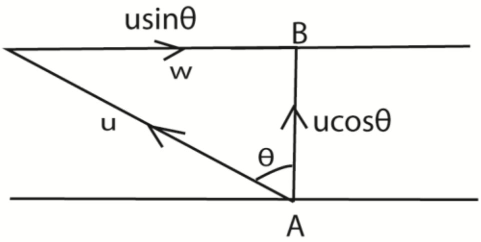
RELATIVE VELOCITY

Crossing a river

There are two cases to consider when crossing a river

Case I: Shortest route

If the water s not still and boatman wishes to cross **directly opposite** to the standing point. In order to cross from point A to point B directly opposite A (perpendicularly), then the course set by the boat must be upstream of the river.



u = speed of the boat in still water

w= speed of running water

At point B: w = usinθ

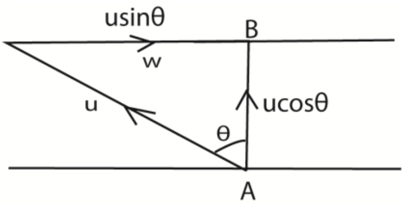
θ = $\sin^{-1}\left(\frac{w}{u}\right)$

θ = is the direction to the vertical but the direction to the ban is $(90 - \theta)^0$

Time taken = $\frac{AB}{u\cos\theta}$

Example 1

A man who can swim at 6km/h in still water would like to swim between two directly opposite point on the river banks of the river 300m flowing at 3km/h. Find the time taken to do this.



AB = 0.3km

θ = $\sin^{-1}\left(\frac{w}{u}\right) = \sin^{-1}\left(\frac{3}{6}\right) = 30^0$

Time taken = $\frac{AB}{u\cos\theta} = \frac{0.3}{6\cos 30}$

time = 0.058hrs = 3.46minute

He must swim at 300 to AB in order to cross directly and it takes him 3.46minutes.

Alternatively

Using Pythagoras theorem

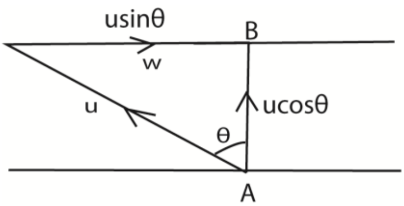
$6^2 = 3^2 + V_{AB}^2$

$V_{AB} = \sqrt{36 - 9}$ = 5.1962km/h

Time = $\frac{AB}{V_{AB}} = \frac{0.3}{5.1963}$ = 0.058hrs

Example 2

Two points A and B are on opposite banks of a river flowing at $\frac{5}{6}ms^{-1}$. A man who can swim at $\frac{25}{18}ms^{-1}$ in still water would like to swim directly from A to B. Find the width of the river if he takes 2minutes to cross the river.



$\theta = \sin^{-1}\left(\frac{w}{u}\right) = \sin^{-1}\left(\frac{5/6}{25/18}\right) = 36.87^0$

Time taken = $\frac{AB}{ucos\theta}$ = $\frac{0.3}{\frac{25}{18}cos36.87}$

AB = 133.333m

Alternatively: using Pythagoras theorem

$\left(\frac{25}{18}\right)^2 = \left(\frac{5}{6}\right)^2 + V_{AB}^2$

$V_{AB}=1.1111m$

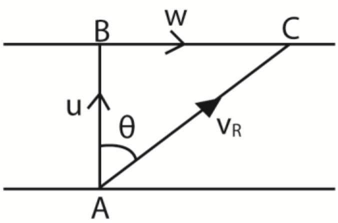
Time taken $=\frac{AB}{V_{AB}}$

2 x 60 = $\frac{AB}{1.1111}$

AB = 133.333m

Case II: the shortest time taken/ as quickly as possible

If the boatman wishes to cross the river as quickly as possible, then he should steer his boat directly from A to B as shown. The river pushes the boat down stream



Time taken to cross the river, t = $\frac{AB}{u}$

Distance covered downstream = wt

Or

Distance downstream = $w\frac{AB}{u}$

tanθ = $\frac{w}{u}$ or θ = $\tan^{-1}\frac{w}{u}$

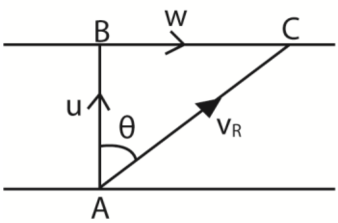
The resultant velocity downstream V_R

$$V_R = \sqrt{w^2 + u^2}$$

Example 3

A man who can swim at $2ms^{-1}$ in still water wishes to swim across a river 120m wide as quickly as possible. If the river flowsat0.5ms-1, find the time the man takes to cross far downstream he travels.

Solution



u = $2ms^{-1}$, w = $0.5ms^{-1}$, AB = 120m

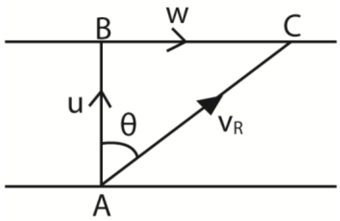
t = $\frac{AB}{u} = \frac{120}{2} = 60s$

Distance = wt = 0.5 x 60 = 30m

Example 4

A boat can travel at 3.5ms^{-1} in still water. A river is 80m wide and the current flows at 2ms^{-1}, calculate

(a) the shortest time to cross the river and the distance downstream the oat is carried.

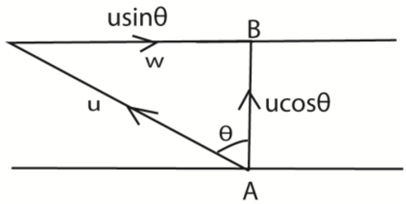


u = 3.5ms^{-1}, w = 2ms^{-1}, AB = 80m

t = $\frac{AB}{u} = \frac{80}{3.5} = 22.95s$

Distance, BC = wt = 2 x 22.95 = 45.8m

(b) the course that must be set to point exactly opposite the starting point and time taken for crossing



course, θ = $\sin^{-1}\frac{w}{u} = \sin^{-1}\frac{2}{3.5} = 34.8^0$

Time for crossing = $\frac{80}{3.5cos34.8}$= 27.8s

u = 3.5ms^{-1}, w = 2ms^{-1}, AB = 80m

Revision exercise 1

1. A man who can row at 0.9ms^{-1} in still water wishes to cross a river of width 1000m as quickly as possible. If the current flows at a rate of 0.3ms^{-1}. Find the time taken for journey. Determine the direction in which he should point the boat and position of the boat where he lands. [111.11s, 71.57^0 to the bank, 333.33 downstream]
2. A man swims at 5kmh^{-1} in still water. Find the time it takes the man to swim across the river250m wide, flowing at 3kmh^{-1}, if he swims so as to cross the river
   (a) the shortest route [225s]
   (b) in the quickest time[180s]
3. A boy can swim in still water at 1ms^{-1}, he swims across the river flowing at 0.6ms^{-1} which is 300m wide, find the time he takes
   (a) if he travels the shortest possible distance [375s]
   (b) if he travels as quickly as possible and the distance downstream, [300s, 180m]
4. A boy wishes to swim across a river 100m wide as quickly as possible. The river flows at 3kmh^{-1} and the boy can swim at 4kmh^{-1} in still water. Find the time that the boy takes to cross the river and how far downstream he travels. [90s, 75m]

Relative motion

It is composed of

(a) relative velocity
(b) Relative path

(a) Relative velocity

This is the velocity a body would have as seen by an observer on another body. Suppose A and B are two moving bodies, the velocity of A relative to B is the velocity of A as it appears to observer on B

It is denoted by $V_{AB} = V_A – V_B$

Note that $V_{AB} \neq V_{BA}$ since $V_{BA} = V_B – V_A$.

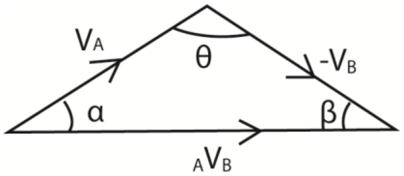
Numerical calculations

There are two methods used in calculations

- Geometric method and
- Vector method

(i) Geometric method.

If V_A and V_B are not given in vector form and the velocity of A relative to B is required, then we can reverse the velocity of B such that $V_{AB} = V_A + (-V_B)$ and the vector triangle is drawn as below.



$V_{AB}^2 = V_A^2 + V_B^2 - 2V_A \times V_B \cos\theta$

and

$$\frac{V_{AB}}{sin\theta} = \frac{V_B}{sin\alpha} = \frac{V_A}{sin\beta}$$

(ii) Vector method

Find components of velocity for each separately

$\therefore V_{AB} = V_A - V_B$

Example 5

Particle A is moving due north at $30ms^{-1}$ and particle B is moving due south at $20ms^{-1}$. Find the velocity of A relative to B.

Solution

$\uparrow V_A = 30ms^{-1}$ and $\downarrow V_B = 20ms^{-1}$

$V_{AB} = V_A - V_B$

$V_{AB} = \binom{0}{30} - \binom{0}{-20} = \binom{0}{50}$

$|V_{AB}| = \sqrt{0^2 + 50^2} = 50ms^{-1}$ due north

Example 6

A particle A has a velocity $(4i + 6j -5k)ms^{-1}$ while B has a velocity of $(-10i – 2j + 6k)ms^{-1}$. Find the velocity of A relative to B.

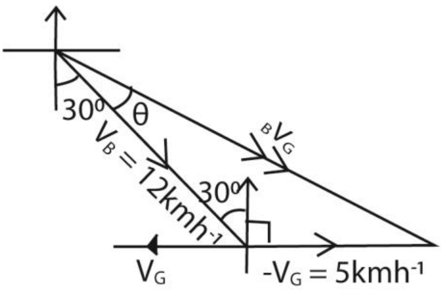
$V_{AB} = V_A - V_B$

$$V_{AB} = \begin{pmatrix} 4 \\ 6 \\ -5 \end{pmatrix} - \begin{pmatrix} -10 \\ -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 14 \\ 8 \\ -11 \end{pmatrix} ms^{-1}$$

Example 7

A girl walks at $5kmh^{-1}$ due west and a boy runs $12kmh^{-1}$ at a bearing of 150^0. Find the velocity of the boy relative to the girl

Method I (geometrical)



$V_{BG}^2 = V_B^2 + V_G^2 - 2V_B \times V_G \cos 120^0$

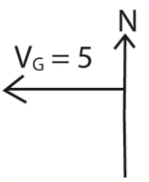
$V_{BG} = \sqrt{5^2 + 12^2 - 2 \, x \, 5 \, x \, 12 \, cos120^0}$ = $15.13ms^{-1}$

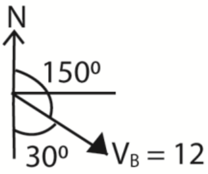
$\frac{5}{sin\theta} = \frac{15.13}{\sin 120}$

$\theta = 16.63^0$

The relative velocity is $15.13ms^{-1}$ at $S46.63^0E$

Method II (Vector)



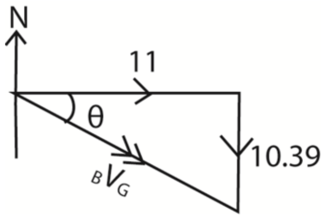


$V_{BG} = V_B - V_G$

$= \begin{pmatrix} 12sin30 \\ -12cos30 \end{pmatrix} - \begin{pmatrix} -5 \\ 0 \end{pmatrix} = \begin{pmatrix} 11 \\ -10.39 \end{pmatrix}$

$|V_{BG}| = \sqrt{11^2 + (-10.39)^2}$

$= 15.13ms^{-1}$

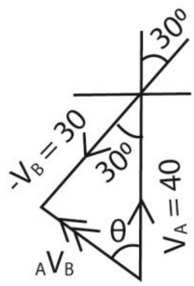


$\theta = \tan^{-1}\frac{10.39}{11}$ = 43.40

The relative velocity is $15.13ms^{-1}$ at $S46.63^0E$

Example 8

Plane A is flying due north at $40kmh^{-1}$ while plane B is flying in the direction $N30^0E$ at $30kmh^{-1}$. Find the velocity of A relative B



$V_{AB}^2 = V_A^2 + V_B^2 - 2V_A \times V_B \cos 30^0$

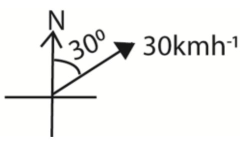
$V_{BG} = \sqrt{40^2 + 30^2 - 2 \, x \, 40 \, x \, 30 \, cos30^0}$

$= 20.53kmh^{-1}$

$\frac{30}{sin\theta} = \frac{20.53}{\sin 30}$; $\theta = 46.94^0$

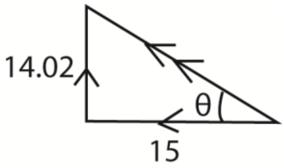
The relative velocity is $20.53kmh^{-1}$ at $N46.9^0W$

Method II (vectors)



$V_{AB} = V_A - V_B = \begin{pmatrix} 0 \\ 40 \end{pmatrix} - \begin{pmatrix} 30sin30 \\ -30cos30 \end{pmatrix} = \begin{pmatrix} -15 \\ 14.02 \end{pmatrix}$

$|V_{BG}| = \sqrt{(-15)^2 + (14.02)^2}$ = $20.53kmh^{-1}$



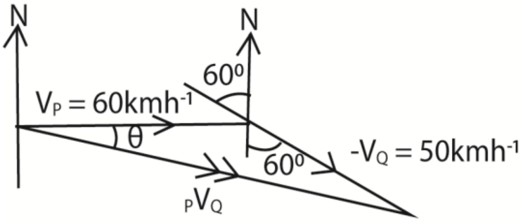
$\theta = \tan^{-1}\frac{14.02}{15} = 43.1^0$

The relative velocity is 20.53kmh^{-1} at N46.9^{0}W

Example 9

Ship P is steering 60kmh^{-1} due east while ship Q is steering in the direction N600W at 50kmh^{-1}. Find the velocity of P relative to Q.

Method I (Geometrical)



$V_{PQ}^2 = V_P^2 + V_Q^2 - 2V_P \times V_Q \cos 150^0$

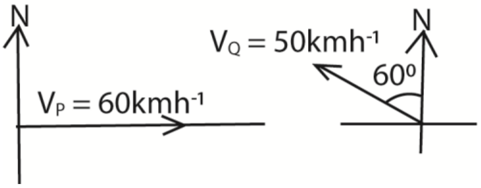
$V_{BG} = \sqrt{60^2 + 50^2 - 2\,x\,60\,x\,50\,cos150^0}$

$= 106.28$kmh^{-1}

$\frac{50}{sin\theta} = \frac{106.28}{\sin 30}$; $\theta = 13.6^0$

The relative velocity is 106.28kmh^{-1} at S76.4^{0}E

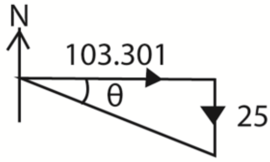
Method II (Vector)



$V_{PQ} = V_P - V_Q$

$= \binom{60}{0} - \binom{-50sin60}{50cos60} = \binom{103.301}{-25}$

$|V_{BG}| = \sqrt{(103.301)^2 + (-25)^2}$ =106.3kmh^{-1}



$\theta = \tan^{-1}\frac{25}{103.301} = 13.6^0$

Direction S(90^0 – 13.6^0)E

The relative velocity is 106.28kmh^{-1} at S76.4^{0}E

Finding true velocity

Example 10

To a cyclist riding due north at 40kmh^{-1}, a steady wind appears to blow from N60^{0}E at 50kmh^{-1}. Find the true velocity of the wind

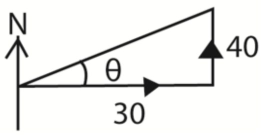
Solution

$V_{WC} = V_W - V_C$

$\binom{30}{0} = V_W - \binom{0}{40}$

$V_W = \frac{30}{40}$

$|V_W| = \sqrt{(30)^2 + (40)^2}$ = 50kmh^{-1}



$\theta = \tan^{-1}\frac{40}{30} = 53.13^0$

Direction N (90 – 15.13)^{0}E = N36.87^{0}E

Example 11

To a motorist travelling due north at 40kmh^{-1}, a steady wind appears to blow from N60^{0}E at 50kmh^{-1}.

(a) find the true velocity of the wind

$V_{WM} = V_W - V_M$

$\begin{pmatrix} 50sin60 \\ -50cos60 \end{pmatrix} = V_W - \begin{pmatrix} 0 \\ 40 \end{pmatrix}$

$V_W = \frac{-43.5}{15}$

$|V_W| = \sqrt{(-43.5)^2 + (15)^2}$ = 46kmh^{-1}

θ =$\tan^{-1}\frac{15}{43.5}$ = 19.02^0

Direction: N71^{0}W

(b) If the wind velocity and direction remain constant but the speed of the motorist is increase, find his speed when the wind appears to be blowing from the direction N45^{0}E.

$V_{WM} = V_W - V_M$

$\begin{pmatrix} -bsin45 \\ -bcos45 \end{pmatrix} = \begin{pmatrix} 46\sin71 \\ -46cos71 \end{pmatrix} - \begin{pmatrix} 0 \\ a \end{pmatrix}$

i components : -bsin45 = 46sin71

b = 61.5096

j components: -bcos45 = -46cos71 – a

a = 58.47kmh^{-1}

Example 12

To a man travelling due north at 10kmh^{-1}, a steady wind appears to blow from East. When he travels in the direction N60^{0}W at 8kmh^{-1}, it appears to come from south. Find the velocity of the wind.

$V_{WM} = V_W - V_M$

$\begin{pmatrix} -a \\ 0 \end{pmatrix} = V_W - \begin{pmatrix} 0 \\ 10 \end{pmatrix}$

$V_W = \begin{pmatrix} -a \\ 10 \end{pmatrix}$ ……………………………..(i)

Also

$V_{WM} = V_W - V_M$

$\begin{pmatrix} 0 \\ b \end{pmatrix} = V_W - \begin{pmatrix} -8sin60 \\ b + 8cos60 \end{pmatrix}$ ….. (ii)

(i) and (ii)

$\begin{pmatrix} -a \\ 10 \end{pmatrix} = \begin{pmatrix} -8sin60 \\ b + 8cos60 \end{pmatrix}$

a = 8sin60 = 4$\sqrt{3}$

10 = b + 8cos60

b = 6

$V_W = \begin{pmatrix} -a \\ 10 \end{pmatrix} = \begin{pmatrix} -4\sqrt{3} \\ 10 \end{pmatrix}$

$|V_W| = \sqrt{(-4\sqrt{3})^2 + 10^2}$ = 12.17kmh^{-1}

$\theta = \tan^{-1}\left(\frac{10}{4\sqrt{3}}\right)$= 55.3^0

Direction: N(90 -53.3)^{0}W= N34.7^{0}W

Example 13

To a cyclist riding due north at 40kmh^{-1}, a steady wind appears to blow eastwards. On reducing his speed to 30kmh$^-$1 but moving in the same direction, the wind appears to come from southwest. Find the velocity of the wind

$V_{WC} = V_W - V_C$

$\begin{pmatrix} a \\ 0 \end{pmatrix} = V_W - \begin{pmatrix} 0 \\ 40 \end{pmatrix}$

$V_W = \begin{pmatrix} a \\ 10 \end{pmatrix}$ ……………………………..(i)

Also

$V_{WC} = V_W - V_C$

$\begin{pmatrix} bsin45 \\ bcos45 \end{pmatrix} = V_W - \begin{pmatrix} 0 \\ 30 \end{pmatrix}$ ….. (ii)

(i) and (ii)

$\binom{a}{40} = \binom{bsin45}{30 + bcos45}$

40 = 30 + bcos45

b = $10\sqrt{2}$

a = bsin45 = $10\sqrt{2}$sin45 = 10

$V_W = \binom{a}{40} = \binom{10}{40}$

$|V_W| = \sqrt{10^2 + 40^2}$ = 41.23kmh^{-1}

$\theta = \tan^{-1}\left(\frac{40}{10}\right)$= 75.96^0

Direction: N(90 -76)^{0}E= N14^{0}E

Exercise 2

1. Car A moving Eastward at 20ms^{-1} and car B is moving Northward at 10ms^{-1}. Find the
   (i) velocity of A relative to B [$10\sqrt{5}ms^{-1}$
   (ii) velocity of B relative to A [$10\sqrt{5}ms^{-1}$
2. A yatch and a trawler leave a harbour at 8am. The yatch travels due west at 10kmh-1 and trawler due east at 20kmh^{-1}
   (i) what is the velocity of the trawler relative to yatch [30kmh^{-1} east]
   (ii) how far apart are the boats at 9.30am [45km]
3. At 10.30am a car travelling at 25ms^{-1} due east overtakes a motor bike travelling at 10ms^{-1} due east. What is the velocity of the car relative to the motor bike and how far apart are the vehicle at 10.30am. [15ms^{-1} east, 900m]
4. Bird A has a velocity of (7i + 3j + 10k)ms^{-1} while bird B has a velocity (6i – 17k)ms^{-1}. Find the velocity of B relative to A [(-i -3j – 27k0ms^{-1}]
5. Joe rides his horse with a velocity $\binom{5}{24}$kmh^{-1} while Jill is riding her horse with velocity $\binom{5}{12}$kmh^{-1}
   (i) Find Joe's velocity as seen by Jill $\left[\binom{5}{24}\text{kmh}^{-1}\right]$
   (ii) What is Jill's velocity as seen by Joe. $\left[\binom{0}{-124}\text{kmh}^{-1}\right]$
6. In EPL football match, a ball is moving at 5ms^{-1}in the direction of N450E and the player is running due north at 8ms^{-1}. Find the velocity of the ball relative to the player.
   [5.69ms^{-1} at S38.38^{0}E]
7. An aircraft is flying at 250kmh^{-1} in direction N60^{0}E and a second aircraft is flying at 200kmh^{-1} in the direction N20^{0}W. Find the velocity of the first aircraft as seen by the pilot of the second aircraft. [292kmh^{-1} at S77.9^{0}E]
8. A ship is sailing southeast at 20kmh^{-1} and a second ship is sailing due west at 25kmh^{-1}. Find the magnitude and direction of the velocity of the first ship relative to the second.
   [41.62kmh^{-1} at S70.13^{0}E]
9. What is the velocity of a cruiser moving at 20kmh^{-1} due to north as seen by an observer on a liner moving at 15kmh^{-1} in the direction N30^{0}W [10.3kmh^{-1} at N46.9^{0}E]
10. A car is being driven at 20ms-1 on a bearing of 0400. Wind is blowing from 3000 with speed of 10ms-1. Find the velocity of the wind as experienced by the driver of the car.
    [48.13ms^{-1} at S18.13^{0}W]
11. An aircraft is moving at 250kmh^{-1} in direction N60^{0}E. The second aircraft is moving at 200kmh^{-1} in a direction N20^{0}W. Find the velocity of the first aircraft as seen by the pilot of the second aircraft. [292kmh^{-1} at S77.9^{0}E]

12. To a pigeon flying with velocity of (-2i + 3j + k)ms^{-1}, a hawk appears to have a velocity of (i – 5j -10k)ms^{-1}.Find the true velocity of the hawk[(-i -2j – 9k)ms^{-1}]
13. To a cyclist riding at 3ms^{-1} due east, the wind appears to come from the south with the speed $3\sqrt{3}ms^{-1}$. Find the true speed and direction of the wind. [6ms^{-1} from S30^{0}W]
14. To the pilot of an aircraft A travelling at300kmh^{-1} due south, it appears that an aircraft B is travelling at 600kmh^{-1} in a direction N60^{0}W. Find the true speed and direction of the aircraft B. [520kmh^{-1} west]
15. Jane is riding her horse at 5kmh^{-1} due north and sees Suzan riding her horse apparently with velocity 4kmh^{-1}, N60^{0}E. Find Suzan's true velocity. [7.81kmh^{-1} N26.3^{0}E]
16. A eagle flying at 8ms^{-1} on a bearing of 240^0 sees a chick apparently running at 5ms^{-1} on bearing300^0. Find true velocity of the chick. [11.4ms^{-1} at 262.4^0]
17. A train is travelling at 80kmh^{-1} in direction N15^{0}E. A passenger on the train observes a plane apparently moving at 125kmh^{-1} in the direction N50^{0}E. Find the true velocity of the plane. [196kmh^{-1} N36.5^{0}E]
18. To an athlete jogging at 12kmh^{-1} on a bearing of N10^{0}E, the wind seems to come from a direction N20^{0}W at 15kmh^{-1}. Find the true velocity of the wind. [7.57kmh^{-1} N72.5^{0}W]
19. To a passenger on a boat which is travelling at 20kmh^{-1} on a bearing 230^0, the wind seems to be blowing from 250^0 as 12kmh^{-1}. Find the true velocity of the wind [9.64kmh^{-1} N24.8^{0}E]
20. On a particular day wind is blowing N30^{0}E at a velocity of 4ms^{-1} and a motorist is driving at 40ms^{-1} in the direction of S60^{0}E.
    (a) Find the velocity of the wind relative to the motorist. [40.2ms^{-1} at N54.28^{0}W]
    (b) If the motorist changes the direction maintaining his speed and the wind appears to blow due east. What is the new direction of the motorist [N85.03^{0}W]
21. A, B and C are three aircrafts. A has velocity (200i + 170j)ms^{-1}. To the pilot of A it appears that B has velocity (50i – 270j)ms^{-1}. To the pilot of B it appears that C has a velocity (50i + 170j)ms^{-1}. Find the velocities of B and C [(250i -100j)ms^{-1}, (300i + 70j)ms^{-1}]
22. To a bird flying due east at 10ms^{-1}, the wind seems to come from south. When the bird alters its direction of flight to N30^{0}E without altering its speed, the wind seems to come from the north-west. Find the true velocity of wind. [10.6ms^{-1} from S69.9^{0}W]
23. To an observer on a trawler moving at 12kmh^{-1} in the direction S30^{0}W, the wind appears to come from N600W. To an observer on a ferry moving at 15kmh^{-1} in a direction S80^{0}E, the wind appears to come from the north. Find the true velocity of the wind. [26.8kmh^{-1} N33.4^{0}W]

b. Relative position

Consider two bodies A and B moving with V_A and V_B from points with position vectors OA and OB respectively.

Position of A at time t is $R_{A(t=t)} = R_{A(t=0)} + t \times V_A$

Position of A at time t is $R_{Bt=t)} = R_{B(t=0)} + t \times V_B$

Position of A relative to B at time t is $R_{ABt=t)} = R_{A(t=0)} – R_{B(t=0)}$

$R_{ABt=t)} = (R_{A(t=0)} + t \times V_A) – (R_{B(t=0)} + t \times V_B)$

$R_{ABt=t)} = (R_{A(t=0)} - R_{B(t=0)}) + (V_{AB})t$

Example 14

The velocities of ships P and Q are (i +6j)kmh^{-1} and (-i + 3j)kmh^{-1}. At a certain instant, the displacement between the two ship is (7i + 4j)km. Find the

(a) Relative velocity of ship P to Q

$V_{PQ} = V_P - V_Q$

$V_{PQ} = \begin{pmatrix}1\\6\end{pmatrix} - \begin{pmatrix}-1\\3\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix} kmh^{-1}$

(b) Magnitude of displacement between ships P and Q after 2hours.

$R_{PQt=t)} = (R_{P(t=0)} - R_{Q(t=0)}) + (V_{PQ})t$

$R_{PQt=t)} = \begin{pmatrix}7\\4\end{pmatrix} + t\begin{pmatrix}2\\3\end{pmatrix}$

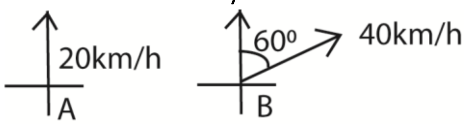
$R_{PQt=2)} = \begin{pmatrix}7\\4\end{pmatrix} + 2\,x\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}11\\10\end{pmatrix} km$

$|R_{PQ}| = \sqrt{11^2 + 10^2} = 14.87km$

Example 15

Two ship A and B move simultaneously with velocities 20kmh^{-1} and 40kmh^{-1}. Ship A moves in the northern direction while ship B moves in N60^0E. Initially ship B is 10km due west of A. Determine

(a) the relative velocity of A to B.

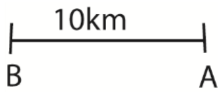


$V_{AB} = V_A - V_B$

$V_{AB} = \begin{pmatrix}0\\20\end{pmatrix} - \begin{pmatrix}40sin60\\40cos60\end{pmatrix} = \begin{pmatrix}-34.641\\0\end{pmatrix}$

$|V_{AB}| = \sqrt{(-34.641)^2 + 0^2} = 34.641$

(b) the position of A relative to B at any time t



$R_{AB(t=t)} = (R_{A(t=0)} - R_{B(t=0)}) + (V_{AB})t$

$R_{AB(t=t)} = \left[\begin{pmatrix}10\\0\end{pmatrix} - \begin{pmatrix}0\\0\end{pmatrix}\right] + t\begin{pmatrix}-34.641\\0\end{pmatrix}$

$R_{AB(t=t)} = \begin{pmatrix}10 - 34.641t\\0\end{pmatrix} km$

Distance and time of closest approach

(Shortest distance and time of shortest distance)

When two particles are moving simultaneously with specific velocities, time will come when they are closest to each other **without** colliding.

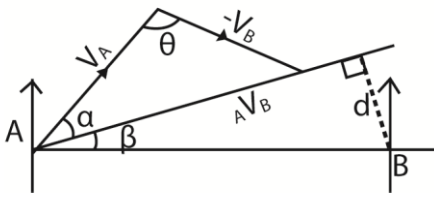
There are three methods used for the distance and time of closest approach, i.e. Geometrical, vector and differential method.

1. vector method
   Consider particle A and B moving with velocities VA and VB from point with position vector OA and OB respectively.
   For minimum distance to be attained then $V_{AB}.R_{AB(t=t)} = 0$. This gives the time.
   Then **shortest distance, d** = $|R_{AB(t=t)}|$
2. Differential method
   The minimum distance is reached when $\frac{d}{dt}|R_{AB(t=t)}|^2 = 0$. This gives time
   Then **shortest distance, d** = $|R_{AB(t=t)}|$

3. Geometrical method

If V_A and V_B are not given in vector form, then the velocity of B is reversed such that $V_{AB} = V_A + (-V_B)$ and the vector triangle is drawn as below.

The shortest distance, d will be perpendicular to V_{AB}



$$V_{AB}{}^2 = V_A{}^2 + V_B{}^2 - 2V_AV_Bcos\theta$$

$$\frac{V_{AB}}{sin\theta} = \frac{V_B}{sin\alpha}$$

Shortest distance, d =ABsinβ

Time to the shortest distance, t = $\frac{ABcos\beta}{V_{AB}}$

Example 16

A particle P starts from rest from a point with position vector (2j + 2k)m with a velocity (j + k)ms^{-1}. A second particle Q starts at the same time from a point whose position vector is (–11i – 2j – 7k)m with a velocity of (2i + j + 2k)ms^{-1}. Find

(i) the shortest distance between the particles

(ii)

(iii) how far each has travelled by this time.

$$V_{PQ} = V_P - V_Q = \begin{pmatrix}0\\1\\1\end{pmatrix} - \begin{pmatrix}2\\1\\2\end{pmatrix} = \begin{pmatrix}-2\\0\\-1\end{pmatrix}$$

$$R_{PQ(t=t)} = \left(R_{P(t=0)} - R_{Q(t=0)}\right) + \left(V_{PQ}\right)t$$

$$R_{PQ(t=t)} = \left[\begin{pmatrix}0\\2\\2\end{pmatrix} - \begin{pmatrix}-11\\-2\\-7\end{pmatrix}\right] + \begin{pmatrix}-2\\0\\-1\end{pmatrix}t$$

$$R_{PQ(t=t)} = \begin{pmatrix}11-2t\\4\\9-t\end{pmatrix}$$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0.$

$$\begin{pmatrix}-2\\0\\-1\end{pmatrix}.\begin{pmatrix}11-2t\\4\\9-t\end{pmatrix} = 0$$

-22 + 4t + 0 -9 + t = 0

t = $\frac{31}{5} = 6.2s$

(i) Then **shortest distance, d** = $\left|R_{PQ(t=t)}\right|$

$$R_{PQ(t=6.2)} = \begin{pmatrix}11-2t\\4\\9-t\end{pmatrix}$$

$$R_{PQ(t=6.2)} = \begin{pmatrix}11-2x\ 6.2\\4\\9-6.2\end{pmatrix} = \begin{pmatrix}-1.4\\4\\2.8\end{pmatrix}$$

$\left|R_{PQ(t=6.2)}\right| = \sqrt{(-1.4)^2 + 4^2 + 2.8^2}$ = 5.08m

(ii) How far each has travelled

$$R_{P(t=t)} = R_{P(t=0)} + (V_P)t$$

$$R_{P(t=6.2)} = \begin{pmatrix}0\\2\\2\end{pmatrix} + 6.2\ x\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\8.2\\8.2\end{pmatrix}$$

$$\left|R_{P(t=6.2)}\right| = \sqrt{0^2 + 8.2^2 + 8.2^2} = 11.6m$$

$$R_{Q(t=6.2)} = \begin{pmatrix}-11\\-2\\-7\end{pmatrix} + 6.2\ x\begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}1.4\\4.2\\5.4\end{pmatrix}$$

$$\left|R_{Q(t=6.2)}\right| = \sqrt{1.4^2 + 4.2^2 + 5.4^2} = 6.8m$$

Method II

$$\frac{d}{dt}\left|R_{AB(t=t)}\right|^2 = 0.$$

$$\left|R_{PQ(t=t)}\right|^2 = \begin{pmatrix}11-2t\\4\\9-t\end{pmatrix}^2$$

$$\left|R_{PQ(t=t)}\right|^2 = (11-2t)^2 + 4^2 + (9-t)^2$$

$$\left|R_{PQ(t=t)}\right|^2 = 218 - 62t + 5t^2$$

$\frac{d}{dt}\left|R_{PQ(t=t)}\right|^2 = \frac{d}{dt}(218 - 62t + 5t^2)$

$\frac{d}{dt}\left|R_{PQ(t=t)}\right|^2 = -62 + 10t = 0$

t = 6.2s

$R_{PQ(t=6.2)} = \begin{pmatrix} 11 - 2x\,6.2 \\ 4 \\ 9 - 6.2 \end{pmatrix} = \begin{pmatrix} -1.4 \\ 4 \\ 2.8 \end{pmatrix}$

$\left|R_{PQ(t=6.2)}\right| = \sqrt{(-1.4)^2 + 4^2 + 2.8^2}$ = 5.08m

Example 17

At 12 noon the position vectors r and velocity vectors v of ship A and ship B are as follows

r_A = (-9i + 6j)km, v_A = (3i + 12j) kmh^{-1} and r_B = (16i + 6j), v_B = (-9i+ 3j)kmh^{-1} respectively

(i) Find how far apart the ships are at noon

$R_{AB(t=0)} = \left(R_{A(t=0)} - R_{B(t=0)}\right)$

$R_{AB(t=0)} = \begin{pmatrix} -9 \\ 6 \end{pmatrix} - \begin{pmatrix} 16 \\ 6 \end{pmatrix} = \begin{pmatrix} -25 \\ 0 \end{pmatrix}$

$\left|R_{AB(t=0)}\right| = \sqrt{(-25)^2 + 0^2} = 25km\ apart$

(ii) Assuming velocities do not change, find the least distance between the ships in the subsequent motion

$V_{AB} = V_A - V_B = \begin{pmatrix} 3 \\ 12 \end{pmatrix} - \begin{pmatrix} -9 \\ 3 \end{pmatrix} = \begin{pmatrix} 12 \\ 9 \end{pmatrix}$

$R_{AB(t=t)} = \left(R_{A(t=0)} - R_{B(t=0)}\right) + (V_{AB})t$

$R_{AB(t=t)} = \begin{pmatrix} -25 \\ 0 \end{pmatrix} + \begin{pmatrix} 12 \\ 9 \end{pmatrix} t = \begin{pmatrix} -25 + 12t \\ 9t \end{pmatrix} km$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0.$

$\begin{pmatrix} 12 \\ 9 \end{pmatrix} . \begin{pmatrix} -25 + 12t \\ 9t \end{pmatrix} = 0$

-300 + 144t + 81t = 0

t = $\frac{4}{3}hours$

Shortest distance, d = $\left|R_{PQ(t=\frac{4}{3})}\right|$

$R_{AB(t=\frac{4}{3})} = \begin{pmatrix} -25 + 12\,x\,\frac{4}{3} \\ 9x\,\frac{4}{3} \end{pmatrix} = \begin{pmatrix} -9 \\ 12 \end{pmatrix} km$

$\left|R_{AB(t=\frac{4}{3})}\right| = \sqrt{(-9)^2 + 12^2}$ = 15km

(iii) Find when their distance of closest approach occurs and the potion vectors of A and B

It occurs at 12.00 + $\frac{4}{3}\ x$ 60 = 1.20pm

how far each travelled

$R_{A(t=t)} = R_{A(t=0)} + V_A t$

$R_{A(t=\frac{4}{3})} = \begin{pmatrix} -9 \\ 6 \end{pmatrix} + \begin{pmatrix} 3 \\ 12 \end{pmatrix} x\frac{4}{3} = \begin{pmatrix} -5 \\ 22 \end{pmatrix} km$

$R_{B(t=t)} = R_{B(t=0)} + V_B t$

$R_{B(t=\frac{4}{3})} = \begin{pmatrix} 16 \\ 6 \end{pmatrix} + \begin{pmatrix} -9 \\ 3 \end{pmatrix} x\frac{4}{3} = \begin{pmatrix} 4 \\ 10 \end{pmatrix} km$

Example 18

At a certain time, the position vectors r and velocity vectors v of ship A and ship B are as follows

r_A = (20j)km V_A = (9i -2j)kmh^{-1} at 14.00hrs

r_B = (i + 4j)km V_B = (4i + 8j)kmh^{-1} at 15.00hrs

Assuming velocities do not change, find

(a) the position vector of A at 15.00hrs

$$R_{A(t=t)} = R_{A(t=0)} + V_A t$$

At 16.00hrs: $R_{A(t=1)} = \begin{pmatrix}0\\20\end{pmatrix} + \begin{pmatrix}9\\-2\end{pmatrix} x\ 1 = \begin{pmatrix}9\\18\end{pmatrix} km$

(b) the least distance between A and B in the subsequent motion

$R_{AB(t=t)} = \left(R_{A(t=0)} - R_{B(t=0)}\right) + (V_{AB})t$

$R_{AB(t=t)} = \left[\begin{pmatrix}9\\18\end{pmatrix} - \begin{pmatrix}1\\4\end{pmatrix}\right] + \left[\begin{pmatrix}9\\-2\end{pmatrix} - \begin{pmatrix}8\\8\end{pmatrix}\right]$t

$R_{AB(t=t)} = \begin{pmatrix}8+5t\\14-10t\end{pmatrix}$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0.$

$\begin{pmatrix}5\\-10\end{pmatrix}.\begin{pmatrix}8+5t\\14-10t\end{pmatrix}$ = 0

t = 0.8hrs

Shortest distance, d = $\left|R_{PQ(t=0.8)}\right|$

$R_{AB(t=0.8)} = \begin{pmatrix}8+5\ x\ 0.8\\14-10\ x\ 0.8\end{pmatrix} = \begin{pmatrix}12\\6\end{pmatrix} km$

$R_{AB(t=0.8)} = \sqrt{12^2+6^2}$ = 13.42km

(c) time at which this least separation occurs.
15.00 + 0.8 x 60 = 15.48hrs

Example 19

At a certain time, the position vectors r and velocity vectors v of ship A and ship B are as follows

r_A = (-2i + 3j)km $\quad$ v_A =(12i -4j)kmh-1 at 11.45am

r_B = (8i + 7j)km $\quad$ v_B = (2i – 14j)kmh-1 at 12.00 noon

Assuming velocities do not change, find

(a) The least distance between A and B in the subsequent motion

OA = $\begin{pmatrix}-2\\3\end{pmatrix}$ and v_A = $\begin{pmatrix}12\\-4\end{pmatrix} kmh^{-1}$

$R_{A(t=t)} = R_{A(t=0)} + V_A t$

12.00: $R_{A(t=\frac{1}{4})} = \begin{pmatrix}-2\\3\end{pmatrix} + \frac{1}{4}\begin{pmatrix}12\\-4\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} km$

$V_{AB} = V_A - V_B = \begin{pmatrix}12\\-4\end{pmatrix} - \begin{pmatrix}2\\-14\end{pmatrix} = \begin{pmatrix}10\\10\end{pmatrix}$

$R_{AB(t=t)} = \left(R_{A(t=0)} - R_{B(t=0)}\right) + (V_{AB})t$

$R_{AB(t=t)} = \left[\begin{pmatrix}1\\2\end{pmatrix} - \begin{pmatrix}8\\7\end{pmatrix}\right] + \begin{pmatrix}10\\10\end{pmatrix} t$

$R_{AB(t=t)} = \begin{pmatrix}-7+10t\\-5+10t\end{pmatrix}$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0.$

$\begin{pmatrix}10\\10\end{pmatrix}.\begin{pmatrix}-7+10t\\-5+10t\end{pmatrix} = 0$

t = 0.6hrs

$R_{AB(t=0.6)} = \begin{pmatrix}-7+10\ x\ 0.6\\-5+10\ x\ 0.6\end{pmatrix}$ =$\begin{pmatrix}-1\\1\end{pmatrix} km$

$\left|R_{AB(t=0.6)}\right| = \sqrt{(-1)^2+1^2}$ =1.4142km

(b) length of time for which A is within range, if ship B has guns within a range of up to 2km

$R_{AB(t=t)} = \begin{pmatrix}-7+10t\\-5+10t\end{pmatrix}$

$\left|R_{AB(t=t)}\right| = 2km$

$\begin{pmatrix}-7+10t\\-5+10t\end{pmatrix}^2 = 2^2$

$(-7+10t)^2 + (-5+10t)^2 = 4$

$100t^2 - 12t + 35 = 0$

t = 0.7hrs or t = 0.5hrs

Time for which they are in range

= 0.7 – 0.5 = 0.2h

Example 20

At 10am, ship A moves with a constant velocity (4i+ 20j) kmh^{-1} and ship B due north of A moves with a constant velocity (-3i – 4j) kmh^{-1}.

(a) Find the velocity of A relative to B

$$V_{AB} = V_A - V_B = \begin{pmatrix}4\\20\end{pmatrix} - \begin{pmatrix}-3\\-4\end{pmatrix} = \begin{pmatrix}7\\24\end{pmatrix} kmh^{-1}$$

(b) If the shortest distance between the two ships is 4.2km. Find the

(i) time to the nearest minute when they are closest together

(ii) original distance apart at 10am

(iii) the bearing of B from A when they are closest together.

Solution

(ii) Let a km be the distance apart at 10am

$R_{AB(t=t)} = \left(R_{A(t=0)} - R_{B(t=0)}\right) + (V_{AB})t$

$R_{AB(t=t)} = \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}0\\a\end{pmatrix}\right] + \begin{pmatrix}7\\24\end{pmatrix}t = \begin{pmatrix}7t\\-a+24t\end{pmatrix} km$

$R_{AB(t=t)} = \begin{pmatrix}7t\\-a+24t\end{pmatrix} km$

$\left|R_{AB(t=t)}\right| = 4.2km$

$\begin{pmatrix}7t\\-a+24t\end{pmatrix}^2 = (4.2)^2$

$(7t)^2 + (-a+24t)^2 = 17.64$

$625t^2 - 48at + a^2 = 17.64$ ………….(i)

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0.$

$\begin{pmatrix}7\\24\end{pmatrix}.\begin{pmatrix}7t\\-a+24t\end{pmatrix} = 0$

49t – 24a + 576t = 0

$a = \frac{625t}{24}$ …………………………………..(ii)

Substituting for a in eqn. (i)

$625t^2 - 48\left(\frac{625t}{24}\right)t + \left(\frac{625t}{24}\right)^2 = 17.64$

$53.1684t^2 = 17.67$

t = $\pm 0.57h = 0.576 \; x \; 60 = 35minute$

(ii) a = $\frac{625t}{24} = \frac{625 \; x \; 0.576}{24}$ = 15km

(iii) $R_{AB(t=0.576)} = \begin{pmatrix}7x \; 0.576\\-15+24 \; x \; 0.576\end{pmatrix} km$

$R_{AB(t=0.576)} = \begin{pmatrix}4.032\\-1.176\end{pmatrix}$

$\theta = \tan^{-1}\left(\frac{1.176}{4.032}\right) = 16.3^0$

$Direction: E16.3^0S$

(c) length of time for which A is within range, if the visibility of ship B is within 12km

$R_{AB(t=t)} = \begin{pmatrix}7t\\-a+24t\end{pmatrix} = \begin{pmatrix}7t\\-15+24t\end{pmatrix} km$

$\left|R_{AB(t=t)}\right| = 12km$

$(7t)^2 + (-a+24t)^2 = 144$

$625t^2 - 720t + 81 = 0$

t = 1.026h or t = 0.126h

Time for which they are in range

= 1.026 – 0.126 = 0.9h

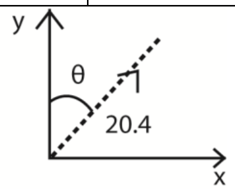
Example 21

At 12 noon a ship A is moving with constant velocity of 20.4kmh^{-1} in the direction $N\theta^0E$ where tan θ = $\frac{1}{5}$. A second ship B is 15km due to north of A. Ship B is moving with constant velocity of 5kmh^{-1} in the direction Sα^0W, where tan α= ¼. If the shortest distance between the ships is 4.2km, find the time to the nearest minute when the distance between the ships is shortest. (12mars)

Table of results

Vector	Magnitude	Direction
V_A	20.4kmh^{-1}	NθE
V_B	5kmh^{-1}	SαW

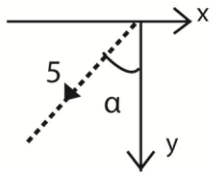
V_A Ξ



$\tan\theta = \frac{1}{5}$; θ = 11.3^0

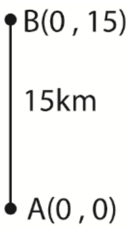
$$V_A = \begin{pmatrix}20.4\sin 11.3^0\\ 20.4\cos\ 11.3^0\end{pmatrix} = \begin{pmatrix}4\\ 20\end{pmatrix}$$

V_B Ξ



$\tan\alpha = \frac{1}{5}$; α = 36.87^0

$$V_{-B} = \begin{pmatrix}-5\sin 36.87^0\\ -5\cos\ 36.87^0\end{pmatrix} = \begin{pmatrix}-3\\ -4\end{pmatrix}$$



$r_A = \int V_A dt = \int \begin{pmatrix}4\\ 20\end{pmatrix} dt = \begin{pmatrix}4t\\ 20t\end{pmatrix} + c$

At t = 0, $r_A = \begin{pmatrix}0\\ 0\end{pmatrix}$ => c = $\begin{pmatrix}0\\ 0\end{pmatrix}$

Hence $r_A = \begin{pmatrix}4t\\ 20t\end{pmatrix}$

$R_B = \int V_B dt = \int \begin{pmatrix}-3\\ -4\end{pmatrix} dt = \begin{pmatrix}-3t\\ -4t\end{pmatrix} + c$

At t = 0, $r_B = \begin{pmatrix}0\\ 15\end{pmatrix}$ => c = $\begin{pmatrix}0\\ 15\end{pmatrix}$

Hence $r_B = \begin{pmatrix}-3t\\ 15-4t\end{pmatrix}$

${}_Ar_B = r_A - r_B = \begin{pmatrix}4t\\ 20t\end{pmatrix} - \begin{pmatrix}-3t\\ 15-4t\end{pmatrix} = \begin{pmatrix}7t\\ 24t-15\end{pmatrix}$

$d_s = |{}_Ar_B| = \sqrt{(7t)^2 + (24t-15)^5}$

but ds = 4.2

⇨ $\sqrt{(7t)^2 + (24t-15)^5} = 4.2$

$\left(\sqrt{(7t)^2 + (24t-15)^5}\right)^2 = 4.2^2$

$(7t)^2 + (24t-15)^5 = 4.2^2$

$47t^2 + 576t^2 - 720t + 225 = 17.64$

$625t^2 - 720t + 207.36 = 0$

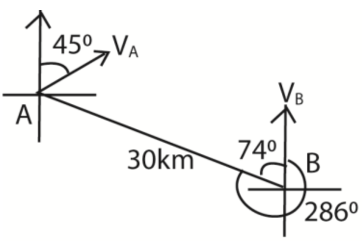
t = $\frac{720 \pm\sqrt{(-720)^2-4(625)(207.36)}}{2(625)}$ = 0.576hours

= 0.576 x 60 = 35minutes

Hence the time at which the distance is shortest is 12:35pm

Example 22

At noon a boat A is 30km from boat B and its direction from B is 2860. A is moving in the North-East direction at 16kmh-1 and B is moving in the north direction at 10kmh-1. Determine when they are closest to each other. What is the distance between them?



$$R_{A(t=0)} = \begin{pmatrix}0\\ 0\end{pmatrix}, R_{B(t=0)} = \begin{pmatrix}30sin74\\ -30cos74\end{pmatrix} km$$

$$V_{AB} = \begin{pmatrix}16sin45\\ 16cos45\end{pmatrix} - \begin{pmatrix}0\\ 10\end{pmatrix} = \begin{pmatrix}11.314\\ 1.314\end{pmatrix} km/h$$

$$R_{AB(t=t)} = \left(R_{A(t=0)} - R_{B(t=0)}\right) + (V_{AB})t$$

$R_{AB(t=t)} = \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}30sin74\\-30cos74\end{pmatrix}\right] + \begin{pmatrix}11.314\\1.314\end{pmatrix}t$

$R_{AB(t=t)} = \begin{pmatrix}-28.838 + 11.314t\\8.269 + 1.314t\end{pmatrix} km$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0.$

$\begin{pmatrix}11.314\\1.314\end{pmatrix}.\begin{pmatrix}-28.838 + 11.314t\\8.269 + 1.314t\end{pmatrix} = 0$

t = 2.43h

$R_{AB(t=2.43)}$

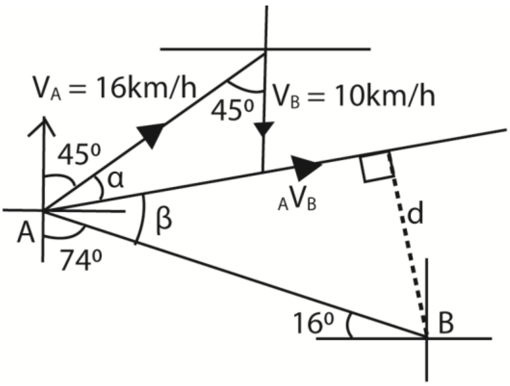
$= \begin{pmatrix}-28.838 + 11.314\ x\ 2.43\\8.269 + 1.314\ x\ 2.43\end{pmatrix} km$

$R_{AB(t=2.43)} = \begin{pmatrix}-1.345\\11.462\end{pmatrix} km$

$|R_{AB(t=2.43)}| = \sqrt{(-1.345)^2 + 11.462^2}$

= 11.54km

Alternatively



$V_{AB}{}^2 = V_A{}^2 + V_B{}^2 - 2V_AV_Bcos45$

$V_{AB} = \sqrt{16^2 + 10^2 - 2x16x10cos45}$

= 11.39km/h

$\frac{10}{sin\alpha} = \frac{11.39}{sin45}$

α = 38.38^0

45 + α + β + 74 = 180^0

β = 22.62^0

d = ABsinβ = 30sin22.62 = 11.54km

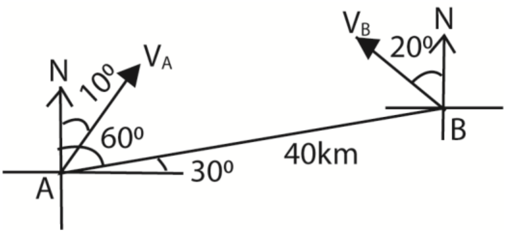
Time, t = $\frac{ABcos\beta}{V_{AB}} = \frac{30cos22.62}{11.39}$ = 2.43h

Time is 2.43h from noon or 2hour and 25.8minutes

It occurs 2.26pm at a distance 11.54km

Example 23

Two planes A and B are both flying above the Pacific Ocean. Plane A is flying on a course of 010^{0}at a speed 300kmh^{-1} and plane B is flying on a course of 340^0 at 200kmh^{-1}. At a certain instant, plane B is 40km from A. Plane A is then on a bearing of 060^0. After what time will they come closest together and what will be their minimum distance apart.



$V_{AB} = V_A - V_B$

$V_{AB} = \begin{pmatrix}300sin10\\300cos10\end{pmatrix} - \begin{pmatrix}-200sin20\\200cos20\end{pmatrix} = \begin{pmatrix}120.4985\\107.5038\end{pmatrix}$

$R_{A(t=0)} = \begin{pmatrix}0\\0\end{pmatrix} km, R_{B(t=0)} = \begin{pmatrix}40cos30\\40sin30\end{pmatrix} km$

$R_{AB(t=t)} = (R_{A(t=0)} - R_{B(t=0)}) + (V_{AB})t$

$R_{AB(t=t)} = \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}40cos30\\40sin30\end{pmatrix}\right] + t\begin{pmatrix}120.4985\\107.5038\end{pmatrix}$

$R_{AB(t=t)} = \begin{pmatrix}-34.641 + 120.4985t\\-20 + 107.5038t\end{pmatrix}$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0$

$\begin{pmatrix}120.4985\\107.5038\end{pmatrix}\begin{pmatrix}-34.641 + 120.4985t\\-20 + 107.5038t\end{pmatrix} = 0$

-6324.2645 + 26076.9555t = 0

t = 0.2425h

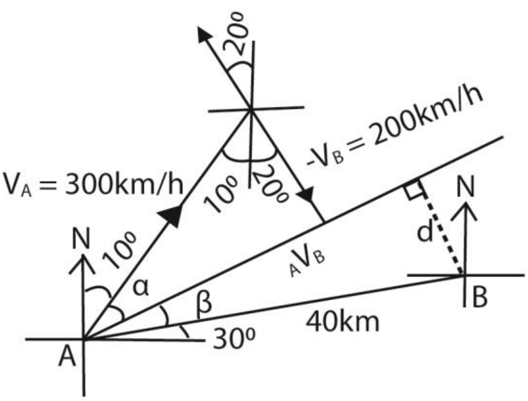
Least distance = $|R_{AB(t=0.2425)}|$

$R_{AB(t=\,0.2425)} =$
$\begin{pmatrix}-34.641 + 120.4985x0.2425\\-20 + 107.5038x0.2425\end{pmatrix}$

$R_{AB(t=0.2425)} = \begin{pmatrix} 5.4202 \\ 6.0692 \end{pmatrix} km$

$|R_{AB(t=0.2425)}| = \sqrt{(5.4202)^2 + (6.0692)^2} = 8.14m$

Alternatively



$V_{AB}{}^2 = V_A{}^2 + V_B{}^2 - 2V_AV_B cos30$

$V_{AB} = \sqrt{300^2 + 200^2 - 2x300x200cos30}$

= 161.484km/h

$\frac{200}{sin\alpha} = \frac{161.484}{sin30}$

α = 38.26⁰

α + β =50

β = 11.74⁰

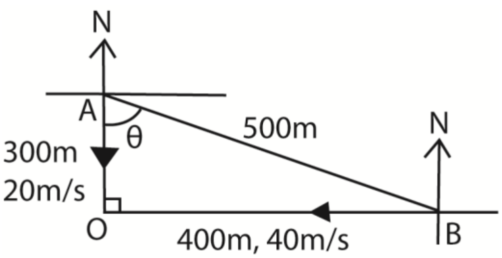
d = ABsinβ = 40sin11.74 = 8.14km

Time, t = $\frac{ABcos\beta}{V_{AB}} = \frac{40cos11.74}{161.484} = 0.2425$h

Example 24

At a given instant two cars are at a distance 300m and 400m from a point of intersection O of two roads crossing at right angles and are approaching O at uniform speeds of 20m/s and 40m/s respectively. Find

(i) Initial distance between the two cars
(ii) shortest distance between the cars
(iii) time taken to reach this point



$AB = \sqrt{300^2 + 400^2} = 500m$

$\theta = \tan^{-1}\left(\frac{400}{300}\right) = 53.1^0$

$R_{A(t=0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} km, R_{B(t=0)} = \begin{pmatrix} 50cos53.1 \\ 50sin53.1 \end{pmatrix} km$

$V_{AB} = V_A - V_B$

$V_{AB} = \begin{pmatrix} 0 \\ -20 \end{pmatrix} - \begin{pmatrix} -40 \\ 0 \end{pmatrix} = \begin{pmatrix} 40 \\ -20 \end{pmatrix} ms^{-1}$

$R_{AB(t=t)} = \left(R_{A(t=0)} - R_{B(t=0)}\right) + (V_{AB})t$

$R_{AB(t=t)} = \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 500cos53.1 \\ 500sin53.1 \end{pmatrix}\right] + t\begin{pmatrix} 40 \\ -20 \end{pmatrix}$

$R_{AB(t=t)} = \begin{pmatrix} -399.842 + 40t \\ 300.21 - 20t \end{pmatrix}$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0$

$\begin{pmatrix} 40 \\ -20 \end{pmatrix} \cdot \begin{pmatrix} -399.842 + 40t \\ 300.21 - 20t \end{pmatrix} = 0$

-21,997.88 +2000t = 0

t = 11s

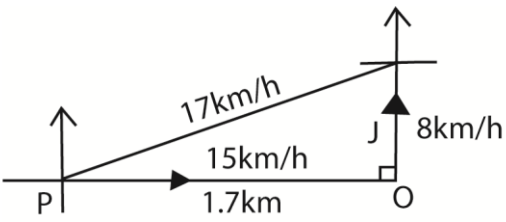
$R_{AB(t=11)} = \begin{pmatrix} -399.842 + 40x11 \\ 300.21 - 20 \, x \, 11 \end{pmatrix} = \begin{pmatrix} 40.158 \\ 80.21 \end{pmatrix} m$

$R_{AB(t=11)} = \sqrt{(40.158)^2 + (80.21)^2}$ =89.701m

Example 25

A road running north-south crosses a road running east-west at a junction O. Initially Paul is on the east-west, 1.7km west of O and is cycling towards O at15mk/h. At the same time John is at O cycling due north at 8km/h. If Paul and John do not alter their velocities, find the

(i) relative velocity of Paul to John
(ii) shortest distance between Paul and John



$R_{P(t=0)} = \binom{0}{0} km, R_{J(t=0)} = \binom{1.7}{0} km$

$V_{PJ} = V_P - V_J$

$V_{PJ} = \binom{15}{0} - \binom{0}{8} = \binom{15}{-8} ms^{-1}$

$|V_{PJ}| = \sqrt{15^2 + (-8)^2}$ = 17km/h

$R_{PJ(t=t)} = (R_{P(t=0)} - R_{J(t=0)}) + (V_{PJ})t$

$R_{PJ(t=t)} = \left[\binom{0}{0} - \binom{1.7}{0}\right] + \binom{15}{-8} t$

$R_{PJ(t=t)} = \binom{-1.7 + 15t}{-8t}$

For minimum distance: $V_{PJ}.R_{PJ(t=t)} = 0$

$\binom{15}{-8}.\binom{-1.7 + 15t}{-8t}$ =0

-25.5 +289t = 0

t = 0.088h

Least distance = $|R_{PJ(t=0.088)}|$

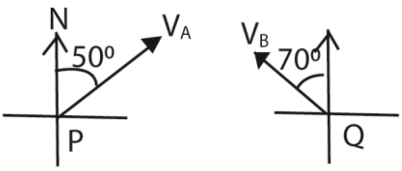
$R_{PJ(t=0.088)} = \binom{-1.7 + 15 \; x \; 0.088}{-8x \; 0.88} = \binom{-0.38}{-0.704}$

$|R_{PJ}| = \sqrt{(-0.38)^2 + (-0.704)^2}$ = 0.8km

Example 26

Two airship P and Q are 100km apart, P being west of Q. Two Helicopters A and B fly simultaneously from P and Q respectively, at 11.00a.m. Helicopter A is flying with a constant speed of 400km/h in the direction N50^{0}E. Helicopter B flying at a constant speed of 500km in the direction N70^{0}W. Find the

(i) Time when the helicopters are closest together.
(ii) closest distance between the helicopters



$R_{A(t=0)} = \binom{0}{0} km, R_{B(t=0)} = \binom{100}{0} km$

$V_{AB} = V_A - V_B$

$V_{AB} = \binom{400sin50}{400cos50} - \binom{-500sin70}{500cos70}$

$V_{AB} = \binom{776.264}{86.105} kms^{-1}$

$R_{AB(t=t)} = (R_{A(t=0)} - R_{B(t=0)}) + (V_{AB})t$

$R_{AB(t=t)} = \left[\binom{0}{0} - \binom{100}{0}\right] + t\binom{776.264}{86.105}$

$R_{AB(t=t)} = \binom{-100 + 776.264t}{86.105t} km$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0$

$\binom{776.264}{86.105}\binom{-100 + 776.264t}{86.105t}.$

609999.869t =77624.4

t = 0.1273h = 0.1273 x 60= 8minutes

Time when they are closest

= 11.00 + 8= 11.08am

Least distance = $\left|R_{AB(t=0.1273)}\right|$

$R_{AB(t=0.1273)}\begin{pmatrix}-100+776.264\ x\ 0.1273\\86.105\ x\ 0.1273\end{pmatrix}km$

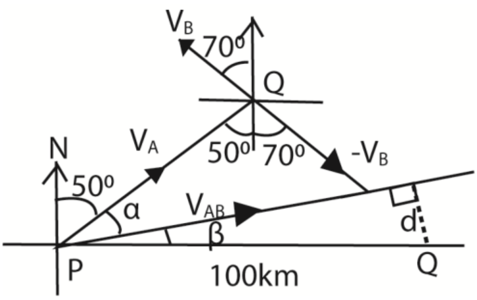
$R_{AB(t=0.1273)}=\begin{pmatrix}-1.182\\10.961\end{pmatrix}km$

$\left|R_{AB(t=0.1273)}\right|=\sqrt{(-1.182)^2+10.961^2}$

=11.0247km

Closest distance =11.025km

Alternatively



$V_{AB}{}^2=V_A{}^2+\ V_B{}^2-2V_AV_Bcos120$

$V_{AB}=\sqrt{400^2+500^2-2x400x500cos120}$

= 781.025km/h

$\frac{500}{sin\alpha}=\frac{781.025}{sin120}$

α = 33.67^0

α + β + 50 =90

β = 6.33^0

d = PQsinβ = 100sin6.33 = 11.025km

Time, t = $\frac{PQcos\beta}{V_{AB}}=\frac{100cos6.33}{781.025}$

t = 0.1273h = 0.1273 x 60= 8minutes

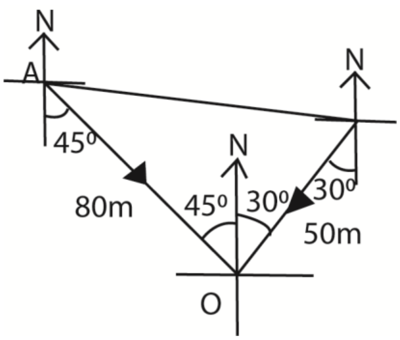
Time when they are closest

= 11.00 + 8= 11.08am

Example 26

Car A is 80m North-West of point O, Car B is 50m N300E of O. Car A is moving at 20m/s on a straight road towards O. Car B is moving at 10m/s on another straight road towards O. Determine the

(i) Initial distance between the cars
(ii) Velocity of A relative to B
(iii) shortest distance between the two cars as they approach O



$R_{A(t=0)}=\begin{pmatrix}-80cos45\\80sin45\end{pmatrix},R_{B(t=0)}=\begin{pmatrix}50sin30\\50cos30\end{pmatrix}$

$R_{AB(t=0)}=\begin{pmatrix}-80cos45\\80sin45\end{pmatrix}-\begin{pmatrix}50sin30\\50cos30\end{pmatrix}=\begin{pmatrix}-81.5685\\13.2673\end{pmatrix}$

$\left[R_{AB(t=0)}\right]=\sqrt{(-81.5685)^2+13.2673^2}$ = 82 .6404m

(ii) $V_A=\begin{pmatrix}20sin45\\-20cos45\end{pmatrix},V_B=\begin{pmatrix}-10sin30\\-10cos30\end{pmatrix}$

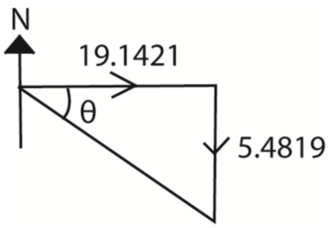
$V_{AB}=V_A-V_B$

$V_{AB}=\begin{pmatrix}20sin45\\-20cos45\end{pmatrix}-\begin{pmatrix}-10sin30\\-10cos30\end{pmatrix}$

$V_{AB}=\begin{pmatrix}19.1421\\-5.4819\end{pmatrix}$

$|V_{AB}|=\sqrt{(19.1421)^2+(-5.4819)^2}$

= 19.9116m/s



θ=$\tan^{-1}\frac{5.4819}{19.1421}$=15.98^0

Direction: E15.98^{0}S

(iii)

$R_{AB(t=t)} = (R_{A(t=0)} - R_{B(t=0)}) + (V_{AB})t$

$R_{AB(t=t)} = \begin{pmatrix}-81.5685\\13.2673\end{pmatrix} + t\begin{pmatrix}19.1421\\-5.4819\end{pmatrix}$

$R_{AB(t=t)} = \begin{pmatrix}-81.5685 + 19.1421t\\13.2673 - 5.4821t\end{pmatrix} km$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0$

$\begin{pmatrix}19.1421\\-5.4819\end{pmatrix}.\begin{pmatrix}-81.5685 + 19.1421t\\13.2673 - 5.4821t\end{pmatrix} = 0$

t = 4.1216s

Least distance = $|R_{AB(t=4.1216s)}|$

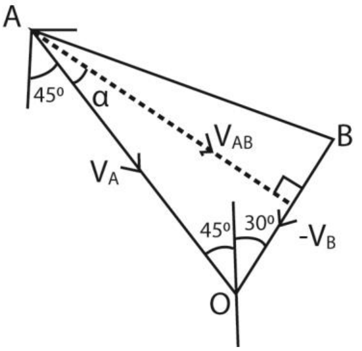
$R_{AB(t=t)} = \begin{pmatrix}-81.5685 + 19.1421\ x\ 4.1216\\13.2673 - 5.4821\ x\ 4.1216\end{pmatrix} m$

$R_{AB(t=t)} = \begin{pmatrix}-2.672\\-9.328\end{pmatrix} m$

$|R_{AB(t=t)}| = \sqrt{(-2.672)^2 + (-9.328)^2}$ =9.728m

Alternatively

Method II: Using geometric approach



Using cosine rule

$|V_{AB}|^2 = 20^2 + 10^2 - 2\ x\ 20\ x10\cos 75^0$

$|V_{AB}| = 19.912ms^{-1}$

$\frac{|V_{AB}|}{\sin 75^0} = \frac{10}{\sin\alpha}$

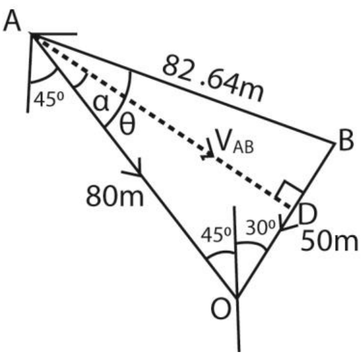
$\alpha = 29.02^0$

$45^0 + 29.02^0 = 74.02^0$

∴ The velocity of A relative to B is 19.912ms^{-1} due S74.02^{0}E

The shortest distance between the two cars as they approach O (04marks)

Using geometrical approach



$\frac{50}{\sin\theta} = \frac{82.64}{\sin 75^0}$; θ = 35.76^0

$< BAD = 35.76 - 29.02 = 6.74^0$

$\sin 6.74^0 = \frac{|r_{AB}|}{AB}$

$_Ar_B$ = 82.64sin 6.74^0 = 9.699m

∴ The shortest distance between the two cars they approach O is 9.699m

Revision exercise 3

1. 1.At 8am ship A and ship Bare 11km apart with B due west of A. a and B move with a constant velocities (-4i +3j)kh/h and (2i + 4j)km/h respectively. Find the the

(i) least distance between the two ship in the subsequent motion[1.81km]

(ii) time to the nearest minute at which this situation occurs[9.47am]

2. 2. At 7.30am, two ship A and B are 8km apart with B due north of A. A and B move with constant velocities (12j)km/h and (-5i)km/h respectively. Find the

(i) least distance between the two ship in the subsequent motion[3.08km]

(ii) time to the nearest minute at which this situation occurs[8.04am]

3. A and B are two tankers at 13.00hrs, tanker B bas position vector of (4i+8j)km relative to A. A and B move with constant velocities (6i + 9j)km/h and (-3i + 6j)km/h respectively. Find the
   (i) least distance between the two ship in the subsequent motion[6.32km]
   (ii) time to the nearest minute at which this situation occurs[13.40hrs]
4. At 12 noon the position vectors r and velocity vectors v of two ship A and B are as follows
   r_A = (5i + j)km, V_A = (7i + 3j)km/h and r_B = (8i + 7j)km, V_B = (2i –j)km/h
   (i) Assuming velocities do not change , find the least distance between the ships in subsequent motion [2.81]
   (ii) Find the time when their distance of closest approach occur [12.57pm]
5. At a certain time, the position vectors r and velocity vectors V of two ship A and B are as follows

   r_A = (3i + j)km, V_A = (2i + 3j)km/h at 11.00am

   r_B = (2i - j)km, V_B = (3i + 7j)km/h at 12.00noon

   Assuming velocities do not change; find the

   (i) The position vector of A at noon [5i + 4j]
   (ii) Distance between the ships at 12.00 noon [5.83km]
   (iii) The least distance between A and B in the subsequent motion[1.7km]
   (iv) Time at which the least separation occurs [1.21pm]

6. At 12 noon, the position vectors r and velocity vectors V of two battle ship A and battle B are as follows

   r_A = (13i + 5j)km, V_A = (3i - 10j)km/h

   r_B = (3i - 5j)km, V_B = (15i + 14j)km/h

   (i) Assuming the velocities do not change, find the least distance between the ships in subsequent motion [4.47km]
   (ii) The battle ship has guns with a range of up to 5km, find the length of time during which the cruiser is within range of the battle ships [10minutes]
7. At tine t = 0 the position vectors r and velocity vectors V of two battle ship A and battle B are as follows

   $$r_A = \begin{pmatrix}1\\2\\3\end{pmatrix}m, V_A = \begin{pmatrix}-6\\0\\1\end{pmatrix}m/s \text{ and } r_B = \begin{pmatrix}4\\-14\\1\end{pmatrix}m, V_A = \begin{pmatrix}-5\\1\\7\end{pmatrix}m/s$$

   Assuming velocities do not change, find

   (i) The potion vectors of B relative to A at time t seconds $\left[\left(\begin{pmatrix}7\\2\\2\end{pmatrix}+\begin{pmatrix}9\\13\\-7\end{pmatrix}t\right)km\right]$
   (ii) The least distance between the ships in the subsequent motion [15.9m]
   (iii) The time taken to the closest distance $\left[\frac{25}{33}s\right]$

8. At time t = 0 the position vectors r and velocity vectors V of two battle ship A and battle B are as follows

   $r_A = (3i + j + 5k)$m, $V_A = (4i + j – 3k)$m/s

   $r_B = (i - 3j + 2k)$ m, $V_B = (i + 2j + 2k)$m/s

   Assuming velocities do not change, find

   (i) The position vector of B relative to A at time t second $\left[\begin{pmatrix} 2+3t \\ 4-t \\ 3-5t \end{pmatrix} m\right]$

   (ii) The value of t when A and B are closed $\left(\frac{13}{35}\right)$

   (iii) Least distance between A and B [4.917m]

9. At tine t = 0 the position vectors r and velocity vectors V of two battle ship A and battle B are as follows

   $r_A = (\beta)m, V_A = \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix} m/s \text{ and } r_B = (2\beta)m, V_A = \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix} m/s$

   Where β is a constant, assuming velocities do not change show that the least distance between the ships in the subsequent motion is $\frac{\beta}{73}$ and their distance of closest approach is $\frac{6\beta\sqrt{2}}{\sqrt{73}}$.

10. A lizard on a wall at point A, has a position vector $r_A = \begin{pmatrix} 65 \\ 40 \\ 0 \end{pmatrix} cm$. At time t= 0 seconds a fly has a position vector $r_F = \begin{pmatrix} 37 \\ 16 \\ 22 \end{pmatrix} cm$ and velocity vector $V_F = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} cm/s$

    If the fly were to continue with this velocity, find the closest distance it would come to the lizard and the value of t when it occurs [$\sqrt{374}m$, 7s]

11. A particle P move with constant velocity (2i + 3j + 8k)m/s passes a point with position vector (6i – 11j + 4k)m. At the same instant particle Q passes through a point whose position vector is (i -2j + 5k)m moving at constant velocity of (3i + 4j- 7k)m/s. Find

    (a) Position of Q relative to P at that instant. [10.344m]

    (b) Shortest distance between the particles [10.32m]

    (c) Time that elapses before the particles are nearest to each other [0.0485s]

12. Two particles P and Q move with constant velocities (4i + j – 2k)m/s and (6i + 3k)m/s respectively. Initially P is at a point whose position vector is (i – 20j + 21k)m and Q is at a point whose position vector is (i + 3k)m. find

    (a) Time for which the distance between P and Q is least [2.2s]

    (b) Distance of P from the origin at the time when the distance between P and Q is least [28.8m]

    (c) Least distance between P and Q [24.14m]

Course of closest approach

If A is to pass as close as possible to B, then velocity of A must ne perpendicular to the relative velocity

$V_{AB}.V_A = 0$

Example 27

Two particles P and Q initially at positions (3i + 2j)m and (13i + 2j)m respectively begin moving. Particle P moves with a constant velocity (2i + 6j)m/s. A second particle Q moves with a constant velocity of (5j)m/s

(a) Find
  (i) Time when the particles are closest together.
  (ii) Bearing of particle O from Q when they are closest to each other.

(b) Given that half the time, the particle are moving closest to each other, particle P reduces its speed to half its original speed, in the direction to approach particle Q and the velocity of particle Q remains unchanged, find the direction of particle P.

Solution

$r_{P(t=0)} = \begin{pmatrix}3\\2\end{pmatrix}m, r_{Q(t=0)} = \begin{pmatrix}13\\2\end{pmatrix}m$

$V_{PQ} = V_P - V_Q = \begin{pmatrix}2\\6\end{pmatrix} - \begin{pmatrix}0\\5\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}m/s$

$r_{PQ(t=t)} = \left(r_{P(t=0)} - r_{Q(t=0)}\right) + \left(V_{PQ}\right)t$

$r_{PQ(t=t)} = \begin{pmatrix}3\\2\end{pmatrix} - \begin{pmatrix}13\\2\end{pmatrix} + \begin{pmatrix}2\\1\end{pmatrix}t = \begin{pmatrix}-10+2t\\t\end{pmatrix}m$

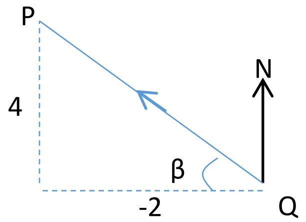
For minimum distance $V_{PQ}.r_{PQ(t=t)}$=0

$\begin{pmatrix}2\\1\end{pmatrix}.\begin{pmatrix}-10+2t\\t\end{pmatrix} = 0$

-20 +4t +t = 0

t = 4s

(ii) $r_{PQ(t=4)} = \begin{pmatrix}-10+2\ x\ 4\\4\end{pmatrix} = \begin{pmatrix}-2\\4\end{pmatrix}$



$\beta = \tan^{-1}\frac{4}{2}$=63.43^0

The bearing of P from Q = (270 + 63.43)

=333.43^0

(b) At t = 2s, $V_P = \begin{pmatrix}1\\3\end{pmatrix}m/s$

Let P move at angle θ to x-axis

$V_{PQ} = \sqrt{10}\frac{cos\theta}{sin\theta} - \begin{pmatrix}0\\5\end{pmatrix} = \begin{pmatrix}\sqrt{10}co\theta\\\sqrt{10}sin\theta - 5\end{pmatrix}$

If P is to approach Q; $8sin\theta$

$\begin{pmatrix}\sqrt{10}co\theta\\\sqrt{10}sin\theta - 5\end{pmatrix}.\begin{pmatrix}\sqrt{10}cos\theta\\\sqrt{10}sin\theta\end{pmatrix}$ = 0

10cos^{2}θ + 10sin^{2}θ -5$\sqrt{10}sin\theta$ =0

θ =$\sin^{-1}\frac{10}{5\sqrt{10}}$ = 39.2^0

Direction: N50.8^{0}E

Example 28

A motor boat B is travelling at a constant velocity of 10m/s due east and motor boat A is travelling at a constant speed of 8m/s. Initially A and B are 600m apart with A due south of B. Find

(a) course that A should set to get close as possible to B

Let A move at an angle θ to x-axis

$V_{AB} = \begin{pmatrix}8cos\theta\\8sin\theta\end{pmatrix} - \begin{pmatrix}10\\0\end{pmatrix} = \begin{pmatrix}8cos\theta - 10\\8sin\theta\end{pmatrix}$

If A is to approach B; $V_{AB}.V_A = 0$

$\begin{pmatrix}8cos\theta\\8sin\theta\end{pmatrix}.\begin{pmatrix}8cos\theta - 10\\8sin\theta\end{pmatrix}$ =0

θ = $\cos^{-1}\frac{64}{80}$ =36.9^0

Direction: N53.1^{0}E or E36.9^{0}N

(ii) Closest distance and time taken for the situation to occur

$r_{A(t=0)} = \begin{pmatrix}0\\0\end{pmatrix} m, r_{B(t=0)} = \begin{pmatrix}0\\600\end{pmatrix} m$

$r_{AB(t=t)} = \left(r_{A(t=0)} - r_{B(t=0)}\right) + (V_{AB})t$

$r_{AB(t=t)} = \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}0\\600\end{pmatrix}\right] + \begin{pmatrix}8cos36.9 - 10\\8sin36.9\end{pmatrix} t$

$r_{AB(t=t)} = \begin{pmatrix}-3.603t\\-600 + 4.803t\end{pmatrix}$

For minimum distance: $V_{AB}.r_{AB(t=t)} = 0$

$\begin{pmatrix}-3.603\\4.803\end{pmatrix}.\begin{pmatrix}-3.603t\\-600 + 4.803t\end{pmatrix}$ =0

t = 80s

$r_{AB(t=80)} = \begin{pmatrix}-3.603 \ x\ 80\\-600 + 4.803x80\end{pmatrix}$

$= \begin{pmatrix}-288.24\\-215.76\end{pmatrix}$

Least distance,d = $\left|r_{AB(t=80)}\right|$

$\left|r_{AB(t=80)}\right| = \sqrt{(-288.24)^2 + (-215.76)^2}$

d = 360m

Example 29

A motor boat B is travelling at constant velocity of 14km/h due north and a motor boat A is travelling at constant speed 12km/h. Initially A and B are 5.2km apart with A due west of B. Find

(i) Couse that A should set in order to get as close as possible to B

Let A move at an angle θ to x-axis

$V_{AB} = \begin{pmatrix}12cos\theta\\12sin\theta\end{pmatrix} - \begin{pmatrix}0\\13\end{pmatrix} = \begin{pmatrix}12cos\theta\\12sin\theta - 13\end{pmatrix}$

If A is to approach B; $V_{AB}.V_A = 0$

$\begin{pmatrix}12cos\theta\\12sin\theta - 13\end{pmatrix}.\begin{pmatrix}12cos\theta\\12sin\theta\end{pmatrix}$=0

θ = $\sin^{-1}\frac{12}{13}$ =67.4^0

Direction: N22.6^{0}E or 67.4^{0}N

(ii) Closest distance and time taken for the situation to occur

$r_{A(t=0)} = \begin{pmatrix}0\\0\end{pmatrix} m, r_{B(t=0)} = \begin{pmatrix}5.2\\0\end{pmatrix} m$

$r_{AB(t=t)} = \left(r_{A(t=0)} - r_{B(t=0)}\right) + (V_{AB})t$

$r_{AB(t=t)} = \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}5.2\\0\end{pmatrix}\right] + \begin{pmatrix}12cos67.4\\12sin67.4 - 13\end{pmatrix} t$

$r_{AB(t=t)} = \begin{pmatrix}-5.2 + 4.612t\\-1.921t\end{pmatrix}$

For minimum distance: $V_{AB}.r_{AB(t=t)} = 0$

$\begin{pmatrix}4.612\\-1.921\end{pmatrix}.\begin{pmatrix}-5.2 + 4.612t\\-1.921t\end{pmatrix}$ =0

t = 0.961h

$r_{AB(t=0.961)} = \begin{pmatrix}-5.2 + 4.612 \ x\ 0.961\\-1.921 \ x\ 0.961\end{pmatrix}$

$= \begin{pmatrix}-0.7725\\-1.8442\end{pmatrix}$

Least distance, d = $\left|r_{AB(t=0.961)}\right|$

$\left|r_{AB(t=0.961)}\right| = \sqrt{(-0.7725)^2 + (-1.8442)^2}$

d = 2km

Revision exercise 4

1. A ship A is moving with constant velocity of 18km/h in a direction N55^{0}E and is initially 6km from a second ship B, the bearing of A from B being N25^{0}W. If B moves with a constant speed of 15km/h. Find

(a) Course that B should set in order to get as close as possible to A [N21.4^{0}E]
(b) Closest distance and time taken for the situation to occur.[4.135km, t=0.437h]

2. Two aircraft A and B are flying at the same altitude with A initially 10km due north of B and flying at constant speed of 300m/s on a bearing of 060^0. If B flies at constant speed of 200m/s, find
   (a) Course that B should set in order to get as close as possible to A [E78.4^{0}N]
   (b) Closest distance and time taken for the situation to occur.[9.79km, t= 9.12s]
3. At 8am two boats A and B are 5.2km apart with A due west of B, and B travelling due north at a steady speed 13km/h. If A travels with a constant speed of 12km/h, show that for A to get as close as possible to B, A should set a course of NθOE where sinθ = $\frac{5}{13}$. Find the closest distance and time at which it occurs. [2km, 8.57 am]
4. Two aircraft A and B are flying at the same altitude with A initially 5km due north of B and B flying at constant speed of 300m/s on bearing of 060^0.If A flies at constant speed of 200m/s, find
   (a) Course that A should set in order to get as close as possible to B [108.2^0]
   (b) Time taken for the situation to occur.[5.4min]

5. A ship A moving with a constant speed of 24km/h in the direction N40^{0}E and is initially 10km from a second ship B, the bearing of A from B being N300W. If B moves with a constant speed of 22km/h; find
   (a) Course that B should set in order to get as close as possible to A [N16.4^{0}E]
   (b) Closest distance and time taken for the situation to occur.[6.89km, 45min]

Interception and collision

Consider two bodies A and B moving with V_A and V_B from points with position vectors OA and OB

Position of A after time t is

$r_{A(t=t)} = r_{A(t=0)} + t\ x\ V_A$

Position of B after time t is

$r_{B(t=t)} = r_{B(t=0)} + t\ x\ V_B$

For collision to occur $r_{A(t=t)} = r_{B(t=t)}$

$r_{A(t=0)} + t\ x\ V_A = r_{B(t=0)} + t\ x\ V_B$

$(r_{A(t=0)} - r_{B(t=0)}) + t(V_{AB})=0$

Hence $r_{AB(t=t)} = 0$

Example 30

The position vectors $r_A = (5i - 3j + 4k)m$ and $r_B = (7i + 5j - 2k)m$ are for two particles with velocities $V_A = (2i + 5j + 3k)m/s$ and $V_B = (-3i - 55j + 18k)m/s$ respectively. Show that if the velocities remain constant, a collision will occur

Solution

$r_{A(t=0)} + t\ x\ V_A = r_{B(t=0)} + t\ x\ V_B$

$$\begin{pmatrix}5\\-3\\4\end{pmatrix} + t\begin{pmatrix}2\\5\\3\end{pmatrix} = \begin{pmatrix}7\\5\\-2\end{pmatrix} + t\begin{pmatrix}-3\\-15\\18\end{pmatrix}$$

$$\begin{pmatrix}-2\\-8\\6\end{pmatrix} = \begin{pmatrix}-5\\-20\\15\end{pmatrix}t$$

Along i direction: -2 = -5t; t = 0.4s

Along j direction: -8= -20t; t = 0.4s

Along k direction: 6 = 15t; t = 0.4s

Since t is the same in all directions, collision occurred

Example 31

At 12 noon the position vectors r and velocity vectors V of two ships A and B are as follows

$r_A = (i + 7j)m, V_A = (6i + 2j)m/s,\ \ r_B = (6i + 4j)m \text{ and } V_B = (-4i + 8j)m/s$

Assuming velocities do not change

(i) Show that collision will occur
(ii) Find the time at which collision occurs
(iii) Find the position vector of the location during collision

Solution

$r_{A(t=0)} + t\ x\ V_A = r_{B(t=0)} + t\ x\ V_B$

$\binom{1}{7} + t\binom{6}{2} = \binom{6}{4} + t\binom{-4}{8}$

$\binom{-5}{3} = \binom{-10}{6}t$

Along the i direction: -5 = -10t; t = 0.5h

Along the j direction: 3 = 6t; t = 0.5h

Since t is the same in all directions

Collision occurred

(ii) time it occurred = 12:00 + 0.5 x 60

= 12:30pm

(iii) How far each had travelled

$r_{A(t=t)} = r_{A(t=0)} + t\ x\ V_A$

$r_{A(t=0.5)} = \binom{1}{7} + 0.5\binom{6}{2} = \binom{4}{8}km$

Example 32

At 11:30am a battle ship is at a place with position vector (-6i + 12j)km and is moving with velocity vector (16i – 4j)km/h. At 12:00 noon a cruiser is at a place with position vector (12i – 15j) and is moving with velocity vector (8i + 16j)km/h. Assuming velocities do not change

(i) Show that collision will occur
(ii) Find the time at which collision occurs
(iii) Find the position vector of the location of collision

$r_{A(t=t)} = r_{A(t=0)} + t\ x\ V_A$

At 12:00: $r_{A(t=0.5)} = \binom{-6}{12} + 0.5\binom{16}{-4} = \binom{2}{10}$

For collision to occur

$r_{A(t=0)} + t\ x\ V_A = r_{B(t=0)} + t\ x\ V_B$

$\binom{2}{10} + t\binom{16}{-4} = \binom{12}{-15} + t\binom{8}{18}$

$\binom{-10}{25} = \binom{-8}{20}t$

Along the i direction: -10 = -8t; t = 1.25h

Along the j direction: 15 = 20t; t = 1.25h

Since t is the same in all directions collision occurred

(ii) time it occurred = 11:30 + 1.25 x 60

= 12:45pm

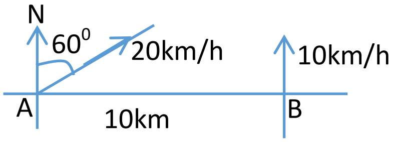
(iii) How far each had travelled

$r_{A(t=t)} = r_{A(t=0)} + t\ x\ V_A$

$r_{A(t=0.5)} = \binom{2}{10} + 1.25\binom{16}{-4} = \binom{22}{5}km$

Example 33

At 12:30 noon two ships A and B are 10km apart with B due east of A. A is travelling N600E at a speed of 12km/h and ship B is travelling due north at 10km/h. Show that, is the two ships do not change their velocities, they collide and fins to the nearest minute when collision occurs.



For collision to occur

$r_{A(t=0)} + t\ x\ V_A = r_{B(t=0)} + t\ x\ V_B$

$\binom{0}{0} + t\binom{20sin60}{20cos60} = \binom{10}{0} + t\binom{0}{10}$

$\binom{10\sqrt{3}}{10}t = \binom{10}{0} + t\binom{0}{10}$

Along the i direction: $10\sqrt{3}t$ = 10;

t = 0.5774h

Along the j direction: 10t = 10t

∴ t =0.5774 x 60 = 35minute

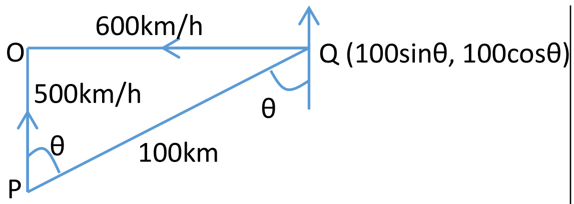
Collision occurred at 12:35minutes

Example 34

Two aircraft P and Q are flying at the same height. P is flying due north at 500km/h while Q is flying due west at 600km/h. When the aircrafts are 10km apart, the pilots realize that they are about to collide. The pilot of P changes Course to 345^0 and maintains the speed of 500km/h. The pilot Q maintains his course but increases speed. Determine the

(i) Distance each aircraft would have travelled if the pilots had not realized that they were about to collide

Solution



θ = $\tan^{-1}\frac{600}{500}$ = 50.2^0

For collision to occur

$r_{P(t=0)} + t\ x\ V_P = r_{Q(t=0)} + t\ x\ V_Q$

$\binom{0}{0} + t\binom{0}{500} = \binom{100sin\theta}{100cos\theta} + t\binom{-600}{0}$

$\binom{0}{0} + t\binom{0}{500} = \binom{100sin50.2}{100cos50.2} + t\binom{-600}{0}$

Along i direction: 0 = 100sin50.2 – 600t

t = 0.128h

Distance moved by P

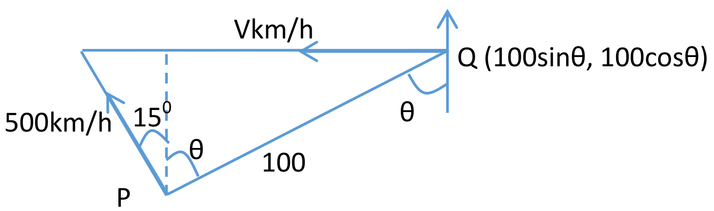
$\binom{0}{0} + 0.128\binom{0}{500} = \binom{0}{64} = 64km$

Distance moved by Q

$\binom{100sin\theta}{100cos\theta} + 0.128\binom{-600}{0}$ =$\binom{0.0284}{64.011}$

=64.011km

(ii) New speed beyond which the aircraft Q must fly in order to avoid collision



For collision to occur

$r_{P(t=0)} + t\ x\ V_P = r_{Q(t=0)} + t\ x\ V_Q$

$\binom{0}{0} + t\binom{-500\sin 15}{500cos15} = \binom{100sin\theta}{100cos\theta} + t\binom{-V}{0}$

$\binom{0}{0} + t\binom{-500\sin 15}{500cos15} = \binom{100sin50.2}{100cos50.2} + t\binom{-V}{0}$

Along j direction: 500cos15t = 64.011

t = 0.1325h

Along i direction:

-500sin15t = 76.8284 – Vt

V x 0.1325 = 76.8284 + 500sin15 x 0.1325

V =709.2837km/h

Course of interception

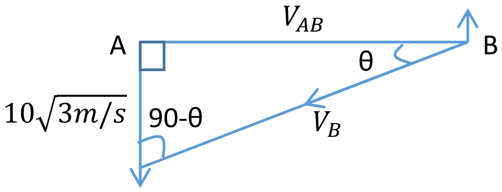
Suppose particle A with speed V_A is to intercept particle B with speed V_B, then

- Draw a sketch diagram showing the initial position and velocities of the two particles
- For interception to occur, the relative velocity must be in the direction of the initial displacement of the particles.

Example 35

At an instant a body A travelling south at $10\sqrt{3}$m/s is 150m west of B. Show that B will intercept A if B is travelling S30^{0}W at 20m/s and find the time that elapses before collision occurs.

Solution



$A = \binom{0}{0}m \qquad B = \binom{150}{0}m$

$V_A = \binom{0}{-10\sqrt{3}}m/s \qquad V_B = \binom{-20cos\theta}{-20sin\theta}m/s$

OA + t x v_A = OB + t x v_B

$\binom{0}{0} + \binom{0}{-10\sqrt{3}}t = \binom{150}{0} + \binom{-20cos\theta}{-20sin\theta}t$

j: 20sinθ =$10\sqrt{3}$

θ = $\sin^{-1}\frac{\sqrt{3}}{2}$ = 60^0

Bearing S30^{0}W

i: t = $\frac{150}{20cos\theta} = \frac{150}{20\cos 60} = 15s$

Alternatively

$\frac{sin90}{20} = \frac{sin\theta}{10\sqrt{3}}$

θ = 60^0: bearing S30^{0}W

Also, $\frac{sin90}{20} = \frac{\sin(90-\theta)}{V_{AB}}$

$V_{AB} = 10m/s$

t = $\frac{AB}{VA_{AB}} = \frac{150}{10} = 15s$

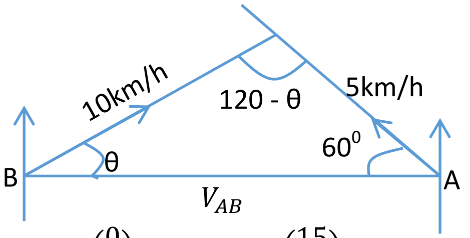
Example 36

At 9:00am two ships A and B are 15km apart with B on a bearing of 270^0 from A. Ship A moves at 5km/h on a bearing of 330^0. If the maximum speed of B is 10km/h. Find the

(i) Direction B should set in order to intercept A as soon as possible

(ii) Time taken for the interception to occur.

Solution



$B = \begin{pmatrix}0\\0\end{pmatrix} km \quad A = \begin{pmatrix}15\\0\end{pmatrix} km$

$V_B = \begin{pmatrix}10cos\theta\\10sin\theta\end{pmatrix} km/h \quad V_A = \begin{pmatrix}-5cos60\\5sin60\end{pmatrix} km/h$

OB + t x v_B = OA + t x v_A

$\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}10cos\theta\\10sin\theta\end{pmatrix} t = \begin{pmatrix}15\\0\end{pmatrix} + \begin{pmatrix}-5cos60\\5sin60\end{pmatrix} t$

j: 10sinθ =$5sin60$

θ = $\sin^{-1}\frac{5sin60}{10}$ = 25.7^0

Bearing E25.7^0N

i: t = $\frac{15}{10cos\theta+2.5} = \frac{15}{2.5+10\cos 25.7} = 1.303h$

t = 1.303 x 60 = 78mins

Alternatively

$\frac{sin60}{10} = \frac{sin\theta}{5}$

θ = 25.7^0: bearing E25.7^0N

Also, $\frac{sin90}{10} = \frac{\sin(120-\theta)}{V_{AB}}$

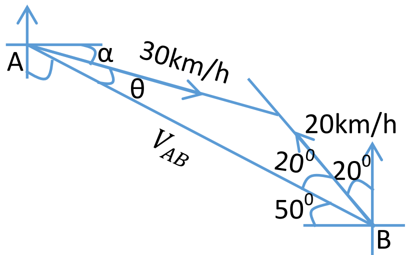
$V_{AB} = 11.515km/h$

t = $\frac{AB}{VA_{AB}} = \frac{15}{11.515}\ x\ 60 = 78minutes$

Example 37

At 12:00 noon two ship A and B are 12km apart with B on a bearing of 140^0 from A. Ship A moves at 30km/h to intercept B which is travelling at 20km/h on a bearing of 340^0. Find the

(i) Direction A should set in order to intercept B (ii) time taken for the interception to occur.



$A = \begin{pmatrix}0\\0\end{pmatrix} km \quad B = \begin{pmatrix}12cos50\\-12sin50\end{pmatrix} km$

$V_A = \begin{pmatrix}30cos\alpha\\30sin\alpha\end{pmatrix} km/h \quad V_B = \begin{pmatrix}-20cos20\\20sin20\end{pmatrix} km/h$

OA + t x v_A = OB + t x v_B

$\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}30cos\alpha\\30sin\alpha\end{pmatrix} t = \begin{pmatrix}12cos50\\-12sin50\end{pmatrix} + \begin{pmatrix}-20cos20\\20sin20\end{pmatrix} t$

i: $30tcos\alpha$ =$12cos50 + -20tcos20$

$t = \frac{7.713}{30cos\alpha+6.84}$ .... (i)

J: $30tsin\alpha$ =$-12sin50 + 20tsin20$

$t = \frac{9.193}{30sin\alpha+18.794}$ .... (ii)

(i) = (ii): $\frac{7.713}{30cos\alpha+6.84} = \frac{9.193}{30sin\alpha+18.794}$

231.39sinα – 275.79cosα = 82.078

But sinα= $\frac{2T}{1+T^2}$ and cosα= $\frac{1-T^2}{1+T^2}$

231.39$\left(\frac{2T}{1+T^2}\right)$ – 275.79$\left(c\frac{1-T^2}{1+T^2}\right)$ = 82.078

357.868T^2 + 462.78T – 193.712 = 0

T = -1.626 or T = 0.333

∴ T = 0.333

sinα= $\frac{2T}{1+T^2}$

α =$\sin^{-1}\frac{2\ x\ 0.333}{1+0.333^0}$ =36.8^0

Bearing: E36.8^0S

$t = \frac{7.713}{30cos\alpha+6.84} = \frac{7.713}{30cos36.8+6.84}$

T = 0.25h=0.25x 60 =15minutes

Alternatively

$\frac{sin20}{30} = \frac{sin\theta}{20}$

θ = 13.2^0:

Bearing (50 – 13,2)^{0}S

E36.8^{0}S or S53.2^{0}E

Also, $\frac{sin20}{30} = \frac{\sin(180-(20+13.2))}{V_{AB}}$

$V_{AB} = 48.03km/h$

t = $\frac{AB}{VA_{AB}} = \frac{12}{48.03}\ x\ 60 = 15minutes$

Revision exercise 5

1. At 12:00 noon two ships A and B are 12km apart with B on a bearing of 250^0 from A. Ship A moves at 4km/h on a bearing of 320^0. If the maximum speed of B is 7km/h, find the
   (a) Direction B should set in order to intercept A [N37.6^{0}E]
   (b) Time taken for interception to occur. [99minutes]
2. Initially two particles A and B are 48m apart with B due north of A. A has a constant velocity of (5i + 4j)m/s and B a constant speed of 13m/s. Find the velocity of Y if it is to intercept A and find the time taken to do so [5i – 12j)m/s, 3s]
3. At 12 noon the position vectors, r and velocity vectors, V of two ships are
   $r_A = \binom{5}{2}km, V_A = \binom{15}{10}km/h\ \ and\ r_B = \binom{7}{7}km, V_A = \binom{9}{-5}km/h$
   Show that if the ships do not alter their velocities, a collision will occur and find the time at which it occurs and the position vector of its location [12.30pm, (10i + $\frac{16}{3}j$)km]
4. At 11.30 a jumbo jet has position vectors (-100i + 220j)km and it is moving with velocity vectors (300i + 400J)km/h. At 11:45am a cargo plane has a position vectors (-60i + 355j)km and is moving with velocity vectors (400i + 300j)km/h. Assuming velocities do not change
   (a) Show that the planes will crash
   (b) Find the time of the crash. [12.06pm]
   (c) Find the position vector of the crash [(80i + 460j)km]
5. At 2pm the position vectors, r and velocity vectors, V of three ships are as follows
   $r_A = (5i + j)km \quad V_A = (9i + 18j)km/h$
   $r_B = (12i + 5j)km \quad V_A = (-12i + 6j)km/h$
   $r_C = (13i - 3j)km \quad V_A = (9i + 12j)km/h$
   Assuming velocities do not change
   (a) Show that Ship A and B will collide and fins when and where collision occur. [2.20pm, (8i + 7j)km]
   (b) Find the position vector of C when A and B collide and find how far C is from the point of collision. [(16i + j)km, 10km]
   (c) When the collision occurs, C immediately changes its course but not its speed and streams directly to the scene. When does C arrive[3.00pm]
6. At 12 noon the position vectors, r and velocity vectors, V of three ships A, B and C are as follows
   $r_A = (10.5i + 6j)km \quad V_A = (9i + 18j)km/h$
   $r_B = (7i + 20j)km \quad V_A = (12i + 6j)km/h$
   $r_C = (10i + 15j)km \quad V_A = (6i + 12j)km/h$
   Assuming velocities do not change
   (a) Show that Ship A and B will collide and fins when and where collision occur. [1:10pm, (21i + 27j)km]
   (b) When the collision occurs, C immediately changes its course but not its speed and streams directly to the scene. When does C arrive[1:300pm]

7. In gulf water, a battleship streaming at 16km/h is5km southwest of a submarine. Find the course which the submarine should set in order to intercept the battle ship, if its speed is 12km/h. [N15^{0}W]
8. A boy hits a ball at 15m/s in a direction S80^{0}W. A girl 45m and S65^{0}W from the boy run at 6m/s to intercept the ball. Find in what direction the girl must rum to intercept the ball as quickly as possible and how long does it take her. [N24.7^{0}E, 2.35s]
9. A helicopter sets off from its bas and flies at 50m/s to intercept a ship which, when the helicopter sets off, is at a distance of 5km on a bearing 335^0 from the base. The ship is travelling at 10m/s on a bearing 095^0. Find the course that the helicopter pilot should set if he is to intercept the ship as quickly as possible and the time interval between the helicopter taking off and its reaching the ship. [N15^{0}W, 92.2s]
10. A life boat sets out a harbour at 9:10pm to go for assistance of a yatch which is, at the time, 5km due north of the harbour and drifting due west at 8km/h. If the life boat travels at 20km/h find:
    (a) Course the life boat should set so as to reach the yatch as quickly as possible [S23.6^{0}W]
    (b) Time when the boat arrives [9:27pm]
11. A coast guard vessel wishes to intercept a yatch suspected of smuggling. At 1am the yatch is 10km due east of the coast guard vessel and travelling due north at 15km/h. If the coast guard vessel travels at 20km/h,
    (a) In which direction should it steer in order to intercept the yatch? [N41.4^{0}E]
    (b) When would this interception occur. [1:45am]
12. The driver of a speed boat travelling at 75km/h wishes to intercept a yatch travelling at 20km/h in a direction N40^{0}E. Initially the speed boat is 10km from the yatch on a bearing S30^{0}E. Find
    (a) Course the speed boat should set so as to reach the yatch as quickly as possible. [N15.5^{0}W]
    (b) Time when the interception occurs[9minutes and 7 seconds]
13. A jet fighter travelling at 30km/h wishes to intercept a plane travelling at 20km/h in a course of 200^0. Initially the plane is 40km away on a bearing of 11^0 from the jet fighter. Find
    (a) Course the jet fighter should set so as to reach the plane as quickly as possible. [S5^{0}E]
    (b) Time taken for interception to occur. [48minutes and 24 seconds]
14. A batman hits a ball at 15m/s in a direction S80^{0}W. A fielder, 45m and S65^{0}W from the batsman, runs at 6m/s to intercept the ball. Assuming the velocities remain unchanged,
    (a) Find what direction the fielder must take to intercept the ball as quickly as possible. [N24.7^{0}E]
    (b) How long did it take him. [2.4s}