit for the first 30 units and sh60 per unit thereafter, make a formula for the cost sh$C$ for E units of electricity, where $E > 30$. Use the formula to find the cost of 100 units.

The first 30 units cost $30 \times$ sh5 $=$ sh150
The remaining $(E - 30)$ units cost sh60 $\times (E - 30) =$ sh$60(E - 30)$

$\therefore$ the total cost sh$C$ is given by
$C = 150 + 60(E - 30)$ or $C = 60E - 1650$ (Note that the units are omitted)

When $E = 100$, $C = 60E - 1650 = 60 \times 100 - 1650 = 6000 - 1650 = 4350$

The cost of 100 units is sh4,350

When building a formula choose letters where possible, such as C and E above, which bring to mind the quantities they represent.

Exercise 47a

1 The cost of a telegram is sh60 for the first 10 words (or less). Each additional word over ten costs sh4. Find a formula for the cost sh$C$ of sending a telegram of w words, where $w > 10$. Use your formula to find the cost of sending a telegram with 30 words.

2 Find a formula for the total surface area S cm² of a cuboid of length l cm, width w cm and height h cm. Find S for a cuboid measuring 10cm by 8cm by 6cm.

3 Karugonjo makes a rectangular enclosure out of barbed wire fencing of total length 120m. If the length of the rectangle is l m, obta.n a formula for the area A m² in terms of l.

4 Find a formula for the area A cm² between two concentric circles of radius R cm and r cm respectively, where $R > r$.

5 A car can be hired by paying a fixed amount of sh5,000 plus sh3,000 per day plus sh50 per km. Find a formula for the cost sh$C$ for hiring this car for d days and running it for k km. Use your formula to find the cost of hiring this car for one week and running it for 1,000km.

6 Find a formula for the area A cm² of an equilateral triangle of side a cm.

7 The thickness of each page of a book is p mm and of each cover c mm. If the book has n pages (or sides) give a formula for the total thickness T mm of the book in terms of n, p and c. What does your formula give for the thickness of this Book where $p = 0{\cdot}1$, and $c = 0{\cdot}35$? (*Hint* First find the value of n for this Book.)

8 A pyramid on a square base of side a cm has four equilateral triangular faces. Find a formula for its volume V cm³ in terms of a.
Hence find a formula for a regular **octahedron** of edge a cm.

9 Obtain a formula for the curved surface area A cm² of a cone in terms of its base radius r cm and slant height l cm. (*Hint* The net of the curved surface of a cone is a sector of a circle.)

47.2 Changing the Subject of a Formula

Consider the distance (s), speed (v), time (t) formula: $s = vt$. The subject of the formula is s since this is given in terms of v and t only. Other arrangements of this formula are:

$v = \frac{s}{t}$ with v the subject $\qquad t = \frac{s}{v}$ with t the subject

Example 1 Make m the subject of the formula $a = 2m - b$

Write down the formula:	$a = 2m - b$
Add b to each side:	$a + b = 2m$
Divide each side by 2:	$\frac{a+b}{2} = m$
Change sides:	$m = \frac{1}{2}(a + b)$

Example 2 Make r the subject of the formula $V = \frac{4}{3}\pi r^3$.

Write down the formula:	$V = \frac{4}{3}\pi r^3$
Multiply by 3:	$3V = 4\pi r^3$
Divide by 4π:	$\frac{3V}{4\pi} = r^3$
Take the cube root and change sides:	$r = \sqrt[3]{\frac{3V}{4\pi}}$

Example 3 Make S the subject of the formula $r = \frac{S-a}{S}$

Write down the formula:	$r = \frac{S-a}{S}$
Multiply by S:	$rS = S - a$

(*NB* If we now divide by r getting $S = \frac{S-a}{r}$, S is not the subject because it appears on both sides of the formula.
To avoid this we bring the terms containing S to one side of the formula.)

Subtract S from each side:	$rS - S = -a$
Factorise:	$S(r - 1) = -a$
Divide by $(r - 1)$:	$S = \frac{-a}{r-1}$
Multiply numerator and denominator by -1:	$S = \frac{a}{1-r}$

Exercise 47b

In **1** to **18** make the bold letter the subject.

1 $A = \frac{1}{2}bh$ **2** $C = 2\pi r$ **3** $I = \frac{PRT}{100}$

4 $y = mx + c$ **5** $A = \pi r^2$ **6** $S = 4\pi r^2$

7 $A = kr^3$ **8** $c = \frac{b^2 - d}{4a}$ **9** $a = \frac{2A - bh}{h}$

10 $a = \dfrac{2A - bh}{h}$

11 $V = \frac{1}{3}\pi r^2 h$

12 $S = 2\pi r(r + h)$

13 $a = \sqrt{(b^2 + c^2)}$

14 $T = 2\pi\sqrt{\dfrac{l}{g}}$

15 $S = \frac{n}{2}[2a + (n - 1)d]$

16 $y = \dfrac{t}{1 - t}$

17 $x = \dfrac{1 - t^2}{1 + t^2}$

18 $A = P\left(1 + \dfrac{r}{100}\right)^n$

19 For $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$ make sin B the subject.

20 For $p^2 = q^2 + r^2 - 2qr\cos P$ make cos P the subject.

47.3 Use of Formulae

Formulae help us to solve problems. However you must:
(i) choose the appropriate formula,
(ii) be clear about which quantity to find and make that the subject of the formula,
(iii) substitute correctly the numerical values into the formula,
(iv) make the correct numerical calculation.

Example Use the formula $v^2 = u^2 + 2gs$ to find s when $v = 25$, $u = -15$ and $g = 10$.

Making s the subject gives: $s = \dfrac{v^2 - u^2}{2g}$

Substitute the numerical values: $s = \dfrac{25^2 - (-15)^2}{2 \times 10} = \dfrac{625 - 225}{20} = \dfrac{400}{20} = 20$

Exercise 47c

1 Find v where $v = u + ft$ and $u = -2$, $f = 3$, $t = 4$.

2 Find s where $s = ut - \frac{1}{2}gt^2$ and $u = 25$, $t = 2$, $g = 10$.

3 Find h where $A = \frac{1}{2}(a + b)h$ and $A = 60$. $a = 12$, $b = 8$.

4 Find r where $V = \pi r^2 h$ and $V = 1{,}100$, $h = 14$, $\pi = \frac{22}{7}$.

5 Find A where $A = \pi(R^2 - r^2)$ and $R = 14{\cdot}5$, $r = 13{\cdot}5$, $\pi = \frac{22}{7}$.
(*Hint* Factorise the difference of two squares first.)

6 Find r where $A = \pi h(R^2 - r^2)$ and $R = 13$, $h = 35$, $A = 2{,}750$, $\pi = \frac{22}{7}$.

7 Use the lens formula $\dfrac{1}{f} = \dfrac{1}{u} + \dfrac{1}{v}$ to find v when $f = 8$ and $u = 18$.

8 Use the simple pendulum formula $T = 2\pi\sqrt{\dfrac{l}{g}}$ to find g when $l = 0{\cdot}56$, $T = 1{\cdot}5$ and $\pi = 3{\cdot}14$.

9 The external surface area of a hollow cylinder closed at one end is 176cm². If the radius of the cylinder is 4cm, calculate the height. (Take $\pi = \frac{22}{7}$)

10 A solid cone has a total surface area of 704cm². If the base radius is 7cm, find
(i) the slant height,
(ii) the perpendicular height. (Take $\pi = \frac{22}{7}$)

48 VARIATION

48.1 Direct Variation

The circumference C of a circle is directly proportional to its diameter d (see **10.1**). We may write $C \propto d$, where $\propto$ means *is proportional to*. We also say that C **varies as** d. If $C \propto d$ it follows that $C = kd$ where k is the constant of proportion. In this case we know that $k = \pi$ and therefore $C = \pi d$.

Example The volume V of a sphere varies as the cube of the radius r. If the volume of a sphere of radius 4cm is 268cm³, find the volume of a sphere of radius 10cm.

$V \propto r^3 \quad \therefore \quad V = kr^3$ where k is a constant

Now substitute the values $V = 268$ and $r = 4$ to find k.

$\therefore \quad 268 = k \times 4^3$ which gives $k = \frac{268}{64} = 4{\cdot}19$ (3 sf) $\quad \therefore \quad V = 4{\cdot}19r^3$

When $r = 10$, $\quad V = 4{\cdot}19 \times 10^3 = 4{,}190$

The volume of a sphere of radius 10cm is approximately 4,190cm³.

48.2 Inverse Variation

Two quantities x and y are in inverse proportion (see **10.2**) if $x \propto \frac{1}{y}$ and we may write $x = \frac{k}{y}$ where k is a constant. In this case y **varies inversely as** x.

Example The pressure P N per cm² varies inversely as the volume V cm³ of gas inside a weather balloon. On release from the meteorological station $V = 15{,}000$ and $P = 12$. Find (i) P when $V = 18{,}000$, (ii) V when $P = 9$.

We are given $P \propto \frac{1}{V} \quad \therefore \quad P = \frac{k}{V}$ where k is a constant

When $V = 15{,}000$, $P = 12$ So $12 = \frac{k}{15000}$ or $k = 12 \times 15{,}000 = 180{,}000$

$\therefore \quad P = \frac{180000}{V}$

(i) when $V = 18{,}000$, $\quad P = \frac{180000}{18000} = 10$

(ii) when $P = 9$, $\quad V = \frac{180000}{P} = \frac{180000}{9} = 20{,}000$

Exercise 48a

1 Given $y \propto x$ and $y = 25$ when $x = 10$, find (i) y when $x = 6$, (ii) x when $y = 14$.

2 Given $y \propto \frac{1}{x}$ and $y = 25$ when $x = 2$, find (i) y when $x = 10$, (ii) x when $y = 2{\cdot}5$.

3 Given $p \propto q^2$ and $p = 18$ when $q = 3$, find (i) p when $q = 6$, (ii) q when $p = 200$.

4 Given $p \propto \frac{1}{q^3}$ and $p = 12{\cdot}5$ when $q = 2$, find (i) p when $q = 0{\cdot}5$, (ii) q when $p = 0{\cdot}1$.

5 For a bus travelling from Tororo to Mbale, the time taken t varies inversely as the average speed v. If $t = 45$ minutes when $v = 60$km/h, show that the constant of proportion is 2,700 and find (i) v when $t = 30$ minutes, (ii) t when $v = 45$km/h.

6 The resistance to motion R of a motor car varies as the square of its speed v. When $v = 50$km/h, $R = 250$N. Find (i) R when $v = 80$km/h, (ii) v when $R = 160$N.

7 If a stone is dropped from a height, the distance it falls varies as the square of the time taken. If the stone falls 5m in 1 second, how far would it fall in 3 seconds?

8 The brightness L of a light source varies inversely as the square of the distance d from it. The brightness of a light bulb is 1,000Lumens per cm^2 from a distance of 2m.
(i) What will be the brightness from a distance of 20m?
(ii) From what distance would the brightness be 40 Lumens per cm^2?

48.3 Joint Variation

In some situations more than two quantities may vary.

Example As a car moves round a circular arc (a bend in the road), it has an acceleration towards the centre of the arc which varies as the square of the speed v and inversely as the radius r of the arc. When $v = 36$km/h and $r = 100$m, $a = 1\text{m/s}^2$.
Find a when $v = 72$km/h and $r = 50$m.

The quantities varying are a, v and r where $a \propto v^2$ and $a \propto \frac{1}{r}$

These may be combined as $a \propto \frac{v^2}{r}$ or $a = \frac{kv^2}{r}$ where k is a constant.

Substitute the given values: $1 = \frac{k \times 36^2}{100}$ $\therefore$ $k = \frac{100}{36 \times 36} = \frac{25}{324}$ $\therefore$ $a = \frac{25v^2}{324r}$

When $v = 72$ and $r = 50$, $a = \frac{25 \times 72 \times 72}{324 \times 50} = 8$

The acceleration towards the centre is 8m/s^2.

48.4 Part Variation

If in the previous Example, a particular bend was being considered, then the radius r would be constant and a would vary as v^2 only. That is $a = cv^2$, where c is the constant of proportion. Putting in the initial data for the 100m bend ie. $a = 1$ when $v = 36$ gives:

$$1 = c \times 36^2 \qquad \therefore \quad c = \frac{1}{1296} \qquad \text{So } a = \frac{v^2}{1296}$$

Suppose the car had a tendency to overturn on this bend if $a > 9\text{m/s}^2$. What would be the maximum safe speed round the bend?

Now $v = \sqrt{(1{,}296a)}$ and when $a = 9$, $v = \sqrt{(1{,}296 \times 9)} = 36 \times 3 = 108$

The maximum safe speed is 108km/h.

Exercise 48b

1 Given $z \propto xy^2$ and $z = 6$ when $x = 3$ and $y = 2$, find (i) z when $x = y = 4$, (ii) x when $z = 12$ and $y = 3$, (iii) y when $z = 50$ and $x = 16$.

2 Given that $p \propto \frac{q}{r^2}$ and $p = 5$ when $q = 8$ and $r = 4$, find (i) p when $q = 14$ and $r = 10$, (ii) q when $p = 4$ and $r = 5$, (iii) r when $p = 6$ and $q = 135$.

3 In a rectangle of constant area, the length varies inversely as the breadth. If the length is 15cm when the breadth is 8cm, find the length when the breadth is 10cm.

4 The time t taken to build a housing estate varies as h, the number of houses built and inversely as m, the number of men who build them.
(i) Write down a formula relating t, h and m.
(ii) If it takes 200 men 9 months to build a 15-house estate, how long would it take 300 men to build a 20-house estate?

5 If 30 chickens lay 72 eggs in 5 days, how many eggs would 50 chickens be expected to lay in 4 days?

6 A car which accelerates from rest moves a distance s which varies as the acceleration f and as the square of the time taken, t. If s = 16m when f = 2m/s² and t = 4s, find s when f = 3m/s² and t = 6s.

7 In Q6, what time (to the nearest second) would it take for the car to move 100m if the acceleration was 4m/s²?

8 A cylinder whose volume is constant has its radius r inversely proportional to the square root of its height h. If r = 32cm when h = 9cm, find r when h = 16cm.

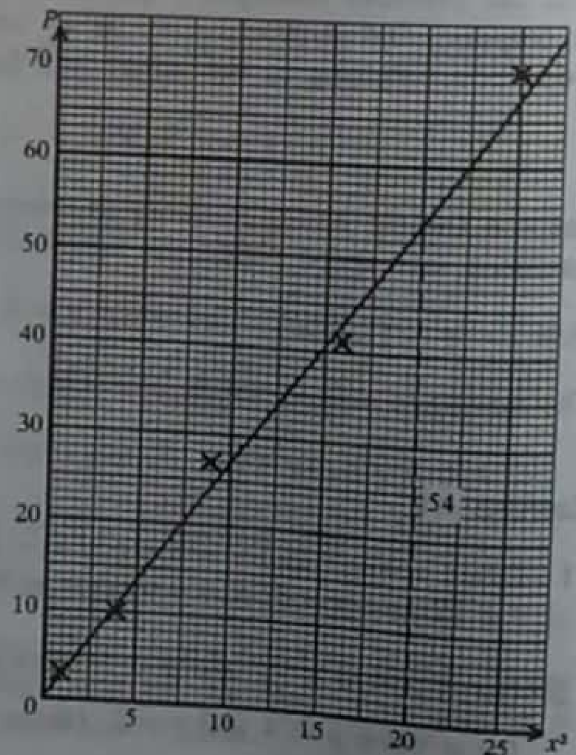
48.5 Graphical Determination of Physical Laws

Example From experiment, the table shows the relationship between two quantities P and x.

x	0	1	2	3	4	5
P	0	3	10	27	41	70

It is thought that $P \propto x^2$. Plot values of P against x^2 on a graph. By drawing the best straight line through these points, find the relation between P and x.

Points (0, 0), (1, 3), (4, 10), (9, 27), (16, 41) and (25, 70) are plotted. If $P \propto x^2$ the points should lie on a straight line through the origin. This is not quite the case because the data (called **empirical data**) was obtained from experiment. Therefore we draw the **best straight line** through the points as shown.



Since $P = kx^2$, the gradient (see **20.3**) of the line gives k. Using the points (0, 0) and (20, 54) on the line we have:

$$\text{Gradient} = k = \frac{54-0}{20-0} = 2{\cdot}7$$

The required relationship is $P = 2{\cdot}7x^2$

Exercise 48c

1 Experimental values for quantities A and x are as follows.

x	1	2	3	4	5	6
A	8·9	4·6	2·8	2·4	1·8	1·6

Assuming $A \propto \frac{1}{x}$, plot A against $\frac{1}{x}$ on a graph. By drawing the best straight line through these points, determine the relation between A and x.

2 It is thought that two quantities y and x have a relation of the form $y = a + bx^2$. Experimental values for x and y are:

x	1	2	3	4	5
y	5·5	7·8	10	15	19·5

By plotting values of y against x^2, draw the best straight line and hence determine the values of a and b.

3 The period T seconds of a small oscillation of a simple pendulum of length l cm is found to be as follows:

Length l cm	10	20	40	60	80	100
Period T s	0·6	0·9	1·3	1·6	1·7	2·0

It is known that the relationship between T and l is of the form $T = k\sqrt{l}$. By plotting T against $\sqrt{l}$ on a graph and drawing the best straight line through these points, determine the value of k.

It is also known that $k = \frac{2\pi}{\sqrt{g}}$ where g cm/s² is the acceleration due to gravity. Use your value of k to obtain a value of g. (Take $\pi = 3{\cdot}14$)

49 GRAPHS

49.1 Drawing Cubics

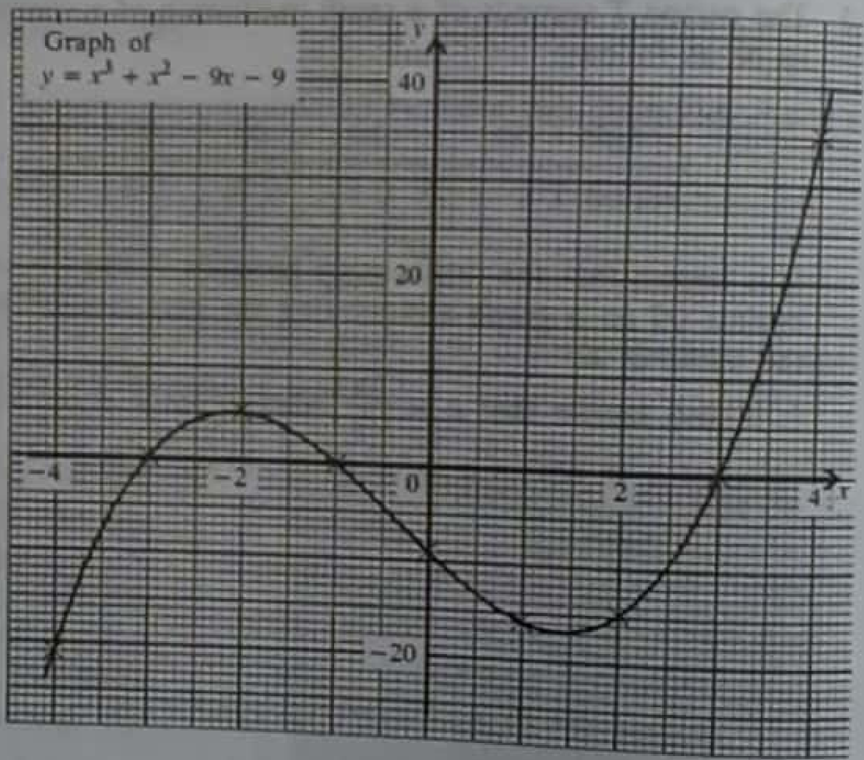
Drawing of linear and quadratic graphs was considered in **20.2** and **35.3** respectively. The following is an example of drawing a cubic graph.

Example Draw the graph of $y = x^3 + x^2 - 9x - 9$ for values of x from -4 to $+4$.

Taking the domain as $\{x : -4 \le x \le 4\}$, draw up a table of values:

x	-4	-3	-2	-1	0	1	2	3	4
x^3	-64	-27	-8	-1	0	1	8	27	64
x^2	16	9	4	1	0	1	4	9	16
$-9x$	36	27	18	9	0	-9	-18	-27	-36
-9	-9	-9	-9	-9	-9	-9	-9	-9	-9
y	-21	0	5	0	-9	-16	-15	0	35

Draw x- and y-axes and choose suitable scales according to the range of values of x and y as shown. Plot the points $(-4, -21)$, $(-3, 0)$, and so on. Intermediate points such as $(-1{\cdot}5, 3{\cdot}4)$ may be found and plotted if necessary. Join with a smooth curve.



Note Axes and labelling should be in ink (or drawn with a ball-point pen). For points and graph use a sharp pencil, preferably 2H.

Exercise 49a

In **1** to **8**, draw the graphs for the given domain.

1 $y = x^3 - 4x$ for $\{x : -3 \le x \le 3\}$

2 $y = 4x - x^3$ for $\{x : -3 \le x \le 3\}$

3 $y = x^3$ for $\{x : -3 \le x \le 4\}$

4 $y = x^3 - 3x^2 - x + 3$ for $\{x : -2 \le x \le 4\}$

5 $y = 3x^2 - x^3$ for $\{x : -1 \le x \le 4\}$

6 $y = x^3 + x^2 - x + 2$ for $\{x : -3 \le x \le 2\}$

7 $y = 1 - x + x^2 - x^3$ for $\{x : -1 \le x \le 2\}$

8 $y = 8 + 4x - 2x^2 - x^3$ for $\{x : -3 \le x \le 3\}$

9 Use your graph of $y = x^3$, drawn in **Q3**, to find an estimate for the cube of (i) $1{\cdot}5$, (ii) $3{\cdot}4$, (iii) $-2{\cdot}7$, and for the cube root of (iv) 10, (v) 40, (vi) -20.

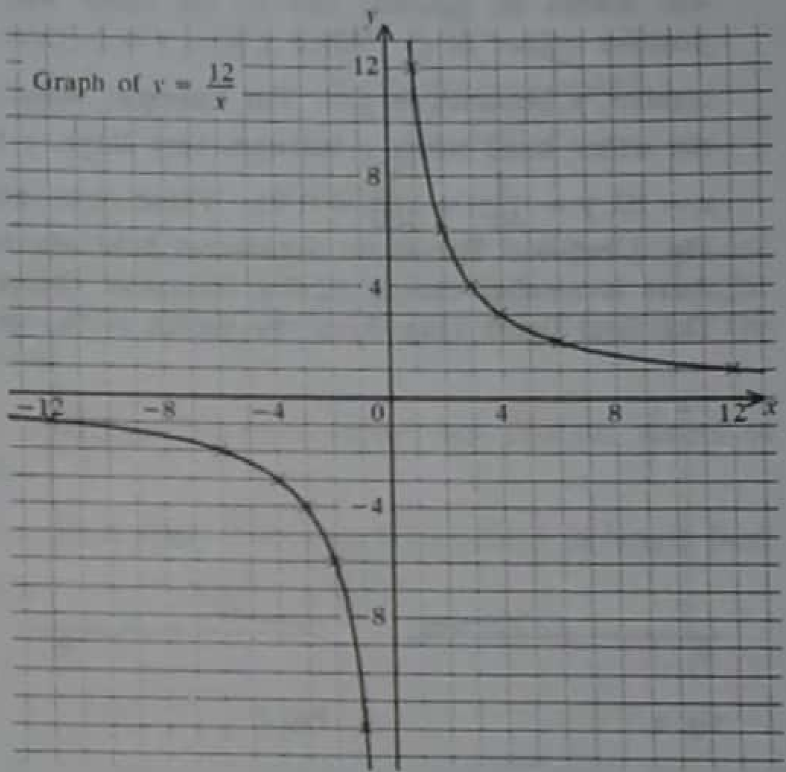
10 Draw the graph of $y = x^3 - x^2 - 7x + 3$ over the domain $\{x : -3 \le x \le 4\}$. Use your graph to find the *three* solutions to the equation $x^3 - x^2 - 7x + 3 = 0$.

Example 1 Draw the graph of (i) $y = \frac{12}{x}$ (ii) $y = x + \frac{12}{x}$ for $\{x : -12 \leq x \leq 12\}$.

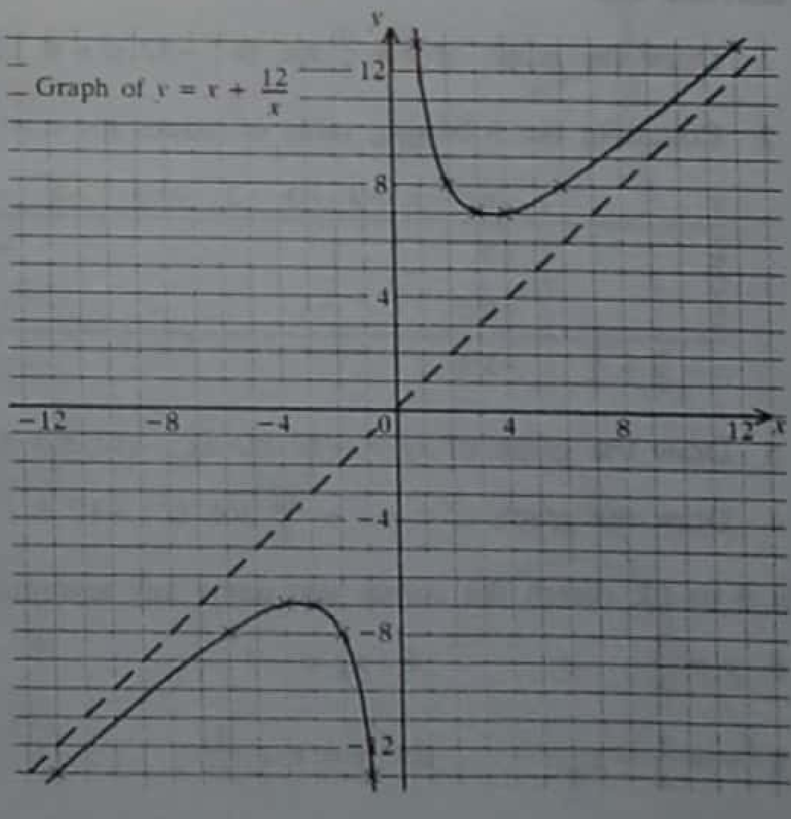
A table of values for both graphs is as follows:

x	-12	-6	-4	-3	-2	-1	1	2	3	4	6	12
$y = \frac{12}{x}$	-1	-2	-3	-4	-6	-12	12	6	4	3	2	1
$y = x + \frac{12}{x}$	-13	-8	-7	-7	-8	-13	13	8	7	7	8	13

(i) Draw x- and y-axes, choosing suitable scales. These should be the *same* on both axes as the x and y values are *symmetrical*. (The curve could just as well be written as $x = \frac{12}{y}$ and so the values of y also extend from -12 to $+12$.)
Note that x may not take the value zero as $12 \div 0$ has no real value. Similarly, no value of x will give y the value zero. Hence the curve approaches but never reaches the x- and y-axes.
A line which a curve approaches but never reaches is called an **asymptote**. Both the x- and y-axes are asympyotes to the curve given by $y = \frac{12}{x}$.
The curve has two **branches** and is known as a **rectangular hyperbola**. The graph (drawn on squared paper) is shown in the first diagram.



(ii) The completed graph is shown in the second diagram. As in (i) above, x may not take the value zero, therefore the y-axis is an asymptote.
As x becomes very large, the value of $\frac{12}{x}$ becomes very small. Hence we have $x + \frac{12}{x} \simeq x$.
So the curve approaches the line $y = x$ for large positive and negative values of x. Hence $y = x$ is an asymptote.



Example 2 Sketch the graph of $y = \frac{1}{x^2}$.

[Note: *sketch* means to show the approximate shape of a curve with special regard to asymptotes or other important features such as where it cuts the x- and y-axes.]

Study the equation $y = \frac{1}{x^2}$. First notice that x may not take the value zero. This indicates that the line $x = 0$ (or the y-axis) is an asymptote. Whether x is positive or negative, x^2 will always be positive and therefore $\frac{1}{x^2}$ will always be positive. Hence y will always be positive and so the curve will lie entirely above the x-axis. Consider positive values of x. When $x = 1$, $y = 1$, thus (1, 1) lies on the curve. As x becomes smaller, y becomes larger. For example, when $x = 0{\cdot}1$, $y = \frac{1}{(0{\cdot}1)^2} = \frac{1}{0{\cdot}01} = 100$. On the other hand when x becomes very large, y becomes very small. For example, when $x = 10$, $y = \frac{1}{10^2} = \frac{1}{100} = 0{\cdot}01$. These observations for positive values of x are shown in Fig. 1 below. The above arguments hold whether x takes positive or negative values, so there is another exactly similar branch of the curve for negative values of x (see Fig. 2). The final figure (Fig. 3) is the sketch of the graph.

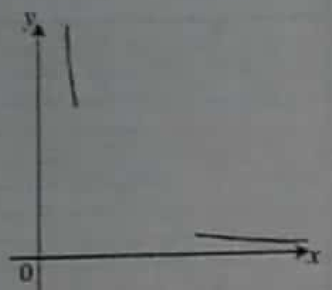


Fig. 1

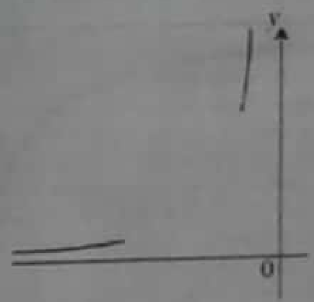


Fig. 2

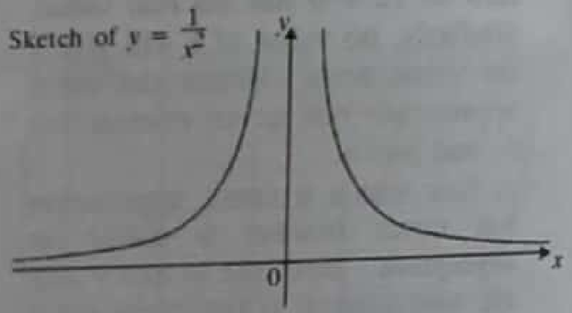


Fig. 3

Exercise 49b

1 Draw the graph of $y = \frac{6}{x}$ for $\{x : -12 \le x \le 12\}$. What are the asymptotes?

2 Complete the following table of values for $y = \frac{x}{2} + \frac{3}{x}$. Draw the graph of this function.

x	-6	-5	-4	-3	-2	$-1{\cdot}5$	-1	$-0{\cdot}6$	$-0{\cdot}5$	$0{\cdot}5$	$0{\cdot}6$	1	$1{\cdot}5$	2	3	4	5	6
$\frac{x}{2}$	-3							$-0{\cdot}3$						1				
$\frac{3}{x}$	$-0{\cdot}5$							-5						$1{\cdot}5$				
$y = \frac{x}{2} + \frac{3}{x}$	$-3{\cdot}5$							$-5{\cdot}3$						$2{\cdot}5$				

3 Draw the graph of $xy = -12$ for $\{x : -12 \le x \le 12\}$. State the asymptotes.

4 Draw the graph of $y = x - \frac{12}{x}$ for $\{x : -12 \le x \le 12\}$. State the asymptotes.

In **5** to **12**, sketch the curves and name any asymptotes.

5 $y = \frac{1}{x}$ **6** $xy = -1$ **7** $y = 1 + \frac{1}{x}$ **8** $y = 1 + \frac{1}{x^2}$

9 $y = \frac{1}{x} - x$ **10** $y = x + \frac{1}{x^2}$ **11** $y = x^2 + \frac{1}{x^2}$ **12** $y = \frac{x}{x^2 + 1}$

50 TANGENTS

50.1 Tangent to a Circle

A straight line which touches a circle at one point is called a **tangent**. The line segment joining the **point of contact** T to the centre O of the circle (ie. the radius OT) is perpendicular to the tangent XY (see Fig. 1).

Fig. 1

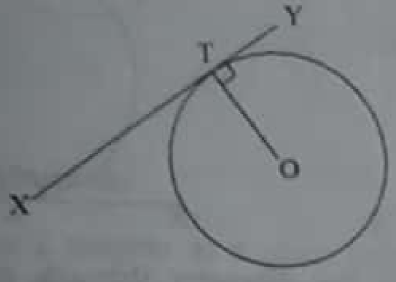
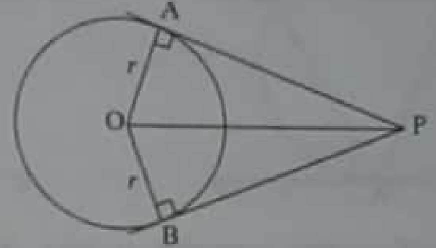


Fig. 2



If two tangents AP and BP are drawn from a point P to a circle, as shown in Fig. 2, they are equal in length, ie. the line segments AP and BP are equal in length. This is because triangles OAP and OBP are congruent (RHS). (See **37.4**) Also OP bisects $\angle APB$. Note that the figure has line symmetry about OP (see **41.3**).

Example If, in Fig. 2, P is 10cm from O and the circle is of radius 6cm find the length of each tangent and the angle between them.

In $\triangle OAP$ by Pythagoras, $AP = \sqrt{(10^2 - 6^2)} = 8$cm. Hence each tangent is of length 8cm.

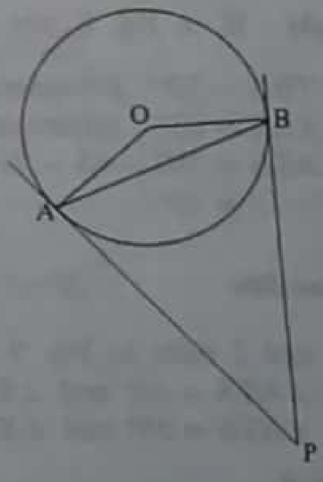
Also in $\triangle OAP$, $\sin \angle OPA = \frac{6}{10} = 0{\cdot}6$ $\therefore$ $\angle OPA = 36{\cdot}9°$

So $\angle BPA = 2 \times 36{\cdot}9° = 73{\cdot}8°$

Exercise 50a

In **1** to **7** refer to Fig. 3.

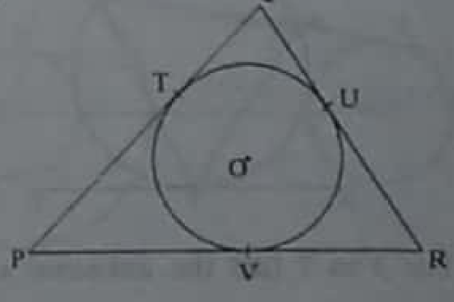
Fig. 3



1 If $\angle APB = 70°$, find $\angle ABP$.
2 If $\angle ABP = 70°$, find $\angle APB$.
3 If $\angle AOB = 100°$, calculate $\angle OAP$, $\angle PAB$ and $\angle APB$.
4 If $\angle APB = 70°$ calculate $\angle AOB$.
5 If OA = 7cm and OP = 25cm calculate AP, BP and $\angle APB$.
6 If AP = 12cm and OP = 15cm find OA and $\angle APB$.
7 If OA = 10cm and $\angle AOB = 120°$ find AP and OP.

In **8** to **10**, O is the centre of the inscribed circle of triangle PQR (see Fig. 4).

Fig. 4



8 If QU = 3cm, RV = 4cm and PT = 5cm, find the lengths of the sides of $\triangle PQR$.
9 If $\angle TQU = 48°$ and $\angle URV = 72°$, find the angles of $\triangle TUV$.
10 If $\angle TOU = 110°$ and $\angle VOU = 160°$, find the angles of $\triangle PQR$.

50.2 Alternate Segment Property

In Fig. 5, the chord AB divides the circle into two **segments**. Line XY is the tangent to the circle at A. The angle between the chord and the tangent is equal to the angle the chord subtends in the **alternate segment**, ie. $\angle BAY = \angle APB$.

Fig. 5

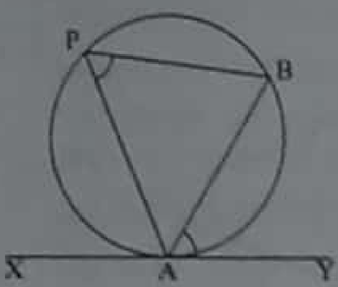


Fig. 6

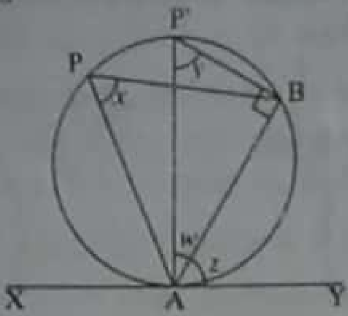
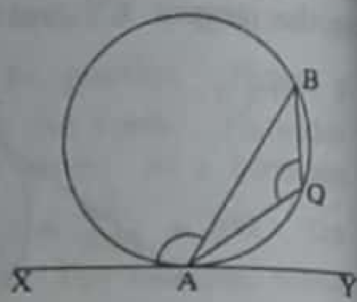


Fig. 7



The proof of this property is seen from Fig. 6 in which the diameter through A has been drawn. This meets the circle again at P'. BP' has also been drawn.
We now reason as follows:

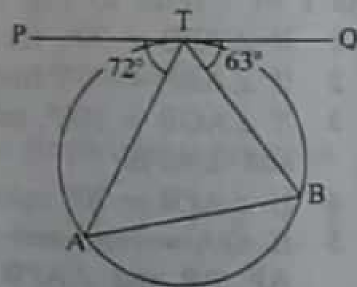
	$x = y$	(same segment)
	$\angle ABP' = 90^\circ$	(angle in a semi-circle)
In $\triangle ABP'$	$y + w + 90^\circ = 180^\circ$	(angle sum property of a $\triangle$)
	$\therefore\ y = 90^\circ - w$	
	$\angle P'AY = 90^\circ$	(tangent perpendicular to radius)
	$\therefore\ z = 90^\circ - w$	
	Hence $y = z$	
	$\therefore\ x = y = z$	$\therefore\ \angle APB = \angle BAY$

The result is true for the obtuse angle BAX made between the tangent and chord (see Fig. 7). For this angle the alternate (or opposite) segment is ABQ and $\angle BAX = \angle AQB$.

Example If, in Fig. 8, the tangent PQ touches the circle at T, find the angles of $\triangle TAB$.

$\angle TBA = 72^\circ$ (alternate segment)
$\angle TAB = 63^\circ$ (alternate segment)
$\angle ATB = 180 - 63 - 72$ (straight line PQ)
$= 45^\circ$

Fig. 8



Exercise 50b

For 1 and 2 refer to Fig. 9.

1 If $\angle ATX = 60^\circ$ and $\angle BTY = 70^\circ$, find the angles of $\triangle TAB$.
2 If $\angle ATB = 55^\circ$ and $\angle XTA = 65^\circ$, find $\angle TAB$.

Fig. 9

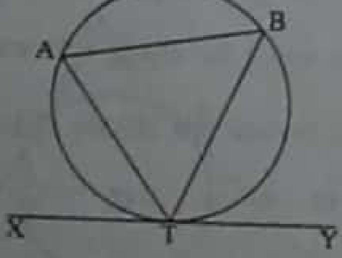


Fig. 10

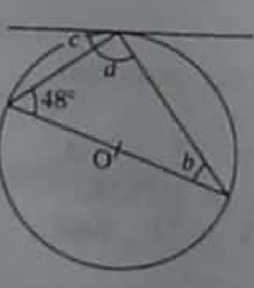
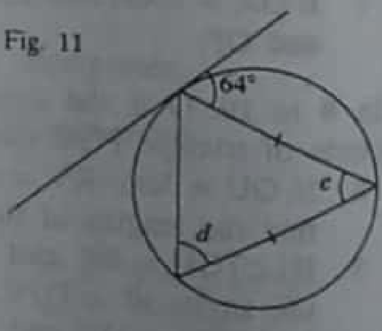


Fig. 11



For 3 to 7 find the unknown angles indicated in Figs. 10 to 14 respectively.

Fig. 12

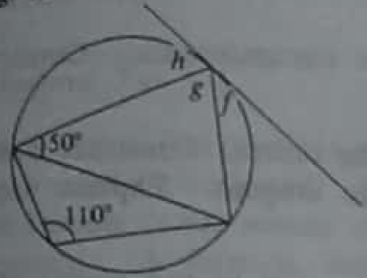


Fig. 13

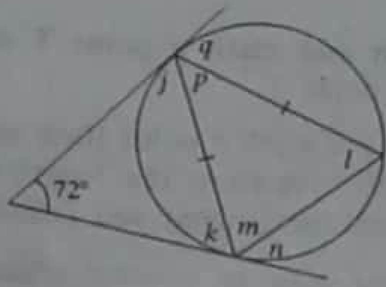
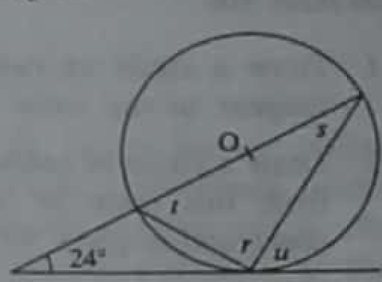


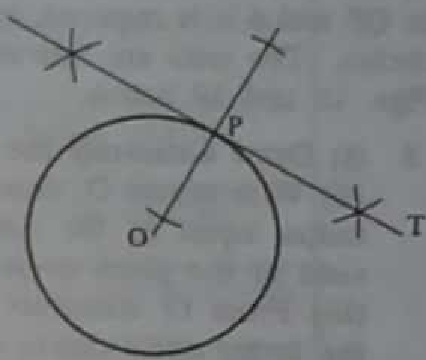
Fig. 14



50.3 Constructions

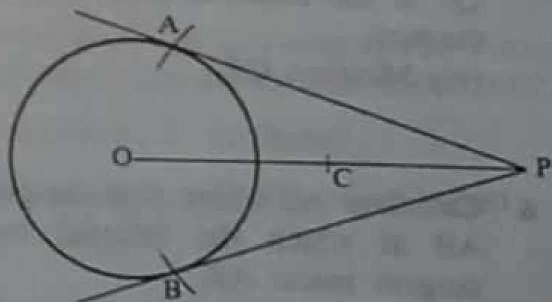
(i) To draw a tangent at a given point P on a given circle (see Fig. 15): join P to the centre O of the circle and construct a line PT through P which is perpendicular to OP (see 23.3). This gives the required tangent as shown.

Fig. 15



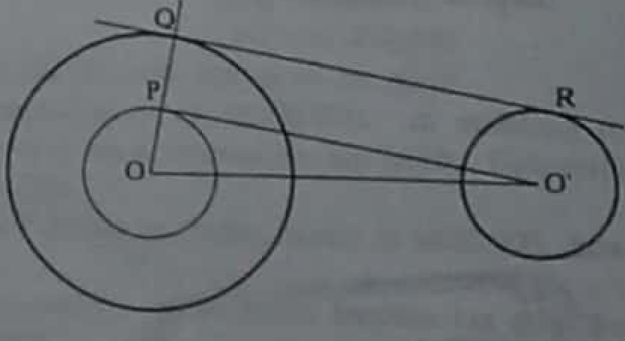
(ii) To draw tangents from a given point P outside a given circle, centre O (see Fig. 16): join O to P and bisect line segment OP (see 23.3) to give C. With centre C and radius OC draw arcs to cut the given circle at A and B. Join PA and PB to give the required tangents as shown.

Fig. 16



(iii) To draw external tangents common to two given circles (see Fig. 17): first join their centres OO'. With a radius equal to the *difference* in the circle radii, OP, draw a circle with centre O, the centre of the larger of the two given circles. Use construction (ii) above to construct O'P, a tangent to the small circle, centre O, from the point O'. Produce OP to cut the large circle at Q. From Q construct the tangent QR to the circle, centre O'. This is the required common tangent as shown.

Fig. 17



Exercise 50c

1 Draw a circle of radius 3cm and mark a point T on the circumference. Construct a tangent to the circle at the point T.

2 Draw a circle of radius 6 cm and mark a point 10cm from the centre. Construct a tangent from this point to the circle. Measure the length of the tangent. Explain why this construction gives 90° between the tangent and radius.

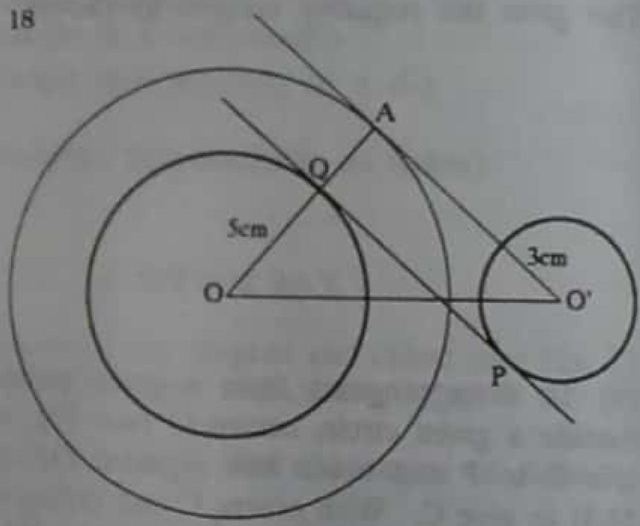
3 On a plain sheet of A4 paper draw an external common tangent to two circles of radii 7cm and 5cm respectively, with their centres 12·5cm apart. Measure the length of the tangent.

4 Draw a common tangent to two circles each of radius 4cm with their centres 10cm apart. Measure the length of the tangent.

In **Q5** and **6** it is required, by two methods, to draw the internal common tangent to two given circles. The radii are 5cm and 3cm respectively and their centres are 12cm apart, as shown in Figs. 18 and 19 below.

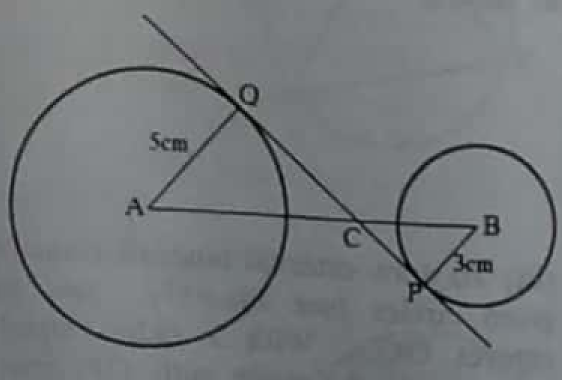
5 (i) Draw accurately the two circles.
(ii) With centre O, draw a circle with radius equal to the *sum* of the two radii of the given circles.
(iii) From O' construct a tangent to the larger circle centre O to touch it at A.
(iv) Join OA to give Q.
(v) From Q construct a tangent to the circle centre O' to give P.
QP is the required internal common tangent.
(vi) Measure QP.

Fig. 18



6 Calculate AC where C is the point on AB at which the internal common tangent meets AB.
(i) Draw the two circles accurately and construct the tangents from C to both circles to obtain P and Q.
(ii) Join P to Q to give the required tangent. Measure PQ.

Fig. 19



51 FINANCE

51.1 Income Tax

In order that a Government of a country may provide services to its people money (called **revenue**) must be collected. **Income tax** is one such source of revenue. Individuals whose **annual income** (total money earned per year) exceeds a certain amount pay this tax to the Government. A formula, which may be adjusted from time to time, is provided by the Government for calculating the tax payable.

The following is an extract from the *1991 Finance Bill* for resident individual rates of tax:

	Total Income	*Tax Rate*
1.	Where total income does not exceed sh480,000	Nil
2.	Where total income exceeds sh480,000 but does not exceed sh620,000	10 per cent of the amount by which total income exceeds sh480,000
3.	Where total income exceeds sh620,000 but does not exceed sh870,000	sh14,000 plus 20 per cent of the amount by which total income exceeds sh620,000
4.	Where total income exceeds sh870,000 but does not exceed sh1,310,000	sh64,000 plus 30 per cent of the amount by which total income exceeds sh870,000
5.	Where total income exceeds sh1,310,000 but does not exceed sh2,620,000	sh196,000 plus 40 per cent of the amount by which total income exceeds sh1,310,000
6.	Where total income exceeds sh2,620,000	sh720,000 plus 50 per cent of the amount by which total income exceeds sh2,620,000

Example Ebwadel earns sh65,000 per month. What is the annual income tax payable?

Annual income = monthly income × 12 = sh65,000 × 12 = sh780,000

Look at the table. This income lies within the amounts specified in paragraph 3 where the *Tax Rate* is "sh14,000 plus 20% of the amount by which income exceeds sh620,000".

Now total income exceeds sh620,000 by sh(780,000 − 620,000) = sh160,000

$\therefore$ Tax payable = sh(14,000 + $\frac{20}{100}$ × 150,000) = sh(14,000 + 32,000) = sh46,000

Ebwadel's annual income tax payable is sh46,000.

Exercise 51a

Use the table of tax rates given in 51.1.

1 Find the tax payable on the following annual incomes.
(i) sh650,000 (ii) sh1,000,000 (iii) sh420,000
(iv) sh560,000 (v) sh870,000 (vi) sh4,000,000

2 Akishule earns sh150,000 per month. What annual income tax does he pay?

3 Ssentongo has a regular annual income from employment of sh300,000. In addition to this he receives rent of sh20,000 per month per house on 8 houses he has built. Calculate his total annual income and the income tax payable.

4 Akabwai arranges to pay his income tax monthly. If his monthly salary is sh50,000, how much income tax does he pay per month?

5 Rwakara pays his income tax quarterly (every 3 months). How much income tax does he pay per quarter if his monthly salary is sh135,000?

51.2 Other Sources of Revenue

Income tax is an example of direct taxation. The government raises more revenue by indirect taxation such as sales tax, import duty, Commercial Transactions Levy (CTL), road tolls, and so on. Also District and local Councils raise revenue from Graduated Tax, property rates and other means. Questions on these topics are contained in the following Exercise.

Exercise 51b

1 An article that would have cost sh550 is subject to 10% sales tax. What will it cost now?
2 The import duty on a certain item valued at sh100,000 is 85%. The new value is subject to 10% sales tax. What is the item worth now?
3 A TV valued at sh400,000 is subject to 80% import duty plus 10% sales tax on top of this. How much revenue does the government receive?
4 In one day of trading, a retailer sells 25 crates of soda. If each crate holds 24 bottles and each bottle is sold for sh300, how much CTL (at 10% of the retail price) is paid to the government?
5 The annual trading licence to run a petrol station in Kampala is sh172,500. In addition there is a minimum storage charge of sh23,000 for up to 10,000 litres of petrol stored plus sh5,750 for every extra 5,000 l stored over this. What annual revenue does the Kampala City Council receive from three petrols stations which store 5,000 l, 10,000 l and 25,000 l of petrol respectively?
6 At a road toll the charge is sh100 per car, sh200 per taxi and sh500 per bus or lorry. How much revenue is collected on a day when 1,500 cars, 230 taxis, 25 buses and 60 lorries pass this toll?
7 Annual property rates in Kampala are reckoned at 10% of the value of the property and are paid quarterly. Ang'a owns a house valued at sh3,500,000. How much does he pay to the Kampala City Council per quarter?
8 The formula used by a District for calculating the annual Graduated Tax (sh$G$) in terms of an individual's annual salary (sh$S$) is as follows:

$$15G = S \quad \text{for} \quad S \leq 180{,}000 \qquad \text{and} \qquad 10G = S - 60{,}000 \quad \text{for} \quad S > 180{,}000$$

The value of G obtained from the formula is truncated (rounded down: see **17.2 Ex 2**) to the nearest sh1,000 to give the Graduated Tax payable.
Calculate the Graduated Tax payable on an annual income of
(i) sh120,000 (ii) sh250,000 (iii) sh85,000 (iv) sh365,000

51.3 Insurance

An individual may be financially protected by means of **insurance**. An **insurance company** is set up to provide this protection or **cover**. Suppose a man builds a house worth sh5,000,000. By paying an agreed annual **premium**, say sh50,000, to an insurance company he may insure this house against the **risk** (probability) of fire, flood, earthquake and other hazards which may damage or destroy it. A written document (**policy**), which contains all the conditions involved, is signed by both parties (the man, called a **client**, and the Company) upon payment of the premium. If the house is destroyed in a manner laid down in the policy then the Company will pay to the man the insured value, ie. sh5,000,000.
In calculating insurance premiums many factors, some of which involve probability, need to be considered. In the Example and Exercise which follow, we shall use the formula

$$P = pA\left(1 + \frac{r}{100}\right)$$

where P is the annual premium payable, p is the probability (see **Topic 56**) that the disaster or other event for which cover is required will happen, A is the insured value and r is the overall percentage profit the Company expects to make.

In practice, the formulae used are considerably more complicated than the one given above.

Example Rwabose insures his camera valued at sh150,000 against loss or theft. The probability that one of these events will happen during the course of a year is estimated to be 0·02 and the Company aims to make a 20% profit. What premium will Rwabose pay?

Using the above formula gives:

$$P = pA\left(1 + \tfrac{r}{100}\right)$$
$$= 0{\cdot}02 \times 150{,}000(1 + 0{\cdot}2)$$
$$= 3{,}600$$

Rwabose pays an annual premium of sh3,600.

Exercise 51c

Use the formula given in **51.3**, where applicable.

1 Odeke insures his household goods valued at sh350,000 against fire and theft. The probability that one of these events will occur is 0·025. What will be his annual premium if the insurance company makes a 30% profit?

2 Mukasa insures his car valued at sh5,000,000 against accident. The insurance company bases its probability estimate on the fact that, out of 250 cars insured in the previous year, 3 had accidents. If the Company looks for a 25% profit, calculate the annual premium payable.

3 Nyakana takes out a house insurance against natural disaster, ie. earthquake, tempest, typhoon, landslide, etc., and against dangers such as strike, riot, civil strife, rebellion and invasion. The chance that any of these disasters will occur in a particular year and that his house will be affected is reckoned to be 1 in 500. If the house is valued at sh15m, what annual premium should Nyakana expect to pay if the insurance company works on a 15% profit?

4 A particular insurance company assesses the cost of insurance for general house contents as sh5 per sh20,000 value of articles insured. Particularly valuable items have to be specified and these are charged at sh9 per sh20,000 value of these items. Find the total cost of insuring sh8,000,000 of general house contents together with sh1,200,000 worth of specified valuable items.

5 To encourage people to drive carefully insurance companies give a *No Claims Bonus* (NCB) if the policy holder had no accident for which a claim was submitted in the previous year. This is reckoned as a percentage discount (see **17.5**) on the calculated annual premium payable. Mukasa of **Q2** has a 40% NCB. He has an accident to his car which the garage estimates will cost sh25,000 to repair. If he claims from the insurance company he will lose his NCB.

(i) What is his net annual premium?

(ii) Assuming he does not have another accident, what amount will Mukasa save when he pays his next annual premium if he pays for the repairs to his car himself?

52 MATRICES

52.1 Definitions

A **matrix** is an ordered array of **elements** that can be described by **rows** and **columns**.

Consider $M = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 9 & 0 & 3 \end{pmatrix}$

M is a 3 by 4 (or 3×4) matrix because it has 3 rows and 4 columns. Matrix M is said to be of **order** 3×4.

A **row matrix** has only one row, for example $(1 \ -2 \ 4)$, being of order 1×3.

A **column matrix** has only one column, for example $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ being of order 3×1.

A **square matrix** has the same number of rows as columns.

A **null** (or **zero**) **matrix** has every element 0, for example $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and is denoted by **0**.

A **unit** (or **identity**) **matrix** is a square matrix with each element in the leading diagonal 1 and every other element 0. Examples are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

These matrices are denoted by $\mathbf{I}_2$ and $\mathbf{I}_3$ respectively.

52.2 Addition and Subtraction

Only matrices of the same order can be added or subtracted by applying these operations to corresponding elements. Examples are:

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 0 & 2 \\ 3 & 8 & 0 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 2 & 2 \\ -2 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+1 & 2+0 & 3+2 & 1+2 \\ 4-2 & 5+0 & 0+1 & 2+1 \\ 3+2 & 8+1 & 0+0 & 3+1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 5 & 3 \\ 2 & 5 & 1 & 3 \\ 5 & 9 & 0 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 0 & 2 \\ 3 & 8 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 2 & 2 \\ -2 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-1 & 2-0 & 3-2 & 1-2 \\ 4-(-2) & 5-0 & 0-1 & 2-1 \\ 3-2 & 8-1 & 0-0 & 3-1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 & -1 \\ 6 & 5 & -1 & 1 \\ 1 & 7 & 0 & 2 \end{pmatrix}$$

52.3 Multiplication by a Scalar

If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ then $3\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 9 & 3 \end{pmatrix}$

This is the same as multiplying each element of the matrix by 3.

52.4 Matrices as Stores of Information

Example 1 The results of 9 matches played by 4 football teams can be shown in a table or stored in a matrix.

Table

Team	Wins	Draws	Losses
A	3	6	0
B	2	1	6
C	4	4	1
D	1	5	3

Matrix

$$\begin{pmatrix} 3 & 6 & 0 \\ 2 & 1 & 6 \\ 4 & 4 & 1 \\ 1 & 5 & 3 \end{pmatrix}$$

Example 2 Three points have coordinates P(1, 2), Q(3, 4) and R(−2, 5). The coordinates can be shown in a table or by a matrix.

Table

Point	P	Q	R
x	1	3	−2
y	2	4	5

Matrix

$$\begin{pmatrix} 1 & 3 & -2 \\ 2 & 4 & 5 \end{pmatrix}$$

Matrices can be used to store information as the above Examples show.

Exercise 52a

1 Give the order of the following matrices.

(i) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (iv) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (v) $(1\ 2\ 3\ 4\ 5)$

(vi) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & -3 & -4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$ (vii) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 3 & 1 & 4 \end{pmatrix}$

In **2** to **7** express as one matrix.

2 $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$ **3** $\begin{pmatrix} 1 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix} + \begin{pmatrix} 5 & 6 & 7 \\ 1 & 2 & 3 \end{pmatrix}$ **4** $\begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 9 & 4 \end{pmatrix} + \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}$

5 $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ **6** $\begin{pmatrix} 1 & 2 & 3 \\ 5 & 7 & 8 \end{pmatrix} - \begin{pmatrix} 5 & 6 & 7 \\ 1 & 2 & 3 \end{pmatrix}$ **7** $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

8 If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$ find (i) $\mathbf{A} + \mathbf{B}$, (ii) $\mathbf{A} - \mathbf{B}$, (iii) $5\mathbf{A}$, (iv) $5\mathbf{B}$, (v) $5(\mathbf{A} + \mathbf{B})$.

9 Find **A** if $\mathbf{A} + \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

10 Find **B** if $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ -1 & -2 & -3 \end{pmatrix} + \mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

11 The football teams of Example 1 of **52.4** played four more matches with the results shown in the matrix. Use matrix addition to give the matrix that describes the number of wins, draws and losses for each team after 13 matches.

$$\begin{pmatrix} 2 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 3 & 1 \end{pmatrix}$$

12 Each of the points P, Q and R of Example 2 of **52.4** are given a translation of $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$.
(i) Give a 2×3 matrix that will effect this translation for the points.
(ii) By adding your matrix to $\begin{pmatrix} 1 & 3 & -2 \\ 2 & 4 & 5 \end{pmatrix}$, find the coordinates of P, Q and R after the translation.

52.5 Multiplication

If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$ then

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1\times5+2\times7 & 1\times6+2\times8 \\ 3\times5+4\times7 & 3\times6+4\times8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

Here, the orders of **A** and **B** were the same, ie. both 2×2. Generally, however, the orders of two matrices are not the same for multiplication to be possible. Study the following:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 1\times7+2\times8+3\times9 \\ 4\times7+5\times8+6\times9 \end{pmatrix} = \begin{pmatrix} 50 \\ 122 \end{pmatrix}$$

Here the orders are 2×3, 3×1 and 2×1. Note that the number of columns of the first matrix equals the number of rows of the second and the resulting matrix (2×1) is composed of the number of rows of the first matrix and the number of columns of the second.

If **A** is an $m \times n$ matrix and **B** an $n \times p$ matrix then it is possible to give the product matrix **C** where

$$\begin{array}{ccccc} \mathbf{A} & \times & \mathbf{B} & = & \mathbf{C} \\ m \times \textcircled{n} & & \textcircled{n} \times p & & m \times p \\ & \text{same} & & & \end{array}$$

Exercise 52b

Form the products in **1** to **5**.

1 $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$ **2** $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ **3** $\begin{pmatrix} 4 & 1 & 2 \\ 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 2 \end{pmatrix}$ **4** $\begin{pmatrix} 6 & 5 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 4 & -5 \\ -5 & 6 \end{pmatrix}$ **5** $\begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 & -5 \\ -3 & 4 \end{pmatrix}$

6 If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ find $\mathbf{A}^2$.

7 If $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ find $\mathbf{B}^2$, $\mathbf{B}^3$ and $\mathbf{B}^n$.

8 If $\mathbf{C} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ show that $\mathbf{C}^2 = \mathbf{I}_2$.

9 If $\mathbf{D} = \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}$ and $\mathbf{E} = \begin{pmatrix} -2 & -2 \\ 3 & 3 \end{pmatrix}$ show that $\mathbf{DE} = \mathbf{0}$.

10 If $\mathbf{F} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ show that $\mathbf{F}^3 + \mathbf{I} = \mathbf{0}$.

11 In Example 1 of **52.4**, 3 points are awarded for a win, 1 for a draw and 0 for a loss. Give this information in a 3×1 matrix. Use matrix multiplication to find out which of A, B, C or D has (i) most, (ii) fewest points after the 9 matches.

12 Using your answer to **Ex 52a Q11**, repeat Q11 to find which of A, B, C or D has (i) most, (ii) fewest points after the 13 matches.

13 A factory makes three products X, Y and Z. The table shows the units of labour, materials and other items needed to produce one of each product.

	Labour	Materials	Other items
X	4	1	2
Y	2	4	1
Z	1	5	2

(i) Represent this data by matrix **P**.
(ii) Labour costs sh200 per unit, materials sh300 per unit and other items cost sh500 per unit. Represent this data by a column matrix **Q**.
(iii) Find **PQ** and hence state the cost of each product X, Y and Z in shillings.

52.6 Inverse of a Matrix

If $\mathbf{A} = \begin{pmatrix} 8 & 5 \\ 3 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & -5 \\ -3 & 8 \end{pmatrix}$ the product **AB** is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or **I** the identity matrix, ie. $\mathbf{AB} = \mathbf{I}$.
Matrix **B** is the **inverse** of **A** for multiplication, and **A** is the inverse of **B**.
The inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{(ad - bc)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, as can be verified by multiplication.
The expression $ad - bc$ is called the **determinant** of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
The inverse of **M** is denoted by $\mathbf{M}^{-1}$ so we write $\mathbf{MM}^{-1} = \mathbf{I}$ or $\mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$.
A matrix whose determinant is zero has no inverse. Why? Such a matrix is called **singular**.
The matrix $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is called the **adjunct** of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Exercise 52c

Find the inverse of each matrix in 1 to 7.

1 $\begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix}$ 2 $\begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix}$ 3 $\begin{pmatrix} 4 & 7 \\ 5 & 9 \end{pmatrix}$ 4 $\begin{pmatrix} 7 & 2 \\ 4 & 1 \end{pmatrix}$ 5 $\begin{pmatrix} 4 & 2 \\ 8 & 4 \end{pmatrix}$ 6 $\begin{pmatrix} -3 & 5 \\ -4 & 2 \end{pmatrix}$ 7 $\begin{pmatrix} 6 & 4 \\ -9 & -6 \end{pmatrix}$

8 If $\begin{pmatrix} n & 4 \\ 16 & n \end{pmatrix}$ has no inverse find possible values of n.

9 If $\begin{pmatrix} n+6 & n \\ 5 & 2 \end{pmatrix}$ has no inverse, find the value of n.

10 If $\begin{pmatrix} n+4 & 3 \\ 4 & n \end{pmatrix}$ is singular, find possible values of n.

11 Find the inverse of $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

52.7 Simultaneous Equations in Two Unknowns

(i) The simultaneous equations

$$5x + 4y = 13$$
$$2x + 3y = 8$$

may be written in matrix form as $\begin{pmatrix} 5 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 \\ 8 \end{pmatrix}$

To solve the equations, **premultiply** both sides by the adjunct of $\begin{pmatrix} 5 & 4 \\ 2 & 3 \end{pmatrix}$ the **coefficient matrix.**

$$\begin{pmatrix} 3 & -4 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 5 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 13 \\ 8 \end{pmatrix}$$
$$\begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \end{pmatrix}$$

This gives $7x = 7$ and $7y = 14$, hence $x = 1$ and $y = 2$.

(ii) The **system** of equations

$$5x + 4y = 13$$
$$2x + 3y = 8$$

when written in matrix form as above, can be solved by premultiplying by the inverse of the coefficient matrix $\begin{pmatrix} 5 & 4 \\ 2 & 3 \end{pmatrix}$.

$$\text{This is } \frac{1}{5\times3 - 4\times2}\begin{pmatrix} 3 & -4 \\ -2 & 5 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & -4 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} \frac{3}{7} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{5}{7} \end{pmatrix}$$

We have $$\begin{pmatrix} 5 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 \\ 8 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} \frac{3}{7} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{5}{7} \end{pmatrix}\begin{pmatrix} 5 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3}{7} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{5}{7} \end{pmatrix}\begin{pmatrix} 13 \\ 8 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

This gives $x = 1$ and $y = 2$, as before.

(iii) For the equations $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$, the coefficient matrix $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ has no inverse since its determinant is zero. This means there is no single (**unique**) solution. Try to solve the simultaneous equations $x + 2y = 5$ and $3x + 6y = 4$ graphically (see **22.2**). It will be found that the lines $\{(x, y) : x + 2y = 5\}$ and $\{(x, y) : 3x + 6y = 4\}$ are parallel. Hence there is no solution since the lines do not intersect. The equations are said to be **inconsistent**.

(iv) For the equations $\begin{pmatrix}1 & 2\\3 & 6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3\\9\end{pmatrix}$, the coefficient matrix $\begin{pmatrix}1 & 2\\3 & 6\end{pmatrix}$ is singular. However, the second equation $3x + 6y = 9$ is three times the first $x + 2y = 3$. The geometrical interpretation of this is that each equation corresponds to the same line. The system is **consistent** and the lines are **coincident** thus giving many solutions, for example (1, 1), (5, −1), (3, 0), …

Exercise 52d

In **1** to **4** use the adjunct matrix method to solve for x and y.

1 $\begin{pmatrix}2 & 1\\3 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}5\\10\end{pmatrix}$

2 $\begin{pmatrix}1 & -1\\1 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}$

3 $\begin{pmatrix}1 & 5\\3 & 14\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}4\\11\end{pmatrix}$

4 $\begin{pmatrix}17 & 11\\10 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-6\\-8\end{pmatrix}$

In **5** to **8** use the inverse matrix method to solve for x and y.

5 $4x - 3y = 5$
$3x - 2y = 4$

6 $x + y = 10$
$x - y = 3$

7 $8x + 7y = 30$
$7x + 6y = 26$

8 $4x - y = -5$
$2x + 5y = 3$

In **9** and **10** solve for x and y and interpret your result geometrically.

9 $4x - 6y = 10$
$2x - 3y = 5$

10 $2x + 5y = 4$
$4x + 10y = 6$

53 TRANSFORMATIONS

53.1 Introduction

A geometrical **transformation** changes a given figure, called the **object**, in a certain defined way. The result of this change gives another figure called the **image**. We say an object is **mapped onto** its image. According to the particular transformation, an object may change in position, size or shape. Any points, lines, angles or other features which do not change in a transformation are called **invariant**. A transformation which does not change the size or shape of an object, ie. one in which the object and image are congruent, is called an **isometry**. Translation, reflection and rotation are isometries and have been introduced in Topics **38**, **41** and **42** respectively. The questions of the first three Exercises revise these transformations.

53.2 Translation

Exercise 53a

1 Translate the triangle PQR whose vertices are P(1, 2), Q(1, 4) and R(2, 4) by the column vector $\binom{3}{-1}$ and give the corresponding coordinates of the image.

2 The point (1, 2) becomes (3, 5) under a translation. State the column vector which describes this translation.

3 Plot the points A(1, 1), B(3, 1) and C(3, 2) and its image A′B′C′ after a translation $\binom{5}{2}$.

4 Give A′B′C′ in **Q3** a translation of $\binom{-3}{1}$ and state the coordinates of its image A″B″C″.

5 What single translation maps triangle ABC of **Q3** onto triangle A″B″C″ of **Q4**?

53.3 Reflection

Exercise 53b

1 Reflect the point (1, −2) in the mirror line $y = 0$ and state the coordinates of the image.

2 Reflect A(2, 4) in the line $\{(x, y) : y = x\}$ to give A′ and then in the line $\{(x, y) : x = 0\}$ to give A″. State the coordinates of A′ and A″.

3 Reflect the point (1, 2) successively in the mirror lines given by the equations $x + y = 0$, $x = 0$ and $y = 1$ and give the coordinates of the final image.

4 Plot the points (1, 1), (3, 1) and (3, 2) and join them to form a triangle. Reflect the triangle successively in the lines whose equations are $y = 0$, $x = 0$, $y = 1$ and $x = 2$ and give the coordinates of the final image.

5 What single translation takes the triangle of **Q4** from its initial to its final position?

53.4 Rotation

Exercise 53c

1 Find the coordinates of the point P(4, 2) after a positive quarter-turn (anti-clockwise rotation of 90°) about O, the origin.

2 Give Q(4, 2) a negative quarter-turn (clockwise rotation of 90°) about (1, 1) and state the coordinates of the image.

3 Draw triangle ABC with A(1, 1), B(3, 1) and C(3, 2) on squared paper. Give the triangle a half-turn about the origin O and state the coordinates of the vertices of the image after the rotation.

4 Give triangle ABC of Q3 a positive quarter-turn about the point (1, 1) and state the coordinates of the vertices of the image.

5 Points A(2, 1), B(4, 1) and C(4, 2) are given a rotation such that the image points are A'(−1, 0), B'(−1, 2) and C'(−2, 2). Join AA' and BB' and construct their mediators to locate the centre of rotation, R. Measure ∠ARA' and hence give the angle of rotation.

53.5 Enlargement

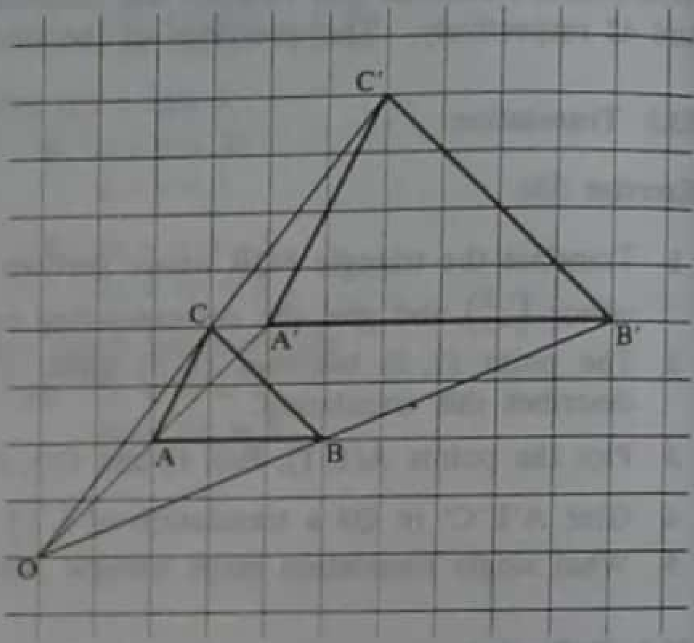
This transformation changes the size and position but not the shape of an object. Thus an object and its image are **similar** (see Topic **37**).

The diagram illustrates an **enlargement** in which △ABC (the object) is mapped onto △A'B'C' (the image). Point O is the **centre of enlargement**. Note that all lines through corresponding points, such as AA', pass through O. By drawing these lines the centre of enlargement is determined.

The **linear scale factor** (LSF) of an enlargement is the ratio of corresponding lengths of the image and object respectively (see **37.1**).

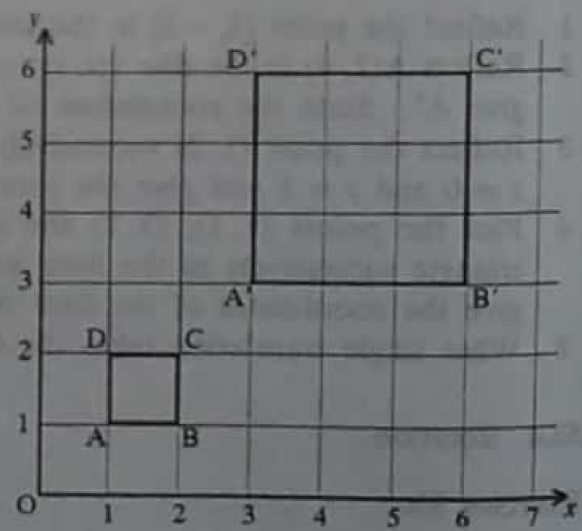
For this enlargement: LSF $= \frac{A'B'}{AB} = \frac{6}{3} = 2$.

The LSF is also given by the ratio of corresponding distances from the centre of enlargement, for example $\frac{OA'}{OA} = 2$.



Exercise 53d

1 In the diagram, the square ABCD is mapped onto A'B'C'D' by an enlargement.
(i) What are the coordinates of the centre of enlargement?
(ii) State the linear scale factor.

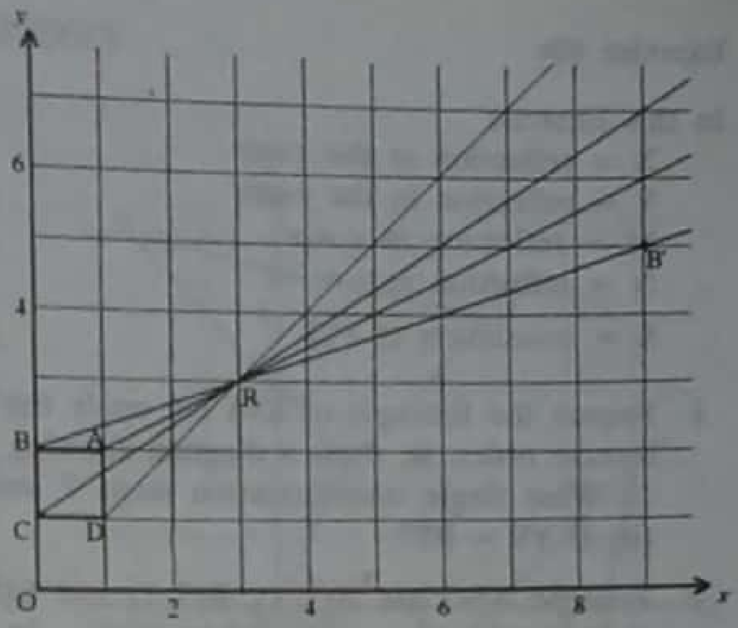


2 A(1, 4), B(3, 4), C(3, 2), D(1, 2) maps onto A'(2, 7), B'(6, 7), C'(6, 3), D'(2, 3) under an enlargement. Find the centre of the enlargement and the linear scale factor.

3 Find the coordinates of the image of A(1, −1), B(1, 1), C(2, 1) after an enlargement of LSF 3 and centre (0, 1).

4 Plot points A(0, 0), B(9, 0), C(9, 9), D(0, 9). Find the coordinates of A', B', C' and D' where A'B'C'D' is the image of ABCD under an enlargement of LSF $\frac{1}{3}$, centre (3, 3).

5 Square ABCD, shown in the diagram, is to be transformed by a **negative enlargement**, centre R(3, 3). The image of B(0, 2) is given as B'(9, 5) as shown.
(i) Copy the diagram on squared paper and complete the image A'B'C'D'.
(ii) State the linear scale factor of this enlargement.



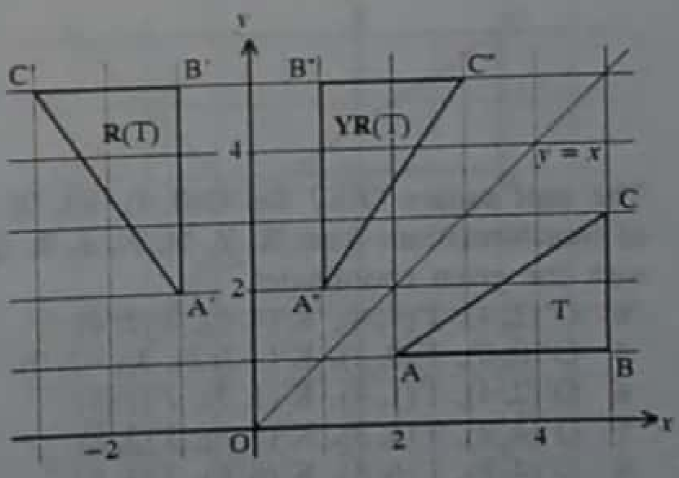
6 $\triangle ABC$ with A(1, 3), B(4, 3), C(4, 1) is mapped onto $\triangle A'B'C'$ by an enlargement with LSF -1, centre O(0, 0).
(i) Draw a diagram on squared paper showing the object and image.
(ii) Describe fully another transformation which would map $\triangle ABC$ onto $\triangle A'B'C'$.

53.6 Combinations of Transformations

Suppose a triangle T undergoes a transformation **P** followed by another transformation **Q**. The image of T under transformation **P** is denoted by **P**(T). The image of **P**(T) under transformation **Q** is denoted by **QP**(T). Note that **QP** means **P** followed by **Q**.

Example A triangle T which has its vertices at A(2, 1), B(5, 1) and C(5, 3) undergoes a transformation **R** followed by a transformation **Y**, where **R** is a positive quarter-turn about the origin and **Y** is a reflection in the y-axis.
(i) Draw a diagram showing T, **R**(T) and **YR**(T).
(ii) Find the single transformation **M** which maps T onto **YR**(T).

(i) The object and images are shown in the diagram on the right.



(ii) Since the triangles T and **YR**(T) are oppositely congruent (see **41.1**) the transformation which maps T onto **YR**(T) must be a reflection.
To find the mirror line, use the method of **41.1 Example 2**. The mediator of the line joining corresponding points, for example AA', is the line whose equation is $y = x$.
Hence $y = x$ is the mirror line.

$\therefore$ **M** is a reflection in $y = x$.

Note we may write **M** = **YR**.

Exercise 53e

In this Exercise

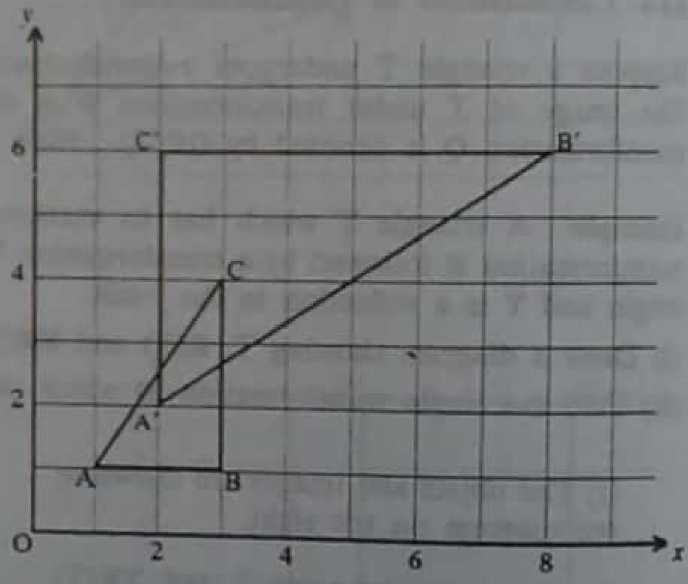
X = reflection in the x-axis
Y = reflection in the y-axis
M = reflection in $y = x$
N = reflection in $y = -x$
S = translation of $\binom{2}{4}$
R = rotation of $+90°$ about $(0, 0)$
Q = rotation of $-90°$ about $(0, 0)$
H = rotation of $180°$ about $(0, 0)$
E = enlargement, centre $(0, 0)$, LSF 2

1 Repeat the Example of **53.6** but apply the transformations **R** and **Y** to triangle T in the reverse order, ie. draw a diagram showing T, **Y**(T) and **RY**(T).
(i) What single transformation maps T onto **RY**(T)?
(ii) Is **YR** = **RY**?

2 Triangle ABC has A(2, 1), B(5, 1) and C(5, 3). Draw △ABC and its image under the given combination of transformations. In each case write down the coordinates of the final image and, except in parts (v) and (vi), describe the single transformation which is equivalent to the given combination.

(i) **XR**	(ii) **RX**	(iii) **HM**	(iv) **MH**
(v) **EQ**	(vi) **QE**	(vii) **SH**	(viii) **HS**
(ix) $\mathbf{R}^2$ (ie. **RR**)	(x) $\mathbf{X}^2$	(xi) **HSH**	(xii) **EHE**

3 In the diagram, give the combination of transformations from **X, Y, M, N, S, R, Q, H, E** which would map △ABC onto △A'B'C'.

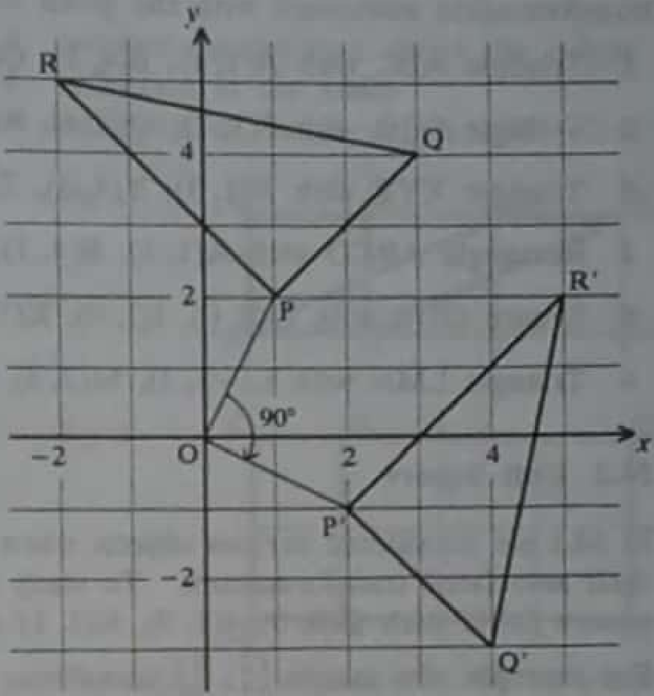


The **unit square** OIKJ has O(0, 0), I(1, 0), K((1, 1), J(0, 1). In **4** to **8** give the combination of transformations from **X, Y, M, N, S, R, Q, H, E** which would map OIKJ onto a final image with the given coordinates.

4 O'(2, 4), I'(4, 4), K'(4, 6), J'(2, 6)
5 O'(0, 0), I'(0, 2), K'(−2, 2), J'(−2, 0)
6 O'(2, 4), I'(1, 4), K'(1, 5), J'(2, 5)
7 O'(4, 2), I'(4, 3), K'(5, 3), J'(5, 2)
8 O'(4, 8), I'(6, 8), K'(6, 6), J'(4, 6)

54.1 The 2 × 2 Transformation Matrix

Consider the triangle PQR shown in the diagram. In **52.4 Example 2** we saw how a matrix is used to store coordinates of points. For the points P(1, 2), Q(3, 4) and R(−2, 5) the matrix is $\begin{pmatrix}1 & 3 & -2\\2 & 4 & 5\end{pmatrix}$. The columns of this matrix $\begin{pmatrix}1\\2\end{pmatrix}$, $\begin{pmatrix}3\\4\end{pmatrix}$ and $\begin{pmatrix}-2\\5\end{pmatrix}$ give the position vectors (see **38.6**) of P, Q and R respectively.



Now consider the following matrix product (see **52.5**):

$$\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}1 & 3 & -2\\2 & 4 & 5\end{pmatrix} = \begin{pmatrix}2 & 4 & 5\\-1 & -3 & 2\end{pmatrix}$$

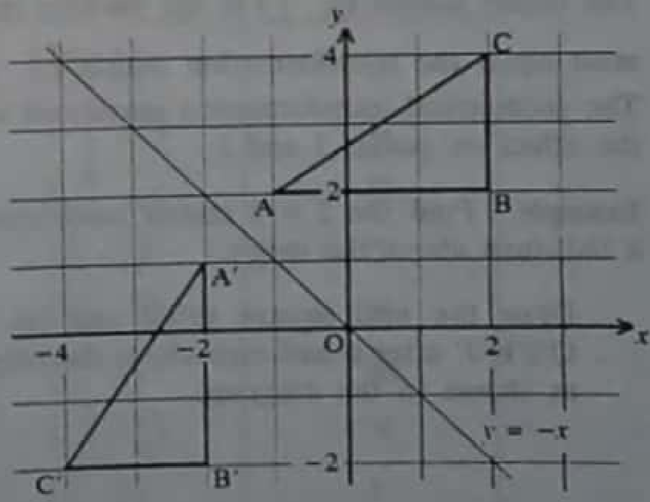
The matrix $\begin{pmatrix}1 & 3 & -2\\2 & 4 & 5\end{pmatrix}$ has been premultiplied by the 2 × 2 matrix $\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$.

The product matrix $\begin{pmatrix}2 & 4 & 5\\-1 & -3 & 2\end{pmatrix}$ stores the coordinates of the points (2, −1), (4, −3) and (5, 2). These points are marked in the diagram and are labelled P′, Q′ and R′ respectively. Upon studying the diagram we see that △PQR is mapped onto △P′Q′R′ by a negative quarter-turn about the origin, O.

The 2 × 2 matrix used for premultiplication, ie. $\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$ is ***associated*** with a negative quarter-turn about the origin.

In general, a 2 × 2 matrix is associated with a transformation in the x-y plane in which the origin is invariant.

Example By considering the △ABC with A(−1, 2), B(2, 2), C(2, 4), find the transformation associated with the matrix $\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$.



Draw △ABC on a grid as shown

The matrix which describes points A, B and C is $\begin{pmatrix}-1 & 2 & 2\\2 & 2 & 4\end{pmatrix}$

Premultiply this matrix by $\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$:

$$\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}\begin{array}{c}\begin{matrix}\mathrm{A} & \mathrm{B} & \mathrm{C}\end{matrix}\\\begin{pmatrix}-1 & 2 & 2\\2 & 2 & 4\end{pmatrix}\end{array} = \begin{array}{c}\begin{matrix}\mathrm{A'} & \mathrm{B'} & \mathrm{C'}\end{matrix}\\\begin{pmatrix}-2 & -2 & -4\\1 & -2 & -2\end{pmatrix}\end{array}$$

(*Note* It is useful to place A, B, ... above corresponding columns as shown.)

The product matrix gives the points A′(−2, 1), B′(−2, −2), C′(−4, −2) and △A′B′C′ is drawn as shown.

Study the object and image to determine the transformation.

The transformation associated with $\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$ is a reflection in $y = -x$.

Exercise 54a

Draw the given object and corresponding image on squared paper and hence determine the transformation associated with the given matrix.

1 Triangle ABC with A(1, 1), B(4, 1), C(4, 3); matrix $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

2 Triangle PQR with P(1, 2), Q(3, 4), R(−2, 5); matrix $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$

3 Triangle XYZ with X(1, 3), Y(4, 3), Z(4, 1); matrix $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$

4 Rectangle ABCD with A(1, 1), B(4, 1), C(4, 3), D(1, 3): matrix $\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$

5 Square OIJK with O(0, 0), I(1, 0), K(1, 1), J(0, 1); matrix $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$

6 Triangle LMN with L(−1, 2), M(2, 3), N(2, −1); matrix $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

54.2 Unit Square

In **54.1** we considered various objects when studying the relation between 2 × 2 matrices and their associated transformations. To study further this relationship we use as object the **unit square** OIKJ with O(0, 0), I(1, 0), K(1, 1) and J(0, 1).

For example, the matrix $\begin{pmatrix}2 & 3\\ 4 & 5\end{pmatrix}$ transforms the unit square as follows:

$$\begin{pmatrix}2 & 3\\ 4 & 5\end{pmatrix}\begin{array}{c}\begin{matrix}\mathrm{O} & \mathrm{I} & \mathrm{K} & \mathrm{J}\end{matrix}\\ \begin{pmatrix}0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\end{pmatrix}\end{array} = \begin{array}{c}\begin{matrix}\mathrm{O'} & \mathrm{I'} & \mathrm{K'} & \mathrm{J'}\end{matrix}\\ \begin{pmatrix}0 & 2 & 5 & 3\\ 0 & 4 & 9 & 5\end{pmatrix}\end{array}$$

Note that $\mathbf{OI} = \begin{pmatrix}1\\ 0\end{pmatrix}$ the **unit vector** along the x-axis transforms into $\begin{pmatrix}2\\ 4\end{pmatrix}$, the first column of the transformation matrix and $\mathbf{OJ} = \begin{pmatrix}0\\ 1\end{pmatrix}$ the unit vector along the y-axis becomes $\begin{pmatrix}3\\ 5\end{pmatrix}$, the second column of this matrix. This becomes particularly clear if we consider the transformation of the points I and J only:

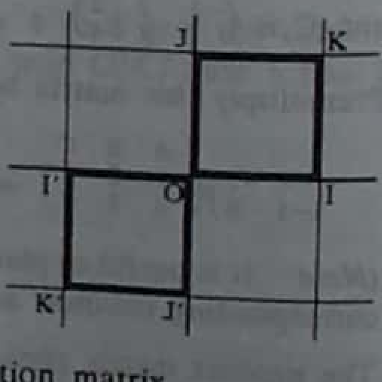
$$\begin{pmatrix}2 & 3\\ 4 & 5\end{pmatrix}\begin{array}{c}\begin{matrix}\mathrm{I} & \mathrm{J}\end{matrix}\\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\end{array} = \begin{array}{c}\begin{matrix}\mathrm{I'} & \mathrm{J'}\end{matrix}\\ \begin{pmatrix}2 & 3\\ 4 & 5\end{pmatrix}\end{array}$$

The object matrix $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ is the identity matrix, $\mathbf{I}_2$ (see **52.1**), and therefore the image matrix must equal the transformation matrix $\begin{pmatrix}2 & 3\\ 4 & 5\end{pmatrix}$.

The geometrical transformation associated with a 2 × 2 matrix can be established by examining the effect on points I and J.

Example Find the 2 × 2 matrix associated with a half-turn about the origin.

Draw the unit square OIKJ and its image O′I′K′J′ after a half-turn about the origin, O, as shown in the diagram.



$\mathbf{OI'} = \begin{pmatrix}-1\\ 0\end{pmatrix}$, which gives the first column of the transformation matrix.

$\mathbf{OJ'} = \begin{pmatrix}0\\ -1\end{pmatrix}$, which gives the second column of the matrix.

Hence the transformation matrix for a half-turn about the origin is $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$.

Exercise 54b

In 1 to 6, by drawing the unit square and its image, find the 2×2 matrix associated with the given transformation.

1 reflection in $y = x$

2 positive quarter-turn about the origin

3 negative quarter-turn about the origin

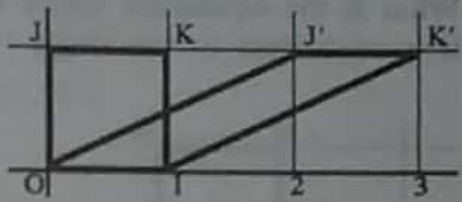
4 reflection in the x-axis

5 enlargement, centre origin, LSF 3

6 enlargement, centre origin, LSF -2

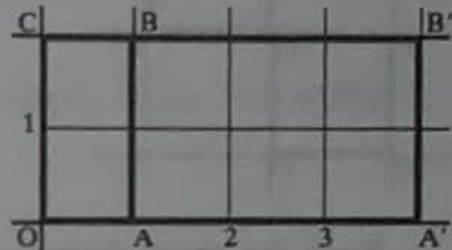
7 In Fig. 1, the unit square OIKJ is mapped onto parallelogram OIK'J' by a transformation called a **shear**.
Find the 2×2 matrix associated with this transformation.

Fig. 1



8 In Fig. 2, the rectangle OABC with O(0, 0), A(1, 0), B(1, 2) and C(0, 2) is mapped onto OA'B'C where A' is (4, 0) and B' is (4, 2) by a transformation called a **stretch**.
Find the matrix of this transformation.

Fig. 2



9 The matrix M where $M = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ can be used to effect a translation in the x-y plane by writing (x, y) as the column vector $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$. Thus $M\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x+a \\ y+b \\ 1 \end{pmatrix}$.
Show that $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ translates (3, 2) onto (5, 3) and use this method to find the image of $(-1, 4)$.

54.3 Combinations of Transformations

We shall use the following list of transformations and their associated matrices, each of which has been established in **54.1** and **54.2**.

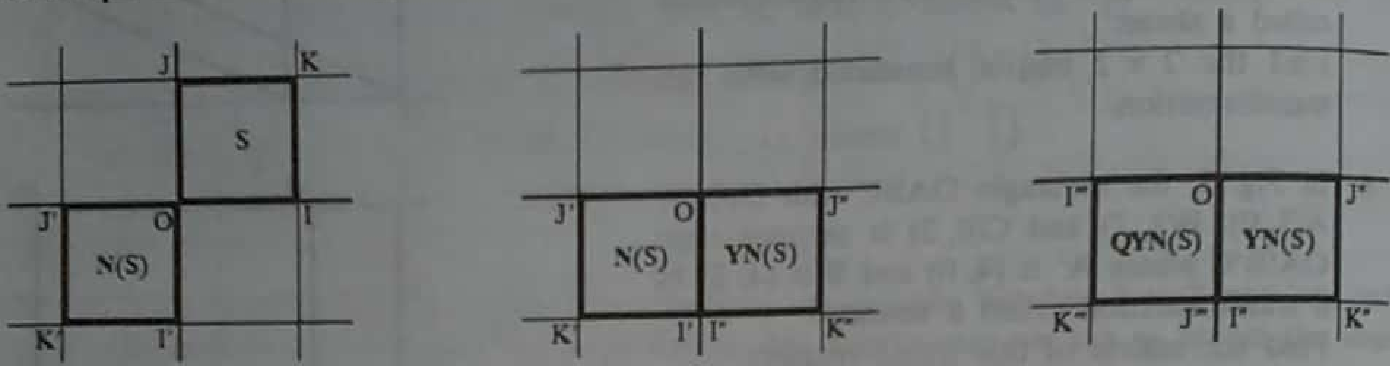
Transformation	Matrix	Symbol
Positive quarter-turn about O	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	**R**
Half-turn about O	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	**H**
Negative quarter-turn about O	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	**Q**
Reflection in x-axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	**X**
Reflection in y-axis	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	**Y**
Reflection in the line $y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	**M**
Reflection in the line $y = -x$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	**N**
Identity	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	**I**

Note The **identity transformation** is one in which an object is mapped onto itself. As an example, see **Exercise 54a Q6**. The object and image are identical in shape, size and position.
The identity matrix **I** is associated with the identity transformation.

An object may undergo two or more transformations (see 53.6). How are the associated matrices related? In the Example of 53.6 we see that **R** followed by **Y** is equivalent to **M** or **YR** = **M**. *This result is true for the associated matrices.*

$$\mathbf{YR} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathbf{M}$$

Example Find the image of the unit square, S, under the combined transformation **QYN**. What is the equivalent single transformation? Find the product matrix **QYN** and verify that it is equivalent to the single transformation.



From the above diagrams we see that the single transformation is a half-turn about O.

$$\mathbf{QYN} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \mathbf{H}$$

H is a half-turn about O.

When given two or more transformations to perform, it is often quicker to find the product of the associated matrices and use this product to determine the final image.

Exercise 54c

R, **H**, **Q**, **X**, **Y**, **M**, **N** and **I** refer to the table of transformations and matrices in **54.3**.

1 Draw a diagram showing the effect of **H** followed by **R** on the unit square OIKJ.
(i) What single transformation would produce the same result?
(ii) Find the product matrix **RH**.
(iii) Is the product matrix of (ii) associated with the single transformation of (i)?

2 Repeat Q1 for transformation **X** followed by **Y**.

In **3** to **10**, (i) draw diagrams showing the effect of the given combination of transformations on the unit square OIKJ, (ii) find the product matrix and draw a single diagram to show the effect of its transformation on the unit square. Compare your results.

3 $\mathbf{R}^2$ **4** $\mathbf{Q}^2$ **5** **YXM** **6** **MNH** **7** **RMN** **8** **RXQ** **9** **MQN** **10** **XYRQ**

54.4 Inverse Mappings

A transformation maps an object onto an image. The **inverse** of a transformation has the opposite effect. It maps the image back onto the object. For example, the inverse of **R**, a positive quarter-turn about the origin (O), is **Q**, a negative quarter-turn about the origin, and we write $\mathbf{Q} = \mathbf{R}^{-1}$. Similarly **R** is the inverse of **Q**, ie. $\mathbf{R} = \mathbf{Q}^{-1}$.

The matrix associated with **R**, a quarter-turn about O, is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The inverse of this matrix (see **52.6**) is $\frac{1}{1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which is the matrix associated with **Q**, a negative quarter-turn about O. You can check this from the list in **54.3**.

Thus, as with a combination of transformations, there is the same correspondence between a transformation and its inverse and the associated matrix and its inverse. The inverse matrix gives the inverse mapping.

In general, a transformation A followed by its inverse A^{-1} is equivalent to the indentity transformation, ie. $A^{-1}A = I$. Also $AA^{-1} = I$. The same applies to the corresponding matrices (see **52.6**).

Exercise 54d

For **1** and **2** refer to the list of transformations and matrices in **54.3**.

1 Draw the unit square S.

(i) Reflect S in the x-axis, ie. find X(S).

(ii) What transformation maps X(S) back onto S? What transformation is X^{-1}?

(iii) What matrix is associated with the transformation X^{-1}?

(iv) Find the matrix product $X^{-1}X$.

2 Repeat Q1 for the tranformations (a) H, (b) M, (c) enlargement, centre origin, LSF 2.

3 You are given the matrix A, where $A = \begin{pmatrix} 2 & -3 \\ 4 & -4 \end{pmatrix}$.

(i) Write down A^{-1}, the inverse of A. (For method, see **52.6**)

(ii) Use the inverse matrix found in (i) to find the coordinates of the point which has the point (9, 16) as its image under the transformation associated with A.

(iii) Find the image of (3, −1) under A.

(iv) Find the matrix A^2. Show that A^2 can be represented in the form kA^{-1} and hence state the value of k.

(v) Find A^3. Give a geometrical description of the transformation associated with the matrix A^3.

4 You are given the matrix M, where $M = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -1 & 7 \end{pmatrix}$.

(i) Draw and label the rectangle with vertices O(0, 0), A(10, 0), B(10, 5) and C(0, 5).

(ii) The points O, P, Q and R are the images of O, A, B and C under the transformation represented by the matrix M. Find, draw and label the quadrilateral OPQR on the same diagram as OABC.

(iii) Draw and label the line s which is invariant (see **53.1**) under the transformation effected by M. Give the equation of s.

(iv) Find the matrix representing the transformation which maps OPQR onto OABC.

(v) Deduce a 2 × 2 matrix representing the single transformation which maps the points O, P, Q and R onto O(0, 0), L(2, 0), M(2, 1) and N(0, 1) respectively.

(vi) Find the image of OPQR under the transformation represented by $\frac{1}{5}M^{-1}$.

5 Find the image of the unit square under the transformation associated with the matrix N, where $N = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$.

(i) Find the area of the image.

(ii) What is the determinant of N? (See **52.6**)

(iii) How is the ratio of the image area to the original area of the unit square related to the determinant of N? Is this a general result?

55 SETS (2)

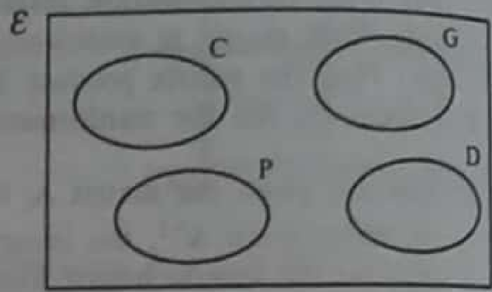
55.1 Universal Set

Consider the following sets:

C = {all cows on Bakara's farm} G = {all goats on Bakara's farm}
P = {all pigs on Bakara's farm} D = {all dogs on Bakara's farm}
A = {all animals on Bakara's farm}

Sets C, G, P and D are all subsets of A (see **5.4**). Set A is called a **universal set** since it contains all the other sets in this example. It may contain other sets such as {cats} and {sheep} providing these animals refer to those on Bakara's farm.

We shall use the symbol $\mathcal{E}$ to denote the universal set. The above information is summarised and illustrated in a Venn diagram (see **5.2**) as shown.



$\mathcal{E}$ = {all animals on Bakara's farm}
C = {all cows} G = {all goats}
P = {all pigs} D = {all dogs}

Notes 1. Once a universal set has been defined, the descriptions of its subsets may be abbreviated as shown. This is because {all cows}, for example, *must* refer to animals on Bakara's farm.

2. In the Venn diagram the universal set is represented by a *rectangle*.

3. In this example the sets C, G, P and D are **disjoint** sets (see **5.6**).

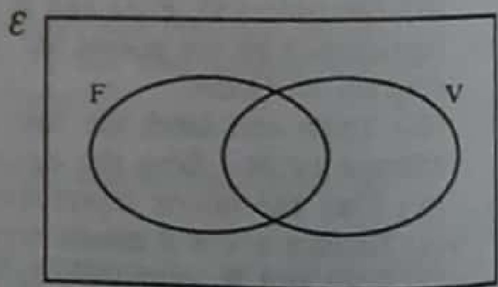
Example Draw a Venn diagram to illustrate the following sets, where F and V are both subsets of the universal set.

$\mathcal{E}$ = {all boys in Namilyango College who play school games}
F = {all footballers} V = {all volley-ball players}

Name two other possible universal sets for F and V.

The Venn diagram is shown on the right.
Note that, although the relation between sets F and V is not known, the **intersection** situation is assumed since it is possible that some boys play *both* games.
Two other universal sets for F and V are:

$\mathcal{E}$ = {all S.4 boys in Namilyango College}
$\mathcal{E}$ = {all ball-players in Uganda}



Exercise 55a

In 1 to 6, draw a Venn diagram to illustrate $\mathcal{E}$, A and B. Give two other possible universal sets for A and B, making them as different from each other as you can.

1 $\mathcal{E}$ = {all natural numbers from 1 to 20}
A = {all even numbers from 4 to 14}
B = {all odd numbers from 9 to 19}

2 $\mathcal{E}$ = {all S.3 pupils at Nabisunsa school}
A = {all pupils who like Mathematics}
B = {all pupils who like English}

3 $\mathcal{E}$ = {all polygons}
A = {all right-angled triangles}
B = {all isosceles triangles}

4 $\mathcal{E}$ = $\{x : 1 \le x \le 20\}$
A = {all multiples of 2}
B = {all multiples of 3}

5 $\mathcal{E}$ = {all teachers at Masindi school}
A = {all teachers who wear glasses}
B = {all teachers who wear size 9 shoes}

6 $\mathcal{E}$ = {all even numbers from 1 to 30}
A = {all numbers divisible by 4}
B = {all numbers divisible by 8}

In 7 to 12, (i) choose a universal set for P and Q, (ii) draw a Venn diagram for $\mathcal{E}$, P and Q.

7 P = {all multiples of 3 between 19 and 41}, Q = {all multiples of 4 between 19 and 41}

8 P = {all students taller than 175cm}, Q = {all students who weigh more than 60kg}

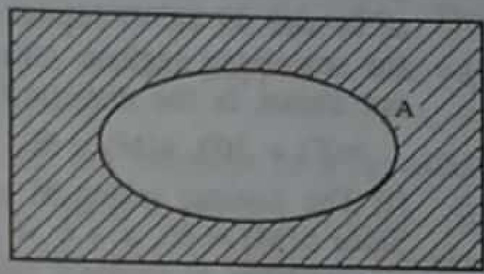
9 P = {all square numbers}, Q = {all triangle numbers} (See **16.2**)

10 P = {all squares}, Q = {all rectangles} **11** P = {all circles}, Q = {all pentagons}

12 P = {all cows with black markings}, Q = {all cows with white markings}

13 A certain school has 100 boys in S.4. Of these, 56 play chess, 67 play draughts and 35 play both these games. Define three sets, naming one of them as the universal set. Draw a Venn diagram entering the number of boys in each region (see Example of **5.5**). Hence find the number of boys who play neither game.

55.2 Complement



Consider sets $\mathcal{E}$ = {natural numbers}, A = {odd numbers}. The Venn diagram is shown on the right. What does the shaded region represent? It represents all the natural numbers which are not odd, ie. {even numbers}. This set is called the **complement** of A and is denoted by A'.

Generally, the complement of a set P (denoted by P' and read as *P complement*) contains all the elements of the universal set which do not belong to P.

Exercise 55b

In 1 to 5, describe or list where possible the complement of A.

1 $\mathcal{E}$ = {all cows on Lusiba's farm}, A = {all black cows}

2 $\mathcal{E}$ = {all countries in Africa}, A = {all countries in East Africa}

3 $\mathcal{E}$ = {all buses, lorries, cars, motorcycles, bicycles}, A = {all cars, bicycles}

4 $\mathcal{E}$ = {all pupils at Gayaza Girls High School}, A = {all girls}

5 $\mathcal{E}$ = {all numbers on the number line}, A = {all rational numbers} (See **26.2**)

In 6 to 10, $\mathcal{E}$ = {letters of the word *universal*}, ie. {u, n, i, v, e, r, s, a, l}. List the elements of the complement of the given set. A Venn diagram may help you.

6 P = {s, l, a, v, e} **7** Q = {r, u, i, n}

8 R = {l, e, a, r, n} ∩ {n, u, r, s, e} **9** S = {l, u, n, a, r} ∪ {s, i, l, v, e, r}

10 Simplify (i) $\mathcal{E}'$, (ii) ∅', (iii) (A')'.

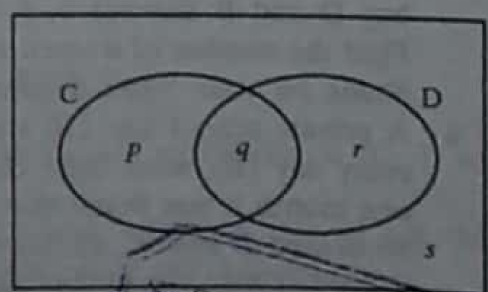
In 11 to 14, $\mathcal{E}$ = {the first 12 natural numbers}, E = {even numbers}, T = {triangle numbers}. Draw a Venn diagram to illustrate the given set and hence list its elements.

11 (E ∩ T)' **12** E' ∪ T **13** (E ∪ T)' **14** E' ∩ T

15 Study your answers to **11** to **14**. What conclusions can you make?

55.3 Problems

Example 1 Solve Ex **55a Q13** above.



$\mathcal{E}$ = {all S.4 boys in the school}

C = {all chess players} D = {all draughts players}

Draw the Venn diagram. The number of boys in each region is given a letter as shown.

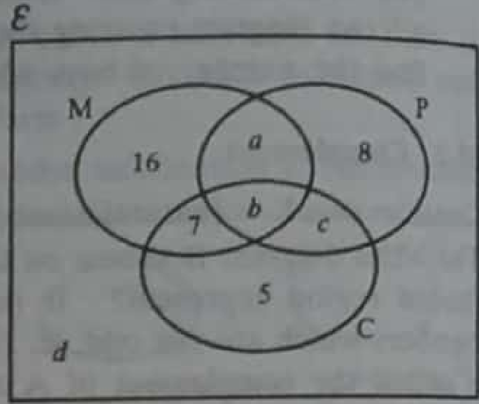
Since 35 boys play *both* games, we have $q = 35$.

$p = 56 - 35 = 21$ $r = 67 - 35 = 32$

$s = 100 - (21 + 35 + 32) = 100 - 88 = 12$

The number of boys who play neither game is 12.

Example 2 One year at St. Aloysius College, Nyapea there were 100 students in A-level taking various subject combinations. Of these, 60 students took Mathematics(M), 45 took Physics(P) and 40 Chemistry(C). Out of the 60 students taking M, 16 did neither P nor C. Of the 45 taking P, 8 did neither M nor C. Of those who took C, 5 took neither M nor P. Seven students took both M and C but not P.
Find the number who took (i) M and P, (ii) P and C but not M, (iii) all three subjects and (iv) none of the three subjects.



Draw the Venn diagram as shown with three mutually-intersecting sets. Use the given information to mark numbers of elements in the appropriate regions. Use a, b, and so on to represent unknown numbers in the remaining regions. The additional information not shown in the diagram is:

$n(\mathcal{E}) = 100$, $n(M) = 60$, $n(P) = 45$, $n(C) = 40$

(i) The number taking M and P is given by $a + b$.
Since $n(M) = 60$, $a + b = 60 - (16 + 7) = 37$.
The number of students taking M and P is 37.

(ii) Since $n(P) = 45$, $c = 45 - (a + b + 8) = 45 - (37 + 8) = 45 - 45 = 0$.
Hence there were no students who took P and C but not M.

(iii) Since $n(C) = 40$, $b = 40 - (c + 7 + 5) = 40 - (0 + 7 + 5) = 40 - 12 = 28$.
Therefore 28 students took all three subjects.

(iv) $d = 100 - (60 + 5 + 8 + c) = 100 - (60 + 13 + 0) = 100 - 73 = 27$
Thus 27 students took none of the three subjects.

Exercise 55c (Keep your solutions for use in Exercise **56b**.)

1 Out of 600 students at Kololo High School, 250 like swimming, 350 like basket-ball and 100 like both swimming and basket-ball. How many students like neither of these sports?

2 In a family of 15 people, 7 eat pork, 9 eat beef and 2 eat neither pork nor beef. Find the number of people in the family who eat both pork and beef.

3 At Mount St. Mary's School, Namagunga, the 120 girls in S.3 may opt to take Physics(P) and/or Chemistry(C). If 60 girls take both P and C, 35 do not take P and 44 do not take C, how many take neither P nor C? Also give (i) $n(P' \cap C)$, (ii) $n(P \cup C')$.

4 Twenty-eight pupils are asked which drink they like out of soda(S), tea(T) and passion fruit(P). It is found that 18 pupils like S, 10 like T and 20 like P. Also, 3 pupils like T and P but not S, 4 like P but neither S nor T and 5 like S but not T nor P. Given that 9 pupils like T and P, find the number who like (i) all three drinks, (ii) S and P but not T, (iii) S and T but not P, (iv) T only, (v) none of the three drinks.

5 One year, 120 women in Kampala were asked what they would buy for their Christmas shopping. From their responses it was found that: 60 women intended to buy dresses(D), 80 would buy belts(B), 40 would buy dresses and shoes(S), 30 would buy B and S, 7 would buy D and B but not S, 5 would buy S only and 4 would buy none of these items. Find the number of women who intended to buy (i) D only, (ii) B only, (iii) all three items. Shade on your Venn diagram the region $D \cap S \cap B'$ and state its number of elements.

6 A private school has 110 students. According to the school rules, students should bathe every day (B), wash their clothes every weekend (W) and iron their clothes (I). During one month it was found that: 96 students obeyed B, 103 obeyed W and 88 obeyed I. Also, 89 obeyed B and W, 82 obeyed B and I and 85 obeyed W and I. Given that every student obeyed at least one of the rules, find the number who obeyed all three. Give $n(B' \cap W' \cap I)$.

56 PROBABILITY

In this Topic you will need some coins and at least two dice. A die can be made by constructing a cube of edge 2cm (see **29.1**) and then marking each face with the numbers 1, 2, 3, 4, 5, 6 respectively.

56.1 Probability by Symmetry

Probability tells us how likely (or unlikely) a future event is to occur. If a die is rolled on a table top, it will stop so that one of the six numbers 1, 2, 3, 4, 5 or 6 will be face up. There are six **possible outcomes** and each is **equally likely** due to the **symmetry** of the cube. The **probability** of getting a 4, written p(4), is one sixth $\left(\frac{1}{6}\right)$ because this can happen in one way out of the six possible outcomes. Similarly, for example, $p(2) = \frac{1}{6}$.
In the above example, the six possible outcomes of rolling a die are members of a set called the **possibility space** (or **sample space**), denoted by $\mathcal{E}$ (see **55.1**). So $\mathcal{E} = \{1, 2, 3, 4, 5, 6\}$. To find the probability of a particular **event** occuring, p(5) say, we find a set E (which is a subset of $\mathcal{E}$) whose members give all ways in which the event can occur. Here $E = \{5\}$.

Now $n(E) = 1$ and $n(\mathcal{E}) = 6$. Hence $p(5) = \dfrac{n(E)}{n(\mathcal{E})} = \frac{1}{6}$, as before.

Name one side of a coin heads (H) and the other side tails (T). If the coin is tossed, either one of the two sides H or T could land face up. What is the probability of getting a head, p(H)?
Here $\mathcal{E} = \{H, T\}$ and $E = \{H\}$, so $p(H) = \dfrac{n(E)}{n(\mathcal{E})} = \frac{1}{2}$ Similarly $p(T) = \frac{1}{2}$

Example 1 If a person tosses a coin twice, what is the probability of getting (i) two heads, (ii) two tails, (iii) a head and a tail?

$\mathcal{E} = \{HH, HT, TH, TT\}$ so $n(\mathcal{E}) = 4$

(i) $E = \{HH\}$, so $n(E) = 1$ $\quad\therefore\quad$ p(two heads) $= \dfrac{n(E)}{n(\mathcal{E})} = \frac{1}{4}$

(ii) $E = \{TT\}$, so $n(E) = 1$ $\quad\therefore\quad$ p(two tails) $= \dfrac{n(E)}{n(\mathcal{E})} = \frac{1}{4}$

(iii) $E = \{HT, TH\}$, so $n(E) = 2$ $\quad\therefore\quad$ p(head and tail) $= \dfrac{n(E)}{n(\mathcal{E})} = \frac{2}{4} = \frac{1}{2}$

(*Note* These probabilities would be the same if two coins were tossed simultaneously.)

In the above Example we see that

$$p(HH) + p(H \text{ and } T) + p(TT) = \tfrac{1}{4} + \tfrac{1}{2} + \tfrac{1}{4} = 1$$

Therefore the sum of all the probabilities in the possibility space is 1.

All probabilities lie between 0 and 1. If the event is impossible, its probability is 0. If an event is certain to occur, its probability is 1.
Because the sum of all the probabilities in the posibility space is 1 it follows that, if x is the probability of an event happening, then $1 - x$ is the probability of it not happening.

Example 2 If a **fair** (or **unbiased**) die is rolled once on a table top, what is the probability of (i) getting a 4, (ii) not getting a 4, (iii) getting a number greater than 4.

There are six equally likely outcomes: $\mathcal{E} = \{1, 2, 3, 4, 5, 6\}$ and $n(\mathcal{E}) = 6$

(i) $E = \{4\}$ so $n(E) = 1$ $\quad\therefore\quad p(4) = \dfrac{n(E)}{n(\mathcal{E})} = \frac{1}{6}$

(ii) $p(\text{not } 4) = 1 - p(4) = 1 - \frac{1}{6} = \frac{5}{6}$

(iii) $E = \{5, 6\}$ so $n(E) = 2$ $\quad\therefore\quad p(5 \text{ or } 6) = \dfrac{n(E)}{n(\mathcal{E})} = \frac{2}{6} = \frac{1}{3}$

56.2 Probability by Survey

What is the probability that the Headmistress will be in her Office at 11am each day except Sunday? The only way of answering this question is to make a note every day for a week, or better still a month, at that time. Hence the probability is based on observation or **survey**, not symmetry. Suppose after making a note the Headmistress is in her Office at that time on 20 out of 24 days. Then the probability of her being in is $\frac{20}{24}$ or $\frac{5}{6}$, ie. $p(\text{In}) = \frac{5}{6}$

Exercise 56a

For each question state whether your estimate of the probability is based on symmetry or survey.

1 A fair die is thrown. What is the probability that the outcome is (i) an odd number, (ii) a number less than 3, (iii) 7?

2 I make a note of the number of days I get two or more letters and find it to be 7 out of 21. What is the probability I get two or more letters next Friday?

3 A box contains 5 red, 6 blue and 7 green beads. One is selected at **random**. What is the probability that it is (i) red, (ii) blue, (iii) green?
(*Random* means each bead is equally likely to be chosen.)

4 A letter is chosen at random from FORMAN AND NYAKAIRU. What is the probability that it is (i) a vowel, (ii) a consonant, (iii) a vowel or a consonant, (iv) the letter C?

5 Out of ten journeys, a taxi was stopped by the police on nine of them. What is the probability that on its next journey the taxi is stopped by the police?

6 A fair die is thrown 180 times. How many fives would you *expect* to obtain?

7 A school canteen operator notes that 5 out of every 7 pupils buying a soda ask for lemonade. What is the probability that the next pupil ordering a soda does *not* ask for lemonade?

8 The possibity space for the sum when two dice are rolled is partly shown.
Copy and complete it.
What is the probability of getting a score of 7?

9 Find the probability that the score when two dice are thrown is greater than 8.

6	7					12
5						
4			7			
3						
2		4				
1	2					
	1	2	3	4	5	6

10 Three dice are thrown together. Why are there 216 possible outcomes? Write down the different ways of obtaining a total of 5 (for example 1 + 1 + 3, 1 + 3 + 1, 2 + 2 + 1, ...). What is the probability of getting a total of 5 when three dice are thrown?

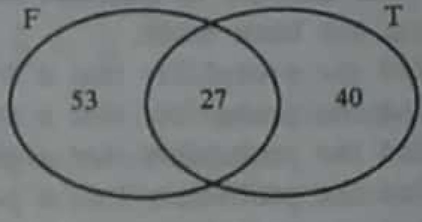
56.3 Probability using Venn Diagrams

Certain probabilities are more easily found using a Venn diagram as the following Examples show.

Example 1 In the Example of 5.5, what is the probability that a pupil chosen at random plays (i) tennis only, (ii) both sports?

First study the two methods of solution on Page 13.

The corresponding Venn diagram is shown on the right.



(i) E = {pupils who play tennis only} and $n(E) = 40$ $n(\mathcal{E}) = 120$

$\therefore$ p(pupils who play tennis only) $= \dfrac{n(E)}{n(\mathcal{E})} = \dfrac{40}{120} = \dfrac{1}{3}$

(ii) E = {pupils who play both sports} and $n(E) = 27$ $n(\mathcal{E}) = 120$

$\therefore$ p(pupils who play both sports) $= \dfrac{n(E)}{n(\mathcal{E})} = \dfrac{27}{120} = \dfrac{9}{40}$

Example 2 In a class of 36 pupils, the probability of a pupil chosen at random being over 1·8m tall is 0·25 and that of a pupil being 16 years old or over is 0·5. Given that there are 3 pupils who are under sixteen but over 1·8m tall, find the probability that a pupil chosen at random is under sixteen but not over 1·8m tall.

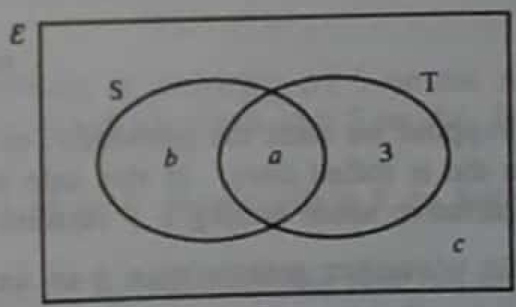
Let $\mathcal{E}$ = {all pupils in the class}
T = {pupils over 1·8m tall}
S = {pupils 16 years old or over}

Now $0{\cdot}25 = \frac{1}{4} = \frac{9}{36}$ and $0{\cdot}5 = \frac{1}{2} = \frac{18}{36}$

From this we have $n(T) = 9$ and $n(S) = 18$

Also $n(\mathcal{E}) = 36$ and $n(T \cap S') = 3$

Draw the Venn diagram as shown.



Since $n(T) = 9$, $a = 9 - 3 = 6$ Since $n(S) = 18$, $b = 18 - 6 = 12$

Since $n(\mathcal{E}) = 36$, $c = 36 - (12 + 6 + 3) = 36 - 21 = 15$

$\therefore$ p(under 16 but not over 1·8m tall) $= \frac{15}{36} = \frac{5}{12}$

Exercise 56b

1 Out of 180 Senior 6 students, 130 wore grey trousers on a certain day, 110 wore a pair of black shoes and all students wore at least one of these items. Find the probability that a student chosen at random was wearing both grey trousers and black shoes.

2 Work through **Exercise 5c Q11**. Find the probability that a pupil selected at random passed in (i) both subjects, (ii) neither subject.

3 Work through **Exercise 5d Q12**. Find the probability that an animal chosen at random from Enwaku's herd has (i) no white markings, (ii) no black markings.

4 A natural number is chosen at random from $\{x : 1 \le x \le 20\}$. Find the probability that it is (i) a multiple of 2, (ii) a multiple of 3, (iii) a multiple of 2 or 3. What is the probability that the number chosen is divisible by 6?

5 A pupil is to be selected at random from an S.4 class. The probability that this pupil will be amongst the top ten in a recent Mathematics quiz for this class is 0·25. How many pupils are in the class?

6 A pupil is selected at random from a class of 40. The probability that the pupil is a girl is 0·5. The probability that the pupil likes Mathematics is 0·9. If one boy does not like Mathematics, how many girls do like Mathematics?

From your answers to **Exercise 55c Q1** to **6**, answer **7** to **12** respectively, in which a random selection has been made.

7 Find the probability that a student likes neither sport.

8 Find the probability that a member of the family eats both pork and beef.

9 Find the probability that a girl takes neither Physics nor Chemistry.

10 Find the probability that a pupil likes none of the three drinks.

11 Find the probability that a woman intends to buy all three items.

12 Find the probability (correct to 2 dp) that a student obeys all three rules.

56.4 Mutually Exclusive Events

In Example 2 of **56.1**, p(5 or 6) = $\frac{1}{3}$. This could have been obtained by adding the separate probabilities:

$$p(5 \text{ or } 6) = p(5) + p(6) = \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{2}{6} = \tfrac{1}{3}$$

This only works because the events of getting a 5 or a 6 when a die is rolled once cannot both occur at the same time, ie. they are **mutually exclusive** events.

If events A and B are mutually exclusive we use the **addition rule** to give

$$p(A \text{ or } B) = p(A) + p(B)$$

Suppose we want the probability of obtaining a number greater than 4 or an odd number when a die is rolled once. In this case we cannot use the above rule. The events are not mutually exclusive since getting a 5 satisfies *both* conditions.

So p(number greater than 4 or an odd number) = p(1, 3, 5 or 6) = $\frac{4}{6} = \frac{2}{3}$

Exercise 56c

1 A fair die is thrown. Calculate the probability of getting (i) an even number, (ii) a 5, (iii) an even number or a 5.

2 A single digit is chosen at random from 1, 2, 3, 4, 5, 6, 7, 8 and 9. What is the probability of getting (i) a number divisible by 3, (ii) a number divisible by 5, (iii) a number divisible by 3 or 5?

3 Two dice are thrown and their sum noted. Find the probability of getting (i) the sum an even number, (ii) the sum an odd number, (iii) the sum an even or an odd number.

4 A box contains 4 red, 3 blue and 3 green beads. What is the probability of selecting at random (i) a blue bead, (ii) a red bead, (iii) a blue or a red bead?

5 The probability that Waibale wins the 100m race is $\frac{1}{4}$ and the probability that Mubiru wins it is $\frac{1}{3}$. What is the probability that Waibale or Mubiru wins the 100m race in which they both compete?

6 Two boxers, Kakwezi and Byara, are due to fight. The outcome of the contest can be a win for either boxer or it can be a draw. The probability that Kakwezi will win is 0·5 and that of Byara winning is 0·3. What is the probability that the match will be drawn?

56.5 Independent Events

A coin is tossed and a die rolled. The twelve equally likely outcomes are:

H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6

The probability of getting a Head *and* a Six is $\frac{1}{12}$, ie. p(H and 6) = $\frac{1}{12}$.
This could have been obtained by multiplying the separate probabilities:

$$p(\text{H and } 6) = p(\text{H}) \times p(6) = \tfrac{1}{2} \times \tfrac{1}{6} = \tfrac{1}{12}$$

This works because the obtaining of a Head does not influence, nor is it connected in any way with a Six showing on the die. These events are called **independent** events.
If C and D are independent events then the probability that both C and D occur is given by the **product rule**:

$$p(\text{C and D}) = p(\text{C}) \times p(\text{D})$$

Example Box A contains 4 red and 3 green beads and Box B contains 5 red and 2 green beads. Charlotte selects a box and then a bead at random from that box. She is twice as likely to choose Box B as Box A. What is the probability that she selects a green bead?

Charlotte may choose [(Box B *and* a green bead)] *or* [(Box A *and* a green bead)]

$$\therefore \quad p(\text{Green}) = [p(\text{Box B}) \times p(\text{Green})] + [p(\text{Box A}) \times p(\text{Green})]$$
$$= \left(\tfrac{2}{3} \times \tfrac{2}{7}\right) + \left(\tfrac{1}{3} \times \tfrac{3}{7}\right) = \tfrac{4}{21} + \tfrac{3}{21} = \tfrac{7}{21} = \tfrac{1}{3}$$

Exercise 56d

1 Die A and die B are thrown together. Find the probability of obtaining (i) 5 on die A and 5 on die B, (ii) 5 on die A and not 5 on die B, (iii) not 5 on die A and 5 on die B, (iv) one 5 only (*Hint* use (ii) and (iii)).

2 A die is thrown twice. Find the probability of (i) a 6 on both throws, (ii) a 5 on both throws, (iii) the same number on both throws.

3 A person chosen at random is equally likely to have his or her birthday this year on any day of the week. (i) What is the probability that it falls on a Monday? (ii) For two people, state the probability that both have their birthdays on a Monday. (iii) For two people, state the probability that they have their birthdays on the same day of the week. (iv) What is the probability of their birthdays not being on the same day of the week?

4 Two dice are thrown. Find the probability that (i) the total score is 7, (ii) the total score is 7 or 11, (iii) the total score is neither 7 nor 11.
In a game, if a player gets a total score of 7 or 11 with one throw of two dice it is called a win. If a player throws the dice twice, calculate the probability that (iv) he wins on both occasions, (v) he wins on neither occasion, (vi) he wins at least once, (vii) he wins just once. (viii) How are your answers to (iv), (v) and (vii) related?

5 In an examination room, 68 candidates were taking Art, 40 Music and 36 French. Previous results indicated 75% would pass Art, 65% Music and 50% French. (i) If a candidate had been picked at random, find the probability that (a) he took Art, (b) he took French and passed, (c) he took Music and failed, (d) he passed his subject whatever subject he took. (ii) Find how many candidates were expected to fail their examination.

6 Box A contains 5 red and 4 blue beads and Box B contains 3 red and 6 blue beads. Rebecca selects a box and then a bead at random from that box. If she picks Box A three times out of four, calculate the probability of her picking a blue bead.

57 LOCI

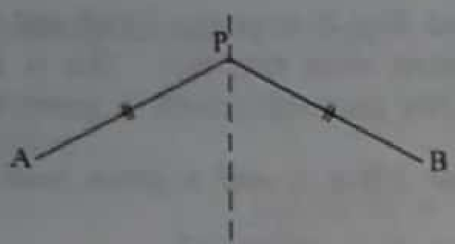
57.1 Definition

A **locus** is a set of points which satisfy a given condition. It can be regarded as a path traced out by a moving point. The locus of a point which moves so that it is always 3cm from a fixed point is a circle in the plane; a sphere in three dimensions. The plural of locus is loci.

57.2 Loci in a Plane

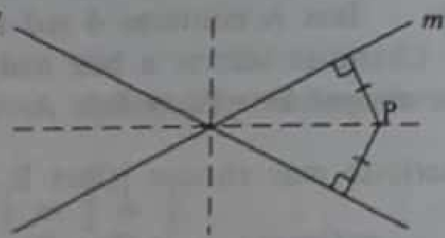
Some of the standard loci are illustrated and described below. In the diagrams A and B are fixed points; l and m are fixed lines; d is a fixed distance; θ is a fixed angle. In each case the set defining the locus is given and the locus itself is shown as a broken line(s) or curve.

(i) {all points P : AP = BP}



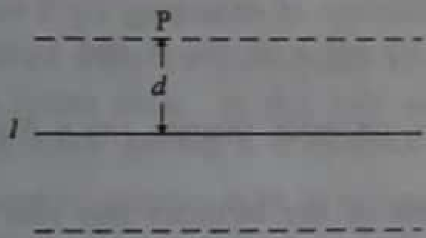
The locus of P is the mediator (perpendicular bisector) of AB.

(ii) {P : P is equidistant from l and m}



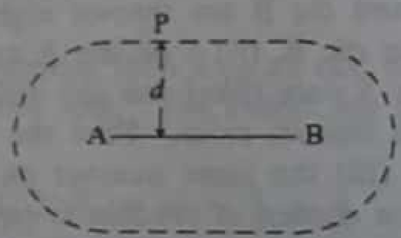
The locus of P is the angle bisectors of the angles made by l and m.

(iii) {P : P is a fixed distance from l}



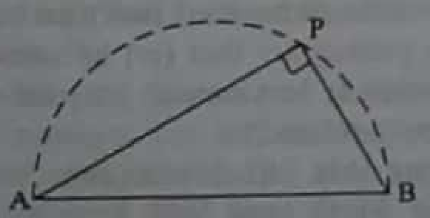
The locus of P is two lines parallel to l.

(iv) {P : P is a fixed distance from $\overline{AB}$}



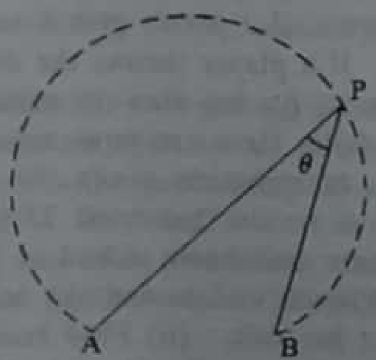
The locus of P is similar to a running track.

(v) {P : $\angle APB = 90°$}



The locus of P is a semi-circle with AB as diameter.
(The complete locus is a full circle.)

(vi) {P : $\angle APB = \theta$}



The locus is an arc of a circle. This is called the *Constant Angle Locus*.

Exercise 57a

In 1 to 5, A and B are fixed points 4cm apart in a plane. In each case construct accurately the complete locus and describe it.

1 {P : AP = 5cm} 2 {P : BP = 3cm} 3 {P : PA = PB}
4 {P : ∠APB = 90°} 5 {P : ∠APB = 30°} (*Hint* ∠ at centre = 2 × 30 = 60°)

In **6** to **10** the locus of P is a **region**. Illustrate the locus by shading the region. (Points A and B are as in **1** to **5**.)

6 {P : AP < 5cm} 7 {P : BP > 3cm} 8 {P : PA < PB}
9 {P : ∠APB > 90°} 10 {P : ∠APB < 30°}

11 Use your loci of **6** and **7** to show the region given by {P : AP < 5cm} ∩ {P : BP > 3cm} using one diagram.

12 Use your loci of **8** and **9** to show the region given by {P : PA < PB} ∩ {P : ∠APB = 90°}.

13 ABCD is a parallelogram in which AB = 9cm, AD = 5cm and ∠BAD = 60°.
(i) Construct ABCD using ruler and compasses only.
(ii) Measure and write down the length of AC.
(iii) Construct the locus {X : X is equidistant from A and C}
(iv) Construct the locus {Y : ∠BYD = 90°}
(v) Point P lies inside ABCD such that AP ≥ PC and ∠BPD ≤ 90°. Indicate clearly, by shading, the region in which the point must lie.

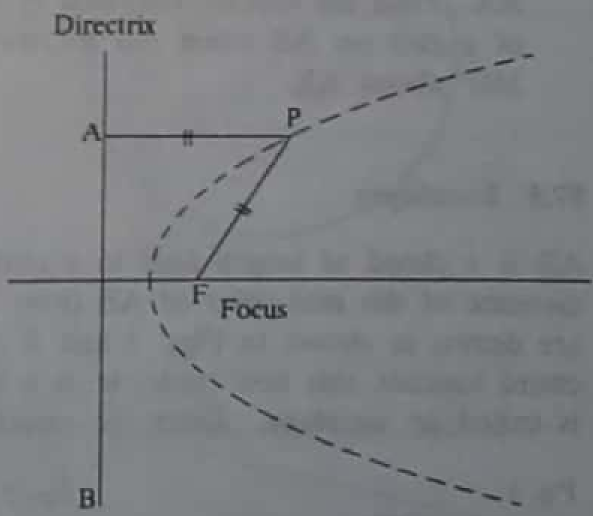
14 Indicate by shading the region in which a point Q must lie if it is inside the unit square (0, 0), (1, 0), (1, 1) and (0, 1) but outside a unit square centre (1, 1).

15 Draw a △ABC and construct the mediators of the three sides to meet at O. With centre O and radius OA draw a circle to pass through the three vertices.

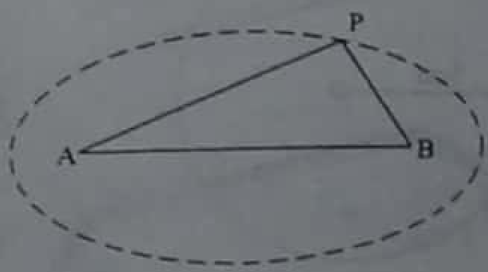
16 Draw a △PQR and construct the angle bisectors to meet at I. With centre I draw a circle to touch the sides of △PQR.

57.3 Other Loci in the Plane

(i) *Parabola* The point P moves so that its distance from a fixed point F, called the *focus* is equal to its distance from a fixed line AB called the *directrix*. The point P traces out a **parabola** as shown on the right.



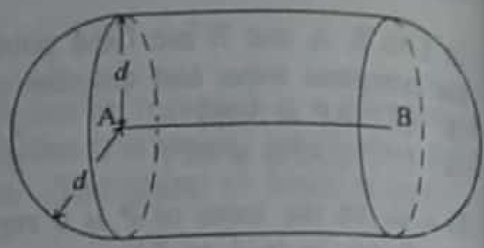
(ii) *Ellipse* The point P moves so that the sum of the distances PA and PB is constant. The point P traces out an **ellipse** as shown.



57.4 Loci in Three Dimensions

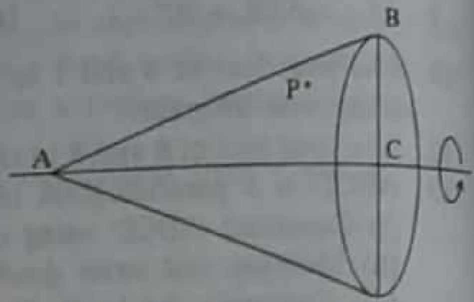
(i) *Cylinder* If in locus (iii) of **57.2**, P was allowed to move in three dimensions, the locus would be a cylinder, with line l as the axis and radius d.

If the axis is a line segment AB (see diagram) the locus is a cylinder with a hemispherical cap at each end.



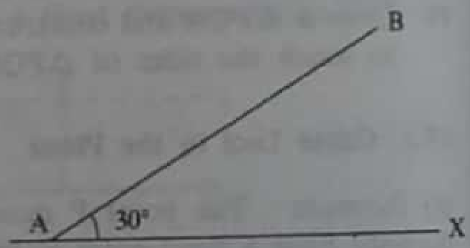
(ii) *Cone* The triangle ABC is rotated about the line AC through 360° (see diagram). The solid so formed is a cone. It may be considered the locus of points P such that $\angle PAC \leq \angle BAC$ and $\angle PCA \leq 90°$, ie.

$$\{P : \angle PAC \leq \angle BAC\} \cap \{P : \angle PCA \leq 90°\}$$



Exercise 57b

1. Draw the locus of points which are *equidistant* from the line $\{(x, y) : x = -4\}$, the directrix, and the point (4, 0), the focus.
2. Take a piece of thin string 10cm long and fix its ends 6cm apart. Using your pencil and keeping the string taut trace out an ellipse.
3. Find the volume enclosed by the locus of points which are 3·5cm from a line segment of length 10cm. (Take π to be $\frac{22}{7}$)
4. The diagram shows a line segment AB of length 14cm inclined at 30° to the line (axis) AX. Find the volume enclosed by the locus of points on AB when AB rotates through 360° about AX.



57.5 Envelopes

AB is a chord of length 8cm in a circle of radius 5cm, centre O (see Fig. 1). What is the distance of the mid-point of AB from the centre of the circle? More and more such chords are drawn as shown in Figs. 2 and 3. Their mid-points lie on a circle of radius 3cm. Each chord touches this new circle, ie. is a tangent to it. A curve formed by a succession of lines is called an **envelope**. Draw the envelope for yourself.

Fig. 1

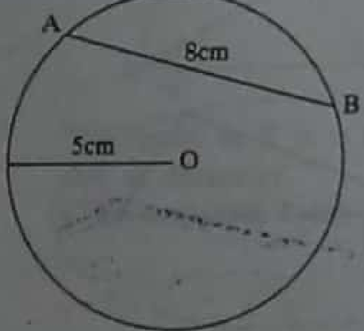


Fig. 2

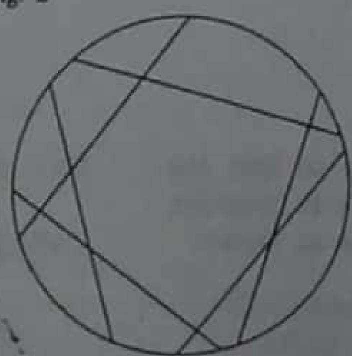
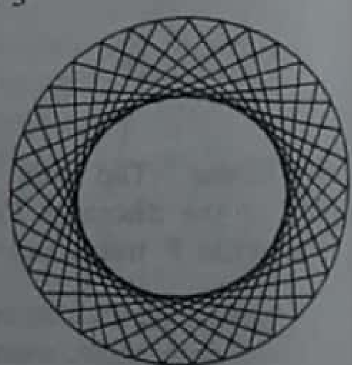


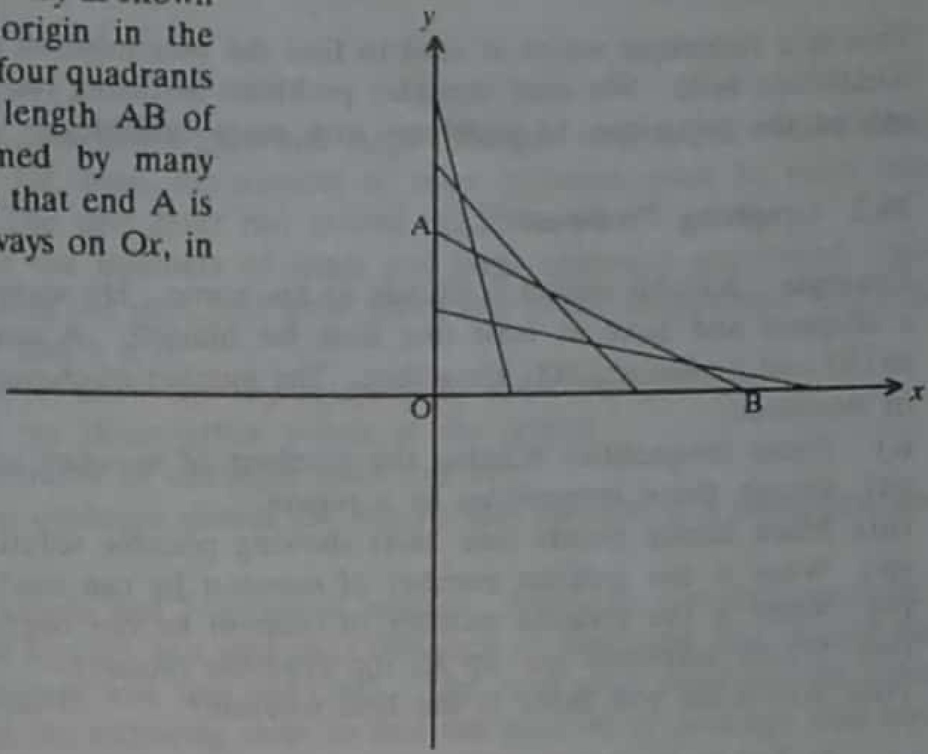
Fig. 3



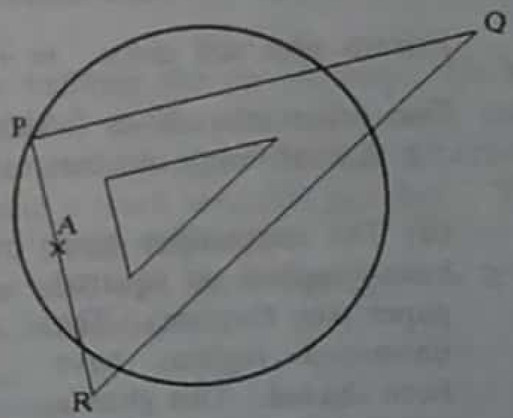
Exercise 57c

The following envelopes are best drawn on plain paper.

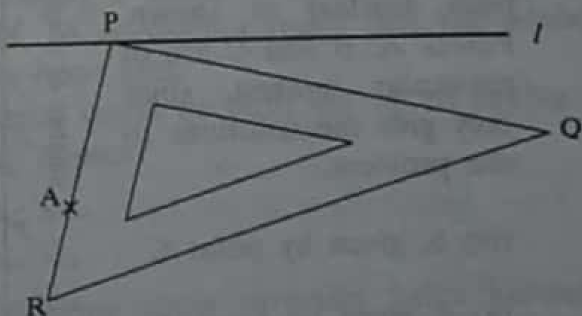
1 Draw perpendicular axes Ox and Oy as shown in the diagram. Have the origin in the centre of your paper so that all four quadrants are available. Using a fixed length AB of 8cm, find the envelope formed by many different positions of AB such that end A is always on Oy and end B is always on Ox, in the four quadrants.



2 Draw a circle of radius 4cm. Mark a point A, 3cm from the centre. Place a set-square PQR so that the right-angle lies on the circle at P and PR passes through A. Draw line PQ. Repeat for many positions of P on the circle with PR passing through A. What is the envelope of the lines PQ?



3 Draw a line l and mark a point A, 3cm below the line. Place set-square PQR so that P is on the line l and PR passes through A. Draw the line PQ. Repeat for many different positions of P both sides of A. What is the name of the envelope formed by the lines PQ?



58 LINEAR PROGRAMMING

58.1 Introduction

This is a technique which is used to find the best solution to a practical problem when certain conditions hold. We shall consider problems involving two variables, although it is possible to extend the technique to problems with more variables.

58.2 Graphing Problems

Example Kajubu invites 6 friends to his home. He wants to give each either a *mandazi* or a *chapatti* and have at least one item for himself. A *mandazi* costs sh100, a *chapatti* costs sh150 and he has sh1,000 altogether. The number of *chapatti* must be at least half the number of *mandazi*.

(i) Form inequalities relating the numbers of *mandazi* and *chapatti* Kajubu buys.
(ii) Graph these inequalities as a region.
(iii) Mark lattice points (see **36.4**) showing possible solutions.
(iv) What is the greatest number of *mandazi* he can buy?
(v) What is the greatest number of *chapatti* he can buy?
(vi) Which solutions use up all the available money?
(vii) Which do you think is the best solution?

(i) Let m be the number of *mandazi* Kajubu buys and c the number of *chapatti*.

Considering the cost gives: $100m + 150c \leq 1{,}000$ which simplifies to $2m + 3c \leq 20$

Friends plus self gives: $m + c \geq 7$ The other condition gives: $c \geq \frac{1}{2}m$

Two other inequalities (which have no effect in this particular example) are $m \geq 0$ and $c \geq 0$, confirming our interest in the positive quadrant.

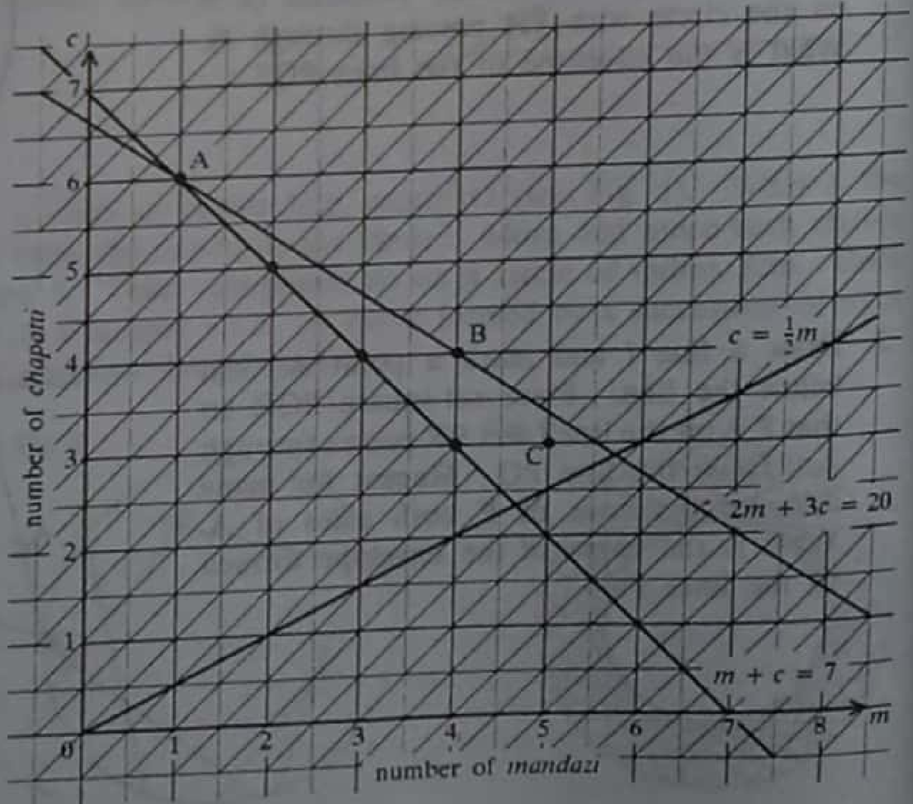
(ii) The inequalities have been graphed on squared paper (see diagram). The unwanted regions have been shaded. This gives a triangular region as shown.

(iii) Six lattice points have been marked as shown. Points A, B and C are of particular interest, since they give the solutions to the problem.

(iv) 5, given by point C

(v) 6, given by point A

(vi) (1, 6) and (4, 4) given by A and B



(vii) (5, 3) given by C

This last solution does not use all the available money, yet Kajubu can buy the maximum number of eight items thus having two for himself.

Exercise 58a

1 Small cabbages at the market cost sh150 each and large cabbages cost sh250. Asta wants to buy at least 10 cabbages. Of these, the number of large cabbages must be more than the number of small ones. The cost must not exceed sh2,250.
(i) If x and y respectively are the numbers of small and large cabbages purchased, show that $3x + 5y \leq 45$ by considering the cost, and write down two other inequalities involving x and y in addition to $x \geq 0$ and $y \geq 0$.
(ii) On squared or graph paper, show the region given by these inequalities.
(iii) Give the coordinates of the three lattice points of the region.
(iv) What is the maximum number of cabbages Asta can buy?
(v) How many small and large cabbages should she buy so that the cost is a minimum and what is this cost?

2 Transport is required for 60 people and 2 tonnes of luggage. Available are pick-ups which carry 10 people and 500kg of luggage and cost sh12,000 each to hire, and taxis which can carry 15 people, 250kg of luggage and cost sh15,000. The total cost must not be more than sh75,000. Work through the following steps to find the number of pick-ups and taxis which should be hired.
(i) Form inequalities in p and t, the number of pick-ups and taxis hired.
(ii) Graph these inequalities and hence find three possible solutions to this problem.
(iii) Which is the best solution?

3 A contractor has to transport 60 tonnes of sand to a building site on Monday. He has 8 pick-ups which can each carry 1 tonne of sand and 5 trucks which can each carry 3 tonnes. The pick-ups make 5 trips per day and the trucks 3 trips. He has 10 drivers available. The cost of a pick-up is sh25,000 per day and a truck sh60,000 per day. The contractor needs to know the number of pick-ups and trucks required.
(i) Let p and t be the number of pick-ups and trucks used and show that $5p + 9t \geq 60$ by considering the delivery of sand. Form five other inequalities.
(ii) Represent these inequalities on one diagram.
(iii) Mark the six lattice points which represent solutions.
(iv) Which three of these give sensible solutions?
(v) Which solution would give the lowest cost?

4 A formula for finding the approximate cost of building a house is: sh200,000 per m^2 for floor area plus sh300,000 per m^2 for window area. A regulation states that the window area of a house must be at least one-sixth of the floor area.
(i) Illustrate with a region the sizes of houses which could be built for up to sh15m.
(ii) What is the floor area of the largest of these houses?

58.3 Optimisation

Finding the best solution to a complex problem involving many variables helps business managers to make decisions maximising profits and minimising costs. To start to understand the procedure, a very simplified version of a real life situation is dealt with. The technique of translating a certain line within a defined region either to maximise or minimise something is called **optimisation**.

Example A small engineering firm produces *posho* grinding machines (P) and machines for processing cattle fodder (C) which are both made from sheet metal. Machine P takes 5 men to produce it and uses 6m² of sheet metal. Machine C takes 3 men to produce it and needs 10m² of sheet metal. There are 37 men and 90m² of sheet metal available to produce these machines. How many of each machine should be made if (i) all the sheet metal is to be used, (ii) all the men are to be employed, (iii) the number of machines to be produced is to be a maximum?

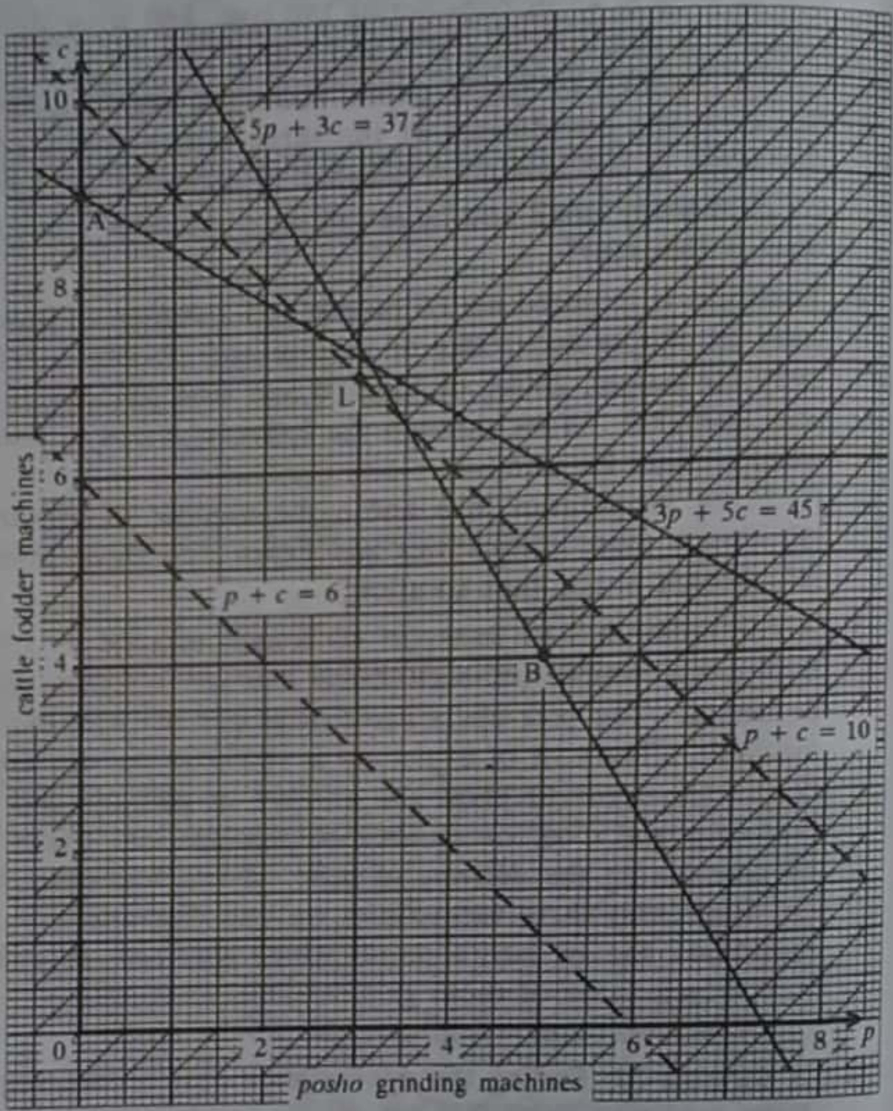
Let p and c be the number of machines P and C produced.

Considering men employed gives: $5p + 3c \leq 37$

Considering sheet metal gives: $3p + 5c \leq 45$

Also $p \geq 0$ and $c \geq 0$

These inequalities give the region shown in the diagram which has been drawn on graph paper.



(i) A(0, 9) is the only lattice point on the *metal* line in the region. Hence the firm should manufacture no *posho* grinding and 9 cattle fodder machines.

(ii) B(5, 4) is the only lattice point on the *men* line in the region. Hence the firm should manufacture 5 *posho* grinding and 4 cattle fodder machines.

(iii) We need the maximum value of $p + c$ in the region. Give $p + c$ any suitable value, say 6. Draw the line $p + c = 6$ (shown broken in the diagram). Translate this line away from the origin so that the value of $p + c$ increases. This is done by using the ruler and set-square method of drawing a parallel line to a given line (see **23.3**). Continue to translate the line away from the origin until it passes through the last lattice point of the region. This is L. The translated line is now $p + c = 10$ as shown. So 10 is the maximum value of $p + c$ in the region. Since L is (3, 7), the firm should produce 3 *posho* grinding and 7 cattle fodder machines.

Note The optimum solution will be given at a vertex of the region or, in the case of integral solutions, near a vertex.

Exercise 58b

1 A farmer has 50ha on which to grow beans and maize and wishes to grow at least 15 ha of each. He has sh12m available for this. The inclusive cost from ploughing to harvesting is sh300,000 per hectare for beans and sh200,000 per hectare for maize.

(i) Let b and m be the number of hectares of beans and maize grown. Write down four inequalities involving b and m and simplify them where possible.

(ii) Graph the region representing the above situation.

(iii) Suppose the profit on beans is sh500,000 per hectare and on maize sh400,000 per hectare. Draw the line $500{,}000b + 400{,}000m = 10{,}000{,}000$, ie. $5b + 4m = 100$. By translating this line away from the origin, find the coordinates of the point of the region such that $500{,}000b + 400{,}000m$ is a maximum. Hence state the number of hectares of beans and maize the farmer should plant to make maximum profit. What is this profit?

(iv) If instead the profit was sh400,000 per hectare on beans and sh500,000 per hectare on maize, how many hectares of each should be grown to maximise the profit? State this profit.

(v) How many hectares of beans and maize should he plant if the profit was sh600,000 and sh300,000 per hectare respectively. What is the profit in this case?

2 Mary is to prepare a meal of rice and meat for her family. Rice costs sh400 per kg and contains 360 calories per 100g. Meat costs sh1,000 per kg and contains 150 calories per 100g. The total mass of the food is to be at least 6kg and it must contain between 12,000 and 18,000 calories.

(i) By considering the caloric content of the food, show that $40 \le 12r + 5m \le 60$ where r and m are the number of kilograms of rice and meat respectively.

(ii) Write down three other inequalities involving r and m.

(iii) Graph the region representing possible proportions of rice and meat.

(iv) Write down an expression for the cost of the meal and, by letting this equal sh6,000, draw a suitable *cost* line.

(v) Translate this line towards the origin and so find the most economical amounts of rice and meat for this meal.

3 A soda factory is to have installed two types of production line P and Q. A production line of type P produces 300 crates of soda per day. It takes 10 men to operate it and uses $60m^2$ of floor space. Type Q produces 200 crates per day, takes 16 men to operate it but uses only $30m^2$ of floor space. If there is $420m^2$ of floor space available for production lines and 120 men are available to operate them, how many of each type should be installed so as to maximise the output?

4 Tomatoes at a certain market are classified as either *large* or *small* and are placed in heaps of four. A heap of four large tomatoes has a mass of 500g and costs sh400. A heap of four small tomatoes has a mass of 200g and costs sh200. For each large heap and every two small heaps purchased, one small tomato is given free. Nyalulu wants to buy at least 40 tomatoes whose total mass is at least 3kg.

(i) By considering the mass he buys, show that $22x + 9y \ge 120$ where x and y are the numbers of large and small heaps purchased respectively.

(ii) Show also that $10x + 9y \ge 80$ and write down two other inequalities involving x and y.

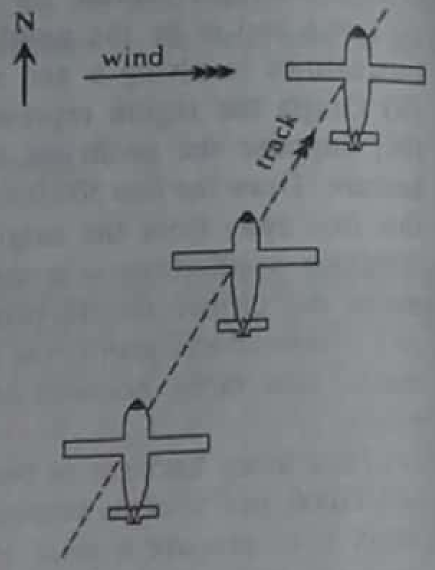
(iii) Represent possible values of x and y as a region.

(iv) By translating a suitable *cost* line, find the numbers of large and small piles of tomatoes Nyalulu should buy to make the cost a minimum.

5 Indicate on graph paper the region in which $7x + 4y \ge 44$, $2x + 9y \ge 36$ and $4x + 7y \le 49$. With the added restriction that x and y are integers, find the greatest and least values of $x + y$ from your graph.

59 NAVIGATION

59.1 Air Navigation



When flying a plane, the pilot has to take into account the motion of the air through which he is flying. On the ground this motion is referred to as *wind*.

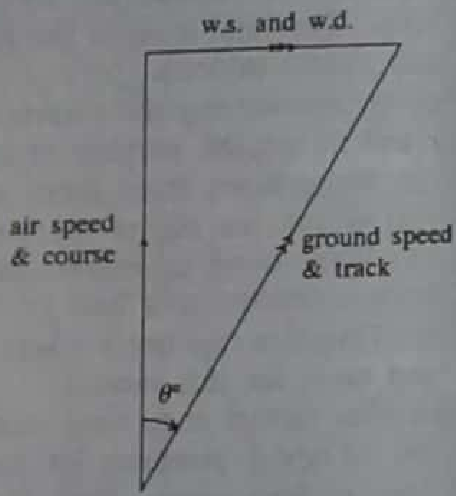
The diagram shows a plane whose pilot has set a **course** which is due North. This is the direction in which the plane is pointing. However, the plane does not move in this direction because there is a wind blowing from the West. The wind blows the plane off course so that it actually moves along the **track** shown as a broken line.

The direction of the track and the associated **ground speed** depend upon:

(i) the **wind direction** (w.d.) and **wind speed** (w.s.)

(ii) the course set and associated **air speed**.

The air speed is the speed with which the plane flies through the air and is indicated on the pilot's instrument panel.



The above quantities are represented by three vectors and are related by the vector triangle shown. (See **38.4**) The angle θ between the course and the track is called the **drift**.

Mark each vector as follows: a single arrow for course, a double arrow for track and a triple arrow for wind direction.

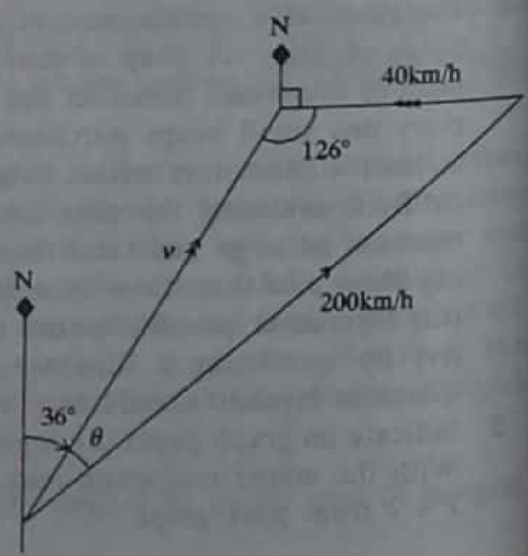
NB The w.d. is the direction *from* which the wind is blowing. For example, an East wind blows *from* the East, ie. towards the West.

Example A plane leaves Entebbe at 7.30 am for Soroti which is at a distance of 230km and on a bearing of 036° (see **25.2**) from Entebbe. There is a wind blowing at 40km/h from the East and the plane's air speed is 200km/h.

(i) What course should the pilot take?

(ii) Calculate the ground speed.

(iii) What is the estimated time of arrival (ETA)?



Sketch the vector triangle of velocities as shown. Note that the track is in the direction of the bearing of Soroti from Entebbe. One angle (126°) of the triangle has been found using this bearing and knowledge of the wind direction. To find θ, the angle of drift and v, the ground speed, make a scale drawing (see **25.3**) or use calculation as follows.

(i) Using the Sine Rule gives $\frac{\sin\theta}{40} = \frac{\sin 126°}{200}$ (See 45.1)

$\therefore \quad \sin\theta = \frac{40\sin 126°}{200} = \frac{40\sin 54°}{200} = \frac{0{\cdot}809}{5} = 0{\cdot}162$ (See 44.3)

$\therefore \quad \theta = 9{\cdot}3°$

The pilot's course is $36° + 9{\cdot}3° = 045{\cdot}3°$.

(ii) The third angle of the triangle $= 180 - (126 + 9{\cdot}3) = 44{\cdot}7°$

$$\frac{v}{\sin 44{\cdot}7°} = \frac{200}{\sin 126°}$$

$$\therefore \quad v = \frac{200\sin 44{\cdot}7°}{\sin 126°}$$

$$= \frac{200\sin 44{\cdot}7°}{\sin 54°}$$

$$= \frac{200 \times 0{\cdot}703}{0{\cdot}809} = 200 \times 0{\cdot}869 = 174 \text{ (3 sf)}$$

No	Log
0·703	$\bar{1}{\cdot}847$
0·809	$\bar{1}{\cdot}908$
0·869	$\bar{1}{\cdot}939$

The plane's ground speed is 174km/h.

(iii) The ground speed is used to calculate T, the time of flight.

$$T = \frac{\text{distance}}{\text{speed}} = \frac{230}{174} = 1{\cdot}32 \text{ hours} \approx 1\text{h } 19\text{min}$$

No	Log
230	2·362
174	2·241
1·32	0·121

The ETA in Soroti is 8.49 am.

Exercise 59a

Use scale drawing or calculation.

1 A pilot flies a plane with an air speed of 180km/h on a course of 045°. A wind is blowing from the East at 40km/h.
(i) Find the ground speed. (*Hint* Use Cosine Rule if calculating.)
(ii) Find the angle of drift. (*Hint* Use Sine Rule if calculating.)
(iii) Find the bearing of the track.

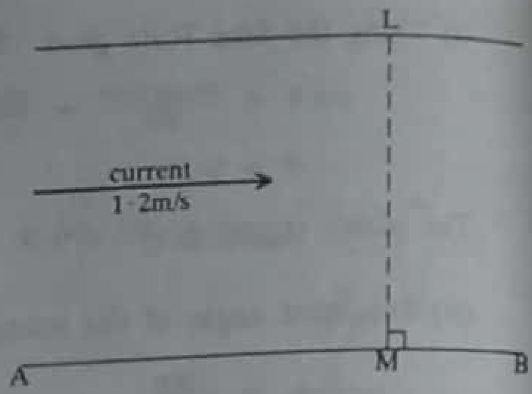
2 A pilot navigating a course of 190° with an air speed of 220km/h finds that he has in fact travelled due South with a ground speed of 200km/h. Find the speed and direction of the wind.

3 A pilot flying a plane with an air speed of 200km/h wishes to fly due North. A wind is blowing from the South East at 30km/h.
Find (i) the course he should take,
(ii) the ground speed.

4 A plane leaves Entebbe at 7 am for Nairobi. The distance and bearing of Nairobi from Entebbe are 510km and 107° respectively. There is a 40km/h wind blowing from 140° and the plane's air speed is 180km/h.
(i) What course should the pilot take?
(ii) Find the ground speed.
(iii) What is the ETA in Nairobi?

5 The plane of Q4 returns to Entebbe the following morning and flies under the same weather conditions. If it leaves Nairobi at 7 am, what time should it arrive in Entebbe?

6 In still water, the Likoni ferry at Mombasa, Kenya has a speed of 3m/s. At the time of crossing from M to L (see diagram) the water (current) in the channel is flowing at 1·2m/s in a direction perpendicular to ML as shown.
(i) At what angle to the shoreline AMB should the ferry captain steer in order to cross the channel directly from M to L?
(ii) If the distance ML is 500m, how long will the crossing take?

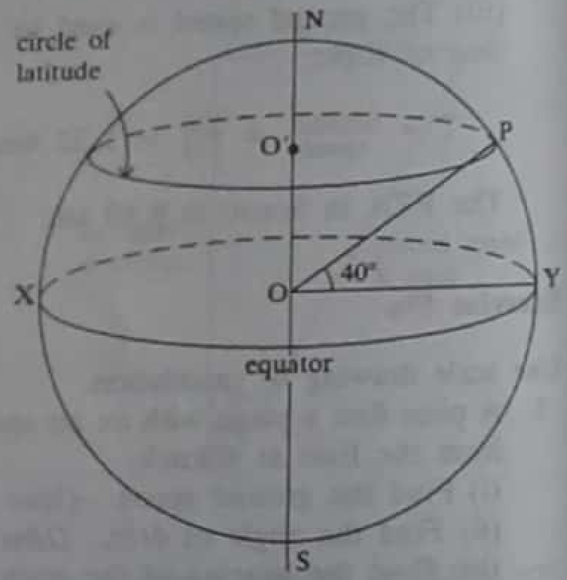


59.2 Great and Small Circles

A circle drawn on the surface of a sphere having its centre at the centre of the sphere is called a **great circle**. Any other circle so drawn which is not a great circle is called a **small circle**. An example of a great circle on the Earth's surface (considered to be a sphere) is the equator.

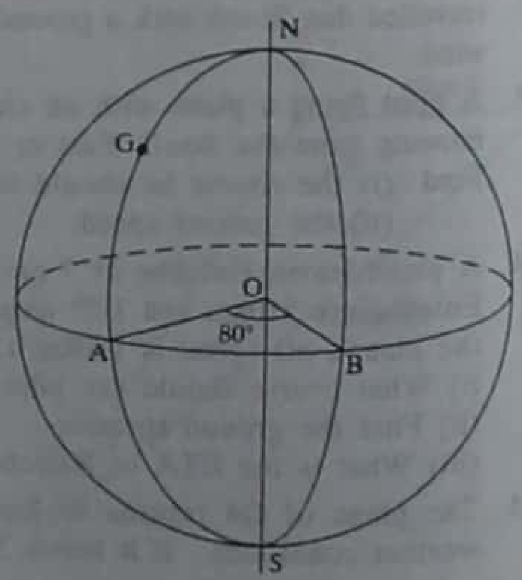
59.3 Latitude and Longitude

The equator is an example of a **circle of latitude**. Other circles of latitude are parallel to this and are sometimes called **parallels** of latitude. The one shown in the diagram has its centre at O' a point on the Earth's **axis** NS. All circles of latitude (apart from the equator) are small circles and are described by the angle POY subtended at the centre of the Earth. If $\angle POY = 40°$, then the circle of latitude is called 40°N since it is North of the equator. Any point on this circle will have a latitude of 40°N.



Circles on the Earth's surface passing through the poles are called **circles of longitude**. These are all great circles. The next diagram shows, in addition to the equator, the halves of two such circles NGAS and NBS, called **meridians of longitude**. The point G represents Greenwich (pronounced Grinidge) in London, England. The meridian NGAS through Greenwich is called the **Greenwich meridian**. It has been internationally agreed that this be used as a reference for other meridians. The meridian NBS is described by angle AOB subtended at the centre of the Earth. If $\angle AOB = 80°$, then this meridian is called 80°E since it is East of Greenwich. Any point on this meridian will have a longitude of 80°E.

Thus the position of any point on the Earth's surface is described by giving its latitude and longitude. For example, the latitude and longitude of Arua is approximately (3°N, 31°E). Latitude and longitude is a coordinate system enabling us to locate points on a sphere, just as x- and y-coordinates locate points in a plane.

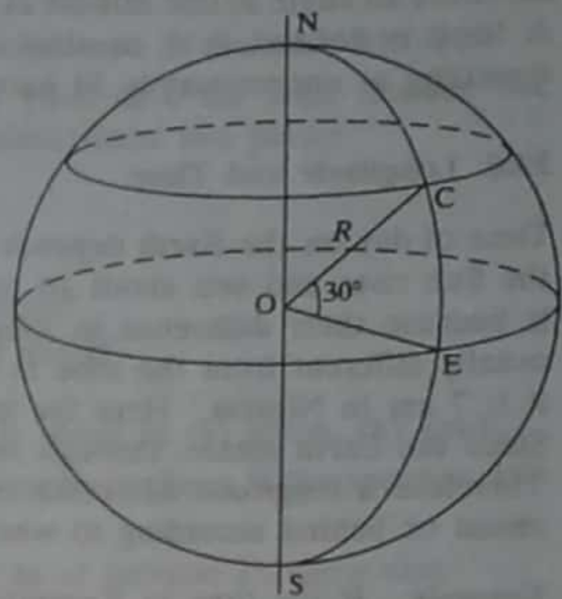


 Distance along Circles

To find such distances use the method of 7.6.

Example 1 Find the distance travelled along the meridian of longitude 32°E in flying from Entebbe (0°, 32°E) to Cairo (30°N, 32°E) given that the radius of the Earth is 6,370km.

The diagram shows the meridian of longitude, NCES, 32°E. Since Cairo (C) has latitude 30°N and Entebbe (E) is on the equator, ∠COE = 30°. Arc CE is part of a great circle whose radius R is 6,370km.



$$\text{Arc CE} = \frac{\theta}{360} \times 2\pi R$$

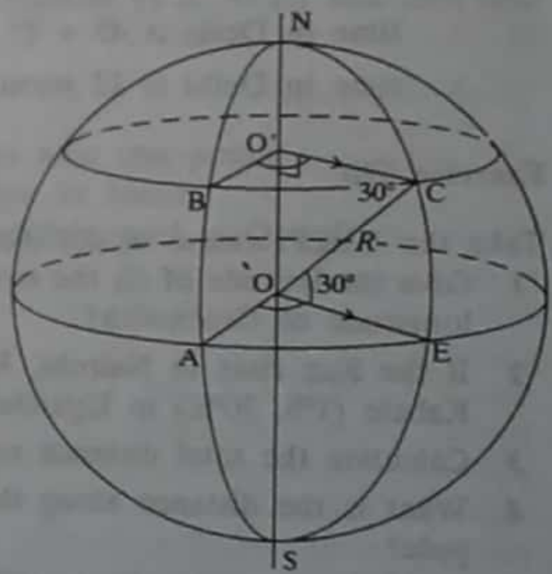
$$= \frac{30}{360} \times 2 \times 3{\cdot}14 \times 6{,}370$$

$$= 3{,}330 \quad \text{using tables}$$

The required distance is 3,330km (3 sf)

Example 2 A plane flies from Cairo (30°N, 32°E) to New Orleans (30°N, 90°W) along the circle of latitude 30°N. Calculate (i) the distance round the small circle of latitude 30°N, (ii) the distance travelled by the plane.

(i) In the diagram, O′ is the centre of the small circle of latitude 30°N and O′C is its radius.



Now O′C is parallel to OE,

∴ ∠O′CO = ∠COE (alternate ∠s) = 30°

In the right-angled triangle O′OC,

O′C = R cos 30° = 6,370 cos 30°

The circumference C of the circle of latitude 30°N is given by

$$C = 2\pi \times O'C$$

$$= 2 \times 3{\cdot}14 \times 6{,}370 \cos 30°$$

$$= 34{,}600 \text{ (3 sf)}$$

Distance round circle of latitude is 34,600km.

(ii) In addition to the meridian of longitude NCES through Cairo (C), the diagram shows the meridian NBAS through New Orleans (B). Since one longitude is to the East of Greenwich and the other is to the West, the difference in longitude is 32 + 90 or 122°. Therefore ∠AOE = 122°. Also ∠BO′C = ∠AOE = 122°.

$$\text{Arc BC} = \frac{\theta}{360} \times \text{circumference} = \frac{122}{360} \times 34{,}600 = 11{,}700 \text{ (3 sf)}$$

The distance travelled by the plane is 11,700km.

Note The radius r of a circle of latitude $\lambda°$ (pronounced *lamda*) is given by $r = R \cos \lambda$, where R is the radius of the Earth.

59.5 Nautical Miles and Knots

An angle of one degree (1°) is divided into sixty **minutes** (60').

So one degree = sixty minutes or $1° = 60'$

A **nautical mile** is the length of an arc of a great circle on the surface of the Earth which subtends an angle of one minute at the centre. From this definition: 1 nautical mile $\simeq$ 1·85km. A **knot** is defined as 1 nautical mile per hour. A car moving at a speed of 100km/h is travelling at approximately 54 knots.

59.6 Longitude and Time

Time of day on the Earth depends upon the longitude of the place in question. For example, the Sun rises and sets about 16 minutes earlier in Tororo than it does in Fort Portal. This is because their difference in longitude is 4°. For this reason the time in one country is usually different from the time in another. For instance, when the time in Uganda is 9 am, it is 7 am in Nigeria. Here the longitude difference between the two countries is about 30°. Since the Earth rotates through 360° in 24 hours, in 1 hour it turns through $360 \div 24 = 15°$. Therefore a longitude difference of 15° represents a time difference of 1 hour. This will be ahead or behind according to whether the movement is East or West.

Example If the time in Kampala (0°, 33°E) is 12 noon, what should be the local time in Delhi (28°N, 78°E)?

The longitude difference is $78 - 33 = 45°$ East

$\therefore$ time in Delhi is $45 \div 15 = 3$ hours ahead of Kampala

$\therefore$ time in Delhi is 12 noon + 3 hours, ie. 3 pm

Exercise 59b

Take the Earth's radius as 6,370km.

1 Give the latitude of (i) the equator, (ii) the North pole, (iii) the South pole. What is the longitude of Greenwich?

2 If the Sun rises in Nairobi, Kenya (1°S, 37°E) at 6.45 am, at what time will it rise in Kabale (1°S, 30°E) in Uganda?

3 Calculate the total distance round the equator.

4 What is the distance along the Greenwich meridian from the South pole to the North pole?

5 A plane flies along the meridian 13°E from Luanda (9°S, 13°E) to Tripoli (33°N, 13°E). Find the distance travelled.

6 Find the distance round the parallel of latitude 49°N.

7 A plane flies along the parallel of latitude 49°N from Paris (49°N, 2°E) to Vancouver (49°N, 123°W).
Find (i) the distance travelled,
(ii) the time difference between these places.

8 A plane flies along the circle of latitude 24°S from Gaberones (24°S, 26°E) in Botswana to Alice Springs (24°S, 134°E) in Australia.
Find (i) the distance travelled,
(ii) the time difference between these places.

9 In **Example 2** of **59.4**, calculate (i) the length of the chord BC, (ii) the angle BOC, (iii) the distance along the great circle from Cairo to New Orleans.
How does this distance compare with the distance along the small circle of latitude?

10 A plane travels from Nairobi to Lagos taking the great circle route. Given that the corresponding arc of this great circle subtends an angle of 35° at the centre of the Earth, find the distance travelled.
Compare your answer with the distance given on the signboard of **Exercise 43c Q13**.

11 New Orleans (30°N, 90°W) and Lhasa (30°N, 90°E) in Tibet have the same latitude but their longitudes differ by 180°. Find the distance between these two places
(i) along the circle of latitude 30°N,
(ii) by the great circle route over the North pole.
Compare your answers.
If it is 12 noon in Lhasa, what time is it in New Orleans?

12 (i) What is the distance round the equator in nautical miles?
(*Hint* Use the definition given in **59.5**.)
(ii) An aircraft flies this distance in 24 hours. Find its speed in (a) knots, (b) km/h.

13 In **Example 1** of **59.4** find the time taken to fly from Entebbe to Cairo if the speed is 450 knots.

14 (i) If C is the circumference in nautical miles of a circle of latitude $\lambda°$, show that

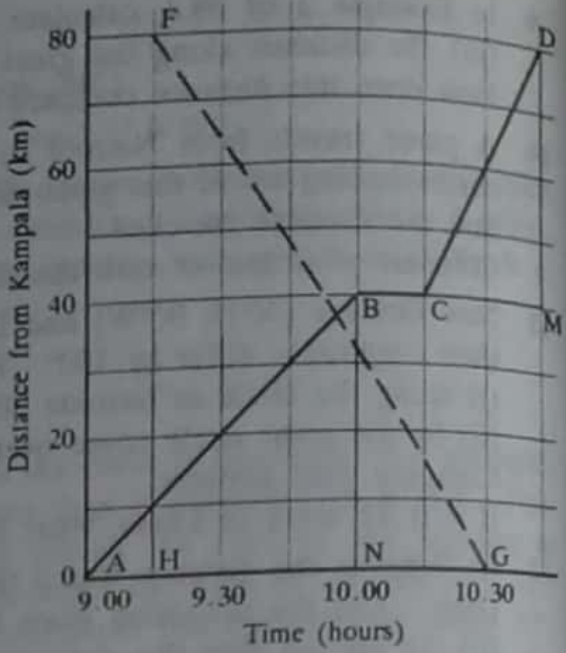
$$C = 21{,}600 \cos \lambda$$

(ii) Find a formula for the length A nautical miles of an arc of this circle which subtends an angle of $\theta°$ at its centre.

15 An aircraft whose speed is 450 knots takes off from Nairobi (1°S, 37°E) and flies due North for 5 hours 4 minutes.
Find (i) the distance travelled in nautical miles,
(ii) the latitude and longitude of the aircraft,
(iii) by looking at an atlas, the name of the town near this point.
From this point the aircraft flies due West for a further 3h 36min.
Find (iv) its latitude and longitude by using the formula obtained in **Q14(ii)**,
(v) the name of the town near this point.

60 KINEMATIC GRAPHS

60.1 Travel Graphs



(i) *Distance-Time*

A taxi leaves Kampala at 9am and travels to Jinja, stopping once on the way at Lugazi. The continuous line ABCD, shown on the grid, is the **distance-time** graph. The line segment AB shows that between 9.00 and 10.00 the taxi travels 40km from Kampala (A) to Lugazi(B). Its **speed** is given by:

$$\frac{\text{distance}}{\text{time}} = \frac{40}{1} = 40\text{km/h}$$

But **gradient** of AB is given by:

$$\frac{BN}{AN} = \frac{40}{1} = 40 \quad (\text{See } \mathbf{20.3})$$

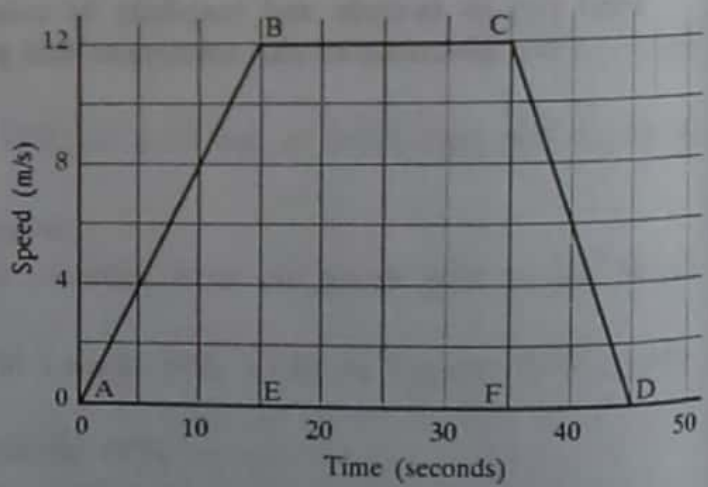
Note that BN represents a *distance* and AN represents a *time*.
Therefore *the* ***gradient*** *of a distance-time graph gives the* ***speed****.*
From 10.00 to 10.15 (BC on the graph) the taxi is stationary at Lugazi. Note that the gradient of BC is zero, ie. speed = 0.
Line segment CD represents the journey from Lugazi(C) to Jinja(D).
Speed from Lugazi to Jinja = gradient of CD = DM ÷ CM = 40 ÷ 0·5 = 80km/h.
The broken line segment FG is the travel graph of a mini-bus. Point F shows that at 9.15 it was 80km from Kampala, ie. at Jinja. Point G indicates that it reaches Kampala at 10.30.

In 1¼h it travels 80km, so its speed is given by: $\frac{\text{distance}}{\text{time}} = \frac{80}{1\frac{1}{4}} = 80 \times \frac{4}{5} = 64\text{km/h}$

Now gradient of FG is: $\frac{FH}{HG} = \frac{-80}{1\frac{1}{4}} = -64$ (See **20.3**)

The *negative* sign indicates that the mini-bus travels in the *opposite* direction to the taxi.

(ii) *Speed-Time*



A train accelerates (see **8.3**) steadily from rest to a speed of 12 metres per second (12m/s) in 15 seconds, stays at that speed for 20 seconds and then steadily slows to rest in another 10 seconds. The diagram shows the **speed-time** graph for the train. The speed increases by 12m/s in 15 seconds, that is at a rate of 12 ÷ 15m/s each second or 0·8m/s per second. The **acceleration** of the train is 0·8m/s per second and we write this as 0·8m/s². Notice that the gradient of AB is 12 ÷ 15.
Therefore *the* ***gradient*** *of a speed-time graph gives the* ***acceleration****.*
The horizontal line segment BC represents the constant speed of 12m/s. Its gradient is zero and so the acceleration is zero. Over the final 10 seconds the speed decreases from 12m/s to zero. The **deceleration** or **retardation** is 12 ÷ 10 or 1·2m/s².
Now, gradient of CD = −12 ÷ 10 = −1·2. The *negative* sign indicates a *retardation*.
Consider again the section BC of the graph, which represents a constant speed of 12m/s for a period of 20 seconds (from 15 seconds to 35 seconds). The distance covered during this time is given by: distance = speed × time = 12 × 20 = 240m.

Now the *area* of rectangle BEFC (called the **area under the graph**) is given by:

$$\text{Area} = \text{length} \times \text{width} = BE \times EF = 12 \times 20 = 240$$

Note that BE represents a *speed* and EF represents a *time*.

Therefore *the area under a speed-time graph gives the* ***distance*** *travelled.*

Now consider the first 15 seconds (from 0s to 15s). To find the distance travelled during this period we calculate the corresponding area under the graph, ie. the area of $\triangle AEB$.

$$\text{Area of } \triangle AEB = \tfrac{1}{2} \times \text{base} \times \text{height} = \tfrac{1}{2} \times AE \times BE = \tfrac{1}{2} \times 15 \times 12 = 90$$

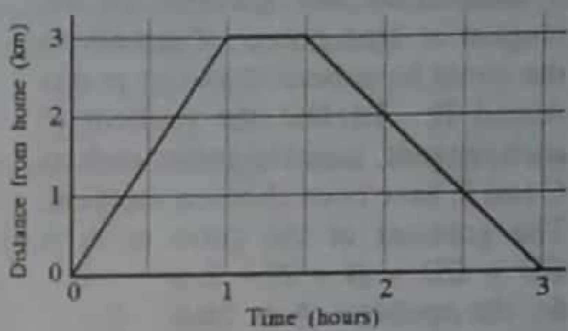
So the distance travelled during the first 15 seconds is 90m.

Similarly for the final 10 seconds (from 35s to 45s), the distance travelled will be represented by the area of $\triangle FDC$. Check that this gives a distance of 60m.

The total distance travelled is represented by the area ABCDFE and is 90 + 240 + 60 or 390m. This could be calculated directly by finding the area of the trapezium ABCD.

Exercise 60a

1 Ilero went for a walk from her home. Her distance-time graph is shown in the diagram.

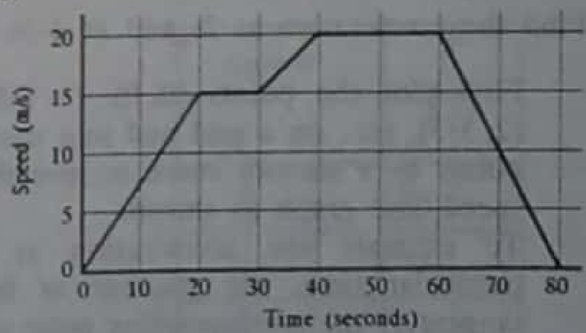


(i) If she left home at 09.40, when did she arrive back?
(ii) How far did Ilero walk altogether?
(iii) For how long did she rest?
(iv) Calculate her outward and return speeds.
(v) What was her slowest speed?

2 Draw a distance-time graph to illustrate each of the following.
(i) Kamugisha cycles for 1h at 15km/h, stops for 1h and then continues in the same direction for 2h at 10km/h.
(ii) A pick-up moves at 30km/h for $1\frac{1}{2}$h and returns at once to its starting point in 1h.

3 A cyclist leaves Soroti at 10.00 and cycles at 15km/h to Kapiri which is 25km away.
(i) At what time does he arrive? (ii) Draw a distance-time graph for his journey.
On the same day a car starts from Soroti at 10.30 and travels to Kapiri at 60km/h. It stays at Kapiri for 20 minutes and then returns to Soroti at the same speed.
(iii) Draw its distance-time graph on the same axes as the cyclist. (iv) At what times, and how far from Soroti does the car pass the cyclist?

4 The diagram shows the speed-time graph of a train. Find



(i) for how long the train travelled at 15m/s,
(ii) its maximum speed,
(iii) the two accelerations,
(iv) the retardation,
(v) the distance travelled in the first 30 seconds,
(vi) the total distance travelled.

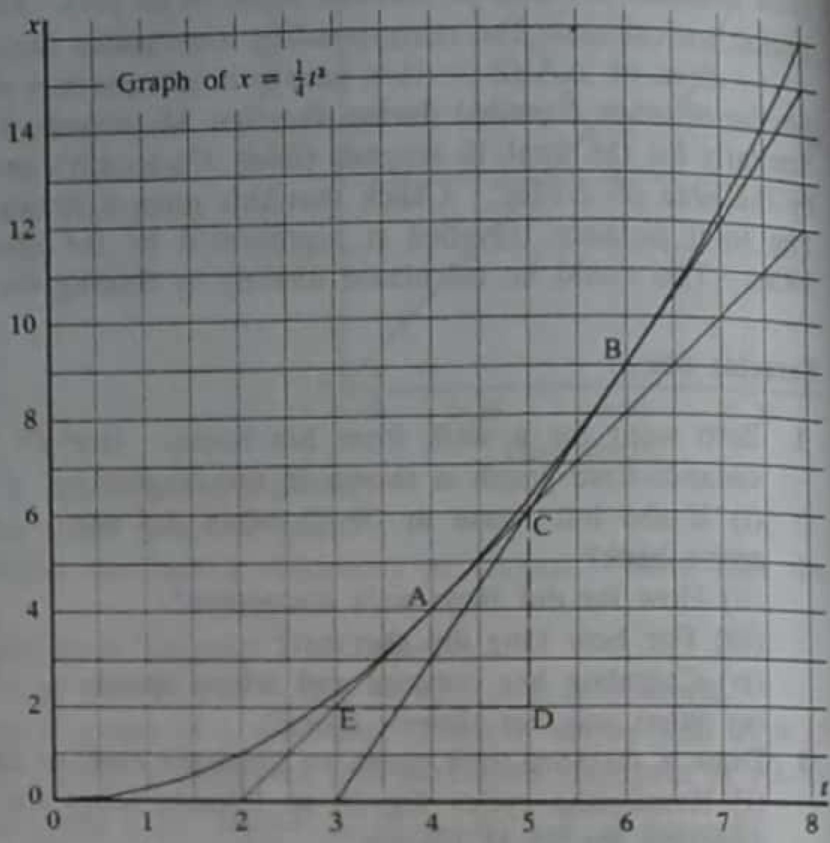
5 A car accelerated steadily from 0 to 8m/s in 20 seconds, remained at that speed for 10 seconds and then slowed steadily to rest in 16 seconds. Draw its speed-time graph. State the acceleration for the first 20 seconds and the deceleration for the last 16 seconds. Calculate the distance travelled in the total 46 seconds.

6 A bus accelerated steadily from rest to 24m/s in 30 seconds, stayed at that speed for 40 seconds and then slowed steadily to rest in 20 seconds.
(i) When was the speed 12m/s? (ii) What was the acceleration?
(iii) What was the retardation? (iv) How far did the bus travel?

60.2 Gradient of a Curve

A car or cyclist is much more likely to be changing speed constantly and so the distance-time graph would therefore not consist of line segments but a curve as in the following example.

A ball is rolled from rest down a slope. After t seconds it is x metres from its starting point where $x = \frac{1}{4}t^2$. The graph of its motion is shown in the diagram.



Since this is a distance-time graph, the gradient gives the speed. Now the gradient of a straight line is constant but that of a curve is not. The gradient at a point on a curve is defined as the gradient of the **tangent** at that point. Tangents to the curve have been drawn at points A and B. To find the gradient of each tangent, suitable points such as C and E have been choosen on them. The gradient of the curve at A is CD ÷ ED = 4 ÷ 2 = 2.

So the speed at A is 2m/s.

Check that the gradient of the curve at B is 3.

The speed of the ball at B is 3m/s.

If the speed-time graph is a curve, as in the following Example, the gradient of the tangent at a point represents the acceleration at that instant.

Example The speed of a train at various times is shown in the following table.

Time in seconds (t)	0	1	2	3	4	5	6
Speed in m/s (v)	0	1·8	3·3	4·5	5·3	5·8	6

Find the acceleration at 2s and at 4·5s.

First plot the points (0, 0), (1, 1·8), (2, 3·3), etc., on a grid and join these points by a smooth curve to give the speed-time graph as shown.

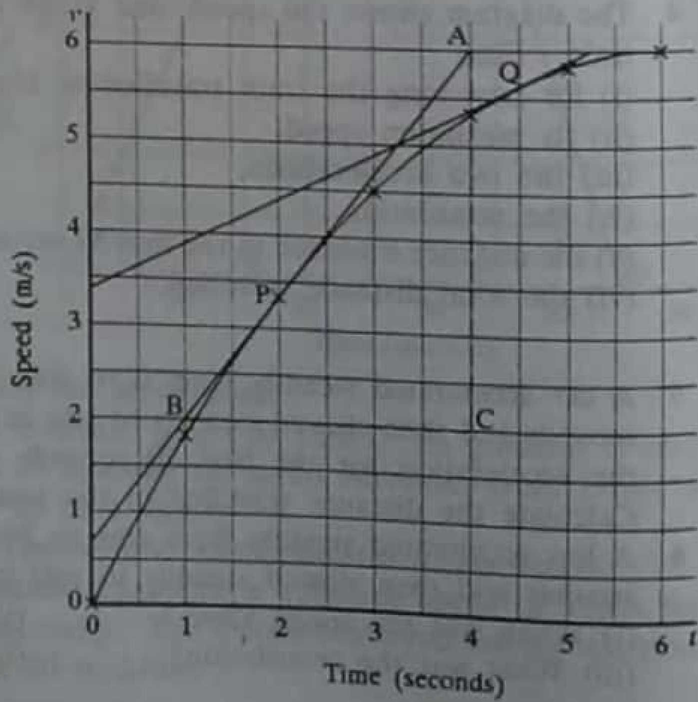
To estimate the acceleration at a particular time, the gradient of the tangent at the corresponding point on the curve must be determined.

Draw the tangent at P where $t = 2$. Suitable points A, B are taken on it.

Gradient = AC ÷ BC = 4 ÷ 3 ≃ 1·3

Acceleration at 2 seconds is 1·3m/s².

Measure the gradient of the tangent at Q to show that the acceleration at 4·5 seconds is approximately 0·5m/s².



Exercise 60b

1 Draw the graph of $x = 6t - t^2$ for $\{t : 0 \leq t \leq 6\}$ with x and t as in **60.2**. Draw tangents to estimate the speed when $t = 1$, 2 and 3 seconds. Comment on your result for $t = 3$.

2 The table gives the distances observed at 1 second intervals for a car starting from rest:

Time (t)	0	1	2	3	4	5	6	7	8	9	10	11	12
Distance (x)	0	1	3	6	10	15	21	28	36	45	55	66	78

Estimate the speed of the car when $t = 4$, 7 and 10 seconds.

3 A train moved according to the information given below.

Time (minutes)	0	1	2	3	4	5	6	7	8	9	10
Distance (metres)	0	5	20	60	120	240	420	580	680	725	740

(i) Find the average speed over (a) the first four minutes, (b) the whole 10 minutes.
(ii) By drawing tangents, estimate the speed at (a) 4 minutes, (b) 8 minutes.
(iii) By drawing a suitable tangent, estimate the greatest speed and when this was reached.

4 The table shows speedometer readings at various times during a car journey.

Time from start in seconds (t)	0	5	10	15	20	25	30	35	40
Speed in m/s (v)	0	11	19	25	30	34	37	39	40

Plot these points on a suitable grid and draw a smooth curve through them. Estimate the acceleration at (i) 10 seconds, (ii) 25 seconds after the start.

5 Draw a speed-time graph from the data given below.

Time in seconds (t)	0	1	2	3	4	5	6	7	8	9	10
Speed in m/s (v)	0	2·2	3·8	4·7	5	4·9	4·4	3·6	2·5	1·3	0

(i) Estimate the acceleration at 3 seconds and the retardation at 7 seconds.
(ii) What is the maximum speed? When is this and what is the acceleration at this time?

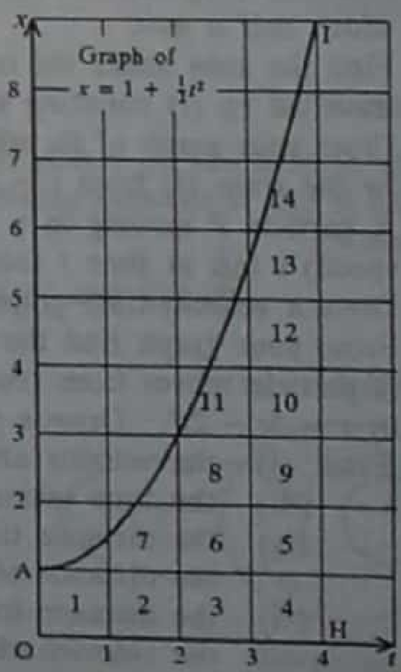
60.3 Area under a Speed-Time Graph

The area under a speed-time graph represents the distance travelled (see **60.1**). If this area is bounded by straight lines, for example a pentagon, it may be found by dividing it into a number of shapes like triangles, rectangles and trapezia whose areas we know. The exact area of a figure bounded by a curve, such as OAIH below, may not be found in this way. We shall discuss two techniques for *estimating* such an area.

(a) *Counting Squares*

Use the following table of values to draw accurately the graph of $x = 1 + \frac{1}{2}t^2$ for $\{t : 0 \leq t \leq 4\}$.
(For clarity the diagram shows the graph drawn on squared paper, although graph paper may be used.)

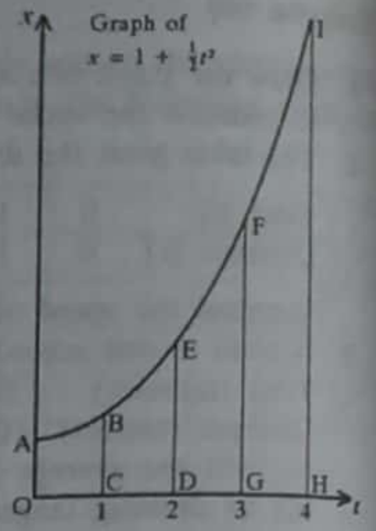
t	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4
x	1	$1\frac{1}{8}$	$1\frac{1}{2}$	$2\frac{1}{8}$	3	$4\frac{1}{8}$	$5\frac{1}{2}$	$7\frac{1}{8}$	9



Find an estimate of the area OAIH by counting unit squares which lie within it. A part square is counted as a whole square if more than half of it lies within the area. The approximate area, by this method is, as shown, 14 square units. If this is a speed-time graph with x in m/s and t in seconds then the area represents a distance travelled of 14m.

(b) *Trapezium Method*

The area OAIH can be estimated by using the approximate trapezia OABC, BCDE, DEFG and FGHI. The width of each 'trapezium' is 1 and the area is calculated using *half the sum of the parallel sides times the width*. In the calculation, *half width* comes out as a common factor:



$$\text{Area} = \tfrac{1}{2} \times 1 \times [(1 + 1\tfrac{1}{2}) + (1\tfrac{1}{2} + 3) + (3 + 5\tfrac{1}{2}) + (5\tfrac{1}{2} + 9)]$$
$$= \tfrac{1}{2} \times 1 \times [(1 + 9) + 2(1\tfrac{1}{2} + 3 + 5\tfrac{1}{2})]$$
$$= \tfrac{1}{2} \times [10 + 20] = 15 \text{ square units}$$

Note that an accurate graph need not be drawn and the **Trapezium Rule** is

$$\text{Area} = \tfrac{1}{2}\text{width} \times (\text{ends} + \text{twice middles})$$

Example Find the distance travelled in the 6 seconds by the train of the Example of **60.2**.

Since each unit on the t-axis represents 1 second and each unit on the v-axis represents 1m/s, each square unit of area represents a distance travelled of 1m.. Using the Trapezium Rule to obtain the area under the curve for the 6 seconds we have:

$$\text{Area} = \tfrac{1}{2} \times \text{width} \times (\text{ends} + \text{twice middles})$$
$$= \tfrac{1}{2} \times 1 \times [0 + 6 + 2(1{\cdot}8 + 3{\cdot}3 + 4{\cdot}5 + 5{\cdot}3 + 5{\cdot}8)]$$
$$= \tfrac{1}{2} \times 1 \times [6 + 2 \times 20{\cdot}7]$$
$$= \tfrac{1}{2} \times 47{\cdot}4 = 23{\cdot}7 \text{ square units}$$

The distance travelled by the train is approximately 24m.

Exercise 60c

1 Redraw the graph of $x = 1 + \frac{1}{2}t^2$ using a scale of 1 square for half a unit on both axes. Use the method of counting squares to estimate the area under the curve from 0 to 4. (Remember four of these 'new' squares represent 1 square unit under the curve.)

2 Use the Trapezium Rule to estimate the area under the above curve using trapezia of width half a unit.

3 Find the area under the curve drawn in **Ex 60b Q4** and hence estimate the total distance travelled by (i) counting squares of side 5 units, (ii) using trapezia of width 5 units.

4 From your graph of **Ex 60b Q5** use the Trapezium Rule to estimate the distance travelled by the train (i) from $t = 2$ to $t = 8$, (ii) over the whole 10 seconds.

5 A particle P moving in a straight line passes through a point O. The **velocity** (directed speed) v m/s at time t seconds after passing through O is given by $v = 3t + 2$. Draw a velocity-time graph for $\{t : 0 \le t \le 3\}$. From your graph find the acceleration of the particle and the total distance it travels.

6 A particle moves from rest in a straight line. Its velocity, v m/s, after t seconds is given by $v = 5t - 2t^2$. Draw a velocity-time graph for $\{t : 0 \le t \le 3\}$.
Find (i) the velocity after 2 seconds, (ii) the acceleration after 3 seconds,
(iii) the time taken for the particle to come to rest again,
(iv) the distance travelled in the first 2 seconds,
(v) the distance of the particle from its starting point when the velocity is zero,
(vi) the distance from its starting point when the acceleration is zero,
(vii) the velocity when the acceleration is zero.

Specimen Examination Papers

For each paper the following instructions apply if sat under examination conditions.

Time allowed: 2½ hours

Answer *all* questions from Section A and not more than *five* from Section B.

Section A carries 40 marks and section B 60 marks. To enable students to assess their performance, the Section A marks are stated for each question. Section B questions carry 12 marks each.

PAPER 1

SECTION A

1 Find the exact value of $\dfrac{0{\cdot}04^2 \times 0{\cdot}8^2}{0{\cdot}2^4}$ (2 marks)

2 Simplify $\dfrac{7x-5}{6} + \dfrac{11-13x}{12}$ (2 marks)

3 A triangle has an area of 30cm^2 and the length of its base is 8cm. Calculate the vertical height of this triangle. (2 marks)

4 Given that θ is an acute angle and $\sin\theta = \frac{4}{5}$ find, without using tables, the value of $\cos(90° - \theta)$. (2 marks)

5 If $x^2 = 0{\cdot}6$, find the base ten logarithm of x correct to three decimal places. (2 marks)

6 From a rectangular sheet of metal, 75cm long and 40cm wide, 28 circular discs, each of radius 5cm are punched. Calculate the area of the sheet which remains. (Take π as $\frac{22}{7}$) (3 marks)

7 Laura, Mary and Nancy share 7,350 shillings in the proportion 7 : 4 : 3. Find the difference in Laura's and Nancy's share. (3 marks)

8 Solve the simultaneous equations

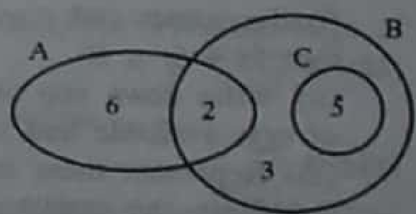
$$5x + 2y = 5$$
$$3x - 0{\cdot}2y = 10$$

(3 marks)

9 A map is drawn on the scale 1 : 50,000. A and B are two points on the map in which the length of line segment AB is 2·12cm. Calculate the distance between A and B on the ground in metres. (3 marks)

10 The quadrilateral OABC has vertices O(0, 0), A(4, 1), B(8, 6) and C(1, 5).
(i) Write down as column vectors **OB** and **AC**.
(ii) Calculate the lengths of $|\mathbf{OB}|$ and $|\mathbf{AC}|$. (3 marks)

11 The Venn diagram illustrates the activities of 30 students.
A = {students who play basketball}
B = {students who play soccer}
C = {students who play for the school team}



(i) How many students play games?
(ii) Find n(A ∪ B).
(iii) Describe in words the members of the set A ∩ B. (3 marks)

12 Given the matrices $M = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$ and $N = \begin{pmatrix} 3 & -4 \\ -6 & 8 \end{pmatrix}$
find (i) $M + N$, (ii) MN, (iii) M^{-1} (3 marks)

13 Calculate 3^1, 3^2, 3^3, 3^4, 3^5 giving your answers in base 6. (3 marks)

14 If $a * b$ denotes the arithmetic mean of two numbers a and b,
calculate (i) $8 * 20$, (ii) $(3 * 7) * 15$. (3 marks)

15 On graph paper draw a diagram to illustrate the region defined by the three inequalities
$$y \leq 5,\ 3y + x \geq 12,\ y \geq x$$ (3 marks)

SECTION B

16 In the equilateral triangle ABC, D is the mid-point of BC and E is the mid-point of AC. The lines AD and BE meet at X. The length of the line segment AB is 10cm. Calculate correct to three significant figures
(i) the lengths of the line segments AD and XD,
(ii) the area of the triangle ABC,
(iii) the area of the quadrilateral XDCE,
(iv) the area of the circle which passes through A, B and C.

17 The vectors OA and OC are given by $\mathbf{OA} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{OC} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ and O is the origin.
(i) Prove that the angle AOC is a right-angle.
(ii) OABC is a rectangle. Use vectors to prove that the coordinates of B are (5, 0).
Under the transformation with matrix $\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$ the points A, B, C are mapped onto A′, B′, C′.
(iii) Find the coordinates of these image points and show that OAA′ is a straight line.
(iv) Show the rectangle OABC and its image OA′B′C′ in a Cartesian diagram.
(v) Calculate the area of the rectangle OABC and the area of its image.

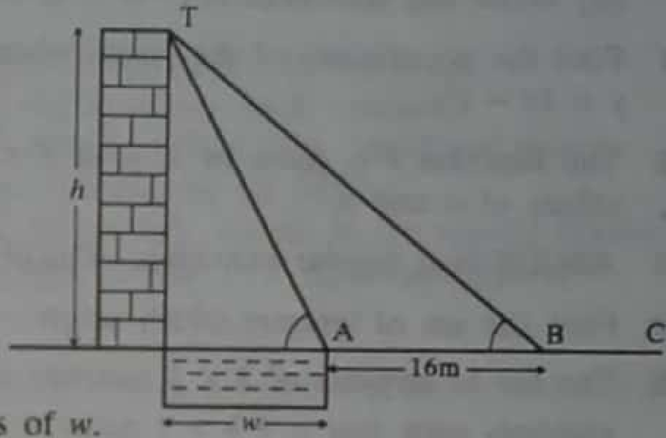
18 A fair die, coloured red, has six faces numbered 1, 1, 1, 2, 2, 3. A second fair die, coloured blue, has six faces numbered 1, 2, 2, 3, 3, 4. If both dice are rolled once, list the 36 possible outcomes.
Calculate the probability that when both dice are thrown
(i) the score on the red die will equal the score on the blue die,
(ii) the score on the red die will be less than the score on the blue die,
(iii) the sum on the two dice will be 5.
(Leave your answers as fractions in their lowest terms.)

19 To make one scone requires 275g of flour, 75g of butter and 3 eggs. One rock cake requires 330g of flour, 150g of butter and 2 eggs. A cook has 3·3kg of flour, 1·2kg of butter and 24 eggs with which to make as many of these cakes as possible.
(i) If x scones and y rock cakes are made, explain why the mass of flour available means that $5x + 6y \leq 60$.
(ii) Write down two other inequalities derived from the mass of butter and the number of eggs available and simplify them.
(iii) Represent these inequalities graphically.
(iv) State the maximum number of scones and cakes that can be made under these conditions and mark the corresponding point on your graph.

20 You are given the functions $f: x \rightarrow x^2 - 9$ and $g: x \rightarrow x + 1$.
(i) Find the values of f(3), g(−8), fg(2).
(ii) Write down the factors of $p^2 - q^2$ and hence or otherwise, express fg in the form $fg: x \rightarrow (x + a)(x + b)$ where a and b are integers.
(iii) Solve the equation $fg(x) = 0$.
(iv) Indicate on a sketch graph the points where the curve $y = fg(x)$ meets the axes.

21 The depth of water in a rectangular swimming bath increases uniformly from 1 metre at the shallow end to 3·5 metres at the deep end. The bath is 25 metres long and 12 metres wide. Calculate the volume of water in the bath, giving your answer in cubic metres. The bath is emptied by means of a cylindrical pipe of internal radius 9cm. The water flows down the pipe at a speed of 3m/s. Calculate the number of litres emptied from the bath in 1 minute, giving your answer to the nearest 10 litres. Take π to be 3·14.

22 A tourist wishes to estimate the height, h metres, of a tower and the width, w metres, of a moat that surrounds it (see diagram). Taking the shortest route, he walks from C to a point A on the bank of the moat. At B he estimates the angle of elevation of the top T of the tower to be 45°. At A he estimates it to be 60°. He measures the distance AB as 16 metres to the nearest metre.



(i) Write down two expressions for h in terms of w.
(ii) Hence calculate the values he obtained for h and w.
(iii) On enquiry at the information office the tourist is told that the height of the tower is 40 metres and the width of the moat is 25 metres.
(a) Calculate the correct angle of elevation at A.
(b) Find the percentage error in the estimate of the height, giving your answer to one significant figure.

PAPER 2

SECTION A

1 Express $\frac{3}{80}$ as an exact decimal. (2 marks)

2 Solve the equation $52(x - 3) + (4x - 5) = 7$ (2 marks)

3 Without using tables find the exact value of $\frac{22 \times 0{\cdot}693}{0{\cdot}99 \times 14}$ (2 marks)

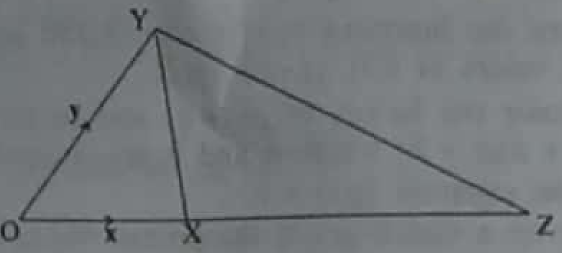
4 Given the formula $\frac{1}{x} = \frac{1}{y} + \frac{1}{z}$ express z in terms of x if $y = 5x$. (2 marks)

5 A pay rise of 9½% was given to a worker earning a wage of sh44,000. Find the wage of the worker after the pay rise. (2 marks)

6 Calculate the area of a circular field whose perimeter is 220 metres. (Take π as $\frac{22}{7}$) (3 marks)

7 Add together the number 1·3, the square root of the number 1·3 and the reciprocal of the number 1·3, giving your answer correct to two significant figures. (3 marks)

8 In the diagram shown $\mathbf{OX} = \mathbf{x}$, $\mathbf{OY} = \mathbf{y}$ and $\mathbf{OZ} = 3\mathbf{OX}$. Express as simply as possible $2\mathbf{OY} + \mathbf{ZY}$ in terms of $\mathbf{x}$ and $\mathbf{y}$.



(3 marks)

9 Given that 0·561 litres of hydrogen has a mass of 0·0502g calculate, correct to two decimal places, the mass of 1 litre of hydrogen. (3 marks)

10 $\mathcal{E}$ = {animals} W = {white animals} R = {rabbits} P = {animals with pink ears}
(i) Use some of the symbols P, R, $\subset$, $\cap$, $\varnothing$, $\neq$ to write the following statement:
"Some rabbits have pink ears."
(ii) Write the statement $R \cap W = \varnothing$ in non-mathematical everyday English. (3 marks)

11 Find the coordinates of the points where the line given by $y = 3$ cuts the curve given by $y = 4x - x^2$. (3 marks)

12 The function f is given by $x \rightarrow ax + b$. Given that $f(2) = 4$ and $f(-1) = -5$, find the values of a and b. (3 marks)

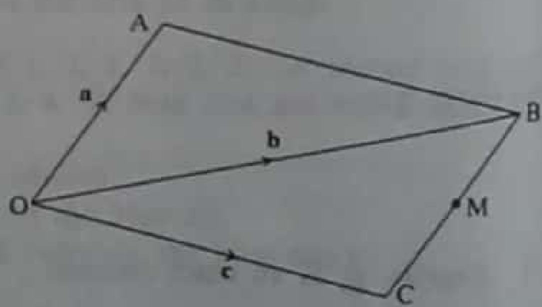
13 ABCDE is a regular pentagon. Calculate the size of $\angle CAD$. (3 marks)

14 Find the set of integers which satisfy $-3 < 2x + 3 \leq 1$. (3 marks)

15 The set M consists of 2×2 matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in which a, b, c and d are real numbers such that $a + b = 1$ and $c + d = 1$.
Two members of the set M are $\mathbf{P} = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 2 & -1 \\ -5 & 6 \end{pmatrix}$.
Determine which of the following are elements of M: (i) $\mathbf{PQ}$, (ii) $\mathbf{P}^2$, (iii) $\mathbf{Q}^{-1}$.
(3 marks)

SECTION B

16 OABC is a parallelogram. M is the mid-point of BC. The position vectors of A, B, C referred to the origin O are **a**, **b**, **c** respectively.
(i) Write down, in terms of **b** and **c**, expressions for the vectors **CB**, **CM**, **OM**.
(ii) Express **AM** in terms of **a**, **b** and **c**.
(iii) P is a point (not shown on the diagram), with position vector **p**, such that $\mathbf{AP} = \frac{2}{3}\mathbf{AM}$.



Express **AP** in terms of the position vectors **a** and **p**, and show that **p** can be expressed in the form $\frac{1}{n}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ where n is an integer and state the value of n.
(iv) Use the fact that $\mathbf{b} = \mathbf{a} + \mathbf{c}$ to find **p** in terms of **b** only. Deduce a result about the position of P.

17 A triangle has vertices at A(−2, −2), B(6, 0) and C(2, 4). The points A′, B′ and C′ are the images of A, B and C under the transformation S whose matrix is $\begin{pmatrix} -0{\cdot}5 & 0 \\ 0 & -0{\cdot}5 \end{pmatrix}$.
(i) Find the coordinates of A′, B′ and C′ and show the triangles ABC and A′B′C′ on one diagram.
(ii) Describe S in words as fully as possible.
(iii) Find the areas of the triangles A′B′C′ and ABC.

18 A salesman sells articles at sh500 each. He sells 2,000 articles in the first week. In the second week he sells 15% more than in the first, and in the third week he sells 10% more than in the second. Express the number of articles sold in the third week as a percentage of the number sold in the first week.
Each week he receives $4\frac{1}{2}$% of the price of each article on each of the first 1,600 sold and 7% of the price of each article he sells in excess of 1,600. Calculate the amount he receives in the third week.
In the fourth week the salesman receives sh58,400. Calculate the number of articles sold in this week.

19 A goldfish bowl is part of a sphere of radius 10cm. Using a reference system as for the Earth, the plane base is at latitude 60°S and the rim is at 60°N. The water level is at 10°N.
(i) An ant on the outside at the equator crawls to the rim by the shortest route. How far does it crawl?
(ii) The ant now falls in and remains stationary on the surface of the water.
Calculate (a) the distance of the ant from the centre of the water surface,
(b) the distance it has fallen to the surface, correct to 2 significant figures.
(iii) A small fish at the centre of the bowl sees the ant on the surface.
Calculate (a) the angle of elevation of the ant from the fish, correct to the nearest degree,
(b) the distance of the ant from the fish, correct to 2 significant figures.
(Take π as 3·14)

20 (i) Calculate values of $y = 8 - x^2$ for $x = 0, 0{\cdot}5, 1, 1{\cdot}5, 2, 2{\cdot}5$ showing your results in a table.
(ii) Draw the graph of $\{(x, y) : y = 8 - x^2\}$ for values of x from 0 to 2·5 inclusive.
(iii) Draw and label the line whose equation is $y = 2x$.
(iv) Shade carefully the parts of the diagram not in the set
$$A = \{(x, y) : x \geq 0,\ y \geq 2x,\ y \leq 8 - x^2\}$$
(v) Draw the straight line with equation $8x + y = 8$, labelling it clearly. Hence mark the point P in A where $8x + y$ has its maximum value and state this value.

21 A boat on the Nile travels up and down the river between two places A and B. The speed of the boat is 10km/h in still water. When there is a current of x km/h flowing, the speed of the boat is $(10 + x)$km/h downstream and $(10 - x)$km/h upstream. The distance between A and B along the river is 15km.
(i) Calculate the total time travelling from A to B and back to A (a) when there is no current flowing, (b) when there is a current of 2km/h flowing.
(ii) When there is a current of x km/h flowing it is found that the total time for this double journey is 3h 12min. Find x.

22 A group of 120 students was given a test in which the maximum mark was 10. A summary of the results is given in the table.

Mark	0	1	2	3	4	5	6	7	8	9	10
Frequency	2	3	7	10	13	17	26	19	12	7	4

(i) Draw a frequency polygon to illustrate this data.
(ii) Calculate the mean mark.
(iii) What is the median mark?
(iv) Find what percentage gained at least 7 marks.

PAPER 3

SECTION A

1 Find the exact value of $2\frac{1}{2} + (\frac{3}{5} \times 1\frac{1}{4}) - 1\frac{1}{8}$ (2 marks)

2 Find how many hours there are between 18.35 hours on 8th June 1992 and 07.35 hours on 11th June 1992. (2 marks)

3 A motorist drives 48km/h for 2 hours and then 60km/h for 1 hour. Find his average speed for the whole journey. (2 marks)

4 An equilateral triangle is inscribed in a circle of radius 5cm. Calculate the length of a side of the triangle giving your answer correct to one place of decimals. (3 marks)

5 Which value of n gives $34_n = 22_{10}$? (2 marks)

6 By sketching a rectangle and suitably dividing it illustrate the algebraic relation

$$(a + b)(a + c) = a^2 + ac + ab + bc$$ (3 marks)

7 $f : x \to 2x + 5 \quad g : x \to x^2 - 25 \quad h : x \to x(x + 5)$

(i) Evaluate f(4) and g(−3).

(ii) Write down and simplify an expression for gf(x).

(iii) Express gf(x) in terms of h(x). (4 marks)

8 Given that $a = b + \frac{c}{kx}$ express x in terms of a, b, c and k, (2 marks)

9 Two boxes each contain seven blue and three red beads. One bead is drawn at random from each box. Find the probability that one bead is blue and one is red. (3 marks)

10 P = {factors of 24} and Q = {factors of 30}. List the elements of P ∩ Q. (2 marks)

11 A shop sold 6,800 articles in the first week and 7,140 articles in the second week. Given that sales are increasing at the same percentage each week, calculate the number of articles sold in the third week. (3 marks)

12 Find the value of x if $4 \times 2^{3(x-1)} = 2^{-x}$ (3 marks)

13 In an election there were two candidates. If 27,180 votes were cast and the winner gained 3,558 votes more than the loser, how many votes did the loser receive? (3 marks)

14 An aircraft which is about to land is 1,000m high and at a horizontal distance of 5,000m from its point of touch down. Assuming that the aircraft descends in a straight line, calculate the angle that its line of descent makes with the horizontal. (3 marks)

15 The following is an extract from a table of income tax rates for non-residents:

	Total Annual Income	*Tax Rate*
1.	Where total income does not exceed sh620,000	10 per cent
2.	Where total income exceeds sh620,000 but does not exceed sh870,000	sh62,000 plus 20 per cent of the amount by which total income exceeds sh620,000
3.	Where total income exceeds sh870,000 but does not exceed sh1,310,000	sh112,000 plus 30 per cent of the amount by which total income exceeds sh870,000

What tax per year should a non-resident pay whose total monthly income is sh100,000? (3 marks)

SECTION B

16 A gardener bought a number of plants for sh19,800. Later he bought four extra plants at a cheaper price for a further sh360 altogether. This second purchase reduced the average cost of all the plants by sh30.
Find the number of plants he bought originally.

17 The points A and B and the foot P of a vertical tower PT are at the corners of a triangle on level ground. The length of AB is 100m, the angle PAB is 59° and angle PBA is 64°. If the angle of elevation of T from A is 30°, use scale drawing, or otherwise, to find
(i) the length of PA
(ii) the height of the tower PT,
(iii) the angle of elevation of T from B.
(Give your answers to the nearest whole number.)

18 The vertices of an isosceles triangle E have coordinates A(0, 5), B(−1, 8), C(4, 8). Draw and label E on a cartesian graph with scales from −2 to 16 on both axes.

Transformation S is defined as S : $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
Draw and label the isosceles triangle S(E) and the point S(A).
Describe fully the single transformation S.

Transformation U is defined as U : $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -1{\cdot}2 & 1{\cdot}6 \\ 1{\cdot}6 & 1{\cdot}2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
Draw and label the isosceles triangle U(E) and the point U(A).
Describe fully in geometrical terms a single transformation that maps the triangle U(E) onto the triangle S(E) so that U(A) maps onto S(A).

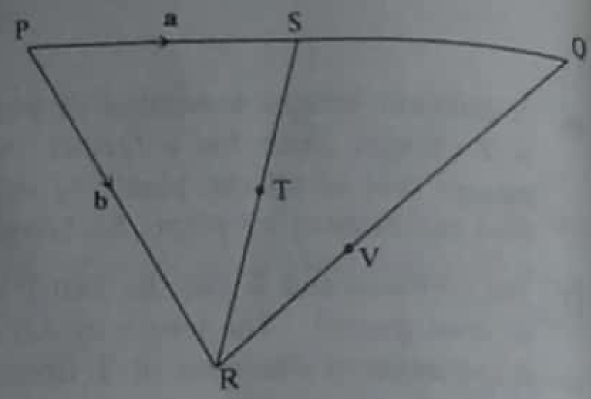
19 A bakery has two factories A and B which carry out x bakings and y bakings respectively over a given period. The following table shows the production at each baking.

	A	B
Large loaves	180	270
Small loaves	100	250
Rolls	200	100

Production at the end of the period must be such that not more than 2,700 large loaves, not less than 1,500 small loaves and not less than 1,400 rolls are produced.
(i) State three inequalities in x and y to satisfy the given conditions and draw appropriate graphs and shade the area which satisfies these conditions.
(ii) Using your graphs estimate the value of x and the value of y to give the minimum number of bakings.
(iii) State the number of bakings possible at Factory B when exactly six bakings are made at Factory A.

20 (a) The length of the side of a square ABCD is 40m. A point P is taken in the side BC so that angle BAP is 32°. Calculate
(i) the length of BP, correct to three significant figures,
(ii) the angle BDP.
(b) Two similar wedding cakes A and B have corresponding linear measurements in the ratio 2 : 3 respectively.
(i) Given that the vertical height of A is 8cm calculate the vertical height of B.
(ii) The area requiring icing on B is 2,700cm^2. Calculate the area requiring icing on A.
(iii) The cost of the flour to make A is sh4,000. What is the corresponding cost for B?

21 In the diagram, PS = **a**, PR = **b** and S is the mid-point of PQ. T is the mid-point of SR and QV = 2VR.
Express the following in terms of **a** and **b**.
(i) SR, (ii) ST, (iii) PT, (iv) PQ, (v) QR, (vi) VR, (vii) PV
Show that PT = kPV where k is a scalar, and find the value of k.
What can you deduce about the points P, T and V?



22 A machine component is made from a cone whose base is of radius 10cm and whose vertical height is 9cm This cone has a hemispherical cavity of radius 4·5cm drilled from its plane base. Calculate, correct to two significant figures
(i) the volume of the original cone,
(ii) the total surface area of the original cone,
(iii) the volume of the machine component,
(iv) the total surface area of the machine component. (Take π as 3·14)

PAPER 4

SECTION A

1 Simplify $\dfrac{2\frac{5}{6} + 2\frac{2}{15} \times \frac{5}{8}}{5\frac{1}{3} - 1\frac{7}{12}}$ (3 marks)

2 Use tables to find the cube root of 0·02597 correct to three decimal places. (2 marks)

3 Describe the transformation represented by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. (2 marks)

4 Calculate the area of a parallelogram whose adjacent sides are 3·9cm and 2·4cm in length and contain an angle of 57°, giving your answer correct to 2 dp. (2 marks)

5 A leopard runs 800 metres in 50 seconds. Calculate its speed in km/h. (2 marks)

6 Solve the equation $\dfrac{3x + 2}{x + 2} = x$ (3 marks)

7 Two simple functions f and g are such that gf : $x \to \dfrac{1}{x + 1}$. Find fg. (2 marks)

8 Find the probability that the sum obtained when two dice are rolled is a prime number. (3 marks)

9 Find BA if A = $\begin{pmatrix} 1 & 0 \\ 1 & -3 \\ 1 & 2 \end{pmatrix}$ and B = $\begin{pmatrix} -1 & 0 & 2 \\ 0 & 2 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (3 marks)

10 AB and BC are two adjacent sides of a regular 12-sided polygon. The perpendicular from C meets AB produced at D. Calculate ∠BCD. (3 marks)

11 P, Q, R and S are points such that PQ = $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$, PR = $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and PS = $\begin{pmatrix} -1 \\ 7 \end{pmatrix}$.
Find QR and RS. What conclusion can you draw about Q, R and S? (3 marks)

12 P is the point (3, 4) and Q is the point (−4, 5). Find the area of triangle OPQ, where O is the origin of coordinates. (3 marks)

13 Solve the equation $3x^2 + 6x - 7 = 0$ giving the roots correct to 2 dp. (3 marks)

14 Draw a Venn diagram showing sets P and Q such that $n(\mathcal{E}) = 24$, $n(P) = 11$, $n(Q) = 7$ and $n(P \cap Q) = 3$. What is $n(P \cup Q)$ and $n(P \cup Q)'$? (3 marks)

15 Without using tables, find the value of $3 \log_{10}5 + 5 \log_{10}2 - \frac{1}{2}\log_{10}16$. (3 marks)

SECTION B

16 In 1989 the total cost of manufacturing an article was sh1,250 and this was divided between the costs of material, labour and transport in the ratio 8 : 14 : 3. In 1990 the cost of material remained unaltered, labour costs increased by 30% and transport costs increased by 20%. Calculate the cost of manufacture of the article in 1990.
In 1991 the cost of manufacture of the article was sh2,030. The cost of material had doubled and transport charges were the same as in 1990. Express the labour costs in 1991 as a percentage of the labour costs in 1989.

17 C is the foot of a vertical tower, CT, 30m high. Points A and B are in the same horizontal plane as C and they are equidistant from C. P is the point on AB which is nearest to C.
Given that the angle of elevation of the top of the tower from P is 48° and that $\angle ACB = 120°$, calculate
(i) the length of CP,
(ii) the length of CB,
(iii) the length of AB,
(iv) the angle of elevation of the top of the tower from B.

18 The table of values given refers to the function $y = Ax^2 + Bx$.

x	−1	−0·5	0	0·5	1	1·5	2	2·5
y	5	2	0	−1	−1	0	2	5

Calculate the values of A and B.
Plot the graph of $y = Ax^2 + Bx$ for values of x from −1 to 2·5 and, by drawing a suitable straight line, obtain the solution to the equation $Ax^2 + Bx = x + 1$.

19 Some students were asked what games they played. 12 played rugby, 16 played squash, 13 played tennis and 8 played none of these games. 3 played both rugby and squash, 5 played both rugby and tennis and 2 played only tennis. Let R, S and T be the sets rugby, squash and tennis players respectively. Let the number playing all three games be x. Draw a Venn diagram and show, in terms of x, the number of students in each region of the diagram in set R. Also show the number in each of the other four regions.
Find the total number of students questioned and state the possible values of x.
A student is chosen at random. Find the probability that this student plays neither rugby nor tennis.

20 (i) Calculate the coordinates of the image of the point A(4, 3) obtained by premultiplying its column vector $\binom{4}{3}$ by the matrix $\begin{pmatrix} 0{\cdot}6 & -0{\cdot}8 \\ 0{\cdot}8 & 0{\cdot}6 \end{pmatrix}$.
(ii) Show that this mapping is a rotation and find the angle of this rotation about the origin, O.
(iii) Calculate the image of B under the same rotation where B is the point (2, 5).
(iv) Hence, by finding the area of its image, state the area of triangle OAB.

21 ABCDE is a five-sided plane figure (not regular) in which BC = AE, CD = DE and $\angle BCD = \angle DEA$.
Prove that (i) BD = AD,
(ii) $\angle ABC = \angle EAB$,
(iii) AC = BE.

22 OABC is a quadrilateral. P, Q and R are points on OA, OB and OC respectively such that OA = 3OP, OB = 5OQ and OC = 2OR.
Taking **OP** = **p**, **OQ** = **q** and **OR** = **r**, express in terms of **p**, **q** and **r** the vectors **AB**, **BC** and **CA**.
Given further that OABC is a parallelogram, show that $3\mathbf{p} - 5\mathbf{q} + 2\mathbf{r} = 0$.
Hence express **PQ** and **QR** in terms of **p** and **q** only.
Show that PQR is a straight line.
Find the ratio PQ : QR.

PAPER 5

SECTION A

1 Simplify $\dfrac{2\frac{4}{5} + 1\frac{1}{4}}{3\frac{3}{5}} - \frac{5}{16}$ (3 marks)

2 Three warning lights flash at intervals of 18, 21 and 28 seconds respectively. Given that they all start flashing together, after how long will they again flash together? (2 marks)

3 Convert a pressure of $1{\cdot}87\text{kg/cm}^2$ to tonnes/m^2. (3 marks)

4 Find the mean height of the following students (given in cm): 157, 152, 160, 145, 162, 148, 159, 143, 163, 151. (2 marks)

5 Convert 194_{10} to base five. (2 marks)

6 A letter is chosen at random from the word PROBABILITY. Find (i) the probability of choosing a vowel, (ii) the probability of choosing a consonant. (2 marks)

7 For the sets P and Q, $n(P) = 15$, $n(Q) = 19$ and $n(P \cap Q) = 7$. Find $n(P \cap Q')$ and $n(P' \cap Q)$. (2 marks)

8 The population of a village increased from 840 to 903 in one year. Express the increase as a percentage of the initial population. (2 marks)

9 When the petrol in the tank of a car is full the mass of the petrol is 30kg. Given that 1cm^3 of petrol has a mass of $0{\cdot}79$g, calculate the volume of the tank in litres giving your answer correct to the nearest litre. (3 marks)

10 Factorise $21x^2 - 8x - 4$. (3 marks)

11 Find the matrix $M = \begin{pmatrix} p & q \\ 0 & s \end{pmatrix}$ given that $M\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $M\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$. (3 marks)

12 A pyramid VABCD on a square base has each of its 8 edges of length 10cm. Calculate the angle between edges VA and VC. (4 marks)

13 Calculate the number of sides of a regular polygon in which each of the interior angles is 165°. (3 marks)

14 A, B and C have position vectors **a**, **b** and **c** respectively relative to the origin O. Give in terms of **a**, **b** and **c** (where appropriate) the vectors CB and $\frac{1}{3}$AC. (2 marks)

15 The four vertices of a quadrilateral are at the points representing the 2nd, 4th, 7th and 11th hours on a circular clock face. Calculate the sizes of the interior angles of the quadrilateral. (4 marks)

SECTION B

16 The values of the three main kinds of foods: protein, fat and carbohydrate, measured in calories per gram, are 4, 9 and 4 respectively. Calculate the number of calories in a man's daily diet which contains 100g of protein, 100g of fat and 400g of carbohydrate.
The man reduces his daily protein to 50g and his daily fat to 64g. Calculate how many grams of carbohydrate he needs each day to have the same number of calories daily as he had before.
The prices for equal masses of these three kinds of food are in the ratio 6 : 3 : 1. Calculate, correct to the nearest whole number, the percentage reduction in this man's daily food bill.

17 Three towns A, B and C lie on a straight road running East from A. B is 3km from A, and C is 11km from A. Another town D lies to the North of this road and is 5km from B and 7km from C. Using scale drawing, or otherwise, find
(i) the distance of D from A,
(ii) the bearing of D from A,
(iii) the time taken by a man to walk at 4km per hour from A to D then to C and then via B back to A.

18 Copy and complete the following table giving, correct to two decimal places, the values of the function $x + \frac{1}{2x}$ for the given values of x.

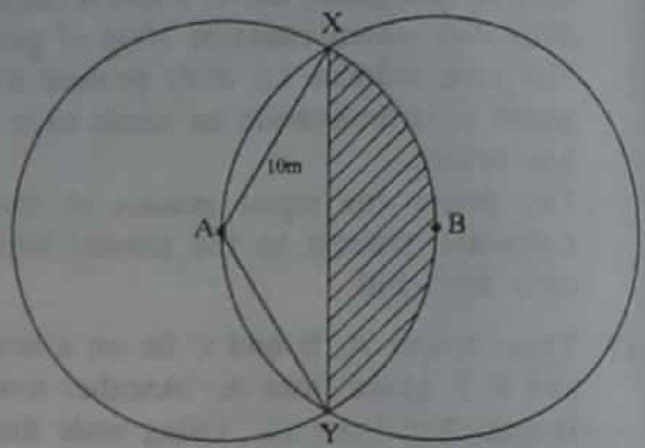
x	0·15	0·25	0·4	0·5	1·0	1·5	2·0	2·5	3·0
$\frac{1}{2x}$	3·33		1·25			0·33			0·17
$x + \frac{1}{2x}$	3·48		1·65			1·83			3·17

Draw the graph of $y = x + \frac{1}{2x}$ for the values of x from 0·15 to 3·0.
From your graph obtain (i) the minimum value of $x + \frac{1}{2x}$
(ii) the values of x for which $x + \frac{1}{2x} = 2$.
By drawing a suitable straight line to intersect the first graph, write down the values of x which satisfy the equation $2x + \frac{1}{2x} = 3$.

19 The set of divisors of a whole number n, including n itself but not 1, is denoted by D(n), so that, for instance D(6) = {2, 3, 6}.
(i) List the elements of the sets D(8), D(20), D(8) ∩ D(20) and D(8) ∪ D(20).
(ii) State the value of r for which D(8) ∩ D(20) = D(r). How is r related to 8 and 20?
(iii) State the least value of s for which D(8) ∪ D(20) ⊂ D(s). How is s related to 8 and 20?

20 A school receives a grant of sh480,000 with which to buy x standard calculators and y super calculators. Given that each standard calculator costs sh15,000 and each super calculator costs sh30,000, form an inequality in x and y and simplify it.
The school decides that the total number of calculators must be at least 20 and that the number of standard calculators must be not more than twice the number of super calculators. From this latter information form two further inequalities in x and y. Draw appropriate graphs and, by shading, indicate clearly the region consisting of points whose coordinates satisfy the three inequalities.
Given that each standard calculator will be used by 5 students and each super calculator by 3 students, estimate the number of standard calculators and the number of super calculators which should be bought for maximum student usage under the given conditions.

21 Two goats are tethered to points A and B which are 10m apart on level ground. Each tether is 10m long. The diagram shows the total area which the two goats can cover between them.
XY is the common chord of the two circles. The segment XBY of the circle, centre A, has been shaded.
Calculate, giving your answers correct to the nearest whole number



(i) $\angle XAY$ in degrees,
(ii) the area of $\triangle AXY$ in square metres,
(iii) the area of the sector AXBY,
(iv) the area of the segment XBY (shaded),
(v) the total area of ground which the two goats can cover. (Take π as 3·14)

22 A cyclist starts from rest and moves along a straight road. His speed is measured at 2-second intervals from the start and is shown in the following table.

Time (t seconds)	0	2	4	6	8	10	12
Speed (v m/s)	0	0·2	1	2·5	4·8	8	12·1

Draw the speed-time graph by plotting v against t and joining the points with a smooth curve.
(i) Draw a tangent to the curve at $t = 6$. Measure and write down the gradient of the tangent. What is your estimate of the acceleration of the cyclist when $t = 6$?
(ii) Estimate the area under your curve from $t = 0$ to $t = 12$, stating clearly how you have made the estimate. Hence give an estimate of the distance travelled by the cyclist during the 12 seconds.

Answers

Exercise 1a Page 1

1 546,239 **2** 20,202,020 **3**(i) seven hundred and ninety eight thousand one hundred and forty-two (ii) one hundred million one hundred and one thousand and one **4** 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47 **5** 53, 59, 61, 67, 71, 73 **6**(i) 1, 2, 4, 8, 16 (ii) 1, 2, 3, 6, 9, 18 (iii) 1, 2, 4, 5, 10, 20 (iv) 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72, 144 (v) 1, 2, 4, 8, 16, 32, 64, 128, 256 **7** 1,650 **8** 8 **9** 45 **10**(i) 6 (ii) 210 (iii) 1,260 (iv) 1,260 (v) equal **11**(i) 18 (ii) 504 (iii) 9,072 (iv) 9,072 (v) equal **12** 35 **13** 300,300 **14** $1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248 = 496$; $1 + 2 + 4 + 8 + 16 + 32 + 64 + 127 + 254 + 508 + 1{,}016 + 2{,}032 + 4{,}064 = 8{,}128$ (The next two perfect numbers are 33,550,336 and 8,589,869,056)

Exercise 1b Page 2

1 18 **2** 30 **3** 13 **4** 6 **5** 22 **6** 14 **7** 12 **8** 3 **9** 9 **10** 19 **11** 3 **12** 18 **13** 39 **14** 0 **15** 14 **16** 1 **17** 36 **18** 13 **19** 45 **20** 64 **21** $(24 - 7) + 1$ **22** $24 - (7 + 1)$ **23** $(36 \times 2) + 7$ **24** $36 \times (2 + 7)$ **25** $60 \div (4 \times 3)$ **26** $(60 \div 4) \times 3$ **27** $(128 \div 8) + (8 \times 2)$ **28** $[128 \div (8 + 8)] \times 2$ **29** $128 \div [(8 + 8) \times 2]$ **30** $[(128 \div 8) + 8] \times 2$

Exercise 2a Page 3

1 0, +2 **2** −6, −4 **3** < **4** < **5** > **6** > **7** > **8** > **9** > **10** < **11** < **12** > **13** < **14** >

Exercise 2b Page 3

1 +8 **2** −3 **3** −8 **4** −2 **5** +10 **6** −2 **7** 0 **8** −2 **9** 0 **10** −4 **11** 60 **12** −91 **13** −54 **14** −7 **15** 6 **16** −23 **17** −2 **18** 98 **19** −16 **20** 31

Exercise 2c Page 4

1 12 **2** −12 **3** −12 **4** −2 **5** 3 **6** −25 **7** 3 **8** −26 **9** −9 **10** 9 **11** $(4 \times -3) + (-2 \times -1)$ **12** $[4 \times (-3 + (-2))] \times -1$ **13** $[(4 \times -3) + (-2)] \times -1$ **14** $4 \times [-3 + (-2 \times -1)]$ **15** $(-32 \div 8) + (8 \times -2)$ **16** $[-32 \div (8 + 8)] \times -2$ **17** $[(-32 \div 8) + 8] \times -2$ **18** $-32 \div [(8 + 8) \times -2]$

Exercise 2d Page 4

1 −sh200,000 **2** −3m/s **3** -10m/s^2 **4** −100m **5** −10min **6** −sh2,300 **7** −17°C **8** +sh7,500 **9** −32°C **10** −260m

Exercise 3a Page 5

1 5, 12, 25, 100 **2** $\frac{7}{4}$, $\frac{21}{5}$, $\frac{35}{6}$, $\frac{50}{7}$ **3** $1\frac{3}{4}$, $4\frac{1}{5}$, $5\frac{2}{7}$, $4\frac{5}{9}$ **4** $\frac{1}{4}$, $\frac{1}{3}$, $\frac{2}{5}$, $\frac{3}{7}$, $\frac{1}{2}$ **5** $\frac{2}{3} = \frac{6}{9} = \frac{8}{12}$; $\frac{4}{8} = \frac{5}{10}$ **6** $\frac{2}{3}$, $\frac{7}{8}$, $\frac{9}{10}$, $\frac{8}{9}$, $\frac{16}{17}$ **7** 12, 8, 18, 20, 9, 14 twenty-fourths **8** (i) $\frac{1}{10}$ (ii) $\frac{3}{5}$ (iii) $\frac{1}{10}$ (iv) $\frac{5}{12}$ (v) $\frac{1}{3}$ (vi) $\frac{1}{16}$

Exercise 3b Page 6

1 $\frac{5}{7}$ **2** $\frac{1}{2}$ **3** $\frac{10}{63}$ **4** $8\frac{1}{4}$ **5** $1\frac{5}{6}$ **6** $\frac{23}{60}$ **7** 2 **8** $\frac{19}{20}$ **9** $1\frac{1}{20}$ **10** $1\frac{13}{84}$

Exercise 3c Page 7

1 $\frac{2}{21}$ **2** $\frac{3}{10}$ **3** $1\frac{1}{2}$ **4** $2\frac{1}{3}$ **5** $4\frac{1}{4}$ **6** $\frac{5}{6}$ **7** $2\frac{2}{3}$ **8** $\frac{7}{39}$ **9** $4\frac{1}{6}$ **10** $1\frac{1}{3}$

Exercise 5e
Page 14

1 T 2 T 3 F 4 T 5 T 6 F 7 F 8 T 9(i) A (ii) B 10 X = Y 11 P ∩ T = {2} 12 E ⊂ F; 8 is a multiple of 4 13 S ⊂ Q 14 43

Exercise 6a
Page 16

1 ninety-one, $9 \times 10^1 + 1 \times 10^0$ 2 one hundred and fifty-four, $1 \times 10^2 + 5 \times 10^1 + 4 \times 10^0$ 3 six hundred and thirty, $6 \times 10^2 + 3 \times 10^1 + 0 \times 10^0$ 4 one thousand and twenty-seven, $1 \times 10^3 + 0 \times 10^2 + 2 \times 10^1 + 7 \times 10^0$ 5 twenty-seven thousand and one, $2 \times 10^4 + 7 \times 10^3 + 0 \times 10^2 + 0 \times 10^1 + 1 \times 10^0$
6 one hundred and one and eleven hundredths, $1 \times 10^2 + 0 \times 10^1 + 1 \times 10^0 + 1 \times 10^{-1} + 1 \times 10^{-2}$ 7 a human being has ten fingers and thumbs (digits) on his hands 8 Roman system; I, V, X, L, C, D, M 9 101,010,101; one hundred and one million ten thousand one hundred and one 10 80,800,008; eighty million eight hundred thousand and eight 11 111,111,111 12 20,202

Exercise 6b
Page 17

1 8_{10} 2 23_{10} 3 63_{10} 4 64_{10} 5 511_{10} 6 512_{10} 7 12_8 8 20_8 9 45_8 10 143_8 11 1673_8 12 3720_8 13 26_8 14 64_8 15 1110_8 16 a base eight numeral cannot have the digit 8 17 652_8 18(i) 350_8 (ii) 4720_8; place a zero at the end of the number being multiplied by 10_8

Exercise 6c
Page 18

1 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010 2 1111111_2, 127_{10} 3 12_{10} 4 16_{10} 5 27_{10} 6 90_{10} 7 129_{10} 8 1101_2 9 11001_2 10 101111_2 11 1101000_2 12 11001010_2 13 1000_2 14 101001_2 15 111111_2 16 111110_2 17 100000000_2 18(i) one and five-eighths (ii) one and eleven-sixteenths 19(i) $10{\cdot}01_2$ (ii) $2{\cdot}2_8$ 20 1000_2 21 101110111_2; $5_8 = 101_2$, $6_8 = 110_2$, $7_8 = 111_2$ 22 100_2, 111_2, 101_2; 100111101_2 23 $4_8 = 100_2$, $2_8 = 010_2$, $3_8 = 011_2$, 100010011_2 24(i) 765_8 (ii) 1635_8 25(i) 1101010_2 (ii) 11100111_2 (iii) 1011_2

Exercise 7a
Page 19

1(i) 70mm (ii) 25mm (iii) 2·5mm 2(i) 0·7cm (ii) 700cm (iii) 7cm 3(i) 1·5m (ii) 2·5m (iii) 59m 4(i) 2·5km (ii) 0·259km (iii) 2·5km

Exercise 7b
Page 19

1(i) 1,000mm² (ii) 260mm² (iii) 200,000mm² 2(i) 9cm² (ii) 80cm² (iii) 0·79cm² 3(i) 50,000cm² (ii) 92,000cm² (iii) 2,000,000cm² 4(i) 90ha (ii) 420ha (iii) 1·5ha 5 19,800ha

Exercise 7c
Page 19

1(i) 0·1*l* (ii) 2·567*l* (iii) 1,000*l* 2(i) 25d*l* (ii) 0·25d*l* (iii) 3·24d*l* 3(i) 1,000cm³ (ii) 242cm³ (iii) 200,000cm³ 4(i) 1*l* (ii) 24*l* (iii) 500*l*

Exercise 7d
Page 20

1(i) 9,200g (ii) 92g (iii) 125g 2(i) 2·4kg (ii) 0·942kg (iii) 0·065kg 3(i) 0·25t (ii) 25t (iii) 0·025t 4(i) 2,500kg (ii) 250kg (iii) 0·25kg

Exercise 7e
Page 20

1 118m, 114m² 2 36m² 3 1·5km, 13·5ha 4 3cm 5 6cm 6 8cm, 16cm 7 216cm³, 216cm² 8 10·5cm 9 135*l* 10 1,110cm³

Exercise 7f
Page 21

1 33cm, 86·6cm² (3 sf) 2(i) 7cm (ii) 154cm² 3 3,850m² 4(i) 14mm (ii) 88mm 5 62·8cm 6 44m 7 29·92cm, 188cm² (3 sf) 8 110km 9 28cm²

Exercise 7g Page 22

1 sh1, sh5; sh5, sh10, sh20, sh50, sh100, sh200, sh500, sh1,000 **2** sh34,500 **3** sh74,400 **4**(i) sh4,900 (ii) sh11,550 (iii) sh16,200,000 **5** sh22,800

Exercise 7h Page 22

1 1h 45min, 1h 45min, 15.05, 03.20, 06.25 next day **2** 696 **3** 40ms, 40,000mcs, 40,000,000ns **4** 2,000,000,000; 120,000,000,000 **5** 5h 20min

Exercise 7i Page 23

1(i) 273°K (ii) 373°K (iii) 250°K (iv) 320°K (v) 1°K **2**(i) 27°C (ii) 127°C (iii) −51°C (iv) −123°C (v) −173°C **3** 717·4°K **4**(i) 45°C (ii) 100°C (iii) 60°C (iv) 150°C (v) 0°C (vi) −40°C **5**(i) 318°K (ii) 373°K (iii) 333°K (iv) 423°K (v) 273°K (vi) 233°K

Exercise 8a Page 24

1 42km/h **2** 4h **3** 108km **4** 2h **5** $58\frac{1}{3}$km/h

Exercise 8b Page 25

1 4m/s² **2** 15m/s **3** $6\frac{2}{3}$s **4**(i) 1·5m/s² (ii) 1·25m/s² (iii) 0·5m/s² (iv) 0·5m/s² (v) 0·8m/s²

Exercise 8c Page 25

1 4·5*l*/min **2** 2 hours **3** 5·4*l* **4** 2min 55s

Exercise 8d Page 25

1 sh750 **2** sh650 **3** sh250 **4** sh900 **5** sh650 **6** sh450 **7** sh450 **8** sh400 **9** sh500 **10** sh850

Exercise 8e Page 26

1 12·25g/cm³ **2** 12·1g/cm³ **3** 11·13g/cm³ **4** 10·16g/cm³ **5** 10·05g/cm³

Exercise 8f Page 26

1 83/km² **2** 150; 950,000; 8,200; 14·5; 500,000; 4,600; 2,800; 230

Exercise 9a Page 27

1 2 : 3 **2** 1 : 5 **3** 1 : 8 **4** 5 : 48 **5** 10 : 9 **6** 1 : 2 **7** 1 : 9 **8** 18 : 1 **9** 24 : 25 **10** 24 : 13 **11**(i) 4 : 3 (ii) 3 : 4 **12**(i) 3 : 2 (ii) 3 : 2 (iii) 9 : 4

Exercise 9b Page 27

1 20, 30 **2** 5, 35 **3** sh6,500, sh3,500 **4** sh1,500, sh3,500, sh5,000 **5** sh400, sh800, sh2,400 **6** sh1,440, sh720, sh240 **7** 10cm, 20cm, 30cm, 40cm **8** 100m, 200m, 300m, 400m, 500m **9** 60cm **10** sh9,000 **11** 300g **12** 14

Exercise 9c Page 28

1 sh7,200 **2** sh5,000 **3** 64cm³ **4** 36cm³ **5** 45cm by 30cm, (i) 2 : 5 (ii) 4 : 25 **6**(i) 3 : 4 (ii) 9 : 16

Exercise 9d Page 28

1 3 : 4 : 8 **2** 14 : 12 : 39 **3** 12 : 20 : 35 **4** 2 : 4 : 6 : 7 **5** 4 : 8 : 12 : 9 : 6 **6** 15 : 8

Exercise 9e Page 28

1 sh1,500 **2** 800cm³ **3** 600m*l* **4** 530g **5** 1·59t **6** 369g **7** sh13,200 **8** 21h **9** sh2,270,000 **10** 82·08h

Exercise 9f Page 29

1 1 : 10,000 **2** 1 **3** 2·5 **4** 5 **5** 1 : 20,000 **6** 4cm **7** 510km

Exercise 10a — Page 30

1 sh750 2 14 3 141km 4 14h 5 100 6 14kg 7 3·75min 8 7,500kg 9 20kg 10 sh4,800 11(i) sh$\frac{500m}{n}$ (iii) sh$\frac{9h}{4}$ (vi) $\frac{10^6 NK}{A}$kg (viii) $\frac{73}{2x}$min (x) $2{\cdot}5x$ bags

Exercise 10b — Page 31

1 4 days 2 40h 3 22·5 days 4 20 taps 5 2h 6 16 days 7 doubled 8 14 minutes 9 30 boxes 10 5 days 11 $1\frac{1}{3}$h 12(i) $\frac{xy}{z}$h (ii) $\frac{st}{v}$days (iii) $\frac{pn}{m}$days (iv) $\frac{cd}{e}$days (v) $\frac{cd}{f}$men

Exercise 11a — Page 32

1(a) $\frac{9}{16}$ (b) $6\frac{1}{4}$ (c) 0·0009 (d) 0·25 (e) 0·0025 (f) 1·21 (g) 1,000,000 (h) 10^8 (i) 36 (j) 0·01 2(a) 15,129 (b) 622,521 (c) 20·7936 (d) 0·974169 (e) 0·00826281 3(a) 289 (b) 961 (c) 1,020 (d) 7,570 (e) 11,000 (f) 9·99 (g) 10·0 (h) 0·0548 (i) 0·000151 (j) 0·00000408 4(a) 25 (b) 49 (c) 36 (d) 4 (e) −7 (f) 1 (g) 625 (h) 49 5 5·5cm, 6·5cm; 30·25cm² and 42·25cm²

Exercise 11b — Page 34

1(a) 12 (b) $\frac{2}{3}$ (c) 0·6 (d) 0·2 (e) $1\frac{1}{2}$ (f) 0·08 (g) 0·01 (h) 100 (i) 10^3 (j) $\frac{1}{10^3}$ 2(a) 24 (b) 45 (c) 105 (d) 77 (e) 504 3(a) 689 (b) 3,017 (c) 1·76 (d) 15·7 (e) 0·296 4(a) 6·24 (b) 2·18 (c) 1·41 (d) 31·6 (e) 0·190 5(a) 2·14 (b) 4·20 (c) 6·75 (d) 13·3 (e) 6·32 (f) 1·41 (g) 2·24 (h) 1·01 (i) 35·6 (j) 114 (k) 0·373 (l) 0·118 (m) 0·0316 (n) 0·0281 (o) 0·00447 6(a) 3·06cm² (b) 428m² (c) 118,000km² (d) 0·00608mm² 7(a) 5·74cm (b) 22·4m (c) 44·7km (d) 0·632mm 8(a) 5 (b) 12 (c) 59 (d) 600 (e) 13 (f) 12 9 5cm

Exercise 12a — Page 36

1 15cm 2 6cm 3 20cm 4 40cm 5 18cm 6 3·5cm 7 25cm 8 2·5cm 9 500m, yes (3, 4, 5 triangle - see 12.3) 10 17cm

Exercise 12b — Page 37

1 5 2 12 3 21 4 13 5 24 6 25 7 17 8 45 9 80 10 25 11 48 12 80 13 yes 14 no 15 no 16 yes

Exercise 12c — Page 37

1 right-angled (R) 2 acute-angled (A) 3 A 4 R 5 obtuse-angled 6 R

Exercise 12d — Page 38

1 7·81cm 2 1·73cm 3 1·41cm 4 6·49cm 5 17cm 6 25cm 7 21cm 8 6·1m 9 210km 10 148km (3sf)

Exercise 13a — Page 39

1 8 2 27 3 125 4 216 5 343 6 512 7 729 8 1,000 9 3,375 10 12·167 11 1,730 12 1·73 13 1,730,000 14 0·00173 15 50,700 16 195 17 4,490 18 88·7 19 0·712 20 0·00286 21 104cm³ 22 37 23 216 24 both 216 25 225, $(1 + 2 + 3 + \ldots + n)^2 = \frac{1}{4}n^2(n + 1)^2$ 26 $1^3 + 3^3 + 5^3 + 7^3 + 9^3 + 11^3 + 13^3 + 15^3 = 8{,}128$ 27 $1^3 + 12^3$, $9^3 + 10^3$

Exercise 13b — Page 40

1 6 2 8 3 9 4 15 5 42 6 1·82 7 2·15 8 4·64 9 3·88 10 57·2 11 0·758 12 0·414 13 0·171 14 0·0974 15 0·0215 16 2·71cm 17 4·31cm (3sf) 18 12·7cm 19 2·39 20 32·7

Exercise 14a — Page 42

1 1,024 2 10,000,000 3 625 4 8 5 1,000,000 6 100,000 7 64 8 −243 9 $\frac{1}{16}$ 10 $3\frac{3}{8}$ 11 512 12 4 13 10,000 14 1 15 75 16 a^4b^2 17 xy 18 $72m^2n^3$

Exercise 7g **Page 22**

1 sh1, sh5; sh5, sh10, sh20, sh50, sh100, sh200, sh500, sh1,000 **2** sh34,500 **3** sh74,400 **4**(i) sh4,900 (ii) sh11,550 (iii) sh16,200,000 **5** sh22,800

Exercise 7h **Page 22**

1 1h 45min, 1h 45min, 15.05, 03.20, 06.25 next day **2** 696 **3** 40ms, 40,000mcs, 40,000,000ns **4** 2,000,000,000; 120,000,000,000 **5** 5h 20min

Exercise 7i **Page 23**

1(i) 273°K (ii) 373°K (iii) 250°K (iv) 320°K (v) 1°K **2**(i) 27°C (ii) 127°C (iii) −51°C (iv) −123°C (v) −173°C **3** 717·4°K **4**(i) 45°C (ii) 100°C (iii) 60°C (iv) 150°C (v) 0°C (vi) −40°C **5**(i) 318°K (ii) 373°K (iii) 333°K (iv) 423°K (v) 273°K (vi) 233°K

Exercise 8a **Page 24**

1 42km/h **2** 4h **3** 108km **4** 2h **5** $58\frac{1}{3}$km/h

Exercise 8b **Page 25**

1 4m/s² **2** 15m/s **3** 10s **4**(i) 1·5m/s² (ii) 1·25m/s² (iii) 0·5m/s² (iv) 0·5m/s² (v) 0·8m/s²

Exercise 8c **Page 25**

1 4·5*l*/min **2** 2 hours **3** 5·4*l* **4** 2min 55s

Exercise 8d **Page 25**

1 sh750 **2** sh650 **3** sh250 **4** sh900 **5** sh650 **6** sh450 **7** sh450 **8** sh400 **9** sh500 **10** sh850

Exercise 8e **Page 26**

1 12·25g/cm³ **2** 12·1g/cm³ **3** 11·13g/cm³ **4** 10·16g/cm³ **5** 10·05g/cm³

Exercise 8f **Page 26**

1 83/km² **2** 150; 950,000; 8,200; 14·5; 500,000; 4,600; 2,800; 230

Exercise 9a **Page 27**

1 2 : 3 **2** 1 : 5 **3** 1 : 8 **4** 5 : 48 **5** 10 : 9 **6** 1 : 2 **7** 1 : 9 **8** 18 : 1 **9** 24 : 25 **10** 24 : 13 **11**(i) 4 : 3 (ii) 3 : 4 **12**(i) 3 : 2 (ii) 3 : 2 (iii) 9 : 4

Exercise 9b **Page 27**

1 20, 30 **2** 5, 35 **3** sh6,500, sh3,500 **4** sh1,500, sh3,500, sh5,000 **5** sh400, sh800, sh2,400 **6** sh1,440, sh720, sh240 **7** 10cm, 20cm, 30cm, 40cm **8** 100m, 200m, 300m, 400m, 500m **9** 60cm **10** sh9,000 **11** 300g **12** 14

Exercise 9c **Page 28**

1 sh7,200 **2** sh5,000 **3** 64cm³ **4** 36cm³ **5** 45cm by 30cm, (i) 2 : 5 (ii) 4 : 25 **6**(i) 3 : 4 (ii) 9 : 16

Exercise 9d **Page 28**

1 3 : 4 : 8 **2** 14 : 12 : 39 **3** 12 : 20 : 35 **4** 2 : 4 : 6 : 7 **5** 4 : 8 : 12 : 9 : 6 **6** 15 : 8

Exercise 9e **Page 28**

1 sh1,500 **2** 800cm³ **3** 600m*l* **4** 530g **5** 1·59t **6** 369g **7** sh13,200 **8** 21h **9** sh2,270,000 **10** 82·08h

Exercise 9f **Page 29**

1 1 : 10,000 **2** 1 **3** 2·5 **4** 5 **5** 1 : 20,000 **6** 4cm **7** 510km

Exercise 17e Page 52

1 sh36,990 2 12·5% 3 sh18,000

Exercise 17f Page 53

1 sh32,000 2 sh20,000 3 10% 4 2 years 5 sh1,600,000, sh1,750,000

Exercise 17g Page 53

1 sh242,000, sh42,000 2 sh116,640, sh16,640 3 sh56,180, sh6,180 4 sh133,100, sh33,100 5 sh364,000 6 sh6,710,000

Exercise 17h Page 54

1 sh80,000, sh26,000 2 sh220,000 3 sh26,250 4 sh87,500

Exercise 18a Page 55

1 $4(x - 3) = 12$ 2 $x + 5 = 2(x - 14 + 5)$ 3 $x + (x + 10) + 2[x + (x + 10)] = 180$ 4 $\frac{x}{4} + \frac{x}{2} + \frac{x}{3} = 4$ 5 $50{,}000 + 120d = 230{,}000$ 6 $5{,}000 + \frac{5}{100} \times 1{,}000n = 17{,}000$ 7 $2b + 3(b - 150) + (b + 300) = 2{,}550$ 8 $u + 2 \times 10 = 26$

Exercise 18b Page 56

1 $6x$ 2 $-2t$ 3 $4ax - x - 4$ 4 $5a + 4b - 3ab$ 5 $3p - 6pq$ 6 $2u^2 - uv$ 7 $2 - 2x - y + x^2 - 3y^2 + xy$ 8 $-3mn - 4m^2n^2$ 9 $pq^2 + qp^2 + 3pq + 1$ 10 $x^3 + x^2y - 3xy^2 - y^3$

Exercise 18c Page 56

1 $12x - 8$ 2 $5x + 12$ 3 $14y + 7$ 4 $4p + 2q$ 5 $3m^2n + 3mn^2$ 6 $ax^2 + 3ay^2 - a$ 7 0 8 $\frac{1}{3}x^3y + \frac{1}{3}xy^3 - \frac{2}{3}x^2y^2$ 9 $2(3x + 2)$ 10 $3(3x + 3y + 1)$ 11 $2mn(mn - 1)$ 12 $2x(2x + 1)$ 13 $5p^2q^2(3 - 2q)$ 14 $\frac{1}{4}xy(3x + 1)$

Exercise 18d Page 57

1 0 2 1 3 14 4 12 5 −414 6 −22 7 72 8 29

Exercise 18e Page 57

1 10 2 10 3 −2 4 −5 5 2 6 4 7 −32 8 34 9(i)(a) −14 (b) −7 (c) −14 (d) −7 (ii) $2a^3 + a^2 - 2a + 1$ 10(i) 6 (ii) 16

Exercise 19a Page 58

1 y/z 2 1 3 r/q 4 m 5 1 6 a^2b^2/c^2 7 $(b/a)^2$ 8 a/d 9 a^3c/b 10 v^2w^4 11 $2xy$ 12 s/t 13 pqr 14 1 15 $(4x)/(3y)$ 16 $4/y$ 17 z/y 18 $(x/y)^2$ 19 y/x 20 q/p

Exercise 19b Page 59

1 $\frac{7x}{12}$ 2 $\frac{4x + 3y}{12}$ 3 $\frac{7}{x}$ 4 $\frac{3y + 4x}{xy}$ 5 $\frac{x^2 + 12}{3x}$ 6 $\frac{xy + 12}{3y}$ 7 $\frac{2x + 1}{x}$ 8 $\frac{3y + 2}{y}$ 9 $\frac{x^2 + 1}{x}$

10 $\frac{2 + x}{1 + x}$ 11 $\frac{-x}{1 + x}$ 12 $\frac{1 - x}{x(x + 1)}$ 13 $\frac{x + 2}{x(x - 1)}$ 14 $\frac{2x + 3}{(x + 1)(x + 2)}$ 15 $\frac{1}{(x + 1)(x + 2)}$

16 $\frac{-5x - 1}{(3x + 1)(2x + 1)}$ 17 $\frac{x + 3}{2(x - 1)}$ 18 $\frac{x}{12}$ 19 $\frac{x^2 - 5x - 2}{2x(x + 1)}$ 20 $\frac{1 - t^2}{1 + t^2}$ 21 $\frac{11 - 6x}{12x}$

22 $\frac{2(x^2 + 1)}{(x - 1)(x + 1)}$ (see 34.3) 23 $\frac{1}{x + 1}$ 24 $\frac{4t}{1 - t^2}$ (see 34.3) 25 $\frac{1}{(x + 1)(x + 3)}$

26 $\frac{x + 2}{x + 3}$ 27 $\frac{1}{x + 4}$ 28 $\frac{2}{3}$ 29 $\frac{1}{2}(x + 3)$ 30 $\frac{a - 2}{a - 1}$ 31 $\frac{1}{2}x$ 32 $\frac{x + 2}{x + 3}$ 33 $\frac{x + 2}{x + 3}$

34 $\frac{1}{2}(x + 2)$ 35 $\frac{x - 2}{2x - 1}$ 36 $\frac{a + b}{a - b}$ 37 $\frac{a - b}{a + b}$ 38 $\frac{2x + 3}{2x - 3}$ 39 $\frac{x - 4}{x - 6}$ 40 $\frac{m + n}{m - n}$

Exercise 20a
Page 61

In 1 to 10 many pairs are possible. The following are chosen to help you draw the graph.
1 (0, 0), (1, 2), (2, 4) **2** (0, 1), (1, 2), (2, 3) **3** (0, −3), (1, −1), (2, 1) **4** (0, 4), (1, 3), (2, 2) **5** (1, 2), (3, 1), (5, 0) **6** (0, 8), (1, 4), (2, 0) **7** (0, 0), (1, −1), (2, −2) **8** (−2, 0), (−2, 1), (−2, 2) **9** (0, 4), (1, 4), (2, 4) **10** (0, −1), (2, −2), (4, −3) **11** $y = 4x + 1$ **12** $x + 3y = 12$ **13** $2y = 3x + 4$ **14** $3x + 4y = 24$

Exercise 20b
Page 61

1 1 **2** 2 **3** $\frac{1}{3}$ **4** −1 **5** $-\frac{1}{2}$ **6** −3 **7** 1 **8** $-\frac{1}{4}$ **9** $\frac{1}{7}$ **10** $-\frac{2}{3}$ **11** 2 **12** 3 **13** 1 **14** −1 **15** $-\frac{1}{2}$ **16** 2 **17** $\frac{2}{3}$ **18** $\frac{3}{2}$ **19** 4 **20** $-\frac{5}{3}$ **21** 0 **22** infinite

Exercise 20c
Page 62

1 3, 2 **2** $\frac{1}{2}$, $2\frac{1}{2}$ **3** $\frac{2}{3}$, $\frac{2}{3}$ **4** $-\frac{1}{2}$, −2 **5** $y = 2x + 5$ **6** $2y = x + 6$ **7** $y = x - 1$ **8** $y + 2x = 0$ **9** $3y + x = 5$ **10** $x + 2y + 5 = 0$ **11** $y = 3x + 1$ **12** $y = x + 2$ **13** $y = 2x - 1$ **14** $y + x = 3$ **15** $y = x + 6$ **16** $2y = x - 4$ **17** $2y = 3x$ **18** $2y + x = 0$ **19** $2y = 3x + 2$ **20** $5x + 2y + 3 = 0$

Exercise 21a
Page 64

1 7 **2** 6 **3** −3 **4** $-\frac{1}{3}$ **5** 3 **6** −2 **7** $-\frac{4}{3}$ **8** 10 **9** 1 **10** 3 **11** −1 **12** −3 **13** 1 **14** 1 **15** 1 **16** 0 **17** −3 **18** 13

Exercise 21b
Page 65

1 6 **2** 23 years **3** 25°, 35°, 120° **4** 6 **5** 1,500km **6** 240 books **7** sh450 **8** 6 **9** sh450 **10** 5·5ha **11** 30km **12** 20km, 10.50am **13** 15 hours

Exercise 22a
Page 66

1 (3, 2) **2** (−2, 3) **3** (3, −1) **4** (−2, −1) **5** $m = 1{\cdot}1$, $n = 0{\cdot}3$ **6** $a = -1{\cdot}7$, $b = 1$

Exercise 22b
Page 67

1 $x = 1$, $y = 2$ or (1, 2) **2** (3, 2) **3** (−2, 1) **4** (−1, −3) **5** (1, −3) **6** $(-\frac{1}{3}, \frac{2}{3})$

Exercise 22c
Page 68

1 $x = 1$, $y = 2$ or (1, 2) **2** (5, −1) **3** (−1, 2) **4** (−4, −1) **5** (3·5, 4·5) **6** $m = -8$, $n = 6$

Exercise 22d
Page 68

1 sh250 **2** sh250 and sh300, ie. sh550 **3** 168km, 100km **4** 9g/cm³, 4g/cm³

Exercise 23a
Page 69

1(i) acute(A) (ii) obtuse(O) (iii) reflex(R) (iv) R (v) A (vi) O (vii) R **2**(i) A (ii) R (iii) R (iv) R (v) O **3**(i) $f = h = 110°$, $g = 70°$ (ii) $e = g = 60°$, $f = h = 120°$ **4** $x = 24°$; 24°, 48°, 72°, 96°, 120° **5** $d = 62{\cdot}5°$ **6**(i) 0° (ii) 90° (iii) 180° (iv) 45° (v) 157·5°

Exercise 23b
Page 70

1 $x = z = 40°$, $y = 140°$ **2** $a = 60°$, $b = 80°$ **3** $c = 130°$, $d = 140°$ **4** $i = 79°$, $j = 34°$, $k = l = m = 67°$ **5** $p = 40°$, $q = 30°$, $n = 110°$ **6** $r = 120°$, $s = 20°$, $t = 140°$ **7** $x = 14°$, $y = 28°$ **8** $u = 40°$, $v = w = 70°$

Exercise 23c
Page 71

2 4·9cm or 5·0cm **3** BD = 5·0cm or 5·1cm, AC = 8·6cm or 8·7cm **4** PQ = 5·2cm
5 this circle is the **circumscribed circle** of triangle ABC and passes through the three vertices A, B, C
6 yes, the **inscribed circle** of triangle ABC which *touches* the three sides
7 yes (the **orthocentre** of triangle ABC)

Exercise 24a Page 72

1 $b = d = 30°$, $c = 150°$, $e = g = 70°$, $h = 110°$, $i = k = 40°$, $j = l = 140°$ 2 $x = 15$; $45°, 60°, 75°$ 3 $x = 35°$, $y = 120°$, $z = 25°$ 4 $60°$ 5 $a = 40°$, $b = 65°$, $c = d = 75°$

Exercise 24b Page 73

1 $x = 18$; $72°, 90°, 108°, 126°, 144°$ 2 $2{,}700°$ 3 20 4 36
5 $x = 20$; $80°, 90°, 120°, 100°, 160°, 170°$

Exercise 24c Page 73

1 36° 2 36°, 36°, 108°; 36°, 72°, 72° 3 30°, 60°, 90° 4 22·5°, 45°, 112·5°; 45°, 67·5°, 67·5°; 22·5°, 67·5°, 90° 5 regular pentagon, hexagon and octagon 6 36cm²

Exercise 25a Page 74

1 135° 2 90° 3 180° 4 315° 5 270° 6 337·5° 7 337·5° 8 180° 9 270° 10 292·5°

Exercise 25b Page 74

1(i) N 67·5° E (ii) 067·5° 2(i) S 45° E (ii) 135° 3(i) N 90° E (this notation is not usually used for the four principal directions) (ii) 090° 4(i) S 22·5° W (ii) 202·5° 5(i) S 67·5° W (ii) 247·5° 6(i) N 45° W (ii) 315° 7(i) N 22·5° W (ii) 337·5° 8(ii) 180° 9(ii) 270° 10(ii) 000° or 360° 11(i) 090° (ii) 060° (iii) 180° (iv) 240° (v) 270° (vi) 000° or 360°

Exercise 25c Page 76

1 145m (nearest 5m) 2(i) 4·5km (ii) 163° (iii) 486ha 3(i) 6·6km (ii) 14·3km (iii) 025° 4(i) 5,000m (ii) 333° or 334° (iii) 7·7km 5 1·7km, 330°

Exercise 26a Page 77

1 $2\sqrt{2}$ 2 $3\sqrt{2}$ 3 $6\sqrt{2}$ 4 $5\sqrt{3}$ 5 $5\sqrt{2}$ 6 2 7 $6\sqrt{3}$ 8 2 9 $3\sqrt{2}$ 10 $3 + \sqrt{2}$
11 $3 + 2\sqrt{2}$ 12 $7 - 4\sqrt{3}$

Exercise 26b Page 78

1 $\sqrt{3}$ 2 $4\sqrt{5}$ 3 $\sqrt{2} - 1$ 4 $4 - \sqrt{10}$ 5 $4 + \sqrt{10}$ 6 $5 + 3\sqrt{3}$ 7 $2 - \sqrt{3}$ 8 $2 + \sqrt{2}$
9 0·707105 10 0·57735 11 0·41421 12 1 13 3·14626 14 0·73205 15 13·9282
16 0·0718 17 2·82842 18 −1·73205 19 0·223607 20 $a = 0$, $b = 8$

Exercise 27a Page 79

1 $\log x + 2\log y$ 2 $4\log x - 3\log y$ 3 $\log x + \frac{1}{2}\log y$ 4 $2\log x - \log y$
5 $\frac{1}{2}(\log x + \log y)$ 6 $\frac{1}{3}\log x - 3\log y$ 7 $1 + \log x - \log y$ 8 $1 - \log x - \log y$
9 $\log y + \frac{1}{2}(1 + 3\log x)$ 10 $\frac{1}{4}(\log x + 3\log y) - 2$

Exercise 27b Page 80

(Answers are correct to 4 sf)
1 19·00 2 3·956 3 0·3562 4 964·3 5 0·4942 6 0·2209 7 0·2983 8 7·970
9 27·88 10 0·7310

Exercise 28a Page 82

1 53·1° 2 45·6° 3 36·9° 4 48·6° 5 10·7cm 6 6·16cm 7 14·5cm 8 5·77cm
9 7·78cm 10 5·6m (2 sf) 11 053°, 5km 12 85m 13 8km 14 643588

Exercise 28b Page 83

1 15·59cm² 2 11·31cm²

Exercise 29a Page 84

9 F + V = E + 2

Exercise 29d

Page 87

1(i) 25cm (ii) 21·1° (iii) 24·2° **2** 8·66cm, 35·3° **3** 10cm; (i) 53·1° (ii) 59·0° or 65·8° **4** 3·5cm **5**(i) 156cm³ (ii) 211cm² **6** HXB = 10m, HYB = 10·3m, HZB = 11·4m; HXB

Exercise 30a

Page 88

1 4cm **2** 13cm **3** 8cm **4** 16cm **5** 7·07cm (3 sf) **6** 2cm or 14cm, depending on whether chords are on same side or opposite sides of centre **7** 20cm **8** 264cm² (3 sf)

Exercise 30b

Page 90

1 130° **2** 56° **3** 224° **4** 102° **5** ∠AOC = 94°, reflex ∠AOC = 266°, ∠ADC = 133° **6** 147° **7** reflex ∠AOC = 214°, ∠AOC = 146°, ∠ABC = 73° **8** 83° **9** $x = 60°$, $y = 60°$, $z = 30°$ **10** $p = 50°$, $q = 130°$, $r = 20°$ **11** 4·45cm²

Exercise 30c

Page 91

1 $r = 75°$ **2** $q = 63°$, $s = 117°$ **3** $d = g = 86°$, $b = 94°$, $c = 100°$, $a = 80°$ **4** $u = 90°$, $w = 60°$, $x = 120°$, $y = z = 30°$ **5** $a = b = 70°$, $c = 110°$, $d = e = 35°$

Exercise 31a

Page 93

1 26cm² **2** 8·75cm² **3** 9·39cm² **4** 12√5 or 26·8cm² **5** 24cm² **6** 30cm² **7** 31·7cm² **8** 36·0cm² **9**(i) 30cm² (ii) 7·5cm **10**(i) 31cm² (ii) 124ha

Exercise 31b

Page 95

1(i) 216cm² (ii) 216cm³ **2**(i) 94·0cm² (ii) 60·0cm³ **3**(i) 384cm² (ii) 384cm³ **4**(i) 748cm² (ii) 1,540cm³ **5**(i) 283cm² (ii) 314cm³ **6**(i) 616cm² (ii) 1,440cm³ **7**(i) 942cm² (ii) 2,090cm³ **8**(i) 21,400m³ (ii) 5,680m² **9**(i) 2,260m³ (ii) 902m² **10** sh150 **11** 3,080cm³ **12** 6·60cm³

Exercise 32a

Page 98

1 I→B, I→M, I→N, I→L, I→G **2** W→L, W→G, T→B, T→M, T→N **3** K→B, K→M, K→N, K→L, K→G **4** B→T, B→K, M→T, M→K, L→W, L→K, G→W, G→K **5** N→T, N→K **6** W↔T, W→K, T→K, B↔M, B→N, M→N, L↔G **7** K→W, K→T, N→B, N→M **8** B↔L, B↔G, M↔L, M↔G, N↔L, N↔G **9** B→I, M→I, N→I, L→I, G→I **10** 2→2, 2→4, 2→6, 2→8, 3→3, 3→6, 3→9, 4→4, 4→8, 5→5, 6→6, 7→7, 8→8, 9→9 **11** 1→0, 2→0, 3→0, 2→1, 3→1, 3→2 **12** 0→0, 0→1, 0→2, 0→3, 1→1, 1→2, 1→3, 2→2, 2→3, 3→3 **13** −2→−1, 0→0, 2→1 4→2, 6→3, 8→4 **14** 0→0, 1→−1, 1→1, 4→−2, 4→2, 9→3 **15** 0→0, −1→1, 1→1, −2→4, 2→4, 3→9 **16** 9→4, 12→4, 13→4, 14→4, 12→6, 13→6, 14→6, 12→7, 13→7, 14→7, 13→9, 14→9 **17** M→L, P→L, R→L, S→L, P→M, M→R, P→R, S→R, P→S **18** L↔M, M↔P, P↔R, R↔S **19** A↔B, A↔F, A↔G, A↔J, B↔F, B↔G, B↔J, G↔J, G↔F, J↔F, C↔E, C↔H, E↔H, D↔I

Exercise 32b

Page 99

1(i) domain = {K, B, A, M, O, J, N}, range = {6, 7, 8, 9} (ii) K→8, B→7, A→8, M→6, O→9, J→7, N→8 (iii) many-one **2**(i) domain = {L, M, P, R, S}, range = {12, 13, 14, 15} (ii) L→12, M→14, P→15, R→13, S→14 (iii) many-one **3** domain = {A, B, C, D}, range = {f, t, v}; A→f, B→t, B→v, C→f, C→v, D→f, D→t, D→v; many-many **4** S→1, C→2, N→3, K→4; domain = {S, C, N, K}, range = {1, 2, 3, 4}; one-one **5** 4→B, 5→A, 5→D, 7→C, 7→F, 9→E; domain = (4, 5, 7, 9}, range = {A, B, C, D, E, F}; one-many **6**(i) many-one (ii) it is very likely there are, but not necessarily **7** {0, 3, 6, 9, 12, 15}, one-one **8** {0, 0·5, 1, 1·5, 2, 2·5}, one-one **9** {1, 3, 5, 7, 9, 11}, one-one **10** {−2, 1, 4, 7, 10, 13}, one-one **11** {0, 1, 2, 3, 4, 5}, one-one **12** range = {0, 1, 4, 9}; ±3→9, ±2→4, ±1→1, 0→0; many-one **13** range = {0, ±1, ±2, ±3, ±4}; 0→0, 1→±1, 4→±2, 9→±3, 16→±4; one-many **14** range = $\{\pm\frac{1}{3}, \pm\frac{1}{2}, \pm 1, \pm 2, \pm 3\}$; $-3 \to -\frac{1}{3}$, $-2 \to -\frac{1}{2}$, $-1 \to -1$, $-\frac{1}{2} \to -2$, $-\frac{1}{3} \to -3$, $\frac{1}{3} \to 3$, $\frac{1}{2} \to 2$, $1 \to 1$, $2 \to \frac{1}{2}$, $3 \to \frac{1}{3}$; one-one; equal

Exercise 32c Page 100

1 (K, 8), (B, 7), (A, 8), (M, 6), (O, 9), (J, 7), (N, 8) **2** (L, 12), (M, 14), (P, 15), (R, 13), (S, 14) **3** (A, f), (B, t), (B, v), (C, f), (C, v), (D, f), (D, t), (D, v) **4** (S, 1), (C, 2), (N, 3), (K, 4) **5** (5, A), (4, B), (7, C), (5, D), (9, E), (7, F) **7** (0, 0), (1, 3), (2, 6), (3, 9), (4, 12), (5, 15) **8** (0, 0), (1, 0·5), (2, 1), (3, 1·5), (4, 2), (5, 2·5) **9** (0, 1), (1, 3), (2, 5), (3, 7), (4, 9), (5, 11) **10** (0, −2), (1, 1), (2, 4), (3, 7), (4, 10), (5, 13) **11** (0, 5), (1, 4), (2, 3), (3, 2), (4, 1), (5, 0) **12** (0, 0), (±1, 1), (±2, 4), (±3, 9) **13** (0, 0), (1, ±1), (4, ±2), (9, ±3), (16, ±4) **14** $(-3, -\frac{1}{3})$, $(-2, -\frac{1}{2})$, $(-1, -1)$, $(-\frac{1}{2}, -2)$, $(-\frac{1}{3}, -3)$, $(\frac{1}{3}, 3)$, $(\frac{1}{2}, 2)$, $(1, 1)$, $(2, \frac{1}{2})$, $(3, \frac{1}{3})$ **15**(i) a↔c, a↔i, a↔k, c↔i, c↔k, i↔k, b↔j, b↔l, b↔d, j↔l, j↔d, l↔d, e↔f, e↔g, e↔h, f↔g, f↔h, g↔h (ii) *Note* some pairs of lines are skew eg. a and j. Such lines are not parallel and do not intersect: a↔b, a↔d, a↔l, a↔j, a↔e, a↔f, a↔g, a↔h, b↔c, b↔k, b↔i, b↔e, b↔f, b↔g, b↔h, c↔d, c↔l, c↔j, c↔e, c↔f, c↔g, c↔h, d↔i, d↔k, d↔e, d↔f, d↔g, d↔h, e↔i, e↔j, e↔k, e↔l, f↔i, f↔j, f↔k, f↔l, g↔i, g↔j, g↔k, g↔l, h↔i, h↔j, h↔k, h↔l, i↔j, i↔l, j↔k, k↔l (iii) same as (i); domain = {a, b, c, d, e, f, g, h, i, j, k, l}, range = {9, 12, 20}; a→20, b→12, c→20, d→12, e→9, f→9, g→9, h→9, i→20, j→12, k→20, l→12; many-one; ordered pairs are (a, 20), (b, 12), (c, 20), (d, 12), (e, 9), (f, 9), (g, 9), (h, 9), (i, 20), (j, 12), (k, 20), (l, 12)

Exercise 33a Page 101

1 yes **2** yes **3** no **4** yes **5** no **6** yes **7** yes **8** yes **9** yes **10** yes **11** yes **12** yes **13** no **14** yes **15** $f(x) = 3x$ **16** $f(x) = x$ **17** $f(x) = 3x - 6$ **18** $f(x) = x^2$ **19** $f(x) = x^3 + 1$ **20**(i) 5 (ii) −3 (iii) −7 (iv) $2y - 3$ **21**(i) 5 (ii) −3 (iii) −1 (iv) 0 (v) $\frac{y+4}{y-4}$ **22**(i) 1 (ii) 5 (iii) 5 (iv) $z^2 + 1$ **23** {1, 2, 4, 8, 16} **24**(i) 8 (ii) 41 **25**(i) 12 (ii) −12

Exercise 33b Page 102

1 $y = x + 1$ **2** $y = 2x - 1$ **3** $y = x$ **4** $x + y = 3$ **5** $y = 3$ **6** $y = x + 2$ **7** $x + y = 3$ **8** $x + 2y = 4$ **9** $x = 4$ **10** $x = -1$ **11** {0, 0·5, 2, 4·5, 8, 12·5}; (0, 0), (±1, 0·5), (±2, 2), (±3, 4·5), (±4, 8), (±5, 12·5) **12** graph, with axes, forms trapezium with vertices at (0, 0), (0, 3), (2, 3), (5, 0)

Exercise 33c Page 103

1(i) 7 (ii) 10 (iii) $2x + 3$ (iv) $2(x + 3)$ **2**(i) 4 (ii) 0 (iii) $3x - 2$ (iv) $3(x - 2)$ **3**(i) 16 (ii) 6 (iii) $(x + 2)^2$ (iv) $x^2 + 2$ **4**(i) 8 (ii) 4 (iii) $x^2 + 2x$ (iv) x^2 **5**(i) 2 (ii) −1 (iii) $2x - 2$ (iv) $2x - 5$ **6**(i) −1 (ii) 4 (iii) $2(x - 1)^2 - 3$ (iv) $2x^2 - 4$ **7**(i) $f(5) = 28$, $g(5) = 64$ (ii) $f(x) = x^2 + 3$, $g(x) = (x + 3)^2$ (iii) no **8**(i) $4x^2 - 18$ (ii) $16x^2 - 64x + 60$ **9**(i) $2a^2 + 16$ (ii) $4a^2 + 8$; $x = \pm 2$ **10**(i) $2a^2 - 1$ (ii) $2a^2 + 8a + 5$; $x = -0{\cdot}75$ **11**(i) $3x + 4$ (ii) $3x + 4$; yes **12**(i) x (ii) x; yes **13**(i) $x + 6$ (ii) $x + 9$ **14**(i) $x - 4$ (ii) $x - 6$ **15**(i) $9x$ (ii) $27x$ **16**(i) x (ii) $5 - x$ **17**(i) x^4 (ii) x^8 **18**(i) x (ii) $\frac{1}{x}$ **19**(i) $\frac{1+x}{2+x}$ (ii) $\frac{2+x}{3+2x}$ **20**(i) $1 - \frac{1}{x}$ (ii) x

Exercise 33d Page 105

1 $\frac{1}{4}x$ **2** $x - 4$ **3** $2x$ **4** $x + 2$ **5** $\pm\sqrt{x}$ **6** x^2 **7** $5 - x$ **8** $\frac{1}{x}$ **9** $x + 5$, $2x$; $\frac{1}{2}(x - 5)$ **10** $x - 4$, $3x$; $\frac{1}{3}(x + 4)$ **11** $\frac{1}{2}x$, $x + 7$; $2x - 7$ **12** $x + 1$, x^2; $\sqrt{(x - 1)}$ **13** $\frac{1}{2}(x + 3)$ **14** $8 - x$ **15** $3/x$ **16** $\frac{1}{3}\sqrt{(4 - x)}$ **17** $f^{-1}(x) = x$ **18** $f^{-1}(x) = \frac{1}{2}(x - 1)$; $ff^{-1}(x) = f^{-1}f(x) = x$; each is identity function **19** in each case $ff^{-1} = f^{-1}f$ = identity function **20** $f^2(x) = \frac{1}{1-x}$; since $ff^2(x) = x$ then $f^2 = f^{-1}$, $f^{-1}(x) = \frac{1}{1-x}$ **21**(iii) $y = 3x - 6$ (iv) $g(x) = 3x - 6$ (v) $f^{-1}(x) = 3x - 6$; $g = f^{-1}$ **22**(a)(iii) $2y = x$ (iv) $g(x) = \frac{1}{2}x$ (v) $f^{-1}(x) = \frac{1}{2}x$; $g = f^{-1}$ (b)(iii) $y = 3x$ (iv) $g(x) = 3x$ (v) $f^{-1}(x) = 3x$; $g = f^{-1}$ (c)(iii) $y = x - 2$ (iv) $g(x) = x - 2$ (v) $f^{-1}(x) = x - 2$; $g = f^{-1}$ (d)(iii) $2y = x + 2$ (iv) $g(x) = \frac{1}{2}(x + 2)$ (v) $f^{-1}(x) = \frac{1}{2}(x + 2)$; $g = f^{-1}$ (e)(iii) $x + 2y = 5$ (iv) $g(x) = \frac{1}{2}(5 - x)$ (v) $f^{-1}(x) = \frac{1}{2}(5 - x)$; $g = f^{-1}$ (f)(iii) $x + y = 5$ (iv) $g(x) = 5 - x$ (v) $f^{-1}(x) = 5 - x$; $g = f^{-1}$ **23** reflect graph in $y=x$; no

Exercise 34a Page 106

1 $x^2 + 2x$ 2 $2x^2 - 2x$ 3 $3x - 6x^2$ 4 $8x - 6x^2$ 5 $x^2 + 5x + 6$ 6 $6x^2 + 17x + 12$
7 $2x^2 + x - 1$ 8 $2 - x - x^2$ 9 $3x^2 - 11x + 6$ 10 $x^2 - 16$ 11 $1 - x^2$ 12 $x^2 + 6x + 9$
13 $x^2 + 4x + 4$ 14 $x^2 - 6x + 9$ 15 $25x^2 - 20x + 4$ 16 $9 + 12x + 4x^2$ 17 $2x^2 + 6x + 4$
18 $3 - 9x + 6x^2$ 19 $x^3 + 6x^2 + 11x + 6$ 20 $8 + 12x - 2x^2 - 3x^3$ 21 $x^2 + 3ax + 2a^2$
22 $2x^2 - 3ax - 2a^2$ 23 $x^2 - a^2$ 24 $2x^3 + ax^2 - 3a^2x$ 25 $x^3 + 6ax^2 + 11a^2x + 6a^3$
26 $a^2 + 2ab + b^2$ 27 $a^2 - 2ab + b^2$ 28 $a^2 - b^2$

Exercise 34b Page 107

1 $p^2 + 2pq + q^2$ 2 $4m^2 - 4mn + n^2$ 3 $a^2 - 9b^2$ 4 $9x^2 + 30xy + 25y^2$ 5 $9x^2 - 24x + 16$
6 $9x^2 - 4$ 7 $1 - 12a + 36a^2$ 8 $x^2 + x + \frac{1}{4}$ 9 $1 - x + \frac{1}{4}x^2$ 10 $\frac{1}{4}x^2 - 1$ 11 $1 - 6x^2 + 9x^4$
12 $x^4 - 4$ 13 $a^3 + 3a^2b + 3ab^2 + b^3$ 14 $a^3 - 3a^2b + 3ab^2 - b^3$ 15 $a^3 + b^3$ 16 $a^3 - b^3$
17 $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$ 18 $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

Exercise 34c Page 108

1 $(x + 3)(x + 4)$ 2 $(x - 3)(x - 4)$ 3 $(x + 3)(x - 2)$ 4 $(x - 3)(x + 2)$ 5 $(2x + 1)(x + 2)$
6 $(3x + 1)(x - 1)$ 7 $(x - 15)(x + 2)$ 8 $(x - 3)(x - 10)$ 9 $(5x + 2)(x + 5)$
10 $(x - 18)(x + 4)$ 11 $(5x - 6)(x + 2)$ 12 $(3x + 2)(2x - 5)$ 13 $(x + 1)^2$ 14 $(x - 4)^2$
15 $(2x - 3)^2$ 16 $(x + 1)(x - 1)$ 17 $(x + 3)(x - 3)$ 18 $(3x + 2)(3x - 2)$

Exercise 35a Page 109

1 2, 3 2 −5, 2 3 −4, 15 4 −12, 2 5 ±2 6 0, 6 7 $-2, \frac{1}{2}$ 8 $\frac{1}{2}$ (twice) 9 $-\frac{3}{2}, \frac{2}{3}$
10 $-\frac{1}{2}, \frac{5}{2}$ 11 −2, 4 12 −2, −1 13 $(x - 3)(x + 2) = x^2 - x - 6 = 0$ 14 $6x^2 + x - 2 = 0$
15 6cm 16 6·5cm, 5cm 17 $x = 8$; 6cm, 8cm, 10cm 18 $h = 7$ 19 $v = 60$

Exercise 35b Page 111

Answers are given to 1 dp where appropriate.
1 −2, 0·5 2 −2·4, 0·9 3 −1·8, 0·3 4 −1·5, 0 5 −1, −0·5 6 no real solutions
7 −0·6, 2·3 8 0·2, 1·4 9 −0·2, 1·8 10 −0·3, 2 11 −0·5, 2·1 12 no real solutions
13(i) −0·7, 2·2 (ii) −0·3, 1·8 (iii) $2y = 3x - 1$; 0·5, 1 14 −0·6, 1·6 15 −2·3, 1·3
16 −0·7, 2·7 17 −2·4, 0·4 18 −1·8, 0·3 19 −0·6, 2·3 20 no real solutions

Exercise 35c Page 111

1(i) $-1 < x < 2{\cdot}5$ (ii) $x < -0{\cdot}5$ or $x > 2$ (iii) $0{\cdot}5 \le x \le 1$ 2(i) $-1{\cdot}5 < x < 0$
(ii) $x < -2{\cdot}5$ or $x > 1$ (iii) $-2 \le x \le 0{\cdot}5$ (iv) $x \le -1$ or $x \ge -0{\cdot}5$ 3 $-0{\cdot}6 < x < 2{\cdot}3$
4 $x < 0{\cdot}2$ or $x > 1{\cdot}4$ 5 $-0{\cdot}2 \le x \le 1{\cdot}8$ 6 $x \le -0{\cdot}3$ or $x \ge 2$ 7 $-0{\cdot}5 < x < 2{\cdot}1$
8 all real values of x

Exercise 36a Page 112

1 $x \le 3$ 2 $x \ge 4$ 3 $x > 3$ 4 $x \ge -1$ 5 $x < 0{\cdot}5$ 6 $x > -1{\cdot}5$ 7 $0 < x < 3$ 8 $-0{\cdot}5 \le x < 0$

Exercise 36c Page 115

4 (0, 3), (0, 4), (1, 2), (1, 3), (2, 2); 5 points 5 (3, 2) 6 (1, 2), (2, 2), (2, 3), (3, 1), (3, 2)
7 $x = 2$, $y = 3$ 8(i) $x > 1$, $y \ge 2x$ (iii) (2, 4) ie. 2 loaves and 4 sodas (iv) sh200

Exercise 36d Page 115

1 $-1 \le x \le 2$ 2 $1 < x < 4$ 3 $x \le -1$ or $x \ge 3$ 4 $x \le -4$ or $x \ge 4$ 5 $-4 < x < 0$
6 $0{\cdot}5 \le x \le 4$ 7 $x < -1$ or $x > 5$ 8 $x \le -3$ or $x \ge 2$ 9 $-3 \le x < 0$ or $x \ge 3$
10 (−1, 1), (0, 1), (1, 1), (1, 2), (2, 2), (3, 3)

Exercise 37a Page 116

1 $z = 24$, $a = 15$ 2 25m 3 $x = 6$cm 4 4cm 5 AX = 4·5cm, BX = 7·5cm

Exercise 37b **Page 117**

1(i) 2 : 3 (ii) 2 : 3; yes; 4 : 9 **2** 48ha **3** 9cm **4** 25 : 9 **5** 4 : 3 **6** 4·5cm² **7** yes; 360kg **8** 1 : 50,000

Exercise 37c **Page 118**

1 yes; (i) 9 : 16 (ii) 27: 64 **2** 192cm³ **3** 40cm³ **4** 648cm³ **5** 62·5*l*

Exercise 37d **Page 119**

1 Δs CBA, EDF are congruent (in the order of the letters) to ΔPQR
2 Δs PQR, XVW **3** SAS **4** equal chords cut off equal arcs which subtend equal angles at the circumference, eg. C = D **5** ASS is not always a case of congruence

Exercise 38a **Page 121**

1 $\binom{3}{2}$ **2** $\binom{4}{5}$ **3** $\binom{3}{-4}$ **4** $\binom{-6}{-6}$ **5** $\binom{-4}{3}$ **6** $\binom{10}{-16}$ **7**(i) $\binom{-4}{-4}$ (ii) L'(−2, −1), M'(2, −1), N'(−1, 0) **8** A(−2, 8), B(−3, 5), C(2, 5) **9** (i) D"(7, 3), E"(7, 5), F"(10, 6) (ii) $\binom{6}{2}$ (iii) $\binom{2}{-1} + \binom{4}{3} = \binom{6}{2}$ **10**(i) $\mathbf{AB} = \binom{5}{3}$, $\mathbf{CD} = \binom{10}{6}$ (ii) $\binom{10}{6} = 2\binom{5}{3}$ (iii)(a) **CD** = 2**AB** (b) **CD** is parallel to **AB**

Exercise 38b **Page 122**

1 $\mathbf{AC} = \mathbf{a} + \mathbf{b}$, $\mathbf{BD} = \mathbf{b} + \mathbf{c}$, $\mathbf{AD} = \mathbf{a} + \mathbf{b} + \mathbf{c}$, $\mathbf{DA} = -(\mathbf{a} + \mathbf{b} + \mathbf{c})$ **3** $\mathbf{x} = \binom{4}{1}$, $\mathbf{y} = \binom{-1}{2}$ $\mathbf{z} = \binom{2}{-2}$ **4** 14°, 117° and 45° respectively **5** $\sqrt{17}$, $\sqrt{5}$ and $2\sqrt{2}$ respectively
6(i) $\mathbf{OA} = \binom{3}{1}$, $\mathbf{OC} = \binom{2}{3}$ (ii) B(5, 4) (iii) parallelogram **7** $\mathbf{d} + \mathbf{e} = \binom{2}{4}$, $2\mathbf{e} = \binom{-4}{2}$, $2\mathbf{e} - \mathbf{d} = \binom{-8}{-1}$, $2\mathbf{e} + 3\mathbf{d} = \binom{8}{11}$, $\frac{1}{2}\mathbf{d} + \frac{3}{2}\mathbf{e} = \binom{-1}{3}$ **8** $\mathbf{OP} = \binom{1}{1}$, $\mathbf{OQ} = \binom{2}{5}$, $\mathbf{PQ} = \binom{1}{4}$, $\mathbf{RS} = \binom{1}{4}$, $\mathbf{QS} = \binom{4}{1}$, $\mathbf{PR} = \binom{4}{1}$ **9**(i) $\binom{-6}{-2}$ (ii) $x = -5, y = 2$ **10** $p = -\frac{3}{2}$, $q = \frac{5}{2}$

Exercise 38c **Page 123**

1 $\mathbf{PQ} = \mathbf{RS} = \binom{1}{4}$ **2** parallelogram since **AB** = **DC**; rhombus since |**AB**| = |**BC**|; square since |**AC**| = |**BD**| **3** $\binom{4}{6} = 2\mathbf{a}$, $\binom{-1}{-1\cdot5} = -\frac{1}{2}\mathbf{a}$, $\binom{-8}{-12} = -4\mathbf{a}$, $\binom{3}{4\cdot5} = \frac{3}{2}\mathbf{a}$ **4** (7, −2)
5 parallel, length AB = $\frac{1}{2}$length CD **6** collinear, length AB = $\frac{1}{3}$length BC
7 $\mathbf{AB} = \mathbf{b} - \mathbf{a}$, $\mathbf{CD} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$, $\mathbf{CD} = \frac{1}{2}\mathbf{AB}$, AB ∥ CD, length CD = $\frac{1}{2}$length AB
8 $\mathbf{BC} = \frac{1}{2}\mathbf{a} - \mathbf{b}$, $\mathbf{BE} = \frac{1}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}$, $\mathbf{OE} = \frac{1}{3}(\mathbf{a} + \mathbf{b})$, $\mathbf{BA} = \mathbf{a} - \mathbf{b}$, $\mathbf{BD} = \frac{1}{2}(\mathbf{a} - \mathbf{b})$, $\mathbf{OD} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$; $\mathbf{OE} = \frac{2}{3}\mathbf{OD}$; O, E, D collinear **9** $\mathbf{OP} = \frac{1}{3}\mathbf{a}$, $\mathbf{AB} = \mathbf{c}$, $\mathbf{OB} = \mathbf{a} + \mathbf{c}$, $\mathbf{OQ} = \frac{1}{5}(\mathbf{a} + \mathbf{c})$, $\mathbf{OR} = \frac{1}{2}\mathbf{c}$, $\mathbf{PQ} = \frac{1}{15}(3\mathbf{c} - 2\mathbf{a})$, $\mathbf{QR} = \frac{1}{10}(3\mathbf{c} - 2\mathbf{a})$; $\mathbf{PQ} = \frac{2}{3}\mathbf{QR}$; hence P, Q, R collinear **10** $\mathbf{OD} = \frac{2}{3}\mathbf{OE}$

Exercise 38d **Page 125**

1 OD : DE = 4 : 3, AD : DB = 3 : 4 **2** 4 : 1 **3** 3 : 5 **4** $\mathbf{PS} = \mathbf{b} - \mathbf{a}$, $\mathbf{OR} = 3\mathbf{a} + 2\mathbf{b}$; $h = \frac{2}{5}$, $k = \frac{1}{5}$; OX : XR = 1 : 4, PX : XS = 2 : 3 **5**(i) AG : GD = 2 : 1, BG : GE = 2 : 1 (ii) $\frac{1}{3}(\mathbf{a} + \mathbf{b})$ (iii) $\frac{1}{2}(\mathbf{a} + \mathbf{b})$ (iv) 2 : 1 (v) medians AD, BE, OF are concurrent (they pass through one point) at G

Exercise 39a **Page 126**

1 $2y = x + 3$ **2** $5x + 2y = 19$ **3** $x + 3y = 1$ **4** $y = x - 2$ **5** $2x = y + 1$ **6** $x + 3y = 2$
7 (−0·5, 2) **8** (1·5, 0·5) **9** (2·5, 1·5)

Exercise 39b Page 127

1 5 **2** 13 **3** 17 **4** $3\sqrt{2}$ **5** $5\sqrt{2}$ **6** $3\sqrt{5}$ **7** $AB = 5\sqrt{2}$, $BC = \sqrt{10}$, $CA = 2\sqrt{10}$; $\angle ACB = 90°$ **8** $PQ = QR = \sqrt{85}$, $RP = \sqrt{90}$ $(= 3\sqrt{10})$

Exercise 39c Page 128

1 parallel **2** perpendicular **3** neither **4** perpendicular **5** $2x - 3y = -5$ **6** $4y - 3x = 10$ **7**(i) $\frac{3}{4}$ (ii) $-\frac{4}{3}$ (iii) $4x + 3y = 19$ (iv) (1, 5) (v) 5 (vi) 25 sq units (vii) B(8, 4), C(5, 8) *or* B(0, −2), C(−3, 2) **8**(i) (2, −3) (ii) WZ: $2y - 3x = 14$, ZY: $3y + 2x = 8$ (iii) W(−4, 1), Y(4, 0) (iv) 26 sq units **9** *outline proofs:* (i) grad AD = grad BC = $\frac{6}{7}$ $\therefore$ AD ∥ BC, grad AB = grad DC = $-\frac{2}{9}$ $\therefore$ AB ∥ DC, AD = AB = $\sqrt{85}$ (ii) grad AC × grad BD = $\frac{1}{4} \times -4 = -1$ $\therefore$ AC⊥BD **10**(i) S(−1, −2) (ii) PQ = PS = $\sqrt{145}$ (iii) mid-point of PR and SQ is at (0·5, 2·5), grad QS × grad PR = $3 \times -\frac{1}{3} = -1$

Exercise 40a Page 130

4 24° **5** 126° **7** 6° **8** 8 and 10 **9** 96° and 72° **10** 84°, 14

Exercise 40b Page 131

1 5 **2** 13·5 **3** 7·1 **4** 5 **5** $\frac{1}{2}(13 + 14) = 13{\cdot}5$ **6** mode = 3, mean = 4·1, median = 4, range = 8 **7** 55·5 **8** 8 **9** 74 + 8 = 82 **10** 5 × 8 = 40 **11** 548 **12** 99·8

Exercise 40c Page 132

1

goals	0	1	2	3	4	5	6
matches	4	9	8	4	3	0	2

2

shoe size	4	5	6	7	8	9	10
number of girls	4	8	11	7	6	3	1

3 intervals are: 1, (2), 2, 4 **5** intervals are: (6), 3, 3, 6, 6

Exercise 40d Page 134

1 mode = 5, median = 4, mean = 4 **2** mode = 0, median = 2, mean = 2·4 **3**(i) 17·95min (ii) $16 + \frac{39}{20} = 17{\cdot}95$min **4** 3·8kg **5** 3·45km

Exercise 41a Page 135

1 (3, −4) **2** (−3, 4) **3** (4, 3) **4** (−4, −3) **5** $y = 1$ **6** (2, 2), (2, 1), (1, 1), (1, 2) **7** 24cm **8** (2, −1), (4, −1), (4, −2) **9** $\theta = 30°$; 60° **10** 40°; $2(30 - \phi)°$

Exercise 41b Page 136

1 1 line **2** 3 lines **3** no lines, unless isosceles **4** 2 lines **5** 4 lines **6** 5 lines **7** 6 lines **8** 8 lines **9** 2 lines **10** 1 line

Exercise 41c Page 137

1 equal **2** isosceles; two sides equal, base angles equal **3** A + B + C = 180° **4** rhombus; AC⊥BD, AC bisects BD **5** kite; 90°

Exercise 42a Page 139

1(i) (2, −5) (ii) (−5, −2) **2** (−4, 3) **3** (5, 0) **4** L′(−3, −1), M′(−6, −1), N′(−3, −3) **5** L′(−1, 3), M′(−1, 6), N′(−3, 3) **6**(iii) $\binom{-2}{2}$

Exercise 42b Page 140

1 (0, 0), +90° **2** (1, −2), +90° **3** (−0·5, 2), 180° **4** (1, 4), 180° **5**(ii) $k = 4$ (iv) $\mathbf{R} = \mathbf{P}^3$ or $\mathbf{R} = \mathbf{QPQ}$ **6**(i) +45° (ii) +90° (iii) B→D, D→F, BD→DF; BD⊥DF, BD = DF (iii) FH (v) $n = 8$ (vi) $m = 8$

Exercise 42c Page 142

1 H (order 2), N (order 2), X (order 4), Z (order 2); H, N, X, Z **2**(i) 3 (ii) 2 (iii) 5 (iv) infinite **3** 4, 5, 3 respectively **5** △ has vertices at (1, −1), (3, −1), (3, −2)

Exercise 43a **Page 143**

1 C 2 B 3 C 4 D 5 A 6 C 7 A 8 C 9 B 10 D

Exercise 43b **Page 144**

1 20,000 2 0·0004 3 500,000 4 2,000 5 sh100m 6 $\frac{2 \times 3{\cdot}14 \times 148600000}{365{\cdot}25 \times 24 \times 60 \times 60} \simeq$ 30km/h

Exercise 43c **Page 144**

1 5 ± 0·5cm, 10% 2 10 ± 0·5km, 5% 3 5·2 ± 0·05cm, 1% 4 24 ± 1cm, 4% 5 500 ± 5kg, 1% 6 75 ± 2·5km/h, 3% 7 4 ± 0·005ha, $\frac{1}{8}$% 8 20 ± 0·05*l*, $\frac{1}{4}$% 9(i) 0·6% (ii) 5%; (ii) 10(i) 0·2, 5% (ii) 0·8, $\frac{0{\cdot}8}{3} \times 100 \simeq 27\%$ 11(i) 10% (ii) 81 to 121cm² (iii) 20% 12 30% 13(i) 0·5km (ii) less than 0·01% (iii) less than 0·001cm (iv) no 14(i) 0·5ft (ii) 2,776·42 ± 0·15m (iii) 0·005% (iv) probably A

Exercise 44a **Page 146**

1 sin 20° ≃ 0·34, cos 20° ≃ 0·94, tan 20° ≃ 0·4 2 0·64, 0·77, 0·8 3 0·87, 0·5, 1·7 4 0·98, 0·17, 5·7 5 0·98, −0·17, −5·7 6 0·87, −0·5, −1·7 7 0·64, −0·77, −0·8 8 0·34, −0·94, −0·4 9 −0·34, −0·94, 0·4 10 −0·64, −0·77, 0·8 11 −0·87, −0·5, 1·7 12 −0·98, −0·17, 5·7 13 −0·98, 0·17, −5·7 14 −0·87, 0·5, −1·7 15 −0·64, 0·77, −0·8 16 −0·34, 0·94, −0·4 17 0, 1, 0, −1, 0 18 1, 0, −1, 0, 1 19 0, 0, 0 20 no, they have infinite values

Exercise 44b **Page 147**

1 0·72 2 0·5 3 −0·5 4 −0·72 5 0·41 6 0·41 8 0·64 9 −0·64 10 −0·64 11 0·64 12 −0·31 13 −0·31 15 1 16 −1 17 1 18 −1 19 −0·7 20 0·7

Exercise 44c **Page 148**

1 sin 115° = 0·906, cos 115° = −0·423, tan 115° = −2·145 2 −0·819, −0·574, 1·428 3 −0·906, 0·423, −2·145 4 −0·174, 0·985, −0·176 5 0·643, 0·766, 0·839 6 −0·866, −0·5, 1·732 7 $\sin 135° = \frac{1}{\sqrt{2}}$, $\cos 135° = -\frac{1}{\sqrt{2}}$, tan 135° = −1 8 $\frac{1}{2}, -\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}$ 9 $-\frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{3}}$ 10 $-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1$ 11 $-\frac{\sqrt{3}}{2}, \frac{1}{2}, -\sqrt{3}$ 12 $-\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}$ 13 14cm² 14 34·7cm² 15 56·4° or 123·6°

Exercise 45a **Page 150**

1 a = 5·65cm, c = 6·13cm, C = 70° 2 b = 10·5cm, B = 67°, c = 11·0cm 3 B = 26·5°, C = 23·5°, c = 6·24cm 4 a = 4·99cm, A = 29·6°, B = 52·4° 5 a = 6·59cm, b = 5·31cm, C = 114° 6 A = 90°, B = 60°, b = 7·79cm 7 q = 7·45cm, Q = 45·8°, R = 74·2° or q = 2·55cm, Q = 14·2°, R = 105·8° 8 674m 9 101km

Exercise 45b **Page 152**

1 a = 8·20cm, B = 43·5°, C = 66·5° 2 a = 11·5cm, B = 29·3°, C = 40·7° 3 A = 48·2°, B = 58·4°, C = 73·4° 4 A = 100·3°, B = 36·2°, C = 43·5° 5 P = 103·1° (not 76·9°) 6 R = 46·9°, P = 103·1°; find the smaller angle first since it cannot be obtuse 7 194km, 322·2° 8 211km, 111·9° 9 90°

Exercise 46a **Page 153**

1 1, $(x + 1)^2$ 2 4, $(x + 2)^2$ 3 16, $(x + 4)^2$ 4 36, $(x + 6)^2$ 5 1, $(x - 1)^2$ 6 4, $(x - 2)^2$ 7 16, $(x - 4)^2$ 8 36, $(x - 6)^2$ 9 $\frac{1}{4}$, $(x + \frac{1}{2})^2$ 10 $\frac{9}{4}$, $(x - \frac{3}{2})^2$ 11 $\frac{1}{16}$, $(x + \frac{1}{4})^2$ 12 $\frac{9}{16}$, $(x - \frac{3}{4})^2$

Exercise 46b **Page 154**

1 −2, 4 2 −4, 2 3 −2, 3 4 −2, 1·5 5 0·59, 3·41 6 −2·41, 0·41 7 −0·80, 8·80 8 0·44, 4·56 9 −0·62, 1·62 10 −0·35, 4·35 11 0·5, 2 12 −1·12, 1·79 13(i) −1, $\frac{2}{3}$ (ii) −1, $\frac{2}{3}$; factorisation 14 we cannot find $\sqrt{(-1)}$ 16 $-\frac{3}{2}$ (twice)

Exercise 46c **Page 155**

1 −3, −1 **2** −1, 3 **3** −3·79, 0·79 **4** 0·21, 4·79 **5** −2, 2·5 **6** 0·56, 4·44 **7** −1·77, −0·23
8 −3·47, −0·19 **9** −1·88, 0·13 **10** $x = \frac{4 \pm \sqrt{(-8)}}{4}$; we cannot find $\sqrt{(-8)}$ **11** $\frac{5}{2}$ (twice)

Exercise 46d **Page 156**

1 (1, 2), (2, 5) **2** (−2, −3), $\left(\frac{2}{3}, \frac{7}{3}\right)$ **3** (4, 3) twice **4** (1, −1), (5, 7) **5** (0, −1), $\left(\frac{8}{5}, \frac{3}{5}\right)$
6 $\left(-\frac{1}{3}, -3\right)$, (1, 1) **7** (0·59, 0·76), (3·41, 9·24) **8** (2, 1) twice; line tangent to curve at (2, 1)

Exercise 47a **Page 157**

1 $C = 60 + 4(w - 10)$; sh140 **2** $S = 2(lw + wh + hl)$; 376 **3** $A = l(60 - l)$
4 $A = \pi(R^2 - r^2)$ **5** $C = 5{,}000 + 3{,}000d + 50k$; sh76,000 **6** $A = \frac{1}{4}\sqrt{3} \times a^2$
7 $T = \frac{1}{2}np + 2c$; $n = 280$, 14·7mm **8** $V = \frac{1}{6}\sqrt{2} \times a^3$; $V = \frac{1}{3}\sqrt{2} \times a^3$ **9** $A = \pi rl$

Exercise 47b **Page 158**

1 $b = \frac{2A}{h}$ **2** $r = \frac{C}{2\pi}$ **3** $T = \frac{100I}{PR}$ **4** $m = \frac{y - c}{x}$ **5** $r = \sqrt{\frac{A}{\pi}}$ **6** $r = \sqrt{\frac{S}{4\pi}}$

7 $x = \sqrt[3]{\frac{A}{k}}$ **8** $d = b^2 - 4ac$ **9** $A = \frac{1}{2}h(a + b)$ **10** $h = \frac{2A}{a + b}$ **11** $r = \sqrt{\frac{3V}{\pi h}}$

12 $h = \frac{S}{2\pi r} - r$ **13** $b = \sqrt{(a^2 - c^2)}$ **14** $l = \frac{gT^2}{4\pi^2}$ **15** $d = \frac{2(S - an)}{n(n - 1)}$ **16** $t = \frac{y}{1 + y}$

17 $t = \sqrt{\left(\frac{1 - x}{1 + x}\right)}$ **18** $r = 100\left(\sqrt[n]{\frac{A}{P}} - 1\right)$ **19** $\sin B = \frac{b \sin A}{a}$

20 $\cos P = \frac{q^2 + r^2 - p^2}{2qr}$

Exercise 47c **Page 159**

1 10 **2** 30 **3** 6 **4** 5 **5** 88 **6** 12 **7** 14·4 **8** 9·82 **9** 5cm **10**(i) 25cm (ii) 24cm

Exercise 48a **Page 160**

1(i) 15 (ii) 5·6 **2**(i) 5 (ii) 20 **3**(i) 72 (ii) 10 **4**(i) 800 (ii) 10 **5**(i) 90km/h (ii) 60min
6(i) 640 Newtons (ii) 40km/h **7** 45m **8**(i) 10 Lumens per cm² (ii) 10m

Exercise 48b **Page 161**

1(i) 32 (ii) $\frac{8}{3}$ (iii) 2·5 **2**(i) 1·4 (ii) 10 (iii) 15 **3** 12cm **4**(i) $t = \frac{kh}{m}$ (ii) 8 months
5 96 **6** 54m **7** 7s **8** 24cm

Exercise 48c **Page 163**

1 $A \simeq \frac{9}{x}$ **2** $y \simeq 5 + 0{\cdot}6x^2$ **3** $k = 0{\cdot}2$; $g = 986$

Exercise 49a **Page 164**

1 cubic cutting x-axis at −2, 0 and 2 **2** cubic cutting x-axis at −2, 0 and 2 **3** cubic through (0, 0) **4** cubic cutting x-axis at −1, 1 and 3, y-axis at 3 **5** cubic touching x-axis at (0, 0) and cutting x-axis at 3 **6** cubic cutting x-axis at −2, y-axis at 2 **7** cubic cutting x-axis at 1, y-axis at 1 **8** cubic touching x-axis at −2 and cutting x-axis at 2, y-axis at 8
9(i) 3·4 (ii) 39·3 (iii) −19·7 (iv) 2·15 (v) 3·42 (vi) −2·71 **10** −2·41, 0·41, 3

Exercise 49b **Page 166**

Asymptotes are: **1** x- and y- axes (the axes) **2** y-axis, $2y = x$ **3** the axes **4** y-axis, $y = x$ **5** the axes **6** the axes **7** y-axis, $y = 1$ **8** y-axis, $y = 1$ **9** y-axis, $y + x = 0$ **10** y-axis, $y = x$ **11** y-axis, $y = x^2$ (a curved asymptote) **12** x-axis (curve cuts this asymptote at (0, 0))

Exercise 50a Page 167

1 55° **2** 40° **3** ∠OAP = 90°, ∠PAB = 50°, ∠APB = 80° **4** 110° **5** AP = BP = 24cm, ∠APB = 32·5° **6** OA = 9cm, ∠APB = 73·7° **7** AP = 17·3cm, OP = 20cm **8** PQ = 8cm, QR = 7cm, RP = 9cm **9** ∠TUV = 60°, ∠UVT = 66°, ∠VTU = 54° **10** ∠P = 90°, ∠Q = 70°, ∠R = 20°

Exercise 50b Page 168

1 ∠TAB = 70°, ∠TBA = 60°, ∠ATB = 50° **2** 60° **3** $a = 90°$, $b = c = 42°$ **4** $d = 64°$, $e = 52°$ **5** $f = 50°$, $g = 70°$, $h = 60°$ **6** $j = k = l = m = q = 54°$, $n = p = 72°$ **7** $r = 90°$, $s = 33°$, $t = u = 57°$

Exercise 50c Page 170

2 8cm; ∠ in semi-circle = 90° and tangent perpendicular to radius **3** 12·3cm **4** 10cm **5**(vi) 8·9cm or 9·0cm **6** AC = 7·5cm (ii) 8·9cm or 9·0cm

Exercise 51a Page 171

1(i) sh20,000 per annum (ii) sh103,000 (iii) nil (iv) sh8,000 (v) sh64,000 (vi) sh1,410,000 **2** sh392,000 **3** sh2,220,000; sh560,000 **4** sh1,000 **5** sh80,000

Exercise 51b Page 172

1 sh605 **2** sh203,500 **3** sh392,000 **4** sh18,000 **5** sh195,500, sh195,500, sh212,750 respectively **6** sh238,500 **7** sh87,500 **8**(i) sh8,000 (ii) sh19,000 (iii) sh5,000 (iv) sh30,000

Exercise 51c Page 173

1 sh11,375 **2** sh75,000 **3** sh34,500 **4** sh2,540 **5**(i) sh45,000 (ii) sh5,000

Exercise 52a Page 175

1(i) 2 × 4 (ii) 3 × 2 (iii) 2 × 2 (iv) 2 × 1 (v) 1 × 5 (vi) 3 × 4 (vii) 3 × 3 **2** $\begin{pmatrix} 2 & 6 \\ 6 & 6 \end{pmatrix}$ **3** $\begin{pmatrix} 6 & 9 & 11 \\ 6 & 8 & 10 \end{pmatrix}$ **4** $\begin{pmatrix} 10 & 10 & 10 \\ 11 & 11 & 11 \\ 11 & 11 & 5 \end{pmatrix}$ **5** $\begin{pmatrix} 0 & -2 \\ -3 & -3 \end{pmatrix}$ **6** $\begin{pmatrix} -4 & -4 & -4 \\ 4 & 5 & 5 \end{pmatrix}$ **7** $\begin{pmatrix} 6 \\ 5 \\ 5 \end{pmatrix}$ **8**(i) $\begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$ (ii) $\begin{pmatrix} -3 & -1 \\ 1 & 3 \end{pmatrix}$ (iii) $\begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix}$ (iv) $\begin{pmatrix} 20 & 15 \\ 10 & 5 \end{pmatrix}$ (v) $\begin{pmatrix} 25 & 25 \\ 25 & 25 \end{pmatrix}$ **9** $\begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}$ **10** $\begin{pmatrix} -1 & -2 & -3 \\ -4 & -5 & 0 \\ 1 & 2 & 3 \end{pmatrix}$ **11** $\begin{pmatrix} 5 & 6 & 2 \\ 5 & 2 & 6 \\ 4 & 6 & 3 \\ 1 & 8 & 4 \end{pmatrix}$ **12**(i) $\begin{pmatrix} 4 & 4 & 4 \\ 2 & 2 & 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 5 & 7 & 2 \\ 4 & 6 & 7 \end{pmatrix}$

Exercise 52b Page 176

1 $\begin{pmatrix} 5 & 8 \\ 13 & 20 \end{pmatrix}$ **2** $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ **3** $\begin{pmatrix} 8 & 17 \\ 12 & 15 \end{pmatrix}$ **4** $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ **5** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **6** $\begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$ **7** $\mathbf{B}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$, $\mathbf{B}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}$, $\mathbf{B}^n = \begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix}$ **11** $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ (i) C (ii) B **12**(i) A (ii) D **13**(i) $\mathbf{P} = \begin{pmatrix} 4 & 1 & 2 \\ 2 & 4 & 1 \\ 1 & 5 & 2 \end{pmatrix}$ (ii) $\mathbf{Q} = \begin{pmatrix} 200 \\ 300 \\ 500 \end{pmatrix}$ (iii) $\mathbf{PQ} = \begin{pmatrix} 2{,}100 \\ 2{,}100 \\ 2{,}700 \end{pmatrix}$; sh2,100, sh2,100, sh2,700 respectively

Exercise 52c Page 177

1 $\begin{pmatrix} 7 & -5 \\ -4 & 3 \end{pmatrix}$ **2** $\begin{pmatrix} 1 & -3 \\ -2 & 7 \end{pmatrix}$ **3** $\begin{pmatrix} 9 & -7 \\ -5 & 4 \end{pmatrix}$ **4** $\begin{pmatrix} -1 & 2 \\ 4 & -7 \end{pmatrix}$ **5** no inverse **6** $\frac{1}{14}\begin{pmatrix} 2 & -5 \\ 4 & -3 \end{pmatrix}$ **7** no inverse **8** ±8 **9** 4 **10** −6 or 2 **11** $\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

Exercise 52d Page 178

1 (2, 1) **2** (1, −1) **3** (−1, 1) **4** (−1, 1) **5** (2, 1) **6** (6·5, 3·5) **7** (2, 2) **8** (−1, 1) **9** many solutions; coincident lines **10** no solutions (equations inconsistent); parallel lines

Exercise 53a Page 179

1 (4, 1), (4, 3), (5, 3) 2 $\binom{2}{3}$ 3 A′(6, 3), B′(8, 3), C′(8, 4) 4 A″(3, 4), B″(5, 4), C″(5, 5) 5 $\binom{2}{3}$

Exercise 53b Page 179

1 (1, 2) 2 A′(4, 2), A″(−4, 2) 3 (2, 3) 4 (5, 3), (7, 3), (7, 4) 5 $\binom{4}{2}$

Exercise 53c Page 179

1 (−2, 4) 2 (2, −2) 3 A′(−1, −1), B′(−3, −1), C′(−3, −2) 4 A′(1, 1), B′(1, 3), C′(0, 3) 5 R(1, −1), +90°

Exercise 53d Page 180

1(i) (0, 0) (ii) 3 2 centre (0, 1), LSF 2 3 A′(3, −5), B′(3, 1), C′(6, 1) 4 A′(2, 2), B′(5, 2), C′(5, 5), D′(2, 5) 5(i) A′(7, 5), C′(9, 7), D′(7, 7) (ii) −2 6(i) A′(−1, −3), B′(−4, −3), C′(−4, −1) (ii) half-turn about O

Exercise 53e Page 182

1 A′(−2, 1), B′(−5, 1), C′(−5, 3); A″(−1, −2), B″(−1, −5), C″(−3, −5) (i) N (ii) no 2(i) A″(−1, −2), B″(−1, −5), C″(−3, −5); N (ii) A″(1, 2), B″(1, 5), C″(3, 5); M (iii) A″(−1, −2), B″(−1, −5), C″(−3, −5); N (iv) A″(−1, −2), B″(−1, −5), C″(−3, −5); N (v) A″(2, −4), B″(2, −10), C″(6, −10); − (vi) A″(2, −4), B″(2, −10), C″(6, −10); − (vii) A″(0, 3), B″(−3, 3), C″(−3, 1); half-turn about (1, 2) (viii) A″(−4, −5), B″(−7, −5), C″(−7, −7); half-turn about (−1, −2) (ix) A″(−2, −1), B″(−5, −1), C″(−5, −3); H (x) A″(2, 1), B″(5, 1), C″(5, 3); identity (see **54.3**) (xi) A″(0, −3), B″(3, −3), C″(3, −1); translation of $\binom{-2}{-4}$ (xii) A″(−8, −4), B″(−20, −4), C″(−20, −12); enlargement centre (0, 0), LSF −4 3 EM or ME 4 SE 5 ER or RE 6 SR 7 MS 8 ESX

Exercise 54a Page 184

1 reflection in x-axis 2 reflection in y-axis 3 half-turn about (0, 0) 4 enlargement centre (0, 0), LSF 2 5 positive quarter-turn about (0, 0) 6 identity (see **54.3**)

Exercise 54b Page 185

1 $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ 2 $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ 3 $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ 4 $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ 5 $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ 6 $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ 7 $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ 8 $\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ 9 (1, 5)

Exercise 54c Page 186

1(i) Q (ii) Q (iii) yes 2(i) H (ii) H (iii) yes 3 H 4 H 5 N 6 I 7 Q 8 Y 9 Q 10 H

Exercise 54d Page 187

1(ii) X^{-1} = reflection in x-axis = X (iii) X (iv) I 2(a)(ii) H^{-1} = half-turn about (0, 0) = H (iii) H (iv) I (b)(ii) M^{-1} = reflection in $y=x$ = M (iii) M (iv) I (c)(ii) enlargement centre (0, 0), LSF 0·5 (iii) $\begin{pmatrix} 0{\cdot}5 & 0 \\ 0 & 0{\cdot}5 \end{pmatrix}$ (iv) I 3(i) $\frac{1}{4}\begin{pmatrix} -4 & 3 \\ -4 & 2 \end{pmatrix}$ (ii) (3, −1) (iii) (9, 16) (iv) $\begin{pmatrix} -8 & 6 \\ -8 & 4 \end{pmatrix}$; $k = 8$ (v) $\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$; enlargement centre (0, 0), LSF 8 4(ii) O(0, 0), P(6, −2), Q(10, 5), R(4, 7) (iii) $2y = x$ (iv) $M^{-1} = \frac{1}{5}\begin{pmatrix} 7 & -4 \\ 1 & 3 \end{pmatrix}$ (v) $\frac{1}{5}M^{-1}$ (iv) O(0, 0), L(2, 0), M(2, 1), N(0, 1) 5 O′(0, 0), I′(3, 1), K′(4, 3), J′(1, 2); (i) 5 sq units (ii) 5 (iii) equal; yes

Exercise 55a Page 188

1 {natural numbers}, {integers} 2 {pupils at primary school}, {candidates for UCE} 3 {triangles}, {geometrical shapes} 4 {x : $10 \le x \le 60$}, {real numbers} 5 {S.4 teachers}, {people in Nebbi} 6 {x : $1 \le x \le 30$}, {rational numbers} 7 {x : $19 < x < 41$} 8 {students} 9 {natural numbers} 10 {quadrilaterals} 11 {geometrical shapes} 12 {cows} 13 12

Exercise 55b Page 189

1 {non-black cows on Lusiba's farm} 2 {countries of Africa which are not in East Africa} 3 {buses, lorries, motorcycles} 4 { } or ∅ 5 {irrational numbers} 6 {r, u, i, n} 7 {s, l, a, v, e} 8 {v, i, s, u, a, l} 9 ∅ 10(i) ∅ (ii) $\mathcal{E}$ (iii) A 11 {1, 2, 3, 4, 5, 7, 8, 9, 11, 12} 12 {1, 2, 3, 4, 5, 7, 8, 9, 11, 12} 13 {5, 7, 9, 11} 14 {5, 7, 9, 11} 15 If A and B are any sets then $(A \cap B)' = A' \cup B'$; $(A \cup B)' = A' \cap B'$ (these are known as *De Morgan's Laws*)

Exercise 55c Page 190

1 100 2 3 3 19; (i) 16 (ii) 104 4(i) 6 (ii) 7 (iii) 0 (iv) 1 (v) 2 5(i) 13 (ii) 43 (iii) 22; 18 6 79; 0

Exercise 56a Page 192

1(i) $\frac{1}{2}$ (ii) $\frac{1}{3}$ (iii) 0; symmetry 2 $\frac{1}{3}$; survey 3(i) $\frac{5}{18}$ (ii) $\frac{1}{3}$ (iii) $\frac{7}{18}$; symmetry 4(i) $\frac{7}{17}$ (ii) $\frac{10}{17}$ (iii) 1 (iv) 0; symmetry 5 0·9; survey 6 $180 \div 6 = 30$; symmetry 7 $\frac{2}{7}$; survey 8

6	7	8	9	10	11	12
5	6	7	8	9	10	11
4	5	6	7	8	9	10
3	4	5	6	7	8	9
2	3	4	5	6	7	8
1	2	3	4	5	6	7
	1	2	3	4	5	6

$\frac{6}{36} = \frac{1}{6}$; symmetry 9 $\frac{10}{36} = \frac{5}{18}$; symmetry
10 $6 \times 6 \times 6 = 216$; $1 + 1 + 3$, $1 + 3 + 1$, $3 + 1 + 1$, $1 + 2 + 2$, $2 + 1 + 2$, $2 + 2 + 1$; $\frac{6}{216} = \frac{1}{36}$; symmetry

Exercise 56b Page 193

1 $\frac{1}{3}$ 2(i) $\frac{14}{25}$ (ii) 0 3(i) $\frac{5}{12}$ (ii) $\frac{1}{4}$ 4(i) 0·5 (ii) 0·3 (iii) 0·65; 0·15 5 40 6 17 7 $\frac{1}{6}$ 8 0·2 9 $\frac{19}{120}$ 10 $\frac{1}{14}$ 11 $\frac{11}{60}$ 12 0·72

Exercise 56c Page 194

1(i) $\frac{1}{2}$ (ii) $\frac{1}{6}$ (iii) $\frac{2}{3}$ 2(i) $\frac{1}{3}$ (ii) $\frac{1}{9}$ (iii) $\frac{4}{9}$ 3(i) $\frac{1}{2}$ (ii) $\frac{1}{2}$ (iii) 1 4(i) $\frac{3}{10}$ (ii) $\frac{4}{10}$ (iii) $\frac{7}{10}$ 5 $\frac{7}{12}$ 6 0·2

Exercise 56d Page 195

1(i) $\frac{1}{36}$ (ii) $\frac{5}{36}$ (iii) $\frac{5}{36}$ (iv) $\frac{5}{18}$ 2(i) $\frac{1}{36}$ (ii) $\frac{1}{36}$ (iii) $\frac{1}{6}$ 3(i) $\frac{1}{7}$ (ii) $\frac{1}{49}$ (iii) $\frac{7}{49} = \frac{1}{7}$; $\frac{6}{7}$ 4(i) $\frac{1}{6}$ (ii) $\frac{2}{9}$ (iii) $\frac{7}{9}$ (iv) $\frac{4}{81}$ (v) $\frac{49}{81}$ (vi) $\frac{32}{81}$ (vii) $\frac{28}{81}$ (viii) $\frac{4}{81} + \frac{49}{81} + \frac{28}{81} = 1$ 5(i)(a) $\frac{17}{36}$ (b) $\frac{1}{8}$ (c) $\frac{7}{72}$ (d) $\frac{95}{144}$ (ii) 49 6 $\frac{1}{2}$

Exercise 57a Page 197

1 circle, centre A, radius 5cm 2 circle, centre B, radius 3cm 3 mediator of AB 4 two semi-circles, ie. a circle, diameter AB 5 two circular arcs, radius 4cm on chord AB 6 region inside circle 7 region outside circle 8 half-plane 9 region inside circle 10 region outside two circular arcs 13(ii) 12·3cm 15 circumscribed circle of △ABC 16 inscribed circle of △PQR

Exercise 57b Page 198

1 parabola, vertex (0, 0) 3 565cm³ (3 sf) 4 622cm³ (3 sf)

Exercise 57c Page 199

1 four-cusped hypocycloid (astroid), whose Cartesian equation is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$ (see front cover) 2 an ellipse 3 a parabola

Exercise 58a Page 201

1(i) $x + y \geq 10$, $y > x$ (iii) (3, 7), (4, 6), (5, 6) (iv) 11, given by (5, 6) (v) 4 small, 6 large given by (4, 6); sh2,100 2(i) people: $2p + 3t \geq 12$, luggage: $2p + t \geq 8$, cost: $4p + 5t \leq 25$, $p \geq 0$, $t \geq 0$ (ii) 3 pick-ups 2 taxis, 5 pick-ups 1 taxi, 6 pick-ups 0 taxis (iii) 3 pick-ups 2 taxis 3(i) $p \leq 8$, $t \leq 5$, $p + t \leq 10$, $p \geq 0$, $t \geq 0$ (iii) (3, 5), (4, 5), (5, 5), (5, 4), ~~(6, 4), (7, 3)~~ (iv) 3 pick-ups 5 trucks, 5 pick-ups 4 trucks, 7 pick-ups 3 trucks (v) 7 pick-ups 3 trucks 4 $2f + 3w \leq 150$, $6w \geq f$; 60m²

Exercise 58b

Page 203

1(i) $b + m \le 50$, $b \ge 15$, $m \ge 15$, $3b + 2m \le 120$ (iii) 20ha beans, 30ha maize; sh22m (iv) 15ha beans, 35ha maize; sh23·5m (v) 30ha beans, 15ha maize; sh22·5m **2**(ii) $r \ge 0$, $m \ge 0$, $r + m \ge 6$ (iv) sh$(400r + 1{,}000m)$ (v) 4·3kg rice, 1·7kg meat **3** 5 of type P, 4 of type Q **4**(ii) $x \ge 0$, $y \ge 0$ (iv) 3 large heaps, 6 small heaps **5** greatest value 10 given by (7, 3), least value 8 given by (4, 4) or (5, 3)

Exercise 59a

Page 205

1(i) 154km/h (ii) 10·6° (iii) 034·4° **2** 41·7km/h from 246° **3**(i) 006·1° (ii) 220km/h **4**(i) 113·9° or 114·0° (ii) 145km/h (iii) 10.31am **5** 9.24am **6**(i) 66° (ii) 3 min

Exercise 59b

Page 208

Answers are correct to 3 sf where applicable.

1(i) 0° (ii) 90°N (iii) 90°S; 0° **2** 7.13am **3** 40,000km **4** 20,000km **5** 4,670km **6** 26,300km **7**(i) 9,120km (ii) 8h 20min, ie. 8 hours (nearest hour) **8**(i) 11,000km (ii) 7h 12min ie. 7h (nearest hour) **9**(i) 9,650km (ii) 98·5° (iii) 10,900km; great circle route 800km shorter **10** 3,890km; shorter **11**(i) 17,300km (ii) 13,300km; great circle route shorter; 12 midnight **12**(i) $360 \times 60 = 21{,}600$ nautical miles (ii)(a) 900 knots (b) 1,670km/h **13** 4 hours **14** $A = 60\theta\cos\lambda$ **15**(i) 2,280 nautical miles (ii) 37°N, 37°E (iii) Gaziantep (iv) 37°N, 3°E (v) Algiers

Exercise 60a

Page 211

1(i) 12.40 (ii) 6km (iii) 30 min (iv) 3km/h; 2km/h (v) 0km/h **3**(i) 11.40 (iv) 10.40, 10km; 11.20, 20km **4**(i) 10s (ii) 20m/s (iii) 0·75m/s², 0·5m/s² (iv) 1m/s² (v) 300m (vi) 1,075m **5** 0·4m/s², 0·5m/s²; 224m **6**(i) at 15s, 80s (ii) 0·8m/s² (iii) 1·2m/s² (iv) 1,560m

Exercise 60b

Page 213

1 4m/s, 2m/s, 0m/s; object momentarily at rest **2** 4·5m/s, 7·5m/s, 10·5m/s **3**(i)(a) 30m/min (b) 74m/min (ii)(a) 90m/min (b) 60m/min (iii) 190m/min, at 6 min **4**(i) 1·4m/s² (ii) 0·7m/s² **5**(i) 0·6m/s², 0·9m/s² (ii) just over 5m/s; 4·3s, 0m/s²

Exercise 60c

Page 214

1 $59 \div 4 = 14{\cdot}75$ sq units **2** 14·75 sq units **3**(i) 43 squares, 1,075m (ii) 1,075m **4**(i) 25·75m (ii) 32·4m **5** 3m/s²; 19·5m **6**(i) 2m/s (ii) −7m/s² (iii) 2·5s (iv) 4·7m (v) 0m and 5·2m (vi) 2·6m (vii) 3·1m/s

Paper 1

Page 215

1 0·64 **2** $\frac{1}{12}(x + 1)$ **3** 7·5cm **4** 0·8 **5** −0·111 or $\bar{1}{\cdot}889$ **6** 800cm² **7** sh2,100 **8** $x = 3$, $y = -5$ **9** 1,060m **10**(i) $\binom{8}{6}$, $\binom{-3}{4}$ (ii) 10, 5 **11**(i) 16 (ii) 16 (iii) {students who play basketball and soccer} **12**(i) $\begin{pmatrix} 6 & -2 \\ -1 & 12 \end{pmatrix}$ (ii) $\begin{pmatrix} -3 & 4 \\ -9 & 12 \end{pmatrix}$ (iii) $\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix}$ **13** 3, 13, 43, 213, 1043 **14**(i) 14 (ii) 10 **16**(i) 8·66cm, 2·89cm (ii) 43·3cm² (iii) 14·4cm² (iv) 105cm² **17**(iii) A′(5, 10), B′(15, 10), C′(10, 0) (v) 10, 100 sq units **18**(i) 0·25 (ii) $\frac{11}{18}$ (iii) 0·25 **19**(ii) $x + 2y \le 16$, $3x + 2y \le 24$ (iv) 10, (4, 6) **20**(i) 0, −7, 0 (ii) $(p + q)(p - q)$, $fg : x \to (x + 4)(x - 2)$ (iii) $x = -4$ or 2, (iv) at (−4, 0), (2, 0), (0, −8) **21** 675m³, 4,580l **22**(i) $h = w \tan 60° = w\sqrt{3}$, $h = (w + 16) \tan 45° = w + 16$ (ii) 21·9, 37·9 (iii)(a) 58° (b) 5%

Paper 2

Page 217

1 0·0375 **2** 3 **3** 1·1 **4** 1·25x **5** sh48,180 **6** 3,850m² **7** 3·2 **8** $3(\mathbf{y} - \mathbf{x})$ **9** 0·09g **10**(i) $R \cap P \neq \varnothing$ (ii) there are no white rabbits **11** (1, 3), (3, 3) **12** $a = 3$, $b = -2$ **13** 36° **14** {−2, −1} **15** all **16**(i) $CB = \mathbf{b} - \mathbf{c}$, $CM = \frac{1}{2}(\mathbf{b} - \mathbf{c})$, $OM = \frac{1}{2}(\mathbf{b} + \mathbf{c})$

(ii) $\mathbf{AM} = -\mathbf{a} + \frac{1}{2}(\mathbf{b} + \mathbf{c})$ (iii) $\mathbf{AP} = \mathbf{p} - \mathbf{a}$, $n = 3$ (iv) $\mathbf{p} = \frac{2}{3}\mathbf{b}$, P lies on OB such that OP : PB = 2 : 1 **17**(i) A′(1, 1), B′(−3, 0), C′(−1, −2) (ii) enlargement, centre (0, 0), LSF −0·5 (iii) 5, 20 sq units **18** 126·5%, sh68,550, sh2,240 **19**(i) 10·5cm (ii)(a) 5cm (b) 6·9cm (iii)(a) 19° (b) 5·3cm **20**(i) 8, 7·75, 7, 5·75, 4, 1·75 (v) P(2, 4), 20 **21**(i)(a) 3hours (b) 3hours 7min 30sec (ii) $x = 2·5$ **22**(ii) 5·6 (iii) 6 (iv) 35%

Paper 3 **Page 221**

1 $2\frac{1}{8}$ **2** 61hours **3** 52km/h **4** 8·7cm **5** $n = 6$ **7**(i) 13, −16 (ii) $4x^2 + 20x$ (iii) $gf(x) = 4h(x)$ **8** $x = \dfrac{c}{k(a - b)}$ **9** 0·42 **10** {1, 2, 3, 6} **11** 7,497 **12** $x = 0·25$ **13** 11,811 **14** 11·3° **15** sh211,000 **16** 44 **17**(i) 107m (ii) 62m (iii) 31° **18** S(A) is (0, 10); enlargement, centre (0, 0), LSF 2; U(A) is (8, 6); reflection in $y = 2x$ **19**(i) $2x + 3y \leq 30$, $2x + 5y \geq 30$, $2x + y \geq 14$ (ii) $x = 5$, $y = 4$ (iii) 4, 5 or 6 **20**(a)(i) 25·0m (ii) 24·4° (b)(i) 12cm (ii) 1,200cm² (iii) sh13,500 **21**(i) $\mathbf{b} - \mathbf{a}$ (ii) $\frac{1}{2}(\mathbf{b} - \mathbf{a})$ (iii) $\frac{1}{2}(\mathbf{a} + \mathbf{b})$ (iv) $2\mathbf{a}$ (v) $\mathbf{b} - 2\mathbf{a}$ (vi) $\frac{1}{3}(\mathbf{b} - 2\mathbf{a})$ (vii) $\frac{2}{3}(\mathbf{a} + \mathbf{b})$; $k = \frac{3}{4}$ P, T, V collinear and PT : TV = 3 : 1 **22**(i) 940cm³ (ii) 740cm² (iii) 750 cm³ (iv) 800cm²

Paper 4 **Page 223**

1 $1\frac{1}{9}$ **2** 0·296 **3** reflection in y-axis **4** 7·85cm² **5** 57·6km/h **6** $x = -1$ or 2 **7** $x \to \dfrac{1}{x} + 1$ **8** $\frac{5}{12}$ **9** $\begin{pmatrix} 1 & 4 \\ 1 & -8 \\ 1 & 3 \\ 1 & 2 \end{pmatrix}$ **10** 60° **11** Both $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$, R is mid-point of QS **12** 15·5 sq units **13** −2·83, 0·83 **14** 15, 9 **15** 3 **16** sh1,490, 150% **17**(i) 27·0m (ii) 54·0m (iii) 93·5m (iv) 29·1° **18** A = 2, B = −3; −0·22, 2·22 **19** x, $3 - x$, $5 - x$, $4 + x$; 35, $x = 0, 1, 2$ or 3; $\frac{3}{7}$ **20**(i) A′(0, 5) (ii) OA = OA′ = 5, 53·1° (iii) B′(−2·8, 4·6) (iv) area △OAB = area △OA′B′ = $\frac{1}{2} \times 5 \times 2·8 = 7$ sq units **21**(i) △BDC ≡ △ADE (SAS) (ii) proof follows from ∠DBA = ∠DAB (isosceles △ADB) and ∠CBD = ∠EAD (iii) △ABC ≡ △BAE (SAS) **22** $\mathbf{AB} = 5\mathbf{q} - 3\mathbf{p}$, $\mathbf{BC} = 2\mathbf{r} - 5\mathbf{q}$, $\mathbf{CA} = 3\mathbf{p} - 2\mathbf{r}$; proof follows from $\mathbf{OA} = \mathbf{CB}$; $\mathbf{PQ} = \mathbf{q} - \mathbf{p}$, $\mathbf{QR} = \frac{3}{2}(\mathbf{q} - \mathbf{p})$, $\mathbf{QR} = \frac{3}{2}\mathbf{PQ}$, PQ : QR = 2 : 3

Paper 5 **Page 225**

1 $\frac{13}{16}$ **2** 4min 12seconds **3** 18·7tonnes/m² **4** 154cm **5** 1234_5 **6**(i) $\frac{4}{11}$ (ii) $\frac{7}{11}$ **7** 8, 12 **8** 7·5% **9** 38l **10** $(7x + 2)(3x - 2)$ **11** $\begin{pmatrix} 3 & -4 \\ 0 & 2 \end{pmatrix}$ **12** 90° **13** 24 **14** $\mathbf{b} - \mathbf{c}$, $\frac{1}{2}(\mathbf{c} - \mathbf{a})$ **15** 75°, 75°, 105°, 105° **16** 2,900 calories, 531g, 21% **17**(i) 7km (ii) 052° (iii) 6h 15min **18** 2, 2·25; 1, 1·5; 0·5, 1·5; 0·25, 2·25; 0·2, 2·7 (i) 1·41 (ii) 0·29, 1·71; $y = 3 - x$: 0·19, 1·31 **19**(i) {2, 4, 8}, {2, 4, 5, 10, 20}, {2, 4}, {2, 4, 5, 8, 10, 20} (ii) $r = 4$, r is HCF of 8 and 20 (iii) $s = 40$, s is LCM of 8 and 20 **20** $x + 2y \leq 32$; $x + y \geq 20$, $x \leq 2y$; maximise $5x + 3y$; 16 standard, 8 super **21**(i) 120° (ii) 43m² (iii) 105m² (iv) 61m² (or 62) (v) 504m² **22**(i) 0·9, 0·9m/s² (ii) 44 sq units (by counting squares), 45·1 sq units (by trapezium rule); 44m or 45m

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