O LEVEL REVISION

1. PHYSICS

2.CHEMISTRY

3.BIOLOGY

4.MATHEMATICS

5. ADDN. MATHS

CONTENTS

TOPIC 1 Physical Quantities, Units and Measurement

Objectives

Candidates should be able to:
(a) show understanding that all physical quantities consist of a numerical magnitude and a unit
(b) recall the following base quantities and their units: mass (kg), length (m), time (s), current (A), temperature (K), amount of substance (mol)
(c) use the following prefixes and their symbols to indicate decimal sub-multiples and multiples of the SI units: nano (n), micro (μ), milli (m), centi (c), deci (d), kilo (k), mega (M), giga (G)
(d) show an understanding of the orders of magnitude of the sizes of common objects ranging from a typical atom to the Earth
(e) state what is meant by scalar and vector quantities and give common examples of each
(f) add two vectors to determine a resultant by a graphical method
(g) describe how to measure a variety of lengths with appropriate accuracy by means of tapes, rules, micrometers and calipers, using a vernier scale as necessary
(h) describe how to measure a short interval of time including the period of a simple pendulum with appropriate accuracy using stopwatches or appropriate instruments

NOTES

1.1 Physical Quantities and SI Units

1. Physical quantities consist of:
   (a) Numerical magnitude – denotes the size of the physical quantity.
   (b) Unit – denotes the physical quantity it is expressing.

2. Physical quantities can be classified into:
   (a) Basic quantities

Basic Quantity	Name of SI Unit	SI Unit
length	metre	m
mass	kilogram	kg
time	second	s
thermodynamic temperature	kelvin	K
amount of substance	mole	mol

   (b) Derived quantities – defined in terms of the basic quantities through equations. SI units for these quantities are obtained from the basic SI units through the equations.

Example 1.1

Density = $\frac{\text{Mass}}{\text{Volume}}$ (Unit for mass: kg, Unit for volume: m^3)

Therefore unit for density = $\frac{kg}{m^3}$ = kg/m^3

3. (a) Units of measurements: SI units are used as standardised units in all measurements in the world. SI is the short form for “International System of Units”.
   (b) Other Units:

Length	Mass	Time
1 km = 1000 m	1 kg = 1000 g	1 h = 60 min
1 m = 100 cm	1 g = 1000 mg	1 min = 60 s
1 cm = 10 mm		

4. Examples of some derived quantities and their units:

Derived Quantity	SI Unit
area	m^2
volume	m^3
density	kg/m^3
speed	m/s

A complete list of key quantities, symbols and units used for the O Level examination can be found in the syllabus.

1.2 Prefixes, Symbols and Orders of Magnitude

1. Physical quantities can be very large, like 23 150 000 000 m, or very small, like 0.000 000 756 m. Writing down such numbers can be time consuming and error-prone. We use prefixes to indicate decimal sub-multiples and multiples of the SI units to make writing such numbers easier.

2. Some prefixes of the SI units are as follows:

Prefix	Multiple	Symbol	Factor	Order of Magnitude
Tera	1 000 000 000 000	T	10^{12}	12
Giga	1 000 000 000	G	10^{9}	9
Mega	1 000 000	M	10^{6}	6
Kilo	1000	k	10^{3}	3
Deci	0.1	d	10^{-1}	–1
Centi	0.01	c	10^{-2}	–2
Milli	0.001	m	10^{-3}	–3
Micro	0.000 001	μ	10^{-6}	–6
Nano	0.000 000 001	n	10^{-9}	–9
Pico	0.000 000 000 001	p	10^{-12}	–12

The ones in bold are specifically required in the syllabus.

Example 1.2

(a) 0.000 0031 m = 3.1 μm = 3.1×10^{-6} m

(b) 0.000 000 0012 s = 1.2 ns = 1.2×10^{-9} s

3. When measurements are too large or too small, it is convenient to express them in standard form as follows:
   $M \times 10^{N}$
   M lies in the range of: $1 \leqslant M < 10$
   N denotes the order of magnitude and is an integer.

4. Orders of magnitude are often being used to estimate numbers which are extremely large to the nearest power of ten.
   E.g.
   (a) Estimate the number of strands of hair on a person's head.
   (b) Estimate the number of breaths of an average person in his lifetime.

5. The following tables show how the orders of magnitude are used to compare some masses and lengths.

Mass/kg	Factor
Electron	10^{-30}
Proton	10^{-27}
Ant	10^{-3}
Human	10^{1}
Earth	10^{24}
Sun	10^{30}

Length/m	Factor
Radius of a proton	10^{-15}
Radius of an atom	10^{-10}
Height of an ant	10^{-3}
Height of a human	10^{0} ($10^{0} = 1$)
Radius of the Earth	10^{7}
Radius of the Sun	10^{9}

Example 1.3

Find the ratio of the height of a human to that of an ant.

Ratio of height of human to that of an ant = $\dfrac{10^{0}}{10^{-3}} = 10^{3} = 1000$.

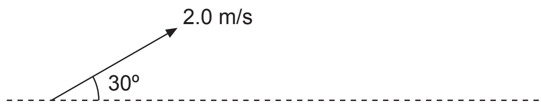
1.3 Scalars and Vectors

1. A scalar quantity – has only magnitude but does not have direction.
   E.g. mass, distance, time, speed, work, power.

2. A vector quantity – has both magnitude and direction.
   E.g. weight, displacement, velocity, acceleration, force.

Example 1.4

The velocity of a particle can be stated as: "speed of particle = 2.0 m/s and it is moving at an angle of 30° above the horizontal".

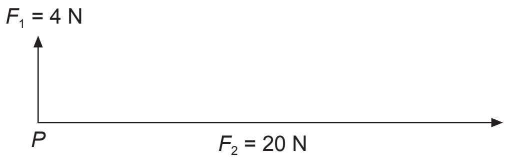


1.4 Addition of Vectors

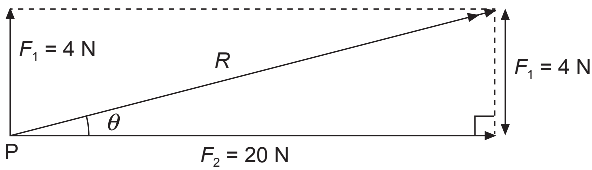
1. Involves magnitude and direction.

Example 1.5

Find the resultant force R at point P due to F_1 and F_2.



Method 1: Trigonometric Method



Using Pythagoras' Theorem:

$R = \sqrt{(F_1)^2 + (F_2)^2}$

$R = \sqrt{4^2 + 20^2} = \sqrt{416}$

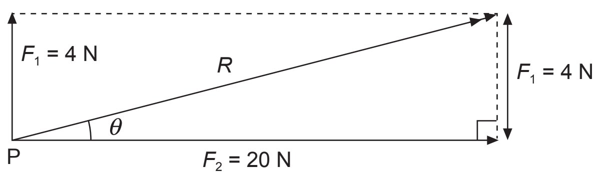
$R = 20.4$ N

R is at an angle θ above the horizontal

$\tan\ \theta = \dfrac{F_1}{F_2} = \dfrac{4}{20} = \dfrac{1}{5}$

$\theta = 11.3°$

Method 2: Graphical Method



(Not drawn to scale)

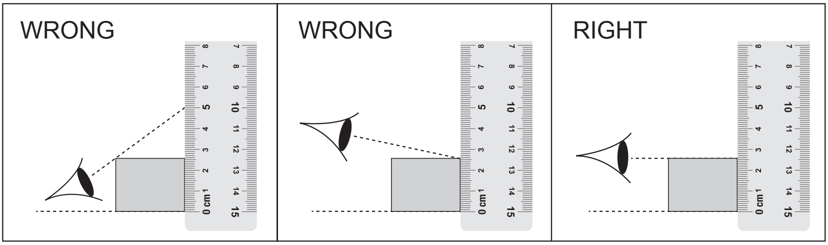
Step 1: Select an appropriate scale
E.g. 1 cm to 2 N.
Step 2: Draw a parallelogram of vectors to scale.
Step 3: Measure the diagonal to find R.
Step 4: Use the protractor to measure angle θ.

1.5 Measurement of Length

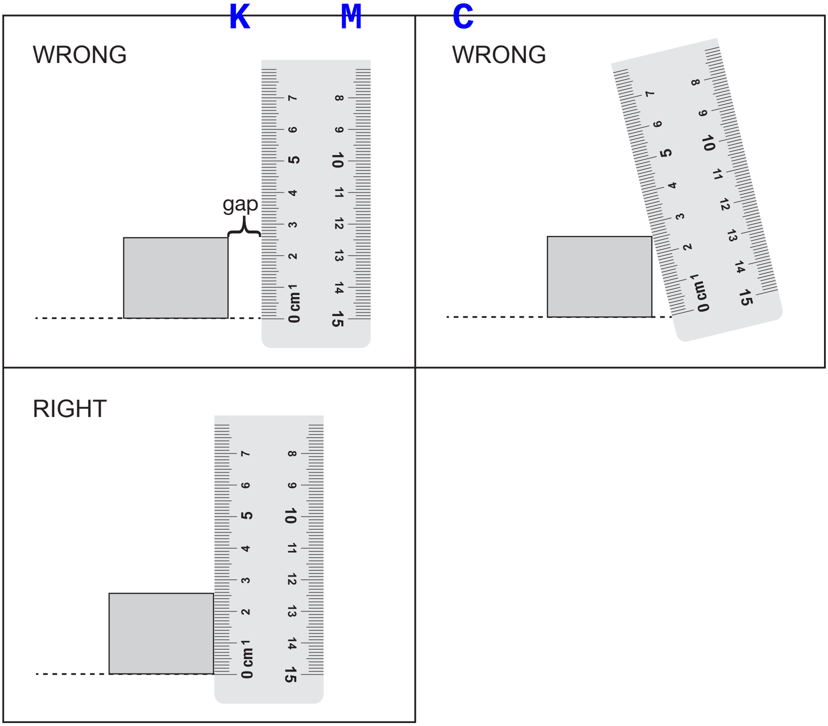
1. Choice of instrument depends on the degree of accuracy required.

Range of length, *l*	Instrument	Accuracy	Example
l > 100 cm	Measuring tape	± 0.1 cm	waistline of a person
5 cm < *l* < 100 cm	Metre rule	± 0.1 cm	height of an object
1 cm < *l* < 10 cm	Vernier calipers	± 0.01 cm	diameter of a beaker
l < 2 cm	Micrometer screw gauge	± 0.001 cm	thickness of a length of wire

2. How parallax errors can occur during measurement:
   (a) incorrect positioning of the eye



   (b) the object is not touching the marking of the scale
   (for measuring tape and metre rule, ensure that the object is **in contact** with the scale)



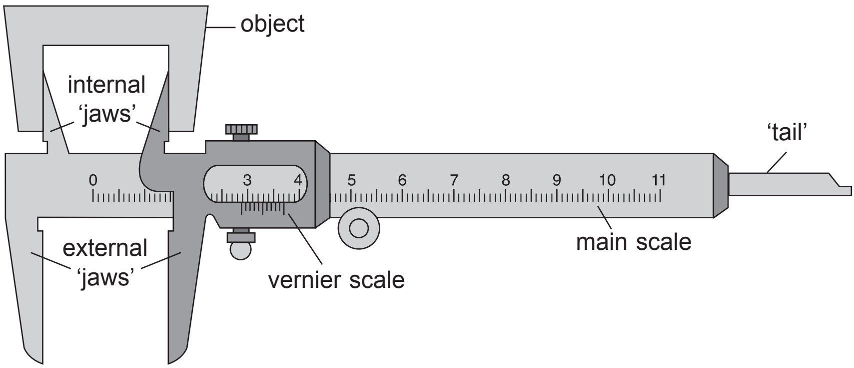
3. A measuring instrument can give precise but not accurate measurements, accurate but not precise measurements or neither precise nor accurate measurements.

   (a) Precision is how close the measured values are to each other but they may not necessarily cluster about the true value. Zero errors and parallax errors affect the precision of an instrument.

   (b) Accuracy is how close a reading is to the true value of the measurement. The accuracy of a reading can be improved by repeating the measurements.

4. **Vernier calipers**

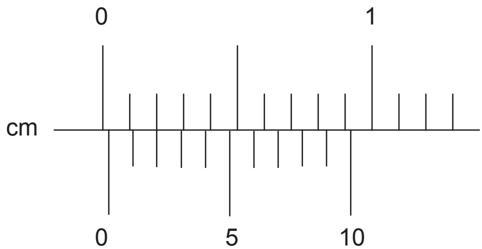
A pair of vernier calipers can be used to measure the thickness of solids and the external diameter of an object by using the external jaws. The internal jaws of the caliper are used to measure the internal diameter of an object. The tail of the caliper is used to measure the depth of an object or a hole. Vernier calipers can measure up to a precision of ±0.01 cm.



Precautions: Check for zero error and make the necessary correction.

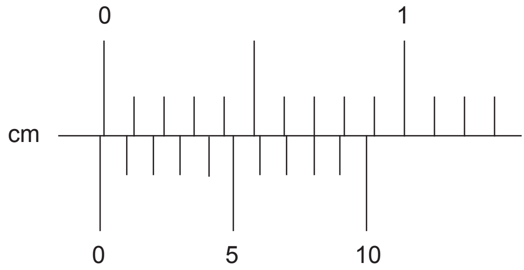
Example 1.6

(a) Positive zero error:



Zero error = +0.02 cm

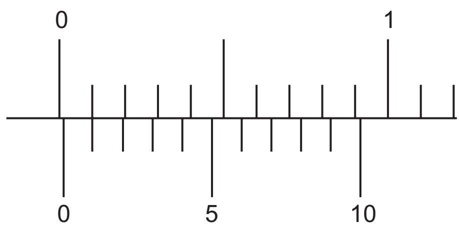
(b) Negative zero error: K M C



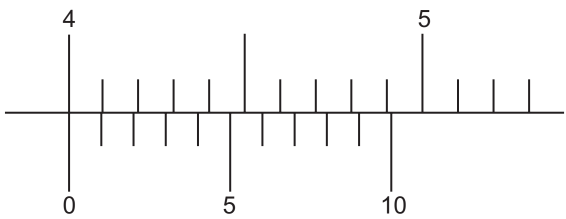
Zero error = –0.02 cm

Note: In (b), the pair of vernier calipers is built with an existing zero error. There is a negative reading without any object between its jaws. The vernier scale is pushed 0.02 cm to the left.

Example 1.7
When the jaws of a pair of vernier calipers are closed, the vernier caliper reading is as shown.



When the same pair of vernier calipers is used to measure the diameter of a beaker, the vernier caliper reading is as shown.

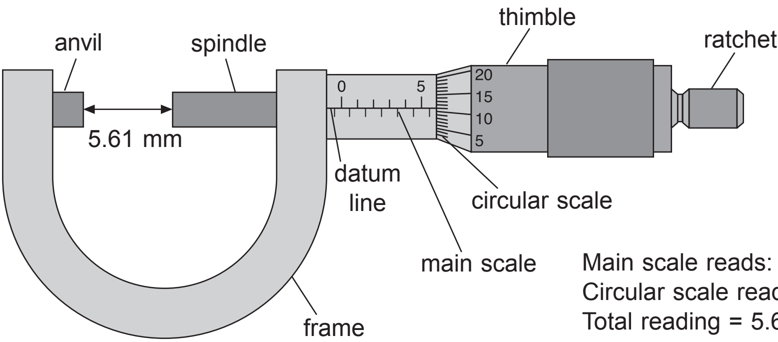


What is the diameter of the beaker?

Solution

Zero Error = +0.01 cm
Reading = 4.00 + 0.01 = 4.01 cm
Actual reading = 4.01 − 0.01 = 4.00 cm

5. Micrometer screw gauge

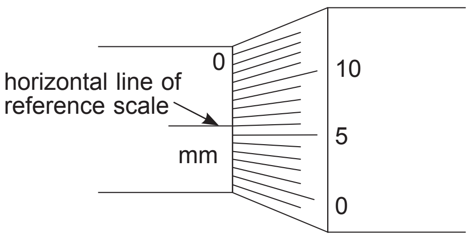


Main scale reads: 5.50 mm
Circular scale reads: 0.11 mm
Total reading = 5.61 mm

Precautions:
(a) Ensure that the jaws of the micrometer screw gauge are completely closed by turning the ratchet until you hear a 'click' sound.
(b) Check that the '0' mark of the thimble scale is completely in line with the horizontal line of the reference scale. If not, there is zero error.

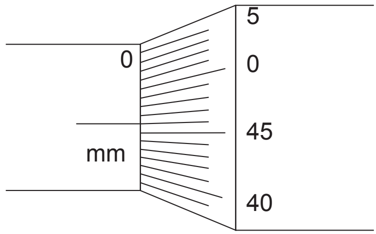
Example 1.8

(a) Positive zero error: '0' mark is below the horizontal line



Zero error = +0.06 mm

(b) Negative zero error: '0' mark is above the horizontal line



Zero error = –0.04 mm

Example 1.9

A micrometer screw gauge is used to measure the thickness of a plastic board. When the jaws are closed without the plastic board in between, the micrometer reading is shown in Fig. (a).

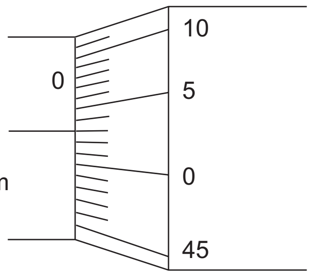


Fig. (a)

With the jaws closed around the plastic board, the micrometer reading is shown in Fig. (b).

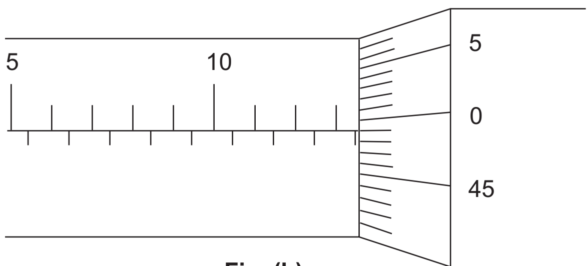


Fig. (b)

What is the thickness of the plastic board?

Solution

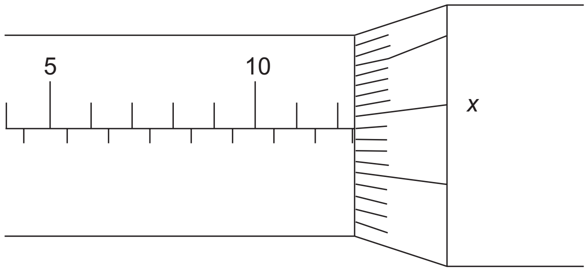
Zero error = +0.03 mm

Reading = 13.5 + 0.49 = 13.99 mm

Actual thickness of plastic board = 13.99 – (+0.03) = 13.96 mm

Example 1.10

The micrometer reading as shown in the figure is 12.84 mm.



What is the value of *x* on the circular scale?

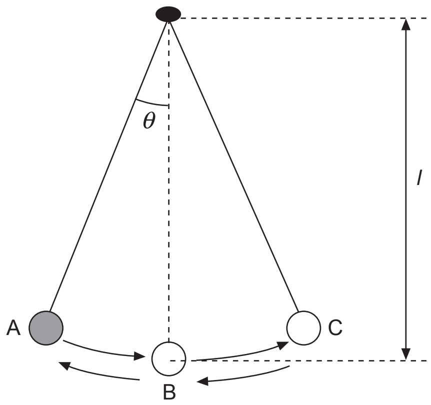
Solution

Reading = 12.5 + reading on the circular scale = 12.84 mm

Reading on the circular scale = 12.84 – 12.5 = 0.34 mm

Since the marking *x* is 1 mark above 0.34 mm, the value of *x* is 35.

6. Period of oscillation of a simple pendulum.



(a) (i) One oscillation – One complete to-and-fro movement of the bob from point A to B to C and back to A.

(ii) Period, *T* – Time taken for one complete oscillation.

(iii) Amplitude – The distance between the rest position of the bob (point B) to the extreme end of the oscillation (either point A or point C).

(b) Steps to find the period of oscillation:

Step 1: Take the total time for 20 oscillations.

Step 2: Repeat **Step 1**.

Step 3: Take the average of the two timings.

Step 4: Divide the average in **Step 3** by 20 to obtain the period.

(c) The period of the pendulum, T, is affected only by its length, l, and the acceleration due to gravity, g.

$$T = 2\pi\sqrt{\frac{T}{g}}$$

T is not affected by the mass of the pendulum bob.

Example 1.11

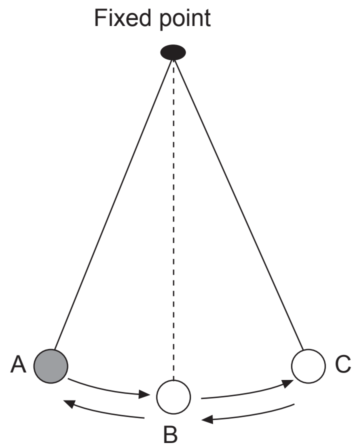
A pendulum swings backwards from B to A and forwards to C passing through B, he middle point of the oscillation. The first time the pendulum passes through B, a stopwatch is started. The thirtieth-time the pendulum passes through B, the stopwatch is stopped and the reading taken is 25.4 seconds. What is the period of the pendulum?

Solution

$$\text{Period} = \frac{\text{Total time taken}}{\text{Number of oscillations}}$$

$$= \frac{25.4}{15}$$

$$= 1.69 \text{ s}$$



TOPIC 2 Kinematics

Objectives

Candidates should be able to:

(a) state what is meant by speed and velocity
(b) calculate average speed using *distance travelled / time taken*
(c) state what is meant by uniform acceleration and calculate the value of an acceleration using *change in velocity / time taken*
(d) interpret given examples of non-uniform acceleration
(e) plot and interpret a displacement-time graph and a velocity-time graph
(f) deduce from the shape of a displacement-time graph when a body is:
  (i) at rest
  (ii) moving with uniform velocity
  (iii) moving with non-uniform velocity
(g) deduce from the shape of a velocity-time graph when a body is:
  (i) at rest
  (ii) moving with uniform velocity
  (iii) moving with uniform acceleration
  (iv) moving with non-uniform acceleration
(h) calculate the area under a velocity-time graph to determine the displacement travelled for motion with uniform velocity or uniform acceleration
(i) state that the acceleration of free fall for a body near to the Earth is constant and is approximately 10 m/s^2
(j) describe the motion of bodies with constant weight falling with or without air resistance, including reference to terminal velocity

NOTES..

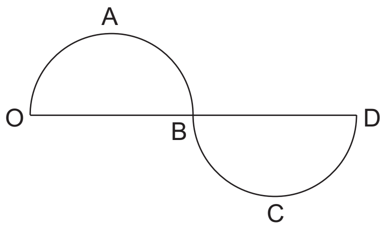
2.1 Distance vs Displacement and Speed vs Velocity

1.

Scalar	Vector
Distance	Displacement
Speed	Velocity

Example 2.1

A car travelled from point O to D along the curved path OABCD.



The distance travelled by the car is OABCD.
The displacement of the car from point O is OD (to the right of O).

2. When measuring/ calculating the displacement of an object, one has to include its starting point.

Example 2.2

Wrong: "The displacement of the bus is 500 m." (*500 m from where?*)

Right: "The displacement of the bus from point A is 500 m in the backward direction." or "The displacement of the bus from point A is –500 m (taking the forward direction as positive)."

3. (a) The formula for calculating speed is

$$\text{Speed} = \frac{\text{Distance travelled}}{\text{Time taken}}$$

(b) $\text{Average speed} = \dfrac{\text{Total distance travelled}}{\text{Total time taken}}$

(c) Velocity is the rate of change of displacement of an object from a fixed point (displacement per unit time).

(d) $\text{Average velocity} = \dfrac{\text{Resultant displacement from a fixed point}}{\text{Total time taken}}$

The average velocity v_{avg} of an object moving through a displacement (Δx) along a straight line in a given time (Δt) is:

$$v_{avg} = \frac{\Delta x}{\Delta t}$$

where $\Delta x = x_{final} - x_{initial}$

$x_{initial}$: Initial displacement from starting point
x_{final} : Final displacement from starting point

2.2 Acceleration

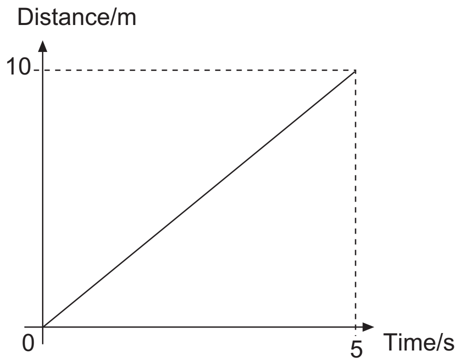
1. Acceleration is the rate of change of velocity.
2. $a = \dfrac{\Delta x}{\Delta t} = \dfrac{v - u}{\Delta t} =$ where v is the final velocity, u is the initial velocity and Δt is the time taken.
3. Acceleration is a vector quantity. (You need to give both the magnitude and direction when writing down the answer.)

2.3 Graph of Distance vs Time

1. The distance-time graph of a moving object along a straight road is used to find its speed.
2. The gradient of the graph gives the speed of the object.

Example 2.3

Object moving at uniform speed



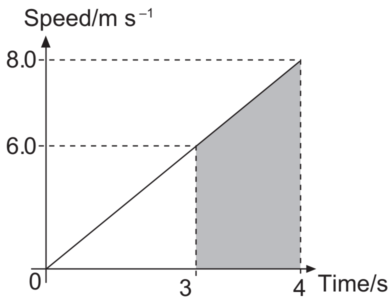
$$\text{Speed} = \frac{10 - 0}{5 - 0} = \frac{10}{5} = 2 \text{ m/s or } 2 \text{ m s}^{-1}$$

2.4 Graph of Speed vs Time

1. The speed-time graph of a moving object along a straight road is used to find:
   (a) Acceleration (Using the gradient of graph)
   (b) Distance travelled (Using the area under the graph)

Example 2.4

Object moving with uniform acceleration:



Acceleration = $\dfrac{8.0 - 0.0}{4 - 0}$ = 2 m/s^2 or 2 m s^{-2}

Distance travelled from t = 0 to t = 4 s

= $\dfrac{1}{2}$(4 – 0)(8.0 – 0.0) = 16 m

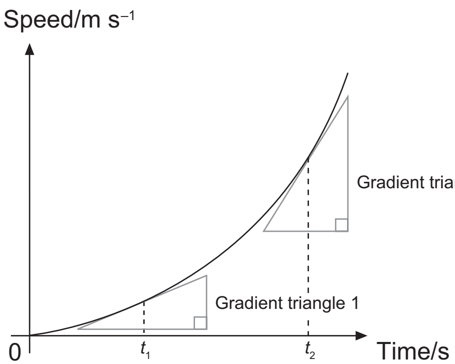
Distance travelled from t = 3 to t = 4 s

= $\dfrac{1}{2}$(4 – 3)(8.0 + 6.0) = 7 m

2. For an object moving with constant acceleration, the speed-time graph is a sloping straight line. A constant acceleration means that speed is increasing at a constant rate.

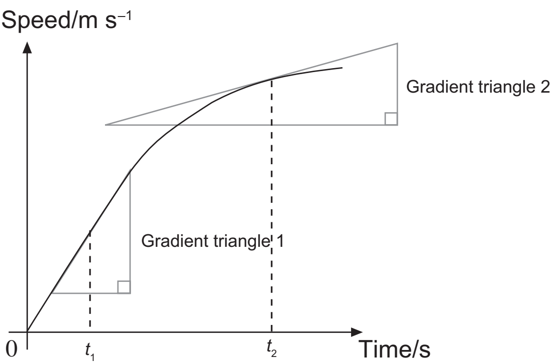
2.5 Interpret Other Speed-Time Graphs (Non-Uniform Acceleration)

1. Increasing acceleration:



Notice that the gradient of the graph becomes steeper.
The gradient of triangle 2 is larger than the gradient of triangle 1.
(Gradient gets more and more positive).
The **speed is increasing** with **increasing acceleration** (increasing rate).
At time = t_1, acceleration = a_1.
At time = t_2, acceleration = a_2.
$a_2 > a_1$

2. Decreasing acceleration:



Notice that the gradient of the graph becomes less steep.
The gradient of triangle 2 is smaller than the gradient of triangle 1.
(Gradient gets less and less positive).
The **speed is increasing** with **decreasing acceleration** (decreasing rate).
At time = t_1, acceleration = a_1.
At time = t_2, acceleration = a_2.
$a_2 < a_1$

2.6 Acceleration Due to Free-Fall

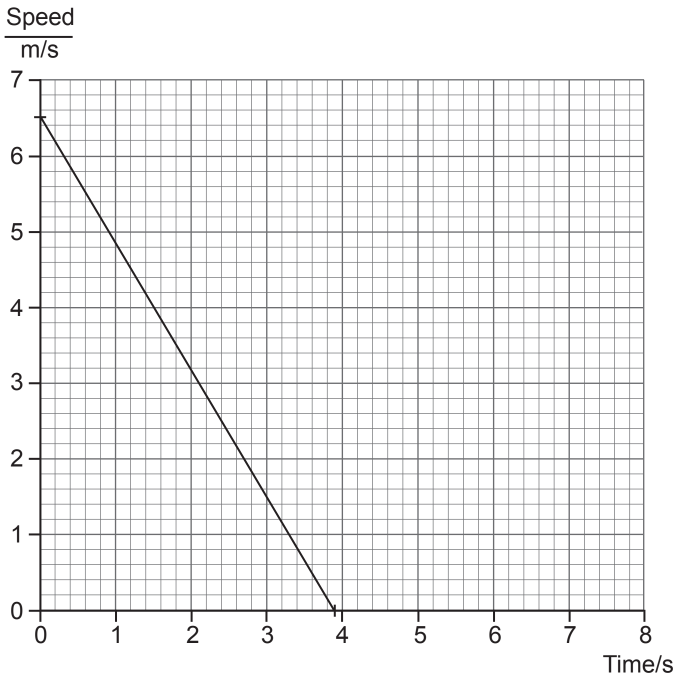
Near the surface of the earth, the acceleration of free fall for an object is constant and is approximately 10 m/s^2. When an object drops from the top of a building, its speed will increase from 0 m/s uniformly at a rate of 10 m/s per second.

2.7 Effect of Air Resistance

1. In real life, a falling object will encounter air resistance on Earth, unless it is moving in a vacuum.
2. Air resistance acts against the motion of the object increasingly to reduce its downward acceleration (NOT SPEED) to zero.
3. When the air resistance increases till it is equal to the weight of a falling object, the acceleration of the object is zero.
4. With zero acceleration, the object will continue falling downward at a constant velocity.
5. The constant velocity of the object is known as “terminal velocity”.

Example 2.5 **K M C**

An astronaut standing on the Moon's surface throws a rock vertically upwards. The figure shows the speed-time graph of the rock where at t = 0 s, the rock just leaves the astronaut's hand. Air resistance on the Moon can be neglected.



(a) (i) What is the time taken for the rock to reach its maximum height?

(ii) What is the total distance travelled by the rock when it returns to its initial position?

(iii) Find the acceleration of the rock.

(b) The rock is then brought back to the Earth's surface and the astronaut repeats the same action as on the Moon. Determine whether the speed-time graph of the rock, when it is thrown on Earth, will be different. Explain your answer.

Solution

(a) (i) From the graph, the time taken for the rock to reach its maximum height is 3.90 seconds.

(ii) Total distance travelled = 2 × area under the graph

$$= \left(\frac{1}{2} \times (6.50 - 0.00) \times (3.90 - 0.00)\right) \times 2$$

$$= 25.4 \text{ m (to 3 s.f.)}$$

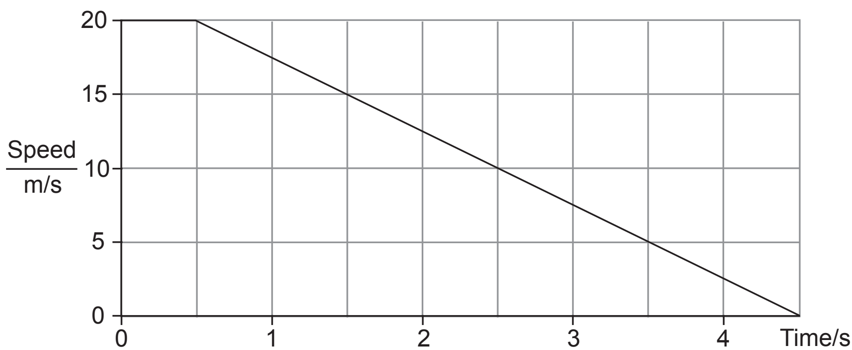
(iii) Acceleration of rock $= \dfrac{6.50 - 0.00}{0.00 - 3.90}$

$= -1.67$ m s^2 or $-$ 1.67 m s^{-2} (to 3 s.f.)

(b) The speed-time graph of the rock on Earth is different because the speed of the rock decreases as it falls from a height. This is due to air resistance. The speed of the rock is decreasing at an increasing rate. The deceleration of the rock increases as the speed decreases. Hence, the gradient of the speed-time graph is steeper initially and becomes gentler after some time. The sketch of the speed-time graph is a curve and not a straight line.

Example 2.6

The graph shows the speed of a car from the time the driver saw an obstacle on the road and applied the brakes till the car came to a stop.



(a) How long did it take the driver to begin applying the brakes after seeing the obstacle?

(b) Calculate the distance travelled
   (i) before the brakes were applied,
   (ii) while the brakes were being applied.

(c) Calculate the average speed of the car.

Solution

(a) The speed remains at 20 m/s for the first 0.5 seconds, so the driver took 0.5 seconds to begin applying the brakes after seeing the obstacle.

(b) (i) Distance travelled before braking

$= 20 \times 0.5$

$= 10$ m

(ii) Distance travelled while the brakes were being applied

$= \frac{1}{2} \times 20 \times (4.5 - 0.5)$

$= 40$ m

(c) Average speed of car $= \dfrac{\text{Total distance travelled}}{\text{Total time taken}}$

$= \dfrac{10 + 40}{4.5}$

$= \dfrac{50}{4.5}$

$= 11.1$ m/s or 11.1 m s^{-1} (to 3 s.f.)

TOPIC 3 Dynamics

Objectives

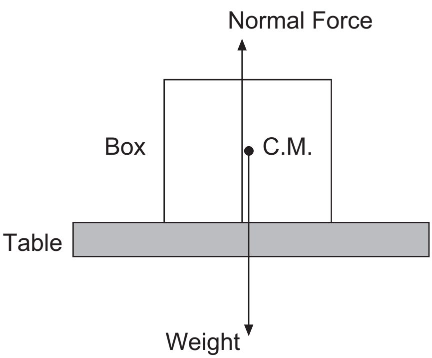
Candidates should be able to:

(a) apply Newton's Laws to:
(i) describe the effect of balanced and unbalanced forces on a body
(ii) describe the ways in which a force may change the motion of a body
(iii) identify action-reaction pairs acting on two interacting bodies (stating of Newton's Laws is not required)
(b) identify forces acting on an object and draw free body diagram(s) representing the forces acting on the object (for cases involving forces acting in at most 2 dimensions)
(c) solve problems for a static point mass under the action of 3 forces for 2-dimensional cases (a graphical method would suffice)
(d) recall and apply the relationship *resultant force = mass × acceleration* to new situations or to solve related problems
(e) explain the effects of friction on the motion of a body

NOTES

3.1 Forces

1. A force (SI unit: Newton, symbol: N) is a push or a pull exerted on a body by another body, i.e. an object resting on a table will have a contact force (normal force) acting on it upwards. This force is equal to its weight.



Note:
1. The Normal Force and Weight arrows are of the same length but in opposite directions.
2. Normal Force arrow starts from the base of box (contact between the box and the table top).
3. Weight starts from the centre of mass of the box, C.M. (indicated by the black dot).

2. Effects of a force on a body:
   (a) Increase/ decrease speed of a body (accelerate/ decelerate)
   (b) Change direction of a moving body

3. Newton's First Law:
   A body will remain stationary or in continuous linear motion unless acted upon by a resultant force.

4. Newton's Second Law:
   Resultant vector sum of forces on body is given by:

$$\boxed{F = ma}$$

   where m is the mass of the body and a is the acceleration of the body in the direction of F.

5. Newton's Third Law:
   For every action, there is an equal and opposite reaction.

3.2 Balanced and Unbalanced Forces

1. Balanced forces: If resultant $F = 0$ N, the body is either stationary or moving with constant velocity.

   Example 3.1
   A parachutist falls to the ground at terminal velocity when his weight is equal to the upward force acting on him due to air resistance. Hence, the resultant force acting on him is zero, i.e. his acceleration is zero.

2. Unbalanced forces: If resultant $F \neq 0$ N,
   (a) a stationary body will start moving,
   (b) a moving body will change its velocity.

3.3 Friction

1. Friction is the force which opposes motion when objects slide over each other. For a moving object, the friction on the object acts in the direction opposite to its motion.

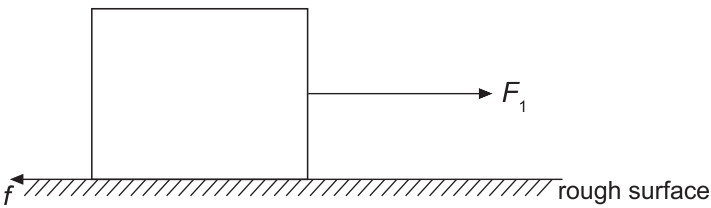
2. Advantages of friction:
   (a) Walking on roads.
   (b) Friction in brake pads and wheels of cars and bicycles.

3. Disadvantages of friction:
   (a) Wears down moving parts of machines.
   (b) For an object moving on a rough surface, more energy is needed to move the object as compared to moving on a smooth surface.
   (c) For an object moving on a rough surface, energy is required for the object to maintain a constant speed. Otherwise, it will slow down and come to a stop.

4. Ways to overcome friction:
   (a) Use lubricant (i.e. graphite or oil) for moving parts of machines.
   (b) Use ball-bearings between moving surfaces.
   (c) Make sure moving parts of machines have very smooth surfaces.

Example 3.2

An object weighing 50 N lies on a rough surface. A constant F_1 force of 12 N acts on the object. The frictional force f acting on the object is 2 N. Find the acceleration of the object. (Take acceleration due to gravity to be 10 m/s^2.)



Solution

Vertically, resultant force = normal force – weight = 0 N

Horizontally, resultant force $R = F_1 - f = 12 - 2 = 10$ N

(Object will only accelerate on horizontal plane)

Mass of object $= \frac{50}{10}$ kg

$= 5$ kg

Using formula:

$F = ma$

$10 = 5a$

$a = 2$ m/s^2

(Object is accelerating at 2 m/s^2 to the right, i.e. in the direction of F.)

Example 3.3 K M C

An object moves in a circular path at a constant speed. Is the object accelerating?

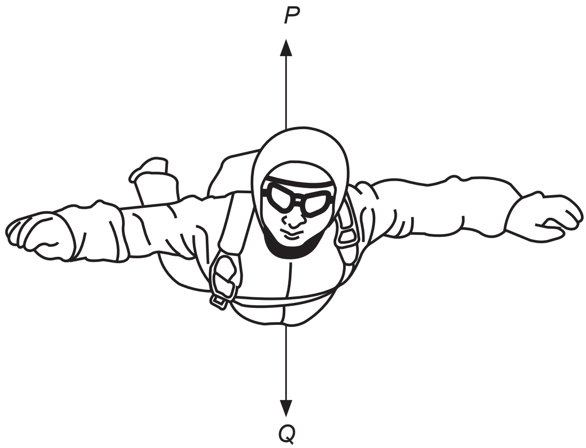
Solution

Yes. Its velocity keeps changing (because direction keeps changing), hence there is a resultant force causing the change. Resultant force acts towards the centre of the circle.

Example 3.4

A skydiver of mass 60 kg falls from rest vertically downwards at a constant velocity.

The figure shows the forces, P and Q, acting on him.



(a) Identify the forces P and Q acting on the skydiver.

(b) Explain why P is acting upwards.

(c) When the skydiver starts to fall from rest, the forces P and Q are unbalanced.

  (i) Find P and Q at $t = 0$ s.

  (ii) Find P and Q when the velocity of the skydiver is uniform.

  (iii) Describe, in terms of the forces acting on the sky diver, why the velocity of the skydiver increases before reaching terminal velocity.

Solution

(a) P is the air resistance on the skydiver and Q is the weight of the skydiver.

(b) Air resistance opposes the motion of the skydiver. Since the skydiver is falling vertically downwards, the air resistance acting on him is in the upward direction to oppose his motion.

(c) Take all forces acting downwards as positive.

(i) $P = 0$ N

$Q = mg = 60 \times 10 = 600$ N

(ii) When the velocity of the skydiver is uniform, he has reached terminal velocity. The resultant force acting on him is 0 N.

$Q - P = 0$

$P = Q = 600$ N

(iii) As a result of unbalanced forces, there will be a non-zero resultant force acting on the skydiver, and it is acting vertically downwards. By Newton's 2nd Law, the skydiver is accelerating downwards. Hence, the velocity of the skydiver increases before it reaches terminal velocity.

TOPIC 4 Mass, Weight and Density

Objectives

Candidates should be able to:

(a) state that mass is a measure of the amount of substance in a body
(b) state that mass of a body resists a change in the state of rest or motion of the body (inertia)
(c) state that a gravitational field is a region in which a mass experiences a force due to gravitational attraction
(d) define gravitational field strength, *g*, as gravitational force per unit mass
(e) recall and apply the relationship *weight* = *mass* × *gravitational field strength* to new situations or to solve related problems
(f) distinguish between mass and weight
(g) recall and apply the relationship *density* = *mass* / *volume* to new situations or to solve related problems

NOTES

4.1 Mass

1. Defined as a measure of the amount of substance in a body. (SI unit: kilogram, symbol: kg)
2. The magnitude of mass depends on the size of the body and the number of atoms in the body.
3. Mass is a scalar quantity.

4.2 Inertia

1. Defined as the resistance of the body to change in its state of rest or motion due to its mass.
2. To overcome inertia of a body, a force has to be applied.
   This force is dependent on the body's mass.

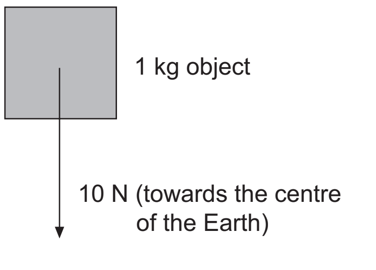
4.3 Gravitational Field Strength

Defined as the gravitational force acting on a body per unit mass.

	Gravitational Field Strength
Earth	$10\ N\ kg^{-1}$
Moon	$1.6\ N\ kg^{-1}$

Note:
These are approximate values for points close to and on the planets' surface.

i.e. on Earth, a force of 10 N is pulling on a 1 kg falling object.



Note:
Since the resultant force on object is 10 N (weight), the acceleration of the object is (by Newton's 2nd Law) $10\ m\ s^{-2}$.

4.4 Weight

1. Defined as the gravitational force W acting on an object of mass m.
2. When a body falls, its gravitational force (weight) can produce an acceleration, g (the acceleration due to gravity).
3. Using Newton's 2nd Law of $F = ma$, we have $W = mg$.
4. Comparison of weight and mass:

		Mass	Weight
1.	definition	the amount of substance in a body	the gravitational pull acting on a body
2.	depends on location	no	yes
3.	measured by using	beam balance	spring balance
4.	unit	kilogram	Newton

4.5 Density

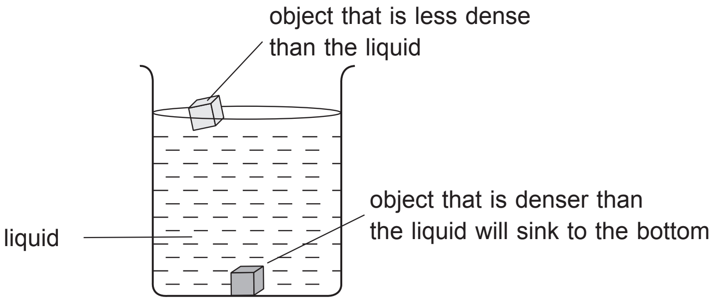
1. The density of a body, ρ, is defined as its mass, m, per unit volume, V.

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}}$$

$$\rho = \frac{m}{V}$$

2. SI Unit: kg m^{-3}

3. For an object to float in a liquid, the object has to be less dense than the liquid. As such, if an object is denser than the liquid, the object will sink in the liquid.



TOPIC 5 Turning Effect of Forces

Objectives

Candidates should be able to:

(a) describe the moment of a force in terms of its turning effect and relate this to everyday examples
(b) recall and apply the relationship *moment of a force (or torque) = force × perpendicular distance from the pivot* to new situations or to solve related problems
(c) state the principle of moments for a body in equilibrium
(d) apply the principle of moments to new situations or to solve related problems
(e) show understanding that the weight of a body may be taken as acting at a single point known as its centre of gravity
(f) describe qualitatively the effect of the position of the centre of gravity on the stability of objects

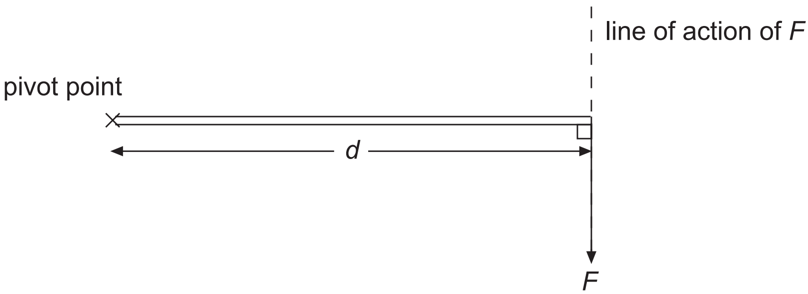
NOTES

5.1 Moment of a Force

1. Moment – the turning effect of a force about a pivoting point
2. Moment of force = $F \times d$

   F: Force

   d: perpendicular distance of line of action of F from pivot



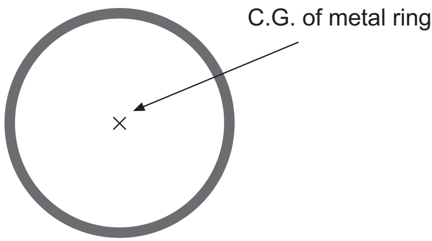
3. SI unit of moment: N m

4. Conditions for object to be in equilibrium:
   (1) The sum of moments about any point is zero. (Principle of Moments)
   (2) The vector sum of forces on object is zero.

5. Principle of Moments:
   For an object in equilibrium, the sum of clockwise moments about any point is equal to the sum of anticlockwise moments about the same point.
   (Resultant moment = 0 N m)

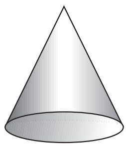
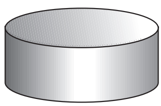
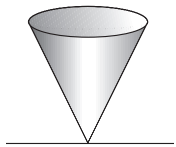
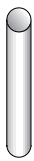
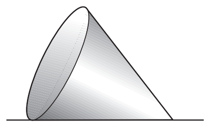
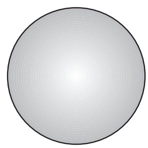
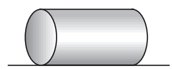
5.2 Centre of Gravity (C.G.)

1. The C.G. of an object is the point where the whole weight appears to act on.
2. The C.G. will not change regardless of how the object is orientated.
3. The C.G. can lie outside an object.
   E.g. C.G. of a metal ring is in the middle of the circle.



5.3 Stability

1. Stability – the ability of an object to retain its original position after being displaced slightly.

	Stable	Unstable	Neutral
Base Area	Large	Small	1 line of contact or point(s) of contact with surface
Height of C.G.	Low	High	–
Slight displacement from equilibrium position	Return to original position	Topple over	Stay in new position
Example	 Cone resting on its base  Cylindrical shape, resting on its base (large base)	 Cone at its vertex  Cylindrical shape, resting on its base (small base)	 Cone resting on its side  Sphere  Cylinder resting on its side

2. The stability of an object can be improved by:
   (a) Lowering its C.G. (Add weights to the object's lower part).
   (b) Increasing the base area of the object.

TOPIC 6 Pressure

Objectives

Candidates should be able to:

(a) define the term pressure in terms of force and area
(b) recall and apply the relationship *pressure* = *force* / *area* to new situations or to solve related problems
(c) describe and explain the transmission of pressure in hydraulic systems with particular reference to the hydraulic press
(d) recall and apply the relationship pressure due to a *liquid column* = *height of column* × *density of the liquid* × *gravitational field strength* to new situations or to solve related problems
(e) describe how the height of a liquid column may be used to measure the atmospheric pressure
(f) describe the use of a manometer in the measurement of pressure difference

NOTES...

6.1 Pressure

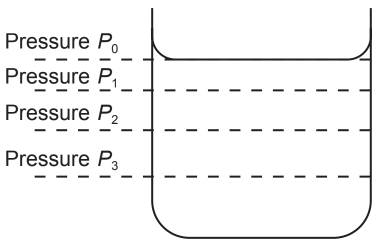
1. Pressure is the force acting per unit area.

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}}$$

2. SI unit: Pascal (Pa) or N m^{-2}

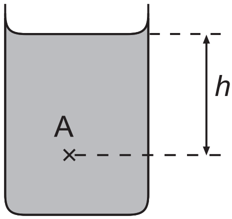
6.2 Liquid Pressure

1. An object immersed in a uniform liquid will experience a pressure which depends only on the height of the liquid above the object.



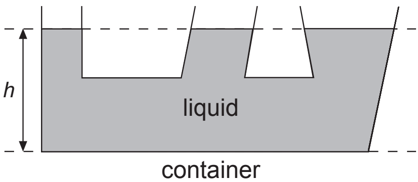
P_0: Atmospheric pressure
Pressure increases with depth: $P_0 < P_1 < P_2 < P_3$

2. Pressure at point A due to the liquid, $P = \rho g h$



Pressure at point A = $P_0 + \rho g h$
P_0 – Atmospheric pressure

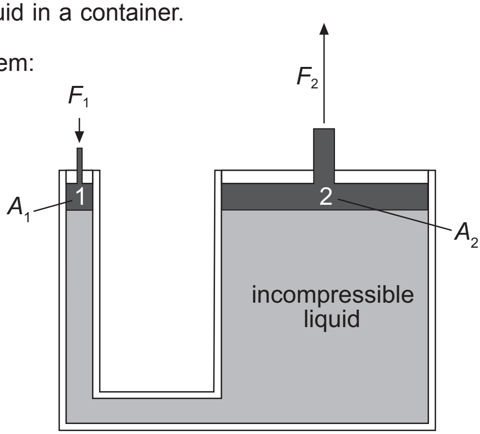
3. When a liquid is at equilibrium, the pressure is the same at any point along the same horizontal surface. Thus the liquid in the container settles at a common height, h.



6.3 Transmission of Pressure in Hydraulic System

1. Pressure can be transmitted in all directions if it is exerted on an incompressible fluid in a container.

2. Components of a hydraulic system:
   - Container with two openings
   - A press
   - A piston
   - Incompressible fluid



Hydraulic System

3. In the above figure, if the two pistons at '1' and '2' have the same area, then the force F_1 exerted on one piston will have the same magnitude as F_2 at the other piston.

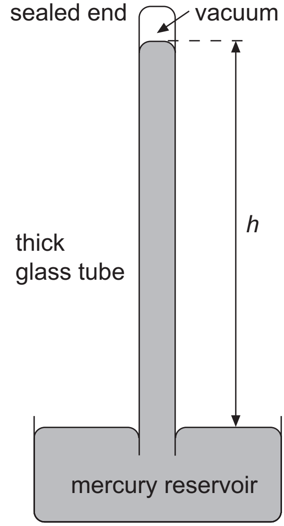
4. If the area A_1 is smaller than area A_2, then the force exerted at '1' will produce a larger force at '2'.

$$\text{Pressure} = P = \frac{F_1}{A_1} = \frac{F_2}{A_2}$$

5. Thus we can use a hydraulic system to lift heavy objects.

6.4 Atmospheric Pressure

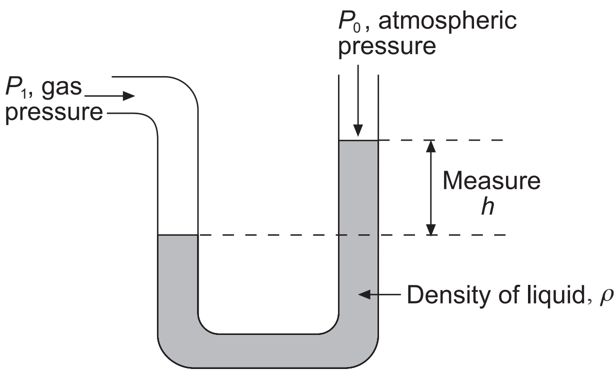
1. Defined as the force per unit area exerted against a surface by the weight of air above that surface.
2. Instrument to measure atmospheric pressure: mercury barometer
3. At sea-level, h = 760 mm.
   Atmospheric pressure recorded as 760 mm Hg.
4. Even if the tube is tilted, h will still remain the same unless it is brought to a different level where the atmospheric pressure is different.



Mercury Barometer

6.5 Manometer

1. The manometer is an instrument that is used to measure gas pressure.
2. Gas pressure, $P_1 = P_0 + \rho gh$ where h – difference in height.



Manometer

TOPIC 7 Energy, Work and Power

Objectives

Candidates should be able to:

(a) show understanding that kinetic energy, potential energy (chemical, gravitational, elastic), light energy, thermal energy, electrical energy and nuclear energy are examples of different forms of energy

(b) state the principle of conservation of energy and apply the principle to new situations or to solve related problems

(c) calculate the efficiency of an energy conversion using the formula efficiency = energy converted to useful output / total energy input

(d) state that kinetic energy $E_k = \frac{1}{2}mv^2$ and gravitational potential energy $E_p = mgh$ (for potential energy changes near the Earth's surface)

(e) apply the relationships for kinetic energy and potential energy to new situations or to solve related problems

(f) recall and apply the relationship *work done* = *force* × *distance moved in the direction of the force* to new situations or to solve related problems

(g) recall and apply the relationship *power* = *work done* / *time taken* to new situations or to solve related problems

NOTES...

7.1 Energy

1. Different forms: kinetic energy (*KE*), elastic potential energy, gravitational potential energy (*GPE*), chemical potential energy, thermal energy.
2. SI unit: Joule (J)
3. Principle of Conservation of Energy: The total energy in a system remains constant and cannot be created or destroyed. It can only be converted from one form to another without any loss in the total energy.

7.2 Gravitational Potential Energy (*GPE*) and Kinetic Energy (*KE*)

1. Take the surface of the Earth to be the reference level ($GPE = 0$).
   GPE of an object of mass m at height h above surface:

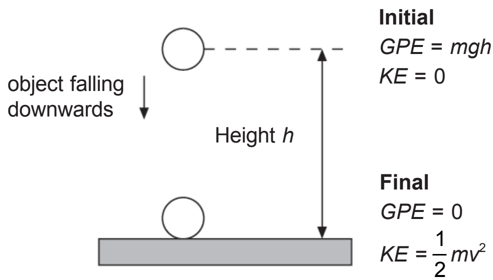
$$\boxed{GPE = mgh}$$

2. KE of a moving object of mass m, with a velocity v is

$$\boxed{KE = \frac{1}{2}mv^2}$$

Example 7.1

For a free-falling object of mass m, its gravitational potential energy is converted into kinetic energy. Take ground level as reference level ($GPE = 0$).



Apply the Principle of Conservation of Energy and assuming there is no air resistance:

Total energy at height h = Total energy at ground level

$$mgh = \frac{1}{2}mv^2$$

Velocity of object, $v = \sqrt{2gh}$

Note: The total energy of the object is constant throughout its fall, not just at the two positions used in the above calculation.
($GPE + KE$ = Total energy = Constant)

7.3 Work

1. Energy is required for an object to do work.
2. Defined as the product of applied force (F) and the distance moved (s) in the direction of the force.

$$\boxed{W = Fs}$$ Unit: J

3. No work is done if the applied force F does not displace the object along the direction of the force.

7.4 Power

1. Defined as the rate of work done.

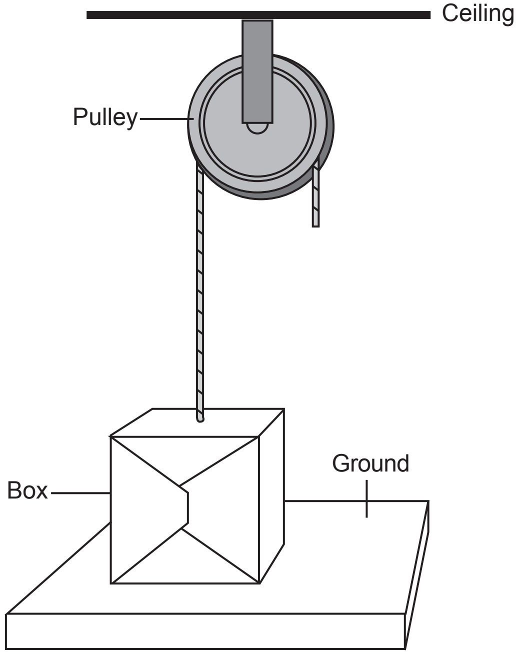
$$\text{Power} = \frac{\text{Work done}}{\text{Time taken}}$$

2. SI unit: Watt (W) or J s^{-1}
3. Efficiency of an energy/ power conversion:

$$\text{Energy} = \frac{\text{Energy converted into useful output}}{\text{Total energy output}} \times 100\%$$
$$= \frac{\text{Useful power output}}{\text{Total power output}} \times 100\%$$

Example 7.2

A box with a mass of 30 kg can be lifted by a light rope threaded through a smooth pulley.



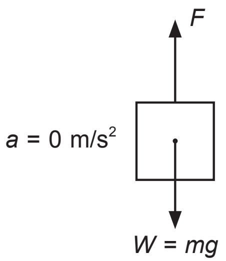
(a) If the box is lifted at a constant speed from the ground to a height of 2.0 m in 4.0 s, what is the power required?

(b) If the box is lifted with a constant acceleration of 1.5 m/s^2 from rest to a height of 3.0 m above the floor, what is the power required?

Take g, the gravitational field strength as 10 N/kg.

Solution

(a) Draw a free body diagram of the box and identify all the forces acting on it.



F – Applied force
W – Weight of the box
R – Resultant force on the box
s – Displacement of the box from the ground

Take forces acting upwards to be positive.

Using Newton's 2nd Law,

$F - mg = 0$

$\therefore F = mg$

$= (30)(10)$

$= 300$ N

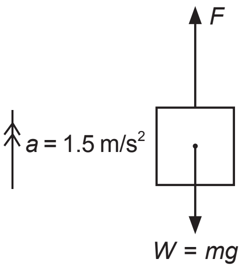
Power required = rate of work done

$$= \frac{Fs}{t}$$

$$= \frac{300 \times 2.0}{4.0}$$

$= 150$ W

(b)



F – Applied force

W – Weight of the box

R – Resultant force on the box

Take forces acting upwards to be positive.
Using Newton's 2nd Law,

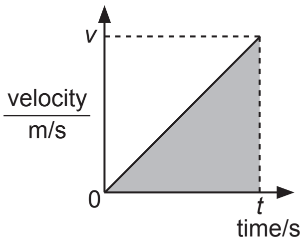
$F - mg = ma$

$F - 300 = 30 \times 1.5$

$F = 45 + 300$

$F = 345$ N

Sketch the speed-time graph of the box to obtain the time taken for the box to move to a height of 3.0 m above the ground.



t – time required for the box to reach a height of 3.0 m.

v – final velocity of the box

From the graph, we can obtain the velocity (gradient of graph) and the total displacement of the box.

Gradient of velocity-time graph,

$$a = \frac{v - u}{t} = \frac{v - 0}{t}$$

$$1.5 = \frac{v}{t}$$

$$v = 1.5t \text{ ------ (1)}$$

Area under the graph (shaded) = Displacement s of box from the ground

$$s = \frac{1}{2} \times \text{Base} \times \text{Height} = \frac{1}{2} \times t \times v$$

$$s = \frac{1}{2}vt$$

$$\frac{1}{2}vt = 3.0 \text{ ------ } (2)$$

Substitute (1) into (2):

$$\frac{1}{2}(1.5t)t = 3.0$$

$$\frac{3}{4}t^2 = 3.0$$

$$t^2 = 4.0$$

$$(t - 2.0)(t + 2.0) = 0$$

$$t = 2.0 \text{ s (since } t > 0)$$

$$\text{Power required} = \frac{345 \times 3.0}{2.0}$$

$$= 518 \text{ W (to 3 s.f.)}$$

TOPIC 8 Kinetic Model of Matter

Objectives

Candidates should be able to:

(a) compare the properties of solids, liquids and gases
(b) describe qualitatively the molecular structure of solids, liquids and gases, relating their properties to the forces and distances between molecules and to the motion of the molecules
(c) infer from Brownian motion experiment the evidence for the movement of molecules
(d) describe the relationship between the motion of molecules and temperature
(e) explain the pressure of a gas in terms of the motion of its molecules
(f) recall and explain the following relationships using the kinetic model (stating of the corresponding gas laws is not required):
   (i) a change in pressure of a fixed mass of gas at constant volume is caused by a change in temperature of the gas
   (ii) a change in volume occupied by a fixed mass of gas at constant pressure is caused by a change in temperature of the gas
   (iii) a change in pressure of a fixed mass of gas at constant temperature is caused by a change in volume of the gas
(g) use the relationships in (f) in related situations and to solve problems (a qualitative treatment would suffice)

NOTES

8.1 States of Matter

The 3 States of Matter

	Solid	Liquid	Gas
Volume	Definite	Definite	Indefinite (Takes the shape and size of container)
Shape	Definite	Indefinite (Takes the shape of container)	Indefinite (Takes the shape of container)
Compressibility	Not compressible	Not compressible	Compressible

	Solid	Liquid	Gas
Arrangement of atoms/molecules	1. Closely packed together 2. Orderly arrangement 3. Held together by large forces	1. Closely packed in clusters of atoms or molecules 2. Atoms/ molecules slightly further apart compared to particles 3. Held together by large forces	1. Atoms or molecules are very far apart and occupy any given space 2. Negligible forces of attraction between atoms/ molecules.
Density	High (Usually)	High	Low
Forces between atoms/ molecules	Very strong	Strong	Very Weak
Movement of atoms/ molecules	Can only vibrate about fixed positions	Able to move pass each other and not confined to fixed positions	Move in random manner independent of each other and at high speed.

Common mistakes:

1. Some substances, such as carbon dioxide, are commonly known to be in gaseous state at room temperature. However, this does not mean that the carbon dioxide molecules move in random motion.

   (Check its state (temperature): solid or gas, etc.)

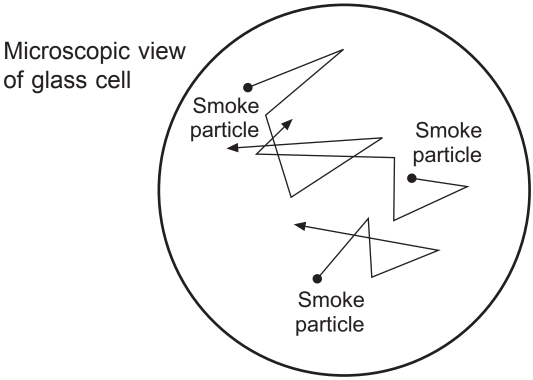
   E.g. Dry ice is a solid which consists of carbon dioxide molecules in an orderly arrangement.

2. ***Not all*** solids have high density, i.e. “ice” is a solid consisting of water molecules arranged orderly in an open hollow structure. Hence, its density is lower than water (liquid) and it can float in water.

8.2 Brownian Motion

The random and irregular motion of gas and liquid molecules.

Experimental observation (using microscope): Smoke particles in a sealed glass cell move about randomly and irregularly, because of bombardment by air molecules in the cell.



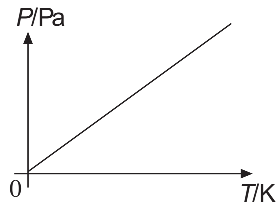
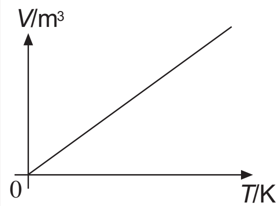
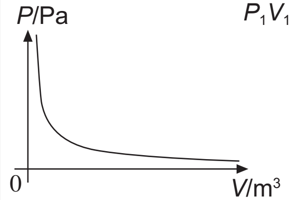
Smoke particles moving randomly

8.3 Pressure of Gas

1. In a sealed container, gas can exert pressure on the walls of the container.
2. The large number of molecules move at high speed, colliding against the container's walls and exerting a force against the wall when they bounce off the walls.
3. The force per unit area exerted by the molecules on the wall is the pressure of the gas on the wall.
4. Gas pressure increases when the
   (a) number of molecules in the container increases,
   (b) speed of molecules increases,
   (c) molecules have larger mass.

8.4 Relationship between Pressure (P), Volume (V) and Temperature (T)

1. For a constant mass of gas:

	P	***V***	***T***	***Relationship***
1.	Increase	Constant	Increase	P is directly proportional to T.  $\frac{P_1}{T_1} = \frac{P_2}{T_2}$
2.	Constant	Increase	Increase	V is directly proportional to T.  $\frac{V_1}{T_1} = \frac{V_2}{T_2}$
3.	Increase	Decrease	Constant	P *is* inversely proportional to V. $P_1V_1 = P_2V_2$ 

Example 8.1

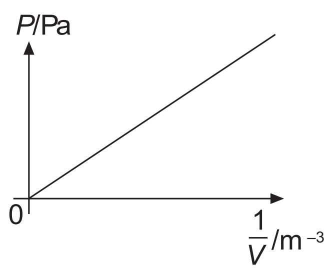
To get a linear graph that shows P is inversely proportional to V, rearrange the equation:

$P_1V_1 = P_2V_2 =$ constant, k

$$P_1 = \frac{k}{V_1}$$

Sketch the graph of P against $\frac{1}{V}$:

y-axis (P), x-axis $\left(\frac{1}{V}\right)$, gradient $= k$



TOPIC 9 Transfer of Thermal Energy

Objectives

Candidates should be able to:
(a) show understanding that thermal energy is transferred from a region of higher temperature to a region of lower temperature
(b) describe, in molecular terms, how energy transfer occurs in solids
(c) describe, in terms of density changes, convection in fluids
(d) explain that energy transfer of a body by radiation does not require a material medium and the rate of energy transfer is affected by:
  (i) colour and texture of the surface
  (ii) surface temperature
  (iii) surface area
(e) apply the concept of thermal energy transfer to everyday applications

NOTES..

9.1 Types of Heat Transfer

1. 3 types of heat transfer: Conduction, Convection, Radiation
2. Transfer of thermal energy is ***always*** from a high temperature region to a low temperature region (Temperature gradient).

	Conduction	Convection	Radiation
Medium	Solids Liquids Gases	Liquids (fluid) Gases (fluid)	Vacuum*
Process	1. Vibration of atoms/ molecules 2. Movement of free electrons (if any, i.e. metals) For solids, their atoms/molecules are in fixed positions	Movement of atoms/ molecules in the form of convection by currents set up by density change in parts of the fluid being heated.	Infrared waves (no medium required)

* Radiation does not require matter to transfer heat, but radiation can travel through matter (through several thousands of metres in air or a few metres in common solids).

9.2 Conduction

1. A direct contact between media is necessary.
2. Metals are the best solid conductors because of their free electrons.
3. Liquids and gases are poor conductors because their molecules are not closely packed together in fixed positions like solids.
4. Application: Use metals to make cooking utensils.

9.3 Convection

1. Molecules/ atoms must be free to move.
2. Set-up of a convection current: The fluid closer to the heat source expands, and its density decreases and the surrounding denser fluid displaces it.
3. Application: Air conditioners are placed near the ceiling because cold air, being denser, will sink to displace the warm air in the room.

9.4 Radiation

1. Factors affecting radiation:
   (a) Colour
   (b) Roughness
   (c) Area exposed to radiation
2. Good radiator/ good absorber of radiation: black, dull surface, with a huge amount of surface area exposed.
3. Poor radiator/ poor absorber of radiation: bright, shiny and polished surface.
4. Application: Greenhouses for growing plants.

9.5 Vacuum Flask

1. Reduces heat transfer in or out through conduction, convection and radiation.
2. Can store and maintain temperature (either hot or cold) of the contents in the flask.

Type of heat transfer	How heat transfer is reduced
Convection	Vacuum between the double glass walls.
Conduction	Vacuum between the double glass walls. Insulated cover and stopper.
Radiation	Shiny silvered inner surface of the glass walls.

TOPIC 10 Temperature

Objectives

Candidates should be able to:

(a) explain how a physical property which varies with temperature, such as volume of liquid column, resistance of metal wire and electromotive force (e.m.f.) produced by junctions formed with wires of two different metals, may be used to define temperature scales

(b) describe the process of calibration of a liquid-in-glass thermometer, including the need for fixed points such as the ice point and steam point

NOTES

10.1 Temperature

1. A measure of the degree of 'hotness' or 'coldness' of a body.
2. SI Unit: Kelvin (K)
3. Commonly-used unit is degree Celsius (°C): θ (K) = θ (°C) + 273.15

10.2 Measurement of Temperature

1. Material for temperature measurement: Substance/ material which possesses temperature-dependent property and thus can change continuously with temperature variations.

2. Temperature-dependent (Thermometric) Properties:

Thermometric Property	Thermometer	Range
Volume of a fixed mass of liquid (**e.g.** mercury or alcohol)	Mercury Alcohol Clinical thermometer	–10 ºC to 110 ºC –60 ºC to 60 ºC 35 ºC to 42 ºC
Electromotive force (e.m.f.) (between hot and cold junctions of two different metals joined together)	Thermocouple	–200 ºC to 60 ºC Common ones
Resistance of metal **e.g.** Platinum	Resistance thermometer	–200 ºC to 1200 º C
Pressure of a fixed mass of gas at constant volume	Constant-volume gas thermometer	Estimated –258 ºC to 1027 ºC

10.3 Temperature Scale

1. Temperature is measured with reference to 2 fixed points:
   (a) Lower Fixed Point or Ice point (0 °C):
   Temperature of pure melting ice at standard atmospheric pressure.
   (b) Upper Fixed Point or Steam point (100 °C):
   Temperature of pure boiling water at standard atmospheric pressure.

2. The length between the 2 fixed points is divided into 100 equal intervals of 1 °C.

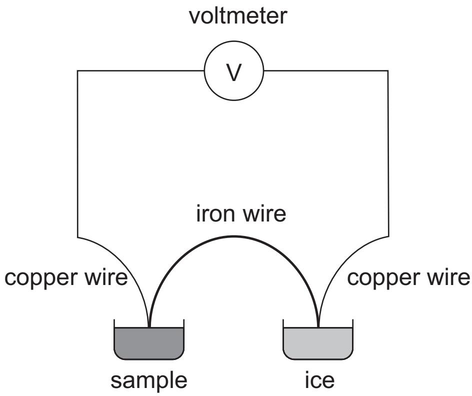
3. Apply the following general formula to calculate temperature of a material:

$$\theta\,°\mathrm{C} = \frac{X_\theta - X_0}{X_{100} - X_0} \times 100\ °\mathrm{C}$$

where:
θ is temperature of material
X_θ is thermometric property at θ
X_{100} is thermometric property at steam point
X_0 is thermometric property at ice point
i.e. for clinical thermometer, X is the length of the mercury thread at temperature θ; for thermocouples, it is the voltmeter reading at temperature θ.

10.4 The Thermocouple

1. To measure the temperature of an unknown substance:
   (a) One junction is kept at a constant temperature (i.e. ice point).
   (b) The other junction is kept at the point where the temperature is to be measured.



2. Advantages:
   (a) Can withstand high temperature with suitable metals.
   (b) Large temperature range. Can measure very low or very high temperatures.
   (c) Junctions used are sharp and pointed and therefore can be used to measure temperature accurately at a point.
   (d) Rapid response to temperature change.

TOPIC 11 Thermal Properties of Matter

Objectives

Candidates should be able to:

(a) describe a rise in temperature of a body in terms of an increase in its internal energy (random thermal energy)
(b) define the terms heat capacity and specific heat capacity
(c) recall and apply the relationship *thermal energy* = *mass* × *specific heat capacity* × *change in temperature* to new situations or to solve related problems
(d) describe melting/ solidification and boiling/ condensation as processes of energy transfer without a change in temperature
(e) explain the difference between boiling and evaporation
(f) define the terms latent heat and specific latent heat
(g) recall and apply the relationship *thermal energy* = *mass* × *specific latent heat* to new situations or to solve related problems
(h) explain latent heat in terms of molecular behaviour
(i) sketch and interpret a cooling curve

NOTES

11.1 Introduction

1. Temperature – a measure of the internal energy of the substance's atoms/ molecules.
2. Increase in temperature – caused by the supply of heat which increases internal energy.
3. Internal energy – sum of kinetic energy and potential energy of the atoms/ molecules.

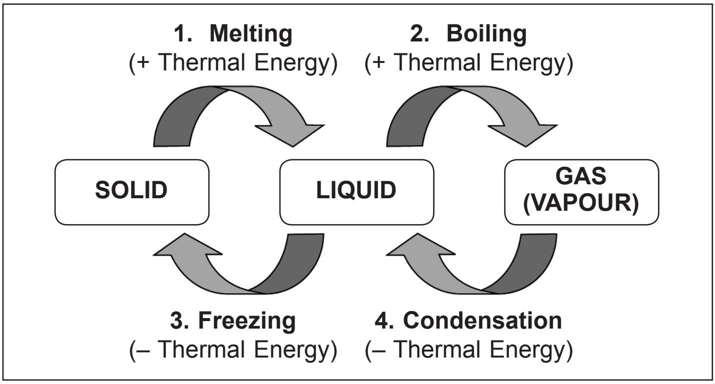
4. Types of internal energy:

State of substance	Type of internal energy
Solid	vibrational kinetic energy + potential energy
Liquid	translational kinetic energy + potential energy
Gas	mainly translational kinetic energy

11.2 Change of States

1. Two main changes occur when heat is supplied to a substance
   (a) Increase in temperature
   (b) Change of state (i.e. solid to liquid)

2. The following chart shows the changes of state (without temperature change) and their corresponding processes involved:



+ Thermal Energy: Heat is absorbed by substance
– Thermal Energy: Heat is removed from substance (Released to surroundings)

1. Melting	(i) Definition: a change of state from solid to liquid without a change in temperature. (ii) Melting point: constant temperature at which a solid melts into a liquid. (iii) Process: Heat absorbed is used to do work to break intermolecular bonds between the atoms/ molecules of the solid. (iv) The reverse process is freezing.
2. Boiling	(i) Definition: a change of state from liquid to gas without a change in temperature. (ii) Boiling point: constant temperature at which a liquid boils. (iii) Process: Heat supplied to the liquid is used to do work in separating the atoms or molecules as well as in pushing back the surrounding atmosphere. (iv) The reverse process is condensation.
3. Freezing	(i) Reverse process of melting. (ii) Definition: a change of state from liquid to solid without a change in temperature. (iii) Freezing point: constant temperature at which a liquid changes to a solid. (iv) Process: Heat is released as the intermolecular bonds are formed when the liquid atoms or molecules come together to form a solid. (v) For a pure substance, the melting point is the same as the freezing point.
4. Condensation	(i) Reverse process of boiling/ evaporation. (ii) Definition: a change of state from gas to liquid without a change in temperature. (iii) Condensation point: constant temperature at which a gas changes to a liquid. (iv) Process: Heat is released as the intermolecular bonds are formed when the gaseous atoms or molecules come together to form a liquid. (v) For a pure substance, the boiling point is the same as the condensation point.

3. Other processes:
   (a) Evaporation (liquid to gas)
   (b) Sublimation (solid to gas)

4. Differences between boiling and evaporation:

Boiling	Evaporation
occurs at a fixed temperature	occurs at any temperature
occurs throughout the liquid	occurs on the surface of substance
bubbles are visible	bubbles are not visible
fast process	slow process
heat is supplied to substance by an energy source	heat is absorbed by substance from the surroundings

5. Factors affecting melting and boiling points of water:

Factor	Melting Point	Boiling Point
Increase Pressure	Lower	Higher
Add Impurities	Lower	Higher

11.3 Heat Capacities and Latent Heat

1. The following terms are used in calculations in this chapter:

Term	SI Units	Definition	Formula
Heat capacity, C	J °C^{-1} or J K^{-1}	Thermal energy needed to increase temperature of substance by 1 °C or 1 K.	$Q = C\Delta\theta$
Specific heat capacity, c	J kg^{-1} °C^{-1} or J kg^{-1} K^{-1}	Thermal energy needed to increase temperature of 1 kg of substance by 1 °C or 1 K.	$Q = mc\Delta\theta$

Term	SI Units	Definition	Formula
Specific latent heat of fusion, l_f	J kg^{-1}	Thermal energy needed to change 1 kg of substance from solid to liquid without temperature change.	$Q = ml_f$
Specific latent heat of vaporisation, l_v	J kg^{-1}	Thermal energy needed to change 1 kg of substance from liquid to gas without temperature change.	$Q = ml_v$

Q – Amount of thermal energy needed (J), $\Delta\theta$ – Change in temperature

2. Comparison between substances of high and low heat capacities

Heat Capacity	Time to cool down/ heat up	Reason
High	Longer	Need to lose more energy (cooling) or absorb more energy (heating).
Low	Shorter	Need to lose less energy (cooling) or absorb less energy (heating).

TOPIC 12 General Wave Properties

Objectives

Candidates should be able to:

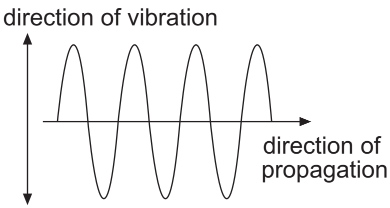
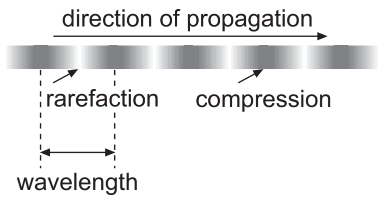
(a) describe what is meant by wave motion as illustrated by vibrations in ropes and springs and by waves in a ripple tank
(b) show understanding that waves transfer energy without transferring matter
(c) define speed, frequency, wavelength, period and amplitude
(d) state what is meant by the term wavefront
(e) recall and apply the relationship *velocity* = *frequency* × *wavelength* to new situations or to solve related problems
(f) compare transverse and longitudinal waves and give suitable examples of each

NOTES

12.1 Introduction

1. Wave motion is the propagation of oscillatory movement or disturbance from one region to another.
2. A wave transfers energy from one place to another without transferring matter.
3. All waves follow the laws of reflection and refraction.
4. Mechanical waves require a medium (i.e. water or air molecules) for propagation.
5. Electromagnetic waves (See Topic 14) are propagations of oscillations in electromagnetic fields. The propagation does not require a medium, thus electromagnetic waves can travel in vacuum.

6. We classify waves in this topic into two types based on their propagation method:
   (a) Transverse
   (b) Longitudinal

Transverse waves	Longitudinal waves
 	
Movement of particles in the medium: Perpendicular to the direction of propagation (movement) of wave	**Movement of particles in the medium:** Parallel to the direction of propagation (movement) of wave
Examples Water waves, electromagnetic waves **Characteristics** 1. The particles oscillate perpendicularly (up and down) to the direction of travel. 2. Peak: Highest point reached by the particle from its neutral position 3. Trough: Lowest point reached by the particle from its neutral position 4. The distance between adjacent particles remains constant, in the direction of the propagation of the wave.	**Examples** Sound wave **Characteristics** 1. The particles oscillate along (to-and-fro) the direction of travel. 2. Compression: Section in which the particles are closest together 3. Rarefaction: Section in which the particles are furthest apart. 4. The distance between adjacent particles varies from a maximum value (furthest apart) to a minimum value (closest together), in the direction of the propagation of the wave.

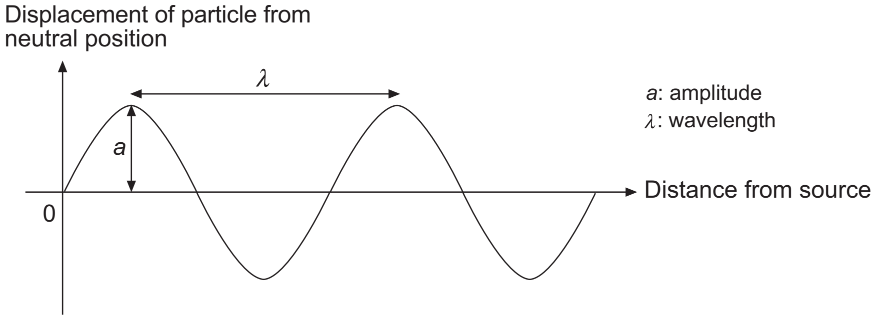
12.2 Terms used to describe a wave

1. For both transverse and longitudinal waves, the particles oscillate about their undisturbed positions (neutral positions). The neutral positions lie along an axis in the direction of wave propagation.

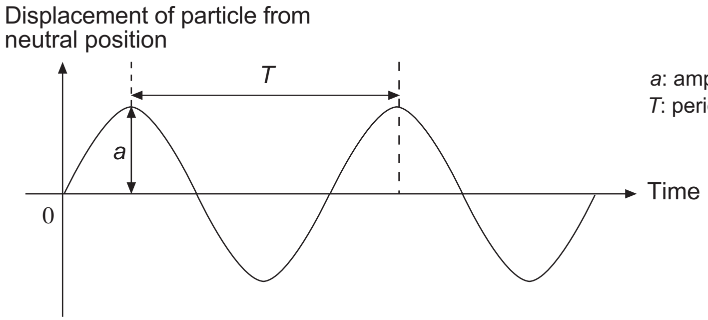
2. The following graphs show sine-curves used to describe the wave terms used for both types of waves.

Note: These are graphs and not transverse waves!

Displacement-distance Graph



Displacement-time Graph



3. Common wave terms:

Term	Transverse waves	Longitudinal waves
Amplitude, a (m)	The maximum displacement of the particle from its neutral position **perpendicular** to the direction of propagation. (i.e. height of crest from neutral position.)	The maximum displacement of the particle from its neutral position **along** the direction of propagation.
Wavelength, λ (m)	The distance between two successive crests or two successive troughs.	The distance between two successive compressions or two successive rarefractions.
Frequency, f (Hz)	The number of complete waves produced in one second.	
Period, T (s)	The time taken to produce one complete wave. **Formula:** $T = \frac{1}{f}$	
Speed, v (m)	The distance moved by any part of the wave in one second. **Formula:** $v = f\lambda$	

12.3 Wavefront

1. A wavefront is a line or surface, in the path of a wave motion, on which all particles are oscillating in phase.

2. There are two types of wavefronts:
   (a) Circular wavefront (close to point source of disturbance)
   (b) Plane wavefront (straight wavefronts far from point source of disturbance)

3. The amplitude of particles along the same wavefront is the same.

TOPIC 13 Light

Objectives

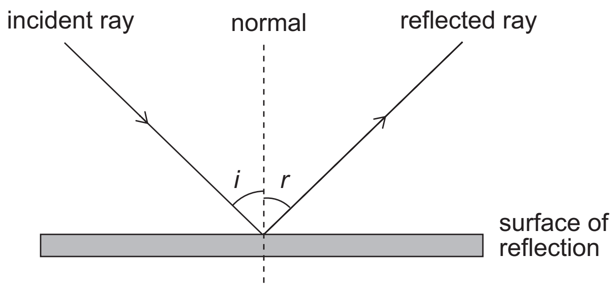
Candidates should be able to:

(a) recall and use the terms for reflection, including *normal*, *angle of incidence* and *angle of reflection*
(b) state that, for reflection, the angle of incidence is equal to the angle of reflection and use this principle in constructions, measurements and calculations
(c) recall and use the terms for refraction, including *normal*, *angle of incidence* and *angle of refraction*
(d) recall and apply the relationship $\frac{\sin i}{\sin r}$ = *constant* to new situations or to solve related problems
(e) define *refractive index* of a medium in terms of the ratio of speed of light in vacuum and in the medium
(f) explain the terms *critical angle* and *total internal reflection*
(g) identify the main ideas in total internal reflection and apply them to the use of optical fibres in telecommunication and state the advantages of their use
(h) describe the action of a thin lens (both converging and diverging) on a beam of light
(i) define the term *focal length* for a converging lens
(j) draw ray diagrams to illustrate the formation of real and virtual images of an object by a thin converging lens

NOTES

13.1 Reflection

1. The diagram below shows a ray of light being reflected from a plane surface.



2. The following terms are commonly used in the reflection of light:

Term	Definition
Normal	Imaginary line perpendicular to the surface of reflection
Angle of incidence, i	Angle between the incident ray and the normal
Angle of reflection, r	Angle between the reflected ray and the normal

3. Laws of reflection:
   (a) Angle i = Angle r
   (b) The incident ray, reflected ray and the normal at the point of incidence all lie on the same plane.

4. Characteristics of an image formed in a plane mirror:
   (a) Upright
   (b) Virtual (Cannot be captured on a screen)
   (c) Laterally inverted
   (d) Same size as the object
   (e) Image distance from the other side of the surface of reflection is the same as the object's distance from the surface of reflection.

13.2 Refraction

1. The diagrams below show a ray of light refracted as it passes from air into glass and from glass into air. Note how the light ray bends in each case.

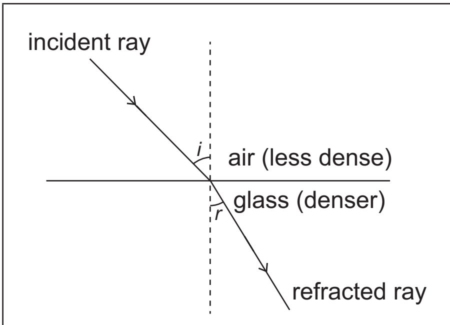


Fig. (a)

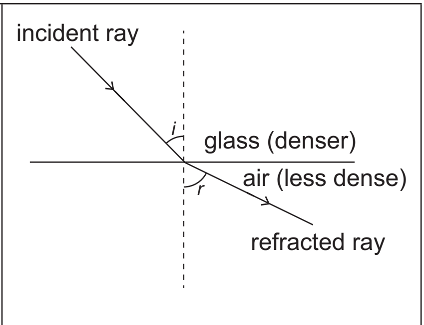


Fig. (b)

2. The following terms are commonly used in refraction:

Term	Definition
Normal	Imaginary line perpendicular to the surface of reflection
Angle of incidence, *i*	Angle between incident ray & normal
Angle of refraction, *r*	Angle between refracted ray & normal
Refractive index of a medium, *n*	Ratio of the speed of light in vacuum to the speed of light in medium
Critical angle	Angle of incidence in a denser medium for which the angle of refraction in the less dense medium is 90°
Total internal reflection	Complete reflection of an incident ray of light within a denser medium surrounded by a less dense medium when the incident angle is greater than the critical angle

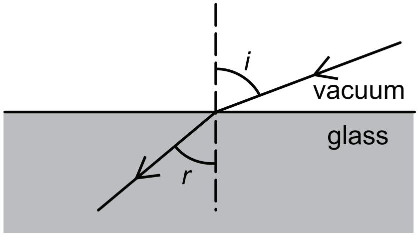
3. Refractive index of vacuum is taken as 1.
   Air has a refractive index of 1.0003 which is very close to 1, but is not equal to 1.

4. Laws of refraction:

(a) The incident ray, refracted ray and the normal at the point of incidence all lie on the same plane.

(b) Snell's Law: $\dfrac{\sin i}{\sin r}$ = constant, for two given media.

***E.g.* 1**: For the light ray passing from a **less dense** medium to a **denser** medium (such as vacuum to glass),

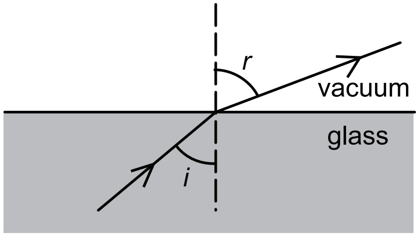


$$\frac{\sin i}{\sin r} = \frac{n_{\text{denser medium}}}{n_{\text{vacuum}}} = \frac{n}{1}$$

$$\frac{\sin i}{\sin r} = n$$

where n is the refractive index of the denser medium.

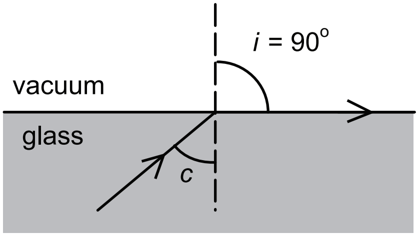
***E.g.* 2**: For the light ray passing from a **denser** medium (such as glass to vacuum) to a **less dense** medium,



$$\frac{\sin i}{\sin r} = \frac{n_{\text{vacuum}}}{n_{\text{denser medium}}} = \frac{1}{n}$$

where n is the refractive index of the denser medium.

***E.g.* 3**: For light ray passing from a denser medium into a less dense medium at a critical angle, $i = c$,



$$\frac{\sin i}{\sin r} = \frac{n_{\text{vacuum}}}{n_{\text{glass}}}$$

$$\frac{n_{\text{vacuum}}}{n_{\text{glass}}} = \frac{\sin c}{\sin 90^\circ} \text{ where } i = c \text{ and } r = 90^\circ$$

$$n_{\text{glass}} = \frac{1}{\sin c}$$

$$\Rightarrow n = \frac{1}{\sin c}$$

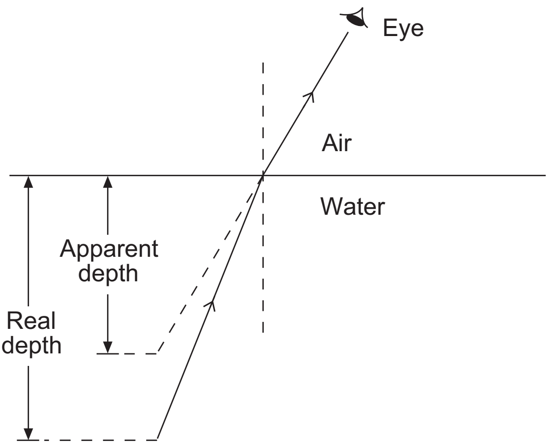
where n is the refractive index of the denser medium.

When other incident angles $i > c$, the incident ray will undergo total internal reflection.

Note that $n_{\text{vacuum}} = 1$ and $n_{\text{air}} = 1.0003$.

5. Refractive index, n, of a medium (i.e. water) can also be calculated as follows:

$$\frac{\text{real depth}}{\text{apparent depth}} = n$$



6. The speed of light is slower in a denser medium as compared to that in a less dense medium.

Example 13.1

A ray of light travels from within a piece of glass into air. The incident angle is 10° and the refractive index of glass is 1.61. Calculate the angle of refraction.

Solution

Refractive index of glass, $n_{glass} = 1.61$

Refractive index of air, $n_{air} = 1.0003$

$$\frac{\sin i}{\sin r} = \frac{n_{air}}{n_{glass}} \; (\textit{Common mistake}: \frac{\sin i}{\sin r} = n_{glass})$$

$$\frac{\sin 10^\circ}{\sin r} = \frac{1.0003}{1.61}$$

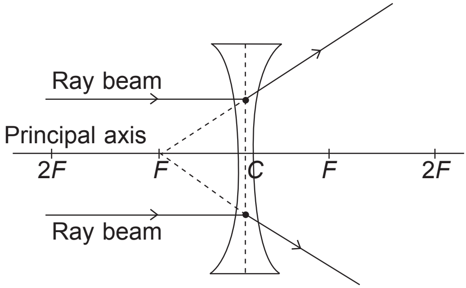
$$\sin r = \frac{1.61 \sin 10^\circ}{1.0003}$$

Angle $r = 16.2^\circ$ (to 1 d.p.)

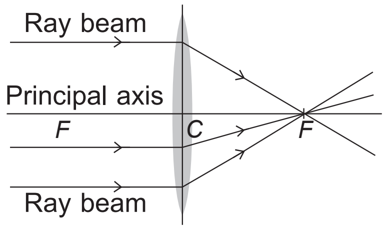
7. An application of total internal reflection: optical fibres to transmit data.
   Principle: The polished surfaces of the fibres are made of a material of suitable refractive index for total internal reflection of light.
   Advantages:
   1. Optical fibres have high electrical resistance, so it can be used near high-voltage equipment safely.
   2. Since optical fibres have lower density than copper, the mass is lower for the same volume of wires. Hence optical fibres are suitable for mobile vehicle applications such as aircrafts where mass and space are concerns.
   3. Optical fibres are resistant to chemical corrosion
   4. Optical fibres do not emit electric fields or magnetic fields since they carry light instead of electrical currents, hence they will not interfere with nearby electronic equipment or themselves be subject to electromagnetic interference.
   5. Since optical fibres are secured, it is difficult to intercept signals without disrupting them, unlike conventional current carrying copper cables.

13.3 Lenses

1. Actions of a thin lens: As shown in the following diagrams, a converging lens converges a beam of light whereas a diverging lens diverges a beam of light.



Diverging lens

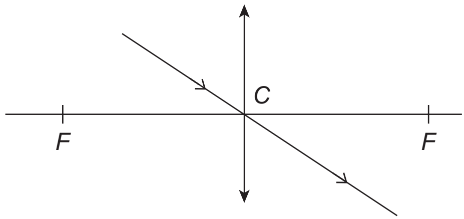
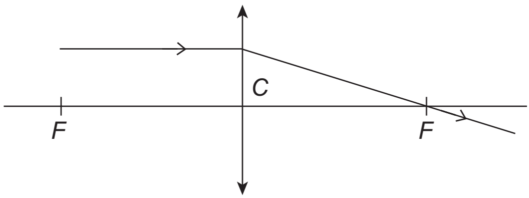
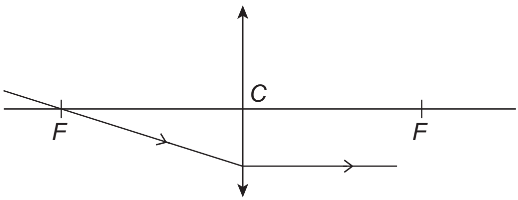


Converging lens

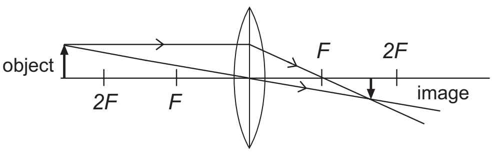
2. The following table summarises the main features of a lens:

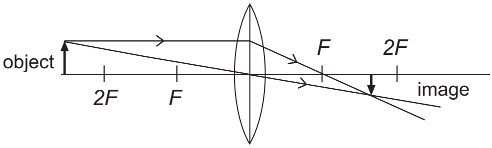
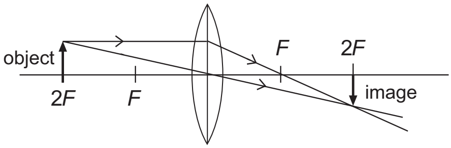
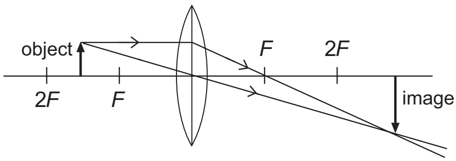
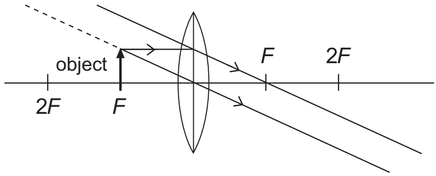
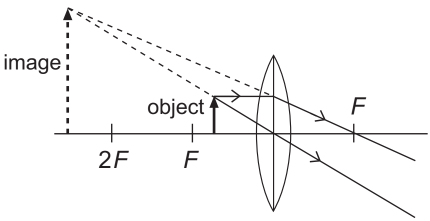
Term	Definition
Focal length, *f*	Distance between the optical centre, *C* and the principal focus *F*.
Optical centre, *C*	Midpoint between the lens' surface on the principal axis. Rays passing through optical centre are not deviated.
Principal axis	Line passing symmetrically through the optical centre of the lens.
Principal focus or Focal point, *F*	Point of convergence for all light rays refracted by the lens.
Focal plane	Plane which passes through *F* and perpendicular to the principal axis.

3. Ray diagrams are drawn to locate the position and the size of an image.

Action of incident ray	Diagram
Ray passing through *C* passes straight through without a change in direction.	Converging lens 
Ray parallel to principal axis passes through lens and changes direction and passes through *F*.	Converging lens 
Ray passing through *F* initially reaches lens and passes out parallel to principal axis.	Converging lens 

4. Types of images formed by a **thin** converging lens

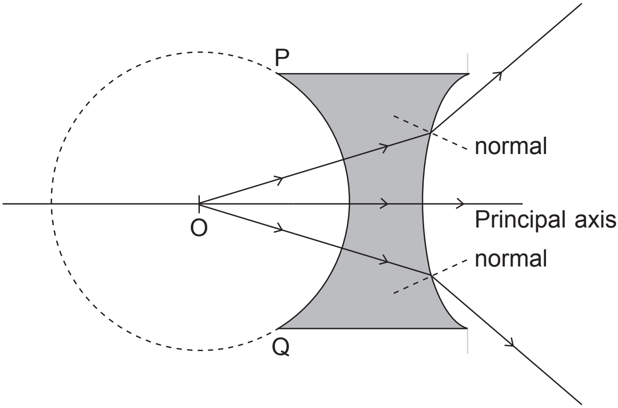
Object distance	Ray diagram	Image characteristics	Application
At infinity		• Real • At *F*	Telescope lens

Object distance	Ray diagram	Image characteristics	Application
Greater than 2*F*		• Inverted • Real • Diminished • Between *F* and 2*F*	Camera lens
At 2*F*		• Inverted • Real • Same size as object • At 2F	Photocopier
Between *F* and 2*F*		• Inverted • Real • Magnified	Projector
At *F*		• Image formed at infinity. (Light rays travel parallel to each other.)	
Less than *F*		• Upright • Enlarged • Virtual (On the same side of the lens as the object.)	Magnifying glass

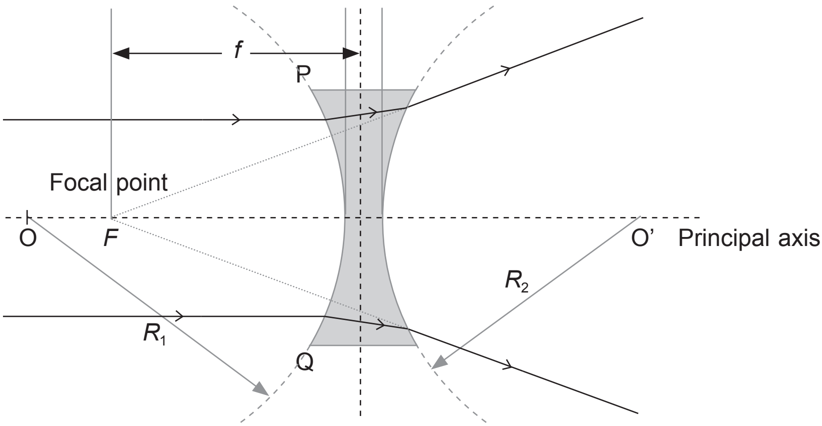
Note: Ray diagrams must ALWAYS have arrows to indicate direction of the ray.

5. Special cases

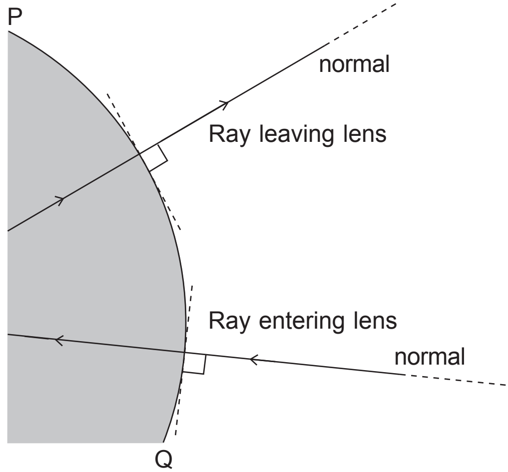
- Diverging lens: Light source from O, centre of curvature of the lens.
  The figure shows part of a diverging lens where one of the faces of the lens PQ is part of a circle with centre O.



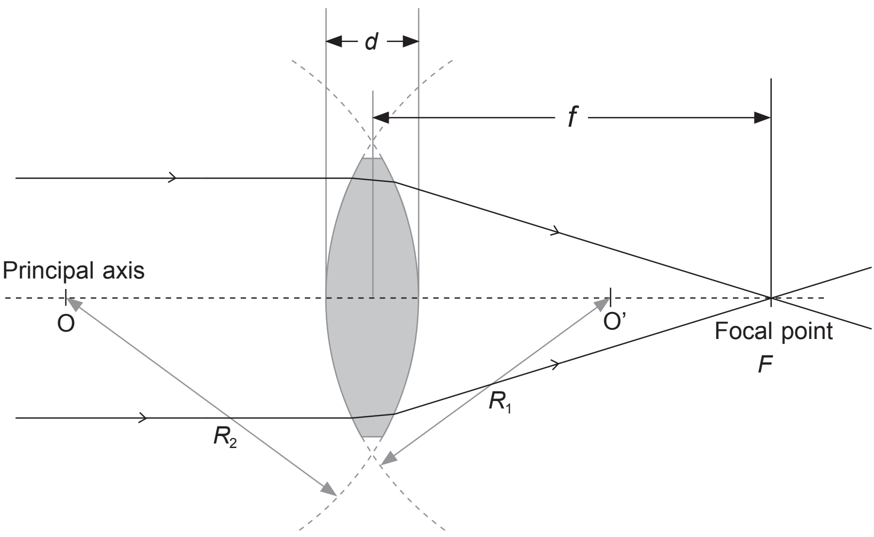
Any light rays drawn from O to PQ will be normal (90°) to the surface PQ because they are moving along the normal line.
Hence any light ray originated from O and entering into the lens PQ will be moving into the lens without changing direction.
A complete diverging lens is shown in the figure below, where O and O' are the centre of the circles (dotted), *F* is the focal point and *f* is the focal length. R_1 and R_2 are the radii of the circles with centres O and O' respectively.



- Converging lens: Light rays entering or leaving the lens will travel along the path of the normal to the lens surface which is a part of a circle. The figure shows part of a converging lens where PQ is part of a circle.



The rays will not change direction because they are moving along the path of the normal line. The path forms an angle of 90° to the surface of the lens.
A complete converging lens is shown in the figure below, where O and O' are the centre of the circles (dotted), *F* is the focal point and *f* is the focal length. R_1 and R_2 are the radii of the circles with centres O and O' respectively.



TOPIC 14 Electromagnetic Spectrum

Objectives

Candidates should be able to:

(a) state that all electromagnetic waves are transverse waves that travel with the same speed in vacuum and state the magnitude of this speed
(b) describe the main components of the electromagnetic spectrum
(c) state examples of the use of the following components:
  (i) radiowaves (e.g. radio and television communication)
  (ii) microwaves (e.g. microwave oven and satellite television)
  (iii) infra-red (e.g. infra-red remote controllers and intruder alarms)
  (iv) light (e.g. optical fibres for medical uses and telecommunications)
  (v) ultra-violet (e.g. sunbeds and sterilisation)
  (vi) X-rays (e.g. radiological and engineering applications)
  (vii) gamma rays (e.g. medical treatment)
(d) describe the effects of absorbing electromagnetic waves, e.g. heating, ionisation and damage to living cells and tissue

NOTES

14.1 Components of the Electromagnetic Spectrum

1. All electromagnetic waves (EM waves) are transverse waves that travel at the speed of light (3×10^8 m/s) in vacuum and slow down in other media.

2. EM waves do not require a medium for propagation.

3. EM waves can be absorbed or emitted by matter.

4. The main components of the electromagnetic spectrum are as follows:

EM Wave	Order of Magnitude of Wavelength, λ/m	Application
γ-ray (Gamma ray)	10^{-3}	Manufacturing: Checking of cracks/ holes in metal plates. Medical: Radiotherapy.

EM Wave	Order of Magnitude of Wavelength, λ/m	Application
X-ray	10^{-10}	Medical: Inspection of bones for signs of fractures.
Ultraviolet (UV)	10^{-8}	Medical: Production of vitamin D in the body.
Visible light spectrum: *Violet* *Indigo* *Blue* *Green* *Yellow* *Orange* *Red* In increasing order of λ	10^{-7}	
Infrared radiation (IR)	10^{-4}	Remote control for television sets.
Microwave	10^{-2}	Microwave oven for cooking.
Radio Wave	10^{-2} to 10^{3}	Telecommunication.

14.2 Harmful Effects of Absorbing EM Waves

1. EM waves transmit radiation energy from one region to another.

2. Radiation may damage living cells and tissues through heating and ionisation.
   (a) Heating: Organic molecules in tissue gain kinetic energy from incident radiation. The energy increase is detected by a temperature rise. When the temperature gets too high, the molecules break apart and the tissue gets cooked.
   (b) Ionisation: Organic molecules absorb energy to break molecular bonds to form ions which can react with neighbouring molecules. This results in destruction or changes to the tissue.

3. Mobile phones emit radiation in the form of electromagnetic waves which can heat up the brain.

4. Too much sun-tanning can lead to an overdose of ultraviolet radiation which can cause skin cancer (i.e. melanoma).

TOPIC 15 Sound

Objectives

Candidates should be able to:

(a) describe the production of sound by vibrating sources
(b) describe the longitudinal nature of sound waves in terms of the processes of compression and rarefaction
(c) explain that a medium is required in order to transmit sound waves and the speed of sound differs in air, liquids and solids
(d) describe a direct method for the determination of the speed of sound in air and make the necessary calculation
(e) relate loudness of a sound wave to its amplitude and pitch to its frequency
(f) describe how the reflection of sound may produce an echo, and how this may be used for measuring distances
(g) define ultrasound and describe one use of ultrasound, e.g. quality control and pre-natal scanning

NOTES

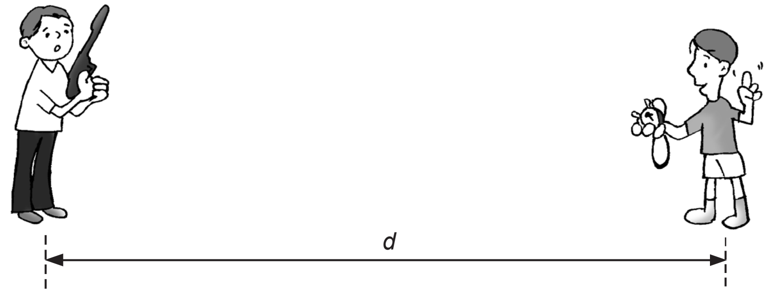
15.1 Production of Sound Waves

1. Sound waves are produced when objects vibrate in a medium.
2. Sound waves are longitudinal waves which require a medium for propagation.

15.2 Medium of Propagation for Sound

1. When sound waves travel in different media, the speed differs.
   Speed of sound in solids > speed of sound in liquids > speed of sound in air.
2. In solids, the atoms are more closely packed together, as compared to liquids and gases. Hence, sound travels the fastest in solids.

15.3 Determining the Speed of Sound



1. To determine the speed of sound, a gunman and a timer can stand apart from each other in an open field at a known distance *d*.

2. The gunman will fire a pistol into the air. The timer will start his stopwatch upon seeing the flash of the pistol and stop the stopwatch when he hears the sound of the pistol. The time interval is recorded as Δt.

3. The speed of sound is calculated as:

$$\boxed{v = \frac{d}{\Delta t}}$$

4. The speed of sound in air is about 330 m/s. Since the human reaction time is about $\frac{2}{3}$ of a second, *d* has to be sufficiently large for the experiment to be accurate.

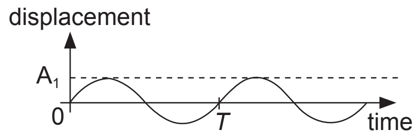
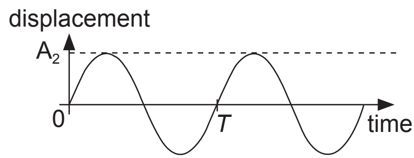
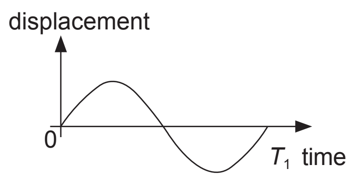
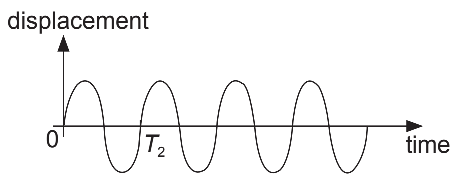
Example 15.1
In a storm, an observer saw a lightning flash, followed by the sound of thunder 4.0 seconds later. Given that the speed of sound in air is approximately 330 m/s, find the observer's distance from where the lightning occurred.

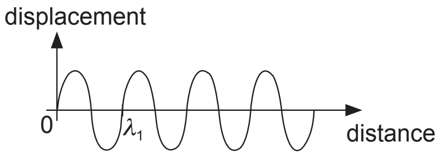
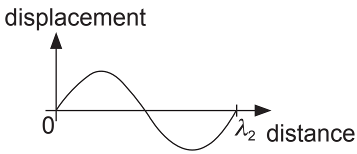
Solution
The lightning flash, which the observer sees, is assumed to reach him immediately after the lightning occurs (speed of light = 3.0×10^8 m/s).
Let the distance of the observer from the lightning be *d*.

$$\Delta t = 4.0 \text{ s}$$
$$330 = \frac{d}{\Delta t} = \frac{d}{4}$$
$$d = 1320 \text{ m}$$

15.4 Characteristics of Sound

1. The following table shows the main characteristics of sound and the factors affecting these characteristics:

Characteristics	Factors
Loudness	Amplitude of a sound wave (a higher amplitude leads to a louder sound)  soft  loud Note: amplitude $A_1 < A_2$
Pitch	Wavelength of a sound wave (a shorter wavelength leads to a higher pitch) Frequency of a sound wave (a higher frequency leads to a higher pitch) From the equation, $v = f\lambda$, since v is constant, we observe that when the wavelength, λ, decreases, the frequency, f, increases. As such, a shorter wavelength leads to a higher frequency which leads to a higher pitch.  low pitch  high pitch

Characteristics	Factors
	Note: $T_1 > T_2$ From the equation, $T = \frac{1}{f}$, we observe that when the period, T, increases, the frequency, f, decreases. As such, a longer period leads to a lower frequency which leads to a lower pitch.  high pitch  low pitch Note: $\lambda_1 < \lambda_2$

15.5 Echoes

1. Echoes are produced when a sound wave is reflected from a surface.
2. The reflected sound (echo) can be heard separately from the original sound if the source of the sound is much closer to the observer than to the reflecting surface.
3. To reduce the effect of echoes in buildings, walls are roughened up with padding and the floors are covered with rugs or carpets. This is to scatter the incident sound wave so that the reflected sound is reduced.
4. Using echo to measure distance.

Example 15.2

A man stood in front of a tall cliff. He fired a pistol into the air and started his stopwatch simultaneously. After 3.0 s, he heard the echo of the pistol shot. Given that the speed of sound is 330 m/s, find his distance from the cliff.

Solution

Let distance of man from cliff be d.

$2d = 330 \times 3.0$

$d = 495$ m

(We used $2d$ because 3.0 s is the time taken for the sound to hit the cliff and be reflected back to the man.)

15.6 Ultrasound

1. Ultrasound is the sound with frequencies that are greater than 20 000 Hz.
2. The audible range of sound for humans is between 20 Hz and 20 000 Hz. Hence humans cannot hear ultrasound.
3. Some applications of ultrasound:
   (a) Pre-natal scan to check the development of babies in womb.
   (b) Used by ships to find depth of seabed.
   (c) Check for cracks in metal pipes that are too small for the naked eye to see.

TOPIC 16 Static Electricity

Objectives

Candidates should be able to:

(a) state that there are positive and negative charges and that charge is measured in coulombs
(b) state that unlike charges attract and like charges repel
(c) describe an electric field as a region in which an electric charge experiences a force
(d) draw the electric field of an isolated point charge and recall that the direction of the field lines gives the direction of the force acting on a positive test charge
(e) draw the electric field pattern between two isolated point charges
(f) show understanding that electrostatic charging by rubbing involves a transfer of electrons
(g) describe experiments to show electrostatic charging by induction
(h) describe examples where electrostatic charging may be a potential hazard
(i) describe the use of electrostatic charging in a photocopier, and apply the use of electrostatic charging to new situations

NOTES

16.1 Atomic Structure

1. Matter is made up of small units called atoms.
2. An atom consists of a positively-charged nucleus surrounded by negatively charged electrons orbiting around the nucleus. The overall charge of an atom is zero.
3. The positively-charged nucleus consists of positively-charged protons held together by neutral particles called neutrons.
4. When excess electrons are added to an atom, the atom becomes negatively charged.
5. When electrons are removed from an atom, the atom becomes positively charged.

16.2 Electric Charges

1. Electric charges are either positive or negative.
2. Like charges repel each other; unlike charges attract each other.
3. Rubbing (charging by friction) causes electrons to be transferred from one object to another. Charge transfer between two objects only involves electron transfer. There is NO MOVEMENT of positive charges (which are the nuclei of the atoms). Otherwise, the solid will deform.
4. Insulators can be charged by rubbing, unlike conductors (metals), because **electrons** are not free to move about in an insulator and thus charges are localised to the surfaces where rubbing occurs.
5. Examples of insulators and the types of charges they gain from rubbing:

Type of insulator rod	Type of cloth used for rubbing	Charges gained by cloth	Charges gained by rod
Cellulose acetate	wool	$-Q$	$+Q$
Glass	silk	$-Q$	$+Q$
Ebonite	fur	$+Q$	$-Q$
Polythene	wool	$+Q$	$-Q$

6. The excess charge, Q, carried away by one body must be equal to the number of electrons removed from the other body. The charges are in multiples of an electron charge, e (-1.6×10^{-19} C) according to the equation:

$\boxed{Q = Ne}$ where N is a whole number.

7. The unit of charge is the coulomb (Symbol: C).
8. Electric charge Q is related to current I and time t by the equation:

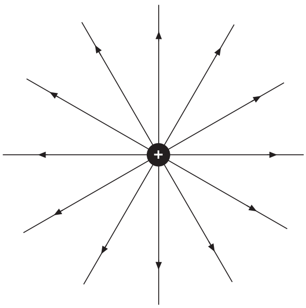
$$\boxed{Q = It}$$

16.3 Concept of Electric Field

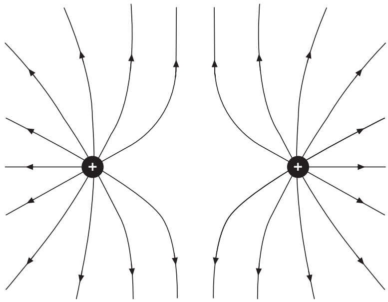
1. An electric field is a region in space in which a unit positive charge experiences a force.
2. Electric field is a vector quantity. The direction of the field is determined by the direction of the force acting on the unit positive charge.
3. An electric field is set up by a charge. When a unit positive charge is brought near a negative charge, the positive charge will experience a force of attraction towards the negative charge and vice versa.

Example 16.1: Examples of field patterns set up by point charges

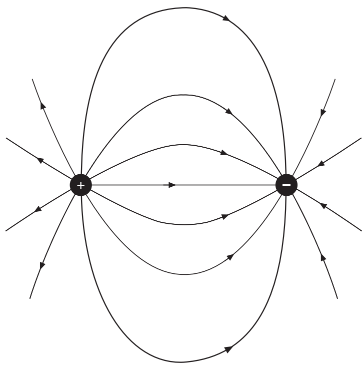
(a) Isolated positive charge



(b) Two equal magnitude, positive charges close to each other



(c) Two charges with equal magnitude but opposite signs



16.4 Hazards of Electrostatic Charging

1. Lightning: A large charge build-up in the clouds due to the friction between water and air molecules results in the ionisation of the air. The ionised air provides a path for conduction of electrons to the ground through tall, pointed objects.
   Remedy: Lightning conductors can be placed at the top of tall buildings to allow electrons to flow steadily from the air to the ground.

2. Fire: An excessive build up of charges due to friction with air can lead to an explosion or a fire in aircrafts.
   Remedy: Tyres are made of slightly conductive rubber to discharge the aircraft when it touches down.

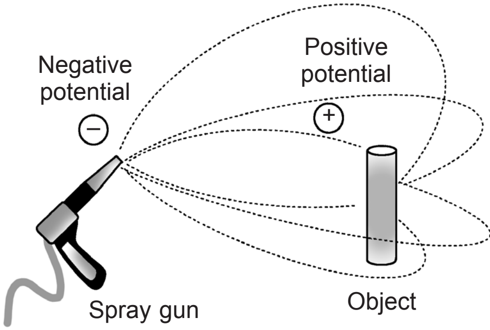
16.5 Some Applications of Electrostatics

1. Spray painting:

   Steps:

   (1) A fixed electric potential difference is maintained between the paint spray nozzle and the object to be painted. (i.e. the nozzle is negatively-charged and the object is positively charged)

   (2) As the paint leaves the nozzle, the droplets are charged.



   (3) Since the droplets all have the same charge, they repel each other so that the paint spreads out evenly.

   (4) The paint droplets are all attracted to the positively-charged object and stick strongly to its surface.

2. Photocopier:

   Steps:

   (1) Positive charges are arranged in a pattern to be copied on the surface of an insulator drum.

   (2) Negatively-charged toner powder is sprinkled on the drum.

   (3) Only the portions of the drum with positive charges allow the toner powder to stick to it to form the image.

   (4) The resultant pattern is then transferred onto the paper and fixed permanently by heat.

TOPIC 17 Current of Electricity

Objectives

Candidates should be able to:

(a) state that current is a rate of flow of charge and that it is measured in amperes
(b) distinguish between conventional current and electron flow
(c) recall and apply the relationship charge = *current* × *time* to new situations or to solve related problems
(d) define electromotive force (e.m.f.) as the work done by a source in driving unit charge around a complete circuit
(e) calculate the total e.m.f. where several sources are arranged in series
(f) state that the e.m.f. of a source and the potential difference (p.d.) across a circuit component is measured in volts
(g) define the p.d. across a component in a circuit as the work done to drive unit charge through the component
(h) state the definition that resistance = *p.d. / current*
(i) apply the relationship $R = V/I$ to new situations or to solve related problems
(j) describe an experiment to determine the resistance of a metallic conductor using a voltmeter and an ammeter, and make the necessary calculations
(k) recall and apply the formulae for the effective resistance of a number of resistors in series and in parallel to new situations or to solve related problems
(l) recall and apply the relationship of the proportionality between resistance and the length and cross-sectional area of a wire to new situations or to solve related problems
(m) state Ohm's Law
(n) describe the effect of temperature increase on the resistance of a metallic conductor
(o) sketch and interpret the I/V characteristic graphs for a metallic conductor at constant temperature, for a filament lamp and for a semiconductor diode

NOTES..

17.1 Conventional Current and Electron Flow

1. Definition of current: the rate of flow of electric charges.

2. Equation: $\boxed{I = \dfrac{Q}{t}}$

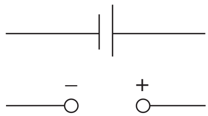
   I is the current (unit: A)
   Q is the charge (unit: C, or equivalent unit: A s)
   t is the time (unit: s)

3. Definition of ampere: 1 ampere is the current carried by 1 coulomb of charge flowing in a circuit in 1 second.

4. The flow of conventional current in a circuit arises from the flow of electrons (negative charges) in the opposite direction.

5. Direct Current (d.c.): A direct current only flows in one direction.

6. Alternating Current (a.c.): An alternating current periodically reverses its direction back and forth.

 d.c. source	 a.c. source

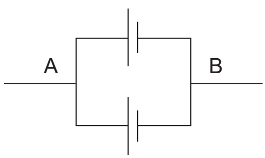
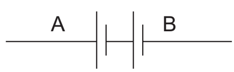
17.2 Electromotive Force (e.m.f.)

1. Definition of electromotive force: The electromotive force of a d.c. source is the work done by the source to drive a unit charge round a closed circuit.

2. Equation: $\boxed{W = QV}$

   W is the work done by source (unit: J)
   Q is the charge (unit: C)
   V is the e.m.f. (unit: V)

3. The following table shows some of the different types of arrangement of 1.5 V cells and the resultant e.m.f.:

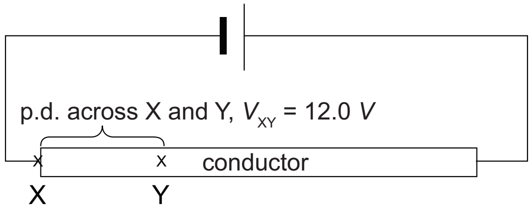
Two cells in parallel Overall e.m.f. across AB = 1.5 V 	**Two cells in series** Overall e.m.f. across AB = 3.0 V 

17.3 Potential Difference (p.d.) K M C

1. Definition of potential difference: The potential difference across a circuit component is the work done to drive a unit charge through the circuit component.

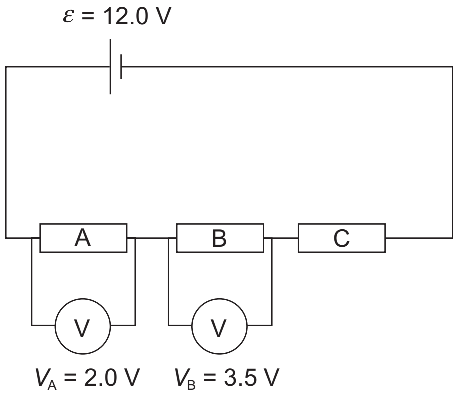
Example 17.1

For a conductor (resistor wire) connected in a closed circuit, the potential difference across two points, X and Y, in part of the conductor is the work done to drive a unit charge across the two points through that part of the conductor.



Example 17.2

The following circuit shows three resistors, A, B and C, connected in series. The potential difference across A and B are given as V_A = 2.0 V and V_B = 3.5 V. Given that the e.m.f. of the battery is 12.0 V, find the potential difference across resistor C.

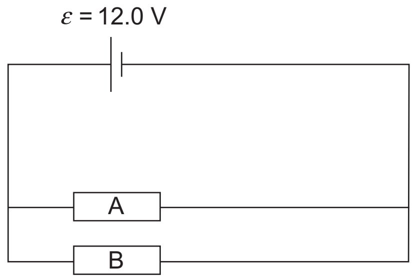


Solution

V_C = 12.0 – 2.0 – 3.5 = 6.5 V

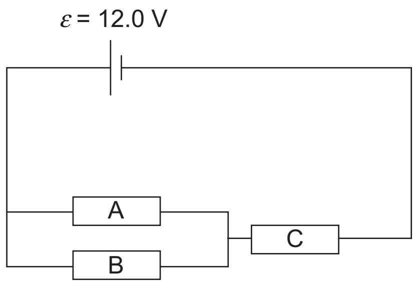
2. In a circuit with two resistors A and B, the potential difference across resistor A is the same as the potential difference across resistor B if the two resistors are arranged in parallel.

   $V_A = V_B = 12.0$ V



Example 17.3

Given that $V_C = 5.0$ V, find the potential difference across A and B.



Solution

Since resistors A and B are arranged in parallel, $V_A = V_B = 12.0 - 5.0 = 7.0$ V.

17.4 Resistance

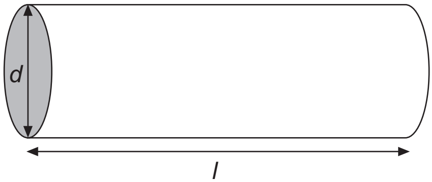
1. Definition of resistance: The ratio of potential difference (V) across the conductor to the current (I) flowing through it.

$$R = \frac{V}{I}$$

   R is resistance of conductor (unit: Ω, equivalent unit: 1 Ω = 1 V A^{-1})
   V is potential difference across the conductor (unit: V)
   I is current through the conductor (unit: A)

2. The resistance of a piece of cylindrical wire R is related to its length l, cross sectional area A and its resistivity, ρ (each type of material has its own resistivity):

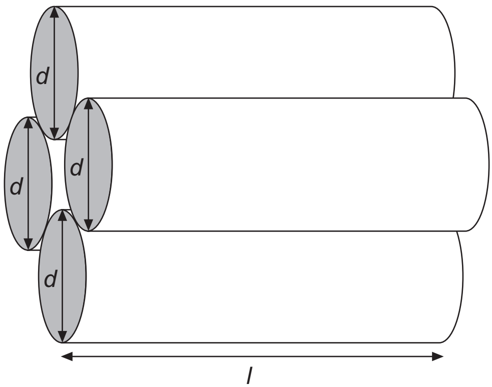
$$R = \frac{\rho l}{A}$$



d – diameter of wire
l – length of wire
Cross-sectional area of wire,

$$A = \pi\left(\frac{d}{2}\right)^2$$

3. Parallel resistors
4 identical resistors are connected in parallel as shown in the diagram.



Effective cross-sectional area = 4 × A = 4A

Effective length of bundle of 4 resistors = l

Effective resistance, R_{eff}

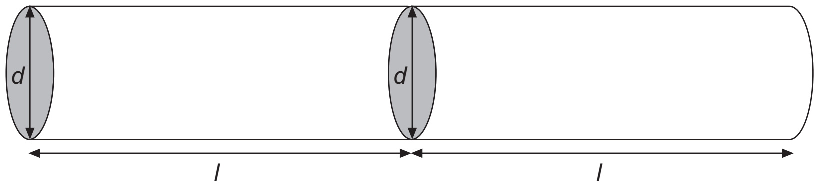
$$= \frac{\rho l}{4A} = \frac{1}{4}R$$

Formula:

$$\frac{1}{R_{elf}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}$$

4. Series resistors **K M C**

2 identical resistors are connected in series as shown in the diagram.



Effective cross-sectional area = A

Effective length of 2 resistors = $2l$

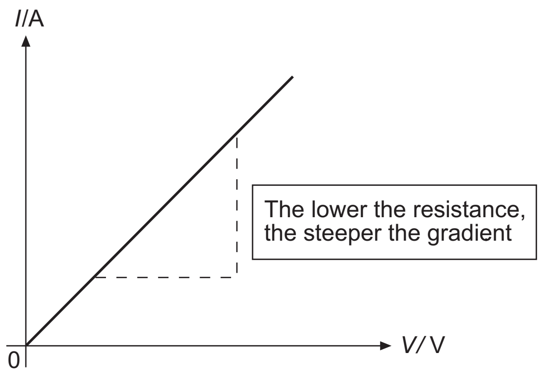
Effective resistance, $R_{eff} = \frac{\rho(2l)}{A} = 2\frac{\rho l}{A} = 2R$

Formula:

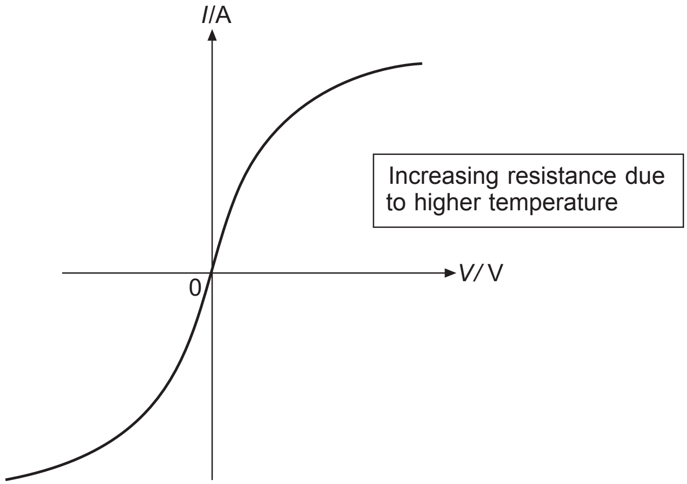
$$\boxed{R_{eff} = R_1 + R_2}$$

5. Ohm's Law: Ohm's law states that the current flowing in a conductor is directly proportional to the potential difference applied across it when all other physical conditions such as temperature are constant.

The I-V graph of an ohmic conductor is as follows:



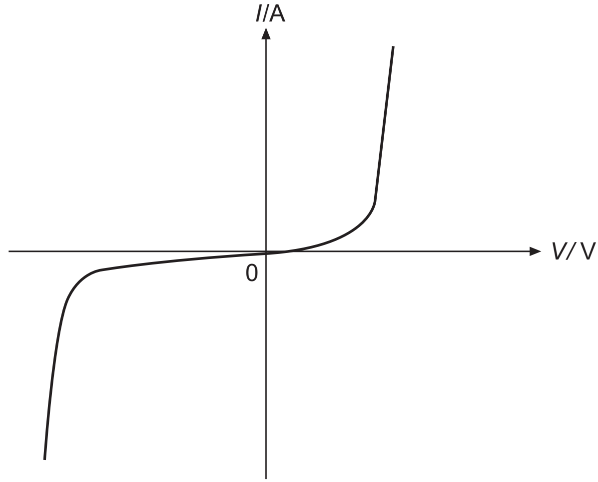
6. For a filament lamp (non-ohmic conductor), its *I-V* graph is not a straight line. As such, it does not obey Ohm's Law. As more current flows in a lamp, its metal filament becomes hotter and atoms in the filament vibrate faster, moving further away from their positions. This leads to an increase in the frequency of collisions with the travelling electrons that hinder their flow, causing more resistance. Hence, the gradient of its graph is fairly constant at low current *I* and potential difference *V*, but with increasing current, the resistance increases (gradient decreases).



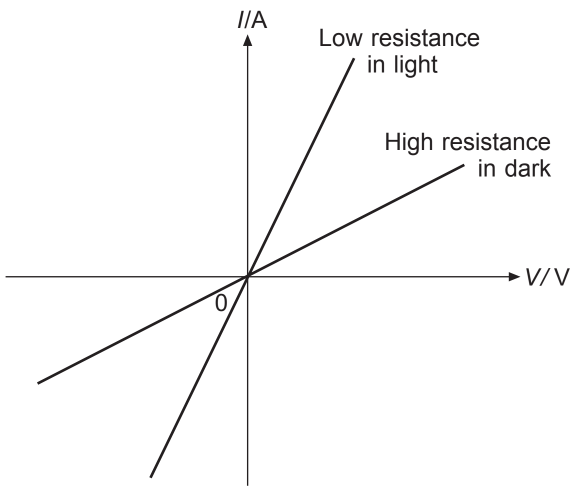
17.5 Diode and Light-dependent Resistor

1. A diode can be used to convert a.c. to d.c. in a process called rectification. A diode is a semiconductor device that allows current to only flow in one direction.

2. The *I*-*V* characteristic graph for the semiconductor diode is shown:



3. A light dependent resistor (LDR) is a semiconductor. When light shines onto the LDR, electrons are released. This increases the number of current-carrying electrons. As the light intensity increases, the current also increases, resulting in a fall in resistance. In the dark, there are no electrons and the current experiences a greater resistance.



TOPIC 18 D.C. Circuits

Objectives

Candidates should be able to:

(a) draw circuit diagrams with power sources (cell, battery, d.c. supply or a.c. supply), switches, lamps, resistors (fixed and variable), variable potential divider (potentiometer), fuses, ammeters and voltmeters, bells, light-dependent resistors, thermistors and light-emitting diodes

(b) state that the current at every point in a series circuit is the same and apply the principle to new situations or to solve related problems

(c) state that the sum of the potential differences in a series circuit is equal to the potential difference across the whole circuit and apply the principle to new situations or to solve related problems

(d) state that the current from the source is the sum of the currents in the separate branches of a parallel circuit and apply the principle to new situations or to solve related problems

(e) state that the potential difference across the separate branches of a parallel circuit is the same and apply the principle to new situations or to solve related problems

(f) recall and apply the relevant relationships, including $R = V/I$ and those for current, potential differences and resistors in series and in parallel circuits, in calculations involving a whole circuit

(g) describe the action of a variable potential divider (potentiometer)

(h) describe the action of thermistors and light-dependent resistors and explain their use as input transducers in potential dividers

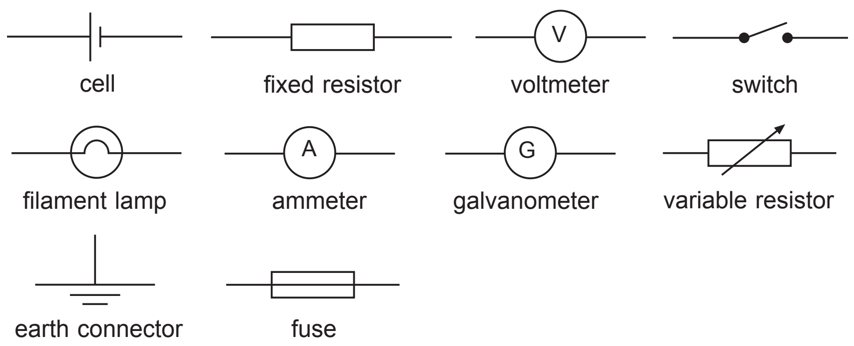
(i) solve simple circuit problems involving thermistors and light-dependent resistors

NOTES...

18.1 Current and Potential Difference in Circuits

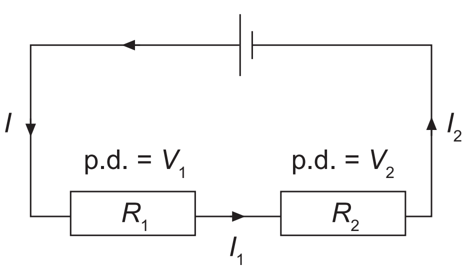
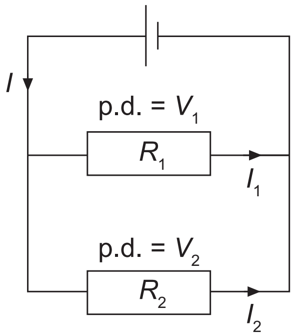
1. Current can only flow in a **closed** circuit.

2. The following table shows some of the electrical symbols used in circuit diagrams:

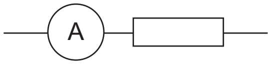
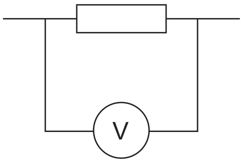


18.2 Series and Parallel Circuits

1. Comparison between a series circuit and a parallel circuit:

Series Circuit	Parallel Circuit
 • Only 1 path for current flow • The current is the same at all points in a series circuit. $I = I_1 = I_2$ • The potential difference across each resistor is different based on their resistance. • The sum of the potential differences across the resistors gives the e.m.f. of the cell. $V = V_1 + V_2$ • Effective resistance: $R_{eff} = R_1 + R_2$	 • There is more than 1 path for the current to flow. • Current: $I = I_1 + I_2$ • The potential difference across each resistor is the same and is equal to the e.m.f. of the cell. $V = V_1 = V_2$ Effective resistance: $\frac{1}{R_{elf}} = \frac{1}{R_1} + \frac{1}{R_2}$

2. Ammeter and Voltmeter

Component	Use	Characteristic
Ammeter	Measures the current flowing through resistor. To be connected in series.  	Very small resistance (so that the potential difference aross it is negligible).
Voltmeter	Measures the potential difference across resistor. To be connected in parallel.  	Very high resistance (so that negligible amount of current will flow through it).

Example 18.1

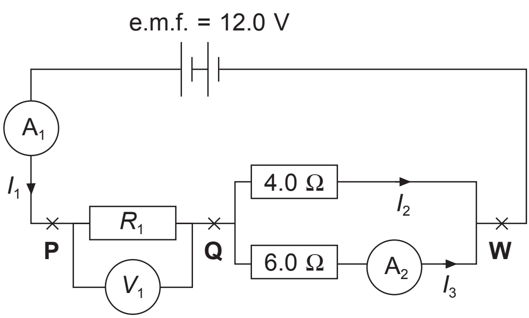
In the following circuit diagram, the effective resistance of the circuit is 5.4 Ω. Find:

(a) the resistance of R_1

(b) the reading of ammeter 1

(c) the voltmeter reading

(d) the reading of ammeter 2



Solution

(a) Effective resistance across **QW** = $\left(\dfrac{1}{4.0} + \dfrac{1}{6.0}\right)^{-1}$ = 2.4 Ω

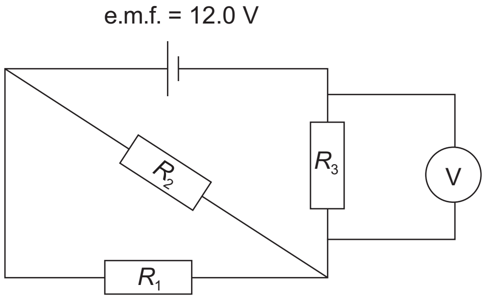
Hence, R_1 = 5.4 – 2.4 = 3.0 Ω

(b) Let the current reading in A_1 be I_1:
Using Ohm's Law: $V = IR$
$12.0 = I_1(5.4)$
$I_1 = 2.222$ A
$= 2.22$ A (to 3 s.f.)

(c) Current through $R_1 = I_1 = 2.222$ A
Potential difference (p.d.) across $R_1 = V$
$V = I_1R_1 = 2.222 \times 3.0$

(d) Let the current reading in A_2 be I_3:
p.d. across **QW** = 12.0 – 6.67 = 5.33 V
$5.33 = I_3(6.0)$
$I_3 = 0.888$ A (to 3 s.f.)

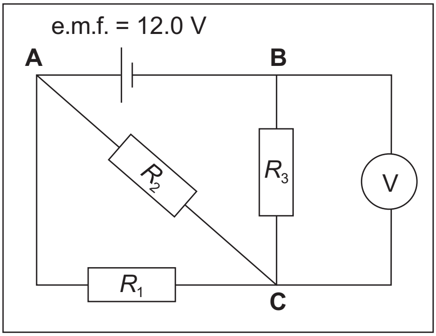
Example 18.2
Three resistors are connected to a 12.0 V battery as shown in the circuit below:



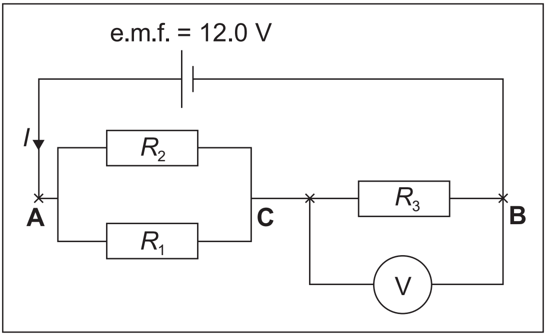
Given that $R_1 = 4.0\ \Omega$, $R_2 = 1.0\ \Omega$, $R_3 = 3.0\ \Omega$, find the voltmeter reading.

Solution K M C

Let us add points **A**, **B**, **C** to the circuit diagram and redraw it.
Observe that R_1 and R_2 are parallel across points **A** and **C**:



Original



Redrawn

Effective resistance across **AC** = $\left(\frac{1}{4.0}+\frac{1}{1.0}\right)^{-1}$ = 0.8 Ω

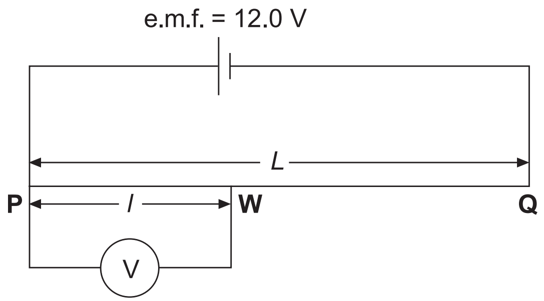
Effective resistance of the whole circuit = 0.8 + R_3 = 0.8 + 3.0 = 3.8 Ω
Let the current through whole circuit be I.
Using Ohm's Law,
12.0 = I × 3.8
I = 3.158 A (to 4 s.f.)
p.d. across R_3 = IR_3 = 3.158 × 3.0 = 9.47 V
Voltmeter reading = 9.47 V (to 3 s.f.)

18.3 Potential Divider Concept

1. Recall that resistance is directly proportional to length:

   $R = \dfrac{\rho l}{A}$.

2. Let us use a uniform wire PQ of length L to replace the box resistors for the circuit below:



Let the resistance of the wire **PQ** be $R_{PQ} = \dfrac{\rho l}{A}$ --------- Equation (1)

Take a point **W** which is the distance l from **P**:

$R_{PW} = \dfrac{\rho l}{A}$ --------- Equation (2)

From Equation (1), $\dfrac{\rho}{A} = \dfrac{R_{PQ}}{L}$. Substitute into Equation (2).

$$R_{PW} = \left(\frac{R_{PQ}}{L}\right) l = \left(\frac{l}{L}\right) R_{PQ}$$

$$\left(\frac{R_{PW}}{R_{PQ}}\right) = \left(\frac{l}{L}\right)$$

Current I through a series circuit is the same.

$$V_{PW} = IR_{PW} = \left(\frac{l}{L}\right) IR_{PQ}$$

Thus,

$$V_{PW} = \left(\frac{l}{L}\right) V$$

When $l = L$, $V_{PW} = V$,

which tells us that (i) as l decreases, V_{PW} also decreases,

(ii) as l increases, V_{PW} also increases,

(iii) $\dfrac{V_{PQ}}{L}$ = constant.

Example 18.3

The wire PQ used in the circuit below has a length of 3.0 m. The resistance of PQ is 4.0 Ω. Find *l* for the voltmeter to register a reading of 4.0 V.

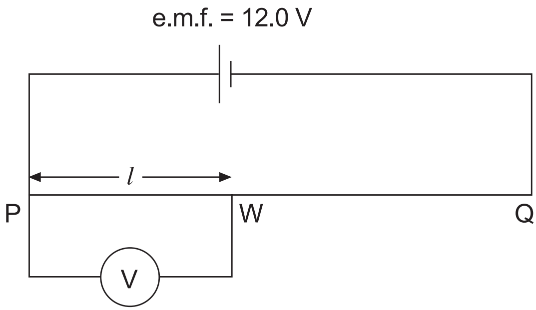
Solution

$$\frac{R_{PW}}{R_{PQ}} = \left(\frac{l}{L}\right)$$

$$\frac{R_{PW}}{R_{PQ}} = \frac{IR_{PW}}{IR_{PQ}} = \left(\frac{l}{L}\right)$$

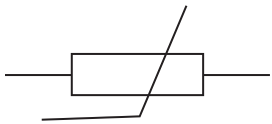
$$\frac{4.0}{12.0} = \left(\frac{l}{3.0}\right)$$

$$l = 1.0 \text{ m}$$



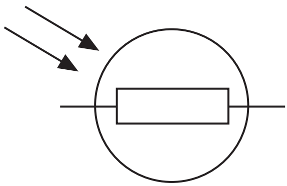
18.4 Thermistors and Light-Dependent Resistors (LDR)

1. A thermistor is a non-ohmic conductor. As it gets hotter, its resistance decreases. Thermistors are used for the control of temperature.



thermistor

2. An LDR is a semiconductor device (cadmium sulphide). Its resistance decreases as the intensity of light on it increases. LDRs are used in illumination control.



LDR

TOPIC 19 Practical Electricity

Objectives

Candidates should be able to:

(a) describe the use of the heating effect of electricity in appliances such as electric kettles, ovens and heaters
(b) recall and apply the relationships $P = VI$ and $E = VIt$ to new situations or to solve related problems
(c) calculate the cost of using electrical appliances where the energy unit is the kW h
(d) compare the use of non-renewable and renewable energy sources such as fossil fuels, nuclear energy, solar energy, wind energy and hydroelectric generation to generate electricity in terms of energy conversion efficiency, cost per kW h produced and environmental impact
(e) state the hazards of using electricity in the following situations:
 (i) damaged insulation
 (ii) overheating of cables
 (iii) damp conditions
(f) explain the use of fuses and circuit breakers in electrical circuits and of fuse ratings
(g) explain the need for earthing metal cases and for double insulation
(h) state the meaning of the terms live, neutral and earth
(i) describe the wiring in a mains plug
(j) explain why switches, fuses, and circuit breakers are wired into the live conductor

NOTES...

19.1 Application of Heating Effects of Electricity

1. Household appliances such as kettles, irons and rice-cookers make use of the heating effect of electric current.

2. Nichrome is chosen as a heating element due to the following advantages:
 (a) cheap
 (b) high resistance
 (c) high melting point
 (d) does not oxidise easily

19.2 Electrical Energy and Power

1. Recall: $W = QV$
   W is the work done by source (unit: J)
   Q is the charge (unit: C)
   V is the e.m.f. (unit: V)

2. Since $Q = It$, we have $W = (It)V = VIt$
   I is the current (unit: A)
   t is the time taken (unit: s)

3. The following table summarises the different forms of the electrical energy equation:

Equation 1	$W = VIt$	
Equation 2	$W = I^2Rt$	by substituting $V = IR$ into (1)
Equation 3	$W = \frac{V^2}{R}t$	by substituting $I = \frac{V}{R}$ into (1)

4. Power, $P = \frac{\text{Work time, } W}{\text{Time, } t}$
   Rearranging, we have $W = Pt$
   Compare with Equations 1, 2 and 3 in the above table:

Equation 1	$P = VI$
Equation 2	$P = I^2R$
Equation 3	$P = \frac{V^2}{R}$

5. SI unit for power: W or J/s

19.3 Calculating Cost of Using Electricity

1. The unit for measuring electrical consumption is the kilowatt-hour (kWh), which is the energy used by an electrical device at a rate of 1000 W in 1 hour.

$$\begin{aligned} 1\text{ kWh} &= 1000\text{ W} \times 1\text{ h} \\ &= 1000 \times 60 \times 60 \\ &= 3\,600\,000\text{ J} \\ &= 3.6 \times 10^6\text{ J} \end{aligned}$$

Example 19.1

Given that electrical energy costs \$0.25 per kWh, find the total cost of running eight 60 W lamps and a 3 kW electrical kettle continuously for 8 minutes.

Solution

Total power = (8 × 60) + (1 × 3000) = 3480 W = 3.48 kW

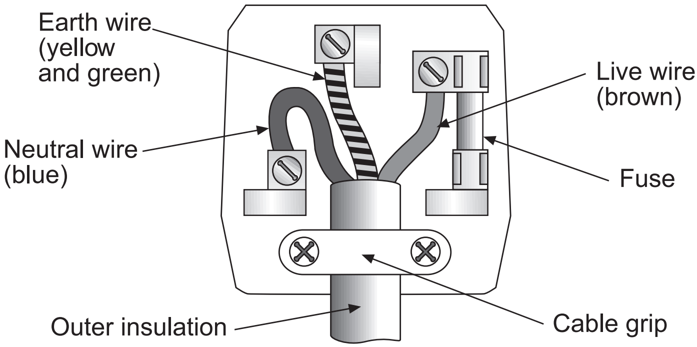
No. of hours of operation $= \dfrac{8}{60} = \dfrac{2}{15}$ h

Total cost $= 3.48 \times \dfrac{8}{60} \times 0.25 = \$0.116 = \$0.12$ (to 2 d.p.)

19.4 Hazards of Using Electricity

1. Electricity is dangerous and can harm people if it is not used properly.
2. Some of the common dangers involved are:
   (a) Handling electrical appliances with wet hands can lead to electric shock.
   (b) Overheated cables can lead to fire.
   e.g. Plugging many appliances to one power point using multiplugs.
   (c) Electrical cables with damaged insulation, especially the live wire, can lead to an electric shock.

19.5 Safe Use of Electricity in the House



1. There are three wires in the household electric cable: live (L), neutral (N) and earth (E).
   (a) All appliances need at least 2 wires (live and neutral) to form a complete circuit.
   (b) The live (L) wire (brown) delivers the current at high voltage from the supply to the appliance. It is the most dangerous, thus switches, fuses and circuit breakers are wired to it instead of the other wires.
   (c) The neutral (N) wire (blue) completes the circuit by forming a path for the current back to the supply. It is usually at 0 V.
   (d) The earth (E) wire (yellow and green) is a low-resistance wire, usually connected to the metal casing of the appliance.
   (e) Earthing (use of earth wire) protects the user from an electric shock if the metal casing should accidentally become live (contacted with bare live wire).
   (f) The large current that flows from the loose live wire through the metal casing and the earth wire will blow the circuit fuse and cut off the supply to the appliance.

2. Fuse
   (a) A fuse is a safety device that is connected to the live wire of an electrical circuit to protect the equipment and wiring against any excessive current flow.
   (b) Characteristics:
   Made of tin-lead alloy with a low melting point.
   Common fuse ratings: 1 A, 2 A, 5 A, 10 A and 13 A.

(c) How does a fuse work?

1. Fuse rating for a fuse in a device must be slightly higher than the current through the device.
2. When the current is too large, the fuse becomes hot and melts (blown fuse), thus cutting off the current flow from the live wire to the device.
3. The blown fuse will have to be replaced by a new one for the device to work again.

3. Switches are used to close and open a circuit. Switching off disconnects the high voltage from an appliance.

4. Double insulation

(a) Double insulation is a safety feature in an electrical appliance that can substitute for an earth wire.

(b) It means that in addition to the first insulation covering the wires, there is a second insulation (**e.g.** plastic casing of a hair dryer).

TOPIC 20 Magnetism

Objectives

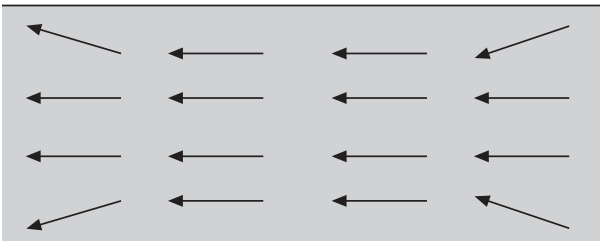
Candidates should be able to:

(a) state the properties of magnets
(b) describe induced magnetism
(c) describe electrical methods of magnetisation and demagnetisation
(d) draw the magnetic field pattern around a bar magnet and between the poles of two bar magnets
(e) describe the plotting of magnetic field lines with a compass
(f) distinguish between the properties and uses of temporary magnets (e.g. iron) and permanent magnets (e.g. steel)

NOTES..

20.1 Laws of Magnetism

1. Properties of Magnets:
   (a) A magnet has two poles where the magnetic forces are the strongest: North pole and South pole.
   (b) Magnets DO NOT exist as monopoles (unlike electric charges).
   (c) We can use arrows to indicate magnetic dipoles in a magnet. The arrowhead indicates North pole.

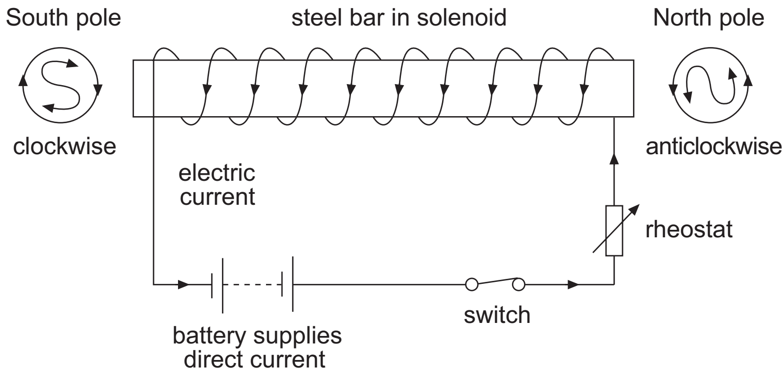


The arrows nearer to the edge are not exactly parallel due to repulsion of like poles.

   (d) The law of magnetism states that like poles repel and unlike poles attract.
   (e) Repulsion is the only way to test if an object is a magnet.

2. Induced magnetism: A magnetic material becomes an induced magnet when placed in a magnetic field, i.e. near a permanent magnet. The magnetic field from the magnet aligns the randomly arranged dipoles in the material.

3. Magnetisation using electricity:
   To magnetise a steel bar, one can place it in a solenoid connected to a d.c. source.
   (a) The magnetic field produced by the solenoid magnetises the steel bar.
   (b) The polarities of the magnetised steel bar depend on the direction of the current.
   (c) If the bar is viewed from one end and the current flows in an anticlockwise direction, then that end will be the North-pole; if clockwise, then that end will be the South-pole.



20.2 Magnetic Properties

1. Examples of magnetic materials: iron, steel, nickel and cobalt.

2. Permanent magnets are magnets that do not lose their magnetism easily. They are made from materials like steel. Steel is an alloy of carbon and iron.

3. The differences between the magnetic properties of iron and steel can be summarised in the table below:

Properties	Iron	Steel
Material	soft	hard
Magnetisation	easy	difficult
Demagnetisation	easy	difficult
Magnetic field strength in solenoid	strong	weak
Magnetism	temporary	permanent

4. Comparison between electromagnet and permanent magnet:

Electromagnet	Permanent magnet
Made of a coil of wire (often with a soft iron core).	Made of hard magnetic material like steel.
Magnetism is temporary. Requires a current through the coil to sustain the magnetic field strength.	Magnetism is permanent. Does not require any electric current to retain magnetic field strength.
Applications: telephone receivers, electric relays, electric bells, circuit breakers and loudspeakers*.	Applications: magnetic doorstops, compasses, motors, dynamos and loudspeakers*

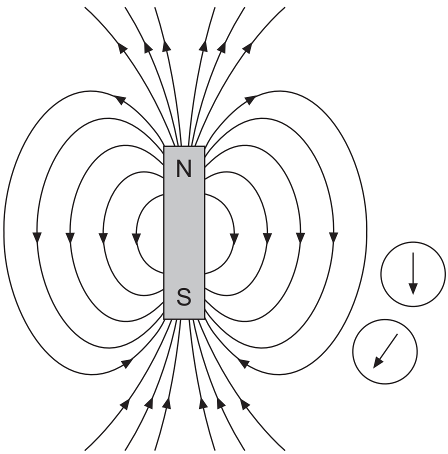
* A loudspeaker uses both an electromagnet and a permanent magnet.

20.3 Magnetic Field

1. A magnetic field is a region in space where magnetic materials experience a force.

2. Magnetic field lines: We draw magnetic field lines to help us visualise the direction of the magnetic forces.

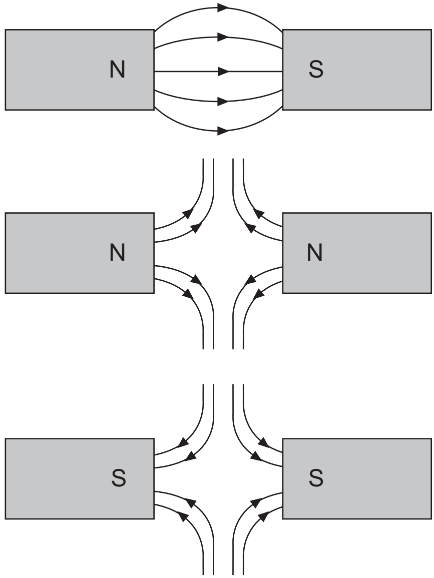
3. A compass can be used to plot the magnetic field lines around a magnet by marking each end of the compass needle with a dot as it is moved from the North pole to the South pole and linking up the dots together to form a solid line. The arrow on the line indicates the direction the compass needle points.



Important:

1. Magnetic field lines always start from North and end at South.
2. Each line is always in a complete closed loop (no matter how big the loop is) unlike electric field lines which can point to infinity.
3. Strength of a magnetic field depends on how close the lines are spaced together. (Closer ⟶ Stronger)

4. The magnetic field lines between like poles and unlike poles are as follows:



TOPIC 21 Electromagnetism

Objectives

Candidates should be able to:

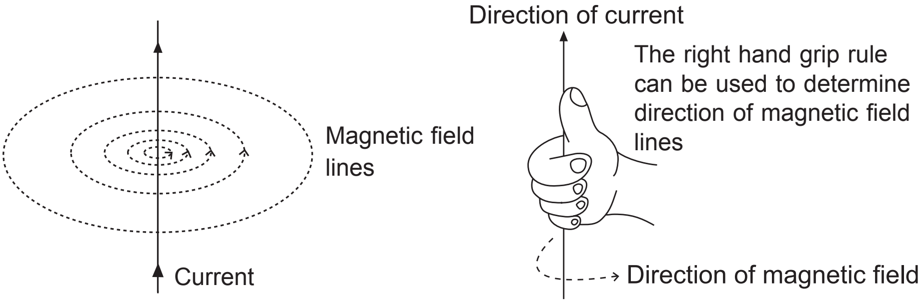
(a) draw the pattern of the magnetic field due to currents in straight wires and in solenoids and state the effect on the magnetic field of changing the magnitude and/or direction of the current
(b) describe the application of the magnetic effect of a current in a circuit breaker
(c) describe experiments to show the force on a current-carrying conductor, and on a beam of charged particles, in a magnetic field, including the effect of reversing
  (i) the current
  (ii) the direction of the field
(d) deduce the relative directions of force, field and current when any two of these quantities are at right angles to each other using Fleming's left-hand rule
(e) describe the field patterns between currents in parallel conductors and relate these to the forces which exist between the conductors (excluding the Earth's field)
(f) explain how a current-carrying coil in a magnetic field experiences a turning effect and that the effect is increased by increasing
  (i) the number of turns on the coil
  (ii) the current
(g) discuss how this turning effect is used in the action of an electric motor
(h) describe the action of a split-ring commutator in a two-pole, single-coil motor and the effect of winding the coil on to a soft-iron cylinder

NOTES..

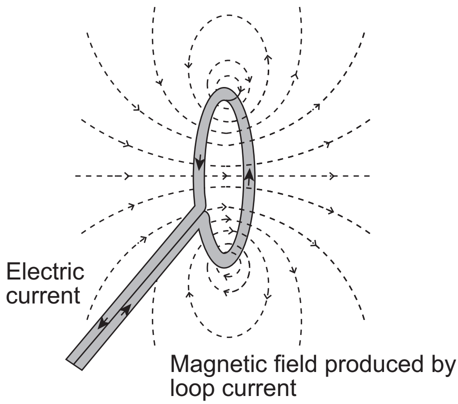
21.1 Magnetic Effect of a Current

1. A current-carrying wire will produce a magnetic field around it. The pattern of the field lines depends on how the wire is shaped.

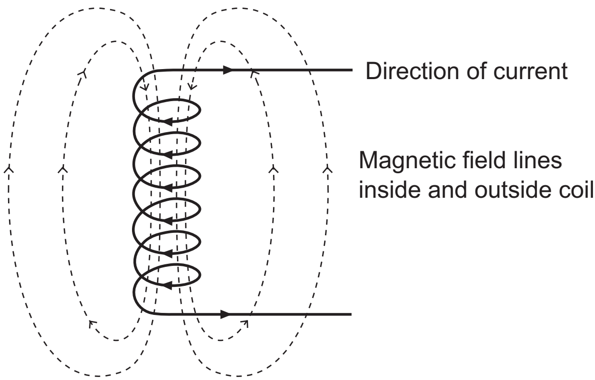
2. For a straight wire, the field lines form concentric circles around the wire as shown (note direction of arrows on field lines):



3. A higher current will result in a stronger magnetic field around the wire.
4. The field pattern of a single turn of circular wire carrying current is as shown:



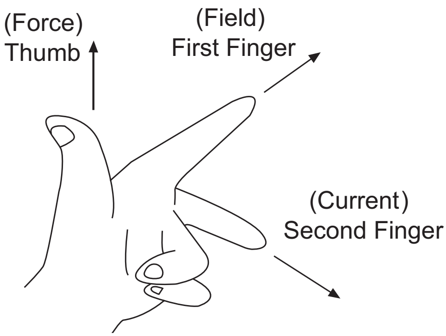
5. A solenoid's magnetic field pattern is as shown:



21.2 Force on a Current-carrying Conductor

1. When a current carrying wire is placed in a magnetic field, it will experience a magnetic force.

2. The direction of the force can be found using Fleming's Left Hand Rule:



Thumb: Direction of force

First Finger: Direction of field

Second Finger: Direction of current in wire

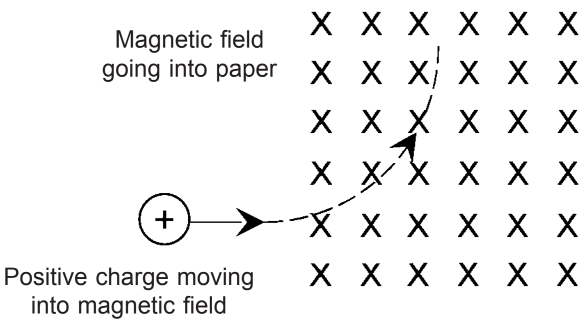
Note: The three fingers must be held at 90° to each other.

3. For a positive charge moving in space, it will behave like a current-carrying wire.

4. For a negative charge, the direction of the current will be **opposite** to its direction of travel.

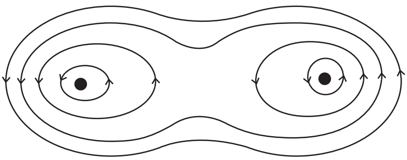
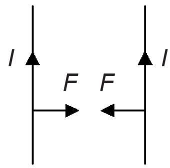
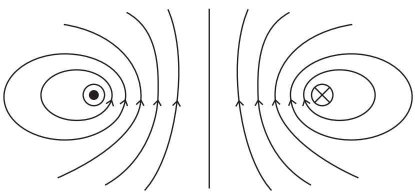
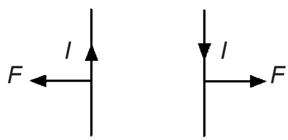
Example 21.1

For a positive charge moving into a magnetic field as shown, it will experience a force to its left; hence its path is curved.



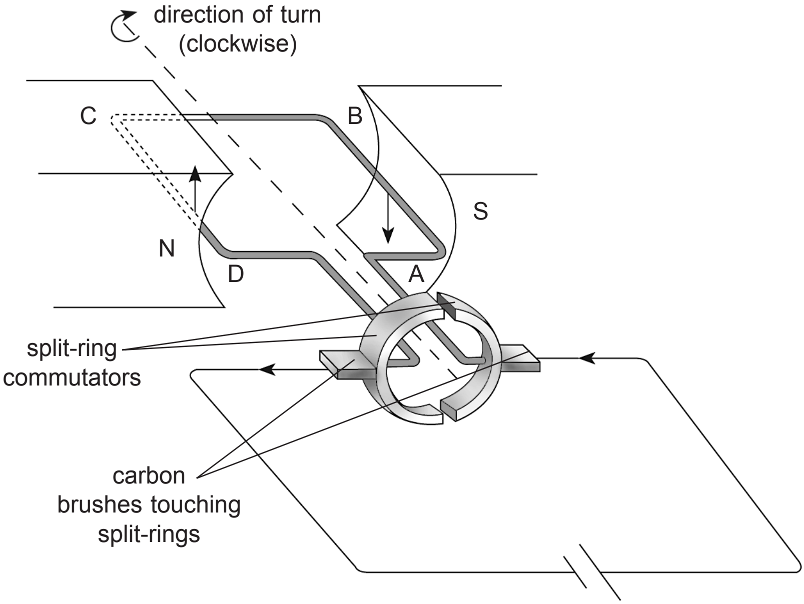
5. Force between two current-carrying wires.

When two wires are carrying current, they will experience mutual forces of attraction or repulsion because each of them will produce a magnetic field which will affect the other. If the currents flow in the same direction, the wires will attract each other; if the currents flow in opposite directions, the wires will repel.

1. Currents in same direction  ● Current coming out of paper	Notice that the field lines are only crowded outside and not in the middle? That is because the field lines cancel out in the middle of the wires. Hence there is an attractive force pulling the two wires together. 
2. Currents in opposite directions  ● Current coming out of paper ⊗ Current going into paper	Notice that the field lines are crowded between the wires. Crowded field lines exert a force sideway against each other. Hence there is a repulsive force pushing the wires apart. 

21.3 D.C. Motor

1. The behaviour of a current-carrying conductor in a magnetic field can be applied in electric motors which convert electrical energy into kinetic energy (i.e. fans).
2. The electric motor makes use of the principle that a current carrying coil will experience a turning effect inside a magnetic field.



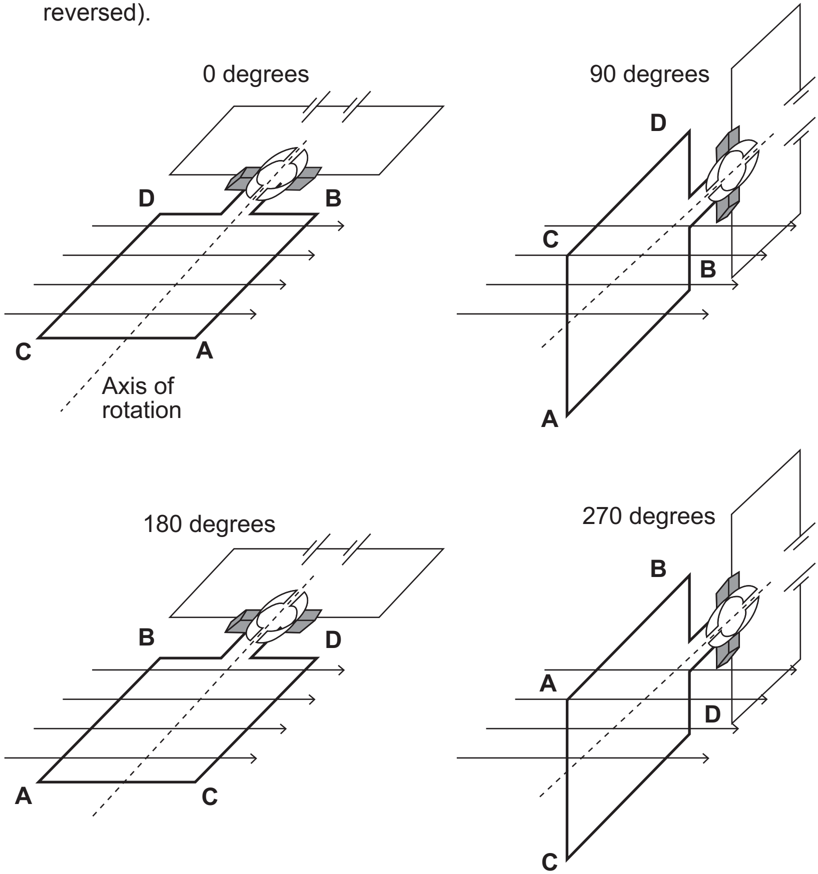
3.

Features	Role
Split-ring commutator	The split in the ring allows direction of current to be reversed in the coil to allow the coil to always rotate in one direction.
Carbon brushes	Carbon (graphite) can conduct electricity and is also a lubricant. It allows the commutator to turn smoothly with minimal friction.

4. Stages of operation
   (a) The carbon brushes make a connection with the coil every 180° turn for current to flow through the coil. In the 0° diagram, the brushes are in contact with the voltage source.
   (b) Current through wire segment C-D interacts with the magnetic field resulting in an upward force (left hand rule). Similarly, current that flows through segment A-B produces a downward force. Both forces are of equal magnitude,but opposite directions (currents in different direction). Thus a turning effect about the axis in the middle of the coil is created.

(c) In the 90° and 270° diagrams, the brushes are not in contact with the voltage source and no force is produced. In these two positions, the rotational kinetic energy of the coil keeps it spinning until the brushes regain contact.

(d) In the 180° diagram, the same thing occurs, but the force on **A-B** is upwards and force on **C-D** is downwards (direction of currents has reversed).



5. The strength of the turning effect can be increased by:
   (a) Increasing strength of magnetic field (use stronger magnets).
   (b) Increasing number of turns of wires in the coil.
   (c) Increasing the area of the coil (Area **ABDC**).
   (d) Increasing the current.
   (e) Adding a soft iron core around which the wires are coiled.

TOPIC 22 Electromagnetic Induction

Objectives

Candidates should be able to:

(a) deduce from Faraday's experiments on electromagnetic induction or other appropriate experiments:
   (i) that a changing magnetic field can induce an e.m.f. in a circuit
   (ii) that the direction of the induced e.m.f. opposes the change producing it
   (iii) the factors affecting the magnitude of the induced e.m.f.

(b) describe a simple form of a.c. generator (rotating coil or rotating magnet) and the use of slip rings (where needed)

(c) sketch a graph of voltage output against time for a simple a.c. generator

(d) describe the use of a cathode-ray oscilloscope (c.r.o.) to display waveforms and to measure potential differences and short intervals of time (detailed circuits, structure and operation of the c.r.o. are not required)

(e) interpret c.r.o. displays of waveforms, potential differences and time intervals to solve related problems

(f) describe the structure and principle of operation of a simple iron-cored transformer as used for voltage transformations

(g) recall and apply the equations $V_P / V_S = N_P / N_S$ and $V_P I_P = V_S I_S$ to new situations or to solve related problems (for an ideal transformer)

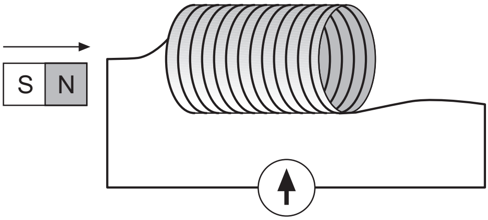
(h) describe the energy loss in cables and deduce the advantages of high voltage transmission

NOTES

22.1 Principles of Electromagnetic Induction

1. Electromagnetic induction: When there is a change in the magnetic flux (magnetic field) linking the conductor, an e.m.f. and hence a current is induced between the ends of the conductor.

2. When the North pole of the bar magnet is moved towards the solenoid, an induced current is generated which produces a North pole at the end of the solenoid facing the magnet. The induced North pole is to oppose the motion of the magnet's North pole. Once the magnet stops moving, the induced current dies down to zero.

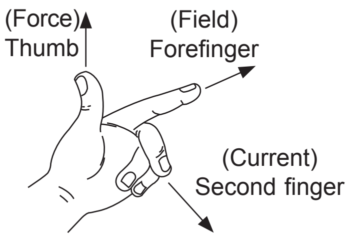
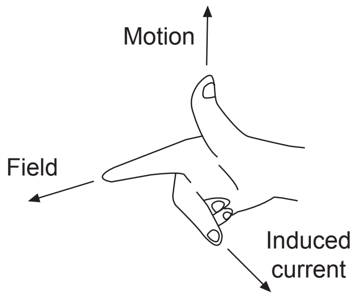


3. Faraday's Law of electromagnetic induction:
   The magnitude of the induced e.m.f. is directly proportional to the rate of change of magnetic flux linking the coil.

4. A larger e.m.f. is produced when:
   (a) the number of turns of wire in solenoid is increased.
   (b) a stronger magnet is used.
   (c) the speed with which magnet is moved towards or away of the solenoid is increased.
   (d) a soft iron core is placed inside the solenoid.

5. Lenz's Law: The direction of induced current sets up a magnetic field to oppose the change in the magnetic flux producing it.

6. Fleming's left hand rule and Fleming's right hand rule

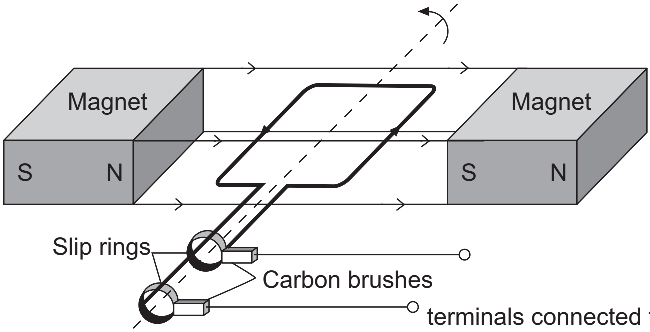
Left Hand	Right Hand
Quantities involved: • Direction of force on conductor • Direction of current • Direction of magnetic field. Given directions of any two of the above three quantities, it is possible to find the direction of the third quantity.  **Thumb:** Direction of force **Forefinger:** Direction of field **Second Finger:** Direction of current in wire Note: The three fingers must be held at 90° to each other.	**Quantities involved:** • Direction of induced current • Direction of magnetic field • Direction of motion Given directions of any two of the above three quantities, it is possible to find the direction of the third quantity.  **Thumb:** Direction of motion **First Finger:** Direction of field **Second Finger:** Direction of induced current in wire Note: The three fingers must be held at 90° to each other.

7. Energy Conversion Process

Dynamo, Generator	Kinetic Energy to Electrical Energy
Motor	Electrical Energy to Kinetic Energy

22.2 A.C. Generator

1.



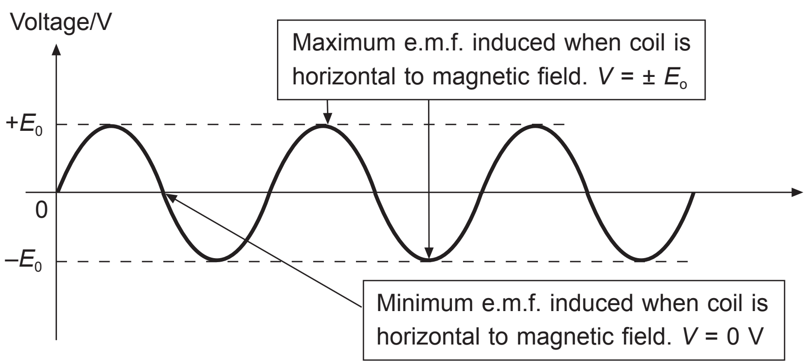
2. An a.c. generator is used to generate electricity. It consists of a coil of rectangular wires situated between two magnets as shown above.

3. When the coil is rotated, it cuts the magnetic field and causes a change in the magnetic flux linkage. As long as the coil keeps on rotating, the rate of change of magnetic flux linking the coil is non-zero and hence, an e.m.f. will be induced in the coil. By Faraday's Law of electromagnetic induction, the magnitude of the e.m.f. that is induced in the coil is directly proportional to the rate of change of magnetic flux linking the coil.

4. Kinetic energy (rotation) is converted into electrical energy.

5. The a.c. generator's coil is connected to two slip-rings which make sliding contact with the carbon brushes at all times (unlike the split-ring commutator used by d.c. motors).

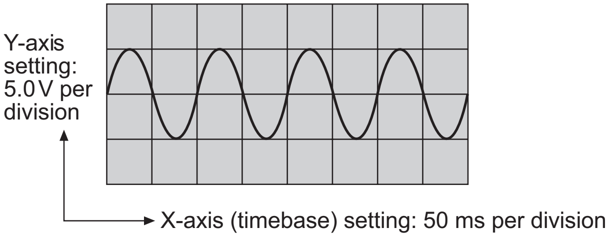
6. The voltage-time graph of the induced e.m.f. is as follows:



22.3 Uses of Cathode-Ray Oscilloscope (c.r.o.) K M C

1. Measure p.d.
2. Display waveforms
3. Measure short time intervals

Example 22.1



Amplitude = one division = 5.0 V

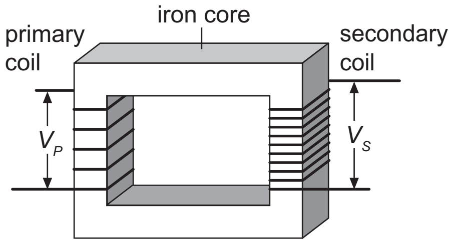
Period, T = 2 divisions = 100 ms = 0.1 s

Frequency, $f = \dfrac{1}{T} = \dfrac{1}{0.1} = 10$ Hz

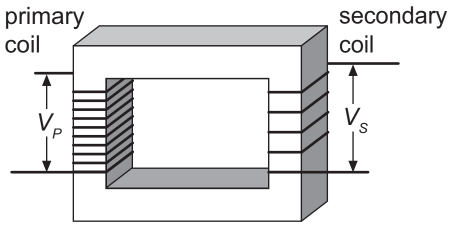
22.4 Principles of Operation of a Transformer

1. The advantage of producing a.c. instead of d.c. at power plants is that a.c. can be stepped up or down to suit household and industries' needs. Household a.c. voltage is stepped down to 240 V.

2. The following diagram shows a step-up and a step-down transformer:



Step-up transformer
$V_P < V_S$
$N_P < N_S$



Step-down transformer
$V_P > V_S$
$N_P > N_S$

3. Principle of operation of a transformer:
a.c. will produce a changing magnetic field. By coiling a primary coil of wires and a secondary coil around an iron core, the changing magnetic field produced by the primary coil will induce an e.m.f. in the secondary coil.

4. For an ideal transformer, we have:
$P_P = P_S \Rightarrow I_P V_P = I_S V_S$

$$\boxed{\frac{I_P}{I_S} = \frac{N_S}{N_P}}$$

Also, $$\boxed{\frac{V_S}{V_P} = \frac{N_S}{N_P}}$$

Note:
1. For practical transformers, if the load on the secondary circuit increases in resistance (more devices connected to the secondary terminals in series), then the amount of power output required will also increase.
2. The power input ($P = I_P V_P$) from the generator is NOT FIXED. Only V_P and V_S are fixed.
   For N2013/P1/Q40, the 230 V has been transmitted over a long distance <u>without transformers</u>.
3. The amount of I_p depends on consumption.

5. Power plants transmit electricity through thick cables at high voltages for the following reasons:
   (a) A higher voltage will mean a lower current travelling in the cable.
   (b) Thick cables (large cross-sectional area) mean the cables have low resistance.

   $$\left(R = \frac{\rho l}{A}\right)$$

   In this way, less power is lost as heat due to heating effect in the cables.

Group																	
I	II											III	IV	V	VI	VII	0
							1 H Hydrogen 1										4 He Helium 2
7 Li Lithium 3	9 Be Beryllium 4											11 B Boron 5	12 C Carbon 6	14 N Nitrogen 7	16 O Oxygen 8	19 F Fluorine 9	20 Ne Neon 10
23 Na Sodium 11	24 Mg Magnesium 12											27 A*l* Aluminium 13	28 Si Silicon 14	31 P Phosphorus 15	32 S Sulfur 16	35.5 C*l* Chlorine 17	40 Ar Argon 18
39 K Potassium 19	40 Ca Calcium 20	45 Sc Scandium 21	48 Ti Titanium 22	51 V Vanadium 23	52 Cr Chromium 24	55 Mn Manganese 25	56 Fe Iron 26	59 Co Cobalt 27	59 Ni Nickel 28	64 Cu Copper 29	65 Zn Zinc 30	70 Ga Gallium 31	73 Ge Germanium 32	75 As Arsenic 33	79 Se Selenium 34	80 Br Bromine 35	84 Kr Krypton 36
85 Rb Rubidium 37	88 Sr Strontium 38	89 Y Yttrium 39	91 Zr Zirconium 40	93 Nb Niobium 41	96 Mo Molybdenum 42	Tc Technetium 43	101 Ru Ruthenium 44	103 Rh Rhodium 45	106 Pd Palladium 46	108 Ag Silver 47	112 Cd Cadmium 48	115 In Indium 49	119 Sn Tin 50	122 Sb Antimony 51	128 Te Tellurium 52	127 I Iodine 53	131 Xe Xenon 54
133 Cs Caesium 55	137 Ba Barium 56	139 La Lanthanum 57 *	178 Hf Hafnium 72	181 Ta Tantalum 73	184 W Tungsten 74	186 Re Rhenium 75	190 Os Osmium 76	192 Ir Iridium 77	195 Pt Platinum 78	197 Au Gold 79	201 Hg Mercury 80	204 T*l* Thallium 81	207 Pb Lead 82	209 Bi Bismuth 83	Po Polonium 84	At Astatine 85	Rn Radon 86
Fr Francium 87	226 Ra Radium 88	227 Ac actinium 89 †															

*58–71 Lanthanoid series
†90–103 Actinoid series

140 Ce Cerium 58	141 Pr Praseodymium 59	144 Nd Neodymium 60	Pm Promethium 61	150 Sm Samarium 62	152 Eu Europium 63	157 Gd Gadolinium 64	159 Tb Terbium 65	163 Dy Dysprosium 66	165 Ho Holmium 67	167 Er Erbium 68	169 Tm Thulium 69	173 Yb Ytterbium 70	175 Lu Lutetium 71
232 Th Thorium 90	Pa Protactinium 91	238 U Uranium 92	Np Neptunium 93	Pu Plutonium 94	Am Americium 95	Cm Curium 96	Bk Berkelium 97	Cf Californium 98	Es Einsteinium 99	Fm Fermium 100	Md Mendelevium 101	No Nobelium 102	Lr Lawrencium 103

Key

a
X
b

a = relative atomic mass
X = atomic symbol
b = proton (atomic) number

TOPIC 1 Kinetic Particle Theory

Objectives

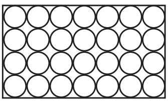
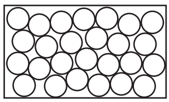
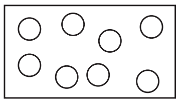
Candidates should be able to:

(a) describe the solid, liquid and gaseous states of matter and explain their interconversion in terms of the kinetic particle theory and of the energy changes involved
(b) describe and explain evidence for the movement of particles in liquids and gases
(c) explain everyday effects of diffusion in terms of particles
(d) state qualitatively the effect of molecular mass on the rate of diffusion and explain the dependence of rate of diffusion on temperature

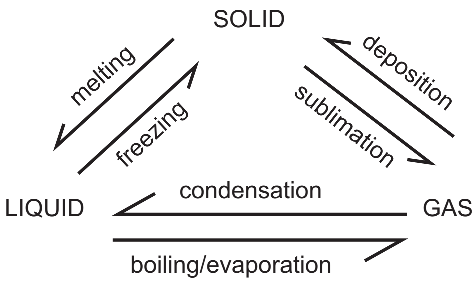
1. Kinetic Particle Theory

All matter is made of particles which are in constant random motion. This accounts for the properties of the three states of matter and the changes of states.

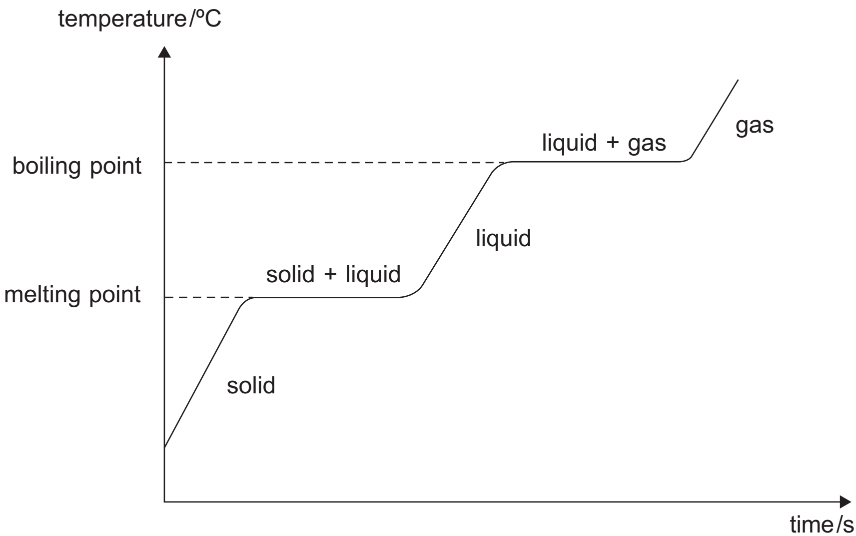
2. Properties of the Three States of Matter

Property	Solid	Liquid	Gas
Structure			
Packing of particles	Tightly packed. Arranged in an orderly manner.	Packed closely together, but not as tightly as in solids. No regular arrangement.	Spaced far apart from each other
Movement of particles	Can only vibrate about fixed positions	Particles slide past each other	Particles move freely at high speeds
Shape	Fixed shape	No fixed shape. Takes on the shape of the container it is in.	No fixed shape. Takes on the shape of the container it is in.
Volume	Fixed volume. Not easily compressed.	Fixed volume. Not easily compressed.	No fixed volume. Easily compressed.

3. Changes of State

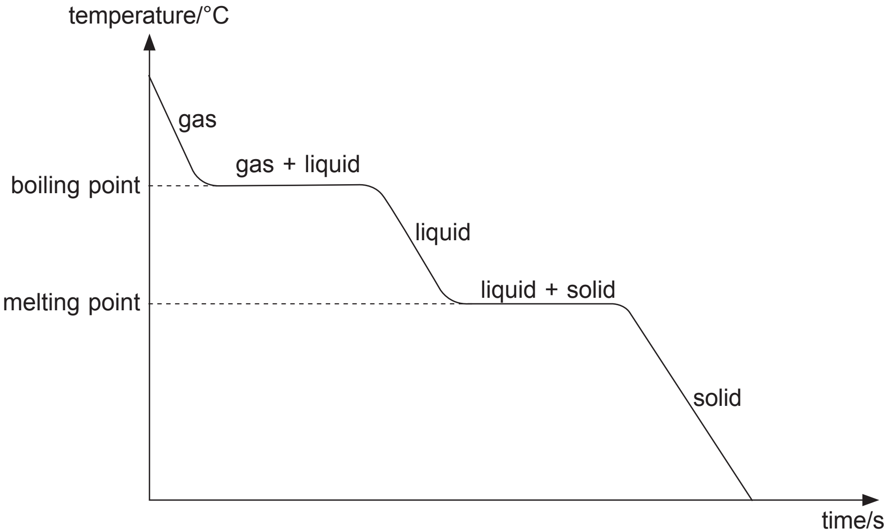


The following diagram shows the temperature change when a substance undergoes changes in state.



At parts where the graph rises, heat is supplied to the substance to raise its temperature. The graph becomes flat when the substance undergoes a change in state. The graph remains flat as heat is taken in to overcome the interactions between the particles.

The following diagram shows the temperature change when a pure substance undergoes cooling.



At parts where the graph falls, heat is given out from the substance to the surroundings and its temperature decreases. The graph becomes flat when the substance undergoes a change in state. The graph remains flat as the particles form bonds, producing heat which is given out to the surroundings.

1. Melting : Occurs at the melting point. Particles absorb heat and vibrate more vigorously, allowing them to overcome the interparticle interactions holding them in fixed positions.

2. Freezing : Occurs at the melting point. Particles release heat and move more slowly. Interparticle interactions are formed and the particles are forced to be held in a fixed and orderly arrangement.

3. Boiling : Occurs at the boiling point. Particles absorb heat and gain more kinetic energy. The particles move fast enough to completely overcome the forces of attraction.

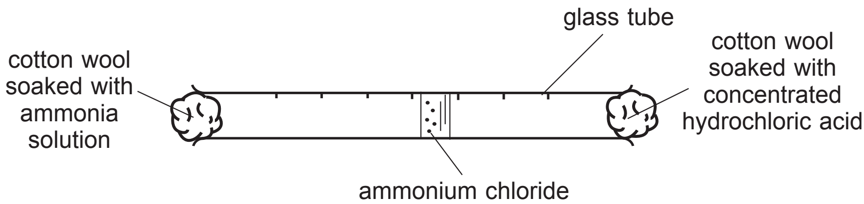
4. Evaporation : Occurs below the boiling point. Particles at the surface gain sufficient energy to escape into the surroundings.

5. Condensation : Occurs at the boiling point. Particles release heat and move more slowly. The forces of attraction are then able to hold the particles closely.

4. Diffusion

Particles of matter move from a region of higher concentration to a region of lower concentration.

Particles with higher mass move more slowly than particles with lower mass. For example, ammonia diffuses at a higher rate than hydrogen chloride since it is lighter (M_r of ammonia = 17, M_r of hydrogen chloride = 36.5).



At higher temperature, the rate of diffusion is greater as the particles have more kinetic energy and can move faster.

TOPIC 2 Experimental Techniques

Objectives

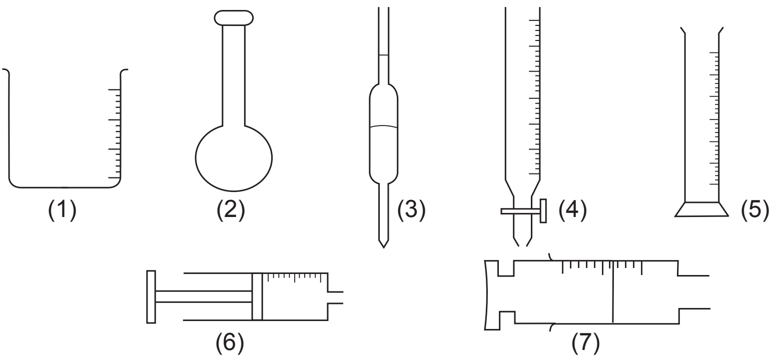
Candidates should be able to:

(a) name appropriate apparatus for the measurement of time, temperature, mass and volume, including burettes, pipettes, measuring cylinders and gas syringes

(b) suggest suitable apparatus, given relevant information, for a variety of simple experiments, including collection of gases and measurement of rates of reaction

1. Measuring Volume

Volumes of solutions have to be frequently measured in chemistry experiments. The following are apparatus for measuring volume.

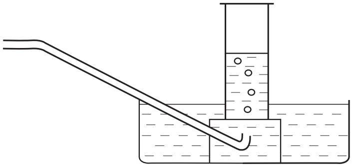


1. Beaker : To measure volumes of liquids approximately according to the graduated marks on the apparatus.
2. Volumetric flask : To accurately measure fixed volumes of liquids when solutions of flask particular concentrations need to be prepared.
3. Pipette : To accurately measure volumes of liquids when a fixed volume of solution is needed for an experiment.
4. Burette : To accurately measure (nearest 0.1 cm^3) volumes of liquids which are used up in an experiment.
5. Measuring cylinder : To measure volumes of liquids with some accuracy (nearest 0.1 cm^3) according to the graduated marks on the apparatus.

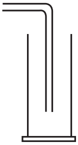
6. Syringe : To measure small volumes of liquids with some accuracy according to the graduated marks on the apparatus.
7. Gas syringe : To accurately measure volumes of gases produced in experiments according to the graduated marks on the apparatus.

2. Collecting Gases Produced

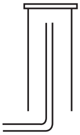
1. Displacement of water: Used to collect gases which are not very soluble in water, such as oxygen and hydrogen.



2. Downward delivery: Used to collect gases which are denser than air, such as carbon dioxide, hydrogen chloride and chlorine.



3. Upward delivery: Used to collect gases which are less dense than air, such as ammonia and hydrogen.



3. Drying Gases Produced

When gases produced need to be obtained dry, the moisture content has to be removed using appropriate drying agents.

1. Fused calcium chloride: This is calcium chloride which has been heated. This can be used to dry gas which does not react with calcium chloride.
2. Concentrated sulfuric acid: This is a common drying agent but it cannot be used to dry gases which are basic.
3. Quick lime: This is a drying agent used to dry basic gases such as ammonia.

TOPIC 3 Methods of Purification

Objectives

Candidates should be able to:

(a) describe methods of separation and purification for the components of mixtures, to include:
- (i) use of a suitable solvent, filtration and crystallisation or evaporation
- (ii) sublimation
- (iii) distillation and fractional distillation
- (iv) use of a separating funnel
- (v) paper chromatography

(b) suggest suitable separation and purification methods, given information about the substances involved in the following types of mixtures:
- (i) solid-solid
- (ii) solid-liquid
- (iii) liquid-liquid (miscible and immiscible)

(c) interpret paper chromatograms including comparison with 'known' samples and the use of R_f values

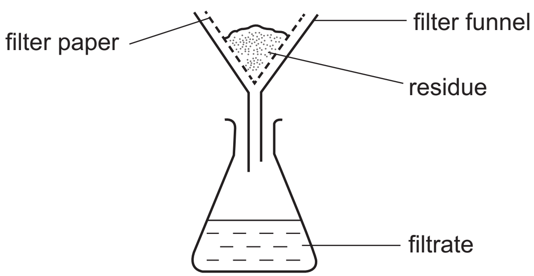
(d) explain the need to use locating agents in the chromatography of colourless compounds

(e) deduce from the given melting point and boiling point the identities of substances and their purity

(f) explain that the measurement of purity in substances used in everyday life, e.g. foodstuffs and drugs, is important

1. Filtration

Filtration is used to separate a mixture of a liquid (or solution) and an insoluble solid. The insoluble solid is collected as the residue while the liquid is collected as the filtrate.

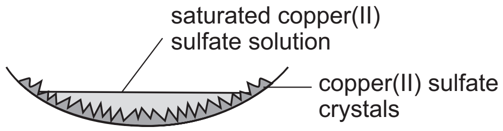


2. Evaporation

This method is used to evaporate off the solvent from a solution to obtain the dissolved substance. This is only applicable to substances that do not decompose upon heating.

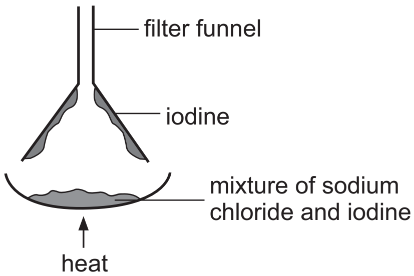
3. Crystallisation

Crystallisation can be used to recover a dissolved substance from its solution. This method is particularly useful for substances that decompose upon heating. This is carried out by heating a solution until it is saturated. The saturated solution is then left to cool, allowing for the substance to crystallise.



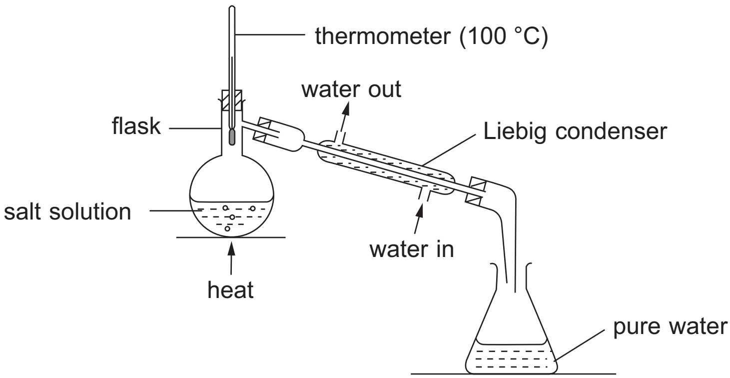
4. Sublimation

This method is used to obtain a solid that sublimes from a solid mixture. Examples of solids that sublime include iodine and naphthalene (found in mothballs).



5. Distillation

Distillation is used to separate a liquid from a mixture. The substances in the mixture must have large differences in boiling points for the pure liquid to be obtained.



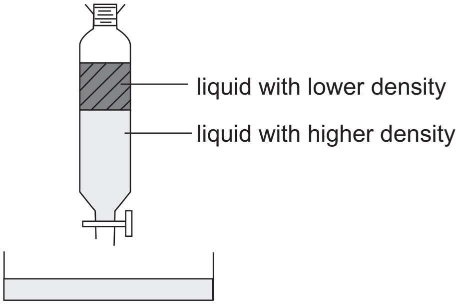
6. Fractional Distillation

In cases where a mixture contains liquids that have relatively close boiling points, fractional distillation is used for purification.

In such mixtures, the vapour produced is a mixture of these substances. The fractionating column aids in separating the vapour into individual components, which allow for the collection of pure substances.

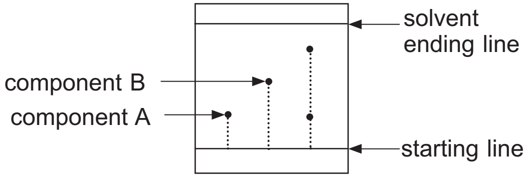
7. Separation using a Separating Funnel

The separating funnel is used to separate a mixture of liquids that have different densities. The liquid with lower density is found in the top layer while the liquid with higher density is found in the bottom layer.



8. Paper Chromatography

This is used in the separation of small quantities of mixtures. The mixture is separated based on the difference in solubility of its components in a particular solvent.



The identity of a component in the mixture can be deduced by comparing the R_f value obtained in the chromatogram with existing R_f values of known substances.

$$R_f \text{ value of a component} = \frac{\text{distance moved by component from the starting line}}{\text{distance moved by solvent from the starting line}}$$

A locating agent is used to expose colourless spots in a chromatogram.

TOPIC 4 Elements and Compounds

Objectives

Candidates should be able to:

(a) describe the differences between elements, compounds and mixtures

1. Elements, Compounds and Mixtures

An element is a substance that cannot be broken down into simpler substances through any chemical or physical means. Elements can exist as atoms or molecules. Each molecule of an element can consist of two or more atoms that are chemically combined.

A compound is a substance that contains two or more elements which are chemically combined in a fixed ratio. It can consist of either molecules or ions. The properties of a compound differ from its constituent elements.

A mixture consists of two or more substances that are mixed together. These substances can be elements or compounds. The ratio of these substances in a mixture is not fixed. The components in a mixture can easily be separated through physical methods.

TOPIC 5 Atoms and Ions

Objectives

Candidates should be able to:

(a) state the relative charges and approximate relative masses of a proton, a neutron and an electron
(b) describe, with the aid of diagrams, the structure of an atom as containing protons and neutrons (nucleons) in the nucleus and electrons arranged in shells (energy levels)
(c) define *proton (atomic) number* and *nucleon (mass) number*
(d) interpret and use symbols such as $^{12}_{6}C$
(e) define the term *isotopes*
(f) deduce the numbers of protons, neutrons and electrons in atoms and ions given proton and nucleon numbers

1. Subatomic Particles

Subatomic Particle	Proton	Neutron	Electron
Mass (amu)	1	1	$\frac{1}{1840}$
Charge	+1	0	–1

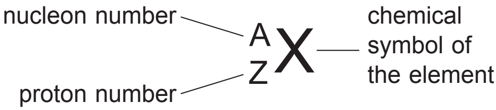
1 atomic mass unit (amu) is approximately 1.67×10^{-27} kg.

Protons and neutrons are found in the nucleus of an atom. They are collectively known as nucleons.

Electrons are found outside the nucleus. They are arranged in shells, also referred to as energy levels, which surround the nucleus.

Isotopes are atoms of the same element that have different numbers of neutrons. They share the same chemical properties but may differ in their physical properties.

2. Chemical Symbol



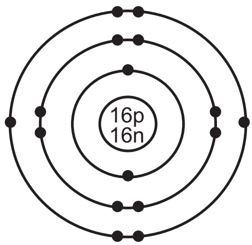
Each element is represented by a unique chemical symbol.

The nucleon number, or the mass number, gives the total number of protons and neutrons in the nucleus of an atom.

The proton number, also called the atomic number, gives the number of protons in the nucleus of an atom. The number of electrons is equal to the number of protons in an atom.

3. Electronic Structure

Electrons are arranged in shells around the nucleus of an atom. The first shell can contain up to 2 electrons and the second shell can hold up to 8 electrons. For simple analysis, it is taken that the third shell holds a maximum of 8 electrons.



Structure of a sulfur atom

Sulfur is represented by the symbol $^{32}_{16}S$, indicating that it has 16 protons and 16 neutrons. The number of neutrons is calculated by subtracting the atomic number from the nucleon number. Since it is electrically neutral, it has 16 electrons as well.

The first electron shell contains 2 electrons, the second shell contains 8 electrons and the third shell contains 6 electrons. The electronic configuration can be written as 2.8.6.

The outermost electron shell is also called the valence electron shell.

TOPIC 6 Chemical Bonding

Objectives

Candidates should be able to:

(a) describe the formation of ions by electron loss/gain in order to obtain the electronic configuration of a noble gas
(b) describe the formation of ionic bonds between metals and non-metals
(c) state that ionic materials contain a giant lattice in which the ions are held by electrostatic attraction
(d) deduce the formulae of other ionic compounds from diagrams of their lattice structures, limited to binary compounds
(e) relate the physical properties (including electrical property) of ionic compounds to their lattice structure
(f) describe the formation of a covalent bond by the sharing of a pair of electrons in order to gain the electronic configuration of a noble gas
(g) describe, using 'dot-and-cross' diagrams, the formation of covalent bonds between non-metallic elements
(h) deduce the arrangement of electrons in other covalent molecules
(i) relate the physical properties (including electrical property) of covalent substances to their structure and bonding

1. Formation of Ions

An atom is most stable when the valence electron shell is completely filled. Atoms of elements either gain or lose electrons to attain a stable electronic configuration.

Non-metals usually gain electrons to form negative ions (anions) while metals usually lose electrons to form positive ions (cations).

The charge of an ion can be found by finding the difference between the number of electrons and the number of protons.

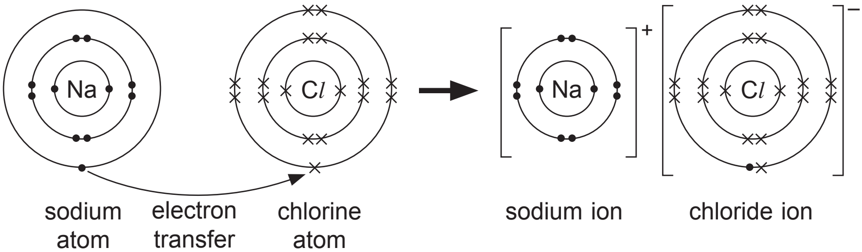
2. Ionic Bonding

This type of bonding takes place between oppositely-charged ions. This usually occurs for compounds made from a metal and a non-metal.

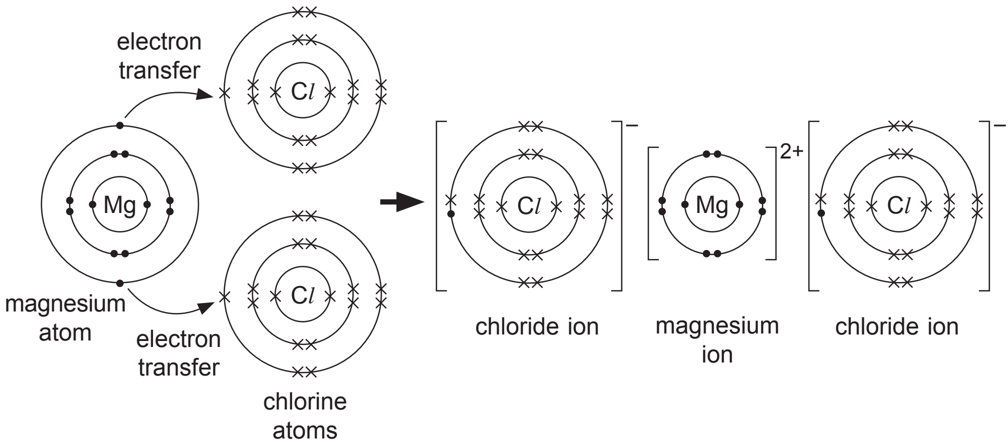
Ionic bonds are formed by electron transfer, where metal atoms donate electrons to non-metal atoms. The ions are arranged in an ionic lattice and are held together by electrostatic forces of attraction.

Two examples of dot-and-cross diagrams that illustrate the formation of ionic bonds are as shown.

1. Sodium (metal) reacts with chlorine (non-metal) to form sodium chloride, $NaCl$



2. Magnesium (metal) reacts with chlorine (non-metal) to form magnesium chloride, $MgCl_2$



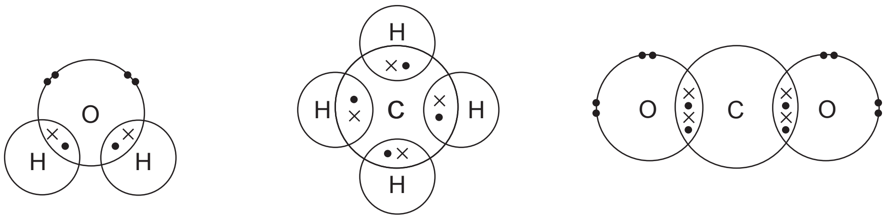
3. Covalent Bonding

Covalent bonds are formed between non-metal atoms. The bond is formed by sharing of electrons between atoms.

A single covalent bond is formed by the sharing of two electrons between two atoms, with the atoms contributing one electron each.

Covalent substances can be found as simple molecules or as large molecules.

Some of the common covalent compounds are shown below with their electron sharing arrangements. Note that only the outermost electrons are used for electron sharing.



water, H_2O | methane, CH_4 | carbon dioxide, CO_2

• electron of oxygen
× electron of hydrogen

• electron of carbon
× electron of hydrogen

• electron of oxygen
× electron of carbon

Covalent bonds are also formed between atoms of the same elements. Hydrogen, oxygen, nitrogen and halogen (Group VII) elements exist as diatomic molecules by forming covalent molecules of two atoms bonded together. The covalent bonds in hydrogen and oxygen molecules are shown below.

hydrogen molecule, H_2 | oxygen molecule, O_2

TOPIC 7 Structure of Matter

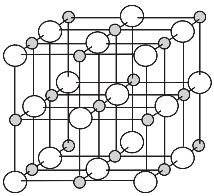
Objectives

Candidates should be able to:

(a) compare the structure of simple molecular substances, e.g. methane; iodine, with those of giant molecular substances, e.g. poly(ethene); sand (silicon dioxide); diamond; graphite in order to deduce their properties
(b) compare the bonding and structures of diamond and graphite in order to deduce their properties such as electrical conductivity, lubricating or cutting action
(c) deduce the physical and chemical properties of substances from their structures and bonding and vice versa
(d) describe metals as a lattice of positive ions in a 'sea of electrons'
(e) relate the electrical conductivity of metals to the mobility of the electrons in the structure

1. Ionic Compounds

In ionic compounds, the positive ions and negative ions are held together by strong electrostatic forces of attraction, forming giant lattice structures.



Ionic compounds have very high melting and boiling points. This is because a lot of energy is required to overcome the strong forces of attraction holding the ions in the lattice together before the compound can melt or boil. Due to their high melting and boiling points, they are usually found as solids at room temperature and pressure.

The melting and boiling points are influenced by the strength of the electrostatic forces of attraction. Magnesium oxide has a higher melting point than sodium chloride. The ions in sodium chloride have charges of +1 and –1, while the ions in magnesium oxide have charges of +2 and –2. The electrostatic forces of attraction are stronger in magnesium oxide, hence more energy is required to melt it.

Ionic compounds conduct electricity when dissolved in water or in molten state, but not when in solid state. In aqueous and molten states, the ions are free to move and hence can conduct electricity. In solid state however, the ions are held in fixed positions in the lattice structure.

2. Simple Molecular Structures

Covalent substances with simple molecular structures consist of small discrete molecules that are held together by weak intermolecular forces of attraction. These forces are also known as van der Waals' force of attraction.

Substances with simple molecular structures have low melting and boiling points as a small amount of energy is required to overcome the weak intermolecular forces of attraction.

The strength of the forces of attraction is dependent on molecular size. Substances with large molecules are held together by stronger intermolecular forces compared to those with small molecules. Therefore, the melting and boiling points of large simple molecules are higher than those of small simple molecules.

These substances do not conduct electricity as they do not have any freely-moving charge carriers.

3. Giant Molecular Structures

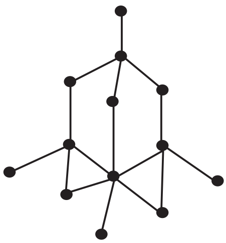
Covalent substances with giant molecular structures consist of an extensive network of atoms held together by covalent bonds.

Substances with giant molecular structures have high melting and boiling points as a lot of energy is required to overcome the strong covalent bonds holding the atoms together.

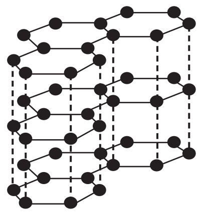
Apart from graphite, giant molecular substances usually do not conduct electricity.

4. Diamond and Graphite

Diamond and graphite are allotropes of carbon which have giant molecular structures. The carbon atoms in these substances are arranged in different manners, hence giving them different properties.



diamond



graphite

Each atom in diamond is covalently bonded to four other atoms. Due to its rigid structure, diamond is a very hard substance and is used for drill tips or cutting tools.

All valence electrons in each carbon atom are used for covalent bonding, therefore diamond cannot conduct electricity.

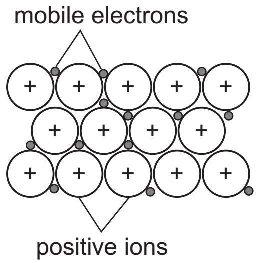
Each atom in graphite is covalently bonded to three other atoms, forming a continuous layer of carbon atoms arranged in hexagons. Graphite consists of many layers of carbon atoms which are held together by weak van der Waals' forces of attraction. These layers of carbon atoms can slide past each other, making graphite a soft and slippery substance. This makes graphite suitable for use as a lubricant.

Each carbon atom in graphite has one free electron since each atom forms only three covalent bonds. These electrons are delocalised along the layer of carbon atoms. The presence of delocalised electrons allows for the conduction of electricity.

Both diamond and graphite have very high melting and boiling points as a lot of energy is required to break the strong covalent bonds holding the carbon atoms together.

5. Metallic Bonding

Atoms in a metal are held by metallic bonding in a giant lattice structure. These atoms lose their valence electrons, which are then delocalised across the metal lattice. The metal lattice structure consists of lattice of positive ions surrounded by a 'sea of electrons'. The electrostatic forces of attraction between the positive ions and the mobile electrons hold the structure together.



Metals are good conductors of electricity and heat due to the presence of mobile electrons.

Metals have high melting and boiling points as a lot of energy is required to overcome the strong electrostatic forces of attraction between the 'sea of electrons' and the lattice of positive ions.

As atoms in metals are packed tightly in layers, they usually have high densities. The neat arrangement of atoms also makes metals malleable and ductile, which means that metals can be shaped by applying pressure and stretched without breaking. Metallic bonding is not affected when a force is applied as the layers of positive ions can slide past each other among the 'sea of mobile electrons'.

TOPIC 8 Writing Formulae and Equations

Objectives

Candidates should be able to:

(a) state the symbols of the elements and formulae of the compounds mentioned in the syllabus
(b) deduce the formulae of simple compounds from the relative numbers of atoms present and vice versa
(c) deduce the formulae of ionic compounds from the charges on the ions present and vice versa
(d) interpret chemical equations with state symbols
(e) construct chemical equations, with state symbols, including ionic equations

1. Chemical Equations

A chemical equation shows the reactants and products in a reaction and may include state symbols, which show the physical states of each substance.

Solid, liquid and gaseous states are represented by the state symbols (s), (l) and (g) respectively. Substances that are dissolved in water to form an aqueous solution are represented with the state symbol (aq).

As atoms cannot be created or destroyed, the number of atoms of each element has to be the same on both sides of the equation, i.e. the equation has to be balanced. Numbers are added in front of the chemical formulae to balance the equation.

A reaction may be described as 'irreversible' or 'reversible'. This can be indicated in a chemical equation by using different arrows. $\rightarrow$ is used for irreversible reactions and $\rightleftharpoons$ is used for reversible reactions.

2. Ionic Equations

Ionic equations are chemical equations that show the reaction involving ions. It is important to note the physical states of all substances in the reaction.

A balanced chemical equation with state symbols is first written. Break this equation down further by writing the dissolved substances in terms of ions. It is useful to note the solubility rules of salts in water before rewriting the equation. The ionic equation is then obtained by cancelling out spectator ions.

Spectator ions remain unchanged at the end of the reaction, showing that they do not take part in the reaction.

TOPIC 9 Stoichiometry and Mole Concept

Objectives

Candidates should be able to:

(a) define relative atomic mass, A_r
(b) define relative molecular mass, M_r, and calculate relative molecular mass (and relative formula mass) as the sum of relative atomic masses
(c) calculate the percentage mass of an element in a compound when given appropriate information
(d) calculate empirical and molecular formulae from relevant data
(e) calculate stoichiometric reacting masses and volumes of gases (one mole of gas occupies 24 dm^3 at room temperature and pressure); calculations involving the idea of limiting reactants may be set
(f) apply the concept of solution concentration (in mol/dm^3 or g/dm^3) to process the results of volumetric experiments and to solve simple problems
(g) calculate % yield and % purity

1. Relative Atomic Mass

The relative atomic mass (A_r) of an atom is the average mass of the atom compared with $\frac{1}{12}$ of the mass of a carbon-12 atom. This value is a ratio and does not have any units.

The relative atomic mass is not always a whole number due to the presence of isotopes (as covered in Topic 5). This value is obtained by taking the average of the relative masses of isotopes of an element based on their natural abundance.

2. Relative Molecular Mass and Relative Formula Mass

The mass of a molecule, which can be a compound or an element, is given by the relative molecular mass (M_r). The relative molecular mass is the average mass of the molecule compared with $\frac{1}{12}$ of the mass of a carbon-12 atom.

This value is the sum of the relative atomic masses of the component atoms as stated in the chemical formula of the molecule.

Ionic compounds do not exist as molecules, therefore it is more accurate to refer to their mass as the relative formula mass. The relative formula mass of an ionic compound is the sum of the relative atomic masses of atoms as stated in its chemical formula.

3. The Mole

1 mole of any substance consists of 6×10^{23} particles. The number 6×10^{23} is called the Avogadro's constant.

The number of moles of a substance can be obtained by dividing the total number of particles by the Avogadro's constant.

$$\text{Number of moles} = \frac{\text{Number of particles}}{6 \times 10^{23}}$$

The mass of 1 mole of substance is given by its molar mass. The molar mass of an element is equal to its relative atomic mass. For a molecular substance, its molar mass is equal to its relative molecular mass. Likewise, the molar mass of an ionic compound is equal to its relative formula mass.

Molar mass has the units g/mol. The number of moles can be obtained by dividing the mass of the substance in grams by its molar mass.

$$\text{Number of moles} = \frac{\text{Mass (g)}}{\text{Molar mass (g/mol)}}$$

4. Percentage Composition of Compounds

The percentage by mass of an element in a compound is given by the following formula.

Percentage by mass of an element in a compound

$$= \frac{\text{Number of atoms of the element} \times A_r \text{ of the element}}{\text{Molar mass (g/mol)}} \times 100\%$$

5. Empirical and Molecular Formulae

The empirical formula of a compound gives the simplest ratio of the number of atoms of each element in the compound. This is found by taking the proportions of atoms of each element and comparing them in terms of moles.

The molecular formula of a compound gives the actual number of atoms of each element in the compound. The molecular formula of a compound is always a multiple of its empirical formula.

Since the molecular formula is always a multiple of the empirical formula, a compound with the empirical formula A_xB_y has a molecular formula of $(A_xB_y)_n$, where n is an integer. The value of n can be found using the following formula.

$$n = \frac{\text{actual relative molecular mass}}{\text{relative molecular mass from empirical formula}}$$

6. Calculations Involving Gases

1 mole of any gas occupies a volume of 24 dm^3 at room temperature and pressure. This volume is also called the molar volume. Recall that 1 dm^3 is equal to 1000 cm^3.

The number of moles of gas is given by dividing the volume of the gas by the molar volume. Note that the calculation applies only at room temperature and pressure.

$$\text{Number of moles} = \frac{\text{Volume of gas (dm}^3\text{)}}{24\ \text{dm}^3}$$

7. Calculations Involving Solutions

Calculating the amount of reactant particles in a solution requires the concentration of the solution. The concentration gives the amount of reactants dissolved per unit volume of a solution. This can be expressed in g/dm^3 or mol/dm^3.

Concentration in g/dm^3 can be converted to mol/dm^3 by using the formula below.

$$\text{Concentration in mol/dm}^3 = \frac{\text{Concentration in g/dm}^3}{\text{Molar mass of reactant in g/mol}}$$

8. Percentage Yield and Percentage Purity

The percentage yield of a reaction is calculated using the theoretical yield and the actual yield.

$$\text{Percentage yield} = \frac{\text{Actual yield}}{\text{Theoretical yield}} \times 100\%$$

The theoretical yield refers to the calculated amount of products, assuming that the reaction goes into completion. The actual yield is the amount of product that forms in the actual reaction.

The percentage purity gives the percentage of a substance in an impure sample.

$$\text{Percentage purity} = \frac{\text{Mass of pure substance in the sample}}{\text{Mass of the sample}} \times 100\%$$

TOPIC 10 Acids and Bases

Objectives

Candidates should be able to:

(a) describe the meanings of the terms *acid* and *alkali* in terms of the ions they produce in aqueous solution and their effects on Universal Indicator
(b) describe how to test hydrogen ion concentration and hence relative acidity using Universal Indicator and the pH scale
(c) describe qualitatively the difference between strong and weak acids in terms of the extent of ionisation
(d) describe the characteristic properties of acids as in reactions with metals, bases and carbonates
(e) state the uses of sulfuric acid in the manufacture of detergents and fertilisers; and as a battery acid
(f) describe the reaction between hydrogen ions and hydroxide ions to produce water, $H^+ + OH^- \rightarrow H_2O$, as neutralisation
(g) describe the importance of controlling the pH in soils and how excess acidity can be treated using calcium hydroxide
(h) describe the characteristic properties of bases in reactions with acids and with ammonium salts
(i) classify oxides as acidic, basic, amphoteric or neutral based on metallic/non-metallic character

1. Physical Properties of Acids

An acid is a substance that dissolves in water to produce hydrogen ions (H^+). Acids have a sour taste, turn blue litmus red and give solutions with pH values below 7.

As hydrogen ions are responsible for the properties of acids, an acid that is not dissolved in water does not show these properties.

Some commonly used acids are hydrochloric acid (HCl), sulfuric acid (H_2SO_4) and nitric acid (HNO_3).

2. Chemical Properties of Acids

Dilute acids react with metals that lie above hydrogen in the reactivity series. The reaction produces salt and hydrogen gas.

zinc + dilute hydrochloric acid → zinc chloride + hydrogen
$Zn(s) + 2HCl(aq) \rightarrow ZnCl_2(aq) + H_2(g)$

Metals such as copper and silver do not react with dilute acids as they are unreactive.

Although lead lies above hydrogen in the reactivity series, it appears to be unreactive in dilute hydrochloric acid and dilute sulfuric acid. This is due to the formation of a layer of insoluble salt around the metal. The layer prevents contact between the acid and the metal, therefore causing the reaction to end prematurely.

Acids react with carbonates (and hydrogen carbonates) to produce salt, water and carbon dioxide.

calcium carbonate + dilute sulfuric acid → calcium sulfate + water + carbon dioxide
$CaCO_3(s) + H_2SO_4(aq) \rightarrow CaSO_4(aq) + H_2O(l) + CO_2(g)$

Acids react with bases to form salt and water. The base could be a metal oxide or an alkali.

aluminium oxide + dilute hydrochloric acid → aluminium chloride + water
$Al_2O_3(s) + 6HCl(aq) \rightarrow 2AlCl_3(aq) + 3H_2O(l)$

sodium hydroxide + dilute hydrochloric acid → sodium chloride + water
$NaOH(aq) + HCl(aq) \rightarrow NaCl(aq) + H_2O(l)$

3. Acid Strength and Concentration

Acid strength is determined by the degree of ionisation of an acid in water.

A strong acid fully ionises in water to form H^+ ions. Such acids include hydrochloric acid, sulfuric acid and phosphoric acid.

$$HCl(aq) \rightarrow H^+(aq) + Cl^-(aq)$$

A weak acid partially ionises in water. The partial dissociation is represented in an equation with a ⇌ symbol. Examples of weak acids include carboxylic acids, such as ethanoic acid (CH_3COOH).

$$CH_3COOH(aq) \rightleftharpoons CH_3COO^-(aq) + H^+(aq)$$

The concentration of an acid depends on the amount of acid dissolved in water. Dissolving a small amount of acid in water gives a dilute acid solution, while dissolving a large amount of acid in water gives a concentrated acid solution.

4. Uses of Sulfuric Acid

Sulfuric acid (H_2SO_4) is an important substance in the chemical industries. It is used in the manufacture of detergents and fertilisers. It is also used in car batteries as an electrolyte.

5. Physical Properties of Bases

A base is a metal oxide or hydroxide that reacts with acids to produce salt and water.

Some bases dissolve in water to produce OH^- ions. These bases are known as alkalis. Examples of such alkalis include sodium hydroxide and calcium hydroxide.

Bases have a bitter taste, turn red litmus blue, and give solutions with pH values above 7.

6. **Chemical Properties of Bases**

Alkalis undergo neutralisation with acids to produce salt and water only. Neutralisation involves the reaction between H^+ and OH^- ions to produce water. This can be described in the following ionic equation.

$$H^+(aq) + OH^-(aq) \rightarrow H_2O(l)$$

Heating alkalis with ammonium salts produces salt, water and ammonia gas.

sodium hydroxide + ammonium chloride → sodium chloride + water + ammonia
$NaOH(aq) + NH_4Cl(aq) \rightarrow NaCl(aq) + H_2O(l) + NH_3(g)$

7. **The pH Scale**

The pH scale is a measure of acidity or basicity of substances that are dissolved in water. This measurement is made based on the relative concentrations of H^+ and OH^- ions present.

The pH scale ranges from 0 to 14. Acids have pH values below 7 while bases have pH values above 7. Neutral solutions have a pH value of 7.

Acids have higher concentrations of H^+ ions compared to OH^- ions. An acid that has a high concentration of H^+ ions will have a lower pH value than an acid with a low concentration of H^+ ions.

Bases have higher concentrations of OH^- ions compared to H^+ ions. A base that has a high concentration of OH^- ions will have a higher pH value than a base with a low concentration of OH^- ions.

8. **pH Indicators**

A pH indicator displays different colours at different pH values.

Universal Indicator is a mixture of pH indicators that gives different colours at different pH values. The table below lists the different colours and the pH range at which they are observed.

pH range	Colour
Below 3	Red
3 to 6	Orange/Yellow
7	Green
8 to 11	Blue
Above 11	Violet

9. Oxides

Oxides are compounds formed from oxygen and another element. These can be categorised into four types of oxides, namely acidic oxides, basic oxides, amphoteric oxides and neutral oxides.

Non-metals usually form acidic oxides. These oxides can dissolve in water to give acids. Acidic oxides react with bases to form salt and water. For example, carbon dioxide reacts with calcium hydroxide to form calcium carbonate and water.

carbon dioxide + calcium hydroxide → calcium carbonate + water
$CO_2(g) + Ca(OH)_2(aq) \rightarrow CaCO_3(s) + H_2O(l)$

Metals usually form basic oxides. Some of these oxides dissolve in water to give alkalis. Basic oxides react with acid to form salt and water. For example, magnesium oxide reacts with sulfuric acid to form magnesium sulfate and water.

magnesium oxide + sulfuric acid → magnesium sulfate + water
$MgO(s) + H_2SO_4(aq) \rightarrow MgSO_4(aq) + H_2O(l)$

Some metals form amphoteric oxides. These oxides display both acidic and basic properties and as such, can react with both acids and bases. Such oxides include aluminium oxide (Al_2O_3), zinc oxide (ZnO) and lead(II) oxide (PbO).

Some non-metals form neutral oxides, which exhibit neither basic nor acidic properties. Instances of such oxides are water (H_2O), carbon monoxide (CO) and nitric oxide (NO).

10. Soil pH

Plants are sensitive to changes in soil pH. The pH levels can be controlled by adding certain chemicals. For acidic soil, bases such as calcium oxide (quicklime) and calcium hydroxide (slaked lime) can be added to neutralise the excess H^+ ions. This process is known as ‘liming’.

As some of these bases are soluble in water, care must be taken to avoid adding excess base as this would increase the soil pH. This would make the soil too alkaline for plant growth.

TOPIC 11 Salts

Objectives

Candidates should be able to:

(a) describe the techniques used in the preparation, separation and purification of salts as examples of some of the techniques specified in Topic 3

(b) describe the general rules of solubility for common salts to include nitrates, chlorides (including silver and lead), sulfates (including barium, calcium and lead), carbonates, hydroxides, Group I cations and ammonium salts

(c) suggest a method of preparing a given salt from suitable starting materials, given appropriate information

(d) describe the use of aqueous sodium hydroxide and aqueous ammonia to identify the following aqueous cations: aluminium, ammonium, calcium, copper(II), iron(II), iron(III), lead(II) and zinc (formulae of complex ions are not required)

(e) describe tests to identify the following anions: carbonate (by the addition of dilute acid and subsequent use of limewater); chloride (by reaction of an aqueous solution with nitric acid and aqueous silver nitrate); iodide (by reaction of an aqueous solution with nitric acid and aqueous silver nitrate); nitrate (by reduction with aluminium in aqueous sodium hydroxide to ammonia and subsequent use of litmus paper) and sulfate (by reaction of an aqueous solution with nitric acid and aqueous barium nitrate)

(f) describe tests to identify the following gases: ammonia (using damp red litmus paper); carbon dioxide (using limewater); chlorine (using damp litmus paper); hydrogen (using a burning splint); oxygen (using a glowing splint) and sulfur dioxide (using acidified potassium manganate(VII))

1. Solubility of Salts

While salts are ionic compounds, not all salts are soluble in water. The solubility of a salt has to be considered before deciding on the method of its preparation.

The following table summarises the solubilities of various common salts at room temperature.

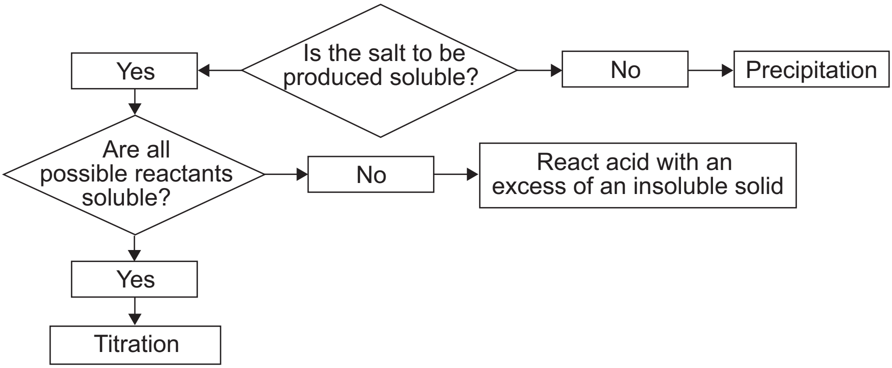
Soluble salts	Insoluble salts
All nitrates	–
All halides (Cl^-, Br^-, I^-) except	Silver halides ($AgCl$, $AgBr$, AgI) and lead(II) halides ($PbCl_2$, $PbBr_2$, PbI_2)
All sulfates (SO_4^{2-}) except	Barium sulfate ($BaSO_4$), lead(II) sulfate ($PbSO_4$) and calcium sulfate ($CaSO_4$)
Ammonium carbonate (NH_4CO_3), sodium carbonate (Na_2CO_3), potassium carbonate (K_2CO_3)	All other carbonates

Note that all sodium, potassium and ammonium salts are soluble.

2. Preparation of Salts

Soluble salts can be prepared by reacting acids with a suitable reagent. These reagents can be a metal, a carbonate, a basic oxide or an alkali. Insoluble salts are prepared through precipitation.

The following flowchart provides a guide to choosing an appropriate method for preparing a salt.



3. Preparing Insoluble Salts

Insoluble salts are prepared through precipitation. This is done through mixing two aqueous solutions, one containing the cation of the salt and another containing the anion of the salt. After mixing the two solutions, the salt can be separated through filtration and purified by washing with distilled water.

4. Preparing Soluble Salts

Soluble salts can be prepared by reacting an acid with an insoluble solid. The insoluble solid can be a metal, a carbonate or a base. This is done by adding an excess of solid reactant to aqueous acid.

acid + metal → salt + hydrogen gas
acid + insoluble carbonate → salt + carbon dioxide + water
acid + insoluble base → salt + water

The excess solid reactant ensures that the acid is completely reacted. Once the reaction is complete, excess solid can be filtered off to obtain a solution of the salt.

This method does not apply to all solid reactants. Reactive metals such as sodium or calcium cannot be used as they react violently with dilute acids, making the reaction dangerous to perform. Unreactive metals such as copper and silver do not react with dilute acids.

Soluble salts can also be prepared by reacting an acid with a soluble reactant through titration. The reactant can be an alkali or a soluble carbonate.

acid + alkali → salt + water
acid + soluble carbonate → salt + carbon dioxide + water

Since both reactants are soluble, exact quantities of each reactant have to be used to avoid contamination of the final product.

The quantities are obtained by performing titration once with a suitable indicator. An indicator is necessary to determine when the reaction is complete as the reactants used are usually colourless. Titration is then repeated without an indicator when the amount of reactants required has been obtained.

Since all sodium, potassium and ammonium salts are soluble, titration is the best method to prepare these salts.

For both methods mentioned, a pure solid sample of the salt can be obtained through crystallisation or evaporating water off the salt solution.

5. Tests for Gases

Gas	Test	Observation
Oxygen, O_2	Place a glowing splint into the test-tube.	The glowing splint relights.
Hydrogen, H_2	Place a lighted splint at the mouth of the test-tube.	The lighted splint extinguishes with a 'pop' sound.
Carbon dioxide, CO_2	Bubble the gas into limewater.	A white precipitate of calcium carbonate forms.
Sulfur dioxide, SO_2	Place a paper soaked with acidified potassium manganate(VII) at the mouth of the test-tube.	The paper turns from purple to colourless.
Chlorine, Cl_2	Place a damp blue litmus paper at the mouth of the test-tube.	The blue litmus turns red and is finally bleached white.
Ammonia, NH_3	Place a damp red litmus paper at the mouth of the test-tube.	The red litmus turns blue.

6. Tests for Cations

Cation	Reaction with aqueous sodium hydroxide	Reaction with aqueous ammonia
Aluminium ion, Al^{3+}	A white precipitate forms. The precipitate dissolves in excess NaOH to give a colourless solution.	A white precipitate forms. The precipitate is insoluble in excess NH_3.
Calcium ion, Ca^{2+}	A white precipitate forms. The precipitate is insoluble in excess NaOH.	No visible change.
Copper(II) ion, Cu^{2+}	A light blue precipitate forms. The precipitate is insoluble in excess NaOH.	A light blue precipitate forms. The precipitate dissolves in excess NH_3 to give a deep blue solution.
Iron(II) ion, Fe^{2+}	A dirty green precipitate forms. The precipitate is insoluble in excess NaOH.	A dirty green precipitate forms. The precipitate is insoluble in excess NH_3.
Iron(III) ion, Fe^{3+}	A reddish-brown precipitate forms. The precipitate is insoluble in excess NaOH.	A reddish-brown precipitate forms.The precipitate is insoluble in excess NH_3.
Lead(II) ion, Pb^{2+}	A white precipitate forms. The precipitate dissolves in excess NaOH to give a colourless solution.	A white precipitate forms. The precipitate is insoluble in excess NH_3.
Zinc ion, Zn^{2+}	A white precipitate forms. The precipitate dissolves in excess NaOH to give a colourless solution.	A white precipitate forms. The precipitate dissolves in excess NH_3 to give a colourless solution.
Ammonium ion, NH_4^+	No precipitate forms. Ammonia gas is produced on warming.	No visible change.

7. Tests for Anions

Anion	Test	Observation
Nitrate ion, NO_3^-	Add aqueous sodium hydroxide and a small piece of aluminium foil, and warm the mixture.	Ammonia gas is released, the gas turns damp red litmus blue.
Carbonate ion, CO_3^{2-}	Add dilute hydrochloric acid.	Carbon dioxide is released, the gas forms a white precipitate when bubbled into limewater.
Chloride ion, Cl^-	Add dilute nitric acid, followed by aqueous silver nitrate.	A white precipitate of silver chloride is produced.
Iodide ion, I^-	Add dilute nitric acid, followed by aqueous lead(II) nitrate.	A yellow precipitate of lead(II) iodide is produced.
Sulfate ion, SO_4^{2-}	Add dilute nitric acid, followed by aqueous barium nitrate.	A white precipitate of barium sulfate is produced.

TOPIC 12 Oxidation And Reduction

Objectives

Candidates should be able to:

(a) define *oxidation* and *reduction* (redox) in terms of oxygen/hydrogen gain/loss
(b) define redox in terms of electron transfer and changes in oxidation state
(c) identify redox reactions in terms of oxygen/hydrogen gain/loss, electron gain/loss and changes in oxidation state
(d) describe the use of aqueous potassium iodide and acidified potassium manganate(VII) in testing for oxidising and reducing agents from the resulting colour changes

1. Oxidation and Reduction

Oxidation can be seen as the gain of oxygen, the loss of hydrogen, the loss of electrons or the increase in oxidation number of a substance.

The reverse occurs in reduction. It can be seen as the loss of oxygen, the gain of hydrogen, the gain of electrons or the decrease in oxidation number of a substance.

2. Calculating Oxidation Numbers

An element has an oxidation state of 0, regardless of whether it is found as individual atoms or in molecules. For example, neon (Ne) and chlorine (Cl_2) have oxidation states of 0.

The sum of oxidation numbers of all atoms in an uncharged compound is 0. For a polyatomic ion, the sum of oxidation numbers of all atoms is equal to its charge.

The oxidation state of an ion is given by its charge. For example, a magnesium ion Mg^{2+} and an oxide ion O^{2-} have oxidation states of +2 and –2 respectively.

Some elements have fixed oxidation numbers in compounds. Oxygen usually has the oxidation state of –2 in its compounds. Hydrogen usually has the oxidation state of +1 in its compounds.

3. Oxidising and Reducing Agents

An oxidising agent is a substance that causes another substance to be oxidised.

The presence of an oxidising agent can be tested using potassium iodide (KI) solution. Iodide ions (I^-) are oxidised to form iodine (I_2) when an oxidising agent is present. The solution turns from colourless to brown.

$$\underset{\text{colourless}}{2I^-(aq)} \rightarrow \underset{\text{brown}}{I_2(aq)} + 2e^-$$

A reducing agent is a substance that causes another substance to be reduced.

The presence of a reducing agent can be tested using acidified potassium manganate(VII) ($KMnO_4$) solution. Manganate(VII) ions are reduced to manganese(II) ions in the presence of a reducing agent. The solution turns from purple to colourless.

$$\underset{\text{purple}}{MnO_4^-(aq)} + 8H^+(aq) + 5e^- \rightarrow \underset{\text{colourless}}{Mn^{2+}(aq)} + 4H_2O(l)$$

Acidified potassium dichromate(VI) solution can also be used to test for the presence of reducing agents. Dichromate(VI) ions ($Cr_2O_7^{2-}$) are reduced to form chromium(III) (Cr^{3+}) ions. The solution turns from orange to green.

$$\underset{\text{orange}}{Cr_2O_7^{2-}(aq)} + 14H^+(aq) + 6e^- \rightarrow \underset{\text{green}}{2Cr^{3+}(aq)} + 7H_2O(l)$$

TOPIC 13 Metals

Objectives

Candidates should be able to:

(a) describe the general physical properties of metals as solids having high melting and boiling points, malleable, good conductors of heat and electricity in terms of their structure
(b) describe alloys as a mixture of a metal with another element
(c) identify representations of metals and alloys from diagrams of structures
(d) explain why alloys have different physical properties to their constituent elements
(e) place in order of reactivity calcium, copper, (hydrogen), iron, lead, magnesium, potassium, silver, sodium and zinc by reference to
  (i) the reactions, if any, of the metals with water, steam and dilute hydrochloric acid,
  (ii) the reduction, if any, of their oxides by carbon and/or by hydrogen
(f) describe the reactivity series as related to the tendency of a metal to form its positive ion, illustrated by its reaction with
  (i) the aqueous ions of the other listed metals
  (ii) the oxides of the other listed metals
(g) deduce the order of reactivity from a given set of experimental results
(h) describe the action of heat on the carbonates of the listed metals and relate thermal stability to the reactivity series
(i) describe the ease of obtaining metals from their ores by relating the elements to their positions in the reactivity series
(j) describe and explain the essential reactions in the extraction of iron using haematite, limestone and coke in the blast furnace
(k) describe steels as alloys which are a mixture of iron with carbon or other metals and how controlled use of these additives changes the properties of the iron
(l) state the uses of mild steel
(m) describe the essential conditions for the corrosion (rusting) of iron as the presence of oxygen and water; prevention of rusting can be achieved by placing a barrier around the metal
(n) describe the sacrificial protection of iron by a more reactive metal in terms of the reactivity series where the more reactive metal corrodes preferentially, e.g. underwater pipes have a piece of magnesium attached to them

1. Physical Properties of Metals

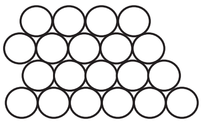
Metals usually have high densities, melting points and boiling points. Some exceptions would be Group I metals (some are less dense than water) and mercury (which is a liquid at room temperature and pressure). Metals are good conductors of heat and electricity, and are often shiny, ductile and malleable.

2. Alloys

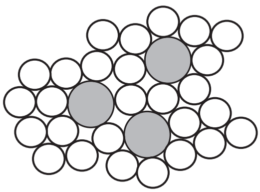
Pure metals are usually not widely used as they are soft and may corrode easily, therefore alloys are used instead. An alloy is a mixture of a metal and one or more elements, which may be metal or non-metal.

A pure metal is soft due to the regular arrangement of atoms in the metal lattice. The atoms are arranged in neat layers which slide past each other easily when a force is applied.

In an alloy however, the arrangement of atoms is disrupted by the presence of atoms of different sizes. This prevents the layers of atoms from sliding easily, making the alloy harder than the pure metal.



pure metal



alloy

Alloying metals helps to change properties to make it more suitable for a particular use. For instance, an alloy of iron and chromium has greater resistance to rusting compared to pure iron.

3. The Reactivity Series

The reactivity series arranges metals in order of reactivity. Metals that are more reactive have a higher tendency of forming ions compared to metals that are less reactive.

most reactive ↑	Potassium	(K)
	Sodium	(Na)
	Calcium	(Ca)
	Magnesium	(Mg)
	Aluminium	(A*l*)
	(Carbon)	(C)
	Zinc	(Zn)
	Iron	(Fe)
	Lead	(Pb)
	(Hydrogen)	(H)
	Copper	(Cu)
	Silver	(Ag)
least reactive	Gold	(Au)

4. Displacement Reactions of Metals

Displacement reaction takes place when a more reactive metal is placed in the salt solution of a less reactive metal. Since the more reactive metal has a higher tendency to form ions, it displaces the less reactive metal from its salt.

For example, when copper metal is placed in a solution of silver nitrate, copper displaces silver from the silver nitrate solution.

copper metal + silver nitrate → copper nitrate + silver

$Cu(s) + 2AgNO_3(aq) \rightarrow Cu(NO_3)_2(aq) + 2Ag(s)$

However, no reaction occurs when a less reactive metal is placed in the salt solution of a more reactive metal. No change is seen when copper metal is placed in magnesium sulfate solution since magnesium is more reactive than copper.

5. **Reactions of Metals with Water**

Metal	Observations	Equation
Potassium	Reacts violently with cold water. Hydrogen gas catches fire and explodes.	Reaction with cold water: $2K(s) + 2H_2O(l) \rightarrow 2KOH(aq) + H_2(g)$
Sodium	Reacts violently with cold water. Hydrogen gas may catch fire.	Reaction with cold water: $2Na(s) + 2H_2O(l) \rightarrow 2NaOH(aq) + H_2(g)$
Calcium	Reacts moderately with cold water.	Reaction with cold water: $Ca(s) + 2H_2O(l) \rightarrow Ca(OH)_2(aq) + H_2(g)$
Magnesium	Reacts slowly with cold water. Hot magnesium reacts violently with steam and burns with a white glow.	Reaction with cold water: $Mg(s) + 2H_2O(l) \rightarrow Mg(OH)_2(aq) + H_2(g)$ Reaction with steam: $Mg(s) + H_2O(g) \rightarrow MgO(s) + H_2(g)$
Aluminium	Reacts readily with steam. Reaction slows down due to the formation of a protective oxide layer.	Reaction with steam: $2Al(s) + 3H_2O(g) \rightarrow Al_2O_3(s) + 3H_2(g)$
Zinc	Hot zinc reacts readily with steam. Zinc oxide produced is yellow when hot and white when cold.	Reaction with steam: $Zn(s) + H_2O(g) \rightarrow ZnO(s) + H_2(g)$
Iron	Hot iron reacts slowly with steam.	Reaction with steam: $3Fe(s) + 4H_2O(g) \rightarrow Fe_3O_4(s) + 4H_2(g)$
Lead	No reaction	-
Copper	No reaction	-
Silver	No reaction	-
Gold	No reaction	-

6. Reactions of Metals with Dilute Hydrochloric Acid

Metal	Observations	Equation
Potassium	Reacts violently	$2K(s) + 2HCl(aq) \rightarrow 2KCl(aq) + H_2(g)$
Sodium	Reacts violently	$2Na(s) + 2HCl(aq) \rightarrow 2NaCl(aq) + H_2(g)$
Calcium	Reacts violently	$Ca(s) + 2HCl(aq) \rightarrow CaCl_2(aq) + H_2(g)$
Magnesium	Reacts readily	$Mg(s) + 2HCl(aq) \rightarrow MgCl_2(aq) + H_2(g)$
Aluminium	Reacts readily	$2Al(s) + 6HCl(aq) \rightarrow 2AlCl_3(aq) + 3H_2(g)$
Zinc	Reacts moderately fast	$Zn(s) + 2HCl(aq) \rightarrow ZnCl_2(aq) + H_2(g)$
Iron	Reacts slowly	$Fe(s) + 2HCl(aq) \rightarrow FeCl_2(aq) + H_2(g)$
Lead	No reaction	-
Copper	No reaction	-
Silver	No reaction	-
Gold	No reaction	-

Reactions of metals with dilute hydrochloric acid can be seen as the displacement of hydrogen in the acid by a more reactive metal.

While lead is higher than hydrogen in the series, it does not react with dilute hydrochloric acid due to the formation of an insoluble layer of lead(II) chloride. The salt acts as a protective layer and prevents the acid from reacting further with the metal.

7. Extraction of Metals

Metals are usually found in nature as ores, which mainly consist of metal oxides. The extraction of a metal from its ore depends on its reactivity. A more reactive metal usually requires tougher methods of extraction compared to a less reactive metal.

Zinc and metals lying below it in the reactivity series can be extracted from their oxides through heating with carbon.

Aluminium and other metals above it in the reactivity series form very stable oxides that are not easily reduced. They can only be extracted from their ores through electrolysis of their molten oxides.

8. Thermal Stability of Metal Carbonates

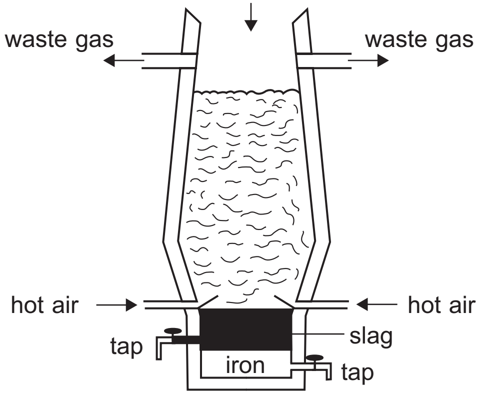
Reactive metals form very stable carbonates which do not decompose easily upon heating. On the other hand, the carbonates of metals which are less reactive are easily decomposed by heat.

Carbonates of potassium and sodium are thermally stable since these metals are found high in the reactivity series. Carbonates of calcium, magnesium, zinc, iron, lead and copper decompose upon heating to form metal oxide and carbon dioxide.

Silver carbonate is the least stable since silver metal is the least reactive. It decomposes completely into silver metal and carbon dioxide.

9. Extraction of Iron

Iron is extracted from its ore, haematite (contains iron(III) oxide, Fe_2O_3), by heating with carbon. Haematite, coke (mainly carbon) and limestone (calcium carbonate, $CaCO_3$) are loaded at the top of the blast furnace while hot air is introduced at the bottom of the furnace.



blast furnace

1. Coke is oxidised by oxygen in the hot air in an exothermic reaction.

   carbon + oxygen → carbon dioxide
   $C(s) + O_2(g) \rightarrow CO_2(g)$

   Carbon dioxide then further reacts with carbon to produce carbon monoxide.

   carbon dioxide + carbon → carbon monoxide
   $CO_2(g) + C(s) \rightarrow 2CO(g)$

2. Carbon monoxide reduces iron(III) oxide in haematite to molten iron. Since iron has high density, it sinks to the bottom of the furnace.

   iron(III) oxide + carbon monoxide → molten iron + carbon dioxide
   $Fe_2O_3(s) + 3CO(g) \rightarrow 2Fe(l) + 3CO_2(g)$

3. Limestone undergoes thermal decomposition.

   calcium carbonate → calcium oxide + carbon dioxide
   $CaCO_3(s) \rightarrow CaO(s) + CO_2(g)$

   Silicon dioxide, an acidic impurity, is removed by reacting with calcium oxide, which is basic in nature.

   calcium oxide + silicon dioxide → calcium silicate (slag)
   $CaO(s) + SiO_2(s) \rightarrow CaSiO_3(l)$

   The reaction forms slag, which floats on top of molten iron. The molten iron and slag are tapped off separately at the bottom of the furnace.

10. Steel

Steel is an alloy of iron and carbon. The properties of steel can be further altered by the addition of other metals and controlling the amounts of these components.

Molten iron produced from the blast furnace is known as pig iron or cast iron and contains many impurities. These impurities can be removed by introducing oxygen through the molten iron. Removal of these impurities leaves behind pure iron, which is also known as wrought iron.

Different percentages of carbon may be added to wrought iron to form steel. Low-carbon steel (or mild steel) is hard and malleable and can be used to make car bodies. High-carbon steel is harder but more brittle than low-carbon steel and is used to make cutting tools.

Stainless steel is a mixture of iron, carbon, chromium and nickel. Addition of these elements makes it more resistant to corrosion and hence, makes it suitable for use as cutlery or surgical tools.

11. Rusting of Iron

Iron corrodes in the presence of water and oxygen to form rust (hydrated iron(III) oxide).

iron + oxygen + water → hydrated iron(III) oxide
$4Fe(s) + 3O_2(g) + 2H_2O(l) \rightarrow 2Fe_2O_3.H_2O(s)$

Rusting can be prevented by painting or covering the metal with a layer of oil. This protects iron from being exposed to oxygen and water.

Sacrificial protection can be used to prevent rusting. A more reactive metal is used as the sacrificial metal and corrodes in place of iron. This is usually done by attaching a block of magnesium or zinc to the iron, or galvanisation, where iron is coated with zinc.

Plating iron with zinc also protects the metal from direct contact with water and oxygen.

TOPIC 14 Electrolysis

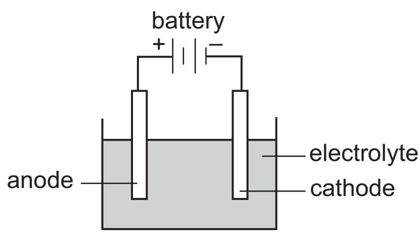
Objectives

Candidates should be able to:

(a) describe electrolysis as the conduction of electricity by an ionic compound (an electrolyte), when molten or dissolved in water, leading to the decomposition of the electrolyte
(b) describe electrolysis as evidence for the existence of ions which are held in a lattice when solid but which are free to move when molten or in solution
(c) describe, in terms of the mobility of ions present and the electrode products, the electrolysis of molten sodium chloride, using inert electrodes
(d) predict the likely products of the electrolysis of a molten binary compound
(e) apply the idea of selective discharge based on
  (i) cations: linked to the reactivity series
  (ii) anions: halides, hydroxides and sulfates
  (iii) concentration effects (In all cases above, inert electrodes are used.)
(f) predict the likely products of the electrolysis of an aqueous electrolyte, given relevant information
(g) construct ionic equations for the reactions occurring at the electrodes during the electrolysis, given relevant information
(h) describe the electrolysis of aqueous copper(II) sulfate with copper electrodes as a means of purifying copper
(i) describe the electroplating of metals
(j) describe the production of electrical energy from simple cells (i.e. two electrodes in an electrolyte) linked to the reactivity series and redox reactions (in terms of electron transfer)

1. Electrolytic Cell

Electrolysis is the use of electricity to break down a compound into its constituents. The process takes place in an electrolytic cell.



The battery provides a source of electricity for reactions to occur. During the process, electrons flow from the positive terminal to the negative terminal of the battery.

The electrodes used in electrolysis conduct electricity. Inert graphite or platinum electrodes are usually used.

The electrode connected to the positive terminal of the battery is the anode and the electrode connected to the negative terminal of the battery is the cathode. Reduction occurs at the cathode while oxidation occurs at the anode.

The electrolyte contains mobile ions which allow for electricity to flow through. It is usually an acid solution, or an ionic compound that is molten or dissolved in water. A solid ionic compound cannot be used as its ions are in fixed positions in the crystal lattice structure.

2. Electrolysis of Molten Ionic Compounds

When an ionic compound is molten, it splits up into positive ions (cations) and negative ions (anions) which are free to move to the cathode and the anode respectively.

At the cathode, electrons are taken in by cations, while at the anode, electrons are lost by anions. To maintain a complete electrical circuit, the number of electrons taken in at the cathode must be the same as the number of electrons lost at the anode.

Since cations take in electrons, they are reduced. Anions are oxidised as they lose electrons.

3. Electrolysis of Solutions of Ionic Compounds

When a solution of an ionic compound is used instead, the autoionisation of water has to be taken into consideration as well.

Water partially dissociates to form hydrogen and hydroxide ions.

water ⇌ hydrogen ion + hydroxide ion
$H_2O \rightleftharpoons H^+ + OH^-$

These ions will compete with those of the ionic compound to be discharged at each of the electrodes.

The ease of discharge of cations can be predicted based on the reactivity series. As reactive metals tend to form ions, their ions are not easily discharged. Ions of less reactive metals have a higher tendency of getting discharged as they accept electrons more easily.

OH^- I^- Br^- Cl^- NO_3^- SO_4^{2-}

$\longrightarrow$

Most likely to be discharged — Least likely to be discharged

Hydroxide ions are most readily discharged in dilute solutions. Nitrates and sulfates are usually not discharged and tend to stay in the solution.

However, when the solution is concentrated, halide ions are preferentially discharged rather than hydroxide ions.

5. Purification of Copper

Copper can be purified by electrolysing copper(II) sulfate solution using copper electrodes. Impure copper is used as the anode while pure copper acts as the cathode.

At the anode, OH^- ions are not discharged since the electrode is not inert. Instead, copper atoms are oxidised to form Cu^{2+} ions.

Anode: $Cu(s) \rightarrow Cu^{2+}(aq) + 2e^-$

The impure copper anode gradually dissolves as the atoms are oxidised. The impurities are left behind to sink to the bottom of the cell as the anode dissolves.

At the cathode, Cu^{2+} ions in the electrolyte are discharged and deposited on the pure copper.

Cathode: $Cu^{2+}(aq) + 2e^- \rightarrow Cu(s)$

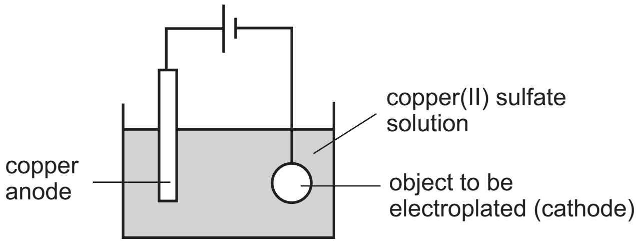
The pure copper cathode gains mass as a layer of pure copper is deposited.

6. Electroplating

Electroplating is done to coat a metal with another metal to improve its appearance or to improve its resistance to corrosion.

The metal used for plating is used as the anode and the object to be electroplated acts as the cathode. The electrolyte used is the salt solution of the metal used for plating.

The plating of an object with copper metal is shown below.



Copper metal acts as the anode as it is used to plate the object. The electrolyte used is a salt solution of its salt (copper(II) sulfate solution) and the object to be plated acts as the cathode.

At the anode, the copper atoms are oxidised into Cu^{2+} ions, which enter the electrolyte. At the cathode, Cu^{2+} ions are discharged and deposited on the object, plating it with copper metal.

7. Simple Cells

Simple cells convert chemical energy into electrical energy. The cell uses two different metals as electrodes and the voltage produced varies depending on the metals used.

In such a cell, the more reactive metal acts as the anode while the less reactive metal acts as the cathode.

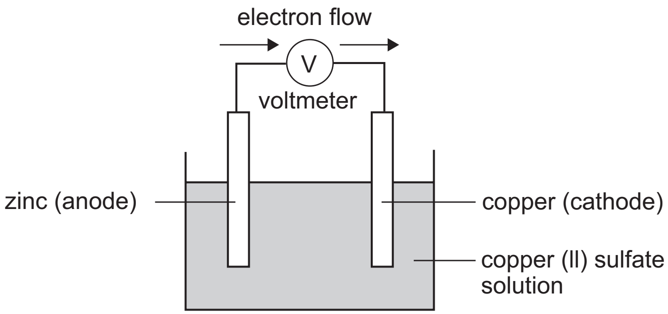
In a zinc-copper cell, zinc metal acts as the anode while copper acts as the cathode.

At the anode, zinc oxidises to form Zn^{2+} ions. In the process, electrons are released and they flow out of the anode through the wire.

Anode: $Zn(s) \rightarrow Zn^{2+}(aq) + 2e^-$

Electrons flow from the zinc anode to the copper cathode, where Cu^{2+} ions in the electrolyte are reduced and deposited as copper metal on the cathode.

Cathode: $Cu^{2+}(aq) + 2e^- \rightarrow Cu(s)$



The zinc anode gradually loses mass as zinc atoms get oxidised while the copper cathode gains mass as Cu^{2+} ions are reduced and deposited. The overall equation of the reaction is obtained by adding the half-equations.

Overall equation: $Zn(s) + Cu^{2+}(aq) \rightarrow Zn^{2+}(aq) + Cu(s)$

A greater voltage is produced when the two metals used are far apart in the reactivity series. A magnesium-copper cell generates a higher voltage than a zinc-copper cell since the difference in reactivity is greater in the magnesium-copper cell.

TOPIC 15 Periodic Table

Objectives

Candidates should be able to:

(a) describe the Periodic Table as an arrangement of the elements in the order of increasing proton (atomic) number
(b) describe how the position of an element in the Periodic Table is related to proton number and electronic structure
(c) describe the relationship between group number and the ionic charge of an element
(d) explain the similarities between the elements in the same group of the Periodic Table in terms of their electronic structure
(e) describe the change from metallic to non-metallic character from left to right across a period of the Periodic Table
(f) describe the relationship between group number, number of valency electrons and metallic/non-metallic character
(g) predict the properties of elements in Group I and Group VII using the Periodic Table
(h) describe lithium, sodium and potassium in Group I (the alkali metals) as a collection of relatively soft, low density metals showing a trend in melting point and in their reaction with water
(i) describe chlorine, bromine and iodine in Group VII (the halogens) as a collection of diatomic non-metals showing a trend in colour, state and their displacement reactions with solutions of other halide ions
(j) describe the elements in Group 0 (the noble gases) as a collection of monatomic elements that are chemically unreactive and hence important in providing an inert atmosphere
(k) describe the lack of reactivity of the noble gases in terms of their electronic structures
(l) describe the transition elements as metals having high melting points, high density, variable oxidation state and forming coloured compounds
(m) state that the elements and/or their compounds are often able to act as catalysts

1. Features of the Periodic Table

Elements are arranged in order of increasing atomic numbers in the Periodic Table. They are organised into horizontal rows known as periods, and vertical columns known as groups.

2. Metals and Non-metals

Elements can be classified as metals or non-metals. There is also a class of elements known as metalloids, which exhibit both metal and non-metal properties. These elements are found along the diagonal line in the Periodic Table.

3. Variations Across a Period

In a period, metals are found on the left side while the non-metals are found on the right side. The metallic character of elements decreases as we move from left to right of a period.

Elements in the same period have the same number of electron shells. The number of electron shells corresponds with the period number of the element. For example, aluminium belongs to Period 3 and has three electron shells.

4. Variations Down a Group

The metallic character of elements increases as we move down a group. This is due to the increase in size of atoms. Valence electrons are further away from the nucleus of the atom and are not as strongly attracted.

Larger atoms in the group will lose their valence electrons more easily than smaller atoms. Therefore, moving down a group, there is an increase in metallic character.

Elements in the same group have the same number of valence electrons. The number of valence electrons each of the elements has corresponds with the group number. For instance, Group I elements (e.g. lithium, sodium, potassium) each have one valence electron each while Group II elements (e.g. boron, magnesium, calcium) have two valence electrons each.

5. Group I: Alkali Metals

Elements in Group I are also known as alkali metals. The atoms of these elements have one valence electron each. These metals are soft and can be cut easily with a knife. They have relatively low melting and boiling points. Their densities are relatively low. Lithium, sodium and potassium have densities lower than water, enabling them to float.

Moving down the group, the melting and boiling points decrease while the densities increase.

Alkali metals are highly reactive metals. They react easily with oxygen and water.

The reactivity of these metals increases as we move down the group. This is due to an increase in the atom size, which means that the valence electrons are further away from the nucleus and are more easily lost.

Alkali metals react with water to form an alkali and hydrogen gas. The trend in reactivity can be observed from their reactions with water. Lithium reacts quickly with cold water, but potassium reacts very violently with cold water.

As alkali metals easily give away their valence electrons, they are strong reducing agents. These metals react with non-metals to form ionic salts which are soluble in water.

6. **Group VII: Halogens**

Elements in Group VII are also known as halogens. Atoms of these elements have seven valence electrons each.

These are non-metals that are found as diatomic molecules (e.g. Cl_2, Br_2, I_2). Since they are found as simple covalent molecules, they have low melting and boiling points. They are coloured substances.

Moving down the group, the melting and boiling points increase. At the same time, the colour intensity of these elements increases. This can be observed from their physical properties at room temperature and pressure. Chlorine is a yellow-green gas, bromine is a reddish-brown liquid and iodine is a black solid.

Halogens are highly reactive non-metals as they only need to gain one electron for a noble gas electronic configuration.

The reactivity decreases as we move down the group. As the atoms increase in size, the force of attraction between the valence shell electrons and the nucleus is weaker. As a result, larger halogens do not gain electrons as easily as smaller ones. Out of the three halogens, chlorine is the most reactive while iodine is the least reactive.

Halogens undergo displacement reaction, where a more reactive halogen displaces a less reactive halogen from its salt. For instance, when chlorine gas is bubbled into sodium bromide solution, bromide ions get displaced. The solution changes from colourless to reddish-brown as bromine molecules are produced in the reaction.

$Cl_2(g) + 2NaBr(aq) \rightarrow 2NaCl(aq) + Br_2(aq)$

As halogens accept electrons easily, they are strong oxidising agents.

7. **Group VIII: Noble Gases**

Elements in Group VIII (or sometimes referred to as Group 0) are known as noble gases. These are inert non-metals which are found as monatomic gases. Their lack of reactivity is due to their complete shell of valence electrons, hence they rarely react to form compounds.

Due to their unreactive nature, noble gases are often used to provide an inert atmosphere. The following table shows some applications of noble gases.

Element	Application
Helium	Weather balloons
Neon	Advertising signs or lights
Argon	Lightbulbs or welding
Krypton	Lasers
Xenon	Photographic flashes or lamps in motion picture projection

8. Transition Elements

Transition elements are a block of metals found between Groups II and III in the Periodic Table. These metals have high melting and boiling points and high densities. Compounds of transition elements are usually coloured.

Transition elements have variable oxidation states. They can form ions of different charges, as opposed to Group I or Group VII elements, which usually form ions of a single charge. For example, iron commonly forms Fe^{2+} and Fe^{3+} ions.

Transition elements and their compounds are good catalysts and are commonly used in industrial processes. For example, nickel is used in the manufacture of margarine (hydrogenation of vegetable oil) and iron is used in the Haber process (manufacture of ammonia).

TOPIC 16 Energy Changes

Objectives

Candidates should be able to:

(a) describe the meaning of enthalpy change in terms of exothermic (ΔH negative) and endothermic (ΔH positive) reactions
(b) represent energy changes by energy profile diagrams, including reaction enthalpy changes and activation energies
(c) describe bond breaking as an endothermic process and bond making as an exothermic process
(d) explain overall enthalpy changes in terms of the energy changes associated with the breaking and making of covalent bonds
(e) describe hydrogen, derived from water or hydrocarbons, as a potential fuel, reacting with oxygen to generate electricity directly in a fuel cell

1. Enthalpy Change

The enthalpy change of a reaction is the amount of energy involved in the reaction and is represented by the symbol ΔH.

2. Exothermic and Endothermic Changes

In an exothermic change, heat is released into the surroundings and this is detected as a rise in temperature. Examples of such processes include condensation, freezing, neutralisation reactions, combustion of fuels and respiration.

In an endothermic change, heat is absorbed from the surroundings and this is detected as a drop in temperature. Examples of such processes include evaporation, melting, dissolving of ammonium chloride, photosynthesis and thermal decomposition.

The enthalpy change of a chemical reaction can be calculated as follows.
ΔH = Total energy of products – Total energy of reactants

For an exothermic reaction, the total energy of the reactants is greater than the total energy of the products, i.e. $\Delta H < 0$.

For an endothermic reaction, the total energy of the products is greater than the total energy of the reactants, i.e. $\Delta H > 0$.

3. Bond Making and Bond Breaking

Energy is released when bonds are made and energy is absorbed when bonds are broken, i.e. bond making is an exothermic process while bond breaking is an endothermic process.

The enthalpy change of a reaction can be calculated by taking the difference between the energy required for bond breaking and the energy required for bond making.

$$\Delta H = \text{Total energy absorbed for bond breaking} - \text{Total energy released for bond making}$$

Since $\Delta H < 0$ for an exothermic reaction, the energy released for bond making is greater than the energy absorbed for bond breaking. For an endothermic reaction where $\Delta H > 0$, the energy absorbed for bond breaking is greater than the energy released for bond making.

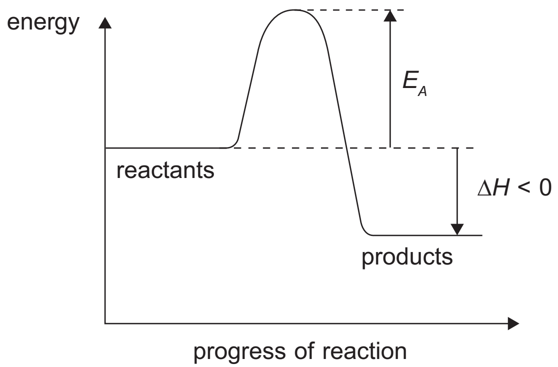
4. Activation Energy

Reactant particles must overcome an energy barrier before they can form products. They must possess a minimum amount of energy for this to occur. The minimum energy required for reactants to form products is the activation energy. Particles that have energy that is equal to or greater than the activation energy will be able to react.

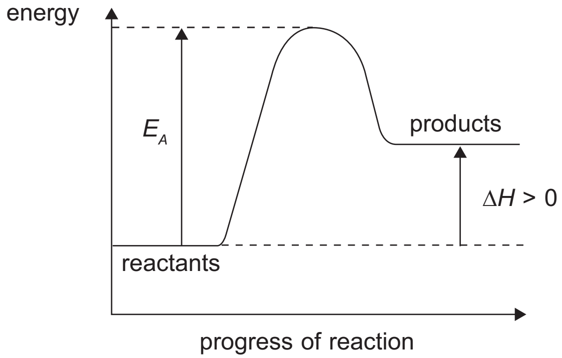
5. Energy Level Diagram

An energy level diagram shows both the enthalpy change and activation energy of a reaction.

For an exothermic reaction, the total energy of the reactants is higher than the total energy of the products.



For an endothermic reaction, the total energy of the products is higher than the total energy of the products.



6. Fuels

Fuels are substances that are burned to release energy. Fossil fuels are commonly used and they undergo complete combustion with excess oxygen to produce carbon dioxide and water.

Apart from combustion of fossil fuels, fuel cells can also be used to release energy. One such fuel cell is a hydrogen-oxygen fuel cell. In such a cell, oxygen is reduced at the anode to form hydroxide ions while hydrogen is oxidised to form water.

Anode: $2H_2(g) + 4OH^-(aq) \rightarrow 4H_2O(l) + 4e^-$

Cathode: $O_2(g) + 2H_2O(l) + 4e^- \rightarrow 4OH^-(aq)$

Overall: $2H_2(g) + O_2(g) \rightarrow 2H_2O$

TOPIC 17 Speed of Reaction

Objectives

Candidates should be able to:

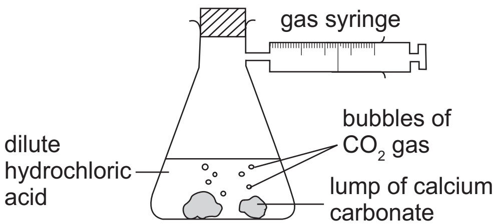
(a) describe the effect of concentration, pressure, particle size and temperature on the speeds of reactions and explain these effects in terms of collisions between reacting particles
(b) define the term *catalyst* and describe the effect of catalysts (including enzymes) on the speeds of reactions
(c) explain how pathways with lower activation energies account for the increase in speeds of reactions
(d) state that some compounds act as catalysts in a range of industrial processes and that enzymes are biological catalysts
(e) suggest a suitable method for investigating the effect of a given variable on the speed of a reaction
(f) interpret data obtained from experiments concerned with speed of reaction

1. Measuring Speed of Reaction

The speed of reaction can be obtained by measuring quantities of reactants or products at regular time intervals. These quantities can then be plotted against time to give an overview of the speed of reaction at different times.

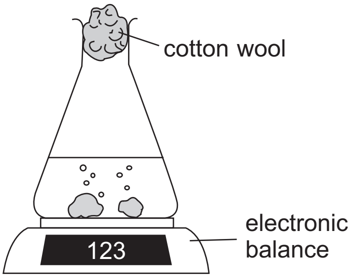
For a reaction that has a gaseous product, the speed of reaction can be obtained by measuring the speed at which gas is produced. This is done by measuring the volume of gas produced at regular time intervals.

The experimental set-up for collecting and measuring the gas is as shown.



The speed of reaction for reactions with gaseous products can also be obtained by measuring the speed at which the reactants are used up. This is done by measuring the mass of the reaction mixture at regular time intervals.

The experimental set-up for measuring the decrease in the mass of the reaction mixture is as follows.



The gradient of the graph gives the speed of reaction. A large gradient (or a steep slope) indicates that the speed of reaction is high while a small gradient (or a gentle slope) indicates a low speed of reaction. When the reaction stops, the gradient is zero.

The graph with values obtained from the amount of reactants has a negative slope. Conversely, the slope of the graph with values obtained from the amount of products has a positive slope.

2. Effective Collisions

Before a reaction can occur, there must be effective collisions between reactant particles. This means that reactant particles must collide in the correct orientation, with the minimum required energy. The speed of reaction is affected by the rate at which effective collisions occur. A higher frequency of effective collisions leads to a greater speed of reaction.

3. Factors Affecting the Speed of Reaction

The concentration of reactants affects the speed of reactions that involve aqueous reactants. With a higher concentration of reactants, the speed of reaction increases. This occurs as there are more reactant particles per unit volume, therefore the frequency of effective collisions increases.

Pressure affects the speed of reactions that involve gases. Increasing the pressure of gaseous reactants can be seen as increasing its concentration as there are more reactants per unit volume. Therefore at higher pressures, the speed of reaction increases since the frequency of effective collisions is higher.

For reactions involving solid reactants, increasing the total exposed surface area results in a greater speed of reaction. This happens when a large piece of reactant is broken into smaller pieces. With a greater surface area exposed, there are more places for other reactants to collide with. This results in a higher frequency of effective collisions.

The speed of reaction increases with an increase in temperature. At lower temperatures, reactant particles move slowly and do not collide with sufficient energy. At higher temperatures, reactant particles have more kinetic energy. They are then able to move faster and collide more frequently. As a result, more particles possess sufficient energy for effective collisions to occur. This results in a higher frequency of effective collisions.

The presence of a catalyst increases the speed of reaction by providing an alternate reaction pathway that has lower activation energy. There will be more reactant particles that possess sufficient energy for effective collisions. Therefore a catalysed reaction would proceed faster than an uncatalysed reaction.

4. Catalysts

A catalyst is a substance that increases the speed of reaction by lowering the energy barrier required for a reaction to proceed. It remains chemically unchanged at the end of a reaction. Only a small amount of catalyst is required to speed up a reaction.

The effect of a solid catalyst can be improved by increasing its surface area. When a catalyst has a smaller particle size, there is a greater surface area on which reactions can take place.

Note that catalysts only lower the activation energy of a reaction, but do not alter the energy of the reactants or products.

Some common catalysts used in industries are as shown in the following table.

Catalyst	Application
Iron	Haber process (manufacture of ammonia)
Platinum	Catalytic converters
Aluminium oxide (alumina) or silicon dioxide (silica)	Cracking of large hydrocarbons
Nickel	Hydrogenation of alkenes

Enzymes are biological catalysts that catalyse biochemical reactions in living organisms. They are proteins and can only catalyse one type or one class of reactions. These catalysts work best at certain temperature and pH ranges.

The fermentation of glucose uses enzymes which are produced from yeast to catalyse the formation of ethanol. The process takes place at approximately 37 °C since these enzymes work best at this temperature. If temperatures are too low, the enzymes would be inactive. On the other hand, if temperatures are too high, the enzymes would be denatured and can no longer catalyse reactions.

TOPIC 18 Ammonia

Objectives

Candidates should be able to:

(a) describe the use of nitrogen, from air, and hydrogen, from the cracking of crude oil, in the manufacture of ammonia
(b) state that some chemical reactions are reversible, e.g. manufacture of ammonia
(c) describe the essential conditions for the manufacture of ammonia by the Haber process
(d) describe the displacement of ammonia from its salts

1. The Haber Process

Ammonia is an important chemical that is manufactured in large amounts through the Haber process. It is produced from nitrogen gas and hydrogen gas. Nitrogen gas is obtained directly from the air and hydrogen gas is obtained from the cracking of large hydrocarbons.

As the formation of ammonia from hydrogen and nitrogen is a reversible process, reaction conditions are controlled to maximise the yield of ammonia.

nitrogen + hydrogen $\rightleftharpoons$ ammonia
$N_2 + 3H_2 \rightleftharpoons 2NH_3$

The production of ammonia is favoured at low temperatures. These low temperatures however, are kinetically unfavourable as the reaction would proceed too slowly. Therefore a relatively high temperature of 450 °C is used.

Higher pressures result in a higher yield of ammonia. Despite this, it is expensive to generate and maintain high pressures and have equipment that can withstand the extreme pressures. Considering these costs, the Haber process usually takes place at 250 atm.

An iron catalyst is also used to further increase the rate of reaction.

2. **Displacement of Ammonia** K M C

Ammonia is displaced when ammonium salts are heated with alkalis.

ammonium chloride + sodium hydroxide → ammonia + sodium chloride + water

$NH_4Cl + NaOH \rightarrow NH_3 + NaCl + H_2O$

3. **Fertilisers**

Nitrogen is needed for the production of proteins for healthy plant growth. While nitrogen is abundant in the air, most plants cannot utilise atmospheric nitrogen. Nitrogen is supplied to plants in the form of ammonium salts and urea.

Ammonium fertilisers cannot be added alongside agricultural lime (calcium hydroxide and calcium oxide) as ammonia would be displaced from ammonium salts. This causes wastage of the fertiliser as ammonia gas cannot be utilised by plants.

TOPIC 19 Air and Atmosphere

Objectives

Candidates should be able to:

(a) describe the volume composition of gases present in dry air as being approximately 78% nitrogen, 21% oxygen and the remainder being noble gases (with argon as the main constituent) and carbon dioxide
(b) name some common atmospheric pollutants
(c) state the sources of these pollutants as
   (i) carbon monoxide from incomplete combustion of carbon-containing substances
   (ii) nitrogen oxides from lightning activity and internal combustion engines
   (iii) sulfur dioxide from volcanoes and combustion of fossil fuels
(d) describe the reactions used in possible solutions to the problems arising from some of the pollutants named in (b)
   (i) the redox reactions in catalytic converters to remove combustion pollutants
   (ii) the use of calcium carbonate to reduce the effect of 'acid rain' and in flue gas desulfurisation
(e) discuss some of the effects of these pollutants on health and on the environment
   (i) the poisonous nature of carbon monoxide
   (ii) the role of nitrogen dioxide and sulfur dioxide in the formation of 'acid rain' and its effects on respiration and buildings
(f) discuss the importance of the ozone layer and the problems involved with the depletion of ozone by reaction with chlorine containing compounds, chlorofluorocarbons (CFCs)
(g) describe the carbon cycle in simple terms, to include
   (i) the processes of combustion, respiration and photosynthesis
   (ii) how the carbon cycle regulates the amount of carbon dioxide in the atmosphere
(h) state that carbon dioxide and methane are greenhouse gases and may contribute to global warming, give the sources of these gases and discuss the possible consequences of an increase in global warming

1. Composition of Air

Dry air consists of approximately 79% nitrogen, 20% oxygen, 0.97% noble gases and 0.03% carbon dioxide. The percentage of water vapour present in the air varies across geographical locations.

2. Fractional Distillation of Air

As air is a mixture of gases, it can be separated into its components through fractional distillation.

Air is first liquefied by cooling and compressing. Liquid air is then boiled and passed through a fractionating column, where the fractions are distilled and tapped off based on their boiling points. The component with the lowest boiling point is distilled first.

Nitrogen boils at –196 °C and is distilled first. Argon and oxygen have boiling points of –186 °C and –183 °C respectively and are distilled later.

3. Carbon Monoxide

Carbon monoxide is a toxic, colourless and odourless gas that is produced from incomplete combustion of carbon-containing substances.

The gas forms a stable compound with haemoglobin in red blood cells, preventing oxygen from binding with it. This deprives cells of oxygen and leads to headaches, fatigue or death.

4. Oxides of Nitrogen

Nitrogen combines with oxygen in various ratios to form different types of oxides. These oxides are collectively known as oxides of nitrogen (or NO_x). Two such oxides include nitrogen monoxide (NO) and nitrogen dioxide (NO_2).

They are formed at high temperatures, such as in a combustion engine or by lightning. These high temperatures encourage nitrogen and oxygen present in the atmosphere to react.

Being acidic oxides, they cause acid rain when dissolved in rain water. When inhaled directly, the gas irritates lung tissues and the eyes.

5. Sulfur Dioxide

Sulfur dioxide is a pungent and colourless gas. It is naturally produced in large quantities during volcanic eruptions. Burning of fossil fuels also releases sulfur dioxide due to the presence of sulfur in fuels.

It is an acidic oxide that dissolves in rain water to form acid rain. Like oxides of nitrogen, inhaling sulfur dioxide causes respiratory problems and irritates the eyes.

6. Unburnt Hydrocarbons and Ozone

Unburnt hydrocarbons are released due to incomplete combustion of fuels. Breathing in these gases may cause cancer.

Ozone is produced when oxides of nitrogen react with unburnt hydrocarbons in the presence of sunlight. At ground level, ozone is a pollutant as breathing the gas causes respiratory problems.

7. **Acid Rain**

While pure water has a pH of 7, rain water is usually slightly acidic in unpolluted areas. This is due to the dissolving of carbon dioxide to form a carbonic acid (a weak acid).

In polluted areas where oxides of nitrogen and sulfur dioxide are present, rain water becomes more acidic. This is due to the formation of strong acids in rain water, such as nitric acid, nitrous acid and sulfurous acid.

Acid rain reacts with structures with metal or carbonates, causing them to corrode and be destroyed. It destroys aquatic habitats by making them too acidic for aquatic life. The soil pH is also greatly lowered, making it unsuitable for plants to survive.

8. **Catalytic Converters**

Catalytic converters are installed in exhaust systems of cars to reduce the emissions of carbon monoxide, oxides of nitrogen and unburnt hydrocarbons. These substances undergo redox reactions in the presence of platinum and rhodium catalysts to form substances that are less harmful.

Carbon monoxide is oxidised to form carbon dioxide.
$2CO(g) + O_2(g) \rightarrow 2CO_2(g)$

Oxides of nitrogen are reduced to form nitrogen.
$2NO(g) + 2CO(g) \rightarrow N_2(g) + 2CO_2(g)$

Unburnt hydrocarbons are further oxidised to form carbon dioxide and water.

9. **Flue Gas Desulfurisation**

Sulfur dioxide is removed from waste gases emitted from industrial processes through flue gas desulfurisation. This involves passing the waste gases through calcium carbonate or calcium oxide. Sulfur dioxide is removed from these gases as it reacts with basic substances.

sulfur dioxide + calcium carbonate → calcium sulfite + carbon dioxide
$SO_2(g) + CaCO_3(s) \rightarrow CaSO_3(s) + CO_2(g)$

sulfur dioxide + calcium oxide → calcium sulfite
$SO_2(g) + CaO(s) \rightarrow CaSO_3(s)$

Calcium sulfite undergoes further oxidation with atmospheric oxygen to form calcium sulfate.

calcium sulfite + oxygen → calcium sulfate
$2CaSO_3(s) + O_2(g) \rightarrow 2CaSO_4(s)$

10. **Ozone Layer**

The ozone layer is found high in the Earth's atmosphere. Although ozone is regarded as an air pollutant at ground level, the ozone layer has an important role of absorbing excess ultraviolet (UV) radiation from the Sun.

It has been found that the ozone layer is depleting and substances such as chlorofluorocarbons (CFCs) contribute to this. CFCs are compounds that contain chlorine, fluorine and carbon.

Chlorine radicals are produced when CFCs are exposed to UV radiation. These free radicals destroy the ozone layer by reacting with ozone molecules.

Ozone depletion would cause more UV radiation to reach the Earth's surface, which becomes a problem as excess UV radiation causes health issues such as skin cancer and formation of eye cataracts.

The use of CFCs has since been banned in most countries. Despite this, ozone depletion will continue as CFCs released long ago are still present in the atmosphere.

11. Carbon Cycle

The level of carbon dioxide in the atmosphere is maintained by the carbon cycle.

Carbon dioxide is released through respiration, combustion of fuels and decay of organic material.

Respiration occurs in living things, and this involves breaking down glucose to obtain energy, carbon dioxide and water.

glucose + oxygen → carbon dioxide + water + energy

$C_6H_{12}O_6(aq) + 6O_2(g) \rightarrow 6CO_2(g) + 6H_2O(l)$ + energy

At the same time, carbon dioxide is removed from the atmosphere by photosynthesis. This occurs in plants and involves converting carbon dioxide and water into glucose and oxygen in the presence of sunlight.

carbon dioxide + water → glucose + oxygen

$6CO_2(g) + 6H_2O(l) \rightarrow C_6H_{12}O_6(aq) + 6O_2(g)$

12. Global Warming

The Earth is kept warm through the greenhouse effect, which is caused by the trapping of heat energy from sunlight by greenhouse gases. Examples of such greenhouse gases are carbon dioxide, methane and water vapour.

Excessive amounts of greenhouse gases result in a stronger greenhouse effect, which leads to global warming. Global warming refers to the increase in global temperatures due to the accumulation of greenhouse gases in the atmosphere.

When greenhouse gases are produced faster than the rate at which they are removed from the atmosphere, an accumulation of these gases result in global warming. Global warming refers to an increase in global temperatures due to high levels of greenhouse gases.

Measures are taken to prevent adding on to the existing amount of greenhouse gases in the atmosphere. The burning of fossil fuels for energy is reduced to reduce emissions of carbon dioxide.

TOPIC 20 Introduction to Organic Chemistry

Objectives

Candidates should be able to:

(a) name natural gas, mainly methane, and petroleum as sources of energy
(b) describe petroleum as a mixture of hydrocarbons and its separation into useful fractions by fractional distillation
(c) name the following fractions and state their uses
   (i) petrol (gasoline) as a fuel in cars
   (ii) naphtha as the feedstock and main source of hydrocarbons used for the production of a wide range of organic compounds in the petrochemical industry
   (iii) paraffin (kerosene) as a fuel for heating and cooking and for aircraft engines
   (iv) diesel as a fuel for diesel engines
   (v) lubricating oils as lubricants and as a source of polishes and waxes
   (vi) bitumen for making road surfaces
(d) describe the issues relating to the competing uses of oil as an energy source and as a chemical feedstock

1. Homologous Series

A homologous series of organic compounds have similar chemical properties as they share the same functional group. The chemical formula of a series can be described with a general formula.

There is a gradual change in the physical properties of the members as we move down a homologous series.

2. Naming Organic Compounds

The prefix of the name of an organic compound gives the number of carbon atoms present in the compound.

Number of carbon atoms	Prefix
1	Meth-
2	Eth-
3	Prop-
4	But-

The suffix of the name of an organic compound tells us which homologous series the compound belongs to.

Homologous series	Suffix
Alkanes	-ane
Alkenes	-ene
Alcohols	-ol
Carboxylic acids	-oic acid

3. Isomers

Isomers are compounds which have the same chemical formula but different structural formulae. They could belong to different homologous series. They usually share the same chemical properties but differ in physical properties.

4. Fractional Distillation of Petroleum

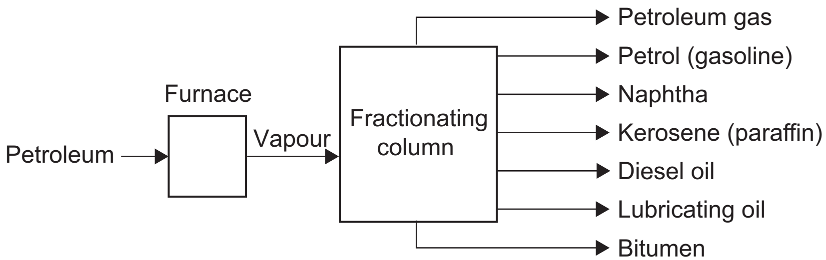
Petroleum is a mixture of hydrocarbons which can be separated into various useful fractions through fractional distillation.

Petroleum is first passed through a furnace to be heated into a vapour. The vapour is then passed through the fractionating column. Since petroleum consists of hydrocarbons of different sizes, the fractions have different boiling points and condense at different temperatures.

The vapour rises up the column where they will condense and be tapped off. Higher parts of the column have lower temperatures while lower parts of the column have higher temperatures.

Since lighter fractions have lower boiling points, they are tapped off at higher parts of the column. Heavier fractions on the other hand, have higher boiling points and are tapped off at lower parts of the column.

A simplified diagram of the fractional distillation of petroleum and the fractions collected is shown.



5. Fractions of Petroleum and Their Uses

Fraction	Boiling point range (°C)	Uses
Petroleum gas	Below 40	Fuel for domestic use (cooking, heating)
Petrol (gasoline)	40 to 75	Fuel for cars
Naphtha	90 to 150	Feedstock for chemical industries
Kerosene (paraffin)	150 to 240	Fuel for aircraft engines; Fuel for cooking and heating
Diesel oil	220 to 250	Fuel for heavy vehicles
Lubricating oil	300 to 350	For lubricating machine parts; Making waxes and polishes
Bitumen	Above 350	For road surfaces; Roofing

As smaller fractions can be used as fuel and chemical feedstock, the demand for smaller fractions is generally higher than that of larger fractions such as bitumen. To meet these demands, larger fractions are cracked to form smaller hydrocarbon molecules.

TOPIC 21 Alkanes and Alkenes

Objectives

Candidates should be able to:

(a) describe an homologous series as a group of compounds with a general formula, similar chemical properties and showing a gradation in physical properties as a result of increase in the size and mass of the molecules, e.g. melting and boiling points; viscosity; flammability
(b) describe the alkanes as an homologous series of saturated hydrocarbons with the general formula C_nH_{2n+2}
(c) draw the structures of branched and unbranched alkanes, C_1 to C_4, and name the unbranched alkanes methane to butane
(d) define *isomerism* and identify isomers
(e) describe the properties of alkanes (exemplified by methane) as being generally unreactive except in terms of combustion and substitution by chlorine
(f) describe the alkenes as an homologous series of unsaturated hydrocarbons with the general formula C_nH_{2n}
(g) draw the structures of branched and unbranched alkenes, C_2 to C_4, and name the unbranched alkenes ethene to butene
(h) describe the manufacture of alkenes and hydrogen by cracking hydrocarbons and recognise that cracking is essential to match the demand for fractions containing smaller molecules from the refinery process
(i) describe the difference between saturated and unsaturated hydrocarbons from their molecular structures and by using aqueous bromine
(j) describe the properties of alkenes (exemplified by ethene) in terms of combustion, polymerisation and the addition reactions with bromine, steam and hydrogen
(k) state the meaning of *polyunsaturated* when applied to food products
(l) describe the manufacture of margarine by the addition of hydrogen to unsaturated vegetable oils to form a solid product

1. Alkanes

Alkanes are saturated hydrocarbons with the general formula C_nH_{2n+2}, where $n \geqslant 1$. Names of alkanes usually end with '-ane'. The first four members of the alkane homologous series are listed in the following table.

Name	Methane	Ethane	Propane	Butane
n	1	2	3	4
Molecular formula	CH_4	C_2H_6	C_3H_8	C_4H_{10}

Moving down the series, the molecular size of the alkanes increases. This means that the intermolecular forces of attraction become stronger. This leads to an increase in melting and boiling points and an increase in viscosity down the series. The flammability however decreases with an increase in molecular size of the alkanes.

2. Chemical Properties of Alkanes

Alkanes are generally unreactive as C – C and C – H bonds are not easily broken. They can only undergo combustion and substitution reactions.

Combustion occurs when an alkane combines with oxygen. The reaction is exothermic and hence, alkanes are used as fuels and are burned for energy.

If the alkane burns in excess oxygen, complete combustion occurs to produce carbon dioxide and water only. If the alkane burns under oxygen-deficient conditions, soot (carbon) and carbon monoxide are produced as well.

Alkanes can only react with halogens through substitution reactions. This occurs in the presence of ultraviolet light. Hydrogen atoms are substituted by halogen atoms in the reaction. The reaction produces a mixture of halogen-containing compounds.

3. Alkenes

Alkenes are unsaturated compounds with the general formula C_nH_{2n}, where $n \geqslant 2$.Note that a 1-carbon alkene cannot exist.

Names of alkenes usually end with '-ene'. The first three members of the alkene homologous series are listed in the following table.

Name	Ethene	Propene	Butene
n	2	3	4
Molecular formula	C_2H_4	C_3H_6	C_4H_8

4. Chemical Properties of Alkenes

Like alkanes, alkenes undergo complete combustion when there is sufficient oxygen to form carbon dioxide and water only. They undergo incomplete combustion to produce soot and carbon monoxide.

Due to the higher carbon-to-hydrogen ratio, alkenes burn with a smokier flame than their corresponding alkanes.

Alkenes are called unsaturated compounds due to the presence of C = C bonds. These bonds allow for alkenes to undergo addition reactions, which is a characteristic of alkenes.

Hydrogen gas can be added to an alkene to obtain an alkane. This reaction occurs at 200 °C in the presence of nickel as a catalyst.

ethene + hydrogen → ethane
$C_2H_4(g) + H_2(g) \rightarrow C_2H_6(g)$

Margarine is produced by adding hydrogen to vegetable oils. Vegetable oil contains many C = C bonds, hence it is described to be polyunsaturated. One hydrogen molecule is added across each C = C bond in this process.

Halogens can be added across the C = C bond at room temperature and pressure to produce halogenoalkanes. An example is the addition of bromine to an alkene.

ethene + bromine → dibromoethane
$C_2H_4(g) + Br_2(aq) \rightarrow C_2H_4Br_2(aq)$

The addition of bromine is used in testing for the presence of unsaturated compounds. Aqueous bromine is reddish-brown and becomes colourless when an unsaturated compound is added.

Alcohols can be produced from the addition of steam to alkenes. This takes place at a temperature of 300 °C and pressure of 60 atm, in the presence of phosphoric(V) acid, which acts as a catalyst.

ethene + steam → ethanol
$C_2H_4(g) + H_2O(g) \rightarrow C_2H_5OH(l)$

Alkene molecules can also react with one another to form a large saturated molecule through addition polymerisation. This process takes place at high temperature and pressure in the presence of a catalyst.

ethene → poly(ethene)

$n\ C_2H_4(g) \rightarrow (C_2H_4)_n(s)$

5. Cracking

Large hydrocarbons can be broken down into smaller molecules through cracking. This process requires aluminium oxide or silicon dioxide as catalyst.

The mixture of large hydrocarbons is passed over the catalyst at a high temperature of about 600 °C. These molecules are then broken down into a mixture of small alkanes and alkenes, and hydrogen is sometimes produced as well.

Small hydrocarbon molecules such as ethene are required as starting materials for petrochemical industries. Cracking is important as it converts larger fractions of petroleum, which are of lower demand, into small hydrocarbons which are in high demand.

In addition, cracking provides the source of hydrogen for the production of ammonia in the Haber process.

TOPIC 22 Alcohols and Carboxylic Acids

Objectives

Candidates should be able to:

(a) describe the alcohols as an homologous series containing the –OH group
(b) draw the structures of alcohols, C_1 to C_4, and name the unbranched alcohols methanol to butanol
(c) describe the properties of alcohols in terms of combustion and oxidation to carboxylic acids
(d) describe the formation of ethanol by the catalysed addition of steam to ethene and by fermentation of glucose
(e) state some uses of ethanol
(f) describe the carboxylic acids as an homologous series containing the $-CO_2H$ group
(g) draw the structures of carboxylic acids methanoic acid to butanoic acid and name the unbranched acids, methanoic acid to butanoic acid
(h) describe the carboxylic acids as weak acids, reacting with carbonates, bases and some metals
(i) describe the formation of ethanoic acid by the oxidation of ethanol by atmospheric oxygen or acidified potassium manganate(VII)
(j) describe the reaction of a carboxylic acid with an alcohol to form an ester
(k) state some commercial uses of esters

1. Alcohols

Alcohols are a homologous series of organic compounds that have the general formula $C_nH_{2n+1}OH$, where $n \geqslant 1$. They have the functional group $-OH$, which is also called the hydroxyl group.

Names of alcohols usually end with '-ol'. The first four members of the alcohol homologous series are listed in the following table.

Name	Methanol	Ethanol	Propanol	Butanol
n	1	2	3	4
Molecular formula	CH_3OH	C_2H_5OH	C_3H_7OH	C_4H_9OH

Alcohols are liquids at room temperature and pressure and are very volatile. As the molecular sizes of the alcohols increases down the series, the forces of attraction between the molecules become stronger. As a result, the melting and boiling points increase with larger molecular size.

Smaller alcohols are miscible in water. As the molecular size of the alcohols increases, solubility in water decreases.

An important member of the homologous series is ethanol, which is used in food and drinks, as a solvent for paints and perfumes, and as fuel.

2. Chemical Properties of Alcohols

Alcohols undergo complete combustion when there is sufficient oxygen to produce carbon dioxide and water.

When heated with oxidising agents such as acidified potassium manganate(VII), alcohols undergo oxidation to form carboxylic acids.

3. Production of Ethanol

Ethanol used for human consumption is usually produced through fermentation of glucose from fruits or grains. The process is carried out with yeast at 37 °C, in the absence of oxygen.

The temperature has to be kept at 37 °C as the enzymes in yeast work best at this temperature. Increasing the temperature would denature the enzymes and cause them to be unable to catalyse the reaction.

Fermentation produces a dilute solution of ethanol. High concentrations of ethanol cannot be obtained directly from this process as the yeast dies when the concentration of ethanol reaches about 15%.

Ethanol is also produced on a larger scale through the addition of steam to ethene. This takes place at 300 °C and 60 atm in the presence of phosphoric(V) acid as a catalyst. This process produces an ethanol solution of higher purity and concentration compared to fermentation.

4. Carboxylic Acids

Alcohols are a homologous series of organic acids that have the general formula $C_nH_{2n+1}COOH$, where $n \geqslant 0$. They have the functional group –COOH, which is also called the carboxyl group.

Names of carboxylic acids usually end with '-oic acid'. The first four members of the carboxylic acid homologous series are listed in the following table.

Name	Methanoic acid	Ethanoic acid	Propanoic acid	Butanoic acid
n	0	1	2	3
Molecular formula	$HCOOH$	CH_3COOH	C_2H_5COOH	C_3H_7COOH

5. Chemical Properties of Carboxylic Acids

Carboxylic acids are weak acids that partially dissociate in water to give hydrogen ions. Due to the presence of these hydrogen ions when carboxylic acids dissolve in water, they undergo reactions of acids.

Carboxylic acids react with metals that lie above hydrogen in the reactivity series to produce salt and water.

ethanoic acid + magnesium → magnesium ethanoate + hydrogen
$2CH_3COOH(aq) + Mg(s) \rightarrow (CH_3COO)_2Mg(aq) + H_2(g)$

Carboxylic acids react with metal carbonates to produce salt, carbon dioxide and water.

ethanoic acid + calcium carbonate → calcium ethanoate + carbon dioxide + water
$2CH_3COOH(aq) + CaCO_3(s) \rightarrow (CH_3COO)_2Ca(aq) + CO_2(g) + H_2O(l)$

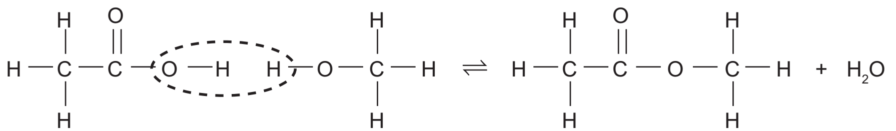
Carboxylic acids react with metal hydroxides to produce salt and water.

ethanoic acid + sodium hydroxide → sodium ethanoate + water
$CH_3COOH(aq) + NaOH(aq) \rightarrow CH_3COONa(aq) + H_2O(l)$

6. Esters

Esters are sweet-smelling liquids that are used as solvents for perfumes or for making artificial food flavourings. They can be produced through esterification from the reaction between alcohols and carboxylic acids.

Esterification requires heating an alcohol and a carboxylic acid with a few drops of concentrated sulfuric acid as a catalyst. This process is a reversible reaction as indicated by the $\rightleftharpoons$ symbol.



ethanoic acid methanol methyl ethanoate water

Apart from acting as a catalyst in the reaction, concentrated sulfuric acid is a dehydrating agent and removes water produced. This also helps speeding up the rate of product formation.

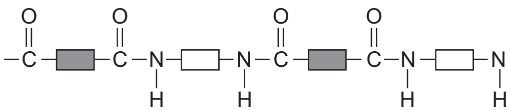
Note that there are two parts to the name of an ester. The first part of the name is taken from the alcohol while the second part is taken from the carboxylic acid from which it is made.

TOPIC 23 Macromolecules

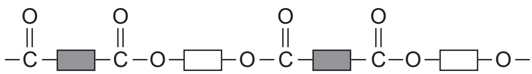
Objectives

Candidates should be able to:

(a) describe macromolecules as large molecules built up from small units, different macromolecules having different units and/or different linkages
(b) describe the formation of poly(ethene) as an example of addition polymerisation of ethene as the monomer
(c) state some uses of poly(ethene) as a typical plastic
(d) deduce the structure of the polymer product from a given monomer and vice versa
(e) describe nylon, a polyamide, and *Terylene*, a polyester, as condensation polymers, the partial structure of nylon being represented as



and the partial structure of *Terylene* as



(f) state some typical uses of man-made fibres such as nylon and *Terylene*
(g) describe the pollution problems caused by the disposal of non-biodegradable plastics

1. Polymers

A polymer is a very large molecule that consists of many smaller molecules joined together by covalent bonds. These smaller units that make up the polymer are also called monomers.

2. Addition Polymerisation

Addition polymerisation occurs for unsaturated monomers. The reaction takes place under high temperature and pressure in the presence of a catalyst.

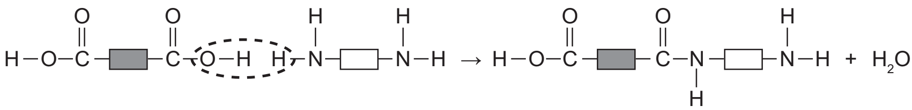
The process involves breaking the C = C bond so that the monomers can form bonds with other monomers. No atom or molecule is lost in this process, so the empirical formula of the addition polymer is the same as its monomer.

Poly(ethene) is produced through the addition polymerisation of ethene. It is used in making plastic bags and clingfilm.

3. Condensation Polymerisation

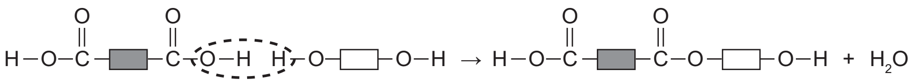
Condensation polymerisation involves joining of monomers with the loss of a small molecule for each linkage formed.

Nylon is a polyamide that has monomers joined together by amide linkages. It is formed from dicarboxylic acids and diamines. The formation of an amide linkage between the carboxyl and amine groups results in the loss of a water molecule.



Terylene is a polyester that has monomers joined together by ester linkages. It is formed from dicarboxylic acids and diols. The formation of an ester linkage between the carboxyl and hydroxyl groups results in the loss of a water molecule.

Nylon and *Terylene* are both used to make clothing, curtains, fishing lines, parachutes and sleeping bags.



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TOPIC 1 Cell Structure and Organisation

Objectives

Candidates should be able to:

(a) identify cell structures (including organelles) of typical plant and animal cells from diagrams, photomicrographs and as seen under the light microscope using prepared slides and fresh material treated with an appropriate temporary staining technique:
- chloroplasts
- cell surface membrane
- cell wall
- cytoplasm
- cell vacuoles (large, sap-filled in plant cells, small, temporary in animal cells)
- nucleus

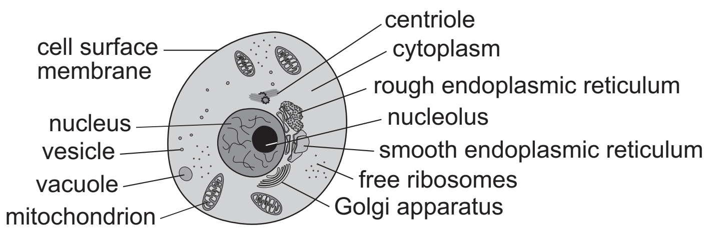
(b) identify the following membrane systems and organelles from diagrams and electron micrographs:
- endoplasmic reticulum
- mitochondria
- Golgi body
- ribosomes

(c) state the functions of the membrane systems and organelles identified above
(d) compare the structure of typical animal and plant cells
(e) state, in simple terms, the relationship between cell function and cell structure for the following:
- absorption – root hair cells
- conduction and support – xylem vessels
- transport of oxygen – red blood cells

(f) differentiate cell, tissue, organ and organ system

1.1 Animal cell

1. The following is a diagram of a generalised animal cell as seen under an electron microscope:



A generalised animal cell

2. The **cell surface membrane** or plasma membrane is a partially permeable membrane surrounding the cytoplasm of the cell. It controls substances entering or leaving the cell.

3. The **cytoplasm** is the gel-like matrix embedded with organelles. It is the site of most cellular activities.

4. The **cell vacuoles** are small fluid-filled spaces bound by a membrane. In animal cells they store water and food substances. They are usually not permanent.

5. The **nucleus** is an organelle surrounded by an envelope called the nuclear envelope. It contains darker bodies called nucleoli (singular: nucleolus) and thread-like structures called chromatin which are made of DNA. The nucleus controls cellular activities such as growth, repair, and cell division.

6. The **endoplasmic reticulum** (ER) is a network of membranes forming tubes and flattened spaces. There are two types of ER:

   (a) The **smooth endoplasmic reticulum** (SER) does not have ribosomes attached to it. It synthesises fats and steroids such as sex hormones. It also contains enzymes that detoxify drugs and poisons.

   (b) The **rough endoplasmic reticulum** (RER) is studded with **ribosomes**. Ribosomes in the cell can either be free ribosomes (i.e. they lie freely in the cytoplasm) or be attached to the membrane of the RER. Ribosomes synthesise proteins.

7. All proteins made in the RER depart in membrane-bound vesicles to the Golgi apparatus.

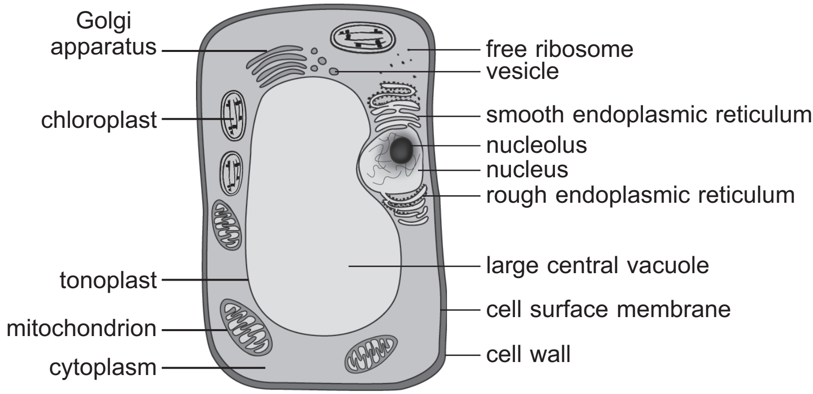
8. The **Golgi apparatus** resembles a stack of flattened disc-shaped spaces surrounded by membranes. It stores, sorts and modifies substances made by the ER, and packages them in vesicles to be secreted out of the cell.

9. The **mitochondria** (singular: mitochondrion) are small elongated organelles with folded inner membranes. Aerobic respiration takes place in the mitochondria. Aerobic respiration is the process where energy is extracted from food substances in the presence of oxygen. This energy is used by the cell to perform cellular activities such as growth and cell division.

10. The **centrioles** are a pair of barrel-shaped structures at right angles to each other. They play a role in cell division. Centrioles are usually absent in plants.

1.2 Plant cell

1. The following is a diagram of a generalised plant cell as seen under an electron microscope:



A generalised plant cell

2. The plant cell contains most of the structures present in an animal cell, with a few differences:

   (a) Instead of many small vacuoles, plant cells have a **large central vacuole** filled with cell sap, surrounded by a membrane called the **tonoplast**. Cell sap is mainly made up of water, with dissolved amino acids and mineral salts. Besides storage, the vacuole also takes in waste products and water.

   (b) Presence of a cellulose **cell wall** – The cell wall is non-living and fully permeable. It protects the cell from injury and gives the cell its shape.

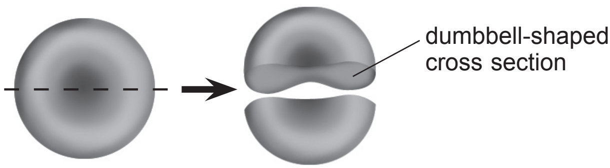
   (c) Presence of **chloroplasts** – Chloroplasts are oval membrane-bound organelles filled with chlorophyll. They are the sites of photosynthesis, which is the process by which plants make food.

   (d) Centrioles are absent.

Note: The structures visible under a light microscope would be: cell membrane, cytoplasm, nucleus, vacuoles, cell wall and chloroplasts.

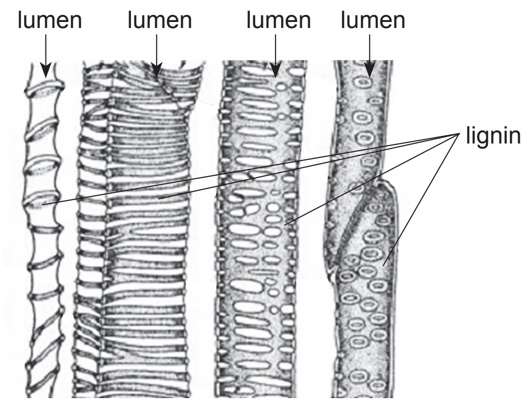
1.3 Adaptation of cells to their functions

1. Red blood cells deliver oxygen to the body tissues via the blood. Adaptations to this function include:
   (a) Red blood cells contain haemoglobin, an oxygen-carrying protein.
   (b) Red blood cells have no nucleus, so they have a flattened biconcave shape with a dumbbell-shaped cross section. This enables them to have a higher surface area to volume ratio for faster diffusion of oxygen. It also allows the cell to be more flexible when squeezing through blood capillaries.



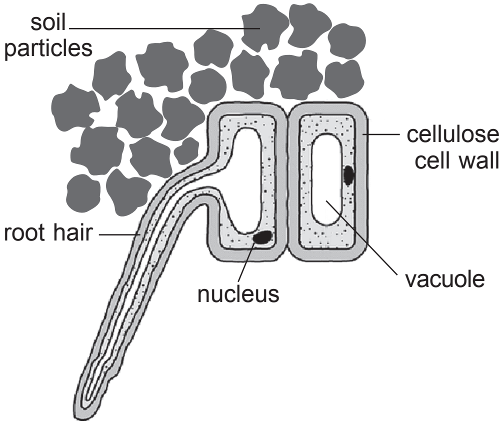
Cross-section of a red blood cell

2. Xylem vessels are elongated hollow tubes that are made of xylem cells linked end to end. Xylem cells are dead at maturity. They conduct water and mineral salts from the roots to the leaves of the plant. They also play a role in mechanical support. Adaptations to these functions include:
   (a) Absence of protoplasm and cross-walls which could impede water flow through the lumen (internal cavity)
   (b) Deposition of lignin on the cell walls which strengthens vessel walls, providing support



Xylem vessels

3. Root hair cells are cells which extend into the soil to absorb water and mineral salts. An adaptation to this function is a long and narrow structure called the root hair, which extends into the soil to absorb water. This increases the surface area to volume ratio of the cell, resulting in faster absorption.



A root hair cell

1.4 Organisation of a multicellular organism

1. The cell is the most basic unit of a living organism that can be classified as living.
2. A group of cells of the same type that are found near each other and carry out the same function comprises a **tissue**.
3. An **organ** is made up of different tissues working together to perform a specific function or a group of functions within an organism. An organ has a distinct shape which allows it to carry out its function well.
4. A group of functionally-related organs form an **organ system**.
5. Many organ systems working together make up a multicellular organism.

TOPIC 2 Movement of Substances

Objectives

Candidates should be able to:

(a) define *diffusion* and describe its role in nutrient uptake and gaseous exchange in plants and humans

(b) define *osmosis* and describe the effects of osmosis on plant and animal tissues

(c) define *active transport* and discuss its importance as an energy-consuming process by which substances are transported against a concentration gradient, as in ion uptake by root hairs and uptake of glucose by cells in the villi

2.1 Diffusion

1. Diffusion is the net (overall) movement of molecules from a region of higher concentration to a region of lower concentration down a concentration gradient. Concentration refers to the number of particles per unit volume.
2. A **concentration gradient** is the difference in concentration between a region of higher concentration of a substance and a region of lower concentration of the substance.
3. When the concentration gradient is steeper, the rate of diffusion will be faster.
4. When a concentration gradient exists, diffusion will take place until the particles are evenly distributed throughout the region.

2.2 Diffusion in biological systems

1. Diffusion is an important mode of nutrient uptake and gaseous exchange in cells.
2. The cell surface membrane is a **partially permeable membrane** that allows gases such as oxygen and carbon dioxide to pass through freely but not some other substances.
3. In cells which undergo respiration, oxygen is continually being used up within the cell. This creates a concentration gradient where oxygen concentration is lower inside the cell than in the surroundings. Thus, dissolved oxygen diffuses into the cell.

4. Carbon dioxide and other waste products are generated by the cell. This sets up a concentration gradient where the concentration of these substances is higher within the cell than outside. Thus, the substances leave the cell by diffusion.
5. In unicellular organisms such as the amoeba, diffusion is an important mode of nutrient uptake.

2.3 Osmosis

1. Osmosis is the net movement of water molecules from a region of higher water potential to a region of lower water potential, through a partially permeable membrane.
2. **Water potential** is a measure of the tendency of water molecules to move from one region to another. Since water is the solvent, forming the volume of a solution, it is not meaningful to think about the concentration of water, i.e. the number of water molecules per unit volume.
3. Water molecules that surround solutes causing them to dissolve are not able to move about freely as they are bound to the solutes. The more concentrated a solution is, the lower the number of freely moving water molecules present, hence the lower the water potential of the solution. As a result, a dilute solution has a higher water potential than a concentrated solution and pure water has the highest water potential.

Example

A U-tube filled with sucrose solutions of different concentrations was set up as shown in Fig. (a). After a few hours, it was observed that the water level in one arm of the U-tube had increased while the water level in the other arm had decreased as shown in Fig. (b). Describe and explain what had taken place in terms of the movement of the particles in the sucrose solutions.

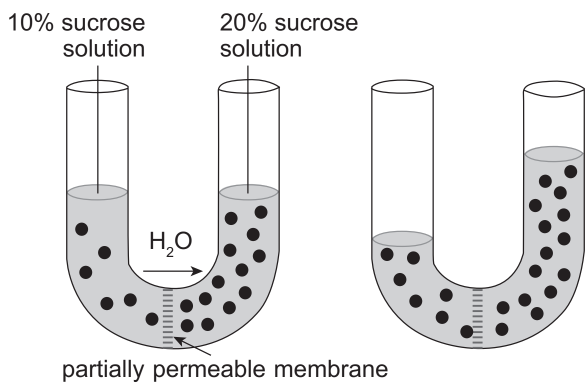
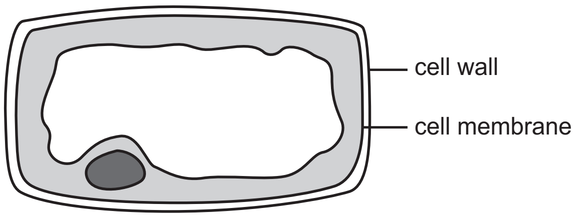


Fig. (a) Fig. (b)

Answer: The 20% sucrose solution is more concentrated than the 10% sucrose solution. Hence, it has a lower water potential as compared to the 10% sucrose solution. The partially permeable membrane does not allow sucrose molecules to pass through as sucrose molecules are too big; it only allows water molecules to pass through. As a result, water will move through the partially permeable membrane by osmosis, from the arm with the 10% sucrose solution (higher water potential) to the arm with the 20% sucrose solution (lower water potential), until the water potentials of the sugar solutions in both arms are the same. The net movement of water molecules is from left to right, hence the right arm has a higher water level at the end of the experiment.

2.4 Osmosis in plant cells

1. Osmosis in living systems refers to the movement of water molecules across the partially permeable cell surface membrane. The cell wall is fully permeable.
2. As the large central vacuole occupies most of the space in a plant cell, the water potential of the cell sap is considered to be the water potential of the plant cell.
3. When a plant cell is immersed in a solution of higher water potential relative to its cell sap, water molecules enter the cell by osmosis.
4. The vacuole increases in size and the expanded cell contents exert pressure on the cell wall.
5. The cellulose cell wall of a plant cell is strong and rigid.
6. The cell wall exerts an opposing pressure on the cell contents, preventing the entry of more water. This prevents the cell from overexpanding and bursting.
7. At this point, the plant cell is very firm or **turgid**. Turgor pressure provides mechanical support for many non-woody plants.

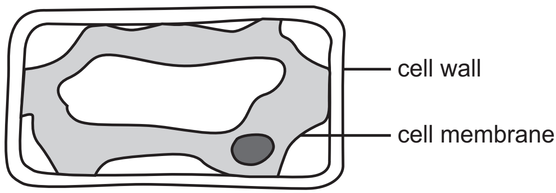


A turgid plant cell

8 . When a plant cell is immersed in a solution with a lower water potential relative to its cell sap, water diffuses out of the cell into the solution by osmosis.

9. The vacuole shrinks and the cell stops exerting pressure on the cell wall. The cell becomes limp or **flaccid**. If it is placed in a solution with a high water potential at this point, turgidity can be restored.

10. If more water leaves the cell, the vacuole and cytoplasm shrink to such an extent that the cell surface membrane pulls away from the cell wall. The phenomenon in which the cell surface membrane pulls away from the cell wall is called **plasmolysis**. This can be lethal if the cell is not quickly transferred to a solution with a higher water potential relative to its cell sap.



A plasmolysed plant cell

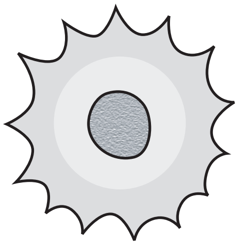
2.5 Osmosis in animal cells

1. When an animal cell is immersed in a solution with a higher water potential relative to its cytoplasm, water diffuses into the cell by osmosis.

2. The cell swells. As more water enters the cell, it swells to such an extent that it bursts. This is because it does not have a cell wall. This process is called **cytolysis**.

An animal cell undergoing cytolysis

3. When an animal cell is immersed in a solution with a lower water potential, relative to its cytoplasm, water diffuses out of the cell by osmosis.
4. The cell shrinks and become dehydrated. In red blood cells, little spikes appear on the cell surface membrane, and the cell is said to have undergone **crenation**. The animal cell will die if it is not removed from the solution.



A crenated red blood cell

2.6 Active transport

1. Active transport is the process in which energy is used to transport substances across a biological membrane against a concentration gradient.
2. The energy used for active transport is obtained through cellular respiration.
3. Uptake of dissolved mineral salts by root hair cells and glucose uptake by cells in the villi of the small intestine are examples of active transport.

TOPIC 3 Biological Molecules

Objectives

Candidates should be able to:
(a) state the roles of water in living organisms
(b) list the chemical elements which make up
- carbohydrates
- fats
- proteins

(c) describe and carry out tests for
- starch (iodine in potassium iodide solution)
- reducing sugars (Benedict's solution)
- protein (biuret test)
- fats (ethanol emulsion)

(d) state that large molecules are synthesised from smaller basic units
- glycogen from glucose
- polypeptides and proteins from amino acids
- lipids such as fats from glycerol and fatty acids

(e) explain enzyme action in terms of the 'lock and key' hypothesis
(f) explain the mode of action of enzymes in terms of an active site, enzyme-substrate complex, lowering of activation energy and enzyme specificity
(g) investigate and explain the effects of temperature and pH on the rate of enzyme catalysed reactions

3.1 Role of water in animals

1. About 70% of the human body consists of water. Water is found in cell cytoplasm, blood, digestive juices, tissue fluid, fluid in joints and contained within organs i.e. spinal cord, the brain, the eyes, gastrointestinal tract, etc.
2. Water moderates body temperature. It has a high specific heat capacity, which means that a lot of energy is required to raise the temperature of water by 1°C. Hence, water helps the cell resist changes in temperature.
3. It plays a role in **evaporative cooling**. Water is a component of sweat, which removes heat from the body when it evaporates.
4. Water is a reactant in certain chemical reactions in the body, such as the **hydrolysis** of food molecules during digestion.
5. Water is a component of body fluids with lubricative or protective properties such as lubricants in joints, coating the stomach lining, mucus in the oesophagus, and cervical mucus in the female reproductive system.

6. Water is an extremely versatile **solvent**. More things dissolve in water than in any other solvent. Because of this property,

   (a) water is the medium in which chemical reactions take place in living organisms, and

   (b) water serves as a **transportation** medium. It transports water-soluble food products from the small intestine to other parts of the body and waste materials from cells to the excretory organs for removal. It transports hormones to the target organs or tissues. Blood is the main transport medium in the body.

3.2 Role of water in plants

1. Water is a key reactant in photosynthetic processes.
2. It provides physical support to the plant in the form of turgor pressure.
3. Water is required to transport dissolved mineral salts from the roots to other parts of the plant through xylem vessels.
4. Water is required to transport sugars made in the leaves to other parts of the plant.

3.3 Simple carbohydrates

1. Carbohydrates are organic molecules made up of carbon, hydrogen and oxygen with the general formula for most carbohydrates being $C_nH_{2n}O_n$.
2. Carbohydrates are classified into 3 main groups: **monosaccharides**, **disaccharides** and **polysaccharides** depending on the number of basic sugar units they have.
3. Monosaccharides are the most basic unit of carbohydrates and are the simplest form of sugars. Common examples are glucose, fructose and galactose.
4. Disaccharides are formed when two monosaccharides undergo a condensation reaction. Common examples are maltose (formed by 2 glucose units), sucrose (1 glucose, 1 fructose) and lactose (1 galactose, 1 glucose).
5. A **condensation reaction** is a chemical reaction when two molecules combine together to form a single molecule with the elimination of a water molecule.
6. A disaccharide can be split into its component monosaccharides by undergoing **hydrolysis** in which a water molecule is added to the disaccharide to break it down into its component monosaccharides. Enzymes are usually required for this process.

3.4 Test for reducing sugars

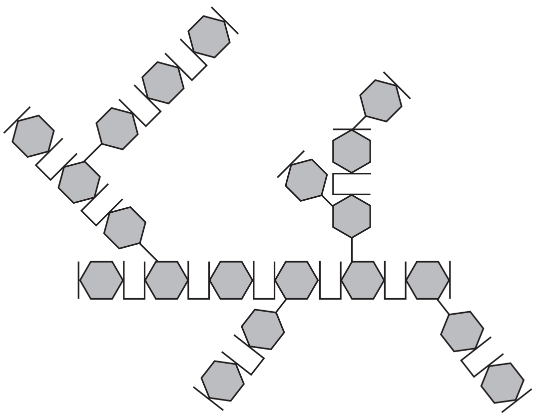
1. The test for reducing sugars is known as the Benedict's test.
2. The main reagent is Benedict's solution which contains copper(II) sulfate.
3. Reducing sugars can reduce copper(II) ions in Benedict's solution to copper(I) in the form of copper(I) oxide, a brick-red precipitate.
4. Reducing sugars are glucose, fructose, galactose, maltose and lactose. Sucrose is not a reducing sugar.
5. Procedure: Add 2 cm^3 of Benedict's solution to 2 cm^3 of sample solution and mix the contents thoroughly. Heat the test tube in a boiling water bath for 5 minutes. If the sample is an insoluble solid, crush it or cut it into small pieces before adding 2 cm^3 of water and 2 cm^3 of Benedict's solution.
6. The colour of the solution changes from green to orange to brick-red with increasing amounts of reducing sugars present.

3.5 Complex carbohydrates

1. Polysaccharides include starch, glycogen and cellulose. They are long chains of glucose molecules linked together in condensation reactions. Each chain may contain thousands of glucose molecules.
2. In starch, the glucose molecules are linked together in long straight chains or branched chains. It is a storage molecule in plants.

A starch molecule

3. In glycogen, the glucose molecules are linked together in highly branched chains. It is a storage molecule in animals and fungi.



A glycogen molecule

4. In cellulose, the glucose molecules are linked in long straight chains. The linkage between the glucose molecules is not the same as that in starch. Cellulose is the tough material found in cell walls of plants. Cellulose is the fibre necessary in a healthy diet.

A cellulose molecule

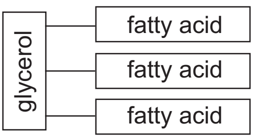
5. Glycogen and starch are the storage forms of glucose in animal and plant cells respectively. This is because
   (a) they are insoluble in water and do not affect water potential in cells,
   (b) they are too large to diffuse out of the cells and thus remain within the cells,
   (c) they have compact shapes, and
   (d) they can be easily hydrolysed into glucose for cellular respiration.

3.6 Test for starch

1. The test for starch is called the iodine test. Iodine is added to the sample and the colour change (if any) is observed.
2. Procedure: Add a few drops of iodine solution to the sample. If the sample contains starch, it will turn blue-black in colour.

3.7 Fats

1. Fats (lipids) are organic molecules made up of carbon, hydrogen and oxygen. There is no general formula for fats. The ratio of hydrogen to oxygen is much higher in fats than in carbohydrates, where the ratio of hydrogen to oxygen is 2 : 1.
2. Fats are made from two types of smaller molecules: glycerol and fatty acids. Each fat molecule contains a glycerol molecule and 3 fatty acids. Each fatty acid is linked to the glycerol backbone in a condensation reaction.



A fat molecule

3. When 3 water molecules are added to a fat molecule with the help of enzymes in a hydrolysis reaction, the fat molecule breaks down into fatty acids and glycerol.

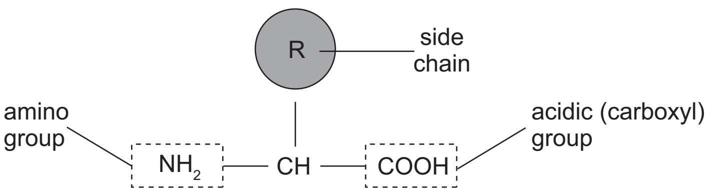
4. Fats are storage molecules that can store a large amount of energy.
5. They are also an important component of cell membranes.
6. Fats are used to make steroids and certain hormones.
7. Fats are also used as insulating material to prevent the loss of body heat.
8. Fat is also a solvent for fat-soluble vitamins.

3.8 Test for fats

1. The test for fats is known as the ethanol emulsion test.
2. Ethanol is added to the sample to allow the fats present in it to dissolve. Water is then added to the ethanolic mixture. Since fats do not dissolve in water, they precipitate out of the solution to give a cloudy white emulsion.
3. Procedure: Add 2 cm^3 of ethanol to the sample in a test tube and shake the contents thoroughly. Add 2 cm^3 of water and mix the contents. If fats are present, a white emulsion will be observed.

3.9 Proteins

1. Proteins are complex organic molecules made up of carbon, hydrogen, oxygen and nitrogen. They may also contain sulfur.
2. In the form of enzymes, proteins participate in all cellular processes and are responsible for almost everything living organisms do.
3. There are tens of thousands of different proteins, each serving a different function and having a unique structure.
4. Proteins are made up of **amino acids**.
5. An amino acid is a molecule with the general structure:



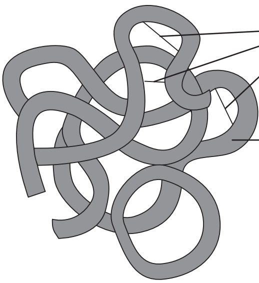
An amino acid

6. There are about 20 different naturally-occurring amino acids which have different side chains (also known as R groups).
7. Amino acids are combined in many different ways to form different protein molecules.

8. Amino acids link up in a condensation reaction to form a **polypeptide** chain. The bonds between the amino acids are known as peptide bonds.

9. Proteins are made of one or more polypeptide chains twisted, folded and coiled into a unique 3-dimensional structure.

10. The bonds between the amino acids, peptide bonds, are strong but the bonds that hold the 3-dimensional coiled structures together are weak and can easily be broken by heat or by changes in pH. Examples of such bonds are hydrogen bonds, ionic interactions and van der Waals interactions.



A protein molecule

11. When these bonds are broken, the protein loses its 3-dimensional conformation. This process is called **denaturation**. Proteins can be denatured if they are heated or placed in an environment with unsuitable pH. Denaturation usually leads to loss of function as proteins require their 3-dimensional shape to function. Denaturation can also cause proteins to lose their solubility and precipitate out of the solution.

12. Many proteins are enzymes, which catalyse chemical reactions within our body.

13. Structural proteins found in muscle cells play a role in movement.

14. Other proteins take part in cell growth, repair and reproduction.

15. Antibodies are proteins in our body that help us fight diseases.

3.10 Test for proteins

1. The test for proteins is known as the biuret test.

2. The main reagents are sodium hydroxide and copper(II) sulfate.

3. Procedure: Add 1 cm^3 of sodium hydroxide solution to 1 cm^3 of sample solution in a test tube and mix thoroughly. Add a few drops of 1% copper(II) sulfate solution dropwise into the mixture, shaking after each drop. Allow the mixture to stand for 5 minutes.

4. If proteins are present, a violet colouration will be observed.

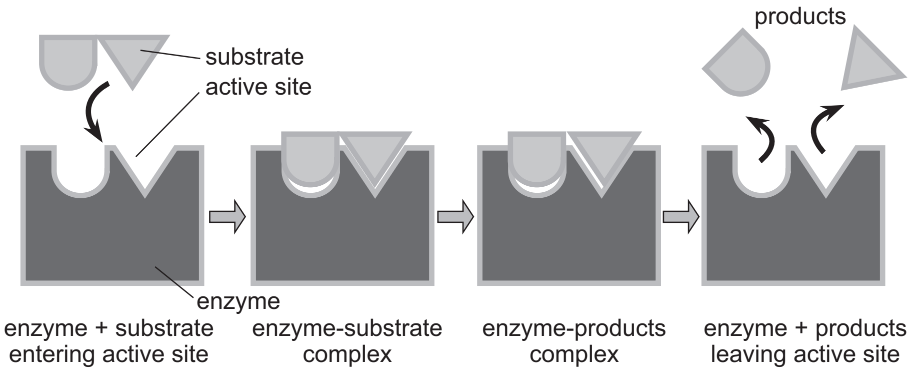
3.11 Enzymes

1. Enzymes are biological **catalysts** that speed up the rate of chemical reactions without being altered in the reaction. They are made of proteins.
2. Enzymes work by lowering the **activation energy** of a chemical reaction. Activation energy is the amount of energy needed for a reaction to take place.
3. Enzymes allow biochemical reactions to take place without drastic conditions such as high temperatures because less heat energy is required to start a reaction.
4. Enzymes can break down or build up biological molecules.
5. Enzymes are required in small amounts because they remain unchanged in the chemical reactions they catalyse and can be reused.
6. They are **substrate-specific**. Substrates are the reactants that an enzyme acts on. Each enzyme can only act on the particular substrate of the reaction they are supposed to catalyse. For example, amylase can only digest starch and not cellulose even though they are both polymers of glucose.
7. Therefore, each enzyme catalyses a different reaction. This is due to its unique 3-dimensional structure.

3.12 'Lock and key' hypothesis

1. The 'lock and key' hypothesis relates enzyme specificity to the presence of active sites. An **active site** is the region on an enzyme molecule that the substrate binds to. It is usually a pocket or groove on the surface of the enzyme that is part of the enzyme's unique 3-dimensional structure.
2. The shape of the active site conforms to the substrate. Only the correct substrate is able to fit into the active site.
3. The process begins when the substrate molecule binds to the active site of the enzyme to form an **enzyme-substrate complex**.
4. The reaction is then catalysed at the active sites to convert the substrate into product molecules.
5. The product molecules depart from the active site, leaving the enzyme free to catalyse another reaction.

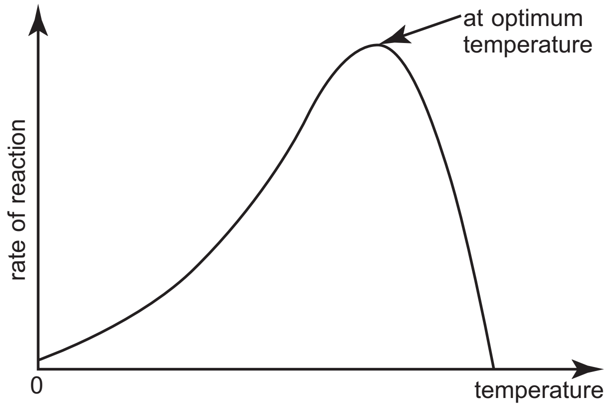
6. The diagram below illustrates the 'lock and key' hypothesis for a reaction in which an enzyme breaks down a substrate molecule into 2 product molecules:



Process of an enzyme-catalysed reaction

3.13 Effects of temperature on the rate of enzyme-catalysed reactions

1. The effects of temperature on the rate of enzyme-catalysed reactions is shown in the graph below:



Effect of temperature on the rate of reaction

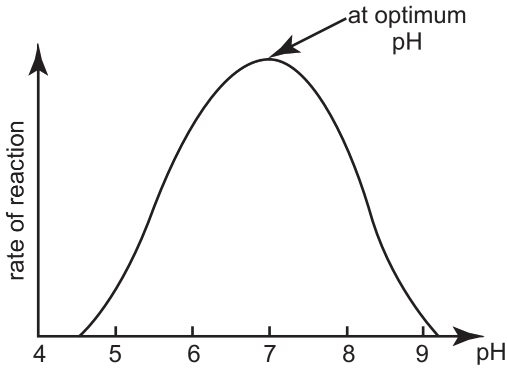
2. At low temperatures, enzymes are inactive and the rate of reaction is very low. Substrate and enzyme molecules have little kinetic energy, hence the frequency of collision is low. In addition, most substrate molecules do not contain sufficient energy to overcome the activation energy required to start a reaction.

3. As temperature increases, the rate of enzyme activity increases. Enzyme activity doubles with every 10°C rise in temperature. This is because the reactants have higher levels of energy, and the substrate molecules are able to collide with active sites more frequently.

4. At the optimum temperature, enzyme activity is the highest.
5. As the temperature increases beyond the optimum temperature, enzyme activity drops sharply. This is because enzymes are made of proteins, which are denatured at high temperatures. The enzyme loses its 3-dimensional structure and active site conformation due to the breaking of the weak bonds that hold the structure together.
6. At extremely high temperatures, the enzyme is completely denatured and the rate of reaction drops to zero.

3.14 Effects of pH on the rate of enzyme-catalysed reactions

1. The graph showing the effects of pH on the rate of enzyme-catalysed reactions is shown in the graph below:



Effect of pH on the rate of reaction of amylase

2. Enzyme activity is the highest at the optimum pH of the enzyme.
3. As the pH increases or decreases from the optimum, enzyme activity sharply decreases. This is because the hydrogen bonds and ionic bonds that hold the 3-dimensional structure are disrupted. The shape of the active site is changed as the enzyme is denatured.
4. At extreme pH levels, the enzyme is completely denatured and the rate of reaction drops to zero.
5. The optimum pH for each enzyme differs. For example, pepsin works best under the acidic conditions in the stomach, while intestinal enzymes work best under alkaline conditions.

TOPIC 4 Nutrition in Humans

Objectives

Candidates should be able to:

(a) describe the functions of main regions of the alimentary canal and the associated organs: mouth, salivary glands, oesophagus, stomach, duodenum, pancreas, gall bladder, liver, ileum, colon, rectum, anus, in relation to ingestion, digestion, absorption, assimilation and egestion of food, as appropriate

(b) describe peristalsis in terms of rhythmic wave-like contractions of the muscles to mix and propel the contents of the alimentary canal

(c) describe the functions of enzymes (e.g. amylase, maltase, protease, lipase) in digestion, listing the substrates and end-products

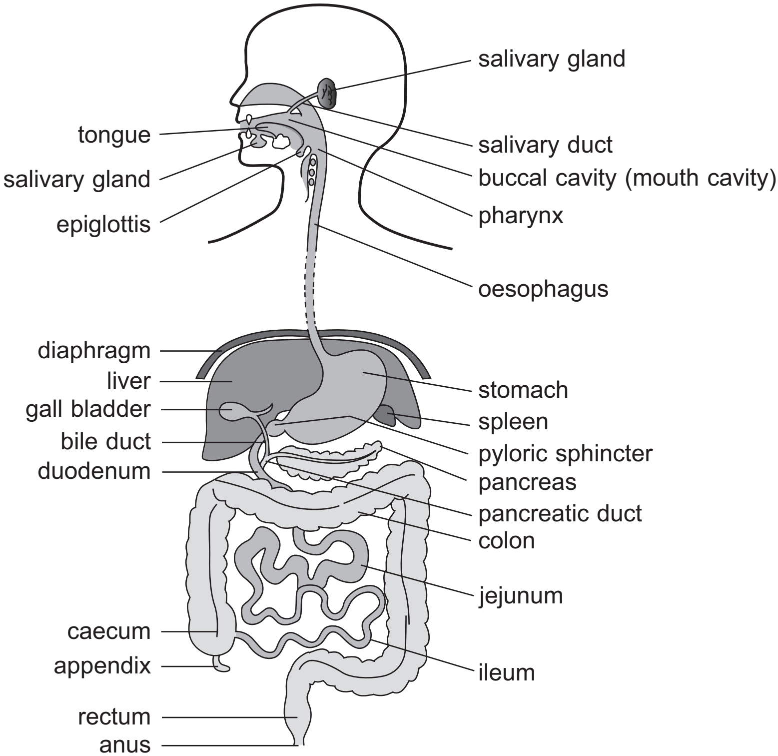
(d) describe the structure of a villus and its role, including the role of capillaries and lacteals in absorption

(e) state the function of the hepatic portal vein as the transport of blood rich in absorbed nutrients from the small intestine to the liver

(f) state the role of the liver in
- carbohydrate metabolism
- fat digestion
- breakdown of red blood cells
- metabolism of amino acids and the formation of urea
- breakdown of alcohol

(g) describe the effects of excessive consumption of alcohol: reduced self-control, depressant, effect on reaction times, damage to liver and social implications

4.1 Overview of the digestive system



The human digestive system

1. Human digestion takes place in the mouth, stomach and small intestine.
2. The alimentary canal consists of the mouth, the oesophagus, the stomach, the small and large intestines and the anus.
3. Other organs associated with digestion include the liver, pancreas, gall bladder and salivary glands.

4.2 The mouth

1. Food enters the body through the mouth, or **buccal cavity**. Physical and chemical digestion takes place in the mouth. In the mouth:
   (a) Teeth start to break the food into smaller pieces. This makes food easier to swallow and also increases the surface area to volume ratio of the food for the digestive enzymes to work on more efficiently.
   (b) Salivary glands secrete saliva which moistens the food and makes it easier to swallow. Saliva also contains salivary amylase, an enzyme which breaks down starch into maltose. The optimum pH of salivary amylase is 7.
   (c) The tongue rolls the food into a **bolus**, which is then swallowed.

4.3 The oesophagus

1. The food passes through the pharynx and enters the **oesophagus**. The oesophagus is a muscular tube that leads to the stomach.
2. It is made up of two layers of smooth muscle. The external layer is the longitudinal muscle and the inner layer is the circular muscle. These muscles found along much of the entire length of the alimentary canal.
3. These muscles contract and relax alternately to cause wave-like contractions known as peristalsis.
4. Food moves along the oesophagus due to peristalsis.
5. Digestion of starch by salivary amylase continues in the oesophagus.

4.4 The stomach

1. The food reaches the stomach, which is a muscular bag with elastic walls.
2. The stomach walls form deep pits that contain gastric glands. These glands secrete mucus which protects the stomach walls. They also secrete gastric acid and pepsinogen.
3. Peristalsis in the stomach churns the food to break the food up and mix it thoroughly with gastric juice.
4. Gastric acid is hydrochloric acid with pH 2. Gastric acid
   (a) stops the activity of salivary amylase by denaturing it,
   (b) changes the inactive form of pepsin, pepsinogen, into the active form, pepsin, and
   (c) kills germs and bacteria.
5. Pepsin is a protease. The optimum pH for pepsin is about 2.

6. Food leaves the stomach in small quantities at regular intervals, and enters the small intestine through the pyloric sphincter as a semi-liquid mass known as chyme. The pyloric sphincter is a ring of muscle at the base of the stomach that allows chyme to pass into the small intestine in small amounts at a time. Allowing the food to pass into the small intestine in small quantities ensures that the food can be completely digested by the enzymes in the intestines. If the person had a heavy meal, the contents of the stomach may be emptied over a period of up to three hours.

4.5 The small intestine

1. The small intestine is divided into three parts: the duodenum, jejunum and ileum.
2. Food is moved through the small intestine by peristalsis.
3. In the duodenum, chyme from the stomach mixes with digestive juices from the pancreas, liver, gall bladder and intestinal glands.
4. The pancreas produces pancreatic juice, which is an alkaline solution containing trypsinogen, pancreatic amylase and pancreatic lipase. Pancreatic juice enters the duodenum through the pancreatic duct.
5. Intestinal juice contains intestinal lipase, enterokinase, erepsin, maltase, lactase, sucrase and several other enzymes.
6. All enzymes in the small intestine have an optimum pH under alkaline conditions.
7. Bile, an alkaline greenish-yellow fluid, is produced by the liver and stored in the gall bladder. It passes into the small intestine through the bile duct. Bile breaks up large fat droplets into smaller fat droplets in a process called emulsification. This increases the surface area to volume ratio of the fats for lipases on work on and speeds up fat digestion.
8. Action of enzymes involved in carbohydrate digestion in the small intestine:

$$\text{starch} \xrightarrow{\text{pancreatic amylase}} \text{maltose}$$

$$\text{maltose} \xrightarrow{\text{maltase}} \text{2 molecules of glucose}$$

$$\text{sucrose} \xrightarrow{\text{sucrase}} \text{glucose + fructose}$$

$$\text{lactose} \xrightarrow{\text{lactase}} \text{glucose + galactose}$$

9. Action of enzymes involved in fat digestion in the small intestine:

$$\text{fats} \xrightarrow{\text{lipase}} \text{3 fatty acids + glycerol}$$

10. Action of enzymes involved in protein digestion in the small intestine:

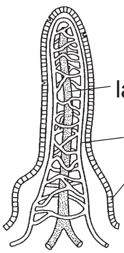
$$\text{trypsinogen} \xrightarrow{\text{enterokinase}} \text{trypsin}$$

$$\text{proteins} \xrightarrow{\text{trypsin}} \text{polypeptides}$$

$$\text{polypeptides} \xrightarrow{\text{peptidases / erepsin}} \text{amino acids}$$

Note: Enterokinase converts the inactive form of trypsin, trypsinogen, into trypsin.

11. Food is completely digested in the small intestine. The jejunum and ileum function mainly to absorb nutrients and water.
12. Nutrients have to be absorbed into the body from the lumen of the small intestine. The small intestine is adapted for this role by having:
    (a) An inner wall with large circular folds
    (b) Finger-like projections on the inner wall called villi
    (c) Each epithelial cell on the villi has smaller projections called microvilli
13. These adaptations increase the surface area of the small intestine, resulting in a larger surface for absorption.
14. The villi have thin walls (one-cell thick) so that food molecules diffuse over a shorter distance.
15. Within each villus is a network of capillaries and a small vessel called a lacteal.
16. Nutrients are absorbed across the wall of the small intestine and into the capillaries or lacteal. The lacteal transports fats away from the small intestine while the capillaries transport sugars and amino acids.



A villus

17. The transport of food away from the small intestine sets up a concentration gradient for diffusion.
18. Glucose and amino acids are absorbed by diffusion or active transport depending on the concentration gradient.
19. Fatty acids and glycerol are absorbed by the epithelial cells of the villi and recombined within those cells to form fats, which are transported into a lacteal.

20. Water is absorbed by passive diffusion throughout the length of the small intestine and mineral salts are absorbed in the ileum.
21. The food eventually leaves the small intestine and enters the large intestine.

4.6 The large intestine

1. The large intestine or colon is shaped like an inverted U and has the function of absorbing the remaining water and mineral salts that have not been absorbed by the small intestine. Note that most of the water that was present in the small intestine (from liquid in ingested food as well as the water content in intestinal mucus and digestive juices) had been absorbed by the small intestine.
2. The undigested waste matter moves along the large intestine by peristalsis, getting progressively drier.
3. The undigested waste matter comprises mainly cellulose, which is indigestible to humans.
4. The waste matter ends up at the rectum where it is stored before it can be eliminated from the body through the anus. The elimination of waste material is called **egestion**.

4.7 Transport of products of digestion

1. As absorption takes place in the small intestine, the blood in the capillaries of the villi becomes very rich in simple sugars and amino acids.
2. The blood capillaries of the villi converge into a large blood vessel called the **hepatic portal vein**, which leads to the liver.
3. The blood from the small intestine travels to the liver via the hepatic portal vein. The composition of blood in this vein varies greatly throughout the day depending on whether absorption of nutrients is occurring in the small intestine.

4.8 Role of the liver in carbohydrate metabolism

1. The liver is involved in carbohydrate metabolism and regulation of blood glucose concentration.
2. When the glucose level in blood is high, the islets of Langerhans in the pancreas secrete insulin, which is a hormone that stimulates the liver cells to convert glucose into glycogen. The liver cells convert excess glucose in the blood from the hepatic portal vein into glycogen, which is stored in the liver.
3. When the glucose level in blood is low, the islets of Langerhans secrete glucagon, which is a hormone that stimulates the liver cells to convert stored glycogen in the liver back into glucose. The glucose is released into the blood leaving the liver, which supplies glucose to the body cells.

4.9 Role of the liver in fat metabolism

1. The liver produces bile, an alkaline liquid which helps fat digestion by emulsifying fats.
2. It oxidises fats to produce energy.
3. It converts excess carbohydrates and proteins to fatty acids and glycerol which are exported and stored as fatty tissue.

4.10 Role of the liver in breakdown of red blood cells

1. Aging red blood cells are removed by the spleen.
2. Haemoglobin from the red blood cells is brought to the liver, where it is broken down. The iron from the haemoglobin is stored in the liver while the other metabolic by-products of the breakdown form bile pigments.

4.11 Role of the liver in protein metabolism

1. The liver is involved in the synthesis of plasma proteins e.g. albumin, and blood clotting factors e.g. fibrinogen.
2. The liver is responsible for the deamination of excess amino acids, which refers to the removal of the amino group ($-NH_2$) from an amino acid.
3. The amino group is converted into ammonia, which is toxic to cells, before it is further converted to urea by enzymes in the liver, and subsequently removed in urine.
4. The remnants of the amino acid are converted to glucose.

4.12 Role of the liver in detoxification

1. The liver breaks down toxic substances for excretion in urine or bile.
2. It also breaks down alcohol to acetaldehyde through the action of an enzyme called alcohol dehydrogenase.
3. Acetaldehyde is then converted to harmless acetic acid by acetaldehyde dehydrogenase.
4. Alcohol irritates oesophageal, stomach and intestinal linings. Excessive alcohol consumption can lead to inflammation and ulcers.
5. Excessive alcohol consumption can also lead to inflammation, scarring and destruction of liver cells.
6. The liver cells are replaced with fibrous scar tissue in a disease called cirrhosis of the liver, leading to loss of liver function.

TOPIC 5 Nutrition in Plants

Objectives

Candidates should be able to:

(a) identify and label the cellular and tissue structure of a dicotyledonous leaf, as seen in transverse section using the light microscope and describe the significance of these features in terms of their functions, such as the
- distribution of chloroplasts in photosynthesis
- stomata and mesophyll cells in gaseous exchange
- vascular bundles in transport

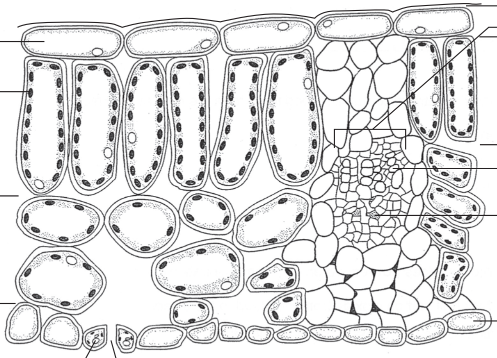
(b) state the equation, in words and symbols, for photosynthesis
(c) describe the intake of carbon dioxide and water by plants
(d) state that chlorophyll traps light energy and converts it into chemical energy for the formation of carbohydrates and their subsequent uses
(e) investigate and discuss the effects of varying light intensity, carbon dioxide concentration and temperature on the rate of photosynthesis (e.g. in submerged aquatic plant)
(f) discuss light intensity, carbon dioxide concentration and temperature as limiting factors on the rate of photosynthesis

5.1 External leaf structure

1. The leaf blade or **lamina** is thin, with a large surface area to volume ratio, increasing sunlight absorbed for photosynthesis and diffusion of carbon dioxide and oxygen.
2. The leaf stalk or **petiole** holds the leaf away from the stem so that the leaf can get more sunlight.
3. The **mid-rib** and **vein network** carry food substances away from the leaves, and water and mineral salts to the leaves.

5.2 Internal leaf structure

1. The diagram below shows the cross section of a dicotyledonous leaf as seen under a microscope:



Cross section of a dicotyledonous leaf

2. The **upper epidermis** is a single layer of irregular, closely packed cells covered by a layer of waxy cuticle. The cuticle protects the epidermis and prevents excessive water loss through evaporation. It is transparent to allow sunlight to pass through. Epidermal cells contain no chloroplasts.

3. **Palisade mesophyll** cells are columnar in shape and closely packed. They contain a lot of chloroplasts to increase absorption of sunlight for photosynthesis.

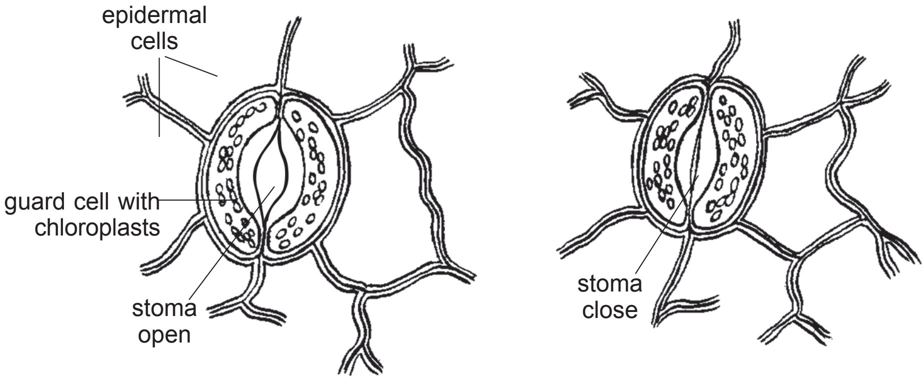
4. **Spongy mesophyll** cells are irregular in shape with numerous intercellular air spaces around them to allow for fast diffusion of carbon dioxide, which enters the leaf through the stomata, to all photosynthetic cells. They contain fewer chloroplasts than palisade mesophyll cells. They are covered with a thin film of moisture so that carbon dioxide can dissolve in it.

5. Within the mesophyll layers are the **vascular bundles** containing xylem and phloem tissue. This brings the transport tissue into close contact with the photosynthetic tissue, allowing water and mineral salts to be distributed to the photosynthetic cells efficiently and food products to be brought to other parts of the plant.

6. The **lower epidermis** is also a single layer of closely-packed cells covered by a layer of waxy cuticle.

7. **Guard cells** are bean-shaped, chloroplast-containing cells located in the lower epidermis. They control the opening and closing of the **stoma** (plural: stomata), the gap between the guard cells. The stomata allow carbon dioxide to diffuse in, oxygen to diffuse out and water vapour to escape.

5.3 Mode of operation of guard cells



Open stoma **Closed stoma**

1. Plants open their stomata during the day for carbon dioxide intake and close their stomata during the night to minimise water loss through transpiration.
2. Guard cells control the opening and closing of stomata through regulation of water potential within themselves.
3. Photosynthesis in guard cell chloroplasts provides the energy for the uptake of potassium ions into the cell.
4. This lowers the water potential within the guard cells, causing water to enter by osmosis.
5. Each guard cell has a thicker cellulose cell wall on the side surrounding the stomata, as compared to the side adjacent to neighbouring epidermal cells. When water enters the cell, the side away from the stoma, being thinner, expands more than the side framing the stoma. This causes the cells to become curved such that the stoma opens.
6. When there is excessive water loss, even during the day, the guard cells lose turgor and become flaccid, causing the stoma to close.

5.4 Intake of carbon dioxide

1. When carbon dioxide within the leaf is used up by photosynthesis, the concentration of carbon dioxide in the leaf becomes lower than that in atmospheric air.
2. Carbon dioxide diffuses into the intercellular air spaces of the spongy mesophyll layer through the stomatal openings.
3. The mesophyll cells exposed to the intercellular air spaces are covered by a thin film of water. Carbon dioxide dissolves in it and diffuses into the cells.

5.5 Intake of water

1. The vascular bundles in the stem pass through the petioles and enter the leaves.
2. Within the leaves, they branch throughout the mesophyll layers, forming leaf veins.
3. Water from the roots travels through the xylem vessels in the vascular bundles and enters the leaves.
4. Once out of the xylem vessels, water travels from cell to cell through osmosis.

5.6 Photosynthesis

1. Photosynthesis is the process by which plants convert carbon dioxide and water into sugars using sunlight as energy in the presence of chlorophyll.
2. Equation for photosynthesis:

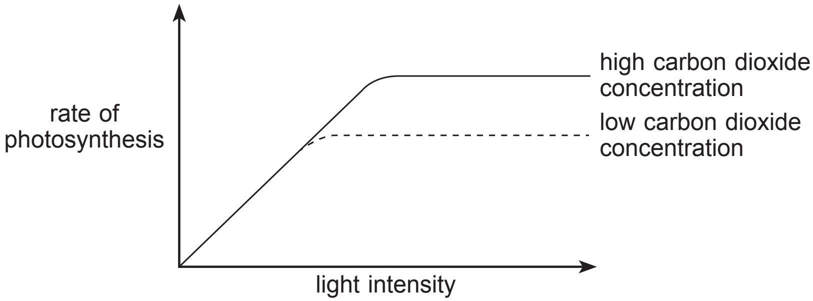
$$6CO_2 + 12H_2O \xrightarrow[\text{chlorophyll}]{\text{light energy}} C_6H_{12}O_6 + 6O_2 + 6H_2O$$

$$\text{carbon dioxide + water} \xrightarrow[\text{chlorophyll}]{\text{light energy}} \text{glucose + oxygen + water}$$

3. Photosynthesis is split into 2 stages: light-dependent stage and light-independent stage.
4. Light-dependent stage:
   (a) Light energy is absorbed by chlorophyll and used to split water into hydrogen and oxygen atoms in a process called **photolysis**.
   (b) The oxygen atoms combine to form oxygen gas which is a product of photosynthesis.
   (c) Other high-energy molecules are generated for use in the light-independent stage to convert carbon dioxide into glucose.
5. Light-independent stage:
   (a) The chemical energy stored during the light reactions as high-energy molecules is used in a series of reactions to convert carbon dioxide into carbohydrate.
   (b) Hydrogen from the light reactions is used as a reducing agent in the process.
   (c) The carbohydrate formed in this stage is converted to glucose and other carbohydrates by enzymes.
   (d) No light energy is required in this stage.

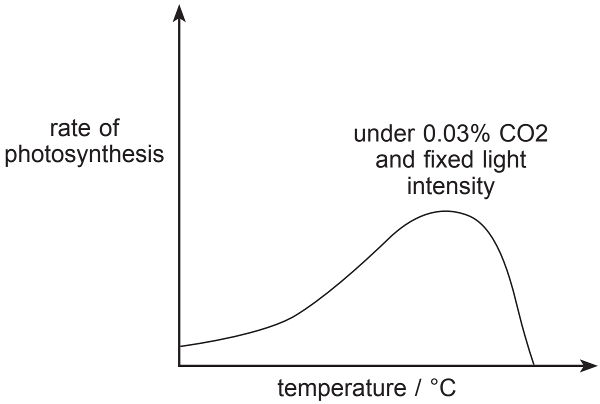
5.7 Limiting factors on rate of photosynthesis

1. **Light intensity**, **carbon dioxide concentration** and **temperature** are limiting factors on the rate of photosynthesis.
2. At a constant temperature and carbon dioxide concentration, the rate of photosynthesis increases with increasing light intensity until it reaches a plateau.
3. When the plateau is reached, light is no longer the limiting factor in the reaction. The concentration of carbon dioxide becomes the limiting factor.
4. Increasing the concentration of carbon dioxide raises the plateau reached.
5. Increasing the temperature over a certain range has little effect at low light intensities but increases the rate of photosynthesis at high light intensities.



Effect of light intensity on the rate of photosynthesis

6. Both light and dark reactions involve enzymes which would be denatured at a high temperature.



Effect of temperature on the rate of photosynthesis

TOPIC 6 Transport in Flowering Plants

Objectives

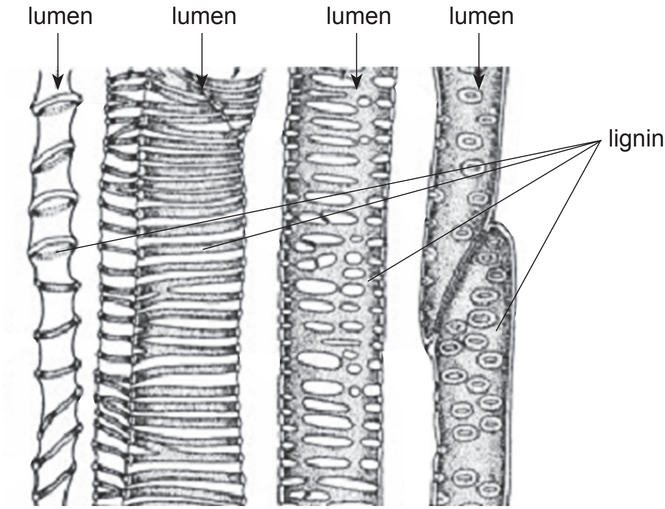
Candidates should be able to:

(a) identify the positions and explain the functions of xylem vessels, phloem (sieve tube elements and companion cells) in sections of a herbaceous dicotyledonous leaf and stem, under the light microscope
(b) relate the structure and functions of root hairs to their surface area, and to water and ion uptake
(c) explain the movement of water between plant cells, and between them and the environment in terms of water potential. (Calculations on water potential are **not** required.)
(d) outline the pathway by which water is transported from the roots to the leaves through the xylem vessels
(e) define the term *transpiration* and explain that transpiration is a consequence of gaseous exchange in plants
(f) describe and explain
- the effects of variation of air movement, temperature, humidity and light intensity on transpiration rate
- how wilting occurs

(g) define the term *translocation* as the transport of food in the phloem tissue and illustrate the process through translocation studies

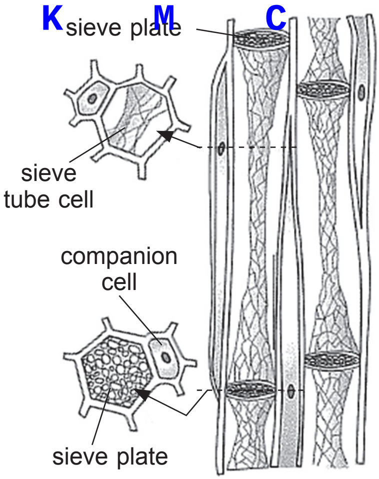
6.1 Transport vessels

1. Vascular tissues of the plant consist of **xylem vessels** and the **phloem**.
2. Xylem vessels are elongated hollow tubes that are made of xylem cells linked end to end. Xylem cells are dead at maturity.
3. Functions of xylem tissue:
   (a) Conduct water and mineral salts from the roots to the leaves
   (b) Mechanical support
4. Adaptations to these functions include:
   (a) Absence of protoplasm and cross-walls which could impede water flow through the lumen (central space)
   (b) Deposition of **lignin** on the cell walls which strengthens vessel walls, providing support



Xylem vessels

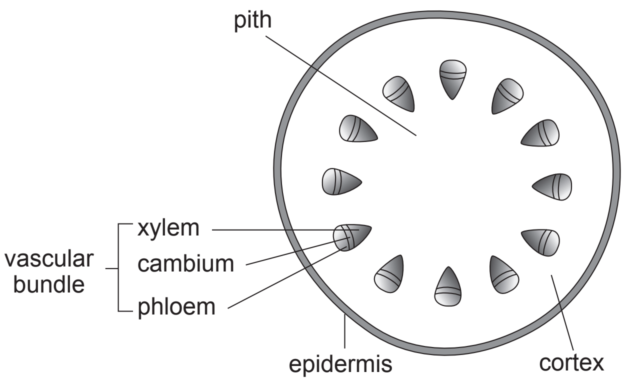
5. The phloem tissue consists of **sieve tube elements** and **companion cells**.
6. Sieve tube elements are elongated thin-walled living cells. They have degenerate protoplasm, which means they lack organelles such as the nucleus, ribosomes and the large central vacuole.
7. Sieve tube elements are arranged end to end, with porous walls called **sieve plates** between them.
8. There is one companion cell closely associated with each sieve tube element. Companion cells contain nuclei, cytoplasm and numerous mitochondria, and are responsible for performing the metabolic functions of the sieve tube elements.
9. The function of the phloem is to conduct sugars and amino acids from the leaves to other parts of the plant.
10. Adaptations to this function include:
    (a) Porous sieve plates that allow uninterrupted flow of food substances through the sieve tubes
    (b) Numerous mitochondria in the companion cells that provide energy for them to help load sieve tube members with sugar



Phloem vessels

6.2 Position of vascular tissue in dicotyledonous stems

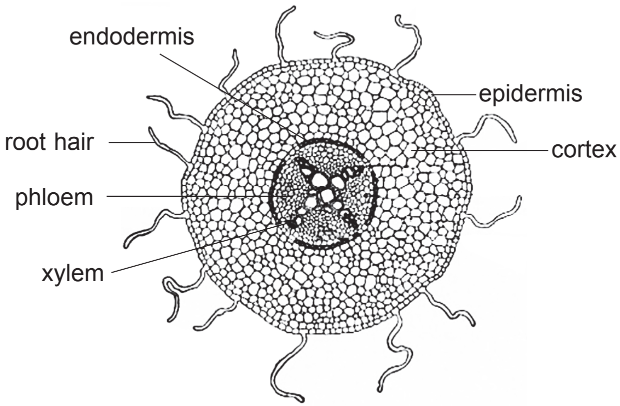
1. In dicotyledonous stems, the vascular bundles are arranged in a ring around a central pith.
2. Between the ring of vascular tissue and the epidermis is the cortex. The epidermis is covered by waterproof cuticle that minimises water loss in the stem.
3. Within the vascular bundles, the phloem tissue is found on the side facing the cortex and the xylem on the side facing the pith. Between the xylem and phloem is a layer called the cambium. Cambium cells can differentiate into new xylem and phloem tissues.
4. Food is stored in the cortex and pith.



Transverse section of a stem

6.3 Position of vascular tissue in dicotyledonous roots

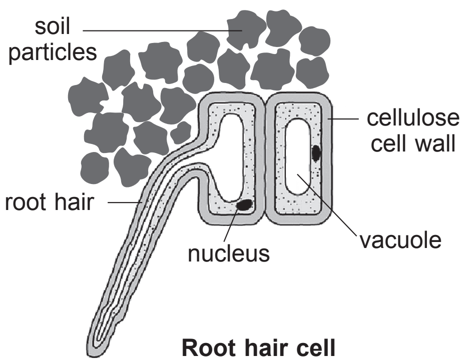
1. The outermost layer of the root is the piliferous layer. It is a single layer of cells bearing root hairs.
2. The layer below the epidermis is called the cortex. It consists of storage tissue.
3. The central region of the root contains xylem and phloem tissues. The xylem radiates from the centre, with phloem tissues alternating between them.



Transverse section of a root

6.4 Root hair cells

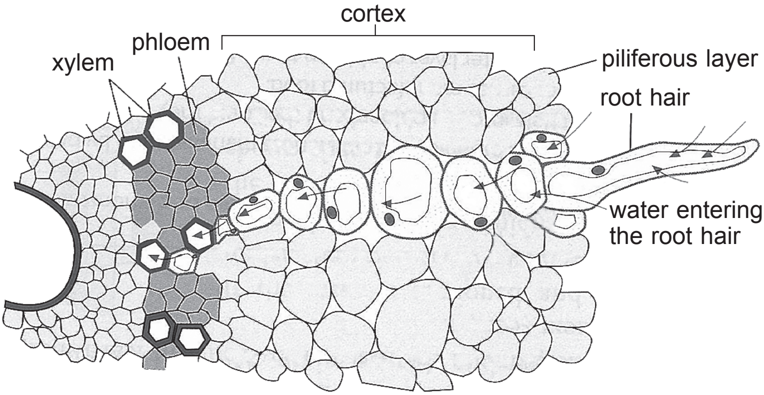
1. Root hairs are tubular outgrowths of root epidermal cells. Each root hair is usually an outgrowth of a single epidermal cell, so they are one-cell thick.
2. Being long and narrow, they have a large surface area to volume ratio for rapid absorption of water and minerals.
3. The cell surface membrane controls the water potential of the cell sap. The cell sap has a lower water potential than the soil solution, causing osmosis to take place.



Root hair cell

6.5 Absorption of water and minerals by root hair cells

1. Soil particles are usually coated with water and dissolved mineral salts.
2. The cell sap in the root hair cells contains sugars and ions that cause it to be at a lower water potential than soil solution.
3. Water moves across the partially permeable cell surface membrane from the soil solution into the cell sap by osmosis.
4. The cell sap now has a higher water potential than the cell sap in the adjoining cell.
5. Water moves across the cell surface membranes into the adjoining cell by osmosis.
6. This process continues until the water enters the xylem vessels and moves up the plant.



The path of water through the root

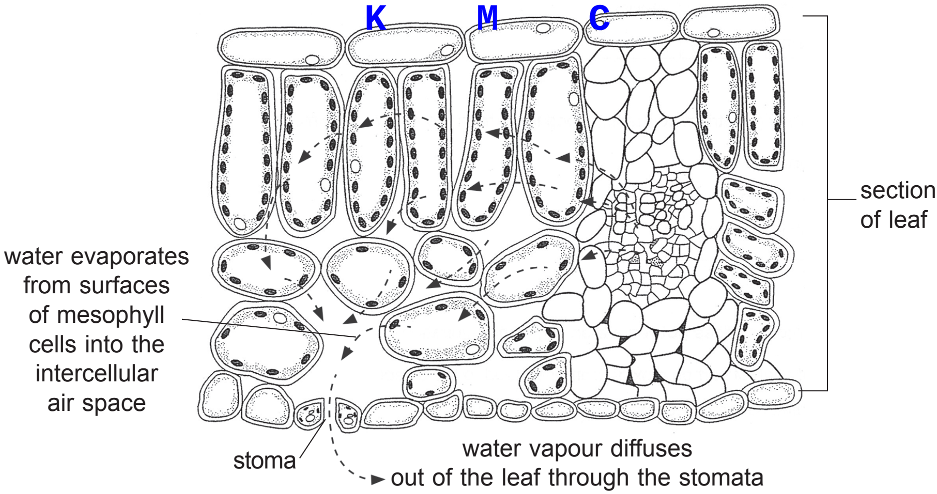
6.6 Transportation of water from the roots to the leaves

1. Water travels from the roots to the leaves against gravity through 3 primary mechanisms:
   (a) Root pressure
   (b) Transpiration
   (c) Capillary action
2. Root cells pump mineral salts into the xylem vessels using active transport. This causes the water potential of the xylem vessels to be lower than the water potential in the cortex cells. Water moves into the xylem vessels by osmosis, creating a pressure that forces water to move upwards. This is called **root pressure**.
3. Root pressure is not the main mechanism for movement of water in most plants as it can only force water to travel a short distance.
4. **Transpiration** is the loss of **water vapour** from the stomata of the leaves through diffusion.

5. The stomata have to be open for carbon dioxide intake due to photosynthesis. This allows the loss of water vapour from the intercellular air spaces in the leaves as the air outside has a lower water vapour concentration than the air inside the leaf. Transpiration is the necessary cost of carbon dioxide intake.
6. However it is also responsible for the **transpiration pull**, which is the main force that causes water to travel upwards in plants.
7. Transpiration pull is the suction force caused by transpiration that pulls water up the xylem.
8. **Capillary action** is the tendency of water to travel up the narrow xylem tubes due to the interactions between water molecules and the xylem walls. This is usually observed in young plants with narrow veins and is not significant in larger plants.

6.7 Factors affecting the rate of transpiration

1. Water vapour in the intercellular air space diffuses out of the stomata.
2. Evaporation from the thin film of water that coats the mesophyll cells replaces the water lost through transpiration.
3. As water evaporates from the mesophyll cells, the water potential of the cell sap decreases. The mesophyll cells absorb water from neighbouring cells closer to the vascular bundles by osmosis. These cells, in turn, absorb water from the xylem vessels.
4. This creates a suction force that pulls the entire column of water up the xylem vessels.
5. Factors affecting transpiration are:
   (a) **Humidity** of the surroundings – Humidity affects the concentration gradient of water vapour between the intercellular air spaces in the leaf and the external environment. The higher the humidity, the higher the concentration of water vapour in the external air. The diffusion gradient for water vapour is less steep so the rate of transpiration is lowered.
   (b) **Air movement** – Wind removes the water vapour that accumulates outside the stomata due to transpiration. This maintains the steep diffusion gradient of water vapour. The rate of transpiration will remain high as long as water vapour is continually being removed by wind.
   (c) **Temperature** – Heat increases the rate of evaporation and also increases the movement of water molecules. The higher the temperature, the higher the rate of evaporation as well as the rate of movement of water vapour, and thus, the higher the rate of transpiration.
   (d) **Light intensity** – Light intensity causes stomatal opening. Since transpiration takes place mainly through the stomata, the rate of transpiration will increase with increased light intensity.
6. **Wilting** takes place when the rate of transpiration exceeds the rate of water intake by the roots. Plant cells lose water and become flaccid.



The path of water in a leaf

6.8 Translocation

1. Translocation is the transport of sugars from the leaves to other parts of the plant. This is done by the phloem tissues. The leaves, which supply sugar, are known as the source while other parts of the plant which require sugar are known as the sink.
2. Energy is required for this process as the mode of uptake of sugars into sieve tube elements in the leaves is active transport.
3. At the end of the sieve tube where sugars are being unloaded for use, sugars are also removed from the sieve tube by active transport.

6.9 Translocation studies

1. To show that translocation occurs in the phloem, radioactive carbon dioxide may be introduced to the plant. After a few hours, slices of tissues are removed from the stems to determine where radioactivity, which indicates the presence of radioactive sugars, first appears.
2. Translocation occurs from source to sink and the direction of the movement may be upwards or downwards. To study the direction of translocation in a plant, a ring of bark, containing the phloem, is cut away from the stem. A few days later, a bulge has formed on top of the cut. This is formed due to an accumulation of phloem sap, as it is unable to move downwards towards the roots.
3. When an aphid is introduced to a plant, it will insert its proboscis into the stem to feed. The rest of the aphid is removed from its proboscis and phloem sap will continue to exude from the free end of the proboscis, which shows that there is pressure in phloem sap. This pressure is formed due active loading of sugar at the source, which will cause water to enter the phloem to generate a region of high pressure, and active unloading of sugar at the sink, which will cause water to exit the phloem, generating a region of low pressure.

TOPIC 7 Transport in Humans

Objectives

Candidates should be able to:

(a) identify the main blood vessels to and from the heart, lungs, liver and kidney
(b) state the role of blood in transport and defence
- red blood cells – haemoglobin and oxygen transport
- plasma – transport of blood cells, ions, soluble food substances, hormones, carbon dioxide, urea, vitamins, plasma proteins
- white blood cells – phagocytosis, antibody formation and tissue rejection
- platelets – fibrinogen to fibrin, causing clotting

(c) list the different ABO blood groups and all possible combinations for the donor and recipient in blood transfusions
(d) relate the structure of arteries, veins and capillaries to their functions
(e) describe the transfer of materials between capillaries and tissue fluid
(f) describe the structure and function of the heart in terms of muscular contraction and the working of valves
(g) outline the cardiac cycle in terms of what happens during systole and diastole. (Histology of the heart muscle, names of nerves and transmitter substances are **not** required.)
(h) describe coronary heart disease in terms of the occlusion of coronary arteries and list the possible causes, such as diet, stress and smoking, stating the possible preventative measures

7.1 Overview of the human circulatory system

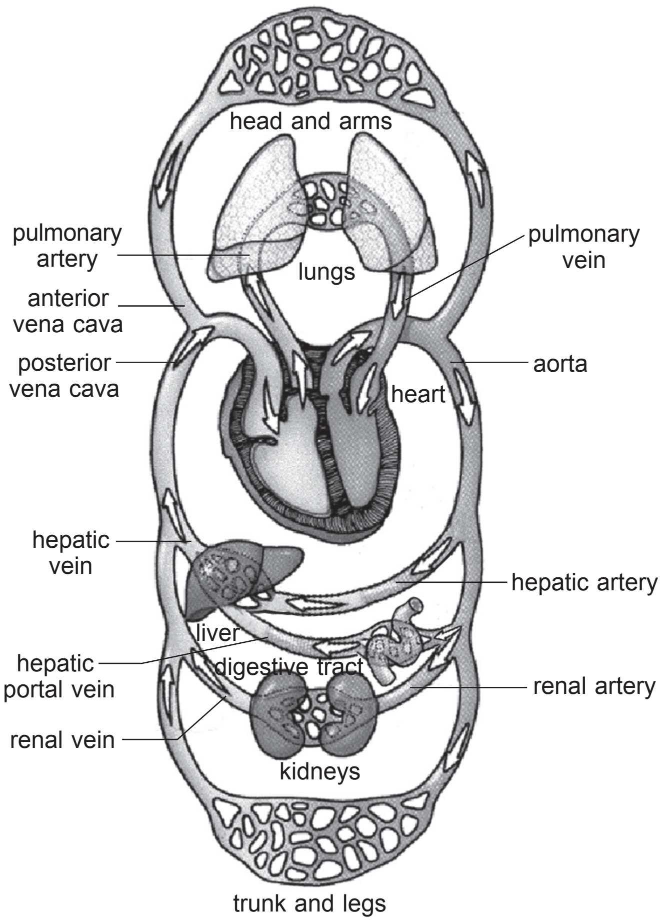
1. The components of the circulatory system are the **heart**, **blood vessels** and **blood**.
2. Blood passes through the heart twice in a complete circuit. This is termed double circulation.
3. Double circulation consists of:

   (a) **Systemic circulation** – Carries oxygenated blood (oxygen-rich) from the heart to all body organs and returns oxygen-poor blood to the heart

   (b) **Pulmonary circulation** – Carries deoxygenated blood (oxygen-poor) from the heart to the lungs for gaseous exchange before returning blood to the heart for transport to the body organs via systemic circulation

4. The three main types of blood vessels are:
   (a) **Arteries** – Vessels that carry blood away from the heart to body organs. Arteries branch into arterioles and then into capillaries.
   (b) **Capillaries** – Microscopic vessels that connect between the arteries and veins. They converge into venules which converge into veins. They form networks called capillary beds that are present in most body tissues.
   (c) **Veins** – Vessels that return blood to the heart
5. The main vessels of the human circulatory system are:
   (a) **Pulmonary arteries** that supply oxygen-poor blood from the heart to the lungs
   (b) **Pulmonary veins** that bring oxygen-rich blood from the lungs to the heart
   (c) **Aorta** that supplies oxygen-rich blood from the heart to the rest of the body. The aorta branches into: **coronary arteries** which supply cardiac tissue, an anterior branch leading to the head and arms and a posterior branch (dorsal aorta) leading to abdominal organs and legs.
   (d) Branches of the **dorsal aorta** include:
      (i) **Hepatic artery** from the heart to the liver
      (ii) Arteries to the alimentary canal
      (iii) **Renal arteries** from the heart to the kidneys
   (e) **Vena cava** consists of an anterior branch which returns blood from the head and arms to the heart and a posterior branch.
   (f) **Posterior vena cava** collects blood from the posterior parts of the body, such as from:
      (i) **Hepatic veins** from the liver to the heart
      (ii) **Renal veins** from the kidneys to the heart

(g) **Hepatic portal vein** transports blood from the alimentary canal to the liver. Blood from the liver is returned to the heart via the hepatic vein.



The human circulatory system

7.2 Components of blood

1. Blood is a connective tissue consisting of 45% cells suspended in 55% plasma.
2. Plasma is a clear yellowish liquid consisting mostly of water. It contains soluble proteins such as albumin and fibrinogen, as well as dissolved substances such as nutrients, waste products and ions.
3. Cellular elements in blood include:
   (a) Red blood cells (**erythrocytes**) which function to transport oxygen. Adaptations to this function are:
      (i) Flattened, biconcave shape without nucleus or organelles at maturity, increasing the surface area to volume ratio for faster diffusion of oxygen
      (ii) Contains haemoglobin, an iron-containing protein which is able to bind reversibly with oxygen
      (iii) Flexibility to turn bell-shaped in order to pass through the narrow lumen of the capillaries
   (b) White blood cells (**leukocytes**) are responsible for fighting infections in the body. There are two main types of white blood cells:
      (i) **Phagocytes** have lobed (bi-lobed, tri-lobed, multi-lobed) nuclei and granular cytoplasm. They engulf and digest foreign particles such as bacteria.
      (ii) **Lymphocytes** have a large rounded nucleus and a small amount of cytoplasm. They produce antibodies to protect the body from pathogens.
   (c) **Platelets** (thrombocytes) are small cell fragments which have no nuclei. They play a role in blood clotting.


Erythrocytes


Phagocyte


Lymphocyte

7.3 Role of blood in transport

1. Blood plasma transports:
   (a) Simple sugars, amino acids, fatty acids and glycerol from the capillaries in the small intestine
   (b) Waste products of metabolism from tissues:
      (i) Carbon dioxide in the form of bicarbonate ions. Carbon dioxide enters the blood from body tissues by diffusion into red blood cells, which contain the enzyme carbonic anhydrase to convert it to hydrogen carbonate. The hydrogen carbonate then diffuses out of red blood cells to be carried in plasma. In the lungs, the reverse occurs.
      (ii) Nitrogenous waste products such as urea, uric acid and creatinine to the kidneys to be removed
   (c) Hormones from the glands to target tissues
   (d) Heat from muscles and liver throughout the body
2. Red blood cells transport:
   (a) Oxygen as oxyhaemoglobin
   (b) A small amount of carbon dioxide bound to haemoglobin

7.4 Transport of oxygen by red blood cells

1. As air enters the lungs, oxygen dissolves in the fluid covering the moist epithelium of the alveoli.
2. The oxygen diffuses into the capillaries of the lungs where they bind reversibly with haemoglobin in red blood cells to form oxyhaemoglobin.
3. When blood is transported to oxygen-poor respiring tissues, oxyhaemoglobin releases its oxygen which then diffuses into tissue cells.

7.5 Immune function of white blood cells

1. **Phagocytosis** refers to the ingestion of harmful foreign particles, bacteria and dead or dying cells by certain types of white blood cells called phagocytes.
2. When phagocytes detect a foreign particle, it engulfs it by stretching itself around the particle and enclosing it. It then digests the particle and kills it.
3. After phagocytosis, these cells die and form pus.
4. **Antibodies** are special proteins found in blood and other bodily fluids that help phagocytes identify and neutralise foreign particles. Antibodies also activate other immune responses.

5. When pathogens enter the blood, they stimulate lymphocytes to produce antibodies.

6. Antibodies may be present in the blood long after infection has been cured, conferring **immunity** to that particular infection.

7.6 Tissue rejection

1. Tissue rejection occurs when the transplanted tissue is not accepted by the body of the transplant recipient.

2. During tissue rejection, the tissues of the transplanted organ are treated as foreign bodies by the recipient's immune system and are attacked by phagocytes. This causes the transplanted tissue to fail.

3. Prevention of tissue rejection:

   (a) Required tissue can be transplanted from genetically-similar donors.

   (b) Tissue can be transplanted from one part of the body to another, e.g. skin grafting, as the tissue will be recognised as the recipient's own tissue.

   (c) Immunosuppressive drugs can be taken to suppress the immune system of the recipient. Associated problems include:

      (i) Lowered resistance to infection

      (ii) Having to continue taking the drugs for their entire lifespan

7.7 Blood clotting

1. The blood clotting process begins at the site of injury when blood vessels are damaged.

2. Platelets are activated, and the damaged tissue and activated platelets release thrombokinase.

3. Thrombokinase converts plasma protein, prothrombin, into thrombin in the presence of calcium and vitamin K.

4. Thrombin converts fibrinogen, a soluble plasma protein, to fibrin, an insoluble protein that forms long threads.

5. Fibrin forms a mesh across the damaged surface and traps red blood cells, forming a clot.

6. The clot prevents further blood loss, and also restricts the entry of pathogens into the blood.

7.8 Blood groups

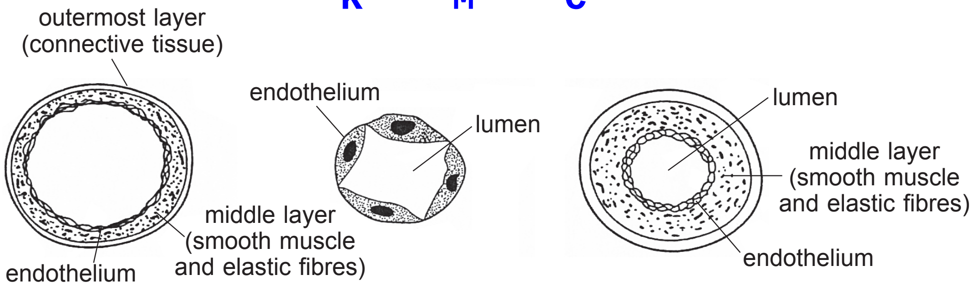
1. There are 4 blood groups: A, B, AB and O. This classification is based on certain proteins present on the surfaces of red blood cells.
2. These proteins can be recognised by antibodies present in the blood plasma as either foreign or 'self'.
3. If they are recognised as foreign, an immune response will be mounted against the foreign blood, resulting in agglutination, where the red blood cells clump together and are marked for phagocytosis.
4. When no agglutination occurs, it shows the blood can be accepted by the recipient.
5. Transfusion results between the different blood groups are shown below:

Donor	Recipient			
	A	**B**	**AB**	**O**
A	Accepted	Rejected	Accepted	Rejected
B	Rejected	Accepted	Accepted	Rejected
AB	Rejected	Rejected	Accepted	Rejected
O	Accepted	Accepted	Accepted	Accepted

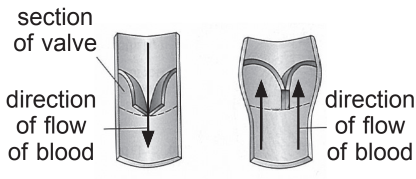
7.9 Blood vessels and their functions

1. Arteries are blood vessels which carry blood away from the heart.
2. They have thick, muscular and elastic walls that can withstand the surge of the high pressure blood pumped out of the heart.
3. The arterial wall is divided into three layers. The outer layer is a protective layer consisting of connective tissue and elastic fibre. The middle layer consists of smooth muscle and more elastic fibres and the innermost layer next to the lumen consists of the **endothelium**, a single layer of flattened cells.
4. All arteries carry oxygenated blood with the exception of the pulmonary arteries.
5. Arteries split up into arterioles which are structurally similar to arteries but smaller in diameter.
6. Arterioles control blood flow into capillary beds by:
   (a) Contracting the smooth muscle layer in the arteriole wall.
   (b) Using sphincters, which are bands of smooth muscle located where arterioles branch into capillaries. Contraction prevents blood flow into capillary beds.

7. Capillaries are microscopic vessels with walls that are only one-cell thick. Their walls consist of a layer of flattened cells called endothelial cells.
8. The endothelium is partially permeable, allowing diffusion to occur.
9. Capillaries branch to form networks called capillary beds, which infiltrate almost all tissues, allowing exchange of substances to take place.
10. The extensive branching increases the total cross-sectional area of the vessels, lowering the blood pressure in the capillaries and hence the rate of blood flow, giving more time for the exchange of substances.
11. Capillaries converge into venules which are small vessels structurally similar to veins.
12. Venules converge to form veins.
13. Similar to arterial walls, the walls of veins consist of three layers.
14. However, the middle wall contains much less smooth muscle and elastic fibres. Hence they are not as thick, muscular or elastic as arteries. Therefore, a vein has a larger lumen as compared to an artery with the same external diameter.
15. The blood pressure in the veins is much lower than that of the arteries. Blood flows more slowly and smoothly so there is no need for thick, muscular and elastic walls.
16. Blood flow through the veins is assisted by the presence of semi-lunar valves and skeletal muscle action.
17. When we move, our skeletal muscles pinch the veins and move blood through them.
18. Blood is prevented from flowing backwards by the semi-lunar valves. Blood moving backwards causes the valves to close.
19. Veins carry blood back to the heart. The exceptions are portal veins, which carry blood between two capillary beds, e.g. the hepatic portal vein.
20. Veins carry deoxygenated blood with the exception of the pulmonary veins.



Vein **Capillary** **Artery**



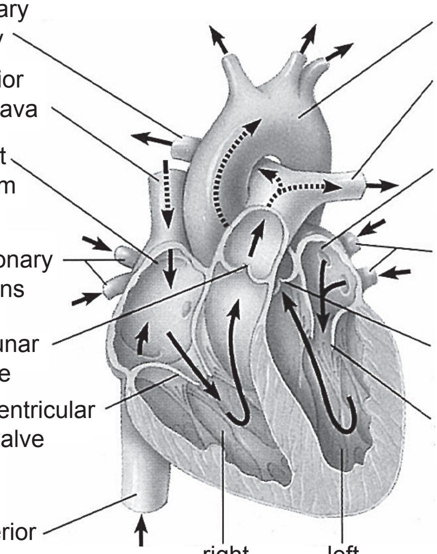
Semi-lunar valves in veins

7.10 Exchange of substances in capillaries

1. Capillaries are found between tissue cells.
2. As blood enters the capillaries, the narrow lumen of the capillaries forces red blood cells to travel in a single line.
3. Rate of blood flow decreases, allowing more time for the exchange of materials between tissue cells and red blood cells.
4. At the arterial end of capillaries, the blood pressure is high, forcing plasma through capillary walls into tissues. Plasma proteins are unable to pass through capillary walls.
5. The solution bathing tissue cells becomes known as tissue fluid, or **interstitial fluid**.
6. There is a higher concentration of nutrients and oxygen in blood than in the interstitial fluid. They diffuse across the endothelium of the capillary into the interstitial fluid, and from there, across the plasma membranes of tissue cells.
7. Waste materials from the tissue cells diffuse into the interstitial fluid, where they are present in higher concentrations than within the blood. They diffuse across the endothelium of the capillary into blood and are transported to excretory organs for removal.

7.11 Structure of the heart

1. A diagram of the heart and its associated blood vessels is shown below:



The human heart

2. The heart is mainly made up of cardiac muscle tissue surrounded by a double-walled sac called a **pericardium**. The inner membrane of the pericardium is connected to the outer layer of the cardiac muscle. Between the two layers is the pericardial fluid, which reduces friction when the heart is beating.
3. The four chambers of the heart are the right and left atria and ventricles.
4. The atria are the upper chambers of the heart, with relatively thin walls. They collect blood returning to the heart and pump it into the ventricles.
5. The ventricles have thick, muscular walls. The left ventricle has thicker walls than the right ventricle, as it has to pump blood to the rest of the body.
6. The right side of the heart pumps deoxygenated blood and the left side pumps oxygenated blood. The septum separating the right and left sides prevent the blood from mixing, so that the maximum amount of oxygen can be carried to the tissues.
7. Between the right atrium and ventricle is a valve called the **tricuspid valve** which consists of three flaps attached to the walls of the right ventricle by cord-like tendons called ***cordae tendineae***.

8. Between the left atrium and left ventricle is a **bicuspid valve** (mitral valve) which consists of two flaps, also attached by *cordae tendineae*.

9. The bicuspid and tricuspid valves are collectively known as atrioventricular valves.

10. Vessels associated with the heart are the anterior and posterior venae cavae, pulmonary veins and artery, aorta and coronary arteries. The coronary arteries are found on the heart surface itself, and supply blood to the heart muscles.

11. Located at the start of the aorta and pulmonary arteries are **semi-lunar valves**.

7.12 Cardiac cycle

1. One complete sequence of pumping and filling of the heart is called the cardiac cycle.

2. The contraction phase is called **systole** and the relaxation phase is called **diastole**.

3. The cycle starts when the whole heart is relaxed. The right atrium receives blood from the venae cavae and the left atrium receives blood from the pulmonary veins.

4. The next stage is **atrial systole**. When the atria contract, atrioventricular valves open and blood flows into the ventricles.

5. Next, the ventricles contract and atrioventricular valves close, producing the 'lub' sound of the heartbeat. This is called **ventricular systole**. The pressure in the ventricles increases, causing the semi-lunar valves in the pulmonary artery and aorta to open. Blood flows into the aorta and pulmonary artery. While the ventricles are contracting, the atria relax in **atrial diastole**.

6. Finally, the ventricles relax. This is called **ventricular diastole**. The semi-lunar valves shut because the ventricles are at a lower blood pressure than the aorta and pulmonary arteries. This causes the 'dub' sound of the heartbeat. The atrioventricular valves open due to the drop in ventricular pressure.

7.13 Coronary heart disease

1. Coronary heart disease occurs when the coronary arteries become blocked (occluded) or narrowed.

2. The heart muscles will no longer be able to receive sufficient oxygen and nutrients.

3. This can cause a **heart attack**. During a heart attack, blood supply to part of the heart muscle is completely cut off due to blockage in the coronary arteries. The affected part dies, which can affect the heart's ability to pump and lead to heart failure.

4. A cause of coronary heart disease is atherosclerosis, in which an artery wall thickens and hardens due to the deposition of plaque, which causes the lumen of the artery to become narrower.
5. The narrowing of the lumen of the arteries causes an increase in blood pressure. This causes arteries to develop rough linings, which increases the likelihood of formation of blood clots inside the arteries. This is known as thrombosis.
6. This obstructs blood flow in the afflicted artery. If it occurs in a coronary artery, a heart attack takes place.
7. Factors that contribute to atherosclerosis include:
   (a) High intake of cholesterol and saturated fats
   (b) Stress
   (c) Smoking
8. Preventive measures include:
   (a) Healthy diet – low in cholesterol and saturated fats
   (b) Not smoking – nicotine increases blood pressure
   (c) Exercising – lowers stress and strengthens the heart

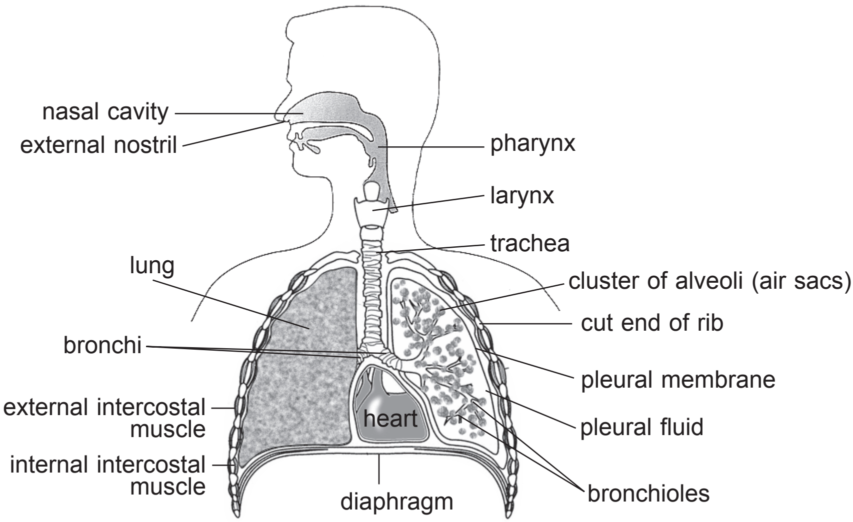
TOPIC 8 Respiration in Humans

Objectives

Candidates should be able to:

(a) identify on diagrams and name the larynx, trachea, bronchi, bronchioles, alveoli and associated capillaries
(b) state the characteristics of, and describe the role of, the exchange surface of the alveoli in gas exchange
(c) describe the removal of carbon dioxide from the lungs, including the role of the carbonic anhydrase enzyme
(d) describe the role of cilia, diaphragm, ribs and intercostal muscles in breathing
(e) describe the effect of tobacco smoke and its major toxic components – nicotine, tar and carbon monoxide, on health
(f) define and state the equation, in words and symbols, for aerobic respiration in humans
(g) define and state the equation, in words only, for anaerobic respiration in humans
(h) describe the effect of lactic acid in muscles during exercise

8.1 Overview of the human respiratory system



The human gas exchange system

1. Breathing is the transport of oxygen from the outside air to the cells, and carbon dioxide from the cells to the outside air. This is not the same as cellular respiration, which is the process by which an organism breaks down food molecules to release energy for life processes.

2. The human respiratory system consists of :

   (a) Nasal passages – Passages leading from the nostrils lined with a moist mucous membrane

   (b) Pharynx – Common passage for the opening of the oesophagus and the trachea

   (c) Larynx – Voice box containing vocal cords

   (d) Trachea – A tube supported by C-shaped cartilage connecting the larynx and the lungs. The C-shaped cartilage prevents the trachea from collapsing as the air pressure in the lungs changes. It branches into two bronchi, one to each lung.

   (e) Bronchi – Branches repeatedly within the lungs to produce numerous finer tubes called bronchioles. The bronchioles at the end of the branching terminate in clusters of air sacs called alveoli. The epithelial lining of the bronchi and trachea are covered with a thin film of mucus and cilia, which are hair-like structures that can move. The mucus traps dust, pollen and other particles and the cilia sweeps it upwards into the pharynx to be swallowed into the oesophagus.

   (f) Lungs – Located in the pleural cavity, they are enclosed by the pleura, a two-layered membrane structure. The inner layer is in contact with the lungs while the other layer adheres to the wall of the chest cavity. The space between the two membranes is known as the pleural space, and it contains a small amount of pleural fluid, which acts as a lubricant when the lungs expand and contract during breathing.

   (g) Related muscles, ribs and diaphragm.

8.2 The thoracic cavity

1. The lungs are protected by the ribs which extend from the backbone to the sternum (breast bone).

2. Two sets of muscles attached to the ribs are involved in breathing. These are the external and internal intercostal muscles. When one set contracts, the other set relaxes.

3. The diaphragm is a sheet of skeletal muscle that forms the bottom wall of the thoracic cavity. When the diaphragm muscles contract, the diaphragm moves downwards. When they relax, the diaphragm moves up again.

4. The intercostal muscles and the diaphragm work together to change the volume of the chest cavity (thoracic cavity).

8.3 Inhalation

1. During inhalation, the diaphragm contracts, flattens and moves downwards.
2. The external intercostal muscles contract while the internal intercostal muscles relax. The ribs move upwards and outwards.
3. The thoracic cavity increases in volume.
4. This causes the air pressure of the lungs to fall below that of the atmosphere.
5. Air rushes into the lungs.
6. During inhalation, air passes through the respiratory passage in the order: nasal cavity, pharynx, larynx, trachea, bronchi, bronchioles, alveoli.

8.4 Exhalation

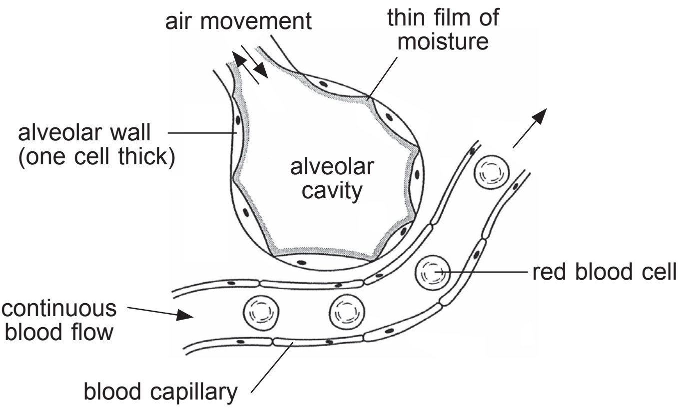
1. During exhalation, the diaphragm relaxes and arches upwards.
2. The internal intercostal muscles contract while the external intercostal muscles relax, moving the ribs downwards and inwards.
3. The thoracic cavity decreases in volume.
4. Air pressure in the lungs is now higher than that of the atmosphere.
5. Air flows out of the lungs until the air pressure in the lungs reaches equilibrium with atmospheric air pressure.

8.5 The alveoli

1. The alveoli are the sites of gas exchange in the lungs.
2. They are present in large quantities, providing a huge surface area for gas exchange.
3. The walls of the alveoli are one-cell thick, resulting in a small distance for diffusion.
4. They are covered with a thin film of water to allow oxygen to dissolve and subsequently diffuse in solution across the cell surface membranes.
5. They are well-supplied with blood capillaries which transport away diffused oxygen and supply carbon dioxide for excretion. The continuous removal of oxygen and the supply of carbon dioxide maintain the respective concentration gradients of these gases.

8.6 Mechanism of oxygen transfer in the alveoli

1. The exchange surface of the alveoli is the thin moist epithelium of the inner surfaces.
2. Capillaries branching from the pulmonary artery supply oxygen-poor blood to the alveoli.
3. Oxygen from the air in the alveoli taken in during inhalation dissolves in the moisture on the lining.
4. The dissolved oxygen diffuses down the concentration gradient across the alveolar wall and the endothelium of the blood capillaries into the oxygen-poor blood.
5. The oxygenated blood leaves the capillaries and enters the pulmonary veins to be carried back to the heart.



A section of an alveolus

8.7 Removal of carbon dioxide

1. 7% of carbon dioxide released during respiration is transported as dissolved carbon dioxide in blood plasma. 23% is transported bound to haemoglobin in red blood cells. 70% is transported as bicarbonate ions in the blood.
2. Mechanism of conversion of carbon dioxide into bicarbonate ions:
   (a) Carbon dioxide from respiring cells diffuses into blood plasma and then into red blood cells.
   (b) An enzyme, carbonic anhydrase, is present in red blood cells. It catalyses the interconversion of carbon dioxide with water to give carbonic acid, which dissociates into bicarbonate ions and hydrogen ions.

$$CO_2 + H_2O \rightleftharpoons H_2CO_3 \rightleftharpoons HCO_3^- + H^+$$

   (c) The hydrogen carbonate ions diffuse into plasma.

3. In the lungs:

   (a) Hydrogen carbonate ions diffuse back into red blood cells where they combine with hydrogen ions released from haemoglobin to form carbonic acid.

   (b) Carbonic acid forms water and carbon dioxide.

   (c) The carbon dioxide diffuses out of the blood into the alveolar space where it is expelled during exhalation.

8.8 Effects of tobacco smoke on health

1. Harmful components of tobacco smoke are:

   (a) Nicotine
      (i) Addictive stimulant that stimulates adrenaline release
      (ii) Increases heart rate and blood pressure
      (iii) Increases risk of stroke, heart attack and impotence

   (b) Carbon monoxide
      (i) Poisonous gas that combines irreversibly with haemoglobin to form carboxyhaemoglobin
      (ii) Reduces efficiency of blood to transport oxygen
      (iii) Increases risk of atherosclerosis
      (iv) Increases risk of thrombosis

   (c) Tar
      (i) Carcinogenic
      (ii) Paralyses cilia lining air passages, reducing effectiveness of dust and irritant removal

   (d) Irritants
      (i) Paralyse cilia lining air passages
      (ii) Increase risk of chronic bronchitis and emphysema

8.9 Chronic bronchitis

1. Chronic bronchitis is caused by irritation to the respiratory lining of the airways, resulting in inflammation.

2. There is increased production of mucus by the epithelium. Cilia on the epithelium become paralysed, unable to remove mucus and foreign particles.

3. Airflow becomes blocked due to swelling and mucus.

4. Symptoms are wheezing, shortness of breath and a persistent cough.

8.10 Emphysema

1. Emphysema is caused by exposure to toxic chemicals, e.g. tobacco smoke.
2. It is a lung disease characterised by the permanent enlargement of air spaces due to a destruction of alveolar walls. This decreases the gas exchange surface area.
3. The lungs lose their elasticity and lose their ability to effectively expel air.
4. Oxygen uptake and carbon dioxide removal is impaired and severe breathlessness is experienced.

8.11 Cellular respiration

1. Cellular respiration is a process by which cells break down food molecules to release energy stored in food.
2. This energy is used to sustain vital life processes.
3. There are two modes of respiration, **aerobic respiration** and **anaerobic respiration**.

8.12 Aerobic respiration

1. Aerobic respiration is the oxidation of glucose molecules in the **presence of oxygen** to release a large amount of energy, with carbon dioxide and water as waste products.
2. The overall equation is:

   $$C_6H_{12}O_6 + 6O_2 \longrightarrow 6CO_2 + 6H_2O + \text{energy}$$

   glucose + oxygen ⟶ carbon dioxide + water + energy
3. Respiration is carried out in a complicated series of reactions involving enzymes.
4. It occurs within the mitochondria of cells.
5. Energy released from respiration is used for:
   - (a) Synthesising complex molecules from simpler molecules i.e. proteins from amino acids, hormones, enzymes
   - (b) Cell growth and division: synthesis of new protoplasm and genetic material
   - (c) Muscular contraction, both voluntary (involving skeletal muscles) and involuntary (cardiac muscle and smooth muscle i.e. heartbeat and peristalsis)
   - (d) Active transport
   - (e) Transmission of nerve impulses
6. Some energy is also released as heat during respiration.

8.13 Anaerobic respiration

1. Anaerobic respiration is the breakdown of glucose molecules in the **absence of oxygen**. Waste products vary from organism to organism. Less energy is released compared to aerobic respiration.
2. Anaerobic respiration in humans primarily occurs in the muscle cells.
3. The preferred mode of respiration in muscle cells is aerobic. However, during periods of strenuous exercise, since there is a limit to the rate of breathing and heart rate, not enough oxygen is available to the muscle cells to sustain aerobic respiration.
4. In such cases, muscle cells respire anaerobically for short durations in order to meet the energy demands of the activity.
5. The equation for anaerobic respiration in humans is:

$$C_6H_{12}O_6 \longrightarrow 2C_3H_6O_3 + \text{energy}$$

$$\text{glucose} \longrightarrow \text{lactic acid} + \text{energy}$$

6. The energy produced by anaerobic respiration supplements the energy produced by aerobic respiration.
7. When anaerobic respiration occurs, there is a build up of **lactic acid** in the muscle cells.
8. This causes **fatigue**. Anaerobic respiration in humans can only be sustained for a short time before the body needs to recover.
9. During the recovery process, more oxygen needs to be taken in. This is evidenced by heavy panting after strenuous exercise.
10. The oxygen taken in is used to restore the body to its resting state. This is done by transporting the lactic acid from the muscles to the liver, where some lactic acid is completely oxidised to carbon dioxide and water to produce energy to convert the remaining lactic acid into glucose.
11. The amount of oxygen required for this process is called the **oxygen debt**.

TOPIC 9 Excretion in Humans

Objectives

Candidates should be able to:
(a) define *excretion* and explain the importance of removing nitrogenous and other compounds from the body
(b) outline the function of the nephron with reference to ultra-filtration and selective reabsorption in the production of urine
(c) outline the role of anti-diuretic hormone (ADH) in osmoregulation
(d) outline the mechanism of dialysis in the case of kidney failure

9.1 Excretion

1. Excretion is the process by which the body removes metabolic waste products and toxic materials.
2. Metabolic processes consist of anabolic processes and catabolic processes.
3. Anabolic processes are 'building-up' processes where larger molecules are synthesised from smaller molecules. Examples include:
   (a) Synthesis of proteins from amino acids
   (b) Synthesis of glycogen from glucose
   (c) Photosynthesis with oxygen as waste material
4. Catabolic processes are 'breaking-down' processes where larger molecules are broken down to form smaller molecules. Examples include:
   (a) Cellular respiration with carbon dioxide and water as by-products
   (b) Deamination of amino acids in the liver with urea as a by-product
   (c) Breakdown of haemoglobin in the liver with bile pigments as by-products
5. Waste products have to be removed because they can be harmful if they accumulate in the body.

6. The waste products of metabolism are excreted by the following organs:

Excretory organs	Excretory products	Excreted as
Lungs	Carbon dioxide	Exhaled air
Kidneys	Excess mineral salts, urea, uric acid, creatinine, excess water	Urine
Skin	Excess mineral salts, small quantities of urea, excess water	Sweat
Liver	Bile pigments	Secreted as bile, leaves the body in faeces

9.2 Overview of the human urinary system

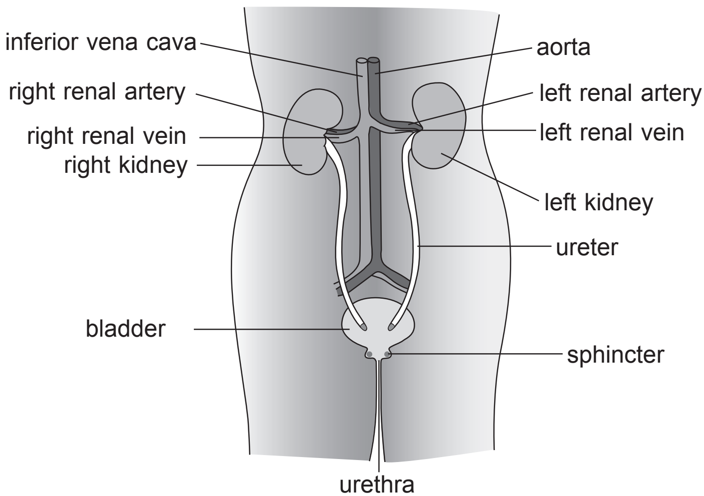
1. The human urinary system consists of :

   (a) The **kidneys**, which are two bean-shaped organs located in the abdominal cavity.

   (b) The **ureters**, which are narrow tubes that emerge from a depression in the concave surface of the kidney called a **hilum**. The ureters connect to the urinary bladder.

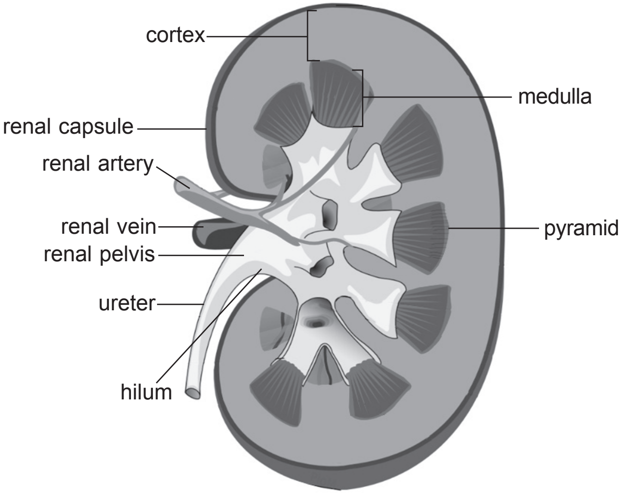
   (c) The **urinary bladder** is an elastic and muscular organ that collects and stores urine excreted by the kidneys. The sphincter muscle at the base of the bladder controls the flow of urine into the urethra. It is controlled by nervous impulses from the brain.

   (d) The **urethra** is a duct that connects the urinary bladder to the outside of the body. Urine passes through this tube to the outside.



The human urinary system

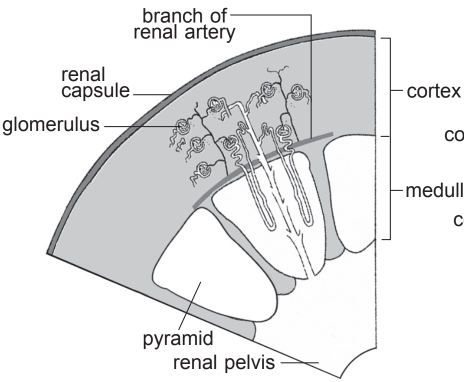
9.3 Structure of a kidney



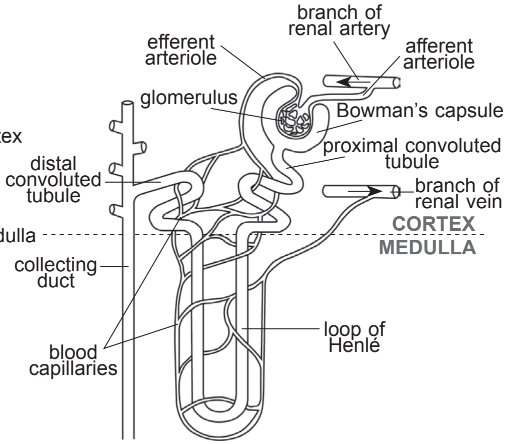
A kidney

1. The kidney is made up of two distinct regions: an outer **cortex** and the inner **medulla**.
2. The cortex is covered by a protective fibrous capsule called the **renal capsule**.
3. The medulla consists of 8 to 18 conical **pyramids**.
4. Across the cortex and medulla are numerous excretory tubules called **nephrons**, as well as **collecting ducts** and their associated blood vessels.
5. Nephrons are the urine-producing units of the kidney.
6. The tips of the pyramids empty urine into an area called the **renal pelvis**. The renal pelvis functions as a funnel collecting urine from all the pyramids to deliver to the ureter.
7. Blood enters each kidney from the **renal artery** and leaves via the **renal vein**, both connected to the kidney at the hilum.

9.4 Structure of a nephron K M C



A section of a kidney



A nephron

1. Components of the nephron are:

   (a) **Glomerulus** – A ball of capillaries that obtains its blood supply from an **afferent arteriole** which branches off the renal artery. It drains into an **efferent arteriole**. The high pressure of the blood in the glomerulus forces water, urea, salts and small solutes through the partially permeable endothelium into the lumen of the **Bowman's capsule** in a process known as **ultrafiltration**.

   (b) Bowman's capsule – The start of the tubular component of a nephron. It surrounds the glomerulus in a cup-like structure. Together, the Bowman's capsule and the glomerulus make up a **renal corpuscle** (Malpighian corpuscle).

   (c) **Proximal convoluted tubule** – A convoluted tubule leading from the Bowman's capsule which straightens up as it passes into the medulla, leading into the loop of Henlé.

   (d) **Loop of Henlé** – Consists of a descending limb, a hairpin turn and an ascending limb. It re-enters the cortex.

   (e) **Distal convoluted tubule** – Convoluted portion of nephron leading from the loop of Henlé, connecting it to the collecting duct.

2. The collecting duct is a tubule into which distal convoluted tubules from several nephrons empty their filtrate. It extends deep into the medulla, opening up into the renal pelvis. It is not considered part of the nephron.

9.5 Urine formation

1. Excess mineral salts, nitrogenous wastes and excess water are excreted through the kidneys through ultrafiltration and selective reabsorption of useful materials.
2. **Ultrafiltration** occurs in the glomerulus. Blood enters the glomerulus through an afferent arteriole from the renal artery. Blood pressure forces water, urea, salts and other small solutes (e.g. glucose, amino acids and vitamins) into the lumen of the Bowman's capsule. Blood cells and large molecules remain in the capillaries.
3. The high blood pressure (high hydrostatic pressure) driving the ultrafiltration in the glomerulus is due to the afferent arteriole having a larger diameter than the efferent arteriole.
4. The endothelium of the glomerular capillaries and the basement membrane of the Bowman's capsule that wraps around the capillaries are partially permeable membranes, thus only small soluble substances are able to pass through.
5. The glomerular filtrate passes from the lumen of the Bowman's capsule into the proximal convoluted tubule.
6. Within this tubule, most of the mineral salts and all of the glucose and amino acids are absorbed through active transport or diffusion. Water is reabsorbed by osmosis.
7. Reabsorption of water continues in the loop of Henlé.
8. Water and salts are reabsorbed in the distal convoluted tubule.
9. Water is reabsorbed from the collecting duct.
10. Excess salts, nitrogenous waste products, excess water and processed drugs and poisons from the liver enter the renal pelvis as urine.

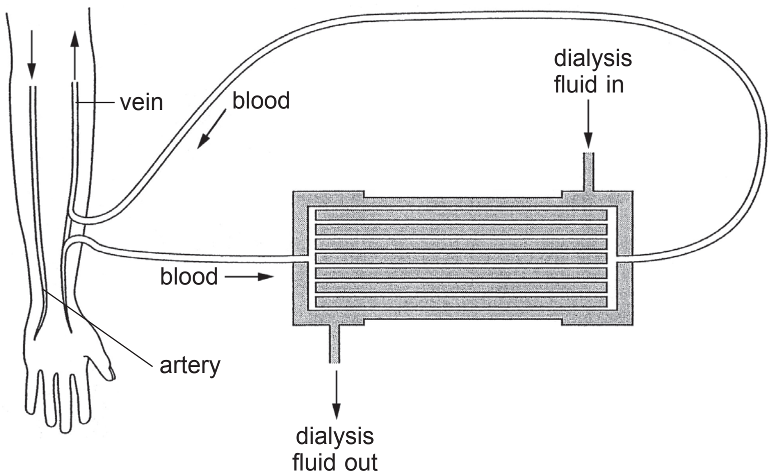
9.6 Kidneys as osmoregulators

1. Osmoregulation is the control of water and mineral salts in the blood.
2. The water potential of blood has to be maintained for proper functioning of the body.
3. Excessive gain in water due to drinking or excessive loss due to diarrhoea or sweating will result in a change in the water potential of blood.
4. Excess water could also cause water to move into cells from tissue fluid by osmosis. This causes the cells to swell and burst.
5. Too little water would cause water to move out of the cells into tissue fluid causing dehydration.
6. Excess water could also lead to an increase in blood pressure due to an increase in volume. This could lead to stroke.

7. The amount of water in blood is controlled by a hormone called **antidiuretic hormone** (ADH).

8. ADH is produced in the hypothalamus of the brain and is stored and released from the pituitary gland.
9. The hypothalamus contains osmoreceptor cells that can monitor the water potential of blood.
10. When blood water potential decreases beyond a certain amount, the pituitary gland is stimulated to secrete more ADH into the blood.
11. ADH works on the distal convoluted tubules and the collecting ducts in the kidneys.
12. It makes the epithelium more permeable to water.
13. This causes more water to be reabsorbed, producing a smaller volume of more concentrated urine.
14. The water potential of blood then returns to regular levels.
15. When the water potential of blood increases beyond normal levels, the osmoreceptor cells in the hypothalamus stimulate the pituitary gland to release less ADH.
16. The epithelium of the kidney tubules and collecting ducts become less permeable to water.
17. Less water is reabsorbed resulting in a larger volume of dilute urine.
18. The water potential of blood returns to normal levels.

9.7 Dialysis



1. The kidneys function to remove waste products, excess water and excess mineral salts.
2. A dialysis machine would have to perform the functions of a kidney.
3. In dialysis, blood is passed over a dialysis membrane of a large surface area which is permeable to small molecules but does not allow proteins to pass through.
4. On the other side of the dialysis membrane is the dialysis fluid, which contains the same concentration of essential substances as the blood plasma, with the exception of metabolic wastes.
5. Substances move from the blood to the dialysis fluid and vice versa through diffusion down a concentration gradient.
6. As blood flows through the tubules immersed in dialysis fluid, metabolic waste diffuses out of the tubing into the fluid.
7. Fresh dialysis fluid is continually supplied during dialysis in order to maintain a low concentration of urea in the fluid as compared to that in blood plasma.
8. The direction of blood flow is opposite to the direction of flow of the dialysis fluid in order to increase the length of exchange surface with the necessary concentration gradients. This is known as countercurrent flow.

TOPIC 10 Homeostasis

Objectives

Candidates should be able to:

(a) define *homeostasis* as the maintenance of a constant internal environment

(b) explain the basic principles of homeostasis in terms of stimulus resulting from a change in the internal environment, a corrective mechanism and negative feedback

(c) identify on a diagram of the skin: hairs, sweat glands, temperature receptors, blood vessels and fatty tissue

(d) describe the maintenance of a constant body temperature in humans in terms of insulation and the role of: temperature receptors in the skin, sweating, shivering, blood vessels near the skin surface and the co-ordinating role of the hypothalamus

10.1 Homeostasis

1. Homeostasis is the maintenance of a constant internal environment. It allows an organism to survive in a changing environment.
2. It involves:
   (a) Thermoregulation – the maintenance of a constant body temperature
   (b) Osmoregulation – the maintenance of a constant water potential and pH
3. Thermoregulation is the maintenance of body temperature within a range that will allow cells to function effectively.
4. Many body processes, including metabolism, involve enzymes, which have an optimal temperature range.
5. Large body temperature changes could affect the rate of cellular respiration or alter membrane properties.
6. Osmoregulation is important because changes in the water potential could affect the direction of osmosis in body cells and the electrolyte balance across cell membranes.
7. Homeostasis involves a process called **negative feedback**. Negative feedback is a corrective mechanism in which the body's response is to restore the normal conditions of the internal environment.

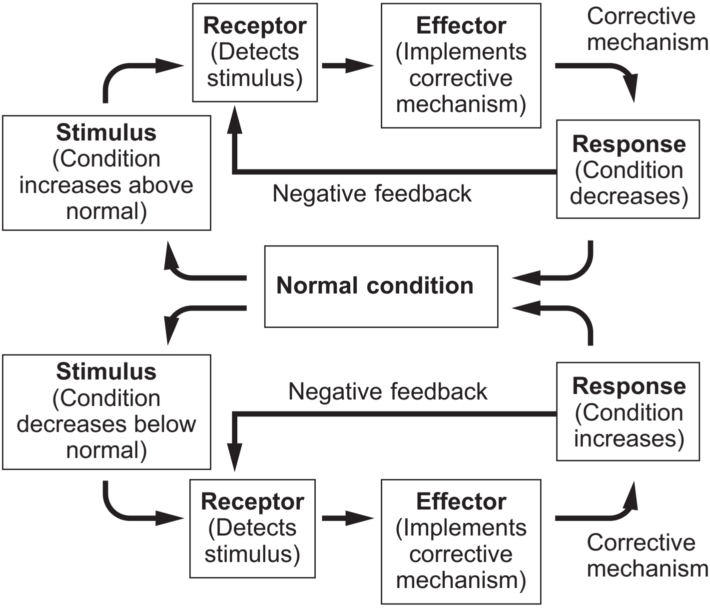
8. Terms involved in negative feedback control mechanism:

(a) Stimulus – A change in internal environment

(b) Receptor – Sense organs that detect the stimulus

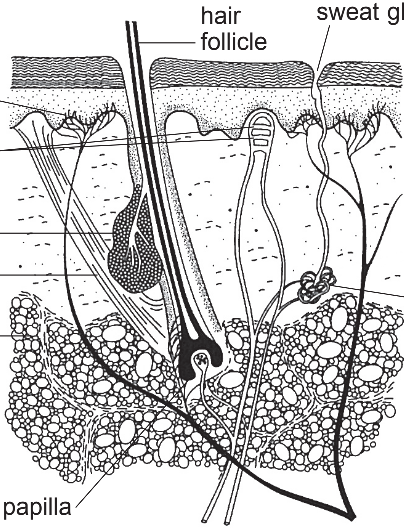
(c) Effector – Effect corrective responses

(d) Response – Condition returns to normal, gives negative feedback to receptor



Negative feedback pathway

10.2 Structure of the skin



Structure of the skin

1. The skin comprises two layers: the epidermis and the dermis.
2. The epidermis is the outermost layer which forms a waterproof and protective covering.
3. The dermis is the layer containing **hair follicles**, **sweat glands**, **sebaceous glands**, **blood vessels**, **mechanoreceptors** and **thermoreceptors**.
4. The arterioles leading to the capillaries in the dermis are controlled by nerves. They respond to stimulation by undergoing **vasoconstriction** and **vasodilation**.
5. Vasoconstriction is the contraction of smooth muscles in the arteriole walls. It decreases the diameter of the blood vessels, reducing blood flow. The skin looks pale when vasoconstriction takes place.
6. Vasodilation is the relaxation of smooth muscles in the arteriole walls. It increases the diameter of the blood vessels, increasing blood flow. The skin becomes flushed when vasodilation takes place.
7. Hairs grow within the hair follicles. Attached to the hair follicles are sebaceous glands which produce sebum, and hair erector muscles, which raise hair.
8. Sweat glands are coiled tubes that secrete sweat through a sweat duct. Secreted sweat contains water, sodium chloride and small amounts of metabolic waste products.
9. Sweat glands are used for body temperature regulation.

10. Nerve endings of sensory neurones enable pressure, pain or temperature changes to be detected.

11. Beneath the skin is a layer which consists of connective tissue and adipose tissue. Adipose cells store fat. This layer serves as insulation and padding.

10.3 Thermoregulation

1. Heat is produced by metabolic activities within the body. Most heat is produced by the liver, the brain, the heart and the contraction of skeletal muscles.
2. Heat can be removed from the body by conduction, convection and radiation if the environmental temperature is lower than the body temperature. Otherwise, heat would be gained.
3. Heat can be removed through evaporation of sweat.
4. The skin participates in thermoregulation through vasoconstriction, vasodilation and sweating.
5. The **hypothalamus** in the brain regulates body temperature by receiving information about temperature changes from thermoreceptors located in the skin and within the hypothalamus itself, and activating mechanisms that promote heat gain or loss.

10.4 Coping with heat gain

1. When the external temperature rises above normal levels, thermoreceptors within the skin send signals to the hypothalamus in the brain. Any corresponding rise in blood temperature is also detected by thermoreceptors located within the hypothalamus itself. The hypothalamus is stimulated to send out nerve impulses to:
   (a) Arterioles in the skin, stimulating vasodilation. Increased blood flow in superficial capillaries causes more heat loss through conduction, convection and radiation.
   (b) Sweat glands, stimulating sweat production. Heat is lost through evaporation of sweat from the skin.
   (c) Hair erector muscles, which relax so that hair follicles lie flat. This ensures that no air is trapped by the hairs as air is a good insulator. This is more evident in animals.
   (d) Lungs, stimulating rapid breathing or panting. Heat is lost through exhaled air. This is also more evident in animals.
2. Body temperature returns to normal.

10.5 Coping with heat loss

1. When the external temperature falls below normal levels, thermoreceptors in the skin send signals to the hypothalamus. A decrease in blood temperature is also detected by thermoreceptors in the hypothalamus. The hypothalamus is stimulated to send out nerve impulses to:

    (a) Arterioles in the skin, stimulating vasoconstriction. Decreased blood flow in superficial capillaries causes less heat loss through conduction, convection and radiation.

    (b) Sweat glands, stopping sweat production

    (c) Hair erector muscles, which constrict so that hair follicles are raised. This traps a layer of air between the hairs which acts as an insulating layer.

    (d) Muscles, causing involuntary and increased contraction of muscles, known as shivering. This increases cellular respiration in muscle cells, producing heat.

2. Body temperature returns to normal.

3. In humans, the always-present layer of adipose tissue beneath the skin acts as insulation.

TOPIC 11 Co-ordination and Response in Humans

Objectives

Candidates should be able to:

(a) state the relationship between receptors, the central nervous system and the effectors
(b) describe the structure of the eye as seen in front view and in horizontal section
(c) state the principal functions of component parts of the eye in producing a focused image of near and distant objects on the retina
(d) describe the pupil reflex in response to bright and dim light
(e) state that the nervous system – brain, spinal cord and nerves, serves to co-ordinate and regulate bodily functions
(f) outline the functions of sensory neurones, relay neurones and motor neurones
(g) discuss the function of the brain and spinal cord in producing a co-ordinated response as a result of a specific stimulus in a reflex action
(h) define a *hormone* as a chemical substance, produced by a gland, carried by the blood, which alters the activity of one or more specific target organs and is then destroyed by the liver
(i) explain what is meant by an endocrine gland, with reference to the islets of Langerhans in the pancreas
(j) state the role of the hormone adrenaline in boosting blood glucose levels and give examples of situations in which this may occur
(k) explain how the blood glucose concentration is regulated by insulin and glucagon as a homeostatic mechanism
(l) describe the signs, such as an increased blood glucose level and glucose in urine, and the treatment of *diabetes mellitus* using insulin

11.1 The human nervous system

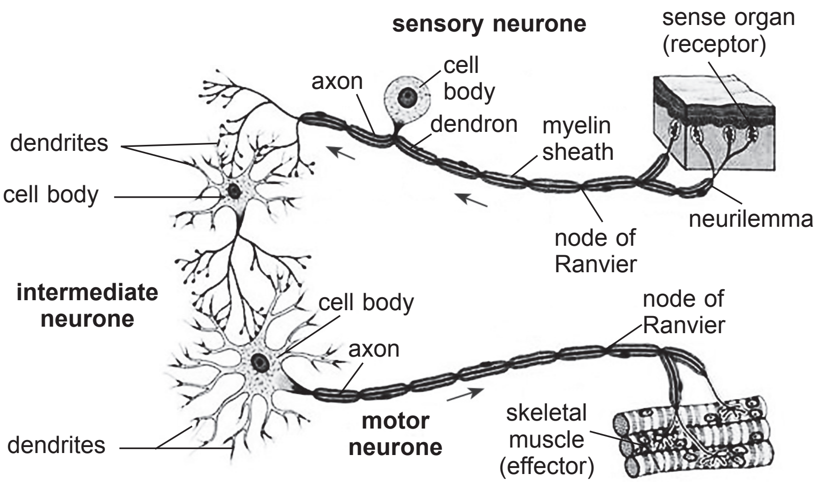
1. The human nervous system consists of:
   (a) Central nervous system (CNS) consisting of the brain and spinal cord
   (b) Peripheral nervous system (PNS) consisting of nerves connecting the central nervous system and the rest of the body. The function of the PNS is to conduct sensory and motor signals between the CNS and the limbs and organs (receptors and effectors).
2. A stimulus is a change in the environment that causes an organism to react. Stimuli are detected by sensory receptors.
3. A response is a change in the body as a result of the stimulus. Effector cells are muscle cells or gland cells, which carry out the response to stimuli.
4. Bodily functions are classified into voluntary actions and involuntary actions.

5. Involuntary actions are actions that cannot be consciously controlled, such as heartbeat, peristalsis, vasoconstriction and reflex actions.
6. Voluntary actions are actions that are consciously controlled.

11.2 Nervous tissue

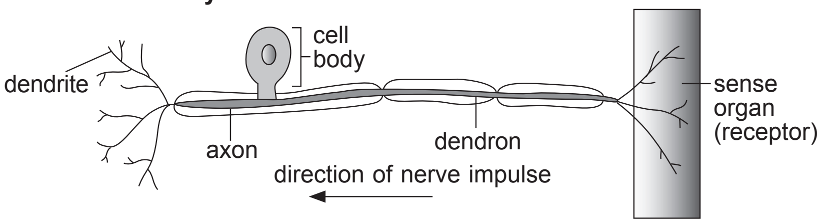
1. Nerve impulses are transmitted by nerves, which are bundles of neurones wrapped in connective tissue.
2. A neurone is a nerve cell.
3. There are three main types of neurones:
   (a) **Sensory neurones** – Respond to stimuli affecting cells of the sensory organ they are found in and relay signals to the CNS
   (b) **Intermediate neurones** (relay neurones) – Transmit nerve impulses from the sensory neurones to the motor neurones; found within the CNS
   (c) **Motor neurones** – Transmit nerve impulses from the CNS to the effector muscle cells or gland cells
4. Neurones share common characteristics:
   (a) A relatively large cell body containing the nucleus and organelles.
   (b) Slender nerve fibres that increase the distance over which nerve impulses can be transmitted. There are two types of nerve fibres.
      (i) **Axons** are long, slender projections that conduct nerve impulses away from the cell body of the neurone.
      (ii) **Dendrons** are branched projections that conduct nerve impulses towards the cell body.
      (iii) At the terminal ends of axons and dendrons, the nerve fibre branches. These branches are known as dendrites. Where the axon is connected to muscles, these branches are also known as motor end plates.

5. The relationship between sensory neurones, the CNS and motor neurones is shown below:



A neural pathway

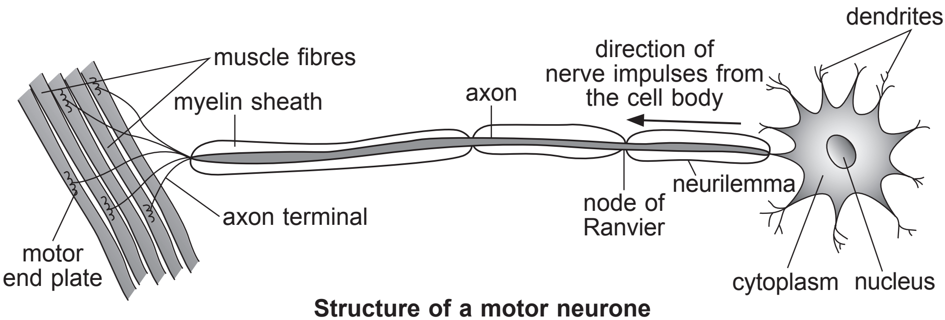
11.3 Structure of a sensory neurone



Structure of a sensory neurone

1. The sensory neurone has a smooth and rounded cell body, a single long dendrite and a short axon. The dendron is structurally similar to an axon and is myelinated.

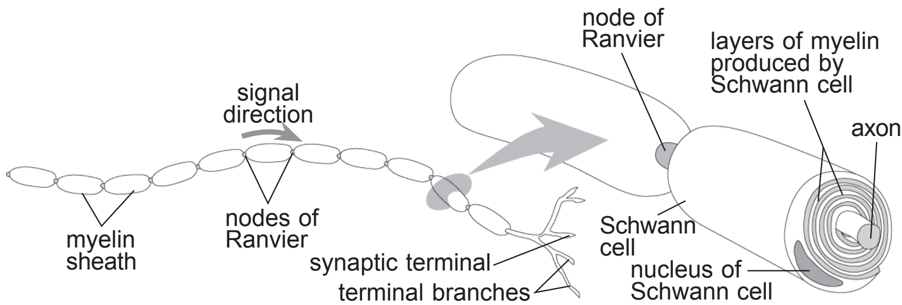
11.4 Structure of a motor neurone



Structure of a motor neurone

1. The motor neurone consists of a cell body and a long thin axon covered by a myelin sheath.
2. Around the cell body are branching dendrites that receive nerve impulses from other neurones and conduct them towards the cell body.
3. The axon conducts signals away from the cell body towards the effector cells.

11.5 Structure of an axon



Structure of an axon

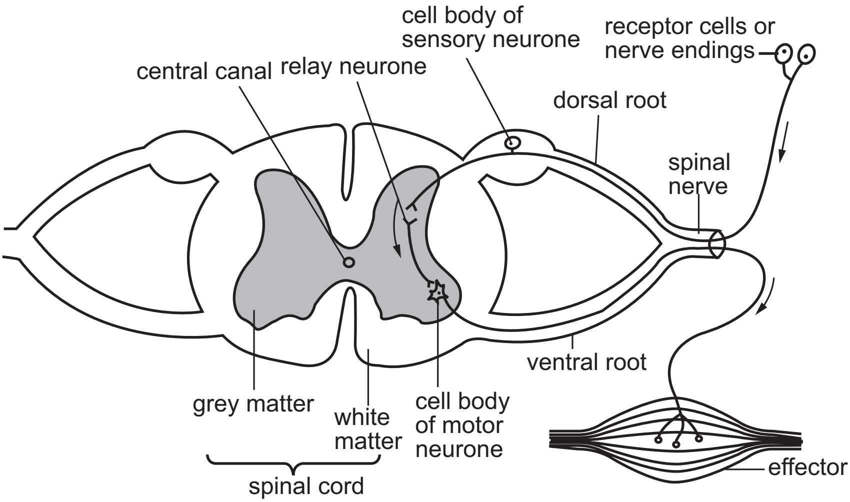
1. In the PNS, supporting cells called Schwann cells form an electrically-insulating layer around axons called the **myelin sheath**; 80% of the myelin sheath consists of lipids.
2. The gaps between adjacent Schwann cells are called **nodes of Ranvier**.
3. The myelin sheath increases the speed at which nerve impulses travel along the axon by allowing nerve impulses to jump from node to node.

11.6 Synapses

1. A synapse is a junction between two neurones or between a neurone and an effector.
2. At a synapse, impulses from the axon of one neurone are transmitted to the dendrites of another neurone or to effector cells.
3. Nerve impulses are transmitted across the tiny space of a synapse by chemicals called **neurotransmitters**.

11.7 Reflex actions

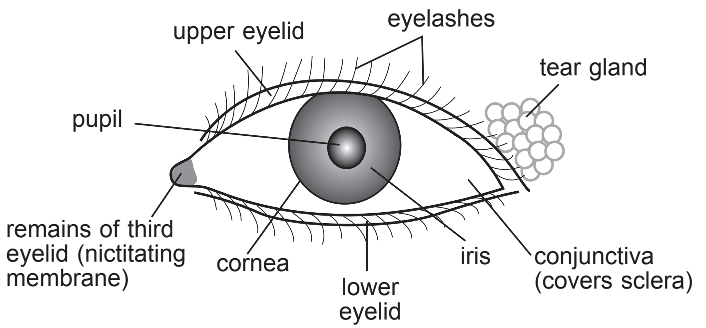
1. Reflex actions are involuntary responses to a specific stimulus. They cannot be consciously controlled.
2. The pathway by which nerve impulses travel during reflex actions is called a **reflex arc**.
3. It consists of:
   (a) Receptor
   (b) Sensory neurone
   (c) Intermediate neurone / relay neurone (located in CNS)
   (d) Motor neurone
   (e) Effector
4. The diagram below shows the reflex arc, the pathway of nervous impulses controlling a reflex response:



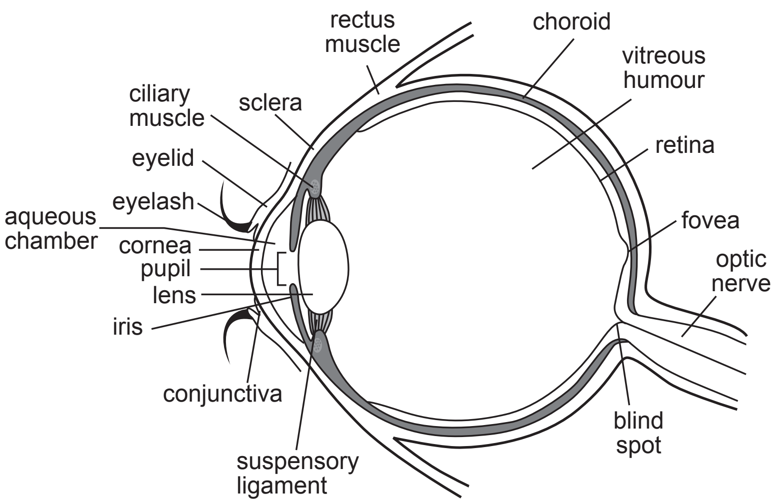
The reflex arc

5. Receptors in the skin detect the stimulus.
6. Nerve impulses are produced which are transmitted by the sensory neurone to the spinal cord.
7. In the spinal cord, the nerve impulses are transmitted across a synapse to an intermediate neurone and then across another synapse to the motor neurone. Nerve impulses are also transmitted to the brain.
8. Nerve impulses travel along the motor neurone to the motor end plate.
9. The nerve impulses stimulate the motor end plate and cause the muscle to contract.

11.8 Structure of the eye



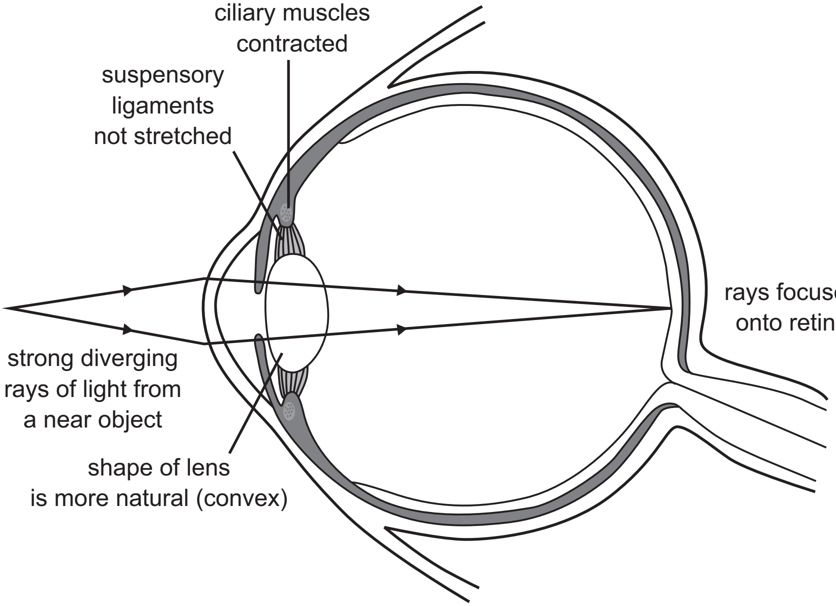
Structures at the front part of the eye



Vertical section of the eye

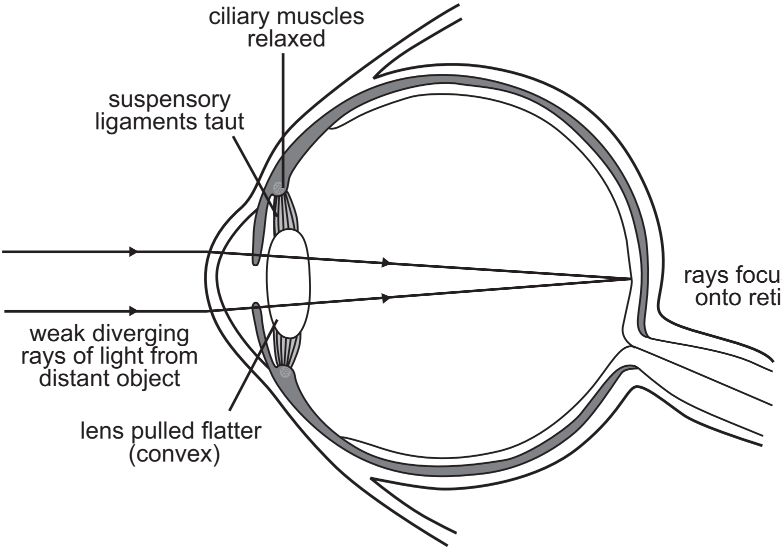
1. **Iris** – Pigmented circular sheet of muscles that control the contraction and dilation of the iris through the contraction and relaxation of the circular muscles and radial muscles
2. **Pupil** – A hole in the middle of the iris which allows light to enter the eye
3. **Sclera** – Tough white outer layer of connective tissue
4. **Conjunctiva** – Thin, transparent mucous membrane that helps to lubricate the eye
5. **Cornea** – Transparent refractive layer covering the iris and pupil. It causes the most of the refraction of light entering the eye. The cornea is continuous with the sclera.
6. **Tear gland** – Gland lying at the upper corner of the eyelid. Secretes tears which lubricate the eye, nourish the cornea and keeps it free from dust.
7. **Choroid** – Black middle layer of the eyeball, between the sclera and retina. Contains blood vessels that supply oxygen and nutrients, and remove metabolic waste products. It is pigmented black to prevent an internal reflection of light.
8. **Retina** – Innermost layer of the eyeball which contains photoreceptors. Photoreceptors are connected to nerve endings from the optic nerve.
9. **Lens** – Transparent biconvex structure that refracts light onto the retina. The lens is flexible and its curvature can be changed. It is responsible for the process of accomodation, a reflex action where the lens is able to change its curvature to focus sharp images on the retina.
10. **Ciliary body** – Contains ciliary muscles which control the curvature of the lens. It is also responsible for producing aqueous humour.
11. **Suspensory ligament** – Connects the ciliary body to the lens
12. **Aqueous humour** – A transparent, water substance filling the space between the cornea and the lens. It keeps the front of the eye firm and helps refract light into the eye.
13. **Vitreous humour** – Clear gel filling the space between the lens and the retina. It keeps the eyeball firm and helps refract light onto the retina.
14. **Fovea** – Yellow pit in the retina where images are usually focused
15. **Optic nerve** – Transmits visual information from the retina to the brain. There are no photoreceptors in the area of the retina where the optic nerve leaves the eye. This area is called the blind spot.

11.9 Focusing on a near object



1. Light rays from a near object enter the eye as diverging rays to fall on the retina.
2. The retina sends impulses to the brain, which sends impulses to the ciliary muscles.
3. The ciliary muscles contract, causing the suspensory ligaments to become slack.
4. The suspensory ligaments relax their pull on the lens. The elastic lens becomes thicker and rounder, causing more refraction of the rays of light, enabling a sharp image to be focused on the retina.

11.10 Focusing on a distant object



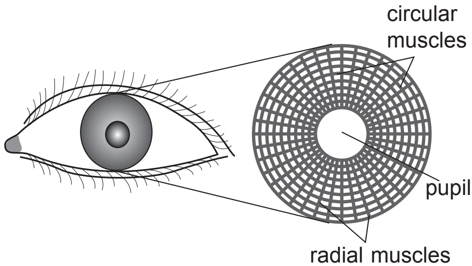
1. Light rays from a distant object enter the eye as almost parallel rays to fall on the retina.
2. The retina sends impulses to the brain, which sends impulses to the ciliary muscles.
3. The ciliary muscles relax, causing the suspensory ligaments to become taut.
4. The suspensory ligaments pull on the lens more. The elastic lens becomes thinner and less curved, causing less refraction of the rays of light, enabling a sharp image to be focused on the retina.

11.11 The pupil reflex

1. The **pupil reflex** is an involuntary action where the pupils contract or dilate in response to changing light intensities.
2. The pupils dilate to allow more light to enter the eye for better vision when light intensity is low, and contract to restrict light entry when light intensity is high as excessive light can damage the retina.
3. The size of the pupil is controlled by two sets of involuntary muscles in the iris called the **circular muscles** and the **radial muscles**.
4. The reflex arc involves these components:
   (a) Light entering the eye falls on the retina.
   (b) Retina sends impulse via optic nerve to the brain. The brain is the organ of the CNS that is nearest to the eye.

(c) The brain sends impulse to the iris muscles.

(d) The circular and radial muscles respond to change the size of the pupil, to adjust to the light conditions.



Structure of the iris

5. When light intensity is high:
   (a) Circular muscles in the iris contract.
   (b) Radial muscles in the iris relax.
   (c) The pupil constricts.

6. When light intensity is low:
   (a) Circular muscles in the iris relax.
   (b) Radial muscles in the iris contract.
   (c) The pupil dilates.

11.13 Hormones

1. A hormone is a chemical substance produced by a gland and carried by the blood, which alters the activity of one or more specific target organs.
2. Hormones are active in minute quantities and are destroyed by the liver and excreted by the kidneys.
3. They affect cellular metabolism and coordinate the growth, development and activity of an organism.
4. Glands are classified into two groups: exocrine glands and endocrine glands.
5. Exocrine glands are glands that secrete their products via ducts. Examples include sweat glands and salivary glands.
6. Endocrine glands are glands that secrete their products directly into the bloodstream. Examples include the pituitary gland, thyroid gland, adrenal gland and the gonads.
7. Some glands are both exocrine and endocrine. An example would be the pancreas, which secretes pancreatic juice via the pancreatic duct, and insulin and glucagon from the islets of Langerhans into the bloodstream.

11.14 The pancreas as an endocrine gland

1. The islets of Langerhans in the pancreas are areas in the pancreas that contain groups of endocrine cells.
2. These cells produce the hormones **insulin** and **glucagon**.
3. Insulin and glucagon are antagonistic hormones that participate in homeostatic control of blood glucose level by negative feedback mechanism.
4. When blood glucose level exceeds the normal level, more insulin is released and acts to lower the glucose level.
5. When blood glucose level falls below the normal level, more glucagon is released and acts to increase the glucose level.
6. Insulin decreases blood glucose concentration by:
   (a) Stimulating body cells to increase glucose uptake by increasing permeability of plasma membranes to glucose
   (b) Stimulating the liver and muscle cells to store glucose in the form of glycogen
   (c) Decreasing production of glucose from glycogen breakdown in the liver
   (d) Decreasing the conversion of fatty acids and amino acids to glucose in the liver
7. Glucagon increases blood glucose concentration by stimulating liver cells to:
   (a) Convert glycogen to glucose
   (b) Convert amino acids and fatty acids to glucose
   (c) Convert lactic acid into glucose

11.15 Diabetes mellitus

1. Diabetes mellitus is a condition in which the body does not produce sufficient insulin or does not respond to insulin.
2. The excess glucose cannot be completely reabsorbed by the kidneys and are excreted in the urine.
3. Symptoms include:
   (a) A persistent high blood glucose concentration
   (b) Presence of glucose in the urine
   (c) Excessive urination, excessive thirst and weight loss

4. Diabetes can cause:
   (a) Poor immune response – increased susceptibility to infections
   (b) Damaged blood vessels leading to vision loss and a decreased sensation in the limbs
   (c) Kidney failure and heart failure
5. Diabetic individuals can control their disease by receiving regular injections and controlling their carbohydrate intake.

11.16 Adrenaline

1. Adrenaline is a hormone produced by the adrenal glands located above the kidneys. It is responsible for the '**fight-or-flight response**' triggered by stress (emotional or physical threats to the organism).
2. In response to stress, the adrenal medulla secretes adrenaline into the blood.
3. The adrenaline travels to target organs, causing:
   (a) Increased conversion of glycogen to glucose in the liver and skeletal muscles
   (b) Increased glucose release into blood by liver cells
   (c) Increased metabolic rate, causing more energy to be released in cellular respiration
   (d) Increased heart rate and volume of blood pumped per unit time, increasing oxygen and glucose supply to muscle cells
   (e) Dilated bronchioles and increased breathing rate and depth, allowing more oxygen to be taken in for cellular respiration
   (f) Decreased blood supply to the digestive system, the kidneys and the skin as vasoconstriction occurs in several body parts, diverting blood supply to the heart, brain and skeletal muscles
   (g) Vasodilation occurring in other body parts, increasing blood supply to these organs
   (h) Dilated pupils, enhancing vision
   (i) Contracted hair erector muscles, producing 'goose pimples'

TOPIC 12 Reproduction

Objectives

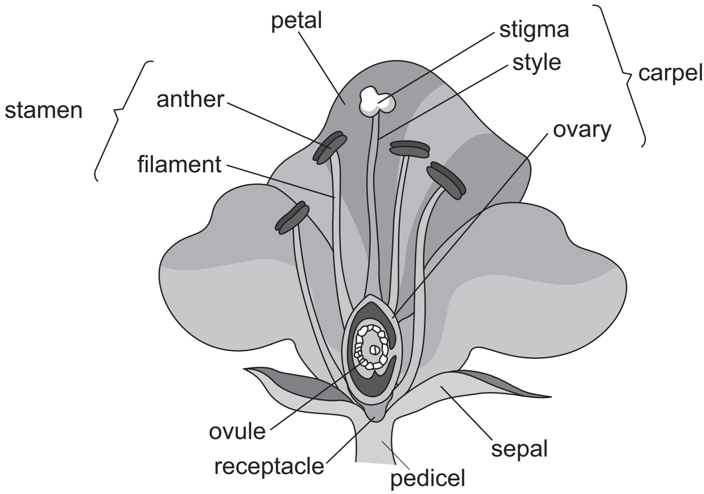
Candidates should be able to:

(a) define *asexual reproduction* as the process resulting in the production of genetically identical offspring from one parent
(b) define *sexual reproduction* as the process involving the fusion of nuclei to form a zygote and the production of genetically dissimilar offspring
(c) identify and draw, using a hand lens if necessary, the sepals, petals, stamens and carpels of one, locally available, named, insect-pollinated, dicotyledonous flower, and examine the pollen grains under a microscope
(d) state the functions of the sepals, petals, anthers and carpels
(e) use a hand lens to identify and describe the stamens and stigmas of one, locally available, named, wind-pollinated flower, and examine the pollen grains using a microscope
(f) outline the process of pollination and distinguish between self-pollination and cross-pollination
(g) compare, using fresh specimens, an insect-pollinated and a wind-pollinated flower
(h) describe the growth of the pollen tube and its entry into the ovule followed by fertilisation (production of endosperm and details of development are **not** required)
(i) identify on diagrams, the male reproductive system and give the functions of: testes, scrotum, sperm ducts, prostate gland, urethra and penis
(j) identify on diagrams, the female reproductive system and give the functions of: ovaries, oviducts, uterus, cervix and vagina
(k) briefly describe the menstrual cycle with reference to the alternation of menstruation and ovulation, the natural variation in its length, and the fertile and infertile phases of the cycle with reference to the effects of progesterone and estrogen only
(l) describe fertilisation and early development of the zygote simply in terms of the formation of a ball of cells which becomes implanted in the wall of the uterus
(m) state the functions of the amniotic sac and the amniotic fluid
(n) describe the function of the placenta and umbilical cord in relation to exchange of dissolved nutrients, gases and excretory products (Structural details are **not** required)
(o) discuss the spread of human immunodeficiency virus (HIV) and methods by which it may be controlled

12.1 Reproduction

1. Reproduction is the biological process by which new organisms are produced to ensure the perpetuation of the species.
2. Reproductive methods are grouped into two main groups: **asexual reproduction** and **sexual reproduction**.
3. Asexual reproduction is when an organism produces a genetically identical offspring without the contribution of genetic material from another organism.
4. Sexual reproduction is when a genetically dissimilar offspring is produced through the fusion of two gametes, one from each parent organism, during the process of fertilisation.
5. Gametes are reproductive cells that contain half the number of chromosomes as a normal body cell.
6. The zygote produced during fertilisation contains genetic material from both parents, and is therefore genetically different from them.

12.2 Structure of an insect-pollinated flower

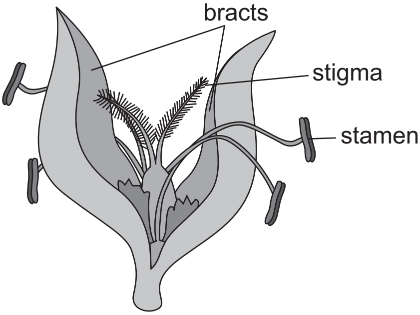


An insect-pollinated flower

1. **Pedicel** (flower stalk) – Modified stem that holds the flower
2. **Receptacle** – The end of the pedicel which holds the parts of the flower
3. **Sepals** – Modified leaves which are green in colour and are found on the outermost ring of floral leaves. They make up the calyx and protect the flower when it is in bud stage.

4. **Petals** – Modified leaves which form the most conspicuous part of the flower; they make up the corolla. They are brightly coloured in insect-pollinated plants and form a platform for insects to land on.
5. **Carpel** – Female reproductive organ. It contains an ovary with one or more ovules and has a sticky tip known as a stigma.
6. **Stigma** – Receptor of pollen grains. Secretes a sugary fluid that stimulates germination of pollen grains.
7. **Style** – Stalk that connects the stigma to the ovary. Holds the stigma in position to trap pollen grains.
8. **Ovary** – Each ovary contains one or more ovules.
9. **Ovule** – Contains female gametes
10. **Stamen** – Male reproductive organ. It consists of an anther and a filament.
11. **Anther** – Contains pollen grains. Pollen grains in insect-pollinated plants are heavy and sticky.
12. **Filament** – Stalk that holds the anther in a suitable position to disperse pollen
13. A flower can have multiple carpels. Multiple carpels form a **pistil**.

12.3 Structure of a wind-pollinated flower



A wind-pollinated flower

1. Flowers are small, dull-coloured, scentless and without nectar.
2. Flower parts are protected by leaf-like structures called bracts.
3. Stamens have long pendulous filaments that hang out of the bracts, exposing anthers to the wind.
4. Stigmas are large, extended and feathery, with a large surface area to trap the small and light pollen grains.

12.4 Differences between insect-pollinated and wind-pollinated flowers

Plant part	Insect-pollinated flower	Wind-pollinated flower
Flower	Large, brightly-coloured petals	Small and dull; flower parts protected by modified leaves called bracts
Scent	Flowers are strong-smelling	Flowers are scentless
Nectar	Present	Absent
Nectar guide	Present	Absent
Stamen	Not pendulous and do not protrude out of the flower	Pendulous and protrude out of the flower
Stigma	Small and compact, do not protrude out of the flower	Large and feathery, protrude out of the flower
Pollen grain	Fairly abundant, large and sticky	Very abundant, small and light

12.5 Pollination

1. Pollination is the transfer of pollen grains from the anther to the stigma, enabling fertilisation.
2. Mechanisms of pollination include **insect pollination** and **wind pollination**.
3. Insect-pollinated flowers contain nectar and have nectar guides which are lines that are visible to insects, guiding them to the location of the nectar.
4. When the insect enters the flower, pollen grains from the anthers stick onto the insect. If pollen grains from a previously-visited flower are present on the insect, they will be transferred to the sticky stigma.
5. Wind-pollinated flowers have their pollen carried away by the wind when the exposed anthers shake in the wind.
6. When the pollen grains come into contact with the large feathery stigmas of another flower, they would be trapped.
7. There are two types of pollination: **self-pollination** and **cross-pollination**.

12.6 Self-pollination

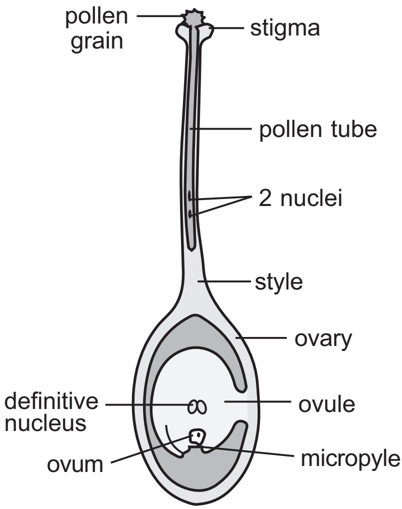
1. Self-pollination is the transfer of pollen grains from the anther to the stigma of the same flower, or from the anther of a flower to the stigma of another flower on the same plant.
2. Factors that promote self-pollination are:
   (a) Bisexual flowers with anthers and stigma maturing at the same time
   (b) Stigma being located directly below the anthers, allowing pollen grains to fall onto it
3. Advantages of self-pollination are:
   (a) Not dependent on external agents of pollination such as insects or wind
   (b) Less wastage of pollen and energy. During wind and insect pollination, a great number of pollen grains are lost as only a few pollen grains come into contact with a stigma of a flower of the same species.
   (c) Only one parent plant is required.
4. A disadvantage of self-pollination is less genetic variation, hence the offspring is less adapted to environmental changes.

12.7 Cross-pollination

1. Cross-pollination is the transfer of pollen grains from the anther of a flower to the stigma of a flower of another plant belonging to the same species.
2. Factors that promote cross-pollination are:
   (a) Plants bearing only male or female flowers. These plants are called dioecious plants.
   (b) In plants with bisexual flowers, the anthers and the stigmas mature at different times.
   (c) Self-incompatibility – When a pollen grain of a flower happens to land on the stigma of the same flower or another flower on the same plant, a biochemical block prevents the pollen grain from germinating.
3. Advantages of cross-pollination are:
   (a) Greater genetic variation, hence the offspring has a higher chance of surviving environmental changes.
   (b) Offspring may have inherited beneficial qualities from both parents.

4. Disadvantages of cross-pollination are:
    (a) Energy-consuming – lots of energy is required to make large amounts of pollen grains.
    (b) A great number of pollen grains are wasted due to the randomness of the dispersal methods.
(c) External agents of pollination i.e. wind, insects are required.
(d) Two parent plants are required.

12.8 Double fertilisation in plants



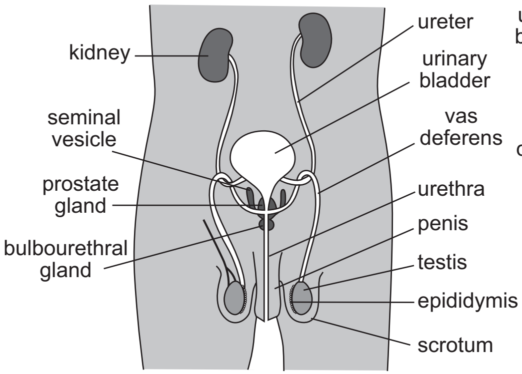
Fertilisation in plants

1. After pollination, a pollen tube grows out of each pollen grain in response to the sugary fluid secreted by the stigma.
2. The cytoplasm and the two nuclei of the pollen grain (generative nucleus and pollen tube nucleus) pass into the pollen tube. The pollen tube nucleus controls the growth of the pollen tube.
3. The pollen tube grows through the cells of the style by secreting enzymes to digest them.
4. The generative nucleus divides to form two male gametes.
5. The pollen tube enters the ovary and then enters the ovule through an opening in the ovule wall called a micropyle and releases the two male gametes.
6. One male gamete fuses with the ovum to form the zygote. The other male gamete fuses with the definitive nucleus to form the endosperm nucleus.

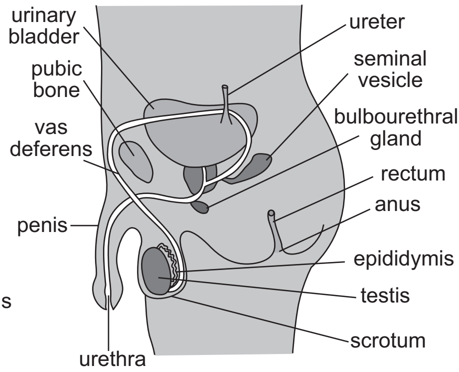
7. The zygote will divide and develop into the embryo. The endosperm nucleus will divide and give rise to the endosperm, a food storage tissue that will nourish the developing embryo.

8. The ovule will develop into a seed and the ovary will develop into a fruit.

12.9 Male reproductive system



Front view of the male reproductive system



Side view of the male reproductive system

1. **Testes** (singular: testis) – The male reproductive organs (**gonads**). Produces sperms (**male gametes**) and male sex hormones e.g. testosterone. Male sex hormones are responsible for development and maintenance of secondary sexual characteristics. Leading from the end of each testis is a narrow tightly-coiled tube called the **epididymis** in which sperms are stored.

2. **Scrotum** – The two testes are held in a pouch-like sac outside the body called the scrotum. The lower temperature in the scrotum is essential for sperm production.

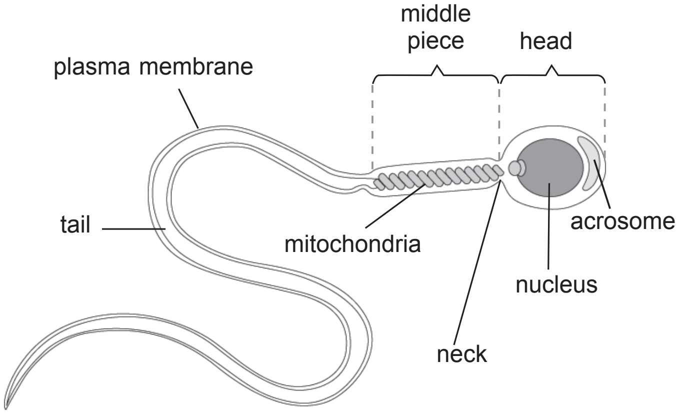
3. **Sperm ducts** – The sperm ducts (**vas deferens**) lead from the epididymis. During ejaculation, they transport sperm from the epididymis to the urethra.

4. **Prostate gland** – The prostate gland is a large gland which secretes directly into the urethra through several small ducts. The fluid contributes to semen. **Semen** is a composition of sperm and fluids from the sex glands containing nutrients and enzymes which nourish and activate the sperm, allowing them to swim actively.

5. **Seminal vesicles** – Ducts from the seminal vesicles join the vas deferens. The seminal vesicles are a pair of glands that secrete a fluid that makes up a proportion of semen.

6. **Cowper's glands** – The Cowper's glands, also known as bulbourethral glands, are a pair of pea-sized glands located beneath the prostate. The fluid produced by the gland contributes to semen.
7. **Urethra** – The urethra is a common passage for urine and semen to pass out of the body. The sphincter muscle at the base of the bladder prevents urine from passing out of the bladder during ejaculation of semen.
8. **Penis** – The penis consists of cylinders of spongy erectile tissue around the urethra. The tissue contains numerous spaces that allow it to fill up with blood. When that happens, the penis becomes erect and hard, allowing it to enter the vagina of a woman during sexual intercourse to deposit semen.

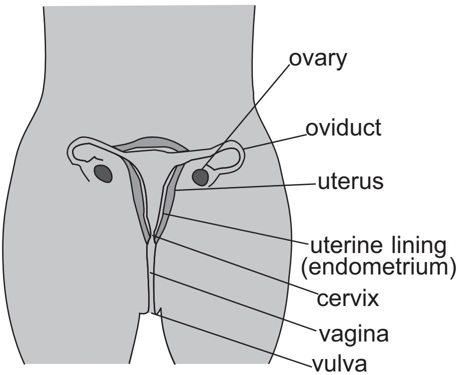
12.10 Spermatozoa



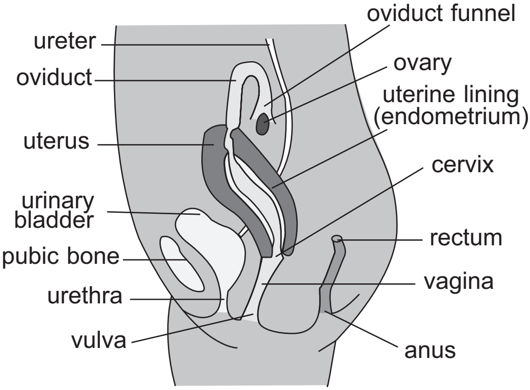
A sperm cell

1. The male gamete, the sperm (singular: spermatozoon, plural: spermatozoa), consists of a head, middle piece and tail.
2. The head contains:
   (a) An acrosome, an enzyme-containing sac. The acrosome contains digestive enzymes which break down the outer membrane of the ovum, allowing for fertilisation
   (b) A small amount of cytoplasm and a large haploid nucleus
3. The middle piece contains numerous mitochondria arranged spirally to provide energy for the sperm to swim to the egg.
4. The tail (flagellum) beats to propel the sperm towards the egg.

12.11 Female reproductive system



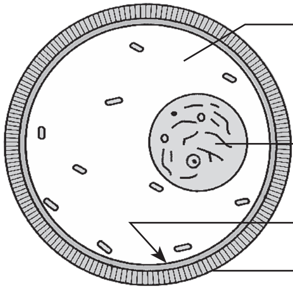
Front view of the female reproductive system



Side view of the female reproductive system

1. **Ovaries** – The female reproductive organs (gonads). Produces ova (singular: ovum) and female sex hormones e.g. estrogen and progesterone. Female sex hormones are responsible for development and maintenance of secondary sexual characteristics. Mature eggs are released from the ovaries into the oviducts.
2. **Oviducts** – The oviduct (fallopian tube) is a narrow muscular tube leading from the ovary to the uterus. The oviduct has a funnel-like opening to make it easier for ova to enter the oviduct. Cilia on the inner lining help move the ovum to the uterus. The ovum is usually fertilised in the oviduct.
3. **Uterus** – The uterus is a thick muscular organ that can stretch as the fetus increases in size during pregnancy. The smooth muscles in the uterine wall contract to expel the fetus during birth. The uterus is lined by a lining called the **endometrium** (uterine lining). The endometrium is richly supplied with blood vessels and is the site of **implantation** of the embryo post-fertilisation. It is broken down every month and flows out of the body in the process called **menstruation**.
4. **Cervix** – The cervix is a circular ring of muscle at the neck of the uterus. It opens into the vagina. It enlarges during birth to allow the passage of the fetus.
5. **Vagina** – The vagina is a thin-walled chamber where sperm is deposited during sexual intercourse. It forms the birth canal through which the baby is born.

12.12 Ovum



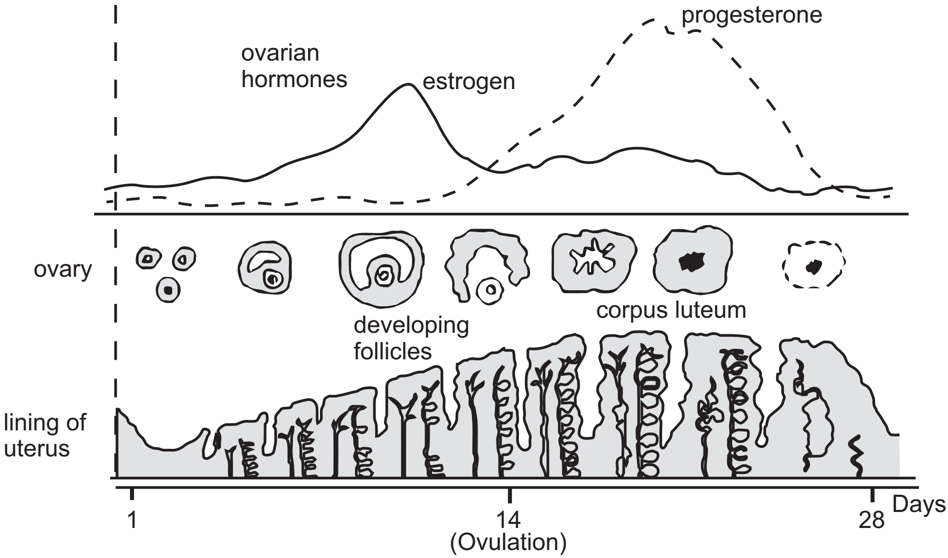
An ovum

1. The female gamete, the ovum, is a large cell containing abundant cytoplasm.
2. It has a large nucleus containing a haploid set of chromosomes.
3. It is surrounded by a plasma membrane and an outer membrane.

12.13 The menstrual cycle

1. The menstrual cycle normally spans over 28 days. There is natural variation in the length of the menstrual cycle, and it can range from 21 to 33 days.
2. **Day 1 to 5**: Menstruation lasts for 5 days. The first day of menstruation is day 1 of the menstrual cycle. The endometrium breaks down and flows out of the body.
3. **Day 6 to 13**: The ovaries secrete estrogen which causes the repair and growth of the endometrium. The endometrium becomes thicker.
4. **Day 14**: A mature ovum is released from the ovaries. Secretion of progesterone is stimulated. The ovum dies after about 1 to 2 days if it is not fertilised.
5. **Day 15 to 28**: Progesterone and estrogen are continually being secreted for continued development and maintenance of the endometrium. Progesterone maintains the endometrium by causing it to become thicker. The endometrium readies for implantation. Towards the end of the cycle, secretion of progesterone and estrogen decline sharply. The endometrium is no longer maintained and disintegrates. It flows out from the uterus together with some blood through the vagina. This marks the beginning of another cycle.
6. The **fertile phase** of the cycle is from day 11 to 17. This is because sperms can survive for 2 to 3 days in the female reproductive system. Sperms deposited in the vagina from day 11 onwards can fertilise the ovum which is released from the ovaries on day 14. The ovum can survive for 1 to 2 days after ovulation; hence fertilisation is possible up till day 17.

7. The rest of the days make up the **infertile phase** of the menstrual cycle. Sexual intercourse during this period is unlikely to result in fertilisation since no ovum is present.



The menstrual cycle

12.14 Fertilisation in humans

1. During sexual intercourse, semen containing sperms is deposited into the vagina of a woman. The fluids from the male sex glands that make up semen provide nutrients and protection for the sperms, as well as a medium for them to swim in.

2. The sperms swim up the oviducts and encounter the ovum.

3. The acrosome of the sperms release enzymes to disperse the layer of cells surrounding the ovum and break down the outer membrane of the ovum.

4. Only 1 sperm will enter the ovum. The plasma membranes of the sperm and the ovum fuse and the sperm nucleus enters the ovum. The plasma membrane of the egg undergoes a change as soon as a single sperm has entered, preventing other sperms from entering.

5. The sperm nucleus fuses with the egg nucleus, forming a fertilised ovum known as a zygote.

6. The remaining sperms eventually die.

12.15 Development of the zygote

1. The cilia on the oviduct lining help move the zygote towards the uterus.
2. In the meantime, the zygote divides many times to form a hollow ball of cells called the embryo.
3. 5 to 7 days after fertilisation, the embryo comes into contact with the endometrium and becomes embedded. This process is known as implantation.
4. Tissues growing out from the embryo invade the endometrium, forming the **placenta**. The placenta is an organ that contains both maternal and embryonic blood vessels. It allows for diffusion between the maternal blood circulation and embryonic blood circulation.
5. The placenta:
   (a) Provides nutrients (glucose, amino acids and mineral salts) and oxygen for the embryo
   (b) Removes waste materials such as urea and carbon dioxide
   (c) Allows protective antibodies to diffuse from maternal blood into embryonic blood
   (d) Provides a barrier preventing maternal blood and embryonic blood from mixing. Reasons for this include:
      (i) Maternal blood pressure is much higher than embryonic blood pressure and would damage vital tissues.
      (ii) The embryo might have a different blood group, resulting in agglutination if mixing of blood occurs.
   (e) Produces progesterone which maintains the endometrium during pregnancy
6. The embryo eventually becomes connected to the placenta by the **umbilical cord**. Embryonic blood travels to the placenta via the arteries of the umbilical cord and returns with oxygen and dissolved food substances via the umbilical vein.
7. A membrane called the **amniotic sac** begins development at the same time as the placenta, and encloses the embryo in a fluid-filled space. The fluid is known as **amniotic fluid**.
8. The amniotic fluid functions to:
   (a) Absorb shock, support and protect the embryo from physical injury
   (b) Lubricate the vagina during birth to reduce friction
   (c) Allow the fetus to move freely during development
9. About 9 weeks after fertilisation, the embryo has developed into a fetus.

12.16 Human Immunodeficiency Virus

1. Acquired Immune Deficiency Syndrome (AIDS) is a disease that can be spread through sexual intercourse.
2. It is caused by a virus called Human Immunodeficiency Virus (HIV).
3. HIV progressively reduces the effectiveness of the infected person's immune system in protecting him from infection.
4. HIV infection progresses to AIDS, the last stage of the infection, in about 9 to 10 years after infection.
5. Symtons of AIDS include:
   (a) Persistent fever, sweat, swollen glands, chills, weakness and weight loss
   (b) Pneumonia
   (c) Tuberculosis
   (d) Chronic diarrhoea
   (e) Brain infection
   (f) Tumours such as Kaposi's sarcoma (cancer of the blood vessels) and cervical cancer in women
6. HIV is transmitted:
   (a) By sexual intercourse with an infected person
   (b) By sharing and reusing contaminated needles during intravenous drug use, tattoos and piercing
   (c) By receiving a blood transfusion from an infected donor
   (d) During pregnancy and childbirth. An infected mother could pass on the disease to her child
7. Spread of HIV can be prevented by:
   (a) Having protected sexual intercourse. A condom reduces the risk of infection.
   (b) Abstinence from sex or having sex with only one partner
   (c) Not sharing objects that could be contaminated with blood or bodily fluids such as hypodermic syringes, razors and toothbrushes
   (d) Screening of blood in a blood bank for HIV infection to reduce chances of transmission during blood transfusions
   (e) Infected mothers should undergo antiretroviral therapies and give birth by caesarean section to minimise risk of transmission to the foetus. Breastfeeding should be avoided after birth.
   (f) Visiting reliable operators for tattoos, piercings or acupuncture where needles are sterilised or disposable

TOPIC 13 Cell Division

Objectives

Candidates should be able to:

(a) state the importance of mitosis in growth, repair and asexual reproduction
(b) explain the need for the production of genetically identical cells
(c) identify, with the aid of diagrams, the main stages of mitosis
(d) state what is meant by *homologous pairs* of chromosomes
(e) identify, with the aid of diagrams, the main stages of meiosis (Names of the sub-divisions of prophase are **not** required)
(f) define the terms *haploid* and *diploid*, and explain the need for a reduction division process prior to fertilisation in sexual reproduction
(g) state how meiosis and fertilisation can lead to variation

13.1 Cell division

1. New cells must be created for growth and repair in organisms.
2. Cell division is the process by which new cells arise.
3. During cell division, a parent cell divides into two or more daughter cells.
4. There are two types of cell division: **mitosis** and **meiosis**.
5. Mitosis takes place in body cells in tissues undergoing growth and repair while meiosis is only involved in the creation of gametes.

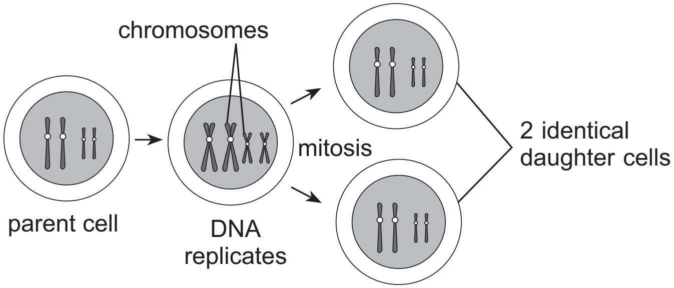
13.2 Chromosomes

1. A chromosome is a single coiled deoxyribonucleic acid (DNA) molecule containing many genes. Genes are sections of DNA that encode genetic instructions.
2. A normal human body cell contains 46 (23 pairs) chromosomes. This number is the **diploid** (2n) number of chromosomes.
3. In gametes, there are only 23 chromosomes. This number is the **haploid** (n) number of chromosomes.
4. The process of DNA replication during cell division must be finely controlled so that the daughter cells produced by mitosis would contain all the genes required for subsequent cell division and differentiation.

5. Errors occurring during DNA replication will be transferred to daughter cells during cell division.
6. This could lead to harmful changes in the genes and affect cellular function.

13.3 Mitosis

1. Mitosis is the process of cell division in which the genetic material of the parent cell is duplicated, producing two daughter cells that are genetically identical to the parent cell.
2. The daughter cells each contain the diploid number of chromosomes.
3. Mitosis is important for growth because genetically identical new cells must be produced during growth.
4. Mitosis is also required for repair. New cells are produced to replace worn-out cells that have been destroyed or shed.
5. Mitosis occurs during asexual reproduction, producing offspring that are genetically identical to the parents as well as to one another.



Mitosis

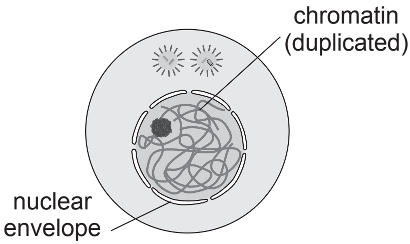
13.4 The cell cycle

1. The cell cycle is the series of events that take place in a cell, resulting in cell division.
2. It consists of a period in which the cell prepares for cell division by accumulating nutrients, increasing its size and number of organelles and replicating its DNA, called the **interphase**, and the actual **mitotic phase**.
3. The mitotic phase consists of mitosis, which is the division of the genetic material, and cytokinesis, which is the division of the cytoplasm.

13.5 Stages of mitosis

Interphase

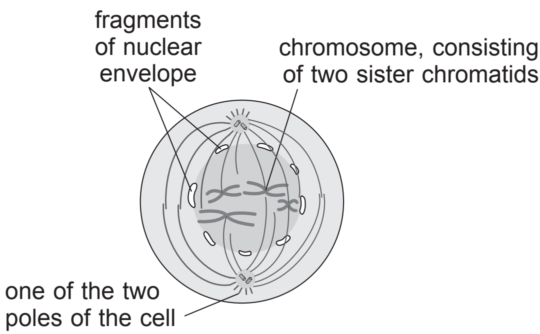
1. During interphase, the cell grows, stores energy and duplicates organelles.
2. The DNA replicates and the total DNA content of the cell doubles. The chromosome number still remains 2n.
3. The chromatin (threads of chromosomes) is in the dispersed state.



Interphase

Prophase

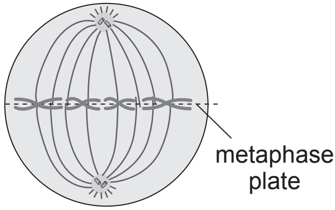
1. The chromatin condenses to form thick strands, which are visible under the light microscope. Each duplicated chromosome appears as two identical sister chromatids joined together at a central region called the centromere. The centromeres form X-shaped structures.
2. The nuclear envelope disappears.



Late prophase

Metaphase

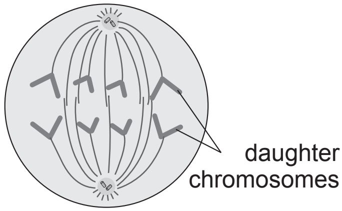
1. The chromosomes line up along the **metaphase plate**, which is an imaginary line equidistant from the two spindle poles.



Metaphase

Anaphase

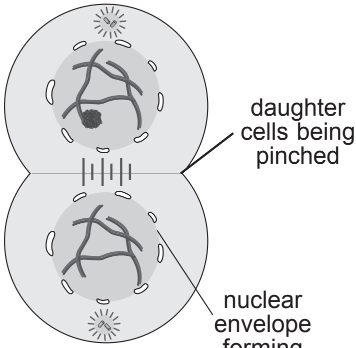
1. Each pair of sister chromatids splits at the centromeres and are pulled to the opposite ends of the cell. Each chromatid is now called a daughter chromosome.
2. At the same time, the opposite ends of the cell move further apart.
3. The two ends of the cell now have equivalent and identical collections of chromosomes.



Anaphase

Telophase

1. Daughter nuclei begin to form at both ends of the cell.
2. The chromosomes in each daughter nucleus uncoil to form chromatin threads.
3. While telophase is taking place, cytokinesis occurs. Cytokinesis is not considered a part of mitosis but is necessary for cell division.



Telophase

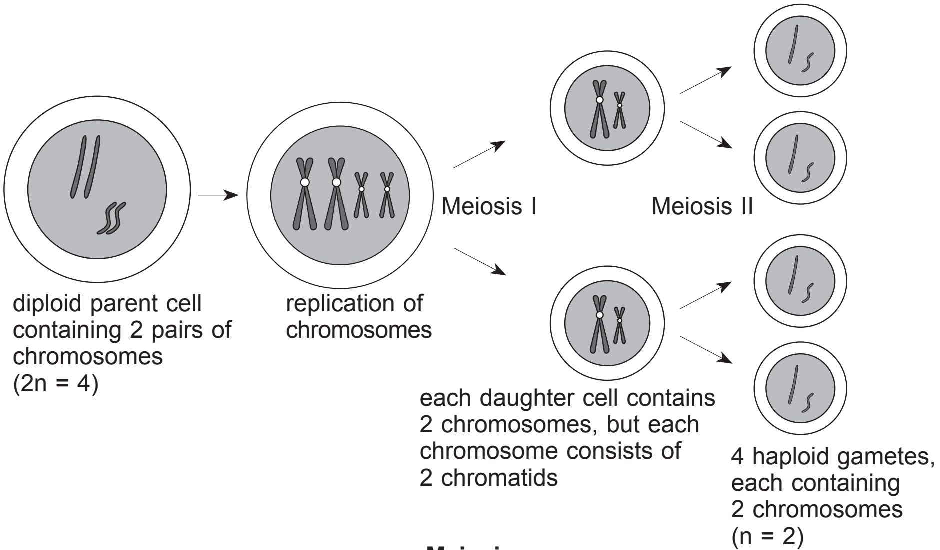
Cytokinesis

1. Cytokinesis, the division of cytoplasm, occurs at the same time the cell is undergoing telophase.
2. The two daughter cells are pinched apart.
3. Each daughter cell has a complete copy of the genome of the parent cell.

13.6 Meiosis

1. Meiosis is the reduction division in cells where the chromosome number in each daughter cell is halved.
2. Normal human body cells contain 2 sets of 23 chromosomes (a maternal set and a paternal set), making a total of 46 chromosomes. Cells containing 2 sets of chromosomes are diploid.
3. Gametes contain only 1 set of chromosomes and are known as haploid cells.
4. Gametes have to be haploid so that when sexual intercourse occurs, 2 haploid gametes can fuse to produce a diploid zygote. The zygote grows and develops by mitosis, preserving its ploidy number and giving rise to a new organism.

5. Meiosis is the process by which haploid gametes are produced.

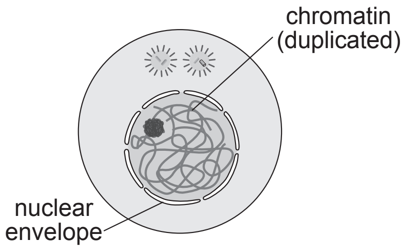


Meiosis

13.7 Stages of meiosis

Interphase

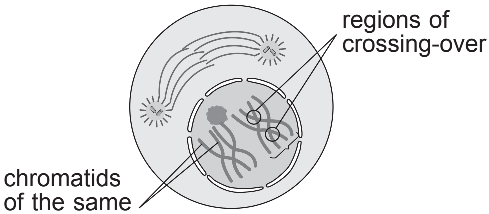
1. Interphase in meiosis is identical to interphase in mitosis.
2. Each of the 46 chromosomes is replicated and exists as two sister chromatids.



Interphase

Prophase I

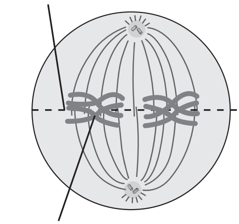
1. Chromatin condenses into chromosomes, which are thick strands that are visible under a light microscope.
2. Homologous chromosomes, one inherited from the father and one inherited from the mother, pair up.
3. **Crossing-over** occurs at many points along the paired chromosomes, where some DNA is exchanged.



Prophase I

Metaphase I

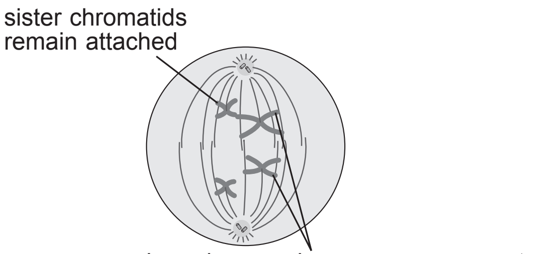
1. The homologous chromosomes line up in pairs along the metaphase plate.
2. One chromosome of each pair of homologous chromosomes ends up on one side of the metaphase plate, while its homologue (also consisting of 2 sister chromatids) is on the other side of the metaphase plate.



Metaphase I

Anaphase I

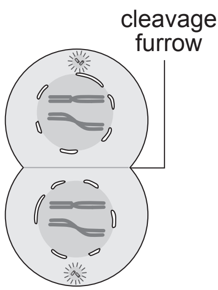
1. Homologous chromosomes separate and move to opposite poles of the cell.
2. The sister chromatids of each chromosome are still attached and move together.



Anaphase I

Telophase I

1. Nuclear membranes form around the chromosomes at each pole of the cell.
2. Cytokinesis occurs.



Telophase I

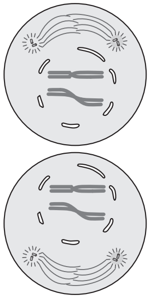
Cytokinesis

1. As in mitosis, cytokinesis involves the formation of a cleavage furrow in animals or a cell plate in plants.

In the second cell division, the sister chromatids are separated. The process is identical to mitosis.

Prophase II

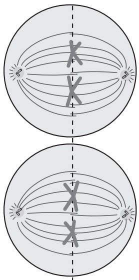
1. Nuclear envelope disappears and chromatin condenses



Prophase II

Metaphase II

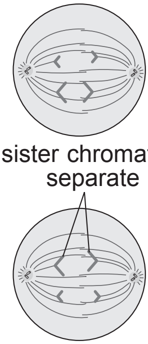
1. The chromosomes are aligned along the metaphase plate.



Metaphase II

Anaphase II

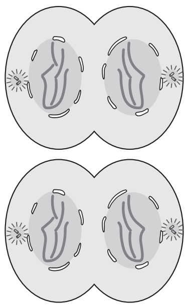
1. The sister chromatids are pulled to opposite poles.



Anaphase II

Telophase II

1. Nuclear envelopes reappear at each pole.
2. Chromosomes uncoil and lengthen.
3. Each daughter cell at the end of meiosis II has a haploid number of unreplicated chromosomes, i.e. half the amount of DNA of a usual body cell.



haploid daughter cells forming

Telophase II

Cytokinesis

Cytokinesis occurs to pinch the daughter cells apart.

13.8 Genetic variation arising from meiosis

1. Genetic variation increases the chances of survival of the species in a changing environment.
2. Variation provides the basis for **natural selection**, a process where, over time, individuals with heritable traits more suitable for the environment are more likely to survive and reproduce, passing on their favourable genes to their offspring.
3. In a changing environment, a larger gene pool (due to genetic variation) is more likely to contain genes that express traits more suitable for the new environment. The species have a higher chance of becoming adapted instead of becoming extinct.
4. Genetic variation arises through 3 processes:
   (a) Independent assortment of chromosomes during metaphase I of meiosis. Independent assortment results in gametes with a random mixture of maternal and paternal chromosomes.
   (b) Crossing-over between homologous chromosomes during prophase I of meiosis. Crossing-over results in genetic recombination, producing chromosomes that have a mixture of maternal and paternal DNA.
   (c) Random fertilisation of gametes. Each gamete has a unique set of 23 chromosomes due to independent assortment and crossing-over in meiosis. Any one male gamete representing one out of the many different possible gene combinations, can fertilise an ovum, also representing one out of the many different possible gene combinations. This will produce variation due to the many different combinations of genes from the male and female gamete.

TOPIC 14 Molecular Genetics

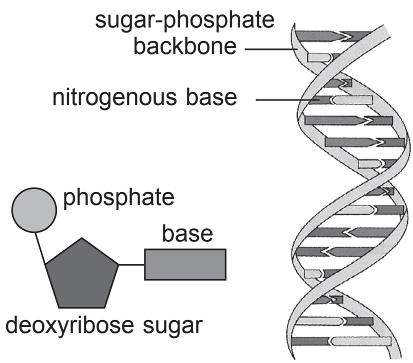
Objectives

Candidates should be able to:

(a) outline the relationship between DNA, genes and chromosomes
(b) state the structure of DNA in terms of the bases, sugar and phosphate groups found in each of their nucleotides
(c) state the rule of complementary base pairing
(d) state that DNA is used to carry the genetic code, which is used to synthesise specific polypeptides (details of transcription and translation are **not** required)
(e) state that each gene is a sequence of nucleotides, as part of a DNA molecule
(f) explain that genes may be transferred between cells. Reference should be made to the transfer of genes between organisms of the same or different species – transgenic plants or animals
(g) briefly explain how a gene that controls the production of human insulin can be inserted into bacterial DNA to produce human insulin in medical biotechnology
(h) discuss the social and ethical implications of genetic engineering, with reference to a named example

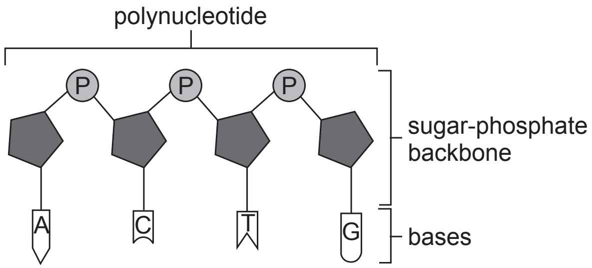
14.1 Structure of DNA

1. Deoxyribonucleic acid (DNA) is a molecule that carries genetic information used in the development and functioning of all organisms.
2. DNA consists of a pair of molecules that are twisted around each other in a shape called a double helix.
3. Each molecule of DNA is a long polymer consisting of basic units called **nucleotides**. Polymers of nucleotides are called **polynucleotides**.
4. Each nucleotide consists of:
   (a) A **deoxyribose** sugar
   (b) A **phosphate group**
   (c) A base containing nitrogen



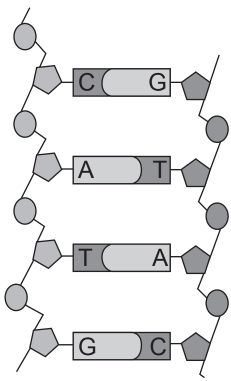
A nucleotide

5. There are four types of **nitrogenous bases**:
   (a) Adenine (A)
   (b) Guanine (G)
   (c) Cytosine (C)
   (d) Thymine (T)
6. The nucleotides polymerise to form a polynucleotide when the deoxyribose sugars of the nucleotides are joined together by phosphate groups, forming the **sugar-phosphate backbone** of the DNA molecule.



A polynucleotide

7. Two strands of polynucleotides wrap around each other to form the double helix structure of DNA. The strands are held together when the nitrogenous bases on one strand form **hydrogen bonds** with the nitrogenous bases on the other strand.
8. **Complementary base pairing** is when each type of base on one strand forms hydrogen bonds with only one type of base on the other strand.
9. **Adenine bonds to only thymine; cytosine bonds to only guanine.**

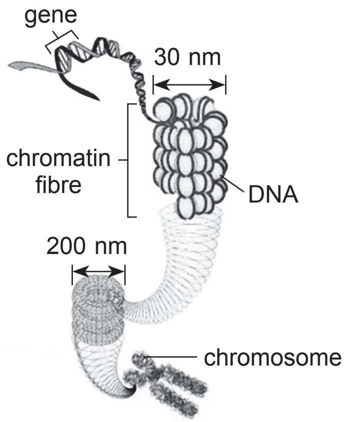


Complementary base pairing

10. Each gene is made up of a unique sequence of nucleotides. A DNA molecule contains many genes along its length.

14.2 Organisation of DNA in cells

1. DNA is wrapped around proteins to form a 'beads on a string' structure.
2. The DNA-wrapped proteins coil to form a chromatin fibre.
3. The chromatin fibres fold and coil further to form the compact structures called chromosomes seen during cell division.



Folding of DNA

14.3 From DNA to phenotype

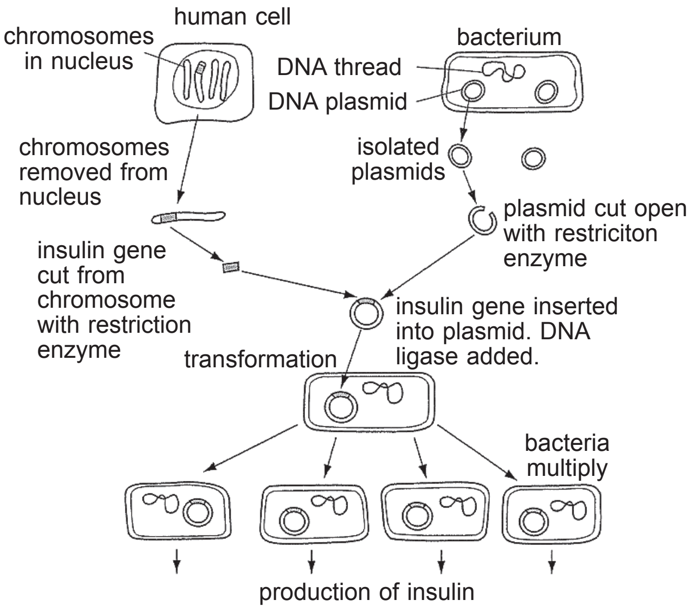
1. A **gene** is a single unit of hereditary information consisting of a specific nucleotide sequence located on the chromosome. Each gene contains information for the production of a **single polypeptide**.
2. These gene products are responsible for every aspect of a living organism e.g. appearance, resistance to specific diseases, biochemical processes necessary for life etc. For example, a gene can affect hair colour by coding for an enzyme involved in the production of hair pigment.

14.4 Genetic engineering

1. Genetic engineering is a technique where genes of interest can be inserted into the genome of a specific organism. For example, genes from bacteria or other plants that confer resistance to diseases or herbicides are inserted into crop plants like soybean or corn in order to make it grow better. Herbicides can be used on weeds growing around the crop plant without killing the crop plant. The genetically-modified plant is known as a **transgenic plant**.
2. Transgenic organisms possess a gene or genes that have been transferred from another species.

14.5 Production of human insulin

1. People suffering from type 1 diabetes mellitus require insulin injections.
2. Genetic engineering is used to produce human insulin from the bacteria *Escherichia coli*.
3. The human insulin gene is first obtained from a human chromosome by cutting it with a restriction enzyme.
4. The plasmid vector is cut with the same restriction enzyme.
5. When the plasmids are mixed with the DNA fragments, they are able to bind as the enzyme cuts both in the same way, generating 'sticky ends' that can join together. DNA ligase is added to the mixture, allowing the cut ends of the DNA to join to form a continuous double strand.
6. The recombinant plasmids are mixed with *E. coli*. Heat shock is applied to the bacteria, opening up pores on the membrane of the bacteria so plasmids enter the bacteria. This process is known as transformation.
7. The bacteria are placed in large steel tanks called fermenters under optimal conditions for growth and reproduction. Features of a fermenter include a nutrient broth containing glucose water and salts, 37°C temperature maintained by a temperature probe, optimal pH maintained by a pH probe, air supply for aeration and a stirrer to mix substances evenly.
8. At the end of fermentation, the bacteria cells are lysed open. Insulin is extracted and purified by crystallisation.



Production of human insulin

14.6 Applications of genetic engineering

1. Genetic engineering has relevance to biological research e.g. genetically-modified (GM) mice are used to study the function of genes.
2. Low-cost, high-yield production of pharmaceutical drugs e.g. insulin, clotting factors for haemophiliacs, human growth hormone.
3. Agriculture, where traits conferred through genetic modification include:
   (a) Survivability in harsh environmental conditions. Areas previously considered unsuitable can be used to grow crops. Crops are also more likely to survive bad weather such as drought.
   (b) Reduced maturation time. Multiple harvests a year translates into increased supply of food.
   (c) Resistance to pests, diseases and herbicides. Crops are less likely to succumb to diseases; farmers are able to use pesticides to remove pests and herbicides to remove weeds without killing the crops.
   (d) Production of toxins that kill pests (bioinsecticides). Farmers save money on pesticides.
   (e) Enhanced nutritional value. Genes coding for vitamin or nutrient production can be inserted into a crop species to yield a more nutritious product.

   Benefits include:
   (a) Lowered cost for farmers since fewer pesticides are used as plants can produce their own. This translates to lower consumer costs and increased accessibility to certain types of food.
   (b) Higher yield since fewer crops are lost to disease or poor environmental conditions.
   (c) GM foods with enhanced nutritional value can be used to supply nutrients to people living in areas without access to certain nutrients in their regular diet.
4. Animal husbandry and aquaculture – GM fish are designed to overproduce growth hormone, resulting in faster growth. This reduces fishing pressure on wild stock.
5. Gene therapy – Gene therapy is the insertion of genes into a person's cells or tissues in order to treat a disease.

14.7 Social and ethical implications of genetic engineering

1. **Potential health concerns** including allergen transfer, transfer of antiobiotic resistance, unknown health effects.
2. **Environmental impact** including transfer of genes to wild plants or weed varieties through cross-pollination, loss of biodiversity, reduced effectiveness of pesticides.
   (a) Genes conferring herbicide tolerance might be transferred to weed varieties, causing the development of herbicide-resistant ‘superweeds’.
   (b) Pesticide-producing GM plants produce pesticides that might indiscriminately kill insects around them, even harmless insects such as butterflies. Such genes crossing over into wild varieties and ending up in a natural environment would have serious ecological implications. This results in a loss of biodiversity and affects the ecological balance.
   (c) There is a concern that insects might build up resistance to pesticides.
3. **Economic impact**
   (a) World food production would be controlled by a few biotechnology companies.
   (b) Increased dependence of developing countries on industrialised countries carrying out genetic research.
   (c) Technology modifying GM plants to produce sterile seeds to minimise the spread of genes into unintended plants and combat patent infringement would result in farmers having to purchase new seeds every year – not financially feasible for farmers in developing countries.
4. **Ethical objections**
   (a) Limitations of modern science to adequately understand the negative effects of GMOs
   (b) Unnatural to mix genes across species, tampering with nature, not respecting natural organisms’ intrinsic values
   (c) Concerns about welfare of GM animals
   (d) GM food labelling is not mandatory in some countries. Consumers might be unaware that they are purchasing and consuming GM products.
   (e) GM food might not have been adequately tested
   (f) Further GM developments might be skewed towards private interests and profit instead of the public welfare

TOPIC 15 Inheritance

Objectives

Candidates should be able to:

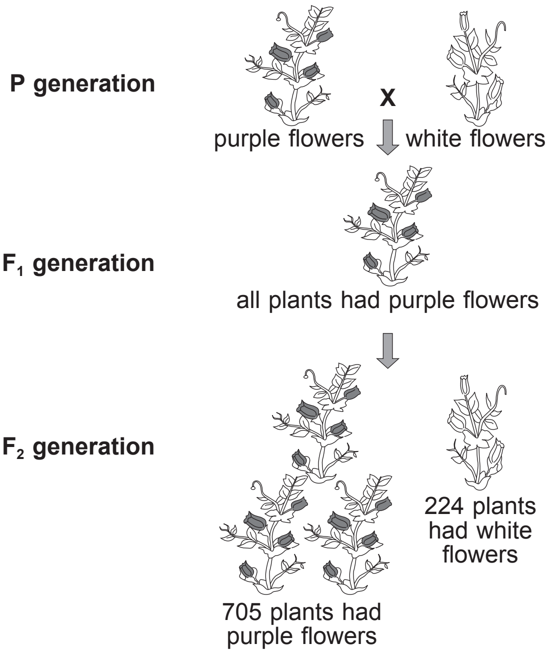
(a) define a *gene* as a unit of inheritance and distinguish clearly between the terms *gene* and *allele*
(b) explain the terms dominant, recessive, codominant, homozygous, heterozygous, phenotype and genotype
(c) predict the results of simple crosses with expected ratios of 3:1 and 1:1, using the terms homozygous, heterozygous, F_1 generation and F_2 generation
(d) explain why observed ratios often differ from expected ratios, especially when there are small numbers of progeny
(e) use genetic diagrams to solve problems involving monohybrid inheritance (Genetic diagrams involving autosomal linkage or epistasis are **not** required)
(f) explain co-dominance and multiple alleles with reference to the inheritance of the ABO blood group phenotypes – A, B, AB, O, gene alleles I^A, I^B and I^O
(g) describe the determination of sex in humans – XX and XY chromosomes
(h) describe mutation as a change in the structure of a gene such as in sickle cell anaemia, or in the chromosome number, such as the 47 chromosomes in the condition known as Down syndrome
(i) name radiation and chemicals as factors which may increase the rate of mutation
(j) describe the difference between continuous and discontinuous variation and give examples of each
(k) state that variation and competition lead to differential survival of, and reproduction by, those organisms best fitted to the environment
(l) give examples of environmental factors that act as forces of natural selection
(m) explain the role of natural selection as a possible mechanism for evolution
(n) give examples of artificial selection such as in the production of economically important plants and animals

15.1 Mendelian genetics

1. Hereditary traits are characteristics that can be passed from one generation to the next.
2. The study of genetics is named after an Austrian monk, Gregor Mendel, who studies heredity in garden pea plants (*Pisum sativum*).
3. Mendel focused on characteristics of pea plants that have an 'either-or' behaviour. For example, the flowers of pea plants are either purple or white, with no intermediates.
4. He started his experiments with **true-breeding** plants. When true-breeding plants self-pollinate, all their offspring are of the same type. True-breeding tall plants will produce tall offspring when self-pollinated.

15.2 Mendel's monohybrid crosses

1. A monohybrid cross begins with:
   (a) Cross-pollination between true-breeding parents (**P generation**)
   (b) P generation produces hybrid offspring (offspring from 2 different varieties), called the **F_1 generation**.
   (c) F_1 hybrids self-pollinate to produce another set of offspring, called the **F_2 generation**.
2. One of Mendel's monohybrid crosses:
   (a) P generation consisted of true-breeding plants producing either purple or white flowers.
   (b) F_1 generation consisted of all purple-flowered plants.
   (c) Self-pollination in the F_1 generation produced an F_2 generation where the ratio of purple-flowered to white-flowered plants is 3 : 1.



3. Important observations:
   (a) F_1 generation **did not possess intermediate traits** between the two parents i.e. flowers produced by F_1 generation were all as purple as flowers produced by the P generation purple-flowered plant, instead of being pale purple – an intermediate between the purple-flowered parent and white-flowered parent.
   (b) Self-pollination in F_1 generation produced offspring (F_2 generation) in which the **white-flowered trait** (not expressed in the F_1 generation) **resurfaced**.

4. Mendel's deductions:

   (a) The heritable factor for white flowers did not disappear in the F_1 generation since it was able to resurface in the F_2 generation.

   (b) Only the purple-flower factor affected flower colour in the F_1 generation. He called the purple-flower trait **dominant** and the white-flower trait **recessive**.

15.3 Mendel's model of heredity

1. Hereditary factors are responsible for transmission of characteristics from one generation to the next. These factors are now called genes.
2. Each characteristic is controlled by a pair of **alleles** (different forms of the same gene) in an organism. For example, flower colour in pea plants is controlled by an allele for purple flowers and an allele for white flowers. In other words, the gene for flower colour has two alleles: purple and white.
3. Each organism inherits one allele from the mother and one allele from the father during sexual reproduction. Each body cell of an organism contains two alleles for each trait.
4. If an organism has two different alleles, then the **dominant allele** will show its effect while the **recessive allele** will have no effect on the organism's appearance.
5. The two alleles will **segregate** during gamete formation. Each gamete will only contain one allele out of the two that are present in the body cells of an organism.

15.4 Glossary of terms involved in genetics

1. **Chromosome** – A chromosome is an organised structure of deoxyribonucleic acid (DNA) and protein that is found in the nuclei of cells. DNA contains genetic information used in the development and functioning of all organisms.
2. **Gene** – A gene is a DNA segment located in a chromosome, which codes for a single unit of inheritance. The place on the chromosome where the gene is located is called the gene locus.
3. **Allele** – Alleles are different versions of the same gene. They are located on the same gene locus in homologous chromosomes.
4. **Phenotype** – An observable characteristic of an organism. It can be physical (appearance), behavioural or physiological. It depends on the genotype of the organism.
5. **Genotype** – The genetic make-up of an organism. The genotype of an organism cannot be easily predicted from the phenotype (appearance) because of the existence of dominant and recessive alleles.

6. **Homozygous** – Each organism inherits two alleles for a given characteristic, one from the mother and one from the father. An organism is said to be homozygous for a given trait when it contains two identical alleles for that trait.
7. **Heterozygous** – An organism is said to be heterozygous for a given trait when it contains two different alleles for the characteristic.
8. **Dominant allele** – A dominant allele is the allele that is fully expressed in the phenotype under both homozygous and heterozygous conditions.
9. **Recessive allele** – A recessive allele is the allele that is only expressed in the phenotype under the homozygous condition. It is masked in the phenotype under heterozygous conditions.

15.5 Explaining Mendelian ratios

Dihybrid cross:

Let *P* represent the dominant allele for purple flowers, and *p*, the recessive allele for white flowers.

P generation phenotype: Purple-flowered White-flowered

P generation genotype: *PP* × *pp*

Gametes: *P* *P* *p* *p*

×	*p*	*p*
P	*Pp*	*Pp*
P	*Pp*	*Pp*

(*PP* × *pp*)

F$_1$ generation genotype: *Pp* *Pp* × *Pp* *Pp*

(self-pollinated)

F$_1$ generation phenotype: **All purple-flowered**

Gametes: *P* *p* *P* *p*

×	*P*	*p*
P	*PP*	*Pp*
p	*Pp*	*pp*

(*Pp* × *Pp*)

F$_2$ generation genotype: *PP* *Pp* *Pp* *pp*

F$_2$ generation phenotype: Purple-flowered Purple-flowered Purple-flowered White-flowered

F$_2$ generation phenotype ratio: **3 purple-flowered : 1 white-flowered**

1. A homozygous dominant plant (*PP*) will only produce gametes containing a single copy of the *P* allele.
2. A homozygous recessive plant (*pp*) will only produce gametes containing a single copy of the *p* allele.

3. When cross-pollination occurs between the two plants, the gametes will combine during fertilisation to produce heterozygous (*Pp*) hybrids.
4. Heterozygous (*Pp*) plants will produce gametes containing either the *P* or the *p* allele in a 1 : 1 ratio.
5. Crossing the F_1 generation will result in a 25% chance of a homozygous dominant offspring, a 50% chance of heterozygous offspring and a 25% chance of a homozygous recessive offspring.
6. When there is a large amount of offspring produced, the observed phenotypic ratio will be approximately 3 : 1.

15.6 Deducing genotype

1. A test cross is used to determine if an individual exhibiting a dominant trait is homozygous or heterozygous for the trait.
2. It is accomplished by crossing the organism with an organism that is homozygous recessive.
3. **Test cross**:

P generation phenotype	Purple-flowered			White-flowered			
P generation genotype		*PP*	×	*pp*			
Gametes	(*P*)	(*P*)		(*p*)	(*p*)		
F_1 generation genotype	*Pp*	*Pp*		*Pp*	*Pp*		
F_1 generation phenotype		**All purple-flowered**					

		pp	
	×	*p*	*p*
PP	*P*	*Pp*	*Pp*
	P	*Pp*	*Pp*

P generation phenotype	Purple-flowered			White-flowered			
P generation genotype		*Pp*	×	*pp*			
Gametes	(*P*)	(*p*)		(*p*)	(*p*)		
F_1 generation genotype	*Pp*	*Pp*		*pp*	*pp*		

		pp	
	×	*p*	*p*
Pp	*P*	*Pp*	*Pp*
	p	*pp*	*pp*

F_1 generation phenotype ratio **1 purple-flowered : 1 white-flowered**

4. A *PP* x *pp* cross produces only *Pp* offspring. Hence, if all the offspring have purple flowers, then the unknown parent must be homozygous dominant for the trait.

5. A *Pp* x *pp* cross produces a 1 : 1 phenotypic ratio. Hence if both purple and white phenotypes appear among the offspring, then the unknown parent must be heterozygous for the trait.

15.7 Dominance

1. **Complete dominance** is when the heterozygote has the same phenotype as the dominant homozygote. The recessive allele present in the heterozygote is masked by the dominant allele.

2. **Co-dominance** is when both alleles contribute equally to the phenotype.

3. An example would be the ABO blood typing system in humans. Human blood groups are determined by 3 alleles for 1 gene: I^A, I^B and I^O.

4. I^O is recessive to both I^A and I^B, while I^A and I^B are codominant when paired together.

5. The various combinations of the alleles and the resultant phenotypes are shown in the table below:

Phenotype (Blood group)	Genotypes
O	I^OI^O
A	I^AI^A or I^AI^O
B	I^BI^B or I^BI^O
AB	I^AI^B

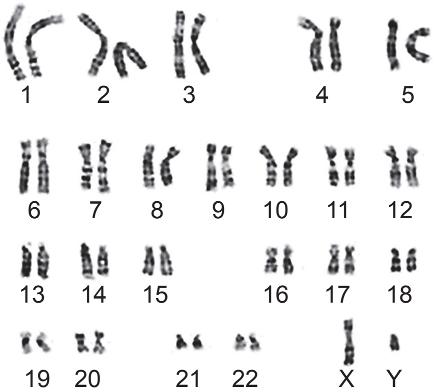
6. The gene for blood group codes for a protein present on the surface of red blood cells, called an antigen. The allele I^A codes for antigen A, I^B codes for antigen B, and no antigen is expressed for allele I^O.

7. For I^AI^B genotype, both antigen A and antigen B are expressed since each of the alleles produces its own antigen. Both alleles contribute to the phenotype, which is blood group AB.

8. The gene for human blood groups is said to have multiple alleles since it exists in more than two alleles.

15.8 Sex determination

1. A karyotype is a picture of a set of chromosomes in a cell. During the preparation of a karyotype, chromosomes are stained and examined under a microscope. A picture is taken and edited to arrange the chromosomes by size.
2. A normal karyotype will show 22 pairs of homologous chromosomes called autosomes, and 1 pair of sex chromosomes.



Human male karyotype

3. It can be used to detect extra or missing pieces of chromosomes that could lead to several congenital conditions.
4. In humans, sex is determined by sex chromosomes. Human sex chromosomes are the X chromosome and the Y chromosome.
5. From the karyotype, it can be seen that the X chromosome is much larger than the Y chromosome.
6. Human males have one X chromosome and one Y chromosome. They have the XY genotype.
7. Human females have two X chromosomes. They have the XX genotype.
8. Genetic diagram for sex determination:

Parents' phenotype: Male Female

Parents' genotype: XY × XX

Gametes: X Y X X

Offspring genotype: XX XX XY XY

Offspring phenotype ratio: **1 female : 1 male**

		XX	
	×	X	X
XY	X	XX	XX
	Y	XY	XY

15.9 Mutation

1. Mutation is a change in gene or chromosomal structure. Mutations that occur in gamete DNA can be passed down to the next generation.
2. Mutations that occur in normal body cells (somatic mutations) are not passed on to the next generation. However, they are responsible for certain types of cancer.
3. Spontaneous mutations can arise during the replication or repair of DNA. The DNA-replication mechanism in our cells normally has high fidelity, but occasional errors might occur.
4. Mutations can also be caused by exposure to mutagens. Mutagens are physical or chemical agents that increase the rate of mutation. Examples of mutagens are ultraviolet radiation, X-rays, radioactive particles such as gamma rays, certain chemicals such as benzene, ethidium bromide and nitrous acid.
5. Gene mutation increases the amount of genetic variation in the gene pool as it introduces new alleles. Some mutations can be favourable.

15.10 Gene mutation

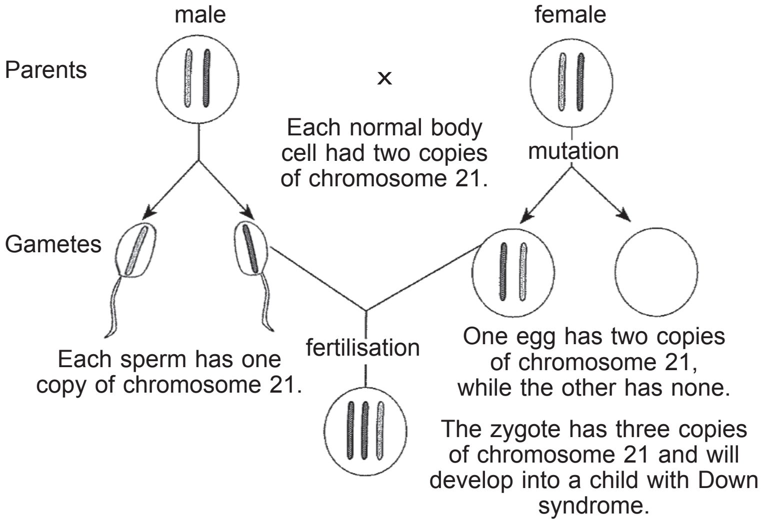
1. An example of a disease caused by gene mutation would be **sickle-cell anaemia**.
2. Sickle-cell anaemia is a blood disorder where red blood cells possess a rigid, sickle shape when oxygen concentration in the blood is low.
3. Normal red blood cells are flexible and can change their shape in order to pass through capillaries. Sickle-shaped red blood cells are not able to do so, blocking up arteries and failing to deliver blood to certain tissues, causing tissue damage.
4. Sickle-cell disease is caused by a mutation in the gene for haemoglobin production. It causes a single amino acid in the normal haemoglobin chain to be replaced by another amino acid. This causes a change in the 3-dimensional shape of the haemoglobin molecule. Hb^S molecules clump together under low oxygen concentration, causing red blood cells to become sickle-shaped.

15.11 Chromosome mutation

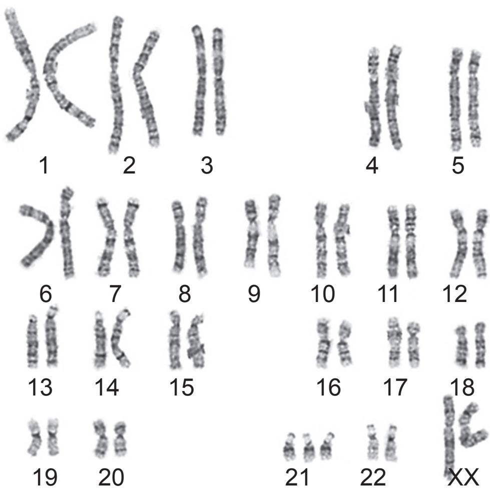
1. **Down syndrome** (trisomy 21) is a condition caused by a chromosome mutation during meiosis (gamete production). It results in mental retardation, and heart and respiratory defects.
2. The zygote inherits 3 copies of chromosome 21 instead of 2, and this mutation is present in all body cells due to mitosis during zygote development. Each body cell of the afflicted individual contains 47 chromosomes instead of the usual 46.
3. This chromosome mutation is far more likely to occur during ovum production than during sperm production.

4. Women above 30 have a higher risk of carrying babies with Down syndrome. Fetal testing is recommended for older mothers to check for Down syndrome in the embryo.

5. The genetic diagram below shows how a zygote with Down syndrome could have been produced.



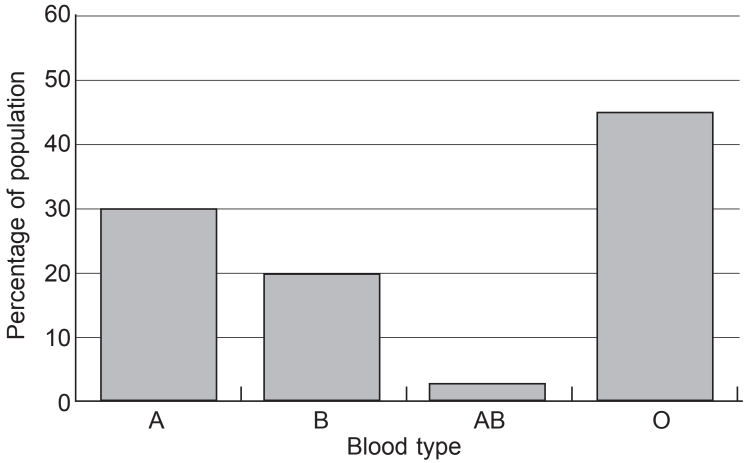
Down syndrome



Karyotype from a female with Down syndrome

15.12 Variation

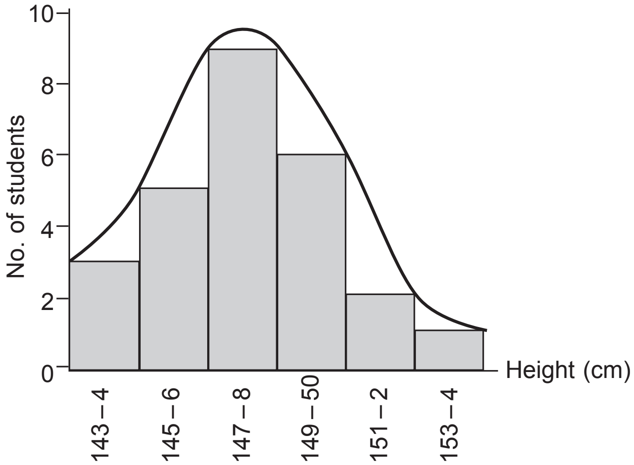
1. Genetic variations are differences in phenotypes between individuals of the same species.
2. In discontinuous variation of a characteristic, individuals possess distinct and separate phenotypes with no intermediates ('either-or' characteristics).
3. Examples of **discontinuous variation** are the flower colour in pea plants (either purple or white), ABO blood types.



Discontinuous variation

4. Discontinuous variation is controlled by alleles of a single gene or a small number of genes and is seldom affected by the environment.
5. In **continuous variation** of a characteristic, an unbroken range of phenotypes exist in the population.

6. Examples include height, skin colour, intelligence and weight, in which many intermediate phenotypes exist.



Continuous variation

7. Intermediate phenotypes are usually more common than extreme phenotypes (i.e. very tall or very short, very dark skin or very pale skin, etc), and when plotted on a graph, a bell-shaped curve is obtained.

8. Continuous variation is caused by the effect of many genes and is often affected by environmental factors.

15.13 Natural selection

1. New alleles arise in a population due to mutation. Independent assortment and crossing over of chromosomes during meiosis, and random fertilisation of gametes give rise to even more variation within the population.

2. Variation in genes results in a wide range of phenotypes in a population. These organisms compete against one another for survival.

3. However, the different varieties of organisms do not have the same chances of survival and reproduction. Some organisms possess more favourable phenotypes, and are better-suited for their environment. These organisms survive better, and reproduce, passing on their favourable traits to their offspring.

4. We say that these favourable traits are 'selected for', because they are present in a higher frequency in the next generation.

5. The differential survival of, and reproduction by organisms best fitted to the environment is known as **natural selection**.

6. **Evolution** is the change in genetic material of a population from one generation to the next. Over time, it can produce major changes in a population that could give rise to a new species.

7. Natural selection is a major mechanism by which evolution takes place because it causes helpful genes to become more common and deleterious genes to become rarer.

8. Environmental factors that act as forces of natural selection could include:

   (a) Disease – Disease-resistant phenotypes would be selected (i.e. sickle-cell trait against malaria).

   (b) Prey – Characteristics conducive to obtaining more food are selected for (e.g. Galapagos finches evolving beaks adapted to particular diets).

   (c) Predators – Methods for evading predators are selected for, such as protective colouration to provide camouflage e.g. in the peppered moth, poison glands (discourage predation), longer legs for faster running (to escape from predators), herd behaviour (animals try to get to the centre of the herd when escaping from predators because it provides protection)

   (d) Mating – Features that are more attractive to females of the same species are selected for (e.g. peacock tail, throat pouches in frigatebirds), as it increases the likelihood of finding a mate and hence reproducing.

15.14 Artificial selection

1. Artificial selection, also known as selective breeding, is the intentional breeding for particular genetic traits.

2. It is used to produce several economically important crops and animals.

3. Traits such as disease-resistance or high quality and yield of crop, tolerance to environmental pressures such as pH, salinity, drought, temperature, tolerance to insects, and tolerance to herbicides are selected for by plant breeders.

4. In animals, traits such as fast-growing, muscular, reproductively-efficient (fertile), good fat marbling (in cattle bred for meat), good milk production (in cows), and good egg production (chickens) are selected.

TOPIC 16 Organisms and their Environment

Objectives

Candidates should be able to:

(a) briefly describe the non-cyclical nature of energy flow
(b) explain the terms producer, consumer and trophic level in the context of food chains and food webs
(c) explain how energy losses occur along food chains, and discuss the efficiency of energy transfer between trophic levels
(d) describe and interpret pyramids of numbers and biomass
(e) describe how carbon is cycled within an ecosystem and outline the role of forests and oceans as carbon sinks
(f) evaluate the effects of
- water pollution by sewage and by inorganic waste
- pollution due to insecticides including bioaccumulation up food chains and impact on top carnivores

(g) outline the roles of microorganisms in sewage treatment as an example of environmental biotechnology
(h) discuss reasons for conservation of species with reference to the maintenance of biodiversity and how this is done, e.g. management of fisheries and management of timber production

16.1 Ecology

1. Ecology is the study of the interactions between organisms and the interactions of these organisms with their environment.

2. Terms related to ecology:

   (a) **Habitat** – The place where an organism lives

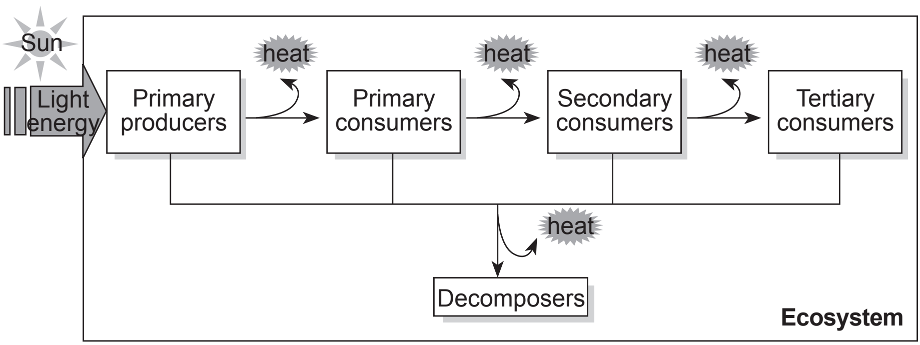
   (b) **Population** – A group of individuals of one species that live in a particular habitat

   (c) **Community** – All the organisms that live in a particular habitat. It consists of populations of organisms that live close enough to interact with one another.

   (d) **Ecosystem** – Consists of a community and its physical environment. Physical factors in the environment that the community interacts with include pH, temperature, light intensity, water and nutrient availability, oxygen / carbon dioxide availability and salinity.

16.2 Energy transfer in an ecosystem

1. A **food chain** is a sequence of energy transfer in the form of food, between organisms in an ecosystem.
2. Each level of the food chain is known as a **trophic level**.
3. **Primary producers** are photosynthetic organisms (autotrophs) that are able to convert light energy from the Sun to chemical energy that can be transferred from one organism to another within the ecosystem. They can also convert inorganic nutrients in the soil to organic nutrients that can be transferred up the food chain.
4. Consumers obtain their energy by consuming other organisms. They occupy a few trophic levels:
   (a) **Primary consumers** feed on primary producers directly. They are herbivores.
   (b) **Secondary consumers** are carnivores that eat herbivores.
   (c) **Tertiary consumers** are carnivores that eat other carnivores.
5. Food chains can be combined to form food webs since some food chains are interconnected.
6. In reality, energy flow in an ecosystem is not so direct. There are many different types of consumers that feed at different trophic levels. For example, parasites and scavengers feed on producers and consumers at every level. Decomposers (bacteria and fungi) obtain their energy from non-living organic material such as faeces, fallen leaves and dead organisms. During decomposition, nutrients from these dead organic matter are released into the soil for plants to use.
7. The flow of energy through an ecosystem:



Flow of energy through an ecosystem

8. Energy enters the ecosystem from the outside. Light energy from the Sun gets converted to chemical energy in producers during photosynthesis. Some of the energy is lost as heat during respiration and other metabolic processes. The rest gets converted into organic matter called **biomass**.

9. The energy moves up the trophic levels as producers get consumed by primary consumers, primary consumers get consumed by secondary consumers etc.

10. Energy is lost at every trophic level as heat in respiration, uneaten organism parts and through waste material. Organisms at each trophic level pass on much less energy to the next trophic level than they receive.

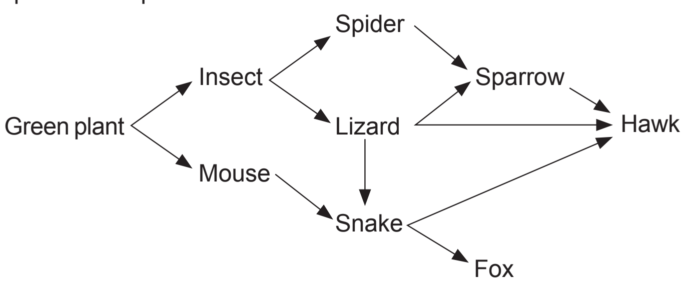
11. Food chains seldom have more than 5 trophic levels as less energy is available at the higher trophic levels.

12. Eventually, all energy supplied to the ecosystem is lost as heat. Energy has to be constantly supplied to the ecosystem from the Sun as heat cannot be recycled into useful forms of energy.

13. Example of a simple food chain:

Grass $\longrightarrow$ Grasshopper $\longrightarrow$ Frog $\longrightarrow$ Snake

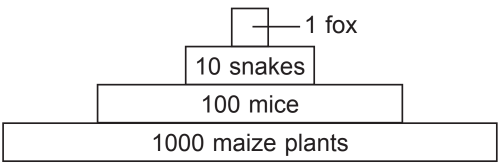
14. Example of a simple food web:



A simple food web

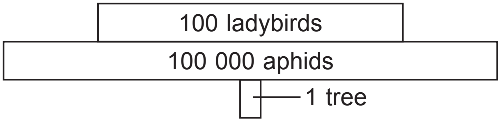
16.3 Pyramids of numbers and biomass

1. A pyramid of numbers shows the population of each trophic level in a food chain. The pyramid of numbers shown below means that at any one time in a given area, there are 1000 maize plants, 100 mice, 10 snakes and 1 fox. The size of each block in the pyramid is proportional to the number of organisms present in that level.

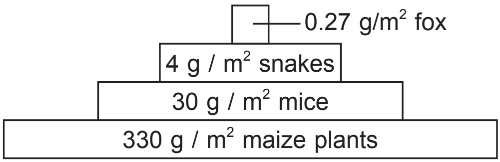


2. The pyramid of numbers can sometimes be inverted.

3. A pyramid of numbers is not an accurate estimate of the amount of energy at each trophic level because the population number does not always correspond to the amount of energy it can transfer to the next trophic level, e.g. a single tree can support a large population of aphids.

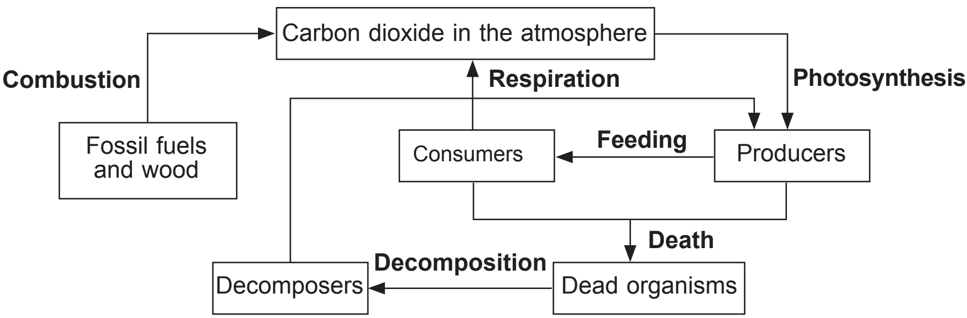


4. A pyramid of biomass shows the dry mass of organisms at each trophic level in a food chain.
5. To calculate biomass, e.g. of 1000 maize plants:
   (a) Dry 20 maize plants in an oven.
   (b) Obtain the average mass of 1 dried maize plant.
   (c) Multiply the average mass by the total number of maize plants in the given area.
6. A typical unit for a biomass pyramid is grams per square metre (g / m^2).



7. In the examples encountered in this chapter, the pyramid of biomass is upright.

16.4 Carbon cycle



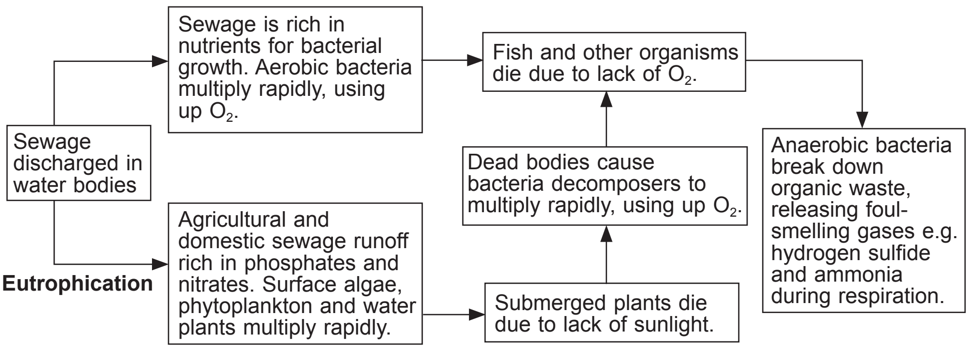
The carbon cycle

1. Green plants convert atmospheric carbon dioxide into glucose during photosynthesis. Within green plants, glucose can be converted to other organic molecules.

2. These carbon compounds are transferred to consumers through the process of feeding.
3. Carbon dioxide is returned to the atmosphere when cellular respiration takes place in living organisms.
4. When the green plants and animals die, decomposers break their organic matter down into carbon dioxide and other simple substances.
5. Fossil fuels are formed from the fossilised remains of dead plants and animals. Carbon compounds from these dead organisms are stored as fossil fuels.
6. When fossil fuels and wood are burnt, carbon dioxide is produced.

16.5 Water pollution

1. Pollution is the contamination of the environment causing harm and damage to the ecosystem. It is usually the result of human activities.
2. Water pollution occurs when pollutants are discharged directly into water without undergoing treatment. A common water pollutant is sewage.
3. Sewage is waste matter from industries and homes. It consists mainly of organic wastes such as detergents, oils and fats, insecticides and herbicides, and debris.
4. Inorganic substances from industrial waste include: leached nutrients and fertilisers (nitrates and phosphates) from farmland, ammonia, sulfur dioxide from power plants, and heavy metals.
5. Some of these pollutants can be directly toxic to the living organisms in the water, causing them to die. Others are carcinogenic and can harm humans who get in contact with the contaminated water.
6. Contaminated water usually encourages growth of microorganisms such as bacteria, parasites (certain protozoa and worms) and viruses. These could lead to diseases such as gastroenteritis, cholera, typhoid and parasitic infection.
7. Other possible outcomes:



Water pollution and eutrophication

16.6 Sewage treatment

1. Environmental biotechnology is when biotechnology is used to treat polluted environments or in environment-friendly processes such as green manufacturing technologies. Sewage treatment is an example of environmental biotechnology.
2. In sewage treatment plants, sewage is drained into settling tanks and sedimentation tanks to allow some of the solid waste to settle and be removed.
3. The sewage then enters the aeration tank, where pure oxygen is bubbled in and bacteria added. The bacteria oxidise carbon compounds to carbon dioxide, oxidise ammonium and nitrogen compounds to nitrates and eventually nitrogen gas, and remove phosphates.
4. The liquid from the aeration tank is then filtered and the solid contents are allowed to settle. Sewage water containing low levels of organic material and suspended matter remains. The sewage water is disinfected to reduce the number of microorganisms in the water before it is discharged back into the environment.
5. The solid matter left behind from the sewage treatment process is known as sludge.
6. Sludge undergoes a process of bacterial digestion to reduce the amount of organic matter and the number of disease-causing microorganisms present.

16.7 Biomagnification

1. **Biological magnification** or bioamplification is the increase in concentration of a substance up a food chain. Successive trophic levels contain high concentrations of the substance.
2. Substances that tend to accumulate up a food chain share one or more of the following characteristics:
   (a) Non-biodegradable or slow biodegradation, so it persists in the environment and can be transported by water to other areas
   (b) Cannot be broken down (detoxified) within organisms
   (c) Cannot be excreted by organisms (insoluble in water)
3. Examples of substances that biomagnify are mercury, arsenic and DDT (dichlorodiphenyltrichloroethane). DDT is a synthetic pesticide used to control mosquitoes. These chemicals are toxic, especially at high concentrations.
4. Each trophic level has to consume a larger biomass than it possesses, from the previous trophic level due to energy loss at every level. Thus, although the toxin present in the lower trophic levels might be small, larger amounts of toxins will accumulate in the higher trophic levels since each top level consumer feeds on a large amount of organisms from the trophic level below it.

5. Case study: DDT
   (a) DDT is non-biodegradable and is transported by water to far-reaching areas.
   (b) It is insoluble in water and cannot be excreted in urine which is water-based.
   (c) It is soluble in lipids and accumulates within the fatty tissues of animals. This process is called **bioaccumulation**, which is the increase in concentration of a substance due to absorption from food and the environment, in the tissues of organisms' bodies.
   (d) The concentration of DDT increases at the higher trophic levels due to **biomagnification**.
   (e) Environmental impact of DDT: DDT is toxic to aquatic life and insects. It is less toxic to mammals but causes eggshell thinning in birds. The eggs are more vulnerable to breakage during incubation, causing a drastic decline in bird reproduction rates. Birds at the top of food chains such as pelicans, ospreys and eagles are particularly affected.

   Note: Biomagnification and bioaccumulation are words that are commonly used interchangeably. However they do not have the same meaning. Bioaccumulation occurs within an organism (within a trophic level) while biomagnification occurs in a food chain (across trophic levels).

16.8 Conservation

1. Conservation is the act of protecting species, their habitats and entire ecosystems from extinction.
2. Conservation covers a wide range of activities. For example, reducing pollution and combating deforestation, preventing global warming, natural resource management and wildlife protection comes under conservation as well.

16.9 Reasons for conservation

1. Ecological value:
   (a) Organisms are interdependent.
      (i) Population fluctuations due to disruption in food chains.
      (ii) Disruption of natural cycles i.e. carbon cycle, water cycle etc.
      (iii) Existence of an organism could be directly dependent on the existence of another. Symbiotic organisms require their host species in order to survive.
   (b) Maintenance of the gene pool when there is a large population decrease in a species, there may be an increase in the chance of inbreeding which gives rise to offspring that are less adapted to environmental changes.

2. Economic value:

(a) Maintain biodiversity

(i) Plants have great medicinal value. Many drugs were derived from plants such as aspirin.

(ii) Animals and plants are both a great source of genetic diversity for plant and animal breeding programs.

(b) Resource management: It is important to manage natural resources such as timber and food sources so that it does not get depleted and we can continue exploiting it profitably.

(c) Ecotourism: It is a source of income for several countries such as Costa Rica, Madagascar, and Kenya.

3. Educational value:

(a) Conservation preserves the existence of species for future generations to study.

(b) Chemicals extracted from plants or animals might be applicable to scientific research in future.

4. Aesthetic value: Conservation preserves natural scenery and wildlife for people to appreciate.

16.10 Conservation in fisheries

1. A fishery is an area with a particular species of fish or aquatic life that is harvested for its commercial value.

2. Wild fisheries are located in the oceans, lakes and rivers, where fish has to be captured or fished. They are prone to overfishing and pollution, which could lead to an imbalance in the ecosystem.

3. Farmed fisheries involve raising fish commercially in tanks. It helps to supply some of the demand for food fish but a great majority of food fish are still obtained from wild fisheries.

4. In order to develop sustainable fisheries so that fish stock is maintained for future fishing, certain measures have been taken:

(a) Many countries have set up ministries or government organisations regulating fishing. These organisations help to control the activities in fisheries by:

(i) Imposing taxes on fishing output

(ii) Vessel licensing, regulating the entry of ships into fishing grounds

(iii) Restrictions on catching techniques such as the prohibition of bottom trawling and dynamite fishing, regulation of fish traps etc.

(iv) Imposing a catch quota

(v) Limiting the period of fishing

(b) Breeding of endangered fish in captivity by private conservation organisations or zoos to be released back into the wild to replenish depleted stock.

16.11 Conservation of forests

1. The forests are the major source of the world's timber. The clearing of forests for timber and land is called deforestation.
2. The indiscriminate logging without sufficient reforestation has led to many environmental and ecological problems such as:
   - (a) The 'slash and burn' practice used to clear forests for agriculture releases a large amount of carbon dioxide which contributes to global warming.
   - (b) Changes in the water cycle resulting in a drier climate. Trees contribute to humidity by transpiration and extract groundwater through their roots to be released into the atmosphere. The loss of this causes climate changes that could lead to desertification.
   - (c) Soil erosion as tree roots are needed to bind soil together.
   - (d) Loss of habitat for many organisms resulting in loss of biodiversity.
3. Forest conservation includes legislation protecting forests from indiscriminate logging such as:
   - (a) Regulating the rate of logging
   - (b) Selective logging where young trees are not cut down
   - (c) Designating land as forest reserves
4. Other conservation practices include reforestation, which is the act of restocking forests which have been depleted. New seedlings are planted to replace trees that have been felled.

CONTENTS

UNIT 1.1 Numbers and the Four Operations

Numbers

1. The set of natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$

2. The set of whole numbers, $W = \{0, 1, 2, 3, \ldots\}$

3. The set of integers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

4. The set of positive integers, $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$

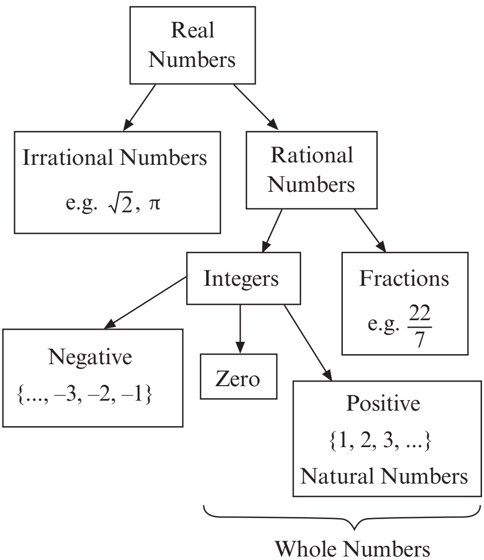
5. The set of negative integers, $\mathbb{Z}^- = \{-1, -2, -3, \ldots\}$

6. The set of rational numbers, $\mathbb{Q} = \{\frac{a}{b}, a, b \in \mathbb{Z}, b \neq 0\}$

7. An irrational number is a number which cannot be expressed in the form $\frac{a}{b}$, where a, b are integers and $b \neq 0$.

8. The set of real numbers $\mathbb{R}$ is the set of rational and irrational numbers.

9.



Example 1

The temperature at the bottom of a mountain was 22 °C and the temperature at the top was –7 °C. Find

(a) the difference between the two temperatures,

(b) the average of the two temperatures.

Solution

(a) Difference between the temperatures $= 22 - (-7)$
$= 22 + 7$
$= 29$ °C

(b) Average of the temperatures $= \frac{22+(-7)}{2}$
$= \frac{22-7}{2}$
$= \frac{15}{2}$
$= 7.5$ °C

Prime Factorisation

10. A prime number is a number that can only be divided exactly by 1 and itself. However, 1 is not considered as a prime number.
e.g. 2, 3, 5, 7, 11, 13, …

11. Prime factorisation is the process of expressing a composite number as a product of its prime factors.

Example 2

Express 30 as a product of its prime factors.

Solution

Factors of 30: 1, 2, 3, 5, 6, 10, 15, 30
Of these, 2, 3, 5 are prime factors.

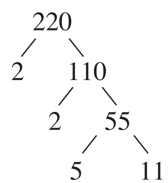
$$\begin{array}{r|l} 2 & 30 \\ \hline 3 & 15 \\ \hline 5 & 5 \\ \hline & 1 \end{array}$$

$\therefore$ $30 = 2 \times 3 \times 5$

Example 3

Express 220 as a product of its prime factors.

Solution



$\therefore$ $220 = 2^2 \times 5 \times 11$

Factors and Multiples

12. The highest common factor (HCF) of two or more numbers is the largest factor that is common to all the numbers.

Example 4

Find the highest common factor of 18 and 30.

Solution

$18 = 2 \times 3^2$
$30 = 2 \times 3 \times 5$

$\text{HCF} = 2 \times 3$
$\quad\;\; = 6$

Example 5

Find the highest common factor of 80, 120 and 280.

Solution

Method 1

$$
\begin{array}{r|lll}
2 & 80, & 120, & 280 \\ \hline
2 & 40, & 60, & 140 \\ \hline
2 & 20, & 30, & 70 \\ \hline
5 & 10, & 15, & 35 \\ \hline
 & 2, & 3, & 7
\end{array}
$$

(Since the three numbers cannot be divided further by a common prime factor, we stop here)

$\text{HCF} = 2 \times 2 \times 2 \times 5$
$\quad\;\; = 40$

Method 2

Express 80, 120 and 280 as products of their prime factors
$80 = 2^3 \times 5$
$120 = 2^3 \times 5$
$280 = 2^3 \times 5 \times 7$

$\text{HCF} = 2^3 \times 5$
$\quad\;\; = 40$

13. The lowest common multiple (LCM) of two or more numbers is the smallest multiple that is common to all the numbers.

Example 6

Find the lowest common multiple of 18 and 30.

Solution

$18 = 2 \times 3^2$
$30 = 2 \times 3 \times 5$

$$\begin{aligned} \text{LCM} &= 2 \times 3^2 \times 5 \\ &= 90 \end{aligned}$$

Example 7

Find the lowest common multiple of 5, 15 and 30.

Solution

Method 1

$$\begin{array}{r|rrr} 2 & 5, & 15, & 30 \\ \hline 3 & 5, & 15, & 15 \\ \hline 5 & 5, & 5, & 5 \\ \hline & 1, & 1, & 1 \end{array}$$

(Continue to divide by the prime factors until 1 is reached)

$$\begin{aligned} \text{LCM} &= 2 \times 3 \times 5 \\ &= 30 \end{aligned}$$

Method 2

Express 5, 15 and 30 as products of their prime factors.

$5 = 1 \times 5$
$15 = 3 \times 5$
$30 = 2 \times 3 \times 5$

$$\begin{aligned} \text{LCM} &= 2 \times 3 \times 5 \\ &= 30 \end{aligned}$$

14. A perfect square is a number whose square root is a whole number.

15. The square of a is a^2.

16. The square root of a is $\sqrt{a}$ or $a^{\frac{1}{2}}$.

Example 8

Find the square root of 256 without using a calculator.

Solution

$$
\begin{array}{r|l}
2 & 256 \\ \hline
2 & 128 \\ \hline
2 & 64 \\ \hline
2 & 32 \\ \hline
2 & 16 \\ \hline
2 & 8 \\ \hline
2 & 4 \\ \hline
2 & 2 \\ \hline
 & 1
\end{array}
$$

(Continue to divide by the prime factors until 1 is reached)

$$
\begin{aligned}
\sqrt{256} &= \sqrt{2^8} \\
&= 2^4 \\
&= 16
\end{aligned}
$$

Example 9

Given that 30k is a perfect square, write down the value of the smallest integer k.

Solution

For $2 \times 3 \times 5 \times k$ to be a perfect square, the powers of its prime factors must be in multiples of 2,
i.e. $k = 2 \times 3 \times 5$
$ = 30$

Cubes and Cube Roots

17. A perfect cube is a number whose cube root is a whole number.

18. The cube of a is a^3.

19. The cube root of a is $\sqrt[3]{a}$ or $a^{\frac{1}{3}}$.

Example 10

Find the cube root of 3375 without using a calculator.

Solution

3	3375
3	1125
3	375
5	125
5	25
5	5
	1

(Continue to divide by the prime factors until 1 is reached)

$$\begin{aligned}\sqrt[3]{3375} &= \sqrt[3]{3^3 \times 5^3} \\ &= 3 \times 5 \\ &= 15\end{aligned}$$

Reciprocal

20. The reciprocal of x is $\dfrac{1}{x}$.

21. The reciprocal of $\dfrac{x}{y}$ is $\dfrac{y}{x}$.

Significant Figures

22. All non-zero digits are significant.

23. A zero (or zeros) between non-zero digits is (are) significant.

24. In a whole number, zeros after the last non-zero digit may or may not be significant,
e.g. $7006 = 7000$ (to 1 s.f.)
$7006 = 7000$ (to 2 s.f.)
$7006 = 7010$ (to 3 s.f.)
$7436 = 7000$ (to 1 s.f.)
$7436 = 7400$ (to 2 s.f.)
$7436 = 7440$ (to 3 s.f.)

Example 11

Express 2014 correct to
(a) 1 significant figure,
(b) 2 significant figures,
(c) 3 significant figures.

Solution

(a) $2014 = \underset{\uparrow \atop \text{1 s.f.}}{2}000$ (to 1 s.f.)

(b) $2014 = \underbrace{20}_{\text{2 s.f.}}00$ (to 2 s.f.)

(c) $2014 = \underbrace{201}_{\text{3 s.f.}}0$ (to 3 s.f.)

25. In a decimal, zeros before the first non-zero digit are not significant,
e.g. 0.006 09 = 0.006 (to 1 s.f.)
0.006 09 = 0.0061 (to 2 s.f.)
6.009 = 6.01 (to 3 s.f.)

26. In a decimal, zeros after the last non-zero digit are significant.

Example 12

(a) Express 2.0367 correct to 3 significant figures.
(b) Express 0.222 03 correct to 4 significant figures.

Solution

(a) 2.0367 = 2.04
(b) $0.222\,03 = 0.\underbrace{2220}_{4\text{ s.f.}}$

Decimal Places

27. Include one extra figure for consideration. Simply drop the extra figure if it is less than 5. If it is 5 or more, add 1 to the previous figure before dropping the extra figure,
e.g. 0.7374 = 0.737 (to 3 d.p.)
5.0306 = 5.031 (to 3 d.p.)

Standard Form

28. Very large or small numbers are usually written in standard form $A \times 10^n$, where $1 \leqslant A < 10$ and n is an integer,
e.g. $1\,350\,000 = 1.35 \times 10^6$
$0.000\,875 = 8.75 \times 10^{-4}$

Example 13

The population of a country in 2012 was 4.05 million. In 2013, the population increased by 1.1×10^5. Find the population in 2013.

Solution

$$
\begin{aligned}
4.05 \text{ million} &= 4.05 \times 10^6 \\
\text{Population in 2013} &= 4.05 \times 10^6 + 1.1 \times 10^5 \\
&= 4.05 \times 10^6 + 0.11 \times 10^6 \\
&= (4.05 + 0.11) \times 10^6 \\
&= 4.16 \times 10^6
\end{aligned}
$$

Estimation

29. We can estimate the answer to a complex calculation by replacing numbers with approximate values for simpler calculation.

Example 14

Estimate the value of $\dfrac{3.49 \times \sqrt{35.7}}{35.1}$ correct to 1 significant figure.

Solution

$$
\begin{aligned}
\frac{3.49 \times \sqrt{35.7}}{35.1} &\approx \frac{3.5 \times \sqrt{36}}{35} \qquad \text{(Express each value to at least 2 s.f.)} \\
&= \frac{3.5}{35} \times \sqrt{36} \\
&= 0.1 \times 6 \\
&= 0.6 \text{ (to 1 s.f.)}
\end{aligned}
$$

Common Prefixes K M C

30.

Power of 10	Name	SI Prefix	Symbol	Numerical Value
10^{12}	trillion	tera-	T	1 000 000 000 000
10^{9}	billion	giga-	G	1 000 000 000
10^{6}	million	mega-	M	1 000 000
10^{3}	thousand	kilo-	k	1000
10^{-3}	thousandth	milli-	m	$0.001 = \frac{1}{1000}$
10^{-6}	millionth	micro-	μ	$0.000\,001 = \frac{1}{1\,000\,000}$
10^{-9}	billionth	nano-	n	$0.000\,000\,001 = \frac{1}{1\,000\,000\,000}$
10^{-12}	trillionth	pico-	p	$0.000\,000\,000\,001 = \frac{1}{1\,000\,000\,000\,000}$

Example 15

Light rays travel at a speed of 3×10^8 m/s. The distance between Earth and the sun is 32 million km. Calculate the amount of time (in seconds) for light rays to reach Earth. Express your answer to the nearest minute.

Solution

$$\begin{aligned} 48 \text{ million km} &= 48 \times 1\,000\,000 \text{ km} \\ &= 48 \times 1\,000\,000 \times 1000 \text{ m} \quad (1 \text{ km} = 1000 \text{ m}) \\ &= 48\,000\,000\,000 \text{ m} \\ &= 48 \times 10^9 \text{ m} \\ &= 4.8 \times 10^{10} \text{ m} \end{aligned}$$

$$\begin{aligned} \text{Time} &= \frac{\text{Distance}}{\text{Speed}} \\ &= \frac{4.8 \times 10^9 \text{ m}}{3 \times 10^8 \text{ m/s}} \\ &= 16 \text{ s} \end{aligned}$$

31. $a + b = b + a$ (Commutative Law)
$a \times b = b \times a$
$(p + q) + r = p + (q + r)$ (Associative Law)
$(p \times q) \times r = p \times (q \times r)$
$p \times (q + r) = p \times q + p \times r$ (Distributive Law)

32. When we have a few operations in an equation, take note of the order of operations as shown.
Step 1: Work out the expression in the **brackets** first. When there is more than 1 pair of brackets, work out the expression in the innermost brackets first.
Step 2: Calculate the **powers** and **roots**.
Step 3: **Divide** and **multiply** from left to right.
Step 4: **Add** and **subtract** from left to right.

Example 16

Calculate the value of the following.
(a) $2 + (5^2 - 4) \div 3$ **(b)** $14 - [45 - (26 + \sqrt{16})] \div 5$

Solution

(a) $2 + (5^2 - 4) \div 3 = 2 + (25 - 4) \div 3$ (Power)
$= 2 + 21 \div 3$ (Brackets)
$= 2 + 7$ (Divide)
$= 9$ (Add)

(b) $14 - [45 - (26 + \sqrt{16})] \div 5 = 14 - [45 - (26 + 4)] \div 5$ (Roots)
$= 14 - [45 - 30] \div 5$ (Innermost brackets)
$= 14 - 15 \div 5$ (Brackets)
$= 14 - 3$ (Divide)
$= 11$ (Subtract)

33. positive number × positive number = positive number
negative number × negative number = positive number
negative number × positive number = negative number
positive number ÷ positive number = positive number
negative number ÷ negative number = positive number
positive number ÷ negative number = negative number

Example 17

Simplify $(-1) \times 3 - (-3)(-2) \div (-2)$.

Solution

$$
\begin{aligned}
(-1) \times 3 - (-3)(-2) \div (-2) &= -3 - 6 \div (-2) \\
&= -3 - (-3) \\
&= 0
\end{aligned}
$$

Laws of Indices

34. Law 1 of Indices: $a^m \times a^n = a^{m+n}$

Law 2 of Indices: $a^m \div a^n = a^{m-n}$, if $a \neq 0$

Law 3 of Indices: $(a^m)^n = a^{mn}$

Law 4 of Indices: $a^n \times b^n = (a \times b)^n$

Law 5 of Indices: $a^n \div b^n = \left(\frac{a}{b}\right)^n$, if $b \neq 0$

Example 18

(a) Given that $5^{18} \div 125 = 5^k$, find k.

(b) Simplify $3 \div 6p^{-4}$.

Solution

(a)

$$
\begin{aligned}
5^{18} \div 125 &= 5^k \\
5^{18} \div 5^3 &= 5^k \\
5^{18-3} &= 5^k \\
5^{15} &= 5^k \\
\therefore k &= 15
\end{aligned}
$$

(b)

$$
\begin{aligned}
3 \div 6p^{-4} &= \frac{3}{6p^{-4}} \\
&= \frac{p^4}{2}
\end{aligned}
$$

Zero Indices

35. If a is a real number and $a \neq 0$, $a^0 = 1$.

Negative Indices

36. If a is a real number and $a \neq 0$, $a^{-n} = \dfrac{1}{a^n}$.

Fractional Indices

37. If n is a positive integer, $a^{\frac{1}{n}} = \sqrt[n]{a}$, where $a > 0$.

38. If m and n are positive integers, $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$, where $a > 0$.

Example 19

Simplify $\dfrac{3y^2}{5x} \div \dfrac{3x^2y}{20xy}$ and express your answer in positive indices.

Solution

$$\begin{aligned}
\frac{3y^2}{5x} \div \frac{3x^2y}{20xy} &= \frac{3y^2}{5x} \times \frac{20xy}{3x^2y} \\
&= \frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{5}}} y^2 x^{-1} \times \frac{\overset{4}{\cancel{20}}}{\underset{1}{\cancel{3}}} x^{-1} \\
&= 4x^{-2}y^2 \\
&= \frac{4y^2}{x^2}
\end{aligned}$$

UNIT 1.2 Ratio, Rate and Proportion

Ratio

1. The ratio of a to b, written as $a : b$, is $a \div b$ or $\frac{a}{b}$, where $b \neq 0$ and $a, b \in \mathbb{Z}^+$.

2. A ratio has no units.

Example 1

In a stationery shop, the cost of a pen is \$1.50 and the cost of a pencil is 90 cents. Express the ratio of their prices in the simplest form.

Solution

We have to first convert the prices to the same units.

$$\$1.50 = 150 \text{ cents}$$

$$\begin{aligned}\text{Price of pen : Price of pencil} &= 150 : 90 \\ &= 5 : 3\end{aligned}$$

Map Scales

3. If the linear scale of a map is $1 : r$, it means that 1 cm on the map represents r cm on the actual piece of land.

4. If the linear scale of a map is $1 : r$, the corresponding area scale of the map is $1 : r^2$.

Example 2

In the map of a town, 10 km is represented by 5 cm.

(a) What is the actual distance if it is represented by a line of length 2 cm on the map?

(b) Express the map scale in the ratio 1 : n.

(c) Find the area of a plot of land that is represented by 10 cm^2.

Solution

(a) Given that the scale is 5 cm : 10 km

= 1 cm : 2 km

Therefore, 2 cm : 4 km

(b) Since 1 cm : 2 km,

$$\left.\begin{array}{l} 1 \text{ cm} : 2000 \text{ m} \\ 1 \text{ cm} : 200\,000 \text{ cm} \end{array}\right\} \text{(Convert to the same units)}$$

Therefore, the map scale is 1 : 200 000.

(c)

1 cm : 2 km

1 cm^2 : 4 km^2

10 cm^2 : $10 \times 4 = 40$ km^2

Therefore, the area of the plot of land is 40 km^2.

Example 3

A length of 8 cm on a map represents an actual distance of 2 km. Find

(a) the actual distance represented by 25.6 cm on the map, giving your answer in km,

(b) the area on the map, in cm^2, which represents an actual area of 2.4 km^2,

(c) the scale of the map in the form 1 : n.

Solution

(a) 8 cm represent 2 km

1 cm represents $\frac{2}{8}$ km = 0.25 km

25.6 cm represents (0.25 × 25.6) km = 6.4 km

(b) 1 cm^2 represents (0.25^2) km^2 = 0.0625 km^2

0.0625 km^2 is represented by 1 cm^2

2.4 km^2 is represented by $\frac{2.4}{0.0625}$ cm^2 = 38.4 cm^2

(c) 1 cm represents 0.25 km = 25 000 cm

∴ Scale of map is 1 : 25 000

Direct Proportion

5. If y is directly proportional to x, then $y = kx$, where k is a constant and $k \neq 0$. Therefore, when the value of x increases, the value of y also increases proportionally by a constant k.

Example 4

Given that y is directly proportional to x, and $y = 5$ when $x = 10$, find y in terms of x.

Solution

Since y is directly proportional to x, we have $y = kx$.
When $x = 10$ and $y = 5$,

$$5 = k(10)$$
$$k = \frac{1}{2}$$

Hence, $y = \frac{1}{2}x$.

Example 5

2 m of wire costs \$10. Find the cost of a wire with a length of h m.

Solution

Let the length of the wire be x and the cost of the wire be y.

$$y = kx$$
$$10 = k(2)$$
$$k = 5$$

i.e. $y = 5x$

When $x = h$,

$$y = 5h$$

$\therefore$ The cost of a wire with a length of h m is \$5h.

Inverse Proportion

6. If y is inversely proportional to x, then $y = \frac{k}{x}$, where k is a constant and $k \neq 0$.

Example 6

Given that y is inversely proportional to x, and $y = 5$ when $x = 10$, find y in terms of x.

Solution

Since y is inversely proportional to x, we have

$$y = \frac{k}{x}$$

When $x = 10$ and $y = 5$,

$$5 = \frac{k}{10}$$

$$k = 50$$

Hence, $y = \dfrac{50}{x}$.

Example 7

7 men can dig a trench in 5 hours. How long will it take 3 men to dig the same trench?

Solution

Let the number of men be x and the number of hours be y.

$y = \dfrac{k}{x}$

$5 = \dfrac{k}{7}$

$k = 35$

i.e. $y = \dfrac{35}{x}$

When $x = 3$,

$y = \dfrac{35}{3}$

$= 11\dfrac{2}{3}$

$\therefore$ It will take $11\dfrac{2}{3}$ hours.

7. To attain equivalent ratios involving fractions, we have to multiply or divide the numbers of the ratio by the LCM.

Example 8

$\frac{1}{4}$ cup of sugar, $1\frac{1}{2}$ cup of flour and $\frac{5}{6}$ cup of water are needed to make a cake. Express the ratio using whole numbers.

Solution

Sugar	:	Flour	:	Water	
$\frac{1}{4}$	:	$1\frac{1}{2}$	:	$\frac{5}{6}$	
$\frac{1}{4} \times 12$	:	$1\frac{1}{2} \times 12$	:	$\frac{5}{6} \times 12$	(Multiply throughout by the LCM, which is 12)
3	:	18	:	10	

UNIT 1.3 Percentage

Percentage

1. A percentage is a fraction with denominator 100,

i.e. $x\%$ means $\dfrac{x}{100}$.

2. To convert a fraction to a percentage, multiply the fraction by 100%,

e.g. $\dfrac{3}{4} \times 100\% = 75\%$.

3. To convert a percentage to a fraction, divide the percentage by 100%,

e.g. $75\% = \dfrac{75}{100} = \dfrac{3}{4}$.

4. New value = Final percentage × Original value

5. Increase (or decrease) = Percentage increase (or decrease) × Original value

6. $$\text{Percentage increase} = \frac{\text{Increase in quantity}}{\text{Original quantity}} \times 100\%$$

$$\text{Percentage decrease} = \frac{\text{Decrease in quantity}}{\text{Original quantity}} \times 100\%$$

Example 1

A car petrol tank which can hold 60 l of petrol is spoilt and it leaks about 5% of petrol every 8 hours. What is the volume of petrol left in the tank after a whole full day?

Solution

There are 24 hours in a full day.
After the first 8 hours,

$$\text{Amount of petrol left} = \frac{95}{100} \times 60$$
$$= 57\ l$$

After the next 8 hours,

$$\text{Amount of petrol left} = \frac{95}{100} \times 57$$
$$= 54.15\ l$$

After the last 8 hours,

$$\text{Amount of petrol left} = \frac{95}{100} \times 54.15$$
$$= 51.4\ l \text{ (to 3 s.f.)}$$

Example 2

Mr Wong is a salesman. He is paid a basic salary and a year-end bonus of 1.5% of the value of the sales that he had made during the year.

(a) In 2011, his basic salary was \$2550 per month and the value of his sales was \$234 000. Calculate the total income that he received in 2011.

(b) His basic salary in 2011 was an increase of 2% of his basic salary in 2010. Find his annual basic salary in 2010.

(c) In 2012, his total basic salary was increased to \$33 600 and his total income was \$39 870.

  (i) Calculate the percentage increase in his basic salary from 2011 to 2012.

  (ii) Find the value of the sales that he made in 2012.

(d) In 2013, his basic salary was unchanged as \$33 600 but the percentage used to calculate his bonus was changed. The value of his sales was \$256 000 and his total income was \$38 720. Find the percentage used to calculate his bonus in 2013.

Solution

(a) Annual basic salary in 2011 $= \$2550 \times 12$
$= \$30\ 600$

Bonus in 2011 $= \dfrac{1.5}{100} \times \$234\ 000$
$= \$3510$

Total income in 2011 $= \$30\ 600 + \$3510$
$= \$34\ 110$

(b) Annual basic salary in 2010 $= \dfrac{100}{102} \times \$2550 \times 12$
$= \$30\ 000$

(c) **(i)** Percentage increase $= \dfrac{\$33\ 600 - \$30\ 600}{\$30\ 600} \times 100\%$
$= 9.80\%$ (to 3 s.f.)

(ii) Bonus in 2012 $= \$39\ 870 - \$33\ 600$
$= \$6270$

Sales made in 2012 $= \dfrac{\$6270}{1.5} \times 100$
$= \$418\ 000$

(d) Bonus in 2013 $= \$38\ 720 - \$33\ 600$
$= \$5120$

Percentage used $= \dfrac{\$5120}{\$256\ 000} \times 100$
$= 2\%$

UNIT 1.4 Speed

Speed

1. Speed is defined as the amount of distance travelled per unit time.

$$\text{Speed} = \frac{\text{Distance Travelled}}{\text{Time}}$$

Constant Speed

2. If the speed of an object does not change throughout the journey, it is said to be travelling at a constant speed.

Example 1

A bike travels at a constant speed of 10.0 m/s. It takes 2000 s to travel from Jurong to East Coast. Determine the distance between the two locations.

Solution

Speed: $v = 10$ m/s
Time: $t = 2000$ s
Distance: $d = vt$
$= 10 \times 2000$
$= 20\ 000$ m or 20 km

Average Speed

3. To calculate the average speed of an object, use the formula

$$\text{Average speed} = \frac{\text{Total distance travelled}}{\text{Total time taken}}.$$

Example 2

Tom travelled 105 km in 2.5 hours before stopping for lunch for half an hour. He then continued another 55 km for an hour. What was the average speed of his journey in km/h?

Solution

$$\text{Average speed} = \frac{105 + 55}{(2.5 + 0.5 + 1)}$$
$$= 40 \text{ km/h}$$

Example 3

Calculate the average speed of a spider which travels 250 m in $1\frac{1}{2}$ minutes. Give your answer in metres per second.

Solution

$1\frac{1}{2}$ min = 90 s

$$\text{Average speed} = \frac{250}{90}$$
$$= 2.78 \text{ m/s (to 3 s.f.)}$$

Conversion of Units

4. Distance:

1 m = 100 cm

1 km = 1000 m

5. Time:

1 h = 60 min

1 min = 60 s

6. Speed:

1 m/s = 3.6 km/h

Example 4

Convert 100 km/h to m/s.

Solution

$$100 \text{ km/h} = \frac{100\,000}{3600} \text{ m/s} \quad (1 \text{ km} = 1000 \text{ m})$$
$$= 27.8 \text{ m/s (to 3 s.f.)}$$

7. Area:

$1 \text{ m}^2 = 10\,000 \text{ cm}^2$

$1 \text{ km}^2 = 1\,000\,000 \text{ m}^2$

$1 \text{ hectare} = 10\,000 \text{ m}^2$

Example 5

Convert 2 hectares to cm^2.

Solution

$$2 \text{ hectares} = 2 \times 10\,000 \text{ m}^2 \quad (\text{Convert to m}^2)$$
$$= 20\,000 \times 10\,000 \text{ cm}^2 \quad (\text{Convert to cm}^2)$$
$$= 200\,000\,000 \text{ cm}^2$$

8. Volume:

$1 \text{ m}^3 = 1\,000\,000 \text{ cm}^3$

$1\ l = 1000 \text{ m}l$

$= 1000 \text{ cm}^3$

Example 6

Convert 2000 cm^3 to m^3.

Solution

Since 1 000 000 cm^3 = 1 m^3,

$$2000 \text{ cm}^3 = \frac{2000}{1\,000\,000}$$
$$= 0.002 \text{ cm}^3$$

9. Mass:

1 kg = 1000 g

1 g = 1000 mg

1 tonne = 1000 kg

Example 7

Convert 50 mg to kg.

Solution

Since 1000 mg = 1 g,

$$50 \text{ mg} = \frac{50}{1000} \text{ g} \qquad \text{(Convert to g first)}$$
$$= 0.05 \text{ g}$$

Since 1000 g = 1 kg,

$$0.05 \text{ g} = \frac{0.05}{1000} \text{ kg}$$
$$= 0.000\,05 \text{ kg}$$

UNIT 1.5 Algebraic Representation and Formulae

Number Patterns

1. A number pattern is a sequence of numbers that follows an observable pattern.

e.g. 1^{st} term, 2^{nd} term, 3^{rd} term, 4^{th} term

1 , 3 , 5 , 7 , …

n^{th} term denotes the general term for the number pattern.

2. Number patterns may have a common difference.

e.g. This is a sequence of even numbers.

$$2 \xrightarrow{+2} 4 \xrightarrow{+2} 6 \xrightarrow{+2} 8 \ldots$$

This is a sequence of odd numbers.

$$1 \xrightarrow{+2} 3 \xrightarrow{+2} 5 \xrightarrow{+2} 7 \ldots$$

This is a decreasing sequence with a common difference.

$$19 \xrightarrow{-3} 16 \xrightarrow{-3} 13 \xrightarrow{-3} 10 \xrightarrow{-3} 7 \ldots$$

3. Number patterns may have a common ratio.

e.g. This is a sequence with a common ratio.

$$1 \xrightarrow{\times 2} 2 \xrightarrow{\times 2} 4 \xrightarrow{\times 2} 8 \xrightarrow{\times 2} 16 \ldots$$

$$128 \xrightarrow{\div 2} 64 \xrightarrow{\div 2} 32 \xrightarrow{\div 2} 16 \ldots$$

4. Number patterns may be perfect squares or perfect cubes.

e.g. This is a sequence of perfect squares.

$$\overset{1^2}{1},\ \overset{2^2}{4},\ \overset{3^2}{9},\ \overset{4^2}{16},\ \overset{5^2}{25} \ldots$$

This is a sequence of perfect cubes.

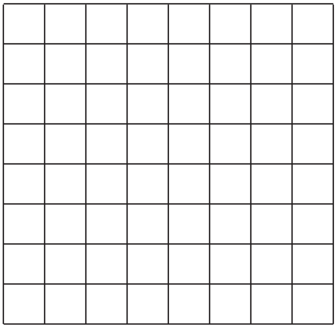
$$\overset{6^3}{216},\ \overset{5^3}{125},\ \overset{4^3}{64},\ \overset{3^3}{27},\ \overset{2^3}{8} \ldots$$

Example 1

How many squares are there on a 8 × 8 chess board?

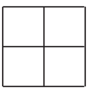
Solution

A chess board is made up of 8 × 8 squares.

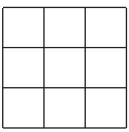


We can solve the problem by reducing it to a simpler problem.

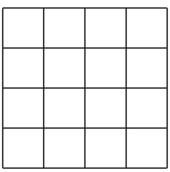
There is 1 square in a 1 × 1 square.



There are 4 + 1 squares in a 2 × 2 square.



There are 9 + 4 + 1 squares in a 3 × 3 square.



There are 16 + 9 + 4 + 1 squares in a 4 × 4 square.

Study the pattern in the table.

Size of square	Number of squares
1×1	$1 = 1^2$
2×2	$4 + 1 = 2^2 + 1^2$
3×3	$9 + 4 + 1 = 3^2 + 2^2 + 1^2$
4×4	$16 + 9 + 4 + 1 = 4^2 + 3^2 + 2^2 + 1^2$
$\vdots$	$\vdots$
8×8	$8^2 + 7^2 + 6^2 + \ldots + 1^2$

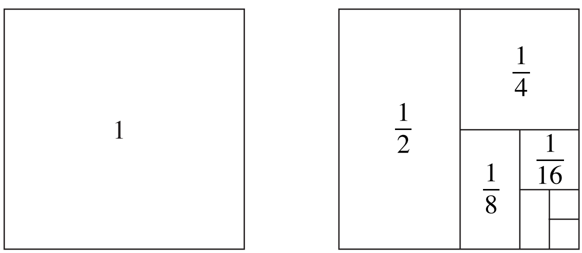
$\therefore$ The chess board has $8^2 + 7^2 + 6^2 + \ldots + 1^2 = 204$ squares.

Example 2

Find the value of $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$.

Solution

We can solve this problem by drawing a diagram.
Draw a square of side 1 unit and let its area, i.e. 1 unit^2, represent the first number in the pattern. Do the same for the rest of the numbers in the pattern by drawing another square and dividing it into the fractions accordingly.



From the diagram, we can see that $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ is the total area of the two squares, i.e. 2 units^2.

$\therefore \; 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 2$

5. Number patterns may have a combination of common difference and common ratio.

e.g. This sequence involves both a common difference and a common ratio.

$$2 \xrightarrow{+3} 5 \xrightarrow{\times 2} 10 \xrightarrow{+3} 13 \xrightarrow{\times 2} 26 \xrightarrow{+3} 29 \ldots$$

$2,\ 5,\ 10,\ 13,\ 26,\ 29 \ldots$

6. Number patterns may involve other sequences.

e.g. This number pattern involves the Fibonacci sequence.

$$0,\ 1 \xrightarrow{0+1} 1 \xrightarrow{1+1} 2 \xrightarrow{1+2} 3 \xrightarrow{2+3} 5 \xrightarrow{3+5} 8 \xrightarrow{5+8} 13 \xrightarrow{8+13} 21,\ 34,\ \ldots$$

$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ \ldots$

Example 3

The first five terms of a number pattern are 4, 7, 10, 13 and 16.

(a) What is the next term?

(b) Write down the nth term of the sequence.

Solution

(a) 19

(b) $3n + 1$

Example 4

(a) Write down the 4^{th} term in the sequence
$2, 5, 10, 17, \ldots$

(b) Write down an expression, in terms of n, for the n^{th} term in the sequence.

Solution

(a) 4^{th} term $= 26$

(b) n^{th} term $= n^2 + 1$

7.
- $kx = k \times x$ (Where k is a constant)
- $3x = 3 \times x$
  $= x + x + x$
- $x^2 = x \times x$
- $kx^2 = k \times x \times x$
- $x^2y = x \times x \times y$
- $(kx)^2 = kx \times kx$

8.
- $\dfrac{x}{y} = x \div y$
- $\dfrac{2 \pm x}{3} = (2 \pm x) \div 3$
  $= (2 \pm x) \times \dfrac{1}{3}$

Example 5

A cuboid has dimensions l cm by b cm by h cm. Find

(i) an expression for V, the volume of the cuboid,

(ii) the value of V when $l = 5$, $b = 2$ and $h = 10$.

Solution

(i) $V = l \times b \times h = lbh$

(ii) When $l = 5$, $b = 2$ and $h = 10$,

$$V = (5)(2)(10)$$
$$= 100$$

Example 6

Simplify $(y \times y + 3 \times y) \div 3$.

Solution

$$(y \times y + 3 \times y) \div 3 = (y^2 + 3y) \div 3$$
$$= \frac{y^2 + 3y}{3}$$

UNIT 1.6 Algebraic Manipulation

Expansion

1. $(a + b)^2 = a^2 + 2ab + b^2$

2. $(a - b)^2 = a^2 - 2ab + b^2$

3. $(a + b)(a - b) = a^2 - b^2$

Example 1

Expand $(2a - 3b^2 + 2)(a + b)$.

Solution

$$\begin{aligned}(2a - 3b^2 + 2)(a + b) &= (2a^2 - 3ab^2 + 2a) + (2ab - 3b^3 + 2b)\\ &= 2a^2 - 3ab^2 + 2a + 2ab - 3b^3 + 2b\end{aligned}$$

Example 2

Simplify $-8(3a - 7) + 5(2a + 3)$.

Solution

$$\begin{aligned}-8(3a - 7) + 5(2a + 3) &= -24a + 56 + 10a + 15\\ &= -14a + 71\end{aligned}$$

Example 3

Solve each of the following equations.

(a) $2(x-3)+5(x-2)=19$

(b) $\frac{2y+6}{9} - \frac{5y}{12} = 3$

Solution

(a)
$$\begin{aligned} 2(x-3)+5(x-2) &= 19 \\ 2x-6+5x-10 &= 19 \\ 7x-16 &= 19 \\ 7x &= 35 \\ x &= 5 \end{aligned}$$

(b)
$$\begin{aligned} \frac{2y+6}{9} - \frac{5y}{12} &= 3 \\ \frac{4(2y+6)-3(5y)}{36} &= 3 \\ \frac{8y+24-15y}{36} &= 3 \\ \frac{-7y+24}{36} &= 3 \\ -7y+24 &= 3(36) \\ 7y &= -84 \\ y &= -12 \end{aligned}$$

Factorisation

4. An algebraic expression may be factorised by extracting common factors,

e.g. $6a^3b - 2a^2b + 8ab = 2ab(3a^2 - a + 4)$

5. An algebraic expression may be factorised by grouping,

e.g.
$$\begin{aligned} 6a + 15ab - 10b - 9a^2 &= 6a - 9a^2 + 15ab - 10b \\ &= 3a(2-3a) + 5b(3a-2) \\ &= 3a(2-3a) - 5b(2-3a) \\ &= (2-3a)(3a-5b) \end{aligned}$$

6. An algebraic expression may be factorised by using the formula $a^2 - b^2 = (a + b)(a - b)$,
e.g. $81p^4 - 16 = (9p^2)^2 - 4^2$
$$= (9p^2 + 4)(9p^2 - 4)$$
$$= (9p^2 + 4)(3p + 2)(3p - 2)$$

7. An algebraic expression may be factorised by inspection,
e.g. $2x^2 - 7x - 15 = (2x + 3)(x - 5)$

$2x$	3	$3x$
x	-5	$-10x$
$2x^2$	-15	$-7x$

Example 4

Solve the equation $3x^2 - 2x - 8 = 0$.

Solution

$$3x^2 - 2x - 8 = 0$$
$$(x - 2)(3x + 4) = 0$$

$x - 2 = 0$ or $3x + 4 = 0$

$x = 2$ $\qquad$ $x = -\dfrac{4}{3}$

Example 5

Solve the equation $2x^2 + 5x - 12 = 0$.

Solution

$$2x^2 + 5x - 12 = 0$$
$$(2x - 3)(x + 4) = 0$$

$$2x - 3 = 0 \quad \text{or} \quad x + 4 = 0$$
$$x = \frac{3}{2} \qquad\qquad x = -4$$

Example 6

Solve $2x + x = \dfrac{12 + x}{x}$.

Solution

$$2x + 3 = \frac{12 + x}{x}$$
$$2x^2 + 3x = 12 + x$$
$$2x^2 + 2x - 12 = 0$$
$$x^2 + x - 6 = 0$$
$$(x - 2)(x + 3) = 0$$
$$\therefore\ x = 2 \text{ or } x = -3$$

x	-2	$-2x$
x	3	$3x$
x^2	-6	x

Addition and Subtraction of Fractions

8. To add or subtract algebraic fractions, we have to convert all the denominators to a common denominator.

Example 7

Express each of the following as a single fraction.

(a) $\frac{x}{3} + \frac{y}{5}$

(b) $\frac{3}{ab^3} + \frac{5}{a^2b}$

(c) $\frac{3}{x-y} + \frac{5}{y-x}$

(d) $\frac{6}{x^2-9} + \frac{3}{x-3}$

Solution

(a) $\frac{x}{3} + \frac{y}{5} = \frac{5x}{15} + \frac{3y}{15}$ (Common denominator = 15)

$$= \frac{5x+3y}{15}$$

(b) $\frac{3}{ab^3} + \frac{5}{a^2b} = \frac{3a}{a^2b^3} + \frac{5b^2}{a^2b^3}$ (Common denominator = a^2b^3)

$$= \frac{3a+5b^2}{a^2b^3}$$

(c) $\frac{3}{x-y} + \frac{5}{y-x} = \frac{3}{x-y} - \frac{5}{x-y}$ (Common denominator = $x - y$)

$$= \frac{3-5}{x-y}$$

$$= \frac{-2}{x-y}$$

$$= \frac{2}{y-x}$$

(d) $\frac{6}{x^2-9} + \frac{3}{x-3} = \frac{6}{(x+3)(x-3)} + \frac{3}{x-3}$

$$= \frac{6+3(x+3)}{(x+3)(x-3)}$$ (Common denominator = $(x + 3)(x - 3)$)

$$= \frac{6+3x+9}{(x+3)(x-3)}$$

$$= \frac{3x+15}{(x+3)(x-3)}$$

Example 8

Solve $\dfrac{4}{3b-6} + \dfrac{5}{4b-8} = 2.$

Solution

$$\frac{4}{3b-6} + \frac{5}{4b-8} = 2 \quad \text{(Common denominator} = 12(b-2))$$

$$\frac{4}{3(b-2)} + \frac{5}{4(b-2)} = 2$$

$$\frac{4(4)+5(3)}{12(b-2)} = 2$$

$$\frac{31}{12(b-2)} = 2$$

$$31 = 24(b-2)$$

$$24b = 31 + 48$$

$$b = \frac{79}{24}$$

$$= 3\frac{7}{24}$$

Multiplication and Division of Fractions

9. To multiply algebraic fractions, we have to factorise the expression before cancelling the common terms. To divide algebraic fractions, we have to invert the divisor and change the sign from ÷ to ×.

Example 9

Simplify.

(a) $\dfrac{x+y}{3x-3y} \times \dfrac{2x-2y}{5x+5y}$

(b) $\dfrac{6p^3}{7qr} \div \dfrac{12p}{21q^2}$

(c) $\dfrac{x+y}{2x-y} \div \dfrac{2x+2y}{4x-2y}$

Solution

(a) $\dfrac{x+y}{3x-3y} \times \dfrac{2x-2y}{5x+5y} = \dfrac{x+y}{3(x-y)} \times \dfrac{2(x-y)}{5(x+y)}$

$$= \frac{1}{3} \times \frac{2}{5}$$

$$= \frac{2}{15}$$

(b) $\dfrac{6p^3}{7qr} \div \dfrac{12p}{21q^2} = \dfrac{6p^3}{7qr} \times \dfrac{21q^2}{12p}$

$$= \frac{3p^2q}{2r}$$

(c) $\dfrac{x+y}{2x-y} \div \dfrac{2x+2y}{4x-2y} = \dfrac{x+y}{2x-y} \times \dfrac{4x-2y}{2x+2y}$

$$= \frac{x+y}{2x-y} \times \frac{2(2x-y)}{2(x+y)}$$

$$= 1$$

Changing the Subject of a Formula

10. The subject of a formula is the variable which is written explicitly in terms of other given variables.

Example 10

Make t the subject of the formula, $v = u + at$.

Solution

To make t the subject of the formula,

$v - u = at$

$$t = \frac{v-u}{a}$$

Example 11

Given that the volume of the sphere is $V = \frac{4}{3}\pi r^3$, express r in terms of V.

Solution

$$V = \frac{4}{3}\pi r^3$$

$$r^3 = \frac{3}{4\pi}V$$

$$r = \sqrt[3]{\frac{3V}{4\pi}}$$

Example 12

Given that $T = 2\pi\sqrt{\frac{L}{g}}$, express L in terms of π, T and g.

Solution

$$T = 2\pi\sqrt{\frac{L}{g}}$$

$$\frac{T}{2\pi} = \sqrt{\frac{L}{g}}$$

$$\frac{T^2}{4\pi^2} = \frac{L}{g}$$

$$L = \frac{gT^2}{4\pi^2}$$

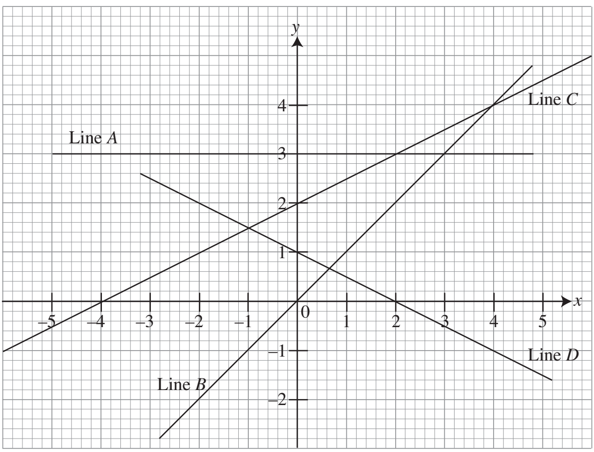
UNIT 1.7 Functions and Graphs

Linear Graphs

1. The equation of a straight line is given by $y = mx + c$, where m = gradient of the straight line and c = y-intercept.
2. The gradient of the line (usually represented by m) is given as
$m = \dfrac{\text{vertical change}}{\text{horizontal change}}$ or $\dfrac{\text{rise}}{\text{run}}$.

Example 1

Find the m (gradient), c (y-intercept) and equations of the following lines.



For Line A:
The line cuts the y-axis at $y = 3$. $\therefore$ $c = 3$.
Since Line A is a horizontal line, the vertical change = 0. $\therefore$ $m = 0$.

For Line B:
The line cuts the y-axis at $y = 0$. $\therefore$ $c = 0$.
Vertical change = 1, horizontal change = 1.

$\therefore$ $m = \dfrac{1}{1} = 1$

For Line C:
The line cuts the y-axis at $y = 2$. $\therefore$ $c = 2$.
Vertical change = 2, horizontal change = 4.

$\therefore$ $m = \dfrac{2}{4} = \dfrac{1}{2}$

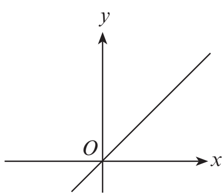
For Line D:
The line cuts the y-axis at $y = 1$. $\therefore$ $c = 1$.
Vertical change = 1, horizontal change = –2.

$\therefore$ $m = \dfrac{1}{-2} = -\dfrac{1}{2}$

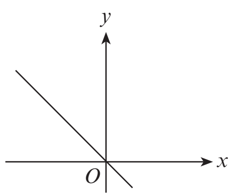
Line	$\boldsymbol{m}$	$\boldsymbol{c}$	**Equation**
A	0	3	$y = 3$
B	1	0	$y = x$
C	$\dfrac{1}{2}$	2	$y = \dfrac{1}{2}x + 2$
D	$-\dfrac{1}{2}$	1	$y = -\dfrac{1}{2}x + 1$

Graphs of $y = ax^n$

3. Graphs of $y = ax$

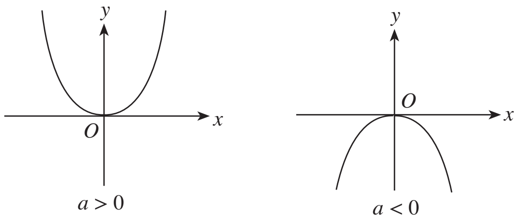


$a > 0$

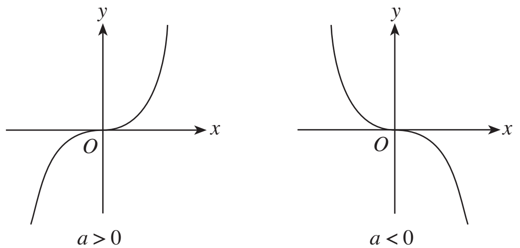


$a < 0$

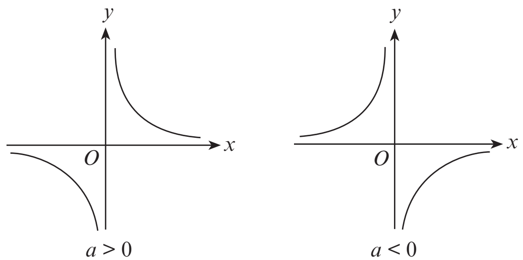
4. Graphs of $y = ax^2$



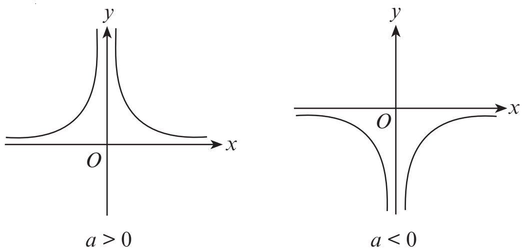
5. Graphs of $y = ax^3$



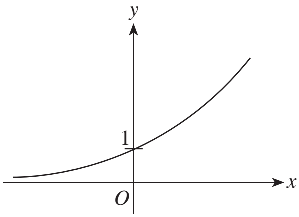
6. Graphs of $y = \dfrac{a}{x}$



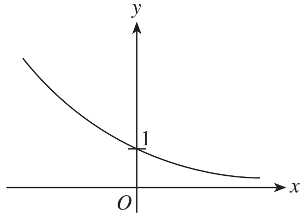
7. Graphs of $y = \dfrac{a}{x^2}$



8. Graphs of $y = a^x$



$a > 1$

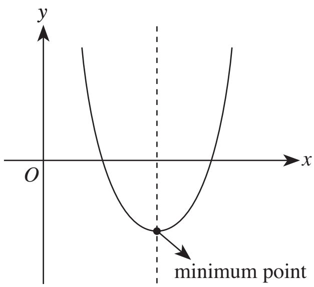


$0 < a < 1$

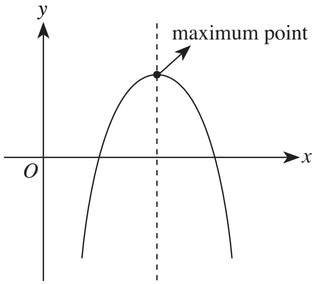
Graphs of Quadratic Functions

9. A graph of a quadratic function may be in the forms $y = ax^2 + bx + c$, $y = \pm(x - p)^2 + q$ and $y = \pm(x - h)(x - k)$.

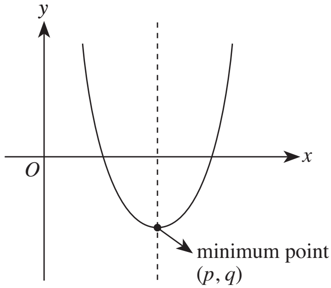
10. If $a > 0$, the quadratic graph has a minimum point.



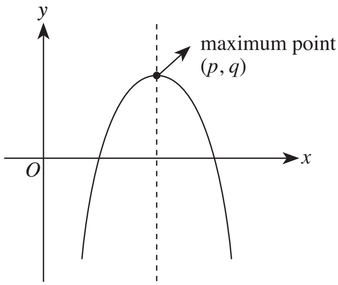
11. If $a < 0$, the quadratic graph has a maximum point.



12. If the quadratic function is in the form $y = (x - p)^2 + q$, it has a minimum point at (p, q).



13. If the quadratic function is in the form $y = -(x - p)^2 + q$, it has a maximum point at (p, q).



14. To find the x-intercepts, let $y = 0$.
To find the y-intercept, let $x = 0$.

15. To find the gradient of the graph at a given point, draw a tangent at the point and calculate its gradient.

16. The line of symmetry of the graph in the form $y = \pm(x - p)^2 + q$ is $x = p$.

17. The line of symmetry of the graph in the form $y = \pm(x - h)(x - k)$ is $x = \dfrac{h + k}{2}$.

Graph Sketching

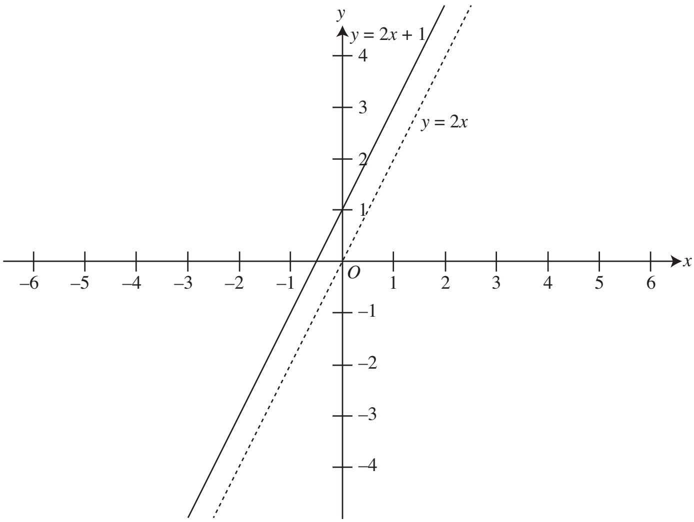
18. To sketch linear graphs with equations such as $y = mx + c$, shift the graph $y = mx$ upwards by c units.

Example 2

Sketch the graph of $y = 2x + 1$.

Solution

First, sketch the graph of $y = 2x$.
Next, shift the graph upwards by 1 unit.



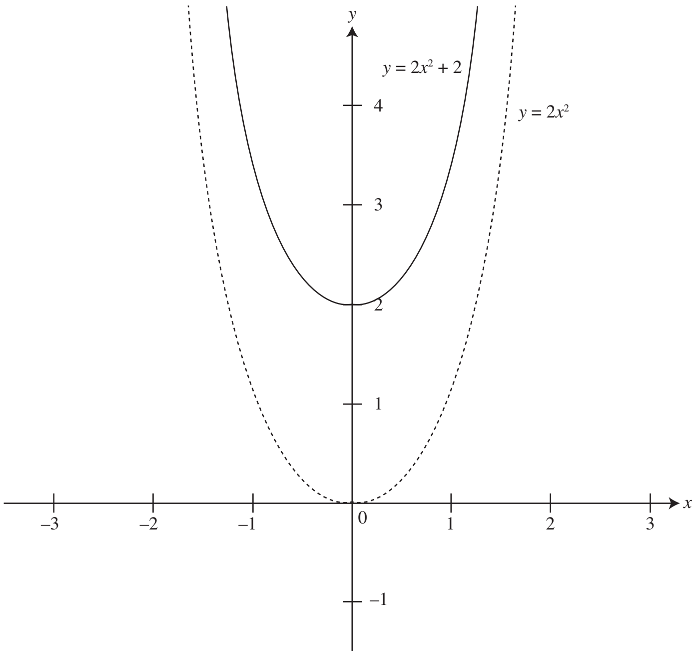
19. To sketch $y = ax^2 + b$, shift the graph $y = ax^2$ upwards by b units.

Example 3

Sketch the graph of $y = 2x^2 + 2$.

Solution

First, sketch the graph of $y = 2x^2$.
Next, shift the graph upwards by 2 units.



Graphical Solution of Equations

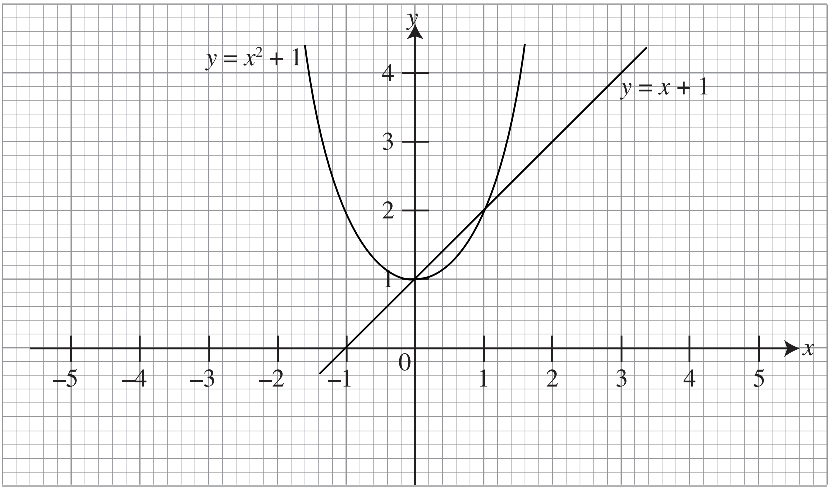
20. Simultaneous equations can be solved by drawing the graphs of the equations and reading the coordinates of the point(s) of intersection.

Example 4

Draw the graph of $y = x^2 + 1$. Hence, find the solutions to $x^2 + 1 = x + 1$.

Solution

x	–2	–1	0	1	2
y	5	2	1	2	5



Plot the straight line graph $y = x + 1$ in the axes above,
The line intersects the curve at $x = 0$ and $x = 1$.
When $x = 0$, $y = 1$.
When $x = 1$, $y = 2$.

$\therefore$ The solutions are $(0, 1)$ and $(1, 2)$.

Example 5

The table below gives some values of x and the corresponding values of y, correct to two decimal places, where $y = x(2 + x)(3 - x)$.

x	–2	–1.5	–1	–0.5	0	0.5	1	1.5	2	2.5	3
y	0	–3.38	–4	–2.63	0	3.13	p	7.88	8	5.63	q

(a) Find the value of p and of q.

(b) Using a scale of 2 cm to represent 1 unit, draw a horizontal x-axis for $-2 \leqslant x \leqslant 4$. Using a scale of 1 cm to represent 1 unit, draw a vertical y-axis for $-4 \leqslant y \leqslant 10$. On your axes, plot the points given in the table and join them with a smooth curve.

(c) Using your graph, find the values of x for which $y = 3$.

(d) By drawing a tangent, find the gradient of the curve at the point where $x = 2$.

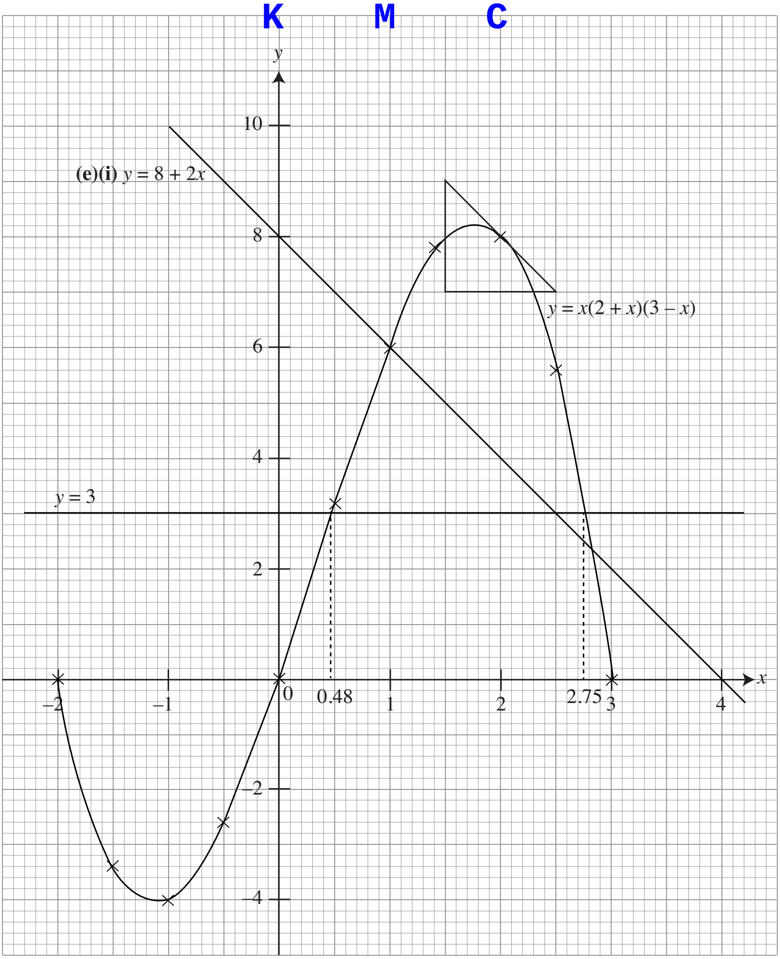
(e) **(i)** On the same axes, draw the graph of $y = 8 - 2x$ for values of x in the range $-1 \leqslant x \leqslant 4$.

(ii) Write down and simplify the cubic equation which is satisfied by the values of x at the points where the two graphs intersect.

Solution

(a) $p = 1(2 + 1)(3 - 1)$
$\quad = 6$
$q = 3(2 + 3)(3 - 3)$
$\quad = 0$

(b)



(c) From the graph, $x = 0.48$ and 2.75 when $y = 3$.

(d) Gradient of tangent $= \dfrac{7-9}{2.5-1.5}$

$= -2$

(e) **(ii)**

$$x(2 + x)(3 - x) = 8 - 2x$$
$$x(6 + x - x^2) = 8 - 2x$$
$$6x + x^2 - x^3 = 8 - 2x$$
$$x^3 - x^2 - 8x + 8 = 0$$

UNIT 1.8 Solutions of Equations and Inequalities

Quadratic Formula

1. To solve the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, use the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example 1

Solve the equation $3x^2 - 3x - 2 = 0$.

Solution

Determine the values of a, b and c.

$a = 3, b = -3, c = -2$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(3)(-2)}}{2(3)}$$

$$= \frac{3 \pm \sqrt{9 - (-24)}}{6}$$

$$= \frac{3 \pm \sqrt{33}}{6}$$

$= 1.46$ or -0.457 (to 3 s. f.)

Example 2

Using the quadratic formula, solve the equation $5x^2 + 9x - 4 = 0$.

Solution

In $5x^2 + 9x - 4 = 0$, $a = 5$, $b = 9$, $c = -4$.

$$x = \frac{-9 \pm \sqrt{9^2 - 4(5)(-4)}}{2(5)}$$

$$= \frac{-9 \pm \sqrt{161}}{10}$$

$= 0.369$ or -2.17 (to 3 s.f.)

Example 3

Solve the equation $6x^2 + 3x - 8 = 0$.

Solution

In $6x^2 + 3x - 8 = 0$, $a = 6$, $b = 3$, $c = -8$.

$$x = \frac{-3 \pm \sqrt{3^2 - 4(6)(-8)}}{2(6)}$$

$$= \frac{-3 \pm \sqrt{201}}{12}$$

$= 0.931$ or -1.43 (to 3 s.f.)

Example 4

Solve the equation $\frac{1}{x+8} + \frac{3}{x-6} = 10$.

Solution

$$\frac{1}{x+8} + \frac{3}{x-6} = 10$$

$$\frac{(x-6)+3(x+8)}{(x+8)(x-6)} = 10$$

$$\frac{x-6+3x+24}{(x+8)(x-6)} = 10$$

$$\frac{4x+18}{(x+8)(x-6)} = 10$$

$$4x + 18 = 10(x + 8)(x - 6)$$

$$4x + 18 = 10(x^2 + 2x - 48)$$

$$2x + 9 = 10x^2 + 20x - 480$$

$$2x + 9 = 5(x^2 + 2x - 48)$$

$$2x + 9 = 5x^2 + 10x - 240$$

$$5x^2 + 8x - 249 = 0$$

$$x = \frac{-8 \pm \sqrt{8^2 - 4(5)(-249)}}{2(5)}$$

$$= \frac{-8 \pm \sqrt{5044}}{10}$$

$$= 6.30 \text{ or } -7.90 \text{ (to 3 s.f.)}$$

Example 5

A cuboid has a total surface area of 250 cm^2. Its width is 1 cm shorter than its length, and its height is 3 cm longer than the length. What is the length of the cuboid?

Solution

Let x represent the length of the cuboid.
Let $x - 1$ represent the width of the cuboid.
Let $x + 3$ represent the height of the cuboid.

$$\begin{aligned}\text{Total surface area} &= 2(x)(x + 3) + 2(x)(x - 1) + 2(x + 3)(x - 1)\\ 250 &= 2(x^2 + 3x) + 2(x^2 - x) + 2(x^2 + 2x - 3) \quad \text{(Divide the equation by 2)}\\ 125 &= x^2 + 3x + x^2 - x + x^2 + 2x - 3\\ 3x^2 + 4x - 128 &= 0\\ x &= \frac{-4 \pm \sqrt{4^2 - 4(3)(-128)}}{2(3)}\\ &= 5.90 \text{ (to 3 s.f.) or } -7.23 \text{ (rejected)}\end{aligned}$$

(Reject –7.23 since the length cannot be less than 0)

$\therefore$ The length of the cuboid is 5.90 cm.

Completing the Square

2. To solve the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$:

Step 1: Change the coefficient of x^2 to 1,

i.e. $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Step 2: Bring $\frac{c}{a}$ to the right side of the equation,

i.e. $x^2 + \frac{b}{a}x = -\frac{c}{a}$

Step 3: Divide the coefficient of x by 2 and add the square of the result to both sides of the equation,

i.e. $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$

Step 4: Factorise and simplify.

i.e. $$\left(x+\frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$= \frac{b^2-4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$= \pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Example 6

Solve the equation $x^2 - 4x - 8 = 0$.

Solution

$$x^2 - 4x - 8 = 0$$
$$x^2 - 2(2)(x) + 2^2 = 8 + 2^2$$
$$(x-2)^2 = 12$$
$$x - 2 = \pm\sqrt{12}$$
$$x = \pm\sqrt{12} + 2$$
$$= 5.46 \text{ or } -1.46 \text{ (to 3 s.f.)}$$

Example 7

Using the method of completing the square, solve the equation $2x^2 + x - 6 = 0$.

Solution

$$2x^2 + x - 6 = 0$$
$$x^2 + \frac{1}{2}x - 3 = 0$$
$$x^2 + \frac{1}{2}x = 3$$
$$x^2 + \frac{1}{2}x + \left(\frac{1}{4}\right)^2 = 3 + \left(\frac{1}{4}\right)^2$$
$$\left(x + \frac{1}{4}\right)^2 = \frac{49}{16}$$
$$x + \frac{1}{4} = \pm\frac{7}{4}$$
$$x = -\frac{1}{4} \pm \frac{7}{4}$$
$$= 1.5 \text{ or } -2$$

Example 8

Solve the equation $2x^2 - 8x - 24$ by completing the square.

Solution

$$2x^2 - 8x - 24 = 0$$
$$x^2 - 4x - 12 = 0$$
$$x^2 - 2(2)x + 2^2 = 12 + 2^2$$
$$(x - 2)^2 = 16$$
$$x - 2 = \pm 4$$
$$x = 6 \text{ or } -2$$

3. **Elimination method** is used by making the coefficient of one of the variables in the two equations the same. Either add or subtract to form a single linear equation of only one unknown variable.

Example 9

Solve the simultaneous equations

$$2x + 3y = 15,$$
$$-3y + 4x = 3.$$

Solution

$2x + 3y = 15$ —— (1)
$-3y + 4x = 3$ —— (2)
(1) + (2):
$(2x + 3y) + (-3y + 4x) = 18$
$6x = 18$
$x = 3$
Substitute $x = 3$ into (1):
$2(3) + 3y = 15$
$3y = 15 - 6$
$y = 3$
$\therefore\ x = 3, y = 3$

Example 10

Using the method of elimination, solve the simultaneous equations

$$5x + 2y = 10,$$
$$4x + 3y = 1.$$

Solution

$5x + 2y = 10$ —— (1)

$4x + 3y = 1$ —— (2)

(1) × 3: $15x + 6y = 30$ —— (3)

(2) × 2: $8x + 6y = 2$ —— (4)

(3) – (4): $7x = 28$

$x = 4$

Substitute $x = 4$ into (2):

$4(4) + 3y = 1$

$16 + 3y = 1$

$3y = -15$

$y = -5$

$\therefore x = 4, y = -5$

4. **Substitution method** is used when we make one variable the subject of an equation and then we substitute that into the other equation to solve for the other variable.

Example 11

Solve the simultaneous equations

$$2x - 3y = -2,$$
$$y + 4x = 24.$$

Solution

$2x - 3y = -2$ —— (1)
$y + 4x = 24$ —— (2)

From (1),

$$x = \frac{-2 + 3y}{2}$$
$$= -1 + \frac{3}{2}y \text{ —— (3)}$$

Substitute (3) into (2):

$$y + 4\left(-1 + \frac{3}{2}y\right) = 24$$
$$y - 4 + 6y = 24$$
$$7y = 28$$
$$y = 4$$

Substitute $y = 4$ into (3):

$$x = -1 + \frac{3}{2}y$$
$$= -1 + \frac{3}{2}(4)$$
$$= -1 + 6$$
$$= 5$$

$\therefore x = 5, y = 4$

Example 12

Using the method of substitution, solve the simultaneous equations

$$5x + 2y = 10,$$
$$4x + 3y = 1.$$

Solution

$5x + 2y = 10$ —— (1)
$4x + 3y = 1$ —— (2)
From (1),
$2y = 10 - 5x$

$$y = \frac{10 - 5x}{2} \quad \text{—— (3)}$$

Substitute (3) into (2):

$$4x + 3\left(\frac{10 - 5x}{2}\right) = 1$$

$$8x + 3(10 - 5x) = 2$$
$$8x + 30 - 15x = 2$$
$$7x = 28$$
$$x = 4$$

Substitute $x = 4$ into (3):

$$y = \frac{10 - 5(4)}{2}$$
$$= -5$$

$\therefore\ x = 4, y = -5$

5. To solve an inequality, we multiply or divide both sides by a positive number without having to reverse the inequality sign,

i.e. if $x \geqslant y$ and $c > 0$, then $cx \geqslant cy$ and $\frac{x}{c} \geqslant \frac{y}{c}$

and if $x > y$ and $c > 0$, then $cx > cy$ and $\frac{x}{c} > \frac{y}{c}$.

6. To solve an inequality, we reverse the inequality sign if we multiply or divide both sides by a negative number,

i.e. if $x \geqslant y$ and $d < 0$, then $dx \leqslant dy$ and $\frac{x}{d} \leqslant \frac{y}{d}$

and if $x > y$ and $d < 0$, then $dx < dy$ and $\frac{x}{d} < \frac{y}{d}$.

Solving Inequalities

7. To solve linear inequalities such as $px + q < r$ whereby $p \neq 0$,

$x < \frac{r-q}{p}$, if $p > 0$

$x > \frac{r-q}{p}$, if $p < 0$

Example 13

Solve the inequality $2 - 5x \geqslant 3x - 14$.

Solution

$$\begin{aligned} 2 - 5x &\geqslant 3x - 14 \\ -5x - 3x &\geqslant -14 - 2 \\ -8x &\geqslant -16 \\ x &\leqslant 2 \end{aligned}$$

Example 14

Given that $\frac{x+3}{5} + 1 < \frac{x+2}{2} - \frac{x+1}{4}$, find

(a) the least integer value of x,

(b) the smallest value of x such that x is a prime number.

Solution

$$\frac{x+3}{5} + 1 < \frac{x+2}{2} - \frac{x+1}{4}$$

$$\frac{x+3+5}{5} < \frac{2(x+2)-(x+1)}{4}$$

$$\frac{x+8}{5} < \frac{2x+4-x-1}{4}$$

$$\frac{x+8}{5} < \frac{x+3}{4}$$

$$4(x+8) < 5(x+3)$$

$$4x + 32 < 5x + 15$$

$$-x < -17$$

$$x > 17$$

(a) The least integer value of x is 18.

(b) The smallest value of x such that x is a prime number is 19.

Example 15

Given that $1 \leqslant x \leqslant 4$ and $3 \leqslant y \leqslant 5$, find

(a) the largest possible value of $y^2 - x$,

(b) the smallest possible value of

(i) $\dfrac{y^2}{x}$,

(ii) $(y - x)^2$.

Solution

(a) Largest possible value of $y^2 - x = 5^2 - 1^2$

$$= 25 - 1$$
$$= 24$$

(b) **(i)** Smallest possible value of $\dfrac{y^2}{x} = \dfrac{3^2}{4}$

$$= 2\frac{1}{4}$$

(ii) Smallest possible value of $(y - x)^2 = (3 - 3)^2$

$$= 0$$

UNIT 1.9 Applications of Mathematics in Practical Situations

Hire Purchase

1. Expensive items may be paid through hire purchase, where the full cost is paid over a given period of time. The hire purchase price is usually greater than the actual cost of the item. Hire purchase price comprises an initial deposit and regular instalments. Interest is charged along with these instalments.

Example 1

A sofa set costs \$4800 and can be bought under a hire purchase plan. A 15% deposit is required and the remaining amount is to be paid in 24 monthly instalments at a simple interest rate of 3% per annum. Find

(i) the amount to be paid in instalment per month,

(ii) the percentage difference in the hire purchase price and the cash price.

Solution

(i) Deposit $= \frac{15}{100} \times \$4800$

$= \$720$

Remaining amount $= \$4800 - \$720$

$= \$4080$

Amount of interest to be paid in 2 years $= \frac{3}{100} \times \$4080 \times 2$ (24 months = 2 years)

$= \$244.80$

Total amount to be paid in monthly instalments $= \$4080 + \$244.80$

$= \$4324.80$

Amount to be paid in instalment per month $= \$4324.80 \div 24$

$= \$180.20$

\$180.20 has to be paid in instalment per month.

(ii) Hire purchase price = \$720 + \$4324.80
= \$5044.80

$$\text{Percentage difference} = \frac{\text{Hire purchase price} - \text{Cash price}}{\text{Cash price}} \times 100\%$$

$$= \frac{5044.80 - 4800}{4800} \times 100\%$$

$$= 5.1\%$$

∴ The percentage difference in the hire purchase price and cash price is 5.1%.

Simple Interest

2. To calculate simple interest, we use the formula

$$I = \frac{PRT}{100},$$

where I = simple interest,
P = principal amount,
R = rate of interest per annum,
T = period of time in years.

Example 2

A sum of \$500 was invested in an account which pays simple interest per annum. After 4 years, the amount of money increased to \$550. Calculate the interest rate per annum.

Solution

$$\frac{500(R)4}{100} = 50$$

$$R = 2.5$$

The interest rate per annum is 2.5%.

Compound Interest

3. Compound interest is the interest accumulated over a given period of time at a given rate when each successive interest payment is added to the principal amount.

4. To calculate compound interest, we use the formula

$$A = P\left(1 + \frac{R}{100}\right)^n,$$

where A = total amount after n units of time,
P = principal amount,
R = rate of interest per unit time,
n = number of units of time.

Example 3

Yvonne deposits \$5000 in a bank account which pays 4% per annum compound interest. Calculate the total interest earned in 5 years, correct to the nearest dollar.

Solution

$$\begin{aligned}\text{Interest earned} &= P\left(1 + \frac{R}{100}\right)^n - P\\ &= 5000\left(1 + \frac{4}{100}\right)^5 - 5000\\ &= \$1083\end{aligned}$$

Example 4

Brian wants to place $10 000 into a fixed deposit account for 3 years. Bank X offers a simple interest rate of 1.2% per annum and Bank Y offers an interest rate of 1.1% compounded annually. Which bank should Brian choose to yield a better interest?

Solution

Interest offered by Bank X: $I = \frac{PRT}{100}$

$$= \frac{(10\,000)(1.2)(3)}{100}$$

$$= \$360$$

Interest offered by Bank Y: $I = P\left(1+\frac{R}{100}\right)^n - P$

$$= 10\,000\left(1+\frac{1.1}{100}\right)^3 - 10\,000$$

$$= \$333.64 \text{ (to 2 d.p.)}$$

$\therefore$ Brian should choose Bank X.

Money Exchange

5. To change local currency to foreign currency, a given unit of the local currency is multiplied by the exchange rate.
e.g. To change Singapore dollars to foreign currency,
Foreign currency = Singapore dollars × Exchange rate

Example 5

Mr Lim exchanged S$800 for Australian dollars. Given S$1 = A$0.9611, how much did he receive in Australian dollars?

Solution

$800 \times 0.9611 = 768.88$

Mr Lim received A$768.88.

Example 6

A tourist wanted to change S$500 into Japanese Yen. The exchange rate at that time was ¥100 = S$1.0918. How much will he receive in Japanese Yen?

Solution

$$\frac{500}{1.0918} \times 100 = 45\ 795.933$$

$$\approx 45\ 795.93$$ (Always leave answers involving money to the nearest cent unless stated otherwise)

He will receive ¥45 795.93.

6. To convert foreign currency to local currency, a given unit of the foreign currency is divided by the exchange rate.
e.g. To change foreign currency to Singapore dollars,
Singapore dollars = Foreign currency ÷ Exchange rate

Example 7

Sarah buys a dress in Thailand for 200 Baht. Given that S$1 = 25 Thai baht, how much does the dress cost in Singapore dollars?

Solution

$200 \div 25 = 8$

The dress costs S$8.

Profit and Loss

7. Profit = Selling price – Cost price

8. Loss = Cost price – Selling price

Example 8

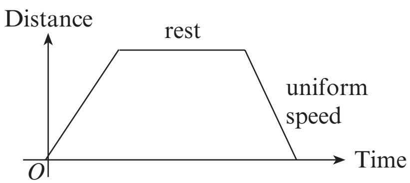
Mrs Lim bought a piece of land and sold it a few years later. She sold the land at \$3 million at a loss of 30%. How much did she pay for the land initially? Give your answer correct to 3 significant figures.

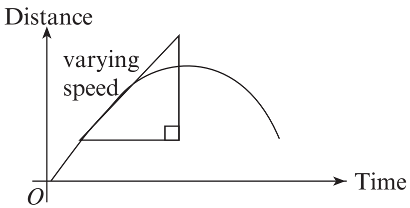
Solution

$3\,000\,000 \times \dfrac{100}{70} = 4\,290\,000$ (to 3 s.f.)

Mrs Lim initially paid \$4 290 000 for the land.

Distance-Time Graphs





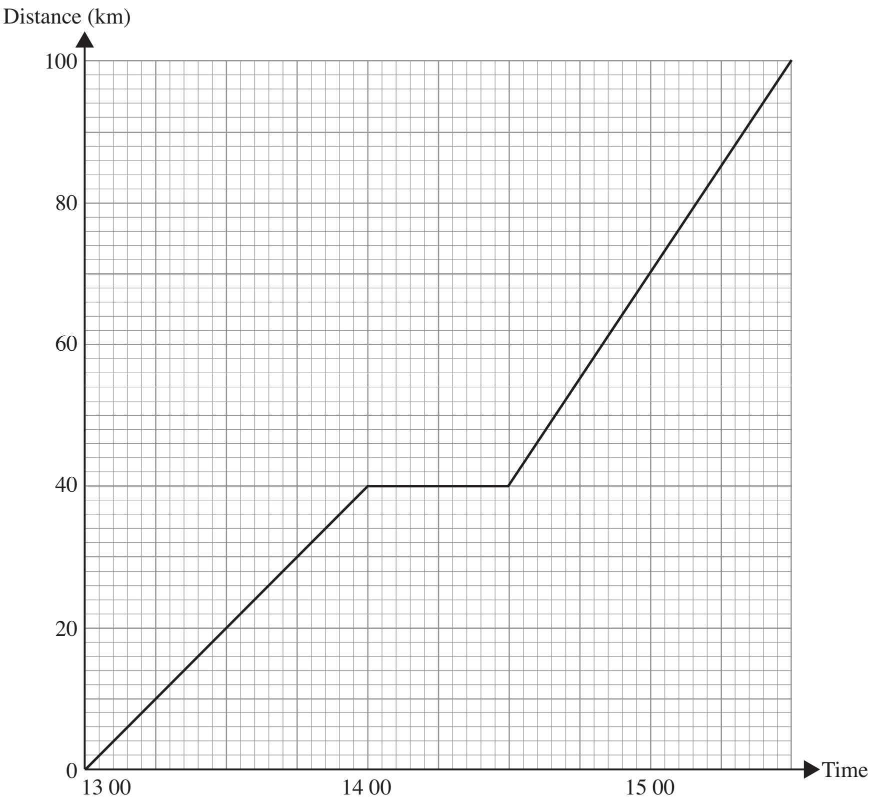
9. The gradient of a distance-time graph gives the speed of the object.

10. A straight line indicates motion with uniform speed.
A curve indicates motion with varying speed.
A straight line parallel to the time-axis indicates that the object is stationary.

11. Average speed $= \dfrac{\text{Total distance travelled}}{\text{Total time taken}}$

Example 9

The following diagram shows the distance-time graph for the journey of a motorcyclist.



(a) What was the distance covered in the first hour?

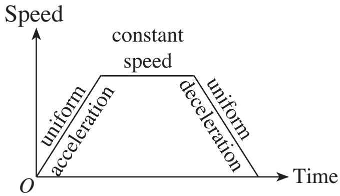
(b) Find the speed in km/h during the last part of the journey.

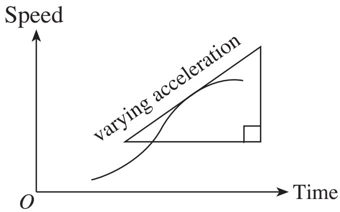
Solution

(a) Distance = 40 km

(b) Speed $= \dfrac{100 - 40}{1}$

$= 60$ km/h





12. The gradient of a speed-time graph gives the acceleration of the object.

13. A straight line indicates motion with uniform acceleration.
A curve indicates motion with varying acceleration.
A straight line parallel to the time-axis indicates that the object is moving with uniform speed.

14. Total distance covered in a given time = Area under the graph

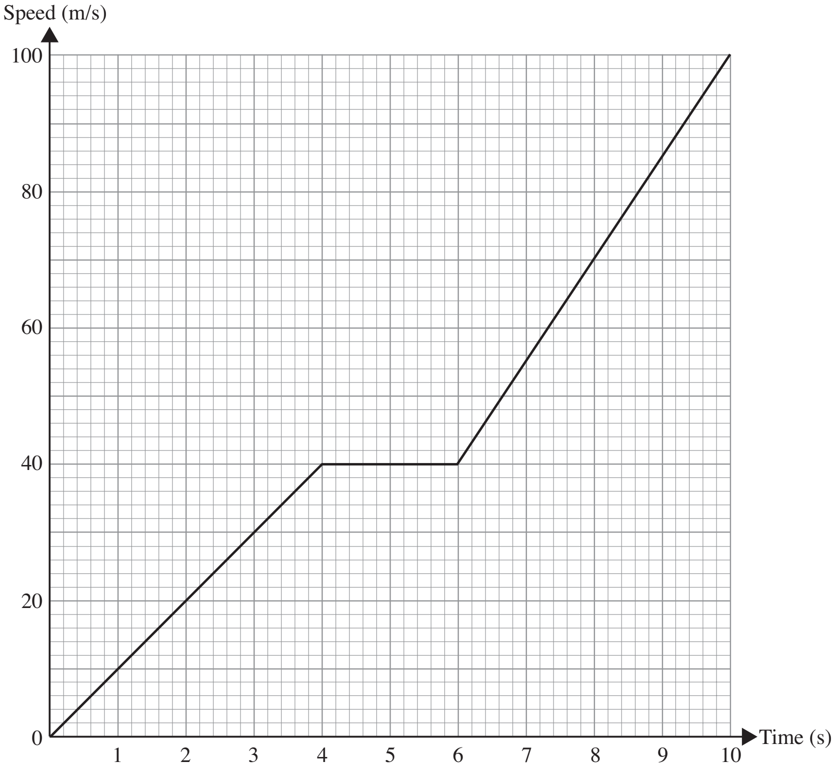
Example 10

The diagram below shows the speed-time graph of a bullet train journey for a period of 10 seconds.

(a) Find the acceleration during the first 4 seconds.

(b) How many seconds did the car maintain a constant speed?

(c) Calculate the distance travelled during the last 4 seconds.



Solution

(a) Acceleration $= \dfrac{40}{4}$

$= 10 \text{ m/s}^2$

(b) The car maintained at a constant speed for 2 s.

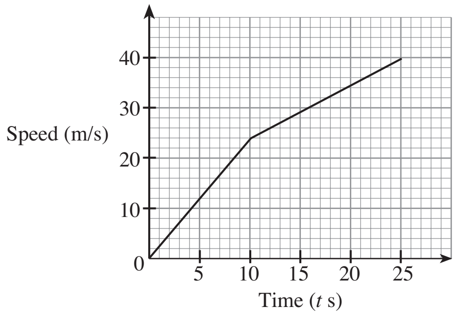
(c) Distance travelled = Area of trapezium

$= \dfrac{1}{2}(100 + 40)(4)$

$= 280 \text{ m}$

Example 11

The diagram shows the speed-time graph of the first 25 seconds of a journey.



Find
(i) the speed when $t = 15$,
(ii) the acceleration during the first 10 seconds,
(iii) the total distance travelled in the first 25 seconds.

Solution

(i) When $t = 15$, speed = 29 m/s.

(ii) Acceleration during the first 10 seconds $= \dfrac{24 - 0}{10 - 0}$

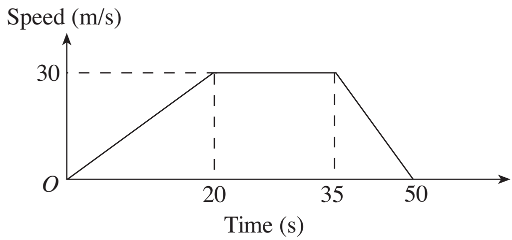
$= 2.4 \text{ m/s}^2$

(iii) Total distance travelled $= \dfrac{1}{2}(10)(24) + \dfrac{1}{2}(24 + 40)(15)$

$= 600 \text{ m}$

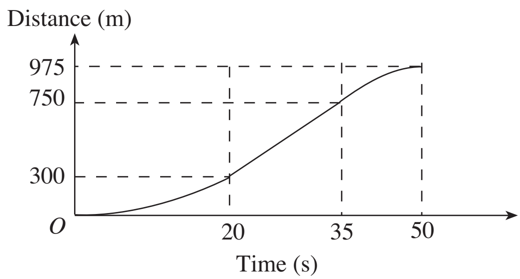
Example 12

Given the following speed-time graph, sketch the corresponding distance-time graph and acceleration-time graph.

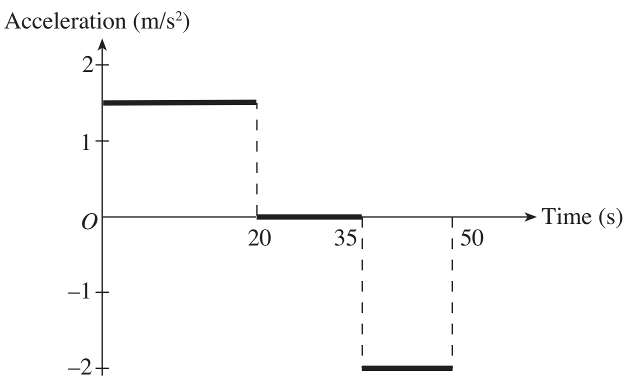


Solution

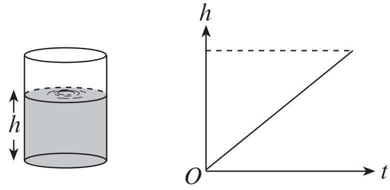
The distance-time graph is as follows:



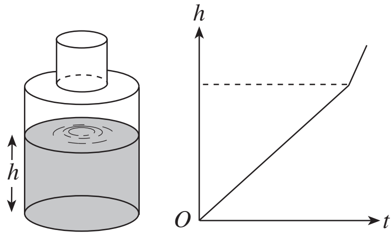
The acceleration-time graph is as follows:



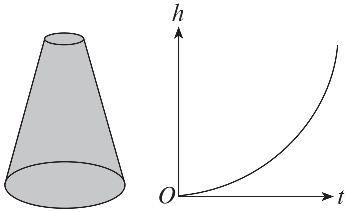
15. If the container is a cylinder as shown, the rate of the water level increasing with time is given as:



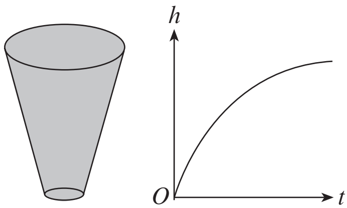
16. If the container is a bottle as shown, the rate of the water level increasing with time is given as:



17. If the container is an inverted funnel bottle as shown, the rate of the water level increasing with time is given as:



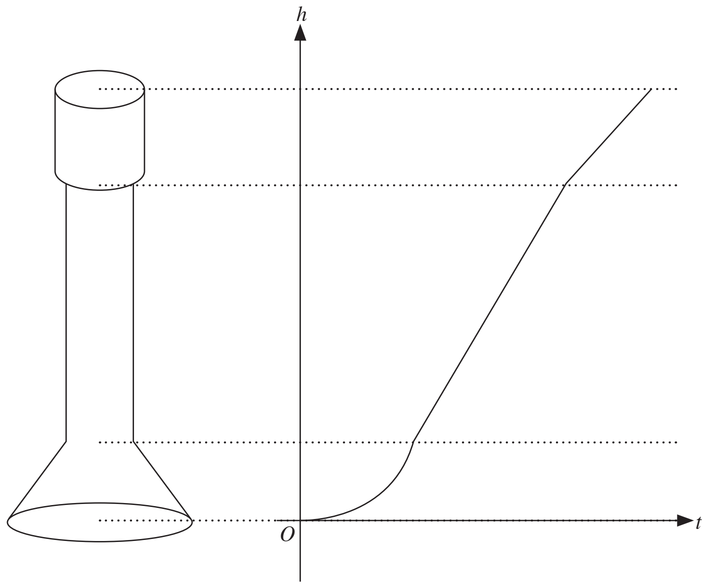
18. If the container is a funnel bottle as shown, the rate of the water level increasing with time is given as:



Example 13

Water is poured at a constant rate into the container below. Sketch the graph of water level (h) against time (t).

Solution



UNIT 1.10 Set Language and Notation

(not included for NA)

Definitions

1. A set is a collection of objects such as letters of the alphabet, people, etc.
The objects in a set are called members or elements of that set.

2. A finite set is a set which contains a countable number of elements.

3. An infinite set is a set which contains an uncountable number of elements.

4. A universal set ξ is a set which contains all the available elements.

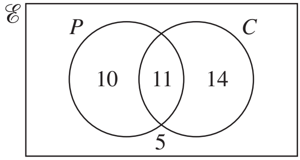
5. The empty set Ø or null set { } is a set which contains no elements.

Specifications of Sets

6. A set may be specified by listing all its members.
This is only for finite sets. We list names of elements of a set, separate them by commas and enclose them in brackets, e.g. $\{2, 3, 5, 7\}$.

7. A set may be specified by stating a property of its elements,
e.g. $\{x$: x is an even number greater than $3\}$.

8. A set may be specified by the use of a Venn diagram.
e.g.



For example, the Venn diagram above represents
ξ = {students in the class},
P = {students who study Physics},
C = {students who study Chemistry}.

From the Venn diagram,
10 students study Physics only,
14 students study Chemistry only,
11 students study both Physics and Chemistry,
5 students do not study either Physics or Chemistry.

Elements of a Set

9. a $\in Q$ means that a is an element of Q.
b $\notin Q$ means that b is not an element of Q.

10. n(A) denotes the number of elements in set A.

Example 1

ξ = {x: x is an integer such that $1 \leqslant x \leqslant 15$}
P = {x : x is a prime number},
Q = {x : x is divisible by 2},
R = {x : x is a multiple of 3}.

(a) List the elements in P and R.
(b) State the value of n(Q).

Solution

(a) $P = \{2, 3, 5, 7, 11, 13\}$
$R = \{3, 6, 9, 12, 15\}$
(b) $Q = \{2, 4, 6, 8, 10, 12, 14\}$
$n(Q) = 7$

Equal Sets

11. If two sets contain the exact same elements, we say that the two sets are equal sets. For example, if $A = \{1, 2, 3\}$, $B = \{3, 1, 2\}$ and $C = \{a, b, c\}$, then A and B are equal sets but A and C are not equal sets.

Subsets

12. $A \subseteq B$ means that A is a subset of B.
Every element of set A is also an element of set B.

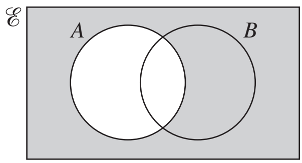
13. $A \subset B$ means that A is a proper subset of B.
Every element of set A is also an element of set B, but A cannot be equal to B.

14. $A \nsubseteq B$ means A is not a subset of B.

15. $A \not\subset B$ means A is not a proper subset of B.

Complement Sets

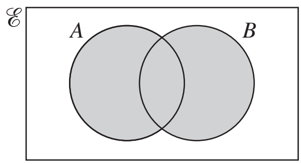
16. A' denotes the complement of a set A relative to a universal set ξ.
It is the set of all elements in ξ except those in A.



The shaded region in the diagram shows A'.

Union of Two Sets

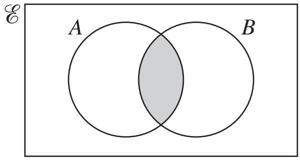
17. The union of two sets A and B, denoted as $A \cup B$, is the set of elements which belong to set A or set B or both.



The shaded region in the diagram shows $A \cup B$.

Intersection of Two Sets

18. The intersection of two sets A and B, denoted as $A \cap B$, is the set of elements which belong to both set A and set B.



The shaded region in the diagram shows $A \cap B$.

Example 2

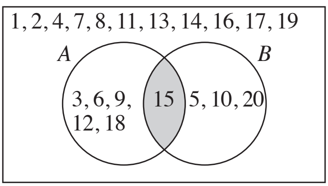
$\xi = \{x: x \text{ is an integer such that } 1 \leqslant x \leqslant 20\}$
$A = \{x : x \text{ is a multiple of } 3\}$
$B = \{x : x \text{ is divisible by } 5\}$

(a) Draw a Venn diagram to illustrate this information.
(b) List the elements contained in the set $A \cap B$.

Solution

$A = \{3, 6, 9, 12, 15, 18\}$
$B = \{5, 10, 15, 20\}$

(a)

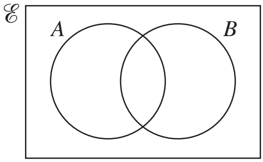


(b) $A \cap B = \{15\}$

Example 3

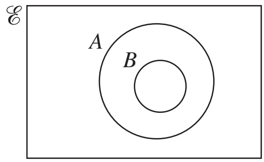
Shade the set indicated in each of the following Venn diagrams.

(a)



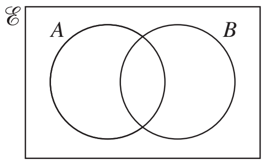
$A \cap B'$

(b)



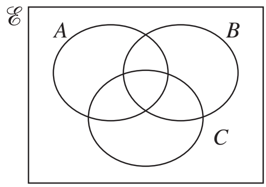
$A' \cap B$

(c)



$(A \cap B)'$

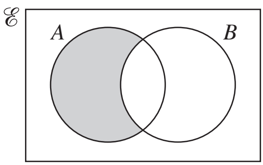
(d)



$A \cap C \cap B$

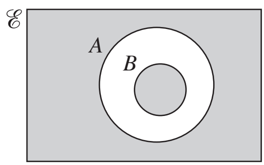
Solution

(a)



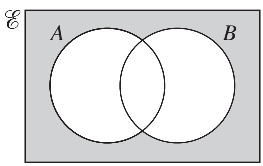
$A \cap B'$

(b)



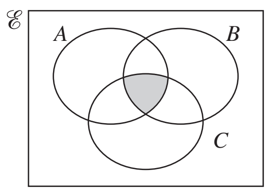
$A' \cap B$

(c)



$(A \cap B)'$

(d)

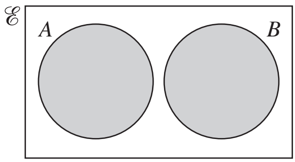


$A \cap C \cap B$

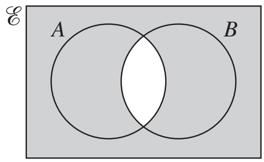
Example 4

Write the set notation for the sets shaded in each of the following Venn diagrams.

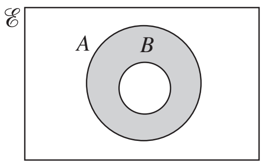
(a)



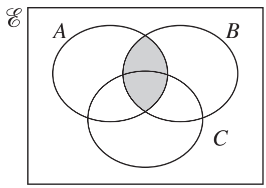
(b)



(c)



(d)



Solution

(a) $A \cup B$
(b) $(A \cap B)'$
(c) $A \cap B'$
(d) $A \cap B$

Example 5

$\xi = \{x \text{ is an integer} : -2 \leqslant x \leqslant 5\}$
$P = \{x : -2 < x < 3\}$
$Q = \{x : 0 < x \leqslant 4\}$

List the elements in
(a) P',
(b) $P \cap Q$,
(c) $P \cup Q$.

Solution

$\xi = \{-2, -1, 0, 1, 2, 3, 4, 5\}$
$P = \{-1, 0, 1, 2\}$
$Q = \{1, 2, 3, 4\}$

(a) $P' = \{-2, 3, 4, 5\}$
(b) $P \cap Q = \{1, 2\}$
(c) $P \cup Q = \{-1, 0, 1, 2, 3, 4\}$

Example 6

$\xi = \{x : x \text{ is a real number: } x < 30\}$
$A = \{x : x \text{ is a prime number}\}$
$B = \{x : x \text{ is a multiple of } 3\}$
$C = \{x : x \text{ is a multiple of } 4\}$

(a) Find $A \cap B$.
(b) Find $A \cap C$.
(c) Find $B \cap C$.

Solution

(a) $A \cap B = \{3\}$
(b) $A \cap C = \varnothing$
(c) $B \cap C = \{12, 24\}$ (Common multiples of 3 and 4 that are below 30)

Example 7

It is given that
ξ = {people on a bus}
A = {male commuters}
B = {students}
C = {commuters below 21 years old}

(a) Express in set notation, students who are below 21 years old.
(b) Express in words, $A' \cap B = \varnothing$.

Solution

(a) $B \subset C$ or $B \cap C$
(b) There are no female commuters who are students.

UNIT 1.11 Matrices

(not included for NA)

Matrix

1. A matrix is a rectangular array of numbers.

2. An example is $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -5 & 8 & 9 \\ 7 & 6 & -1 & 0 \end{pmatrix}$.

This matrix has 3 rows and 4 columns. We say that it has an order of 3 by 4.

3. In general, a matrix is defined by an order of $r \times c$, where r is the number of rows and c is the number of columns.
1, 2, 3, 4, 0, –5, … , 0 are called the elements of the matrix.

Row Matrix

4. A row matrix is a matrix that has exactly one row.

5. Examples of row matrices are $\begin{pmatrix} 12 & 4 & 3 \end{pmatrix}$ and $\begin{pmatrix} 7 & 5 \end{pmatrix}$.

6. The order of a row matrix is $1 \times c$, where c is the number of columns.

Column Matrix

7. A column matrix is a matrix that has exactly one column.

8. Examples of column matrices are $\begin{pmatrix} 0.5 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$.

9. The order of a column matrix is $r \times 1$, where r is the number of rows.

Square Matrix

10. A square matrix is a matrix that has exactly the same number of rows and columns.

11. Examples of square matrices are $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ and $\begin{pmatrix} 6 & 0 & -2 \\ 5 & 8 & 4 \\ 0 & 3 & 9 \end{pmatrix}$.

12. Matrices such as $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{pmatrix}$ are known as diagonal matrices as all the elements except those in the leading diagonal are zero.

Zero Matrix or Null Matrix

13. A zero matrix or null matrix is one where every element is equal to zero.

14. Examples of zero matrices or null matrices are $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

15. A zero matrix or null matrix is usually denoted by **0**.

Identity Matrix

16. An identity matrix is usually represented by the symbol **I**. All elements in its leading diagonal are ones while the rest are zeros.

e.g. 2×2 identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3×3 identity matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

17. Any matrix **P** when multiplied by an identity matrix **I** will result in itself, i.e. $\mathbf{PI} = \mathbf{IP} = \mathbf{P}$

Addition and Subtraction of Matrices

18. Matrices can only be added or subtracted if they are of the same order.

19. If there are two matrices **A** and **B**, both of order $r \times c$, then the addition of **A** and **B** is the addition of each element of **A** with its corresponding element of **B**,

i.e. $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$

and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a-e & b-f \\ c-g & d-h \end{pmatrix}$.

Example 1

Given that $\mathbf{P} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$ and $\mathbf{P} + \mathbf{Q} = \begin{pmatrix} 3 & 2 & 5 \\ 4 & 5 & 5 \end{pmatrix}$, find **Q**.

Solution

$$\mathbf{Q} = \begin{pmatrix} 3 & 2 & 5 \\ 4 & 5 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

Multiplication of a Matrix by a Real Number

20. The product of a matrix by a real number k is a matrix with each of its elements multiplied by k,

i.e. $k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$

Multiplication of Two Matrices

21. Matrix multiplication is only possible when the number of columns in the matrix on the left is equal to the number of rows in the matrix on the right.

22. In general, multiplying a $m \times n$ matrix by a $n \times p$ matrix will result in a $m \times p$ matrix.

23. Multiplication of matrices is not commutative, i.e. $\mathbf{AB} \neq \mathbf{BA}$.

24. Multiplication of matrices is associative, i.e. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$, provided that the multiplication can be carried out.

Example 2

Given that $\mathbf{A} = \begin{pmatrix} 5 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, find $\mathbf{AB}$.

Solution

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 5\times1+6\times3 & 5\times2+6\times4 \end{pmatrix} \\ &= \begin{pmatrix} 23 & 34 \end{pmatrix} \end{aligned}$$

Example 3

Given $\mathbf{P} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 2 & 5 \\ 3 & 4 \\ 1 & 2 \end{pmatrix}$, find $\mathbf{PQ}$.

Solution

$$\begin{aligned} \mathbf{PQ} &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 3 & 4 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1\times2+2\times3+3\times1 & 1\times5+2\times4+3\times2 \\ 2\times2+3\times3+4\times1 & 2\times5+3\times4+4\times2 \end{pmatrix} \\ &= \begin{pmatrix} 11 & 19 \\ 17 & 30 \end{pmatrix} \end{aligned}$$

Example 4

At a school's food fair, there are 3 stalls each selling 3 different flavours of pancake – chocolate, cheese and red bean. The table illustrates the number of pancakes sold during the food fair.

	Stall 1	Stall 2	Stall 3
Chocolate	92	102	83
Cheese	86	73	56
Red bean	85	53	66

The price of each pancake is as follows:

Chocolate: \$1.10
Cheese: \$1.30
Red bean: \$0.70

(a) Write down two matrices such that, under matrix multiplication, the product indicates the total revenue earned by each stall. Evaluate this product.

(b) **(i)** Find $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 92 & 102 & 83 \\ 86 & 73 & 56 \\ 85 & 53 & 66 \end{pmatrix}$.

(ii) Explain what your answer to **(b)(i)** represents.

Solution

(a) $\begin{pmatrix} 1.10 & 1.30 & 0.70 \end{pmatrix}\begin{pmatrix} 92 & 102 & 83 \\ 86 & 73 & 56 \\ 85 & 53 & 66 \end{pmatrix} = \begin{pmatrix} 272.50 & 244.20 & 210.30 \end{pmatrix}$

(b) **(i)** $\begin{pmatrix} 263 & 228 & 205 \end{pmatrix}$

(ii) Each of the elements represents the total number of pancakes sold by each stall during the food fair.

Example 5

A BBQ caterer distributes 3 types of satay – chicken, mutton and beef to 4 families. The price of each stick of chicken, mutton and beef satay is \$0.32, \$0.38 and \$0.28 respectively.
Mr Wong orders 25 sticks of chicken satay, 60 sticks of mutton satay and 15 sticks of beef satay.
Mr Lim orders 30 sticks of chicken satay and 45 sticks of beef satay.
Mrs Tan orders 70 sticks of mutton satay and 25 sticks of beef satay.
Mrs Koh orders 60 sticks of chicken satay, 50 sticks of mutton satay and 40 sticks of beef satay.

(i) Express the above information in the form of a matrix **A** of order 4 by 3 and a matrix **B** of order 3 by 1 so that the matrix product **AB** gives the total amount paid by each family.

(ii) Evaluate **AB**.

(iii) Find the total amount earned by the caterer.

Solution

(i) $\mathbf{A} = \begin{pmatrix} 25 & 60 & 15 \\ 30 & 0 & 45 \\ 0 & 70 & 25 \\ 60 & 50 & 40 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 0.32 \\ 0.38 \\ 0.28 \end{pmatrix}$

(ii) $\mathbf{AB} = \begin{pmatrix} 25 & 60 & 15 \\ 30 & 0 & 45 \\ 0 & 70 & 25 \\ 60 & 50 & 40 \end{pmatrix} \begin{pmatrix} 0.32 \\ 0.38 \\ 0.28 \end{pmatrix} = \begin{pmatrix} 35 \\ 22.2 \\ 33.6 \\ 49.4 \end{pmatrix}$

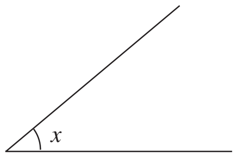
(iii) Total amount earned = \$35 + \$22.20 + \$33.60 + \$49.40
= \$140.20

UNIT 2.1 Angles, Triangles and Polygons

Types of Angles

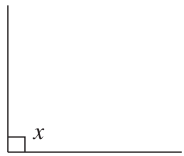
1. In a polygon, there may be four types of angles – acute angle, right angle, obtuse angle and reflex angle.

Acute angle



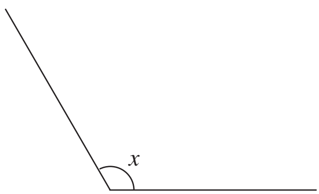
$x < 90°$

Right angle



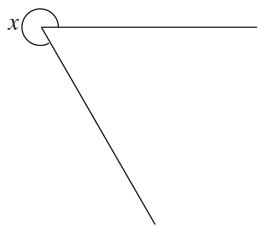
$x = 90°$

Obtuse angle



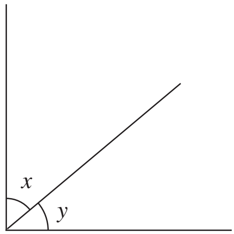
$90° < x < 180°$

Reflex angle



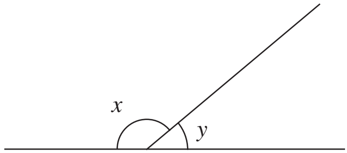
$180° < x < 360°$

2. If the sum of two angles is 90°, they are called complementary angles.



$\angle x + \angle y = 90°$

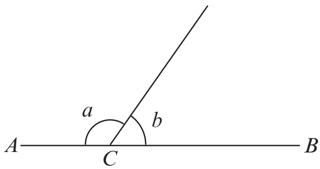
3. If the sum of two angles is 180°, they are called supplementary angles.



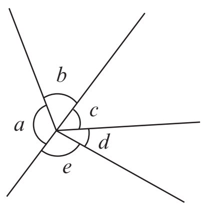
$\angle x + \angle y = 180°$

Geometrical Properties of Angles

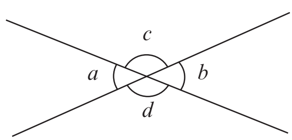
4. If ACB is a straight line, then $\angle a + \angle b = 180°$ (adj. ∠s on a str. line).



5. The sum of angles at a point is 360°, i.e. $\angle a + \angle b + \angle c + \angle d + \angle e = 360°$ (∠s at a pt.).



6. If two straight lines intersect, then $\angle a = \angle b$ and $\angle c = \angle d$ (vert. opp. ∠s).

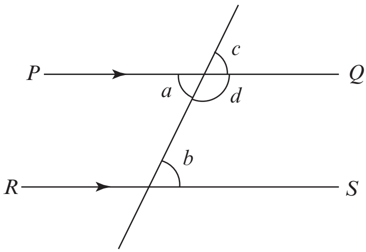


7. If the lines PQ and RS are parallel, then

$\angle a = \angle b$ (alt. $\angle$s),

$\angle c = \angle b$ (corr. $\angle$s),

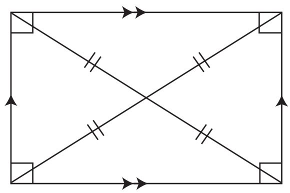
$\angle b + \angle d = 180°$ (int. $\angle$s).



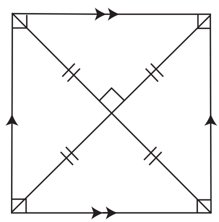
Properties of Special Quadrilaterals

8. The sum of the interior angles of a quadrilateral is 360°.

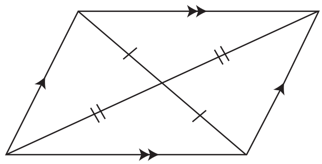
9. The diagonals of a rectangle bisect each other and are equal in length.



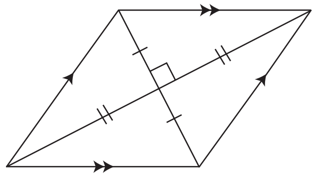
10. The diagonals of a square bisect each other at 90° and are equal in length. They bisect the interior angles.



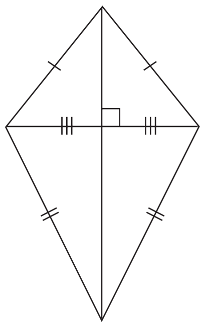
11. The diagonals of a parallelogram bisect each other.



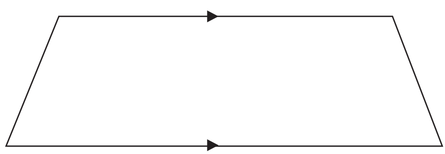
12. The diagonals of a rhombus bisect each other at 90°. They bisect the interior angles.



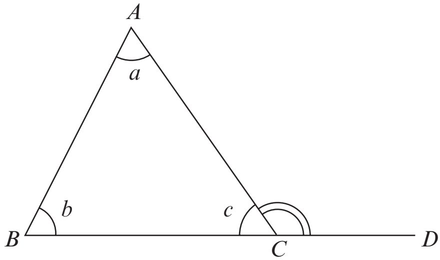
13. The diagonals of a kite cut each other at 90°. One of the diagonals bisects the interior angles.



14. At least one pair of opposite sides of a trapezium are parallel to each other.



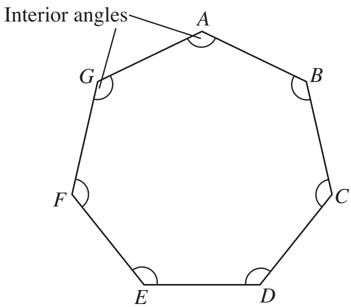
15. The sum of the interior angles of a triangle is 180°, i.e. $\angle a + \angle b + \angle c = 180°$ ($\angle$ sum of Δ).



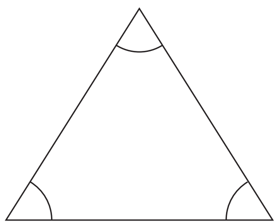
16. If the side BC of ΔABC is produced to D, then $\angle ACD = \angle a + \angle b$ (ext. $\angle$ of Δ).

17. The sum of the interior angles of an n-sided polygon is $(n - 2) \times 180°$.

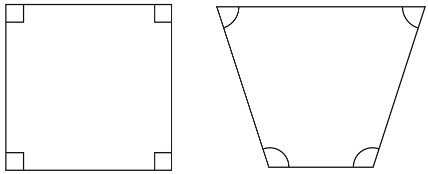
Each interior angle of a regular n-sided polygon = $\dfrac{(n-2)\times 180°}{n}$



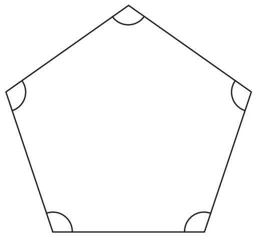
(a) A triangle is a polygon with 3 sides.
Sum of interior angles = $(3 - 2) \times 180° = 180°$



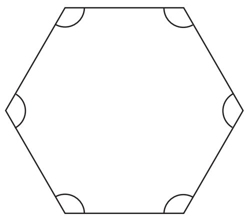
(b) A quadrilateral is a polygon with 4 sides.
Sum of interior angles = $(4 - 2) \times 180° = 360°$



(c) A pentagon is a polygon with 5 sides.
Sum of interior angles = $(5 - 2) \times 180° = 540°$

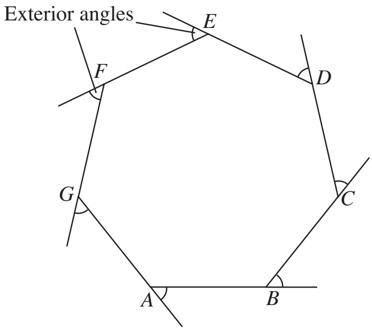


(d) A hexagon is a polygon with 6 sides.
Sum of interior angles = $(6 - 2) \times 180° = 720°$



18. The sum of the exterior angles of an n-sided polygon is 360°.

Each exterior angle of a regular n-sided polygon $= \dfrac{360°}{n}$



Example 1

Three interior angles of a polygon are 145°, 120° and 155°. The remaining interior angles are 100° each. Find the number of sides of the polygon.

Solution

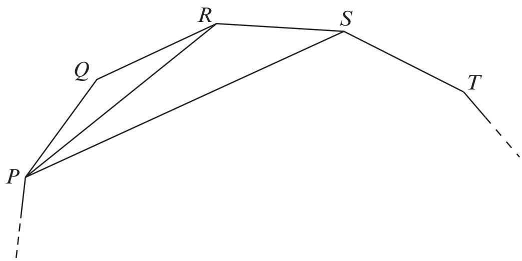
$$360° - (180° - 145°) - (180° - 120°) - (180° - 155°) = 360° - 35° - 60° - 25°$$
$$= 240°$$

$$\frac{240°}{(180° - 100°)} = 3$$

Total number of sides = 3 + 3
= 6

Example 2

The diagram shows part of a regular polygon with n sides. Each exterior angle of this polygon is $24°$.



Find

(i) the value of n,

(ii) $P\hat{Q}R$,

(iii) $P\hat{R}S$,

(iv) $P\hat{S}R$.

Solution

(i) Exterior angle $= \dfrac{360°}{n}$

$$24° = \frac{360°}{n}$$

$$n = \frac{360°}{24°}$$

$$= 15$$

(ii) $P\hat{Q}R = 180° - 24°$

$= 156°$

(iii) $P\hat{R}Q = \dfrac{180° - 156°}{2}$

$= 12°$ (base $\angle$s of isos. Δ)

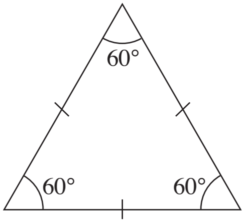
$P\hat{R}S = 156° - 12°$

$= 144°$

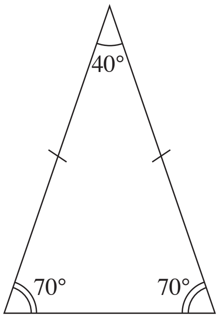
(iv) $P\hat{S}R = 180° - 156°$

$= 24°$ (int. $\angle$s, QR // PS)

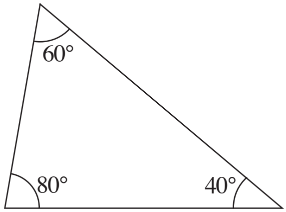
19. In an equilateral triangle, all three sides are equal. All three angles are the same, each measuring 60°.



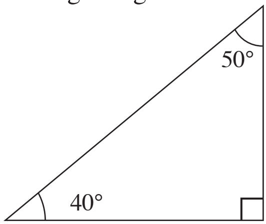
20. An isosceles triangle consists of two equal sides. Its two base angles are equal.



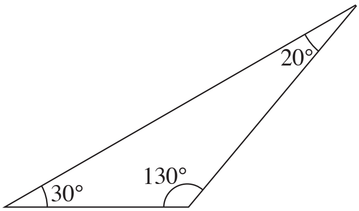
21. In an acute-angled triangle, all three angles are acute.



22. A right-angled triangle has one right angle.



23. An obtuse-angled triangle has one angle that is obtuse.

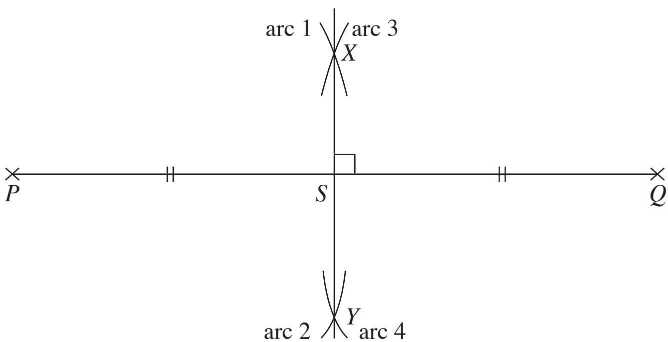


Perpendicular Bisector

24. If the line XY is the perpendicular bisector of a line segment PQ, then XY is perpendicular to PQ and XY passes through the midpoint of PQ.

Steps to construct a perpendicular bisector of line PQ:

1. Draw PQ.
2. Using a compass, choose a radius that is more than half the length of PQ.
3. Place the compass at P and mark arc 1 and arc 2 (one above and the other below the line PQ).
4. Place the compass at Q and mark arc 3 and arc 4 (one above and the other below the line PQ).
5. Join the two intersection points of the arcs to get the perpendicular bisector.



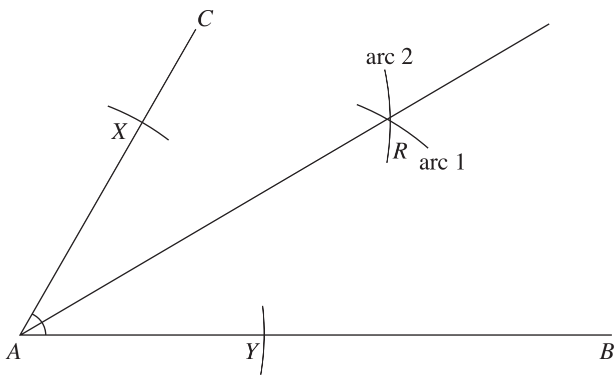
25. Any point on the perpendicular bisector of a line segment is equidistant from the two end points of the line segment.

Angle Bisector

26. If the ray AR is the angle bisector of $B\hat{A}C$, then $C\hat{A}R = R\hat{A}B$.

Steps to construct an angle bisector of $C\hat{A}B$:

1. Using a compass, choose a radius that is less than or equal to the length of AC.
2. Place the compass at A and mark two arcs (one on line AC and the other AB).
3. Mark the intersection points between the arcs and the two lines as X and Y.
4. Place compass at X and mark arc 2 (between the space of line AC and AB).
5. Place compass at Y and mark arc 1 (between the space of line AC and AB) that will intersect arc 2. Label the intersection point R.
6. Join R and A to bisect $C\hat{A}B$.



27. Any point on the angle bisector of an angle is equidistant from the two sides of the angle.

Example 3

Draw a quadrilateral $ABCD$ in which the base $AB = 3$ cm, $A\hat{B}C = 80°$, $BC = 4$ cm, $B\hat{A}D = 110°$ and $B\hat{C}D = 70°$.

(a) Measure and write down the length of CD.

(b) On your diagram, construct

  (i) the perpendicular bisector of AB,

  (ii) the bisector of angle ABC.

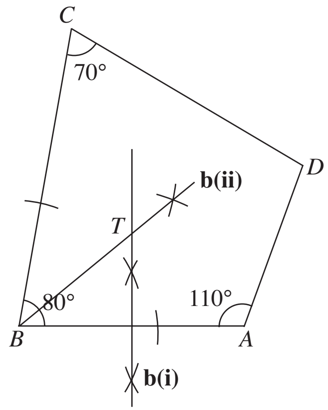
(c) These two bisectors meet at T.
Complete the statement below.

The point T is equidistant from the lines ____________ and ____________

and equidistant from the points ____________ and ____________ .

Solution

(a) $CD = 3.5$ cm

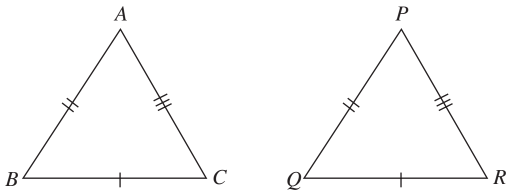


(c) The point T is equidistant from the lines AB and BC and equidistant from the points A and B.

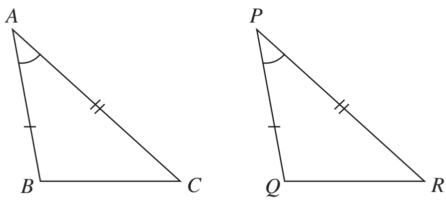
UNIT 2.2 Congruence and Similarity

Congruent Triangles

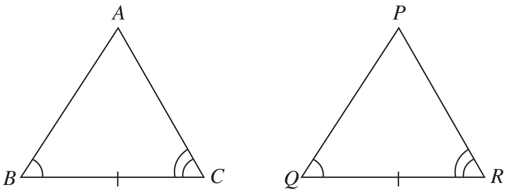
1. If $AB = PQ$, $BC = QR$ and $CA = RP$, then ΔABC is congruent to ΔPQR (SSS Congruence Test).



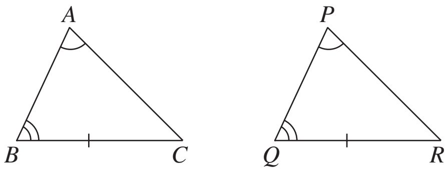
2. If $AB = PQ$, $AC = PR$ and $B\hat{A}C = Q\hat{P}R$, then ΔABC is congruent to ΔPQR (SAS Congruence Test).



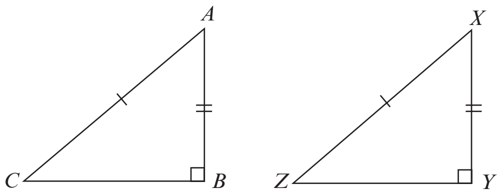
3. If $A\hat{B}C = P\hat{Q}R$, $A\hat{C}B = P\hat{R}Q$ and $BC = QR$, then ΔABC is congruent to ΔPQR (ASA Congruence Test).



If $B\hat{A}C = Q\hat{P}R$, $A\hat{B}C = P\hat{Q}R$ and $BC = QR$, then ΔABC is congruent to ΔPQR (AAS Congruence Test).



4. If $AC = XZ$, $AB = XY$ or $BC = YZ$, and $A\hat{B}C = X\hat{Y}Z = 90°$, then ΔABC is congruent to ΔXYZ (RHS Congruence Test).



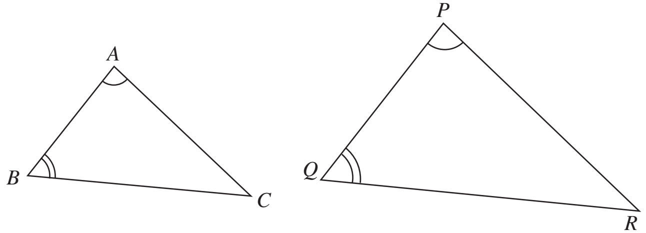
Similar Triangles

5. Two figures or objects are similar if:

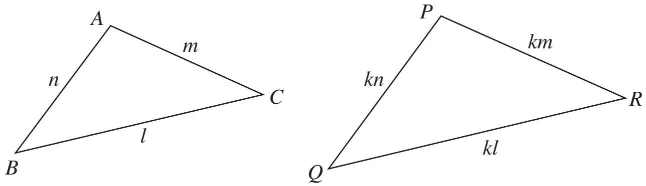
(a) the corresponding sides are proportional and,

(b) the corresponding angles are equal.

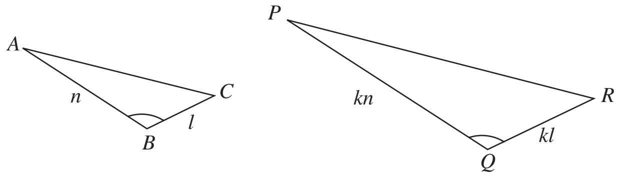
6. If $B\hat{A}C = Q\hat{P}R$ and $A\hat{B}C = P\hat{Q}R$, then ΔABC is similar to ΔPQR (AA Similarity Test).



7. If $\dfrac{PQ}{AB} = \dfrac{QR}{BC} = \dfrac{RP}{CA}$, then ΔABC is similar to ΔPQR (SSS Similarity Test).



8. If $\dfrac{PQ}{AB} = \dfrac{QR}{BC}$ and $A\hat{B}C = P\hat{Q}R$, then ΔABC is similar to ΔPQR (SAS Similarity Test).

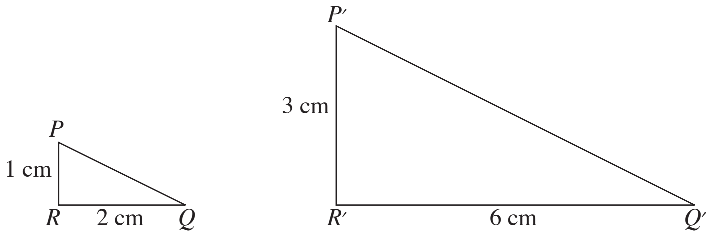


Scale Factor and Enlargement

9. Scale factor = $\dfrac{\text{Length of image}}{\text{Length of object}}$

10. We multiply the distance between each point in an object by a scale factor to produce an image. When the scale factor is greater than 1, the image produced is greater than the object. When the scale factor is between 0 and 1, the image produced is smaller than the object.

e.g. Taking ΔPQR as the object and $\Delta P'Q'R'$ as the image,
$\Delta P'Q'R'$ is an enlargement of ΔPQR with a scale factor of 3.



Scale factor = $\dfrac{P'R'}{PR} = \dfrac{R'Q'}{RQ} = 3$

If we take $\Delta P'Q'R'$ as the object and ΔPQR as the image,

ΔPQR is an enlargement of $\Delta P'Q'R'$ with a scale factor of $\dfrac{1}{3}$.

Scale factor = $\dfrac{PR}{P'R'} = \dfrac{RQ}{R'Q'} = \dfrac{1}{3}$

Similar Plane Figures

11. The ratio of the corresponding sides of two similar figures is $l_1 : l_2$.
The ratio of the area of the two figures is then $l_1^2 : l_2^2$.

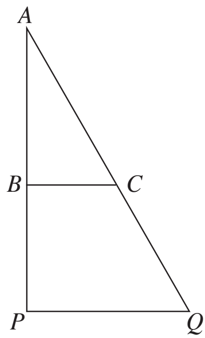
e.g. ΔABC is similar to ΔAPQ.

$$\frac{AB}{AP} = \frac{AC}{AQ} = \frac{BC}{PQ}$$

$$\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta APQ} = \left(\frac{AB}{AP}\right)^2$$

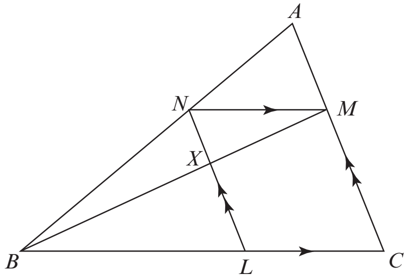
$$= \left(\frac{AC}{AQ}\right)^2$$

$$= \left(\frac{BC}{PQ}\right)^2$$



Example 1

In the figure, NM is parallel to BC and LN is parallel to CA.



(a) Prove that ΔANM is similar to ΔNBL.

(b) Given that $\dfrac{AN}{NB} = \dfrac{2}{3}$, find the numerical value of each of the following ratios.

(i) $\dfrac{\text{Area of }\Delta ANM}{\text{Area of }\Delta NBL}$

(ii) $\dfrac{NM}{BC}$

(iii) $\dfrac{\text{Area of trapezium }BNMC}{\text{Area of }\Delta ABC}$

(iv) $\dfrac{NX}{MC}$

Solution

(a) Since $A\hat{N}M = N\hat{B}L$ (corr. $\angle$s, $MN // LB$) and $N\hat{A}M = B\hat{N}L$ (corr. $\angle$s, $LN // MA$), ΔANM is similar to ΔNBL (AA Similarity Test).

(b) **(i)**

$$\frac{\text{Area of }\Delta ANM}{\text{Area of }\Delta NBL} = \left(\frac{AN}{NB}\right)^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

(ii) ΔANM is similar to ΔABC.

$$\therefore \frac{NM}{BC} = \frac{AN}{AB} = \frac{2}{5}$$

(iii) $$\frac{\text{Area of } \Delta ANM}{\text{Area of } \Delta ABC} = \left(\frac{NM}{BC}\right)^2 = \left(\frac{2}{5}\right)^2 = \frac{4}{25}$$

$$\frac{\text{Area of trapezium } BNMC}{\text{Area of } \Delta ABC} = \frac{\text{Area of } \Delta ABC - \text{Area of } \Delta ANM}{\text{Area of } \Delta ABC} = \frac{25-4}{25} = \frac{21}{25}$$

(iv) ΔNMX is similar to ΔLBX and $MC = NL$.

$$\frac{NX}{LX} = \frac{NM}{LB} = \frac{NM}{BC - LC} = \frac{2}{3}$$

i.e. $$\frac{NX}{NL} = \frac{2}{5}$$

$$\therefore \frac{NX}{MC} = \frac{2}{5}$$

Example 2

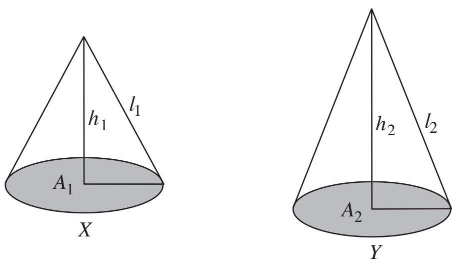
Triangle A and triangle B are similar. The length of one side of triangle A is $\frac{1}{4}$ the length of the corresponding side of triangle B. Find the ratio of the areas of the two triangles.

Solution

Let x be the length of triangle A.
Let $4x$ be the length of triangle B.

$$\frac{\text{Area of triangle } A}{\text{Area of triangle } B} = \left(\frac{\text{Length of triangle } A}{\text{Length of triangle } B}\right)^2$$
$$= \left(\frac{x}{4x}\right)^2$$
$$= \frac{1}{16}$$

Similar Solids



12. If X and Y are two similar solids, then the ratio of their lengths is equal to the ratio of their heights,

i.e. $\frac{l_1}{l_2} = \frac{h_1}{h_2}$.

13. If X and Y are two similar solids, then the ratio of their areas is given by

$$\frac{A_1}{A_2} = \left(\frac{h_1}{h_2}\right)^2 = \left(\frac{l_1}{l_2}\right)^2.$$

14. If X and Y are two similar solids, then the ratio of their volumes is given by

$$\frac{V_1}{V_2} = \left(\frac{h_1}{h_2}\right)^3 = \left(\frac{l_1}{l_2}\right)^3.$$

Example 3

The volumes of two glass spheres are 125 cm^3 and 216 cm^3. Find the ratio of the larger surface area to the smaller surface area.

Solution

Since $\dfrac{V_1}{V_2} = \dfrac{125}{216}$,

$$\frac{r_1}{r_2} = \sqrt[3]{\frac{125}{216}}$$

$$= \frac{5}{6}$$

$$\frac{A_1}{A_2} = \left(\frac{5}{6}\right)^2$$

$$= \frac{25}{36}$$

$\therefore$ The ratio is 36 : 25.

Example 4

The surface area of a small plastic cone is 90 cm^2. The surface area of a similar, larger plastic cone is 250 cm^2. Calculate the volume of the large cone if the volume of the small cone is 125 cm^3.

Solution

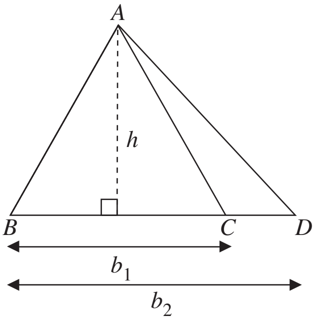
$$\frac{\text{Area of small cone}}{\text{Area of large cone}} = \frac{90}{250}$$
$$= \frac{9}{25}$$
$$\frac{\text{Area of small cone}}{\text{Area of large cone}} = \left(\frac{\text{Radius of small cone}}{\text{Radius of large cone}}\right)^2$$
$$= \frac{9}{25}$$
$$\frac{\text{Radius of small cone}}{\text{Radius of large cone}} = \frac{3}{5}$$
$$\frac{\text{Volume of small cone}}{\text{Volume of large cone}} = \left(\frac{\text{Radius of small cone}}{\text{Radius of large cone}}\right)^3$$
$$\frac{125}{\text{Volume of large cone}} = \left(\frac{3}{5}\right)^3$$

Volume of large cone = 579 cm^3 (to 3 s.f.)

15. If X and Y are two similar solids with the same density, then the ratio of their masses is given by $\frac{m_1}{m_2} = \left(\frac{h_1}{h_2}\right)^3 = \left(\frac{l_1}{l_2}\right)^3$.

Triangles Sharing the Same Height

16. If ΔABC and ΔABD share the same height h, then



$$\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta ABD} = \frac{\frac{1}{2}b_1h}{\frac{1}{2}b_2h}$$

$$= \frac{b_1}{b_2}$$

Example 5

In the diagram below, PQR is a straight line, $PQ = 2$ cm and $PR = 8$ cm. ΔOPQ and ΔOPR share the same height, h. Find the ratio of the area of ΔOPQ to the area of ΔOQR.

Solution

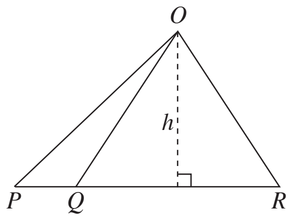
Both triangles share the same height, h.

$$\frac{\text{Area of } \Delta OPQ}{\text{Area of } \Delta OQR} = \frac{\frac{1}{2} \times PQ \times h}{\frac{1}{2} \times QR \times h}$$

$$= \frac{PQ}{QR}$$

$$= \frac{2}{6}$$

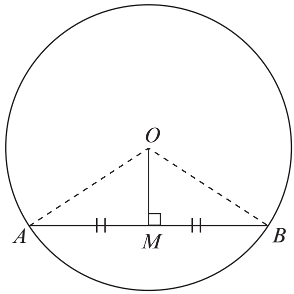
$$= \frac{1}{3}$$



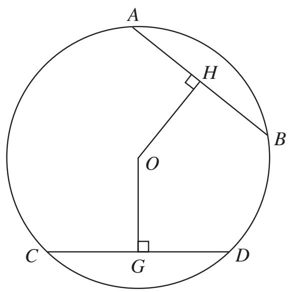
UNIT 2.3 Properties of Circles

Symmetric Properties of Circles

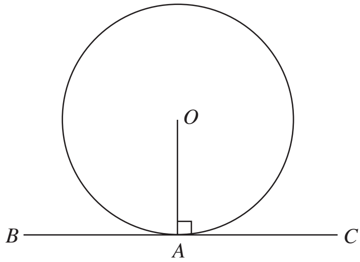
1. The perpendicular bisector of a chord AB of a circle passes through the centre of the circle, i.e. $AM = MB \Leftrightarrow OM \perp AB$.



2. If the chords AB and CD are equal in length, then they are equidistant from the centre, i.e. $AB = CD \Leftrightarrow OH = OG$.



3. The radius *OA* of a circle is perpendicular to the tangent at the point of contact, i.e. $O\hat{A}C = 90°$.

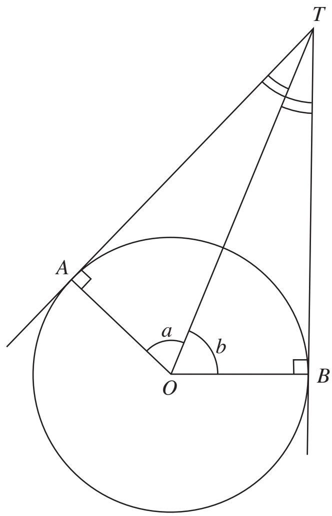


4. If *TA* and *TB* are tangents from *T* to a circle centre *O*, then

(i) $TA = TB$,

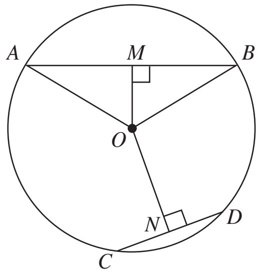
(ii) $\angle a = \angle b$,

(iii) *OT* bisects $A\hat{T}B$.



Example 1

In the diagram, O is the centre of the circle with chords AB and CD. $ON = 5$ cm and $CD = 5$ cm. $O\hat{C}D = 2O\hat{B}A$. Find the length of AB.



Solution

Radius $= OC = OD$

Radius $= \sqrt{5^2 + 2.5^2}$

$= 5.590$ cm (to 4 s.f.)

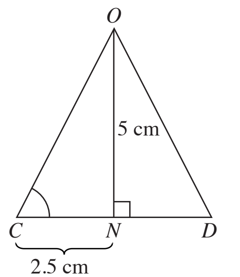
$\tan O\hat{C}D = \dfrac{ON}{AN}$

$= \dfrac{5}{2.5}$

$O\hat{C}D = 63.43°$ (to 2 d.p.)

$O\hat{B}A = 63.43° \div 2$

$= 31.72°$



$OB =$ radius

$= 5.59$ cm

$\cos O\hat{B}A = \dfrac{BM}{OB}$

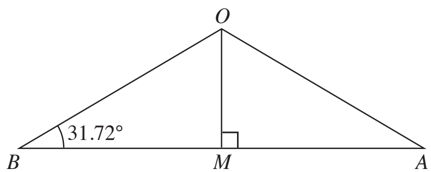
$\cos 31.72° = \dfrac{BM}{5.59}$

$BM = 5.59 \times \cos 31.72°$

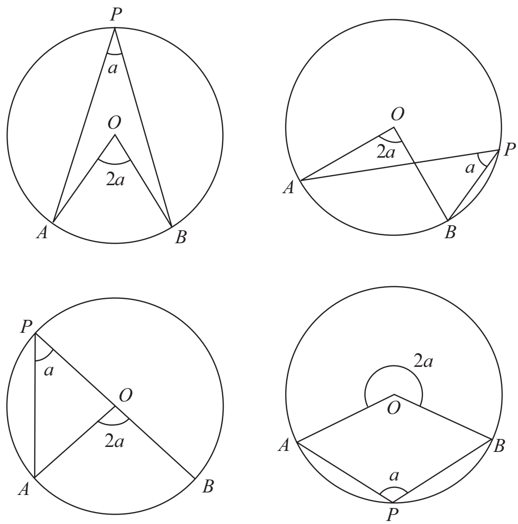
$= 4.755$ cm (to 4 s.f.)

$AB = 2 \times 4.755$

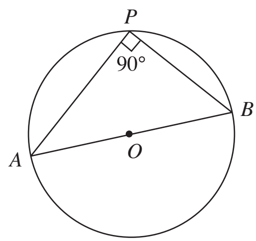
$= 9.51$ cm



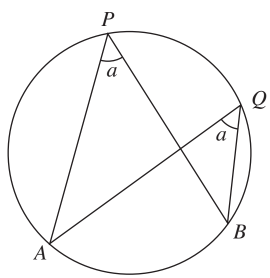
5. An angle at the centre of a circle is twice that of any angle at the circumference subtended by the same arc, i.e. $\angle AOB = 2 \times \angle APB$.



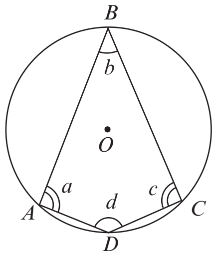
6. An angle in a semicircle is always equal to 90°, i.e. AOB is a diameter $\Leftrightarrow \angle APB = 90°$.



7. Angles in the same segment are equal, i.e. $\angle APB = \angle AQB$.



8. Angles in opposite segments are supplementary, i.e. $\angle a + \angle c = 180°$; $\angle b + \angle d = 180°$.

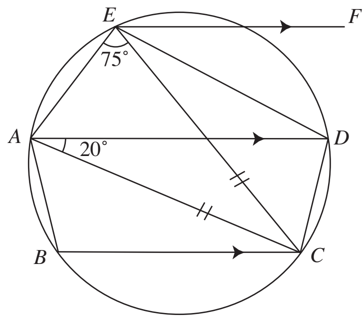


Example 2

In the diagram, A, B, C, D and E lie on a circle and $AC = EC$. The lines BC, AD and EF are parallel. $A\hat{E}C = 75°$ and $D\hat{A}C = 20°$.

Find

(i) $A\hat{C}B$, **(ii)** $A\hat{B}C$,
(iii) $B\hat{A}C$, **(iv)** $E\hat{A}D$,
(v) $F\hat{E}D$.



Solution

(i) $A\hat{C}B = D\hat{A}C$
$= 20°$ (alt. $\angle$s, BC // AD)

(ii) $A\hat{B}C + A\hat{E}C = 180°$ ($\angle$s in opp. segments)
$A\hat{B}C + 75° = 180°$
$A\hat{B}C = 180° - 75°$
$= 105°$

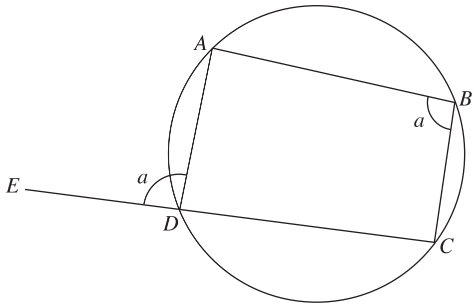
(iii) $B\hat{A}C = 180° - 20° - 105°$ ($\angle$ sum of Δ)
$= 55°$

(iv) $E\hat{A}C = A\hat{E}C = 75°$ (base $\angle$s of isos. Δ)
$E\hat{A}D = 75° - 20°$
$= 55°$

(v) $E\hat{C}A = 180° - 2 \times 75°$
$= 30°$ ($\angle$ sum of Δ)
$E\hat{D}A = E\hat{C}A = 30°$ ($\angle$s in same segment)
$F\hat{E}D = E\hat{D}A = 30°$ (alt. $\angle$s, EF // AD)

9. The exterior angle of a cyclic quadrilateral is equal to the opposite interior angle, i.e. $\angle ADC = 180° - \angle ABC$ ($\angle$s in opp. segments)

$\angle ADE = 180° - (180° - \angle ABC) = \angle ABC$

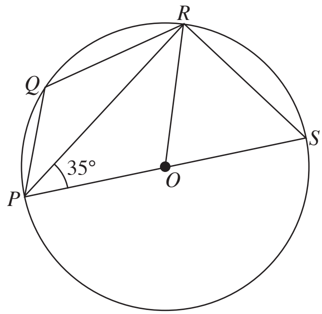


Example 3

In the diagram, O is the centre of the circle and $\angle RPS = 35°$. Find the following angles:

(a) $\angle ROS$,

(b) $\angle ORS$.



Solution

(a) $\angle ROS = 70°$ ($\angle$ at centre = 2 $\angle$ at circumference)

(b) $\angle OSR = 180° - 90° - 35°$ (rt. $\angle$ in a semicircle)

$= 55°$

$\angle ORS = 180° - 70° - 55°$ ($\angle$ sum of Δ)

$= 55°$

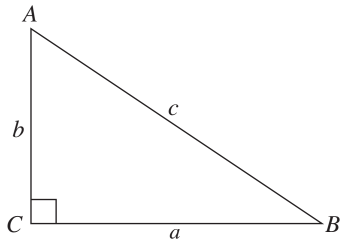
Alternatively,

$\angle ORS = \angle OSR$

$= 55°$ (base $\angle$s of isos. Δ)

UNIT 2.4 Pythagoras' Theorem and Trigonometry

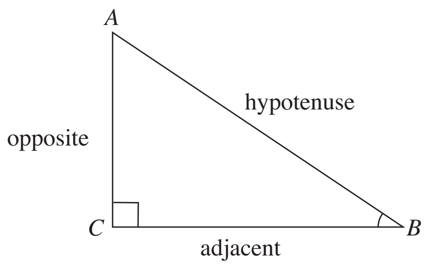
Pythagoras' Theorem



1. For a right-angled triangle ABC, if $\angle C = 90°$, then $AB^2 = BC^2 + AC^2$, i.e. $c^2 = a^2 + b^2$.

2. For a triangle ABC, if $AB^2 = BC^2 + AC^2$, then $\angle C = 90°$.

Trigonometric Ratios of Acute Angles



3. The side opposite the right angle C is called the hypotenuse. It is the longest side of a right-angled triangle.

4. In a triangle ABC, if $\angle C = 90°$,

then $\dfrac{AC}{AB} = \dfrac{\text{opp}}{\text{adj}}$ is called the sine of $\angle B$, or $\sin B = \dfrac{\text{opp}}{\text{hyp}}$,

$\dfrac{BC}{AB} = \dfrac{\text{adj}}{\text{hyp}}$ is called the cosine of $\angle B$, or $\cos B = \dfrac{\text{adj}}{\text{hyp}}$,

$\dfrac{AC}{BC} = \dfrac{\text{opp}}{\text{adj}}$ is called the tangent of $\angle B$, or $\tan B = \dfrac{\text{opp}}{\text{adj}}$.

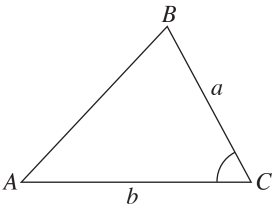
Trigonometric Ratios of Obtuse Angles

5. When θ is obtuse, i.e. $90° < \theta < 180°$,

$$\sin\theta = \sin(180° - \theta),$$
$$\cos\theta = -\cos(180° - \theta).$$
$$\tan\theta = -\tan(180° - \theta).$$

Area of a Triangle

6. Area of $\Delta ABC = \dfrac{1}{2}ab\sin C$



Sine Rule

7. In any ΔABC, the Sine Rule states that $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ or $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

8. The Sine Rule can be used to solve a triangle if the following are given:

- two angles and the length of one side; or
- the lengths of two sides and one non-included angle.

Cosine Rule

9. In any ΔABC, the Cosine Rule states that

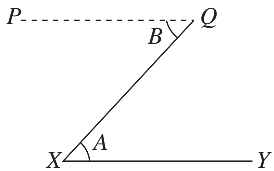
$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

or

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

10. The Cosine Rule can be used to solve a triangle if the following are given:
- the lengths of all three sides; or
- the lengths of two sides and an included angle.

Angles of Elevation and Depression

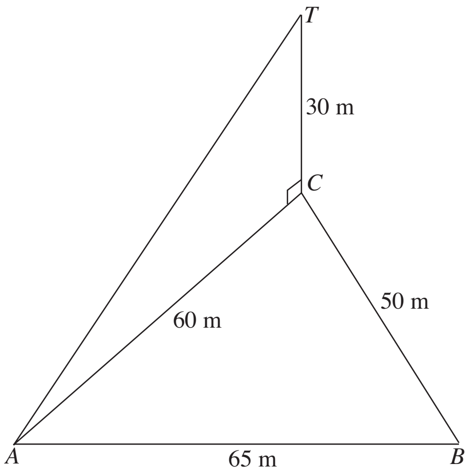


11. The angle A measured from the horizontal level XY is called the angle of elevation of Q from X.

12. The angle B measured from the horizontal level PQ is called the angle of depression of X from Q.

Example 1

In the figure, A, B and C lie on level ground such that $AB = 65$ m, $BC = 50$ m and $AC = 60$ m. T is vertically above C such that $TC = 30$ m.



Find

(i) $A\hat{C}B$,

(ii) the angle of elevation of T from A.

Solution

(i) Using cosine rule,

$$AB^2 = AC^2 + BC^2 - 2(AC)(BC)\cos A\hat{C}B$$

$$65^2 = 60^2 + 50^2 - 2(60)(50)\cos A\hat{C}B$$

$$\cos A\hat{C}B = \frac{1875}{6000}$$

$$A\hat{C}B = 71.8° \text{ (to 1 d.p.)}$$

(ii) In ΔATC,

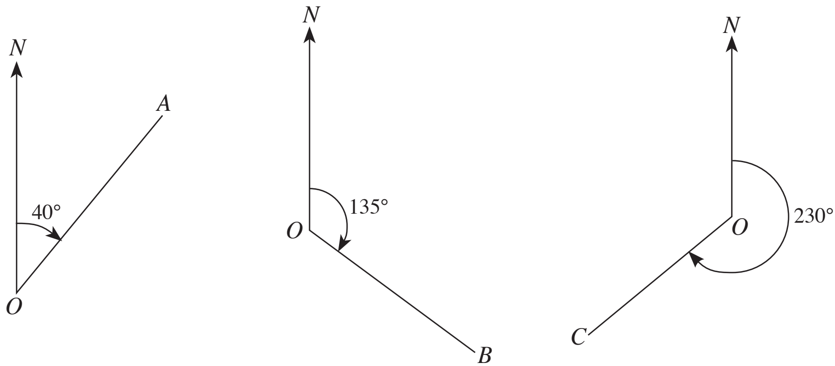
$$\tan T\hat{A}C = \frac{30}{60}$$

$$T\hat{A}C = 26.6° \text{ (to 1 d.p.)}$$

$\therefore$ Angle of elevation of T from A is 26.6°

13. The bearing of a point A from another point O is an angle measured from the north, at O, in a clockwise direction and is written as a three-digit number.

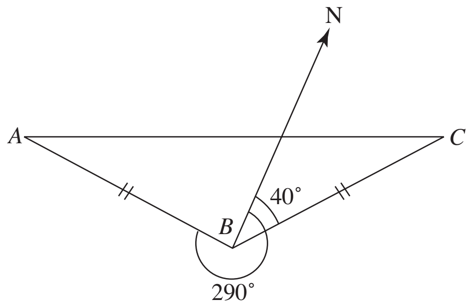
e.g.



The bearing of A from O is 040°.
The bearing of B from O is 135°.
The bearing of C from O is 230°.

Example 2

The bearing of A from B is 290°. The bearing of C from B is 040°. $AB = BC$.

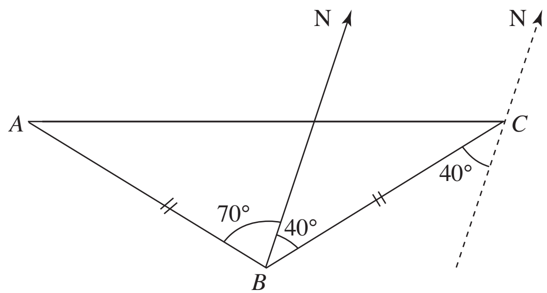


Find

(i) $B\hat{C}A$,

(ii) the bearing of A from C.

Solution



(i) $B\hat{C}A = \dfrac{180° - 70° - 40°}{2}$

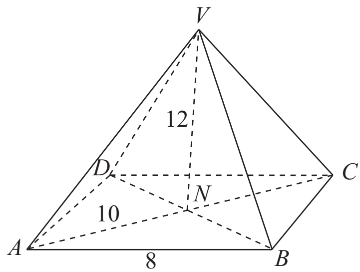
$= 35°$

(ii) Bearing of A from $C = 180° + 40° + 35°$

$= 255°$

14. The basic technique used to solve a three-dimensional problem is to reduce it to a problem in a plane.

Example 3



The figure shows a pyramid with a rectangular base, $ABCD$, and vertex V. The slant edges VA, VB, VC and VD are all equal in length and the diagonals of the base intersect at N. $AB = 8$ cm, $AC = 10$ cm and $VN = 12$ cm.

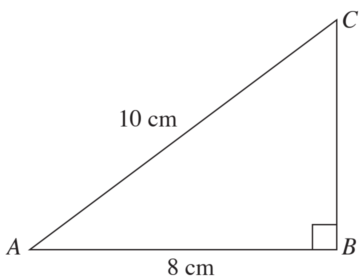
(i) Find the length of BC.

(ii) Find the length of VC.

(iii) Write down the tangent of the angle between VN and VC.

Solution

(i)



Using Pythagoras' Theorem,

$AC^2 = AB^2 + BC^2$

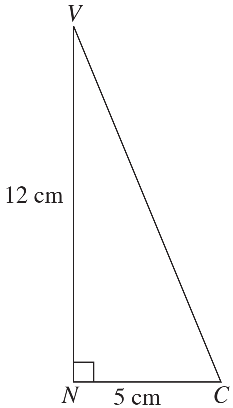
$10^2 = 8^2 + BC^2$

$BC^2 = 36$

$BC = 6$ cm

(ii) $CN = \frac{1}{2}AC$

$= 5$ cm



Using Pythagoras' Theorem,

$VC^2 = VN^2 + CN^2$

$= 12^2 + 5^2$

$= 169$

$VC = 13$ cm

(iii) The angle between VN and VC is $C\hat{V}N$.

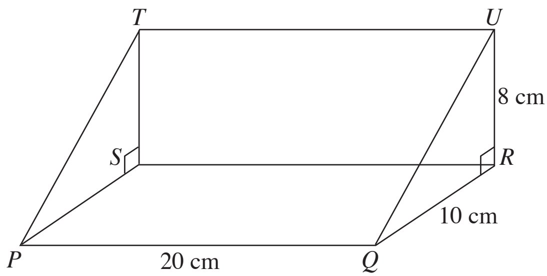
In ΔVNC,

$\tan C\hat{V}N = \frac{CN}{VN}$

$= \frac{5}{12}$

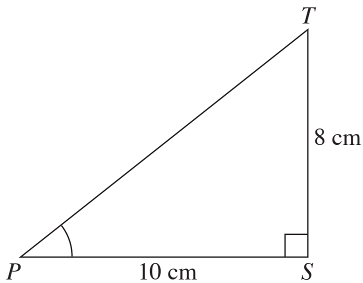
Example 4

The diagram shows a right-angled triangular prism.
Find
(a) $S\hat{P}T$,
(b) PU.



Solution

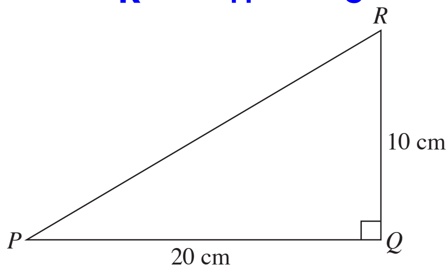
(a)



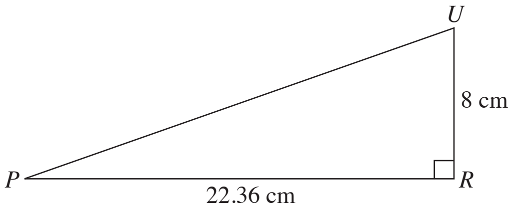
$$\tan S\hat{P}T = \frac{ST}{PS} = \frac{8}{10}$$

$$S\hat{P}T = 38.7° \text{ (to 1 d.p.)}$$

(b)



$$PR^2 = PQ^2 + QR^2$$
$$= 20^2 + 10^2$$
$$= 500$$
$$PR = 22.36 \text{ cm (to 4 s.f.)}$$



$$PU^2 = PR^2 + UR^2$$
$$= 22.36^2 + 8^2$$
$$PU = 23.7 \text{ cm (to 3 s.f.)}$$

UNIT 2.5 Mensuration

Perimeter and Area of Figures

1.

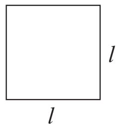
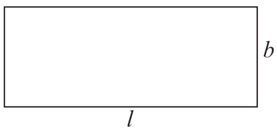
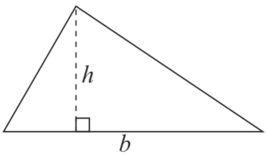
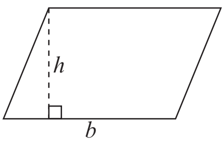
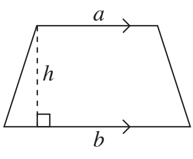
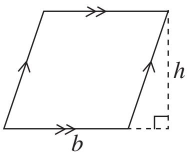
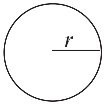
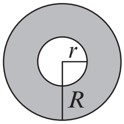
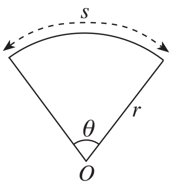
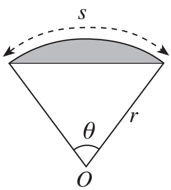
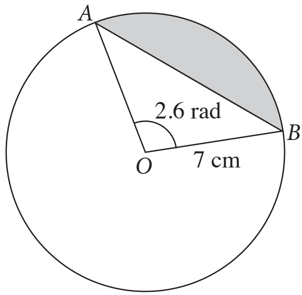
Figure	Diagram	Formulae
Square		Area = l^2 Perimeter = $4l$
Rectangle		Area = $l \times b$ Perimeter = $2(l + b)$
Triangle		Area = $\frac{1}{2} \times$ base $\times$ height $= \frac{1}{2} \times b \times h$
Parallelogram		Area = base $\times$ height $= b \times h$
Trapezium		Area = $\frac{1}{2}(a + b)\, h$
Rhombus		Area = $b \times h$

Figure	Diagram	Formulae
Circle		Area $= \pi r^2$ Circumference $= 2\pi r$
Annulus		Area $= \pi(R^2 - r^2)$
Sector		Arc length $s = \dfrac{\theta°}{360°} \times 2\pi r$ (where θ is in degrees) $= r\theta$ (where θ is in radians) Area $= \dfrac{\theta°}{360°} \times \pi r^2$ (where θ is in degrees) $= \dfrac{1}{2} r^2\theta$ (where θ is in radians)
Segment		Area $= \dfrac{1}{2} r^2(\theta - \sin\theta)$ (where θ is in radians)

Example 1

In the figure, O is the centre of the circle of radius 7 cm. AB is a chord and $\angle AOB = 2.6$ rad. The minor segment of the circle formed by the chord AB is shaded.



Find

(a) the length of the minor arc AB,

(b) the area of the shaded region.

Solution

(a) Length of minor arc $AB = 7(2.6)$

$= 18.2$ cm

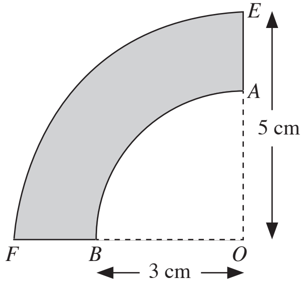
(b) Area of shaded region = Area of sector AOB – Area of ΔAOB

$$= \frac{1}{2}(7)^2(2.6) - \frac{1}{2}(7)^2 \sin 2.6$$

$= 51.1$ cm^2 (to 3 s.f.)

Example 2

In the figure, the radii of quadrants *ABO* and *EFO* are 3 cm and 5 cm respectively.



(a) Find the arc length of *AB*, in terms of π.

(b) Find the perimeter of the shaded region. Give your answer in the form $a + b\pi$.

Solution

(a) Arc length of $AB = \dfrac{3\pi}{2}$ cm

(b) Arc length $EF = \dfrac{5\pi}{2}$ cm

$$\text{Perimeter} = \frac{3\pi}{2} + \frac{5\pi}{2} + 2 + 2$$
$$= (4 + 4\pi) \text{ cm}$$

Degrees and Radians

2. π radians = 180°

$$1 \text{ radian} = \frac{180°}{\pi}$$

$$1° = \frac{\pi}{180°} \text{ radians}$$

3. To convert from degrees to radians and from radians to degrees:

$$\begin{tikzcd}
\text{Degree} \arrow[r, bend left, "\times \frac{\pi}{180°}"] & \text{Radian} \arrow[l, bend left, "\times \frac{180°}{\pi}"]
\end{tikzcd}$$

Conversion of Units

4. **Length**

1 m = 100 cm

1 cm = 10 mm

Area

$1 \text{ cm}^2 = 10 \text{ mm} \times 10 \text{ mm}$

$\quad = 100 \text{ mm}^2$

$1 \text{ m}^2 = 100 \text{ cm} \times 100 \text{ cm}$

$\quad = 10\,000 \text{ cm}^2$

Volume

$1 \text{ cm}^3 = 10 \text{ mm} \times 10 \text{ mm} \times 10 \text{ mm}$

$\quad = 1000 \text{ mm}^3$

$1 \text{ m}^3 = 100 \text{ cm} \times 100 \text{ cm} \times 100 \text{ cm}$

$\quad = 1\,000\,000 \text{ cm}^3$

1 litre = 1000 ml

$\quad = 1000 \text{ cm}^3$

5.

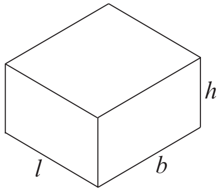
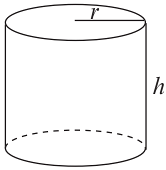
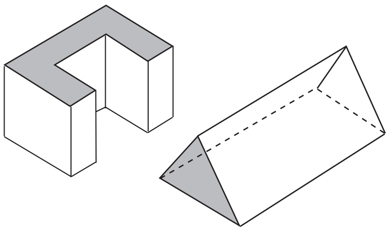
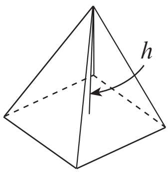
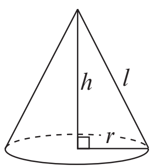
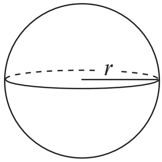
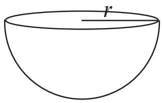
Figure	**Diagram**	**Formulae**
Cuboid		Volume = $l \times b \times h$ Total surface area = $2(lb + lh + bh)$
Cylinder		Volume = $\pi r^2 h$ Curved surface area = $2\pi rh$ Total surface area = $2\pi rh + 2\pi r^2$
Prism		Volume = Area of cross section × length Total surface area = Perimeter of the base × height + 2(base area)
Pyramid		Volume = $\frac{1}{3} \times$ base area $\times h$
Cone		Volume = $\frac{1}{3}\pi r^2 h$ Curved surface area = πrl (where l is the slant height) Total surface area = $\pi rl + \pi r^2$

Figure	Diagram	Formulae
Sphere		Volume $= \frac{4}{3}\pi r^3$ Surface area $= 4\pi r^2$
Hemisphere		Volume $= \frac{2}{3}\pi r^3$ Surface area $= 2\pi r^2 + \pi r^2$ $= 3\pi r^2$

Example 3

(a) A sphere has a radius of 10 cm. Calculate the volume of the sphere.

(b) A cuboid has the same volume as the sphere in part **(a)**. The length and breadth of the cuboid are both 5 cm. Calculate the height of the cuboid.

Leave your answers in terms of π.

Solution

(a) Volume $= \dfrac{4\pi(10)^3}{3}$

$= \dfrac{4000\,\pi}{3}\ \text{cm}^3$

(b) Volume of cuboid $= l \times b \times h$

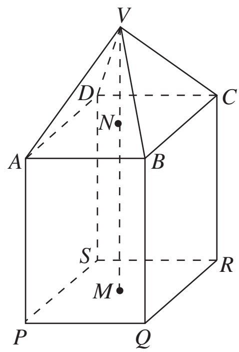
$\dfrac{4000\,\pi}{3} = 5 \times 5 \times h$

$h = \dfrac{160\pi}{3}$ cm

Example 4

The diagram shows a solid which consists of a pyramid with a square base attached to a cuboid. The vertex V of the pyramid is vertically above M and N, the centres of the squares $PQRS$ and $ABCD$ respectively. $AB = 30$ cm, $AP = 40$ cm and $VN = 20$ cm.

(a) Find
  (i) VA,
  (ii) $V\hat{A}C$.
(b) Calculate
  (i) the volume,
  (ii) the total surface area
  of the solid.



Solution

(a) **(i)** Using Pythagoras' Theorem,

$$AC^2 = AB^2 + BC^2$$
$$= 30^2 + 30^2$$
$$= 1800$$
$$AC = \sqrt{1800} \text{ cm}$$
$$AN = \frac{1}{2}\sqrt{1800} \text{ cm}$$

Using Pythagoras' Theorem,

$$VA^2 = VN^2 + AN^2$$
$$= 20^2 + \left(\frac{1}{2}\sqrt{1800}\right)^2$$
$$= 850$$
$$VA = \sqrt{850}$$
$$= 29.2 \text{ cm (to 3 s.f.)}$$

(ii) $V\hat{A}C = V\hat{A}N$

In ΔVAN,

$$\tan V\hat{A}N = \frac{VN}{AN}$$

$$= \frac{20}{\frac{1}{2}\sqrt{1800}}$$

$$= 0.9428 \text{ (to 4 s.f.)}$$

$$V\hat{A}N = 43.3° \text{ (to 1 d.p.)}$$

(b) **(i)** Volume of solid = Volume of cuboid + Volume of pyramid

$$= (30)(30)(40) + \frac{1}{3}(30)^2(20)$$

$$= 42\ 000 \text{ cm}^3$$

(ii) Let X be the midpoint of AB.

Using Pythagoras' Theorem,

$VA^2 = AX^2 + VX^2$

$850 = 15^2 + VX^2$

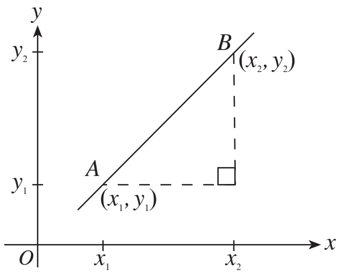
$VX^2 = 625$

$VX = 25$ cm

Total surface area $= 30^2 + 4(40)(30) + 4\left(\frac{1}{2}\right)(30)(25)$

$= 7200 \text{ cm}^2$

UNIT 2.6 Coordinate Geometry



Gradient

1. The gradient of the line joining any two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ or $\dfrac{y_1 - y_2}{x_1 - x_2}$.

2. Parallel lines have the same gradient.

Distance

3. The distance between any two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 1

The gradient of the line joining the points $A(x, 9)$ and $B(2, 8)$ is $\frac{1}{2}$.

(a) Find the value of x.

(b) Find the length of AB.

Solution

(a) Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$$\frac{8-9}{2-x} = \frac{1}{2}$$

$$-2 = 2 - x$$

$$x = 4$$

(b) Length of $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{(2-4)^2 + (8-9)^2}$$

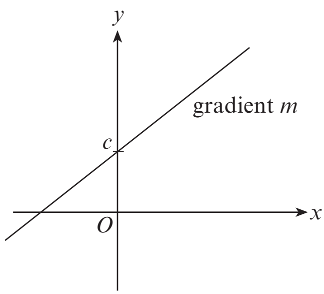
$$= \sqrt{(-2)^2 + (-1)^2}$$

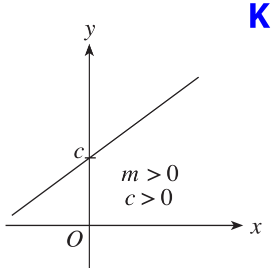
$$= \sqrt{5}$$

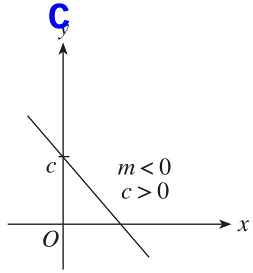
$$= 2.24 \text{ (to 3 s.f.)}$$

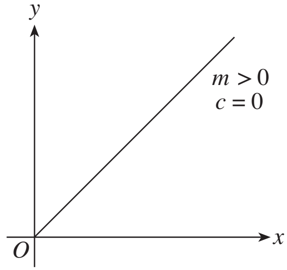
Equation of a Straight Line

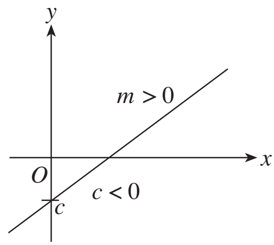
4. The equation of the straight line with gradient m and y-intercept c is $y = mx + c$.

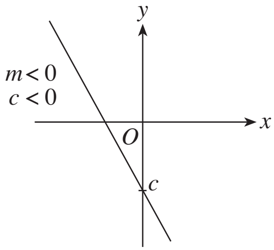


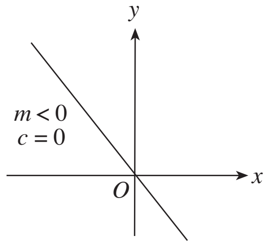

y
c
m > 0
c > 0
O
x

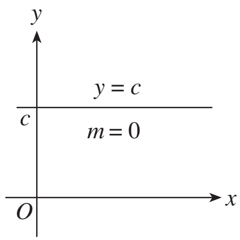

y
c
m < 0
c > 0
O
x

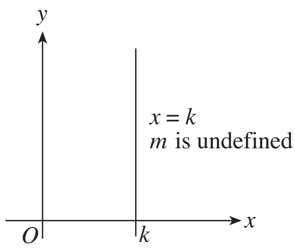

y
m > 0
c = 0
O
x


y
m > 0
O
x
c < 0
c


y
m < 0
c < 0
O
x
c


y
m < 0
c = 0
O
x


y
y = c
c
m = 0
O
x


y
x = k
m is undefined
O
k
x

Example 2

A line passes through the points $A(6, 2)$ and $B(5, 5)$. Find the equation of the line.

Solution

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{5-2}{5-6}$$
$$= -3$$

Equation of line: $y = mx + c$

$y = -3x + c$

To find c, we substitute $x = 6$ and $y = 2$ into the equation above. (We can find c by substituting the coordinates of any point that lies on the line into the equation)

$2 = -3(6) + c$

$c = 20$

$\therefore$ Equation of line: $y = -3x + 20$

5. The equation of a horizontal line is of the form $y = c$.

6. The equation of a vertical line is of the form $x = k$.

Example 3

The points A, B and C are $(8, 7)$, $(11, 3)$ and $(3, –3)$ respectively.

(a) Find the equation of the line parallel to AB and passing through C.

(b) Show that AB is perpendicular to BC.

(c) Calculate the area of triangle ABC.

Solution

(a) Gradient of $AB = \dfrac{3-7}{11-8}$

$$= -\frac{4}{3}$$

$$y = -\frac{4}{3}x + c$$

Substitute $x = 3$ and $y = –3$:

$$-3 = -\frac{4}{3}(3) + c$$

$$-3 = -4 + c$$

$$c = 1$$

$\therefore$ Equation of line: $y\ -\dfrac{4}{3}x + 1$

(b) $AB = \sqrt{(11-8)^2 + (3-7)^2}$

$= 5$ units

$BC = \sqrt{(3-11)^2 + (-3-3)^2}$

$= 10$ units

$AC = \sqrt{(3-8)^2 + (-3-7)^2}$

$= \sqrt{125}$ units

Since $AB^2 + BC^2 = 5^2 + 10^2$

$= 125$

$= AC^2$,

Pythagoras' Theorem can be applied.

$\therefore$ AB is perpendicular to BC.

(c) Area of $\Delta ABC = \dfrac{1}{2}(5)(10)$

$= 25$ units^2

K M C

UNIT 2.7 Vectors in Two Dimensions

(not included for NA)

Vectors

1. A vector has both magnitude and direction but a scalar has magnitude only.

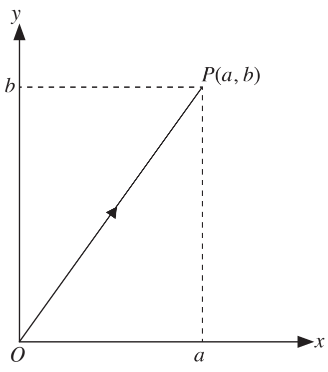
2. A vector may be represented by $\overrightarrow{OA}$, $\underset{\sim}{a}$ or $\mathbf{a}$.

Magnitude of a Vector

3. The magnitude of a column vector $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ is given by $|\mathbf{a}| = \sqrt{x^2 + y^2}$.

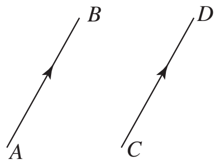
Position Vectors

4. If the point P has coordinates (a, b), then the position vector of P, $\overrightarrow{OP}$, is written as $\overrightarrow{OP} = \begin{pmatrix} a \\ b \end{pmatrix}$.



Equal Vectors

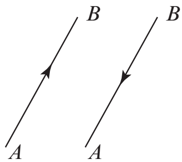
5. Two vectors are equal when they have the same direction and magnitude.



$\overrightarrow{AB} = \overrightarrow{CD}$

Negative Vector

6. $\overrightarrow{BA}$ is the negative of $\overrightarrow{AB}$.
$\overrightarrow{BA}$ is a vector having the same magnitude as $\overrightarrow{AB}$ but having direction opposite to that of $\overrightarrow{AB}$.
We can write $\overrightarrow{BA} = -\overrightarrow{AB}$ and $\overrightarrow{AB} = -\overrightarrow{BA}$.



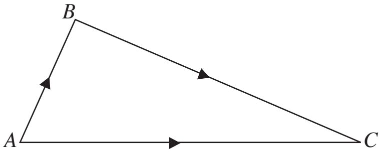
Zero Vector

7. A vector whose magnitude is zero is called a zero vector and is denoted by **0**.

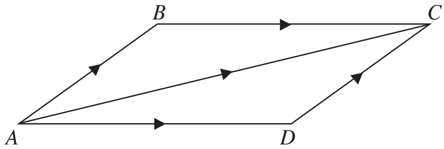
Sum and Difference of Two Vectors

8. The sum of two vectors, **a** and **b**, can be determined by using the Triangle Law or Parallelogram Law of Vector Addition.

9. Triangle law of addition: $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

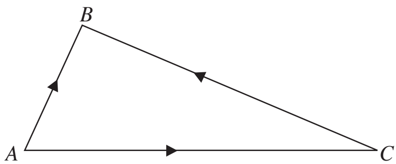


10. Parallelogram law of addition: $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$



11. The difference of two vectors, **a** and **b**, can be determined by using the Triangle Law of Vector Subtraction.

$\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{CB}$



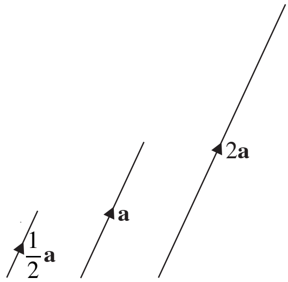
12. For any two column vectors $\mathbf{a} = \begin{pmatrix} p \\ q \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} r \\ s \end{pmatrix}$,

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p+r \\ q+s \end{pmatrix}$$

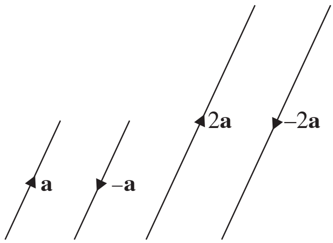
and $\mathbf{a} - \mathbf{b} = \begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p-r \\ q-s \end{pmatrix}$.

Scalar Multiple

13. When $k > 0$, $k\mathbf{a}$ is a vector having the same direction as that of $\mathbf{a}$ and magnitude equal to k times that of $\mathbf{a}$.

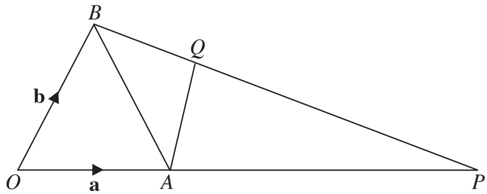


14. When $k < 0$, $k\mathbf{a}$ is a vector having a direction opposite to that of $\mathbf{a}$ and magnitude equal to k times that of $\mathbf{a}$.



Example 1

In $\Delta OAB, \overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.
O, A and P lie in a straight line, such that $OP = 3OA$.
Q is the point on BP such that $4BQ = PB$.



Express in terms of **a** and **b**,

(i) $\overrightarrow{BP}$,

(ii) $\overrightarrow{QB}$.

Solution

(i) $\overrightarrow{BP} = -\mathbf{b} + 3\mathbf{a}$

(ii) $\overrightarrow{QB} = \frac{1}{4}\overrightarrow{PB}$

$= \frac{1}{4}(-3\mathbf{a} + \mathbf{b})$

Parallel Vectors

15. If $\mathbf{a} = k\mathbf{b}$, where k is a scalar and $k \neq 0$, then $\mathbf{a}$ is parallel to $\mathbf{b}$ and $|\mathbf{a}| = k|\mathbf{b}|$.

16. If $\mathbf{a}$ is parallel to $\mathbf{b}$, then $\mathbf{a} = k\mathbf{b}$, where k is a scalar and $k \neq 0$.

Example 2

Given that $\begin{pmatrix} -15 \\ 9 \end{pmatrix}$ and $\begin{pmatrix} w \\ -3 \end{pmatrix}$ are parallel vectors, find the value of w.

Solution

Since $\begin{pmatrix} -15 \\ 9 \end{pmatrix}$ and $\begin{pmatrix} w \\ -3 \end{pmatrix}$ are parallel,

let $\begin{pmatrix} -15 \\ 9 \end{pmatrix} = k\begin{pmatrix} w \\ -3 \end{pmatrix}$, where k is a scalar.

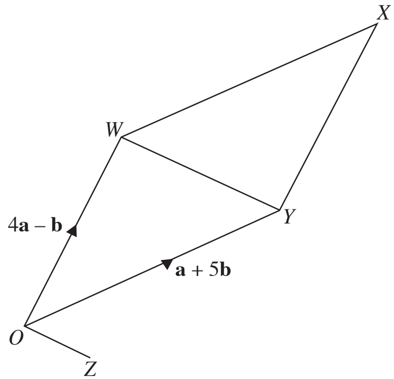
$$9 = -3k$$
$$k = -3$$

i.e. $\quad -15 = -3w$

$$w = 5$$

Example 3

Figure $WXYO$ is a parallelogram.



(a) Express, as simply as possible, in terms of **a** and/or **b**,

(i) $\overrightarrow{XY}$,

(ii) $\overrightarrow{WY}$.

(b) Z is the point such that $\overrightarrow{OZ} = -\mathbf{a} + 2\mathbf{b}$.

(i) Determine if $\overrightarrow{WY}$ is parallel to $\overrightarrow{OZ}$.

(ii) Given that the area of triangle OWY is 36 units^2, find the area of triangle OYZ.

Solution

(a) **(i)**
$$\begin{aligned}\overrightarrow{XY} &= -\overrightarrow{OW}\\ &= \mathbf{b} - 4\mathbf{a}\end{aligned}$$

(ii)
$$\begin{aligned}\overrightarrow{WY} &= \overrightarrow{WO} + \overrightarrow{OY}\\ &= \mathbf{b} - 4\mathbf{a} + \mathbf{a} + 5\mathbf{b}\\ &= -3\mathbf{a} + 6\mathbf{b}\end{aligned}$$

(b) **(i)**
$$\begin{aligned}\overrightarrow{OZ} &= -\mathbf{a} + 2\mathbf{b}\\ \overrightarrow{WY} &= -3\mathbf{a} + 6\mathbf{b}\\ &= 3(-\mathbf{a} + 2\mathbf{b})\end{aligned}$$

Since $\overrightarrow{WY} = 3\overrightarrow{OZ}, \overrightarrow{WY}$ is parallel to $\overrightarrow{OZ}$.

(ii) ΔOYZ and ΔOWY share a common height h.

$$\begin{aligned}\frac{\text{Area of } \Delta OYZ}{\text{Area of } \Delta OWY} &= \frac{\frac{1}{2} \times OZ \times h}{\frac{1}{2} \times WY \times h}\\ &= \frac{\left|\overrightarrow{OZ}\right|}{\left|\overrightarrow{WY}\right|}\\ &= \frac{1}{3}\\ \frac{\text{Area of } \Delta OYZ}{36} &= \frac{1}{3}\end{aligned}$$

$\therefore$ Area of $\Delta OYZ = 12$ units^2

17. If $m\mathbf{a} + n\mathbf{b} = h\mathbf{a} + k\mathbf{b}$, where m, n, h and k are scalars and $\mathbf{a}$ is parallel to $\mathbf{b}$, then $m = h$ and $n = k$.

Collinear Points

18. If the points A, B and C are such that $\overrightarrow{AB} = k\overrightarrow{BC}$, then A, B and C are collinear, i.e. A, B and C lie on the same straight line.

19. To prove that 3 points A, B and C are collinear, we need to show that:

$\overrightarrow{AB} = k\,\overrightarrow{BC}$ or $\overrightarrow{AB} = k\,\overrightarrow{AC}$ or $\overrightarrow{AC} = k\,\overrightarrow{BC}$

Example 4

Show that A, B and C lie in a straight line.

$\overrightarrow{OA} = p$ $\quad$ $\overrightarrow{OB} = q$ $\quad$ $\overrightarrow{BC} = \frac{1}{3}p - \frac{1}{3}q$

Solution

$$\overrightarrow{AB} = q - p$$

$$\begin{aligned} \overrightarrow{BC} &= \frac{1}{3}p - \frac{1}{3}q \\ &= -\frac{1}{3}(q - p) \\ &= -\frac{1}{3}\overrightarrow{AB} \end{aligned}$$

Thus $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$. Hence, the three points lie in a straight line.

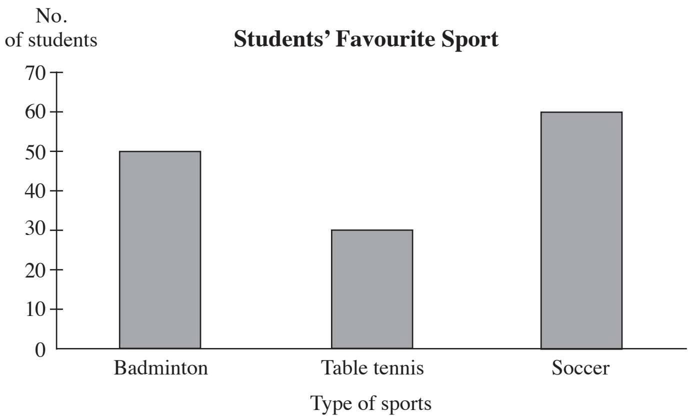
UNIT 3.1 Data Handling

Bar Graph

1. In a bar graph, each bar is drawn having the same width and the length of each bar is proportional to the given data.

2. An advantage of a bar graph is that the data sets with the lowest frequency and the highest frequency can be easily identified.

e.g.



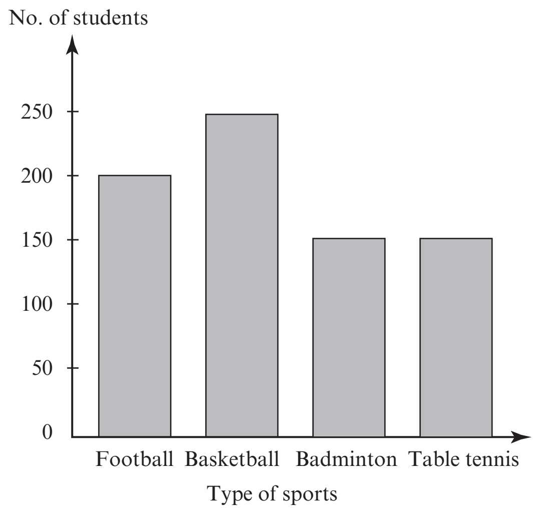
Example 1

The table shows the number of students who play each of the four types of sports.

Type of sports	**No. of students**
Football	200
Basketball	250
Badminton	150
Table tennis	150
Total	**750**

Represent the information in a bar graph.

Solution



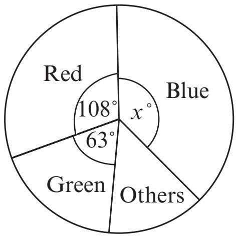
Pie Chart

3. A pie chart is a circle divided into several sectors and the angles of the sectors are proportional to the given data.

4. An advantage of a pie chart is that the size of each data set in proportion to the entire set of data can be easily observed.

Example 2

Each member of a class of 45 boys was asked to name his favourite colour. Their choices are represented on a pie chart.



(i) If 15 boys said they liked blue, find the value of x.
(ii) Find the percentage of the class who said they liked red.

Solution

(i) $x = \frac{15}{45} \times 360$
$= 120$

(ii) Percentage of the class who said they liked red $= \frac{108°}{360°} \times 100\%$
$= 30\%$

Histogram

5. A histogram is a vertical bar chart with no spaces between the bars (or rectangles). The frequency corresponding to a class is represented by the area of a bar whose base is the class interval.

6. An advantage of using a histogram is that the data sets with the lowest frequency and the highest frequency can be easily identified.

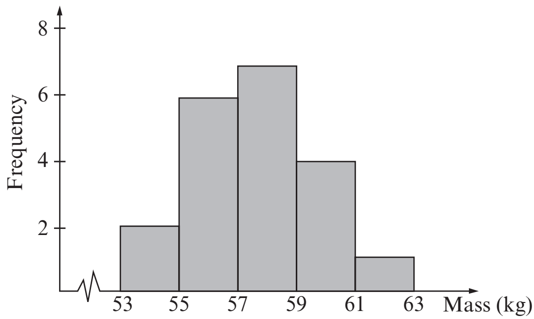
Example 3

The table shows the masses of the students in the school's track team.

Mass (m) in kg	Frequency
$53 < m \leqslant 55$	2
$55 < m \leqslant 57$	6
$57 < m \leqslant 59$	7
$59 < m \leqslant 61$	4
$61 < m \leqslant 63$	1

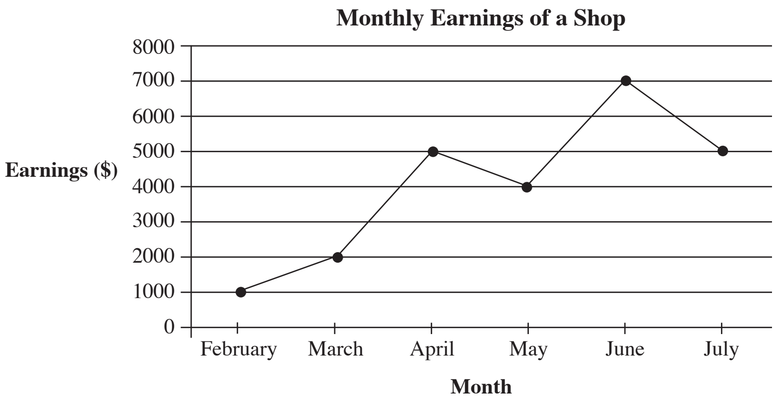
Represent the information on a histogram.

Solution



7. A line graph usually represents data that changes with time. Hence, the horizontal axis usually represents a time scale (e.g. hours, days, years).

e.g.



UNIT 3.2 Data Analysis

Dot Diagram

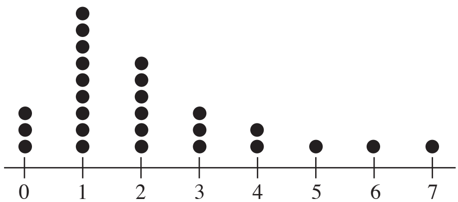
1. A dot diagram consists of a horizontal number line and dots placed above the number line, representing the values in a set of data.

Example 1

The table shows the number of goals scored by a soccer team during the tournament season.

Number of goals	0	1	2	3	4	5	6	7
Number of matches	3	9	6	3	2	1	1	1

The data can be represented on a dot diagram.



Example 2

The marks scored by twenty students in a placement test are as follows:

42	42	49	31	34	42	40	43	35	38
34	35	39	45	42	42	35	45	40	41

(a) Illustrate the information on a dot diagram.

(b) Write down

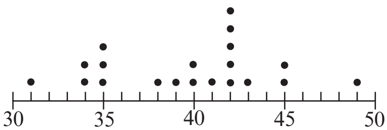
   (i) the lowest score,

   (ii) the highest score,

   (iii) the modal score.

Solution

(a)



(b) **(i)** Lowest score = 31

   (ii) Highest score = 49

   (iii) Modal score = 42

2. An advantage of a dot diagram is that it is an easy way to display small sets of data which do not contain many distinct values.

Stem-and-Leaf Diagram

3. In a stem-and-leaf diagram, the stems must be arranged in numerical order and the leaves must be arranged in ascending order.

4. An advantage of a stem-and-leaf diagram is that the individual data values are retained.

e.g. The ages of 15 shoppers are as follows:

32	34	13	29	38
36	14	28	37	13
42	24	20	11	25

The data can be represented on a stem-and-leaf diagram.

Stem	Leaf
1	1 3 3 4
2	0 4 5 8 9
3	2 4 6 7 8
4	2

Key: 1 | 3 means 13 years old
The tens are represented as stems and the ones are represented as leaves.
The values of the stems and the leaves are arranged in ascending order.

Stem-and-Leaf Diagram with Split Stems

5. If a stem-and-leaf diagram has more leaves on some stems, we can break each stem into two halves.

e.g. The stem-and-leaf diagram represents the number of customers in a store.

Stem	Leaf
4	0 3 5 8
5	1 3 3 4 5 6 8 8 9
6	2 5 7

Key: 4 | 0 means 40 customers

The information can be shown as a stem-and-leaf diagram with split stems.

Stem	Leaf
4	0 3 5 8
5	1 3 3 4
5	5 6 8 8 9
6	2 5 7

Key: 4 | 0 means 40 customers

Back-to-Back Stem-and-Leaf Diagram

6. If we have two sets of data, we can use a back-to-back stem-and-leaf diagram with a common stem to represent the data.

e.g. The scores for a quiz of two classes are shown in the table.

Class *A*	55	98	67	84	85	92	75	78	89	64
	72	60	86	91	97	58	63	86	92	74
Class *B*	56	67	92	50	64	83	84	67	90	83
	68	75	81	93	99	76	87	80	64	58

A back-to-back stem-and-leaf diagram can be constructed based on the given data.

Leaves for Class *B*	Stem	Leaves for Class *A*
8 6 0	5	5 8
8 7 7 4 4	6	0 3 4 7
6 5	7	2 4 5 8
7 4 3 3 1 0	8	4 5 6 6 9
9 3 2 0	9	1 2 2 7 8

Key: 58 means 58 marks

Note that the leaves for Class *B* are arranged in ascending order from the right to the left.

Measures of Central Tendency

7. The three common measures of central tendency are the mean, median and mode.

Mean

8. The mean of a set of n numbers $x_1, x_2, x_3, \ldots, x_n$ is denoted by $\bar{x}$.

9. For ungrouped data,

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} = \frac{\sum x}{n}.$$

10. For grouped data,

$$\bar{x} = \frac{\sum fx}{\sum f}$$

where f is the frequency of data in each class interval and x is the mid-value of the interval.

Median

11. The median is the value of the middle term of a set of numbers arranged in ascending order.

Given a set of n terms, if n is odd, the median is the $\left(\frac{n+1}{2}\right)^{\text{th}}$ term;

if n is even, the median is the average of the two middle terms.

e.g. Given the set of data: 5, 6, $\downarrow$7, 8, 9.
There is an odd number of data.
Hence, median is 7.

e.g. Given a set of data 5, 6, $\downarrow$ 7, 8.
There is an even number of data.
Hence, median is 6.5.

Example 3

The table records the number of mistakes made by 60 students during an exam.

Number of students	24	x	13	y	5
Number of mistakes	5	6	7	8	9

(a) Show that $x + y = 18$.
(b) Find an equation of the mean, given that the mean number of mistakes made is 6.3. Hence, find the values of x and y.
(c) State the median number of mistakes made.

Solution

(a) Since there are 60 students in total,

$$24 + x + 13 + y + 5 = 60$$
$$x + y + 42 = 60$$
$$x + y = 18$$

(b) Since the mean number of mistakes made is 6.3,

$$\text{Mean} = \frac{\text{Total number of mistakes made by 60 students}}{\text{Number of students}}$$

$$6.3 = \frac{24(5)+6x+13(7)+8y+5(9)}{60}$$

$6.3(60) = 120 + 6x + 91 + 8y + 45$

$378 = 256 + 6x + 8y$

$6x + 8y = 122$

$3x + 4y = 61$

To find the values of x and y, solve the pair of simultaneous equations obtained above.

$x + y = 18$ —— (1)

$3x + 4y = 61$ —— (2)

$3 \times (1)$:

$3x + 3y = 54$ —— (3)

$(2) - (3)$: $y = 7$

When $y = 7$, $x = 11$.

(c) Since there are 60 students, the 30$^{\text{th}}$ and 31$^{\text{st}}$ students are in the middle. The 30$^{\text{th}}$ and 31$^{\text{st}}$ students make 6 mistakes each. Therefore, the median number of mistakes made is 6.

Mode

12. The mode of a set of numbers is the number with the highest frequency.

13. If a set of data has two values which occur the most number of times, we say that the distribution is bimodal.

e.g. Given a set of data: 5, 6, 6, 6, 7, 7, 8, 8, 9.
6 occurs the most number of times.
Hence, the mode is 6.

Standard Deviation

14. The standard deviation, s, measures the spread of a set of data from its mean.

15. For ungrouped data,

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n}} \quad \text{or} \quad s = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2}$$

16. For grouped data,

$$s = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} \quad \text{or} \quad s = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$$

Example 4

The following set of data shows the number of books borrowed by 20 children during their visit to the library.

0, 2, 4, 3, 1, 1, 2, 0, 3, 1
1, 2, 1, 1, 2, 3, 2, 2, 1, 2

Calculate the standard deviation.

Solution

Represent the set of data in the table below.

Number of books borrowed	0	1	2	3	4
Number of children	2	7	7	3	1

Standard deviation

$$= \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$$

$$= \sqrt{\frac{0^2(2)+1^2(7)+2^2(7)+3^2(3)+4^2(1)}{20} - \left(\frac{0(2)+1(7)+2(7)+3(3)+4(1)}{20}\right)^2}$$

$$= \sqrt{\frac{78}{20} - \left(\frac{34}{20}\right)^2}$$

$= 1.00$ (to 3 s.f.)

Example 5

The mass, in grams, of 80 stones are given in the table.

Mass (m) in grams	**Frequency**
$20 < m \leqslant 30$	20
$30 < m \leqslant 40$	30
$40 < m \leqslant 50$	20
$50 < m \leqslant 60$	10

Find the mean and the standard deviation of the above distribution.

Solution

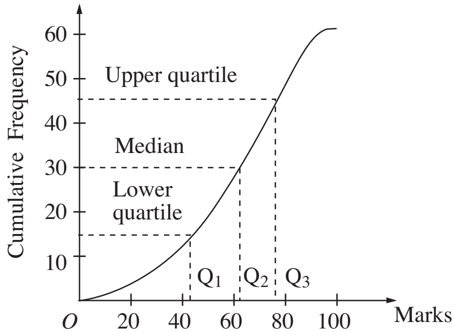
Mid-value (x)	**Frequency (f)**	fx	fx^2
25	20	500	12 500
35	30	1050	36 750
45	20	900	40 500
55	10	550	30 250

$$\text{Mean} = \frac{\sum fx}{\sum f}$$
$$= \frac{500 + 1050 + 900 + 550}{20 + 30 + 20 + 10}$$
$$= \frac{3000}{80}$$
$$= 37.5 \text{ g}$$

$$\text{Standard deviation} = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$$
$$= \sqrt{\frac{12\,500 + 36\,750 + 40\,500 + 30\,250}{80} - \left(\frac{3000}{80}\right)^2}$$
$$= \sqrt{1500 - 37.5^2}$$
$$= 9.68 \text{ g (to 3 s.f.)}$$

Cumulative Frequency Curve

17. The following figure shows a cumulative frequency curve.



18. Q_1 is called the lower quartile or the 25^{th} percentile.

19. Q_2 is called the median or the 50^{th} percentile.

20. Q_3 is called the upper quartile or the 75^{th} percentile.

21. $Q_3 - Q_1$ is called the interquartile range.

Example 6

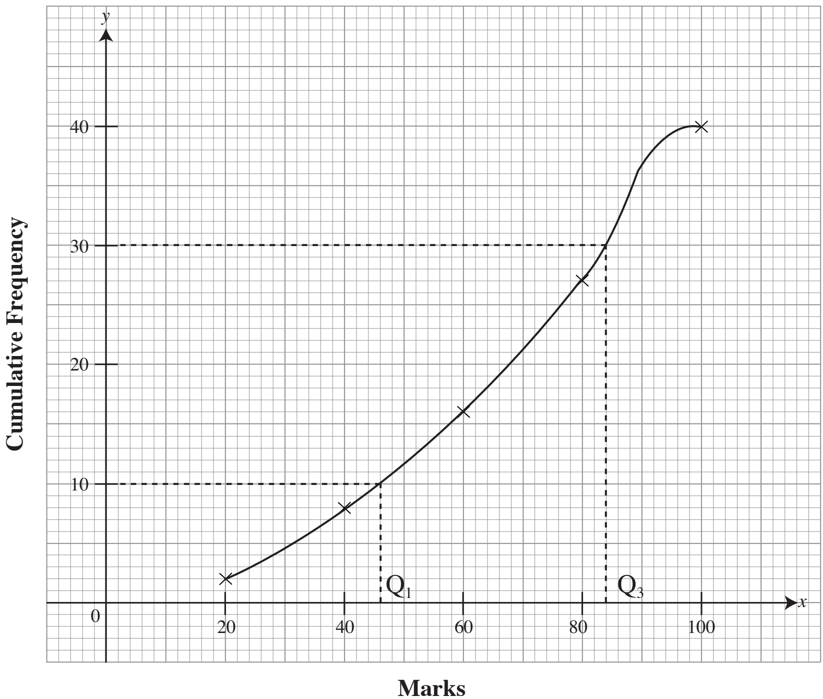
The exam results of 40 students were recorded in the frequency table below.

Mass (m) in grams	**Frequency**
$0 < m \leqslant 20$	2
$20 < m \leqslant 40$	4
$40 < m \leqslant 60$	8
$60 < m \leqslant 80$	14
$80 < m \leqslant 100$	12

Construct a cumulative frequency table and the draw a cumulative frequency curve. Hence, find the interquartile range.

Solution

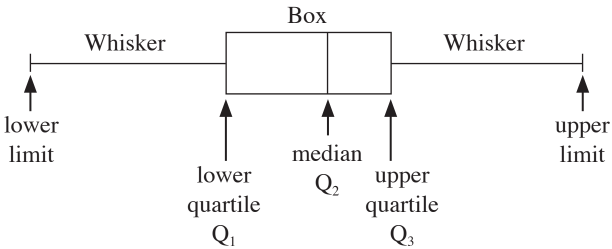
Mass (m) in grams	Cumulative Frequency
$x \leqslant 20$	2
$x \leqslant 40$	6
$x \leqslant 60$	14
$x \leqslant 80$	28
$x \leqslant 100$	40



Lower quartile = 46
Upper quartile = 84
Interquartile range = 84 – 46
= 38

Box-and-Whisker Plot

22. The following figure shows a box-and-whisker plot.



23. A box-and-whisker plot illustrates the range, the quartiles and the median of a frequency distribution.

UNIT 3.3 Probability

1. Probability is a measure of chance.

2. A sample space is the collection of all the possible outcomes of a probability experiment.

Example 1

A fair six-sided die is rolled. Write down the sample space and state the total number of possible outcomes.

Solution

A die has the numbers 1, 2, 3, 4, 5 and 6 on its faces,
i.e. the sample space consists of the numbers 1, 2, 3, 4, 5 and 6.

Total number of possible outcomes = 6

3. In a probability experiment with m equally likely outcomes, if k of these outcomes favour the occurrence of an event E, then the probability, P(E), of the event happening is given by

$$P(E) = \frac{\text{Number of favourable outcomes for event } E}{\text{Total number of possible outcomes}} = \frac{k}{m}.$$

Example 2

A card is drawn at random from a standard pack of 52 playing cards.
Find the probability of drawing

(i) a King,

(ii) a spade.

Solution

Total number of possible outcomes = 52

(i) $P(\text{drawing a King}) = \frac{4}{52}$ (There are 4 Kings in a deck.)

$= \frac{1}{13}$

(ii) $P(\text{drawing a Spade}) = \frac{13}{52}$ (There are 13 spades in a deck.)

Properties of Probability

4. For any event E, $0 \leqslant P(E) \leqslant 1$.

5. If E is an impossible event, then $P(E) = 0$, i.e. it will never occur.

6. If E is a certain event, then $P(E) = 1$, i.e. it will definitely occur.

7. If E is any event, then $P(E') = 1 - P(E)$, where $P(E')$ is the probability that event E does not occur.

Mutually Exclusive Events

8. If events A and B cannot occur together, we say that they are mutually exclusive.

9. If A and B are mutually exclusive events, their sets of sample spaces are disjoint, i.e. $P(A \cup B) = P(A \text{ or } B) = P(A) + P(B)$.

Example 3

A card is drawn at random from a standard pack of 52 playing cards. Find the probability of drawing a Queen or an ace.

Solution

Total number of possible outcomes = 52

P(drawing a Queen or an ace) = P(drawing a Queen) + P(drawing an ace)

$$= \frac{4}{52} + \frac{4}{52}$$

$$= \frac{2}{13}$$

Independent Events

10. If A and B are independent events, the occurrence of A does not affect that of B, i.e. $P(A \cap B) = P(A \text{ and } B) = P(A) \times P(B)$.

Possibility Diagrams and Tree Diagrams

11. Possibility diagrams and tree diagrams are useful in solving probability problems. The diagrams are used to list all possible outcomes of an experiment.

12. The sum of probabilities on the branches from the same point is 1.

Example 4

A red fair six-sided die and a blue fair six-sided die are rolled at the same time.

(a) Using a possibility diagram, show all the possible outcomes.

(b) Hence, find the probability that

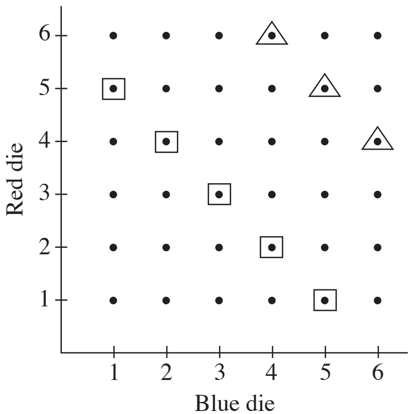
(i) the sum of the numbers shown is 6,

(ii) the sum of the numbers shown is 10,

(iii) the red die shows a '3' and the blue die shows a '5'.

Solution

(a)



(b) Total number of possible outcomes = 6 × 6 = 36

(i) There are 5 ways of obtaining a sum of 6, as shown by the squares on the diagram.

$\therefore$ P(sum of the numbers shown is 6) = $\frac{5}{36}$

(ii) There are 3 ways of obtaining a sum of 10, as shown by the triangles on the diagram.

$$\therefore \text{P(sum of the numbers shown is 10)} = \frac{3}{36} = \frac{1}{12}$$

(iii) P(red die shows a '3') = $\frac{1}{6}$

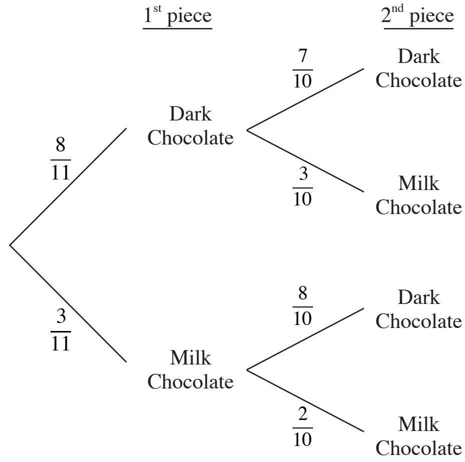
P(blue die shows a '5') = $\frac{1}{6}$

$$\text{P(red die shows a '3' and blue die shows a '5')} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

Example 5

A box contains 8 pieces of dark chocolate and 3 pieces of milk chocolate. Two pieces of chocolate are taken from the box, without replacement. Find the probability that both pieces of chocolate are dark chocolate.

Solution



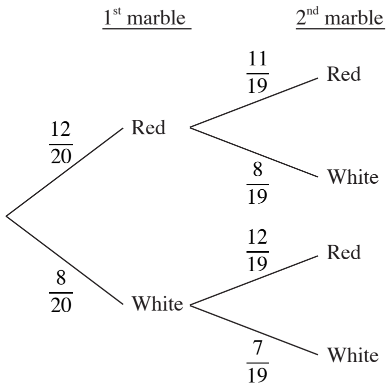
$$\text{P(both pieces of chocolate are dark chocolate)} = \frac{8}{11} \times \frac{7}{10}$$
$$= \frac{28}{55}$$

Example 6

A box contains 20 similar marbles. 8 marbles are white and the remaining 12 marbles are red. A marble is picked out at random and not replaced. A second marble is then picked out at random.
Calculate the probability that
(i) both marbles will be red,
(ii) there will be one red marble and one white marble.

Solution

Use a tree diagram to represent the possible outcomes.



(i) P(two red marbles) $= \dfrac{12}{20} \times \dfrac{11}{19}$

$= \dfrac{33}{95}$

(ii) P(one red marble and one white marble)

$= \left(\dfrac{12}{20} \times \dfrac{8}{19}\right) + \left(\dfrac{8}{20} \times \dfrac{12}{19}\right)$ (A red marble may be chosen first, followed by a white marble, and vice versa)

$= \dfrac{48}{95}$

Example 7

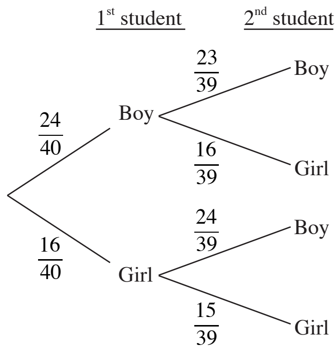
A class has 40 students. 24 are boys and the rest are girls. Two students were chosen at random from the class. Find the probability that

(i) both students chosen are boys,

(ii) a boy and a girl are chosen.

Solution

Use a tree diagram to represent the possible outcomes.



(i) P(both are boys) $= \dfrac{24}{40} \times \dfrac{23}{39}$

$= \dfrac{23}{65}$

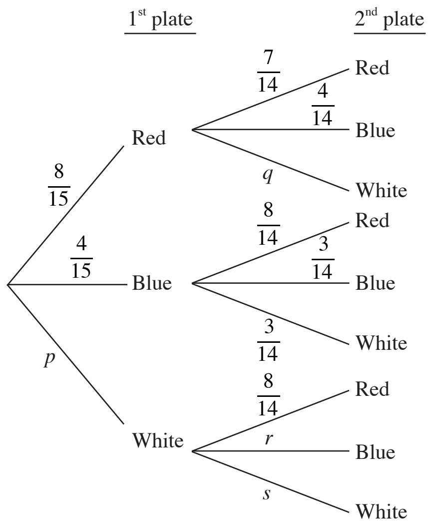
(ii) P(one boy and one girl) $= \left(\dfrac{24}{40} \times \dfrac{16}{39}\right) + \left(\dfrac{16}{40} \times \dfrac{24}{39}\right)$

$= \dfrac{32}{65}$

(A boy may be chosen first, followed by the girl, and vice versa)

Example 8

A box contains 15 identical plates. There are 8 red, 4 blue and 3 white plates. A plate is selected at random and not replaced. A second plate is then selected at random and not replaced. The tree diagram shows the possible outcomes and some of their probabilities.



(a) Find the values of p, q, r and s.

(b) Expressing each of your answers as a fraction in its lowest terms, find the probability that

  (i) both plates are red,

  (ii) one plate is red and one plate is blue.

(c) A third plate is now selected at random. Find the probability that none of the three plates is white.

(a) $p = 1 - \frac{8}{15} - \frac{4}{15}$

$= \frac{1}{5}$

$q = 1 - \frac{7}{14} - \frac{4}{14}$

$= \frac{3}{14}$

$r = \frac{4}{14}$

$= \frac{2}{7}$

$s = 1 - \frac{8}{14} - \frac{2}{7}$

$= \frac{1}{7}$

(b) **(i)** P(both plates are red) $= \frac{8}{15} \times \frac{7}{14}$

$= \frac{4}{15}$

(ii) P(one plate is red and one plate is blue) $= \frac{8}{15} \times \frac{4}{14} + \frac{4}{15} \times \frac{8}{14}$

$= \frac{32}{105}$

(c) P(none of the three plates is white) $= \frac{8}{15} \times \frac{11}{14} \times \frac{10}{13} + \frac{4}{15} \times \frac{11}{14} \times \frac{10}{13}$

$= \frac{44}{91}$

MATHEMATICAL FORMULAE

Compound interest

$$\text{Total amount} = P\left(1+\frac{r}{100}\right)^n$$

Mensuration

$$\text{Curved surface area of a cone } = \pi r l$$

$$\text{Surface area of a sphere } = 4\pi r^2$$

$$\text{Volume of a cone } = \frac{1}{3}\pi r^2 h$$

$$\text{Volume of a sphere } = \frac{4}{3}\pi r^3$$

$$\text{Area of triangle } ABC = \frac{1}{2}ab\sin C$$

$$\text{Arc length} = r\theta, \text{ where } \theta \text{ is in radians}$$

$$\text{Sector area} = \frac{1}{2}r^2\theta, \text{ where } \theta \text{ is in radians}$$

Trigonometry

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Statistics

$$\text{Mean} = \frac{\sum fx}{\sum f}$$

$$\text{Standard deviation} = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$$

CONTENTS

UNIT 1 Simultaneous Equations, Polynomials and Partial Fractions

Simultaneous Linear Equations

1. The solution(s) of a pair of linear and/or non-linear equations correspond to the coordinates of the intersection point(s) of the graphs.

2. A pair of simultaneous linear equations is of the form

$$ax + by = p$$
$$cx + dy = q,$$

where

a, b, c and d are constants,

x and y are variables to be determined.

3. There is usually one solution to a pair of simultaneous linear equations.

4. Methods of solving simultaneous linear equations:

- Elimination (covered in 'O' level Mathematics)
- Substitution (covered in 'O' level Mathematics)
- Matrix method (not in syllabus)
- Graphical method (covered in 'O' level Mathematics)

5. The methods most commonly used to solve simultaneous linear equations are

- **Elimination**
  The coefficient of one of the variables is made the same in both equations. The equations are then either added or subtracted to form a single linear equation with only one variable.

Example 1

Solve the simultaneous equations

$$2x + 3y = 15$$
$$-3y + 4x = 3$$

Solution

$2x + 3y = 15$ —— (1)
$-3y + 4x = 3$ —— (2)

(1) + (2):

$$(2x + 3y) + (-3y + 4x) = 18$$
$$6x = 18$$
$$x = 3$$

When $x = 3$, $y = 3$.

- **Substitution**
  A variable is made the subject of the chosen equation. This equation is then substituted into the equation that was not chosen to solve for the variable.

Example 2

Solve the simultaneous equations

$$2x - 3y = -2,$$
$$y + 4x = 24.$$

Solution

$2x - 3y = -2$ —— (1)
$y + 4x = 24$ —— (2)

From (1):

$$x = \frac{-2 + 3y}{2}$$

$$x = -1 + \frac{3}{2}y \text{ —— (3)}$$

Substitute (3) into (2):

$$y + 4\left(-1 + \frac{3}{2}y\right) = 24$$
$$y - 4 + 6y = 24$$
$$7y = 28$$
$$y = 4$$

When $y = 4$, $x = 5$.

Simultaneous Non-Linear Equations

6. A non-linear equation is **not** of the form $ax + by = p$.

7. Methods of solving simultaneous non-linear equations:
- Substitution
- Graphical method (covered in ‘O’ level Mathematics)

8. The method most commonly used to solve simultaneous non-linear equations is

- Substitution

9. The substitution method:

Step 1: Use the linear equation to express one of the variables in terms of the other.
Step 2: Substitute it into the non-linear equation.
Step 3: Substitute the value(s) obtained in Step 2 into the linear equation to obtain the value of the other variable.

Example 3

Solve the following pair of simultaneous equations.

$$3y = x + 3$$
$$y^2 = 13 + 2x$$

Solution

$3y = x + 3$ —— (1)
$y^2 = 13 + 2x$ —— (2)

From (1):

$y = \frac{1}{3}x + 1$ —— (3) (Use the linear equation to express y in terms of x.)

Substitute (3) into (2):

$$\left(\frac{1}{3}x + 1\right)^2 = 13 + 2x$$

$$\frac{1}{9}x^2 + \frac{2}{3}x + 1 = 13 + 2x$$
$$\frac{1}{9}x^2 - \frac{4}{3}x - 12 = 0$$
$$x^2 - 12x - 108 = 0$$
$$(x - 18)(x + 6) = 0$$
$$x = 18 \text{ or } x = -6$$

When $x = 18$, $y = 7$. (Substitute the values of x into the linear equation to obtain
When $x = -6$, $y = -1$. the corresponding values of y.)

$\therefore x = 18, y = 7$ or $x = -6, y = -1$

Example 4

Solve the simultaneous equations

$$x^2 - 2y^2 = -17,$$
$$x - y = -4.$$

Solution

$x^2 - 2y^2 = -17$ —— (1)

$x - y = -4$ —— (2)

From (2),

$y = x + 4$ —— (3) (Use the linear equation to express y in terms of x.)

Substitute (3) into (1):

$$\begin{aligned} x^2 - 2(x+4)^2 &= -17 \\ x^2 - 2x^2 - 16x - 32 &= -17 \\ -x^2 - 16x - 15 &= 0 \\ x^2 + 16x + 15 &= 0 \\ (x+1)(x+15) &= 0 \\ x = -1 \text{ or } x &= -15 \end{aligned}$$

(Substitute the linear equation into the non-linear equation.)

(Factorise the quadratic expression.)

When $x = -1$, $y = 3$.

When $x = -15$, $y = -11$.

(Substitute the values of x into the linear equation to obtain the corresponding values of y.)

$\therefore x = -1, y = 3$ or $x = -15, y = -11$

Example 5

The line $2x + y = 5$ meets the curve $x^2 + y^2 + x + 12y - 29 = 0$ at the points A and B. Find the coordinates of A and B.

Solution

$$2x + y = 5 \quad — (1)$$
$$x^2 + y^2 + x + 12y - 29 = 0 \quad — (2)$$

From (1),
$y = 5 - 2x$ — (3) (Use the linear equation to express y in terms of x.)

Substitute (3) into (2):

$$x^2 + (5 - 2x)^2 + x + 12(5 - 2x) - 29 = 0$$
$$x^2 + 25 - 20x + 4x^2 + x + 60 - 24x - 29 = 0$$

(Substitute the linear equation into the non-linear equation.)

$$5x^2 - 43x + 56 = 0$$
$$(5x - 8)(x - 7) = 0$$

(Factorise the quadratic expression.)

$$x = 1\frac{3}{5} \text{ or } x = 7$$

When $x = 1\frac{3}{5}$, $y = 1\frac{4}{5}$.
When $x = 7$, $y = -9$.

(Substitute the values of x into the linear equation to obtain the corresponding values of y.)

$\therefore$ The coordinates of A and B are $\left(1\frac{3}{5}, 1\frac{4}{5}\right)$ and $(7, -9)$.

Definitions

10. A polynomial in x is a mathematical expression of a sum of terms, each of the form ax^n, where a is a constant and n is a non-negative integer. It is usually denoted as f(x).
i.e. $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$

11. Examples of polynomials include $x^3 + 2x - 1$, $6x^4 - \frac{1}{2}x^2$ and $-0.2x + x^2 + 5x^3$.
Examples of non-polynomials include $2x^2 + \frac{1}{x}$, $4 - \sqrt{x}$ and $x + x^{\frac{2}{3}}$.

12. $a_n, a_{n-1}, \ldots, a_0$ are coefficients.
a_0 is also called the constant term.

13. The degree (or order) of a polynomial in x is given by the highest power of x.
For example, the degree of $6x^3 - 2x^2 + x - 8$ is 3 and the degree of $1 - x + 5x^4$ is 4.

14. The value of f(x) at $x = c$ is f(c).
For example, if $f(x) = 2x^3 + x^2 - x - 4$, then the value of f($x$) at $x = 1$ is
$f(1) = 2(1)^3 + 1^2 - 1 - 4 = -2$.

Identities

15. An identity is an equation in which the expression on the LHS (left-hand side) is equal to the expression on the RHS (right-hand side).

16. Methods of finding the unknown constants in an identity:
- By substitution of special values of x
- By comparing coefficients

Example 6

It is given that for all values of x, $2x^3 + 5x^2 - x - 2 = (Ax + 3)(x + B)(x - 1) + C$.
Find the values of A, B and C.

Solution

Let $x = 1$: $2(1)^3 + 5(1)^2 - 1 - 2 = (A + 3)(1 + B)(1 - 1) + C$
$$C = 4$$
(Letting x be 1 leaves us with 1 unknown, C.)

Let $x = 0$: $2(0)^3 + 5(0)^2 - 0 - 2 = (0 + 3)(0 + B)(0 - 1) + 4$
$$B = 2$$
(Letting x be 0 leaves us with 1 unknown, B.)

Comparing coefficients of x^3,
$$A = 2$$
$\therefore A = 2, B = 2$ and $C = 4$

Example 7

Given that $2x^3 + 3x^2 - 14x - 5 = (2x - 3)(x + 3)Q(x) + ax + b$, where $Q(x)$ is a polynomial, find the value of a and of b.

Solution

Let $x = -3$: $2(-3)^3 + 3(-3)^2 - 14(-3) - 5 = -3a + b$
$$10 = -3a + b$$
$$3a - b = -10 \quad \text{— (1)}$$

Let $x = \frac{3}{2}$: $2\left(\frac{3}{2}\right)^3 + 3\left(\frac{3}{2}\right)^2 - 14\left(\frac{3}{2}\right) - 5 = \frac{3}{2}a + b$
$$-\frac{25}{2} = \frac{3}{2}a + b$$
$$3a + 2b = -25 \quad \text{— (2)}$$

(2) − (1): $3b = -15$
$$b = -5$$
$$a = -5$$
$\therefore a = -5, b = -5$

Long Division

17. When $3x^3 + 4x^2 - 6x + 3$ is divided by $x - 1$,

- the dividend is $3x^3 + 4x^2 - 6x + 3$
- the quotient is $3x^2 + 7x + 1$
- the divisor is $x - 1$
- the remainder is 4.

$$
\begin{array}{rrl}
 & 3x^2 + 7x + 1 & \longleftarrow \text{Quotient} \\
\text{Divisor} \longrightarrow x-1 & \overline{)\,3x^3 + 4x^2 - 6x + 3} & \longleftarrow \text{Dividend} \\
 & \underline{-(3x^3 - 3x^2)} & \\
 & 7x^2 - 6x + 3 & \\
 & \underline{-(7x^2 - 7x)} & \\
 & x + 3 & \\
 & \underline{-(x - 1)} & \\
 & 4 & \longleftarrow \text{Remainder}
\end{array}
$$

18. Dividend = Quotient × Divisor + Remainder

19. The order of the remainder is always at least one degree less than that of the divisor.

20. The process of long division is stopped when the degree of the remainder is less than the degree of the divisor.

Synthetic Method

21. The synthetic method can be used to divide a polynomial by a linear divisor.

To divide $3x^3 + 4x^2 - 6x + 3$ by $x - 1$,

$$
\begin{array}{r|rrrrl}
1 & 3 & 4 & -6 & 3 & \longleftarrow \text{Coefficients of the Dividend} \\
 & & 3 & 7 & 1 & \\ \hline
 & \multicolumn{3}{c}{\underbrace{3 \quad\; 7 \quad\; 1}_{\text{Coefficients of the Quotient}}} & \underset{\substack{\uparrow \\ \text{Remainder}}}{4} &
\end{array}
$$

22. The Remainder Theorem states that when a polynomial $f(x)$ is divided by $ax - b$, the remainder is $f\left(\frac{b}{a}\right)$.

23. If $f(x)$ is divided by a quadratic divisor, then the remainder is a linear function or a constant.

Example 8

Find the remainder when $x^3 - 2x^2 + 3x - 1$ is divided by $x - 1$.

Solution

Let $f(x) = x^3 - 2x^2 + 3x - 1$.

By Remainder Theorem,

$$\begin{aligned}\text{The remainder is } f(1) &= (1)^3 - 2(1)^2 + 3(1) - 1\\ &= 1 - 2 + 3 - 1\\ &= 1\end{aligned}$$

Example 9

Given that $f(x) = ax^3 - 8x^2 - 9x + b$ is exactly divisible by $3x - 2$ and leaves a remainder of 6 when divided by x, find the value of a and of b.

Solution

Since $f\left(\frac{2}{3}\right) = 0$,

$$a\left(\frac{2}{3}\right)^3 - 8\left(\frac{2}{3}\right)^2 - 9\left(\frac{2}{3}\right) + b = 0$$

$$\frac{8}{27}a - \frac{32}{9} - 6 + b = 0$$

$$8a - 96 - 162 + 27b = 0$$

$$8a + 27b = 258 \quad \text{— (1)}$$

Since $f(0) = 6$,

$$a(0)^3 - 8(0)^2 - 9(0) + b = 6$$

$$b = 6 \quad \text{— (2)}$$

Substitute $b = 6$ into (1):

$$8a + 27(6) = 258$$

$$a = 12$$

$\therefore a = 12, b = 6$

Example 10

Given that $f(x) = 6x^3 + 7x^2 - x + 3$, find the remainder when $f(x)$ is divided by $x + 1$.

Solution

Method 1: Long division

$$
\begin{array}{r|l}
 & 6x^2 + x - 2 \\
\hline
x+1 & 6x^3 + 7x^2 - x + 3 \\
 & \underline{-(6x^3 + 6x^2)} \\
 & \qquad x^2 - x + 3 \\
 & \qquad \underline{-(x^2 + x)} \\
 & \qquad\qquad -2x + 3 \\
 & \qquad\qquad \underline{-(-2x - 2)} \\
 & \qquad\qquad\qquad 5
\end{array}
$$

$\therefore$ The remainder is 5.

Method 2: Synthetic method

$$
\begin{array}{r|rrrr}
-1 & 6 & 7 & -1 & 3 \\
 & & -6 & -1 & 2 \\
\hline
 & 6 & 1 & -2 & 5
\end{array}
$$

$\therefore$ The remainder is 5.

Method 3: Remainder Theorem

$f(x) = 6x^3 + 7x^2 - x + 3$

$f(-1) = 6(-1)^3 + 7(-1)^2 - (-1) + 3$

$= 5$

$\therefore$ The remainder is 5.

Factor Theorem

24. The Factor Theorem states that when a polynomial $f(x)$ is divided by $ax - b$ and that $f\left(\frac{b}{a}\right) = 0$, then $ax - b$ is a factor of $f(x)$.

25. Conversely, if $ax - b$ is a factor of $f(x)$, then $f\left(\frac{b}{a}\right) = 0$ and $f(x)$ is divisible by $ax - b$.

Example 11

Given that $x + 2$ is a factor of $x^3 + ax^2 - x + 4$, calculate the value of a.

Solution

Let $f(x) = x^3 + ax^2 - x + 4$.

Since $x + 2$ is a factor of $f(x)$, by Factor Theorem,

$$\begin{aligned} f(-2) &= 0 \\ (-2)^3 + a(-2)^2 - (-2) + 4 &= 0 \\ -8 + 4a + 2 + 4 &= 0 \\ -2 + 4a &= 0 \\ a &= \frac{1}{2} \end{aligned}$$

Example 12

Prove that $x + 2$ is a factor of $4x^3 - 13x + 6$. Hence solve the equation $4x^3 - 13x + 6 = 0$.

Solution

Let $f(x) = 4x^3 - 13x + 6$. (To prove that $x + 2$ is a factor of $f(x)$, we need to show that $f(-2) = 0$.)

$$\begin{aligned} f(-2) &= 4(-2)^3 - 13(-2) + 6 \\ &= 0 \end{aligned}$$

$\therefore x + 2$ is a factor of $4x^3 - 13x + 6$.

Now $f(x) = 4x^3 - 13x + 6 = (x + 2)(4x^2 + kx + 3)$, where k is a constant.

Comparing coefficients of x^2,

$$\begin{aligned} 0 &= 8 + k \\ k &= -8 \end{aligned}$$

i.e. $f(x) = (x + 2)(4x^2 - 8x + 3)$

$\quad\quad = (x + 2)(2x - 1)(2x - 3)$

To solve $4x^3 - 13x + 6 = 0$,

$$(x + 2)(2x - 1)(2x - 3) = 0$$

$\therefore x = -2$ or $x = \frac{1}{2}$ or $x = \frac{3}{2}$

Example 13

Given that $4x^3 + ax^2 + bx + 2$ is exactly divisible by $x^2 - 3x + 2$, find the value of a and of b. Hence sketch the graph of $y = 4x^3 + ax^2 + bx + 2$ for the values of a and b found.

Solution

Let $f(x) = 4x^3 + ax^2 + bx + 2$.
Since $x^2 - 3x + 2 = (x - 1)(x - 2)$,
$f(x)$ is exactly divisible by $(x - 1)(x - 2)$, (Factorise the quadratic divisor.)
i.e. $f(1) = 0$ and $f(2) = 0$.
When $f(1) = 0$,

$$\begin{aligned} 4(1)^3 + a(1)^2 + b(1) + 2 &= 0 \\ 4 + a + b + 2 &= 0 \\ a + b &= -6 \quad \text{— (1)} \end{aligned}$$

When $f(2) = 0$,

$$\begin{aligned} 4(2)^3 + a(2)^2 + b(2) + 2 &= 0 \\ 32 + 4a + 2b + 2 &= 0 \\ 4a + 2b &= -34 \\ 2a + b &= -17 \quad \text{— (2)} \end{aligned}$$

(2) – (1):

$a = -11$

$b = 5$

$\therefore a = -11, b = 5$

$f(x) = 4x^3 - 11x^2 + 5x + 2 = (x^2 - 3x + 2)(px + q)$

Comparing coefficients of x^3,

$p = 4$

Comparing constants,

$2 = 2q$

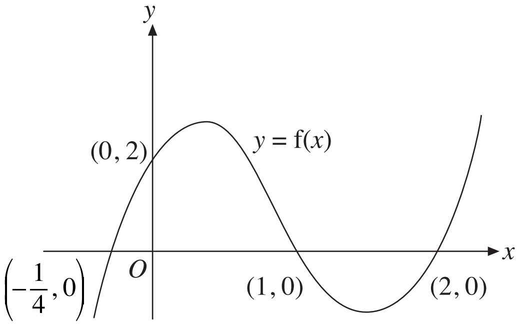
$q = 1$

$$\begin{aligned} f(x) &= (x^2 - 3x + 2)(4x + 1) \\ &= (x - 1)(x - 2)(4x + 1) \end{aligned}$$

When $f(x) = 0$,

$x = 1$ or $x = 2$ or $x = -\frac{1}{4}$. (It is a good practice to find the intercepts with the coordinate axes before sketching the graph.)

When $x = 0$,
$f(0) = 2$.



Factorisation of Cubic Expressions

26. A cubic expression is of the form $ax^3 + bx^2 + cx + d$.

27. Cubic expressions are factorised into:
- 3 linear factors, i.e. $(px + q)(rx + s)(tx + u)$, or
- 1 linear and 1 quadratic factor, i.e. $(px + q)(rx^2 + sx + t)$, where $rx^2 + sx + t$ cannot be factorised into 2 linear factors

28. Methods of factorising cubic expressions:
- Trial and error
- Long division
- Synthetic method
- Comparing coefficients

29. Sum and difference of cubes:
- Sum of cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
- Difference of cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Solving Cubic Equations

30. To solve the equation $f(x) = 0$,

Step 1: Factorise $f(x)$ using the Factor Theorem.

Step 2: Use the synthetic method or compare coefficients to factorise $f(x)$ completely.

Step 3: Equate each factor to zero and use general solution where necessary.

Example 14

Solve the equation $2x^3 + x^2 - 5x + 2 = 0$.

Solution

Let $f(x) = 2x^3 + x^2 - 5x + 2$.

$$\begin{aligned} f(1) &= 2 + 1 - 5 + 2 \\ &= 0 \end{aligned}$$

$\therefore$ $(x - 1)$ is a factor of $f(x)$.

By long division,

$$\begin{array}{r} 2x^2 + 3x - 2 \\ x - 1\overline{)2x^3 + x^2 - 5x + 2} \\ \underline{-(2x^3 - 2x^2)} \\ 3x^2 - 5x + 2 \\ \underline{-(3x^2 - 3x)} \\ -2x + 2 \\ \underline{-(-2x + 2)} \\ 0 \end{array}$$

$$\begin{aligned} f(x) &= (x - 1)(2x^2 + 3x - 2) \\ &= (x - 1)(2x - 1)(x + 2) \end{aligned}$$

When $f(x) = 0$,

$x = 1$ or $x = \dfrac{1}{2}$ or $x = -2$.

Example 15

In the cubic polynomial f(x), the coefficient of x^3 is 4 and the roots of f(x) = 0 are 3, $\frac{1}{2}$ and –4.

(i) Express f(x) as a cubic polynomial in x with integer coefficients.

(ii) Find the remainder when f(x) is divided by $2x - 5$.

(iii) Solve the equation $f\left(\sqrt{x}\right) = 0$.

Solution

(i) Since the roots of f(x) = 0 are 3, $\frac{1}{2}$ and –4, the factors of f(x) are $x - 3$, $2x - 1$ and $x + 4$.

Given also that the coefficient of x^3 is 4, $f(x) = 2(x - 3)(2x - 1)(x + 4)$

$$= 4x^3 + 2x^2 - 50x + 24$$

(ii) $$f\left(\frac{5}{2}\right) = 4\left(\frac{5}{2}\right)^3 + 2\left(\frac{5}{2}\right)^2 - 50\left(\frac{5}{2}\right) + 24$$

$$= -26$$

∴ The remainder is –26.

(iii) Since $f(x) = 2(x - 3)(2x - 1)(x + 4)$,

$f\left(\sqrt{x}\right) = 2\left(\sqrt{x} - 3\right)\left(2\sqrt{x} - 1\right)\left(\sqrt{x} + 4\right)$ (Note that x is replaced with $\sqrt{x}$.)

When $f\left(\sqrt{x}\right) = 0$,

$$2\left(\sqrt{x} - 3\right)\left(2\sqrt{x} - 1\right)\left(\sqrt{x} + 4\right) = 0$$

$\sqrt{x} - 3 = 0$ or $2\sqrt{x} - 1 = 0$ or $\sqrt{x} + 4 = 0$

$\sqrt{x} = 3$ $\quad$ $\sqrt{x} = \frac{1}{2}$ $\quad$ $\sqrt{x} = -4$ (no real solution)

$x = 9$ $\quad$ $x = \frac{1}{4}$

∴ $x = 9$ or $x = \frac{1}{4}$

31. An algebraic fraction is the ratio of two polynomials of the form $\frac{\mathrm{P}(x)}{\mathrm{D}(x)}$, where P($x$) and D($x$) are polynomials in x.

Proper and Improper Fractions

32. If the degree of P(x) is less than the degree of D(x), $\frac{\mathrm{P}(x)}{\mathrm{D}(x)}$ is a proper fraction.

33. If the degree of P(x) is more than or equal to the degree of D(x), $\frac{\mathrm{P}(x)}{\mathrm{D}(x)}$ is an improper fraction.

34. From an improper algebraic fraction $\frac{\mathrm{P}(x)}{\mathrm{D}(x)}$, we can make use of long division to obtain $\frac{\mathrm{P}(x)}{\mathrm{D}(x)} = \mathrm{Q}(x) + \frac{\mathrm{R}(x)}{\mathrm{D}(x)}$, where Q($x$) is a polynomial and $\frac{\mathrm{R}(x)}{\mathrm{D}(x)}$ is a proper algebraic fraction.

35. To express a compound algebraic fraction into partial fractions:

Step 1: Determine if the compound fraction is proper or improper. If it is improper, perform long division (or use the synthetic method if the denominator is linear).

Step 2: Ensure that the denominator is completely factorised.

Step 3: Express the proper fraction in partial fractions according to the cases below.

Step 4: Solve for unknown constants by substituting values of x and/or comparing coefficients of like terms and/or using the "Cover-Up Rule".

Rules of Partial Fractions

36.

Case	Denominator of fraction	Algebraic fraction	Expression used
1	Linear factors	$\frac{mx+n}{(ax+b)(cx+d)}$	$\frac{A}{ax+b}+\frac{B}{cx+d}$
2	Repeated linear factors	$\frac{mx+n}{(ax+b)(cx+d)^2}$	$\frac{A}{ax+b}+\frac{B}{cx+d}+\frac{C}{(cx+d)^2}$
3	Quadratic factor which cannot be factorised	$\frac{mx+n}{(ax+b)(x^2+c^2)}$	$\frac{A}{ax+b}+\frac{Bx+C}{x^2+c^2}$

Example 16

Express $\dfrac{7-2x}{x^2+x-6}$ in partial fractions.

Solution

First factorise the denominator to get the algebraic fraction in the form of $\dfrac{mx+n}{(ax+b)(cx+d)}$.

$x^2+x-6=(x+3)(x-2)$

$$\frac{7-2x}{x^2+x-6}=\frac{7-2x}{(x+3)(x-2)}$$

Then, let $\dfrac{7-2x}{(x+3)(x-2)}=\dfrac{A}{x+3}+\dfrac{B}{x-2}$.

Multiply throughout by $(x+3)(x-2)$,

$$7-2x=A(x-2)+B(x+3)$$

Let $x=2$: $7-2(2)=5B$ (Substituting $x=2$ leaves us with 1 unknown, B.)

$$B=\frac{3}{5}$$

Let $x=-3$: $7-2(-3)=A(-5)$ (Substituting $x=-3$ leaves us with 1 unknown, A.)

$$A=-\frac{13}{5}$$

$$\therefore\ \frac{7-2x}{x^2+x-6}=-\frac{13}{5(x+3)}+\frac{3}{5(x-2)}$$

Example 17

Express $\dfrac{x^4+9}{x^3+3x}$ in partial fractions.

Solution

First we need to perform long division on $\dfrac{x^4+9}{x^3+3x}$.

$$\begin{array}{r l}
 & \quad x \\
x^3+3x \,\big) & x^4+0x^2+9 \\
 & \underline{-(x^4+3x^2)} \\
 & \qquad -3x^2+9
\end{array}$$

$$\frac{x^4+9}{x^3+3x} = x + \frac{-3x^2+9}{x^3+3x}$$

$$= x + \frac{-3x^2+9}{x(x^2+3)}$$

Let $\dfrac{-3x^2+9}{x(x^2+3)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+3}$.

Multiply throughout by $x(x^2+3)$,

$-3x^2+9 = A(x^2+3)+(Bx+c)x$

Let $x = 0 : 9 = 3A$

$A = 3$

Comparing coefficients of x^2,

$-3 = A + B$

$= 3 + B$

$B = -6$

Comparing coefficients of x,

$C = 0$

$$\therefore \frac{x^4+9}{x^3+3x} = x + \frac{3}{x} - \frac{6x}{x^2+3}$$

Example 18

Express $\dfrac{2x^3 - 2x^2 - 24x - 7}{x^2 - x - 12}$ in partial fractions.

Solution

By long division,

$$
\begin{array}{r}
2x \phantom{{}- 2x^2 - 24x - 7} \\
x^2 - x - 12 \overline{\smash{\big)}\, 2x^3 - 2x^2 - 24x - 7} \\
\underline{-(2x^3 - 2x^2 - 24x) \phantom{{}- 7}} \\
-7
\end{array}
$$

$$
\begin{aligned}
\frac{2x^3 - 2x^2 - 24x - 7}{x^2 - x - 12} &= 2x + \frac{-7}{x^2 - x - 12} \\
&= 2x + \frac{-7}{(x-4)(x+3)}
\end{aligned}
$$

Let $\dfrac{-7}{(x-4)(x+3)} = \dfrac{A}{x-4} + \dfrac{B}{x+3}$. (Ignore the $2x$ when expressing $\dfrac{-7}{(x-4)(x+3)}$ into its partial fractions.)

Multiply throughout by $(x - 4)(x + 3)$,

$$-7 = A(x + 3) + B(x - 4)$$

Let $x = 4$: $-7 = 7A$

$$A = -1$$

Let $x = -3$: $-7 = -7B$

$$B = 1$$

$$\therefore \frac{2x^3 - 2x^2 - 24x - 7}{x^2 - x - 12} = 2x - \frac{1}{x-4} + \frac{1}{x+3}$$

Example 19

Express $\dfrac{8x^2 - 5x + 2}{(3x+2)(x^2+4)}$ in partial fractions.

Solution

Let $\dfrac{8x^2 - 5x + 2}{(3x+2)(x^2+4)} = \dfrac{A}{3x+2} + \dfrac{Bx+C}{x^2+4}$.

(Note that $x^2 + 4$ cannot be factorised into 2 linear factors.)

Multiply throughout by $(3x + 2)(x^2 + 4)$,

$8x^2 - 5x + 2 = A(x^2 + 4) + (Bx + C)(3x + 2)$

Let $x = -\dfrac{2}{3}$: $\dfrac{80}{9} = \dfrac{40}{9}A$

$A = 2$

Let $x = 0$: $2 = 4A + 2C$

$= 4(2) + 2C$

$2C = -6$

$C = -3$

Comparing coefficients of x^2,

$8 = A + 3B$

$= 2 + 3B$

$3B = 6$

$B = 2$

$$\therefore \frac{8x^2 - 5x + 2}{(3x+2)(x^2+4)} = \frac{2}{3x+2} + \frac{2x-3}{x^2+4}$$

Example 20

Express $\dfrac{9-4x}{(2x+3)(x-1)^2}$ in partial fractions.

Solution

Let $\dfrac{9-4x}{(2x+3)(x-1)^2} = \dfrac{A}{2x+3} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$.

Multiply throughout by $(2x+3)(x-1)^2$,

$$9 - 4x = A(x-1)^2 + B(x-1)(2x+3) + C(2x+3)$$

Let $x = 1$: $5 = 5C$

$$C = 1$$

Let $x = -\dfrac{3}{2}$: $15 = \dfrac{25}{4}A$

$$A = \frac{12}{5}$$

Comparing coefficients of x^2,

$$0 = A + 2B$$

$$0 = \frac{12}{5} + 2B$$

$$2B = -\frac{12}{5}$$

$$B = -\frac{6}{5}$$

$$\therefore \frac{9-4x}{(2x+3)(x-1)^2} = \frac{12}{5(2x+3)} - \frac{6}{5(x-1)} + \frac{1}{(x-1)^2}$$

37. Cover-Up Rule

The "Cover-Up Rule" is a method to find the unknown numerators of partial fractions.

Given that $\dfrac{P(x)}{(ax+b)(cx+d)} = \dfrac{A}{ax+b} + \dfrac{B}{cx+d}$, where $P(x)$ is a linear polynomial,

$$A = \frac{P\left(-\frac{b}{a}\right)}{c\left(-\frac{b}{a}\right)+d} \text{ and } B = \frac{P\left(-\frac{d}{c}\right)}{a\left(-\frac{d}{c}\right)+b}.$$

Example 21

Express $\dfrac{3x-1}{(x+3)(x-2)}$ in partial fractions.

Solution

Let $\dfrac{3x-1}{(x+3)(x-2)} = \dfrac{A}{x+3} + \dfrac{B}{x-2}$.

Method 1: Substitution

$$\frac{3x-1}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

Multiply throughout by $(x+3)(x-2)$,

$$3x-1 = A(x-2) + B(x+3)$$

Let $x = 2$: $5 = 5B$ (Letting x be 2 leaves us with 1 unknown, B.)

$$B = 1$$

Let $x = -3$: $-10 = -5A$ (Letting x be -3 leaves us with 1 unknown, A.)

$$A = 2$$

$$\therefore \frac{3x-1}{(x+3)(x-2)} = \frac{2}{x+3} + \frac{1}{x-2}$$

Method 2: Cover-Up Rule

Using the Cover-Up Rule,

$$A = \frac{3(-3)-1}{-3-2} \quad \text{and } B = \frac{3(2)-1}{2+3}$$

$$= 2 \qquad\qquad = 1$$

$$\therefore \frac{3x-1}{(x+3)(x-2)} = \frac{2}{x+3} + \frac{1}{x-2}$$

UNIT 2 Quadratic Equations, Inequalities and Modulus Functions

Relationships between the Roots and Coefficients of a Quadratic Equation

1. If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$,

$$\text{Sum of roots, } \alpha + \beta = -\frac{b}{a}$$

$$\text{Product of roots, } \alpha\beta = \frac{c}{a}$$

i.e. $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

2. In general,

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

3. Some useful identities

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
(ii) $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$
(iii) $\alpha^4 - \beta^4 = (\alpha^2 + \beta^2)(\alpha + \beta)(\alpha - \beta)$
(iv) $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$
(v) $\alpha^3 - \beta^3 = (\alpha - \beta)[(\alpha + \beta)^2 - \alpha\beta]$
(vi) $\alpha^3 + \beta^3 = (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta]$

Example 1

The roots of the quadratic equation $2x^2 - 5x = 4$ are α and β.
Find

(i) $\alpha^2 + \beta^2$, **(ii)** $\dfrac{\alpha}{2\beta} + \dfrac{\beta}{2a}$.

Solution

From $2x^2 - 5x = 4$, we have $2x^2 - 5x - 4 = 0$.

$$\alpha + \beta = -\frac{-5}{2} = \frac{5}{2}$$

$$\alpha\beta = \frac{-4}{2} = -2$$

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$$= \left(\frac{5}{2}\right)^2 - 2(-2)$$

$$= \frac{41}{4}$$

(ii) $\dfrac{\alpha}{2\beta} + \dfrac{\beta}{2a} = \dfrac{\alpha^2 + \beta^2}{2a\beta}$

$$= \frac{41}{4} \div 2(-2)$$

$$= \frac{41}{16}$$

Example 2

Using your answers in Example 1, form a quadratic equation with integer coefficients whose roots are $\dfrac{\alpha}{2\beta}$ and $\dfrac{\beta}{2a}$.

Solution

Sum of new roots, $\dfrac{\alpha}{2\beta} + \dfrac{\beta}{2\alpha} = -\dfrac{41}{16}$

Product of new roots, $\dfrac{\alpha}{2\beta} \times \dfrac{\beta}{2\alpha} = \dfrac{1}{4}$

$\therefore$ New equation is $x^2 - \left(-\dfrac{41}{16}\right)x + \dfrac{1}{4} = 0$

i.e. $16x^2 + 41x + 4 = 0$.

Example 3

If α and β are the roots of the equation $2x^2 + 5x - 12 = 0$, where $\alpha > \beta$, find the value of each of the following.

(i) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ **(ii)** $\alpha^2 + \beta^2$

Solution

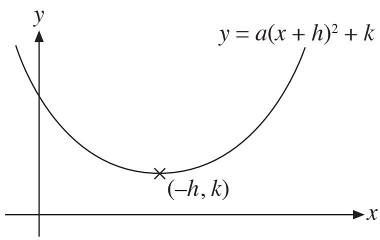
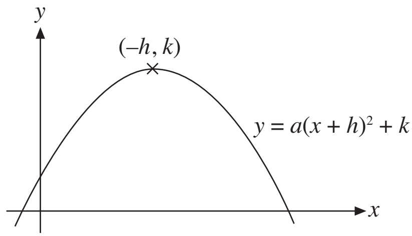
$$\alpha + \beta = -\frac{b}{a} \qquad\qquad \alpha\beta = \frac{c}{a}$$
$$= -\frac{5}{2} \qquad\qquad = -\frac{12}{2}$$
$$= -2\frac{1}{2} \qquad\qquad = -6$$

(i) $$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$$
$$= \frac{-2\frac{1}{2}}{-6}$$
$$= \frac{5}{12}$$

(ii) $$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$
$$= \left(-2\frac{1}{2}\right)^2 - 2(-6)$$
$$= 18\frac{1}{4}$$

Maximum and Minimum Values of Quadratic Functions

4. The quadratic function $ax^2 + bx + c$ can be expressed as $a(x + h)^2 + k$.

<table>
<tr><th>$a > 0$</th><th>$a < 0$</th></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>Minimum value = k, when $x = -h$
Minimum point: $(-h, k)$</td><td>Maximum value = k, when $x = -h$
Maximum point: $(-h, k)$</td></tr>
</table>

Sketching of Quadratic Graphs

5. Method of sketching a quadratic graph:

Step 1: Determine the shape of the graph from a.

Step 2: Express the function as $a(x + h)^2 + k$ to get the coordinates of the maximum or minimum point.

Step 3: Substitute $x = 0$ to find the y-intercept.

Step 4: Substitute $y = 0$ to find the x-intercept(s), if the roots are real.

Example 4

Sketch the function $y = x^2 - 1$.

Solution

Step 1: Since $y = x^2 - 1$ is a quadratic function and a is positive, the graph is U-shaped.

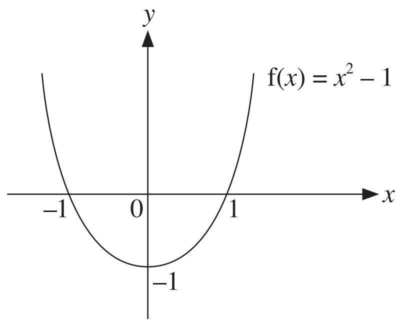
Step 2: Comparing with the form $a(x + h)^2 + k$, we get $a = 1$, which is greater than 0 so it has a minimum point.

From the function $y = x^2 - 1$, $h = 0$, $k = -1$.
$\therefore$ Minimum point $= (0, -1)$

(We can express $x^2 - 1$ as $(x + 0)^2 + (-1)$ and compare with the form $a(x + h)^2 + k$.)

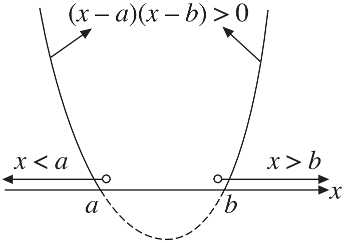
Step 3: When $x = 0$, $f(x) = -1$.

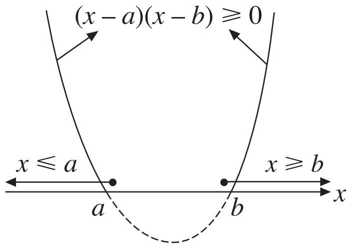
Step 4: When $y = 0$, $x = 1$ and -1.



6. If $(x-a)(x-b) > 0$,
then $x < a$ or $x > b$.

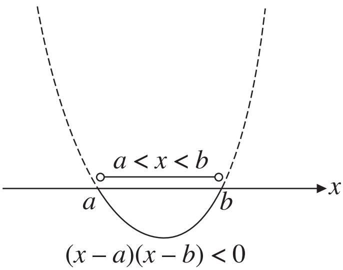
If $(x-a)(x-b) \geqslant 0$,
then $x \leqslant a$ or $x \geqslant b$.

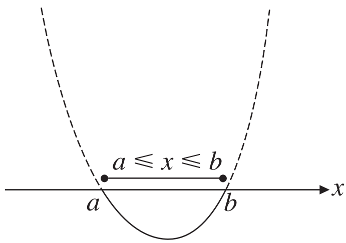




7. If $(x-a)(x-b) < 0$,
then $a < x < b$.

If $(x-a)(x-b) \leqslant 0$,
then $a \leqslant x \leqslant b$.





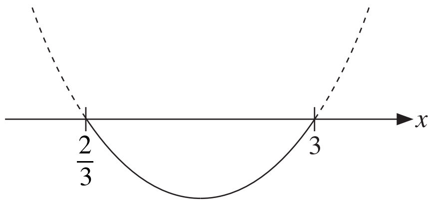
Example 5

Find the range of values of x for which $3x^2 - 4x + 6 \leqslant 7x$.

Solution

$$3x^2 - 4x + 6 \leqslant 7x$$
$$3x^2 - 11x + 6 \leqslant 0$$
$$(3x - 2)(x - 3) \leqslant 0$$

(When solving quadratic inequalities, ensure that the RHS of the inequality is zero before factorising the expression on the LHS.)



$\therefore$ Range of values of x is $\frac{2}{3} \leqslant x \leqslant 3$

Roots of a Quadratic Equation

8. The roots of a quadratic equation $ax^2 + bx + c = 0$ are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

9. $b^2 - 4ac$ is called the discriminant.

10. A quadratic equation has no real roots when $b^2 - 4ac < 0$.
Given a quadratic expression $ax^2 - bx + c$,
it is found that:
given that $b^2 - 4ac < 0$ and $a > 0$, $ax^2 - bx + c > 0$ for all real values of x, and
given that $b^2 - 4ac < 0$ and $a < 0$, $ax^2 - bx + c < 0$ for all real values of x.

Example 6

Is the quadratic expression $5x^2 + 4x + 1$ greater than zero for all real values of x?

Solution

$$\begin{aligned} \text{Discriminant} &= 4^2 - 4(5)(1) \\ &= -4 \end{aligned}$$

Since $b^2 - 4ac < 0$ and $a > 0$, $5x^2 + 4x + 1 > 0$ for all real values of x.

Example 7

Find the range of values of k for which the equation $2x^2 + 5x - k = 0$ has no real roots.

Solution

$2x^2 + 5x - k = 0$

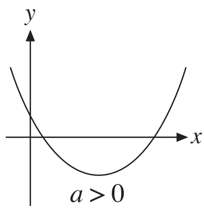
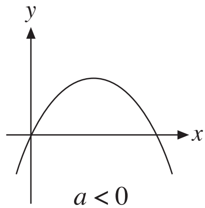
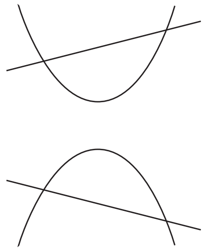
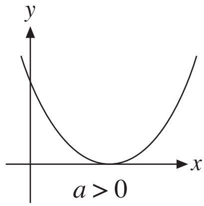
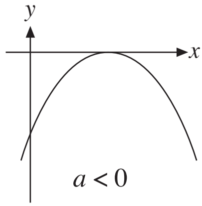
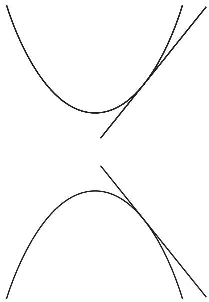
$a = 2, b = 5, c = -k$

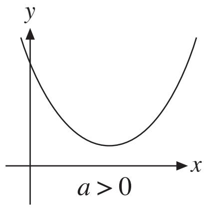
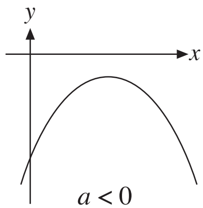
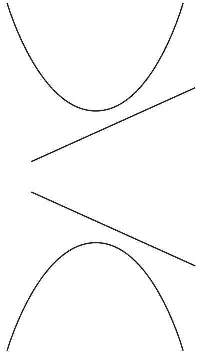
For the equation to have no real roots,

$$\begin{aligned} b^2 - 4ac &< 0 \\ 5^2 - 4(2)(-k) &< 0 \\ 8k &< -25 \\ k &< -\frac{25}{8} \end{aligned}$$

Conditions for the Intersection of a Line and a Quadratic Curve

11.

$b^2 - 4ac$	**Nature of roots**	**Intersection of $y = ax^2 + bx + c$ with the x-axis**	**Intersection of quadratic curve with a straight line**
> 0	2 real and distinct roots	$y = ax^2 + bx + c$ cuts the x-axis at 2 distinct points  $a > 0$  $a < 0$	Line intersects the curve at two distinct points 
$= 0$	2 real and equal roots	$y = ax^2 + bx + c$ touches the x-axis  $a > 0$  $a < 0$	Line is a tangent to the curve 

< 0	No real roots	$y = ax^2 + bx + c$ lies entirely above or entirely below the x-axis i.e. curve is always positive ($a > 0$) or always negative ($a < 0$)  $a > 0$  $a < 0$	Line does not intersect the curve 

Example 8

Find the range of values of k given that the straight line $y = x - k$ cuts the curve $y = kx^2 + 9x$ at two distinct points.

Solution

$y = x - k$ —— (1)

$y = kx^2 + 9x$ —— (2)

Substitute (1) into (2): (Substitute (1) into (2) to obtain a quadratic equation in x.)

$$x - k = kx^2 + 9x$$

$$kx^2 + 8x + k = 0$$

Since the straight line cuts the curve at two distinct points,

Discriminant > 0

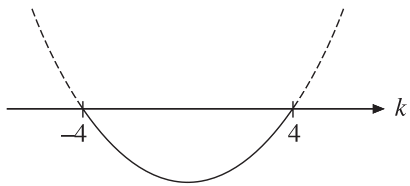
$$8^2 - 4(k)(k) > 0$$

$$64 - 4k^2 > 0$$

$$k^2 - 16 < 0$$

$$(k + 4)(k - 4) < 0$$

(Remember to invert the inequality sign when dividing by a negative number.)



$\therefore$ Range of values of k is $-4 < k < 4$

Example 9

Find the range of values of m for which the line $y = 5 - mx$ does not intersect the curve $x^2 + y^2 = 16$.

Solution

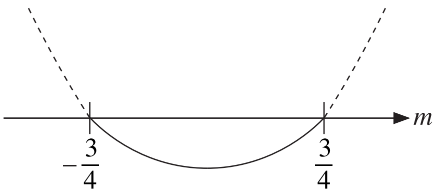
$y = 5 - mx$ —— (1)
$x^2 + y^2 = 16$ —— (2)

Substitute (1) into (2):

$$x^2 + (5 - mx)^2 = 16$$
$$x^2 + m^2x^2 - 10mx + 25 = 16$$
$$(1 + m^2)x^2 - 10mx + 9 = 0$$

Since the line does not intersect the curve,

$$\text{Discriminant} < 0$$
$$(-10m)^2 - 4(1 + m^2)(9) < 0$$
$$100m^2 - 36 - 36m^2 < 0$$
$$64m^2 - 36 < 0$$
$$16m^2 - 9 < 0$$
$$(4m + 3)(4m - 3) < 0$$



$\therefore$ Range of values of m is $-\dfrac{3}{4} < m < \dfrac{3}{4}$

Absolute Valued Functions

12. The absolute value of a function f(x), i.e. $|f(x)|$, refers to the numerical value of f(x).

13. $|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geqslant 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$

14. $|f(x)| \geqslant 0$ for all values of x.

Example 10

Solve $|4x - 3| = 2x$.

Solution

$$|4x - 3| = 2x$$

$$\begin{aligned} 4x - 3 &= 2x & \quad \text{or} \quad & & 4x - 3 &= -2x \\ 2x &= 3 & & & 6x &= 3 \\ x &= \frac{3}{2} & & & x &= \frac{1}{2} \end{aligned}$$

$$\therefore\ x = \frac{3}{2} \text{ or } x = \frac{1}{2}$$

Example 11

Solve $|2x - 3| = 15$.

Solution

$$|2x - 3| = 15$$

$$\begin{aligned} 2x - 3 &= 15 & \quad \text{or} \quad & & 2x - 3 &= -15 \\ 2x &= 18 & & & 2x &= -12 \\ x &= 9 & & & x &= -6 \end{aligned}$$

$$\therefore\ x = 9 \text{ or } x = -6$$

Example 12

Solve $|2x-5|=|4-x|$.

Solution

$$|2x-5|=|4-x|$$

$$\begin{aligned} 2x-5&=4-x \quad &\text{or}\quad 2x-5&=-(4-x)\\ 3x&=9 & &=-4+x\\ x&=3 & x&=1 \end{aligned}$$

$\therefore\ x=3$ or $x=1$

Example 13

Solve $|x^2-3|=2x$.

Solution

$$|x^2-3|=2x$$

$$\begin{aligned} x^2-3&=2x \quad &\text{or}\quad x^2-3&=-2x\\ x^2-2x-3&=0 & x^2+2x-3&=0\\ (x-3)(x+1)&=0 & (x+3)(x-1)&=0\\ x=3 \text{ or } x&=-1 & x=-3 \text{ or } x&=1 \end{aligned}$$

Checking the solutions, $x=3$ or $x=1$. (Substitute your answers into the original equation to check for any extraneous solutions.)

Example 14

Solve $\left|2x^2 - 5x\right| = x$.

Solution

$$\left|2x^2 - 5x\right| = x$$

$$\begin{aligned} 2x^2 - 5x &= x & &\text{or} & 2x^2 - 5x &= -x \\ 2x^2 - 6x &= 0 & & & 2x^2 - 4x &= 0 \\ 2x(x-3) &= 0 & & & 2x(x-2) &= 0 \\ x = 0 \text{ or } x &= 3 & & & x = 0 \text{ or } x &= 2 \end{aligned}$$

$\therefore\ x = 0, x = 2$ or $x = 3$

Graphs of $y = \left|f(x)\right|$

15. Method of sketching the graph of $y = \left|f(x)\right|$:

Step 1: Sketch the graph of $y = f(x)$.

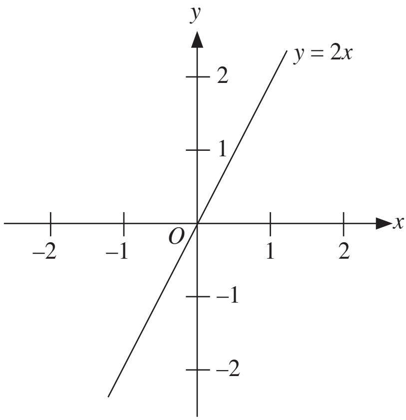
Step 2: The part of the graph below the x-axis is reflected in the x-axis.

Example 15

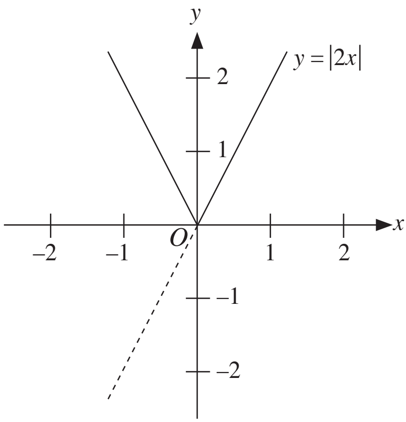
Sketch the graph of $y = 2x$.
Hence, sketch the graph of $y = |2x|$.

Solution

Sketch the graph $y = 2x$.



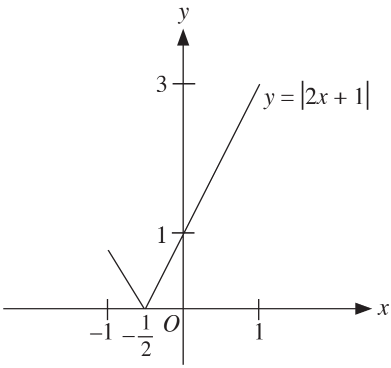
To draw $y = |2x|$, reflect the part of the graph that lies below the x-axis.



Example 16

Sketch the graph of $y = |2x + 1|$ for the domain $-1 \leqslant x \leqslant 1$ and state the corresponding range.

Solution



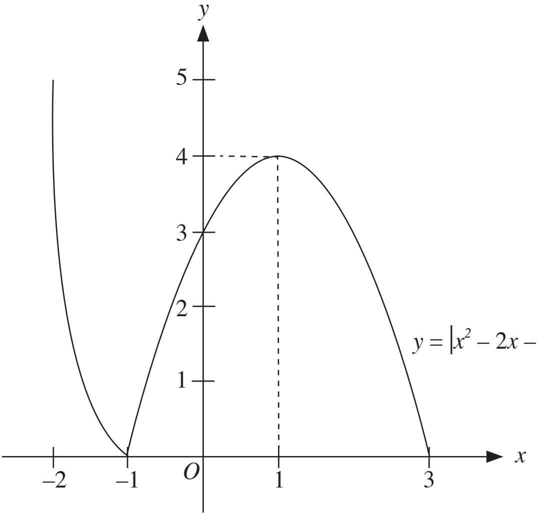
(It is necessary to find the coordinates of the critical points.)

$\therefore$ Range is $0 \leqslant y \leqslant 3$

Example 17

Sketch the graph of $y = |x^2 - 2x - 3|$ for $-2 \leqslant x \leqslant 3$. State the corresponding range.

Solution



(Note that $x^2 - 2x - 3$
$= (x - 3)(x + 1)$.)

$\therefore$ Range is $0 \leqslant y \leqslant 5$

16. If a function is defined as $y = a|bx + c| + d$,

(a) If $a > 0$, it is a V-shaped graph.
If $a < 0$, it is an inverted V-shaped graph.

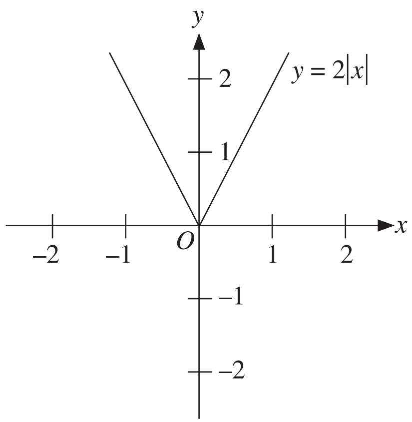
(b) If $d > 0$, $y = a|bx + c|$ is translated up by d units.
If $d < 0$, $y = a|bx + c|$ is translated down by $|d|$ units.

Example 18

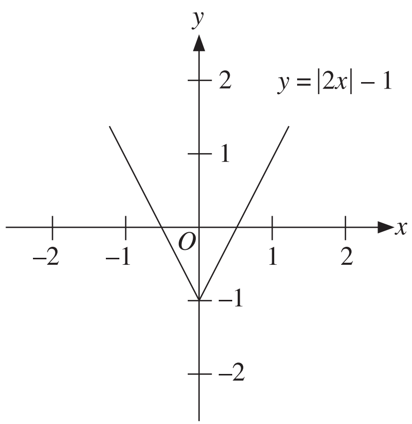
Sketch the graph of $y = 2|x| - 1$.

Solution

Step 1: Sketch the graph $y = 2|x|$.



Step 2: Translate the graph down by 1 unit.



UNIT 3 Binomial Theorem

Binomial Theorem

1. For a positive integer n,

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n$$

where $\binom{n}{r} = \dfrac{n!}{r!(n-r)!} = \dfrac{n(n-1)\ldots n(n-r+1)}{r!}$

2. Number of terms in the expansion of $(a+b)^n$ is $n+1$

3. Special case:
When $a = 1$,

$$(1+b)^n = 1 + \binom{n}{1}b + \binom{n}{2}b^2 + \ldots + \binom{n}{r}b^r + \ldots + b^n$$

Example 1

Find the value of k and of n given that $(1 + kx)^n = 1 + 48x + 1008x^2 + \dots$.

Solution

$$(1 + kx)^n = 1 + \binom{n}{1}(kx) + \binom{n}{2}(kx)^2 + \dots$$ (Use the expansion of $(1 + b)^n$.)

$$= 1 + nkx + \frac{n(n-1)}{2}k^2x^2 + \dots$$

By comparing coefficients,

x: $nk = 48$ (Compare coefficients to obtain a pair of simultaneous equations.)

$$k = \frac{48}{n} \quad \text{— (1)}$$

$$x^2: \frac{n(n-1)k^2}{2} = 1008$$

$$n(n-1)k^2 = 2016 \quad \text{— (2)}$$

Substitute (1) into (2):

$$n(n-1)\frac{2304}{n^2} = 2016$$

$$2304n - 2304 = 2016n$$

$$n = 8$$

$$k = 6$$

$\therefore\ k = 6, n = 8$

Example 2

Write down the first 4 terms in the expansion of $(1 + 2x)^7$ in ascending powers of x.
Hence, find the coefficient of x^3 in the expansion of $(1 + 2x + 3x^2)(1 + 2x)^7$.

Solution

$$(1 + 2x)^7 = 1 + \binom{7}{1}(2x) + \binom{7}{2}(2x)^2 + \binom{7}{3}(2x)^3 + \dots$$ (Use the expansion of $(1 + b)^n$.)

$$= 1 + 14x + 84x^2 + 280x^3 + \dots$$

$$(1 + 2x + 3x^2)(1 + 2x)^7 = (1 + 2x + 3x^2)(1 + 14x + 84x^2 + 280x^3 + \dots)$$

$$= \dots + 280x^3 + 168x^3 + 42x^3 + \dots$$ (There is no need to obtain terms other than x^3.)

$$= \dots + 490x^3 + \dots$$

$\therefore$ Coefficient of x^3 is 490

The notation $n!$

4. $n! = n \times (n - 1) \times (n - 2) \times \dots \times 3 \times 2 \times 1$

5. Some useful rules:

- $\binom{n}{0} = 1$
- $\binom{n}{1} = n$
- $\binom{n}{2} = \dfrac{n(n-1)}{2!}$
- $\binom{n}{3} = \dfrac{n(n-1)(n-2)}{3!}$
- $\binom{n}{n} = 1$

Example 3

Find the value of n given that, in the expansion of $(3 + 2x)^n$, the coefficients of x^2 and x^3 are in the ratio 3 : 4.

Solution

$$(3 + 2x)^n = \ldots + \binom{n}{2}3^{n-2}(2x)^2 + \binom{n}{3}3^{n-3}(2x)^3 + \ldots$$

$$= \ldots + \binom{n}{2}3^{n-2}(4x^2) + \binom{n}{3}3^{n-3}(8x^3) + \ldots$$

$$\frac{\binom{n}{2}3^{n-2}(4)}{\binom{n}{3}3^{n-3}(8)} = \frac{3}{4}$$

$$\frac{\frac{\cancel{n(n-1)}}{2}.3}{\frac{\cancel{n(n-1)}(n-2)}{6}.2} = \frac{3}{4}$$

$$n = 8$$

Example 4

Find the first 4 terms in the expansion of $(1 + 2x)^7$ in ascending powers of x. Use your result to estimate the value of 1.02^7.

Solution

$$(1 + 2x)^7 = 1 + \binom{7}{1}(2x) + \binom{7}{2}(2x)^2 + \binom{7}{3}(2x)^3 + \ldots$$

$$= 1 + 14x + 84x^2 + 280x^3 + \ldots$$

Let $(1 + 2x)^7 = 1.02^7$, then $x = 0.01$.

$1.02^7 = 1 + 14(0.01) + 84(0.01)^2 + 280(0.01)^3 + \ldots$

$= 1.148\ 68$

Example 5

Expand $\left(1+\frac{x}{2}\right)^5$ in ascending powers of x. Hence, deduce the expansion of

(i) $\left(1-\frac{x}{2}\right)^5$, **(ii)** $\left(1+\frac{x}{2}\right)^5 + \left(1-\frac{x}{2}\right)^5$.

Using your answers in **(i)** and **(ii)**, find the exact value of $1.05^5 + 0.95^5$.

Solution

$$\left(1+\frac{x}{2}\right)^5 = 1 + \binom{5}{1}\left(\frac{x}{2}\right) + \binom{5}{2}\left(\frac{x}{2}\right)^2 + \binom{5}{3}\left(\frac{x}{2}\right)^3 + \binom{5}{4}\left(\frac{x}{2}\right)^4 + \binom{5}{5}\left(\frac{x}{2}\right)^5$$

$$= 1 + \frac{5}{2}x + \frac{5}{2}x^2 + \frac{5}{4}x^3 + \frac{5}{16}x^4 + \frac{1}{32}x^5$$

(i) $\left(1-\frac{x}{2}\right)^5 = \left[1+\left(-\frac{x}{2}\right)\right]^5$

$$= 1 - \frac{5}{2}x + \frac{5}{2}x^2 - \frac{5}{4}x^3 + \frac{5}{16}x^4 - \frac{1}{32}x^5$$

(ii) $\left(1+\frac{x}{2}\right)^5 + \left(1-\frac{x}{2}\right)^5 = \left[1 + \frac{5}{2}x + \frac{5}{2}x^2 + \frac{5}{4}x^3 + \frac{5}{16}x^4 + \frac{1}{32}x^5\right]$

$$+ \left[1 - \frac{5}{2}x + \frac{5}{2}x^2 - \frac{5}{4}x^3 + \frac{5}{16}x^4 - \frac{1}{32}x^5\right]$$

$$= 2 + 5x^2 + \frac{5}{8}x^4$$

Let $\left(1+\frac{x}{2}\right)^5 + \left(1-\frac{x}{2}\right)^5 = 1.05^5 + 0.95^5$.

By inspection,

$$x = 0.1$$

$$\therefore\ 1.05^5 + 0.95^5 = 2 + 5(0.1)^2 + \frac{5}{8}(0.1)^4$$

$$= 2.050\,0625$$

Example 6

Write down the expansion of $(1+p)^6$ in ascending powers of p. Hence, find the first 3 terms in the expansion of $(1+2x+2x^2)^6$ in ascending powers of x. Use your result to find the value of $1.002\ 002^6$ correct to 6 decimal places.

Solution

$$(1+p)^6 = 1 + \binom{6}{1}p + \binom{6}{2}p^2 + \binom{6}{3}p^3 + \binom{6}{4}p^4 + \binom{6}{5}p^5 + \binom{6}{6}p^6$$
$$= 1 + 6p + 15p^2 + 20p^3 + 15p^4 + 6p^5 + p^6$$

By comparing $(1+p)^6$ with $(1+2x+2x^2)^6$,

$$p = 2x + 2x^2$$

$$\begin{aligned}(1+2x+2x^2)^6 &= 1 + 6(2x+2x^2) + 15(2x+2x^2)^2 + \dots \\ &= 1 + 12x + 12x^2 + 60x^2 + \dots \\ &= 1 + 12x + 72x^2 + \dots\end{aligned}$$

(The first 3 terms consist of the constant, the term in x and the term in x^2.)

$$1.002\ 002 = 1 + 2(0.001) + 2(0.001)^2$$

Let $x = 0.001$.

$$\therefore\ 1.002\ 002^6 = 1 + 12(0.001) + 72(0.001)^2 + \dots$$
$$= 1.012\ 072 \text{ (to 6 d.p.)}$$

General Term

6. The $(r+1)^{\text{th}}$ term is $\binom{n}{r}a^{n-r}b^r$.

Example 7

Find the 8th term in the expansion of $(3+x)^{12}$ in ascending powers of x.

Solution

$(r+1)^{\text{th}}$ term $= \binom{12}{r} 3^{12-r} x^r$

8^{th} term $= (7+1)^{\text{th}}$ term

$= \binom{12}{7} 3^{12-7} x^7$

$= 792\,(3^5)\,x^7$

$= 192\,456x^7$

Example 8

In the expansion of $(1+x)^n$ in ascending powers of x, the coefficient of the third term is 21. Find the value of n.

Solution

In the expansion of $(1+x)^n$, the $(r+1)^{\text{th}}$ term is $\binom{n}{r} x^r$.

Hence, in the expansion of $(1+x)^n$, the third term is $\binom{n}{2} x^2 = \frac{n(n-1)}{2} \times x^2$.

$\therefore$ The coefficient of the third term is:

$$\frac{n(n-1)}{2} = 21$$

$$n(n-1) = 42$$

$$n^2 - n - 42 = 0$$

$$(n+6)(n-7) = 0$$

$n = -6$ or $n = 7$ (Since n is a positive integer, reject $n = -6$.)

$\therefore\ n = 7$

Term Independent of x

7. Term independent of x refers to the constant term.

Example 9

Find the term independent of x in the expansion of $\left(x^2 + \frac{1}{x}\right)^{12}$.

Solution

Using $T_{r+1} = \binom{n}{r} a^{n-r} b^r$ (Recall the formula for the general term.)

$$T_{r+1} = \binom{12}{r} (x^2)^{12-r} \left(\frac{1}{x}\right)^r$$

$$= \binom{12}{r} x^{24-2r} \left(\frac{1}{x^r}\right)$$

$$= \binom{12}{r} x^{24-3r}$$

$24 - 3r = 0$ (Term independent of x refers to the constant term, i.e. x^0.)

$r = 8$

$\therefore$ Term independent of x is $\binom{12}{8} x^{24-3(8)} = 495$

Example 10

Find the term independent of x in the expansion of $(2x + 3)^4$.

Solution

In the expansion of $(2x + 3)^4$, the $(r + 1)^{\text{th}}$ term is $\binom{4}{r} (2x)^{4-r} 3^r$.

For the term independent of x,

$4 - r = 0$

$r = 4$

$\therefore$ Term independent of x is $\binom{4}{4} (2x)^{4-4} 3^4 = 81$

UNIT 4 Indices, Surds and Logarithms

Rules of Indices

1. (a) $a^m \times a^n = a^{m+n}$
   (b) $a^m \div a^n = a^{m-n}$
   (c) $(a^m)^n = a^{mn}$
   (d) $a^0 = 1$, provided $a \neq 0$
   (e) $a^{-n} = \frac{1}{a^n}$
   (f) $a^{\frac{1}{n}} = \sqrt[n]{a}$
   (g) $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$
   (h) $(a \times b)^n = a^n \times b^n$
   (i) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, provided $b \neq 0$

Example 1

Simplify $81^{\frac{3}{2}} \times 2^6 \div 6^3$.

Solution

$$
\begin{aligned}
81^{\frac{3}{2}} \times 2^6 \div 6^3 &= (3^4)^{\frac{3}{2}} \times 2^6 \div 6^3 \\
&= 3^6 \times 2^6 \div 6^3 \\
&= (3 \times 2)^6 \div 6^3 \\
&= 6^3 \\
&= 216
\end{aligned}
$$

Example 2

Simplify each of the following.

(i) $3^{2n} \times 15^{3n} \div 5^n$

(ii) $\dfrac{25 \times 5^{n-2}}{5^n - 5^{n-1}}$

Solution

(i) $3^{2n} \times 15^{3n} \div 5^n = 3^{2n} \times 3^{3n} \times 5^{3n} \div 5^n$ (Recall that $(a \times b)^n = a^n \times b^n$.)

$= 3^{5n} \times 5^{2n}$

(ii) $\dfrac{25 \times 5^{n-2}}{5^n - 5^{n-1}} = \dfrac{5^2 \times 5^{n-2}}{5^{n-1}(5-1)}$ (5^{n-1} is a common factor in the denominator.)

$= \dfrac{5^n}{4(5^{n-1})}$

$= \dfrac{5}{4}$

Definition of a Surd

2. A surd is an irrational root of a real number, e.g. $\sqrt{2}$ and $\sqrt{3}$.

Operations on Surds

3. **(a)** $\sqrt{a} \times \sqrt{a} = a$

(b) $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

(c) $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$

(d) $m\sqrt{a} + n\sqrt{a} = (m+n)\sqrt{a}$

(e) $m\sqrt{a} - n\sqrt{a} = (m-n)\sqrt{a}$

Example 3

Simplify $\sqrt{8} \div \sqrt{2}$.

Solution

$\sqrt{8} \div \sqrt{2} = \sqrt{\dfrac{8}{2}}$

$= \sqrt{4}$

$= 2$

Example 4

Given that $\left(a+\sqrt{2}\right)\left(3+b\sqrt{2}\right)=8+5\sqrt{2}$, find the possible values of a and of b.

Solution

$$\begin{aligned}\left(a+\sqrt{2}\right)\left(3+b\sqrt{2}\right) &= 3a+ab\sqrt{2}+3\sqrt{2}+2b\\ &= 3a+2b+(ab+3)\sqrt{2}\end{aligned}$$

By comparing,

$3a+2b=8$ — (1) (Equate the rational terms and the irrational terms to obtain
$ab+3=5$ — (2) 2 equations.)

From (2),

$ab=2$

$b=\dfrac{2}{a}$ — (3)

Substitute (3) into (1):

$$3a+2\left(\frac{2}{a}\right)=8$$

$$3a^2-8a+4=0$$

$$(3a-2)(a-2)=0$$

$$a=\frac{2}{3} \quad \text{or} \quad a=2$$

$$b=3 \qquad\quad b=1$$

$\therefore\ a=\dfrac{2}{3}$, $b=3$ or $a=2$, $b=1$

Conjugate Surds

4. $a\sqrt{m}+b\sqrt{n}$ and $a\sqrt{m}-b\sqrt{n}$ are conjugate surds.

5. $\left(a\sqrt{m}+b\sqrt{n}\right)\left(a\sqrt{m}-b\sqrt{n}\right)=a^2m-b^2n$, which is a rational number.

6. The product of a pair of conjugate surds is always a rational number.

Example 5

Simplify $\left(\sqrt{3}+3\sqrt{2}\right)\left(\sqrt{3}-3\sqrt{2}\right)$.

Solution

$$\begin{aligned}\left(\sqrt{3}+3\sqrt{2}\right)\left(\sqrt{3}-3\sqrt{2}\right) &= \left(\sqrt{3}\right)^2 - 9\left(\sqrt{2}\right)^2 \\ &= 3-9(2) \\ &= -15\end{aligned}$$

Rationalising the Denominator

7. To rationalise the denominator of a surd is to make the denominator a rational number.

(a) $$\begin{aligned}\frac{\sqrt{b}}{\sqrt{a}} &= \frac{\sqrt{b}}{\sqrt{a}} \times \frac{\sqrt{a}}{\sqrt{a}} \\ &= \frac{\sqrt{ab}}{a}\end{aligned}$$

(b) $$\begin{aligned}\frac{1}{\sqrt{a}+\sqrt{b}} &= \frac{1}{\sqrt{a}+\sqrt{b}} \times \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}} \\ &= \frac{\sqrt{a}-\sqrt{b}}{a-b}\end{aligned}$$

Example 6

Rationalise the denominator of $\dfrac{3\sqrt{2}+2}{3\sqrt{2}+3}$.

Solution

$$\begin{aligned}\frac{3\sqrt{2}+2}{3\sqrt{2}+3} &= \frac{3\sqrt{2}+2}{3\sqrt{2}+3} \times \frac{3\sqrt{2}-3}{3\sqrt{2}-3} \\ &= \frac{\left(3\sqrt{2}\right)\left(3\sqrt{2}\right)+6\sqrt{2}-9\sqrt{2}-6}{\left(3\sqrt{2}\right)^2-3^2} \\ &= \frac{12-3\sqrt{2}}{9} \\ &= \frac{4-\sqrt{2}}{3}\end{aligned}$$

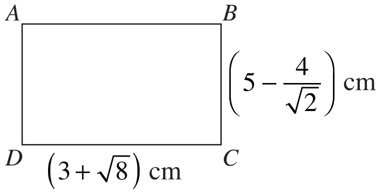
Example 7

The sides of rectangle $ABCD$ are $\left(3+\sqrt{8}\right)$ cm and $\left(5-\dfrac{4}{\sqrt{2}}\right)$ cm in length.

Express, in the form of $a + b\sqrt{2}$, where a and b are integers,

(i) the area of the rectangle in cm^2,

(ii) the area of a square in cm^2, given that AC is one of its sides.



Solution

(i) Area of rectangle $ABCD = \left(3+\sqrt{8}\right)\left(5-\dfrac{4}{\sqrt{2}}\right)$

$$= 15 - \frac{12}{\sqrt{2}} + 5\sqrt{8} - 4\sqrt{4}$$

(Rationalise the denominator of $\dfrac{12}{\sqrt{2}}$.)

$$= 15 - \frac{12\sqrt{2}}{2} + 5\left(2\sqrt{2}\right) - 8$$

$$= \left(7 + 4\sqrt{2}\right)\ \text{cm}^2$$

(ii) Area of square $= AC^2$

$$= \left(3+\sqrt{8}\right)^2 + \left(5-\frac{4}{\sqrt{2}}\right)^2$$

(Apply Pythagoras' Theorem.)

$$= 9 + 6\sqrt{8} + 8 + 25 - \frac{40}{\sqrt{2}} + 8$$

$$= 50 + 12\sqrt{2} - 20\sqrt{2}$$

$$= \left(50 - 8\sqrt{2}\right)\ \text{cm}^2$$

Common Logarithms and Natural Logarithms

8. $\log_{10}$ is called the common logarithm and it is represented by lg.

9. $\log_e$ is called the natural logarithm and it is represented by ln.

Laws of Logarithms

10. **(a)** $\log_a x + \log_a y = \log_a xy$ **(b)** $\log_a x - \log_a y = \log_a \dfrac{x}{y}$

(c) $\log_a x^r = r \log_a x$

More Formulae on Logarithms

11. **(a)** $\log_a b = \dfrac{\log_c b}{\log_c a}$ (Change of Base Formula)

(b) $\log_a 1 = 0$

(c) $\log_a a = 1$

(d) $a^{\log_a y} = y$ (For $\log_a y$ to be a real number, $y > 0$)

(e) $\log_a a^x = x$

Example 8

Simplify $\log_3 81 - \log_5 125 + \log_{\sqrt{2}} 8$.

Solution

$$\begin{aligned}
\log_3 81 - \log_5 125 + \log_{\sqrt{2}} 8 &= \log_3 3^4 - \log_5 5^3 + \frac{\log_2 8}{\log_2 \sqrt{2}} \\
&= 4 \log_3 3 - 3 \log_5 5 + \frac{3\log_2 2}{\frac{1}{2}\log_2 2} \\
&= 4 - 3 + 6 \\
&= 7
\end{aligned}$$

12. Given $a^x = b$,

- If b can be expressed as a power of a, e.g. $b = a^y$, then $a^x = a^y \Rightarrow x = y$.
- If b cannot be expressed as a power of a,
  - take common logarithms on both sides, i.e. $x \lg a = \lg b \Rightarrow x = \dfrac{\lg b}{\lg a}$, or
  - take natural logarithms on both sides if $a = \mathrm{e}$, i.e. $x \ln \mathrm{e} = \ln b \Rightarrow x = \ln b$

Example 9

Solve the exponential equation $9 = 3^{4x}$.

Solution

$$9 = 3^{4x}$$
$$3^2 = 3^{4x}$$
$$2 = 4x$$
$$x = \frac{1}{2}$$

Example 10

Solve the equation $3\mathrm{e}^y - 5 = 2\mathrm{e}^{-y}$.

Solution

$$3\mathrm{e}^y - 5 = 2\mathrm{e}^{-y}$$
$$3\mathrm{e}^y - 5 - \frac{2}{\mathrm{e}^y} = 0$$
$$3(\mathrm{e}^y)^2 - 5\mathrm{e}^y - 2 = 0 \quad \text{(Multiply by } \mathrm{e}^y\text{.)}$$

Let $w = \mathrm{e}^y$.

$$3w^2 - 5w - 2 = 0$$
$$(3w + 1)(w - 2) = 0$$
$$w = -\frac{1}{3} \quad \text{or} \quad w = 2$$
$$\mathrm{e}^y = -\frac{1}{3} \qquad \mathrm{e}^y = 2 \quad \text{(Note that } \mathrm{e}^y > 0\text{.)}$$
$$\text{(no solution)} \qquad y = \ln 2$$
$$= 0.693 \text{ (to 3 s.f.)}$$

13. Given $p(a^{2x}) + q(a^{x}) + r = 0$,
Step 1: Substitute $u = a^x$ to get a quadratic equation $pu^2 + qu + r = 0$.
Step 2: Solve for u and deduce the value(s) of x.

Example 11

Solve the exponential equation $2^{2x+1} = 6(2^x) - 4$.

Solution

$$2^{2x+1} = 6(2^x) - 4$$
$$(2^x)^2(2) = 6(2^x) - 4$$

Let $2^x = y$.

$$2y^2 = 6y - 4$$
$$2y^2 - 6y + 4 = 0$$
$$y^2 - 3y + 2 = 0$$
$$(y-2)(y-1) = 0$$
$$y = 2 \quad \text{or} \quad y = 1$$
$$2^x = 2^1 \qquad 2^x = 2^0$$
$$x = 1 \qquad x = 0$$

Example 12

Without using a calculator, solve the equation $9^x - \frac{28}{3}(3^x) + 3 = 0$.

Solution

$$9^x - \frac{28}{3}(3^x) + 3 = 0$$
$$3(9^x) - 28(3^x) + 9 = 0$$
$$3(3^x)^2 - 28(3^x) + 9 = 0$$ (Ensure that the exponential terms have the same base.)

Let $y = 3^x$.

$$3y^2 - 28y + 9 = 0$$
$$(3y-1)(y-9) = 0$$ (Factorise the quadratic expression.)
$$y = \frac{1}{3} \quad \text{or} \quad y = 9$$
$$3^x = \frac{1}{3} \qquad 3^x = 9$$
$$x = -1 \quad \text{or} \quad x = 2$$

Solving Logarithmic Equations

14. To solve logarithmic equations,

Step 1: Change the bases of the logarithmic functions to the same base. We usually choose the smaller as the final base.

Step 2: Use one of the following methods to solve the equations.

(a) If $\log_a x = \log_a y$, then $x = y$ and vice versa.

(b) If $\log_a x = b$, then $x = a^b$.

(c) Use the laws of logarithms to combine the terms into the forms described in method **(a)** or **(b)**.

Example 13

Solve the equation $\log_a 32x - \log_a (2x^2 + x - 54) = 3 \log_a 2$.

Solution

$$\log_a 32x - \log_a (2x^2 + x - 54) = 3 \log_a 2$$

$$\log_a \frac{32x}{2x^2 + x - 54} = \log_a 2^3$$

$$\frac{32x}{2x^2 + x - 54} = 8$$

$$32x = 16x^2 + 8x - 432$$

$$16x^2 - 24x - 432 = 0$$

$$2x^2 - 3x - 54 = 0$$

$$(2x + 9)(x - 6) = 0$$

$$x = -\frac{9}{2} \text{ (rejected) or } x = 6$$

(Substitute your answers into the original equation to check if any solution needs to be rejected.)

Example 14

Solve the equation $\log_3 (x + 2) = 5$.

Solution

$$\begin{aligned} \log_3 (x + 2) &= 5 \\ x + 2 &= 3^5 \\ x &= 241 \end{aligned}$$

Example 15

(a) Solve the equation $\lg (6x + 4) - \lg (x - 6) = 1$.
(b) Find the value of x given that $e^{x - e} = 10$.

Solution

(a) $\lg (6x + 4) - \lg (x - 6) = 1$

$$\begin{aligned} \lg \frac{6x+4}{x-6} &= 1 \\ \frac{6x+4}{x-6} &= 10 \quad \text{(Change to the exponential form.)} \\ 6x + 4 &= 10x - 60 \\ 4x &= 64 \\ x &= 16 \end{aligned}$$

(b)
$$\begin{aligned} e^{x - e} &= 10 \\ x - e &= \ln 10 \quad \text{(Use ln instead of lg because ln e = 1.)} \\ x &= e + \ln 10 \\ &= 5.02 \text{ (to 3 s.f.)} \end{aligned}$$

Example 16

Solve the simultaneous equations

$$e\sqrt{e^x} = e^{2y},$$
$$\log_4 (x + 2) = 1 + \log_2 y.$$

Solution

$$e\sqrt{e^x} = e^{2y} \quad \text{— (1)}$$
$$\log_4 (x + 2) = 1 + \log_2 y \quad \text{— (2)}$$

From (1),

$$e^1 e^{\frac{x}{2}} = e^{2y}$$
$$e^{1 + \frac{x}{2}} = e^{2y}$$
$$1 + \frac{x}{2} = 2y$$
$$x = 4y - 2 \quad \text{— (3)}$$

From (2),

$$\frac{\log_2 (x + 2)}{\log_2 4} = 1 + \log_2 y$$ (Apply the Change of Base Formula.)

$$\log_2 (x + 2) = 2 + 2\log_2 y$$
$$\log_2 (x + 2) - \log_2 y^2 = 2$$ (Rearrange the logarithmic terms to one side of the equation.)

$$\log_2 \frac{x + 2}{y^2} = 2$$
$$\frac{x + 2}{y^2} = 4$$
$$x = 4y^2 - 2 \quad \text{— (4)}$$

Substitute (3) into (4):

$$4y - 2 = 4y^2 - 2$$
$$4y^2 - 4y = 0$$
$$4y(y - 1) = 0$$
$$y = 0 \quad \text{or} \quad y = 1$$
$$x = -2 \text{ (rejected)} \quad x = 2$$

(Substitute your answers into the original equations to check if any solutions need to be rejected.)

$\therefore x = 2, y = 1$

Example 17

At the beginning of 1980, the number of mice in a colony was estimated at 50 000. The number increased so that, after n years, the number would be $50\ 000 \times e^{0.05n}$. Estimate

(i) the population of the mice, correct to the nearest thousand, at the beginning of the year 2000;

(ii) the year during which the population would first exceed 100 000.

Solution

(i) At the beginning of year 2000, $n = 20$.

$\therefore$ Population of the mice $= 50\ 000 \times e^{0.05(20)}$

$\approx 136\ 000$ (to the nearest thousand)

(ii) Let $50\ 000 \times e^{0.05n} = 100\ 000$

$$e^{0.05n} = 2$$

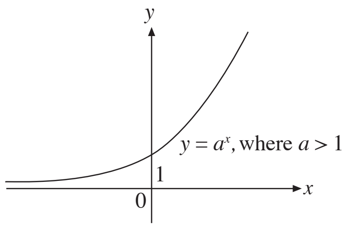
$$0.05n = \ln 2$$

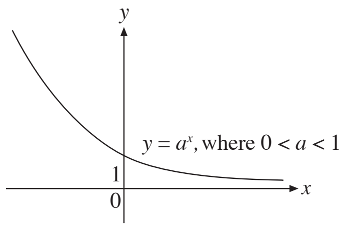
$$n = 13.86$$

$\therefore$ The population will exceed 100 000 in the year 1993.

Graphs of Exponential Functions

15. Graphs of $y = a^x$

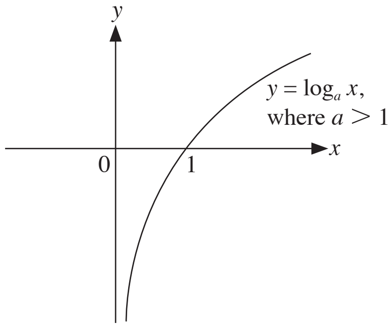


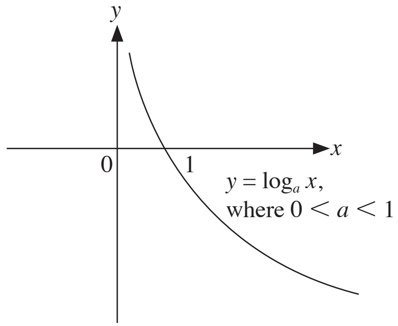


The graph of $y = a^x$ must pass through the point (0, 1) because $a^0 = 1$.

Graphs of Logarithmic Functions

16. Graphs of $y = \log_a x$





Example 18

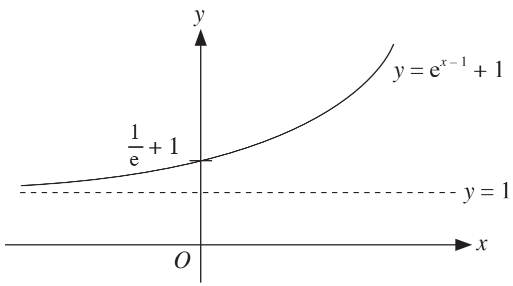
Sketch the graph of each of the following functions.

(a) $y = e^{x-1} + 1$ **(b)** $y = 2e^{1-3x}$

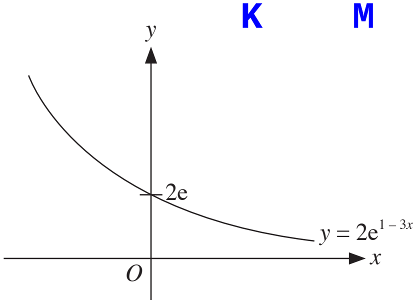
(c) $y = \ln(2x - 3)$ **(d)** $y = \ln(5 - 3x)$

Solution

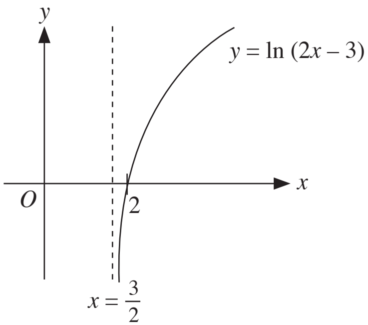
(a)



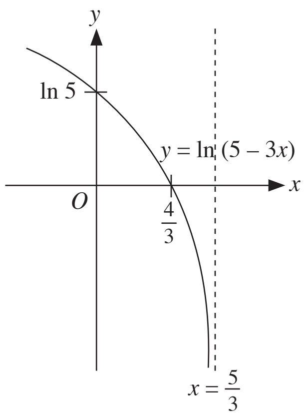
(b)


y
2e
$y = 2e^{1-3x}$
O
x

(c)


y
$y = \ln(2x - 3)$
O
2
x
$x = \frac{3}{2}$

(d)

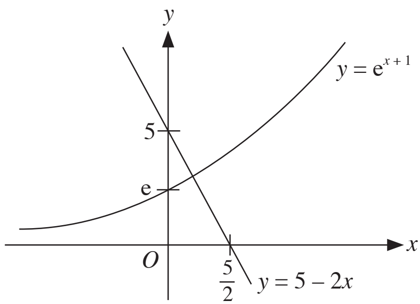

y
ln 5
$y = \ln(5 - 3x)$
O
$\frac{4}{3}$
x
$x = \frac{5}{3}$

Example 19

Sketch the graph of $y = e^{x+1}$. By drawing a suitable straight line on the same graph, find the number of solutions of the equation $x + 1 = \ln(5 - 2x)$.

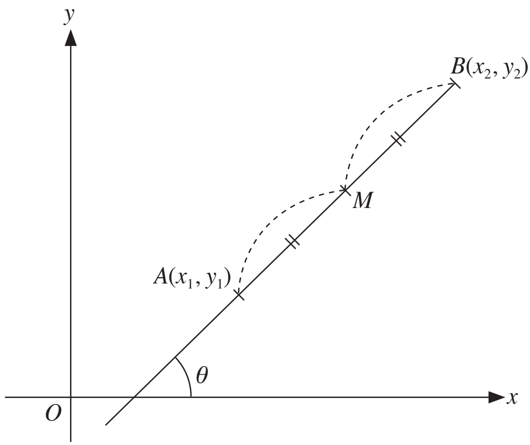
Solution

$x + 1 = \ln(5 - 2x)$
$e^{x+1} = 5 - 2x$
Draw $y = 5 - 2x$.



$\therefore$ There is 1 solution.

UNIT 5 Coordinate Geometry



Distance between 2 Points

1. Length of $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

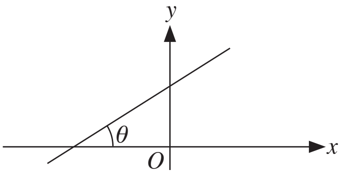
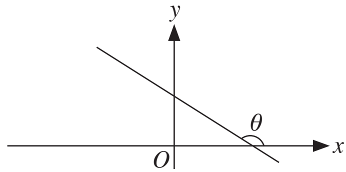
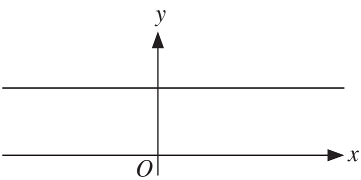
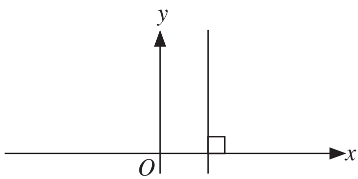
Midpoint of 2 Points

2. Midpoint of $AB = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Gradient of Line and Collinear Points

3. Gradient of AB, $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2} = \tan\theta$

4.

<table>
<tr>
<td>$m > 0$
i.e. $0° < \theta < 90°$
(line slopes upwards)
</td>
<td>$m < 0$
i.e. $90° < \theta < 180°$
(line slopes downwards)
</td>
</tr>
<tr>
<td>$m = 0$
i.e. $\theta = 0°$
(horizontal line)
</td>
<td>m is undefined
i.e. $\theta = 90°$
(vertical line)
</td>
</tr>
</table>

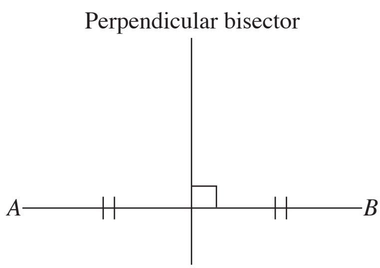
5. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear, then gradient of AB = gradient of BC = gradient of AC and area of $\Delta ABC = 0$.

Parallel and Perpendicular Lines

6. Given that two lines l_1 and l_2 have gradients m_1 and m_2 respectively,

- l_1 is parallel to l_2 if $m_1 = m_2$;
- l_1 is perpendicular to l_2 if $m_1 m_2 = -1$.

7. The perpendicular bisector of a line AB is defined as a line passing through the midpoint of AB, cutting it into two equal halves and it is also perpendicular to AB.



8. Gradient form: $y = mx + c$, where m is the gradient and c is the y-intercept

9. Intercept form: $\frac{x}{a} + \frac{y}{b} = 1$, where a and b are the intercepts the line makes on the x-axis and y-axis respectively

10. General form: $Ax + By + C = 0$, where A, B and C are constants

Example 1

Find the equation of the perpendicular bisector of AB, where A is $(3, 10)$ and B is $(7, 2)$.

Solution

$$\text{Midpoint of } AB = \left(\frac{3+7}{2}, \frac{10+2}{2}\right) = (5, 6)$$

(The perpendicular bisector of AB passes through the midpoint of AB.)

$$\text{Gradient of } AB = \frac{10-2}{3-7} = -2$$

Gradient of perpendicular bisector $= \frac{1}{2}$ $\quad$ $(m_1 m_2 = -1)$

Equation of perpendicular bisector:

$$y - 6 = \frac{1}{2}(x - 5)$$

$$y = \frac{1}{2}x + \frac{7}{2}$$

(To use $y - y_1 = m(x - x_1)$, we require the gradient and the coordinates of a point on the line.)

Example 2

A line segment joins $P(5, 7)$ and $Q(x, y)$. The midpoint of the line segment is $(4, 2)$. Find the coordinates of Q and the equation of the perpendicular bisector of PQ.

Solution

$$\left(\frac{5+x}{2}, \frac{7+y}{2}\right) = (4, 2)$$

$$\frac{5+x}{2} = 4 \qquad \frac{7+y}{2} = 2$$

$$5 + x = 8 \qquad 7 + y = 4$$

$$x = 3 \qquad y = -3$$

$\therefore\ Q(3, -3)$

$$\text{Gradient of } PQ = \frac{7-(-3)}{5-3} = 5$$

$$\text{Gradient of perpendicular bisector} = -\frac{1}{5}$$

Equation of perpendicular bisector:

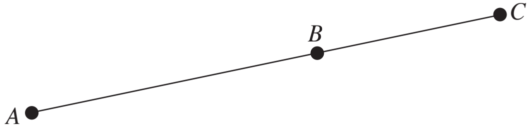
$$\frac{y-2}{x-4} = -\frac{1}{5}$$

$$5y - 10 = -x + 4$$

$$y = \frac{14}{5} - \frac{1}{5}x$$

Collinear Points

11.



- From the diagram, three points A, B and C lie on the same line. We can say that they are collinear.
- To show that the points are collinear, determine 2 of the 3 gradients of the line segments AB, AC and BC. The gradients must be equal, i.e. $m_{AB} = m_{AC}$, $m_{AC} = m_{BC}$ or $m_{AB} = m_{BC}$.

Area of Polygons

12. If $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3), \ldots$, and $N(x_n, y_n)$ form a polygon, then

$$\text{Area of polygon} = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & \ldots & x_n & x_1 \\ y_1 & y_2 & y_3 & \ldots & y_n & y_1 \end{vmatrix}$$

$$= \frac{1}{2}(x_1y_2 + x_2y_3 + \ldots + x_ny_1 - x_2y_1 - x_3y_2 - \ldots - x_1y_n)$$

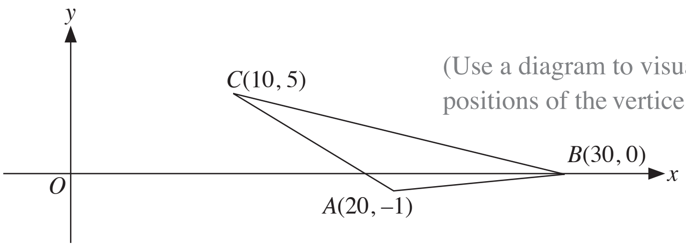
13. If $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ form a triangle ABC, then

$$\text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix}$$

14. Vertices must be taken in a cyclic and anticlockwise order.

Example 3

Find the area of a triangle with coordinates $A(20, -1)$, $B(30, 0)$ and $C(10, 5)$.



(Use a diagram to visualise the relative positions of the vertices of the triangle.)

Solution

$$\text{Area of triangle } ABC = \frac{1}{2}\begin{vmatrix} 30 & 10 & 20 & 30 \\ 0 & 5 & -1 & 0 \end{vmatrix}$$

$$= \frac{1}{2}\left|(150 - 10) - (100 - 30)\right|$$

$$= \frac{1}{2}\left|(140 - 70)\right|$$

$$= 35 \text{ units}^2$$

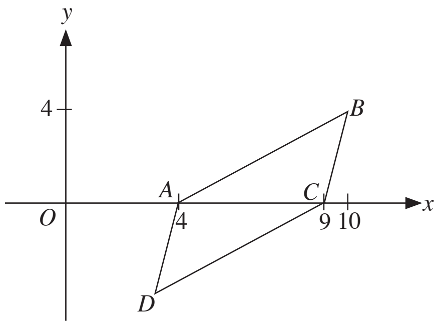
Example 4

A triangle has vertices $A(4, 0)$, $B(10, 4)$ and $C(9, 0)$. Given that $ABCD$ is a parallelogram, find

(i) the coordinates of the point D,

(ii) the area of the parallelogram $ABCD$.

Solution



(Use a sketch to help you visualise the position of D.)

(i) Let the coordinates of D be (x, y).

Midpoint of BD = Midpoint of AC (The diagonals of a parallelogram bisect each other.)

$$\left(\frac{10+x}{2}, \frac{4+y}{2}\right) = \left(\frac{4+9}{2}, \frac{0+0}{2}\right)$$

$$\frac{10+x}{2} = \frac{4+9}{2}, \quad \frac{4+y}{2} = \frac{0+0}{2}$$

$$x = 3 \qquad y = -4$$

$\therefore D(3, -4)$

(ii) Area of parallelogram $ABCD$ (Remember to take the vertices in a cyclic and anticlockwise order.)

$$= \frac{1}{2}\begin{vmatrix} 4 & 3 & 9 & 10 & 4 \\ 0 & -4 & 0 & 4 & 0 \end{vmatrix}$$

$$= \frac{1}{2}(-16 + 36 + 36 - 16)$$

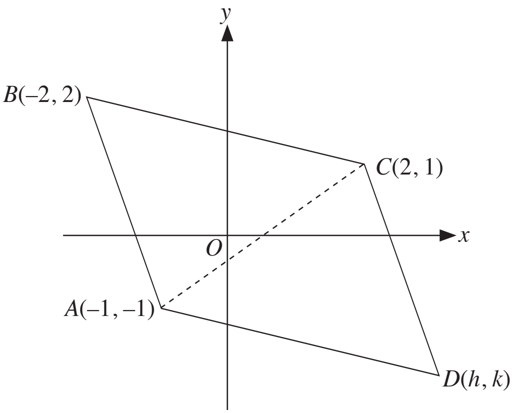
$= 20$ units^2

Example 5

$A(-1, -1)$, $B(-2, 2)$ and $C(2, 1)$ are three vertices of a parallelogram $ABCD$. Find the midpoint of AC. Hence, find the coordinates of D.

Solution

Let the coordinates of D be (h, k).



(Use a sketch to help you visualise the position of D.)

$$\text{Midpoint of } AC = \left(\frac{-1+2}{2}, \frac{-1+1}{2}\right)$$
$$= \left(\frac{1}{2}, 0\right)$$

Midpoint of AC = Midpoint of BD

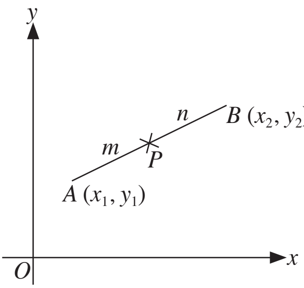
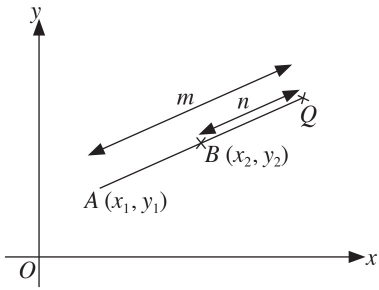
$$\left(\frac{-2+h}{2}, \frac{2+k}{2}\right) = \left(\frac{1}{2}, 0\right)$$

$$\frac{-2+h}{2} = \frac{1}{2}, \quad \frac{2+k}{2} = 0$$

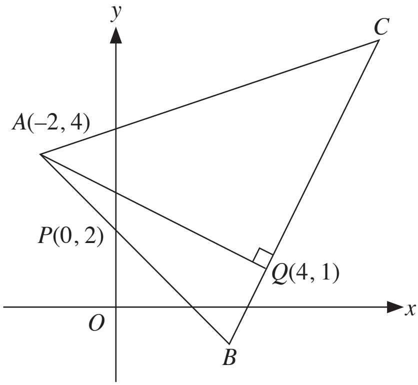
$$h = 3 \qquad k = -2$$

$\therefore\ D(3, -2)$

15.

Internal point of division	External point of division
Let the point P divide the line AB internally in the ratio $m : n$, then P is the point $\left(\dfrac{nx_1 + mx_2}{m+n}, \dfrac{ny_1 + my_2}{m+n}\right)$.  	Let the point Q divide the line AB externally in the ratio $m : n$, then Q is the point $\left(\dfrac{mx_2 - nx_1}{m-n}, \dfrac{my_2 - ny_1}{m-n}\right)$. 

Example 6



The diagram shows a triangle ABC in which A is the point $(-2, 4)$. The side AB cuts the y-axis at $P(0, 2)$. The point $Q(4, 1)$ lies on BC and the line AQ is perpendicular to BC. Find

(i) the equation of BC,

(ii) the coordinates of B.

Given further that Q divides BC internally in the ratio 1 : 3, find

(iii) the coordinates of C,

(iv) the area of triangle ABC.

Solution

(i) Gradient of $AQ = \dfrac{4-1}{-2-4}$

$= -\dfrac{1}{2}$

Gradient of $BC = 2$

Equation of BC: $y - 1 = 2(x - 4)$

$y = 2x - 7 \quad -(1)$

(To use $y - y_1 = m(x - x_1)$, we require the gradient and the coordinates of a point on the line.)

(ii) Gradient of $AB = \dfrac{4-2}{-2-0}$

$= -1$

Equation of AB: $y = -x + 2 \quad -(2)$ (We can use $y = mx + c$ because we know that the y-intercept is 2.)

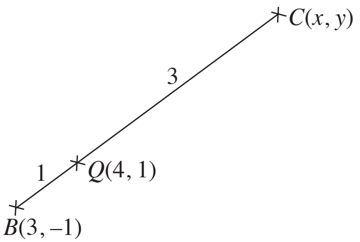
Solving (1) and (2), (Since B lies on AB and BC, we solve the equations of these 2 lines simultaneously.)

$x = 3$

$y = -1$

$\therefore B(3, -1)$

(iii)



(Use a sketch to help you visualise the position of C.)

Let the coordinates of C be (x, y).
Using Ratio Theorem,

$$\left(\frac{3(3)+1(x)}{3+1}, \frac{3(-1)+1(y)}{3+1}\right) = (4, 1)$$

$$\frac{9+x}{4} = 4, \frac{y-3}{4} = 1$$

$$x = 7 \qquad y = 7$$

$\therefore C(7, 7)$

(iv) Area of ΔABC

$$= \frac{1}{2}\begin{vmatrix} -2 & 3 & 7 & -2 \\ 4 & -1 & 7 & 4 \end{vmatrix}$$

(Remember to take the vertices in a cyclic and anticlockwise order.)

$$= \frac{1}{2}(2 + 21 + 28 - 12 + 7 + 14)$$

$= 30$ units^2

UNIT 6 Further Coordinate Geometry

Equation of a Circle

1. Standard form

$$(x-a)^2+(y-b)^2=r^2$$

where (a, b) is the centre and r is the radius

2. General form

$$x^2+y^2+2gx+2fy+c=0$$

where $(-g, -f)$ is the centre and $\sqrt{g^2+f^2-c}$ is the radius

Example 1

Find the equation of the circle with centre $(1, 2)$ and radius of 8.

Solution

Equation of circle:

$(x-1)^2+(y-2)^2=8^2$

$(x-1)^2+(y-2)^2=64$

Example 2

A circle has centre (–1, 1) and passes through (2, 5).
(i) Find the equation of the circle.
(ii) Determine if (3, 3) lies on the circumference of the circle.

Solution

(i) Radius, $r = \sqrt{(2+1)^2 + (5-1)^2}$ (To find the equation of the circle, we need the radius and the coordinates of the centre.)
$= 5$

$\therefore$ Equation of circle is $(x + 1)^2 + (y - 1)^2 = 25$

$$x^2 + 2x + 1 + y^2 - 2y + 1 = 25$$
$$x^2 + y^2 + 2x - 2y - 23 = 0$$

(ii) Substitute $x = 3$, $y = 3$ into $(x + 1)^2 + (y - 1)^2$:
$$(3 + 1)^2 + (3 - 1)^2 = 20$$

$\therefore$ (3, 3) does not lie on the circle. (In fact, (3, 3) lies inside the circle.)

Example 3

A circle has the equation $x^2 + y^2 - 10x + 6y + 9 = 0$.
Find the coordinates of the centre and radius of the circle.

Solution

$$x^2 + y^2 - 10x + 6y + 9 = 0$$
$$(x - 5)^2 - 25 + (y + 3)^2 - 9 + 9 = 0$$
$$(x - 5)^2 + (y + 3)^2 = 5^2$$

Coordinates of centre = (5, –3), radius = 5

Example 4

Find the coordinates of the centre and the radius of a circle with the equation $x^2 + y^2 - 2x - 4y + 5 = 64$.

Solution

$$x^2 + y^2 - 2x - 4y + 5 = 64$$
$$x^2 + y^2 - 2x - 4y - 59 = 0$$

Comparing this with $x^2 + y^2 + 2gx + 2fy + c = 0$

$2g = -2 \qquad 2f = -4 \qquad c = -59$

$g = -1 \qquad f = -2$

Centre of circle $(-g, -f) = (1, 2)$

$$\text{Radius of circle} = \sqrt{g^2 + f^2 - c}$$
$$= \sqrt{(-1)^2 + (-2)^2 + 59}$$
$$= 8$$

Example 5

Find the radius and the coordinates of the centre of the circle $2x^2 + 2y^2 - 3x + 4y + 1 = 0$.

Solution

$$2x^2 + 2y^2 - 3x + 4y + 1 = 0$$
$$x^2 + y^2 - \frac{3}{2}x + 2y + \frac{1}{2} = 0$$
$$\left(x - \frac{3}{4}\right)^2 - \frac{9}{16} + (y+1)^2 - 1 + \frac{1}{2} = 0$$
$$\left(x - \frac{3}{4}\right)^2 + (y+1)^2 = \frac{17}{16}$$

$\therefore$ Radius $= \dfrac{\sqrt{17}}{4}$, coordinates of centre $= \left(\dfrac{3}{4}, -1\right)$

Example 6

Show that the line $4y = x - 3$ touches the circle $x^2 + y^2 - 4x - 8y + 3 = 0$.
Hence, find the coordinates of the point of contact.

Solution

$$4y = x - 3 \quad \text{— (1)}$$
$$x^2 + y^2 - 4x - 8y + 3 = 0 \quad \text{— (2)}$$

From (1),

$$y = \frac{x-3}{4} \quad \text{— (3)}$$

Substitute (3) into (2):

$$x^2 + \left(\frac{x-3}{4}\right)^2 - 4x - 8\left(\frac{x-3}{4}\right) + 3 = 0$$
$$x^2 + \frac{x^2 - 6x + 9}{16} - 4x - 2x + 6 + 3 = 0$$
$$16x^2 + x^2 - 6x + 9 - 64x - 32x + 96 + 48 = 0$$
$$17x^2 - 102x + 153 = 0$$
$$x^2 - 6x + 9 = 0$$

$$\text{Discriminant} = (-6)^2 - 4(1)(9)$$
$$= 0$$

$\therefore$ The line is a tangent to the circle.

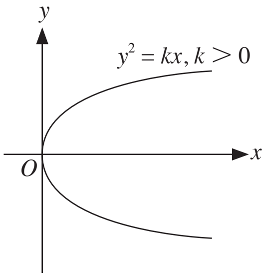
Solving $x^2 - 6x + 9 = 0$,

$$(x-3)^2 = 0$$
$$x = 3$$
$$y = 0$$

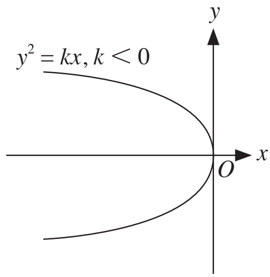
$\therefore$ Point of contact is $(3, 0)$

3. Graphs of the form $y^2 = kx$, where k is a real number

(a) $k > 0$

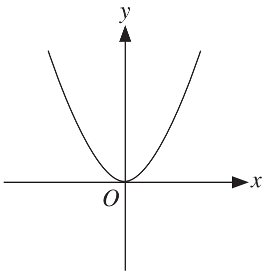


(b) $k < 0$

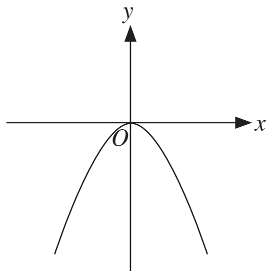


4. Graphs of $y = ax^n$

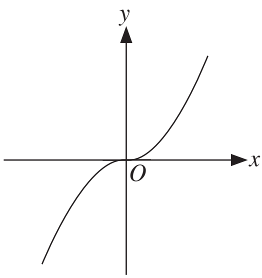
(a) n is even and $a > 0$
e.g. $y = 3x^2$



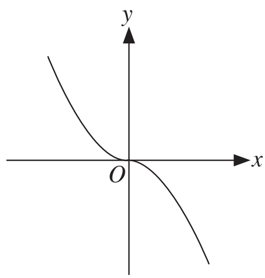
(b) n is even and $a < 0$
e.g. $y = -3x^2$



(c) n is odd and $a > 0$
e.g. $y = 2x^3$

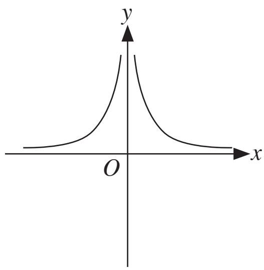


(d) n is odd and $a < 0$
e.g. $y = -2x^3$

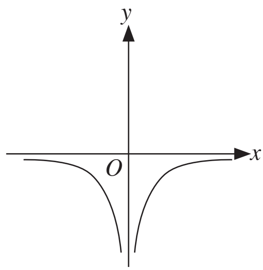


5. Graphs of $y = ax^{-n}$

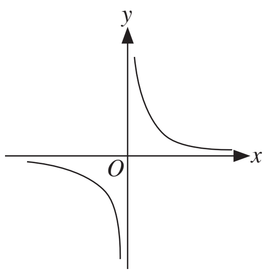
(a) n is even and $a > 0$
e.g. $y = 3x^{-2}$



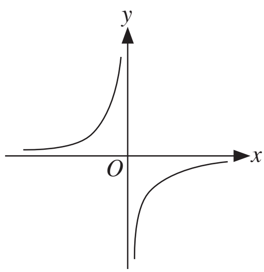
(b) n is even and $a < 0$
e.g. $y = -3x^{-2}$



(c) n is odd and $a > 0$
e.g. $y = 3x^{-3}$



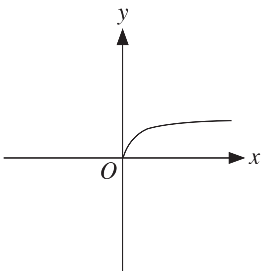
(d) n is odd and $a < 0$
e.g. $y = -3x^{-3}$



6. Graphs of $y = ax^{\frac{1}{n}}$

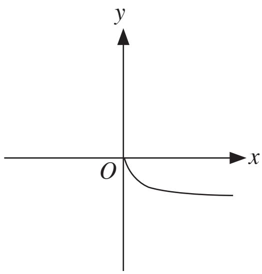
(a) n is even and $a > 0$

e.g. $y = 4x^{\frac{1}{6}}$



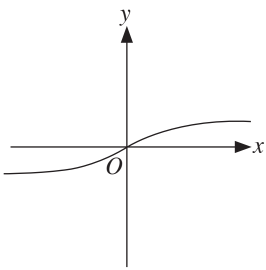
(b) n is even and $a < 0$

e.g. $y = -4x^{\frac{1}{6}}$



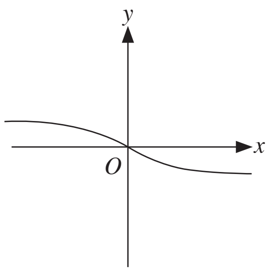
(c) n is odd and $a > 0$

e.g. $y = 3x^{\frac{1}{7}}$



(d) n is odd and $a < 0$

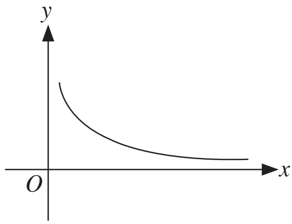
e.g. $y = -3x^{\frac{1}{7}}$



7. Graphs of $y = ax^{-\frac{1}{n}}$

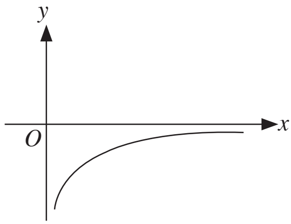
(a) n is even and $a > 0$

e.g. $y = 4x^{-\frac{1}{10}}$



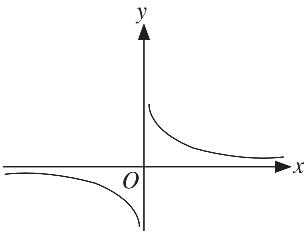
(b) n is even and $a < 0$

e.g. $y = -4x^{-\frac{1}{10}}$



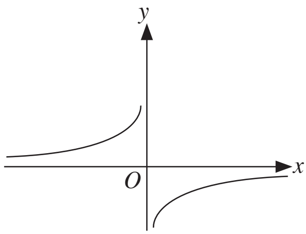
(c) n is odd and $a > 0$

e.g. $y = 0.5x^{-\frac{1}{7}}$



(d) n is odd and $a < 0$

e.g. $y = -0.5x^{-\frac{1}{7}}$

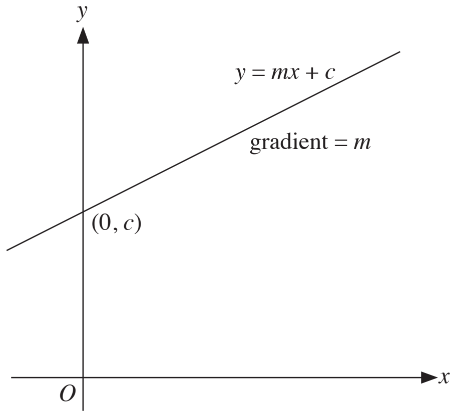


UNIT 7 Linear Law

(not included for NA)

The Linear Law

1.



If the variables x and y are related by the equation $y = mx + c$, then a graph of the values of y plotted against their respective values of x is a straight line graph.

The straight line has a gradient m and it cuts the vertical axis at the point $(0, c)$.

2. Linear Law is used to reduce non-linear functions to the linear form $y = mx + c$.

3. To reduce confusion, we sometimes denote the horizontal axis as X and the vertical axis as Y, i.e. $Y = mX + c$.

4. Some of the common functions and their corresponding X and Y are shown in the table.

Function		X	Y
1. $y = ax^n + b$		x^n	y
2. $y = \dfrac{a}{x^n} + b$		$\dfrac{1}{x^n}$	y
3. $\dfrac{1}{y} = ax^n + b$		x^n	$\dfrac{1}{y}$
4. $y = a\sqrt[n]{x} + b$		$\sqrt[n]{x}$	y
5. $y = a\sqrt{x} + \dfrac{b}{\sqrt{x}}$	i.e $y\sqrt{x} = ax + b$	x	$y\sqrt{x}$
6. $xy = \dfrac{a}{x} + bx$	i.e. $x^2y = bx^2 + a$ or $y = \dfrac{a}{x^2} + b$	x^2 $\dfrac{1}{x^2}$	x^2y y
7. $x = bxy + ay$	i.e. $\dfrac{x}{y} = bx + a$ or $\dfrac{1}{y} = \dfrac{a}{x} + b$	x $\dfrac{1}{x}$	$\dfrac{x}{y}$ $\dfrac{1}{y}$
8. $\dfrac{a}{x} + \dfrac{b}{y} = n$ or $ay + bx = nxy$	i.e. $\dfrac{1}{y} = \left(-\dfrac{a}{b}\right)\dfrac{1}{x} + \dfrac{n}{b}$ i.e. $y = \left(\dfrac{a}{n}\right)\dfrac{y}{x} + \dfrac{b}{n}$	$\dfrac{1}{x}$ $\dfrac{y}{x}$	$\dfrac{1}{y}$ y
9. $y = ax^2 + bx + n$	i.e. $\dfrac{y-n}{x} = ax + b$	x	$\dfrac{y-n}{x}$

Function		X	Y
10. $y = a^2x^2 + 2abx + b^2$	i.e. $\sqrt{y} = ax + b$ or $\sqrt{y} = -ax - b$	x x	$\sqrt{y}$ $\sqrt{y}$
11. $y = \dfrac{a}{x - b}$	i.e. $\dfrac{1}{y} = \dfrac{1}{a}x - \dfrac{b}{a}$	x	$\dfrac{1}{y}$
12. $y = ax^b$	i.e. $\lg y = b \lg x + \lg a$ or $\ln y = b \ln x + \ln a$	$\lg x$ $\ln x$	$\lg y$ $\ln y$
13. $y = ax^b + n$	i.e. $\lg (y - n) = b \lg x + \lg a$	$\lg x$	$\lg (y - n)$
14. $y = ab^x$	i.e. $\lg y = x \lg b + \lg a$	x	$\lg y$
15. $y^n = \dfrac{b^x}{a}$	i.e. $\lg y = x\left(\dfrac{\lg b}{n}\right) - \dfrac{\lg a}{n}$	x	$\lg y$
16. $ya^x = b + n$	i.e. $\lg y = (-\lg a)x + \lg (b + n)$	x	$\lg y$
17. $y^b = 10^{2x + a}$	i.e. $\lg y = \dfrac{2}{b}x + \dfrac{a}{b}$	x	$\lg y$
18. $y^b = e^{2x + a}$	i.e. $\ln y = \dfrac{2}{b}x + \dfrac{a}{b}$	x	$\ln y$

Example 1

The variables x and y are related in such a way that when $y - 3x$ is plotted against x^2, a straight line passing through (2, 1) and (5, 7) is obtained. Find

(i) y in terms of x,

(ii) the values of x when $y = 62$.

Solution

With reference to the sketch graph and using Y to represent $y - 3x$ and X to represent x^2, the equation of the straight line is $\dfrac{Y-1}{X-2} = \dfrac{7-1}{5-2}$.

i.e. $Y - 1 = 2(X - 2)$

$Y = 2X - 3$

(i) $y - 3x = 2x^2 - 3$

$y = 2x^2 + 3x - 3$

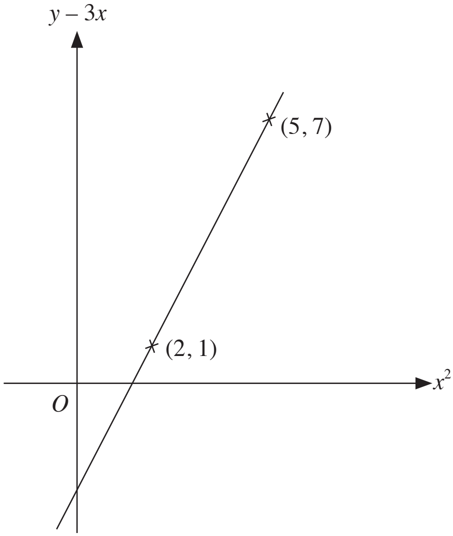
(ii) When $y = 62$,

$2x^2 + 3x - 3 = 62$

$2x^2 + 3x - 65 = 0$

$(x - 5)(2x + 13) = 0$

$x = 5$ or $x = -\dfrac{13}{2}$



Example 2

It is known that x and y are related by the formula $xy = a + bx$, where a and b are constants.

x	2	4	6	8	10
y	38	21.3	15.8	13.1	11.5

Express this equation in a form suitable for drawing a straight line graph. Draw this graph for the given data and use it to estimate the value of a and of b.

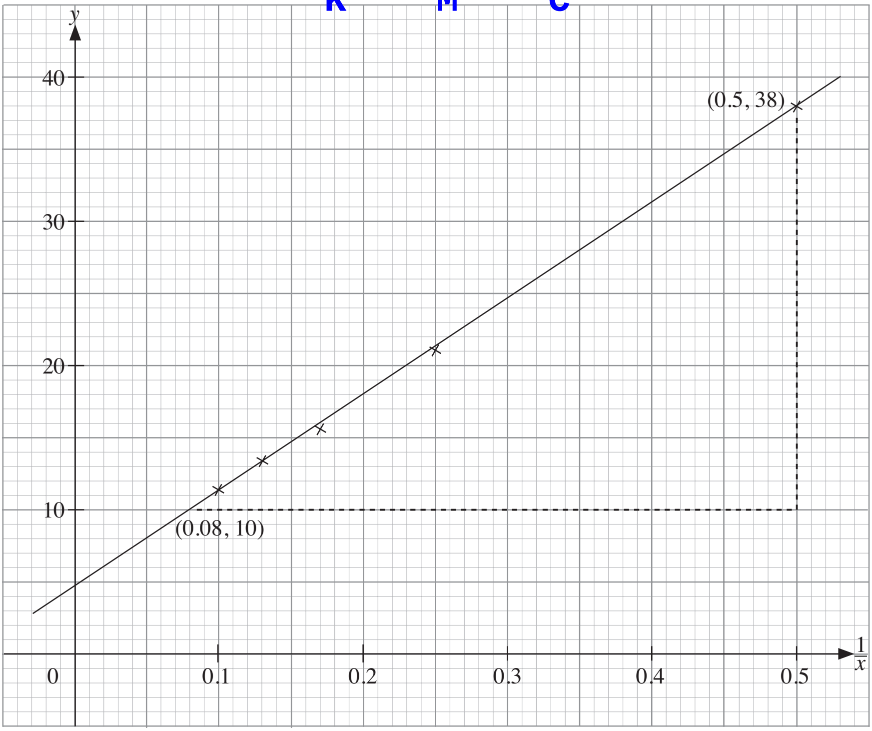
Solution

Since $xy = a + bx$,

$$y = \frac{a}{x} + b.$$

Plot y against $\frac{1}{x}$, gradient $= a$, vertical-axis intercept $= b$.

x	2	4	6	8	10
y	38	21.3	15.8	13.1	11.5
$\frac{1}{x}$	0.50	0.25	0.17	0.13	0.10



From the graph, vertical-axis intercept = 4.8

Using (0.08, 10) and (0.5, 38), gradient = $\dfrac{38 - 10}{0.5 - 0.08}$

= 66.7 (to 3 s.f.)

$\therefore\ a = 66.7,\ b = 4.8$

Example 3

The volume (V) of a container and the height of the container (x) are connected by an equation of the form $V = hk^x$, where h and k are constants.

x	1	2	3	4	5
V	12.6	25.1	50.1	90.0	199.5

(a) Express $V = hk^x$ in a form suitable for drawing a straight line graph.
(b) Plot this straight line graph and use it to estimate the value of h and of k.

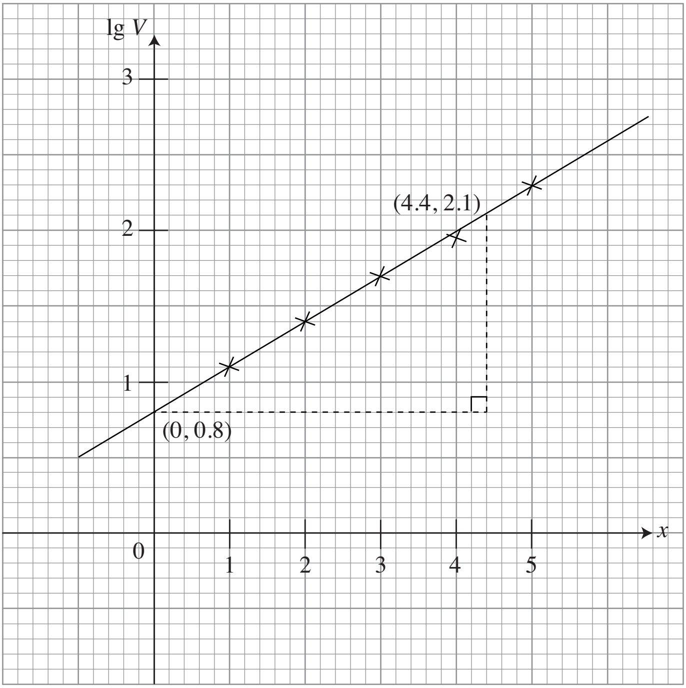
Solution

(a)

$$V = hk^x$$
$$\lg V = (\lg k)x + \lg h$$
$$Y = mX + c$$
$$\text{Gradient} = \lg k$$
$$Y\text{-intercept} = \lg h$$

(b)

x	1	2	3	4	5
V	12.6	25.1	50.1	90.0	199.5
lg V	1.10	1.40	1.70	1.95	2.30



From the graph,

vertical-axis intercept = 0.8

$$\lg h = 0.8$$
$$h = 6.31 \text{ (to 3 s.f.)}$$

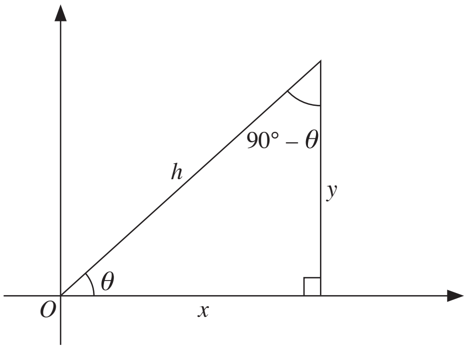
Using (0, 0.8) and (4.4, 2.1),

$$\text{gradient} = \frac{2.1 - 0.8}{4.4 - 0}$$
$$= 0.295 \text{ (to 3 s.f.)}$$
$$\lg k = 0.295$$
$$k = 1.97 \text{ (to 3 s.f.)}$$

UNIT 8 Trigonometric Functions and Equations

Basic Trigonometric Ratios

1.



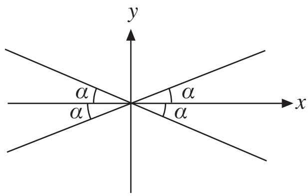
- $\sin\theta = \dfrac{y}{h}$
- $\cos\theta = \dfrac{x}{h}$
- $\tan\theta = \dfrac{y}{x}$
- $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta} = \dfrac{h}{y}$
- $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{h}{x}$
- $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{x}{y}$

Complementary Angles

2.
- $\sin(90° - \theta) = \cos\theta$
- $\tan(90° - \theta) = \cot\theta$
- $\sec(90° - \theta) = \operatorname{cosec}\theta$
- $\cos(90° - \theta) = \sin\theta$
- $\cot(90° - \theta) = \tan\theta$
- $\operatorname{cosec}(90° - \theta) = \sec\theta$

Basic Angle (or Reference Angle)

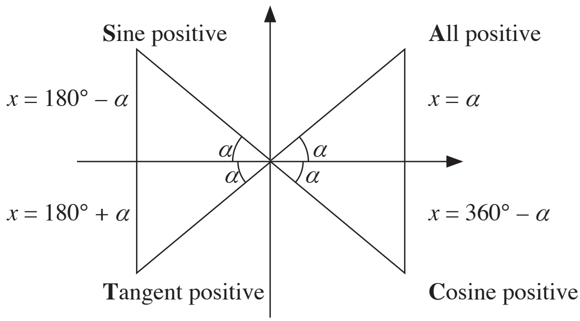
3. The basic angle, α, is the acute angle between a rotating radius about the origin and the x-axis.



- $\sin(-\theta) = -\sin\theta$
- $\cos(-\theta) = \cos\theta$
- $\tan(-\theta) = -\tan\theta$

Signs of Trigonometric Ratios in the Four Quadrants

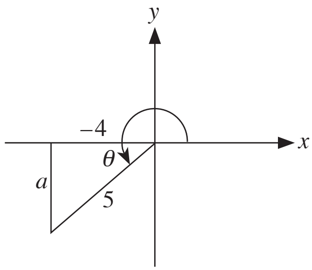
4.



Example 1

Given that $\cos\theta = -\frac{4}{5}$ and that $180° < \theta < 270°$, find the value of $\sin\theta$ and $\tan\theta$.

Solution



θ lies in the 3rd quadrant.

$(-4)^2 + a^2 = 5^2$

$a^2 = 25 - 16$

$= 9$

$a = -3$ (Since a lies in the negative y-axis, $a < 0$.)

$\sin\theta = -\frac{3}{5}$

$\tan\theta = \frac{3}{4}$

Example 2

Given that $\sec \alpha = \frac{17}{15}$ and that α is an acute angle, find the value of each of the following.

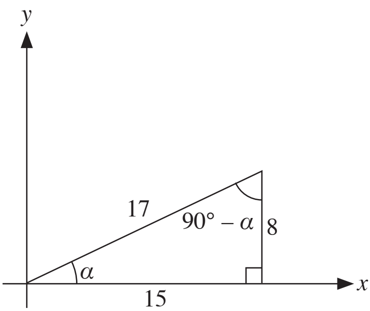
(i) $\sin \alpha$

(ii) $\tan (90° - \alpha)$

(iii) $\cos (180° - \alpha)$

Solution

$\sec \alpha = \frac{17}{15}$ i.e. $\cos \alpha = \frac{15}{17}$ (Recall that $\sec x = \frac{1}{\cos x}$.)



(Use Pythagoras' Theorem to find the length of the side opposite α.)

(i) $\sin \alpha = \frac{8}{17}$

(ii) $\tan (90° - \alpha) = \frac{15}{8}$

(iii) $\cos (180° - \alpha) = -\cos \alpha$

$$= -\frac{15}{17}$$

Example 3

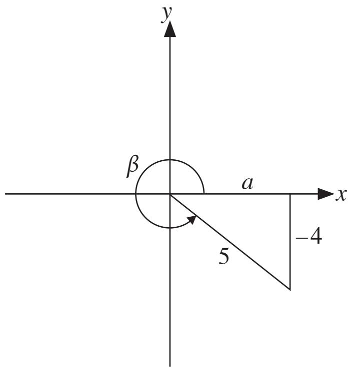
Given that $270^\circ \leqslant \beta < 360^\circ$ and $\sin\beta = -\frac{4}{5}$, find the value of each of the following without using a calculator.

(i) $\cos\beta$

(ii) $\tan\beta$

Solution

β lies in the 4th quadrant.



$$4^2 + a^2 = 5^2$$
$$a^2 = 25 - 16$$
$$= 9$$
$$a = 3$$

(Since a lies in the positive x-axis, $a > 0$.)

(i) $\cos\beta = \frac{3}{5}$

(ii) $\tan\beta = -\frac{4}{3}$

5.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	1	0
$30° = \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$45° = \frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60° = \frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90° = \frac{\pi}{2}$	1	0	Undefined
$180° = \pi$	0	-1	0
$270° = \frac{3\pi}{2}$	-1	0	Undefined
$360° = 2\pi$	0	1	0

Example 4

Find all the angles between 0° and 360° inclusive which satisfy each of the following equations.

(i) $5\sin^2 x - 6\sin x\cos x = 0$

(ii) $1 + 2\sin\left(\frac{3y}{2} + 15°\right) = 0$

Solution

(i) $5\sin^2 x - 6\sin x\cos x = 0$ (Do not make the mistake of dividing throughout

$\sin x\,(5\sin x - 6\cos x) = 0$ by sin x, as you will then be short of answers.)

$5\sin x - 6\cos x = 0$

$5\sin x = 6\cos x$

$\tan x = \frac{6}{5}$ (Recall that $\tan x = \frac{\sin x}{\cos x}$.)

$\alpha = 50.19°$ (to 2 d.p.)

$x = 50.2°, 230.2°$ (to 1 d.p.)

$\sin x = 0$

$x = 0°, 180°, 360°$

$\therefore x = 0°, 50.2°, 180°, 230.2°, 360°$

(ii) $1 + 2\sin\left(\frac{3y}{2} + 15°\right) = 0$

$\sin\left(\frac{3y}{2} + 15°\right) = -\frac{1}{2}$

$\alpha = 30°$

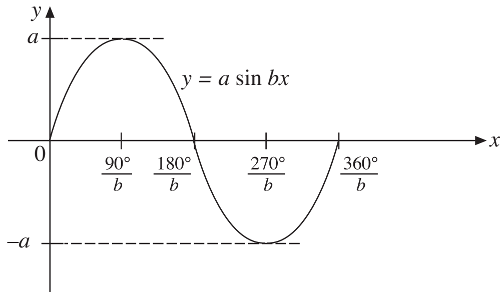
$\frac{3y}{2} + 15° = 210°,\ 330°$ (The required angles are in the 3rd and 4th quadrants.)

$\frac{3y}{2} = 195°,\ 315°$

$\therefore y = 130°, 210°$

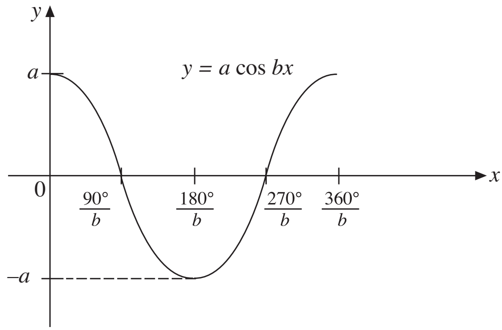
6. $y = a \sin bx$

- amplitude = a
- period = $\dfrac{360°}{b}$



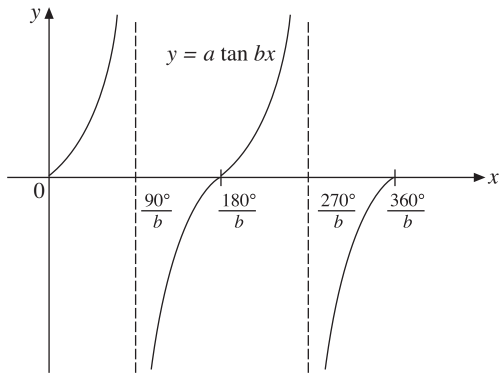
7. $y = a \cos bx$

- amplitude = a
- period = $\dfrac{360°}{b}$



8. $y = a \tan bx$

- period = $\dfrac{180°}{b}$



9. To sketch the graphs of $y = a \sin bx + c$ or $y = a \cos bx + c$ or $y = a \tan bx + c$:

Step 1: Draw the graph of $y = a \sin bx$ or $y = a \cos bx$ or $y = a \tan bx$.

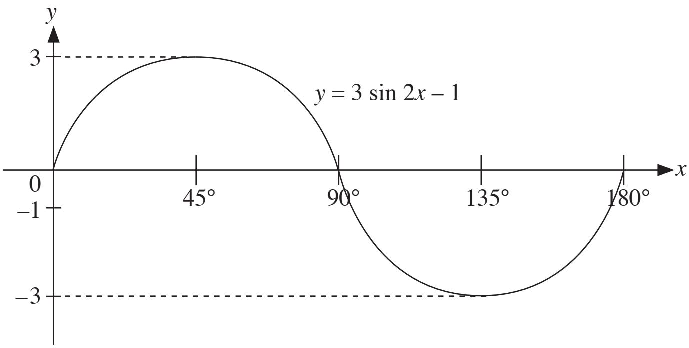
Step 2: If $c > 0$, shift the graph up by c units.
If $c < 0$, shift the graph down by $|c|$ units.

Example 5

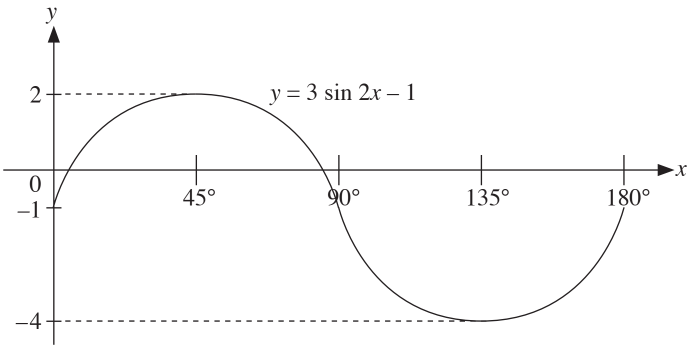
Sketch the graph of $y = 3 \sin 2x - 1$ in the domain $0° \leqslant x \leqslant 180°$.

Solution

First, sketch $y = 3 \sin 2x$.
It has an amplitude of 3 and period of 180°.



Next we shift the graph down by 1 unit.

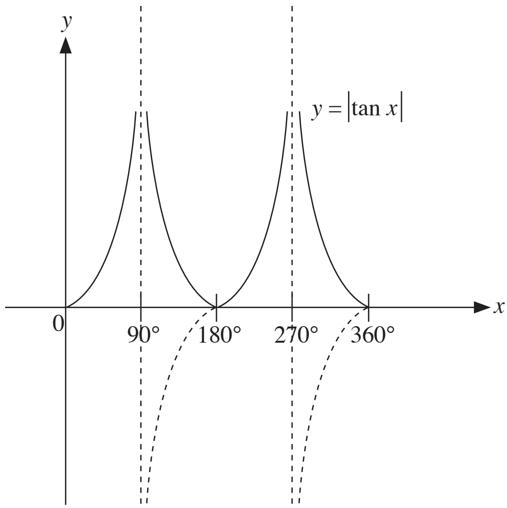


Example 6

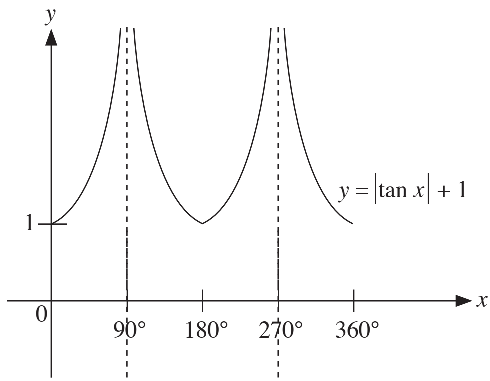
Sketch the graph of $y = |\tan x| + 1$ for $0° \leqslant x \leqslant 360°$.

Solution

First, sketch $y = |\tan x|$.



Next, shift the graph up by 1 unit.



Example 7

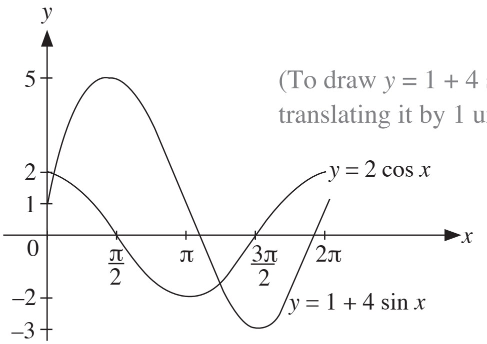
Sketch on the same diagram, for $0 \leqslant x \leqslant 2\pi$, the graphs of

(i) $y = 1 + 4 \sin x$,

(ii) $y = 2 \cos x$.

Hence, deduce the number of roots of the equation $2 \cos x = 1 + 4 \sin x$ for $0 \leqslant x \leqslant 2\pi$.

Solution

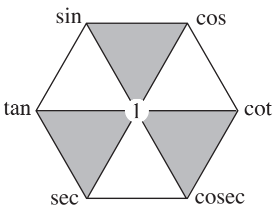


(To draw $y = 1 + 4 \sin x$, we first draw $y = 4 \sin x$, before translating it by 1 unit upwards.)

From the graph, there are 2 roots. (The question is asking for the number of intersection points between the graphs.)

Fundamental Identities

10.



11. $\tan \theta = \dfrac{\sin \theta}{\cos \theta}$

12. $\cot \theta = \dfrac{\cos \theta}{\sin \theta}$

13. $\sec \theta = \dfrac{1}{\cos \theta}$

14. $\operatorname{cosec} \theta = \dfrac{1}{\sin \theta}$

UNIT 9 Trigonometric Identities and Formulae

Fundamental Identities

1. $\sin^2 A + \cos^2 A = 1$

2. $\tan^2 A + 1 = \sec^2 A$

3. $\cot^2 A + 1 = \operatorname{cosec}^2 A$

Example 1

Prove that $\dfrac{1 + \sin x}{\sin x \cos x} = \tan x + \cot x + \sec x$.

Solution

$$\begin{aligned} \text{RHS} &= \tan x + \cot x + \sec x \\ &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} + \frac{1}{\cos x} \\ &= \frac{\sin^2 x + \cos^2 x + \sin x}{\sin x \cos x} \quad \text{(Use the identity } \sin^2 x + \cos^2 x = 1.\text{)} \\ &= \frac{1 + \sin x}{\sin x \cos x} = \text{LHS (proven)} \end{aligned}$$

Example 2

Show that $\dfrac{1}{1+\cos\theta}+\dfrac{1}{1-\cos\theta}=2\operatorname{cosec}^2\theta$.

Solution

$$\begin{aligned}\text{LHS} &= \frac{1}{1+\cos\theta}+\frac{1}{1-\cos\theta}\\ &= \frac{1-\cos\theta+1+\cos\theta}{1-\cos^2\theta}\\ &= \frac{2}{\sin^2\theta} \quad \text{(Use the identity } \sin^2\theta+\cos^2\theta=1.\text{)}\\ &= 2\operatorname{cosec}^2\theta = \text{RHS (proven)} \quad \text{(Recall that } \operatorname{cosec}\theta=\frac{1}{\sin\theta}.\text{)}\end{aligned}$$

Example 3

Prove that $\sec^4\theta-\sec^2\theta=\tan^2\theta+\tan^4\theta$.

Solution

$$\begin{aligned}\text{LHS} &= \sec^4\theta-\sec^2\theta\\ &= \sec^2\theta\,(\sec^2\theta-1)\\ &= (1+\tan^2\theta)(\tan^2\theta) \quad \text{(Use the identity } \tan^2\theta+1=\sec^2\theta.\text{)}\\ &= \tan^2\theta+\tan^4\theta = \text{RHS (proven)}\end{aligned}$$

Example 4

Find all the angles between 0° and 360° inclusive which satisfy the equation

$$3\tan^2 y + 5 = 7\sec y.$$

Solution

$$3\tan^2 y + 5 = 7\sec y$$

$$3(\sec^2 y - 1) + 5 = 7\sec y \quad \text{(Use } \tan^2 y + 1 = \sec^2 y \text{ to obtain a quadratic equation in } \sec y.)$$

$$3\sec^2 y - 7\sec y + 2 = 0$$

$$(3\sec y - 1)(\sec y - 2) = 0$$

$$3\sec y - 1 = 0 \quad \text{or} \quad \sec y - 2 = 0$$

$$\sec y = \frac{1}{3} \qquad \sec y = 2$$

$$\cos y = 3 \text{ (no solution)} \qquad \cos y = \frac{1}{2}$$

$$\alpha = 60°$$

$$\therefore\ y = 60°, 300°$$

Compound Angle Formulae

4. $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

5. $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

6. $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

Example 5

Find all the angles between 0° and 360° which satisfy the equation

$$5 \sin (x + 60°) = \cos (x - 30°).$$

Solution

$$5 \sin (x + 60°) = \cos (x - 30°)$$

$$5[\sin x \cos 60° + \cos x \sin 60°] = \cos x \cos 30° + \sin x \sin 30°$$

$$5\left[\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x\right] = \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x$$

$$5 \sin x + 5\sqrt{3} \cos x = \sqrt{3} \cos x + \sin x$$

$$4 \sin x = -4\sqrt{3} \cos x$$

$$\tan x = -\sqrt{3} \quad \text{(Recall that } \tan x = \frac{\sin x}{\cos x}\text{.)}$$

$$\alpha = 60°$$

$$\therefore x = 120°, 300° \quad (x \text{ lies in the } 2^{\text{nd}} \text{ and } 4^{\text{th}} \text{ quadrants.)}$$

Special Identities

7. $\sin (\theta + 2n\pi) = \sin \theta$, where n is an integer

8. $\cos (\theta + 2n\pi) = \cos \theta$, where n is an integer

9. $\tan (\theta + 2n\pi) = \tan \theta$, where n is an integer

10. $\sin (90° \pm \theta) = \cos \theta$

11. $\cos (90° \pm \theta) = \mp\sin \theta$

12. $\tan (90° \pm \theta) = \mp\cot \theta$

13. $\sin (180° \pm \theta) = \mp\sin \theta$

14. $\cos (180° \pm \theta) = -\cos \theta$

15. $\tan (180° \pm \theta) = \pm \tan \theta$

Double Angle Formulae

16. $\sin 2A = 2 \sin A \cos A$

17. $\cos 2A = \cos^2 A - \sin^2 A$

$= 2\cos^2 A - 1$

$= 1 - 2\sin^2 A$

18. $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

Example 6

Given that $\tan \theta = -\dfrac{4}{3}$ and $270° < \theta < 360°$, find the value of

(i) $\cos(-\theta)$,

(ii) $\cos(90° - \theta)$,

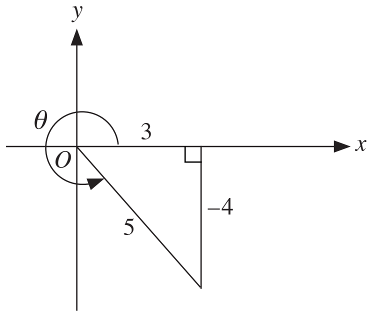
(iii) $\sin(180° + \theta)$,

(iv) $\sin 2\theta$.

Solution

(i) $\cos(-\theta) = \cos\theta$

$= \dfrac{3}{5}$



(ii) $\cos(90° - \theta) = \sin\theta$

$= -\dfrac{4}{5}$

(iii) $\sin(180° + \theta) = -\sin\theta$

$= \dfrac{4}{5}$

(iv) $\sin 2\theta = 2\sin\theta\cos\theta$

$= 2\left(-\dfrac{4}{5}\right)\left(\dfrac{3}{5}\right)$

$= -\dfrac{24}{25}$

Example 7

Given that $\cos 2x = \dfrac{127}{162}$ and $270° \leqslant 2x \leqslant 360°$, find the value of

(i) $\cos x$,

(ii) $\sin x$.

Solution

(i) Since $270° \leqslant 2x \leqslant 360°$, then $135° \leqslant x \leqslant 180°$.

$\therefore$ x lies in the 2nd quadrant.

$$\cos 2x = \frac{127}{162}$$

$$2\cos^2 x - 1 = \frac{127}{162}$$

$$2\cos^2 x = \frac{289}{162}$$

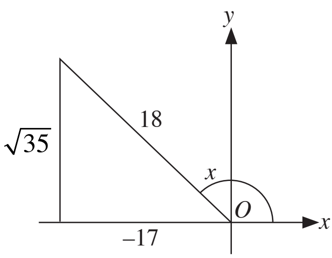
$$\cos^2 x = \frac{289}{324}$$

$$\cos x = \pm\sqrt{\frac{289}{324}}$$

$$= \pm\frac{17}{18}$$

$$\therefore \cos x = -\frac{17}{18}$$ (cos $x < 0$ since x lies in the 2nd quadrant.)

(ii)



$$\sin x = \frac{\sqrt{35}}{18}$$

Half Angle Formulae

Replace A with $\frac{A}{2}$ in the Double Angle Formulae.

19. $\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$

20. $\cos A = \cos^2 \frac{A}{2} \quad \sin^2 \frac{A}{2}$

$$= 2 \cos^2 \frac{A}{2} - 1$$

$$= 1 - 2 \sin^2 \frac{A}{2}$$

21. $\tan A = \dfrac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}$

R-Formulae

$$\left.\begin{aligned}
&\textbf{22.}\quad a \sin \theta + b \cos \theta = R \sin (\theta + \alpha)\\
&\textbf{23.}\quad a \sin \theta - b \cos \theta = R \sin (\theta - \alpha)\\
&\textbf{24.}\quad a \cos \theta + b \sin \theta = R \cos (\theta - \alpha)\\
&\textbf{25.}\quad a \cos \theta - b \sin \theta = R \cos (\theta + \alpha)
\end{aligned}\right\} \text{where } R = \sqrt{a^2 + b^2} \text{ and } \tan \alpha = \frac{b}{a}$$

26. For the expression $a \sin \theta \pm b \cos \theta$ or $a \cos \theta \pm b \sin \theta$,

- Maximum value = R
- Minimum value = $-R$

Example 8

Using the R-formula, find the maximum and minimum values of $6 \sin x - 5 \cos x$ for values of x, where $0° < x < 360°$.

Solution

$6 \sin x - 5 \cos x = R \sin (x - \alpha)$

$R = \sqrt{6^2 + 5^2} = \sqrt{61}$

$\alpha = \tan^{-1} \left(\frac{5}{6}\right)$

$\quad = 39.8°$

$\therefore\ 6 \sin x - 5 \cos x = \sqrt{61} \sin (x - 39.8°)$

Minimum value $= -\sqrt{61}$ (when $\sin (x - 39.8°) = -1$)
Maximum value $= \sqrt{61}$ (when $\sin (x - 39.8°) = 1$)

Example 9

Solve $3 \sin 2x + 2 \sin x = 0$ for $0° \leqslant x \leqslant 360°$.

Solution

$$3 \sin 2x + 2 \sin x = 0$$
$$3(2 \sin x \cos x) + 2 \sin x = 0$$
$$3 \sin x \cos x + \sin x = 0$$
$$\sin x (3 \cos x + 1) = 0$$

$\sin x = 0$ or $3 \cos x + 1 = 0$

$x = 0°, 180°, 360°$ $\qquad \cos x = -\frac{1}{3}$

$\alpha = 70.53°$

$x = 109.5°, 250.5°$ (The required angles are in the 2nd and 4th quadrants.)

Example 10

By expressing $4 \cos x - 3 \sin x$ in the form $R \cos (x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$,

(i) obtain the maximum value of $4 \cos x - 3 \sin x + 5$ and the corresponding value of x,

(ii) solve the equation $4 \cos x - 3 \sin x = 2.5$ for values of x between 0 and 2π inclusive.

Solution

$4 \cos x - 3 \sin x = R \cos (x + \alpha)$

$R = \sqrt{4^2 + 3^2} = 5$

$\tan \alpha = \frac{3}{4}$

$\alpha = 0.6435$ (to 4 s.f.)

$\therefore\ 4 \cos x - 3 \sin x = 5 \cos (x + 0.644)$

(i) Maximum value $= 5 + 5$ (Maximum value of $4 \cos x - 3 \sin x$ is 5)

$= 10$

Maximum value occurs when $\cos (x + 0.6435) = 1$,

i.e. $x + 0.6435 = 2\pi$

$x = 5.64$ (to 3 s.f.)

(ii) $4 \cos x - 3 \sin x = 2.5$

$5 \cos (x + 0.6435) = 2.5$ (Use the expression obtained earlier to solve the equation.)

$\cos (x + 0.6435) = 0.5$

$\alpha = \frac{\pi}{3}$

$x + 0.6435 = 1.047, 5.235$ (to 4 s.f.) ($x + 0.6435$ lies in the 1st and 4th quadrants.)

$x = 0.404, 4.59$ (to 3 s.f.)

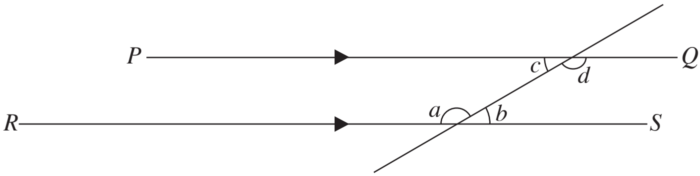
UNIT 10 Proofs in Plane Geometry

(not included for NA)

Useful Properties and Concepts that are learnt in O Level Mathematics

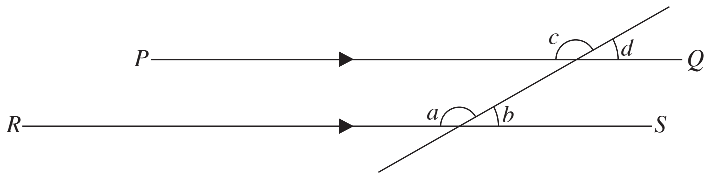
1. Angle Properties

(a) Alternate angles between parallel lines are equal



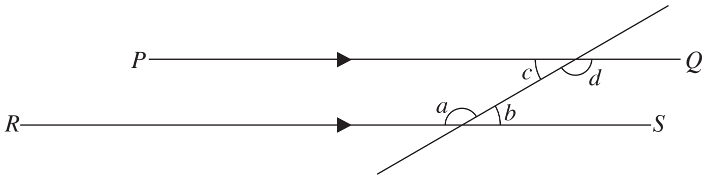
Since PQ // RS, $\angle a = \angle d$ and $\angle b = \angle c$

(b) Corresponding angles between parallel lines are equal



Since PQ // RS, $\angle a = \angle c$ and $\angle b = \angle d$

(c) Interior angles between parallel lines are supplementary



Since PQ // RS, $\angle a + \angle c = 180°$ and $\angle b + \angle d = 180°$

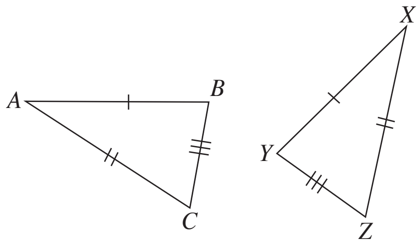
2. Properties of Congruent Triangles

- Corresponding sides are equal in length.
- Corresponding angles are equal.

3. Congruence Tests for Triangles

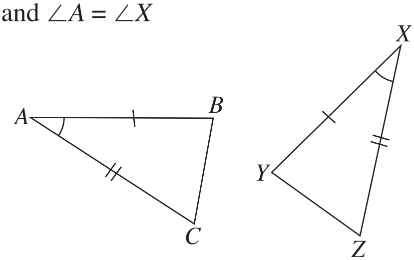
(i) SSS

$AB = XY$, $AC = XZ$ and $BC = YZ$



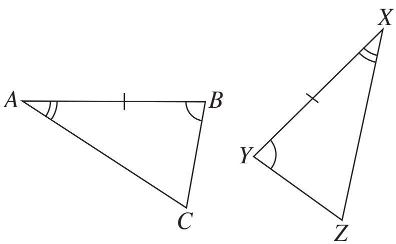
(ii) SAS

$AB = XY$, $AC = XZ$ and $\angle A = \angle X$



(iii) AAS or ASA

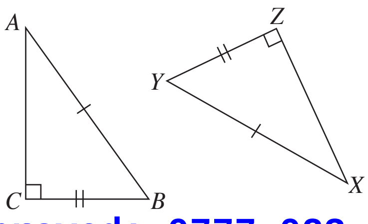
$AB = XY$, $\angle A = \angle X$ and $\angle B = \angle Y$



(iv) RHS

Only applicable for right-angled triangles.

$BC = YZ$, $AB = XY$ and $\angle C = \angle Z = 90°$

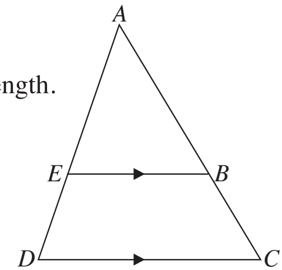


4. Properties of Similar Triangles

- All corresponding angles are equal.
- All corresponding sides are proportional in length.

$$\frac{AE}{AD} = \frac{AB}{AC} = \frac{EB}{DC}$$

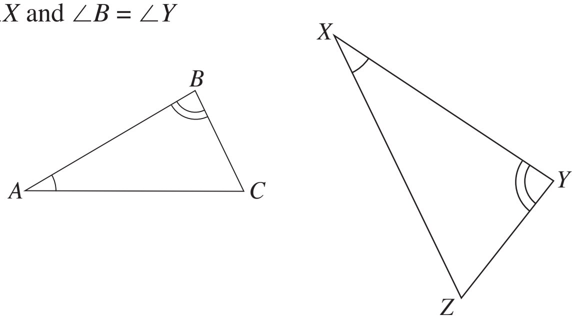
$$\frac{\text{Area of } \Delta AEB}{\text{Area of } \Delta ADC} = \left(\frac{AE}{AD}\right)^2$$



5. Similarity Tests for Triangles

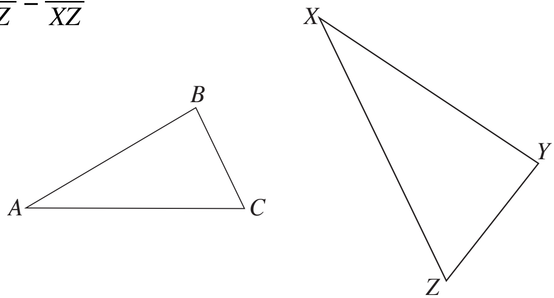
(i) AA

$\angle A = \angle X$ and $\angle B = \angle Y$



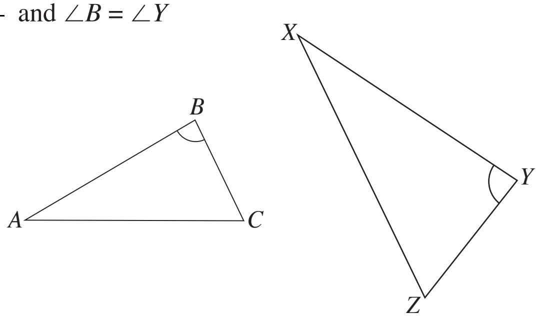
(ii) SSS

$$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$$



(iii) SAS

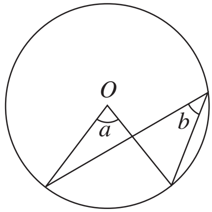
$$\frac{AB}{XY} = \frac{BC}{YZ} \text{ and } \angle B = \angle Y$$



6. Circles

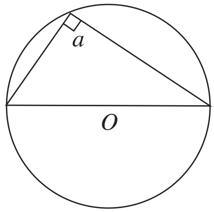
(a) $\angle$ at centre = 2$\angle$ at circumference

An angle at the centre is **twice** any angle at the circumference subtended by the **same arc**, i.e. $\angle a = 2\angle b$.



(b) Rt. $\angle$ in a semicircle

Every angle at the circumference subtended by the diameter of a circle is a **right angle**, i.e. $\angle a = 90°$.

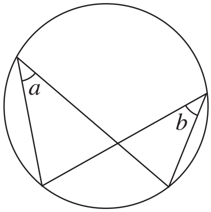


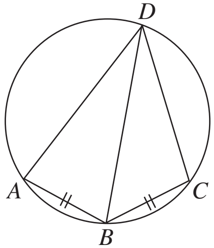
(c) $\angle$s in the same segment

Angles in the same segment of a circle are **equal**, i.e. $\angle a = \angle b$.

Or

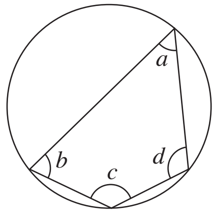
If $AB = BC$, then $\angle ADB = \angle BDC$.





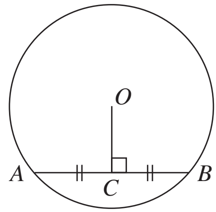
(d) $\angle$s in opp. segments are supplementary

In a cyclic quadrilateral, the opposite angles are **supplementary**, i.e. $\angle a + \angle c = 180°$ and $\angle b + \angle d = 180°$.



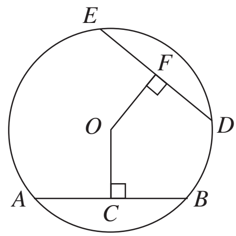
(e) $\perp$ bisector of a chord passes through the centre of the circle

A straight line drawn from the centre to bisect a chord is **perpendicular** to the chord, i.e. $OC \perp AB \Leftrightarrow AC = BC$.



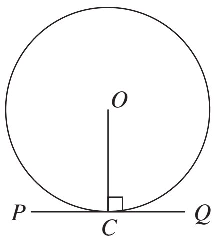
(f) Equal chords are equidistant from the centre

Chords which are equidistant from the centre are equal, i.e. $AB = DE \Leftrightarrow OC = OF$. ($\Delta OAB \equiv \Delta ODE$)



(g) Tangent $\perp$ radius

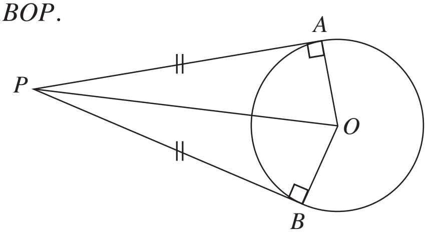
A tangent to a circle is **perpendicular** to the radius drawn to the point of contact, i.e. $OC \perp PQ$.



(h) Tangents from an external point

(i) Tangents drawn to a circle from an external point are equal, i.e. $PA = PB$.

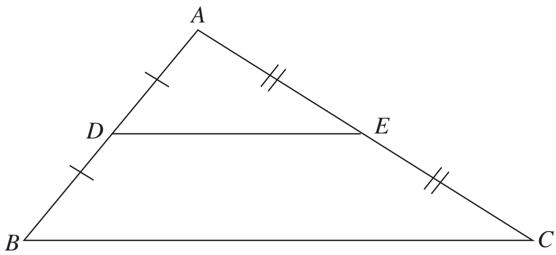
(ii) The line joining the external point to the centre of the circle bisects the angle between the tangents,
i.e. $\angle APO = \angle BPO$ and $\angle AOP = \angle BOP$.
($\Delta OAP \equiv \Delta OBP$)



7. Midpoint Theorem for Triangles

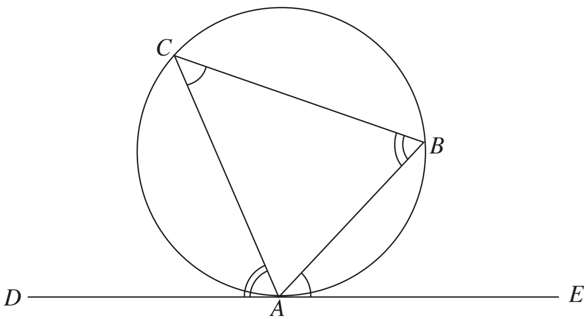
In ΔABC, if D and E are the midpoints of the sides AB and AC respectively, then

$DE \parallel BC$ and $DE = \dfrac{1}{2} BC$.



8. Tangent-chord Theorem (Alternate Segment Theorem)

The angle between a tangent and a chord meeting the tangent at the point of contact is equal to the inscribed angle on the opposite side of the chord,
i.e. $\angle BAE = \angle BCA$ and $\angle CAD = \angle CBA$.

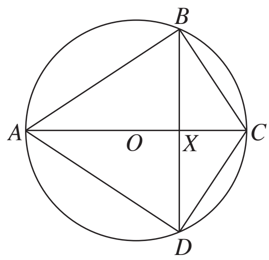


Example 1

The diagram shows a circle, centre O, with diameter AC and $AB = AD$. AC and BD intersect at X.

(a) Prove that ΔABC and ΔADC are congruent.

(b) Prove that BD is perpendicular to AC.



Solution

(a) $AB = AD$

AC is a common side for the two triangles. (The hypotenuse of both triangles are the same.)

$\angle ABC = \angle ADC = 90°$ (rt. $\angle$ in a semicircle)

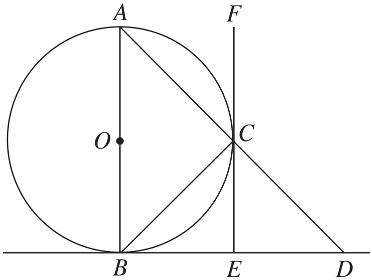
ΔABC is congruent to ΔADC (RHS congruence).

(b) Since ΔABC is congruent to ΔADC and they share the same base (AC), $BX = DX$.

Since AC passes through the centre of the circle, $AC \perp BD$ ($\perp$ bisector of a chord passes through the centre of the circle).

Example 2

In the figure, BD and FE are tangents to the circle, centre O. BED is a tangent to the circle at B and ACD is a straight line. $\angle CED = 90°$.



Prove that

(i) $\angle ABC = \angle ECD$,

(ii) ΔABD is similar to ΔBCD.

Solution

(i) $\angle ECD = \angle ACF$ (vert. opp. $\angle$s)

$\angle ACF = \angle ABC$ ($\angle$s in alt. segments)

$\angle ABC = \angle ECD$

(ii) In ΔABD and ΔBCD,

$\angle BAD = \angle CBD$ ($\angle$s in alt. segments)

$\angle ABD = 90°$ (Tangent $\perp$ radius)

$\angle BCD = \angle BCA = 90°$ (rt. $\angle$ in a semicircle)

i.e. $\angle ABD = \angle BCD$

ΔABD is similar to ΔBCD (AA Similarity Test).

UNIT 11 Differentiation and its Applications

Formulae

1. $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$

2. $\dfrac{\mathrm{d}}{\mathrm{d}x}(ax^n) = anx^{n-1}$

3. $\dfrac{\mathrm{d}}{\mathrm{d}x}(k) = 0$

Addition/Subtraction Rules

4. If $y = u(x) \pm v(x)$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}}{\mathrm{d}x}[u(x)] \pm \dfrac{\mathrm{d}}{\mathrm{d}x}[v(x)]$

Example 1

Differentiate $2x^3 - 8x^2 + \dfrac{1}{x^2} - 4$ with respect to x.

Solution

$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(2x^3 - 8x^2 + \dfrac{1}{x^2} - 4\right)$

$= \dfrac{\mathrm{d}}{\mathrm{d}x}(2x^3 - 8x^2 + x^{-2} - 4)$ (Change $\dfrac{1}{x^2}$ to x^{-2}.)

$= 6x^2 - 16x - 2x^{-3}$

$= 6x^2 - 16x - \dfrac{2}{x^3}$

Chain Rule

5. If y is a function of u, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

6. $\frac{d}{dx}\left[(ax+b)^n\right] = an(ax+b)^{n-1}$

7. In general, $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} \times f'(x)$

Example 2

Differentiate $\sqrt{5-4x^2}$ with respect to x.

Solution

$$\frac{d}{dx}\left(\sqrt{5-4x^2}\right)$$

$$= \frac{d}{dx}\left(5-4x^2\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\left(5-4x^2\right)^{-\frac{1}{2}}(-8x) \qquad \text{(Chain Rule)}$$

$$= -\frac{4x}{\sqrt{5-4x^2}}$$

Product Rule

8. If $y = uv$, where u and v are functions of x, then $\frac{dy}{dx} = u\frac{dv}{dx} \times v\frac{du}{dx}$

Example 3

Differentiate $2x(3x^3 - 2)^3$ with respect to x.

Solution

$$\frac{d}{dx}\left[2x(3x^3 - 2)^3\right]$$

$= 2x(3)(3x^3 - 2)^2(9x^2) + 2(3x^3 - 2)^3$ (Product Rule and Chain Rule)

$= 2(3x^3 - 2)^2(27x^3 + 3x^3 - 2)$ (Take out common factors.)

$= 2(3x^3 - 2)^2(30x^3 - 2)$

$= 4(3x^3 - 2)^2(15x^3 - 1)$

Quotient Rule

9. If $y = \dfrac{u}{v}$, where u and v are functions of x, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Example 4

Differentiate $\dfrac{3x^2 + 4}{\sqrt{2x + 5}}$ with respect to x.

Solution

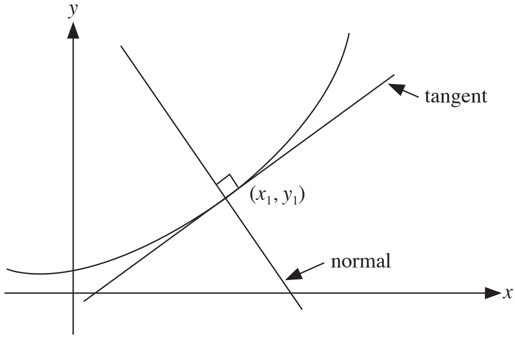
$$\frac{d}{dx}\left[\frac{3x^2 + 4}{\sqrt{2x + 5}}\right]$$

$$= \frac{6x\sqrt{2x + 5} - (3x^2 + 4)\left(\frac{1}{2}\right)(2x + 5)^{-\frac{1}{2}}(2)}{2x + 5} \quad \text{(Quotient Rule)}$$

$$= \frac{6x(2x + 5) - (3x^2 + 4)}{(2x + 5)^{\frac{3}{2}}}$$

$$= \frac{9x^2 + 30x - 4}{\sqrt{(2x + 5)^3}}$$

10. Equation of a straight line: $y - y_1 = m(x - x_1)$



11. To find the equation of a tangent, we need:

Gradient of tangent, $m = \dfrac{dy}{dx}$

Coordinates of a point that lies on the tangent, (x_1, y_1)

12. To find the equation of a normal, we need:

Gradient of tangent $= \dfrac{dy}{dx}$

Gradient of normal $= -1 \div \dfrac{dy}{dx}$

Coordinates of a point that lies on the normal, (x_1, y_1)

Example 5

A curve has the equation $y = x^2 + 3x$.

(i) Find the equation of the tangent to the curve at $(1, 4)$.

(ii) Find the equation of the normal to the curve at $(1, 4)$.

Solution

(i) **Step 1:** Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 2x + 3$$

Step 2: Substitute $x = 1$ into $\frac{dy}{dx}$ to find the gradient of the tangent.

$$\begin{aligned}\frac{dy}{dx} &= 2(1) + 3\\ &= 5\end{aligned}$$

Step 3: Find the equation of the tangent.

$$\begin{aligned}y - 4 &= 5(x - 1)\\ y - 4 &= 5x - 5\\ y &= 5x - 1\end{aligned}$$

(ii) **Step 1:** Find the gradient of the normal.

$$\begin{aligned}\text{Gradient of normal} &= -\frac{1}{\text{Gradient of tangent}}\\ &= -\frac{1}{5}\end{aligned}$$

Step 2: Find the equation of the normal.

$$\begin{aligned}y - 4 &= -\frac{1}{5}(x - 1)\\ 5y - 20 &= -x + 1\\ 5y &= -x + 21\end{aligned}$$

Example 6

The equation of a curve is $y = \dfrac{5}{1 - 3x}$. Find

(i) $\dfrac{dy}{dx}$,

(ii) the equation of the tangent to the curve at $x = 2$,

(iii) the equation of the normal to the curve at $x = 2$.

Solution

(i) $y = \dfrac{5}{1 - 3x}$

$$\frac{dy}{dx} = \frac{(1 - 3x)(0) - 5(-3)}{(1 - 3x)^2}$$

$$= \frac{15}{(1 - 3x)^2}$$

(ii) When $x = 2$,

$y = -1$

$$\frac{dy}{dx} = \frac{3}{5}$$

$\therefore$ Equation of tangent: $y + 1 = \dfrac{3}{5}(x - 2)$

$$y = \frac{3}{5}x - \frac{11}{5}$$

(iii) Gradient of normal $= -\dfrac{5}{3}$ $\quad (m_1 m_2 = -1)$

$\therefore$ Equation of normal: $y + 1 = -\dfrac{5}{3}(x - 2)$

$$y = -\frac{5}{3}x + \frac{7}{3}$$

Connected Rates of Change

13. If $\dfrac{dx}{dt}$ is the rate of change of x with respect to time t and $y = f(x)$, then the rate of change of y with respect to t is given by $\dfrac{dy}{dt} = \dfrac{dy}{dx} \times \dfrac{dx}{dt}$.

14. A positive rate of change is an increase in the magnitude of the quantity involved as the time increases.

15. A negative rate of change is a decrease in the magnitude of the quantity involved as the time increases.

Example 7

Two variables, x and y, are related by the equation $y = \dfrac{x}{3x+7}$. Find the rate of change of x at the instant when $x = 1$, given that y is changing at a rate of 3.5 units/s at this instant.

Solution

$$y = \frac{x}{3x+7}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(3x+7)(1) - x(3)}{(3x+7)^2}\\ &= \frac{3x+7-3x}{(3x+7)^2}\\ &= \frac{7}{(3x+7)^2}\end{aligned}$$

Using $\dfrac{dy}{dt} = \dfrac{dy}{dx} \times \dfrac{dx}{dt}$,

$$3.5 = \frac{7}{(3+7)^2} \times \frac{dx}{dt}$$

$$\frac{dx}{dt} = 50 \text{ units/s}$$

Example 8

A cube has sides of x cm. Its volume, V cm^3, is expanding at a rate of 30 cm^3/s. Find the rate of change of x of the cube when the volume is 64 cm^3.

Solution

$$V = x^3$$
$$\frac{dV}{dx} = 3x^2$$
When $V = 64$, $x = 4$.
When $x = 4$,
$$\frac{dV}{dx} = 3(4)^2$$
$$= 48$$
Given that $\frac{dV}{dt} = 30$,
$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$
$$30 = 48 \times \frac{dx}{dt}$$
$$\frac{dx}{dt} = 0.625 \text{ cm/s}$$

UNIT 12 Further Applications of Differentiation

Increasing/Decreasing Functions

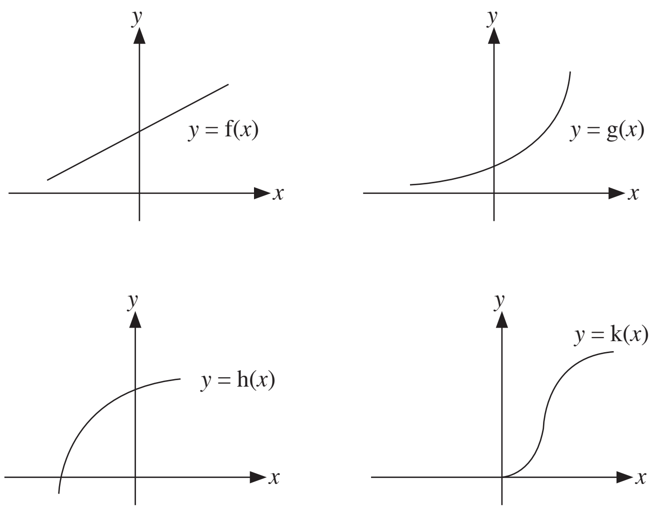
1.

Function in x	y	$f(x)$
First derivative	$\frac{dy}{dx}$	$f'(x)$
Second derivative	$\frac{d^2y}{dx^2}$	$f''(x)$
Third derivative	$\frac{d^3y}{dx^3}$	$f'''(x)$

2. If y is an increasing function (y increases as x increases), the gradient is positive,

i.e. $\frac{dy}{dx} > 0$.

e.g.



Example 1

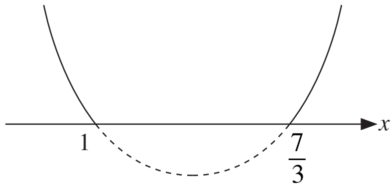
Find the set of values of x for which $f(x) = 2x^3 - 10x^2 + 14x + 5$ is an increasing function.

Solution

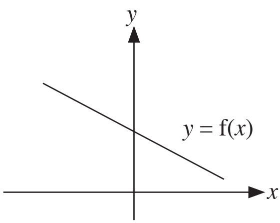
$f(x) = 2x^3 - 10x^2 + 14x + 5$
$f'(x) = 6x^2 - 20x + 14$

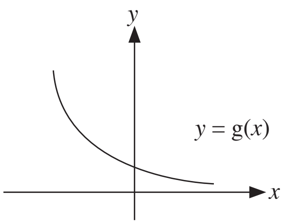
When $f'(x) > 0$,
$6x^2 - 20x + 14 > 0$
$3x^2 - 10x + 7 > 0$
$(3x - 7)(x - 1) > 0$

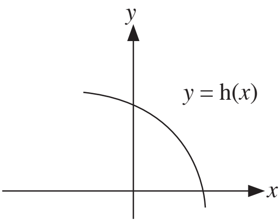
$x < 1$ or $x > \frac{7}{3}$

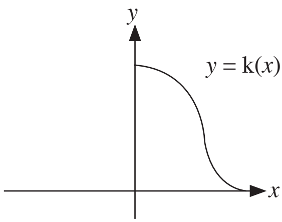


3. If y is a decreasing function (y decreases as x increases), the gradient is negative, i.e. $\frac{dy}{dx} < 0$.









Example 2

Find the set of values of x for which $y = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x$ is a decreasing function.

Solution

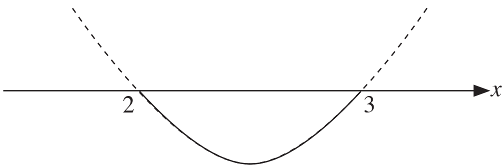
$$y = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x$$

$$\frac{dy}{dx} = x^2 - 5x + 6$$

For y to be a decreasing function, $\frac{dy}{dx} < 0$.

$$x^2 - 5x + 6 < 0$$

$$(x - 3)(x - 2) < 0$$



$\therefore\ 2 < x < 3$

Stationary Points

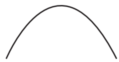
4. If a point (x_0, y_0) is a stationary point of the curve $y = \mathrm{f}(x)$, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = x_0$, i.e. the gradient of the tangent at $x = x_0$ is zero.

5. A stationary point can be a maximum point, a minimum point or a point of inflexion.

Determining the Nature of Stationary Points

6. **First Derivative Test:** Use $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Maximum point

	x^-	x_0	x^+
$\dfrac{\mathrm{d}y}{\mathrm{d}x}$	> 0	0	< 0
slope	/	–	\
stationary point			

Minimum point

	x^-	x_0	x^+
$\dfrac{\mathrm{d}y}{\mathrm{d}x}$	< 0	0	> 0
slope	\	–	/
stationary point			

Point of inflexion

	x^-	x_0	x^+
$\dfrac{\mathrm{d}y}{\mathrm{d}x}$	> 0	0	> 0
slope	/	–	/
stationary point			

Point of inflexion

	x^-	x_0	x^+
$\dfrac{\mathrm{d}y}{\mathrm{d}x}$	< 0	0	< 0
slope	\	–	\
stationary point			

7. **Second Derivative Test:** Use $\frac{d^2y}{dx^2}$.

- If $\frac{d^2y}{dx^2} < 0$, the stationary point is a maximum point.
- If $\frac{d^2y}{dx^2} > 0$, the stationary point is a minimum point.
- If $\frac{d^2y}{dx^2} = 0$, the stationary point can be a maximum point, a minimum point or a point of inflexion. Use the First Derivative Test to determine the nature.

Problems on Maxima and Minima

8. **Step 1:** Find a relationship between the quantity to be maximised or minimised and the variable(s) involved.
   Step 2: If there is more than one variable involved, use substitution to reduce it to one independent variable only.
   Step 3: Find the first derivative of the expression obtained above.
   Step 4: Equate the first derivative to zero to obtain the value(s) of the variable.
   Step 5: Check the nature of the stationary point.
   Step 6: Find the required maximum or minimum value of the quantity.

Example 3

A curve has the equation $y = 3(x + 1)^2$. Find the coordinates of the stationary point and deduce the nature of the stationary point.

Solution

$$\frac{dy}{dx} = 6(x + 1)$$

Let $\frac{dy}{dx} = 0$,

$6(x + 1) = 0$

$x = -1$

When $x = -1$, $y = 0$.

To find the nature of the stationary point, we perform the First Derivative Test.

x	-1.1	-1	-0.9
$\frac{dy}{dx}$	< 0	0	> 0
slope			
stationary point			

$(-1, 0)$ is a minimum point.

Example 4

It is given that $y = \dfrac{16}{x^4}$ and that $z = x^2 + 2y$. Given that x is positive, find the value of x and of y that makes z a stationary value and show that in this case, z has a minimum value.

Solution

$y = \dfrac{16}{x^4}$ – (1)

$z = x^2 + 2y$ – (2)

Substitute (1) into (2): (Express z in terms of one variable.)

$$z = x^2 + \frac{32}{x^4}$$
$$= x^2 + 32x^{-4}$$
$$\frac{dz}{dx} = 2x - 128x^{-5}$$
$$= 2x - \frac{128}{x^5}$$

When $\dfrac{dz}{dx} = 0$,

$$2x - \frac{128}{x^5} = 0$$
$$2x = \frac{128}{x^5}$$
$$x^6 = 64$$
$$x = \pm 2$$

Given that x is positive,

$x = 2$

$y = 1$ (Substitute $x = 2$ into (1) to obtain the value of y.)

$$\frac{d^2z}{dx^2} = 2 + 640x^{-6}$$
$$= 2 + \frac{640}{x^6}$$

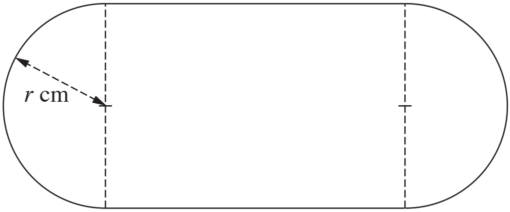
(Use the Second Derivative Test to show that z has a minimum value.)

When $x = 2$,

$$\frac{d^2z}{dx^2} = 12 > 0$$

$\therefore$ z has a minimum value.

Example 5



The diagram shows a rectangle of length x cm and 2 semicircles each of radius r cm. The perimeter of the figure is 400 cm and the area of the rectangle is A cm^2.

(a) Show that $A = 400r - 2\pi r^2$.

(b) Find an expression for $\dfrac{dA}{dr}$.

(c) Calculate

(i) the value of r for which A is a maximum,

(ii) the maximum value of A.

Solution

(a) Given that the perimeter is 400 cm,

$2x + 2\pi r = 400$

$x = 200 - \pi r$

$A = x(2r)$

$= 2r(200 - \pi r)$

$= 400r - 2\pi r^2$ (proven)

(As A is expressed in terms of r only, we make use of the perimeter to obtain an equation involving x and r, before substituting it into A.)

(b) $\dfrac{dA}{dr} = 400 - 4\pi r$

(c) **(i)** When $\frac{dA}{dr} = 0$,

$400 - 4\pi r = 0$

$r = \frac{100}{\pi}$

$\frac{d^2A}{dr^2} = -4\pi < 0$ (Use the Second Derivative Test to check that A is a maximum.)

$\therefore$ A is a maximum.

(ii) When $r = \frac{100}{\pi}$,

$$A = 400\left(\frac{100}{\pi}\right) - 2\pi\left(\frac{100}{\pi}\right)^2$$

$$= \frac{40\ 000}{\pi} - \frac{20\ 000}{\pi}$$

$$= \frac{20\ 000}{\pi}$$

$= 6370$ (to 3 s.f.)

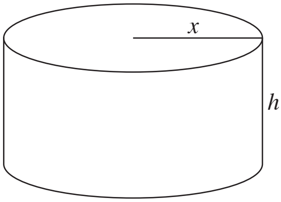
$\therefore$ The maximum value of A is 6370 (to 3 s.f.).

Example 6

A cylinder, which is made using a thin sheet of metal, has a volume of 500 cm^3, radius of x cm and height of h cm.

(a) Express h in terms of x and hence, express the total surface area, A cm^2, in terms of x.

(b) Find the value of x for which A will be a minimum.



Solution

(a)
$$V = \pi x^2 h$$
$$\pi x^2 h = 500$$
$$h = \frac{500}{\pi x^2}$$
$$A = 2\pi x^2 + 2\pi x\left(\frac{500}{\pi x^2}\right)$$
$$= 2\pi x^2 + \frac{1000}{x}$$

(b)
$$A = 2\pi x^2 + 1000x^{-1}$$
$$\frac{dA}{dx} = 4\pi x - 1000x^{-2}$$
$$= 4\pi x - \frac{1000}{x^2}$$

To find the minimum value of A, $\frac{dA}{dx} = 0$.

$$4\pi x - \frac{1000}{x^2} = 0$$
$$x^3 = \frac{250}{\pi}$$
$$x = \sqrt[3]{\frac{250}{\pi}}$$
$$= 4.30 \text{ (to 3 s.f.)}$$

UNIT 13 Differentiation of Trigonometric, Logarithmic & Exponential Functions and their Applications

(not included for NA)

Differentiation of Trigonometric Functions

1. Ensure that your calculator is in the radian mode.

2. $\dfrac{d}{dx}(\sin x) = \cos x$ $\qquad$ $\dfrac{d}{dx}(\cos x) = -\sin x$

$\dfrac{d}{dx}(\tan x) = \sec^2 x$ $\qquad$ $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

$\dfrac{d}{dx}(\cot x) = -\text{cosec}^2\, x$ $\qquad$ $\dfrac{d}{dx}(\text{cosec}\, x) = \text{cosec}\, x \cot x$

3. $\dfrac{d}{dx}\left[\sin(Ax + B)\right] = A\cos(Ax + B)$

$\dfrac{d}{dx}\left[\cos(Ax + B)\right] = -A\sin(Ax + B)$

$\dfrac{d}{dx}\left[\tan(Ax + B)\right] = A\sec^2(Ax + B)$

Example 1

Differentiate each of the following with respect to x.

(a) $3\sin(2x + 1)$ $\qquad$ **(b)** $(2x + 1)\cos 3x$

(c) $x^3 \tan(3x + 2)$

Solution

(a) $\dfrac{d}{dx}\left[3\sin(2x + 1)\right] = 3[2\cos(2x + 1)]$

$= 6\cos(2x + 1)$

(b) $\dfrac{d}{dx}(2x + 1)\cos 3x = (2x + 1)(3)(-\sin 3x) + \cos 3x\,(2)$ (Product Rule)

$= -3(2x + 1)\sin 3x + 2\cos 3x$

(c) $\dfrac{d}{dx}\left[x^3\tan(3x + 2)\right] = x^3(3)\sec^2(3x + 2) + \tan(3x + 2)(3x^2)$ (Product Rule)

$= 3x^2\left[x\sec^2(3x + 2) + \tan(3x + 2)\right]$

Example 2

Find the gradient of the curve $y = x \sin x$ at the point where $x = 1$.

Solution

$$y = x \sin x$$

$$\frac{dy}{dx} = x \cos x + \sin x \qquad \text{(Gradient of curve refers to } \frac{dy}{dx}.)$$

When $x = 1$,

$$\frac{dy}{dx} = \cos 1 + \sin 1 \qquad \text{(Radian mode)}$$

$$= 1.38 \text{ (to 3 s.f.)}$$

$\therefore$ Gradient of curve at $x = 1$ is 1.38

---

4. $$\frac{d}{dx}\left[\sin^n x\right] = n \sin^{n-1} x \cos x$$

$$\frac{d}{dx}\left[\cos^n x\right] = -n \cos^{n-1} x \sin x$$

$$\frac{d}{dx}\left[\tan^n x\right] = n \tan^{n-1} x \sec^2 x$$

5. $$\frac{d}{dx}\left[\sin^n (Ax + B)\right] = An \sin^{n-1} (Ax + B) \cos (Ax + B)$$

$$\frac{d}{dx}\left[\cos^n (Ax + B)\right] = -An \cos^{n-1} (Ax + B) \sin (Ax + B)$$

$$\frac{d}{dx}\left[\tan^n (Ax + B)\right] = An \tan^{n-1} (Ax + B) \sec^2 (Ax + B)$$

In general,

6. $$\frac{d}{dx}\left[\sin^n f(x)\right] = n \sin^{n-1} f(x) \times \frac{d}{dx}\left[\sin f(x)\right]$$

$$\frac{d}{dx}\left[\cos^n f(x)\right] = n \cos^{n-1} f(x) \times \frac{d}{dx}\left[\cos f(x)\right]$$

$$\frac{d}{dx}\left[\tan^n f(x)\right] = n \tan^{n-1} f(x) \times \frac{d}{dx}\left[\tan f(x)\right]$$

Example 3

Differentiate each of the following with respect to x.

(a) $\cos^2 (1 - 3x)$

(b) $3 \tan^3 (2x - \pi)$

(c) $\sin^2 (3x + 2) \cos x^2$

Solution

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos^2 (1 - 3x)\right] = 2 \cos (1 - 3x)\left[-(-3) \sin (1 - 3x)\right]$

$= 6 \cos (1 - 3x) \sin (1 - 3x)$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left[3 \tan^3 (2x - \pi)\right] = 3\left[(3) \tan^2 (2x - \pi)\right]\left[(2) \sec^2 (2x - \pi)\right]$ (Chain Rule)

$= 18 \tan^2 (2x - \pi) \sec^2 (2x - \pi)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}[\sin^2 (3x + 2) \cos x^2]$

$= (2) \sin (3x + 2) (3) \cos (3x + 2) (\cos x^2) + \sin^2 (3x + 2) (2x) (-\sin x^2)$

$= 6 \sin (3x + 2) \cos (3x + 2) \cos x^2 - 2x \sin^2 (3x + 2) \sin x^2$ (Product Rule and Chain Rule)

Differentiation of Logarithmic Functions

7. $\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}$

8. $\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln (ax + b)\right] = \frac{a}{ax + b}$

9. In general, $\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln \mathrm{f}(x)\right] = \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$, where $\mathrm{f}'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}(x)\right]$.

10. As far as possible, make use of the laws of logarithms to simplify logarithmic expressions before finding the derivatives.

Example 4

Differentiate each of the following with respect to x.

(a) $\ln (3x + 1)$

(b) $\ln (2x^2 + 5)^3$

(c) $\ln\left(\dfrac{2x}{3x^2 + 4}\right)$

(d) $\ln\left(\dfrac{8 + 4x}{3x - 5}\right)$

(e) $\ln [x (5x^3 - 2)]$

(f) $x^3 \ln (4x - 1)$

Solution

(a) $\dfrac{d}{dx}\left[\ln (3x + 1)\right] = \dfrac{3}{3x + 1}$

(b)
$$\frac{d}{dx}\left[\ln (2x^2 + 5)^3\right] = \frac{d}{dx}\left[3 \ln (2x^2 + 5)\right]$$
$$= 3\left(\frac{4x}{2x^2 + 5}\right) \quad \text{(Power Law of Logarithms)}$$
$$= \frac{12x}{2x^2 + 5}$$

(c)
$$\frac{d}{dx}\left[\ln\left(\frac{2x}{3x^2 + 4}\right)\right] = \frac{d}{dx}\left[\ln 2x - \ln (3x^2 + 4)\right]$$
$$= \frac{2}{2x} - \frac{6x}{3x^2 + 4}$$
$$= \frac{1}{x} - \frac{6x}{3x^2 + 4}$$

(d)
$$\frac{d}{dx}\left[\ln\left(\frac{8 + 4x}{3x - 5}\right)\right] = \frac{d}{dx}\left[\ln (8 + 4x) - \ln (3x - 5)\right]$$
$$= \frac{4}{8 + 4x} - \frac{3}{3x - 5}$$
$$= \frac{1}{2 + x} - \frac{3}{3x - 5}$$

(e) $\frac{d}{dx}\ln\left[x(5x^3-2)\right] = \frac{d}{dx}\left[\ln x + \ln(5x^3-2)\right]$

$$= \frac{1}{x} + \frac{15x^2}{5x^3-2}$$

(f) $\frac{d}{dx}\left[x^3\ln(4x-1)\right] = x^3\left(\frac{4}{4x-1}\right) + 3x^2\ln(4x-1)$

$$= \frac{4x^3}{4x-1} + 3x^2\ln(4x-1)$$

Example 5

Two variables, x and y, are related by the equation $y = \frac{\ln x}{3x+7}$. Find the rate of change of x at the instant when $x = 1$, given that y is changing at a rate of 0.18 units/s at this instant.

Solution

$$y = \frac{\ln x}{3x+7}$$

$$\frac{dy}{dx} = \frac{(3x+7)\left(\frac{1}{x}\right) - 3\ln x}{(3x+7)^2}$$

$$= \frac{3x+7-3x\ln x}{x(3x+7)^2}$$

Using $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$,

$$0.18 = \frac{3(1)+7-3(1)\ln 1}{1(3+7)^2} \times \frac{dx}{dt}$$

$$\frac{dx}{dt} = 1.8 \text{ units/s}$$

Example 6

x and y are related by the equation $y = \dfrac{\ln 2x}{3x^2}$. Find the rate of change of y at the instant when $y = 0$, given that x is changing at a rate of 2 units/s at this instant.

Solution

$$y = \frac{\ln 2x}{3x^2}$$

$$= \frac{1}{3}x^{-2} \ln 2x$$

$$\frac{dy}{dx} = \frac{1}{3}(-2)x^{-3} \ln 2x + \frac{1}{3}x^{-2}\left(\frac{2}{2x}\right) \quad \text{(Product Rule)}$$

$$= -\frac{2 \ln 2x}{3x^3} + \frac{1}{3x^3}$$

$$= \frac{1 - 2 \ln 2x}{3x^3}$$

When $y = 0$,

$\ln 2x = 0$

$2x = e^0$

$x = \dfrac{1}{2}$

Using $\dfrac{dy}{dt} = \dfrac{dy}{dx} \times \dfrac{dx}{dt}$,

$$\frac{dy}{dt} = \left[\frac{1 - 2 \ln 2\left(\frac{1}{2}\right)}{3\left(\frac{1}{2}\right)^3}\right] \times 2$$

$$= 5\frac{1}{3} \text{ units/s}$$

Differentiation of Exponential Functions

11. $\dfrac{d}{dx}(e^x) = e^x$

12. $\dfrac{d}{dx}(e^{ax+b}) = ae^{ax+b}$

13. In general, $\dfrac{d}{dx}(e^{f(x)}) = f'(x)e^{f(x)}$, where $f'(x) = \dfrac{d}{dx}[f(x)]$.

Example 7

Differentiate each of the following with respect to x.

(a) e^{2-3x}

(b) x^2e^{4x}

(c) $\dfrac{e^{3x}}{x^2+1}$

(d) $\dfrac{e^{\sin x}+1}{e^{\cos x}}$

Solution

(a) $\dfrac{d}{dx}(e^{2-3x}) = -3e^{2-3x}$

(b) $\dfrac{d}{dx}(x^2e^{4x}) = x^2(4e^{4x}) + 2xe^{4x}$ (Product Rule)

$= 2xe^{4x}(2x+1)$

(c) $\dfrac{d}{dx}\left[\dfrac{e^{3x}}{x^2+1}\right] = \dfrac{(x^2+1)(3)e^{3x} - e^{3x}(2x)}{(x^2+1)^2}$ (Quotient Rule)

$= \dfrac{3(x^2+1)e^{3x} - 2xe^{3x}}{(x^2+1)^2}$

$= \dfrac{e^{3x}\left[3(x^2+1) - 2x\right]}{(x^2+1)^2}$

(d) $\dfrac{d}{dx}\left(\dfrac{e^{\sin x}+1}{e^{\cos x}}\right) = \dfrac{e^{\cos x}(\cos x e^{\sin x}) - (e^{\sin x}+1)e^{\cos x}(-\sin x)}{(e^{\cos x})^2}$ (Quotient Rule)

$= \dfrac{e^{\cos x}e^{\sin x}\cos x + (e^{\sin x}+1)e^{\cos x}\sin x}{e^{2\cos x}}$

$= \dfrac{e^{\cos x}\left[e^{\sin x}\cos x + (e^{\sin x}+1)\sin x\right]}{e^{2\cos x}}$

$= \dfrac{e^{\sin x}\cos x + (e^{\sin x}+1)\sin x}{e^{\cos x}}$

Example 8

The equation of a curve is $y = e^x \cos x$, where $0 < x < \pi$.
Find the x-coordinate of the stationary point of the curve.

Solution

$$y = e^x \cos x$$

$$\frac{dy}{dx} = e^x(-\sin x) + \cos x\,(e^x)$$

$$= e^x(\cos x - \sin x)$$

When $\frac{dy}{dx} = 0$,

$e^x(\cos x - \sin x) = 0$

$e^x = 0$ (no solution) or $\cos x - \sin x = 0$

$$\cos x = \sin x$$

$$\tan x = 1$$

$$x = \frac{\pi}{4}$$

Example 9

Given that the equation of a curve is $y = e^{\frac{1}{2}x} + \dfrac{4}{e^{\frac{1}{2}x}}$,

(i) find the coordinates of the stationary point on the curve,

(ii) determine the nature of the stationary point.

Solution

(i) $$y = e^{\frac{1}{2}x} + \frac{4}{e^{\frac{1}{2}x}}$$

$$= e^{\frac{1}{2}x} + 4e^{-\frac{1}{2}x}$$

$$\frac{dy}{dx} = \frac{1}{2}e^{\frac{1}{2}x} + 4\left(-\frac{1}{2}\right)e^{-\frac{1}{2}x}$$

$$= \frac{1}{2}e^{\frac{1}{2}x} - 2e^{-\frac{1}{2}x}$$

When $\frac{dy}{dx} = 0,$

$$\frac{1}{2}e^{\frac{1}{2}x} - 2e^{-\frac{1}{2}x} = 0$$

$$\frac{1}{2}e^{\frac{1}{2}x} = 2e^{-\frac{1}{2}x}$$

$$\frac{e^{\frac{1}{2}x}}{e^{-\frac{1}{2}x}} = 4$$

$$e^{x} = 4$$

$$x = \ln 4$$

$$y = e^{\frac{1}{2}\ln 4} + \frac{4}{e^{\frac{1}{2}\ln 4}}$$

$$= 2 + \frac{4}{2}$$

$$= 4$$

$\therefore$ Coordinates of stationary point are $(\ln 4, 4)$

(ii) $\frac{d^2y}{dx^2} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)e^{\frac{1}{2}x} - (2)\left(-\frac{1}{2}\right)e^{-\frac{1}{2}x}$

$$= \frac{1}{4}e^{\frac{1}{2}x} + e^{-\frac{1}{2}x}$$

When $x = \ln 4,$

$$\frac{d^2y}{dx^2} = 1 > 0$$

$\therefore$ The stationary point is a minimum.

UNIT 14 Integration

Integration

1. If $y = \mathrm{f}(x)$, then $\int y \, \mathrm{d}x = \int \mathrm{f}(x) \, \mathrm{d}x$.

2. If $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{g}(x)$, then $\int \mathrm{g}(x) \, \mathrm{d}x = y + c$, where c is an arbitrary constant.

Formulae and Rules

3. $\int k \, \mathrm{d}x = kx + c$, where k is a constant

4. $\int ax^n \, \mathrm{d}x = \dfrac{ax^{n+1}}{n+1} + c$, where $n \neq -1$

5. $\int (ax+b)^n \, \mathrm{d}x = \dfrac{(ax+b)^{n+1}}{a(n+1)} + c$, where $n \neq -1$

6. $\int [\mathrm{f}(x) \pm \mathrm{g}(x)] \, \mathrm{d}x = \int \mathrm{f}(x) \, \mathrm{d}x \pm \int \mathrm{g}(x) \, \mathrm{d}x$

Example 1

Find

(a) $\int 5 \, \mathrm{d}x$,

(b) $\int 3x^5 \, \mathrm{d}x$,

(c) $\int (2x^3 - 3x + 6) \, \mathrm{d}x$,

(d) $\int x\left(3x^2 + \dfrac{7}{x}\right) \mathrm{d}x$,

(e) $\int 3(2x-5)^6 \, \mathrm{d}x$.

Solution

(a) $\int 5 \, dx = 5x + c$

(b) $\int 3x^5 \, dx = \dfrac{3x^{5+1}}{5+1} + c$

$= \dfrac{1}{2}x^6 + c$

(c) $\int (2x^3 - 3x + 6) \, dx = \dfrac{2x^{3+1}}{3+1} - \dfrac{3x^{1+1}}{1+1} + 6x + c$

$= \dfrac{1}{2}x^4 - \dfrac{3}{2}x^2 + 6x + c$

(d) $\int x\left(3x^2 + \dfrac{7}{x}\right) dx = \int (3x^3 + 7) \, dx$ (Multiply x into the terms in the bracket before doing the integration.)

$= \dfrac{3x^{3+1}}{3+1} + 7x + c$

$= \dfrac{3}{4}x^4 + 7x + c$

(e) $\int 3(2x - 5)^6 \, dx = \dfrac{3(2x-5)^{6+1}}{(6+1)(2)} + c$ (It is not necessary to find the expansion of $(2x - 5)^6$.)

$= \dfrac{3}{14}(2x - 5)^7 + c$

Example 2

Find the equation of the curve which passes through the point (2, 10) and for which $\dfrac{dy}{dx} = 3x^2 - \dfrac{4}{x^2}$.

Solution

$\dfrac{dy}{dx} = 3x^2 - \dfrac{4}{x^2}$

$= 3x^2 - 4x^{-2}$

$y = \int (3x^2 - 4x^{-2}) \, dx$

$= x^3 + 4x^{-1} + c$

$= x^3 + \dfrac{4}{x} + c$

When $x = 2, y = 10,$

$10 = 2^3 + \frac{4}{2} + c$

$c = 0$

$\therefore$ Equation of the curve is $y = x^3 + \frac{4}{x}$

Integration of Trigonometric Functions

7. $\int \sin x \, dx = -\cos x + c$

8. $\int \cos x \, dx = \sin x + c$

9. $\int \sec^2 x \, dx = \tan x + c$

10. $\int \sin (Ax + B) \, dx = -\frac{1}{A} \cos (Ax + B) + c$

11. $\int \cos (Ax + B) \, dx = \frac{1}{A} \sin (Ax + B) + c$

12. $\int \sec^2 (Ax + B) \, dx = \frac{1}{A} \tan (Ax + B) + c$

Example 3

Find

(a) $\int \cos(5x+3)\,dx$, **(b)** $\int 3\sin(3x-1)\,dx$,

(c) $\int 2\sec^2(8-3x)\,dx$.

Solution

(a) $\int \cos(5x+3)\,dx = \frac{1}{5}\sin(5x+3)+c$

(b) $\int 3\sin(3x-1)\,dx = 3\left[\frac{-\cos(3x-1)}{3}\right]+c$

$= -\cos(3x-1)+c$

(c) $\int 2\sec^2(8-3x)\,dx = 2\left[\frac{\tan(8-3x)}{-3}\right]+c$

$= -\frac{2}{3}\tan(8-3x)+c$

(Note that $\int 2\sec^2(8-3x)\,dx \neq -\frac{2}{9}\sec^3(8-3x)+c$)

13. Methods of Integrating Trigonometric Functions:

- Use trigonometric identities e.g. $1+\tan^2 x = \sec^2 x$
- Use double angle formulae e.g. $\cos 2x = 2\cos^2 x - 1$ or $\cos 2x = 1 - 2\sin^2 x$

Example 4

Find

(a) $\int 4\tan^2 3x\,dx$, **(b)** $\int \sin x\cos x\,dx$,

(c) $\int 6\cos^2\frac{x}{2}\,dx$.

Solution

(a) $\int 4\tan^2 3x\,dx = 4\int(\sec^2 3x - 1)\,dx$

$$= 4\left[\frac{1}{3}\tan 3x - x\right] + c$$

$$= \frac{4}{3}\tan 3x - 4x + c$$

(b) $\int \sin x\cos x\,dx = \frac{1}{2}\int 2\sin x\cos x\,dx$

$$= \frac{1}{2}\int \sin 2x\,dx$$

$$= \frac{1}{2}\left[\frac{-\cos 2x}{2}\right] + c$$

$$= -\frac{1}{4}\cos 2x + c$$

(c) $\int 6\cos^2\frac{x}{2}\,dx = 3\int 2\cos^2\frac{x}{2}\,dx$

$$= 3\int(\cos x + 1)\,dx \quad \left(\cos A = 2\cos^2\frac{A}{2} - 1\right)$$

$$= 3[\sin x + x] + c$$

$$= 3\sin x + 3x + c$$

Example 5

Prove that $(2 \cos \theta - \sin \theta)^2 = \frac{3}{2} \cos 2\theta - 2 \sin 2\theta + \frac{5}{2}$.

Hence, find $\int (2 \cos x - \sin x)^2 \, dx$.

Solution

$$\begin{aligned}
\text{LHS} &= (2 \cos \theta - \sin \theta)^2 \\
&= 4 \cos^2 \theta + \sin^2 \theta - 4 \sin \theta \cos \theta \\
&= 4\left(\frac{1 + \cos 2\theta}{2}\right) + \left(\frac{1 - \cos 2\theta}{2}\right) - 2(2 \sin\theta \cos\theta) \\
&= 2 + 2 \cos 2\theta + \frac{1}{2} - \frac{1}{2} \cos 2\theta - 2 \sin 2\theta \\
&= \frac{3}{2} \cos 2\theta - 2 \sin 2\theta + \frac{5}{2} \\
&= \text{RHS (shown)}
\end{aligned}$$

$$\begin{aligned}
\int (2 \cos x - \sin x)^2 \, dx &= \int \left(\frac{3}{2} \cos 2x - 2 \sin 2x + \frac{5}{2}\right) dx \\
&= \frac{\frac{3}{2} \sin 2x}{2} - \frac{(-2 \cos 2x)}{2} + \frac{5}{2}x + c \\
&= \frac{3}{4} \sin 2x + \cos 2x + \frac{5}{2}x + c
\end{aligned}$$

Integration of $\frac{1}{ax + b}$

14. $\int \frac{1}{x} \, dx = \ln|x| + c$

15. $\int \frac{1}{ax + b} \, dx = \frac{1}{a} \ln|ax + b| + c$

16. In general, $\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + c$

Example 6

Find

(a) $\displaystyle\int \frac{5}{3x+5}\,\mathrm{d}x$, (b) $\displaystyle\int \frac{4}{2-3x}\,\mathrm{d}x$,

(c) $\displaystyle\int \frac{4x^2+3x^4}{2x^3}\,\mathrm{d}x$.

Solution

(a) $$\int \frac{5}{3x+5}\,\mathrm{d}x = 5\int \frac{1}{3x+5}\,\mathrm{d}x$$

$$= \frac{5}{3}\int \frac{3}{3x+5}\,\mathrm{d}x$$

$$= \frac{5}{3}\ln(3x+5) + c$$

(Manipulate the expression to obtain one in the form $\dfrac{\mathrm{f}'(x)}{\mathrm{f}(x)}$.)

(b) $$\int \frac{4}{2-3x}\,\mathrm{d}x = 4\int \frac{1}{2-3x}\,\mathrm{d}x$$

$$= \frac{4}{-3}\int \frac{-3}{2-3x}\,\mathrm{d}x$$

$$= -\frac{4}{3}\ln(2-3x) + c$$

(c) $$\int \frac{4x^2+3x^4}{2x^3}\,\mathrm{d}x = \int\left(\frac{4x^2}{2x^3} + \frac{3x^4}{2x^3}\right)\mathrm{d}x$$

$$= \int\left(\frac{2}{x} + \frac{3}{2}x\right)\mathrm{d}x$$

$$= 2\ln x + \frac{3}{4}x^2 + c$$

Example 7

Express $\dfrac{2x+4}{(x+1)(x-2)}$ in partial fractions. Hence, find $\displaystyle\int \frac{2x+4}{(x+1)(x-2)}\,dx$.

Solution

Let $\dfrac{2x+4}{(x+1)(x-2)} = \dfrac{A}{x+1} + \dfrac{B}{x-2}$.

By Cover-Up Rule,

$A = -\dfrac{2}{3}$ and $B = \dfrac{8}{3}$

$$\frac{2x+4}{(x+1)(x-2)} = -\frac{2}{3(x+1)} + \frac{8}{3(x-2)}$$

$$\int \frac{2x+4}{(x+1)(x-2)}\,dx = \int \left[-\frac{2}{3(x+1)} + \frac{8}{3(x-2)}\right] dx$$

$$= -\frac{2}{3}\ln(x+1) + \frac{8}{3}\ln(x-2) + c$$

Example 8

Find $\int \frac{x+15}{(x-2)(x+3)}\,dx$.

Solution

Let $\frac{x+15}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$.

By Cover-up Rule,

$A = \frac{17}{5}$ and $B = -\frac{12}{5}$

$$\frac{x+15}{(x-2)(x+3)} = \frac{17}{5(x-2)} - \frac{12}{5(x+3)}$$

$$\int \frac{x+15}{(x-2)(x+3)}\,dx = \int \left[\frac{17}{5(x-2)} - \frac{12}{5(x+3)}\right] dx$$

$$= \frac{17}{5}\ln(x-2) - \frac{12}{5}\ln(x+3) + c$$

Integration of e^x

17. $\int e^x\,dx = e^x + c$

18. $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$

Example 9

Find

(a) $\int e^{3x}\,dx$,

(b) $\int e^{2x+3}\,dx$,

(c) $\int 6e^{\frac{x}{3}}\,dx$,

(d) $\int \frac{e^{3x-1}-4}{2e^x}\,dx$.

Solution

(a) $\int e^{3x}\,dx = \frac{1}{3}e^{3x} + c$

(b) $\int e^{2x+3}\,dx = \frac{1}{2}e^{2x+3} + c$

(c) $$\int 6e^{\frac{x}{3}}\,dx = \frac{6e^{\frac{x}{3}}}{\frac{1}{3}} + c$$

$$= 18e^{\frac{x}{3}} + c$$

(d) $$\int \frac{e^{3x-1}-4}{2e^x}\,dx = \int\left(\frac{e^{3x-1}}{2e^x} - \frac{4}{2e^x}\right)dx$$

$$= \int\left(\frac{1}{2}e^{2x-1} - 2e^{-x}\right)dx$$

$$= \frac{\frac{1}{2}e^{2x-1}}{2} - \frac{2e^{-x}}{-1} + c$$

$$= \frac{1}{4}e^{2x-1} + \frac{2}{e^x} + c$$

UNIT 15 Applications of Integration

Definite Integrals

1. $$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

2. $$\int_a^a f(x)\,dx = 0$$

3. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

4. $$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

5. $$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

6. $$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

Example 1

Evaluate

(a) $\displaystyle\int_1^3 \left(x^2 - \frac{10}{x^2} + 3\right)dx$ **(b)** $\displaystyle\int_1^4 \sqrt{5x-4}\,dx$

Solution

(a) $$\int_1^3 \left(x^2 - \frac{10}{x^2} + 3\right)dx = \int_1^3 \left(x^2 - 10x^{-2} + 3\right)dx$$

$$= \left[\frac{1}{3}x^3 + 10x^{-1} + 3x\right]_1^3$$

$$= \left[\frac{1}{3}x^3 + \frac{10}{x} + 3x\right]_1^3$$

$$= \left[9 + \frac{10}{3} + 9\right] - \left[\frac{1}{3} + 10 + 3\right]$$

$$= 8$$

(b) $$\int_1^4 \sqrt{5x-4}\,dx = \int_1^4 (5x-4)^{\frac{1}{2}}\,dx$$

$$= \left[\frac{(5x-4)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)(5)}\right]_1^4$$

$$= \frac{2}{15}\left[(5x-4)^{\frac{3}{2}}\right]_1^4$$

$$= \frac{2}{15}\left[16^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$$

$$= \frac{42}{5}$$

Example 2

Given that $\int_1^5 f(x)\,dx = 10$, find the value of each of the following.

(i) $\int_5^1 f(x)\,dx$

(ii) $\int_1^5 2f(x)\,dx$

(iii) $\int_1^4 \left[f(x) + 3\sqrt{x}\right] dx + \int_4^5 f(x)\,dx$

Solution

(i) $\int_5^1 f(x)\,dx = -10$

(ii)
$$\begin{aligned}\int_1^5 2f(x)\,dx &= 2\int_1^5 f(x)\,dx\\ &= 2(10)\\ &= 20\end{aligned}$$

(iii)
$$\begin{aligned}\int_1^4 \left[f(x) + 3\sqrt{x}\right] dx + \int_4^5 f(x)\,dx &= \int_1^4 f(x)\,dx + 3\int_1^4 x^{\frac{1}{2}}\,dx + \int_4^5 f(x)\,dx\\ &= \int_1^5 f(x)dx + 3\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4\\ &= 10 + 2\left[x^{\frac{3}{2}}\right]_1^4\\ &= 10 + 2\left[4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]\\ &= 24\end{aligned}$$

Example 3

Find $\dfrac{d}{dx}\left(\dfrac{1}{9-2x^2}\right)$ and hence find the value of $\displaystyle\int_1^2 \frac{12x}{(9-2x^2)^2}\,dx$.

Solution

$$\frac{d}{dx}\left(\frac{1}{9-2x^2}\right) = \frac{(9-2x^2)(0)-1(-4x)}{(9-2x^2)^2}$$

$$= \frac{4x}{(9-2x^2)^2}$$

$$\int_1^2 \frac{12x}{(9-2x^2)^2}\,dx = 3\int_1^2 \frac{4x}{(9-2x^2)^2}\,dx$$ (Make use of the answer in the first part of the question.)

$$= 3\left[\frac{1}{9-2x^2}\right]_1^2$$

$$= 3\left[\frac{1}{1}-\frac{1}{7}\right]$$

$$= \frac{18}{7}$$

Example 4

Evaluate each of the following.

(a) $\displaystyle\int_0^{\frac{\pi}{3}} 3\sin 3x\,dx$ **(b)** $\displaystyle\int_0^{\frac{\pi}{4}} (\sec^2 x + 2\cos x)\,dx$

Solution

(a) $$\int_0^{\frac{\pi}{3}} 3\sin 3x\,dx = 3\left[\frac{-\cos 3x}{3}\right]_0^{\frac{\pi}{3}}$$

$$= -\left[\cos 3x\right]_0^{\frac{\pi}{3}}$$

$$= -\left[\cos\pi - \cos 0\right]$$

$$= -\left[-1-1\right]$$

$$= 2$$

(b) $\int_0^{\frac{\pi}{4}} (\sec^2 x + 2\cos x)\,dx = [\tan x + 2\sin x]_0^{\frac{\pi}{4}}$

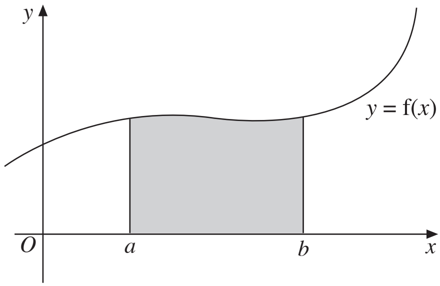
$$= \left[\tan\frac{\pi}{4} + 2\sin\frac{\pi}{4}\right] - [\tan 0 + 2\sin 0]$$

$$= 1 + \sqrt{2}$$

Area bounded by the *x*-axis

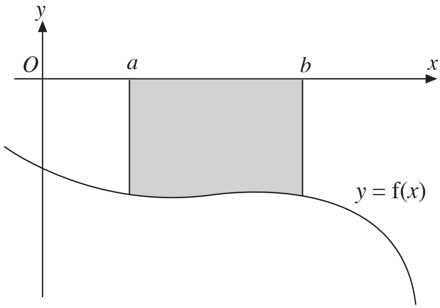
7. For a region above the *x*-axis:

Area bounded by the curve $y = f(x)$, the lines $x = a$ and $x = b$ and the x-axis is $\int_a^b f(x)\,dx$.



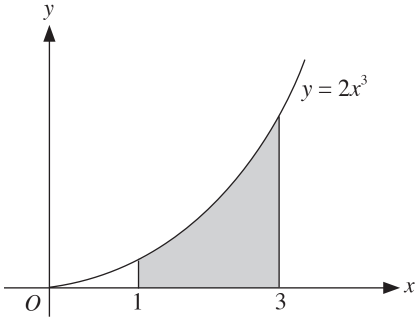
8. For a region below the *x*-axis:

Area bounded by the curve $y = f(x)$, the lines $x = a$ and $x = b$ and the x-axis is $\left|\int_a^b f(x)\,dx\right|$.



Example 5

Find the area of the shaded region bounded by the curve $y = 2x^3$, the x-axis and the lines $x = 1$ and $x = 3$.

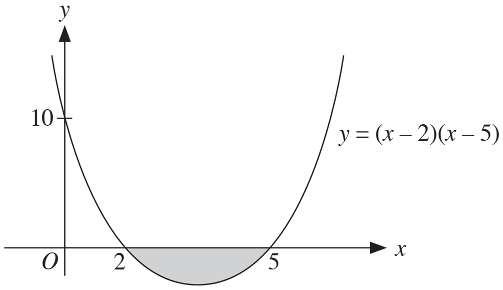


Solution

$$\text{Area of shaded region} = \int_1^3 y\,\mathrm{d}x$$
$$= \int_1^3 2x^3\,\mathrm{d}x$$
$$= \left[\frac{1}{2}x^4\right]_1^3$$
$$= \frac{1}{2}[3^4 - 1^4]$$
$$= 40 \text{ units}^2$$

Example 6

The figure shows part of the curve $y = (x - 2)(x - 5)$. Find the area of the shaded region.



Solution

$$\int_2^5 (x-2)(x-5)\,dx = \left|\int_2^5 (x^2 - 7x + 10)\,dx\right|$$

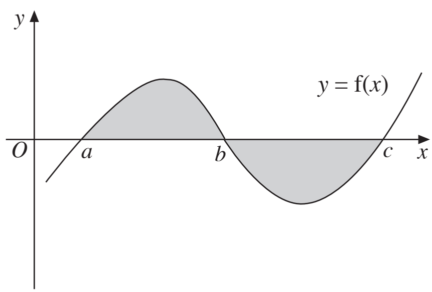
$$= \left|\left[\frac{x^3}{3} - \frac{7x^2}{2} + 10x\right]_2^5\right|$$

$$= \left|\left[\frac{5^3}{3} - \frac{7(5)^2}{2} + 10(5)\right] - \left[\frac{2^3}{3} - \frac{7(2)^2}{2} + 10(2)\right]\right|$$

$$= 4.5 \text{ units}^2$$

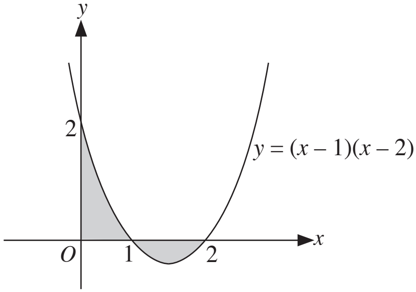
9. For an area enclosed above and below the x-axis:
Area bounded by the curve $y = f(x)$ and the x-axis as shown below is

$$\int_a^b f(x)\,dx + \left|\int_b^c f(x)\,dx\right|$$



Example 7

The diagram shows part of the curve $y = (x-1)(x-2)$.
Find the area bounded by the curve and the x-axis.



Solution

Area of shaded region $= \int_0^1 (x-1)(x-2)\,dx + \left|\int_1^2 (x-1)(x-2)\,dx\right|$

$$= \int_0^1 (x^2 - 3x + 2)\,dx + \left|\int_1^2 (x^2 - 3x + 2)\,dx\right|$$

$$= \left[\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x\right]_0^1 + \left|\left[\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x\right]_1^2\right|$$

$$= \left[\frac{5}{6} - 0\right] + \left|\frac{2}{3} - \frac{5}{6}\right|$$

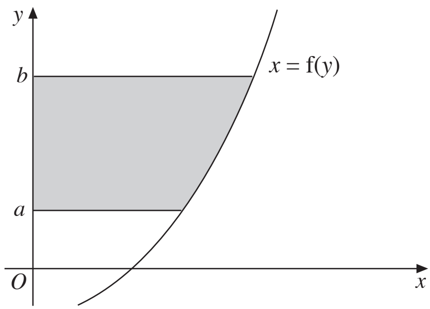
$$= 1 \text{ unit}^2$$

Area bounded by the y-axis

10. For a region on the right side of the y-axis:

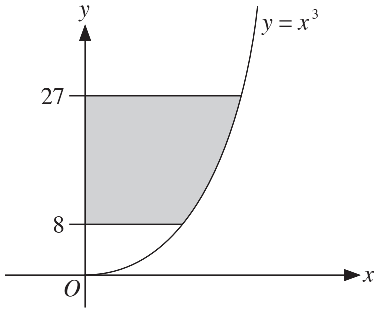
Area bounded by the curve $x = f(y)$, the lines $y = a$ and $y = b$ and the y-axis is

$\int_a^b f(y)\,dy$.



Example 8

The figure shows part of the curve $y = x^3$. Find the area of the shaded region.



Solution

$$y = x^3$$

$$x = \sqrt[3]{y}$$

$$= y^{\frac{1}{3}}$$

Area of shaded region $= \displaystyle\int_8^{27} x\,dy$

$$= \int_8^{27} y^{\frac{1}{3}}\,dy$$

$$= \left[\frac{y^{\frac{4}{3}}}{\frac{4}{3}}\right]_8^{27}$$

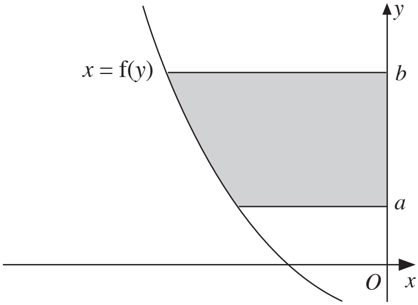
$$= \left[\frac{3}{4}\left(\sqrt[3]{y}\right)^4\right]_8^{27}$$

$$= \frac{3}{4}[81 - 16]$$

$$= 48.75 \text{ units}^2$$

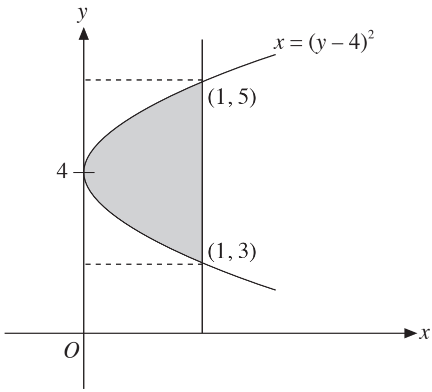
11. For a region on the left side of the y-axis:

Area bounded by the curve $x = \mathrm{f}(y)$, the lines $y = a$ and $y = b$ and the y-axis is $\left| \int_a^b \mathrm{f}(y)\, \mathrm{d}y \right|$.



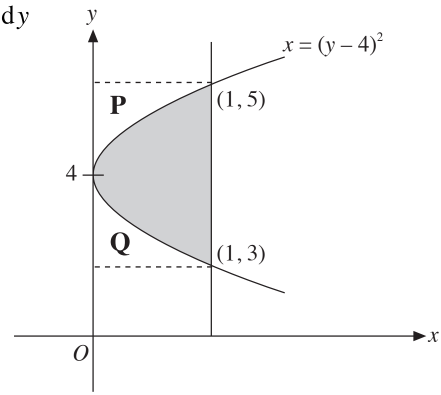
Example 9

Calculate the area of the shaded region shown in the figure.



Solution

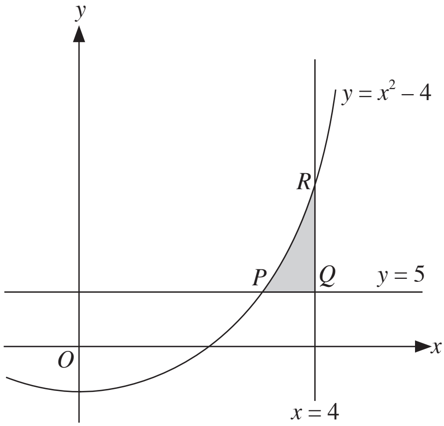
$$\text{Area of } (\mathbf{P} + \mathbf{Q}) = \int_3^5 (y-4)^2 \, dy$$
$$= \left[\frac{(y-4)^3}{3}\right]_3^5$$
$$= \frac{1}{3} - \left(-\frac{1}{3}\right)$$
$$= \frac{2}{3} \text{ units}^2$$



$$\text{Area of shaded region} = \text{Area of rectangle} - \text{Area of } (\mathbf{P} + \mathbf{Q})$$
$$= (1)(2) - \frac{2}{3}$$
$$= \frac{4}{3} \text{ units}^2$$

Example 10

The diagram shows the curve $y = x^2 - 4$. It cuts the line $y = 5$ at $P(3, 5)$. The line $x = 4$ intersects the curve at $R(4, 16)$. Find the area of the shaded region PQR.



Solution

$$\text{Area of } PQR = \int_3^4 \left[(x^2 - 4) - 5\right] \mathrm{d}x$$

$$= \int_3^4 (x^2 - 9)\, \mathrm{d}x$$

$$= \left[\frac{1}{3}x^3 - 9x\right]_3^4$$

$$= \left[-\frac{44}{3} - (-18)\right]$$

$$= 3\frac{1}{3} \text{ units}^2$$

UNIT 16 Kinematics

(not included for NA)

Relationship between Displacement, Velocity and Acceleration

1. Differentiation:

$$\begin{tikzcd}[column sep=huge]
\text{displacement, } s \arrow[r, bend left, "\frac{ds}{dt}"] & \text{velocity, } v \arrow[l, bend left] \arrow[r, bend left, "\frac{dv}{dt} = \frac{d^2s}{dt^2}"] & \text{acceleration, } a \arrow[l, bend left]
\end{tikzcd}$$

Integration: $s = \int v \, dt$ $\quad$ $v = \int a \, dt$

Common Terms used in Kinematics

2. Displacement, s, is defined as the distance moved by a particle in a specific direction.

3. Velocity, v, is defined as the rate of change of displacement with respect to time. v can take on positive or negative values.

4. Acceleration, a, is defined as the rate of change of velocity with respect to time. a can take on positive or negative values.
When $a > 0$, acceleration occurs.
When $a < 0$, deceleration occurs.

5.

Initial	$t = 0$
At rest	$v = 0$
Stationary	$v = 0$
Particle is at the fixed point	$s = 0$
Maximum/minimum displacement	$v = 0$
Maximum/minimum velocity	$a = 0$

6. Average speed = $\dfrac{\text{Total distance travelled}}{\text{Total time taken}}$

7. To find the distance travelled in the first n seconds:
Step 1: Let $v = 0$ to find t.
Step 2: Find s for each of the values of t found in step 1.
Step 3: Find s for $t = 0$ and $t = n$.
Step 4: Draw the path of the particle on a displacement-time graph.

Example 1

A particle moves in a straight line in such a way that, t seconds after passing through a fixed point O, its displacement from O is s m. Given that $s = 2 - \dfrac{4}{t+2}$, find

(i) expressions, in terms of t, for the velocity and acceleration of the particle,
(ii) the value of t when the velocity of the particle is 0.25 m s^{-1},
(iii) the acceleration of the particle when it is 1 m from O.

Solution

(i) $s = 2 - \dfrac{4}{t+2}$

$v = \dfrac{ds}{dt} = \dfrac{4}{(t+2)^2}$ (Apply the Chain Rule of Differentiation)

$a = \dfrac{dv}{dt} = -\dfrac{8}{(t+2)^3}$

(ii) When $v = 0.25$,

$$\frac{4}{(t+2)^2} = 0.25$$
$$(t+2)^2 = 16$$
$$t + 2 = \pm 4$$
$$t = 2 \text{ or } t = -6 \text{ (rejected)}$$

(The negative value of t is rejected since time cannot be negative.)

(iii) When $s = 1$,

$$2 - \frac{4}{t+2} = 1$$
$$\frac{4}{t+2} = 1$$
$$t + 2 = 4$$
$$t = 2$$

Substitute $t = 2$ into $a = -\frac{8}{(t+2)^3}$:

$$a = -\frac{8}{(2+2)^3}$$
$$= -\frac{1}{8}$$

$\therefore$ Acceleration of the particle when it is 1 m from O is $-\frac{1}{8}$ m s^{-2}.

Example 2

A particle moves in a straight line such that its displacement, s m from a fixed point A, is given by $s = 2t + 3\sin 2t$, where t is the time in seconds after passing point A. Find

(i) the initial position of the particle,

(ii) expressions for the velocity and acceleration of the particle in terms of t,

(iii) the time at which the particle first comes to rest.

Solution

(i) $s = 2t + 3\sin 2t$

When $t = 0$, $s = 2(0) + 3\sin 2(0) = 0$.

$\therefore$ The particle is initially at point A.

(ii) $s = 2t + 3\sin 2t$

$$v = \frac{ds}{dt}$$
$$= 2 + 6\cos 2t$$
$$a = \frac{dv}{dt}$$
$$= -12\sin 2t$$

(iii) When $v = 0$,

$$2 + 6\cos 2t = 0$$
$$\cos 2t = -\frac{1}{3}$$
$$2t = 1.91 \text{ (to 3 s.f.)}$$
$$t = 0.955$$

Example 3

A stone that was initially at rest was thrown from the ground into the air, rising at a velocity of $v = 40 - 10t$, where t is the time taken in seconds.

(i) Find the maximum height reached by the stone.

(ii) Find the values of t when the particle is 35 m above the ground.

Solution

(i)
$$s = \int v \, dt$$
$$= \int (40 - 10t) \, dt$$
$$= 40t - 5t^2 + c$$

When $t = 0$, $s = 0$ $\quad \therefore c = 0$

$s = 40t - 5t^2$

At maximum height,

$v = 0$ $\quad$ ($v = 0$ at maximum displacement.)

$40 - 10t = 0$

$t = 4$

When $t = 4$, $s = 40(4) - 5(4)^2 = 80$.

$\therefore$ The maximum height reached by the stone is 80 m.

(ii) When $s = 35$,

$$40t - 5t^2 = 35$$
$$5t^2 - 40t + 35 = 0$$
$$t^2 - 8t + 7 = 0$$
$$(t - 1)(t - 7) = 0$$
$$t = 1 \quad \text{or} \quad t = 7$$

$\therefore$ The particle is 35 m above the ground when $t = 1$ and $t = 7$.

Example 4

A particle moves in a straight line so that, t seconds after passing through a fixed point O, its velocity, v cm s^{-1}, is given by $v = 8t - 3t^2 + 3$. The particle comes to instantaneous rest at the point P. Find

(i) the value of t for which the particle is instantaneously at rest,

(ii) the acceleration of the particle at P,

(iii) the distance OP,

(iv) the total distance travelled in the time interval $t = 0$ to $t = 4$.

Solution

(i) When $v = 0$,

$$8t - 3t^2 + 3 = 0$$
$$3t^2 - 8t - 3 = 0$$
$$(3t + 1)(t - 3) = 0$$
$$t = -\frac{1}{3} \text{ (rejected)} \quad \text{or} \quad t = 3$$

(ii) $v = 8t - 3t^2 + 3$

$a = 8 - 6t$

When $t = 3$, $a = -10$

$\therefore$ Acceleration of the particle at P is -10 cm s^{-2}.

(iii) $s = \int v \, \mathrm{d}t$

$$= \int (8t - 3t^2 + 3) \, \mathrm{d}t$$
$$= 4t^2 - t^3 + 3t + c$$

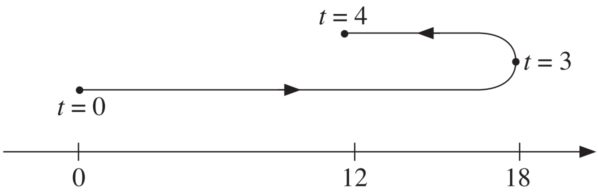
When $t = 0$, $s = 0$ $\quad \therefore c = 0$

$\therefore s = 4t^2 - t^3 + 3t$

When $t = 3$, $s = 18$

$\therefore OP = 18$ cm

(iv) When $t = 0$, $s = 0$.
When $t = 3$, $s = 18$.
When $t = 4$, $s = 12$.



(Draw the path of the particle on a displacement-time graph.)

$\therefore$ Total distance travelled $= 18 + (18 - 12)$
$= 24$ cm

Example 5

A particle moving in a straight line passes a fixed point O with a velocity of 4 m s^{-1}. The acceleration of the particle, a m s^{-2}, is given by $a = 2t - 5$, where t is the time after passing O. Find

(i) the values of t when the particle is instantaneously at rest,

(ii) the displacement of the particle when $t = 2$.

Solution

(i) $v = \int a \, dt$

$= \int (2t - 5) \, dt$

$= t^2 - 5t + c$

When $t = 0$, $v = 4$ $\quad \therefore c = 4$

$\therefore v = t^2 - 5t + 4$

When the particle is instantaneously at rest, $v = 0$.

$t^2 - 5t + 4 = 0$

$(t - 4)(t - 1) = 0$

$t = 4 \quad$ or $\quad t = 1$

(ii) $s = \int v \, dt$

$= \int (t^2 - 5t + 4) \, dt$

$= \frac{1}{3}t^3 - \frac{5}{2}t^2 + 4t + c_1$

When $t = 0$, $s = 0$ $\quad \therefore c_1 = 0$

$\therefore s = \frac{1}{3}t^3 - \frac{5}{2}t^2 + 4t$

When $t = 2$,

$s = \frac{1}{3}(2)^3 - \frac{5}{2}(2)^2 + 4(2)$

$= \frac{2}{3}$

$\therefore$ Displacement of the particle is $\frac{2}{3}$ m

Example 6

A particle starts at rest from a fixed point O and travels in a straight line so that, t seconds after leaving point O on the line, its acceleration, a m s^{-2}, is given by $a = 2 \cos t - \sin t$. Find

(i) the value of t when the particle first comes to an instantaneous rest,

(ii) the distance travelled by the particle in the first 3 seconds after leaving O.

Solution

(i) $a = 2 \cos t - \sin t$

$v = \int a \, dt$ (Recall that when a particle is at instantaneous rest, $v = 0$.)

$= \int (2 \cos t - \sin t) \, dt$

$= 2 \sin t + \cos t + c$

When $t = 0$, $v = 0$ $\quad \therefore c = -1$ (Note that $\cos 0 = 1$.)

$\therefore v = 2 \sin t + \cos t - 1$

When $v = 0$,

$$2\sin t + \cos t - 1 = 0$$

$$2\sin t + \cos t = 1 \quad \text{(R-formula is needed to solve this equation.)}$$

$$\sqrt{5}\sin(t + 0.4636) = 1$$

$$\sin(t + 0.4636) = \frac{1}{\sqrt{5}}$$

basic angle, $\alpha = 0.4636$ (to 4 s.f.)

$$t + 0.4636 = 0.4636, 2.677$$

$$t = 0, 2.21 \text{ (to 3 s.f.)}$$

$\therefore$ The particle first comes to an instantaneous rest when $t = 2.21$.

(ii) $s = \int v \, dt$

$$= \int (2\sin t + \cos t - 1)\, dt$$

$$= \sin t - 2\cos t - t + d$$

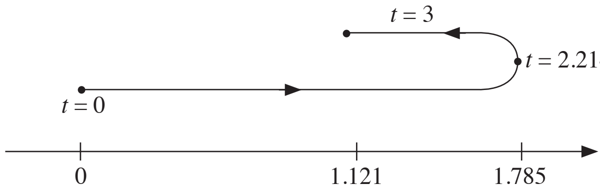
When $t = 0$, $s = 0$ $\quad \therefore d = 2$

$\therefore s = \sin t - 2\cos t - t + 2$

When $t = 0$, $s = 0$.

When $t = 2.214$, $s = 1.785$.

When $t = 3$, $s = 1.121$.



$\therefore$ Total distance travelled $= 1.785 + (1.785 - 1.121)$

$$= 2.45 \text{ m (to 3 s.f.)}$$

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2\sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2}bc \sin A$$