NUMERICAL METHODS

By, Bossa

Location of real roots

The range where the root of an equation lie can be located using the following methods

(i) sign change
(ii) Graphical method

(a) Sign change method

Example 1

Show that equation $x^3 + 6x^2 + 9x + 2 = 0$ has a root between -1 and 0

Solution

$f(x) = x^3 + 6x^2 + 9x + 2$

$f(-1) = (1)^3 + 6(1)^2 + 9(-1) + 2 = -14$

$f(0) = (0)^3 + 6(0)^2 + 9(0) + 2 = 2$

Since there is a sign change the root lies between 0 and -1.

Example 2

Show that the equation $e^{2x}\sin x - 1 = 0$ has a root between 0 and 1

Solution

Note that in trigonometric function the calculator must be in radian mode

f(x) =$e^{2x}\sin x - 1$

f(0) =$e^{2(0)}\sin 0 - 1$ = -1

f(1) =$e^{2}\sin 1 - 1$ = 5.2177

Since there is a sign change the root lies between 0 and 1.

(b) Using graphical method

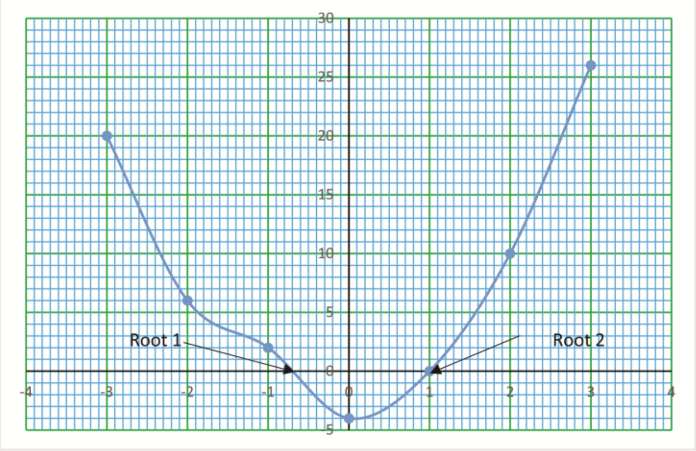
One or two graph(s) can be drawn to locate the root.

(i) Single graph method
When one graph is drawn then the root lies between the two points where the curve crosses the xaxis.

Example 3

Using a suitable graph locate the interval over which the root of the equation $3x^2 + x - 4 = 0$ lie.

x	-3	-2	-1	0	1	2	3
f(x)	20	6	-2	-4	0	10	26

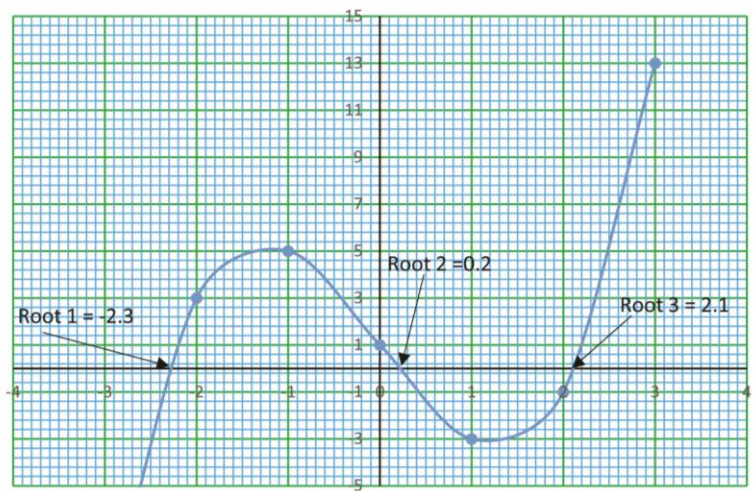


The root lies between -1 and 1

Example 4

Show graphically that there is positive real root of equation $x^3 - 5x + 1 = 0$

x	-3	-2	-1	0	1	2	3
f(x)	-11	3	5	1	-3	-1	13



(ii) Double graph method

When two graphs are drawn, the root lies between the points where the two curves meet.

Note

(i) Both curves must have a consistent scale and should be labelled.

(ii) A line must be drawn using a ruler while a curve must be drawn using a freehand

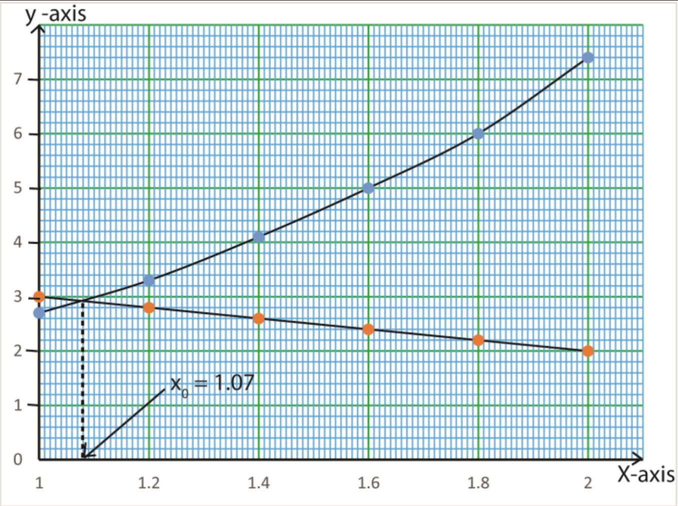
(iii) Both graphs must be labelled

(iv) The initial approximation of the root must be located and indicated in the graph

Example 5

By plotting graph of y = e^x and y = 4 - x on the same axes, show the root of the equation e^x +x - 4 = 0 lie between 1 and 2

x	1	1.2	14	1.6	1.8	2.0
y = e^x	2.7	3.3	4.1	5.0	6.0	7.4
y = 4 - x	3.0	2.8	2.6	2.4	2.2	2

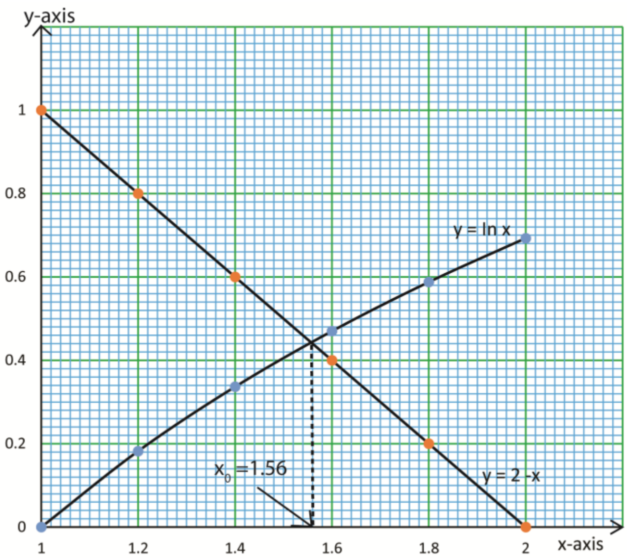


Therefore the root(1.07) lies between 1 and 2.

Example 6

Show that the equation lnx + x – 2 = 0 has a real root between x = 1 and x = 2

x	1	1.2	1.4	1.6	1.8	2.0
y = lnx	0	0.1823	0.3365	0.4700	0.5878	0.6731
y = 2-x	1	0.8	0.6	0.4	0.2	0

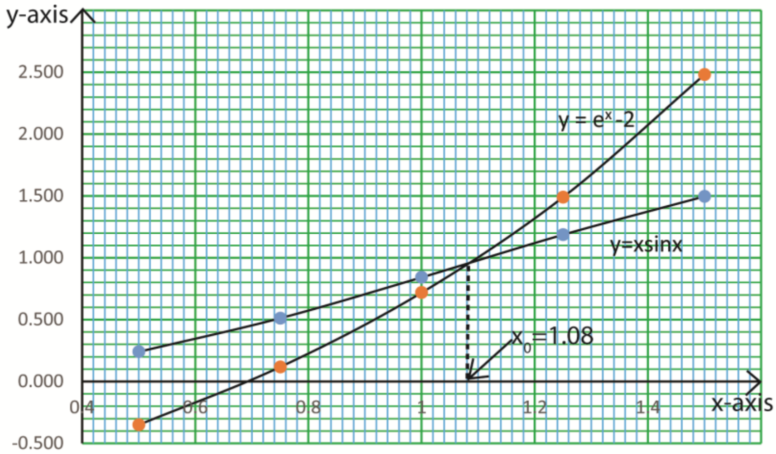


Therefore the root lies between x =1 and x =2

Example 7

By plotting graphs y = e^x -2 an y = x sin x on the same axis show that the root of the equation e^x – 2 –xsinx = 0 lies between x = 0.5 and x = 1.5

x	0.5	0.75	1.00	1.25	1.5
y = xsinx	0.240	0.511	0.841	1.186	1.496
y = e^x-2	-0.351	0.117	0.718	1.490	2.481



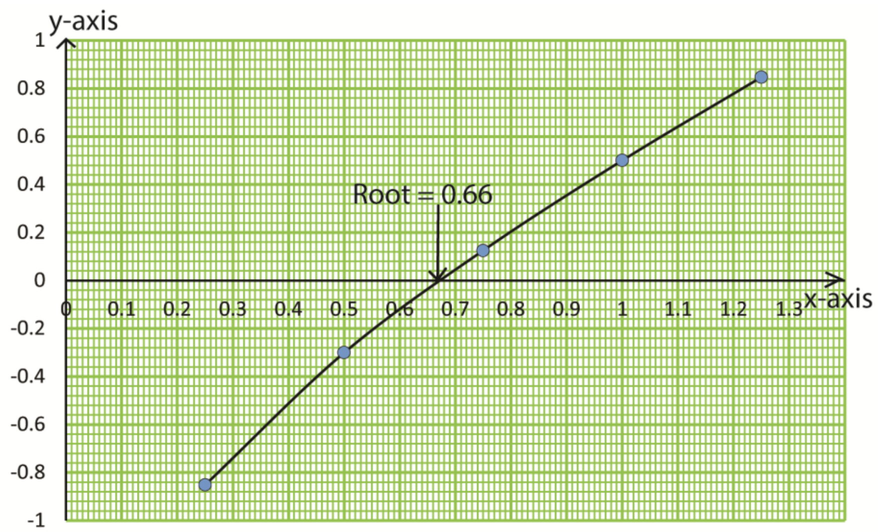
Example 8

Show graphically the equation x + logx = 0.5 has only one real root that lie between 0.5 and 1.

Solution

let y = x + logx - 0.5

x	0.25	0.5	0.75	1.00	1.25
y	-.852	-0.301	0.125	0.5	0.847



Therefore the root (0.66) lies between 0.5 and 1

Revision exercise 1

1. By sketching graphs of y = 2x and y = tanx show that the equation 2x = tanx has only one root between x = 1.1 and 1.2. Use linear interpolation to find the value of the root correct to 2dp.
2. Given the equation y = sinx - $\frac{x}{3}$, show by plotting two suitable graphs on the same axis that positive root lies between $\frac{2\pi}{3}$ and $\frac{5\pi}{6}$.
3. Show graphically that the positive real root of the equation $2x^2 + 3x - 3 = 0$ lies between 0 and 1 [0.7]
4. Use a graphical method to show that the equation $e^x - x - 2 = 0$ has only one real root between 2 and -1 by drawing two graphs y = e^x and y = x + 2 [-1.8]
5. On the same axes, draw graphs of y = 3 – 3x and y = $2x^2$ to show that the root of the equation $2x^2 + 3x\ \ 3 = 0$ lies between -3 and -2 [-2.2]
6. Show graphically that the positive real root of the equation $x^3 - 3x - 1 = 0$, lies between 1 and 2 [1.6]
7. on the same axes, draw graph y = 3x -1 and y = x^3 to show that the root of the equation $X^3 - 3x - 1 = 0$lies between 0 and 1.[0.35]
8. Using suitable graphs and plotting them on the same axes. Find the root of the equation $e^{2x}\sin x - 1 = 0$, in the interval x = 0.1 and x = 0.8. [0.44]
9. Show graphically that equation $e^{-x} = x$ has only one real root between 0.5 and 1. [0.56]
10. Show graphically that equation $e^x = -2x + 2$ has only one real root between 0 and 1.0.
11. on the same axes, draw graphs of y = 9x -4 and y = x^3 show that the root of equation $x^3 - 9x + 4 = 0$ lie between 2.5 and 3
12. Show that the positive real root of equation $4 + 5x^2 - x^3 = 0$ lies between 5 and 6.
13. On the same axes, draw graphs of y = x +1 and y = tanx to show that the equation tanx - x -1 = 0 lie between 1 and 1.5.
14. Using suitable graphs and plotting them on the same axes, find the roots of the equation $5e^x = 4x + 6$ in the interval x = 2 and x = -1.
15. On the same axes, draw graphs of y = 2x + 1 and y = $\log_e(x+2)$ to show that the root of equation $\log_e(x+2) - 2x - 1 = 0$ lies between 1 and 0.
16. Using suitable graphs and plotting them on the same axes, find the real root of the equation $9\log_{10} x = 2(X - 1)$ in the interval x = 3 and x =4.

Method of solving for roots

The following methods can be used

(a) Interpolation

Example 9

Show that the equation $x^4 - 12x^2 + 12 = 0$ has root between 1 and 2. Hence use linear interpolation to get the first approximation of the root.

Solution

$f(x) = x^4 - 12x^2 + 12$

$f(1) = 1^4 - 12(1)^2 + 12 = 1$

$f(2) = 2^4 - 12(2)^2 + 12 = -20$

Since there is a sign change,

then the root lies between

1 and 2.

x	1	x_0	2
f(x)	1	0	-20

$$\frac{x_0-1}{0-1} = \frac{2-1}{-20-1}$$

$x_0 = 1.05$

Example 10

Show that the equation $2x - 3\cos\left(\frac{x}{2}\right) = 0$ has a root between 1 and 2.Hence use linear interpolation twice to get the approximation of the root.

solution

Note: for trigonometric functions the

calculator must be strictly in radian mode

$f(x) = 2x - 3\cos\left(\frac{x}{2}\right)$

$f(1) = 2 \times 1 - 3\cos\left(\frac{1}{2}\right) = -0.633$

$f(2) = 2 \times 1 - 3\cos\left(\frac{2}{2}\right) = 2.379$

Since there is a sign change,

then the root lies between

1 and 2

x	1	x_0	2
f(x)	-0.633	0	2.379

$$\frac{x_0-1}{0--0.633} = \frac{2-1}{2.379--0.633}$$

$x_0 = 1.2102$

x	1.2102	x_0	2
f(x)	-0.047	0	2.379

$$\frac{x_0-1}{0--0.047} = \frac{2-1}{2.379--0.047}$$

$x_0 = 1.226$

Example 11

Show that the equation $3x^2 + x - 5 = 0$ has a real root between x = 1 and x = 2. Hence use linear interpolation twice to calculate the root to 2 dp.

Solution

$f(x) = 3x^2 + x - 5$

$f(1) = 3(1)^2 + 1 - 5 = 1$

$f(2) = 3(2)^2 + 2 - 5 = 9$

Since there is a sign change,

then the root lies between

x	1	x_0	2
f(x)	-1	0	9

$$\frac{x_0-1}{0--1} = \frac{2-1}{9--1}$$

$x_0 = 1.1$

x	1.1	x_0	2
f(x)	-0.27	0	9

$$\frac{x_0-1.1}{0--0.27} = \frac{2-1.1}{9--0.27}$$

$x_0 = 1.13$

(b) General iterative method

This involves generating equation by splitting the original equation into several equations by making x the subject.

Example 12

Given $x^2 + 4x - 2 = 0$. Find the possible equations for estimating the roots

Solution

Let x_{n+1} be a better approximation

x_n be the next approximation

$x_{n+1} = \frac{2}{x_n} - 4$ | $x_{n+1} = \sqrt{(2 - 4x_n)}$ | $x_{n+1} = \frac{2-x^2}{4}$

Example 13

Given $f(x) = x^3 - 3x - 12 = 0$. Generate equations in form of $x_{n+1} = g(x_n)$ that can be used to solve the equation $f(x) = 0$

Solution

Let x_{n+1} be a better approximation

x_n be the next approximation

$x_{n+1} = \frac{x_n^3-12}{3}$ | $x_{n+1} = \sqrt[2]{(3x_n + 12)} = \frac{12}{x_n^2-3}$ | $x_{n+1} = \sqrt{\left(3 + \frac{12}{x_n}\right)} = \frac{3x_n+12}{x_n^2}$

Testing for convergence

From the several iterative equations obtained, the equation whose $|f^1(x_n| < 1$ is the one which converges the correct root.

Example 14

Given the two iterative formulas

(i) $x_{n+1} = \frac{x_n^3-1}{5}$ (ii) $x_{n+1} = \sqrt{\left(5 + \frac{1}{x_n}\right)}$

Using $x_0 = 2$ deduce a more suitable formula for solving the equation. Hence find the root correct to 2dp

$x_{n+1} = \frac{x_n^3-1}{5}$

$f(x_n) = x_{n+1} = \frac{x_n^3-1}{5}$; $f^1(x_n) = \frac{3x_n^2}{5}$

$f^1(2) = \frac{3(2)^2}{5} = 2.4$

since $|f^1(2)| > 1$ it will not converge

$x_{n+1} = \sqrt{\left(5 + \frac{1}{x_n}\right)}$

f(x_n) = $\sqrt{\left(5 + \frac{1}{x_n}\right)}$; $f^1(x_n) = -\frac{1}{2}{x_n}^{-2}\left(5 + \frac{1}{x_n}\right)$

$f^1(2) = -\frac{1}{2}(2)^{-2}\left(5 + \frac{1}{2}\right)$ = -0.0533

since $|f^1(2)|$<1 it will converge so this equation gives the root

$x_{n+1} = \sqrt{\left(5 + \frac{1}{x_n}\right)}$, $|e|$= 0.005, x_0 = 2

x_1 = $\sqrt{\left(5 + \frac{1}{2}\right)}$ = 2.3452

$|x_1 - x_0|$ = 2.3452 – 2 = 0.3452 > 0.005

x_2 = $\sqrt{\left(5 + \frac{1}{2.3452}\right)}$ = 2.3295

$|x_2 - x_1|$ = 2.3452 – 2.3295 = 0.0157 > 0.005

x_3 = $\sqrt{\left(5 + \frac{1}{2.3295}\right)}$ = 2.3301

$|e|$=$|2.3301 - 2.3295|$= 0.0006 < 0.005

Hence root is 2.33

Example 15

Show that the iterative formula for solving the equation x^3 = x + 1 is $x_{n+1} = \sqrt{\left(1 + \frac{1}{x_n}\right)}$ starting with x_0 = 1 find the solution of the equations to 3sf.

Solution

$x_{n+1} = \sqrt{\left(1 + \frac{1}{x_n}\right)}$ $|e|$= 0.005, x_0 = 1

$x_1 = \sqrt{\left(1 + \frac{1}{1}\right)}$ =1.41421

$|x_1 - x_0|$ = $|1.41421 - 1|$ = 0.41421 > 0.005

$x_2 = \sqrt{\left(1 + \frac{1}{1.41421}\right)}$ =1.30656;

$|x_2 - x_1|$ = $|1.30656 - 1.41421|$ = 0.10765 > 0.005

$x_3 = \sqrt{\left(1 + \frac{1}{1.30656}\right)}$ = 1.32869

$|x_3 - x_2|$ = $|1.32869 - 1.30656|$ =0.03691 > 0.005

$x_4 = \sqrt{\left(1 + \frac{1}{1.32869}\right)}$ =1.32389

$|e|$=$|1.32389 - 1.32869|$= 0.0048 < 0.005

Hence the root is 1.32

Example 16

Given two iterative formulae I and II (shown below) for calculating the positive root of the quadratic equation f(x) = 0

$x_{n+1} = \frac{1}{2}(x_n^2 - 1)$ and $x_{n+1} = \frac{1}{2}\left(\frac{x_n^2+1}{x_n-1}\right)$ for n = 1, 2, 3 ....

Taking x_0 = 2.5, use each formula thrice to two decimal places to decide which is the more suitable formula. Give a reason for your answer.

Solution

Iterative formula $x_{n+1} = \frac{1}{2}(x_n^2 - 1)$

x_0 = 2.5

$x_1 = \frac{1}{2}(2.5^2 - 1)$ = 2.625

$|x_1 - x_0|$ = 0.125

$x_2 = \frac{1}{2}(2.625^2 - 1)$ 2.99453125

$|x_2 - x_1|$ = 0.3200125

$x_3 = \frac{1}{2}(2.99453125^2 - 1)$ = 3.837432861

$|x_3 - x_2|$ = 0.89212036

Iterative formula $x_{n+1} = \frac{1}{2}\left(\frac{x_n^2+1}{x_n-1}\right)$

x_0 = 2.5

$x_1 = \frac{1}{2}\left(\frac{2.5^2+1}{2.5-1}\right)$ = 2.416666667

$|x_1 - x_0|$ = 0.083333

$x_2 = \frac{1}{2}\left(\frac{2.416666667^2+1}{2.416666667-1}\right)$ = 2.414215686

$|x_2 - x_1|$ = 0.002450781

$x_2 = \frac{1}{2}\left(\frac{2.414215686^2+1}{2.414215686-1}\right)$ = 2.414215686

$|x_3 - x_2|$ = 0.000002124

The more suitable formula is $x_{n+1} = \frac{1}{2}\left(\frac{x_n^2+1}{x_n-1}\right)$.

Because the absolute difference between $x_3 - x_2$ is less than absolute error, where as in the first formula the absolute difference between$x_3 - x_2$ is greater than absolute error. In all the 2nd formula converge whereas the first formula diverges.

Example 17

(a) (i) Show that the equation $e^x - 2x - 1 = 0$ has a root between x = 1 and x = 1.5.
(ii) Use linear interpolation to obtain an approximation for the root

(b) (i) Solve the equation in (a)(i), using each formula below twice
Take the approximation in (a)(i) as the initial value
Formula I: $x_{n+1} = \frac{1}{2}(e^{x_n} + 1)$.
Formula II: $x_{n+1} = \frac{e^{x_n}(x_n-1)+1}{e^{x_n}-2}$
(ii) Deduce with a reason which of the two formulae is appropriate for solving the given equation in (a)(i). Hence write down a better approximate root, correct to two decimal places

Solution

(a) (i) using sign change method
let $f(x) = e^x - 2x - 1$
$f(1) = e^1 - 2(1) - 1 = -2.817$
$f(1.5) = e^{1.5} - 2(1.5) - 1 = 0.4817$

Since f(1).f(1.5)< 0, the root lies between x = 1 and x = 1.5

(a)(ii) Extract

1	x_0	1.5
-0.2817	0	0.4817

$\frac{x_0-1}{0--0.2817} = \frac{1.5-1}{0.4817--0.2817}$; $x_0 = 1.1845$

Hence the approximation to the root is 1.18 (2 dp)

(b)(i)

Solution

formula 1: $x_{n+1} = \frac{1}{2}(e^{x_n} + 1)$

$x_0 = 1.18$

$x_1 = \frac{1}{2}(e^{1.18} + 1)$= 2.1272

$|x_1 - x_0|$ = 0.9472

$x_2 = \frac{1}{2}(e^{2.127187} + 1)$= 4.6956

$|x_2 - x_1|$ = 2.5684

formula 2: $x_{n+1} = \frac{e^{x_n}(x_n-1)+1}{e^{x_n}-2}$

$x_0 = 1.18$

$x_1 = \frac{e^{1.18}(1.18-1)+1}{e^{1.18}-2}$= 1.2642

$|x_1 - x_0|$ = 0.0842

$x_2 = \frac{e^{1.2642}(1.2642-1)+1}{e^{1.2642}-2}$= = 1.2565

$|x_2 - x_1|$ = 0.0077

Formula 1, the sequence 1.18, 2.1272, 4.6956 diverge, hence the formula is not suitable

Formula 2, the sequence 1.18, 1.2642, 1.2565 converge, hence the formula is suitable solving the equation

A better approximation = 1.26 (2 dp)

Revision exercise 2

1. Given the following iterative formula
   (i) $x_{n+1} = 5 - \frac{3}{x_n}$ (ii) $x_{n+1} = \frac{1}{5}(x_n^2 + 3)$

   Taking x_0 = 5 deduce a more suitable iterative formula for solving the equation
2. Show that the iterative formula for solving the equation $x^2 -5x + 2 = 0$ can be written in two ways as $x_{n+1} = 5 - \frac{2}{x_n}$ or $x_{n+1} = \frac{x_n^2+2}{5}$.
   Starting with x_0 = 4, deduce the more suitable formula for the equation and hence find the root correct to 2 dp [4.56]

3. Show that the iterative formula for solving the equation $x^3 - x - 1 = 0$ is $x_{n+1} = \sqrt{\left(1 + \frac{1}{x_n}\right)}$.
   Starting with xo = 1 find the root of the equation correct to 3 s.f. [1.33]
4. (a) Show that the iterative formula for solving the equation $2x^2 - 6x - 3 = 0$ is $x_{n+1} = \frac{2x_n^2+3}{4x_n+6}$
   (b) Show that the positive root for $2x^2 - 6x - 3 = 0$ lies between 3 and 4. find the root correct to 2 decimal places [3.44]
5. (a) If b is the first approximation to the root of equation x2 = a, show that the second approximation to the root is given by $\frac{b+\frac{a}{b}}{2}$. Hence taking b = 4, estimate $\sqrt{17}$ correct to 3 dp [4.123]
   (b) Show that the positive real root of the equation $x^2 - 17 = 0$ lies between 1.5 and 1.8. Hence use the formula in (a) above to determine the root to 3 dp

(c) Newton Raphson's Method

It is given by $x_{n+1} = x_n - \left[\frac{f(x_n)}{f^1(x_n)}\right]$ n = 1, 2, 3 …

Example 18

Use Newton Raphson's method to find the root of equation $x^3 + x - 1 = 0$ using x0 = 0.5 as the initial approximation, correct your answer to 2 decimal places

Solution

$f(x) = x^3 + x - 1$, $f^1(x) = 2x^2 + 1$

$x_{n+1} = x_n - \left[\frac{(x^3+x-1)}{3x^2+1}\right]$

$x_1 = 0.5 - \left[\frac{((0.5)^3 + 0.5 - 1)}{3(0.5)^2+1}\right]$ =0.7142

$|x_1 - x_0| = 0.7142 - 0.5 = 0.2142 > 0.005$

$x_2 = 0.7142 - \left[\frac{((0.7142)^3 + 0.7142 - 1)}{3(0.7142)^2+1}\right]$ =0.6831

$|x_2 - x_1| = |0.6831 - 0.7142|$

$= 0.0311 > 0.005$

$x_2 = 0.6831 - \left[\frac{((0.6831)^3 + 0.6831 - 1)}{3(0.6831)^2+1}\right]$ =0.6824

$|3 - x_2| = |0.6824 - 0.6831|$

$= 0.0007 < 0.005$

∴ Root = 0.68

Example 19

Show that the equation 5x - 3cos2x = 0 has a root between 0 and 1. Hence use Newton Raphson's method to find the root of equation correct to 2 decimal places using $x_0 = 0.5$.

Solution

Using sign change method to locate

The roots. Note for trigonometric

functions the calculator is used

in radians mode

f(x) =5x - 3cos2x

f(0) = 5(0) – 3cos2(0) = -3

f(1) = 5(1) – 3cos2(1) = 2.455

Since there is change sign the root lies between x = 0 and x = 1 f(x) = 5x - 3cos2x, $f^1(x)$ = 5 + 6sin2x

$x_{n+1} = x_n - \left[\frac{(5x_n - 3\cos 2x_n)}{5 + 6\sin 2x_n}\right]$

$x_0 = 0.5$, $|e| = 0.005$

$x_1 = 0.5 - \left[\frac{(5(0.5) - 3\cos 2(0,5)}{5 + 6\sin 2(0.5)}\right] = 0.4125$

$|x_1 - x_0| = |0.4125 - 0.5| = 0.0875 > 0.005$

$x_1 = 0.4125 - \left[\frac{(5(0.4125) - 3\cos 2(0.4125)}{5 + 6\sin 2(0.4125)}\right] = 0.4096$

$|x_2 - x_1| = |0.4096 - 0.4125| = 0.0029 < 0.005$

∴ Root = 0.41

Example 20

Use Newton Raphson's iterative formula to show that the cube root of a number N is given by $\frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)$. Hence taking $x_0 = 2.5$ determine $\sqrt[3]{10}$ correct to 3 dp.

Solution

$x = N^{\frac{1}{3}}$

$x^3 - N = 0$

$f(x) = x^3 - N$; $f^1(x) = 3x^2$

$x_{n+1} = x_n - \left[\frac{(x_n{}^3 - N)}{3x_n{}^2}\right] = \frac{x_n(3x_n{}^2) - (x_n{}^3 - N)}{3x_n{}^2}$

$= \frac{2x_n{}^3 + N}{3x_n{}^2} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right).$

$x_0 = 2.5$, N = 10, $|e| = 0.005$

$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)$

$x_1 = \frac{1}{3}\left(2(2.5) + \frac{N}{2.5^2}\right) = 2.2$

$|x_1 - x_0| = |2.2 - 2.5| = 0.3 > 0.005$

$x_2 = \frac{1}{3}\left(2(2.2) + \frac{N}{2.2^2}\right) = 2.1554$

$|x_2 - x_1| = |2.1554 - 2.2| = 0.0446 > 0.005$

$x_3 = \frac{1}{3}\left(2(2.1554) + \frac{N}{2.1554^2}\right) = 2.1544$

$|x_3 - x_2| = |2.1544 - 2.1554| = 0.001 < 0.005$

∴ Root = 2.154

Example 21

(a) Show that the equation x -3sinx = 0 has a root between 2 and 3. (03marks)

f(x) = x – 3sinx
f(2) = 2 – 3sin 2 = -0.7279
f(3) = 3 – 3sin3 = 2.5766
since f(2).f(3) = -1.8755<0
there exist a root of x-3sinx =0 between 2 and 3

(b) Show that Newton- Raphson iterative formula for estimating the root of the equation in (a) is given by

$$X_{n+1} = \frac{3(\sin x_n - x_n \cos x_n)}{1 - 3\cos x_n}, n = 0, 1, 2 \ldots.$$

Hence find the root of the equation corrected to 2 decimal places (09 marks)

f'(x) = 1 – 3cos x

$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$

$= x_n - \frac{x_n - 3\sin x_n}{1-3\cos x_n}$

$= \frac{x_n - 3x_n\cos x_n - x_n + 3\sin x_n}{1-3\cos x_n}$

$x_{n+1} = \frac{3(\sin x_n - x_n \cos x_n}{1-3\cos x_n}$

Taking $x_0 = \frac{2+3}{2} = 2.5$

$x_1 = \frac{3(\sin 2.5 - 2.5\cos 2.5}{1-3\cos 2.5} = 2.293$

Error = $|2.293 - 2.5| = 0.207 > 0.005$

$x_2 = \frac{3(\sin 2.293 - 2.5\cos 2.293}{1-3\cos 2.293} = 2.279$

Error = $|2.279 - 2.293| = 0.014 > 0.005$

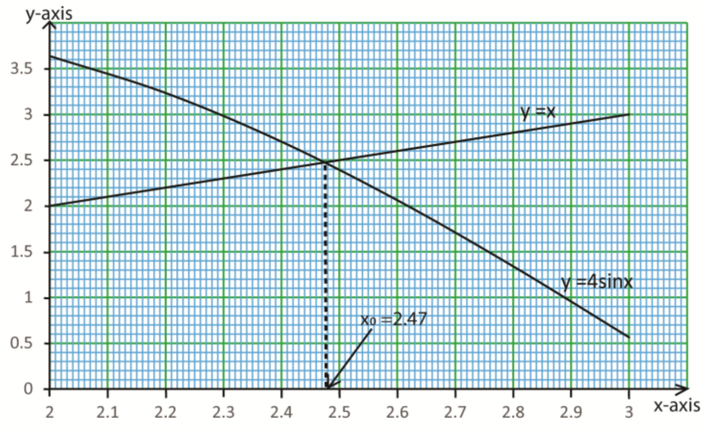
$x_3 = \frac{3(\sin 2.279 - 2.5\cos 2.279}{1-3\cos 2.279} = 2.279$

Error = $|2.279 - 2.279| = 0.000 < 0.005$

∴ root = 2.279 = 2.28(2D)

Example 22

(a) On the same axis, draw graphs of y = x and y = 4sinx to show that the root of the equation x-4sinx = 0 lies between x= 2 and x = 3



Therefore the root (2.47) lies between x = 2 and x = 3

(b) Use Newton Raphson's method to calculate the root of the equation x – 4sinx = 0, taking approximate root in (a) as the initial approximation to the root. correct your answer to 3 decimal places.

f(x) = x – 4sinx

f'(x) = 1 – 4cos x

$x_{n+1} = x_n - \frac{x_n - 4\sin x_n}{1-4\cos x_n}$

$Taking\ x_0 = 2.47$

$x_1 = 2.47 - \frac{2.47 - 4\sin 2.45}{1-4\cos 2.47} = 2.4746$

$Error = |2.4746 - 2.47| = 0.0046 > 0.0005$

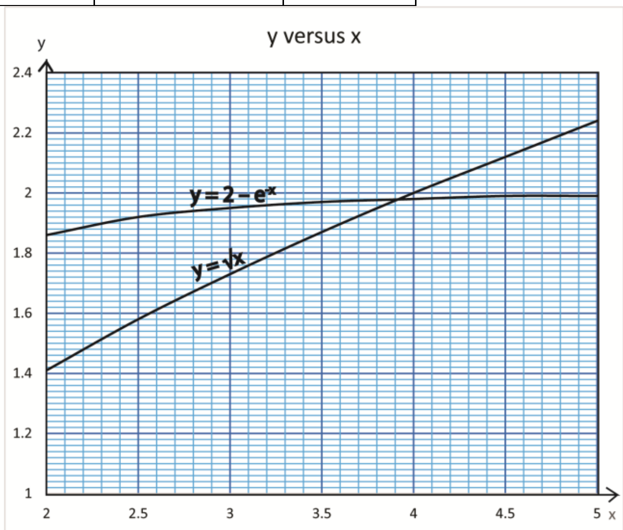
$x_2 = 2.4746 - \frac{2.4746 - 4\sin 2.4546}{1-4\cos 2.4746} = 2.4746$

$Error = |2.4746 - 2.4746| = 0.000 < 0.0005$ ∴ the root = 2.475 (3D)

Example 23

(a) Draw on the same axes the graphs of the curves y = 2 – e^{-x} and y = $\sqrt{x}$values 2≤ x ≤ 5. (04marks)

x	y = 2 – e^{-x}	y = $\sqrt{x}$
2.0	1.86	1.41
2.5	1.92	1.58
3.0	1.95	1.73
3.5	1.97	1.87
4.0	1.98	2.00
4.5	1.99	2.12
5.0	1.99	2.24



(b) Determine from your graph the interval within which the roots of the equation
$e^{-x} + \sqrt{x-2=0}$ lies

Hence, use Newton-Raphson's method to find the root of the equation correct to 3 decimal places (07marks)

Root lies between 3.9 and 4

f(x) = 2 – e^{-x} - $\sqrt{x}$

f'(x) = $e^{-x} - \frac{1}{2\sqrt{x}}$

f(x$_n$) = $e^{-x_n} - \frac{1}{2\sqrt{x_n}}$

$x_{n+1} = x_n - \frac{2- e^{x_n} - \sqrt{x_n}}{2e^{-x_n}\sqrt{x_n}-1}$

$x_0 = \frac{3.9+4}{2} = 3.95$

$x_1 = 3.95 - \frac{2\sqrt{3.95}(2- e^{-3.95} - \sqrt{3.95})}{2e^{-3.95}\sqrt{3.95}-1} = 3.9211$

$Error = |3.9211 - 3.95| = 0.0289$

$x_2 = 3.9211 - \frac{2\sqrt{3.9211}(2- e^{-3.9211} - \sqrt{3.9211})}{2e^{-3.9211}\sqrt{3.9211}-1} = 3.9211$

∴ Root = 3.921 (3dp)

Example 24

Given the equation $X^3 - 6x^2 + 9x + 2 = 0$

(a) Show that the equation has a root between -1 and 0.

Let $f(x) = X^3 - 6x^2 + 9x + 2$

$f(-1) = (-1)^3 - 6(-1)^2 + 9(-1) + 2$

$= -1 - 6 - 9 + 2 = -14$

$f(0) = 0 + 0 + 0 + 2$

$= 2$

$f(-1).f(0) = -14 \times 2 = -28$

since $f(-1).f(0) < 0$; the root exist between -1 and 0.

(b) (i) Show that the Newton Raphson formula approximating the root of the equation is given by $X_{n+1} = \frac{2}{3}\left[\frac{x_n^3 - 3x_n^2 - 1}{x_n^2 - 4x_n + 3}\right]$

$f(x) = X^3 - 6x^2 + 9x + 2$

$f(x_n) = x_n^3 - 6x_n^2 + 9x_n + 2$

$f'(x_n) = 3x_n^2 - 12x_n + 9$

$x_{n+1} = x_n - \left(\frac{x_n^3 - 6x_n^2 + 9x_n + 2}{3x_n^2 - 12x_n + 9}\right)$

$= \frac{x_n(3x_n^2 - 12x_n + 9) - (x_n^3 - 6x_n^2 + 9x_n + 2)}{3x_n^2 - 12x_n + 9}$

$= \frac{(3x_n^3 - 12x_n^2 + 9x_n) - (x_n^3 - 6x_n^2 + 9x_n + 2)}{3x_n^2 - 12x_n + 9}$

$= \frac{2x_n^3 - 6x_n^2 - 2}{3x_n^2 - 12x_n + 9}$

$= \frac{2}{3}\left[\frac{x_n^3 - 3x_n^2 - 1}{x_n^2 - 4x_n + 3}\right]$

(ii) Use the formula in (b)(i) above, with initial approximation of $x_0 = -0.5$, to find the root of the given equation correct to two decimal places

Taking x = -0.5

$x_1 = \frac{2}{3}\left[\frac{(-0.5)^3 - 3(-0.5)^2 - 1}{(-0.5)^2 - 4(-0.5) + 3}\right]$ = -0.2381

$|e| = |0.2381 - (-0.5)|$ = 0.2619

$x_2 = \frac{2}{3}\left[\frac{(-0.2381)^3 - 3(-0.2381)^2 - 1}{(-0.2381)^2 - 4(-0.2381) + 3}\right]$ = -0.1968

$|e| = |0.1968 - (-0.2381)|$ = 0.0.0413

$x_3 = \frac{2}{3}\left[\frac{-0.1968^3 - 3(-0.1968)^2 - 1}{(-0.1968)^2 - 4(-0.1968) + 3}\right]$ = -0.1958

$|e| = |-0.1958 - (-0.1968)|$ = 0.001< 0.005

Hence the root = -0.20 (2D)

Revision Exercise 3

1. Using the Newton Raphson's formula, show that the reciprocal of a number N is $x_n(2 - Nx_n)$
2. Use Newton Raphson's iterative formula to show that the cube root of a number N is given by $\frac{1}{3}\left(2x_n + \frac{N}{x_n{}^2}\right)$. Hence use the iterative formula to find $\sqrt[3]{96}$ correct to3 decimal places. use $x_0 = 5$. [4.579]
3. (a) Show that the equation 3x3 + x -5 = 0 has real root between x = 1 and x = 2.

   (b) Using linear interpolation, find the first approximation for this root to 2dp. [1.04]

(c) Using Newton Raphson's method twice find the value of this root correct to 2 dp. [1.09]

4. (a) Show graphically that there is a positive real root of equation xe^{-x} -2x + 5 = 0 between x = 2 and x = 3
(b) Using Newton Raphson's method, find this root correct to 1 dp. [2.6]
5. Using the iterative formula for NRM, show that the fourth root of a number N is $\frac{3}{4}\left(x_n + \frac{N}{3{x_n}^3}\right)$. Starting with x_0 = 2.5 show that $(45.7)^{\frac{1}{4}}$ = 2.600 (3dp)
6. On the same axes, draw graphs of y = x^3 and y = 2x + 5. Using NRM twice find the positive root of the equation x^3 - 2x – 5 = 0 correct to 2 decimal places. [2.09]
7. (a) Show that the Newton Raphson's formula for finding the smallest positive root of the equation 3tanx + x = 0 is $\frac{6x_n - 3\sin 2x_n}{6 + 2\cos 2x_n}$
(b) By sketching the graphs of y = tans, y = $\frac{-x}{3}$ 0r otherwise, find the first approximation to the required root and use it to find the actual root correct to 3 dp . [2.456]
8. (a) Show that the root of the equation f(x) = e^x + x^3 – 4x = 0 has a root between x =1 and x = 2
(b) Use the Newton Raphson's method to find the root of equation in (a) correct to 2 decimal places. [x_0 = 1, root =1.12]
9. (a) Show that the iterative formula for approximation of the root of f(x) = 0 by NRM process for the equation xe^x + 5x – 10 = 0 is $x_{n+1} = \frac{{x_n}^2 e^{x_n} + 10}{x_n e^{x_n} + e^{x_n} + 5}$.
(b) Show that the root of the equation in (i) above lies between x = 1 and x = 2. Hence find the root of the equation correct to 2 dp. [1.20]
10. (a) Use a graphical method to find a first approximation to the real root of x^3 + 2x -2 = 0.
(b) Use the Newton Raphson's method to find the root of the equation in (a) correct to 2 dp. [0.77]
11. (a) Show that equation x = In (8-x) has a root between x= 1 and x = 2.
(b) Use the Newton Raphson's method to find the root of the equation in (a) correct to 2 decimal places [1.82]
12. (a) Use graphical method to find the first approximation to the root of x^3 – 3x + 4 = 0. [-2]
(b) Use NRM to find the root of the equation in (a) correct to 2 d.p. [-2.20]
13. Show graphically that equation e^x + x -4 = 0 has only one root between x = 1 and x = 2. Use NRM to find the approximation of the equation correct to 3dp. [1.07]
14. Show that the NRM for approximating the K^{th} root of a number N is given by $\frac{1}{K}\left((K-1)x_n + \frac{N}{{x_n}^{K-1}}\right)$. Hence use your formula to find the positive square root of 67 correct to 4 s.f. [8.185].
15. (a) Show that equation x^3 + 3x – 9 = 0 has a root between x= 1 and x = 2.
(b) Use the Newton Raphson's method to find the root of the equation in (a) correct to 2 One places [1.6]
16. (a) Show graphically that there is a positive real root of equation xe^{-x} -2x - 1 = 0 between x = 1 and x = 2
(b) Using Newton Raphson's method, find this root for the equation in (a) correct to 2 dp. [1.26]
17. (a) Show that equation 2x – 3cos $\left(\frac{x}{2}\right)$ = 0 has a root between x= 1 and x = 2.
(b) Use the Newton Raphson's method to find the root of the equation in (a) correct to one places [1.23]

18. (a) If a is the first approximation to the root of the equation $x^5 - b = 0$, show that the second approximation is given by $\frac{4a+\frac{b}{a^4}}{5}$.
    (b) Show that the positive real root of the equation $x^5 - 17 = 0$ lies between 1.5 and 1.8. Hence use the formula in (a) above to determine the root to 3 decimal places. [1.762]
19. (a)(i) On the same axes, draw graphs of $y = x^2$ and y = cox x for $0 \le x \le \frac{\pi}{2}$ at intervals of $\frac{\pi}{8}$.
    (ii) Use your graphs, to find to 1 decimal place an approximate root of the equation $x^2 - \cos x = 0$ [0.8]
    (b) Use the NRM to calculate the root of the equation $x^2 - \cos x = 0$ taking the approximate root in (a) as the initial approximation. Correct your answer to 3 dp. [0.824]
20. (a) (i) Draw on same axes the graphs of equation y = xsinx and $y = e^x - 2$ for $0 \le x \le 1.5$.
    (ii) Use your graphs to find an approximate root of the equation $2 - e^x + x\sin x =$ [1.1]

(c) Use the Newton Raphson's method to find the root of the equation in (a)(ii) correct to three decimal places [1.085]

21. Show graphically that equation $e^x + x - 8 = 0$ has only one real root between x = 1 and x = 2. Use NRM to find approximation of x = In(x 8) correct to 3 dp [1.821]
22. Draw using the same axes, graphs of $y = x^2$ and y = sin2x for $0 \le x \le \frac{\pi}{2}$. From the graphs obtain to one decimal place an approximation of the non-zero root of the equation $x^2 - \sin 2x = 0$. Using NRM, calculate to 2 dp a more suitable approximation. [0.97]
23. Given the equation In(1 + 2x) – x = 0.
    (i) show the root of the equation above lies between x = 1 and x = 1.5
    (ii) Use NRM twice to estimate the root of the equation, correct to 2 dp. [1.26]

Errors

An error, commonly known as absolute error is the absolute difference between exact value and approximate value.

Source of errors

(a) Rounding off

These errors that arise as a result of simply approximating the exact value of different numbers.

Example 1

Round off the following numbers to the given number of decimal places or significant figures.

(i) 3.896234 to 4 dp [3.8962]
(ii) $\frac{2}{3}$ to 3dp [0.667]
(iii) 5.002570 to 3s.f [5.00]

(iv) 0.00652673bto4 s.f [0.006527]

(v) 7.00214 to 4 s.f. [7.002]

(vi) 5415678 to 3 s.f. [5420000]

(b) Truncation

These occur when an infinite number is terminated/cutoff (without rounding off] at some point.

Example 2

Truncate the following number to the given number of decimal places (d.p) or significant figures. s.f.

(i) 4.56172 to 2dp [4.56] (ii) $\frac{2}{3}$ to 3dp [0.666] (iii) 1.345618 to 4 s.f. [1.345]

Common terms used

(a) Error or absolute error
If x represent an approximate value of X and Δx is the error approximation
$|Error| = |exact\ value - approximate\ value|$
$|\Delta x| = |X - x|$

Example 3

Round off 32.5263 to 2 dp and determine the absolute error.

Solution

X = 32.5263, x = 32.53

$|\Delta x| = |X - x|$ = $|32.5263 - 32.53|$ = 0.0037

(b) Relative error

Relative error = $\frac{absolute\ error}{exact\ value} = \frac{|\Delta x|}{X} = \frac{|X-x|}{X}$

(c) Percentage error or percentage relative error

Percentage relative error = $\frac{absolute\ error}{exact\ value} x\ 100\% = \frac{|\Delta x|}{X} x\ 100\% = \frac{|X-x|}{X}\ x\ 100\%$

Example 4

Find the percentage error in rounding off $\sqrt{3}$ 2 dp

Solution

X = $\sqrt{x}$ = 1.732050808, x = 1.73

Percentage error = $\frac{|X-x|}{X}\ x\ 100\%$ = $\frac{|1.732050808 - 1.73|}{1.732050808}\ x\ 100\%$ = 0.118%

(d) Error bound or minimum possible error in an approximated number

This depends on the number of decimal places the number is rounded to. If the number is rounded to n dp, then the maximum possible error in that number is = 0.5 x 10^{-n}.

Example 4

If a student weighs 50kg. Find the range where his weight lies

Solution

n = 0 dp, e = 0.5 x 10^{-0} = 0.5

Range = 50 $\pm$ 0.5 =(49.5, 50.5)

Example 5

If x is given to stated level of accuracy stat the lower and upper bounds of x

(a) 6.45

n = 2 dp, e = 0.5 x 10^{-2} = 0.005

Lower bound = 6.45 – 0.005 = 6.445

upper bound = 6.45 + 0.005 = 6.455

(b) 0.278

n = 3 dp, e = 0.5 x 10^{-3} = 0.0005

(c) Lower bound = 0.278 – 0.0005 = 0.2775

upper bound = 0.278 + 0.0005 = 0.2785

Example 6

A value of w = 150.58m was obtained when measuring the width of the football pitch. Given that the relative error in this value as 0.07%, find the limit within which the value w lies.

% relative error = $\frac{|\Delta w|}{w}\ x 100\%$

0.07 =$\frac{|\Delta w|}{150.58}\ x\ 100$

$|\Delta w|$ = 0.105

Lower limit = 150.58 – 0.105 = 150 .475

Upper limit = 150.58 + 0.105 = 150 .685

Absolute error in an operation

When the minimum and maximum value is known then.

absolute error = $\frac{1}{2}[maximum\ value - minimum\ value]$

Absolute error in addition

Given two numbers a and b with errors Δa + Δb

$(a + b)_{max} = a_{max} + b_{max} = (a + \Delta a) + (b + \Delta b)$

$(a + b)_{min} = a_{min} + b_{min} = (a - \Delta a) + (b - \Delta b)$

Absolute error in subtraction

Given two numbers a and b with errors Δa + Δb

$(a - b)_{max} = a_{max} - b_{min} = (a + \Delta a) - (b - \Delta b)$

$(a + b)_{min} = a_{min} - b_{max} = (a - \Delta a) + (b + \Delta b)$

Example 7

Given that a = 2.453, b = 6.79, find the limits and hence absolute error of

(i) a + b

solution

a = 2.453, Δa = 0.0005 and b = 6.79, Δb = 0.005

$(a + b)_{max} = a_{max} + b_{max} = (a + \Delta a) + (b + \Delta b)$

= (2.453 + 0.0005) + (6.79 + 0.005) = 9.2485

$(a + b)_{min} = a_{min} + b_{min} = (a - \Delta a) + (b - \Delta b)$

= (2.453 - 0.0005) + (6.79 - 0.005) = 9.2375

lower limit = 9.2375; upper limit = 9.2485

absolute error $=\frac{1}{2}[9.2485 - 9.2375]$ = 0.0055

(ii) a - b

solution

$(a + b)_{max} = a_{max} - b_{min} = (a + \Delta a) - (b - \Delta b)$

= (2.453 + 0.0005) - (6.79 - 0.005) = -4.3315

$(a + b)_{min} = a_{min} - b_{max} = (a - \Delta a) - (b + \Delta b)$

= (2.453 - 0.0005) - (6.79 + 0.005) = -4.3425

lower limit = -4.3425; upper limit = -4.3315

absolute error $=\frac{1}{2}[-4.3315 - -4.3425]$ = 0.0055

Absolute error in multiplication

Given two numbers a and b with errors $\Delta a + \Delta b$
$(ab)_{max} = a_{max}b_{max} = (a + \Delta a)(b+\Delta b)$
$(ab)_{min} = a_{min}b_{min} = (a - \Delta a)(b - \Delta b)$

Example 8

Given that a = 4.617, and b = 3.65 find the absolute error in ab

solution

a = 4.617, Δa= 0.0005, b = 3.65, Δb = 0.005

$(ab)_{max} = a_{max}b_{max} = (a + \Delta a)(b+\Delta b) = (4.617 + 0.0005)(3.65b+ 0.005) =16.87696$

$(ab)_{min} = a_{min}b_{min} = (a - \Delta a)(b - \Delta b) = (4.617 - 0.0005)(3.65b - 0.005)=16.82853$

absolute error = $\frac{1}{2}[maximum\ value - minimum\ value] = \frac{1}{2}(16.87696 - 16.82853)$ =0.02422

Example 9

Given that a = 4.617, and b = -3.65 find the

(i) Limits of values where ab lies
Solution

a = 4.617, Δa= 0.0005, b = -3.65, Δb = 0.005

$(ab)_{max} = a_{max}b_{max} = (4.617 + 0.0005)(-3.65b+ 0.005) = -16.83079$
$(ab)_{min} = a_{min}b_{min} = (4.617 - 0.0005)(-3.65b - 0.005 = -16.87331$
Lower limit = -16.87331; upper limit = -16.83079

(ii) the interval of values where ab lies
(-16.87331, -16.83079)

(iii) the absolute error
Absolute error =$\frac{1}{2}[-16.83079 - \ -16.87331]$= 0.02126

Absolute error n division

Given two numbers a and b with errors $\Delta a + \Delta b$

$$\left(\frac{a}{b}\right)_{max} = \frac{a_{max}}{b_{min}} = \frac{(a+\Delta a)}{(b-\Delta b)}$$
$$\left(\frac{a}{b}\right)_{min} = \frac{a_{min}}{b_{max}} = \frac{(a-\Delta a)}{(b+\Delta b)}$$

Example 10

Given a = 1.26, b = 0.435. Find the absolute error of

(i) Range of value where $\frac{a}{b}$ lies

$$\left(\frac{a}{b}\right)_{max} = \frac{a_{max}}{b_{min}} = \frac{(1.25+0.005)}{(0.435-0.0005)} = 2.91139$$

$\left(\frac{a}{b}\right)_{min} = \frac{a_{min}}{b_{max}} = \frac{(1.25-0.005)}{(0.435+0.0005)} = 2.88175$

Range of values is (2.88175, 2.91139)

(ii) Absolute error

$=\frac{1}{2}(2.91139 - 2.88175) = 0.01482$

Example 11

(a) Given that y = e^x and x = 0.62correct to two decimal places, find the interval within which the exact value of y lies. (05marks)

$e_x = 0.005$
$y_{max} = e^{0.625} = 1.8682$
$y_{min} = e^{0.615} = 1.8497$
The interval = (1.8497, 1.8682)

(b) Show that the maximum possible relative error in ysin^2x is

$\left|\frac{\Delta y}{y}\right| + 2\cot x\,|\Delta x|$, where$\Delta x$ and Δy are errors in x and y respectively

Hence find the percentage error in calculating ysin^2x if y = 5.2 $\pm$ 0.05 and x = $\frac{\pi}{6} \pm \frac{\pi}{360}$ (07 marks)

z = y$\sin^2$x

$e_z = \Delta y sin^2x + 2y\Delta x cos\ x sinx$

$\frac{e_z}{z} = \frac{\Delta y sin^2x}{y\sin^2 x} + \frac{2y\Delta x cos\ x sinx}{y\sin^2 x}$

$\left|\frac{e_z}{z}\right| = \left|\frac{\Delta y}{y} + 2cotx.\Delta x\right|$

$\leq \left|\frac{\Delta y}{y}\right| + 2cotx.|\Delta x|$

$\therefore$ Maximum possible error is $\left|\frac{\Delta y}{y}\right| + 2cotx.|\Delta x|$

$percentage\ error = \left[\frac{0.05}{5.2} + 2\cot\frac{\pi}{6} . \left|\frac{\pi}{360}\right|\right] x\ 100\%$ = 3.9845%

Example 12

Two numbers A and B have maximum possible error e_a and e_b respectively.

(a) Write an expression for the maximum possible error in their sum
Maximum possible error = $|e_a||e_b|$

(b) If A = 2.03 and B = 1.547, find the maximum possible error in A + B (05marks)
$e_a = 0.005, e_b = 0.0005$
$|e_{(A+B)}| = |0.005| + |0.0005|$
= 0.0055

Example 13

Given that y = $\frac{1}{x} + x$ and x = 2.4 correct to one decimal place, find the limits within which y lies. (05marks)

Error in 2.4 = $\frac{1}{2} x \frac{1}{10} = 0.05$
y_{max} = 2.45 + $\frac{1}{2.35} = 2.8755$
y_{min} = 2.35 + $\frac{1}{2.45} = 2.7582$
∴ the limits are [2.7582, 2.8755]

Example 14

The numbers X = 1.2, Y = 1.33 and Z = 2.245 have been rounded off to the given decimal places. find the maximum possible value of $\frac{Y}{Z-X}$ correct to 3 decimal places

Maximum value = $\frac{(Y+\Delta Y)}{(Z-\Delta Z)-(X+\Delta X)} = \frac{(1.33+0.005)}{(2.245-0.0005)-(1.2+0.05)}$ = 1.342

Revision exercise 1

1. Given the numbers x = 2.678 and y = 0.8765 measured the nearest possible decimal places indicated.
   (i) state the maximum possible error in x ans y [Δx = 0.0005, Δy = 0.00005]
   (ii) find the limits within which the product xy lie [2.3467, 2.3478]
   (iii) determine the maximum possible error in xy [0.000572]

2. The length, width and height of water all rounded off to 3.65m, 2.14m and 2.5m respectively. Determine the least and greatest amount of water the tank can contain [19.066, 19.992]
3. Given that the values x = 4, y = 6 and z = 8 each has been approximate to the nearest integer. find the maximum and minimum values of
   (i) $\frac{y}{x}$ [1.85714, 1.22222]
   (ii) $\frac{z-x}{y}$ [0.90909, 0.46154]
   (iii) $(x+y)z$ [93.5, 67.5]

Error propagation

Triangular inequality state that $|a \pm b| \leq |\Delta a| + |\Delta b|$

Addition

Consider two numbers X and Y are approximated by x and y with errors Δx and Δy.

$|e_{x+y}| = |(x+\Delta x) + (y+\Delta y) - (x+y)|$

$= |\Delta x + \Delta y| = |\Delta x| + |\Delta y|$

R.E$_{max}$ = $\left[\left|\frac{\Delta x}{X+Y}\right| + \left|\frac{\Delta y}{X+Y}\right|\right]$

Alternatively

absolute error = $\frac{1}{2}[\max - min]$

$= \frac{1}{2}[(x+\Delta x) + (y+\Delta y)] - [(x-\Delta x) + (y-\Delta y)]$

$|e_{x+y}|=|\Delta x + \Delta y|= |\Delta x| + |\Delta y|$

R.E$_{max}$ = $\left[\left|\frac{\Delta x}{X+Y}\right| + \left|\frac{\Delta y}{X+Y}\right|\right]$

Example 15

Given numbers x = 7.824 and y = 3.36 rounded to the given number of decimal places. Find the limits within which (x + y) lies

Solution

Δx = 0.0005, Δy = 0.005

$|e_{x+y}|=|\Delta x| + |\Delta y|$= 0.0005 + 0.005 = 0.0055

working value (x + y) = 7.824 + 3.36 = 10.184

Upper limit = 10.184 + 0.0055 = 10.1895

Lower limit = 10.184 - 0.0055 = 10.1785

Alternatively

$(x + y)_{max}$ = 7.8245 + 3.365 =10.1895

$(x + y)_{min}$ = 7.8235 + 3.355 = 10.1785

Example 16

If x = 4.95 and y = 2.2 are each rounded off to the given number of decimal places. Calculate

(i) The percentage error in x + y

Solution

Δx = 0.005, Δy = 0.05

%error = $\left[\left|\frac{\Delta x}{X+Y}\right| + \left|\frac{\Delta y}{X+Y}\right|\right] x\ 100\%$ = $\left[\left|\frac{0.005}{4.95+2.2}\right| + \left|\frac{0.05}{4.95+2.2}\right|\right] x\ 100\%$ = 0.769

Alternatively

Working value x + y = 4.95 + 2.2 = 7.15

$|e_{x+y}|=|\Delta x| + |\Delta y|$= 0.005 + 0.05 = 0.055

% error = $\frac{0.055}{7.15} x\ 100\%$ = 0.769

(ii) Find the limit within which (x + y) is expected to lie. Give your answer to two decimal places.
Upper limit = 7.15 + 0.055 = 7.21; lower limit = 7.15 – 0.055 = 7.10
Alternatively
Upper limit = 4.955 + 2.25 = 7.21; lower limit = 4.945 + 2.15 = 7.10

Subtraction

Consider two numbers X and Y are approximated by x and y with errors Δx and Δy.

$|e_{x-y}| = |(x + \Delta x) - (y + \Delta y) - (x - y)|$

$=|\Delta x - \Delta y|= |\Delta x| + |\Delta y|$

R.E$_{max}$ = $\left[\left|\frac{\Delta x}{X-Y}\right| + \left|\frac{\Delta y}{X-Y}\right|\right]$

Alternatively

absolute error = $\frac{1}{2}[\max - min]$

$=\frac{1}{2}\{[(x + \Delta x) - (y - \Delta y)] - [(x + \Delta x) - (y + \Delta y)]\}$

$|e_{x-y}|$ = $|\Delta x + \Delta y|$= $|\Delta x| + |\Delta y|$

R.E$_{max}$ = $\left[\left|\frac{\Delta x}{X-Y}\right| + \left|\frac{\Delta y}{X-Y}\right|\right]$

Example 17

Given number x = 6.375 and y 4.46 rounded off to the given number of decimal places. Find the limit within which (x – y) lies

Solution

Δx = 0.0005, Δy = 0.005

$|e_{x-y}|$ = $|\Delta x| + |\Delta y| =$ $|0.0005| + |0.005|$ = 0.0055

working value = 6.375 – 4.46 = 1.915

Upper limit = 1.915 + 0.0055 = 1.9205; Lower limit = 1.915 – 0.0055 = 1.9095

Alternatively

$(x – y)_{max}$ = 6.3755 – 4.455 = 1.9205

$(x – y)_{min}$ = 6.3745 – 4.465 = 1.9095

Example 18

If x = 1.563 and y = 9.87 are each rounded off to the given number of decimal places. Calculate

(i) the percentage error in (x – y)

% error = $\left[\left|\frac{\Delta x}{X-Y}\right| + \left|\frac{\Delta y}{X-Y}\right|\right] x\ 100\% = \left[\left|\frac{0.0005}{1.563-9.87}\right| + \left|\frac{0.005}{1.563-9.87}\right|\right] x100\%$ = 0.0662

Alternatively

Working value = x – y = 1.563 9.87 = -8.307

$|e_{x-y}|$ = $|\Delta x| + |\Delta y| = |0.0005| + |0.005|$ =0.0055

$\%\ error = \frac{0.0055}{-8.307}\ x\ 100\% = 0.0662$

(ii) the limit within which (x – y) is expected to lie. Give your answer to three decimal places

Upper limit = -8.307 + 0.0055 = -8.302

Lower limit = -8.307 - 0.0055 = -8.313

Alternatively

$(x – y)_{max}$ = 1.5635 – 9.865 =-8.302

$(x – y)_{min}$ =1.5625 – 9.875 =-8.313

Multiplication

Consider two numbers X and Y are approximated by x and y with errors Δx and Δy.

$|e_{xy}| = |(x + \Delta x)(y + \Delta y) - (xy)|$

$= |xy + y\Delta x + x\Delta y + \Delta x\Delta y - xy|$

Since Δx and Δy are very small, $\Delta x\Delta y \approx 0$

$|e_{xy}| = |y\Delta x + x\Delta y| = |y\Delta x| + |x\Delta y|$

R.E$_{max}$ = $\left|\frac{y\Delta x}{xy}\right| + \left|\frac{x\Delta y}{xy}\right|$

$= \left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|$

Alternatively

absolute error = $\frac{1}{2}|\max - min|$

$= \frac{1}{2}\left[(x + \Delta x)(y + \Delta y) - [(x - \Delta x)(y - \Delta y)]\right]$

$|e_{xy}| = |y\Delta x + x\Delta y| = |y\Delta x| + |x\Delta y|$

R.E$_{max}$ = $\left|\frac{y\Delta x}{xy}\right| + \left|\frac{x\Delta y}{xy}\right|$

Example 19

Given numbers x = 6.375 and y = 4.46 rounded off to eh given number of decimal places. Find the limit within which (xy) lies

Solution

Δx =0.0005 Δy = 0.005

$|e_{xy}| = |y\Delta x| + |x\Delta y| = |6.375\ x\ 0.005| + |4.46\ x\ 0.0005|$ = 0.0341

working value = xy = 6.375 x 4.46 = 28.4325

Upper limit = 28.4325 + 0.0341 =28.4666

Lower limit = 28.4325 - 0.0341 = 28.3984

Alternatively

$(xy)_{max}$ = 6.3755 x 4.465 =28.4666

$(xy)_{min}$ = 6.3745 x 4.455 = 28.3984

Example 20

If x = 1.563 and y = 9.87 are each rounded off to the given number of decimal places. Calculate

(i) Percentage error in (xy)

Δx = 0.0005, Δy = 0.005

% error $=\left\{\left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|\right\} x\ 100\% = \left\{\left|\frac{0.0006}{1.563}\right| + \left|\frac{0.005}{9.87}\right|\right\} x\ 100\% = 0.0826$

Alternatively

Working value = 1.563 x 9.87 = 15.4268

$|e_{xy}|$ = $|y\Delta x| + |x\Delta y|$ = 9.87 x 0.0005 + 1.563 x 0.005 =0.0128

$$\%\ error = \frac{0.0128}{15.4268}\ x\ 100\% = 0.0826$$

(ii) the limit within which (xy) is expected to lie. Give your answer to three decimal places.

Upper limit = 15.4268 + 0.0128 = 15.440

Lower limit =15.4268 - 0.0128 =15.414

Alternatively

Upper limit = 1.5635 x 9.875 = 15.440

Lower limit = 1.5625 x 9.865 =15.414

Division

Consider two numbers X and Y are approximated by x and y with errors Δx and Δy.

$\left|ex_{/y}\right| = \left|\frac{x+\Delta x}{y+\Delta y} - \frac{x}{y}\right| = \left|\frac{xy+y\Delta x-x\Delta y-xy}{y^2+y\Delta y}\right|$

$= \left|\frac{y\Delta x-x\,\Delta y}{y^2\left(1+\frac{\Delta y}{y}\right)}\right|$

Since Δx and Δy are very small, then $\frac{\Delta y}{y} \approx 0$

$\left|ex_{/y}\right| = \left|\frac{y\Delta x-x\,\Delta y}{y^2}\right|$

$\left|ex_{/y}\right| \leq \frac{|y\Delta x|+|x\,\Delta y|}{|y^2|}$

$e_{max} = \frac{|y\Delta x|+|x\,\Delta y|}{|y^2|}$

$R.E_{max} = \frac{|y\Delta x|-|x\,\Delta y|}{|y^2|} \div \frac{x}{y}$

$R.E_{max} = \left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|$

Alternatively

absolute error = $\frac{1}{2}|\max - min|$

$= \frac{1}{2}\left|\frac{(x+\Delta x)}{(y-\Delta y)} - \frac{(x-\Delta x)}{(y+\Delta y)}\right|$

$ex_{/y} = \left|\frac{x\Delta y+y\Delta x}{y^2-\Delta y^2}\right|$

Since Δx and Δy are very small, then $\Delta y^2 \approx 0$

$\left|ex_{/y}\right| = \left|\frac{y\Delta x-x\,\Delta y}{y^2}\right|$

$\left|ex_{/y}\right| \leq \frac{|y\Delta x|+|x\,\Delta y|}{|y^2|}$

$e_{max} = \frac{|y\Delta x|+|x\,\Delta y|}{|y^2|}$

$R.E_{max} = \frac{|y\Delta x|-|x\,\Delta y|}{|y^2|} \div \frac{x}{y}$

$R.E_{max} = \left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|$

Example 21

Given numbers x = 5.794 and y = 0.28 rounded off to the given number of decimal places. Find limit within which $\frac{x}{y}$ lies

Solution

Δx = 0.0005, Δy = 0.005

$\left|e_{x/y}\right| = \frac{|y\Delta x|+|x\,\Delta y|}{|y^2|}$

$= \frac{|0.28\ x\ 0.0005|+|5.794\ x\ 0.005|}{|0.28^2|}$

= 0.3713

Working value $= \frac{x}{y} = \frac{5.794}{0.28} = 20.6929$

Upper limit = 20.6929 + 0.3713 =21.0642

Lower limit =20.6929 - 0.3713 = 20 .3198

Alternatively

Upper limit = $\frac{5.7945}{0.275}$ =21.079

Lower limit = $\frac{5.7935}{0.285}$ = 20.3281

Example 22

If x = 7.37 and y = 2.00 are each rounded off to the given number of decimal places. Calculate

(i) Percentage error

% error = $=\left\{\left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|\right\} x100\% = \left\{\left|\frac{0.005}{7.37}\right| + \left|\frac{0.005}{2.00}\right|\right\} x100\% = 0.318$

Alternatively

$\left|e_{x/y}\right| = \frac{|y\Delta x|+|x\,\Delta y|}{|y^2|} = \frac{|2.00\ x\ 0.005|+|7.37\ x\ 0.005|}{|2.00^2|} = 0.0117$

Working value $= \frac{x}{y} = \frac{7.37}{2.00} = 3.685$

% error $= \frac{0.0117}{3.685}\ x\ 100 = 0.318$

(ii) the limit within which $\left(\frac{x}{y}\right)$ is expected to lie. Give your answer to three decimal places.

Upper limit = $3.685 + 0.318$ = 3.697

Lower limit = $3.685 - 0.318$ = 3.673

Alternatively

Upper limit = $\frac{7.375}{1.995} = 3.697$

Lower limit = $\frac{7.365}{2.005} = 3.673$

Error in functions

Given a function f(x) with a maximum possible error Δx.

Absolute error, $|e| = |\Delta x| f^1(x)$

Maximum possible relative error, R.E = $\frac{|\Delta x| f^1(x)}{f(x)}$

Example 23

Find the absolute error and maximum relative error in each of the following functions

(i) $y = x^4$

$|e| = |\Delta x| f^1(x) = 4x^3|\Delta x|$

R.E = $\frac{|\Delta x| f^1(x)}{f(x)} = \frac{4x^3|\Delta x|}{x^4} = \frac{4|\Delta x|}{x}$

(ii) $y = x^{\frac{3}{2}}$

$|e| = |\Delta x| f^1(x) = \frac{3}{2}x^{\frac{1}{2}}|\Delta x|$

R.E = $\frac{|\Delta x| f^1(x)}{f(x)} = \frac{\frac{3}{2}x^{\frac{1}{2}}|\Delta x|}{x^{\frac{3}{2}}} = \frac{3}{2}\frac{|\Delta x|}{x}$

(iii) $y = \sin x$

$|e| = |\Delta x| f^1(x) = \cos x|\Delta x|$

R.E = $\frac{|\Delta x| f^1(x)}{f(x)} = \frac{\cos x|\Delta x|}{\sin x} = |\Delta x||\cot x|$

Example 24

Given that the error in measuring an angle is 0.4^0. find the maximum possible error and relative error in tanx if $x = 60^0$.

Solution

$|e| = |\Delta x| f^1(x) = (1 + \tan^2 x)|\Delta x|$

$|e| = (1 + \tan^2 60)\left|\frac{0.4}{180}\pi\right| = 0.0280$

R.E = $\frac{0.0280}{\tan 60} = 0.0162$

Error in a function that has more variables

Given a function f(x, y) with a maximum possible error Δx and Δy respectively

Absolute error, $|e| = |\Delta x| f^1(x) + |\Delta y| f^1(y)$

Maximum possible relative error = $\frac{|\Delta x| f^1(x) + |\Delta y| f^1(y)}{f(x,y)}$

Example 25

Given that X and Y are rounded off to give x and y with error Δx and Δy respectively. Show that the maximum relative error recorded in x^4y is given by $4\left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|$

Solution

$|e| = |\Delta x| f^1(x) + |\Delta y| f^1(y) = |\Delta x 4x^3 y| + \Delta y x^4$

$|e| \leq 4|x^3y||\Delta x| + |x^4||\Delta y|$

$|e_{max}| = 4|x^3y||\Delta x| + |x^4||\Delta y|$

R.E = $\frac{4|x^3y||\Delta x| + |x^4||\Delta y|}{x^4y}$

$= 4\left|\frac{\Delta x}{x}\right| + \left|\frac{\Delta y}{y}\right|$

Example 26

Show that the maximum possible relative error in ysin^2x is

$\left|\frac{\Delta y}{y}\right| + 2\cot x\,|\Delta x|$, where$\Delta x$ and Δy are errors in x and y respectively

Hence find the percentage error in calculating ysin^2x if y = 5.2 $\pm$ 0.05 and x = $\frac{\pi}{6} \pm \frac{\pi}{360}$ (07 marks)

z = y$\sin^2$x

$e_z = \Delta y sin^2x$ + 2y$\Delta x cos\ xsinx$

$\frac{e_z}{z} = \frac{\Delta y sin^2x}{\text{ysin}^2\text{x}} + \frac{2\text{y}\Delta x cos\ xsinx}{\text{ysin}^2\text{x}}$

$\left|\frac{e_z}{z}\right| = \left|\frac{\Delta y}{y} + 2cotx.\Delta x\right|$

$\leq \left|\frac{\Delta y}{y}\right| + 2cotx.|\Delta x|$

$\therefore$ Maximum possible error is $\left|\frac{\Delta y}{y}\right| + 2cotx.|\Delta x|$

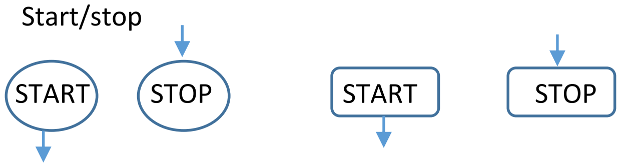
$percentage\ error = \left[\frac{0.05}{5.2} + 2\cot\frac{\pi}{6}.\ \left|\frac{\pi}{360}\right|\right]\ x\ 100\%$ = 3.9845%

Flowcharts in mathematics

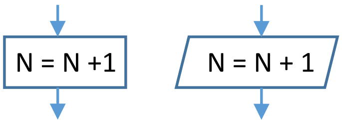
A flow chart is a diagram comprising of systematic steps followed in order to solve a problem.

Shapes used

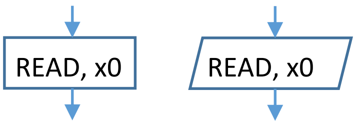
1. Start/stop



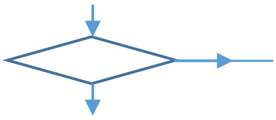
2. OPERATIONASSIGNMENT



This indicates that the new number is obtained by adding one to the previous N



3. Decision box



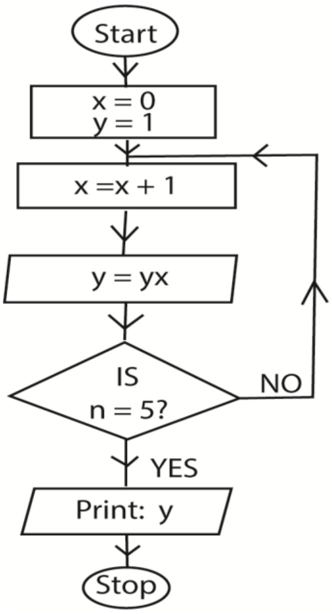
Note: all other shapes can be interchanged except for the decision box

Dry run or trace

This is the method of predicting the outcome of a given flow chart using a table

Example 1

Perform a dry run and state the purpose of the flowchart



Solution

Dry run

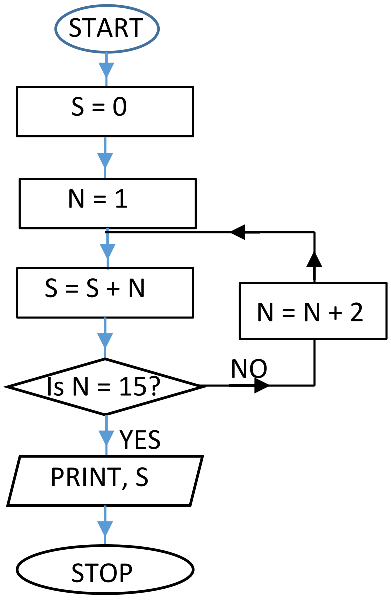
x	y
0	1
1	1
2	2
3	6
4	24
5	120
6	720

Purpose is to compute and print 6!

Relationship is y = x!

Example 2

Study the flow chart below and perform dry run of the flowchart



Solution

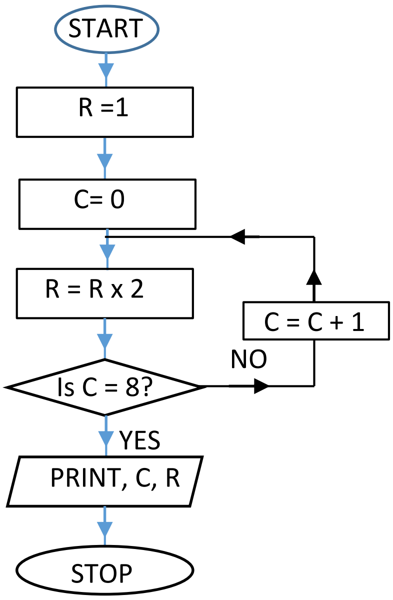
Dry run

N	S	Is N = 15?
1	1	NO
3	4	NO
5	9	NO
7	16	NO
9	25	NO
11	36	NO
13	49	NO
15	64	YES

Purpose is to compute and print the first 8 square numbers

Example 3

Perform a dry run and state the purpose of the flowchart



Solution

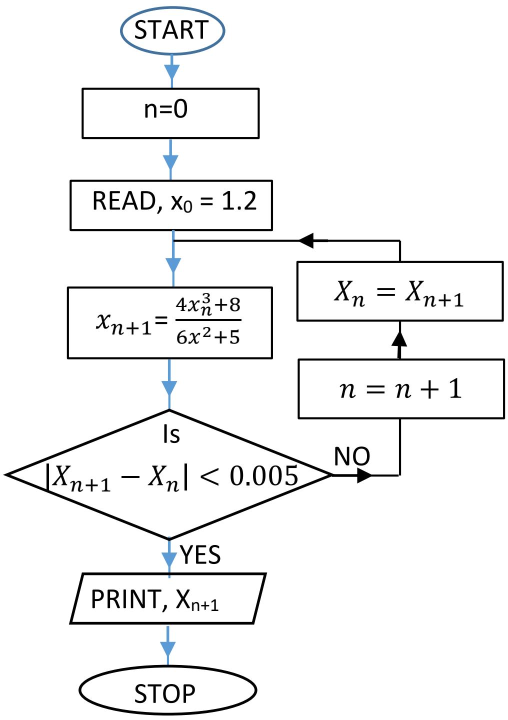
Dry run

C	R	Is C = 8?
0	1	NO
1	2	NO
2	4	NO
3	8	NO
5	32	NO
6	64	NO
7	128	NO
8	256	YES

Purpose is to compute and print 2^8

Example 4

The flowchart below is used to read the root of the equation $2x^3 + 5x – 8 = 0$



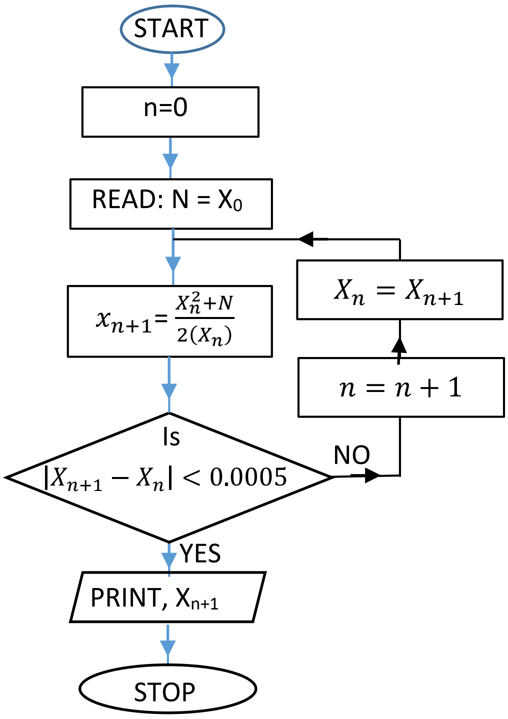
Carry out a dry run of the flow chart and obtain the root of $2x^3 + 5x – 8 = 0$ with an error less than 0.005

N	X_n	X_{n+1}	$\lvert X_{n+1} - X_n \rvert$
0	1.2	1.0933	0.1067
1	1.0933	1.0867	0.0066
2	1.0867	1.0866	0.001

Root is 1.087

Example 5

Study the flowchart below



(i) Carry out a dry run of the flowchart, taking N = 20, X0 = 4 and obtain the root of correct to 3dp.

(ii) State its purpose

Solution

N	X_n	X_{n+1}	$\lvert X_{n+1} - X_n \rvert$
0	4.0	4.5	0.5
1	4.5	4.4722	0.0278
2	4.4722	4.4721	0.0001

Root is 4.472

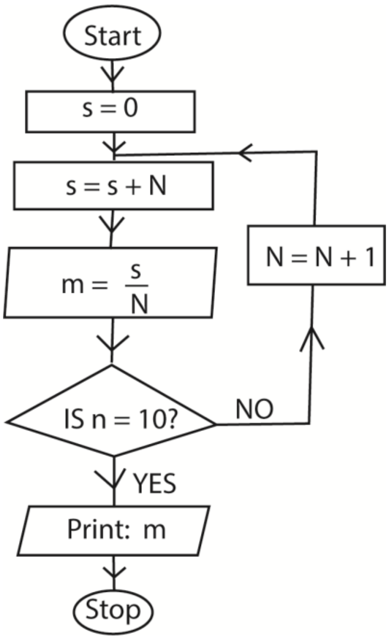
(ii) to print the square root of a number N

Constructing flowcharts

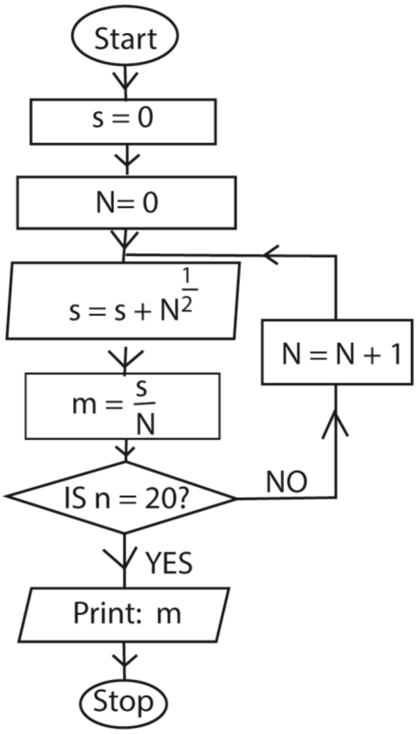
1. Draw a flow chart that reads and prints the mean of the first ten counting numbers

Solution

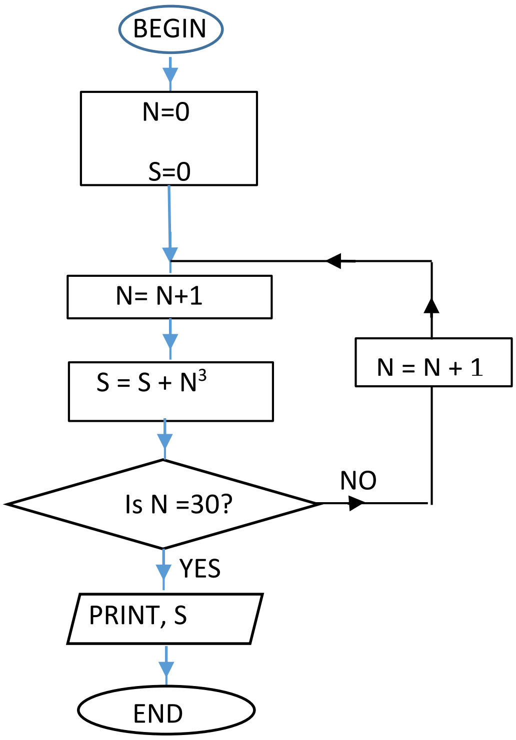
Let S be sum and m the mean



2. Draw a flowchart for computing and printing the mean of the square roots of the first 20 natural numbers

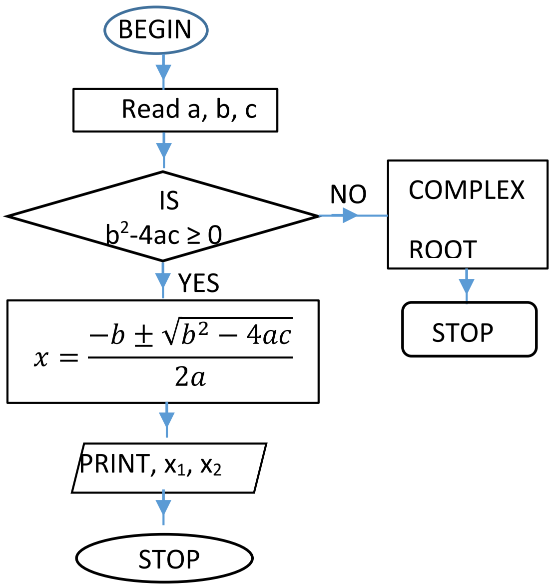


3. Draw a flowchart that computes and prints the sum of the cubes of the first 30 natural numbers



4. Draw a flowchart that computes the root of the equation ax2 + bx + c = 0

Solution



Newton Raphson's method and Flowcharts

Example 6

(a) Show that the iterative formula based on Newton Raphson's method for approximating the root of the equation 2Inx – x + 1 = 0 is given by

$x_{n+1} = x_n\left(\frac{2Inx_n-1}{x_n-2}\right)$, n = 0, 1,2 ………………………. (03marks)

f(x) = 2Inx – x + 1

$f'(x) = \frac{2}{x} - 1$

also

$f(x_n) = 2Inx_n - x_n + 1$

$f'(x_n) = \frac{2}{x_n} - 1$

Using $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

By substitution, we get

$$x_{n+1} = x_n - \frac{2Inx_n - x_n+1}{\left(\frac{2}{x_n}-1\right)}$$

$$= \frac{x_n\left(\frac{2}{x_n}-1\right) - 2Inx_n - x_n+1}{\left(\frac{2}{x_n}-1\right)}$$

$$= \frac{x_n(2-x_n) - x_n(2Inx_n - x_n+1)}{(2-x_n)}$$

$$= \frac{x_n(2-x_n - 2Inx_n + x_n-1)}{(2-x_n)}$$

$$= \frac{x_n(1- 2Inx_n)}{(2-x_n)}$$

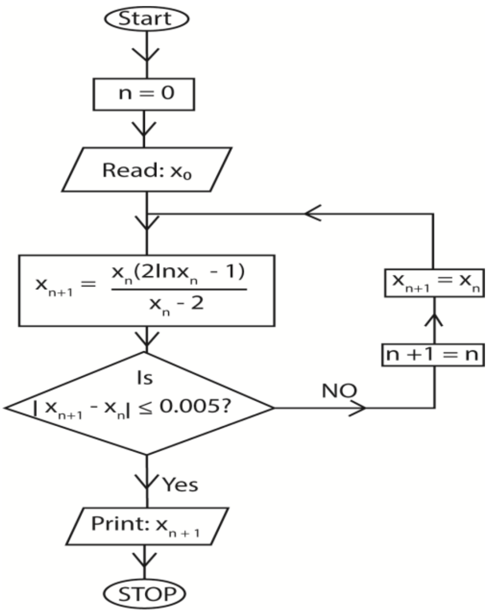
$$= \frac{-x_n(\ 2Inx_n-1)}{-(\ x_n-2)}$$

$$= \frac{x_n(\ 2Inx_n-1)}{(\ x_n-2)}$$

(b) Draw a flow chart that:

(i) reads the initial approximation x_0 of the root

(ii) computes and prints the root correct to two decimal places, using the formula in (a) (05marks)



(ii) Taking x_0 = 3.4, perform a dry run to find the root of the equation (04marks)

Dry run

n	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	3.4	3.51548	0.11548
1	3.51548	3.51286	0.00262
2	3.51286	3.51286	0.0000

Example 7

(a) Show that the Newton-Raphson formula for finding the root of the equation x = $N^{\frac{1}{5}}$ is given by $X_{n+1} = \frac{4X_n^5+N}{5X_n^4}$, n = 0, 1, 2, ... (04marks)

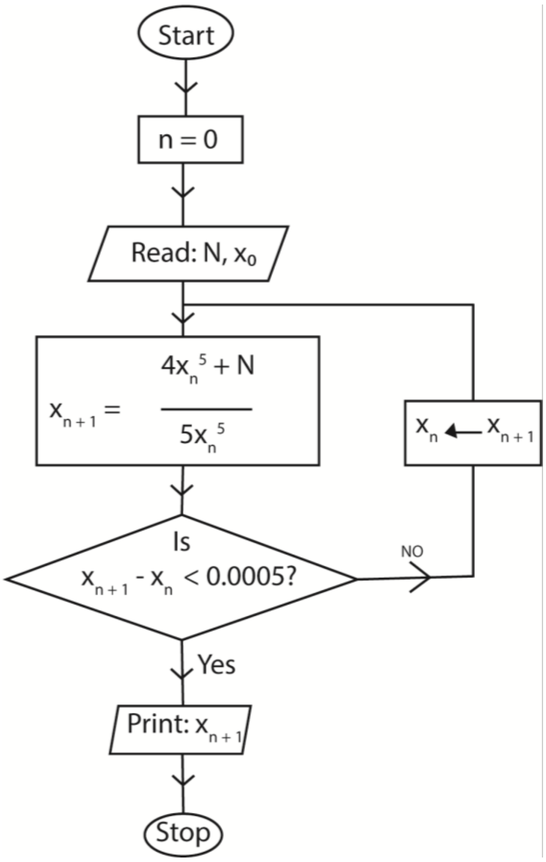
$x = N^{\frac{1}{5}}$
$x^5 = N$
$x^5 - N = 0$
Let f(x) = $x^5 - N$
$f(x_n) = x_n^5 - N$
$f'(x_n) = 5x_n^4$

Using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^5-N}{5x_n^4} = \frac{5x_n^5 - x_n^5 - N}{5x_n^4}$$

$$x_{n+1} = \frac{4x_n^5+N}{5x_n^4}$$, n = 0, 1, 2 ...

(b) Construct a flow chart that
(i) reads N and the first approximation x_0.
(ii) computes the root to three decimal places
(iii) Prints the root (x_n) and the number of iteration (n) (05marks)



(c) Taking N = 50, x_0 = 2.2, perform a dry run for the flow chart. Give your root correct to three decimal places.(03marks)

N = 50, x_o = 2.2

n	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	2.2	2.18688	0.01312
1	218688	2.18672	0.00016

Root = 2.187(3D)

Example 8

(a) Show that the iterative formula based on Newton Raphson's method for solving the equation

Inx + x – 2 = 0 is given by

$X_{n+1} = \frac{x_n(3-Inx_n)}{1+x_n}, n = 0, 1,2 ....$ (04marks)

$let\ f(x) = Inx + x - 2$

$f(x_n) = Inx_n + x_n - 2$

$f'(x) = \frac{1}{x_n} + 1 = \frac{1+x_n}{x_n}$

Using N.R.M

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$= x_n - \left(\frac{Inx_n + x_n - 2}{\frac{1+x_n}{x_n}}\right)$

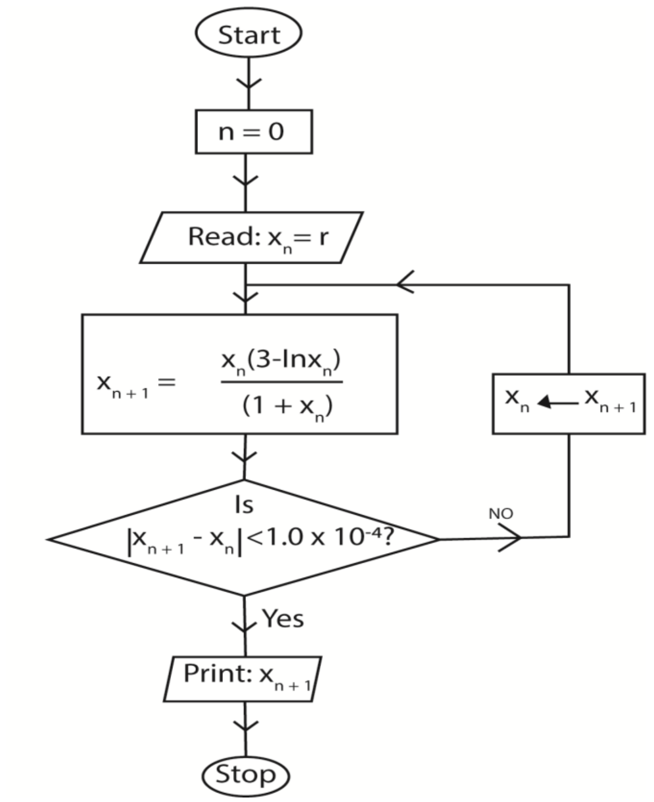
$= \frac{x_n}{1} - \frac{x_n(Inx_n + x_n - 2)}{1+x_n}$

$= \frac{x_n(1+x_n - Inx_n - x_n + 2)}{1+x_n}$

(b)(i) Construct a flow chart that ;

- reads the initial approximation as r

- computes using the interactive formula in (a), and prints the root of equation Inx + x -2 = 0, when the error is less than 1.0 x 10^{-4}.



(ii) Perform a dry run of the flow chart when r = 1.6. (08marks)

n	x_n	X_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	1.6	1.5569	0.0431
1	1.5569	1.5571	0.0002
2	1.5571	1.5571	0.0000

Hence the root = 1.557(3D)

Example 9

(a) Show that iterative formula based on Newton Raphson's method for approximating the sixth root of a number N is given by $x_{n+1} = \frac{1}{6}\left(5x_n + \frac{N}{{x_n}^5}\right)$

(b) Draw a flowchart that

(i) Reads N and the initial approximation x_0 of the root

(ii) computes and prints the root to three decimal places

(c) Taking N = 60, x0 = 1.9, perform a dry run for the flow chart, give your root correct to three decimal places.

Solution

$x = N^{\frac{1}{6}}$

$x^6 = N$

$x^6 - N = 0$

Let f(x) = $x^6 - N$

$f(x_n) = x_n^6 - N$

$f'(x_n) = 6x_n^5$

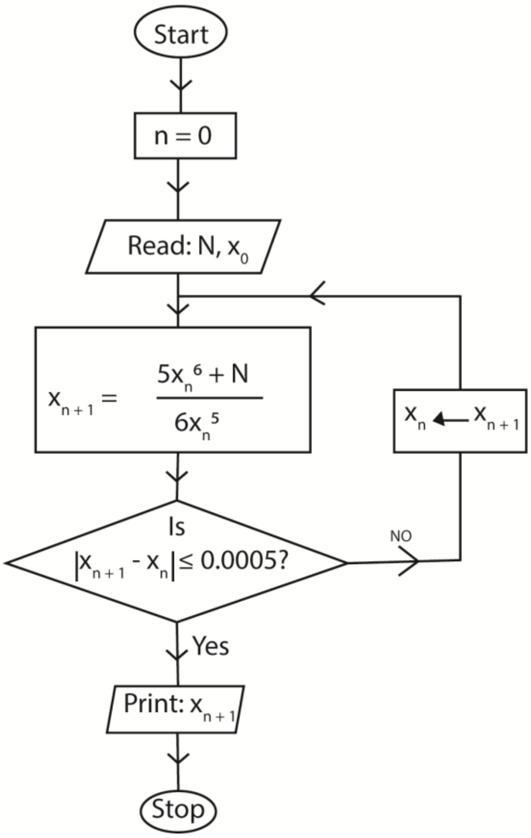
Using NRM

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^6 - N}{6x_n^5}$$

$$= \frac{x_n(6x_n^5) - (x_n^6 - N)}{6x_n^5}$$

$$x_{n+1} = \frac{5x_n^6 + N}{6x_n^5}, \text{ n = 0, 1, 2 …}$$

(b) Flowchart



(c) Dry run

n	x_n	x_{n+1}	$\lvert x_{n+1} - x_n\rvert$
0	1.9	1.9872	0.082
1	1.9872	1.9787	0.0085
2	1.9787	1.9786	0.0001

Example 10

(a) Show that the iterative formula based on Newton Raphson's method for approximating the fourth root of a number N is given by $x_{n+1} = \frac{3}{4}\left(x_n + \frac{N}{3{x_n}^3}\right)$.

(b) Draw a flowchart that
  (i) Records N and initial approximation x_0 of the root
  (ii) computes and prints the root after four iterations.

(c) Taking N = 39.0, x0 = 2.0, perform a dry run for the flowchart, give your root correct to three decimal places

Solution

$x = N^{\frac{1}{4}}$
$x^4 = N$
$x^4 - N = 0$
Let f(x) = $x^4 - N$
$f(x_n) = x_n^4 - N$
$f'(x_n) = 4x_n^3$

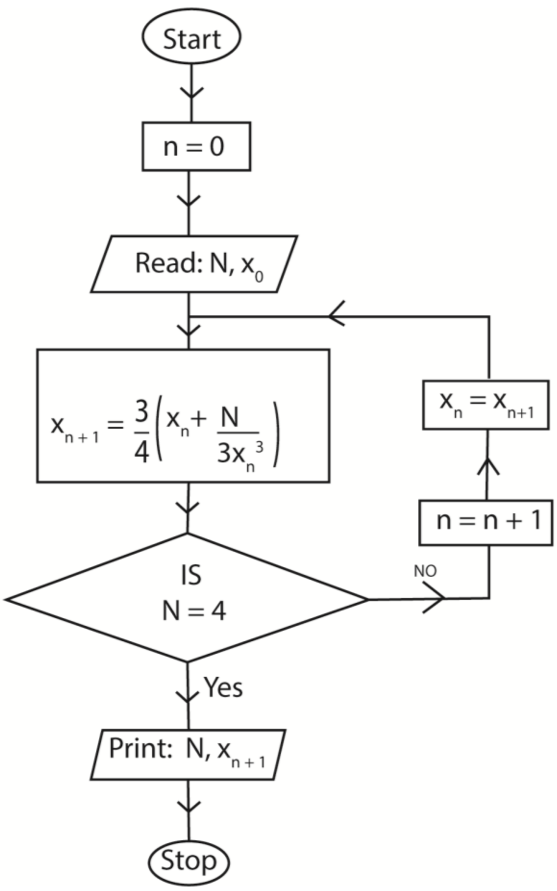
Using NRM

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - N}{4x_n^3}$$

$$= \frac{x_n(4x_n^3) - (x_n^4 - N)}{4x_n^3}$$

$$x_{n+1} = \frac{3x_n^4 + N}{4x_n^3} = \frac{3}{4}\left(x_n + \frac{N}{3{x_n}^3}\right),\ n = 0, 1, 2 \ldots$$

(b) Flowchart



(c) Dry run

n	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	2.0	2.7188	0.7188
1	2.7188	2.5242	0.1945
2	2.5242	2.4994	0.0249
3	2.4994	4.4990	0.0004

Example 11

(a) Show that the iterative formula based on Newton's Raphson's method for finding the natural logarithm of a number N is given by $x_{n+1} = \frac{e^{x_n}(x_n-1)+N}{e^{x_n}}$, n = 0, 1, 2, …

(b) Draw a flowchart that
 (i) Records N and initial approximation x_0 of the root
 (ii) computes and prints the natural logarithm after four iteration and gives natural logarithm to 3 decimal places.

(c) Taking N = 10, x_0 = 2, perform a dry run for the flowchart, give your root correct to three decimal places

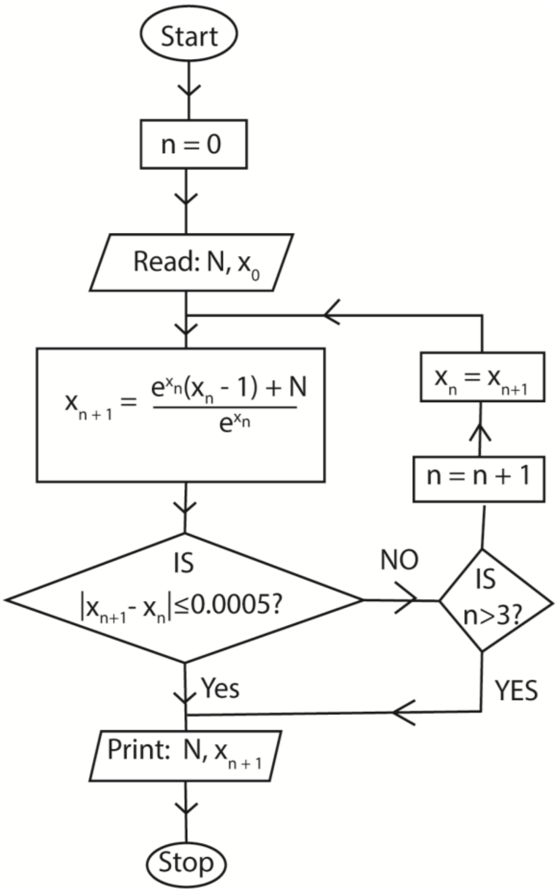
Solution

(a) x = lnN; e^x = N => $e^x - N = 0$
 $f(x) = e^x - N$; $f^1(x) = e^x$
 $x_{n+1} = x_n - \left(\frac{e^{x_n}-N}{e^x}\right)$

$= \frac{x_n e^x-(e^{x_n}-N)}{e^x}$

$= \frac{e^x(x_n-1)+N}{e^x}$

(b) Flowchart



(c) Dry run

n	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	2.0	2.3533	0.3533
1	2.3533	2.3039	0.0494
2	2.3039	2.3026	0.0013
3	2.3026	2.3026	0.0000

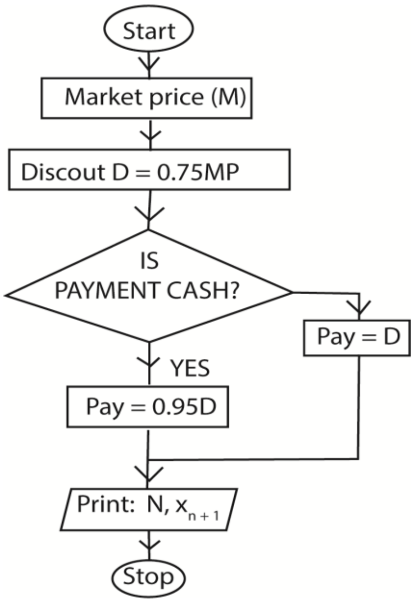
Example 12

A shop offers a 25% discount on all items in their store and a second discount of 5% for paying cash.

(a) Construct a flowchart for the above information

(b) perform a dry run for (i) a shoe of 75,000/= cash and (ii) a shirt of 45,000/= credit

(a) Flowchart



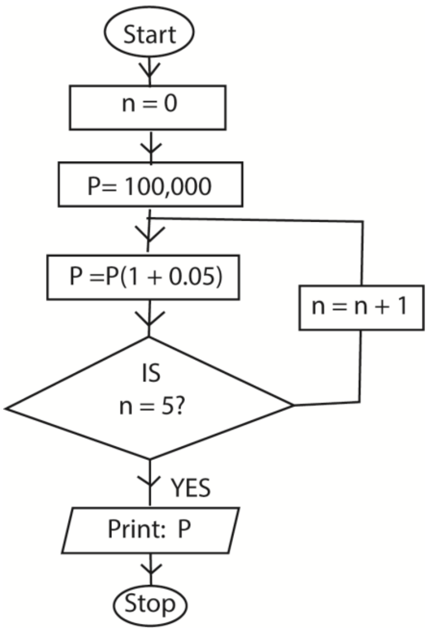
(c) dry run

MP	D= 0.75MP	Pay	Cash = 0.95D	Credit = D
75,000	56,250	cash	53437.50	--------
45,000	33,750	credit	-----------	33750

Example 13

Given that a man deposited 100,000/= to a bank which gives a compound interest of 5%. Draw a flowchart to compute the amount of money accumulated after 5 years and perform a dry run for the flowchart.

Flowchart



Dry run

n	P	A
0	100,000	100,000
1	100,000	105,000
2	105,000	110,250
3	110,250	115,762.0
4	115,762,5	121,550.625
5	121,550.625	127,628.1563

Revision Exercise

1. (a) show that the iterative formula based on Newton Raphson's method for approximating the cube root of a number N is given by $x_{n+1} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right), n = 0, 1, 2, \ldots$
   (b) Draw a flowchart that
      (i) reads N and the initial approximation x_0 of the root
      (ii) computes and prints the root to three decimal places.
   (c) Taking N = 54, x_0 = 3.7, perform a dry run for the flowchart, give your root to three decimal places [3.780]
2. (a) show that the iterative formula based on Newton Raphson's method for approximating the fourth root of a number N is given by $x_{n+1} = \frac{3}{4}\left(x_n + \frac{N}{3x_n^3}\right), n = 0, 1, 2, \ldots$
   (b) Draw a flowchart that
      (i) reads N and the initial approximation x_0 of the root
      (ii) computes and prints the root to two decimal places.
   (c) Taking N = 35, x_0 = 2.3, perform a dry run for the flowchart, give your root to two decimal places. [2.43]
3. (a) show that the iterative formula based on Newton Raphson's method for finding the root of a number $N^{\frac{1}{5}}$ is given by $x_{n+1} = \left(\frac{4x_n^5+N}{5N_n^4}\right), n = 0, 1, 2, \ldots$
   (c) Draw a flowchart that
      (i) reads N and the initial approximation x_0 of the root
      (ii) computes and prints the root to three decimal places.
   (d) Taking N = 50, x_0 = 2.2, perform a dry run for the flowchart, give your root to three decimal places [2.187]
4. (a) show that the iterative formula based on Newton Raphson's method for approximating the cube root of 2Inx – x + 1 = 0 is given by $x_{n+1} = x_n\left(\frac{2Inx_n-1}{x_n-2}\right), n = 0, 1, 2, \ldots$
   (b) Draw a flowchart that
      (i) reads N and the initial approximation x_0 of the root
      (ii) computes and prints the root
   (c) Taking x_0 = 3.4, perform a dry run for the flowchart, give your root to three decimal places
5. (a) show that the iterative formula based on Newton Raphson's method for approximating the cube root of Inx + x – 2 = is given by $x_{n+1} = x_n\left(\frac{3-Inx_n}{1+x_n}\right), n = 0, 1, 2, \ldots$
   (b) Draw a flowchart that
      (iii) reads N and the initial approximation r of the root
      (iv) computes and prints the root of the equation, when the error is less than 10. x 10^{-4}.
   (c) Taking r = 1.6, perform a dry run for the flowchart, give your root to three decimal places
6. (a) show that the iterative formula based on Newton Raphson's method for approximating the cube root of x = In(x +2) is given by $x_{n+1} = \frac{e^{x_n}(x_n-1)+2}{e^{x_n}-1} n = 0, 1, 2, \ldots$
   (b) Draw a flowchart that
      (i) reads the initial approximation x_0 of the root
      (ii) computes and prints the root to three decimal places
   (c) Taking x_0 = 1.2, perform a dry run for the flowchart, give your root to three decimal places
7. (a) show that the iterative formula based on Newton Raphson's method for finding the natural logarithm of number N is given by $x_{n+1} = \frac{e^{x_n}(x_n-1)+N}{e^{x_n}}, n = 0, 1, 2, \ldots$
   (b) Draw a flowchart that

(i) reads N and the initial approximation x_0 of the root
(ii) computes and prints the root to two decimal places.

(c) Taking N = 45, x_0 = 3.5, perform a dry run for the flowchart, give your root to two decimal places [3.81]

8. (a) show that the iterative formula based on Newton Raphson's method for finding the root of the $2x^3 + 5x - 8$ is given by $x_{n+1} = \left(\frac{4x_n^3 + 8}{6x_n^2+5}\right), n = 0, 1, 2, \ldots$

(b) Draw a flowchart that
(i) reads N and the initial approximation α of the root
(ii) computes and prints the root when the error is less than 0.001.

(c) Taking α= 1.1, perform a dry run for the flowchart, give your root to three decimal places [1.087]

9. A shop offers a 25% discount on all items in their store and a second discount of 5% for paying cash.

(a) Construct a flowchart for the above information
(b) perform a dry run for (i) a radio of 125,000/= cash and (ii) a T.V of 340,000/= credit [89.062.50, 255,0000]

10. Given that a man deposited 120,000/= to a bank which gives a compound interest of 15%. Draw a flowchart to compute the amount of money accumulated after 4 years and perform a dry run for the flowchart. [209,880.75/=]

Linear interpolation and extrapolation

Linear interpolation

Deals with computations of values that lie within a given range

Example 1

The table below shows values of a function f(x)

x	1.8	2.0	2.2	2.4
f(x)	0.532	0.484	0.436	0.384

Find values of (i) f(1.88) (ii) x corresponding to f(x) = 0.4

Solution

1.8	1.88	2.0
0.532	y	0.484

$\frac{y-0.532}{1.88-1.8} = \frac{0.884-0.532}{2.0-1.8}$

y = 0.513

(ii)

2.2	x_0	2.4
0.436	0.4	0.384

$\frac{x_0-2.2}{0.4-0.436} = \frac{2.4-2.2}{0.384-0.436}$

x_0 =2.34

Example 2

Given the table below

x	9	10	11	12
f(x)	2.66	2.42	2.18	1.92

Using linear interpolation find

(i) f(x) when x = 10.15 (ii) f^{-1} (2.02)

Solution

10	10.15	11
2.42	y	2.18

$\frac{y-2.42}{10.15-10} = \frac{2.18-2.42}{11-10}$

y =2.384

11	x_0	12
2.18	2.02	1.92

$\frac{x_0-11}{2.02-2.18} = \frac{12-11}{1.92-2.18}$

x_0 = 11.62

Example 3

Given table below

x^0	40.0^0	40.4^0	40.8^0	50.4^0
$\sin x^0$	0.6428	0.6481	0.6534	0.7705

Find (i) $\sin 40.5^0$ (ii) $\sin^{-1} 0.6445$

Solution

40.4^0	40.5^0	40.8^0
0.6481	y	0.6534

$$\frac{y-0.6481}{40.5-40.4} = \frac{0.6534-0.6481}{40.8-40.4}$$

y = 0.6494

40.0^0	x_0	40.4^0
0.6428	0.6445	0.6481

$$\frac{x_0-40.0}{0.6445-0.6428} = \frac{40.4-40.0}{0.6481-0.6428}$$

$x_0 = 40.13$

Linear extrapolation

This deals with computation of values that lie outside given values

Example 4

Given the table below

x	2.2	2.6	3.1
x^3	10.648	17.576	29.791

Find 3.4^3

2.6	3.1	3.4
17.576	29.791	y

$$\frac{y-29.791}{3.4-3.1} = \frac{29.791-17.576}{3.1-2.6}$$

y = 37.12

Example 5

The table below is an extract from table of sec x

$x = 60^0$	0’	12’	24’	36’	48’
$sec\, x$	2.0000	2.0122	2.0245	2.0371	2.0498

Use linear interpolation to determine

(i) sec $60^0 15’$

(ii) angle whose secant is 2.0436 [$60^0 42’$]

Solution

(i)

12’	15’	24’
2.0122	y	2.0245

$$\frac{y-2.0122}{15-12} = \frac{2.0245-2.0122}{24-12}$$

y = 2.03065

(ii)

36’	x	48’
2.0371	2.0436	2.0498

$$\frac{x-36}{2.0436-2.0371} = \frac{48-36}{2.0498-2.0371}$$

x = 42’ hence angle = $60^0 42’$

Example 6

The table below shows the values of a function f(x)

x	1.8	2.0	2.2	2.4
f(x)	0.532	0.484	0.436	0.384

Use linear interpolation to find the value of

(i) F(2.08)

1.8	2.08	2.0
0.532	f(x)	0.484

$$\frac{f(x)-0.436}{2.08-2.0}=\frac{0.436-0.484}{2.2-2.0}$$

$$\frac{f(x)-0.436}{0.08}=\frac{-0048}{0.2}$$

f(x) = 0.4648 or 0.465 (3D)

(ii) x corresponding to f(x) = 0.5 (05marks)

1.8	x	2.0
0.532	0.5	0.484

$$\frac{0.5-0.532}{x-1.8}=\frac{0.484-0.532}{2.0-18}$$

$$\frac{-0.032}{x-1.8}=\frac{-0.048}{0.2}$$

x = 1.9333 or 1.9 (1D)

Example 7

Given the table below,

x	0	10	20	30
y	6.6	2.9	-0.1	-2.9

Use linear interpolation to find

(a) y when x = 16

Extract

x	10	16	20
y	2.9	y_0	-0.1

$$\frac{y_0-2.9}{16-10}=\frac{-0.1-2.9}{20-10}$$

$$\frac{y_0-2.9}{6}-\frac{-3.0}{10}$$

y_0 = 1.1

hence when x = 16, y =1.1

(b) x when y = -1

Extract

x	20	x_0	30
y	-0.1	-1	-2.9

$$\frac{x_0-20}{-1-(-0.1)}=\frac{30-20}{-2.9-(-0.1)}$$

x_0 = 23.2

Hence when y = -1; x = 23.2

Example 8

The table below shows the values of a function f(x) for given values of x.

x	9	10	11	12
f(x)	2.66	2.42	2.18	1.92

Use linear interpolation or extrapolation to find

(a) f(10.4)

Extract

10	10.4	11
2.42	f(x)	2.1

Using gradient approach

$\frac{2.18-f(x)}{11-10.4} = \frac{2.18-2.42}{11-10}$

$\frac{2.18-f(x)}{0.6} = \frac{-0.24}{1}$

f(x) = 2.18 + 0.24 x 0.6 = 2.324

(b) the value of x, corresponding to f(x) = 1.46 (05marks)

Extract

x	12	11
1.46	1.92	2.18

Using gradient approach

$\frac{2.18-1.46}{11-x} = \frac{2.18-1.92}{11-12}$

$\frac{0.72}{11-x} = \frac{0.26}{-1}$

X = 11 - $\frac{-1\ x\ 0.72}{0.26}$ = 13.769

Revision exercise

1. Table below is an extract from the table of cos x

x	0^0	10^0	20^0	30^0	40^0	50^0
Cos x	0.1736	0.1708	0.1679	0.1650	0.1622	0.1593

Use linear interpolation to determine: (i) cos 80^0 36’ [0.1633] $\cos^{-1}(0.1685)$ [$80^0$18’]

2. The table below shows variation of temperature with time in a certain experiment.

Time (s)	0	120	240	360	480	600
Temperature (^{0}C)	100	80	76	65	50	48

Use linear interpolation to determine

(i) value of ^{0}C corresponding to 400s [62^0C]

(ii) time at which the temperature is 77^0C [192s]

3. The table below shows the value of a function In(x) for given values of x

x	1.4	1.5	1.6	1.7
In(x)	0.3365	0.4055	0.4700	0.5306

Using linear interpolation or extrapolation, find

(i) In(1.66) [0.5064] (ii) find value of x corresponding to In(x) = 0.400 [1.492]

4. The table below shows variation of temperature with time in certain experiment.

Time (s)	0	10	15	20	30
Temperature (^{0}C)	80	70.2	65.8	61.9	54.2

Use linear interpolation to determine

(i) value of θ^0 corresponding to T= 18s [63.5^0C]

(ii) Time T at which the temperature θ^0 = 60^0C [22.5s]

5. Given the table below

x	-1.0	-0.5	-1.4
y	-1.0	-2.2	-3.7

Using linear interpolation or linear extrapolation to find

(i) y when x = 0.5 [-4.6] (ii) x when y = -4.5 [0.458]

6. In an examination, scaling is done such that candidate A who originally scored 35% gets 50% and candidate B with 40% gets 65%, determine the original mark for candidate C whose new mark is 80% [45%]
7. The table below is an extract of $\log_{10} x$

x	80.00	80.20	80.50	80.80
$\log_{10} x$	1.9031	1.9042	1.9058	1.9074

Using linear interpolation find

(i) $\log_{10} 80.759$[1.9072]

(ii) the number whose logarithm is 1.90388 [80.14]

8. The table below shows the values of a function f(x) for given values of x

x	2	3	4	5
$f(x)$	3.88	5.11	8.14	11.94

Use linear interpolation to determine

(i) f(2.15) [4.06]

(ii) the value of x corresponding to f(10.6) [4.68]

9. The table below shows distance in km a truck moves with a given amount of fuel in litres (l)

Distance (km)	20	28	33	42
$Fuel\ (l)$	10	13	21	24

Use linear interpolation or extrapolation to find

(i) How far the truck can move on 27.5l of fuel [52.5km]

(ii) the amount of fuel required to cover29.8km [15.88l]

10. The table below shows the values of a continuous f(t) with respect to t

t	0	0.3	0.6	1.2	1.6
$f(t)$	2.72	3.00	3.32	4.06	4.95

Use linear interpolation or extrapolation, find

(i) f(t) when t = 0.9 [3.69]

(ii) the value of t corresponding to f(t) = 4.48 [1.48]

11. The table below shows the delivery charges by courier company

Mass (g)	200	400	600
$charges\ (shs.)$	700	1200	300

Use linear interpolation or extrapolation, find

(i) the delivery charge of a parcel weighing 352g [1080]

(ii) mass of a parcel whose delivery charge is shs. 3,300 [633.33kg]

12. The table below shows the cost y shillings for hiring a motor cycle for a distance x kilometres.

Distance x (km)	10	20	30	40
Cost (shs.)	2800	3600	4400	5200

Use linear interpolation or extrapolation, find

(i) the cost of hiring the motor cycle for distance of 45km [shs. 5600]

(ii) distance travelled if he paid shs. 4000 [25km]

13. The table below shows the values of a function f(x) for given values of x

x	0.4	0.6	0.8
f(x)	-0.9613	-0.5108	-0.2231

Use linear interpolation to determine f^{-1}(-0.4308} correct to 2 decimal places [0.66]

14. The table below shows how T caries with S

T	-2.9	-0.1	2.9	3.1
S	30	20	12	9

Use linear interpolation/extrapolation to estimate values of

(a) T when S = 26 [-1.78] (ii) S when T = 3.4 [4.5]

15. The table below shows the commuter bus fare from stages A to B, C, D and E

Stage	A	B	C	D	E
Distance (km)	0	12	16	19	23
Fare (shs)	0	1300	1700	2200	2500

(a) Jane boarded from A and stopped at a place 2km after E. How much did she pay? (03marks) [shs. 2650]

(b) Okello paid shs 2000. How far from A did the bus leave him? (02marks) [17.km]

16. The table below shows the value of x and corresponding values of a function f(x)

The table below shows how T caries with S

x	0.3	0.6	0.9	1.2
f(x)	3.00	3.22	3.69	4.06

Use linear interpolation/extrapolation to estimate values of

(i) f(x) when x = 0.4 [3.0733] (ii) x when f(x) = 3.82 [1.0054]