Given that angle CAB = 54^o, find the values of x *and* y.

Solution

Since AB is a diameter, the angle ACB is right angle.

$$So \quad x + 54^0 + 90^0 = 180^0$$
$$\therefore \ x = 180^0 - 146^0 = \underline{\underline{36^0}}$$

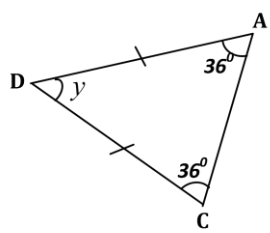
Since DA is a tangent and AB the diameter, angle DAB is right angle.

$$\Rightarrow \quad < DAC + 54^0 = 90^0$$
$$\therefore \ < DAC = 90^0 - 54^0 = \underline{\underline{36^0}}$$

Since DA and DC are tangents from the same point to the circle, the <DAC and <DCA are equal.

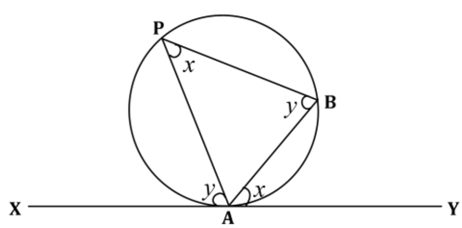
$$\Rightarrow \quad < DAC = < DCA = 36^0$$

Considering triangle ADC



$$So \quad y + 36^0 + 36^0 = 180^0$$
$$\therefore \ y = 180^0 - 72^0 = \underline{\underline{108^0}}$$

18.10 Alternate – segment theorem

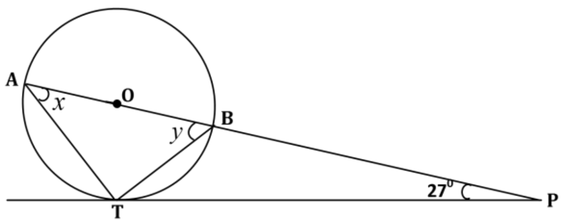


If XY is a tangent to the circle at A, the angle between the tangent and the chord is equal to the angle the chord subtends in the alternate segment i.e.

$$< BAY = < BPA$$
$$Also:$$
$$< PAX = < PBA$$

Example

In the diagram below, O is the centre of the circle and PT is a tangent to the circle at T. the angle TPB = 27^o.



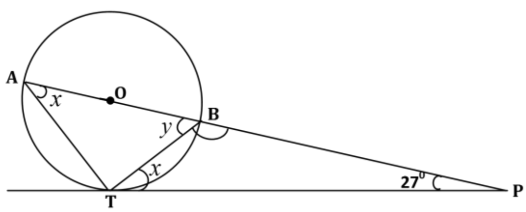
Find angles marked:

i. x

ii. y

Solution

Since AB is the diameter of the circle, <ATB = 90^o

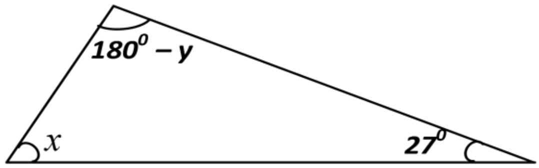


Considering triangle ABT:

$$x + y + 9^0 = 180^0$$
$$\therefore x + y = 90^0 \text{...(1)}$$

By the alternate segment theorem, $< PTB = < TAB = x.$

Considering triangle BTO:



$x + 180^0 - y + 27^0 = 180^0$

$\therefore x - y = -27^0$..(2)

Solving (1) and (2) simultaneously:

$Eqn(2) + eqn(1)$

$x + y = 90^0$

$^{-}\ \underline{x - y = -27^0}$

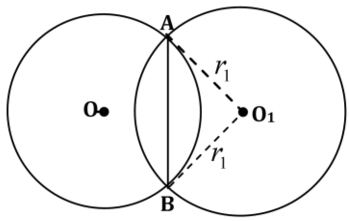
$2x = 63^0 \;\therefore x = 63/2 \quad \underline{\underline{= 31.5^0}}$

And $y = 90^0 - x \Rightarrow y = 90^0 - 31.5^0 = \underline{\underline{58.5^0}}$

18.11 Intersection of circles

When two circles intersect, they share a chord known as common chord. If we know at least one of the angles subtended at the centre of one of the circles by the chord and the radius of the same circle, we can find the length o f the chord.

Consider two circles centre O and O_1 intersecting at A and B.



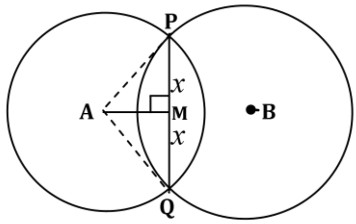
The length AB is therefore a common chord to the two circles. The following examples will illustrate how to calculate the length of the common chord.

Example

Two circles, centre A and B intersect at P and Q. circle center AA has a radius 6.5cm, and the angle subtended by PQ at A is 100^o. Calculate the length of PQ.

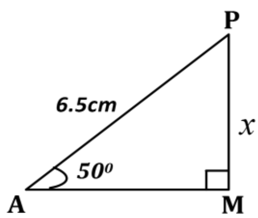
Solution

$Let\ PM = x = MQ$



$< PAQ = 100^0 \Rightarrow < PAM = \frac{100}{2} = 50^0$

Considering triangle APM:



$$\sin 50^0 = \frac{PM}{AP} = \frac{x}{6.5}$$

$$\Rightarrow x = 6.5 \times \sin 50^0 = 4.98cm$$

$$\therefore PQ = 2 \times 4.98 = \underline{\underline{9.96cm}}$$

Example

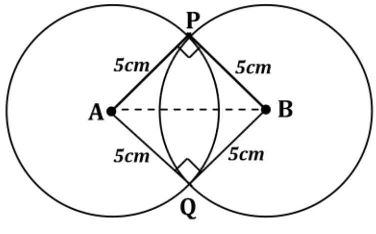
Two equal circles of radius 5cm intersect at right angles.

i. Find the distance between the two centers of the circles.

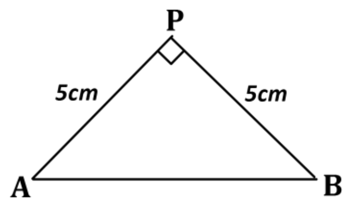
ii. Calculate the area of the common region of the circles.

Solution

i.



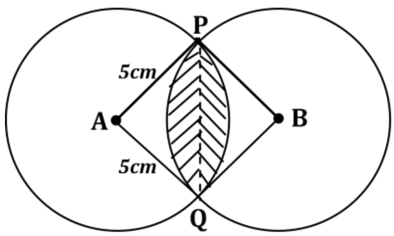
Taking triangle ABP



$AB^2 = 5^2 + 5^2$

$\therefore AB = \sqrt{25+25} \quad = 5\sqrt{2} \quad = \underline{\underline{7.07cm}}$

ii. *Area of the common region of the circle is the shaded part.*



$Area\ of\ triangle\ APQ = \frac{1}{2} \times 5 \times 5 = 12.5cm$

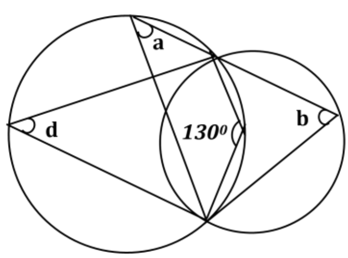
$Areaof\ \sec tor\ APQ = \frac{1}{2} \times 3.14 \times 5^2 = 19.625cm^2$

$Area\ of\ half\ shaded\ region = 19.625 - 12.5 = 7.125cm^2$

$\therefore Area\ of\ common\ region\ of\ the\ circle = 2 \times 7.125 = \underline{\underline{14.25cm^2}}$

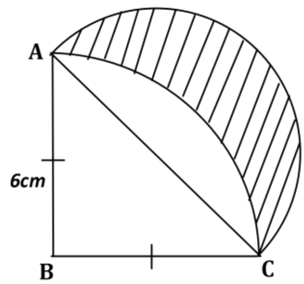
18.12 Miscellaneous Exercise

1. In the diagram below, O is the centre of the circle ABC. Angle AOC = 140°.



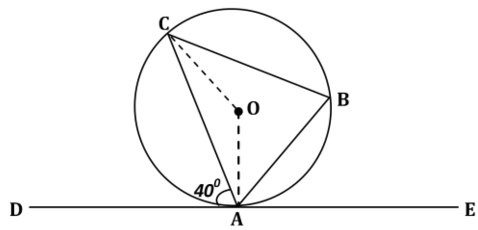
Write down the values of **a**, ***b***, and ***d***.

2. In the diagram below, ABC is an isosceles right–angled triangle.



The shaded area is bounded by two circular arcs. The outer arc is a semi – circle with AC as diameter and the inner arc is a quarter of a circle with centre B. Find the area of the shaded region.

3. In the diagram below, DAE is a tangent to the circle centre O at A. angle CAD = 40^o.

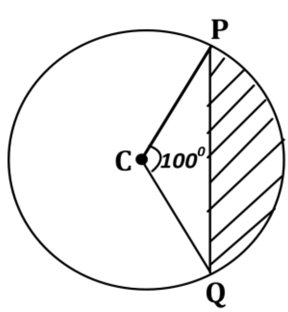


Find:

i. angle OCA

ii. angle ABC.

4. In the figure below, C is the centre of the circle of radius 21m.

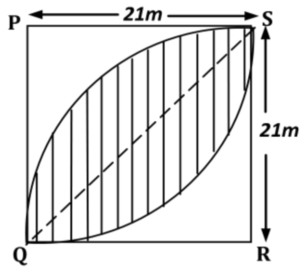


Calculate:

i. length PQ

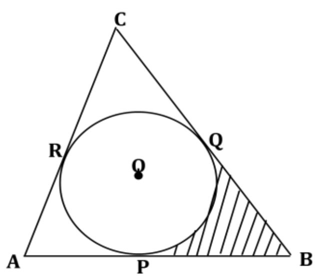
ii. the perimeter and area of the shaded part

5. In the figure below, PQRS is a square of side 21m, PQS and RQS are quadrants.



Taking π as $\frac{22}{7}$, calculate the area of the shaded part.

6. In the diagram below, ABC is an isosceles triangle in which $\overrightarrow{AC} = \overrightarrow{AB} = 8cm \text{ and } \overrightarrow{BC} = 10cm.$ The circle PQR with centre O touches the sides of the triangle at points, P Q and R.



Given that, the points C, O and P are in the same straight line such that $\overrightarrow{PO} = 3\overrightarrow{OC}$.

Calculate:

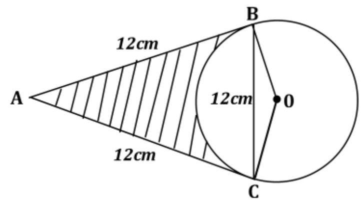
i. the radius of the circle

ii. the area of triangle ABC

iii. the area of the circle

iv. the area of the shaded portion

7. In the figure below, AB and AC are tangents to the circle at points B and C respectively. O is the centre of the circle. Given that $\overrightarrow{AB} = \overrightarrow{BC} = 12cm.$



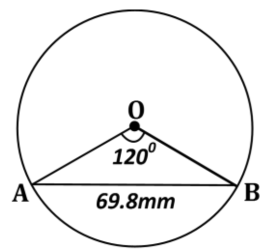
Determine:

i. the obtuse angle BOD

ii. the radius of the circle

iii. the area of minor sector BOC and hence the area of the shaded region

8. In the figure below, AB is a chord of the circle whose centre is O. angle AOB is 120^o and AB = 69.8mm of the circle.

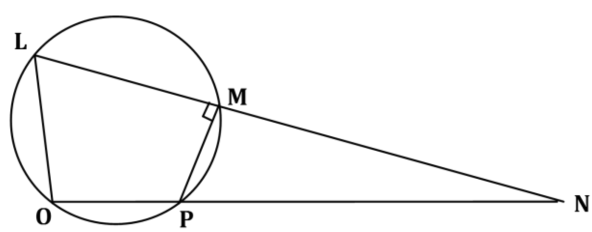


Find the radius of the circle.

(Give your answer correct to 3 significant figures)

9. In the figure below

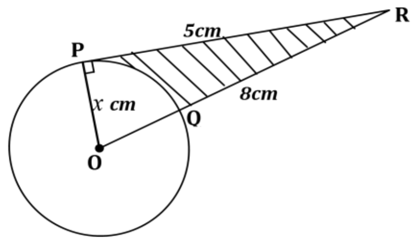
$\overrightarrow{OL} = 4.5cm, \overrightarrow{PM} = 3cm, \overrightarrow{NM} = 4cm \ and \ \overrightarrow{LN} = 7.5cm.$



Find:

i. lengths ON and OP
ii. the radius of the circle
iii. area of OLMP

10. In the figure below, O is the centre of the circle. PR is the tangent and OR intersects the circle at Q.



Given that $\overrightarrow{RQ} = 8cm,\ \overrightarrow{PR} = 5cm,\ and\ \overrightarrow{OP} = x\,cm.$

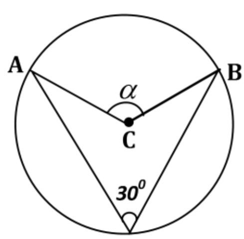
a) i. Express the length OR in terms of x.

ii. Find x.

b) Calculate:

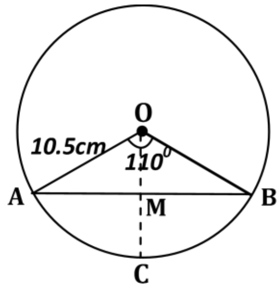
i. the area of the shaded region
ii. Angle subtended by arc PQ at the centre.

11. In the figure below, C is the centre of the circle.



Calculate the length of the chord AB and the angle marked α

12. In the figure below, OACB is a sector of a circle centre O and radius 10.5cm. Angle AOB = 110^{o}.



Calculate:

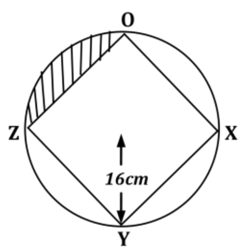
a) the length of CM

b) the length of arc BC

c) area of the minor segment cut off by the chord.

d) the perimeter of the minor segment ACB

13. In the diagram below OXYZ is a square drawn inside a circle of radius 16cm as shown in the diagram.



Calculate the perimeter and area of the shaded part.

19 AREAS AND VOLUMES OF SOLIDS

The solids under consideration include the following:

❖ Prisms

❖ Pyramid

❖ Cone

❖ Sphere

❖ Pipe

19.1 Surface Area of Solids

Definition:

Surface area of a solid is the sum of the areas of all the surfaces of the solid.

19.2 Surface area of a prism:

A prism is a solid, which has uniform cross-section. This includes:

* Rectangular prism i.e. cubes and cuboids.
* Triangular prism
* Circular prisms e.g. cylinders, etc.

The surface area of a prism is found as follows:

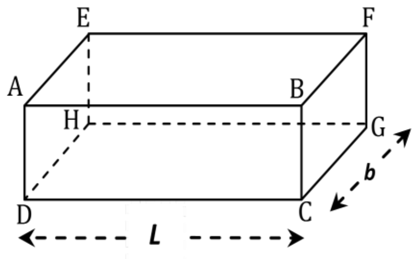
i) Find the area of cross –section and multiply it by 2.

ii) Find the area of each rectangular side face and add up these areas. For the case of a cylinder, find the area of the curved surface.

iii) Add up the results to get the surface area of the prism.

19.3 Surface area of a cuboid

A cuboid is a solid with six faces. Pairs of opposite faces are identical and equal in size.



Faces ABCD and EFGH, AEHD and BFGC, AEFB and DHGC are pairs of identical faces.

- $Area\ of\ \ face\ ABCD = area\ of\ \ face\ EFGH = lh$
  $\therefore Area\ of\ \ faces\ ABCD\ and\ EFGH = lh + lh = 2lh$

- $Area\ of\ \ face\ AEHD = area\ of\ \ face\ BFGC = bh$
  $\therefore Area\ of\ \ faces\ AEHD\ and\ BFGC = bh + bh = 2bh$

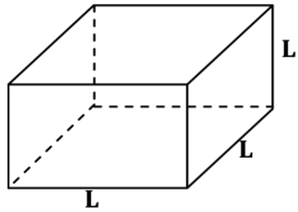
- $Area\ of\ \ face\ AEFB = area\ of\ \ face\ DHGC = lb$
  $\therefore Area\ of\ \ faces\ AEFB\ and\ DHGC = lb + lb = 2lb$

$$\boxed{\begin{aligned}\therefore Total\ \ surface\ \ area\ \ of\ \ the\ \ cuboid\ &= 2lh + 2bh + 2lb \\ &= 2(lh + bh + lb)\end{aligned}}$$

A cuboid is also referred to as rectangular block or simply a box.

19.4 Surface area of a cube

A cube is a solid with six identical square faces.



To find the surface area of the cube, we find the area of one face and multiply it by 6, i.e.

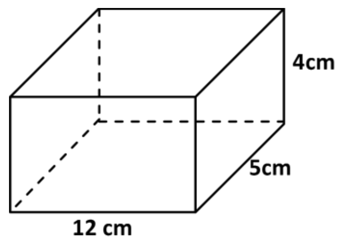
$Total\ surface\ area = 6 \times l \times l$

$$\boxed{\begin{array}{l} Surface\ area\ of\ a\ cube = 6l^2 \\ where\ \ l\ is\ the\ \ lengh\ \ of\ \ side \end{array}}$$

Example

Calculate the surface area of a cuboid measuring 12 cm × 5cm × 4cm.

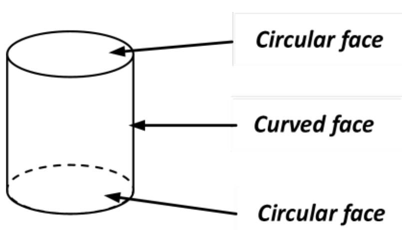
Solution



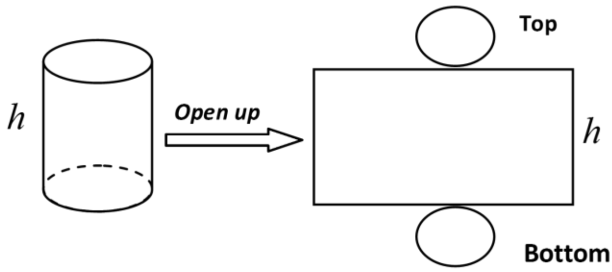
$$\begin{aligned} Surface\ area\ of\ the\ cuboid &= 2(lh + bh + lb) \\ &= 2(12 \times 4 + 5 \times 4 + 12 \times 5) \\ &= \underline{\underline{256cm^2}} \end{aligned}$$

19.5 Surface area of a cylinder

A cylinder has three surfaces: Two are circular and one is curved.



The figure below shows the shape obtained when a hollow cylinder, radius r, height h is opened up and laid out flat.



The curve surface becomes a rectangle measuring $2\pi r$ *by h unts*.

$$There\ for: Area\ of\ top = \pi r^2$$
$$Area\ of\ bottom = \pi r^2$$
$$Area\ of\ curved\ surface\ (rectangle) = 2\pi rh$$

> *Thus*:
> $Total\ surface\ area\ of\ a\ closed\ cylinder = \pi r^2 + \pi r^2 + 2\pi rh = 2\pi r(r+h)$

Note:

1. If the cylinder is hollow and has one open end, then there are only two surfaces, i.e. the curved surface and the bottom surface. In this case, $Total\ surface\ area\ of\ the\ cylinder = 2\pi rh + \pi r^2$

2. However, if the cylinder is open ended, then there is only one surface. In this case, $Surface\ area\ of\ the\ cylinder = 2\pi rh$

Example

A closed cylindrical container has a diameter of 3.2cm and height 4.9cm. Find the area of the material used to make the cylinder. Express your answer to 4sf. (Take π *as* 3.142).

Solution

$$Surface\ area\ a\ closed\ cylinder = 2\pi rh + 2\pi r^2, \quad r = {}^{3.2}\!/_{2} = 1.6cm$$
$$= 2 \times 3.13 \times 1.6(1.6 + 4.9)$$
$$= \underline{\underline{65.35cm^2 (4.s.f)}}$$

Example

A very thin sheet of metal is used to make a cylinder of radius 5cm and height 14cm. Using $\pi = 3.142$, find the total area of the sheet that is needed to make:

a) A closed cylinder

b) A cylinder that is open on one end.

Solution

a) $r = 5cm, h = 14cm, \pi = 3.142$

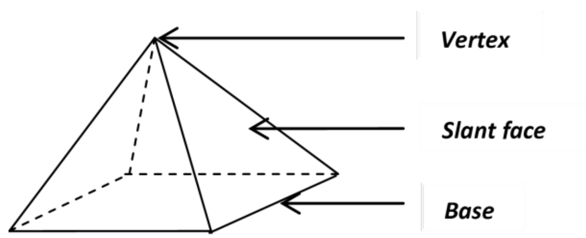
$$\begin{aligned} \textit{For a closed cylinder; surface area} &= 2\pi r(r+h) \\ &= 2 \times 3.142 \times 5(5+14) \\ &= \underline{\underline{596.98cm^2}} \end{aligned}$$

b) For a cylinder open on one end:

$$\begin{aligned} \textit{Surface area} &= 2\pi rh + \pi r^2 \\ &= 2 \times 3.142 \times 5 \times 14 + 3.142 \times 5^2 \\ &= \underline{\underline{518.43cm^2}} \end{aligned}$$

19.6 Surface area of a pyramid

Below is the structure of the pyramid.

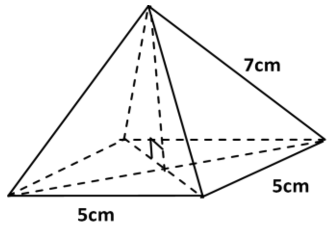


The surface area of a pyramid is obtained as the sum of the areas of the slant faces and the base. Each slanting face is an isosceles triangle.

The following examples illustrate how to obtain the surface areas of a pyramid.

Example

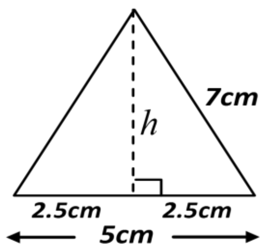
Find the surface area of the pyramid shown below.



Solution

$Area\ of\ the\ base = 5 \times 5 = 25cm^2$

Each slanting face is an isosceles triangle of height h



From Pythagoras theorem:

$$5^2 = h^2 + 2.5^2$$
$$\therefore h^2 = 49 - 6.25 = 42.75$$
$$\Rightarrow h = \sqrt{42.75} = 6.538cm$$

$So\ now\ area\ of\ each\ slanting\ face = \frac{1}{2}bh = \frac{1}{2} \times 5 \times 6.538 = 16.345cm^2$

$But\ there\ are\ four\ slanting\ faces.$

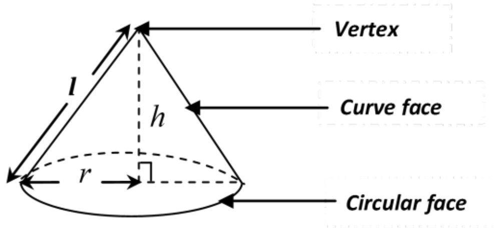
$\therefore Total\ area\ of\ the\ slantig\ faces = 4 \times 16.345 = 65.38cm^2$

$\Rightarrow Total\ surface\ area\ of\ the\ pyramid = 25 + 65.38$

$$= \underline{\underline{90.4cm^2}}$$

19.7 Surface area of a cone

A closed cone has two surfaces; the curved surface and a circular face



If **r** is the radius of the circular face and **l** is the length of the slant edge, the:

$$Area\ of\ a\ curved\ surface = \pi rl$$

$$Area\ of\ circular\ face = \pi r^2$$

$$\therefore Total\ surface\ area\ of\ a\ closed\ cone = \pi r^2 + \pi rl$$

$$\boxed{Surface\ area\ of\ a\ closed\ cone = \pi r(r + l)}$$

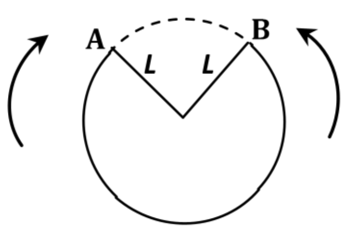
The height **h** of the cone can be obtained by applying Pythagoras theorem. Considering the figure above:

$$l^2 = h^2 + r^2 \Rightarrow h^2 = l^2 - r^2$$

$$\therefore h = \sqrt{l^2 - r^2}$$

19.8 Formation of a cone

A cone can be formed from any section of a circle. Consider a circle of radius *L* shown below.



If a section AB is cut out of the circle and folded in the direction of the arrow, then a cone whose circumference of the base equal to the length of arc AB is formed. Its slanting edge is equal to the radius of the circle.

Note:

If the cone is open, then it has only one surface, which is the curved surface. In this case, its area is simply given by:

$$\text{Surface area of a open cone} = \pi rl$$

Example

A section of a circle of radius 10cm having an angle of 100^o is bent to form a cone.

a) Find the length of the arc of the section

b) Determine the surface area of the cone.

Solution

a) $\text{Length of the arc subtending an angle } \theta \text{ at the centre} = 2\pi l \times \dfrac{\theta}{360}$

$\text{But } \theta = 100^0,\ l = \text{radius} = 10cm$

$\Rightarrow \text{Length of the arc} = 2 \times 3.14 \times \dfrac{100}{360} = \underline{\underline{17.45cm}}$

b) $\text{Surface area of a cone} = \pi r(l + r)$

$\text{where } l - \text{length of the slanting edge and } r - \text{radius of the base}$

$\text{Slanting edge of the cone} = \text{radius of the circle from which it is formed.}$

$\text{Length of the sector} = \text{circumferance of the base of the cone}$

$\Rightarrow 2\pi r = 17.45$

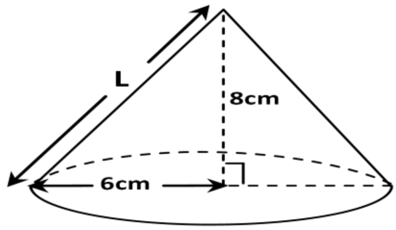
$\therefore r = \dfrac{17.45}{2 \times 3.14} = 2.78cm \quad \text{and } l = 10cm$

$\therefore \text{Surface area of the cone} = \pi r(l + r) = 3.14 \times 2.78(2.78 + 10) = \underline{\underline{111.5cm^2}}$

Example

A cone of base radius 6cm and height 8cm is slit and laid out flat into a section of circle. What angle does the section subtend at the centre?

Solution



By Pythagoras theorm,

$L = \sqrt{6^2 + 8^2} = \sqrt{100} = 10cm$

The slanting edge L of the cone = radius of the sector of the circle

$\therefore$ *radius of the sector* $= 10cm$

The circumferance of the base of the cone $= 2\pi r$ *and* $r = 6cm$

$= 2 \times 3.14 \times 6$

$= 37.68cm$

Let θ *be the angle of the sector, then the circumfereance of the sector is given by*

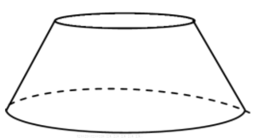
$2\pi L \times \frac{\theta}{360}$

$\Rightarrow 37.68 = 2\pi L \times \frac{\theta}{360}, \quad L = 10cm$

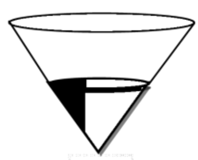
$\therefore \theta = \frac{37.68 \times 360}{2 \times 3.14 \times 10} \quad = \underline{\underline{216^0}}$

19.9 The Frustum

A frustum is obtained by chopping part of a cone. It is a figure in the shape of a bucket or a lampshade as depicted below.



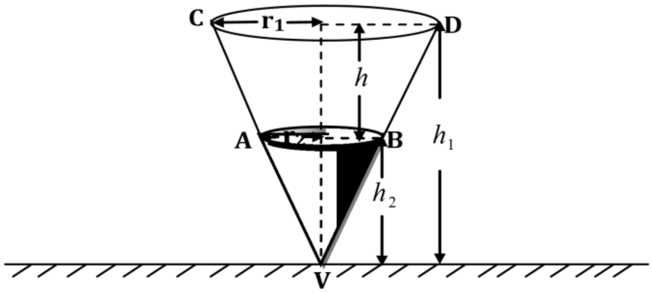
a)



b)

The shaded part of figure b) shows portion of the cone, which has been cut off.

To find the area of the frustum, we apply properties of enlargement to the cone by considering VAB as VCD.



We could also consider VCD as the image of VAB under enlargement. The linear scale factor, which maps VAB onto VCD, is:

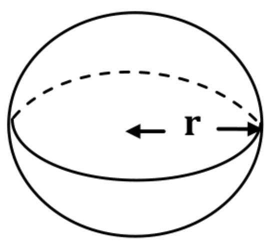
$$\frac{h_1}{h_2} = \frac{r_1}{r_2}, \quad but\ h_1 = h_2 + h \quad \Rightarrow h_2 = h_1 - h$$

$$\therefore \frac{h_1}{h_1 - h} = \frac{r_1}{r_2} \quad \Rightarrow h_1 r_2 = r_1(h_1 - h)$$

$$\Rightarrow h_1 = \frac{r_1 h}{r_1 - r_2}$$

19.10 Surface area of a sphere

The figure below represents a solid sphere of radius **r** units.



The surface area of a sphere is given by below.

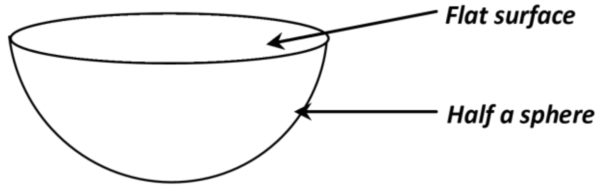
$$\boxed{Surface\ area\ of\ a\ sphere = 4\pi r^2}$$

NB:

The proof for this formula is beyond the scope of this course.

19.11 Surface area of a hemisphere

A hemisphere is half of a sphere.



$$Area\ of\ a\ hemisphere = area\ of\ the\ flat\ surface + area\ of\ half\ of\ the\ sphere$$

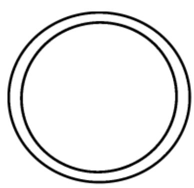
$$= \pi r^2 + \frac{4\pi r^2}{2}$$

$$= \pi r^2 + 2\pi r^2 = 3\pi r^2$$

$$\boxed{Surface\ area\ of\ hemisphere = 3\pi r^2}$$

19.12 Area of a ring

A ring is a circular object with a hole at its centre. Below is a structure of a ring.



If **R** is the radius of the larger circle and **r** is the radius of the smaller circle, then area of the ring is given by:

$$Area\ of\ the\ ring = area\ of\ the\ \text{larger}\ circle + area\ of\ the\ smaller\ circle$$

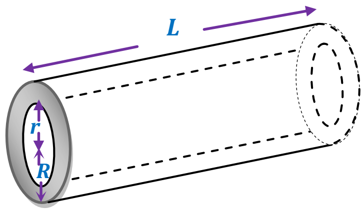
$$= \pi R^2 - \pi r^2$$

$$= \pi (R^2 - r^2), \quad but\ R^2 - r^2 = (R + r)(R - r)$$

$$\boxed{\therefore\ Area\ of\ the\ ring = \pi (R + r)(R - r)}$$

19.13 Surface area of a pipe

Consider a pipe of length L with outer radius R and internal radius as shown below.



The hollow pipe has a uniform cross section, which is a ring.

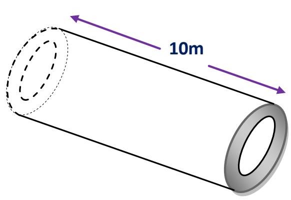
$$\begin{array}{lcl} \text{Total surface area} & = & \text{area of two rings} + \text{curved surface area} \\ \text{of the pipe} & & \text{at both ends} \qquad\qquad \text{of the pipe} \end{array}$$

$$\begin{aligned} &\text{Area of the ring at one end} &&= \pi(R^2 - r^2) \\ &\therefore \text{Area of the ring at both ends} &&= 2\pi(R^2 - r^2) \\ &\text{Area of the curved surface} &&= 2\pi Rl \end{aligned}$$

$$\boxed{\therefore \text{Surface area of the pipe} = 2\pi(R^2 - r^2) + 2\pi Rl}$$

Example

The figure below shows a cylindrical water main, which is 10cm long. The pipe has an inner radius of 30cm and outer radius of 37cm.



Calculate the total surface area of the pipe.

Solution

$$\begin{aligned}
&\textit{Area of the ring at one end} &&= \pi(R^2 - r^2), \quad R = 37cm, r = 30cm \\
& &&= \tfrac{22}{7}(37^2 - 30^2) = 1474cm^2 \\
&\therefore \textit{Area of the ring at both ends} &&= 2 \times 1474 = 2948cm^2 \\
&\textit{Area of the curved surface} &&= 2\pi Rl, \quad R = 37cm, l = 10cm \\
& &&= 2 \times \tfrac{22}{7} \times 37 \times 10 = 2325.7cm^2
\end{aligned}$$

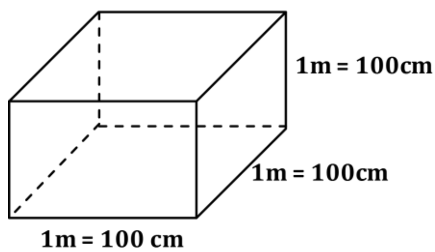
$$\textit{So total surface area of the pipe} = 2948 + 2325.7 = \underline{\underline{5273.7cm^2}}$$

19.14 VOLUME OF SOLIDS

19.14.1 Definition:

Volume is the amount of space occupied by an object.

A unit cube is used as the basic unit of volume. The SI unit of volume is the cubic meter (m^3). Consider a unit cube below i.e. a cube of side 1m.



$$\textit{The volume of the cube} = 1m \times 1m \times 1m = 100cm \times 100cm \times 100cm$$

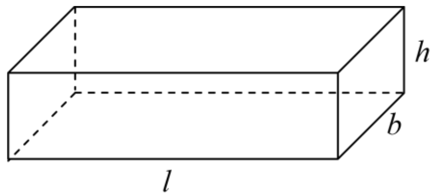
$$\Rightarrow \quad 1m^3 = 1{,}000{,}000cm^3$$

Since $1m^3$ is large for ordinary use, volumes are often measured using cm^3.

$$\boxed{\begin{aligned} 1m^3 &= 1{,}000{,}000\,cm^3 = 1.0 \times 10^6\,cm^3 \\ \therefore 1cm^3 &= \frac{1}{1{,}000{,}000} = 1.0 \times 10^{-6}\,m^3 \end{aligned}}$$

19.14.2 Volume of a cuboid

Consider a cuboid of length l, breadth b, and height h, as shown below.



$$\text{Volume of a cuboid} = l \times b \times h$$

19.14.3 Volume of a cuboid

A cube is just a special cuboid with *length = breadth = height*

$$\text{Volume of a cube} = l \times l \times l = l^3$$

Example

A rectangular tank has 70cm^3 of water. If the tank is 5cm long and the height of water is 4cm, what is the width of the tank?

Solution

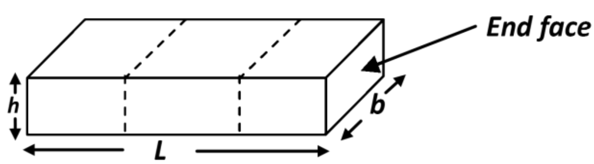
$\text{The volume of water} = l \times b \times h \quad \text{but } V = 70m^3, l = 5m, h = 4m, b = ?$

$\Rightarrow \quad 70 = 5 \times 4 \times b$

$\therefore b = {}^{70}\!/_{20} = \underline{\underline{3.5m}}$

19.14.4 Uniform cross – section

Consider the cuboid shown below, the shaded part is known as the end face.



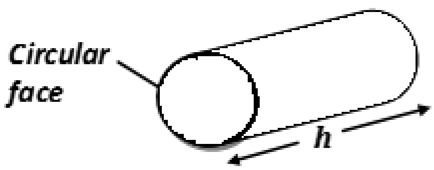
If the cuboid is sliced along the dotted lines, each slice will be parallel and identical to the end face. Such faces are known as cross section of the solid.

$$A\ cuboid\ has\ a\ uniform\ cross\ section\ of\ area = bh. \quad But\ volume = lbh$$
$$\therefore Volume = l \times area\ of\ cross\ section$$

There are many solids, which have uniform cross sections that are not rectangular; their volumes are calculated in the same way.

19.14.5 Volume of a cylinder

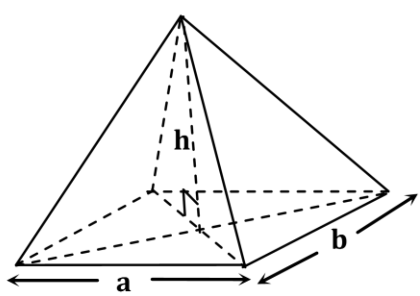
A cylinder is a solid whose uniform cross section is a circular surface.



$$Area\ of\ cross\ section = \pi r^2. \quad where\ r\ is\ the\ radius\ of\ the\ circular\ face$$
$$Volume\ of\ the\ cylinder = area\ of\ circular\ face \times height$$

$$\boxed{\therefore Volume\ of\ a\ cylinder = \pi r^2 h}$$

19.14.6 Volume of a pyramid



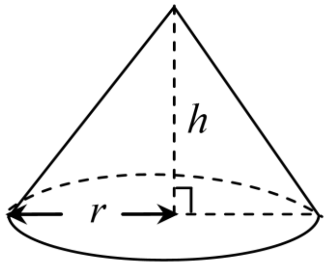
20

$$Volume\ of\ a\ pyramid = \tfrac{1}{3} \times area\ of\ base \times height$$
$$But\ area\ of\ the\ base\ A = a \times b$$

$$\boxed{\therefore Volume\ of\ a\ pyraid = \tfrac{1}{3} abh}$$

20.1.1 Volume of a cone

A cone may be considered as a pyramid with a circular base.



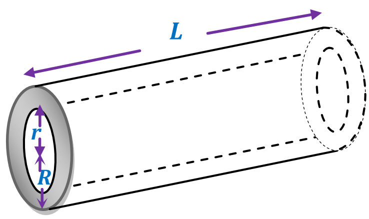
$$Volume\ of\ a\ cone = \tfrac{1}{3} \times area\ of\ base \times height$$

$$But\ area\ of\ the\ base\ A = \pi r^2$$

$$\boxed{\therefore Volume\ of\ a\ cone = \tfrac{1}{3}\pi r^2 h}$$

20.1.2 Volume of a pipe

If **R** is the outer radius and **r** is the internal radius of the pipe of length **L**.



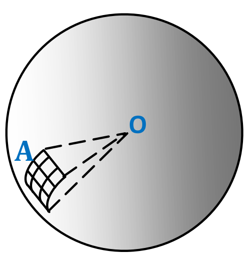
$$Volume\ of\ a\ the\ pipe = area\ of\ cross\ section \times length\ of\ the\ pipe$$

$$But\ area\ of\ cross\ section = \pi(R^2 - r^2)$$

$$\boxed{\therefore Volume\ of\ a\ pipe = \pi(R^2 - r^2)l}$$

20.1.3 Volume of a sphere

Let **A** represents a small square area on the surface of a sphere of radius r, centre O.



If **A** is very small, then it is almost flat. Therefore, the solid formed by joining the vertices of **A** to the centre **O** is a small pyramid of height equal to **r**.

$$Volume\ of\ a\ small\ pyraid = \tfrac{1}{3}Ar$$

If there are **n,** such small pyramids in the sphere with base areas, $\mathbf{A_1}$, $\mathbf{A_2}$, $\mathbf{A_3}$ ….$\mathbf{A_n}$.

$$Their\ volumes\ are;\ \tfrac{1}{3}A_1r, \tfrac{1}{3}A_2r, \tfrac{1}{3}A_3r.......\tfrac{1}{3}A_nr$$
$$Total\ volumes = \tfrac{1}{3}A_1r + \tfrac{1}{3}A_2r + \tfrac{1}{3}A_2r...... + \tfrac{1}{3}A_nr = \tfrac{1}{3}r\left(A_1 + A_2 + A_3 + ... + A_n\right)$$

For the whole surface of the sphere, the sum of all their base area is $4\pi r^2$ i.e.

$$A_1 + A_2 + A_3 + ... + A_n = 4\pi r^2$$

$$Hence\ total\ volume\ V\ of\ a\ sphere = \tfrac{1}{3}r \times 4\pi r^2$$

$$\boxed{\therefore Volume\ of\ a\ sphere = \tfrac{4}{3}\pi r^3}$$

20.1.4 Volume of a hemisphere

Since a hemisphere is half of a sphere, its volume is equal to half of the volume of a sphere, i.e.

$$\boxed{Volume\ of\ a\ hemisphere = \tfrac{2}{3}\pi r^3}$$

Example

A solid hemisphere of radius 5.8cm has density of 10.5g/cm^3.

Calculate:

a) Volume of the solid

b) Mass in kg of the solid

Solution

a) *For volume of solid:*

$$Volume\ of\ a\ hemisphere = \tfrac{2}{3}\pi r^3, \quad r = 5.8cm$$
$$= \tfrac{2}{3} \times 3.142 \times 5.8^3$$
$$= \underline{\underline{408.7cm^3 (4sf)}}$$

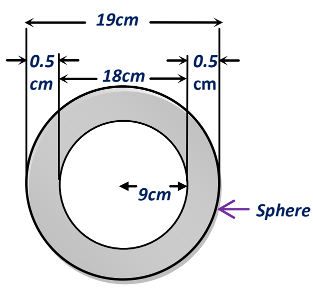
b) *For mass of solid:*

$$From: Density = \frac{Mass}{Volume}$$
$$\Rightarrow Mass = Density \times Volume$$
$$= 10.5 \times 408.7g$$
$$= \frac{4291.35}{1000}$$
$$= \underline{\underline{4.291kg(4sf)}}$$

Example

A hollow sphere has an internal diameter of 18cm and thickness 0.5cm. Find the volume of the material used in making the sphere.

Solution

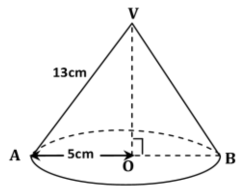


$$Internal\ diameter = 18cm. \Rightarrow Internal\ radius = {}^{18}\!/_{2} = 9cm$$
$$External\ diameter = 18 + 0.5 + 0.5 = 19cm. \Rightarrow External\ diameter = {}^{19}\!/_{2} = 9.5cm$$

$$Volume\ of\ the\ shaded\ part = volume\ of\ material\ used\ to\ make\ the\ sphere$$
$$= Volume\ of\ the\ whole\ sphere - Volume\ of\ unshaded\ part$$
$$= \frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^2$$
$$= \frac{4}{3} \times 3.142(9.5^3 - 9^3)$$
$$= \underline{\underline{537.81cm^3}}$$

Example

The figure below shows a right circular cone AVB. The radius of the base is 5cm and the slanting edge 13cm.



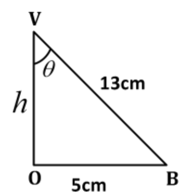
Calculate:

a) Angle VAB
b) Volume of the cone
c) Total surface area of the cone. (Take $\pi = 3.142$)

Solution

*Let **h** be the height of the cone and **θ** be angle OVB.*

Considering triangle VOB:



$$\sin\theta = 5/13 \quad \Rightarrow \theta = \sin^{-1}(5/13) = 22.6^0$$

$$But\ angle\ AVB = 2\theta = 2\times 22.6 = \underline{\underline{45.2^0}}$$

a) *Using Pythagoras theorem;*

$$h^2 + 5^2 = 13^2$$

$$\therefore h = \sqrt{169-25} = \sqrt{144} = 12cm$$

$$Volume\ of\ the\ cone = \tfrac{1}{3}\pi r^2 h, \quad r = 5cm$$

$$= \tfrac{1}{3}\times 3.142\times 5^2 \times 12$$

$$= \underline{\underline{3145.2cm^2}}$$

b) $Total\ surface\ area\ of\ cone = Area\ of\ curved\ surface + Area\ of\ circular\ base$

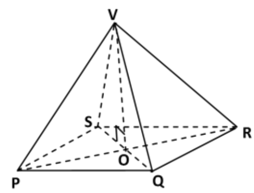
$$Areaofcircularface = \pi\ r^2 = 3.142\times 5^2 = 78.55cm^2$$

$$Area\ of\ curved\ surface = \pi rl = 3.142\times 5\times 13 = 204.23cm^2$$

$$\therefore Total\ surface\ area\ of\ cone = 204.23 + 78.55 = \underline{\underline{282.78cm^2}}$$

Example

The figure below shows a pyramid with a rectangular base PQRS. Given that ***PQ*** = 12m, ***QR*** = 9m and ***VO*** = 10m.



Calculate:

a) The length:
   i. PR
   ii. VR

b) The surface area of the pyramid

c) The volume of the pyramid

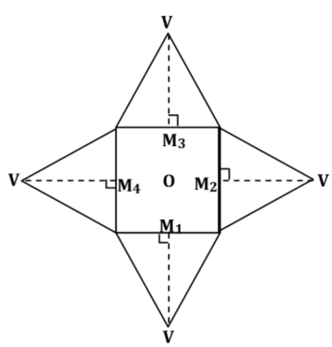
Solution

a) i) $\bar{PR} = \sqrt{PQ^2 + QR^2} = \sqrt{12^2 + 9^2} = \underline{\underline{15m}}$

ii) *VO is perpendicular to the base PQRS*

$$\begin{aligned}\bar{VR} &= \sqrt{VO^2 + QR^2}, \quad but\ OR = \tfrac{1}{2}PR = 7.5m \\ &= \sqrt{10^2 + 7.5^2} \\ &= \underline{\underline{12.5m}}\end{aligned}$$

b) *The figure below shows a net obtained by opening the pyramid at the vertex V. M_1, M_2, M_3 and M_4 are the midpoints of* ***PQ***, ***QR***, ***SR***, *and* ***SP*** *respectively.*



$But\ \bar{VS} = \bar{VR} = \bar{VQ} = \bar{VP} \quad \Rightarrow \bar{M_1V} = \bar{M_3V}\ \ and\ \bar{M_2V} = \bar{M_4V}$

$By\ Pythagoras\ theorem:$

$\bar{M_1V} = \sqrt{(VQ^2 - M_1Q^2)} = \sqrt{12.5^2 - 6^2} = 10.97m \quad \Rightarrow \bar{M_3V} = 10.97m$

$Also:$

$\bar{M_2V} = \sqrt{(VQ^2 - M_2Q^2)} = \sqrt{12.5^2 - 4.5^2} = 11.66m \Rightarrow \bar{M_4V} = 11.66m$

$$\begin{aligned} Surface\ area\ of\ the\ pyramid &= \tfrac{1}{2}PQ \times \bar{M_1V} + \tfrac{1}{2}QR \times \bar{M_2V} + \tfrac{1}{2}SR \times \bar{M_3V} + \tfrac{1}{2}SP \times \bar{M_4V} + PQ \times QR \\ &= \tfrac{1}{2}(12 \times 10.97 + 9 \times 11.66 + 12 \times 10.97 + 9 \times 11.66) + 12 \times 9 \\ &= \underline{\underline{344.31m^2}} \end{aligned}$$

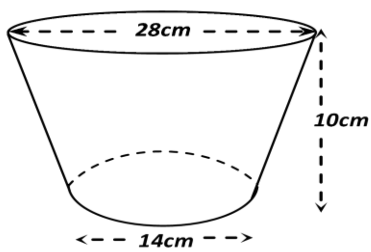
c) $Volume\ of\ the\ pyramid = \tfrac{1}{3} \times area\ of\ the\ base \times h$

$But\ area\ of\ the\ base = a \times b, \quad a = 12m, b = 9m$

$\therefore Volume\ of\ the\ pyramid = \tfrac{1}{3} \times 12 \times 9 \times 10 = \underline{\underline{360m^3}}$

Example

The figure below shows a bucket with a top diameter 28cma and bottom diameter 14cm. the bucket is 10cm deep.

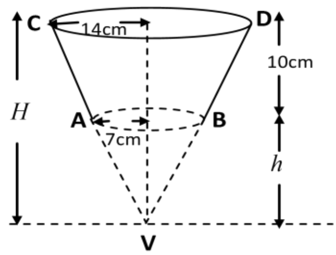


Calculate:

a) The capacity of the bucket in liters

b) The area of the plastic sheet required to make 200 such buckets, taking 5% extra for overlapping and wastage.

Solution



a) *The linear scale factor of the enlargement mapping the cone VAB to VCD is given by*

$$\frac{10+h}{h}=\frac{14}{7}=2 \quad \Rightarrow 10+h=2h \quad \therefore h=10cm$$

$$\therefore Volume\ of\ cone\ VCD=\tfrac{1}{3}\pi r^2 H, \quad but\ H=10+h=20cm$$

$$=\tfrac{1}{3}\times\tfrac{22}{7}\times 14^2\times 20$$

$$=4106.7cm^3$$

$$\frac{10+h}{h}=\frac{14}{7}=2 \quad \Rightarrow 10+h=2h \quad \therefore h=10cm$$

$$\therefore Volume\ of\ cone\ VCD=\tfrac{1}{3}\pi r^2 H, \quad but\ H=10+h=20cm$$

$$=\tfrac{1}{3}\times\tfrac{22}{7}\times 14^2\times 20$$

$$=4106.7cm^3$$

b)

$$Area\ of\ curve\ surface\ of\ cone\ VCD=\pi rl,\ r=14cm,\ l=20cm$$

$$=\tfrac{22}{7}\times 14\times 20=880cm^2$$

$$Area\ of\ curve\ surface\ of\ cone\ VAB=\pi rl,\ r=7cm,\ l=10cm$$

$$=\tfrac{22}{7}\times 7\times 10=220cm^2$$

$$\therefore Area\ of\ curve\ surface\ of\ the\ bucket=880-220=660cm^2$$

$$\Rightarrow Total\ area\ of\ the\ bucket=660+\tfrac{22}{7}\times 7^2=814cm^2$$

$$Total\ area\ of\ the\ plastic\ material\ required$$

$$to\ make\ an\ open\ bucket=814\times\frac{105}{100}=\underline{\underline{854.7cm^2}}$$

20.1.5 Miscellaneous exercise:

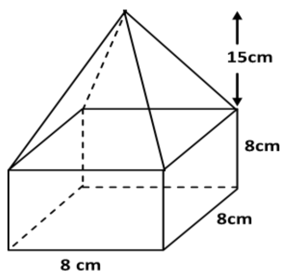
1. A solid cylinder has a radius of 18cm and height 15cm. a conical hole of radius r is drilled in the cylinder on one of the end faces. The conical hole is 12cm deep. If the material removed from the hole is 9% of the volume of the cylinder.

   Find:

   a) The surface area of the hole

   b) The radius of the spherical ball made out of the material.

2. The diagram below shows solid which comprises of a cube surmounted with a pyramid.

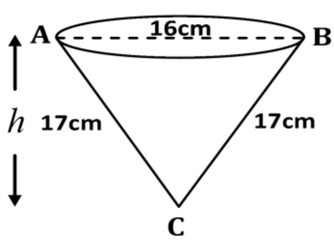


   Calculate:

   a) The surface area of the resulting solid.

   b) The volume of the solid formed.

3. The figure below shows a right circular cone ABC of vertical height h and slant side AB =BC = 17cm, and base diameter AB = 16cm.

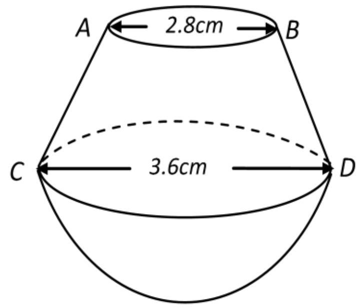


   Find:

   i. h

   ii. The capacity of the cone. Use $\pi = 3.142$

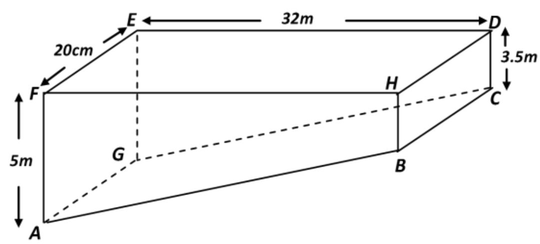
4. The diagram below represents a tank for storing water consisting of a frustum of a cone fastened to a hemisphere centre. AB = 2.8m and CD = 3.6m. The perpendicular height between AB and CD is 2.1cm.



a) Calculate the volume of water in the tank when it is full, giving your answer to the nearest m^3.

b) The cost of running water includes a fixed charge of shs. 150 plus shs. 50 per thousand liters used per month. If a family uses one full such tank of water per month, calculate the bill for this family in a month.

5. The diagram below shows a swimming pool 20m wide and 32m long. The pool is 3.5m deep at the shallow end and 5m deep at the deeper end.

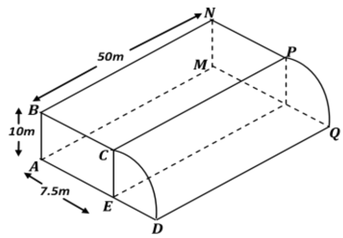


Calculate:

a) Volume and

b) Surface area of the swimming pool.

6. The diagram below shows a shed with uniform cross section. ABCD consists of a rectangle ABCE and a quadrant of a circle ECD with E as the centre.



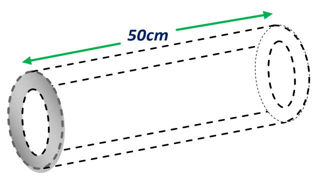
Calculate:

a) The area of cross section ABCD

b) The volume of the shed

c) Area of BCDQPN.

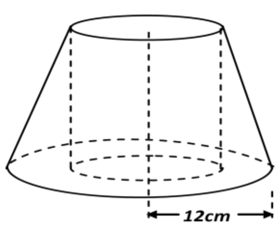
7. The figure below shows a hollow pipe of external diameter 16mm, internal diameter 10mm and length 50cm.



a) Calculate the surface area (in cm^2) of the pipe correct to two decimal places.

b) What would be the surface area of a similar pipe of length 150 cm, external diameter 48mm and internal diameter 30mm?

8. The figure below shows a right circular cone whose original height was 20cm, below part of it was cut-off. The radius of the base is 12cm and that of the top is 8cm. a circular hole of 8cm was drilled through the centre of the solid as shown in the diagram below.



Calculate the volume of the remaining solid. (Use $\pi = 3.142$).

21 LINES AND PLANES IN 3-DIMENSIONS

21.1 Introduction:

Some objects have dimensions of length, width, and height, which are all at right angle to one another. Measurement on such objects can therefore be taken in three dimensions and such objects are known as three – dimensional objects.

Example of such objects includes the following:

- A box
- A cone
- A cylinder
- A pyramid

21.2 Some common term used:

21.2.1 *Lines*

A line is a set of points, which is straight and extends indefinitely in two directions, i.e.

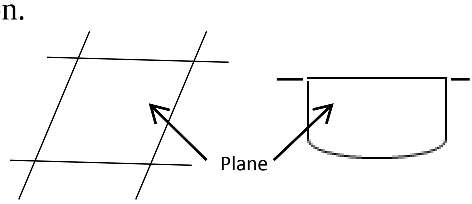
A line segment on the other hand is part of a line with two definite ends, i.e.

21.2.2 *Collinear points*

These are points lying on a single straight line. Non–collinear points on the other hand are any three or more points that do not lie on a straight line.

21.3 A plane

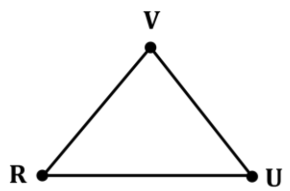
A plane is a set of points in a flat surface and extends indefinitely in all directions. However, when bounded by straight lines or curves it is called a region.



21.3.1 Determination of a plane

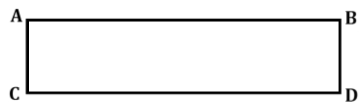
A plane is uniquely determined by:

a) Any three non – collinear points i.e.



The plane RUV is formed by points **R**, **U,** and V.

b) Two parallel lines, e.g.

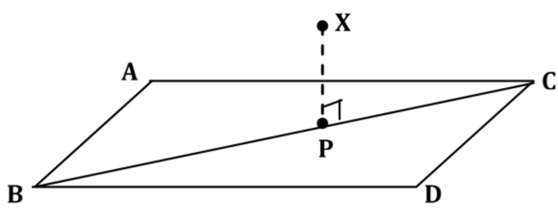


The plane ABCD is determined by the lines AB and CD.

21.4 Projection of the point and the line

21.4.1 Projection of the point onto the plane or line

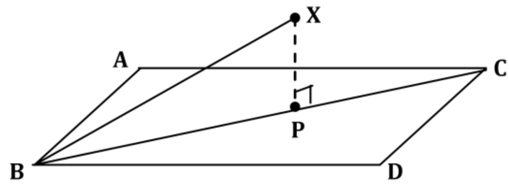
The projection of a point onto a plane or a line is the foot of the perpendicular from the point to the plane or line, i.e.



From the diagram above, P is the projection of point X onto the plane ABCD or to line AC.

21.4.2 Projection of a line onto the plane

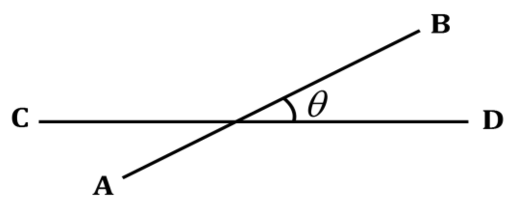
Consider the line AX below.



The projection of the line AX onto the plane ABCD is the line AP.

21.4.3 Angle between two lines

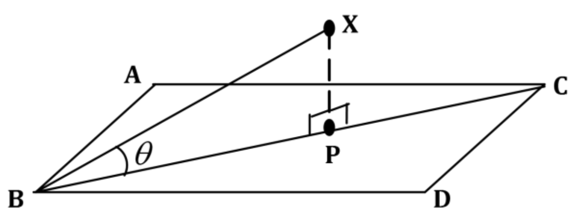
Angle between two lines is defined as the acute angle formed at their point of intersection. Consider lines CD and AB below.



The acute angle between these lines is θ.

21.4.4 Angle between a line and a plane

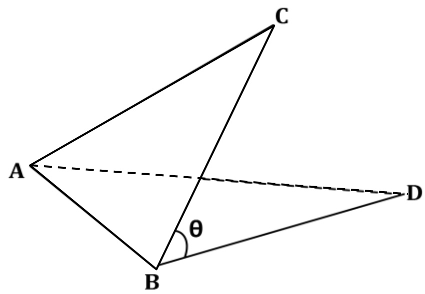
Angle between a line and a plane is defined as the angle between the line and its projection onto the plane. Consider the line AX and its projection AP onto the plane ABCD below.



The angle between the line AX and the plane ABCD is equal to θ.

21.4.5 Angle between two planes

The angle between two planes is the angle between any two lines, one in each plane, which meet on and at right angles to the line of intersection of the planes. Consider planes ABC and ABD intersecting at AB as shown below.



The angle between the planes ABC and ABD is the same as the angle between the line BC and BD, which is equal to θ.

21.5 Calculating distances and angles

In three-dimensional geometry, unknown lengths and angles can in most cases be determined by solving right –angled triangle. It is therefore advisable to sketch the triangle separately from the solid.

Example

A rectangular–based pyramid with vertex V is such that each of the edges VA, VB, VC, VD is 26cm long. The dimensions of the base are AB =CD =16cm and AD =BC =12cm.

Calculate:

a) The height of the pyramid

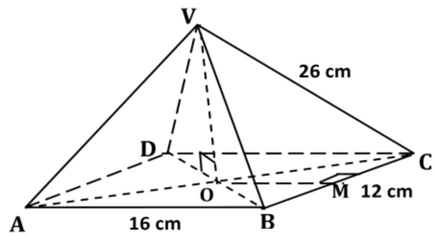
b) The angle between the edges AD and VC

c) The angle between the base and the face VBC

d) The angle between the base and slant edge.

Solution

a)



In triangle ABC:

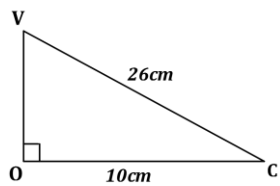
$AC^2 = AB^2 + BC^2$, *by Pythagoras theorm*

$= 16^2 + 12^2 = 400$

$\therefore AC = \sqrt{400} = 20cm$

$OC = \frac{1}{2}AC = \frac{20}{2} = 10cm$

Considering triangle VOC



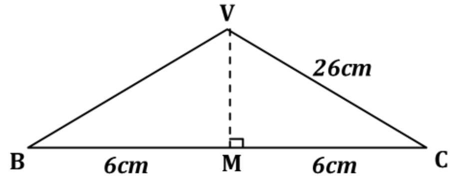
By Pythagoras theorem, the height VO is calculated as follows:

$VO^2 = VC^2 - OC^2 = 26^2 - 10^2 = 576$

$\therefore VO = \sqrt{576} = 24cm$

Therefore, the height VO of the pyramid is 24cm.

b) *AD and VC are skew lines (lines which are not parallel and do not meet). We therefore translate AD to BC to form the required angle VCB.*

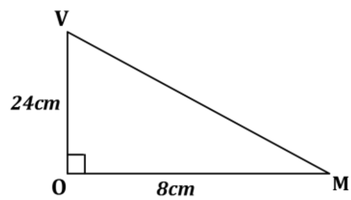


M is the midpoint of BC. From triangle VMC:

$$\cos < VCB = \frac{6}{26} = 0.2308$$
$$\therefore < VCB = \cos^{-1}(0.2308) \; = \; \underline{\underline{76.66^0}}$$

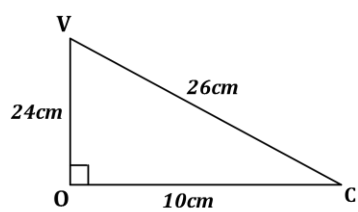
c) *BC is the line of intersection between the two planes and M is the midpoint of BC. VM and OM are lines in the plane, which are both perpendicular to BC. Thus angle VMO is the angle between the base and face VBC.*

Considering triangle VMO:



$$\tan < VMO = \frac{24}{8} = 3$$
$$\therefore < VMO = \tan^{-1}(3) \; = \; \underline{\underline{71.57^0}}$$

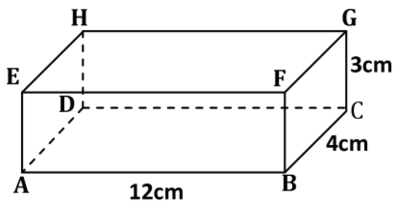
d) *Since VO is perpendicular to the base, VCO is one of the angles between the base and a slant edge. Considering triangle VCO:*



$$\tan < VCO = \frac{VO}{CO} = \frac{24}{10} = 2.4$$
$$\therefore < VCO = \tan^{-1}(2.4) \; = \; \underline{\underline{67.89^0}}$$

Example

ABCDEFGH is a cuboid with dimensions as shown in the figure below



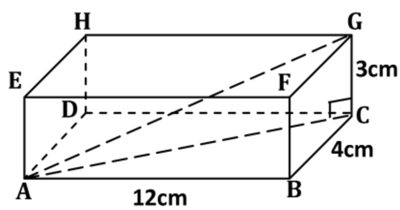
Calculate:

a) The length of AG

b) The angle that AG makes with plane BCGF

c) The shortest distance between line BF and plane ACG

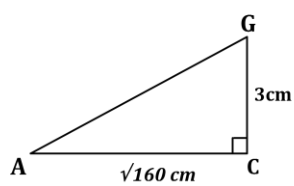
Solution



a) *The length AG can be calculated by considering triangle AGC. But we need the length AC first. This can be calculated from triangle ABC as follows:*

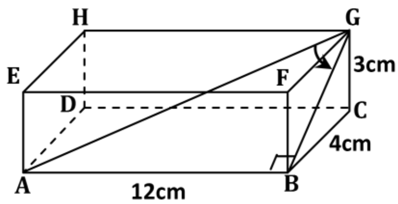
$$AC^2 = AB^2 + BC^2, \text{ by Pythagoras theorem}$$
$$= 12^2 + 4^2 = 160$$
$$\therefore AC = \sqrt{160} \text{ cm}$$

Now considering triangle AGC:



$$AG^2 = AC^2 + CG^2 = \left(\sqrt{160}\right)^2 + 3^2 = 169$$
$$\therefore AG = \sqrt{169} = \underline{\underline{13cm}}$$

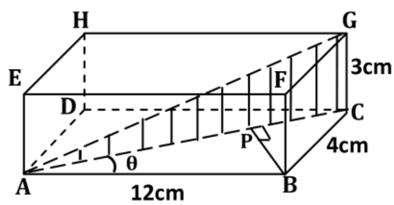
b) *The angle that AG makes with plane BCGF is <AGB since BG is the projection of AG onto plane BCGF.*



From triangle ABG:

$$\tan < ABG = \frac{AB}{GB} = \frac{12}{5} = 2.4$$
$$\therefore < ABG = \tan^{-1}(2.4) \;=\; \underline{\underline{67.38^0}}$$

c) *The shortest distance between a line and a plane is the distance between a point on the line and its projection onto the plane. For the above case, consider the diagram below:*



BP is the shortest distance between line BF and plane ACG.

$$\frac{PB}{AB} = \sin\theta$$
$$\Rightarrow PB = AB\sin\theta,\ \ AB = 12cm$$

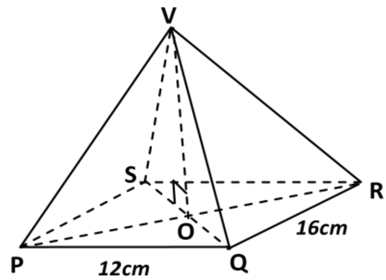
From triangle ABC:

$$\tan\theta = \frac{BC}{AB} = \frac{4}{12} = 1/3$$
$$\therefore \theta = \tan^{-1}\left(1/3\right) = \ 18.43^0$$

$$\therefore PB = 12 \times \sin 18.43^0 = \underline{\underline{3.79cm}}$$

21.6 Miscellaneous Exercise

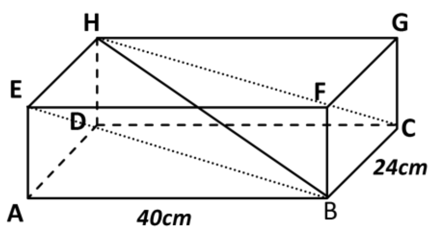
1. The figure below shows a right pyramid standing on a horizontal rectangular base PQRS. Given that PQ =12cm, QR = 16cm and V is 24cm vertically above the horizontal base PQRS



Fine:

i. The length of VQ
ii. The angle between VQ and the horizontal base
iii. The angle between the planes VPQ and VSR.

2. The diagram below shows a cuboid 40cm by 24cm by 18cm.



Calculate:

i. The length of the diagonal HB
ii. The angle between this diagonal and the base ABCD
iii. The angle between planes EBCH and ABCD

3. VEFGH is a right pyramid with a rectangular base EFGH and vertex V. O is the centre of the base and M is the point on OV such that $\bar{OM} = \frac{1}{3}\bar{OV}$. It is given that EF = 8cm, FG = 6cm,VE = VG = 15cm.

a) Find;

i. Length EO

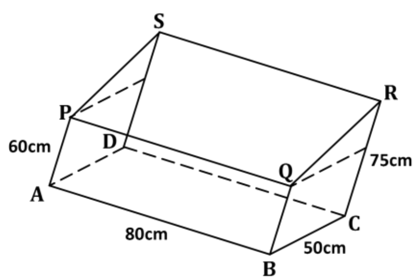
ii. The vertical height OV of the pyramid

b) Find the angle between the opposite slant faces;

i. VEH and VFG

ii. VEF and VHG

4. The figure below shows a cage in which base ABCD and roof PQRS are both rectangular. AP, BQ, CR, and DS are perpendicular to the base.



Calculate:

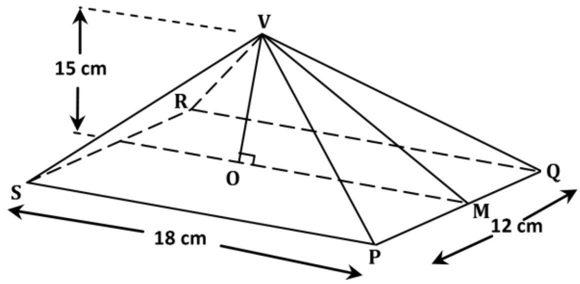
a) The length QR

b) The angle QRC

c) The angle between planes ABCD and PQRS

d) The inclination of PR to the horizontal.

5. The figure below shows a right pyramid on a rectangular base PQRS.



M is the midpoint of PQ. O is the centre of PQRS. Given that PQ = 12cm, QR = 18cm and VO = 15cm.

Calculate:

a) The length of VM and VQ

b) The angle between VP and the base

c) The angle between VPQ and the base.