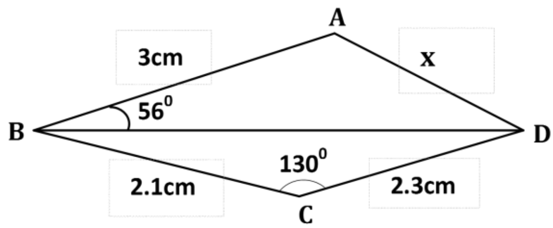
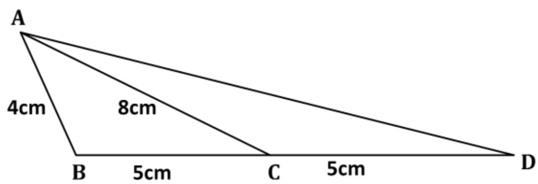
2. Given the diagram below



Use it to find x.

3. In the figure below, AB = 4cm, BC = 5cm, AC = 8cm and CD = 5cm

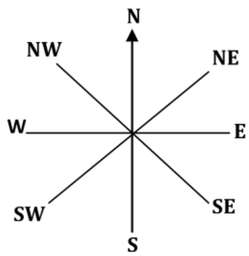


Find the length AD.

13 BEARING

13.1 Introduction:

Directions are described using north(N), south(S),east(E), and west(W) and north-east(NE), south-east(SE), south-west(SW) and north-west(NW).

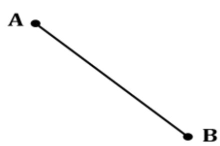


13.2 Ways of giving bearing:

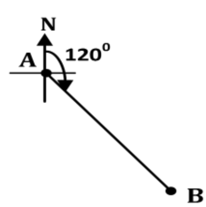
There are two ways of giving bearing. These include the following; true bearing and campus bearing.

13.2.1 True bearing

Here bearing is given as the amount of angle turned clockwise from facing true north. For instance, suppose that you are required to give the bearing of point **B** from **A** below.

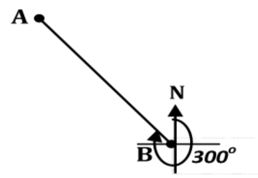


In this case, you will stand at A while facing north, and then you turn clockwise until you are facing point B directly. You then measure the angle you have turned. This gives you the bearing of B from A. Suppose the angle measured is 120^o as shown.



The bearing of point **B** from point **A** is therefore 120^{o}.

Similarly, the bearing of point A from point B can be obtained in the same manner.



The bearing of point A from point B is therefore 300^{o}

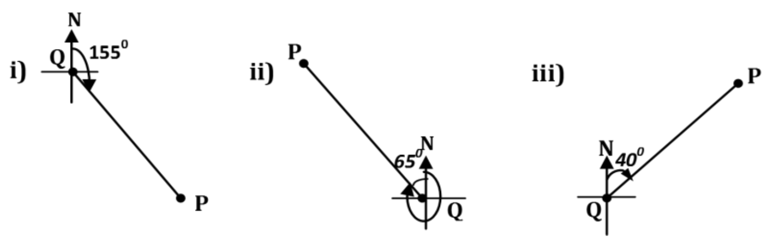
Note:

For precision, three-figure bearing is normally used. E.g. 060^{o}, 075^{o}, 090^{o}, 180^{o}, 270^{o}, etc.

Example

a) State the bearing of P from Q and

b) Also, state the bearing of Q from P in each of these diagrams.



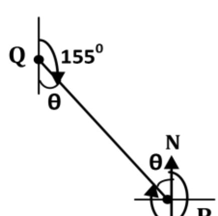
Solution

a) *i) The bearing of* P *from* Q = 155^0

ii) The bearing of P *from* Q = $360^0 - 65^0 = 295^0$

iii) The bearing of P *from* Q = 040^0

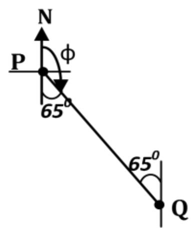
b) *i)*



$$\theta = 180^0 - 55^0$$
$$= 125^0$$

Therefore, the bearing of Q *from* P $= 360^0 - 125^0 = 235^0$

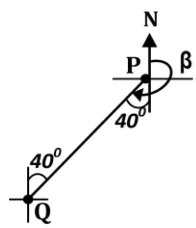
ii)



$$\phi = 180^0 - 65^0$$
$$= 115^0$$

The bearing of Q from P $= \phi = 115^0$

iii)

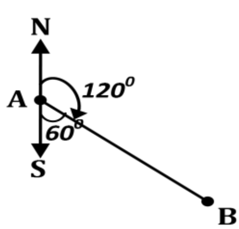


$$\beta = 180^0 + 40^0$$
$$= 220^0$$

Therefore the bearing of Q *from* P $= \beta = 220^0$

13.2.2 Compass bearing

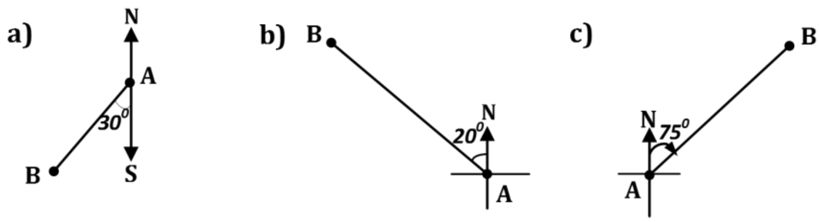
Bearing can be given by measuring the angle east or west from north or south. Consider the diagram below.



Therefore, the bearing of B from A is S60^o E. this bearing is known as compass bearing.

Example

By use of compass bearing, state the bearing of B from A from the following figures.



Solution

a) $S30^0W$

b) $N20^0W$

c) $N75^0E$

Example

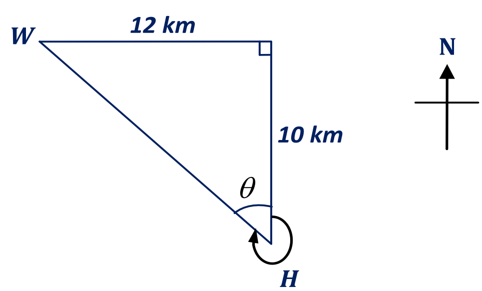
Mr. Okello walked 10km north from his home and then 12km west to the market.

a) What is the distance between Okello's house and the market?

b) What is the bearing of the market from Okello's house?

Solution

a) *First, sketch the diagram. Let H and M stand for Okello's house and Market respectively.*



This is a right-angled triangle. The distance between the market and Okello's house is ***WH***.

$$\therefore HW = \sqrt{10^2 + 12^2} = \underline{\underline{15.6\,km}}$$

b) *The bearing of M from H is equal to* $360^0 - \theta$

$But\ \theta = ?$

$From:\ \tan\theta = \frac{12}{10} = \frac{6}{5}$

$\Rightarrow\ \theta\ =\ \tan^{-1}\left(6/5\right)\ =\ \underline{\underline{50.2^0}}$

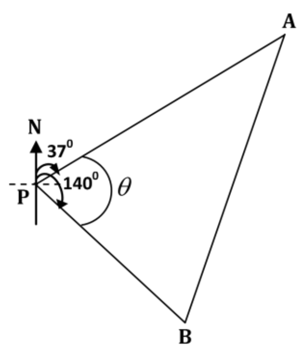
Therefore the bearing of M from H $= 360^0 - 50.2^0 = \underline{\underline{309.8^0}}$

Example

Two boats, **A** and **B** leave a port at 07:00h. Boat **A** travels at 25km/h on a bearing of 037^o, boat **B** travels at 15km/h on a bearing of 140^o. After 3 hours, how far is **A** from **B**?

Solution

Sketch:



$\theta = 140^0 - 37^0 = 103^0$

For boat A:

The distance ***A*** *has travelled in 3 hours is equal to PA*

$From: Distance\ =\ speed \times time,\quad Speed = 25km/h, and\ time = 3hrs$

$\therefore PA\ =\ 25 \times 3 = \underline{\underline{75km}}$

For boat B:

*The distance **B** has travelled in 3 hours is equal to PB*

$Speed = 15km/h, and\ time = 3hrs$

$\therefore PB = 15 \times 3$

$= \underline{\underline{45km}}$

*The distance of **A** from **B** after 3 hours is equal to AB. Using the cosine rule:*

$AB^2 = PB^2 + PA^2 - 2 \times PA \times PB \times \cos\theta$

$AB^2 = 45^2 + 75^2 - 2 \times 75 \times 45 \cos 103^0 = 9168.42$

$\therefore AB = \sqrt{9168.42} = \underline{\underline{95.75km}}$

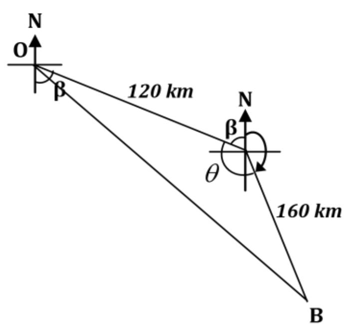
Example

An aeroplane flies 120km in the direction 113^0, then turns and flies 160km in the direction 156^o. Find its distance from the starting point.

Solution

Sketch

*Let **O** be the starting point and **B** the ending point.*



Required to find distance OB.

$\beta = 180^0 - 113^0 = 67^0$

$And\ \theta + 156^0 + \beta = 360^0$

$\therefore \theta = 360^0 - (156^0 + 67^0) = 137^0$

Using the cosine rule:

$OB^2 = 120^2 + 160^2 - 2\times120\times160\times\cos137^0$

$OB^2 = 68083.98$

$\therefore OB = \sqrt{68083.98} = \underline{\underline{260.9\ km}}$

Example

Two planes start from an airport at the same time. One plane flies west at 400km per hour while the other flies at 500km per hour on a bearing of 040^0. What is the distance between the two planes after 15minnutes?

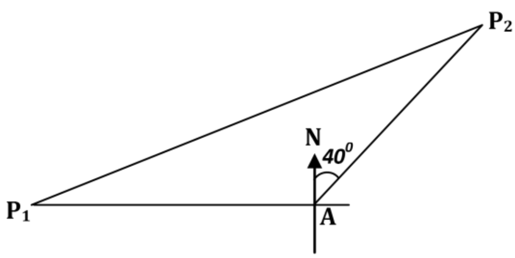
Solution

Let: **A** *be the air port*

P_1 *be position of plane to the west and* **U_1** *its speed.*

P_2 *be position of plane on a bearing of 040^o and* **U_2** *its speed.*

Sketch



Here the speed is given in km/h and time is given in minutes. This implies that, we have to convert 15minutes into hours

$60\ \text{minutes} = 1\ hour$

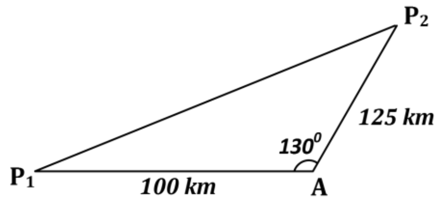
$1\text{minute} = 1hour$

$\therefore 15\,\text{minutes} = \frac{1}{60}\times15 = \frac{1}{4}hour.$ *So time* $T = 0.25hour$

$Distance\ AP_2 = U_2\times T = 500\times0.25 = 125km$

$Distance\ AP_1 = U_1\times T = 400\times0.25 = 100km$

The distance between the two planes after 15 minutes is P_1P_2



Using cosine rule:

$$P_1P_2^{\ 2} = 100^2 + 125^2 - 2 \times 100 \times 125 \times \cos 130^0$$

$$P_1P_2^{\ 2} = 41694.69$$

$$\therefore P_1P_2 = \sqrt{41694.69} = \underline{\underline{204.2\ km}}$$

Example

A boat sails from a port A in the direction 050^0 a distance of 5km to B. from B it sailed on a bearing of 330^0 a distance of 3km to C.

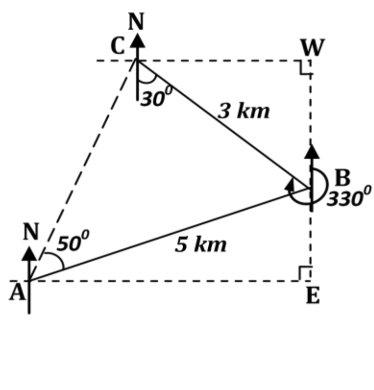
Draw a sketch of the boat's route and use it to calculate:

a) the eastward distance of B from A

b) the westward distance of C from B

Solution

Sketch



a) *Using triangle ABE, AE represents the eastward distance of B from A.*

$$< BAE = 40^0$$

$$Cos40^0 = \frac{AE}{5}, \quad \Rightarrow AE = 5\cos 40^0 = 3.83$$

$$\therefore Eastward\ distance\ of\ B\ from\ A = 3.83km$$

b) *Using triangle BCW, CW represents the Westward distance of C from W.*

$$< CBW = 30^0$$

$$Sin30^0 = \frac{CW}{3}, \quad \Rightarrow CW = 3\cos 40^0 = 1.5$$

$$\therefore Westward\ distance\ of\ C\ from\ B = 1.5km$$

13.2.3 Obtaining the bearing and distance of a point by use of scale drawing

To obtain the bearing or distance of one point from another, you can follow the steps below.

* First, make a rough sketch of the interpreted information.
* Choose a scale for yourself in case the scale to use is not given.
* Convert all the distances given in kilometers to centimeters using your scale.
* Draw the diagram of the sketch to scale using the distances in centimeters.
* Measure the distance of a point you have been asked to obtain using your ruler and then convert the distance you have obtain in centimeters to kilometers using your scale.
* Use your protractor to obtain the bearing of certain point.

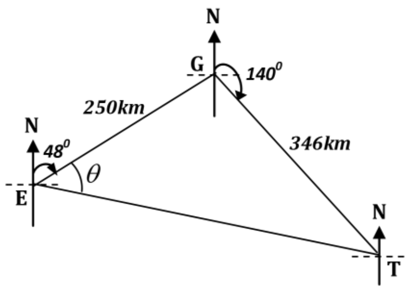
Example

An aeroplane flew from Entebbe to Gulu a distance of 250km and then to Tororo a distance of 346km. the bearing of Gulu from Entebbe was 048^o and the bearing of Tororo from Gulu was 140^o.

By scale drawing, and sing a scale of 1cm to represent 50km, find the distance and direction of Tororo from Entebbe.

Solution

Sketch



$\underline{Scale}:$

$1cm = 50\,km$

$\therefore 1km = \frac{1}{50}km$

$From\ Entebbe\ to\ Gulu:$

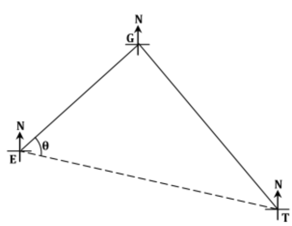
$250\,km = \frac{1}{50} \times 250 = 5.0\,cm$

$Gulu\ to\ Tororo:$

$346km = \frac{1}{50} \times 346 = 6.9\,cm$

<u>Steps to follow:</u>

- *Beginning E, measure 48^o clockwise using a protractor and draw a line along this direction from E. Measure a length of 5.0cm using a ruler along this line from E to G and do the same from G to T.*



- *Measure the length ET using a ruler.*

$$ET = 8.3cm$$
$$\Rightarrow ET = 8.3 \times 50 = 415km$$

Therefore, the distance from Tororo to Entebbe is 415km.

- *Also, measure angle* θ *using a protractor.*

$$\theta = 57^0$$
$$\therefore \textit{the bearing of Tororo from Entebbe} = 48^0 + 57^0 = 105^0$$

Example

Four towns XYZW are situated such that X is 20km in a direction N65^{o}E from Y, Z is 24km in the direction S48^{o}E from X while W is 27km in a direction S39^{o}W from Z.

a) By mean of scale drawing, find the respective locations of the towns.

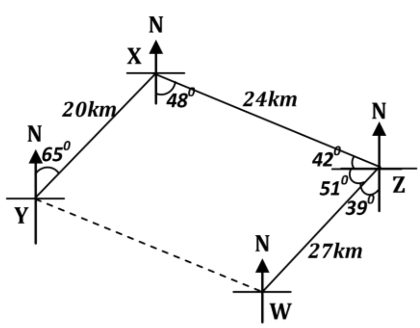
b) Using your drawing, find the distance and bearing of W from Y.

Solution

a) *In this case, the scale is not given; more so, the distances between these towns are not so large. This therefore implies that you have to choose the scale to use by yourself.*

Let 1cm = 5km

Sketch



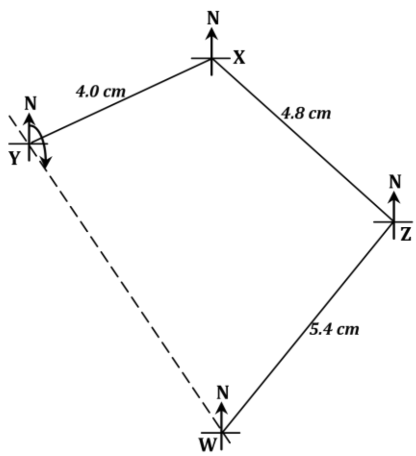
$Scale:$

$XY = 20\,km = 20/5 = 4\,cm$

$XZ = 24km = 24/5 = 4.8\,cm$

$ZW = 27km = 27/5 = 5.4cm$

Scale drawing



b) $The\ distance\ of\ W\ from\ Y = 6.8cm = 6.8 \times 5 = 34km$

$The\ bearing\ of\ W\ from\ Y = 147^0\ OR\ S33^0\ E$

Example

Tree points A, B, and C are 10km, 15km and 25km from an observation point O, on bearings 100^o, 240^0, 240^o, and 000^o from O respectively.

a) Find by scale drawing, the:

i. bearing of C from A

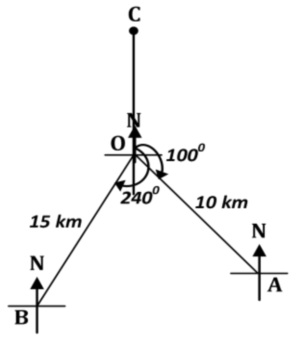
ii. bearing of C from B

iii. distances AB and BC.

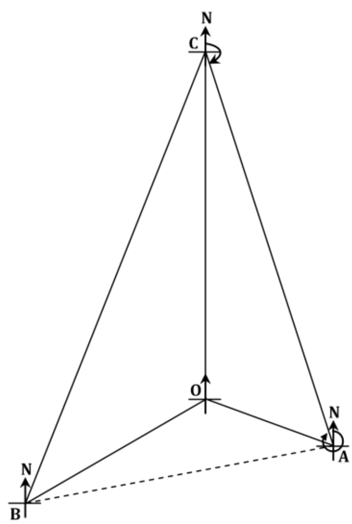
b) If a cyclist is to steadily ride his bicycle from O to C via B at a speed of 12.5km/h. determine how long he would take to travel to C.

Solution

a) *Sketch*



$$Scale:$$
$$Let\ 1cm = 3\ km$$
$$10\,km = {}^{10}\!\!/_{3} = 3.3\,cm$$
$$15km = {}^{15}\!\!/_{3} = 5.0\,cm$$
$$25km = {}^{25}\!\!/_{3} = 8.3cm$$



i. *The bearing of C from A is* 340^o

ii. *The bearing of C from B is* 022^o

iii. $Distance\ AB = 7.8cm = 7.8 \times 3 = 23.4km$

$Distance\ BC = 11.7cm = 11.7 \times 3 = 35.1km$

b) $Total\ distance = OB + BC = 15km + 35.1km = 50.1km$

$$Time\ taken = \frac{total\ distance}{speed}, \quad Speed = 12.5km$$

$$= \frac{50.1}{12.5} = 4.008$$

$$= \underline{\underline{4hrs}}$$

13.3 Miscellaneous exercise

1. A, B and C are points on the same level. Points B and C are 100km and 150km respectively from point A. the bearing of B from A is 225^o and that of C from A is 140^o.

   a) Represent this information on a sketch diagram.

   b) From the sketch, find:
      i. the distance of B from C.
      ii. the bearing of B from C

2. Four boats P, Q R, and S are anchored on a bay such that boat Q is 180meters on a bearing of 075^0 from P, boat R is 240 meters on a bearing of 165^o from Q, boat S is 185 meters to the south of P and due west of R.

   a) Draw a sketch diagram to show the positions of P, Q, R, and S.

   b) Without using a scale diagram, calculate:

      i. the distance PR to 3 significant figures.
      ii. the bearing of P from R

3. A helicopter flies 540km from station A to station B on bearing 060^o. From station B, it travels 465km to C on a bearing of 150^o. From C it heads for station D 360km away on a bearing 265^o.

   a) Draw to scale a diagram showing the route of the helicopter. (Use the scale: 1cm to represent 50km)

   b) From your diagram, determine the distance and bearing of station A from station D.

c) Determine how long it would take the helicopter travelling at a speed of 400km/h to travel direct from station A to station C.

4. In a sports field, four points A, B, C and D are such that B is due south of A and due west of D. AB = 10.8m, BD = 18.8m, CD = 16.6m, $<BAD = 60^0$, $<CDB = 40^0$ and $<BCD = 80^0$.

   A vertical pole erected at D has at D has a height of 4.8cm.

   a) Draw a sketch of the relative positions of the points on the sports field.

   b) Using a scale of 1cm to represent 2cm, draw an accurate diagram to show the relative position of the points and the pole and hence, find:

      i. distances BC and AD
      ii. bearing of B from C
      iii. angle of elevation of the top of the pole from B.

   c) If an athlete runs from point A through points B, C and D and back to A in 16 second, find the athlete's average speed.

5. Three points A, B, and c are on the same horizontal level and are such that B is 150km from A on a bearing of 060^o. The bearing of C from A is 125^o and the bearing of C from B is 60^o.

   a) By scale drawing using 1cm to represent 25km,find the distance of C from:
      i. A
      ii. B

   b) An aeroplane flies from A on a bearing of 340^0 at 300km/h. After 40minutes of flying; the pilot changes the course at point D and flies directly to C at the same speed. Include in your diagram in (a) above the route of the plane. Hence find:

      i. The time (in hours), the plane takes to travel from A to reach C.

      ii. The bearing of D from C

6. A helicopter flies from Moroto due south for 300km. It then flies on a bearing of 255^o for 350km. from there; it flies on a bearing of 020^o for 400km.
    a) Draw an accurate diagram showing the journey of the helicopter using a scale of 1cm to represent 50km.
    b) From your diagram, find the distance and bearing of Moroto from the final position of the helicopter.
    c) Given that the helicopter flies at a steady speed of 200km/h, find how long the whole journey took.

14 MATRICES OF TRANSFORMATIONS

14.1 Definition:

Transformation means a change of position or size or shape or all.

14.2 Common terms used:

Below are some of the most frequently used terms under transformation:

1. Object

   This is the initial figure (shape) formed before transformation has taken place.

2. Image

   This is the figure (shape) obtained when an object has undergone transformation.

3. Congruent (identical)

   This is when two figures have the same size and shape. Therefore, if the shape and size of an object is the same as that of its image, we then say that the object and the image are congruent or identical.

4. Invariant

   This term is used to describe a situation when there is no change in position, size, and shape after transformation. Therefore, if the position, size, and shape of an object are not changed when it has been subjected to transformation, then the position, size, and shape of the image is the same as that of the object. In this case, we say that the object and the image are invariant.

5. Mirror line

   This is a line of symmetry from where reflection of object takes place.

6. Scale factor

   This is a scalar quantity which when operated on an object, can either increase or decrease its size.

Ways by which an object can be transformed

An object can undergo transformation by the following ways:

- Translation
- Transformation by matrix multiplication
- Reflection
- Rotation
- Enlargement
- Combined transformation

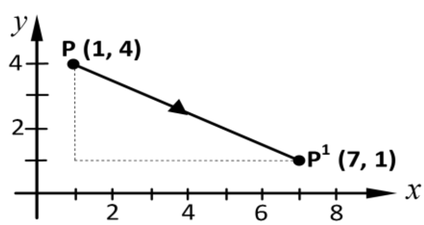
14.3 Translation

14.3.1 Definition:

Translation is the displacement of an object in a specified direction without turning. In other words, it is a movement, which has length and direction. It is described using coordinates or a column matrix.

Example

Suppose point P (1, 4) has been displaced (translated) to its image point P^1 (7, 1). To describe the translation, we need to compare the coordinates of P and P^1.



NB:

Movement to the right and movement upwards are defined as positive while movement to the left and movement downwards are defined as negative.

Now let us consider the displacement of P along the x – and y – axis;

Along the x –axis, P has moved 7 – 1 = 6 spaces (units)

Along the y –axis, P has moved 1 – 4 = -3 spaces (units).

This can shortly be written as $\begin{pmatrix} 6 \\ -3 \end{pmatrix}$ which is a 2×1 matrix (column matrix) and it is known as matrix of translation denoted by T.

$$P^1 - P = T$$

$$I.e. \quad \begin{pmatrix} 7 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$$

Generally therefore, the Image point (**I**), Object point (**O**) and the matrix of Translation (**T**) are related by the expression below.

$$\text{Image point} = \textit{Translation matrix} \quad + \quad \textit{Object point}$$

$$\boxed{I = T + O}$$

This formula can be used to obtain image point given the object point and matrix of translation.

Example

Given translation $T = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$. Find the image of triangle ABC with vertices A (0, 2), B (-3, 4) and C (2, 6) under **T**.

Solution

Let the image of ABC be $A^1B^1C^1$

$$\textit{From}: \text{Image} = \textit{Translation} + \textit{Object}$$

$$A^1 = \begin{pmatrix} -2 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \end{pmatrix}$$

$$B^1 = \begin{pmatrix} -2 \\ 5 \end{pmatrix} + \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 9 \end{pmatrix}$$

$$C^1 = \begin{pmatrix} -2 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 11 \end{pmatrix}$$

$$\therefore \; A(0, 2) \longrightarrow A^1(-2, 7)$$

$$B(-3, 4) \longrightarrow B^1(-5, 9)$$

$$C(2, 6) \longrightarrow C^1(0, 11)$$

Example

Given two matrices of translations as

$$T = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \; and \; K = \begin{pmatrix} -4 \\ 5 \end{pmatrix}. \; A = (0, 0), \; B = (0, 3) \; and \; C = (3, 3)$$

Find the image of triangle ABC under;

a) **T**

b) 2**T**

c) **K**

d) **K + T**

Solution

a) *Let the image of ABC be* $A^1B^1C^1$

$$From : I = T + O, T = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$A^1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$B^1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

$$C^1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

$$\therefore A(0,0) \longrightarrow A^1(2,3)$$
$$B(0,3) \longrightarrow B^1(2,6)$$
$$C(3,3) \longrightarrow C^1(5,6)$$

b)

$$2T = 2\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$A^1 = \begin{pmatrix} 4 \\ 6 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$B^1 = \begin{pmatrix} 4 \\ 6 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$$

$$C^1 = \begin{pmatrix} 4 \\ 6 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \end{pmatrix}$$

$$\therefore\ A(0, 0) \longrightarrow A^1(4, 6)$$
$$B(0, 3) \longrightarrow B^1(4, 9)$$
$$C(3, 3) \longrightarrow C^1(7, 9)$$

c)

$$k = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$$

$$A^1 = \begin{pmatrix} -4 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$$

$$B^1 = \begin{pmatrix} -4 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \end{pmatrix}$$

$$C^1 = \begin{pmatrix} -4 \\ 5 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \end{pmatrix}$$

$$\therefore\ A(0,0) \longrightarrow A^1(-4,5)$$
$$B(0,3) \longrightarrow B^1(-4,8)$$
$$C(3,3) \longrightarrow C^1(-1,8)$$

d)

$$K + T = \begin{pmatrix} -4 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$$

$$A^1 = \begin{pmatrix} -2 \\ 8 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$$

$$B^1 = \begin{pmatrix} -2 \\ 8 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 11 \end{pmatrix}$$

$$C^1 = \begin{pmatrix} -2 \\ 8 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 11 \end{pmatrix}$$

$$\therefore\ A(0, 0) \longrightarrow A^1(-2, 8)$$
$$B(0, 3) \longrightarrow B^1(-2, 11)$$
$$C(3, 3) \longrightarrow C^1(1, 11)$$

Example

The image of PQR is P^1 (0, 0), Q^1 (-2, 4) and R^1 (3, 4). Find the coordinates of PQR and hence sketch PQR and P^1 Q^1 R^1 on the same diagram. Take the translation vectors as $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

Solution

$$From: I = T + O, \Rightarrow O = I - T, \; and \; T = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$\Rightarrow P = P^1 - T$$

$$\therefore \; P = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

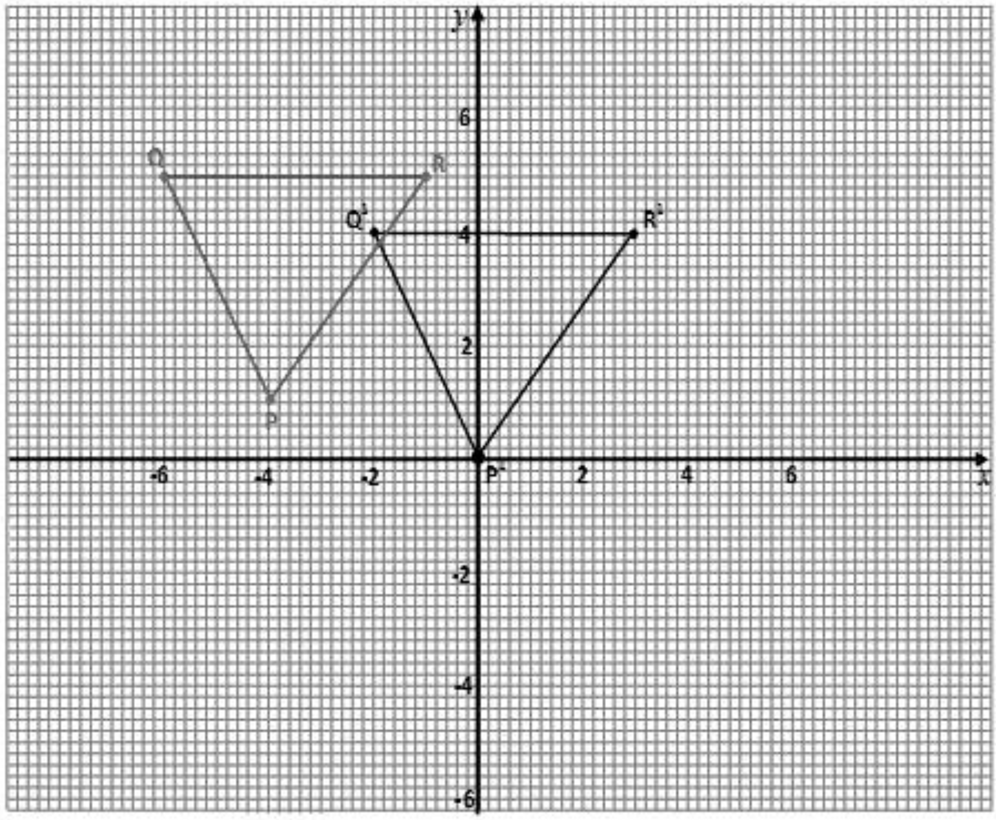
$$\Rightarrow Q = Q^1 - T, \quad Q^1 = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$\therefore \; Q = \begin{pmatrix} -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -6 \\ 5 \end{pmatrix}$$

$$\Rightarrow R = R^1 - T, \quad R^1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\therefore \; R = \begin{pmatrix} 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

$$\therefore P(-4,1) \longrightarrow P^1(0,0)$$
$$Q(-6,5) \longrightarrow Q^1(-2,4)$$
$$R(-1,5) \longrightarrow R^1(3,4)$$



14.4 Matrix of Transformation

Consider point P (x, y). Its position vector is $\begin{pmatrix} x \\ y \end{pmatrix}$. Let it be pre-multiplied i.e. multiplied from the left by the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, i.e

$$MP = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

We say that the matrix **M** has transformed the point P (x, y) to a new point P^1 (x^1, y^1).

$$\Rightarrow \quad x^1 = ax + by$$
$$y^1 = cx + dy$$

The matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is known as the matrix of transformation.

Generally therefore given the object point (P) and the matrix of transformation (M), the image point (P^1) is calculated from the relation given below.

$$\text{Image point} = \text{Martrix of transformation} \times \text{Object point}$$

$$\boxed{P^1 = MP}$$

NB:

- In order to obtain the image point, the object point must be multiplied from the left by the matrix of transformation.
- If the matrix of transformation M is an identity matrix, i.e. $M = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then the new point is left unchanged i.e. invariant.

Example

The point P (4, 5) is transformed by matrix $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Find the image of P.

Solution

Let the image of P be P^1

$$P^1 = MP$$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 4+10 \\ 12+20 \end{pmatrix} = \begin{pmatrix} 14 \\ 32 \end{pmatrix}$$

$$\therefore P^1 = \begin{pmatrix} 14 \\ 32 \end{pmatrix}$$

$$\Rightarrow P\ (4, 5) \rightarrow P\ (14, 32)$$

Example

a) P = (6, 12) is the image of the point K under the transformation $T = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$

Find the coordinates of K.

b) Find the matrix of transformation which transformed W (1, 1) onto W^1 (1, 1) and Y (3, 2) onto Y^1 (4, 3).

Solution

a) $P = TK$

$Let\ K = \begin{pmatrix} a & b \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 6 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a+4b \\ a+3b \end{pmatrix}$$

$\therefore\ 6 = 2a + 4b$....................................(1)

$12 = a + 3b$.......................................(2)

$Eqn(2) - eqn(1)$

$a + 3b = 12$

$^{-}\underline{a + 2b = 3}$

$b = 9\ and\ a = 12 - 3b\ \Rightarrow a = 12 - 27 = -15$

$\therefore\ \underline{\underline{K = (-15, 9)}}$

b) $Let\ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be the matrix that transformed W and Y to W^1 and Y^1 respectively.*

$W^1 = MW,\quad W^1(1,1),\quad W(1,1)$

$$\Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix}$$

$\therefore\ 1 = a + b$......................................(1)

$1 = c + d$.......................................(2)

$Also \quad Y^1 = MY, \quad Y^1(4, 3), \quad Y(3, 2)$

$$\Rightarrow \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3a+2b \\ 3c+2d \end{pmatrix}$$

$\therefore \; 4 = 3a + 2b$......................................(3)

$3 = 3c + 2d$.......................................(4)

Solving eqn(1) and eqn(3)

$2(a + b = 1)$

$\underline{3a + 2b = 4}$

$2a + 2b = 2$

$^{-}\underline{3a + 2b = 4}$

$-a = -2 \Rightarrow a = 2$

$b = 1 - a \Rightarrow b = -1$

Also solving eqn(2) and eqn(4)

$2(c + d = 1)$

$\underline{3c + 2d = 3}$

$2c + 2d = 2$

$^{-}\underline{3c + 2d = 3}$

$-c = -1 \Rightarrow c = 1$

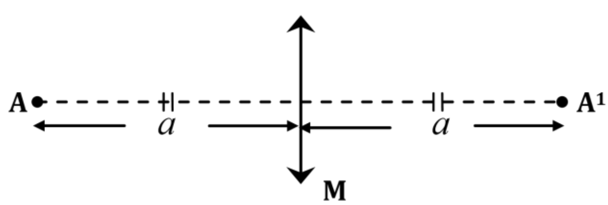
$d = 1 - c \Rightarrow d = 2$

$$\therefore M = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

14.5 Reflection:

For an object to be reflected, there must be a mirror line. A mirror line as stated earlier is a line from where reflection of object takes place.

If point **A** is to reflected along the mirror line M, then its image $\mathbf{A^1}$ will be formed to the left of the mirror line and **A** and $\mathbf{A^1}$ are of equal distance from the mirror line M.



Below are the general properties of reflection:

- Points, which are on the mirror line, are their own images i.e. they are invariant.

- The distance of the image from the mirror line is equal to the distance of the object from the mirror line.

- The mirror line M bisects the angle between the object and its image.

- In reflection, an object and its image are oppositely congruent i.e. lengths and angle remain same but direction is reversed.

a) Finding the image of an object by scale drawing

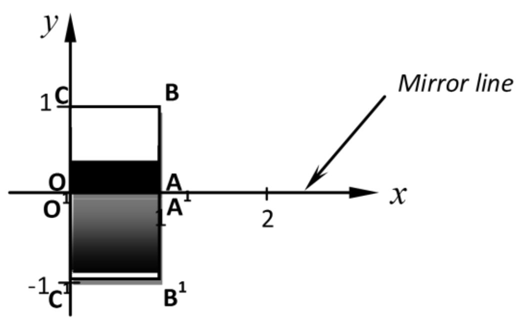
Here we shall use the properties of reflection to find the image of an object.

Example

Find the image of square OABC with O (0, 0), A (1, 0), B (1, 1) and C (0, 1) after reflection along the x –axis.

Solution

X-axis is the mirror line



- *Points O (0, 0) and A (1, 0) which are on the mirror line are their own images i.e. they are invariant.*
- *Points B (1, 1) and C (0, 1) which are not on the mirror line are displaced 1 unit below the x –axis (mirror line).*

$$\Rightarrow O^1 = (0, 0) \quad A^1 = (1, 0)$$
$$B^1 = (1, 1) \quad C^1 = (0, -1)$$
$$OABC \xrightarrow{x-axis} O^1A^1B^1C^1$$

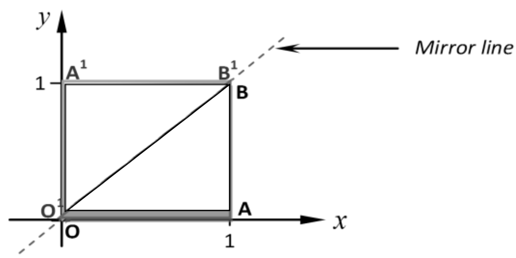
Example

a) Find the image of triangle OAB with O (0, 0), A (1, 0) and B (1, 1) after reflection along the line $y = x$.

b) Find the image A B C after a reflection along the line $y = -x$

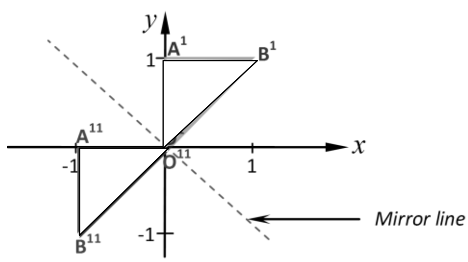
Solution

a)



$A^1 = (0, 1)$
$B^1 = (1, 1)$
$O^1 = (0, 0)$

b)



$A^{11} = (-1, 0)$
$B^{11} = (-1, -1)$
$O^{11} = (0, 0)$

Example

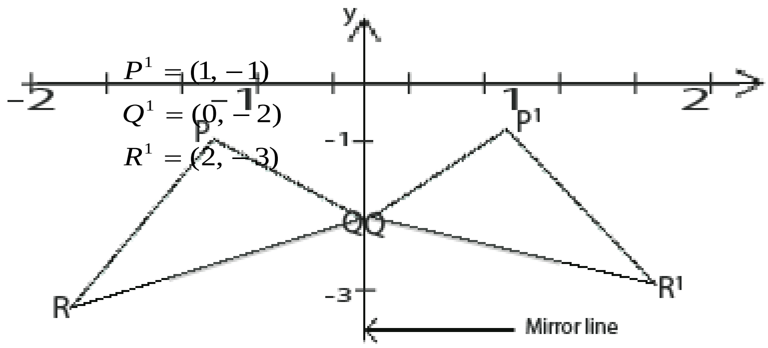
Find the image of triangle PQR with P (-1, -1), Q (0, -2) and R (-2, -3) after a reflection along the y –axis.

Solution

$P^1 = (1, -1)$

$Q^1 = (0, -2)$

$R^1 = (2, -3)$



General case:

If point P (x, y) has been reflected along the following mirror lines;

i. $y - axis$

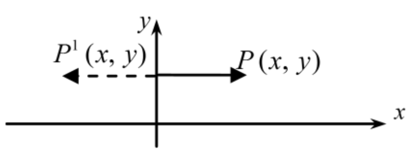
ii. $x - axis$

iii. $y = x$

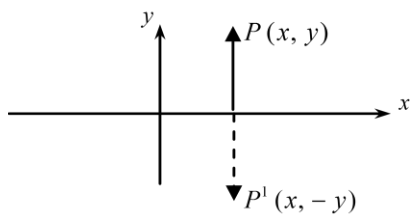
iv. $y = -x$

Then, point P (x, y) would have its image as follows:

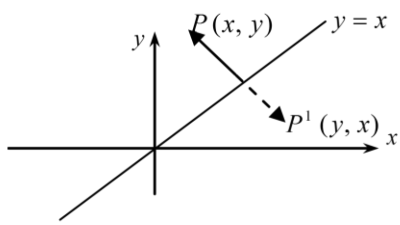
i. $P\,(x,\, y) \xrightarrow{y-axis} P^1(-x,\, y)$



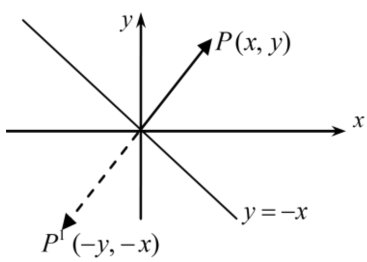
ii. $P\,(x,\, y) \xrightarrow{x-axis} P^1(x, -y)$



iii. $P\ (x,\ y) \xrightarrow{y=x} P^1(y,\ x)$



iv. $P\ (x,\ y) \xrightarrow{y=-x} P^1(-y, -x)$

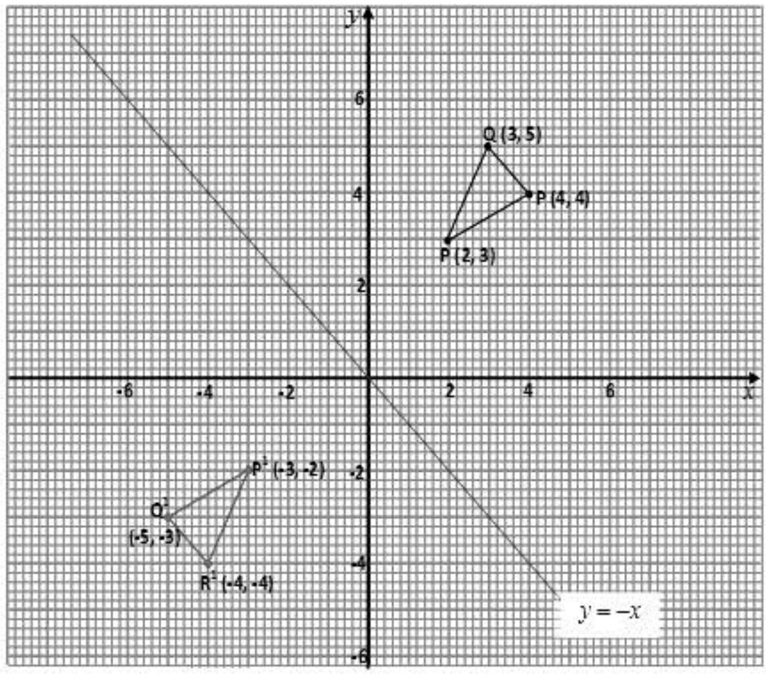


Example

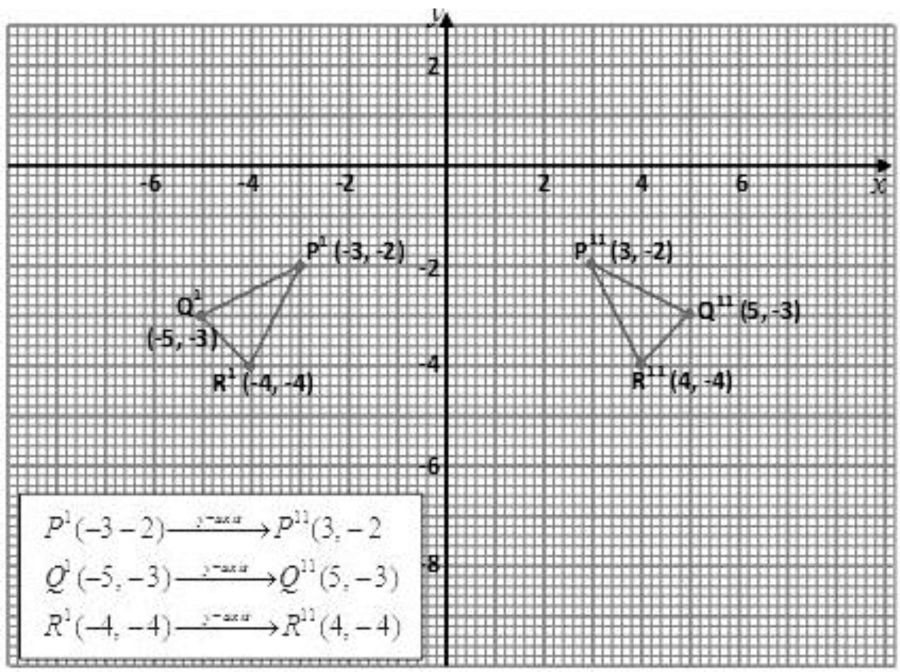
a. Find the image of triangle PQR with P (2, 3), Q (3, 5) and R (4, 4) after a reflection along the line $y = -x$

b. Also find the image of $P^1\ Q^1\ R^1$ when reflected along the y –axis.

a) **Graphical solution**



b) **Graphical solution**



b) Finding the image of an object by use of calculation

Here we need to know the matrix of reflection along the given mirror line. Thereafter, we multiply the object from the left by the matrix of reflection to obtain the image.

14.6 Matrix of reflection along x –axis

Consider two points P (1, 0) and Q (0, 1) being reflected along x –axis.

$$Fom:\quad P(x, y) \xrightarrow{x-axis} P^1(x, -y)$$
$$\Rightarrow \quad P(1, 0) \xrightarrow{x-axis} P^1(1, 0)$$
$$Q(0, 1) \xrightarrow{x-axis} Q^1\ (0, -1)$$

$$Let\ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} be\ the\ matrix\ of\ reflection$$

$$\therefore P^1 = MP\ also\quad Q^1 = MQ$$

$$\begin{array}{ccccc} & P^1 \quad Q^1 & & & P \quad Q \\ \Rightarrow & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & = & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \therefore \begin{matrix} a = 1 & b = 0 \\ c = 0 & d = -1 \end{matrix}$$

$$\therefore M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Therefore the matrix of reflection along the x –axis is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Activity:

Show that the matrix of reflection along;

i. $y - axis\ is \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

ii. $y = x\ is \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

iii. $y = -x\ is \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Example

Find the image of OABC with O (0, 0), A (1, 0), B (1, 1) and C (0, 1) being reflected along the x –axis.

Solution

Method 1:

The matrix of reflection along the x –axis is $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\Rightarrow O^1A^1B^1C^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{array}{c} \begin{array}{cccc} O & A & B & C \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} O^1 & A^1 & B^1 & C^1 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix} \end{array}$$

$\therefore O^1 = (0, 0)$

$A^1 = (1, 0)$

$B^1 = (1, -1)$

$C^1 = (0, -1)$

Method 2:

$$From: \quad P\,(x, y) \xrightarrow{x-axis} P^1(x, -y)$$
$$O(0, 0) \xrightarrow{x-axis} O^1\,(0, 0)$$
$$A(1, 0) \xrightarrow{x-axis} A^1\,(1, 0)$$
$$B(1, 1) \xrightarrow{x-axis} B^1\,(1, -1)$$
$$C\,(0, 1) \xrightarrow{x-axis} C^1\,(0, -1)$$

Example

a) Find A and B the images of A and B respectively under the reflection in the x –axis with A (1, 3) and B (3, 7).

b) Find:

i) The equation of AB

ii) The equation of A^1 B^1

Solution

a) *The matrix of reflection along x –axis* $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$A^1 = MA \text{ and } B^1 = MB. \quad A = (1, 3), B = (3, 7)$

$A^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$\therefore A(1, 3) \longrightarrow A^1\,(1, -3)$

$B^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$

$\therefore B(3, 7) \longrightarrow B^1(3, -7)$

b) i) ***For AB:***

$A = (1, 3), B = (3, 7)$

$\text{Gradient of } AB = \frac{7-3}{3-1} = \frac{4}{2} = 2$

$\text{From } y = mx + c, m = 2 \text{ and using point } A(1, 3), x = 1, y = 3$

$\Rightarrow \quad 3 = 2(1) + c \quad \therefore c = 1$

$\therefore \underline{\underline{y = 2x + 1}}$

ii) ***For A¹ B¹:***

$A^1 = (1, -3), B^1 = (3, -7)$

$\text{Gradient of } A^1B^1 = \frac{-7-^-3}{3-1} = \frac{-4}{2} = -2$

$\text{From } y = mx + c, m = -2 \text{ and using point } A(1, -3), x = 1, y = -3$

$\Rightarrow \quad -3 = -2(1) + c \quad \therefore c = -1$

$\therefore \underline{\underline{y + 2x + 1 = 0}}$

14.7 Rotation

This involves change in position of points when they are turned about a fixed point known as centre of rotation. Centre of rotation is a single fixed point under rotation. Nevertheless, every other point under rotation moves along an arc of a circle with this centre.

When a point changes when it is turned about the centre of rotation, the line from the point through the centre of rotation turns through an angle known as angle of rotation. Angle of rotation therefore is the angle through which a given point has been shifted from its initial position when turned about the centre of rotation.

14.7.1 General properties of rotation:

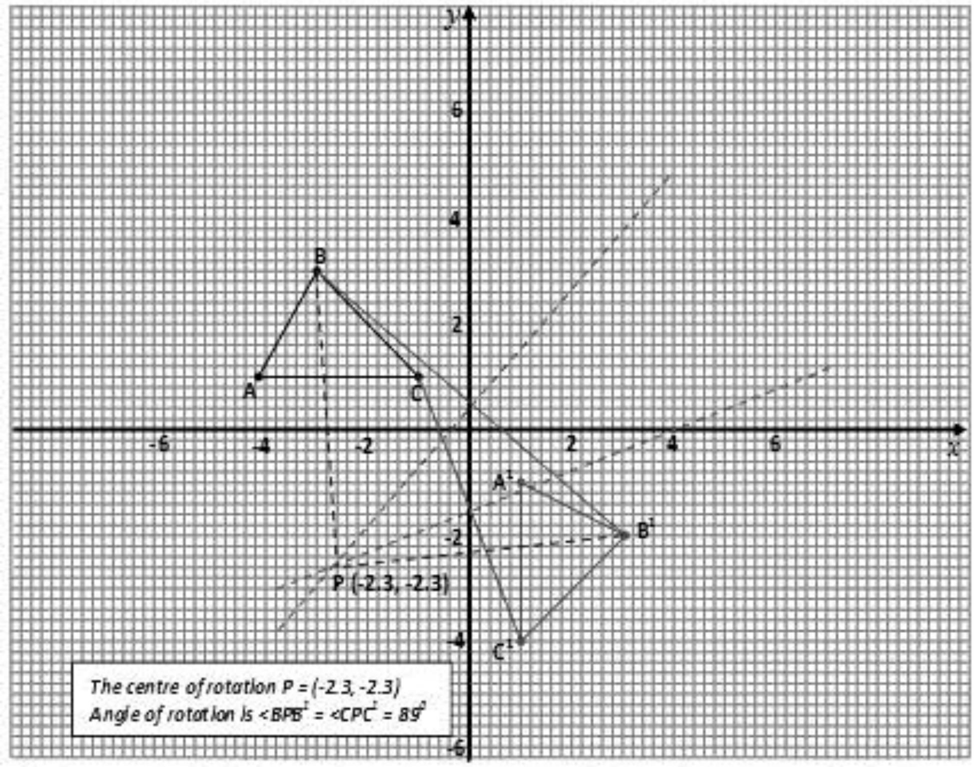
- The image is directly congruent to the object
- The distance of an image point from the centre of rotation is equal to the distance of the corresponding object point from the centre of rotation.
- Each line on the object through the centre of rotation turns through an angle equal to the angle of rotation.
- The centre of rotation is the intersection of the perpendicular bisectors of any two lines from the object to the corresponding image. For instance, given triangle ABC and its image A^1 B^1 C^1, the centre of rotation is the intersection of the perpendicular bisectors of any two of the line segments AA^1, BB^1, and CC^1.

a) *Obtaining the centre and angle of rotation*

When given an object and its image, the centre and angle of rotation can be obtained. The following example will illustrate this concept. Consider triangle ABC with vertices A (-4, 1), B (-3, 3) and C (-1, 1) being rotated on triangle $A^1B^1C^1$ with vertices A^1 (1, -1), B^1 (3, -2) and C^1 (1, -4). Determine the centre and angle of rotation.

In finding the centre and angle of rotation, the following steps should be followed:

- Plot the image $A^1B^1C^1$ and the object ABC on the same graph paper.
- Join BB^1 and then construct its perpendicular bisector.
- Join CC^1 and also construct its perpendicular bisector.
- The point P where the perpendicular bisectors meet gives the centre of rotation.
- The angle of rotation is $<BPB^1$ or $<CPC^1$. It is measured using a protractor.



b) *Finding the image of an object by scale drawing given the angle and centre of rotation*

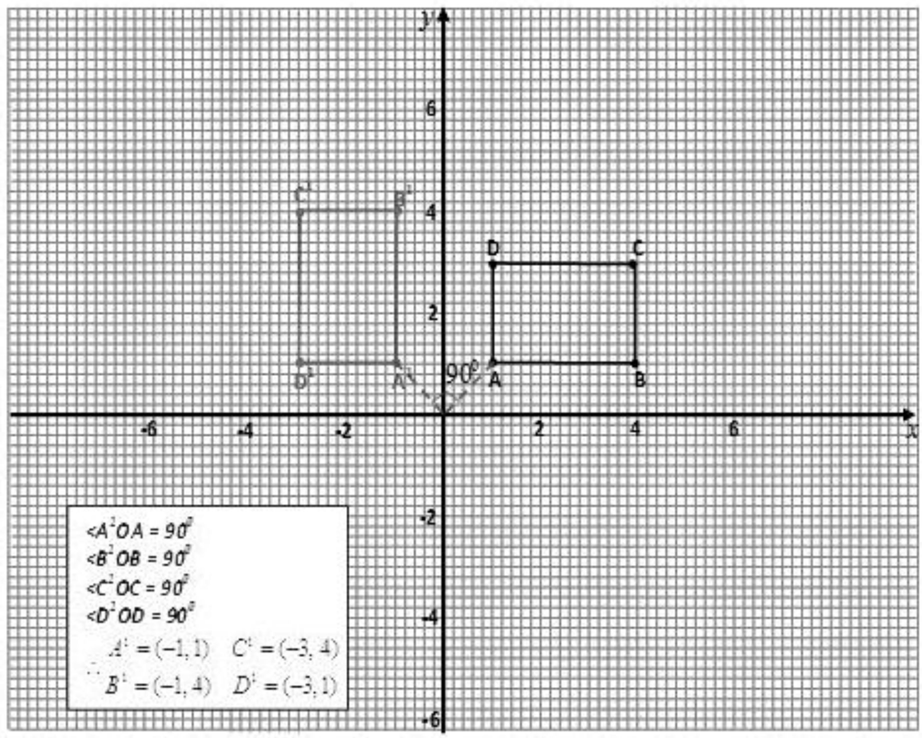
We can find the image of an object if we know the object point, angle of and centre of rotation.

NB:

- Rotation is defined as positive when it is anti-clockwise and negative when it is clockwise.
- A positive rotation through an angle θ is the same as negative rotation of through an angle of (360^o – θ) about the same centre.

Example

Use graph paper to obtain the image of rectangle ABCD with vertices A (1, 1), B (4, 1), C (4, 3) and D (1, 3) when rotated through 90^o anti clockwise with O (0, 0) as the centre of rotation.



c) *Finding the image of an object by calculation*

Here we need to know the matrix of rotation. The object is then multiplied from the left by the matrix of rotation and the result obtained gives the image point. That is to say, if M is the matrix of rotation and point P is the object point, then the image point P^1 is calculate from the expression below.

$$P^1 = MP$$

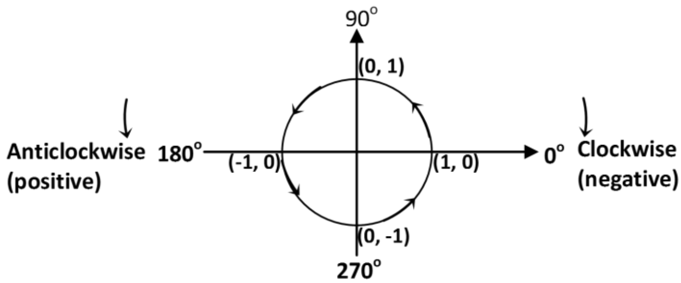
Generally, the matrix of rotation M is given the expression below:

$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Where θ is the angle of rotation

14.7.2 Common terms associated with rotation

Consider a unit circle i.e. a circle of radius 1 and centre (0, 0) as shown below.



Clockwise rotation is negative and anticlockwise rotation is positive. Below are some common terms associated with rotation.

1 *Quarter turn:*
This is the same as turning through 90^0 anticlockwise. I.e. quarter turn = 90^0.

2 *Half turn:*
This is the same as turning through180^o anticlockwise. I.e. Half turn = 180^o.

3 *Three quarter turn:*
This is the same as turning through 270^o anticlockwise. I.e. three –quarter turn = 270^o.

4 *Negative quarter turn:*
This is the same as positive three –quarter turn. I.e. negative–quarter turn = -90^o.

Example

Find the matrix of rotation through:

a) Quarter turn

b) Half turn

c) Three quarter turn

d) Negative quarter turn

e) 30^o

Solution

$$Matrices\ of\ rotation\ M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

a) *Quarter turn: θ =90°*

$$\therefore M = \begin{pmatrix} \cos 90^0 & -\sin 90^0 \\ \sin 90^0 & \cos 90^0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

b) *Half turn: θ =180°*

$$\therefore M = \begin{pmatrix} \cos 180^0 & -\sin 180^0 \\ \sin 180^0 & \cos 180^0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

c) *Three–quarter turn = negative quarter turn: θ = 270°*

$$\therefore M = \begin{pmatrix} \cos 270^0 & -\sin 270^0 \\ \sin 270^0 & \cos 270^0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

d) *Negative Quarter turn: θ = -90°*

$$\therefore M = \begin{pmatrix} \cos -90^0 & -\sin -90^0 \\ \sin -90^0 & \cos -90^0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

e) *For 30°:*

$$\therefore M = \begin{pmatrix} \cos 30^0 & -\sin 30^0 \\ \sin 30^0 & \cos 30^0 \end{pmatrix},\quad but\ \sin 30^0 = \tfrac{1}{2} \quad and\ \cos 30 = \tfrac{\sqrt{3}}{2}$$

$$= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{pmatrix}$$

Example

Find the image of A (3, 4) after a rotation through an angle of 30°.

Solution

The matrix of rotation for 30° is

$$M = \begin{pmatrix} \cos 30^0 & -\sin 30^0 \\ \sin 30^0 & \cos 30^0 \end{pmatrix} = \begin{pmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{pmatrix}$$

$$A^1 = MA = \begin{pmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0.598 \\ 4.964 \end{pmatrix}$$

$$\therefore A^1 = (0.598, 4.964)$$