Example

Find the image of P (3, 00, Q (4, 2) and R (-3, 0) after being rotated about the origin through $3/4$ turn. Hence, sketch the object and its image.

Solution

$3/4$ turn is the same as turning through 270^o. The matrix of rotation is:

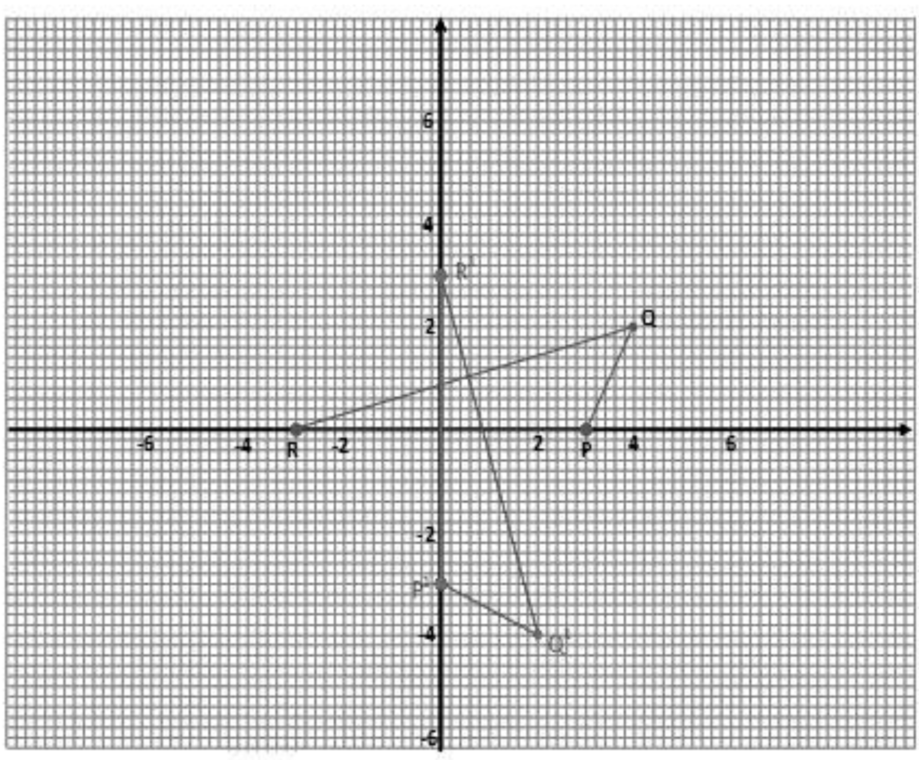
$$\therefore M = \begin{pmatrix} \cos 270^0 & -\sin 270^0 \\ \sin 270^0 & \cos 270^0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Rightarrow P^1Q^1R^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{array}{c} \begin{array}{ccc} P & Q & R \end{array} \\ \begin{pmatrix} 3 & 4 & -3 \\ 0 & 2 & 0 \end{pmatrix} \end{array} = \begin{pmatrix} 0 & 2 & 0 \\ -3 & -4 & 3 \end{pmatrix}$$

$P(3, 0) \longrightarrow P^1(0, 3)$

$Q(4, 2) \longrightarrow Q^1(2, -4)$

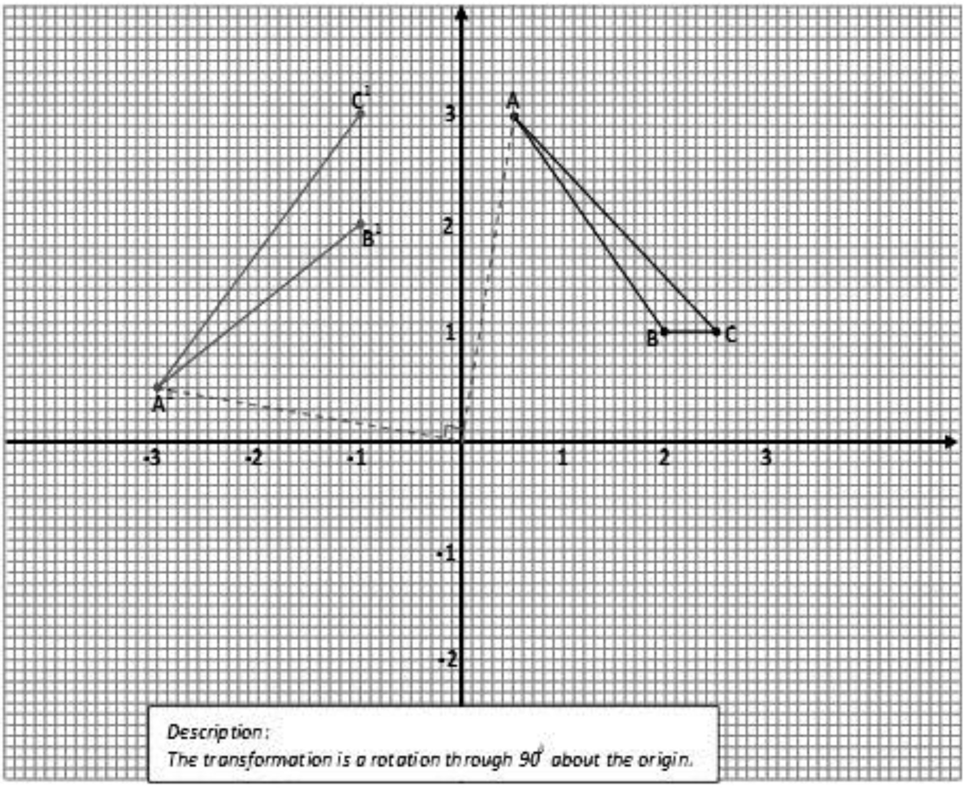
$R(-3, 0) \longrightarrow R^1(0, 3)$



Example

A triangle with vertices A (1, 3), B (2, 1) and C (3, 1) is mapped onto another triangle with vertices A^1 (-3, 1), B (-1, 2) and C (-1, 3). Describe this transformation and find its matrix.

Solution



$$\text{Let the matrix of transformation } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$From: P^1 = MP$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} A^1 & B^1 & C^1 \end{array} \\ \begin{pmatrix} a+3b & 2a+b & 3a+b \\ c+3d & 2c+d & 3c+d \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} A^1 & B^1 & C^1 \end{array} \\ \begin{pmatrix} -3 & -1 & -1 \\ 1 & 2 & 3 \end{pmatrix} \end{array}$$

$$\Rightarrow \begin{array}{lll} a+3b = -3 \text{............(1)} & 2a+b = -1 \text{..............(3)} & 3a+b = -1 \text{.............(5)} \\ c+3d = 1 \text{.............(2)} & 2c+d = 2 \text{.................(4)} & 3c+d = 3 \text{...............(6)} \end{array}$$

$Equation(5) - eqn(3)$	$Equation(6) - eqn(4)$
$3a + b = -1$	$3c + d = 3$
$^{-}\underline{2a + b = -1}$	$^{-}\underline{2c + d = 2}$
$a = 0,$	$c = 1,$
$3a + b = -1 \quad \Rightarrow b = -1$	$3c + d = 3 \quad \Rightarrow d = 0$

$$\therefore M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

17.3 Enlargement:

We say that an object has been enlarged when its size has been increased. An enlargement is a transformation, which results in an image, such that:

* All its lengths and the corresponding lengths on the object bear a constant ration known as scale factor.
* Its angles are equal to the corresponding angles on the object. In order words the object its image are similar.

To describe an enlargement, we need to know its centre of enlargement and its scale factor. A scale factor is a factor by which the size of a given object changes.

17.3.1 General properties of enlargement

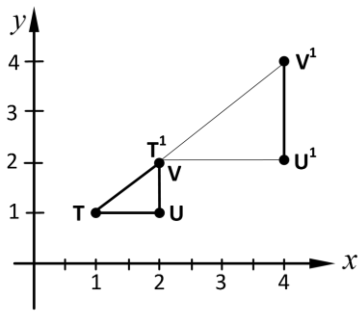
- An object and point, its image and centre of enlargement are collinear.
- For any point A on an object, $\bar{OA^1} = k\bar{OA}$, where k is a scale factor. For instance if k = 3, then this mean that $\bar{OA^1}$ is 3 times $\bar{OA}$ *i.e.* $\bar{OA^1} = 3\bar{OA}$.
- The centre of enlargement is the only point that remains fixed irrespective of the scale factor.
- If the linear scale factor is k, the area scale factor is k^2 and if the enlargement results in the formation of solid object, then the volume scale factor is k^3.

17.3.2 Obtaining the matrix of enlargement

Consider triangle TUV with vertices T (1, 1), U (2, 1) and V (2, 2) being enlarged with scale factor 2 and centre of enlargement O (0, 0). This means that every point on the triangle increases by a factor of 2, i.e.

$$T^1 = 2\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}, \quad U^1 = 2\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix}, \quad V^1 = 2\begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}4\\4\end{pmatrix}$$

The size of triangles TUV and $T^1U^1V^1$ can be compared by drawing them on the same graph.



17.3.3 Matrix of enlargement:

Let the matrix of enlargement $M = \begin{pmatrix}a & b\\c & d\end{pmatrix}$

$$From \quad P^1 = MP$$

$$\begin{array}{ccc} T^1 & U^1 & V^1 \end{array} \qquad\qquad \begin{array}{ccc} T & U & V \end{array}$$

$$\Rightarrow \begin{pmatrix}2 & 4 & 4\\2 & 2 & 4\end{pmatrix} = \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}1 & 2 & 2\\1 & 1 & 2\end{pmatrix}$$

$$\begin{pmatrix}2 & 4 & 4\\2 & 2 & 4\end{pmatrix} = \begin{pmatrix}a+b & 2a+b & 2a+2b\\c+d & 2c+d & 2c+2d\end{pmatrix}$$

$$a+b = 2 \text{............}(1) \qquad c+d = 2\text{............}(4)$$
$$2a+b = 4\text{............}(2) \qquad 2c+d = 2\text{............}(5)$$
$$2a+2b = 4\text{..........}(3) \qquad 2c+2d = 4\text{..........}(6)$$

$Equation(2)-eqn(1)$	$Equation(5)-eqn(4)$
$2a+b=4$	$2c+d=2$
$^{-}\underline{a+b=2}$	$^{-}\underline{c+d=2}$
$a=2,$	$c=0,$
$b=0$	$d=2$

$\therefore The\ matrix\ of\ enlargement\ M, scale\ factor\ 2\ centre\ (0,0)\ is\ M=\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}$

Generally, the matrix of enlargement is given by:

$$M=\begin{pmatrix}k & 0\\0 & k\end{pmatrix}$$

$Where\ k\ is\ the\ scale\ factor$

Note:

a) If k > 0, then the object and its image lie on the same side of the centre of enlargement.

b) If k < 0, then the object and its image are on opposite sides of the centre of enlargement.

c) If $k=\pm 1$, then the object and its image are congruent

d) If 0 < k < 1, then the image is smaller than the object.

Example

The vertices of quadrilateral ABCD have coordinates A (2, 3), B (-3, 4), C (-5, -1) and D (4, -5).

Find the images of the vertices of the quadrilateral under enlargement with centre (0,) and :

a) With the matrices:

i. $\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}$

ii. $\begin{pmatrix}0.5 & 0\\0 & 0.5\end{pmatrix}$

b) With matrix $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

In both a) and b) show the images and the object on the same graph paper.

Solution

a) $\text{Image} = \text{matrix of enlargement} \times \text{object}$

i.
$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 2 & -3 & -5 & 4 \\ 3 & 4 & -1 & -5 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} A^1 & B^1 & C^1 & D^1 \end{array} \\ \begin{pmatrix} 4 & -6 & -10 & 8 \\ 6 & 8 & -2 & -10 \end{pmatrix} \end{array}$$

$A(2, 3) \longrightarrow A^1(4, 6)$

$B(-3, 4) \longrightarrow B^1(-6, 8)$

$C(-5, -1) \longrightarrow C^1(-10, -2)$

$D(4, -5) \longrightarrow D^1(8, -10)$

ii.
$$\begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 2 & -3 & -5 & 4 \\ 3 & 4 & -1 & -5 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} A^{11} & B^{11} & C^{11} & D^{11} \end{array} \\ \begin{pmatrix} 1 & -1.5 & -2.5 & 2 \\ 1.5 & 2 & -0.5 & -2.5 \end{pmatrix} \end{array}$$

$A(2, 3) \longrightarrow A^{11}(1, 1.5)$

$B(-3, 4) \longrightarrow B^{11}(-1.5, 2)$

$C(-5, -1) \longrightarrow C^{11}(-2.5, -0.5)$

$D(4, -5) \longrightarrow D^{11}(2, -2.5)$

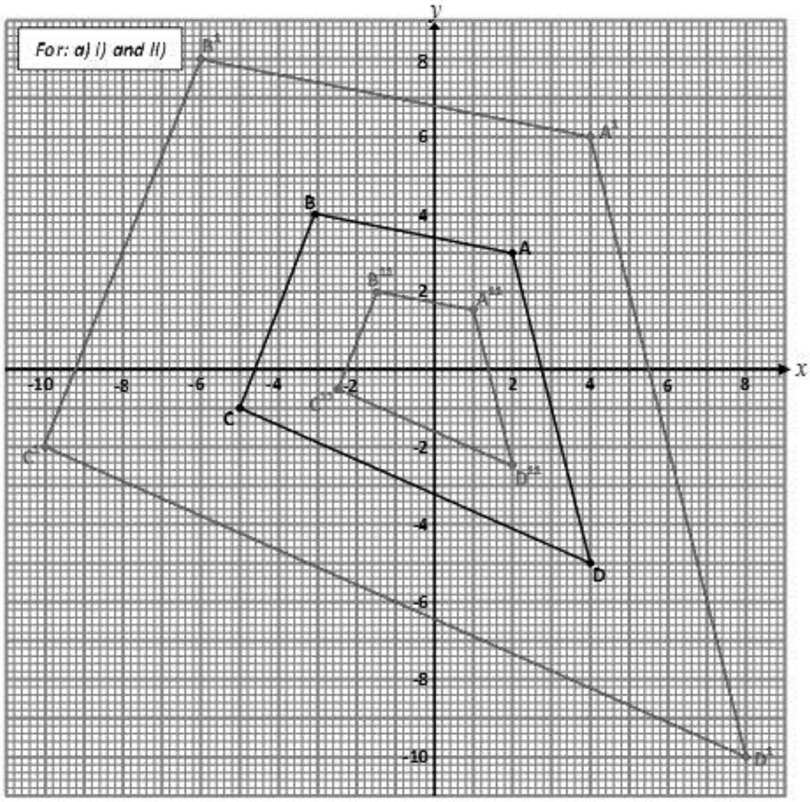
b)
$$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 2 & -3 & -5 & 4 \\ 3 & 4 & -1 & -5 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} A^{111} & B^{111} & C^{111} & D^{111} \end{array} \\ \begin{pmatrix} -4 & 6 & 10 & -8 \\ -6 & -8 & 2 & 10 \end{pmatrix} \end{array}$$

$A(2, 3) \longrightarrow A^{111}(-4, -6)$

$B(-3, 4) \longrightarrow B^{111}(6, -8)$

$C(-5, -1) \longrightarrow C^{111}(10, 2)$

$D(4, -5) \longrightarrow D^{111}(-8, 10)$


For: a) i) and ii)
y
x
A
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8
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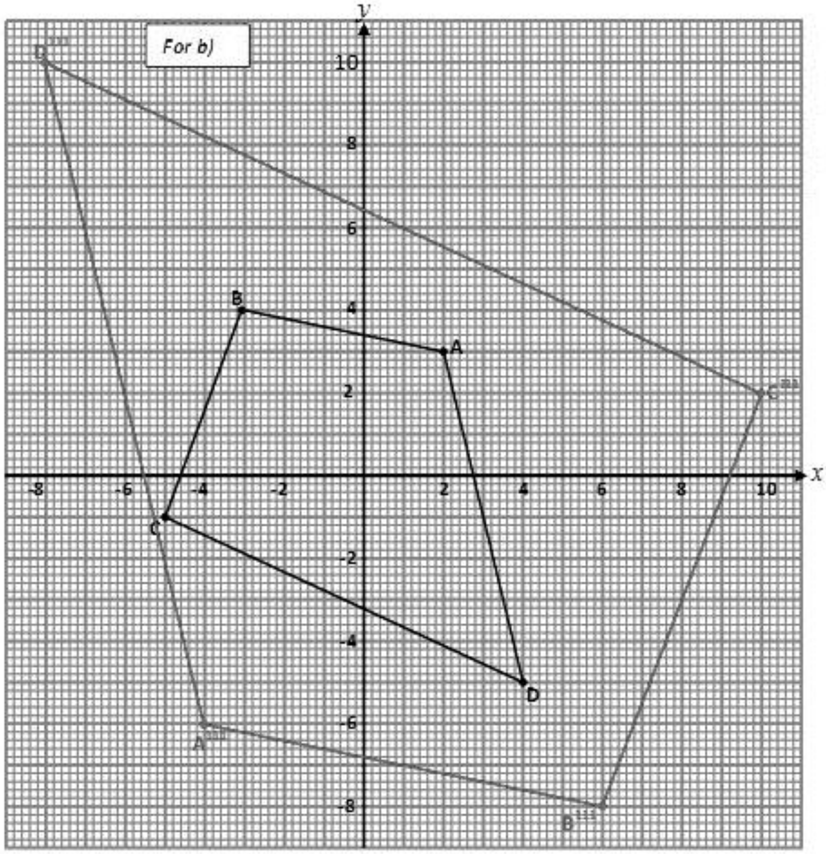


17.3.4 Centre of enlargement (C.E)

The centre of enlargement can be calculated from the expression below

$$C.E = \frac{1}{k-1}(kO - I)$$

$Where \; : k - scale \quad factor$

$\quad O - object \quad position$

$\quad I - image \quad position$

Example

The image of point A (5, 2) under an enlargement scale factor –3 is A^1 (1, 6). Determine the coordinates of the centre of enlargement.

Solution

$$From: \quad C.E = \frac{1}{k-1}(kO - I), \quad k = -3, \quad O = A = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad I = A^1 = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

$$\therefore C.E = \frac{1}{^{-}3-1}\left({}^{-}3\begin{pmatrix} 5 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 6 \end{pmatrix}\right) = \frac{1}{-4}\begin{pmatrix} ^{-}15-1 \\ ^{-}6-6 \end{pmatrix} = \frac{-1}{4}\begin{pmatrix} -16 \\ -12 \end{pmatrix} = \begin{pmatrix} {}^{-}16/{}^{-}12 \\ {}^{-}12/{}^{-}4 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\Rightarrow \quad C.E = (4, 3)$$

17.4 Inverse transformation:

If M is the matrix of transformation that maps an object P onto an image point P^1, then the transformation which maps P^1 back onto P is called inverse of M written as M^{-1}, i.e.

$$If \;\; P^1 = MP, then:$$

$$P = M^{-1}P^1$$

This expression is useful in obtaining the object point given the image point and the transformation matrix.

Example

Under the enlargement $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ the image of triangle ABC has vertices $A^1(4, 2)$, $B^1(4, 4)$ and $C^1(8, 4)$. Find the coordinates of the vertices of the object triangle ABC.

Solution

$$Let \; M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\det M = 4, \quad adjoin \; of \; M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\therefore M^{-1} = \frac{1}{4}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$From \quad P = M^{-1}P^1$$

$$\Rightarrow \quad \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{array}{c} \begin{array}{ccc} A^1 & B^1 & C^1 \end{array} \\ \begin{pmatrix} 4 & 4 & 8 \\ 2 & 4 & 4 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 2 & 2 & 4 \\ 1 & 2 & 2 \end{pmatrix} \end{array}$$

Therefore, the coordinates of the vertices of triangle ABC are A (2, 1), B (2, 2), and C (4, 2).

17.5 Combined transformation:

This is when the object is performed with more than one matrices of translation. Here, you will be required to be in position to:

i. Obtain the image of an object under combined transformation.

ii. State a single transformation that would map the object onto the image obtained after combined transformation.

iii. Obtain a single matrix, which is equivalent to the combined matrices of transformation.

Example

Triangle ABC with vertices A (1, 1), B (1, 4), and C (3, 4) is given a positive quarter turn about the origin to give triangle $A^1B^1C^1$. This is then followed by a reflection along the x –axis giving the image of $A^1B^1C^1$ as $A^{11}B^{11}C^{11}$.

a) State the coordinates of triangles:

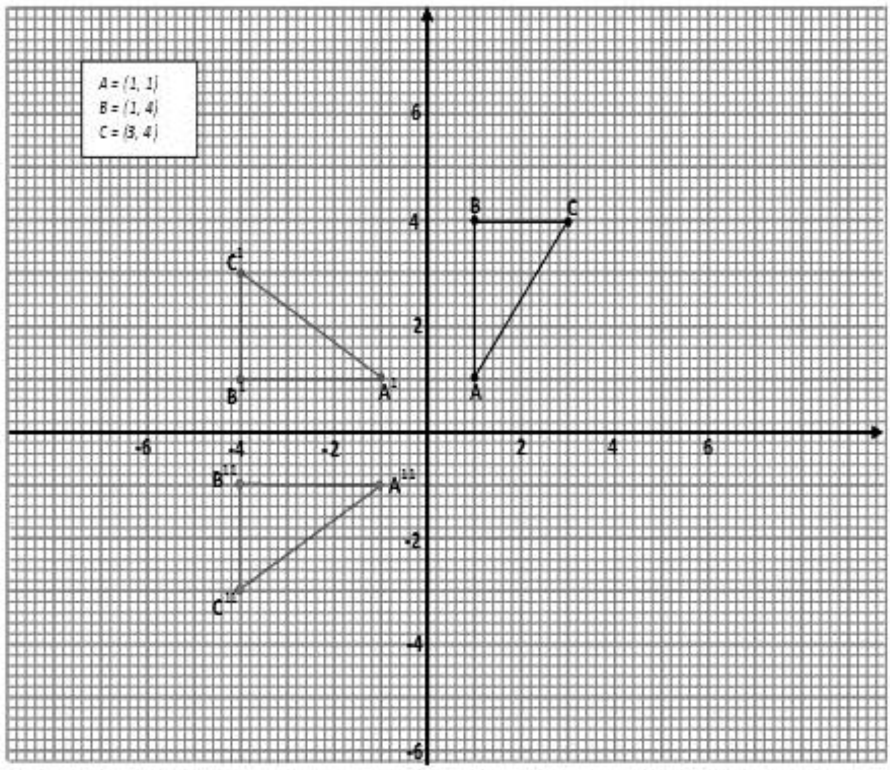
i. $A^1B^1C^1$

ii. $A^{11}B^{11}C^{11}$

b) i) What single transformation maps triangle ABC onto triangle $A^{11}B^{11}C^{11}$?

ii) What is the matrix of this transformation?

Solution



a)
$$A^1 = (-1, 1) \qquad A^{11} = (-1, -1)$$
$$B^1 = (-4, 1) \quad and \quad B^{11} = (-4, -1)$$
$$C^1 = (-4, 3) \qquad C^{11} = (-4, -3)$$

b) i) *A single transformation that would maps triangle ABC directly onto triangle $A^{11}B^{11}C^{11}$ is the reflection along the line* $y = -x$

ii) ***Method 1***

*Let **Q** be the matrix for quarter turn and **X** be the matrix for reflection along x –axis.*

For Q:

$$From\ Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \theta = 90^0 \ for\ positive\ quarter\ turn$$

$$= \begin{pmatrix} \cos 90^0 & -\sin 90^0 \\ \sin 90^0 & \cos 90^0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

<u>For X :</u>

For the reflection along $x-axis$:

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The matrix of this transformation $M = XQ$ but <u>not</u> $M = QX$. *This is because* **Q** *was performed first on triangle ABC followed by* ***X,*** *i.e.*

$$A^{11}B^{11}C^{11} = XQABC$$

$$\Rightarrow M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \therefore M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Which is the same as the matrix of reflection along the line $y = -x$

<u>Method 2</u>

Let this matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\Rightarrow A^{11}B^{11}C^{11} = MABC$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 1 & 1 & 3 \\ 1 & 4 & 4 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} A^{11} & B^{11} & C^{11} \end{array} \\ \begin{pmatrix} -1 & -4 & -4 \\ -1 & -1 & -3 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} a+b & a+4b & 3a+4b \\ c+d & c+4d & 3c+4d \end{pmatrix} = \begin{pmatrix} -1 & -4 & -4 \\ -1 & -1 & -3 \end{pmatrix}$$

$$a+b = -1..........(1) \qquad c+d = -1..........(4)$$

$$\Rightarrow a+4b = -4........(2) \qquad c+4d = -1.........(5)$$

$$3a+4b = -4.......(3) \qquad 3c+4d = -3.........(6)$$

Equation$(2)-eqn(1)$

$$a+4b = -4$$

$$^{-}\,\underline{a+b = -1}$$

$$3b = -3, \quad \Rightarrow b = -1$$

$$a+b = -1 \Rightarrow a = 0$$

Equation$(5)-eqn(4)$

$$c+4d = -1$$

$$^{-}\,\underline{c+d = -1}$$

$$3d = 0, \quad \Rightarrow d = 0$$

$$c+d = -1 \Rightarrow c = -1$$

$$\therefore M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

General case:

- If **X** and **Q** represent transformations, then **XQ** means perform **Q** first followed by **X**.
- Similarly if **X**, **Q** and **R** are transformations, then **QRX** means **X** is performed first then **R** and finally **Q** in that order.

NB:

Make sure the order mention above is always followed.

- $\mathbf{M}^2$ is the same as **MM**, i.e. **M** followed by **M**.
- $(\mathbf{QM})^{-1} = \mathbf{M}^{-1}\mathbf{Q}^{-1}$
- Remember that: $\mathbf{XQ} \neq \mathbf{QX}$

17.6 Relationship between the area of an object and its image Area scale factor:

This is defined as the ratio of the area of the image to the area of its object, i.e.

$$Area\ scale\ \ factor\ = \frac{Area\ of\ \ image}{Area\ of\ \ object}$$

17.7 Area of an image:

Under any transformation with matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$Area\ of\ \ image = magitude\ of\ \ \det M \times Area\ of\ \ the\ object$$
$$I.e.$$
$$Area\ of\ \ image = |ad - bc| \times area\ of\ \ the\ object.\quad Where,\ ad - bc = \det M$$

$$\Rightarrow Area\ sacle\ \ factor\ = \frac{\det M \times Area\ of\ \ object}{Area\ of\ \ object} = \det M$$
$$\therefore Area\ scale\ \ factor\ is\ the\ same\ as\ \text{determin}ant\ of\ \ the\ operator.$$

NB:

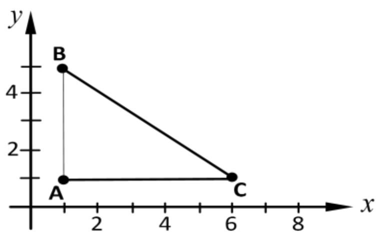
If the $\det M$ is negative, then we have to ignore the negative sign.

Example

Triangle ABC with coordinates A (1, 1), B (1, 5) and C (6, 1) undergoes a transformation represented by matrix $\begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix}$.

Find the area of the image.

Solution



$$Area\ of\ triangle\ ABC = \tfrac{1}{2}bh, \qquad b = 6-1 = 5\,units,\ h = 5-1 = 4\,units$$
$$= \tfrac{1}{2}\times 5\times 4 = 10\,sq\ units$$

$$M = \begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix} \quad \det M = 3\times 2 - 3\times 1 = 3$$

$$\therefore Area\ of\ triangle\ A^1B^1C^1 = \det M \times Area\ of\ triangle\ ABC$$
$$= 3\times 10$$
$$= \underline{\underline{30\ sq\ units}}$$

Summary:

1. Under the transformation of translation, reflection, and rotation, the size is always preserved meaning that the object and its image are identical (congruent). The three transformations above are therefore known as ***isometrics.***

2. Non –isometric transformations on the other hand are transformation for which the object changes position, the size, and sometimes the shape. Enlargement is an example of this transformation.

17.8 Miscellaneous exercise

1. Triangle ABC has vertices A (-4, 1), B (-1, 1) and C (-3, 4). T is the transformation with matrix $T = \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}$

   a) Find the image of ABC under **T**.

   b) Sketch triangle ABC and its image $A^1 B^1 C^1$ after this transformation.

2. Point P (a, b) has been transformed by the transformation with matrix $\begin{pmatrix} 1 & 0 \\ 4 & 5 \end{pmatrix}$. The image of P (a, b) is P^1 (2, 9).

   Find the value of ***a*** and ***b***.

3. A triangle with coordinates A (2, 3), B (6, 3) and C (4, 6) is given a transformation represented by matrices $M = \begin{pmatrix} 0 & -3 \\ -1 & 2 \end{pmatrix}$ *and* $N = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$ to form $A^1 B^1 C^1$ and $A^{11} B^{11} C^{11}$ respectively.

   a) Find the coordinates of $A^1 B^1 C^1$ and $A^{11} B^{11} C^{11}$.

   b) Find a single matrix that maps ABC onto $A^{11} B^{11} C^{11}$.

   c) Find a single matrix that maps $A^{11} B^{11} C^{11}$ back to ABC.

   d) Find the area of triangle $A^{11} B^{11} C^{11}$.

4. Find the image of triangle A (1, 1) B (5, 1) and C (5, 3) after being reflected in the;

   i. X –axis

   ii. Y – axis.

5. Under a rotation X = (5, 3) is mapped onto X^1 = (-2, 5) and Y = (4, 6) is mapped onto Y^1 = (-5, 4). Find by a diagram the centre and angle of rotation as accurately as possible.

6. After a rotation, the image of P (3,0) and Q (4, 2) are P^1(-3, 0) and Q^1(-5, 1) respectively. Find the centre and angle of rotation.

7. ABC has vertices A (-5, 2), B (-1, 2) and C (3, 4). The image of triangle ABC under a rotation is the triangle $A^1B^1C^1$ with A^1 (2, 9), B^1 (2, 5) and C^1 (4, 1). Find the centre and angle of rotation.

8. a) Find the image of B (-4, 5) under a rotation about the origin of:

    i. -270^o

    ii. 45^o

    iii. 37.4^o

   b) Find the transformation matrix for a rotation of:

    i. $^{+}70^0$ about the origin

    ii. 38^o clockwise

9. The points A (-2, 1), B (-2, 4), C (1, 4), and D (1, 1) are vertices of a square ABCD. The images of A, B, C and D under a reflection in the line $x - y = 0$ are A^1, B^1, C^1 and D^1 are then mapped onto the points A^{11}, B^{11}, C^{11} and D^{11} respectively by a positive quarter turn about the origin.

   a) Draw square ABCD and its images $A^1B^1C^1D^1$ and $A^{11}B^{11}C^{11}D^{11}$.

   b) State the coordinates of the vertices of:

    i. $A^1B^1C^1D^1$

    ii. $A^{11}B^{11}C^{11}D^{11}$

10. A triangle XYZ has vertices X (1, 0), Y (3, 0) and Z (3, 4). The triangle is given a positive quarter turn about O (0, 0) to be mapped onto triangle $X^1Y^1Z^1$. The image $X^1Y^1Z^1$ is then reflected along the line $x + y = 0$ to be mapped onto triangle $X^{11}Y^{11}Z^{11}$.

    a) Plot and draw on a graph triangle X Y Z and its images $X^1Y^1Z^1$ and $X^{11}Y^{11}Z^{11}$ respectively.

    b) Using your graph, state the coordinates of the vertices of triangle $X^1Y^1Z^1$ and $X^{11}Y^{11}Z^{11}$.

11. a) PQRS is a square which has been transformed into image ABCD with vertices A (0, 0), B (6, 0), C (6, 6) and D (0, 6) by an enlargement centre (0, 0) and matrix $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

   i. Find the coordinates of PQRS

   ii. Sketch PQRS and its image on the same diagram.

   b) PQRS is transformed by an enlargement centre (0, 0) and matrix $T = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$. Sketch the image of PQRS on the same diagram as in 11 a) ii) above.

12. Unit square OABC, with O = (0, 0), A = (1, 0), B = (1, 1) and C = (0, 1) is transformed by a positive quarter turn about the origin onto OXYZ.

   a) Find the coordinates of the vertices of OXYZ

   b) OXYZ is enlarged with matrix $\begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}$ centre (0, 0). Find the area of the image of OXYZ.

13. The points P (0, 2), Q (1, 4), and R (2, 2) are vertices of triangle PQR. The images of P, Q, and R under a reflection in the line $x - y = 0$ are P^1, $Q^{1,}$ and P^1 respectively. The points P^1, Q^1 and R^1 are then mapped onto the points P^{11} Q^{11} and R^{11} respectively under an enlargement with scale factor -2 and centre of enlargement O (0, 0).

   a) Write down the matrix for the:

   i. Reflection

   ii. Enlargement.

   b) Determine the coordinates of the points:

   i. P^1Q^1 and R^1

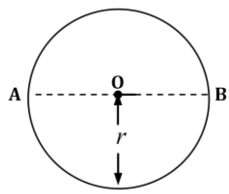
   ii. $P^{11}Q^{11}$ and R^{11}

   c) Find a single matrix of transformation that would map triangle PQR onto P 11Q11R11.

18 THE CIRCLE

18.1 Definition

A circle is a set of points, which are at the same distance from a fixed point.

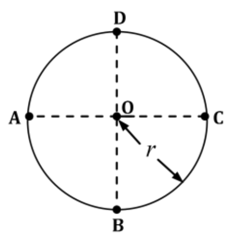


The fixed point **O** is known as the centre of the circle while the constant distance, **r** is known as the radius. The line **AB** from one point of the circle to the other point of the circle through the centre of the circle is known as the diameter and it is twice the radius, i.e.

$$Diameter\,, d = 2r$$

18.2 Circumference of a circle

This is the total distance round the circle



The distance A to B to C to D and back to A is equal to the circumference of the circle of radius r above. The circumference, C of the circle is given by:

$$C = 2\pi r$$
$$where \;\; r \; is \; radius \;\; of \;\; the \;\; circle$$
$$Or \; :$$
$$C = \pi d$$
$$where \;\; d \; is \; diameter \;\; of \;\; the \;\; circle \;\; = 2r$$

Example

Calculate the circumference of the circle whose radius is 7cm.

Solution

$$C = 2\pi r, \quad r = 7cm, \pi = \tfrac{22}{7}$$

$$\Rightarrow C = 2 \times \frac{22}{\cancel{7}} \times \cancel{7}$$

$$= \underline{\underline{44cm}}$$

Example

What is the diameter of a circle whose circumference is 300cm?

Solution

$$C = \pi d, \quad \pi = \tfrac{22}{7}$$

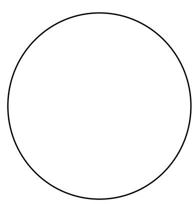
$$\Rightarrow d = \frac{C}{\pi}, \quad C = 300cm,$$

$$= 300 \div \frac{22}{7}$$

$$= \underline{\underline{95.45cm}}$$

18.3 Area of a circle

Consider the circle below of radius r.



The area of a circle is the same as the area of the shaded part and is given by:

$$Area, A = \pi r^2$$

$$: Or$$

$$Area = \frac{\pi d^2}{4} \quad where \;\; r = \tfrac{d}{2}$$

Example

Calculate the area of the circle with diameter 0.5m.

Solution

$$Area\ of\ \ a\ circle = \frac{\pi d^2}{4}$$

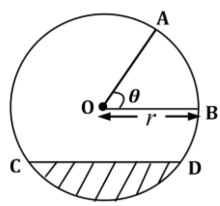
$$= \frac{\frac{22}{7} \pm \times (0.5)^2}{4}$$

$$= \underline{\underline{0.196m^2}}$$

18.4 Chord, arc, and sector

18.4.1 Chord:

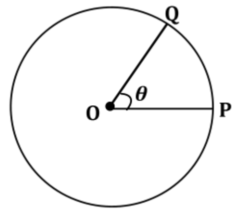
A chord is any straight line joining any two points on the circumference of the circle. Consider a circle of radius r and centre O as shown below.



- The length CD is known as the chord
- The shaded part is known as the minor segment
- The length AB is known as an arc.
- The part OAB is known as the sector.
- θ is the angle subtended at the centre of the circle by an arc AB.

18.4.2 Length of an arc:

Consider a circle of radius r, and entre O and that an arc PQ subtends an angle θ at the centre of the circle.



The circumference of the circle subtends an angle of 360^o at the centre of the circle, Hence:

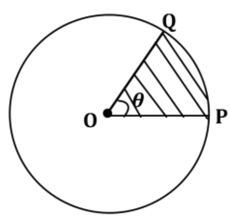
$$\frac{The\ length\ of\ arc\ PQ}{Circumferance\ of\ the\ circle} = \frac{\theta}{360^0}, \quad but\ circumferance\ of\ circle = 2\pi r$$

$$\Rightarrow \quad \frac{Length\ of\ arc\ PQ}{2\pi r} = \frac{\theta}{360^0}$$

$$\boxed{\therefore Length\ of\ an\ arc\ PQ = \frac{\theta}{360^0} \times 2\pi r}$$

18.4.3 Area of the sector:

The area of the sector of a circle can be obtained in a similar way to the length of an arc of a circle.



The sector OPQ subtends an angle θ at O and the area of the circle subtends an angle 360^o at O, thus:

$$\frac{Area\ of\ sector\ OPQ}{Area\ of\ the\ circle} = \frac{\theta}{360^0}, \quad but\ area\ of\ circle = \pi r^2$$

$$\Rightarrow \quad \frac{Area\ of\ sector\ OPQ}{\pi r^2} = \frac{\theta}{360^0}$$

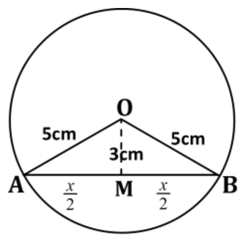
$$\therefore \text{Area of sector } OPQ = \frac{\theta}{360^0} \times \pi r^2$$

Example

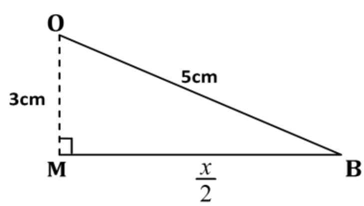
In a circle of radius 5cm, calculate the length of the chord, which is 3cm from the centre.

Solution

Let x be the length of the chord



Extracting triangle OMB



By using Pythagoras theorm

$$OB^2 = OM^2 + MB^2$$

$$5^2 = 3^2 + \left(x/2\right)^2$$

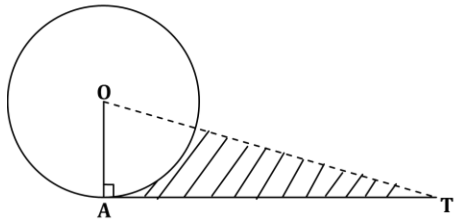
$$\Rightarrow \quad x^2/4 = 25 - 9 = 16$$

$$\therefore x = \sqrt{16 \times 4} = 8cm$$

So, the length of the chord is $8cm$

Example

TA is a tangent to the circle, centre O, and radius 6cm.

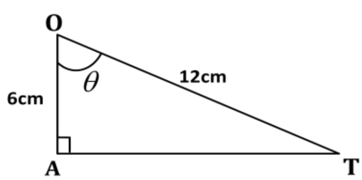


Given that, OT = 12cm. Calculate:

a) Length AT
b) Angle AOT
c) Area of the shaded part

Solution

a) Extracting triangle OAT



By using Pythagoras theorm

$$OA^2 + AT^2 = OT^2$$
$$6^2 + AT^2 = 12^2$$
$$\Rightarrow \quad AT^2 = 144 - 36 = 108$$
$$\therefore AT = \sqrt{108} = \underline{\underline{10.39cm}}$$

b) Let angle AOT be θ

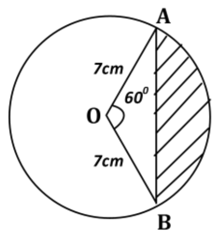
$$Cos\theta = {}^{6}\!/_{12} = {}^{1}\!/_{2}$$
$$\therefore \theta = \cos^{-1}({}^{1}\!/_{2}) = 60^0$$
$$\therefore \underline{\underline{< AOT = 60^0}}$$

c) $Area\ of\ the\ shaded\ part = Area\ of\ triangle\ AOT - Area\ of\ monor\ sector\ AOB$

$$Area\ of\ triangle\ AOT = \tfrac{1}{2} \times AT \times AO = \tfrac{1}{2} \times 10.39 \times 6 = 31.18cm^2$$
$$Area\ of\ \text{minor}\ sector\ AOB = \frac{\theta}{360^0} \times \pi r^2 = \frac{60^0}{360^0} \times 3.41 \times 6^2 = 18.85cm^2$$
$$\therefore Area\ of\ the\ shaded\ part = 31.18 - 18.85 = \underline{\underline{12.33cm^2}}$$

Example

The diagram below shows a circle with an arc, which subtends an angle of 60^o at the centre of the circle of radius 7cm.



a) Find the area of the circle.

b) Find the area of the minor sector AOB.

c) Find the area of the major sector AOB.

d) Find the length of the minor arc AB.

e) Find the length of the major arc AB.

f) Calculate the area of the shaded segment.

Solution

a) $Area\ of\ \ a\ circle = \pi r^2, \quad r = 7cm$

$$= \frac{22}{7} \times 7^2$$

$$= \underline{\underline{154cm^2}}$$

b) $Area\ of\ \text{minor sector}\ AOB\ = \ \frac{\theta}{360^0} \times \pi r^2, \quad \theta = 60^0$

$$= \frac{60}{360} \times \frac{22}{7} \times 7^2$$

$$= \underline{\underline{25.67cm^2}}$$

c) $Area\ of\ \ major\ \text{sector}\ AOB\ = \ \frac{300}{360^0} \times \frac{22}{7} \times 7^2, \quad \text{since}\ \theta = 360^0 - 60^0 = 300^0$

$= \underline{\underline{128.33cm^2}}$

d) $Legth\ of\ the\ arc\ = \dfrac{\theta}{360^0} \times 2\pi r$

$For\ \text{minor}\ arc,\ \theta = 60^0$

$\therefore the\ length\ of\ \text{minor}\ arc\ AB = \dfrac{60}{360} \times 2 \times \dfrac{22}{7} \times 7 = \underline{\underline{7.33cm}}$

e) $For\ major\ arc, \theta = 300^0$

$\therefore the\ length\ of\ major\ arc\ AB = \dfrac{300}{360} \times 2 \times \dfrac{22}{7} \times 7 = \underline{\underline{36.67cm}}$

f) $Area\ of\ the\ shaded\ segment = Area\ of\ \text{minor sector}\ AOB - Area\ of\ triangle\ OAB$

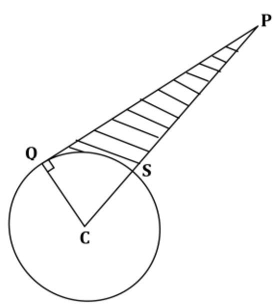
$$= 25.67 \quad - \quad \frac{1}{2} r^2 \sin 60^0$$

$$= 25.67 \quad - \quad \frac{1}{2} \times 7^2 \times \frac{\sqrt{3}}{2}$$

$$= \underline{\underline{4.45cm^2}}$$

Example

In the figure below, C is a centre of a circle of radius 20m, PQ is a tangent to the circle and angle CPQ = 40^o,

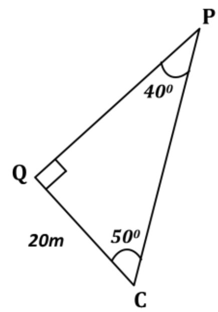


Calculate:

i. the length QP
ii. the perimeter of the shaded region
iii. the area of the shaded region

Solution

i. *Extracting triangle QCP*



$$\frac{CP}{QP} = \tan 40^0$$

$$\therefore QP = \frac{CQ}{\tan 40^0} = \frac{20}{\tan 40^0} = \underline{\underline{23.84m}}$$

ii. $The\ length\ of\ the\ \text{minor}\ arc\ QS \quad = \quad \frac{\theta}{360^0} \times 2\pi r, \quad \theta = 90^0 - 40^0 = 50^0$

$$\therefore \overrightarrow{QS} \quad = \quad \frac{50}{360} \times 2 \times \frac{22}{7} \quad = \quad 17.46m$$

$$The\ length\ \overrightarrow{PS} = \overrightarrow{PC} - \overrightarrow{CS}, \quad but\ \overrightarrow{PC} = \frac{QP}{\cos 40^0}$$

$$\therefore \overrightarrow{PS} \quad = \quad \frac{23.84}{\cos 40^0} - 20 \quad = \quad 11.12m$$

$$\therefore Perimeter of the shaded region = \overrightarrow{PQ} + \overrightarrow{QS} + \overrightarrow{PS} = 23.84 + 17.46 + 11.2 = \underline{\underline{52.4m}}$$

iii. *Area of the shaded part = Area of triangle QCP – Area of minor sector QSC*

$$But\ area\ of\ \text{minor sec}tor\ QSC \quad = \quad \frac{\theta}{360^0} \times \pi r^2, \quad \theta = 50^0, and r = 20m$$

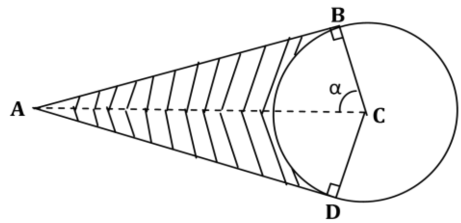
$$= \quad \frac{50}{360} \times \frac{22}{7} \times 20^2 \quad = \quad 174.6om$$

$$Area\ of\ triangle\ PQC \quad = \quad \frac{1}{2} bh \quad = \quad \frac{1}{2} \times 20 \times 23.84 \quad = 2\ \ 38.4m^2$$

$$\therefore Area\ of\ shaded\ part = 238.4 - 174.6 = \underline{\underline{63.8m}}$$

Example

In the diagram below, PQ and PR are tangents to a circle of radius 15cm and centre C.

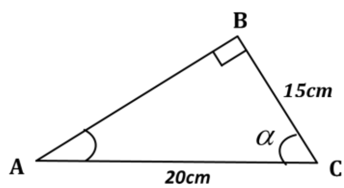


If AC = 20cm, calculate:

i. the angle marked α
ii. the length AB
iii. the area of the shaded part

Solution

i. *Extracting triangle ABC*



Since AB is the tangent to the circle and BC is the radius,
$< ABC = 90^0$

$$\Rightarrow \frac{BC}{AC} = \cos\alpha \quad \Rightarrow \frac{15}{20} = \cos\alpha$$

$$\therefore \alpha = \cos^{-1}\left(\frac{15}{20}\right) = \underline{\underline{41.4^0}}$$

ii. $\frac{AB}{BC} = \tan\alpha \quad \Rightarrow AB = BC\tan 41.4^0 \;\; \therefore AB = 15 \times 0.88 = \underline{\underline{13.2cm}}$

iii. *Area of shaded part = Area of ABCD – Area of minor sector BCD*

$$Area\ of\ triangle\ ABC = \frac{1}{2}bh = \frac{1}{2}\times 150\times 13.2 = 99cm^2$$

$$\therefore Total\ area\ of\ quadrilateral = 2\times 99 = 198cm^2$$

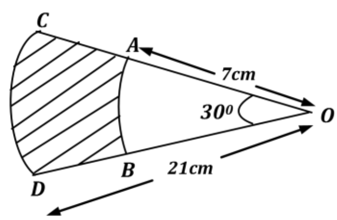
$$And\ area\ of\ \text{minor sector}\ BDC = \frac{\theta}{360}\times \pi r^2,\quad but\ \theta = 2\alpha = 2\times 41.4^0 = 82.8^0$$

$$= \frac{82.8}{360}\times\frac{22}{7}\times 15^2 = 162.6cm^2$$

$$\therefore Area\ of\ shaded\ part = 198.0 - 162.6 = \underline{\underline{35.4cm^2}}$$

Example

In the figure below, O is a centre of two arcs AB and CD with a central angle of 30^o.



Calculate:

a) the perimeter of the shaded part and

b) the area of the shaded part

Solution

a) $$The\ length\ of\ moinor\ arc\ AB = 2\pi r\times\frac{\theta}{360},\quad \theta = 30^0, r = 7cm$$

$$= 2\times\frac{22}{7}\times 7\times\frac{30}{360},\quad = 3.67cm$$

$$The\ length\ of\ moinor\ arc\ CD = 2\pi r\times\frac{30}{360},\quad , r = 21cm$$

$$= 2\times\frac{22}{7}\times 21\times\frac{30}{360},\quad = 11cm$$

$$\therefore Perimeter\ of\ shaded\ region = length\ of\ arc\ AB + length\ of\ arc\ CD + \vec{AC} + \vec{BD}$$

$$= 3.67 + 11 + (21-7) + (21-7)$$

$$= \underline{\underline{42.7cm}}$$

b) $Area\ of\ shaded\ region = area\ of\ \text{sector}\ CDO - area\ of\ \text{sector}\ ABO$

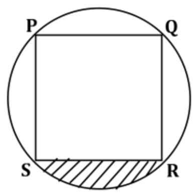
$$= \pi r_1^2 \times \frac{30}{360} - \pi r_2^2 \times \frac{30}{360}, \quad r_1 = 21cm,\ r_2 = 7cm$$

$$= \frac{22}{7} \times 21^2 \times \frac{30}{360} - \frac{22}{7} \times 7^2 \times \frac{30}{360}$$

$$= \underline{\underline{102.6cm^2}}$$

Example

The figure below shows a square PQRS inscribed in a circle of radius 21cm.



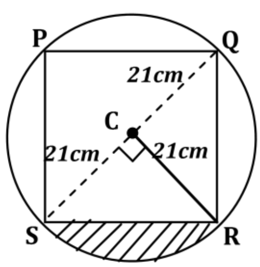
Calculate:

a) the length of the side of the square

b) the area of the shaded region

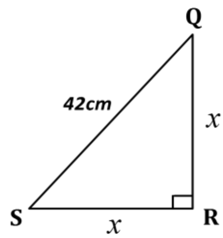
Solution

a)



$SQ = 2 \times 21 = 42cm$

Let x be the length of the side of the square. Extracting triangle SQR:



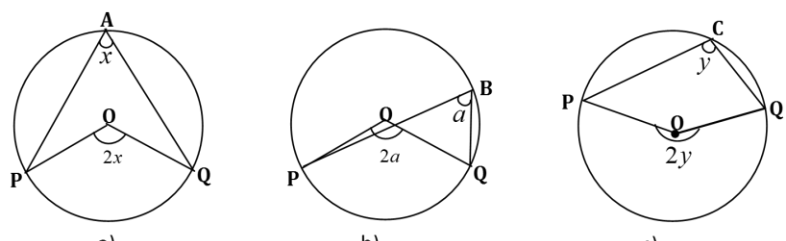
Using Pythagoras theorem:

$$42^2 = x^2 + x^2$$
$$1764 = 2x^2$$
$$\therefore \ x = \sqrt{\frac{1764}{2}} = \underline{\underline{29.7cm}}$$

b) $$\text{Area of shaded part} = \text{area of minor sector } CSR - \text{area of triangle } CSR$$
$$= \pi r^2 \times \frac{90}{360} - \frac{1}{2} \times \overrightarrow{CR} \times \overrightarrow{SC}$$
$$= \frac{22}{7} \times 21^2 \times \frac{90}{360} - \frac{1}{2} \times 21 \times 21$$
$$= \underline{\underline{126cm^2}}$$

18.5 Angle properties of a circle

1. The angle subtended by the arc of a circle at the centre is twice the angle it subtends at the circumference. The following diagrams illustrate this property.

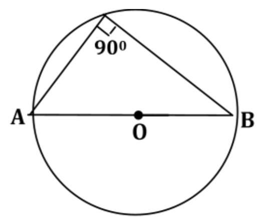


a) $< PAQ = \frac{1}{2} < POQ$ b) $< PBQ = \frac{1}{2} < POQ$ c) $< PCQ = \frac{1}{2} \text{obtuse} < POQ$

In cases a) and b), minor arc **PQ** subtends angles x and a respectively on the circumference. Therefore the angles subtended by the respective arcs at the centre are $2x$ and $2a$.

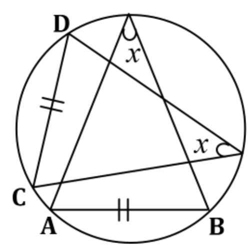
In case c), the major arc **PQ** subtends angle y on the circumference and hence the corresponding angle subtended at the centre by the same arc **PQ** is $2y$.

2. The angle subtended by the diameter at the circumference of the circle is 90^o.



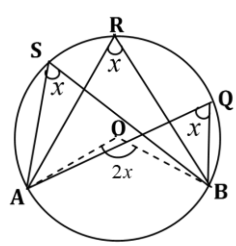
AB is the diameter

3. Equal chords subtend equal angles at the circumference.

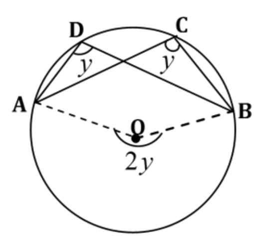


i.e. CD = AB

4. An arc of a circle subtends equal angles at the circumference.



a)



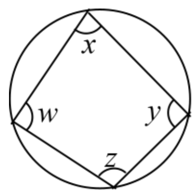
b)

In figure a) above, the minor arc AB subtends $<AQB$, $<ARB$, and $<ASB$ at the circumference and $<AOB$ at the centre of the circle. Therefore $< AQB =< ARB =< ASB = x$ and $< AOB = 2x$

In figure b) above, the major arc AB subtends $<ACB$ and $<ADB$ at the circumference and $<AOB$ at the centre of the circle. Therefore, $< ACB =< ADB = y$ and obtuse $< AOB = 2y$

18.6 Cyclic Quadrilaterals

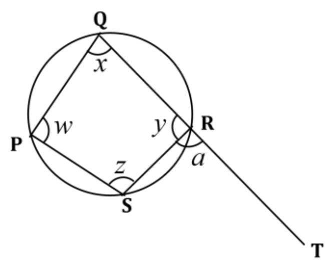
This is a quadrilateral whose all of its four vertices lie on the circumference of a cycle, figure below.



Angle x is opposite $< z$ and $< y$ is opposite $< w$.

18.7 Angle properties of a cyclic quadrilateral

Consider cyclic quadrilateral shown below.



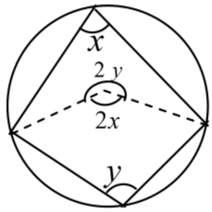
The cyclic quadrilateral has the following angle properties.

1. Opposite angles are supplementary i.e. add up to 180^o

$$w + y = 180^0$$
$$Also : x + z = 180^0$$
$$\Rightarrow \quad w + y = x + z = 180^0$$

Proof:

Consider two angles x and y subtended at the circumference by the minor arc AB and major arc AB respectively.



$$\therefore \ 2y + 2x = 360^0$$
$$\Rightarrow \ y + x = 180^0$$

2. The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

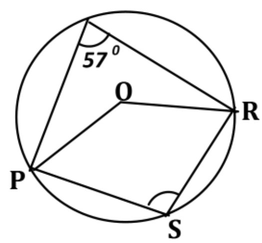
$$\Rightarrow \ w = a$$

Furthermore, sum of exterior angles add up to 180^o

$$\Rightarrow \ y + a = 180^0$$

Example

In the diagram below, O is the centre of the circle.



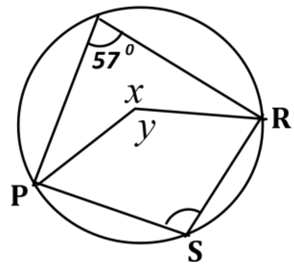
Find the following:

a) <POR

b) The reflect angle POR

c) <PSR

Solution

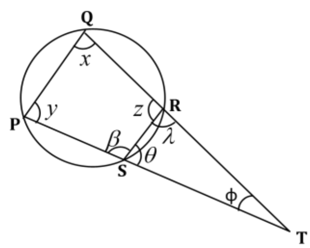


a) $<POR = x = 2\times 57 = 114^0$

b) *The reflect angle* $POR = y = 360^0 - x = 360^0 - 114^0 = 246^0$

c) $<PSR = \frac{1}{2}\times reflect <POR = \frac{1}{2}\times 246 = 123^0$

Example



In the figure above, $\phi = 20^o$ and $\lambda = 40^o$.

Find:

i. $<\theta$

ii. $<z$

iii. $<y$

iv. $<\beta$

v. $<x$

Solution

i. Considering triangle SRT

$\theta + \lambda + \phi = 180^0$, $but \phi = 20^0$, $\lambda = 40^0$

$\Rightarrow \theta + 40^0 + 20^0 = 180^0$

$\therefore \theta = 180^0 - 60^0 = \underline{\underline{120^0}}$

ii. *From sum of interior angle and exterior angle = 180^o.*

$\Rightarrow \; z + \lambda = 180^0$

$\therefore \; z = 180^0 - 40^0 = \underline{\underline{140^0}}$

iii. *From the property of the cyclic quadrilateral i.e. sum of opposite angles are supplementary.*

$\Rightarrow \; y + z = 180^0$

$\therefore \; y = 180^0 - 140^0 = \underline{\underline{40^0}}$

iv. *For* $< \beta$

$\beta + \theta = 180^0$

$\therefore \; \beta = 180^0 - 120^0 = \underline{\underline{160^0}}$

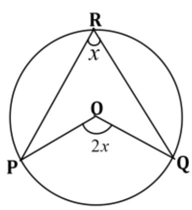
v. *For* $< x$

$x + \beta = 180^0$

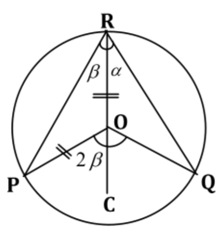
$\therefore \; x = 180^0 - 160^0 = \underline{\underline{120^0}}$

18.8 Angle at the centre of a circle

Consider the angle x being subtended by an arc PQ on the circumference of the circle. The angle therefore subtended at the centre of the circle by an arc PQ is $2x$ as shown. See figure below.

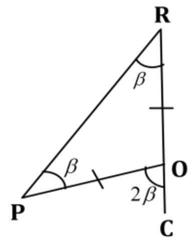


If **RO** is joined and extended to C as shown below



$\beta + \alpha = x$

OR is the radius of the circle and similarly **OP**. therefore, **OR** = **PO** implying that triangle PRO is an isosceles triangle. Now consider triangle PRO.

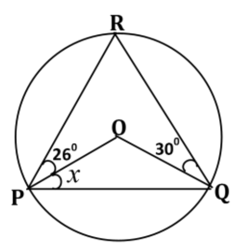


$$\therefore < OPR = < ORP = \beta$$
$$Also \quad < POC = < OPR + < ORP = \beta + \beta = 2\beta$$

$$In\ the\ same\ way,\ < QOC = 2\alpha$$
$$< POC + < QOC = 2\beta + 2\alpha = 2(\beta + \alpha) = 2x$$
$$I.e. < POQ = 2 < PRQ$$

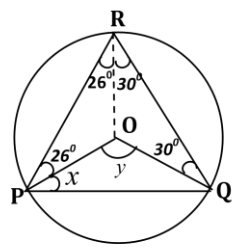
Example

In the figure below, O is the centre of the circle. Angle ABO = 26^o and angle OCA = 30^o.



Calculate the size of angle marked x.

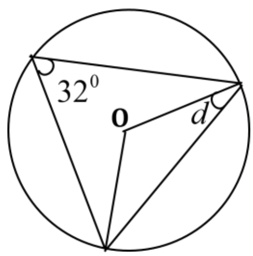
Solution



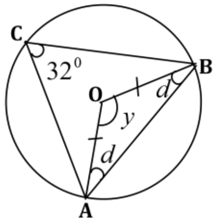
$$< BAC = 26^0 + 30^0 = 56^0$$
$$\therefore Angle\ y = 2 \times 56^0 = 112^0$$
$$< OAC =< OCA = x$$
$$\Rightarrow x + x + y = 180^0$$
$$\Rightarrow 2x = 180^0 - 112^0$$
$$\therefore x = \frac{68^0}{2} = \underline{\underline{34^0}}$$

Example

Find the angle marked **d** of the circle below, given that O is the centre of the circle.



Solution

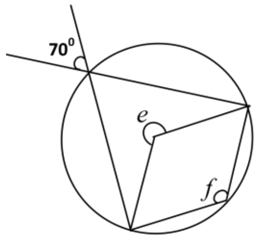


Since AO = OB = radius of the circle, <OAB = <OBA = d

$$\Rightarrow d + d + y = 180^0, but\ y = 2 \times 32^0 = 64^0 \quad \therefore d = {}^{116}\!/_{2} = \underline{\underline{58^0}}$$
$$\Rightarrow 2d = 180^0 - 64^0 = 116^0$$

Example

Consider the diagram below; O is the centre of the circle.

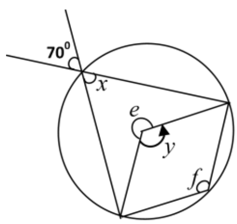


Calculate the size of angles marked:

i. e

ii. f

Solution



i. *Angle x is vertically opposite 70^o, therefore* $x = 70^0$

$Angle\ y = 2x = 2 \times 70^0 = 140^0$

$\therefore e + y = 360^0$

$e = 360^0 - 140^0 = \underline{\underline{220^0}}$

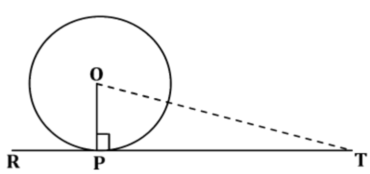
ii. *Also angle* $e = 2f$

$\Rightarrow f = \frac{e}{2} = \frac{220^0}{2} = \underline{\underline{110^0}}$

18.9 Tangent to the circle

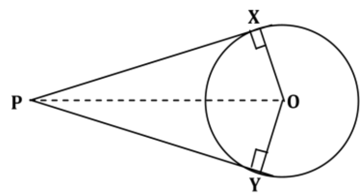
Definition:

A tangent is a line, which just touches the circle at only one point and makes an angle of 90^0 with the radius of the circle.



RT is a tangent to the circle at point P.

For any given point, we can draw two tangents to the circle. The diagram below shows two tangents PX and PY drawn from P to the circle centre O.

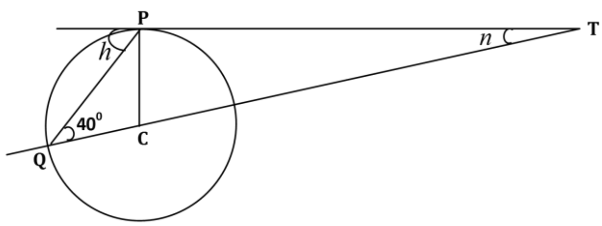


The lengths of tangents to the circle from the same external point are equal.

$$\therefore \overrightarrow{PX} = \overrightarrow{PY}$$

Example

In the diagram below TP is a tangent to the circle with centre C and <PQC = 40^o.

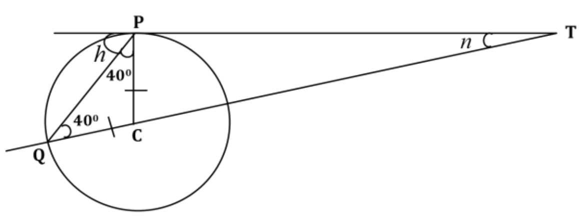


Find ***h*** and ***n***.

Solution

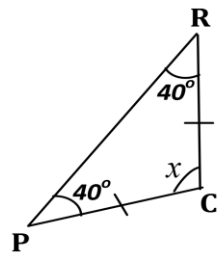
Since QC = CP = radius, this implies that, <Q = <P = 40^o

Since P is a tangent to the circle, <CPT = 90^o



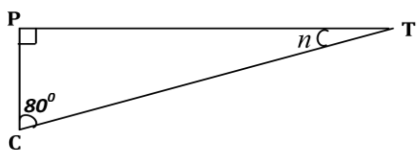
$$\Rightarrow\ h+40^0=90^0$$
$$\therefore\ h=90^0-40^0=\underline{\underline{50^0}}$$

Extracting triangle PQC:



$$x+40^0+40^0=180^0$$
$$\therefore x=180^0-80^0=100^0$$
$$\therefore < PCT=180^0-100^0=80^0 \quad (\textit{angle on a straight line})$$

Considering triangle PCT:



$$n+80^0+90^0=180^0$$
$$\therefore\ n=180^0-170^0=\underline{\underline{10^0}}$$

Example

In the diagram below, AB is the diameter of the circle and DA and DC are tangents to the circle at A and C respectively.