

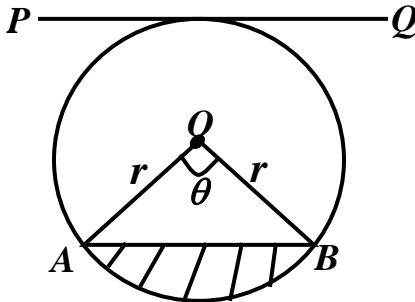
S.4 MATHEMATICS SELF STUDY

DAY 1

PARTS OF A CIRCLE

Summary:

1. The following are the main parts of a circle with centre O



(i) AB is called an arc

(ii) OAB is called a sector

(iii) line AB is called a chord

(iv) line PQ is called a tangent

(v) The shaded part is called a segment (vi) θ is the angle subtended by an arc

(vii) $OA = OB = \text{radius of the circle}$

2. **Theorem:** Since $OA = OB$, then OAB is an isosceles triangle with its line of symmetry bisecting chord AB

3. The following formulas are used in relation to the above circle:

(i) Circumference = $2\pi r$ or πd (ii) Area = πr^2

(iii) Arc length $AB = \frac{\theta}{360} \times 2\pi r$ (iv) Sector area $OAB = \frac{\theta}{360} \times \pi r^2$

(v) Segment area = sector area - triangle area

EXAMPLES:

1. Find the length of an arc which subtends an angle of 126° at the centre of a circle of radius **14cm**. Find also the length of the major arc

2. A chord is **8cm** away from the centre of a circle of radius **17cm**. Find the:

(i) length of the chord

(ii) size of the angle subtended by the chord at the centre

3. A chord of length **10cm** is **12cm** away from the centre of the circle. Find the:

(i) radius of the circle

(ii) length of the minor arc

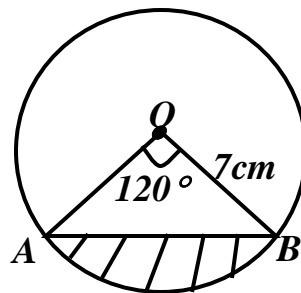
(iii) area of the minor sector

(iv) area of the minor segment

4. A sector of a circle of radius **12cm** has an angle of **150°** at the centre. The sector is folded to form a cone. Find the radius of the circular end of the cone

5. Find the distance between two parallel chords of lengths **32cm** and **24cm** which lie on opposite sides of the centre in a circle of radius **20cm**

6. in the circle below, sector **OAB** has a radius of **7cm** and subtends an angle of **120°** at its centre **O**.



Find the:

(i) shortest distance of chord **AB** from the centre

(ii) perimeter of the shaded segment

(iii) perimeter of the region enclosed between chord **AB** and the major arc

(iv) area of the shaded segment

7. The length of the common chord of two intersecting circles of radius **10cm** and **17cm** is **6cm**. Find the:

(i) angle subtended by the chord at the centre of the two circles

(ii) area common to the two circles

8. Two equal circles of radius **5cm** intersect at right angles. Find the:

(i) distance between the centres of the two circles

(ii) area common to the two circles

9. A sector of a circle of radius **25cm** has an angle of **100.8°** at the centre. The sector is folded to form a cone. Find the:

(i) radius of the circular end of the cone

(ii) height of the cone

(iii) volume of the cone

(iv) total surface area of the cone

HINT: (iv) $T.S.A = \pi r l$ since its circular end is open

10. The minor segment of a circle has a height of **4cm** and a chord of length **16cm**. Find the:

(i) radius of the circle

(ii) area of the segment

11. A dog tied on a rope **5m** long is tethered to a tree **3m** from a straight path. For what distance along the path is one in danger of being bitten by the dog?

ACTIVITY

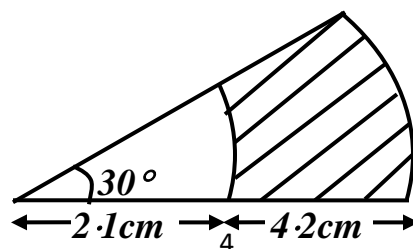
1. A chord of length **70cm** subtends an angle of **120°** at the centre of the circle. Find the:

(i) radius of the circle

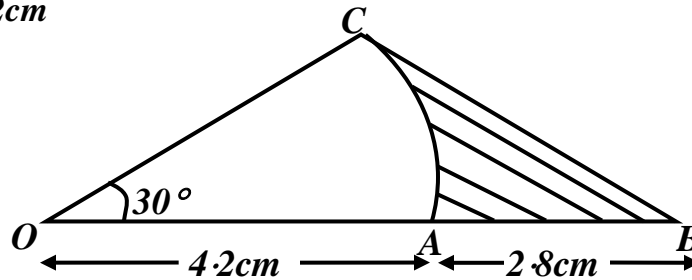
(ii) distance of the chord from the centre

(iii) area of the minor segment

2. A chord of length **6cm** is **4cm** away from the centre of the circle. Find the circumference of the circle
3. A dog tied on a rope **2.5m** long is tethered to a tree **2m** from a straight path. For what distance along the path is one in danger of being bitten by the dog?
4. A sector of a circle of radius **10cm** has an angle of **100°** at the centre. Find the
 - (i) perimeter of the sector
 - (ii) area of the minor segment
5. A chord of length **6cm** makes an angle of **40°** with the radius of the circle. Find the circumference of the circle
6. A chord **3.5cm** away from the centre of the circle subtends an angle of **120°** at its centre. Find the area of the major segment
7. The chord of a circle of radius **10cm** subtends an angle of **120°** at its centre. Find the perimeter of the region enclosed between the chord and the major arc
8. Find the distance between two parallel chords of lengths **32cm** and **24cm** which lie on the same side of the centre in a circle of radius **20cm**
9. A sector of a circle of radius **12.5cm** has an angle of **100.8°** at the centre. The sector is folded to form a cone. Find the:
 - (i) radius of the circular end of the cone
 - (ii) height of the cone
 - (iii) volume of the cone
 - (iv) total surface area of the cone
10. Find the distance between two parallel chords of lengths **24cm** and **10cm** which lie on opposite sides of the centre in a circle of radius **13cm**
11. In the figure below, find the shaded area bounded by two concentric arcs



12. In the triangle **OBC** below, arc **AC** subtends an angle of 30° at the centre **O** of a circle of radius **4.2cm**



If **AB = 2.8cm**, find the area of the shaded region

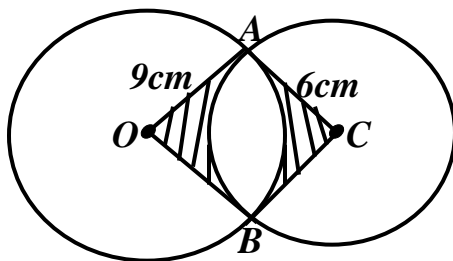
13. The length of the common chord of two intersecting circles of radius **28cm** and **20cm** is **30cm**. Find the:

- (i) angle subtended by the chord at the centre of the two circles
- (ii) area common to the two circles

14. The distance between the centres of two intersecting equal circles of radius **5cm** is **8cm**. Find the:

- (i) length of the common chord of the two circles
- (ii) area common to the two circles

15. Two circles with centres **O** and **C** and radius **9cm** and **6cm** intersect at points **A** and **B** as shown

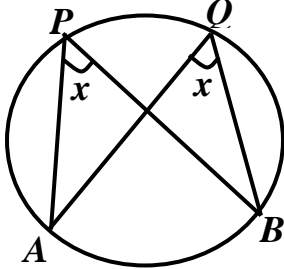
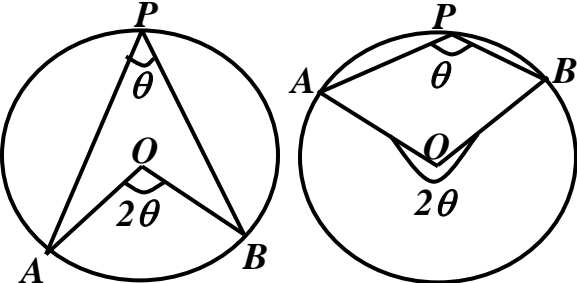
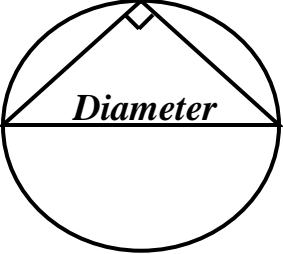
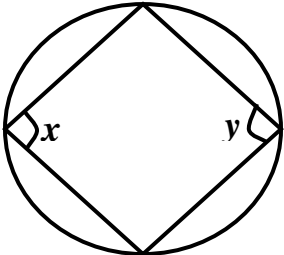


Given that the distance between **O** and **C** is **13cm**, find the:

- (i) reflex angle **AOB**
- (ii) length of chord **AB**
- (iii) area of the shaded region

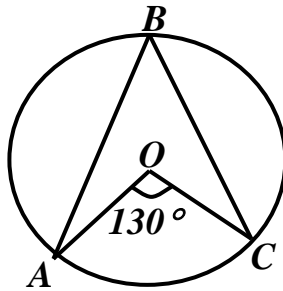
DAY TWO

ANGLES IN A CIRCLE

Angles diagram	Circle theorems
	<p>The angles subtended at the circumference by the same arc are equal $\therefore \angle APB = \angle AQB$ In short: Angles in the same segment are equal</p>
	<p>The angle subtended at the centre by an arc is twice the angle it subtends at its circumference $\therefore \angle AOB = 2 \angle APB$ In short: Angle at the centre is twice the angle at the circumference</p>
	<p>The angles subtended at the circumference by a semi-circle is 90°</p>
	<p>The opposite angles of a cyclic quadrilateral add up to 180° $\therefore x + y = 180^\circ$</p>

EXAMPLES:

1. In the figure below, O is the centre of the circle and $\angle AOC = 130^\circ$. Find $\angle ABC$



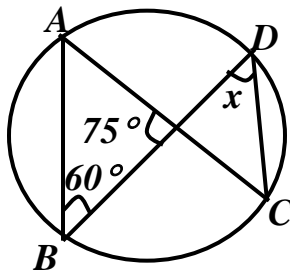
Soln:

$$\angle AOC = 2\angle ABC$$

$$\Rightarrow 130^\circ = 2\angle ABC$$

$$\therefore \angle ABC = 65^\circ$$

2. In the circle below, find the size of angle marked x



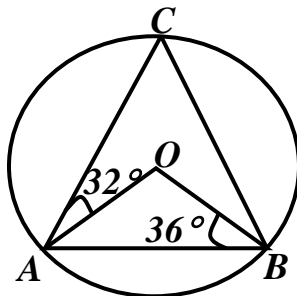
Soln:

$$\angle BAC = x \text{ (angles in the same segment)}$$

$$\Rightarrow x = 180^\circ - (60^\circ + 75^\circ) \text{ (angles in a triangle)}$$

$$\therefore x = 45^\circ$$

3. In the circle below O is its centre $\angle ABO = 36^\circ$ and $\angle OAC = 32^\circ$. Find $\angle ACB$ and $\angle OBC$



Soln:

$\triangle AOB$ is isosceles since $OA = OB$

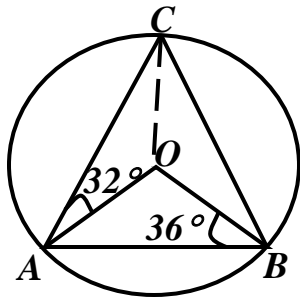
$$\Rightarrow \angle AOB = 180^\circ - (2 \times 36^\circ) = 108^\circ \text{ (angles in a triangle)}$$

$$\angle AOB = 2\angle ACB$$

$$\Rightarrow 108^\circ = 2\angle ACB$$

$$\therefore \angle ACB = 54^\circ$$

Also:



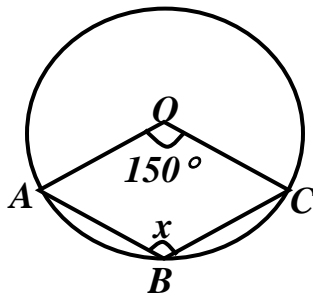
$$\Rightarrow \angle AOC = 180^\circ - (2 \times 32^\circ) = 116^\circ \text{ (angles in a triangle)}$$

$$\angle BOC = 360^\circ - (108^\circ + 116^\circ) = 136^\circ \text{ (angles at a point)}$$

$$\Rightarrow 2\angle OBC = 180^\circ - 136^\circ \text{ (angles in a triangle)}$$

$$\therefore \angle OBC = 22^\circ$$

4. In the circle below O is its centre and $\angle AOC = 150^\circ$. Find the size of angle marked x



Soln:

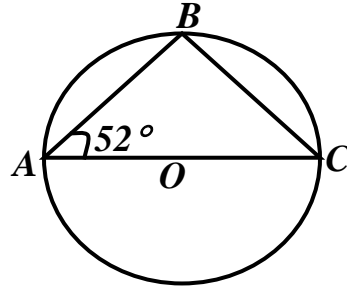
$$\text{Reflex angle } \angle AOC = 360^\circ - 150^\circ = 210^\circ \text{ (angles at a point)}$$

$$\Rightarrow \angle AOC = 2\angle ABC$$

$$210^\circ = 2\angle ABC$$

$$\therefore \angle ABC = 105^\circ$$

5. In the figure below, O is the centre of the circle of radius 7cm and $\angle BAC = 52^\circ$.



Find the:

(i) size of angle ACB

(ii) lengths of AB and BC

Soln:

(i) $\angle ABC = 90^\circ$ (angle in a semi-circle)

$$\therefore \angle ACB = 180^\circ - (90^\circ + 52^\circ) = 38^\circ \text{ (angles in a triangle)}$$

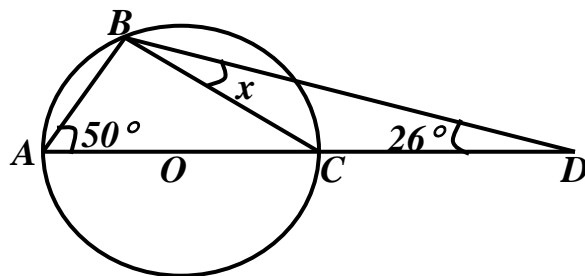
$$(ii) \cos 52 = \frac{AB}{14}$$

$$\therefore AB = 8.6193\text{cm}$$

$$\text{Also } \sin 52 = \frac{BC}{14}$$

$$\therefore BC = 11.0322\text{cm}$$

6. In the circle below O is its centre. Find the size of angle marked x

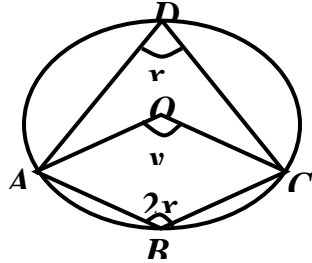


Soln:

(i) $\angle ABC = 90^\circ$ (angle in a semi-circle)

$$\therefore x = 180^\circ - (90^\circ + 50^\circ + 26^\circ) = 14^\circ \text{ (angles in a triangle)}$$

7. In the circle below O is its centre. Find the size of the angles marked x and y



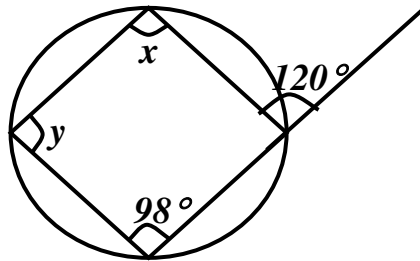
Soln:

$$x + 2x = 180^\circ \text{ (angles in a cyclic quadrilateral)}$$

$$\therefore x = 60^\circ$$

$$\text{Also } y = 2x = 2(60^\circ) = 120^\circ$$

8. In the circle below, find the size of the angles marked x and y



Soln:

$$x + 98^\circ = 180^\circ \text{ (angles in a cyclic quadrilateral)}$$

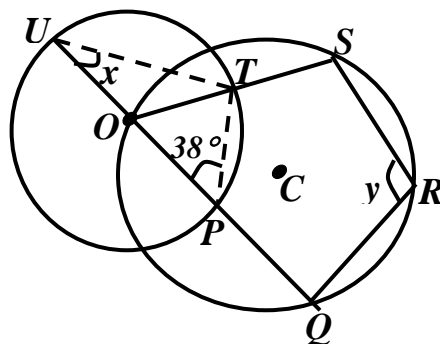
$$\therefore x = 82^\circ$$

$$\text{Also } y + 60^\circ = 180^\circ \text{ (angles in a cyclic quadrilateral)}$$

$$\therefore y = 120^\circ$$

9. In the diagram below, C and O are centres of two intersecting circles. $\angle CPT = 38^\circ$

Find the size of the angles marked x and y



Soln:

(i) $\angle UTP = 90^\circ$ (angle in a semi-circle)

$\therefore x = 180^\circ - (90^\circ + 38^\circ) = 52^\circ$ (angles in a triangle)

Also: $\angle POT = 180^\circ - (2 \times 38^\circ) = 104^\circ$ (angles in a triangle)

$y + 104^\circ = 180^\circ$ (angles in a cyclic quadrilateral)

$\therefore y = 76^\circ$

10. The points $P(-2, -1)$, $Q(h, 7)$ and $R(-3, 6)$ lie on a circle with diameter PQ .

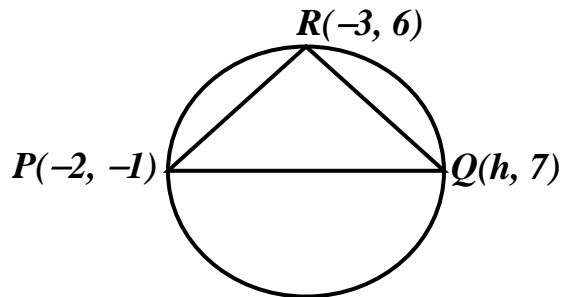
(i) State with a reason the size of angle PRQ

(ii) Show that $h = 4$

(iii) Find the coordinates of the centre and radius of the circle

Soln:

(i)



$\angle PRQ = 90^\circ$ (angle in a semi-circle)

(ii) **Hint:** This could be done using gradient method

$$\text{Gradient of } PR = \frac{6 - (-1)}{-3 - (-2)} = -7$$

$$\text{Gradient of } QR = \frac{7 - 6}{h - (-3)} = \frac{1}{h + 3}$$

For perpendicular lines, $-7 \times \left(\frac{1}{h + 3} \right) = -1$

$\therefore h = 4$

(iii)**Hint:** Centre is the midpoint of the diameter and radius is half the diameter

$$\text{Centre} = \left(\frac{-2+4}{2}, \frac{-1+7}{2} \right) = (2, 3)$$

$$\text{Radius} = \sqrt{(4-2)^2 + (7-3)^2} = 5 \text{ units}$$

ACTIVITY

1. The vertices of an equilateral triangle **A, B** and **C** lie on a circle of radius **8cm**. Find the:

(i) distance of any side of the triangle from the centre of the circle

(ii) length of the side of the triangle

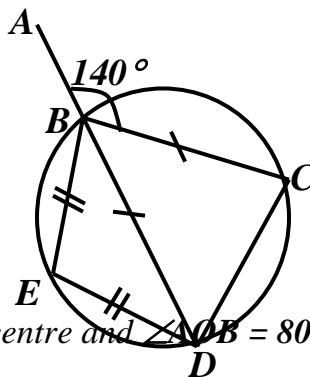
2. The vertices of an equilateral triangle of side **12cm** lie on a circle. Find the:

(i) radius of the circle

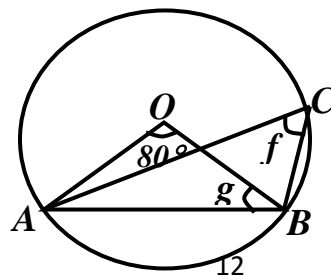
(ii) distance of any side of the triangle from the centre of the circle

(iii) area of the segments cut off by the triangle

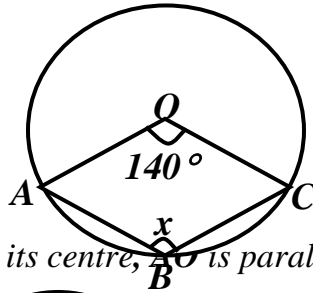
3. In the circle below, **BC = BD**, **BE = DE** and $\angle ABC = 140^\circ$. Find the size of angle **EDB**



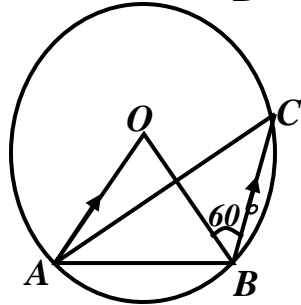
4. In the circle below **O** is its centre and $\angle AOB = 80^\circ$. Find the size of the angles marked **f** and **g**



5. In the circle below O is its centre and $\angle AOC = 150^\circ$. Find the size of angle marked x



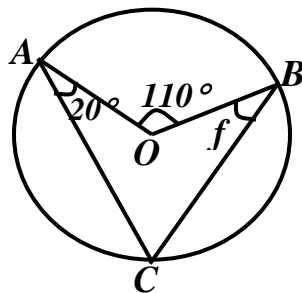
6. In the circle below O is its centre, AO is parallel to BC and $\angle OBC = 60^\circ$.



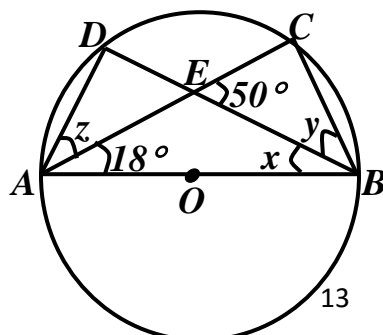
Find the size of angle:

(i) $\angle AOB$ (ii) $\angle ACB$ (iii) $\angle CAB$

7. In the circle below O is its centre, $\angle AOB = 110^\circ$ and $\angle OAC = 20^\circ$. Find the size of angle marked f



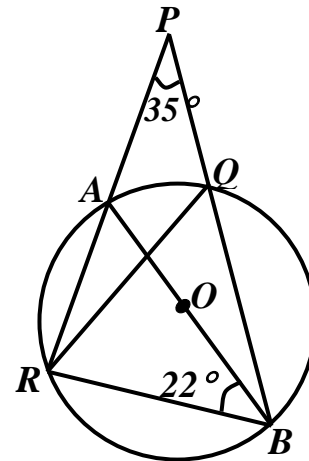
8. In the figure below AB is the diameter of a circle centre O . $\angle BAC = 18^\circ$ and $\angle BEC = 50^\circ$. Find the size of the angles marked x , y and z



9. In the figure below AB is the diameter of a circle centre O . $\angle APQ = 35^\circ$ and $\angle ABR = 22^\circ$.

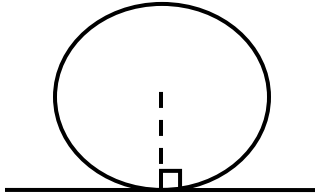
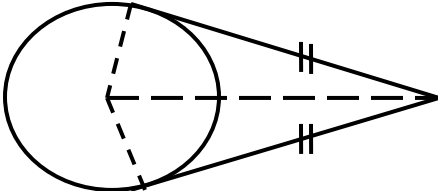
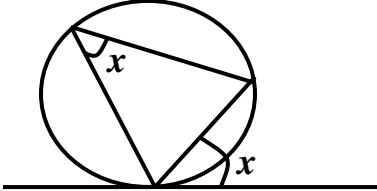
Find the size of angle:

(i) $\angle ABQ$ (ii) $\angle QRA$ (iii) $\angle AOR$



DAY 3

TANGENT PROPERTIES

Tangent diagrams	Circle theorems
	<p>The angle between a tangent and the radius is 90°</p>
	<p>Tangents from an external point are equal in length</p>
	<p>The angle between a tangent and a chord is equal to the angle the chord subtends in the opposite segment (alternate segment)</p>

EXAMPLES:

1. A tangent from **T** to a circle, centre **O** and radius **7cm** touches the circle at **P**. If **OT = 25cm**, find the:

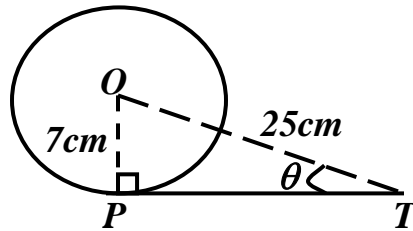
(i) length of **PT**

(ii) size of angle **PTO**

(iii) area of **PTO** that lies outside the circle

Soln:

(i)



$\angle OPT = 90^\circ$ (angle between a tangent and the radius)

$$\therefore PT = \sqrt{25^2 - 7^2} = 24\text{cm}$$

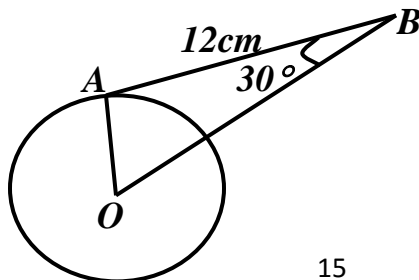
$$(ii) \sin \theta = \frac{7}{25}$$

$$\therefore \theta = 16.26^\circ$$

(iii) $\angle POT = 180^\circ - (90^\circ + 16.26^\circ) = 73.74^\circ$ (angles in a triangle)

$$\therefore \text{Required area} = \frac{1}{2} \times 24 \times 7 - \frac{73.74}{360} \times \frac{22}{7} \times 7^2 = 52.4557\text{cm}^2$$

2. In the circle below **O** is its centre. **AB = 12cm** is a tangent to the circle at **A** and $\angle OBA = 30^\circ$.



Find the:

(i) length of **OB**

(ii) radius of the circle

Soln:

(i) $\angle OAB = 90^\circ$ (angle between a tangent and the radius)

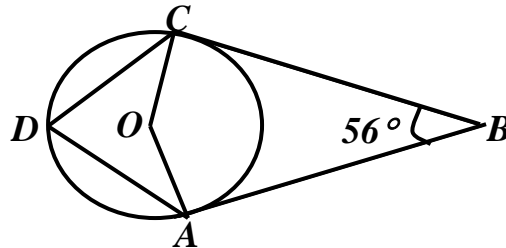
$$\Rightarrow \cos 30 = \frac{12}{OB}$$

$$\therefore OB = 13.8564 \text{ cm}$$

$$(ii) \tan 30 = \frac{OA}{12}$$

$$\therefore OA = 6.9282 \text{ cm}$$

3. In the circle below **O** is its centre. **AB** and **CB** are tangents to the circle and $\angle ABC = 56^\circ$.



Find the size of angle **ADC**

Soln:

$$\angle AOC + 90^\circ + 90^\circ + 56^\circ = 360^\circ \text{ (angles in a quadrilateral)}$$

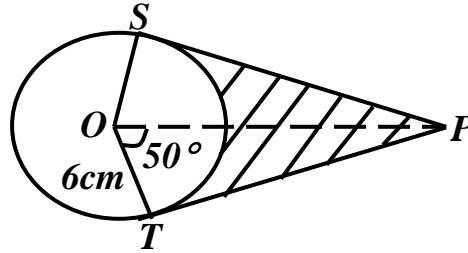
$$\therefore \angle AOC = 124^\circ \text{ (angles in a triangle)}$$

$$\text{If } \angle AOC = 2\angle ADC$$

$$\Rightarrow 124^\circ = 2\angle ADC$$

$$\therefore \angle ADC = 62^\circ$$

4. In the circle below O is its centre. PT and PS are tangents to the circle of radius **6cm** and $\angle POT = 50^\circ$.



Find the area of the shaded region

Soln:

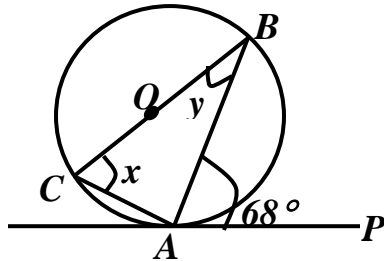
(i) $\angle OTP = 90^\circ$ (angle between a tangent and the radius)

$$\Rightarrow \tan 50 = \frac{PT}{6}$$

$$\therefore PT = 7.1505 \text{ cm}$$

$$\therefore \text{Shaded area} = 2 \left(\frac{1}{2} \times 7.1505 \times 6 \right) - \frac{100}{360} \times \frac{22}{7} \times 6^2 = 11.4744 \text{ cm}^2$$

5. In the circle below O is its centre. AP is a tangent to the circle and $\angle PAB = 68^\circ$.



Find the size of the angles marked x and y

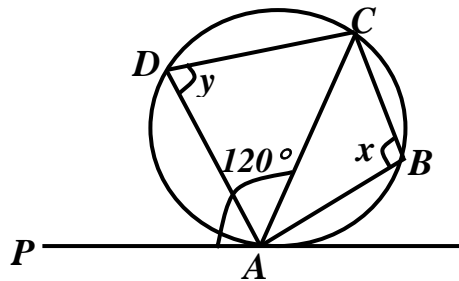
Soln:

$x = 68^\circ$ (angle in alternate segments)

$\angle BAC = 90^\circ$ (angle in a semi-circle)

$$\therefore y = 180^\circ - (90^\circ + 68^\circ) = 22^\circ \text{ (angles in a triangle)}$$

6. In the circle below AP is a tangent to the circle and $\angle PAC = 120^\circ$.



Find the size of the angles marked x and y

Soln:

$$x = 120^\circ \text{ (angle in alternate segments)}$$

$$\text{Also: } y + 120^\circ = 180^\circ \text{ (angles in a cyclic quadrilateral)}$$

$$\therefore y = 60^\circ$$

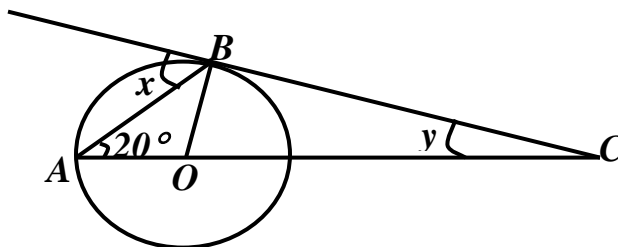
ACTIVITY

1. The tangents at A and B on a circle of radius 7cm intersect at T , and C is any point on the major arc AB . If $\angle ATB = 48^\circ$, find the:

(i) size of angle ACB

(ii) area bounded by the tangents and the minor arc AB

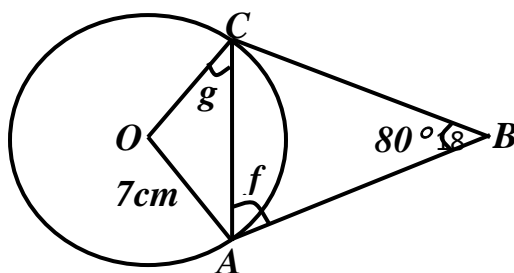
2. In the circle below O is its centre. BC is a tangent to the circle and $\angle BAO = 20^\circ$.



Find the size of the angles marked x and y

3. The angles of a triangle are 50° , 60° and 70° , and a circle touches the sides at A , B and C . Find the angles of triangle ABC

4. In the circle below O is its centre. AB and CB are tangents to the circle of radius 7cm and $\angle ABC = 80^\circ$.



Find the:

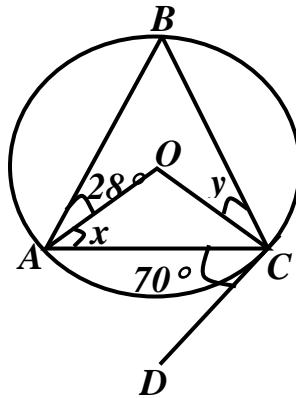
(i) size of the angles marked f and g

(ii) size of the reflex angle AOC

(iii) area bounded by the tangents and the minor arc AC

5. In the circle below O is its centre. DC is a tangent to the circle $\angle BAO = 28^\circ$

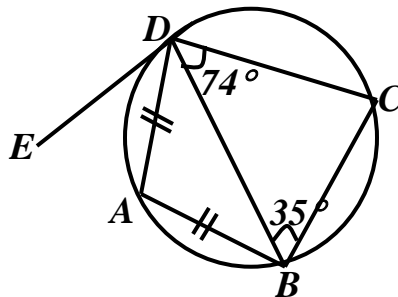
and $\angle ACD = 70^\circ$



Find the size of the angles marked x and y

6. In the circle below ED is a tangent to the circle, $AB = AD$, $\angle BDC = 74^\circ$

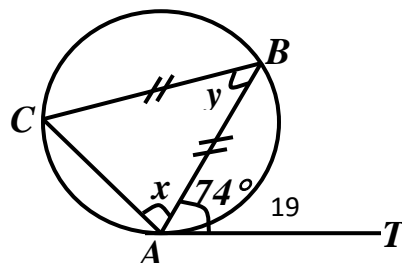
and $\angle DBC = 35^\circ$



Find the size of angle:

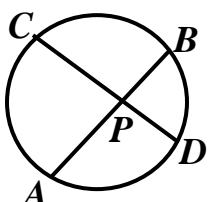
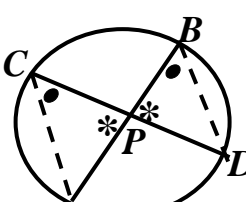
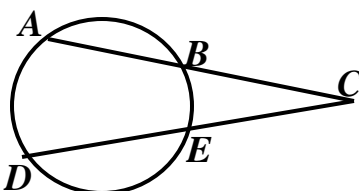
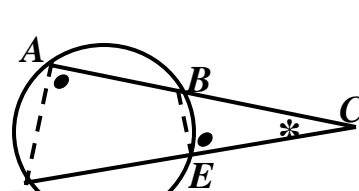
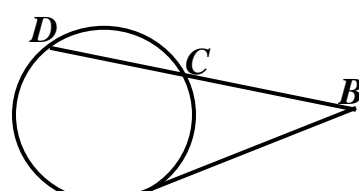
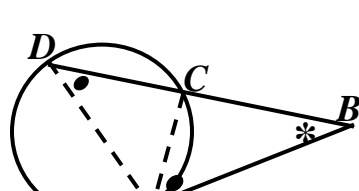
(i) $\angle DAB$ (ii) $\angle BDE$ (iii) $\angle DBA$ (iv) $\angle EDA$

7. In the circle below AT is a tangent to the circle, $AB = CB$ and $\angle BAT = 74^\circ$. Find the size of the angles marked x and y



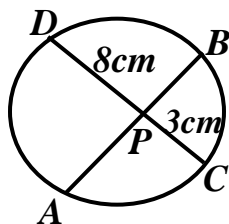
DAY 4

CIRCLES AND SIMILAR TRIANGLES

Circle diagrams	Analysis diagrams	Conclusion
 <p>“ intersecting chords theorem ”</p>		<p>Triangles ACP and PBD are similar</p> $\Rightarrow \frac{AP}{PD} = \frac{CP}{PB}$ $\therefore AP \cdot PB = CP \cdot PD$ <p>“ Each length is measured from the meeting point ”</p>
 <p>“ intersecting secants theorem ”</p>		<p>Triangles ACD and ECB are similar</p> $\Rightarrow \frac{AC}{CE} = \frac{DC}{CB}$ $\therefore CA \cdot CB = CD \cdot CE$ <p>“ Each length is measured from the meeting point ”</p>
 <p>“ intersecting tangent-secant theorem ”</p>		<p>Triangles ABC and ABD are similar</p> $\Rightarrow \frac{AB}{BC} = \frac{DB}{AB}$ $\therefore BA^2 = BD \cdot BC$ <p>“ Each length is measured from the meeting point ”</p>

EXAMPLES:

1. In the circle below chords **AB** and **CD** intersect at **P**. If **CP = 3cm**, **DP = 8cm** and **AB = 10cm**, find the length of **AP**



Soln:

If $AP = x$, $PB = 10 - x$

Using $AP \cdot PB = CP \cdot PD$ (intersecting chords theorem)

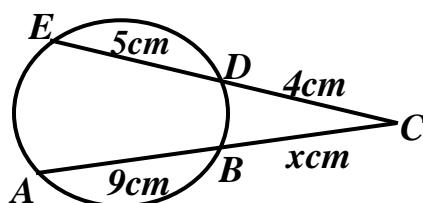
$$\Rightarrow x(10 - x) = 3(8)$$

$$x^2 - 10x + 24 = 0$$

$$x = \frac{10 \pm \sqrt{100 - 96}}{2}$$

$$\therefore x = 6\text{cm} \text{ or } 4\text{cm}$$

2. In the circle below chords AB and ED are produced to intersect at C . If $CD = 4\text{cm}$, $ED = 5\text{cm}$, $AB = 9\text{cm}$ and $BC = x\text{cm}$,



find the:

(i) value of x

(ii) ratio of the areas of triangle ACE to that of BCD

(iii) area of $ABDE$ if the area of triangle ACE is 54cm^2

Soln:

(i) Using $CB \cdot CA = CD \cdot CE$ (Each length from C)

$$\Rightarrow x(x + 9) = 4(9)$$

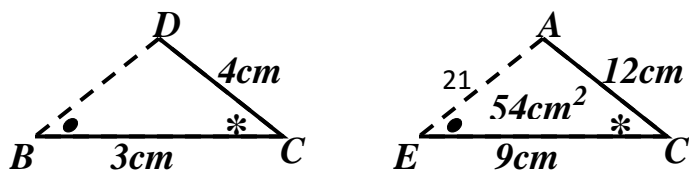
$$x^2 + 9x - 36 = 0$$

$$x = \frac{-9 \pm \sqrt{81 + 144}}{2}$$

$$x = 3 \text{ or } -12$$

$$\therefore x = 3$$

(ii) Using similar triangles ACE and BCD



$$\frac{\text{Area } ACE}{\text{Area } BCD} = \left(\frac{12}{4}\right)^2 = 9$$

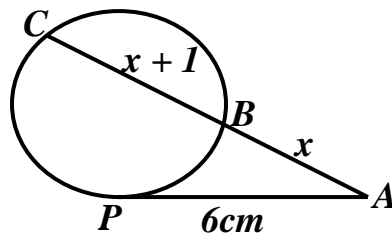
\therefore Required ratio = **9:1**

$$(iii) \frac{\text{Area } BCD}{54} = \left(\frac{4}{12}\right)^2$$

$$\therefore \text{Area } BCD = 6\text{cm}^2$$

$$\Rightarrow \text{Required area} = 54 - 6 = 48\text{cm}^2$$

3. In the circle below secant ABC intersects tangent AP at A . If $AP = 6\text{cm}$, $AB = x\text{cm}$ and $BC = (x + 1)\text{cm}$, find the value of x



Soln:

Using $AB \cdot AC = AP^2$ (Each length from A)

$$\Rightarrow x(2x + 1) = 6^2$$

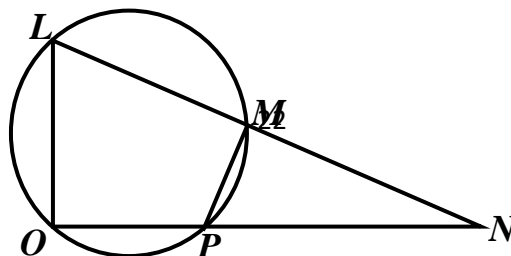
$$2x^2 + x - 36 = 0$$

$$x = \frac{-1 \pm \sqrt{1 + 288}}{4}$$

$$x = 4 \text{ or } -4.5$$

$$\therefore x = 4$$

4. In the circle below $OL = 4.5\text{cm}$, $PM = 3\text{cm}$, $NM = 4\text{cm}$ and $LN = 7.5\text{cm}$.



Find the:

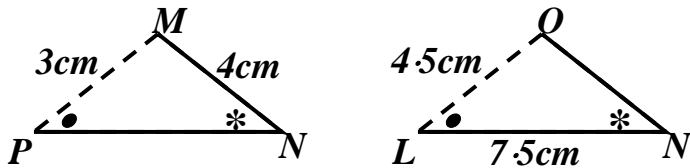
(i) lengths of **ON** and **OP**

(ii) radius of the circle

(iii) area of **OLMP**

Soln:

(i) Using similar triangles **PMN** and **OLN**,



$$\Rightarrow \frac{ON}{4} = \frac{4.5}{3}$$

$$\therefore ON = 6cm$$

$$\text{Also: } \frac{PN}{7.5} = \frac{3}{4.5}$$

$$\therefore PN = 5cm$$

$$\Rightarrow OP = ON - PN = 6 - 5 = 1cm$$

(ii) **PMN** is a right triangle based on its dimensions ($\angle PMN = 90^\circ$)

$\Rightarrow \angle POL = 90^\circ$, thus **LP** is a diameter (angle in a semi-circle)

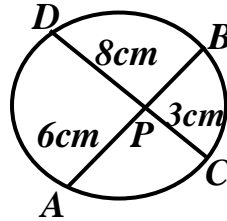
$$\therefore LP = \sqrt{4.5^2 + 1^2} = 4.6098cm$$

$$\Rightarrow \text{Required radius} = \frac{1}{2} \times 4.6098 = 2.3049cm$$

$$(iii) \text{ Required area} = \frac{1}{2} \times 6 \times 4.5 - \frac{1}{2} \times 4 \times 3 = 7.5cm^2$$

ACTIVITY

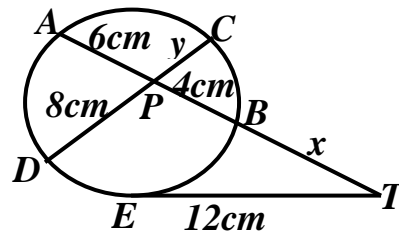
1. In the circle below chords **AB** and **CD** intersect at **P**. If **CP = 3cm**, **DP = 8cm** and **AP = 6cm**,



find the:

- (i) length of **PB**
- (ii) area of triangle **BPC** if the area of triangle **APD** is 20cm^2
- (iii) ratio of the areas of triangle **APC** to that of **BPD**

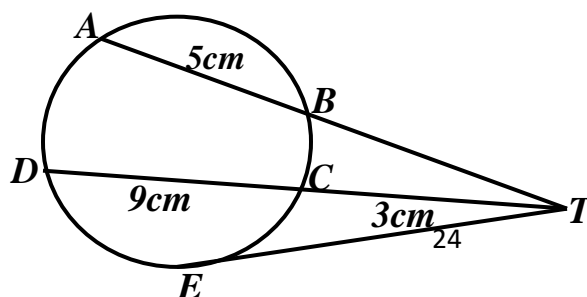
2. In the circle below **TE = 12cm** is a tangent to the circle at **E**. Chords **AB** and **CD** intersect at **P** and **AB** is produced at **T**. If **AP = 6cm**, **PB = 4cm**, **BT = xcm**, **CP = ycm** and **DP = 8cm**,



find the:

- (i) values of **x** and **y**
- (ii) ratio of the areas of triangle **APD** to that of **BPC**

3. In the circle below **TE** is a tangent to the circle at **E**. Chords **AB** and **CD** are produced to intersect at **T**. If **AB = 5cm**, **DC = 9cm** and **CT = 3cm**, find the lengths of **BT** and **ET**



DAY 5

COMMERCIAL ARITHMETIC

Currency conversion

Currency is the money system in use in a country, for example, Uganda shillings (Ush.), Kenya shillings (Ksh.), US dollar (US \$), Pound sterling, e.t.c.

The currency of another country, known as, foreign currency, can be bought and sold at a given exchange rate

Example

- (a) Given that 1 US dollar is equivalent to Ush. 1800, what is the value, in US \$, of Ush. 54,000?
- (b) Convert US \$ 50 to Uganda shillings (Ush.) if US\$ 1 = Ush. 1750. Solution
- (a) \$ 1 is equivalent to Ush. 1,800.

Therefore, Ush. 1 will be equivalent to $\$ \frac{1}{1,800}$

Hence, Ush. 54,000 is equivalent to $\$ \frac{1}{1,800} \times 54,000 = \30 .

- (b) US\$ 50 = Ush. (1750 \times 50) = Ush. 87,500.

ACTIVITY

1. A man in Nairobi received Ush, 500,000 which he exchanged for Kenya shillings at the rate of Ksh. 1 to Ush. 22. How many Kenya shillings did he receive?
2. Given that 1 Japanese Yen = Ush. 1,300, convert 2,800 Japanese Yen to Uganda shillings.
3. A tourist had 12,000 US dollars. He exchanged this money for UK sterling pounds and then for Uganda shillings. How many Uganda shillings did he get given that 1 UK pound sterling = 1.4711 US dollars and 1 Sterling pound = Ush. 2,780.
4. Musa visited Kenya for a holiday. He had Ush.1,500,000 The exchange rate was Ush. 23 to Ksh. 1. He spent Ksh, 60,000 during his stay in Kenya and changed the remainder back to Uganda shillings at the rate of Ksh. 1 = Ush. 24.8. How many Uganda shillings did he receive?

5. A firm in France bought coffee worth Ush. 35 000 000 from Uganda. How much did the firm pay in Euros given that:
1 Euro = 0.9461 US dollars and 1 US dollar = Ush. 1 950.
6. A Ugandan bought a car from Japan for Ush. 6 000 000. How much did he pay for the car in Japanese Yen if 1 Japanese Yen = 1.13 US dollars and 1 US dollar = Ush. 1 720.
7. A Forex Bureau buys one US dollar at Ush.1,900 and sells one Pound Sterling at Ush.3,450. Atim wants to exchange 3,000 US dollars to pound sterling. How many pounds sterling will she get?

PROFIT AND LOSS

Cost price is the price at which goods are bought. It is also called the buying price or purchase price. Selling price is the price at which goods are sold by the second seller (retailer).

If the selling price is more than the cost price, we say that a profit has been made. Thus,
profit = selling price – cost price.

Sometimes the cost price is more than the selling price. We call the difference between the two a loss. Thus, loss = cost price – selling price.

Percentage profit and loss

It is common to express the profit or loss as a percentage. Usually the percentage profit or loss is calculated as a percentage of the cost price.

Example

A trader buys goods for sh. 300 000 and sells them for sh. 360 000. Calculate the percentage profit.

Solution

Cost price = sh. 300 000

Selling price = sh. 360 000

Profit = sh. 360 000 – sh. 300 000
= sh. 60 000

$$\text{Percentage profit} = \frac{60000}{300000} \times 100\% \\ = 20\%.$$

Example

A shopkeeper sells a dress for sh. 23 000 thereby making a profit of 15%. Calculate the cost price of the dress.

Solution

Selling price = sh. 23 000

This is 115% of the cost price which is 100%

$$\text{Therefore, cost price} = \frac{100}{115} \times 23000 = \text{sh. } 20\,000.$$

Alternatively, let the cost price be sh. X

Selling price = sh. 23 000

Percentage profit = 15%

$$\begin{aligned} & \frac{23000 - x}{x} = \frac{15}{100} \\ \text{Therefore, } & \frac{23000 - x}{x} \times 100 = 15 \end{aligned}$$

$$\begin{aligned} 23000 - x &= \frac{15}{100}x \\ 23000 &= \frac{115}{100}x \\ 115x &= 2300000 \\ x &= \frac{2300000}{115} = 20000 \end{aligned}$$

Therefore, cost price = sh. 20 000.

ACTIVITY

1. A man bought a bicycle at sh. 58 000. He sold it at a profit of 10%. What was his selling price?
2. A dealer buys an item for sh. 8 000. He wishes to make a profit of 35%. What should the selling price be?
3. A radio that cost sh. 15 000 was sold at a loss of 18%. What was the selling price?

4. A shopkeeper sells a bag for sh. 32 000, making a loss of 20%. What is the cost price?
5. By selling a table for sh. 31 500, a shopkeeper makes a profit of 15%. Calculate the actual profit.
6. An agent buys 200 items at a total cost of sh. 600 000. She sells 150 of them at a profit of 25% and the remainder at a loss of 8%. Find the amount of her net profit and express it as a percentage of the initial cost of all the items.
7. A retailer buys 30 boxes of strawberries at sh. 3000 each and sells 27 boxes at 30% profit. How much profit does he make?
8. At a sale, the marked prices were reduced by 10%.
 - (a) How much would a buyer have to pay for a shirt marked sh. 25 000?
 - (b) If a customer pays sh. 48 000 for a dress. What is the marked price?

DAY 6

Discount

Sometimes a business reduces a small fraction of the selling price if the customer either pays cash or buys a lot of goods. This reduction is called discount. Most of the time, discounts are expressed as a percentage of the original price.

Example

A television set is selling at sh. 150 000. A customer is offered 4% discount for paying cash. How much does the customer pay for the TV?

Solution

Discount = 4% of sh. 150 000
 = sh. 6 000.

The customer pays sh. 150 000 – sh. 6 000 = sh. 144 000

Commission

Sometimes firms use agents who are not actually their employees to sell their goods. The agent is usually paid a commission for selling the goods. Commission is normally given as a percentage of the sales made.

Example

A saleslady sold goods worth sh. 750 000. She was paid 2% commission on the sales. How much commission did she get?

Value of goods sold = sh. 750 000

$$\begin{aligned}\text{Commission paid is} &= \frac{2}{100} \times 750000 \\ &= \text{sh. } 15\,000\end{aligned}$$

Interest

Simple interest

If you borrow money from a bank or other financial institution then you will have to pay interest (the charge paid for borrowing) in addition to your repayments. The money borrowed or lent is called the principal. When interest is paid at fixed intervals, yearly, half-yearly, quarterly or monthly, the principal is said to be lent (or borrowed) at simple interest.

The interest is calculated on the original principal only. The investor receives interest at regular periods, the principal remains the same. Simple interest is calculated using the following formula

$$\begin{aligned}\text{Interest} &= \frac{P \times R \times T}{100}\end{aligned}$$

Where, P = principal, R = rate of interest per annum (%); T = time (in years).

Note that the units for R and T must be consistent, i.e. If R is per annum, T must be in years,

If R is per month, T must be in months, e.t.c.

When the simple interest for any given time is added to the principal, the sum is called the amount at simple interest for that time. Amount = Principal + Interest, i.e. $A = P + I$.

Example

Find the simple interest on sh. 25 000 for 3.5 years at 18% per annum.

Solution

$$I = \frac{P \times R \times T}{100} = \frac{25000 \times 18 \times 3.5}{100} \\ = \text{sh. } 15\,750.$$

Example

Find the simple interest on sh. 20 000 for $1\frac{3}{4}$ years at $1\frac{1}{2}$ % per month. Find

also, the amount after $1\frac{3}{4}$ years.

Solution

Time = $1\frac{3}{4}$ years $\times 12 = 21$ months; $P = \text{sh. } 20\,000$; $R = 1\frac{1}{2}$ % per month

$$\text{Therefore, } \frac{20000 \times 1\frac{1}{2} \times 21}{100} = \text{sh. } 6\,300.$$

Amount = Principal + Interest

$$= 20\,000 + 6\,300 = \text{sh. } 26\,300.$$

Compound interest

In most financial institutions, interest is added to the money borrowed or lent and then the interest is calculated on this total amount for the next period. Adding the interest is known as compounding the interest, or just compound interest.

$$\text{Compound interest} = \text{Final amount} - \text{original principal.}$$

Note: Simple interest is the same for each period, compound interest becomes greater for successive periods.

Example

Calculate the compound interest on sh. 2 000 for 2 years at 8% per annum.

Solution

First year: Principal = 2 000

Interest = 160 calculated as

$$I = \frac{2000 \times 8 \times 1}{100} = 160$$

Amount = 2000 + 160 = 2 160 Second

year:

Principal = 2 160

Interest = 172.80, calculated as

$$I_2 = \frac{2160 \times 8 \times 1}{100} = 172.80$$

Amount = 2 160 + 172.80 = 2 332.80

Compound interest = Amount – Principal

$$= 2\,332.80 - 2\,000 = \text{sh. } 332.80$$

Alternatively, the compound interest can be calculated using the following formula:

$$A = P \left(1 + \frac{R}{100} \right)^n$$

where, A is the amount after n years; P = principal; R is the rate % p.a. and n is the number of years. If interest is added half yearly, the value of R is half of the given R% p.a. value and n is doubled.

So, when P = 2 000, R = 8 and n = 2,

$$\text{Amount} = 2000(1.08)^2 = 2\,332.80$$

Interest = Amount – principal

$$\begin{aligned} &= 2\,332.80 - 2000 \\ &= \text{sh. } 332.80 \end{aligned}$$

Appreciation and depreciation

Appreciation is a term used to describe an increase in value of an asset. On the other hand, depreciation is used to mean a decrease in value of an asset. The increase or decrease in value is usually expressed as a percentage of the original value.

Note: The depreciation is calculated as a percentage of the remainder after the first period's depreciation has been subtracted.

For example, if a car depreciates at the rate of 10% per year, car costing

Sh. 5 000 000 will depreciate by $\frac{10}{100} \times 5\,000\,000$ after 1 year.

Therefore, its value after 1 year = 5 000 000 – 500 000 = sh. 4 500 000.

After 2 years its depreciation will be $\frac{10}{100} \times 4\,500\,000 = 450\,000$

Therefore, its value after 2 years will be 4 500 000 – 450 000 = sh. 4 050 000.

(The value of the car after 2 years is not 5 000 000 – 50 000 – 50 000 = 4 900 000)

ACTIVITY

- As a result of a civil war, the population of a town decreased by 4% of its total population of 425 000 at the beginning of the year. Then after, it increased by 2% of its size at the beginning of the year for consecutive five years. Calculate, to the nearest thousand, its population at the end of this period.
Calculate also, correct to one decimal place, the percentage increase over this period.

2. Find the amount to which sh. 10 000 accumulates in 12 years at 9% per annum compound interest.
3. A savings and credit society granted a short-term loan of sh. 65 000 at 20% per month simple interest. Calculate the interest after 3 months.
4. The amount of an investment at 4% p.a. compound interest after 6 months is sh. 54 200. Calculate the principal.
5. A man loans a friend sh. 500 000 for 2 years at 7% compound interest . How much will he receive when the loan is repaid?
6. A car rental company hires out cars as follows: sh. 23 000 per day and sh. 1 000 per kilometer covered. They offer a discount of 40 km free each day of hire. A man hires a car for 5 days and drives for 350 km. Calculate the total cost.
7. The price of a car when new is sh. 14 000 000. After one year its market value depreciates by 15%. In each subsequent year it depreciates by 10% of its value by the beginning of the year. Find its value to the nearest shillings, at the end of three years.
8. A business man had sh. 1 200 000 and divided it in the ratio 3:2. He used the larger amount to buy a car, and invested the remainder in a bank which paid simple interest at a rate of 8% per annum. After 18 months, he sold the car at 30% less than what he bought it at. He also withdrew his money and interest from the bank. Calculate:
 - (a) the amount he invested in the bank.
 - (b) the amount for which he sold the car.
 - (c) the total amount he withdrew from the bank.
 - (d) the percentage of the sh. 1 200 000 he had after selling the car.
9. A car valued at sh. 8 900 000 is supposed to depreciate each year at 10% of its value at the beginning of the year. Find its value after three years.
10. A plot of land bought for sh. 5 000 000 appreciated by 12% in the first year and subsequently for 2 more years at 10%. Find its value after 3 years.
11. A man is offered a choice for his salary:
 - (a) Starting salary: Ush. 15 000 000 per year with an annual increase of 10% of the salary at the beginning of the year for three years.

- (b) Starting salary: Ush. 16 500 000 per year with an annual increase of 5% of the salary at the beginning of the year for 3 years.
- (i) what will be his total earnings in the first 3 years in each case?
- (ii) What will be his salary at the beginning of the fourth year in each case?
12. The value of the machinery in a factory depreciates each year by 25% of the value at the beginning of the year. If it was valued at sh. 750 000 after 3 years, find its value when new.
13. For a fixed deposit for 4 years, a bank offers the rates of interest: 10% in the first year, 12% in the second year and 15% in each subsequent year. Find the sum to which a fixed deposit of sh. 800 000 will amount after 4 years.
14. The price of a car was increased by 10% to sh.8,800,000.
- (i) What was the original price of the car?
- (ii) If Tom bought this car at this increased price and sold it a year later at 20% discount. Express Tom's selling price as a percentage of the original price.
15. Find the rate of compound interest levied on sh.8,000 when it has been deposited for 3 years and earned sh.27,000.

Hire Purchase

This is a system of payment where a customer is allowed to buy an item by paying part of the price in cash and then making a fixed payment each month for a number of months.

The first payment is called deposit or down payment, the monthly fixed payment is called monthly installment.

The hire purchase price is usually more than the marked price.

Example

The marked price of a gas cooker is sh. 450 000. A dealer charges 20% more under hire purchase. If the deposit is sh. 30 000, calculate the amount of monthly installments if there are 12 equal installments.

Solution

Marked price = sh. 450 000.

Hire purchase price = sh. 120% of sh. 450 000

$$= \frac{120}{100} \times 450000$$

$$= \text{sh. } 540\,000$$

$$\text{Deposit} = \text{sh. } 30\,000$$

$$\begin{aligned}\text{Monthly installments} &= \frac{540\,000 - 30\,000}{12} \\ &= \text{sh. } 42\,500\end{aligned}$$

Example

A colour TV set is available under hire purchase on payment of a deposit of sh. 20 000 and ten equal monthly installments of sh. 20 000 each. If the cash price is sh. 200 000, calculate what percentage goes the dealer charge extra over the cash price?

Solution

$$\text{Deposit} = 20\,000$$

$$\text{Installments} = 10 \times 20\,000 = 200\,000$$

$$\text{Hire purchase price} = \text{Deposit} + \text{Installments}$$

$$= 20\,000 + 200\,000 = 220\,000$$

$$\text{Cash price} = 200\,000$$

$$\text{Extra payment} = 220\,000 - 200\,000 = \text{sh. } 20\,000.$$

$$\text{Therefore, percentage} = \frac{20\,000}{200\,000} \times 100\% = 10\% \text{ of cash price.}$$

ACTIVITY

A sewing machine is sold under hire purchase: a deposit of sh. 25 000 and 12 monthly installments of sh. 16 500 each. If the hire purchase is 18% higher than the cash price, determine the cash price.

1. A dealer marks a price sh. 4 800 of an article. He charges 10% extra under hire purchase with a deposit of sh.800 and four equal monthly installments. Calculate the amount of monthly installments.

2. The cash price of a refrigerator is sh. 290 000 and its hire purchase price is 15% higher under 12 monthly installments of sh. 25 000 each. Determine the amount of deposit.

3. A dealer displays the following label on an article: Cash or down payment: sh. 250 000.
12 equal monthly installments of sh. 500 000.

If he is charging 25% higher than the cash price, determine the cash price of the article

4. The cash price of a bicycle is sh. 70 000. If the same is bought under HP terms, then there is a deposit of 10% and 12 monthly installments of sh. 7 000. Find the difference between the two prices.

5. A scientific calculator is marked at sh. 45 000. Under hire purchase it is available for a down payment of sh. 20 000 and six monthly installments of sh. 6 000 each. Calculate the
 - (a) hire purchase price
 - (b) extra amount paid over the cash price.

6. The marked price of a article is sh. 25 000. James bought it by paying a deposit of sh. 5 000 and a number of equal monthly installments of sh. 2 500. If the hire purchase price is 20% higher than the marked price, calculate the number of installments.

7. A cash discount of 10% is allowed on a refrigerator whose marked price is sh. 200 000. Hire Purchase terms are: a deposit of 20%, and 12 monthly installments of sh. 19 500 each. Find the difference between the cash price and the payment required under HP terms

Income Tax

This is a tax levied on peoples' incomes. Income is a payment received by someone who gets involved in a legal gainful activity. Examples include: Profit, salary, wages, interest, commission, fees, rent, overtime pay e.t.c.

Gross income is the total income that an individual receives from wages, salary, leave pay, overtime pay, medical allowance, transport allowance e.t.c.

Taxable income is the income on which income tax is levied. It is arrived at by deducting from the gross income the amount of allowances. Thus, taxable income = gross income – allowances

Example

In a certain country income tax is levied as follows: A person's monthly gross Income has certain allowances deducted from it before it is subjected to taxation.

Each child below ten years sh. 5 000

Each child above ten years but less than 18 years – sh. 7 000

Married man: sh. 18 000

Transport allowance: sh. 17 000

David earns sh. 450 000 per month, he is married with 3 children of ages between 2 and 10 years, 2 children above twelve but less than 18 years.

Calculate:

- (i) the taxable income of David
- (ii) the income tax he paid; if the rates are:

Taxable income (sh)	Rate(%)
0 – 100 000	10
100 001 – 200 000	20
200 001 – 300 000	30
300 001 – 400 000	45
400 001 - and above	50

Solution

- (i) David's allowances
 - Marriage 18 000
 - Transport 17 000

$$3 \text{ children} \quad 3 \times 5\,000 = 15\,000$$

$$\underline{2 \text{ children} \quad 2 \times 7\,000 = 14\,000}$$

$$\text{Total allowances} \quad 64\,000$$

$$\text{Taxable income} = 450\,000 - 64\,000$$

$$= \text{sh. } 386\,000$$

(ii) income tax (sh.)

$$100\,000 \quad 100\,000 \times \frac{10}{100} = 10\,000$$

$$100\,000 \quad 100\,000 \times \frac{20}{100} = 20\,000$$

$$100\,000 \quad 100\,000 \times \frac{30}{100} = 30\,000$$

$$86\,000 \quad 86\,000 \times \frac{45}{100} = 38\,700$$

$$\text{Total income tax} = \text{sh. } 98\,700$$

Example

The table below shows the income tax rate of a certain country for government employees. This is applied after the allowances have been already deducted.

Taxable income (sh)	Rate(%)
0 – 100 000	0
100 001 – 200 000	5
200 001 – 300 000	10
300 001 – 450 000	20
450 001 - 550 000	30
550 001 – and above	50

An employee has a gross monthly income of sh. 600 000 and is entitled to the following allowances.

Marriage: sh. 120 000 per annum

Housing and transport: 10% of the gross monthly income. Medical care: sh. 240 000 per annum.

Calculate:

- (i) the amount an employee pays as monthly income tax. (ii) the net monthly income.

Solution

- (i) Allowances:

$$\text{Marriage} \quad 120\,000 \div 12 = 10\,000$$

$$\text{Housing and transport} \quad \frac{10}{100} \times 600\,000 = 60\,000$$

$$\text{Medical care} \quad \text{sh. } 240\,000 \div 12 = 20\,000$$

$$\text{Total allowances} = 90\,000$$

$$\text{Taxable income} = \text{monthly gross income} - \text{monthly allowances}$$

$$= 600\,000 - 90\,000$$

$$= \text{sh. } 510\,000.$$

Income (sh.):	tax (sh.)
100 000	0

$$100\,000 \quad \frac{5}{100} \times 100\,000 = 5\,000$$

$$100\,000 \quad \frac{10}{100} \times 100\,000 = 10\,000$$

$$150\,000 \quad \frac{20}{100} \times 150\,000 = 30\,000$$

$$60\,000 \quad \frac{30}{100} \times 60\,000 = 18\,000$$

$$\text{Total monthly income tax} = 63\,000.$$

Therefore, the total monthly income tax is sh. 63 000

- (ii) Net monthly income = gross monthly income – income tax.

$$= 600\,000 - 63\,000$$

$$= \text{sh. } 537\,000.$$

ACTIVITY

1. In a certain country, income tax is computed after deducting the following allowances:

TYPE OF ALLOWANCE	AMOUNT (USH.)
Marriage	10 000
Single	4 000
Each child above 10 but below 20 years	3 000
Each child under 10 years	2 000

Omoja is married with 3 children, two below 10 years of age and the other child 12 years old. Mbili is single but has two dependants aged 11 and 15 years. Each month Omoja and Mbili earn gross incomes of sh. 130 000 and sh. 120 000 respectively. The income tax is calculated as follows:

Ush.	%age tax
1 st : 01 - 10 000	20
Next: 10 001 – 50 000	15
Rest: 50 001 – and above	10

- (a) Calculate the:
 - (i) taxable income for Omoja and Mbili,
 - (ii) income tax for Omoja and Mbili,
 - (b) Express the total income tax for each man as percentage of their respective taxable incomes.
2. In a certain country, tax is levied on income per month as follows:

Income per month (sh)	Tax Rate(%)
0 – 50 000	10
50 001 – 100 000	15
100 001 – 150 000	20
150 001 – 200 000	30
200 001 - 250 000	35
250 001 – 300 000	40
300 001 – 350 000	45
350 001 – and above	55

Tom's gross monthly income is sh. 766 000.

The allowances given to him are:

Housing allowance:	sh. 10 000 pe month
Marriage allowance:	sh. 919 200 per annum.
Medical allowances:	sh. 50 000 per month
Transport allowances:	sh. 120 000 per annum
Insurance premium:	sh. 72 000 per annum

Tom is married with 5 children: 2 above 10 years but below 18 years, 1 is below 10 years while 2 are above 18 years. The rate per child are as below:

Age	rate:
Below 10 years	sh. 3 000
Between 10 and 18 years	sh. 4 000
Above 18 years	sh. 5 000

(a) Calculate:

- (i) Tom's taxable income,
- (ii) the income tax Tom pays.

(b) Express his income tax as a percentage of his monthly income.

3. (a) Adikini bought a TV set for which the cash price was shs. 599,000.

She bought the TV set on hire purchase terms and had to pay an extra sh. 71 000.

If she made eight equal monthly instalments, how much did she pay per month?

(b) Mukasa wants to buy a house which is priced at sh. 56,000,000. A deposit of 25% of the value of the house is required. A bank will lend him the rest of the money at a compound interest of 15% per annum and payable after two years.

Calculate the:

- (i) deposit Mukasa must make.
- (ii) amount of money Mukasa will have to pay the bank after two years.
- (iii) total money which Mukasa will spend to buy the house.