Solution :

S.No.	Given equation	Expressed as $ax + by + c = 0$	Value of a, b, c		
			a	b	c
1	$x = -5$	$1.x + 0.y + 5 = 0$	1	0	5
2	$y = 2$	$0.x + 1.y - 2 = 0$	0	1	−2
3	$2x = 3$	---	---	---	---
4	$5y = -3$	----	----	---	---

TRY THIS

1. Express the following linear equations in the form of ax + by + c = 0 and indicate the values of a, b, c in each case?

   i) $3x + 2y = 9$ ii) $-2x + 3y = 6$ iii) $9x - 5y = 10$

   iv) $\frac{x}{2} - \frac{y}{3} - 5 = 0$ v) $2x = y$

EXERCISE - 6.1

1. Express the following linear equation in the form of ax+by+c=0 and indicate the values of a, b and c in each case.

   i) $8x + 5y - 3 = 0$ ii) $28x - 35y = -7$ iii) $93x = 12 - 15y$

   iv) $2x = -5y$ v) $\frac{x}{3} + \frac{y}{4} = 7$ vi) $y = \frac{-3}{2}x$

   vii) $3x + 5y = 12$

2. Write each of the following in the form of ax + by + c = 0 and find the values of a, b and c

   i) $2x = 5$ ii) $y - 2 = 0$ iii) $\frac{y}{7} = 3$ iv) $x = \frac{-14}{13}$

3. Express the following statements as a linear equation in two variables.
   (i) The sum of two numbers is 34.
   (ii) The cost of a ball pen is ₹ 5 less than half the cost of a fountain pen.
   (iii) Bhargavi got 10 more marks than double the marks of Sindhu.
   (iv) The cost of a pencil is ₹ 2 and a ball point pen is ₹ 15. Sheela pays ₹ 100 for the pencils and pens she purchased.
   (v) Yamini and Fatima of class IX together contributed ₹ 200/- towards the Prime Minister's Relief Fund.
   (vi) The sum of a two digit number and the number obtained by reversing the order of its digits is 121. If the digits in unit's and ten's place are 'x' and 'y' respectively.

6.3 Solution of a Linear Equation in two variables

You know that linear equation in one variable has a unique solution.

What is the solution of the equation $3x - 4 = 8$?

Consider the equation $3x - 2y = 5$.

What can we say about the solution of this linear equation in two variables? Do we have only one value in the solution or do we have more ? Let us explain.

Can you say $x = 3$ is a solution of this equation?

Let us check, if we substitute $x = 3$ in the equation

We get $3(3) - 2y = 5$

$$9 - 2y = 5$$

i.e., Still we cannot find the solution of the given equation. So, to know the solution, besides the value of 'x' we also need the value of 'y'. we can get value of y from the above equation $9 - 2y = 5$. $\Rightarrow$ $2y = 4$ or $y = 2$

The values of x and y which satisfy the equation $3x - 2y = 5$, are $x = 3$ and $y = 2$. Thus to statisfy, a linear equation in two variables we need two values, one value for 'x' and one value for y.

Therefore any pair of values of 'x' and 'y' which satisfy the linear equation in two variables is called its solution.

We observed that x = 3, y = 2 is a solution of $3x - 2y = 5$. This solution is written as an ordered pair (3, 2), first writing the value for 'x' and then the value for 'y'. Are there any other solutions for the equation? Pick a value of your choice say x = 4 and substitute it in the equation 3x - 2y = 5. Then the equation reduces to $12 - 2y = 5$. Which is an equation in one variable. On solving this we get.

$$y = \frac{12-5}{2} = \frac{7}{2}, \quad \text{so} \left(4, \frac{7}{2}\right) \text{ is another solution, of } 3x - 2y = 5$$

Do you find some more solutions for $3x - 2y = 5$? Check if $(1, -1)$ is another solution?

Thus for a linear equation in two variables we can find many solutions.

Note : An easy way of getting two solutions is put $x = 0$ and get the corresponding value of 'y'. Similarly we can put $y = 0$ and obtain the corresponding value of 'x'.

TRY THIS

Find 5 more pairs of values that are solutions for the above equation.

Example 6. Find four different solutions of $4x + y = 9$. (Complete the table wherever necessary)

Solution :

S.No.	Choice of a value for variable x or y	Simplification		Solution
1.	$x = 0$	$4x + y = 9$	$\Rightarrow 4 \times 0 + y = 9$ $\Rightarrow y = 9$	(0,9)
2.	$y = 0$	$4x + y = 9$	$\Rightarrow 4x + 0 = 9$ $\Rightarrow 4x = 9$ $\Rightarrow x = 9/4$	$\left(\frac{9}{4}, 0\right)$
3.	$x = 1$	$4x + y = 9$	$\Rightarrow 4 \times 1 + y = 9$ $\Rightarrow 4 + y = 9$ $\Rightarrow y = 5$	——
4.	$x = -1$		——	$(-1, 13)$

$\therefore$ (0, 9), $\left(\frac{9}{4}, 0\right)$, (1, 5) and (– 1, 13) are some of the solutions for the above equation.

Example-7. Check which of the following are solutions of an equation $x + 2y = 4$? (Complete the table wherever necessary)

i) $(0, 2)$ ii) $(2, 0)$ iii) $(4, 0)$ (iv) $(\sqrt{2}, -3\sqrt{2})$

v) $(1, 1)$ vi) $(-2, 3)$

Solution : We know that if we get LHS = RHS when we substitute a pair in the given equation, then it is a solution.

The given equation is $x + 2y = 4$

S. No	Pair of Values	Value of LHS	Value of RHS	Relation between LHS and RHS	Solution/ not Solution
1.	$(0, 2)$	$x + 2y = 0 + (2 \times 2)$ $= 0 + 4 = 4$	4	$\therefore$ LHS=RHS	$\therefore (0, 2)$ is a Solution
2.	$(2, 0)$	$x + 2y = 2 + (2 \times 0)$ $= 2 + 0 = 2$	4	......	(0, 2) is a Not a Solution
3.	$(4, 0)$	$x + 2y = 4 + (2 \times 0)$ $= 4 + 0 = 4$	4	LHS = RHS	___
4.	$(\sqrt{2}, -3\sqrt{2})$	$x + 2y = \sqrt{2} + 2(-3\sqrt{2})$ $= \sqrt{2} - 6\sqrt{2}$ $= -5\sqrt{2}$	___	LHS $\neq$ RHS	$(\sqrt{2}, -3\sqrt{2})$ Not a Solution
5.	$(1, 1)$	___	4	LHS $\neq$ RHS	(1, 1) Not a Solution
6.	___	$x + 2y = -2 + (2 \times 3)$ $= -2 + 6 = 4$	4	LHS = RHS	$(-2, 3)$ is a Solution

Example-8. If $x = 3, y = 2$ is a solution of the equation $5x - 7y = k$, find the value of k and write the resultant equation.

Solution : If $x = 3, y = 2$ is a solution of the equation

$$5x - 7y = k \text{ then } 5 \times 3 - 7 \times 2 = k$$
$$\Rightarrow 15 - 14 = k$$
$$\Rightarrow 1 = k$$
$$\therefore k = 1$$

The resultant equation is $5x - 7y = 1$.

Example-9. If $x = 2k + 1$ and $y = k$ is a solutions of the equation $5x + 3y - 7 = 0$, find the value of k.

Solution : It is given that $x = 2k + 1$ and $y = k$ is a solution of the equation $5x + 3y - 7 = 0$ by substituting the value of x and y in the equation we get.

$\Rightarrow 5(2k + 1) + 3k - 7 = 0$

$\Rightarrow 10k + 5 + 3k - 7 = 0$

$\Rightarrow 13k - 2 = 0$ (this is the linear equation in one variable).

$\Rightarrow 13k = 2$

$\therefore k = \dfrac{2}{13}$

Exercise - 6.2

1. Find three different solutions of the each of the following equations.

   i) $3x + 4y = 7$ ii) $y = 6x$ iii) $2x - y = 7$

   iv) $13x - 12y = 25$ v) $10x + 11y = 21$ vi) $x + y = 0$

2. If (0, a) and (b, 0) are the solutions of the following linear equations. Find 'a' and 'b'.

   i) $8x - y = 34$ ii) $3x = 7y - 21$ iii) $5x - 2y + 3 = 0$

3. Check which of the following is solution of the equation $2x - 5y = 10$

   i) (0, 2) ii) (0, –2) iii) (5, 0) iv) $(2\sqrt{3}, -\sqrt{3})$ v) $\left(\frac{1}{2}, 2\right)$

4. Find the value of k, if $x = 2, y = 1$ is a solution of the equation $2x + 3y = k$. Find two more solutions of the resultant equation.

5. If $x = 2 - \alpha$ and $y = 2 + \alpha$ is a solution of the equation $3x - 2y + 6 = 0$ find the value of 'α'. Find three more solutions of the resultant equation.
6. If $x = 1, y = 1$ is a solution of the equation $3x + ay = 6$, find the value of 'a'.
7. Write five different linear equations in two variables and find three solutions for each of them?

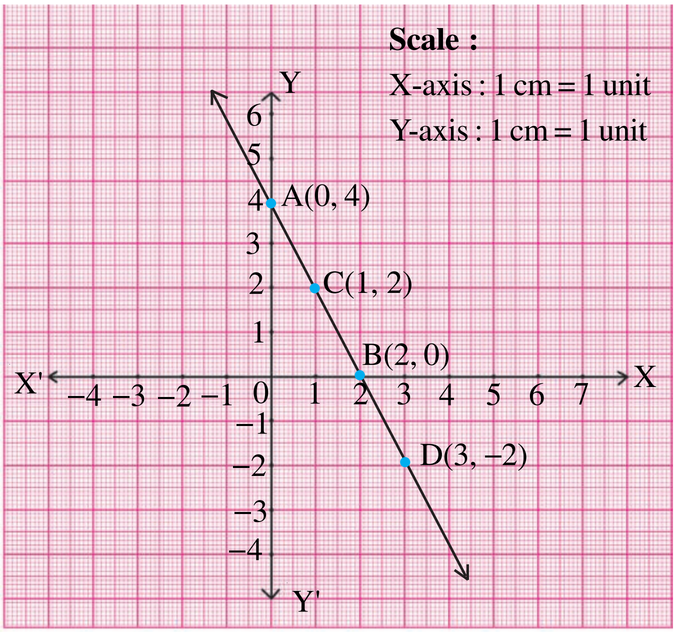
6.4 GRAPH OF A LINEAR EQUATION IN TWO VARIABLES

We have learnt that each linear equation in two variables has many solutions. If we take possible solutions of a linear equation, can we represent them on the graph? We know each solution is a pair of real numbers that can be expressed as a point in the graph.

Consider the linear equation in two variables $4 = 2x + y$. It can also be expressed as $y = 4 - 2x$. For this equation we can find the value of 'y' for a particular value of x. For example if $x = 2$ then $y = 0$. Therefore (2, 0) is a solution. In this way we find as many solutions as we can. Write all these solutions in the following table by writing the value of 'y' against the corresponding value of x.

Table of solutions:

x	$y = 4 - 2x$	(x, y)
0	$y = 4 - 2(0) = 4$	(0, 4)
2	$y = 4 - 2(2) = 0$	(2, 0)
1	$y = 4 - 2(1) = 2$	(1, 2)
3	$y = 4 - 2(3) = -2$	(3, –2)



We see for each value of x there is one value of y. Let us take the value of 'x' along the X-axis. and take the value of y along the Y-axis. Let us plot the points (0, 4), (2, 0), (1, 2) and (3, -2)on the graph. If we join any of these two points we obtain a straight line AD.

If $x = 0$;

$y = 4 - 2x = 4 - 2(0) = 4$

If $x = 2$

$y = 4 - 2(2) = 0$

Do all the other solutions also lie on the line AB?

Now pick any other point on the line say (4,–4). Is this a solution?

Pick up any other point on this line AD and check if its coordinates satisfy the equation or not?

Now take any point not on the line AD say (1, 1). Is it satisfy the equation?

Can you find any point that is not on the line AD but satisfies the equation?

Let us list our observations:

1. Every solution of the linear equation represents a point on the line of the equation.
2. Every point on this line is a solution of the linear equation.
3. Any point that does not lie on this line is not a solution of the equation and vice a versa.
4. The collection of points that give the solution of the linear equation is the graph of the linear equation.

We notice that the graphical representation of a linear equation in two variables is a straight line. Thus, $ax + by + c = 0$(a and b are not simultaneously zero) is called a linear equation in two variables.

6.4.1 How to draw the graph of a linear equation

Steps :

1. Write the linear equation.
2. Put $x = 0$ in the given equation and find the corresponding value of y.
3. Put $y = 0$ in the given equation and find the corresponding value of 'x'.
4. Write the values of x and its corresponding value of y as coordinates of x and y respectively as (x, y) form.
5. Plot the points on the graph paper.
6. Join these points.

Thus line drawn is the graph of linear equation in two variables. However to check the correctness of the line it is better to take more than two points. To find more solutions take different values for 'x' substitute them in the given equation and find the corresponding values of 'y'.

TRY THESE

Take a graph paper, plot the point (2, 4), and draw a line passing through it. Now answer the following questions.

1. Can you draw another line that passes through the point (2, 4).
2. How many such lines can be drawn?
3. How many linear equations in two variables exist for which (2, 4) is a solution?

Example-10. Draw the graph of the equation $y - 2x = 4$ and then answer the following.

(i) Does the point (2, 8) lie on the line? Is (2, 8) a solution of the equation? Check by substituting (2, 8) in the equation.

(ii) Does the point (4, 2) lie on the line? Is (4, 2) a solution of the equation? Check algebraically also.

(iii) From the graph find three more solutions of the equation and also three more which are not solutions.

Solution : Given $y - 2x = 4 \Rightarrow y = 2x + 4$

Table of Solutions

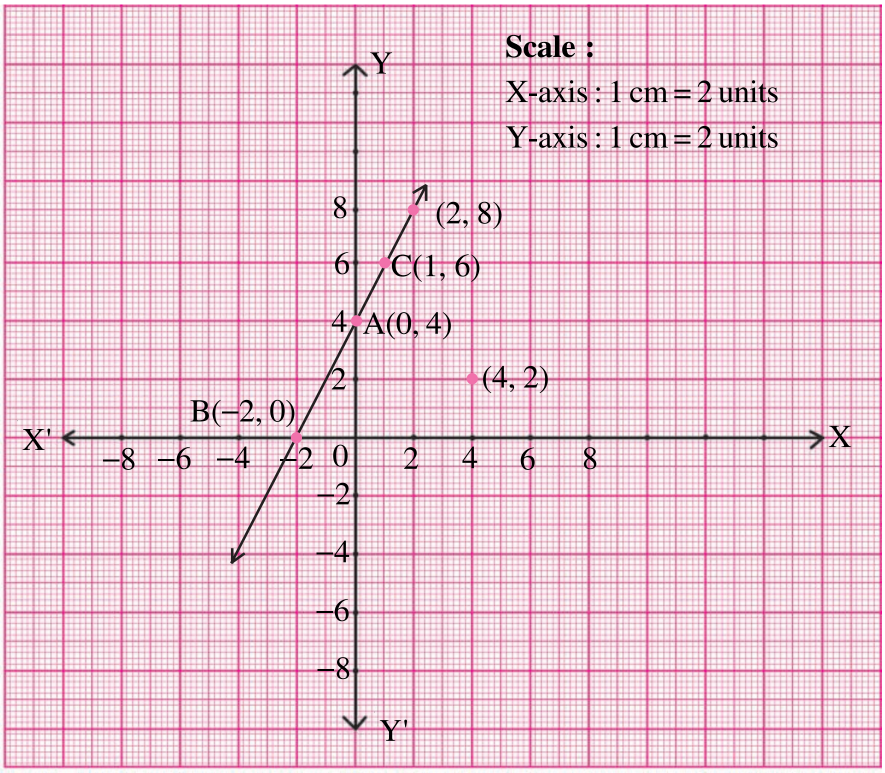
x	$y = 2x + 4$	(x, y)	Point
0	$y = 2(0) + 4 = 4$	(0, 4)	A(0, 4)
–2	$y = 2(-2)+4 = 0$	(–2, 0)	B(–2, 0)
1	$y = 2(1) + 4 = 6$	(1, 6)	C(1, 6)

Plotting the points A, B and C on the graph paper and join them to get the straight line BC as shown in graph sheet. This line is the required graph of the equation $y - 2x = 4$.

(i) Plot the point (2, 8) on the graph paper. From the graph it is clear that the point (2, 8) lies on the line.

Checking algebraically: On substituting (2, 8) in the given equation, we get

LHS $= y - 2x = 8 - 2x2 = 8 - 4 = 4 =$ RHS, So (2, 8) is a solution



(ii) Plot the point (4, 2) on the graph paper. You find that (4, 2) does not lie on the line.

Checking algebraically: By substituting (4, 2) in the given equation we have

$$\text{LHS} = y - 2x = 2 - 2\times 4 = 2 - 8 = -6 \neq \text{RHS, so (4, 2) is not a solution.}$$

(iii) We know that every point on the line is a solution of the given equation. So, we can take any three points on the line as solutions of the given equation. Eg:(-4, -4). And we also know that the point which is not on the line is not a solution of the given equation. So we can take any three points which are not on the line as not solutions of $y - 2x = 4$.
eg : (i) (1, 5); ....; ....

Example-11. Draw the graph of the equation $x - 2y = 3$.

From the graph find (i) The solution (x, y) where $x = -5$

(ii) The solution (x, y) where $y = 0$

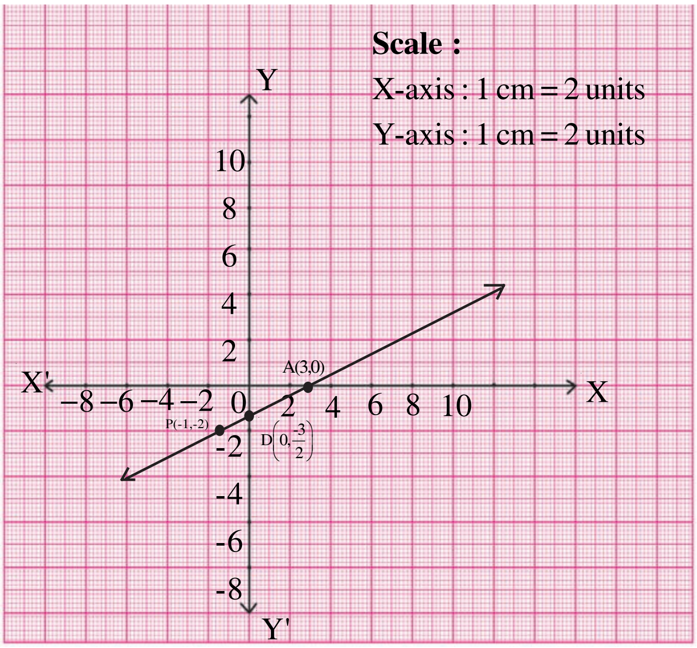
(iii) The solution (x, y) where $x = 0$

Solution : We have $x - 2y = 3 \Rightarrow y = \dfrac{x-3}{2}$

Table of Solutions

x	$y = \frac{x-3}{2}$	(x, y)	Point
3	$y = \frac{3-3}{2} = 0$	(3, 0)	A
1	$y = \frac{1-3}{2} = -1$	(1, – 1)	B
– 1	$y = \frac{-1-3}{2} = -2$	(– 1, – 2)	C

Plotting the points A, B, C on the graph paper and on joining them we get a straight line as shown in the following figure. This line is the required graph of the equation $x - 2y = 3$



(i) We have to find a solution (x, y) where $x = -5$, that is we have to find a point which lies on the straight line and whose x-coordinate is '– 5'. To find such a point we draw a line parallel to y-axis at $x = -5$. (in the graph it is shown as dotted line). This line meets the graph at 'P' from there we draw another line parallel to X-axis meeting the Y-axis at $y = -4$.

The coordinates of $P = (-5, -4)$

Since $P(-5, -4)$ lies on the straight line $x - 2y = 3$, it is a solution of $x - 2y = 3$.

(ii) We have to find a solution (x, y) where $y = 0$.

Since $y = 0$, this point (x, 0) lies on the X-axis. Therefore we have to find a point that lies on the X-axis and on the graph of $x - 2y = 3$.

From the graph it is clear that (3, 0) is the required point.

Therefore, the solution is (3, 0).

(iii) We have to find a solution (x, y) where $x = 0$.

Since $x = 0$ this point (0, y) lies on the Y-axis. Therefore we have to find a point that lies on the Y-axis and on the graph of $x - 2y = 3$.

From the graph it is clear that $\left(0, \frac{-3}{2}\right)$ is this point.

Therefore, the solution is $\left(0, \frac{-3}{2}\right)$.

EXERCISE - 6.3

1. Draw the graph of each of the following linear equations.

   i) $2y = -x + 1$ ii) $-x + y = 6$ iii) $3x + 5y = 15$ iv) $\frac{x}{2} - \frac{y}{3} = 3$

2. Draw the graph of each of the following linear equations and answer the following question.

   i) $y = x$ ii) $y = 2x$ iii) $y = -2x$ iv) $y = 3x$ v) $y = -3x$

   i) Are all these equations of the form $y = mx$, where m is a real number?

   ii) Are all these graphs passing through the origin?

   iii) What can you conclude about these graphs?

3. Draw the graph of the equation $2x + 3y = 11$. Find the value of y when $x = 1$ from the graph.

4. Draw the graph of the equation $y - x = 2$. Find from the graph

   i) the value of y when $x = 4$

   ii) the value of x when $y = -3$

5. Draw the graph of the equation $2x+3y=12$. Find the solutions from the graph

   i) Whose y-coordinate is 3

   ii) Whose x-coordinate is -3

6. Draw the graph of each of the equations given below and also find the coordinates of the points where the graph cuts the coordinate axes

   i) $6x - 3y = 12$ ii) $-x + 4y = 8$ iii) $3x + 2y + 6 = 0$

7. Rajiya and Preethi two students of Class IX together collected ₹ 1000 for the Prime Minister Relief Fund for victims of natural calamities. Write a linear equation and draw a graph to depict the statement.

8. Gopaiah sowed wheat and paddy in two fields of total area 5000 square meters. Write a linear equation and draw a graph to represent the same?

9. The force applied on a body of mass 6 kg. is directly proportional to the acceleration produced in the body. Write an equation to express this observation and draw the graph of the equation.

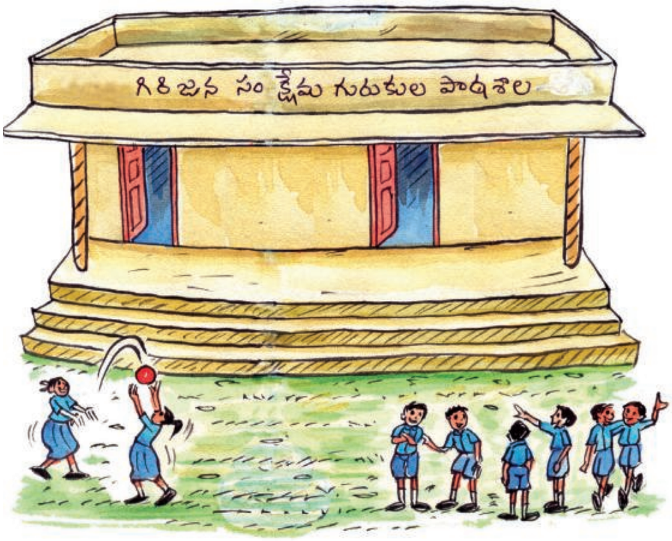
10. A stone is falling from a mountain. The velocity of the stone is given by $V = 9.8t$. Draw its graph and find the velocity of the stone '4' seconds after start.

Example-12. 25% of the students in a school are girls and others are boys. Form an equation and draw a graph for this. By observing the graph, answer the following :

(i) Find the number of boys, if the number of girls is 25.

(ii) Find the number of girls, if the number of boys is 45.

(iii) Take three different values for number of boys and find the number of girls. Similarly take three different values for number of girls and find the number of boys?



Solution : Let the number of girls be 'x' and number of boys be 'y', then

Total number of students = x + y

According to the given information

Number of girls = 25% of the students

$$x = 25\% \text{ of } (x + y)$$

$$= \frac{25}{100} \text{ of } (x + y) = \frac{1}{4}(x + y)$$

$$x = \frac{1}{4}(x + y)$$

$$4x = x + y$$

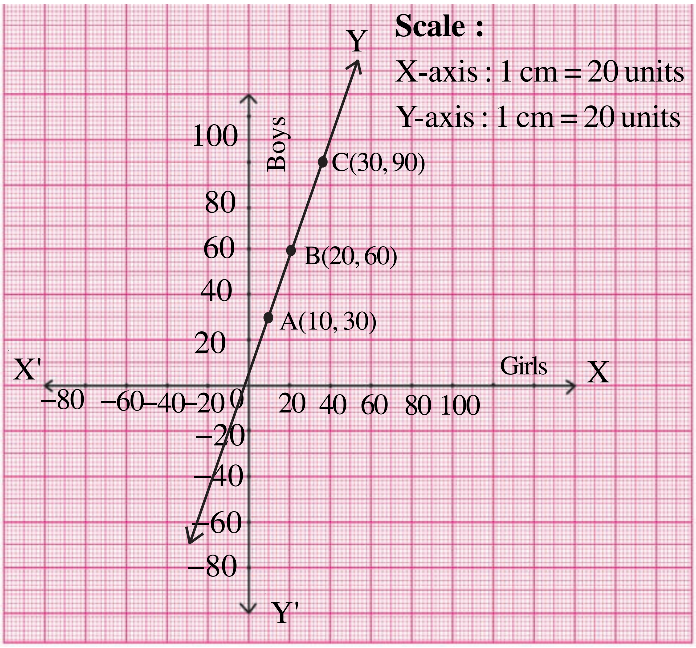
$$3x = y$$

The required equation is $3x = y$ or $3x - y = 0$.

Table of Solutions

x	y = 3x	(x, y)	Point
10	30	(10, 30)	A
20	60	(20, 60)	B
30	90	(30, 90)	C

Plotting the points A, B and C on the graph and on joining them we get the straight line as shown in the following figure.



From the graph we find that

(i) If the number of girls is 25 then the number of boys is 75.

(ii) If the number of boys is 45, then the number of girls is 15.

(iii) Choose the number you want for girls and find the corresponding number of boys. Similarly choose the numbers you want for boys and find the corresponding number of girl. Here do you observe the graph and equation. The line is passing through the origin and if the equation which is in the form $y = mx$ where m is a real number the line passes through the origin.

Example-13. For each graph given below, four linear equations are given. Out of these find the equation that represents the given graph.

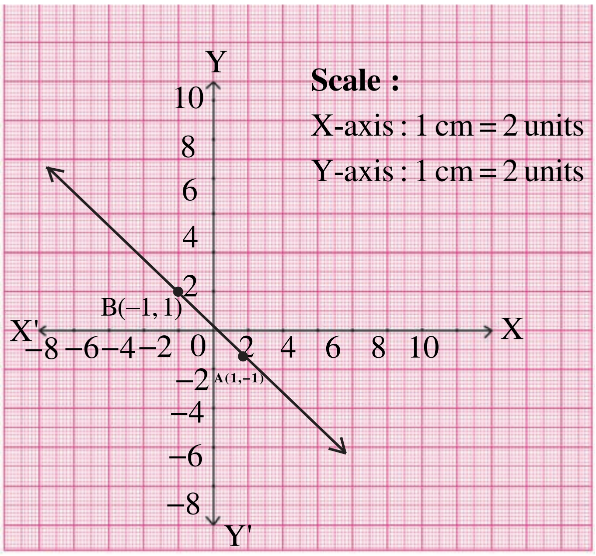
(i) Equations are

A) $y = x$

B) $x + y = 0$

C) $y = 2x$

D) $2 + 3y = 7x$



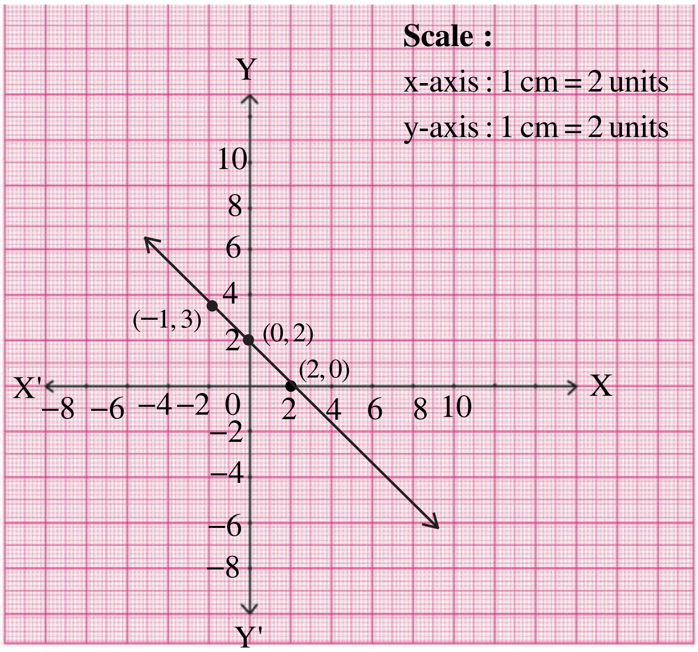
(ii) Equations are

A) $y = x + 2$

B) $y = x - 2$

C) $y = -x + 2$

D) $x + 2y = 6$



Solution :

(i) From the graph we see $(1, -1)$ $(0, 0)$ $(-1, 1)$ lie on the same line. So these are the solutions of the required equation i.e. if we substitute these points in the required equation it should be satisfied. So, we have to find an equation that shoul be satisfied by these pairs. If we substitute $(1, -1)$ in the first equation $y = x$ it is not satisfied. So $y = x$ is not the required equation.

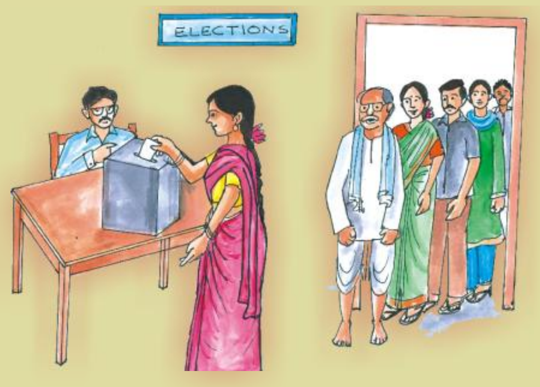
Putting $(1, -1)$ in $x + y = 0$ we find that it satisfies the equation. In fact all the three points satisfy the second equation. So $x + y = 0$ is the required equation.

We now check whether $y = 2x$ and $2 + 3y = 7x$ are also satisfied by $(1, -1)$ $(0, 0)$ and $(-1, 1)$. We find they are not satisfied by even one of the pairs, leave alone all three. So, they are not the required equations.

(ii) The points on the line are $(2, 0)$, $(0, 2)$ and $(-1, 3)$. All these points don't satisfy the first and second equation. Let us take the third equation $y = -x + 2$. If we substitute the above three points in the equation, it is satisfied. So required equation is $y = -x + 2$. Check whether these points satisfies the equation $x + 2y = 6$.

Exercise - 6.4

1. In a election 60% of voters cast their votes. Form an equation and draw the graph for this data. Find the following from the graph.
   (i) The total number of voters, if 1200 voters cast their votes
   (ii) The number votes cast, if the total number of voters are 800



[**Hint:** If the number of voters who cast their votes be 'x' and the total number of voters be 'y' then $x = 60\%$ of y.]

2. When Rupa was born, her father was 25 years old. Form an equation and draw a graph for this data. From the graph find
   (i) The age of the father when Rupa is 25 years old.
   (ii) Rupa's age when her father is 40 years old.
3. An auto charges ₹ 15 for first kilometer and ₹ 8 each for each subsequent kilometer. For a distance of 'x' km. an amount of ₹ 'y' is paid.
   Write the linear equation representing this information and draw the graph. With the help of graph find the distance travelled if the fare paid is ₹ 55? How much would have to be paid for 7 kilometers?
4. A lending library has fixed charge for the first three days and an additional charges for each day thereafter. John paid ₹ 27 for a book kept for seven days. If the fixed charges be ₹ x and subsequent per day charges be ₹ y, then write the linear equation representing the above information and draw the graph of the same. From the graph, find fixed charges for the first three if additional charges for each day thereafter is ₹4. Find additional charges for each day thereafter if the fixed charges for the first three days of ₹7.

5. The parking charges of a car in Hyderabad Railway station for first two hours is ₹ 50 and ₹10 for each subsequent hour. Write down an equation and draw the graph. Find the following charges from the graph

   (i) For three hours (ii) For six hours

   (iii) How many hours did Rekha park her car if she paid ₹ 80 as parking charges?

6. Sameera was driving a car with uniform speed of 60 kmph. Draw distance-time graph. From the graph find the distance travelled by Sameera in

   (i) $1\frac{1}{2}$ hours (ii) 2 hours (iii) $3\frac{1}{2}$ hours

7. The ratio of molecular weight of Hydrogen and Oxygen in water is 1:8. Set up an equation between Hydrogen and Oxygen and draw its graph. From the graph find the quantity of Hydrogen if Oxygen is 12 grams. And quantity of oxygen if hydrogen is $\frac{3}{2}$ gms.?

   [**Hint :** If the quantities of hydrogen and oxygen are 'x' and 'y' respectively, then $x : y = 1:8 \Rightarrow 8x = y$]

8. In a mixture of 28 litres, the ratio of milk and water is 5:2. Set up the equation between the mixture and milk. Draw its graph. By observing the graph find the quantity of milk in the mixture.

   [**Hint:** Ratio between mixture and milk $= 5 + 2 : 5 = 7 : 5$]

9. In countries like USA and Canada temperature is measured in Fahrenheit where as in countries like India, it is measured in Celsius. Here is a linear equation that converts Fahrenheit to Celsius $F = \left(\frac{9}{5}\right)C + 32$

   (i) Draw the graph of the above linear equation having Celsius on x-axis and Fahrenheit on Y-axis.

   (ii) If the temperature is 30°C, what is the temperature in Fahrenheit?

   (iii) If the temperature is 95°F, what is the temperature in Celsius?

   (iv) Is there a temperature that has numerically the same value in both Fahrenheit and Celsius? If yes find it?

6.5 EQUATION OF LINES PARALLEL TO X-AXIS AND Y-AXIS

Consider the equation $x = 3$. If this is treated as an equation in one variable x, then it has the unique solution $x = 3$ which is a point on the number line

However when treated as an equation in two variables and plotted on the coordinate plane it can be expressed as $x + 0.y - 3 = 0$

This has infinitely many solutions, let us find some of them. Here the coefficient of y is zero. So for all values of y, x becomes 3.

Table of solutions

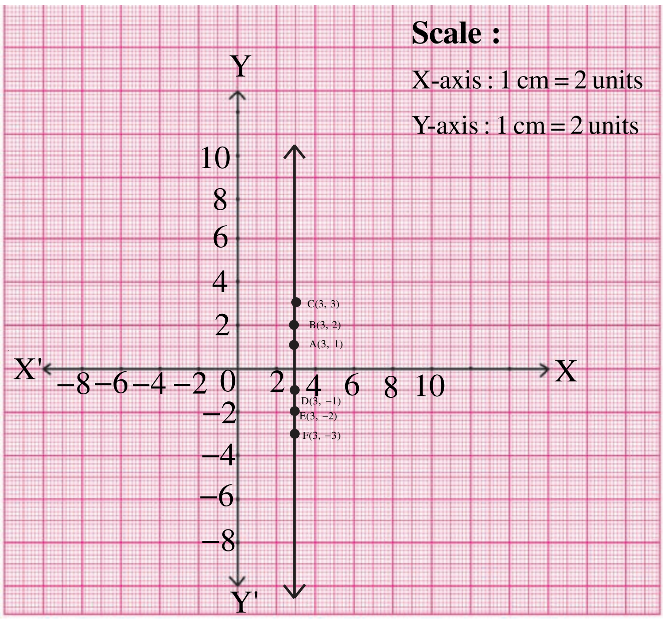
x	3	3	3	3	3	3	……
y	1	2	3	–1	–2	–3	…..
(x, y)	(3, 1)	(3, 2)	(3, 3)	(3, –1)	(3, –2)	(3, –3)	…..
Points	A	B	C	D	E	F	…..

From the table it is clear that this equation has infinitely many solutions of the form (3, a) where a is any real number.

Now draw the graph using the above solutions. What do you notice from the graph?

Is it a straight line? Whether it is any line or axes? The line drawn is a straight line and is parallel to Y-axis?

What is the distance of this line from the y-axis?



Thus the graph of $x = 3$ is a line parallel to the y-axis at a distance of 3 units to the right of it.

DO THIS

1. i) Draw the graph of following equations.

   a) $x = 2$ b) $x = -2$ c) $x = 4$ d) $x = -4$

   ii) Are the graphs of all these equations parallel to Y-axis?

   iii) Find the distance between the graph and the Y-axis in each case

2. i) Draw the graph of the following equations

   a) $y = 2$ b) $y = -2$ c) $y = 3$ d) $y = -3$

   ii) Are all these parallel to the X-axis?

   iii) Find the distance between the graph of the line and the X-axis in each case

From the above observations we can conclude the following:

1. The graph of $x = k$ is a line parallel to Y-axis at a distance of k units and passing through the point (k, 0)
2. The graph of $y = k$ is a line parallel to X-axis at a distance of k units and passing through the point (0, k)

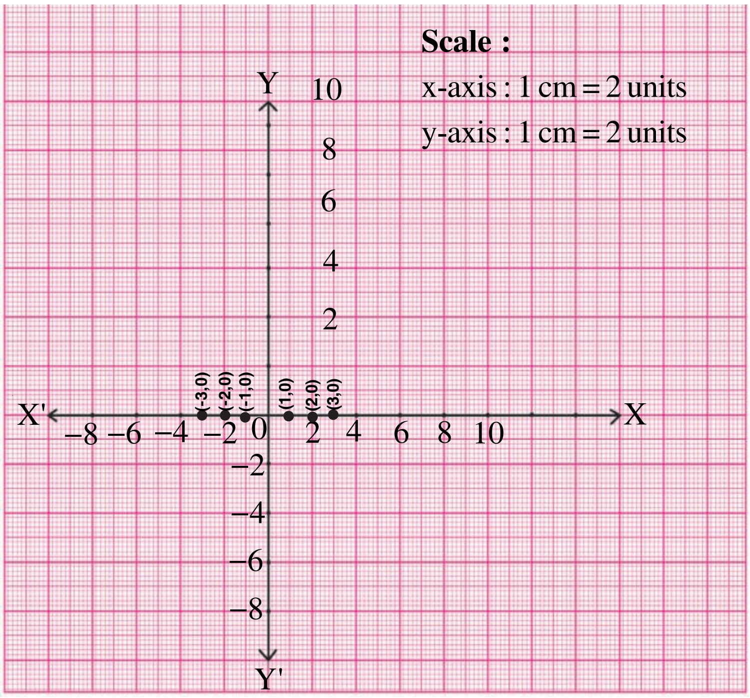
6.5.1 Equation of the X-axis and the Y-axis:

Consider the equation $y = 0$. It can be written as $0.x + y = 0$. Let us draw the graph of this equation.

Table of solutions

x	1	2	3	–1	–2	…...
y	0	0	0	0	0	…..
(x, y)	(1, 0)	(2, 0)	(3, 0)	(–1, 0)	(–2, 0)	…..
Points	A	B	C	D	E	…..

By plotting all these points on the graph paper, we get the following figure. From the graph what do we notice?



We notice that all these points lie on the X-axis and y-coordinate of all these points is '0'.

Therefore the equation $y = 0$ represents X-axis. In other words the equation of the X-axis is $y = 0$.

TRY THESE

Find the equation of Y-axis.

EXERCISE - 6.5

1. Give the graphical representation of the following equation.

   a) On the number line and b) On the Cartesian plane

   i) $x = 3$ ii) $y + 3 = 0$ iii) $y = 4$ iv) $2x - 9 = 0$

   v) $3x + 5 = 0$

2. Give the graphical representation of $2x - 11 = 0$ as an equation in

   i) one variable ii) two variables

3. Solve the equation $3x + 2 = 8x - 8$ and represent the solution on

   i) the number line   ii) the Cartesian plane

4. Write the equation of the line parallel to X-axis, and passing through the point

   i) (0, –3)   ii) (0, 4)   iii) (2, –5)   iv) (3, 4)

5. Write the equation of the line parallel to Y-axis and passing through the point

   i) (–4, 0)   ii) (2, 0)   iii) (3, 5)   iv) (–4, –3)

6. Write the equation of three lines that are

   (i) parallel to the X-axis   (ii) parallel to the Y-axis.

WHAT WE HAVE DISCUSSED

1. If a linear equation has two variables then it is called linear equation in two variables.
2. Any pair of values of 'x' and 'y' which satisfy the linear equation in two variables is called its solution.
3. A linear equation in the two variables has many solutions.
4. The graph of every linear equation in two variables is a straight line.
5. An equation of the form y = mx represents a line passes through the origin.
6. The graph of $x = k$ is a line parallel to Y - axis at a distance of k units and passes through the point (k, 0).
7. The graph of y = k is a line parallel to X-axis at a distance of k units and passes through the point (0, k).
8. Equation of X-axis is y = 0.
9. Equation of Y-axis is x = 0.

07 Triangles

7.1 Introduction

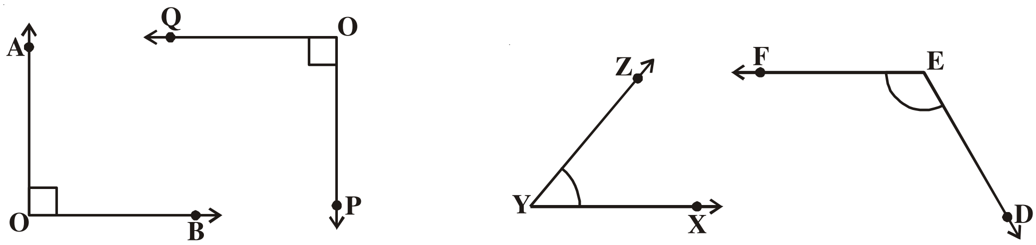
We have drawn figures with lines and curves and studied their properties. Do you remember, how to draw a line segment of a given length? All line segments are not same in size, they can be of different lengths. We also draw circles. What measure, do we need and have been used to draw a circle? It is the radius of the circle. We also draw angles equal to the given angle.

We know if the lengths of two line segments are equal then they are congruent.

$\overline{AB} \cong \overline{CD}$ (Congruent)

$\overline{PQ} \ncong \overline{RS}$ (Non-congruent)

Two angles are congruent, if their angle measure is same.



$\angle AOB \cong \angle POQ$ (Congruent)

$\angle XYZ \ncong \angle DEF$ (Non-congruent)

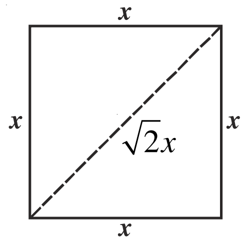
From the above examples we can say that to make or check whether the figures are same in size or not we need some specific information about the measures describing these figures.

Let's consider a square : What is the minimum information required to say whether two squares are of the same size or not?

Satya said- "I only need the measure of the side of the given squares. If the sides of given squares are equal then the squares are of identical size".

Siri said "that is right but even if the diagonals of the two squares are equal then we can say that the given squares are identical and are same in size".

Do you think both of them are right?



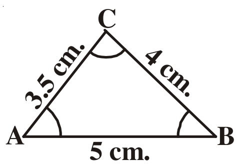
Recall the properties of a square. You can't make two different squares with sides having same measures. Can you? And the diagonals of two squares can only be equal when their sides are equal. See the given figure:

The figures that are same in shape and size are called congruent figures ('Congruent' means equal in all aspects). Hence squares that have sides with same measure are congruent and also with equal diagonals are congruent.

Note : In general, sides decide sizes and angles decide shapes.

We know if two squares are congruent and we trace one out of them on a paper and place it on other one, it will cover the other exactly.

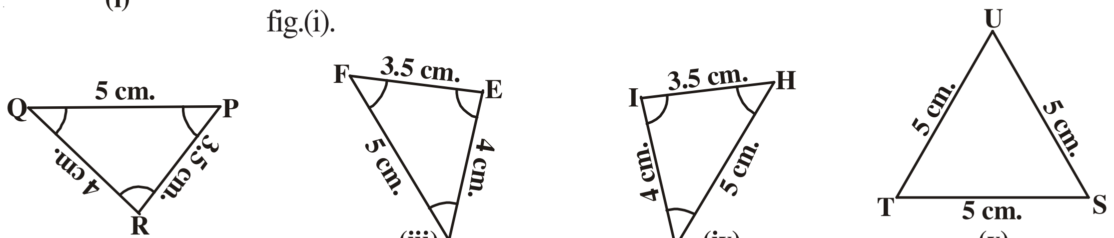
Then we can say that sides, angles, diagonals of one square are respectively equal to the sides, angles and diagonals of the other square.



(i)

Let us now consider the congruence of two triangles. We know that if two traingles are congruent then the sides and angles of one triangle are equal to the corresponding sides and angles of the other triangle.

Which of the triangles given below are congruent to triangle ABC in fig.(i).



If we trace these triangles from fig.(ii) to (v) and try to cover ΔABC. We would observe that triangles in fig.(ii), (iii) and (iv) are congruent to ΔABC while ΔTSU in fig.(v) is not congruent to ΔABC.

If ΔPQR is congruent to ΔABC, we write $\Delta PQR \cong \Delta ABC$.

Notice that when $\Delta PQR \cong \Delta ABC$, then sides of ΔPQR covers the corresponding sides of ΔABC equally and so do the angles.

That is, PQ covers AB, QR covers BC and RP covers CA; $\angle P$ covers $\angle A$, $\angle Q$ covers $\angle B$ and $\angle R$ covers $\angle C$. Also, there is a one-one correspondence between the vertices. That is, P corresponds to A, Q to B, R to C. This can be written as

P $\leftrightarrow$ A, Q $\leftrightarrow$ B, R $\leftrightarrow$ C

Note that under order of correspondence, $\Delta PQR \cong \Delta ABC$; but it will not be correct to write $\Delta QRP \cong \Delta ABC$ as we get QR = AB, RP = BC and QP = AC which is incorrect for the given figures.

Similarly, for fig. (iii),

FD $\leftrightarrow$ AB, DE $\leftrightarrow$ BC and EF $\leftrightarrow$ CA

and F $\leftrightarrow$ A, D $\leftrightarrow$ B and E $\leftrightarrow$ C

So, $\Delta FDE \cong \Delta ABC$ but writing $\Delta DEF \cong \Delta ABC$ is not correct.

Now you give the correspondence between the triangle in fig.(iv) and ΔABC.

So, it is necessary to write the correspondence of vertices correctly for writing of congruence of triangles.

Note that **corresponding parts of congruent triangles** are **equal** and we write in short as 'CPCT' for *corresponding parts of congruent triangles.*

DO THIS

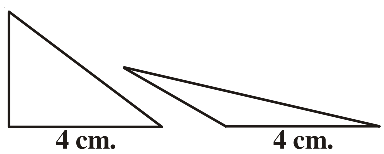
1. There are some statements given below. Write whether they are true or false :
   - i. Two circle are always congruent. ( )
   - ii. Two line segments of same length are always congruent. ( )
   - iii. Two right angle triangles are sometimes congruent. ( )
   - iv. Two equilateral triangles with their sides equal are always congruent. ( )
2. Which minimum measurements do you require to check if the given figures are congruent:
   - i. Two rectangles
   - ii. Two rhombuses

7.2 CRITERIA FOR CONGRUENCE OF TRIANGLES

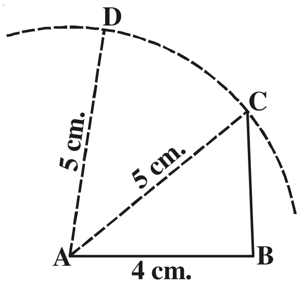
You have learnt the criteria for congruency of triangle in your earlier class.

Is it necessary to know all the three sides and three angles of a triangle to make a unique triangle?

Draw two triangles with one side 4 cm. Can you make two different triangles with one side of 4 cm? Discuss with your friends. Do you all get congruent triangles? You can draw types of triangles if one side is given say 4 cm.

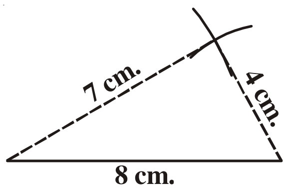


Now take two sides as 4 cm. and 5 cm. and draw as many triangles as you can. Do you get congruent triangles?



We can make different triangles even with these two given measurements.

Now draw triangles with sides 4 cm., 7 cm. and 8 cm.



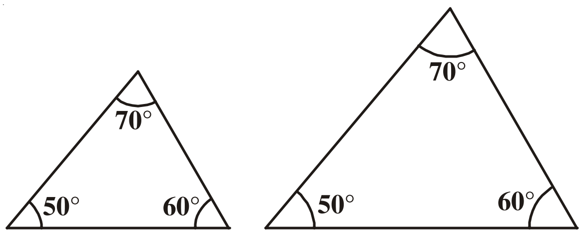
Can you draw two different triangles?

You find that with measurement of these three sides, we can make a unique triangle. If at all you draw the triangles with these dimensions they will be congruent to this unique triangle.

Now take three angles of your choice, of course The sum of the angles must be 180°. Draw two triangles for your choosen angle measurement.

Mahima finds that she can make different triangles by using three angle measurement.

$\angle A = 50°$, $\angle B = 70°$, $\angle C = 60°$



So it seems that knowing the 3 angles in not enough to make a specific triangle.

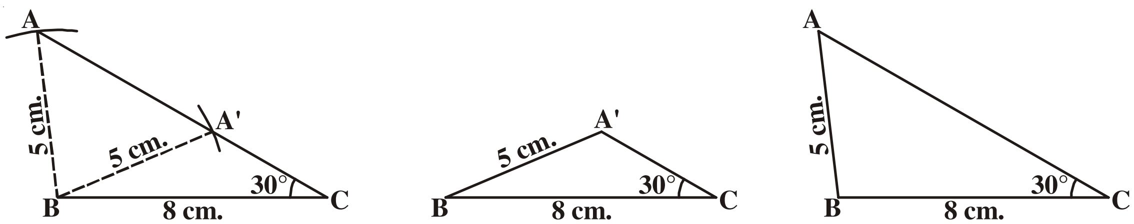
Sharif thought that if two angles are given then he could easily find the third one by using the property of sum of the angles is triangle. So measures of two angles is enough to draw the triangle but not uniquely. Hence giving 3 or 2 angles is not adequate. We need at least three specific and independent measurements (elements) to make a unique triangle.

Now try to draw two distinct triangles with each sets of these three measurements:

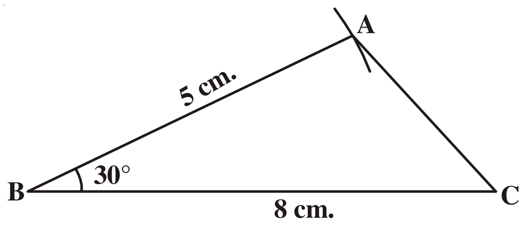
i. ΔABC where AB = 5 cm., BC = 8 cm., $\angle C = 30°$

ii. ΔABC where AB = 5 cm., BC = 8 cm., $\angle B = 30°$

(i) Are you able to draw a unique triangle with the given measurements, draw and check with your friends.



Here we can draw two different triangles ΔABC and $\Delta A'BC$ with given measurements. Now draw two triangles with given measurements (ii). What do you observe? They are congruent triangles. Aren't they?



In the other words you can draw a unique triangle with the measurements given in case(ii).

Have you noticed the order of measures given in case (i) & case (ii) ? In case (i) two sides and one angle are given which is not an included angle but in case (ii) included angle is given along with two sides. Thus given two sides and one angle i.e. three independent measures is not the only criteria to make a unique triangle. But the order of given measurements to construct a triangle also plays a vital role in making a unique triangle.

7.3 Congruence of Triangles

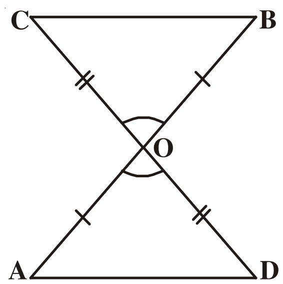
The above has an implication for checking the congruency of triangles. If we have two triangles with one side equal or two triangles with all 3 angles equal, we can not conclude that triangles are congruent as there are more than one triangle possible with these specifications. Even when we have two sides and an angle equal we cannot say that the triangles are congruent unless the angle is between the given sides. We can say that the SAS (side angle side) congruency rule holds but not SSA or ASS.

We take this as the first criterion for congruency of triangles and prove the other criteria through this.

Axiom (SAS congruence rule): *Two triangles are congruent if two sides and the included angle of one triangle are equal to the two sides and the included angle of the other triangle.*

Example-1. In the given Figure AB and CD are intersecting at 'O', OA = OB and OD = OC. Show that

(i) $\Delta AOD \cong \Delta BOC$ and (ii) $AD \parallel BC$.



Solution : (i) you may observe that in ΔAOD and ΔBOC,

$OA = OB$ (given)

$OD = OC$ (given)

Also, since $\angle AOD$ and $\angle BOC$ form a pair of vertically opposite angles, we have

$\angle AOD = \angle BOC$.

So, $\Delta AOD \cong \Delta BOC$ (by the SAS congruence rule)

(ii) In congruent triangles AOD and BOC, the other corresponding parts are also equal.

So, $\angle OAD = \angle OBC$ and these form a pair of alternate angles for line segments AD and BC.

Therefore $AD \parallel BC$

Example-2. AB is a line segment and line l is its perpendicular bisector. If a point P lies on l, show that P is equidistant from A and B.

Solution : Line $l \perp$ AB and passes through C which is the mid-point of AB

We have to show that PA = PB.

Consider ΔPCA and ΔPCB.

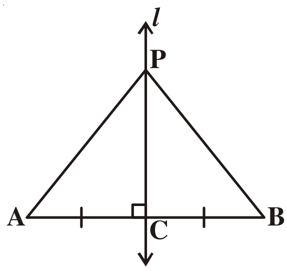
We have AC = BC (C is the mid-point of AB)

$\angle PCA = \angle PCB = 90°$ (Given)

PC = PC (Common)

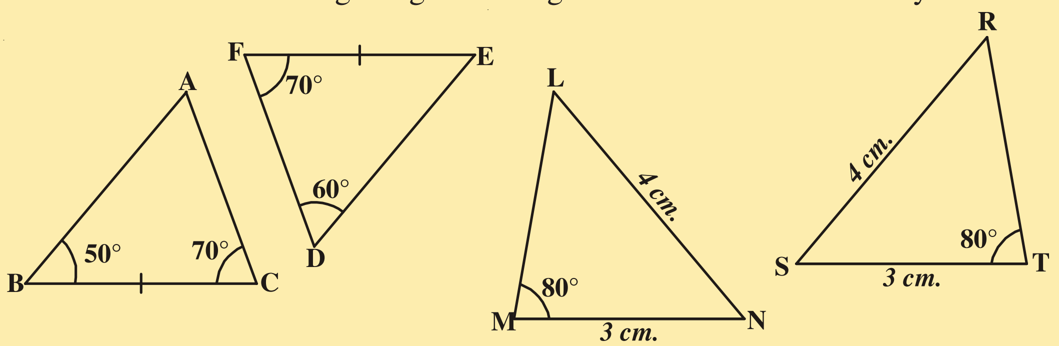
So, $\Delta PCA \cong \Delta PCB$ (SAS rule)

and so, PA = PB, as they are corresponding sides of congruent triangles.



Do These

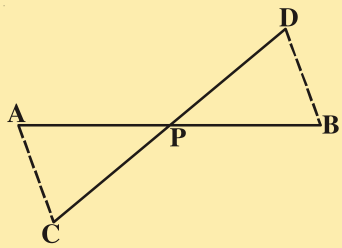
1. State whether the following triangles are congruent or not? Give reasons for your answer.



(i) (ii)

2. In the given figure, the point P bisects AB and DC. Prove that

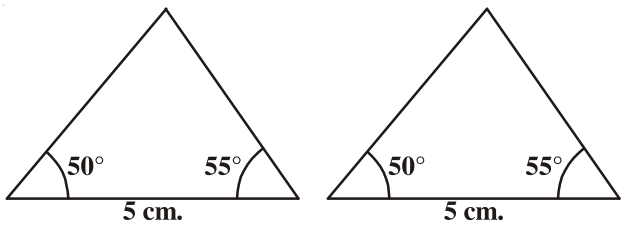
$\Delta APC \cong \Delta BPD$



7.3.1 Other Congruence Rules

Try to construct two triangles in which two of the angles are 50° and 55° and the side on which both these angles lie being 5cm.

Cut out these triangles and place one on the other. What do you observe? You will find that both the triangles are congruent. This result is the angle-side-angle criterion for congruence and is written as ASA criterion you have seen this in earlier classes. Now let us state and prove the result. Since this result can be proved, it is called a theorem and to prove it, we use the SAS axiom for congruence.



Theorem 7.1 (ASA congruence rule) : Two triangles are congruent, if two angles and the included side of one triangle are equal to two angles and the included side of the other triangle.

Given: In ΔABC and ΔDEF

$$\angle B = \angle E,\ \angle C = \angle F \text{ and } \overline{BC} = \overline{EF}$$

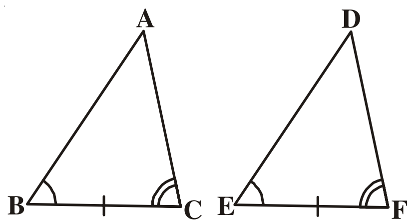
Required To Prove (RTP): $\Delta ABC \cong \Delta DEF$

Proof: There will be three posibilities. The possiblities between $\overline{AB}$ and $\overline{DE}$ are either $\overline{AB} > \overline{DE}$ or $\overline{DE} > \overline{AB}$ or $\overline{DE} = \overline{AB}$.

We will consider all these cases and see what does it mean for ΔABC and ΔDEF.

Case (i): Let $\overline{AB} = \overline{DE}$ Now what do we observe?

Consider ΔABC and ΔDEF



$\overline{AB} = \overline{DE}$ (Assumed)

$\angle B = \angle E$ (Given)

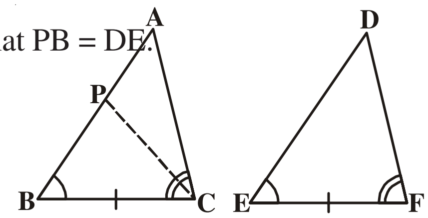
$\overline{BC} = \overline{EF}$ (Given)

So, $\Delta ABC \cong \Delta DEF$ (By SAS congruency axiom)

Case (ii): The second possibility is $AB > DE$.

So, we can take a point P on AB such that PB = DE.

Now consider ΔPBC and ΔDEF



$\overline{PB} = \overline{DE}$ (by construction)

$\angle B = \angle E$ (given)

$\overline{BC} = \overline{EF}$ (given)

So, $\Delta PBC \cong \Delta DEF$ (by SAS congruency axiom)

Since the triangles are congruent their corresponding parts will be equal

So, $\angle PCB = \angle DFE$

But, $\angle ACB = \angle DFE$ (given)

So $\angle ACB = \angle PCB$ (from the above)

Is this possible?

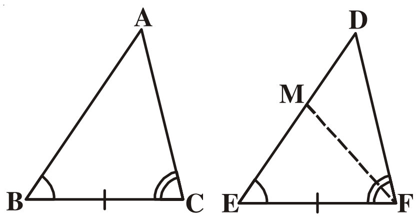
This is possible only if P coincides with A

(or) $\overline{BA} = \overline{ED}$

So, $\Delta ABC \cong \Delta DEF$ (By SAS congruency axiom)

(**Note :** We have shown above that if $\angle B = \angle E$ and $\angle C = \angle F$ and $\overline{BC} = \overline{EF}$ then $\overline{AB} = \overline{DE}$ and the two triangles are congruency by SAS rule).

Case (iii): The third possibility is $\overline{AB} < \overline{DE}$



We can choose a point M on DE such that ME = AB and repeating the arguments as given in case (ii), we can conclude that $\overline{AB} = \overline{DE}$ and so, $\Delta ABC \cong \Delta DEF$. Look at the figure and try to prove it yourself.

Suppose, now in two triangles two pairs of angles and one pair of corresponding sides are equal but the side is not included between the corresponding equal pairs of angles. Are the triangles still congruent? You will observe that they are congruent. Can you reason out why?

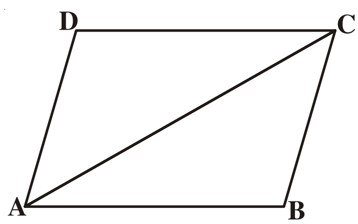
You know that the sum of the three angles of a triangle is 180°. So if two pairs of angles are equal, the third pair is also equal (180° – sum of equal angles).

So, two *triangles are congruent if any two pairs of angles and one pair of corresponding sides are equal.* We may call it as the **AAS Congruence Rule.** Let us now take some more examples.

Example-3. In the given figure, AB|| DC and AD|| BC

Show that $\Delta ABC \cong \Delta CDA$

Solution : Consider ΔABC and ΔCDA



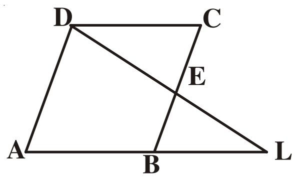
$\angle BAC = \angle DCA$ (alternate interior angles)

AC = CA (common side)

$\angle BCA = \angle DAC$ (alternate interior angles)

$\Delta ABC \cong \Delta CDA$(by ASA congruency)

Example-4. In the given figure, AL || DC, E is mid point of BC. Show that $\Delta EBL \cong \Delta ECD$



Solution : Consider ΔEBL and ΔECD

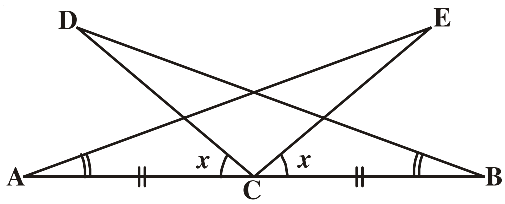
$\angle BEL = \angle CED$ (vertically opposite angles)

BE = CE (since E is mid point of BC)

$\angle EBL = \angle ECD$ (alternate interior angles)

$\Delta EBL \cong \Delta ECD$ (by ASA congruency)

Example-5. Use the information given in the adjoining figure, to prove :



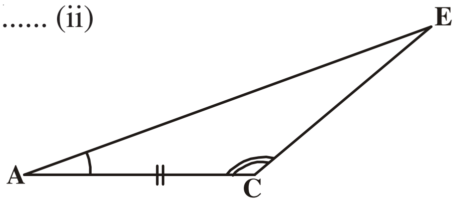
(i) $\Delta DBC \cong \Delta EAC$

(ii) DC = EC.

Solution : Let $\angle ACD = \angle BCE = x$

$\therefore \angle ACE = \angle DCE + \angle ACD = \angle DCE + x$ ...... (i)

$\therefore \angle BCD = \angle DCE + \angle BCE = \angle DCE + x$ ...... (ii)



From (i) and (ii), we get : $\angle ACE = \angle BCD$

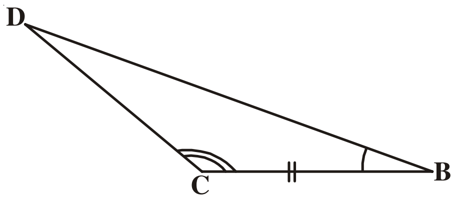
Now in ΔDBC and ΔEAC,

$\angle ACE = \angle BCD$ (proved above)

BC = AC [Given]

$\angle CBD = \angle CAE$ [Given]

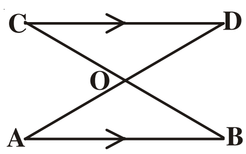
$\Delta DBC \cong \Delta EAC$ [By A.S.A]



since $\Delta DBC \cong \Delta EAC$

DC = EC.(by CPCT)

Example-6. Line-segment AB is parallel to another line-segment CD. O is the mid-point of AD.



Show that (i) $\Delta AOB \cong \Delta DOC$ (ii) O is also the mid-point of BC.

Solution : (i) Consider ΔAOB and ΔDOC.

$\angle ABO = \angle DCO$ (Alternate angles as AB || CD and BC is the transversal)

$\angle AOB = \angle DOC$ (Vertically opposite angles)

OA = OD (Given)

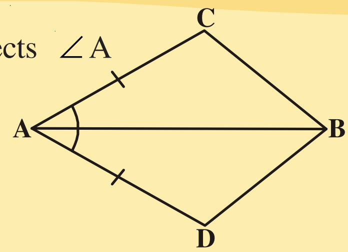
Therefore, $\Delta AOB \cong \Delta DOC$ (AAS rule)

(ii) $OB = OC$ (CPCT)

So, O is the mid-point of BC.

Exercise - 7.1

1. In quadrilateral ACBD, $AC = AD$ and AB bisects $\angle A$. Show that $\Delta ABC \cong \Delta ABD$. What can you say about BC and BD?

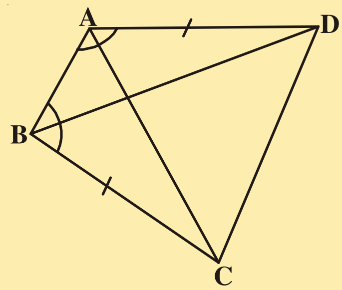


2. ABCD is a quadrilateral in which $AD = BC$ and $\angle DAB = \angle CBA$. Prove that

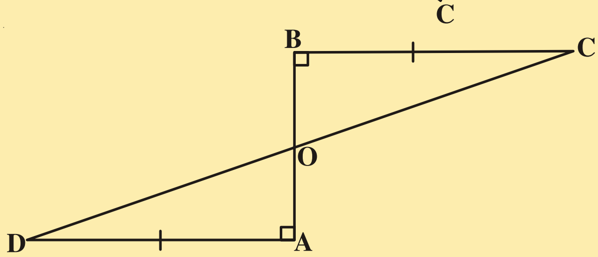
   (i) $\Delta ABD \cong \Delta BAC$

   (ii) $BD = AC$

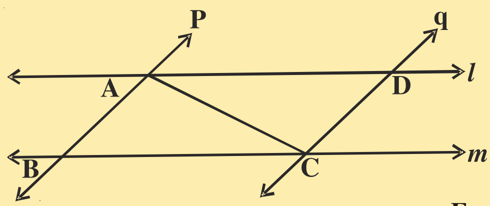
   (iii) $\angle ABD = \angle BAC$



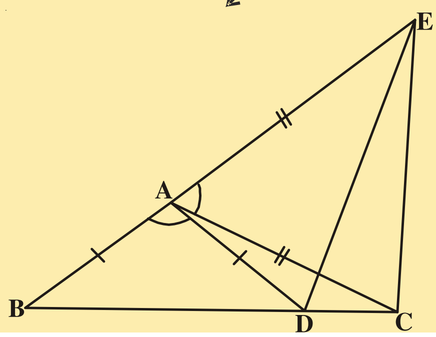
3. AD and BC are equal and perpendiculars to a line segment AB. Show that CD bisects AB.



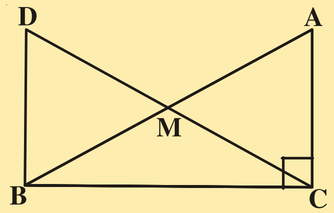
4. l and m are two parallel lines intersected by another pair of parallel lines p and q. Show that $\Delta ABC \cong \Delta CDA$



5. In the adjacent figure, $AC = AE$, $AB = AD$ and $\angle BAD = \angle EAC$. Show that $BC = DE$.



6. In right triangle ABC, right angle is at C, M is the mid-point of hypotenuse AB. C is joined to M and produced to a point D such that DM = CM. Point D is joined to point B (see figure). Show that :



(i) $\Delta AMC \cong \Delta BMD$

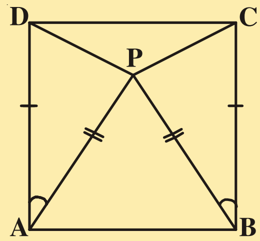
(ii) $\angle DBC$ is a right angle

(iii) $\Delta DBC \cong \Delta ACB$

(iv) $CM = \frac{1}{2} AB$

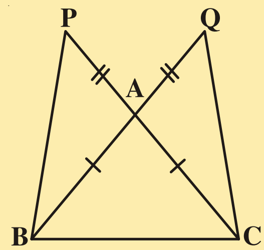
7. In the adjacent figure ABCD is a square and ΔAPB is an equilateral triangle. Prove that $\Delta APD \cong \Delta BPC$.

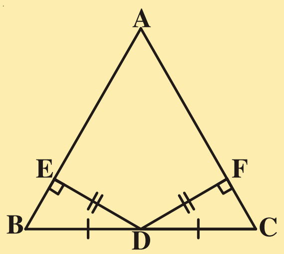
(**Hint :** In ΔAPD and ΔBPC $\overline{AD} = \overline{BC}$, $\overline{AP} = \overline{BP}$ and $\angle PAD = \angle PBC = 90^o - 60^o = 30^o$]



8. In the adjacent figure ΔABC is isosceles as $\overline{AB} = \overline{AC}$, $\overline{BA}$ and $\overline{CA}$ are produced to Q and P such that $\overline{AQ} = \overline{AP}$. Show that $\overline{PB} = \overline{QC}$

(**Hint :** Compare ΔAPB and ΔAQC)





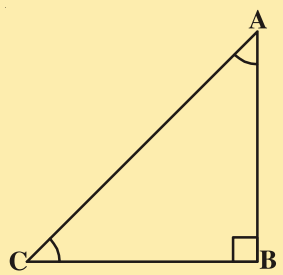
9. In the adjacent figure ΔABC, D is the midpoint of BC. $DE \perp AB$, $DF \perp AC$ and DE = DF. Show that $\Delta BED \cong \Delta CFD$.

10. If the bisector of an angle of a triangle also bisects the opposite side, prove that the triangle is isosceles.

11. In the given figure ABC is a right triangle and right angled at B such that $\angle BCA = 2\angle BAC$.

Show that hypotenuse AC = 2BC.

(**Hint** : Produce CB to a point D that BC = BD)

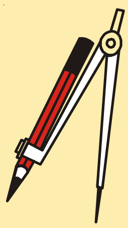


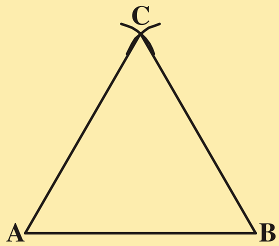
7.4 Some properties of a triangle

In the above section you have studied two criteria for the congruence of triangles. Let us now apply these results to study some properties related to a triangle whose two sides are equal.

ACTIVITY

i. To contruct a triangle using compass, take any measurement and draw a line segment AB. Now open a compass with sufficient length and put it on point A and B and draw an arc. Which type of triangle do you get? Yes this is an isosceles triangle. So, ΔABC in figure is an isosceles triangle with AC = BC. Now measure ∠A and ∠B. What do you observe?



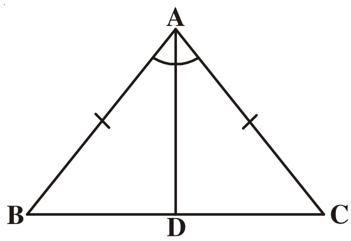


ii. Cut some isosceles triangles.

Now fold the triangle so that two congruent sides fit precisely one on top of the other. What do you notice about ∠A and ∠B?

You may observe that in each such triangle, the angles opposite to the equal sides are equal. This is a very important result and is indeed true for any isosceles triangle. It can be proved as shown below.

Theorem-7.2 : Angles opposite to equal sides of an isosceles triangle are equal.



This result can be proved in many ways. One of the proofs is given here.

Given: ΔABC is an isosceles triangle in which AB = AC.

RTP: ∠B = ∠C.

Construction: Let us draw the bisector of ∠A and let D be the point of intersection of this bisector of ∠A and BC.

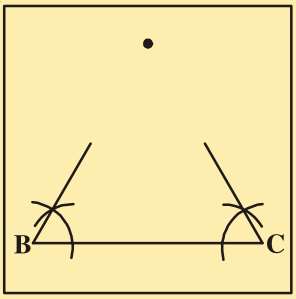
Proof : In ΔBAD and ΔCAD,

AB = AC	(Given)
∠BAD = ∠CAD	(By construction)
AD = AD	(Common)
So, ΔBAD ≅ ΔCAD	(By SAS congruency axiom)
So, ∠ABD = ∠ACD	(By CPCT)
i.e., ∠B = ∠C	(Same angles)

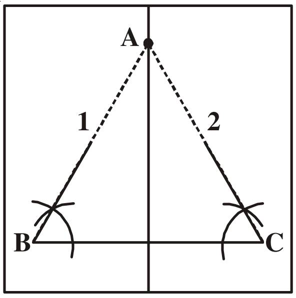
Is the converse also true? That is "If two angles of any triangle are equal, can we conclude that the sides opposite to them are also equal?"

ACTIVITY

1. On a tracing paper draw a line segment BC of length 6cm.
2. From vertices B and C draw rays with angle 60^0 each. Name the point A where they meet.
3. Fold the paper so that B and C fit precisely on top of each other. What do you observe? Is AB = AC?



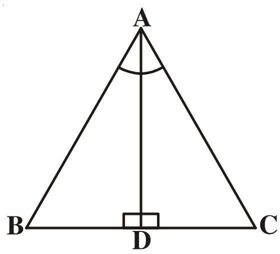
Repeat this activity by taking different angles for ∠B and ∠C. Each time you will observe that the sides opposite to equal angles are equal. So we have the following



Theorem-7.3 : The sides opposite to equal angles of a triangle are equal.

This is the converse of previous Theorem. The student is advised to prove this using ASA congruence rule.

Example-7. In ΔABC, the bisector AD of A is perpendicular to side BC Show that AB = AC and ΔABC is isosceles.



Solution : In ΔABD and ΔACD,

∠BAD = ∠CAD (Given)

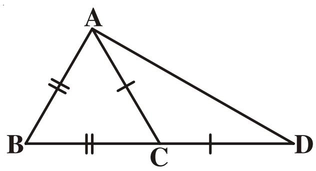
AD = AD (Common side)

∠ADB = ∠ADC = 90° (Given)

So, ΔABD ≅ ΔACD (ASA rule)

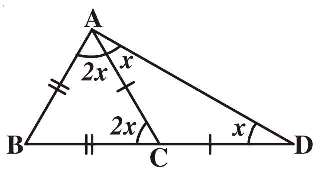
So, AB = AC (CPCT)

or, ΔABC is an isosceles triangle.



Example-8. In the adjacent figure, AB = BC and AC = CD.

Prove that : ∠BAD : ∠ADB = 3 : 1.



Solution : Let ∠ADB = x

In ΔACD, $AC = CD$

$\Rightarrow \angle CAD = \angle CDA = x$

and, ext. $\angle ACB = \angle CAD + \angle CDA$

$= x + x = 2x$

$\Rightarrow \angle BAC = \angle ACB = 2x$. ($\because$ In ABC, AB = BC)

$\therefore \angle BAD = \angle BAC + \angle CAD$

$= 2x + x = 3x$

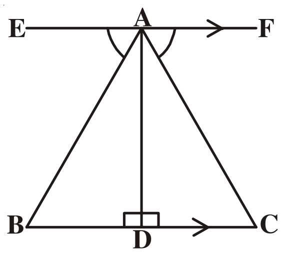
And, $\dfrac{\angle BAD}{\angle ADB} = \dfrac{3x}{x} = \dfrac{3}{1}$

i.e., $\angle BAD : \angle ADB = 3 : 1$.

Hence Proved.

Example-9. In the given figure, AD is perpendicular to BC and EF $\parallel$ BC, if $\angle EAB = \angle FAC$, show that triangles ABD and ACD are congruent.

Also, find the values of x and y if $AB = 2x + 3$, $AC = 3y + 1$, $BD = x$ and $DC = y + 1$.



Solution : AD is perpendicular to EF

$\Rightarrow \angle EAD = \angle FAD = 90^0$

$\angle EAB = \angle FAC$ (given)

$\Rightarrow \angle EAD - \angle EAB = \angle FAD - \angle FAC$

$\Rightarrow \angle BAD = \angle CAD$

In ΔABD and ΔACD

$\angle BAD = \angle CAD$ [proved above]

$\angle ADB = \angle ADC = 90^0$ [Given AD is perpendicular on BC]

and $AD = AD$ (Common side)

$\therefore \Delta ABD \cong \Delta ACD$ [By ASA]

Hence proved.

$\angle ABD = \angle ACD \Rightarrow AB = AC$ and $BD = CD$ [By C.P.C.T]

$\Rightarrow 2x + 3 = 3y + 1$ and $x = y + 1$

$\Rightarrow 2x - 3y = -2$ and $x - y = 1$

Substituting $x = 1 + y$: $2(1 + y) - 3y = -2$

$2 + 2y - 3y = -2$

$-y = -2 - 2$

$-y = -4$

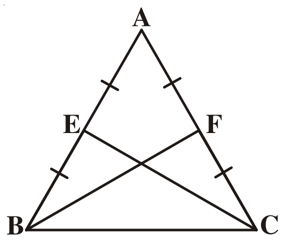
Substituting $y = 4$ in $x = 1 + y$

$x = 1 + 4$

$x = 5$

Example-10. E and F are respectively the mid-points of equal sides AB and AC of ΔABC (see figure)

Show that BF = CE



Solution : In ΔABF and ΔACE,

AB = AC (Given)

$\angle A = \angle A$ (common angle)

AF = AE (Halves of equal sides)

So, $\Delta ABF \cong \Delta ACE$ (SAS rule)

Therefore, BF = CE (CPCT)

Example-11. In an isosceles triangle ABC with AB = AC, D and E are points on BC such that BE = CD (see figure) Show that AD = AE,

Solution : In ΔABD and ΔACE,

AB = AC (given) .... (1)

$\angle B = \angle C$ (Angles opposite to equal sides) .... (2)

Also, BE = CD

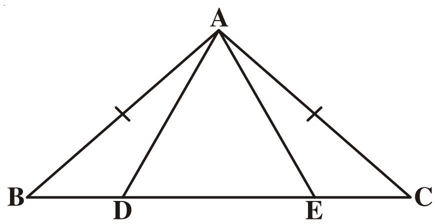
So, BE − DE = CD − DE

That is, BD = CE (3)

So, $\Delta ABD \cong \Delta ACE$

(Using (1), (2), (3) and SAS rule).

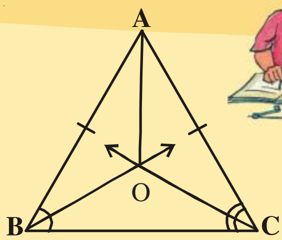
This gives AD = AE (CPCT)



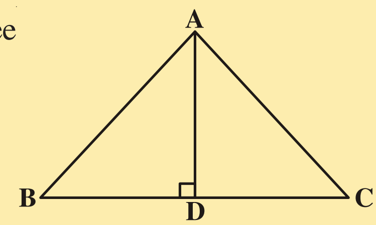
EXERCISE - 7.2

1. In an isosceles triangle ABC, with AB = AC, the bisectors of $\angle B$ and $\angle C$ intersect each other at O. Join A to O. Show that :

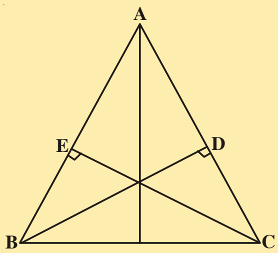
   (i) OB = OC (ii) AO bisects $\angle A$

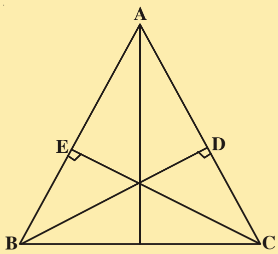


2. In ΔABC, AD is the perpendicular bisector of BC (See adjacent figure). Show that ΔABC is an isosceles triangle in which AB = AC.



3. ABC is an isosceles triangle in which altitudes BD and CE are drawn to equal sides AC and AB respectively (see figure) Show that these altitudes are equal.



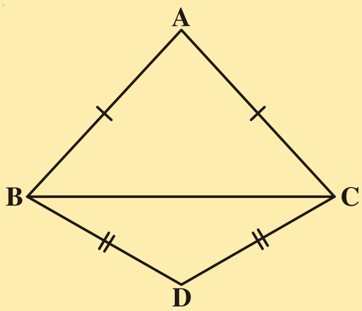


4. ABC is a triangle in which altitudes BD and CE to sides AC and AB are equal (see figure) . Show that

   (i) $\Delta ABD \cong \Delta ACE$

   (ii) $AB = AC$ i.e., ABC is an isosceles triangle.

5. ΔABC and ΔDBC are two isosceles triangles on the same base BC (see figure). Show that $\angle ABD = \angle ACD$.



7.5 Some more criteria for congruency of triangles

Theorem 7.4 (SSS congruence rule) : Through construction we have seen that SSS congruency rule hold. This theorem can be proved using a suitable construction.

In two triangles, if the three sides of one triangle are respectively equal to the corresponding three sides of another triangle, then the two triangles are congruent.

• **Proof for SSS Congruence Rule**

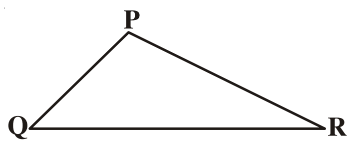
Given: ΔPQR and ΔXYZ are such that $PQ = XY$, $QR = YZ$ and $PR = XZ$

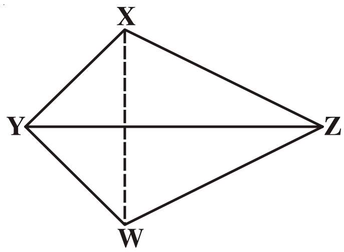
To Prove : $\Delta PQR \cong \Delta XYZ$

Construction : Draw YW such that $\angle ZYW = \angle PQR$ and $WY = PQ$. Join XW and WZ

Proof: In ΔPQR and ΔWYZ

$QR = YZ$	(Given)
$\angle PQR = \angle ZYW$	(Construction)
$PQ = YW$	(Construction)
$\therefore \Delta PQR \cong \Delta WYZ$	(SAS congruence axiom)





$\Rightarrow \angle P = \angle W$ and $PR = WZ$ (CPCT)

$PQ = XY$ (given) and $PQ = YW$ (Construction)

$\therefore XY = YW$

Similarly, $XZ = WZ$

In ΔXYW, $XY = YW$

$\Rightarrow \angle YWX = \angle YXW$ (In a triangle, equal sides have equal angles opposite to them)

Similarly, $\angle ZWX = \angle ZXW$

$\therefore \angle YWX + \angle ZWX = \angle YXW + \angle ZXW$

$\Rightarrow \angle W = \angle X$

Now, $\angle W = \angle P$

$\therefore \angle P = \angle X$

In ΔPQR and ΔXYZ

$PQ = XY$

$\angle P = \angle X$

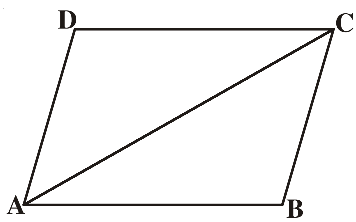
$PR = XZ$

$\therefore \Delta PQR \cong \Delta XYZ$ (SAS congruence criterion)

Let us see the following example based on it.

Example-12. In quadrilateral ABCD, $AB = CD$, $BC = AD$ show that $\Delta ABC \cong \Delta CDA$

Consider ΔABC and ΔCDA



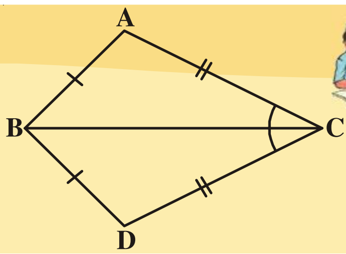
$AB = CD$ (given)

$AD = BC$ (given)

$AC = CA$ (common side)

$\Delta ABC \cong \Delta CDA$ (by SSS congruency rule)

Do This



1. In the adjacent figure ΔABC and ΔDBC are two triangles such that $\overline{AB} = \overline{BD}$ and $\overline{AC} = \overline{CD}$. Show that $\Delta ABC \cong \Delta DBC$.

You have already seen that in the SAS congruency axiom, the pair of equal angles has to be the included angle between the pairs of corresponding equal sides and if not so, two triangles may not be congruent.

ACTIVITY

Construct a right angled triangle with hypotenuse 5 cm. and one side 3 cm. long. How many different triangles can be constructed? Compare your triangle with those of the other members of your class. Are the triangles congruent? Cut them out and place one triangle over the other with equal side placed on each other. Turn the triangle if necessary what do you observe? You will find that two right triangles are congruent, if side and hypotenuse of one triangle are respectively equal to the corereseponding side and hypotenuse of other triangle.

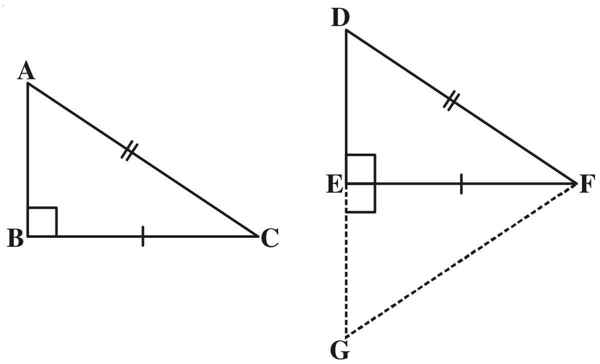
Note that the right angle is not the included angle in this case. So we arrive at the following congruency rule.

Theorem 7.5 (RHS congruence rule) : If in two right triangles the hypotenuse and one side of one triangle are equal to the hypotenuse and one side of the another triangle, then the two triangles are congruent.

Note that RHS stands for right angle - hypotenuse-side.

Let us prove it.

Given: Two right triangles, ΔABC and ΔDEF in which $\angle B = 90^o$ and $\angle E = 90^o$; $AC = DF$ and $BC = EF$.



To prove: $\Delta ABC \cong \Delta DEF$

Construction: Produce DE to G So that EG = AB. Join GF.

Proof:

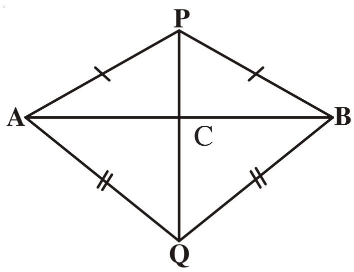
In ΔABC and ΔGEF, we have

$AB = GE$	(By construction)
$\angle B = \angle FEG$	(Each angle is a right angle (90^0))
$BC = EF$	(Given)
$\Delta ABC \cong \Delta GEF$	(By SAS criterion of congruence)
So $\angle A = \angle G \ldots (1)$	(CPCT)

$AC = GF \dots (2)$	(CPCT)
Further, AC=GF and AC=DF	(From (2) and Given)
Therefore DF = GF	(From the above)
So, $\angle D = \angle G \dots (3)$	(Angles opposite to equal sides are equal)
we get $\angle A = \angle D \dots (4)$	(From (1) and (3))
Thus, in ΔABC and ΔDEF $\angle A = \angle D$,	(From (4))
$\angle B = \angle E$	(Given)
So, $\angle A + \angle B = \angle D + \angle E$	(on adding)
But $\angle A + \angle B + \angle C = 180^0$ and	(angle sum property of triangle)
$\angle D + \angle E + \angle F = 180^0$	
$180 - \angle C = 180 - \angle F$	($\angle A + \angle B = 180^0 - \angle C$ and $\angle D + \angle E = 180^0 - \angle F$)
So, $\angle C = \angle F, \dots (5)$	(Cancellation laws)
Now, in ΔABC and ΔDEF, we have	
$BC = EF$	(given)
$\angle C = \angle F$	(from (5))
$AC = DF$	(given)
$\Delta ABC \cong \Delta DEF$	(by SAS axiom of congruence)

Example-13. AB is a line - segment. P and Q are points on either side of AB such that each of them is equidistant from the points A and B (See Fig). Show that the line PQ is the perpendicular bisector of AB.

Solution : You are given PA = PB and QA = QB and you have to show that PQ is perpendicular on AB and PQ bisects AB. Let PQ intersect AB at C.



Can you think of two congruent triangles in this figure ?

Let us take ΔPAQ and ΔPBQ.

In these triangles,

AP = BP (Given)

AQ = BQ (Given)

PQ = PQ (Common side)

So, $\Delta PAQ \cong \Delta PBQ$ (SSS rule)

Therefore, $\angle APQ = \angle BPQ$ (CPCT).

Now let us consider ΔPAC and ΔPBC.

You have : AP = BP (Given)

$\angle APC = \angle BPC$ ($\angle APQ = \angle BPQ$ proved above)

$PC = PC$ (Common side)

So, $\Delta PAC \cong \Delta PBC$ (SAS rule)

Therefore, $AC = BC$ (CPCT) .... (1)

and $\angle ACP = \angle BCP$ (CPCT)

Also, $\angle ACP + \angle BCP = 180°$ (Linear pair)

So, $2\angle ACP = 180°$

or, $\angle ACP = 90°$ .... (2)

From (1) and (2), you can easily conclude that PQ is the perpendicular bisector of AB.

[Note that, without showing the congruence of ΔPAQ and ΔPBQ, you cannot show that $\Delta PAC \cong \Delta PBC$ even though $AP = BP$ (Given)

$PC = PC$ (Common side)

and $\angle PAC = \angle PBC$ (Angles opposite to equal sides in ΔAPB)

It is because these results give us SSA rule which is not always valid or true for congruence of triangles as the given angle is not included between the equal pairs of sides.]

Let us take some more examples.

Example-14. P is a point equidistant from two lines *l* and *m* intersecting at point A (see figure). Show that the line AP bisects the angle between them.

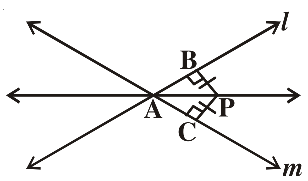
Solution : You are given that lines *l* and *m* intersect each other at A.

Let PB is perpendicular on *l* and

$PC \perp m$. It is given that $PB = PC$.

You need to show that $\angle PAB = \angle PAC$.

Let us consider ΔPAB and ΔPAC. In these two triangles,



$PB = PC$ (Given)

$\angle PBA = \angle PCA = 90°$ (Given)

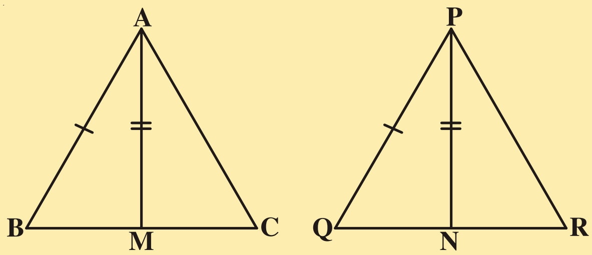
$PA = PA$ (Common side)

So, $\Delta PAB \cong \Delta PAC$ (RHS rule)

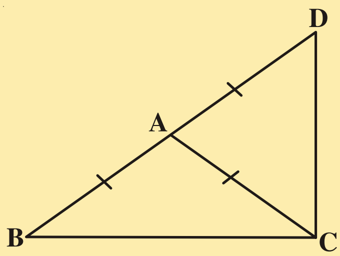
So, $\angle PAB = \angle PAC$ (CPCT)

EXERCISE - 7.3

1. AD is an altitude of an isosceles triangle ABC in which AB = AC.
   Show that, (i) AD bisects BC (ii) AD bisects $\angle A$.
2. Two sides AB, BC and median AM of one triangle ABC are respectively equal to sides PQ and QR and median PN of ΔPQR (See figure). Show that:

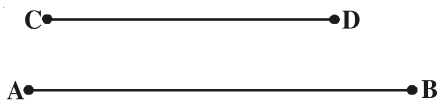
   (i) $\Delta ABM \cong \Delta PQN$
   (ii) $\Delta ABC \cong \Delta PQR$
3. BE and CF are two equal altitudes of a triangle ABC. Using RHS congruence rule, prove that the triangle ABC is isosceles.
4. ΔABC is an isosceles triangle in which AB = AC. Show that $\angle B = \angle C$.
   (Hint : Draw AP $\perp$ BC) (Using RHS congruence rule)
5. ΔABC is an isosceles triangle in which AB = AC. Side BA is produced to D such that AD = AB (see figure). Show that $\angle BCD$ is a right angle.

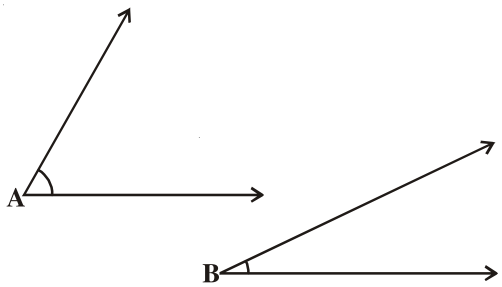
6. ABC is a right angled triangle in which $\angle A = 90^0$ and AB = AC. Show that $\angle B = \angle C$.
7. Show that the angles of an equilateral triangle are 60^0 each.

7.6 INEQUALITIES IN A TRIANGLE

So far, you have been studying the equality of sides and angles of a triangle or triangles. Sometimes, we do come across unequal figures and we need to compare them. For example, line segment AB is greater in length as compared to line segment CD in figure (i) and $\angle A$ is greater than $\angle B$ in following figure (ii).



(i)



(ii)

Let us now examine whether there is any relation between unequal sides and unequal angles of a triangle. For this, let us perform the following activity:

Activity

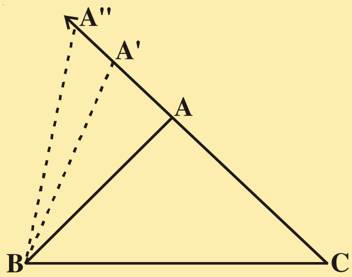
1. Draw a triangle ABC mark a point A' on CA produced (new position of it)

So, $A'C > AC$ (Comparing the lengths)

Join A' to B and complete the triangle $A'BC$.

What can you say about $\angle A'BC$ and $\angle ABC$?

Compare them. What do you observe?



Clearly, $\angle A'BC > \angle ABC$

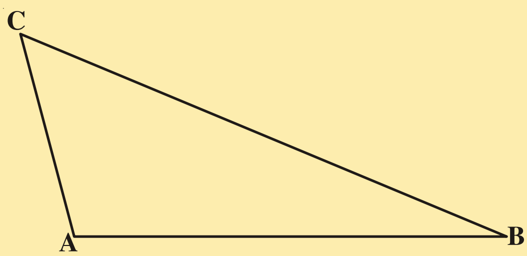
Continue to mark more points on CA (extended) and draw the triangles with the side BC and the points marked.

You will observe that as the length of the side AC is increases (by taking different positions of A), the angle opposite to it, that is, $\angle B$ also increases.

Let us now perform another activity-

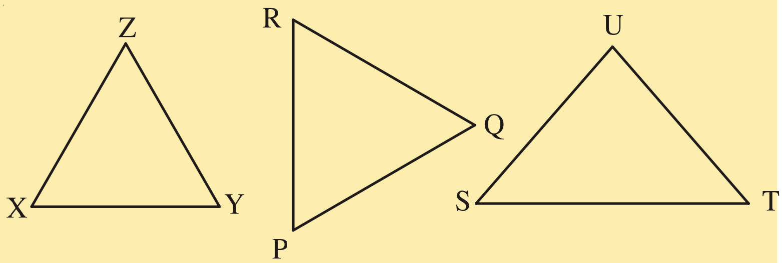
2. Construct a scalene triangle ABC (that is a triangle in which all sides are of different lengths). Measure the lengths of the sides.

Now, measure the angles. What do you observe?



In ΔABC Figure, BC is the longest side and AC is the shortest side.

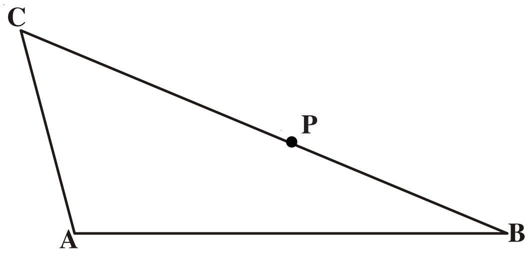
Also, $\angle A$ is the largest and $\angle B$ is the smallest.



Measure angles and sides of each of the above triangles, what is the relation between a side and its opposite angle when campared with another pair?

Theorem-7.6 : If two sides of a triangle are unequal, the angle opposite to the longer side is larger (or greater).

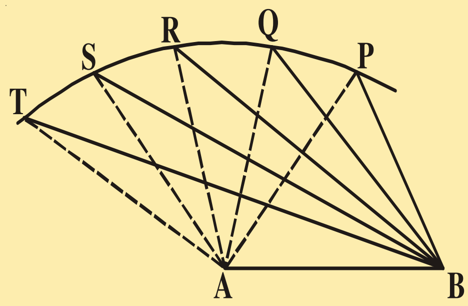
You may prove this theorem by taking a point P on BC such that CA = CP as shown in adjacent figure.



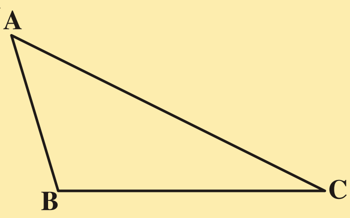
Now, let us do another activity:

ACTIVITY

Draw a line-segment AB. With A as centre and some radius, draw an arc and mark different points say P, Q, R, S, T on it.



Join each of these points with A as well as with B (see figure). Observe that as we move from P to T, $\angle A$ is becoming larger and larger. What is happening to the length of the side opposite to it? Observe that the length of the side is also increasing; that is $\angle TAB > \angle SAB > \angle RAB > \angle QAB > \angle PAB$ and $TB > SB > RB > QB > PB$.

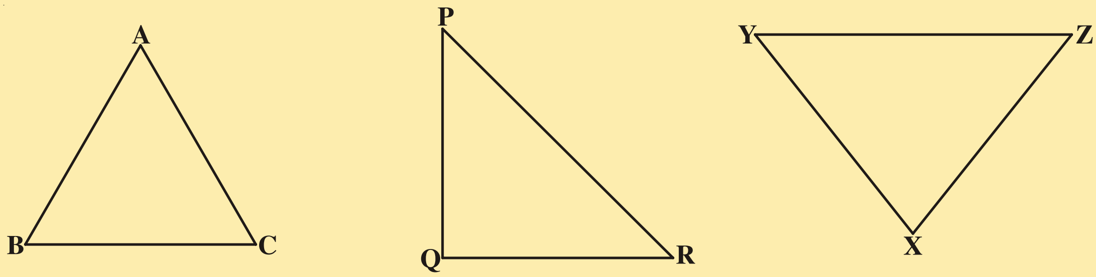


Now, draw any triangle with all angles unequal to each other. Measure the lengths of the sides (see figure).

Observe that the side opposite to the largest angle is the longest. In figure, $\angle B$ is the largest angle and AC is the longest side.

Repeat this activity for some more triangles and we see that the converse of the above Theorem is also true.

Measure angles and sides of each triangle given below. What relation you can visualize for a side and its opposite angle in each triangle.



In this way, we arrive at the following theorem.

Theorem -7.7 : In any triangle, the side opposite to the larger (greater) angle is longer.

This theorem can be proved by the method of contradiction.

DO THIS

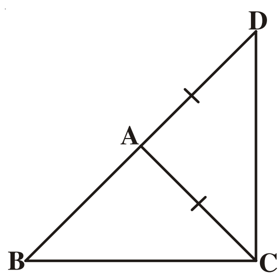
Now draw a triangle ABC and measure its sides. Find the sum of the sides AB + BC, BC + AC and AC + AB, compare it with the length of the third side. What do you observe?

You will observe that AB + BC > AC,

BC + AC > AB and AC + AB > BC.

Repeat this activity with other triangles and with this you can arrive at the following theorem:

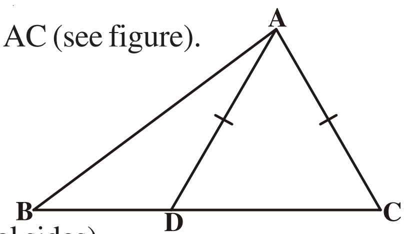
Theorem-7.8 : The sum of any two sides of a triangle is greater than the third side.



In adjacent figure, observe that the side BA of ΔABC has been produced to a point D such that AD = AC. Can you show that $\angle BCD > \angle BDC$ and BA + AC > BC? Have you arrived at the proof of the above theorem.

Let us take some examples based on these results.

Example-15. D is a point on side BC ΔABC such that AD = AC (see figure). Show that AB > AD.



Solution : In ΔDAC,

AD = AC (Given)

So, $\angle ADC = \angle ACD$ (Angles opposite to equal sides)

Now, $\angle ADC$ is an exterior angle for ΔABD.

So, $\angle ADC > \angle ABD$

or, $\angle ACD > \angle ABD$

or, $\angle ACB > \angle ABC$

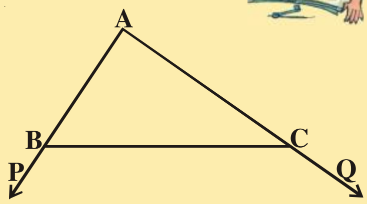
So, $AB > AC$ (Side opposite to larger angle in ΔABC)

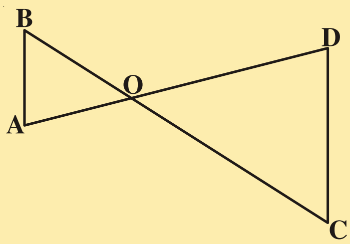
or, $AB > AD$ $(AD = AC)$

EXERCISE - 7.4

1. Show that in a right angled triangle, the hypotenuse is the longest side.

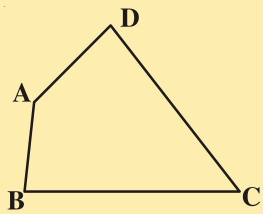
2. In adjacent figure, sides AB and AC of ΔABC are extended to points P and Q respectively.
Also, $\angle PBC < \angle QCB$. Show that $AC > AB$.

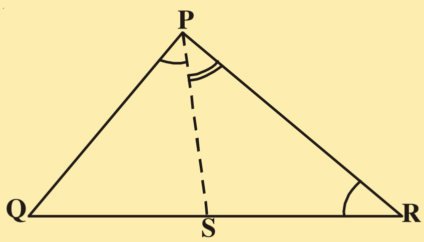




3. In adjacent figure, $\angle B < \angle A$ and $\angle C < \angle D$. Show that $AD < BC$.

4. AB and CD are respectively the smallest and longest sides of a quadrilateral ABCD (see adjacent figure).
Show that $\angle A > \angle C$ and $\angle B > \angle D$.





5. In adjacent figure, $PR > PQ$ and PS bisects $\angle QPR$. Prove that $\angle PSR > \angle PSQ$.

6. If two sides of a triangle measure 4cm and 6cm find all possible measurements (positive Integers) of the third side. How many distinct triangles can be obtained?

7. Try to construct a triangle with 5cm, 8cm and 1cm. Is it possible or not? Why? Give your justification?

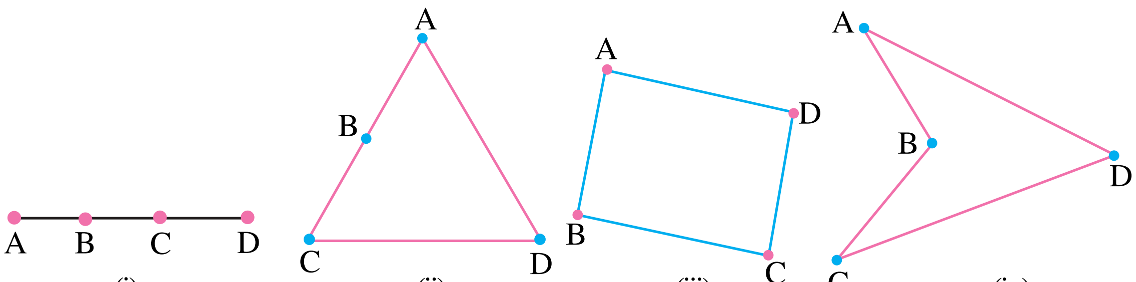
WHAT WE HAVE DISCUSSED

- Figures which are identical i.e. having same shape and size are called congruent figures.
- Three independent elements to make a unique triangle.
- Two triangles are congruent if the sides of one triangle are equal to the sides of another triangle and the corresponding angles in the two triangles are equal.
- Also, there is a one-one correspondence between the vertices.
- In Congruent triangles corresponding parts are equal and we write in short 'CPCT' for corresponding parts of congruent triangles.
- SAS congruence rule: Two triangles are congruent if two sides and the included angle of one triangle are equal to the corresponding two sides and the included angle of the other triangle.
- ASA congruence rule: Two triangles are congruent if two angles and the included side of one triangle are equal to two angles and the included side of other triangle.
- Angles opposite to equal sides of an isosceles triangle are equal.
- Conversely, sides opposite to equal angles of a triangle are equal.
- SSS congruence rule: If three sides of one triangle are equal to the three sides of another triangle, then the two triangles are congruent.
- RHS congruence rule: If in two right triangles the hypotenuse and one side of one triangle are equal to the hypotenuse and corresponding side of the other triangle, then the two triangles are congruent.
- If two sides of a triangle are unequal, the angle opposite to the longer side is larger.
- In any triangle, the side opposite to the larger angle is longer.
- The sum of any two sides of a triangle is greater than the third side.

08 Quadrilaterals

8.1 Introduction

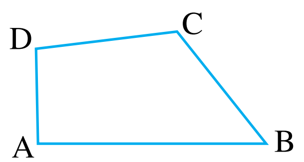
You have learnt many properties of triangles in the previous chapter with justification. You know that a triangle is a figure obtained by joining three non-collinear points in pairs. Do you know which figure you obtain with four points in a plane ? Note that if all the points are collinear, we obtain a line segment (Fig. (i)), if three out of four points are collinear, we get a triangle (Fig(ii)) and if any three points are not collinear, we obtain a closed figure with four sides (Fig (iii), (iv)), we call such a figure as a quadrilateral.



(i) (ii) (iii) (iv)

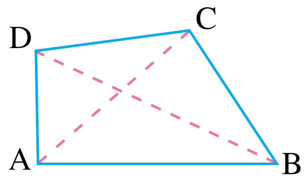
You can easily draw many more quadrilaterals and identify many around you. The Quadrilateral formed in Fig (iii) and (iv) are different in one important aspect. How are they different?

In this chapter we will study quadrilaterals only of type (Fig (iii)). These are convex quadrilaterals.



A quadrilateral is a simple closed figure bounded by four line segments in a plane.

The quadrilateral ABCD has four sides AB, BC, CD and DA, four vertices are A, B, C and D. $\angle A$, $\angle B$, $\angle C$ and $\angle D$ are the four angles formed at the vertices.



When we join the opposite vertices A, C and B, D (in the fig.) AC and BD are the two diagonals of the Quadrilateral ABCD.

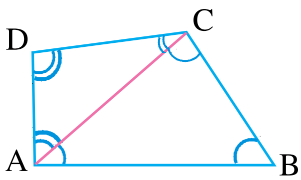
8.2 Properties of a Quadrilateral

There are four angles in the interior of a quadrilateral. Can we find the sum of these four angles? Let us recall the angle sum property of a triangle. We can use this property in finding sum of four interior angles of a quadrilateral.

ABCD is a quadrilateral and AC is a diagonal (see figure).

We know the sum of the three angles of ΔABC is,

$\angle CAB + \angle B + \angle BCA = 180^\circ$ ...(1) (Angle sum property of a triangle)



Similarly, in ΔADC,

$$\angle CAD + \angle D + \angle DCA = 180^\circ \qquad ...(2)$$

Adding (1) and (2), we get

$$\angle CAB + \angle B + \angle BCA + \angle CAD + \angle D + \angle DCA = 180^\circ + 180^\circ$$

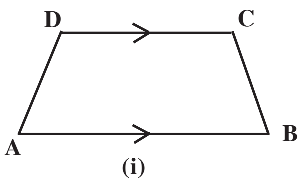
Since $\angle CAB + \angle CAD = \angle A$ and $\angle BCA + \angle DCA = \angle C$

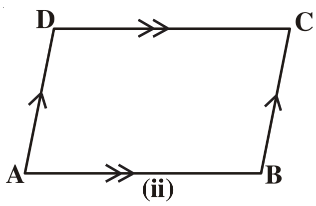
So, $\angle A + \angle B + \angle C + \angle D = 360^\circ$

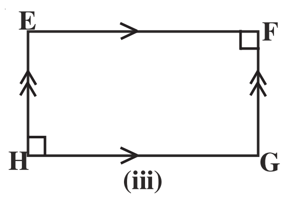
i.e the sum of four angles of a quadrilateral is 360° or 4 right angles.

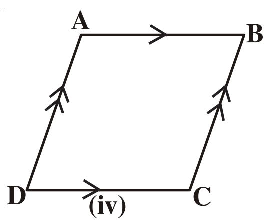
8.3 Different types of Quadrilaterals

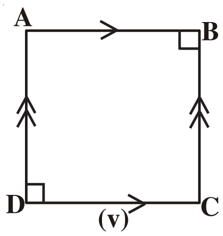
Look at the quadrilaterals drawn below. We have come across most of them earlier. We will quickly consider these and recall their specific names based on their properties.

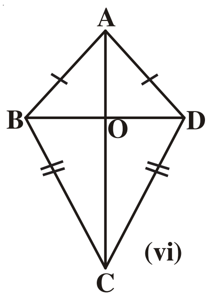












We observe that

- In fig. (i) the quadrilateral ABCD had one pair of opposite sides AB and DC parallel to each other. Such a quadrilateral is called a trapezium.

  If in a trapezium non parallel sides are equal, then the trapezium is an isosceles trapezium.
- In fig. (ii) both pairs of opposite sides of the quadrilateral are parallel such a quadrilateral is called a parallelogram. Fig.(iii), (iv) and (v) are also parallelograms.
- In fig. (iii) parallelogram EFGH has all its angles as right angles. It is a rectangle.
- In fig. (iv) parallelogram has its adjacent sides equal and is called a Rhombus.
- In fig. (v) parallelogram has its adjacent sides equal and angles of 90° this is called a square.
- The quadrilateral ABCD in fig.(vi) has the two pairs of adjacent sides equal, i.e. AB = AD and BC = CD. It is called a kite.

Consider what Nisha says:

A rhombus can be a square but all squares are not rhombuses.

Lalita Adds

All rectangles are parallelograms but all parallelograms are not rectangles.

Which of these statements you agree with?

Give reasons for your answer. Write other such statements about different types of quadrilaterals.

Illustrative examples

Example-1. ABCD is a parallelogram and $\angle A = 60^o$. Find the remaining angles.

Solution : The opposite angles of a parallelogram are equal.

So in a parallelogram ABCD

$$\angle C = \angle A = 60^o \text{ and } \angle B = \angle D$$

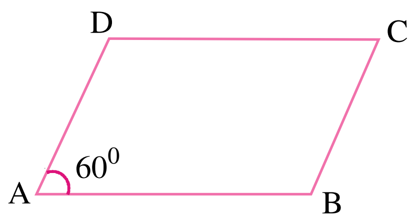
and the sum of consecutive angles of parallelogram is equal to 180^o.

As $\angle A$ and $\angle B$ are consecutive angles

$$\angle D = \angle B = 180^o - \angle A$$

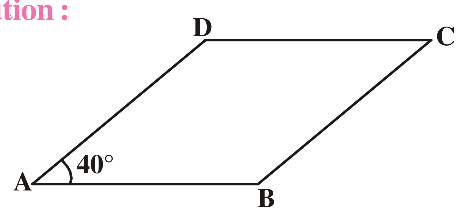
$$= 180^o - 60^o = 120^o.$$

Thus the remaining angles are $120^o, 60^o, 120^o$.



Example-2. In a parallelogram ABCD, $\angle DAB = 40^o$ find the other angles of the parallelogram.

Solution :



ABCD is a parallelogram

$\angle DAB = \angle BCD = 40°$ and AD || BC

As sum of consecutive angles

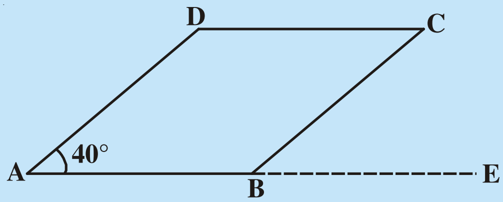
$\angle CBA + \angle DAB = 180°$

$\therefore \ \angle CBA = 180 - 40°$

$= 140°$

From this we can find $\angle ADC = 140°$ and $\angle BCD = 40°$

TRY THIS



Extend AB to E. Find $\angle CBE$. What do you notice. What kind of angles are $\angle ABC$ and $\angle CBE$?

Example-3. Two adjacent sides of a parallelogram are 4.5 cm and 3 cm. Find its perimeter.

Solution : Since the opposite sides of a parallelogram are equal the other two sides are 4.5 cm and 3 cm.

Hence, the perimeter = 4.5 + 3 + 4.5 + 3 = 15 cm.

Example-4. In a parallelogram ABCD, the bisectors of the consecutive angles $\angle A$ and $\angle B$ intersect at P. Show that $\angle APB = 90^o$.

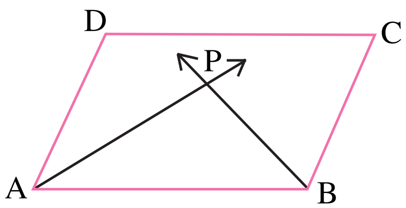
Solution : ABCD is a parallelogram $\overrightarrow{AP}$ and $\overrightarrow{BP}$ are bisectors of consecutive angles, $\angle A$ and $\angle B$.

As, the sum of consecutive angles of a parallelogram is supplementary,

$\angle A + \angle B = 180^o$

$\frac{1}{2}\angle A + \frac{1}{2}\angle B = \frac{180}{2}$

$\Rightarrow \angle PAB + \angle PBA = 90°$



In Δ APB,

$\angle PAB + \angle APB + \angle PBA = 180°$ (angle sum property of triangle)

$\angle APB = 180^o - (\angle PAB + \angle PBA)$

$= 180^o - 90^o$

$= 90^o$

Hence proved.

Exercise - 8.1

1. State whether the statements are True or False.
   - (i) Every parallelogram is a trapezium ( )
   - (ii) All parallelograms are quadrilaterals ( )
   - (iii) All trapeziums are parallelograms ( )
   - (iv) A square is a rhombus ( )
   - (v) Every rhombus is a square ( )
   - (vi) All parallelograms are rectangles ( )
2. Complete the following table by writing (YES) if the property holds for the particular Quadrilateral and (NO) if property does not holds.

Properties	Trapezium	Parallelogram	Rhombus	Rectangle	square
a. Only one pair of opposite sides are parallel	YES				
b. Two pairs of opposite sides are parallel					
c. Opposite sides are equal					
d. Opposite angles are equal					
e. Consecutive angles are supplementary					
f. Diagonals bisect each other					
g. Diagonals are equal					
h. All sides are equal					
i. Each angle is a right angle					
j. Diagonals are perpendicular to each other.					

3. ABCD is trapezium in which AB $\parallel$ CD. If AD = BC, show that $\angle A = \angle B$ and $\angle C = \angle D$.
4. The four angles of a quadrilateral are in the ratio 1: 2:3:4. Find the measure of each angle of the quadrilateral.
5. ABCD is a rectangle AC is diagonal. Find the nature of ΔACD. Give reasons.

8.4 Parallelogram and their Properties

We have seen parallelograms are quadrilaterals. In the following we would consider the properties of parallelograms.

Activity

Cut-out a parallelogram from a sheet of paper again and cut along one of its diagonal. What kind of shapes you obtain? What can you say about these triangles?

Place one triangle over the other. Can you place each side over the other exactly. You may need to turn the triangle around to match sides. Since, the two traingles match exactly they are congruent to each other.

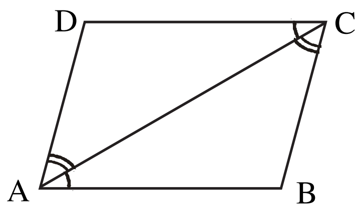
Do this with some more parallelograms. You can select any diagonal to cut along.

We see that each diagonal divides the parallelogram into two congruent triangles.

Let us now prove this result.

Theorem-8.1 : A diagonal of a parallelogram divides it into two congruent triangles.

Proof: Consider the parallelogram ABCD.



Join A and C. AC is a diagonal of the parallelogram.

Since AB || DC and AC is transversal

$\angle DCA = \angle CAB$. (Interior alternate angles)

Similarly DA || CB and AC is a transversal therefore $\angle DAC = \angle BCA$.

We have in ΔACD and ΔCAB

$\angle DCA = \angle CAB$ and $\angle DAC = \angle BCA$

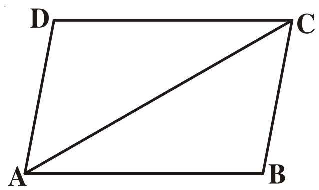
also AC = CA. (Common side)

Therefore $\Delta ABC \cong \Delta CDA$.

This means that the two traingles are congruent by A.S.A. rule (angle, side and angle). This means that diagonal AC divides the parallelogram in two congruent traingles.

Theorem-8.2 : In a parallelogram, opposite sides are equal.

Proof: We have already proved that a diagonal of a parallelogram divides it into two congruent triangles.



Thus in figure $\Delta ACD \cong \Delta CAB$

We have therefore AB = DC and $\angle CBA = \angle ADC$

also AD = BC and $\angle DAC = \angle ACB$

$\angle CAB = \angle DCA$

$\therefore \angle ACB + \angle DCA = \angle DAC + \angle CAB$

i.e. $\angle DCB = \angle DAB$

We thus have in a parallelogram

i. The opposite sides are equal.

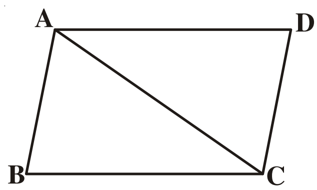
ii. The opposite angles are equal.

It can be noted that with opposite sides of a convex quadrilateral being parallel we can show the opposite sides and opposite angles are equal.

We will now try to show if we can prove the converse i.e. if the opposite sides of a quadrilateral are equal, then it is a parallelogram.

Theorem-8.3 : If each pair of opposite sides of a quadrilateral is equal, then it is a parallelogram.

Proof : Consider the quadrilateral ABCD with AB = DC and BC = AD.



Draw a diagonal AC.

Consider ΔABC and ΔCDA

We have BC = AD, AB = DC and AC = CA (Common side)

So $\Delta ABC \cong \Delta CDA$ (why?)

Therefore $\angle BCA = \angle DAC$ with AC as transversal

or AB || DC ...(1)

Since $\angle ACD = \angle CAB$ with CA as transversal

We have BC || AD ...(2)

Therefore, ABCD is a parallelogram. By (1) and (2)

You have just seen that in a parallelogram both pairs of opposite sides are equal and conversely if both pairs of opposite sides of a quadrilateral are equal, then it is a parallelogram.

Can we show the same for a quadrilateral for which the pairs of opposite angles are equal?

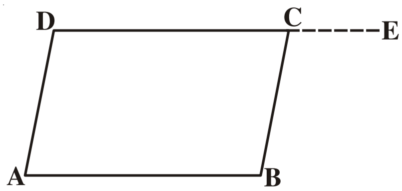
Theorem-8.4 : In a quadrilateral, if each pair of opposite angles are equal then it is a parallelogram.

Proof: In a quadrilateral ABCD, $\angle A = \angle C$ and $\angle B = \angle D$ then prove that ABCD is a parallelogram.

We know $\angle A + \angle B + \angle C + \angle D = 360°$

$$\angle A + \angle B = \angle C + \angle D = \frac{360°}{2}$$

i.e. $\angle A + \angle B = 180°$



Extend DC to E

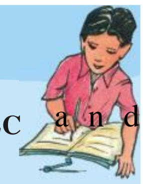
$\angle C + \angle BCE = 180°$ hence $\angle BCE = \angle ADC$

If $\angle BCE = \angle D$ then $AD \parallel BC$ (Why?)

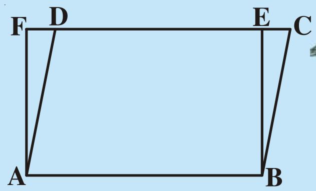
With DC as a transversal

We can similarly show $AB \parallel DC$ or ABCD is a parallelogram.

EXERCISE - 8.2



1. In the adjacent figure ABCD is a parallelogram and ABEF is a rectangle show that $\Delta AFD \cong \Delta BEC$.



2. Show that the diagonals of a rhombus divide it into four congruent triangles.
3. In a quadrilateral ABCD, the bisector of $\angle C$ and $\angle D$ intersect at O.

   Prove that $\angle COD = \frac{1}{2}(\angle A + \angle B)$

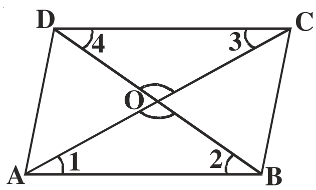
8.5 DIAGONALS OF A PARALLELOGRAM

Theorem-8.5 : The diagonals of a parallelogram bisect each other.

Proof: Draw a parallelogram ABCD.

Draw both of its diagonals AC and BD to intersect at the point 'O'.

In ΔOAB and ΔOCD



Mark the angles formed as $\angle 1, \angle 2, \angle 3, \angle 4$

$\angle 1 = \angle 3$ ($AB \parallel CD$ and AC transversal)

$\angle 2 = \angle 4$ (Why) (Interior alternate angles) and AB = CD (Property of parallelogram)

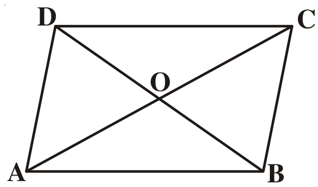
By A.S.A congruency property

$\Delta OCD \cong \Delta OAB$

CO= OA, DO = OB or diagonals bisect each other.

Hence we have to check if the converse is also true. Converse is if diagonals of a quadrilateral bisect each other then it is a parallelogram.

Theorem-8.6 : If the diagonals of a quadrilateral bisect each other then it is a parallelogram.



Proof: ABCD is a quadrilateral.

AC and BD are the diagonals intersect at 'O',

such that OA = OC and OB = OD.

Prove that ABCD is a parallelogram.

(**Hint** : Consider ΔAOB and ΔCOD. Are these congruent? If so then what can we say?)

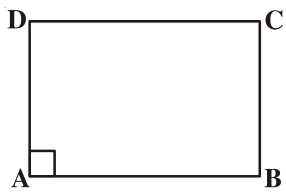
8.5.1 More geometrical statements

In the previous examples we have showed that starting from some generalisation we can find many statements that we can make about a particular figure(Parallelogram). We use previous results to deduce new statements. Note that these statements need not be verified by measurements as they have been shown as true in all cases.

Such statements that are deduced from the previously known and proved statements are called corollary. A corollary is a statement, the truth of which follows readily from an established theorem.

Corollary-1 : Show that each angle of a rectangle is a right angle.

Solution : Rectangle is a parallelogram in which one angle is a right angle.



ABCD is a rectangle. Let one angle is $\angle A = 90^o$

We have to show that $\angle B = \angle C = \angle D = 90^o$

Proof : Since ABCD is a parallelogram,

thus $AD \parallel BC$ and AB is a transversal

so $\angle A + \angle B = 180^o$ (Interior angles on the same side of a transversal)

as $\angle A = 90^o$ (given)

$\therefore \angle B = 180^o - \angle A$

$= 180^o - 90^o = 90^o$

Now $\angle C = \angle A$ and $\angle D = \angle B$ (opposite angles of parallelogram)

So $\angle C = 90^o$ and $\angle D = 90^o$.

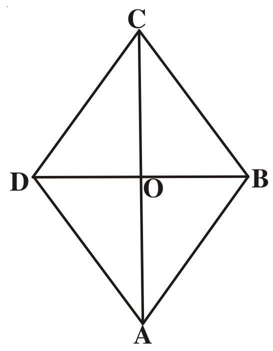
Therefore each angle of a rectangle is a right angle.

Corollary-2 : Show that the diagonals of a rhombus are perpendicular to each other.

Proof : A rhombus is a parallelogram with all sides equal.

ABCD is a rhombus, diagonals AC and BD intersect at O

We want to show that AC is perpendicular to BD



Consider ΔAOB and ΔBOC

$OA = OC$ (Diagonals of a parallelogram bisect each other)

$OB = OB$ (common side to ΔAOB and ΔBOC)

$AB = BC$ (sides of rhombus)

Therefore $\Delta AOB \cong \Delta BOC$ (S.S.S rule)

So $\angle AOB = \angle BOC$

But $\angle AOB + \angle BOC = 180^\circ$ (Linear pair)

Therefore $2\angle AOB = 180^\circ$

or $\angle AOB = \dfrac{180^\circ}{2} = 90^\circ$

Similarly $\angle BOC = \angle COD = \angle AOD = 90^\circ$ (Same angle)

Hence AC is perpendicular on BD

So, the diagonals of a rhombus are perpendicular to each other.

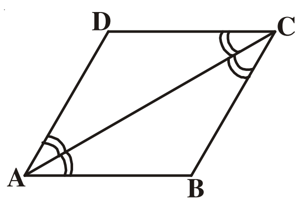
Corollary-3 : In a parallelogram ABCD, if the diagonal AC bisects the angle A, then ABCD is a rhombus.

Proof : ABCD is a parallelogram

Therefore $AB \parallel DC$. AC is the transversal intersects $\angle A$ and $\angle C$

So, $\angle BAC = \angle DCA$ (Interior alternate angles) ...(1)

$\angle BCA = \angle DAC$ ...(2)



But it is given that AC bisects $\angle A$

So $\angle BAC = \angle DAC$

$\therefore$ $\angle DCA = \angle DAC$ ...(3)

Thus AC bisects $\angle C$ also

From (1), (2) and (3), we have

$\angle BAC = \angle BCA$

In ΔABC, $\angle BAC = \angle BCA$ means that $BC = AB$ (isosceles triangle)

But $AB = DC$ and $BC = AD$ (opposite sides of the parallelogram ABCD)

$\therefore$ $AB = BC = CD = DA$

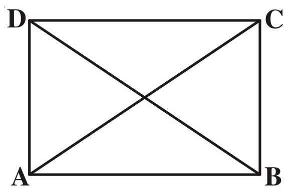
Hence, ABCD is a rhombus.

Corollary-4 : Show that the diagonals of a rectangle are of equal length.

Proof : ABCD is a rectangle and AC and BD are its diagonals

We want to know $AC = BD$

ABCD is a rectangle, means ABCD is a parallelogram with all its angles equal to right angle.



Consider the triangles ΔABC and ΔBAD,

$AB = BA$ (Common)

$\angle B = \angle A = 90^{o}$ (Each angle of rectangle)

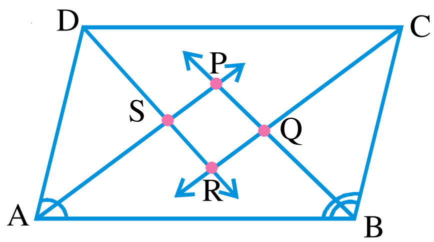
$BC = AD$ (opposite sides of the rectangle)

Therefore, $\Delta ABC \cong \Delta BAD$ (S.A.S rule)

This implies that $AC = BD$

or the diagonals of a rectangle are equal.

Corollary-5 : Show that the angle bisectors of a parallelogram form a rectangle.



Proof : ABCD is a parallelogram. The bisectors of angles $\angle A$, $\angle B$, $\angle C$ and $\angle D$ intersect at P, Q, R, S to form a quadrilateral. (See adjacent figure)

Since ABCD is a parallelogram, $AD \parallel BC$. Consider AB as transversal intersecting them then $\angle A + \angle B = 180^{o}$(Consecutive angles of Parallelogram)

We know $\angle BAP = \frac{1}{2}\angle A$ and $\angle ABP = \frac{1}{2}\angle B$ [Since $\overrightarrow{AP}$ and $\overrightarrow{BP}$ are the bisectors of $\angle A$ and $\angle B$ respectively]

$$\Rightarrow \frac{1}{2}\angle A + \frac{1}{2}\angle B = \frac{1}{2}\times 180^{o}$$

Or $\angle BAP + \angle ABP = 90^{o}$ ...(1)

But In ΔAPB,

$$\angle BAP + \angle APB + \angle ABP = 180^\circ \text{ (Angle sum property of the triangle)}$$

So $\angle APB = 180^\circ - (\angle BAP + \angle ABP)$

$\Rightarrow \angle APB = 180^\circ - 90^\circ$ (From (1))

$= 90^\circ$

We can see that $\angle SPQ = \angle APB = 90^\circ$

Similarly, we can show that $\angle CRD = \angle QRS = 90^\circ$ (Same ang

But $\angle BQC = \angle PQR$ and $\angle DSA = \angle PSR$ (Why?)

$\therefore \angle PQR = \angle QRS = \angle PSR = \angle SPQ = 90^\circ$

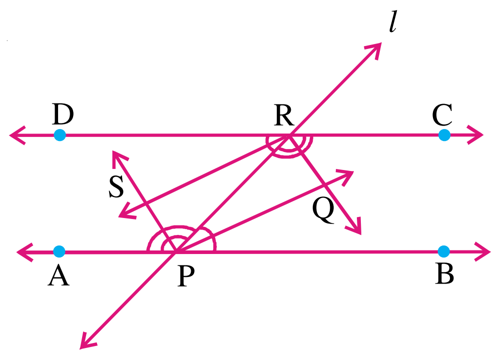
Hence PQRS has all the four angles equal to 90°.

We can therefore say PQRS is a rectangle.

THINK, DISCUSS AND WRITE

1. Show that the diagonals of a square are equal and **right bisectors** of each other.
2. Show that the diagonals of a rhombus divide it into four congruent triangles.

Some Illustrative examples



Example-5. $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$ are two parallel lines and a transversal l, intersects $\overleftrightarrow{AB}$ at P and $\overleftrightarrow{DC}$ at R. Prove that the bisectors of the interior angles form a rectangle.

Proof : $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$, l is the transversal intersecting $\overleftrightarrow{AB}$ at P and $\overleftrightarrow{DC}$ at R respectively.

Let $\overrightarrow{PQ}$, $\overrightarrow{RQ}$, $\overrightarrow{RS}$ and $\overrightarrow{PS}$ are the bisectors of $\angle RPB$, $\angle CRP$, $\angle DRP$ and $\angle APR$ respectively.

$\angle BPR = \angle DRP$ (Interior Alternate angles) ...(1)

But $\angle RPQ = \frac{1}{2}\angle BPR$ ($\because \overrightarrow{PQ}$ is the bisector of $\angle BPR$)

also $\angle PRS = \frac{1}{2}\angle DRP$ ($\because \overrightarrow{RS}$ is the bisector of $\angle DPR$). ...(2)

From (1) and (2)

$\angle RPQ = \angle PRS$

These are interior alternate angles made by $\overline{PR}$ with the lines $\overrightarrow{PQ}$ and $\overrightarrow{RS}$

$\therefore \quad \overrightarrow{PQ} \parallel \overrightarrow{RS}$

Similarly

$\angle PRQ = \angle RPS$, hence $\overrightarrow{PS} \parallel \overrightarrow{RQ}$

Therefore PQRS is a parallelogram ... (3)

We have $\angle BPR + \angle CRP = 180^o$ (interior angles on the same side of the transversal l with line $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$)

$\frac{1}{2}\angle BPR + \frac{1}{2}\angle CRP = 90^o$

$\Rightarrow \angle RPQ + \angle PRQ = 90^o$

But in ΔPQR,

$\angle RPQ + \angle PQR + \angle PRQ = 180^o$ (three angles of a triangle)

$\angle PQR = 180^o - (\angle RPQ + \angle PRQ)$

$= 180^o - 90^o = 90^o$ ... (4)

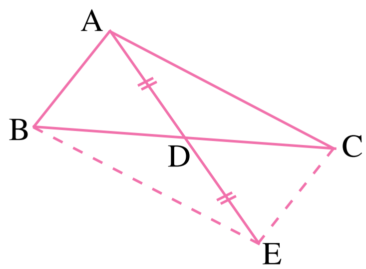
From (3) and (4)

PQRS is a parallelogram with one of its angles as a right angle.

Hence PQRS is a rectangle

Example-6. In a triangle ABC, AD is the median drawn on the side BC is produced to E such that AD = ED prove that ABEC is a parallelogram.

Proof : AD is the median of ΔABC



Produce AD to E such that AD = ED

Join BE and CE.

Now In Δ^s ABD and ECD

BD = DC (D is the midpoints of BC)

$\angle ADB = \angle EDC$ (vertically opposite angles)

AD = ED (Given)

So $\Delta ABD \cong \Delta ECD$ (SAS rule)

Therefore, AB = CE (CPCT)

also $\angle ABD = \angle ECD$

These are interior alternate angles made by the transversal $\overleftrightarrow{BC}$ with lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CE}$.

$\therefore \overleftrightarrow{AB} \parallel \overleftrightarrow{CE}$

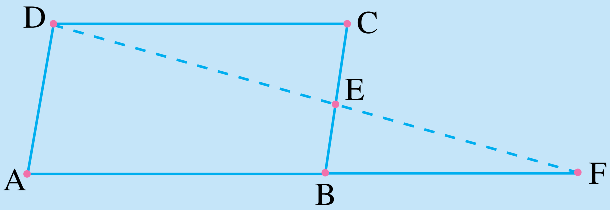
Thus, in a Quadrilateral ABEC,

$AB \parallel CE$ and $AB = CE$

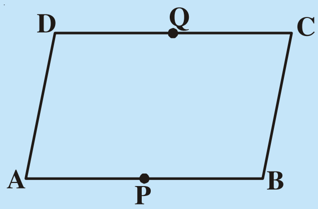
Hence ABEC is a parallelogram.

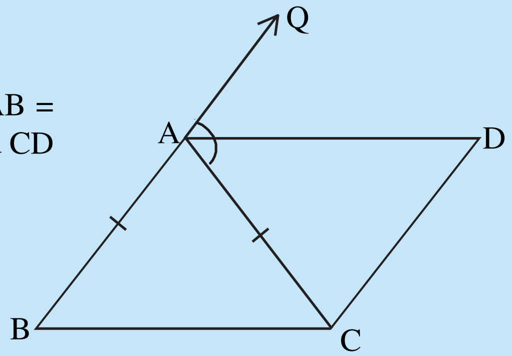
Exercise - 8.3

1. The opposite angles of a parallelogram are $(3x - 2)^o$ and $(x + 48)^o$. Find the measure of each angle of the parallelogram.
2. Find the measure of all the angles of a parallelogram, if one angle is 24^o less than the twice of the smallest angle.
3. In the adjacent figure ABCD is a parallelogram and E is the midpoint of the side BC. If DE and AB are produced to meet at F, show that $AF = 2AB$.



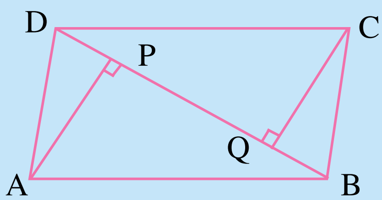
4. In the adjacent figure ABCD is a parallelogram P and Q are the midpoints of sides AB and DC respectively. Show that PBCQ is also a parallelogram.





5. ABC is an isosceles triangle in which AB = AC. AD bisects exterior angle QAC and CD $\parallel$ BA as shown in the figure. Show that
   (i) $\angle DAC = \angle BCA$
   (ii) ABCD is a parallelogram

6. ABCD is a parallelogram AP and CQ are perpendiculars drawn from vertices A and C on diagonal BD (see figure) show that
   (i) $\Delta APB \cong \Delta CQD$
   (ii) $AP = CQ$



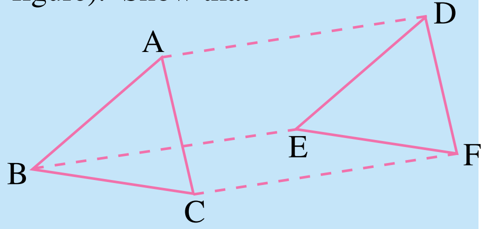
7. In Δ^s ABC and DEF, AB || DE; BC = EF and BC || EF. Vertices A, B and C are joined to vertices D, E and F respectively (see figure). Show that

(i) ABED is a parallelogram

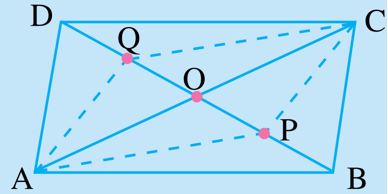
(ii) BCFE is a parallelogram

(iii) AC = DF

(iv) $\Delta ABC \cong \Delta DEF$



8. ABCD is a parallelogram. AC and BD are the diagonals intersect at O. P and Q are the points of tri section of the diagonal BD. Prove that CQ || AP and also AC bisects PQ (see figure).



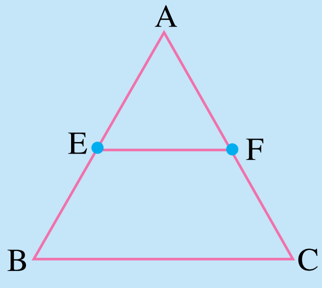
9. ABCD is a square. E, F, G and H are the mid points of AB, BC, CD and DA respectively. Such that AE = BF = CG = DH. Prove that EFGH is a square.

8.6 The Midpoint Theorem of Triangle

We have studied properties of triangle and of a quadrilateral. Let us try and consider the midpoints of the sides of a triangle and what can be derived from them.

Try This

Draw a triangle ABC and mark the midpoints E and F of two sides of triangle $\overline{AB}$ and $\overline{AC}$ respectively. Join the point E and F as shown in the figure.



Measure EF and the third side BC of the triangle. Also measure $\angle AEF$ and $\angle ABC$.

We find $\angle AEF = \angle ABC$ and $\overline{EF} = \frac{1}{2}\overline{BC}$

As these are corresponding angles made by the transversal AB with lines EF and BC, we say EF || BC.

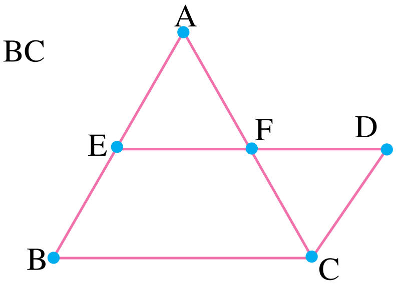
Repeat this activity with some more triangles.

So, we arrive at the following theorem.

Theorem-8.7 : The line segment joining the midpoints of two sides of a triangle is parallel to the third side and also half of it.

Given : ABC is a triangle with E and F as the midpoints of AB and AC respectively.

We have to show that : (i) $EF \parallel BC$ (ii) $EF = \frac{1}{2}BC$



Proof:- Join EF and extend it, and draw a line parallel to BA through C to meet to produced EF at D.

In Δ^s AEF and ΔCDF

AF = CF (F is the midpoint of AC)

$\angle AFE = \angle CFD$ (vertically opposite angles.)

and $\angle AEF = \angle CDF$ (Interior alternate angles as $CD \parallel BA$ with transversal ED.)

By A.S.A congruency rule

$\therefore \Delta AEF \cong \Delta CDF$ ASA congruency rule

Thus AE = CD and EF = DF (CPCT)

We know AE = BE

Therefore BE = CD

Since $BE \parallel CD$ and BE = CD, BCDE is a parallelogram.

So $ED \parallel BC$

$\Rightarrow EF \parallel BC$

As BCDE is a parallelogram, ED = BC(how ?) ($\because$ DF = EF)

we have shown FD = EF

$\therefore$ 2EF = BC

Hence $EF = \frac{1}{2}BC$

We can see that the converse of the above statement is also true. Let us state it and then see how we can prove it.

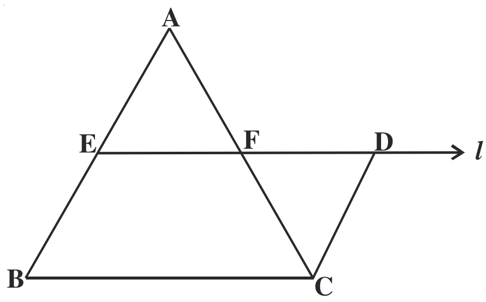
Theorem-8.8 : The line drawn through the midpoint of one of the sides of a triangle and parallel to another side will bisect the third side

Proof: Draw ΔABC. Mark E as the mid point of side AB. Draw a line l passing through E and parallel to BC. The line intersects AC at F.

Construct $CD \parallel BA$

We have to show AF = CF

Consider ΔAEF and ΔCFD



$\angle EAF = \angle DCF$ (BA $\parallel$ CD and AC is transversal) (How ?)

$\angle AEF = \angle D$ (BA $\parallel$ CD and ED is transversal) (How ?)

We can not prove the congruence of the triangles as we have not shown any pair of sides in the two triangles as equal.

To do so we consider EB $\parallel$ DC

and ED $\parallel$ BC

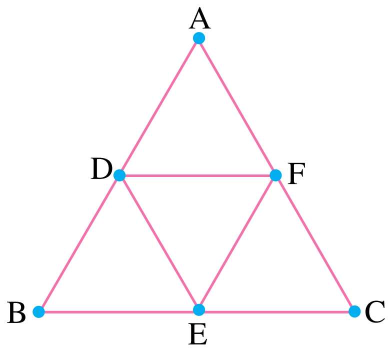
Thus EDCB is a parallelogram and we have BE = DC.

Since BE = AE we have AE = DC.

Hence $\Delta AEF \cong \Delta CFD$

$\therefore$ AF = CF

Some more examples



Example-7. In ΔABC, D, E and F are the midpoints of sides AB, BC and CA respectively. Show that ΔABC is divided into four congruent triangles, when the three midpoints are joined to each other. (ΔDEF is called medial triangle)

Proof : D, E are midpoints of $\overline{AB}$ and $\overline{BC}$ of triangle ABC respectively

so by Mid-point Theorem,

DE $\parallel$ AC

Similarly DF $\parallel$ BC and EF $\parallel$ AB.

Therefore ADEF, BEFD, CFDE are all parallelograms

In the parallelogram ADEF, DF is the diagonal

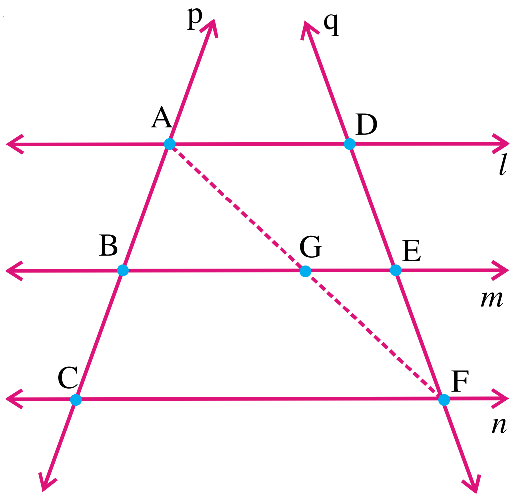
So $\Delta ADF \cong \Delta DEF$ (Diagonal divides the parallelogram into two congruent triangles)

Similarly $\Delta BDE \cong \Delta DEF$

and $\Delta CEF \cong \Delta DEF$

So, all the four triangles are congruent.

We have shown that a triangle ABC is divided in to four congruent traingles by joining the midpoints of the sides.



Example-8. l, m and n are three parallel lines intersected by the transversals p and q at A, B, C and D,E, F such that they make equal intercepts AB and BC on the transversal p. Show that the intercepts DE and EF on q are also equal.

Proof : We need to connect the equality of AB and BC to comparing DE and EF. We join A to F and call the intersection point with 'm' as G.

In ΔACF, AB = BC (given)

Therefore B is the midpoint of AC.

and BG $\|$ CF (how ?)

So G is the midpoint of AF (By the theorem).

Now in Δ AFD, we can apply the same reason as G is the midpoint of AF and GE $\|$ AD,

E is the midpoint of DF.

Thus DE = EF.

Hence l, m and n cut off equal intersects on q also.

Example-9. In the Fig. AD and BE are medians of ΔABC and BE $\|$ DF. Prove that

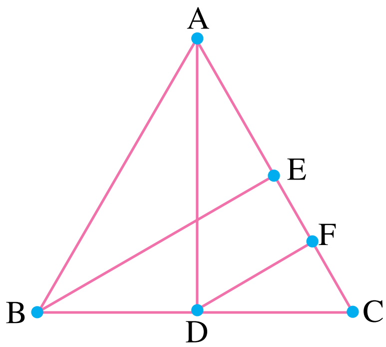
$$CF = \frac{1}{4} AC.$$

Proof : If Δ ABC, D is the midpoint of BC and BE $\|$ DF; By Theorem F is the midpoint of CE.

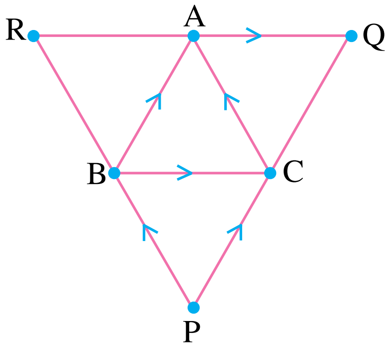
$$\therefore\ CF = \frac{1}{2}CE$$

$$= \frac{1}{2}\left(\frac{1}{2}AC\right) \quad \text{(How ?)}$$

Hence $CF = \frac{1}{4}AC$.



Example-10. ABC is a triangle and through A, B, C lines are drawn parallel to BC, CA and AB respectively intersecting at P, Q and R. Prove that the perimeter of ΔPQR is double the perimeter of ΔABC.



Proof : AB $\parallel$ QP and BC $\parallel$ RQ So ABCQ is a parallelogram.

Similarly BCAR, ABPC are parallelograms

$\therefore$ BC = AQ and BC = RA

$\Rightarrow$ A is the midpoint of QR

Similarly B and C are midpoints of PR and PQ respectively.

$\therefore AB = \frac{1}{2}PQ; \quad BC = \frac{1}{2}QR$ and $CA = \frac{1}{2}PR$ (How)

(State the related theorem)

Now perimeter of $\Delta PQR = PQ + QR + PR$

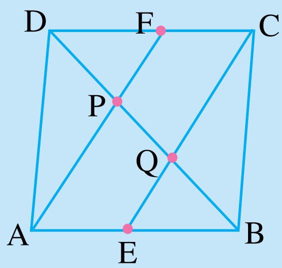
$= 2AB + 2BC + 2CA$

$= 2(AB + BC + CA)$

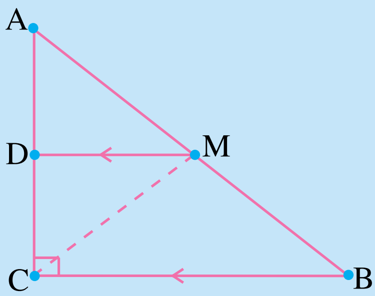
$= 2$ (perimeter of ΔABC).

EXERCISE - 8.4

1. ABC is a triangle. D is a point on AB such that $AD = \frac{1}{4}AB$ and E is a point on AC such that $AE = \frac{1}{4}AC$. If DE = 2 cm find BC.
2. ABCD is quadrilateral E, F, G and H are the midpoints of AB, BC, CD and DA respectively. Prove that EFGH is a parallelogram.
3. Show that the figure formed by joining the midpoints of sides of a rhombus successively is a rectangle.
4. In a parallelogram ABCD, E and F are the midpoints of the sides AB and DC respectively. Show that the line segments AF and EC trisect the diagonal BD.



5. Show that the line segments joining the midpoints of the opposite sides of a quadrilateral and bisect each other.
6. ABC is a triangle right angled at C. A line through the midpoint M of hypotenuse AB and Parallel to BC intersects AC at D. Show that
   (i) D is the midpoint of AC
   (ii) MD $\perp$ AC
   (iii) $CM = MA = \frac{1}{2}AB$.

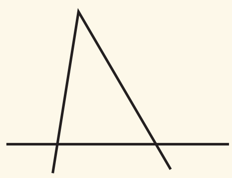


WHAT WE HAVE DISCUSSED

1. A quadrilateral is a simple closed figure formed by four line segments in a plane.
2. The sum of four angles in a quadrilateral is 360^0 or 4 right angles.
3. Trapezium, parallelogram, rhombus, rectangle, square and kite are special types of quadrilaterals
4. Parallelogram is a special type of quadrilateral with many properties. We have proved the following theorems.
   a) The diagonal of a parallelogram divides it into two congruent triangles.
   b) The opposite sides and angles of a parallelogram are equal.
   c) If each pair of opposite sides of a quadrilateral are equal then it is a parallelogram.
   d) If each pair of opposite angles are equal then it is a parallelogram.
   e) Diagonals of a parallelogram bisect each other.
   f) If the diagonals of a quadrilateral bisect each other then it is a parallelogram.
5. Mid point theorm of triangle and converse
   a) The line segment joining the midpoints of two sides of a triangle is parallel to the third side and also half of it.
   b) The line drawn through the midpoint of one of the sides of a triangle and parallel to another side will bisect the third side.

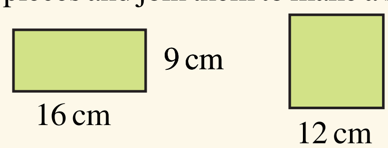
Brain teaser

1. Creating triangles puzzle



Add two straight lines to the above diagram and produce 10 triangles.

2. Take a rectangular sheet of paper whose length is 16 cm and breadth is 9 cm. Cut it in to exactly 2 pieces and join them to make a square.



09 Statistics

9.1 Introduction

One day Ashish visited his mathematics teacher at his home. At that time his teacher was busy in compiling the information which he had collected from his ward for the population census of India.

Ashish : Good evening sir, it seems you are very busy. Can I help you in your work, Sir?

Teacher : Ashish, I have collected the household information for census i.e. number of family members has, their age group, family income, type of house they live and other data.

Ashish : Sir, what is the use of this information?

Teacher : This information is useful for government in planning the developmental programmes and allocation of resources.

Ashish : How does government use this large information?

Teacher : The Census Department compiles this massive data and by using required data handling tools analyse the data and interpretes the results in the form of information. Ashish, you must have learnt basic statistics (data handling) in your earlier classes, didn't you?

Like Ashish we too come across a lot of situations where we see information in the form of facts, numerical figures, tables, graphs etc. These may relate to price of vegetables, city temperature, cricket scores, polling result and so on. These facts or figures which are numerical or otherwise collected with a definite purpose are called **'data'**. Extraction of meaning from the data is studied in a branch of mathematics called **statistics**.

Lets us first revise what we have studied in statistics (data handling) in our previous classes.

9.2 Collection of Data

The primary activity in statistics is to collect the data with some purpose. To understand this let us begin with an exercise of collecting data by performing the following activity.

Activity

Divide the students of your class into four groups. Allot each group the work of collecting one of the following kinds of data:

i. Weights of all the students in your class.

ii. Number of siblings that each student have.

iii. Day wise number of absentees in your class during last month.

iv. The distance between the school and home of every student of your class.

Let us discuss how these students have collected the required information?

1. Have they collected the information by enquiring each student directly or by visiting every house personally by the students?
2. Have they got the information from source like data available in school records?

In first case when the information was collected by the investigator (student) with a definite objective, the data obtained is called **primary data** (as in (i), (ii), (iv)).

In the above task (iii) number of absentees in the last month could only be known by school attendance register. So here we are using data which is already collected by class teachers. This is called secondary data. The information collected from a source, which had already been recorded, say from registers, is called **secondary data**.

Do This

Which of the following are primary and secondary data?

i. Collection of the data about enrollment of students in your school for a period from 2001 to 2010.

ii. Height of students in your class recorded by physical education teacher.

9.3 Presentation of Data

Once the data is collected, the investigator has to find out ways to present it in the form which is meaningful, easy to understand and shows its main features at a glance. Let us take different situation where we need to present the data.

Consider the marks obtained by 15 students in a mathematics test out of 50 marks:

25, 34, 42, 20, 39, 50, 28, 30, 50, 11, 20, 42, 45, 40, 7.

The data in this form is called raw data.

From the given data you can easily identify the minimum and maximum marks. You also remember that the difference between the minimum and maximum marks is called the **range** of given data.

Here minimum and maximum marks are 7 and 50 respectively.

So the range = 50 – 7 = 43,

From the above we can also say that our data lies from 7 to 50.

Now let us answer the following questions from the above date.

i. Find the middle value of the given data.

ii. Find how many children got 60% or more marks in the mathematics test.

Discussion

(i) Ikram said that the middle value of the data is 25 because the exam was conducted for 50 marks. What do you think?

Mary said that it is not the middle value of the data. In this case we have marks of 15 students as raw data, so after arranging the data in ascending order,

7, 11, 20, 20, 25, 28, 30, 34, 39, 40, 42, 42, 45, 50, 50

we can say that the 8th term is the middle term and it is 34.

(ii) You already know how to find 60% of 50 marks (i.e. $\frac{60}{100} \times 50 = 30$).

You find that there are 9 students who got 60% or more marks (i.e. 30 marks or more).

When the number of observations in a data are too many, presentation of the data in ascending or descending order can be quite time consuming. So we have to think of an alternative method.

See the given example.

Example-1. Consider the marks obtained by 50 students in a mathematics test for a total marks of 10.

5, 8, 6, 4, 2, 5, 4, 9, 10, 2, 1, 1, 3, 4, 5,
8, 6, 7, 10, 2, 1, 1, 3, 4,4, 5, 8, 6, 7, 10,
2,8, 6, 4, 2, 5, 4, 9, 10, 2, 1, 1, 3, 4, 5,
8, 6,4, 5, 8

Marks	Tally Marks	No of students
1	卌 |	6
2	卌 |	6
3	|||	3
4	卌 ||||	9
5	卌 ||	7
6	卌	5
7	||	2
8	卌 |	6
9	||	2
10	||||	4
	Total	50

The data is tabulated by using the tally marks, as shown in table.

Recall that the number of students who have obtained a certain number of marks is called the frequency of those marks. For example, 9 students got 4 marks each. So the frequency of 4 marks is 9.

Here in the table, tally marks are useful in tabulating the raw data.

Sum of all frequencies in the table gives the total number of observations of the data.

As the actual observations of the data are shown in the table with their frequencies, this table is called '**Ungrouped Frequency Distribution Table**' or '**Table of Weighted Observations**'.

Activity

Make frequency distribution table of the initial letters of that denotes surnames of your classmates and answer the following questions.

(i) Which initial letter occured mostly among your classmates?

(ii) How many students names start with the alphabat 'I'?

(iii) Which letter occured least as an initial among your classmates?

Suppose for specific reason, we want to represent the data in three categories (i) how many students need extra classes, (ii) how many have an average performance and (iii) how many did well in the test. Then we can make groups as per the requirement and grouped frequency table as shown below.

Class interval (marks)	Category	Tally marks	No. of students
1 - 3	(Need extra class)	卌 卌 卌	15
4 - 5	(Average)	卌 卌 卌 |	16
6 - 10	(Well)	卌 卌 卌 ||||	19

To classify the data according to the requirement or if there are large number of observations. We make groups to condense it. Let's take one more example in which group and frequency make us easy to understand the data.

Example-2. The weight (in grams) of 50 oranges, picked at random from a basket of oranges, are given below:

35, 45, 55, 50, 30, 110, 95, 40, 70, 100, 60, 80, 85, 60, 52, 95, 98, 35, 47, 45, 105, 90, 30, 50, 75, 95, 85, 80, 35, 45, 40, 50, 60, 65, 55, 45, 30, 90, 115, 65, 60, 40, 100, 55, 75, 110, 85, 95, 55, 50

To present such a large amount of data and to make sense of it, we make groups like 30-39, 40-49, 50-59, ... 100-109, 110-119. (since our data is from 30 to 115). These groups are called 'classes' or class-intervals, and their size is called length of the class or class width, which is 10 in this case. In each of these classes the least number is called the lower limit and the greatest number is called the upper limit, e.g. in 30-39, 30 is the 'lower limit' and 39 is the 'upper limit'.

(Oranges weight) Class interval	Tally marks	(Number of oranges) Frequency
30 - 39	卌 |	6
40 - 49	卌 |||	8
50 - 59	卌 ||||	9
60 - 69	卌 |	6
70 - 79	|||	3
80 - 89	卌	5
90 - 99	卌 ||	7
100-109	|||	3
110 - 119	|||	3
	Total	50

Presenting data in this form simplifies and condenses data and enables us to observe certain important features at a glance. This is called a **grouped frequency distribution table**.

We observe that the classes in the table above are non-overlapping i.e. 30-39, 40-49 ... no number is repeating in two class intervals. Such classes are called inclusive classes. Note that we could have made more classes of shorter size, or lesser classes of larger size also. Usually if the raw data is given the range is found (Range = Maximum value − Minimum value). Based on the value of ranges with convenient, class interval length, number of classes are formed. For instance, the intervals could have been 30-35, 36-40 and so on.

Now think if weight of an orange is 39.5 gm. then in which interval will we include it? We cannot include 39.5 either in 30-39 or in 40-49.

In such cases we construct real limits (or boundaries) for every class.

Classes	Class boundaries
20 - 29	19.5 - 29.5
30 - 39	29.5 - 39.5
40 - 49	39.5 - 49.5
50 - 59	49.5 - 59.5
60 - 69	59.5 - 69.5
70 - 79	69.5 - 79.5
80 - 89	79.5 - 89.5
90 - 99	89.5 - 99.5
100 - 109	99.5-109.5
110 - 119	109.5-119.5
120 - 129	119.5-129.5

Average of upper limit of a class interval and lower limit of the next class interval becomes the upper boundary of the class. The same becomes the lower boundary of the next class interval.

Similarly boundaries of all class intervals are calculated. By assuming a class interval before the first class and next class interval after the last class, we calculate the lower boundary any of the first and upper boundary any of the last class intervals.

Again a problem arises that whether 39.5 has to be included in the class interval 29.5-39.5 or 39.5 - 49.5? Here by convention, if any observation is found to be equivalent to upper boundary of a particular class, then that particular observation is considered under next class, but not that of the particular class.

So 39.5 belongs to 39.5 - 49.5 class interval.

The classes which are in the form of 30-40, 40-50, 50-60, ... are called over lapping classes and called as exclusive classes.

If we observe the boundaries of inclusive classes, they are in the form of exclusive classes. The difference between upper boundary and lower boundary of particular class given the length of class interval. Length class interval of 90 – 99 is (i.e. 99.5 – 89.5 = 10) 10.

Example-3. The relative humidity (in %) of a certain city for a September month of 30 days was as follows:

98.1	98.6	99.2	90.3	86.5	95.3	92.9	96.3	94.2	95.1
89.2	92.3	97.1	93.5	92.7	95.1	97.2	93.3	95.2	97.3
96.0	92.1	84.9	90.0	95.7	98.3	97.3	96.1	92.1	89

(i) Construct a grouped frequency distirubtion table with classes 84-86, 86,-88 etc.

(ii) What is the range of the data?

Solution : (i) The grouped frequency distribution table is as follows-

Relative humidity	Tally marks	Number of days
84-86	$\vert$	1
86-88	$\vert$	1
88-90	$\vert\vert$	2
90-92	$\vert\vert$	2
92-94	$\cancel{\vert\vert\vert\vert}\ \vert\vert$	7
94-96	$\cancel{\vert\vert\vert\vert}\ \vert$	6
96-98	$\cancel{\vert\vert\vert\vert}\ \vert\vert$	7
98-100	$\vert\vert\vert\vert$	4

[Note:- 90 comes in interval 90-92 likewise 96 comes in 96-98 class interval]

(ii) Range 99.2 - 84.9 = 14.3 (vary for different places).

Exercise - 9.1

1. Write the mark wise frequencies in the following frequency distribution table.

Marks	Up to5	Up to6	Up to7	Up to8	Up to9	Up to10
No of students	5	11	19	31	40	45

2. The blood groups of 36 students of IX class are recorded as follows.

A	O	A	O	A	B	O	A	B	A	B	O
B	O	B	O	O	A	B	O	B	AB	O	A
O	O	O	A	AB	O	A	B	O	A	O	B

Represent the data in the form of a frequency distribution table. Which is the most common and which is the rarest blood group among these students?

3. Three coins were tossed 30 times simultaneously. Each time the number of heads occurring was noted down as follows;

1	2	3	2	3	1	1	1	0	3	2	1
2	2	1	1	2	3	2	0	3	0	1	2
3	2	2	3	1	1						

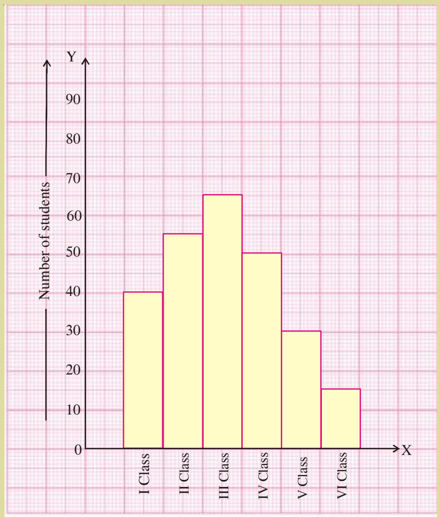
Prepare a frequency distribution table for the data given above.

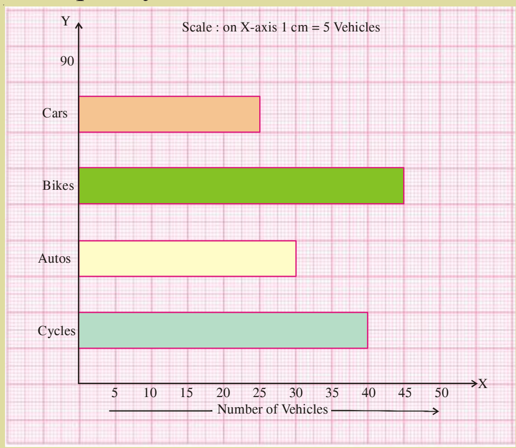
4. A TV channel organized a SMS(Short Message Service) poll on prohibition on smoking, giving options like A – complete prohibition, B – prohibition in public places only, C – not necessary. SMS results in one hour were A B A B C B

A	B	B	A	C	C	B	B	A	B
B	A	B	C	B	A	B	C	B	A
B	B	A	B	B	C	B	A	B	A
B	C	B	B	A	B	C	B	B	A
B	B	A	B	B	A	B	C	B	A
B	B	A	B	C	A	B	B	A	

Represent the above data as grouped frequency distribution table. How many appropriate answers were received? What was the majority of peoples' opinion?

5. Represent the data in the adjacent bar graph as frequency distribution table.





6. Identify the scale used on the axes of the adjacent graph. Write the frequency distribution from it.

7. The marks of 30 students of a class, obtained in a test (out of 75), are given below:

   42, 21, 50, 37, 42, 37, 38, 42, 49, 52, 38, 53, 57, 47, 29

   59, 61, 33, 17, 17, 39, 44, 42, 39, 14, 7, 27, 19, 54, 51.

   Form a frequency table with equal class intervals. (**Hint :** one of them being 0-10)

8. The electricity bills (in rupees) of 25 houses in a locality are given below. Construct a grouped frequency distribution table with a class size of 75.

   170, 212, 252, 225, 310, 712, 412, 425, 322, 325, 192, 198, 230, 320, 412,

   530, 602, 724, 370, 402, 317, 403, 405, 372, 413

9. A company manufactures car batteries of a particular type. The life (in years) of 40 batteries were recorded as follows:

2.6	3.0	3.7	3.2	2.2	4.1	3.5	4.5
3.5	2.3	3.2	3.4	3.8	3.2	4.6	3.7
2.5	4.4	3.4	3.3	2.9	3.0	4.3	2.8
3.5	3.2	3.9	3.2	3.2	3.1	3.7	3.4
4.6	3.8	3.2	2.6	3.5	4.2	2.9	3.6

   Construct a grouped frequency distribution table with exclusive classes for this data, using class intervals of size 0.5 starting from the interval 2 - 2.5.

9.4 Measures of central tendency

Consider the following situations:

Case-1 : In a hostel 50 students usually eat 200 idlies in their breakfast. How many more idlies does the mess incharge make if 20 more students joined in the hostel.

Case-2 : Consider the wages of staff at a factory as given in the table. Which salary figure represents the whole staff:

Staff	1	2	3	4	5	6	7	8	9	10
Salary in ₹ (in thousands)	12	14	15	15	15	16	17	18	90	95

Case-3 : The different forms of transport in a city are given below. Which is the popular means of transport?

1. Car 15%
2. Train 12%
3. Bus 60%
4. Two wheeler 13%

In the first case, we will usually take an average (mean), and use it to resolve the problem. But if we take average salary in the second case then it would be 30.7 thousands. However, verifying the raw data suggests that this mean value may not be the best way to accurately reflect the typical salary of a worker, as most workers have their salaries between 12 to 18 thousands. So, median (middle value) would be a better measure in this situation. In the third case mode (most frequent) is considered to be a most appropriate option. The nature of the data and its purpose will be the criteria to go for average or median or mode among the measures of central tendency.

Think, Discuss and Write

1. Give 3 situations, where mean, median and mode are separately appropriate and counted.

   Considier a situation where fans of two cricketers Raghu and Gautam claim that their star score better than other. They made comparison on the basis of last 5 matches.

Matches		1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}
Runs	Raghu	50	50	76	31	100
made	Gautam	65	23	100	100	10

Fans of both the players added the runs and calcuated the averages as follows.

Raghu's average score $= \dfrac{307}{5} = 61.4$

Gautam average score $= \dfrac{298}{5} = 59.6$

Since Raghu's average score was more than Gautam's, Raghu fan's claimed that Raghu performed better than Gautam, but Gautam fans did not agree. Gautam fan's arranged both their scores in descending order and found the middle score as given below:

Raghu	100	76	50	50	31
Gautam	100	100	65	23	10

Then Gautam fan's said that since his middle-most score is 65, which is higher than Raghu middle-most score, i.e. 50 so his performance should be rated better.'

But we may say that Gautam made two centuries in 5 matches and so he may be better.

Now, to settle the dispute between Raghu's and Gautam's fans, let us see the three measures adopted here to make their point.

The average score they used first is the mean. The 'middle' score they used in the argument is the Median. Mode is also a measure to compare the performance by considering the scores repeated many times. Mode score of Raghu is 50. Mode score of Gautam is 100. Of all these three measures which one is appropriate in this context?

Now let us first understand mean in details.

9.4.1 Arithmetic Mean

Mean is the 'sum of observations of a data divided by the number of observations'. We have already discussed about computing arithmetic mean for a raw data.

$$\text{Mean } \bar{x} = \frac{\text{Sum of observations}}{\text{Number of observations}} \text{ or } \bar{x} = \frac{\Sigma x_i}{n}$$

9.4.1.1 Mean of Raw Data

Example-4. Rain fall of a place in a week is 4cm, 5cm, 12cm, 3cm, 6cm, 8cm, 0.5cm. Find the average rainfall per day.

Solution : The average rainfall per day is the arithmetic mean of the above observations.

Given rainfall through a week are 4cm, 5cm, 12cm, 3cm, 6cm, 8cm, 0.5cm.

Number of observations (n) = 7

$$\text{Mean } \bar{x} = \frac{\sum x_i}{n} = \frac{x_1 + x_2 + x_3 + ...x_n}{n}$$ Where $x_1, x_2 \,.....\, x_n$ are n observation

and $\bar{x}$ is their mean $= \dfrac{4+5+12+3+6+8+0.5}{7} = \dfrac{38.5}{7} = 5.5$ cm.

Example-5. If the mean of 10, 12, 18, 13, P and 17 is 15, find the value of P.

Solution : We know that Mean $\bar{x} = \dfrac{\sum x_i}{n}$

$$15 = \frac{10+12+18+13+P+17}{6}$$

$90 = 70 + P$

$P = 20.$

9.4.1.2 Mean of Ungrouped frequency distribution

Consider this example; Weights of 40 students in a class are given in the following frequency distribution table.

Weights in kg (x)	30	32	33	35	37	41
No of students (f)	5	9	15	6	3	2

Find the average (mean) weight of 40 students.

From the table we can see that 5 students weigh 30 kg., each. So sum of their weights is $5 \times 30 = 150$ kg. Similarly we can find out the sum of weights with each frequency and then their total. Sum of the frequencies gives the number of observations in the data.

$$\text{Mean } (\bar{x}) = \frac{\text{Sum of all the observations}}{\text{Total number of observations}}$$

$$\text{So Mean} = \frac{5\times30+9\times32+15\times33+6\times35+3\times37+2\times41}{5+9+15+6+3+2} = \frac{1336}{40} = 33.40 \text{ kg.}$$

If observations are $x_1, x_2, x_3, x_4, x_5, x_6$ and corresponding frequencies are $f_1, f_2, f_3, f_4, f_5, f_6$ then we may write the above expression as

Mean $\bar{x} = \dfrac{f_1x_1 + f_2x_2 + f_3x_3 + f_4x_4 + f_5x_5 + f_6x_6}{f_1 + f_2 + f_3 + f_4 + f_5 + f_6} = \dfrac{\sum f_i x_i}{\sum f_i}$

$\therefore \bar{x} = \dfrac{\sum f_i x_i}{\sum f_i}$

Example-6. Find the mean of the following data.

x	5	10	15	20	25
f	3	10	25	7	5

Solution :

Step-1 : Calculate $f_i \times x_i$ of each row

Step-2 : Find the sum of frequencies (Σf_i)

and sum of the $f_i \times x_i$ ($\Sigma f_i x_i$)

Step-3 : Calculate $\bar{x} = \dfrac{\Sigma f_i x_i}{\Sigma f_i} = \dfrac{755}{50} = 15.1$

x_i	f_i	$f_i x_i$
5	3	15
10	10	100
15	25	375
20	7	140
25	5	125
	$\Sigma f_i = 50$	$\Sigma f_i x_i = 755$

Example-7. If the mean of the following data is 7.5 , then find the value of 'A'.

Marks	5	6	7	8	9	10
No. of Students	3	10	17	A	8	4

Solution :

Sum of frequencies $(\Sigma f_i) = 42 + \text{A}$

Sum of the $f_i \times x_i$ $(\Sigma f_i x_i) = 306 + 8\text{A}$

Mean $\bar{x} = \dfrac{\Sigma f_i xi}{\Sigma f_i}$

Given Arithmetic Mean $= 7.5$

So, $7.5 = \dfrac{306 + 8\text{A}}{42 + \text{A}}$

$306 + 8\text{A} = 315 + 7.5\text{A}$

Marks (x_i)	No. of Students (f_i)	$f_i x_i$
5	3	15
6	10	60
7	17	119
8	A	8A
9	8	72
10	4	40
	42+A	306+8A

$8A - 7.5\,A = 315 - 306$

$0.5\,A = 9$

$A = 18$

9.4.1.3 Mean of ungrouped frequency Distribution by Deviation method

Example-8. Find the arithmetic mean of the following data:

x	10	12	14	16	18	20	22
f	4	5	8	10	7	4	2

Solution :

(i) Simple Method

Thus in the case of ungrouped frequency distribution, you can use the formula,

$$\bar{x} = \frac{\sum_{i=1}^{7} f_i x_i}{\sum_{i=1}^{7} f_i} = \frac{622}{40} = 15.55$$

x_i	f_i	$f_i x_i$
10	4	40
12	5	60
14	8	112
16	10	160
18	7	126
20	4	80
22	2	44
	$\sum_{i=1}^{7} f_i = 40$	$\sum_{i=1}^{7} f_i x_i = 622$

(ii) Deviation Method

In this method we assume one of the observations which is convenient as assumed mean. Suppose we assume '16' as a mean, be A = 16 the deviation of other observations from the assumed mean are given in table.

x_i	f_i	$d_i = x_i - A$	$f_i d_i$
10	4	−6	−24
12	5	−4	−20
14	8	−2	−16
16 A	10	0	0
18	7	+2	+14
20	4	+4	+16
22	2	+6	+12
	40		−60+42=−18

Sum of frequencies $= 40$

Sum of the $f_i \times d_i$ products $= -60 + 42$

$\Sigma f_i d_i = -18$

Mean $\bar{x} = A + \frac{\Sigma f_i d_i}{\Sigma f_i} = 16 + \frac{-18}{40}$

$= 16 - 0.45$

$= 15.55$

9.4.2 Median

Median is the middle observation of a given raw data, when it is arranged in an order (ascending / descending). It divides the data into two groups of equal number, one part comprising all values greater than median and the other part comprising values less than median.

We have already discussed in the earlier classes that median of a raw data with observations, arranged in order is calculated as follows.

When the data as 'n' number of observations and if 'n' is odd, median is $\left(\frac{n+1}{2}\right)^{th}$ observation.

When n is even, median is the average of $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ observations

TRY THESE

1. Find the median of the scores 75, 21, 56, 36, 81, 05, 42
2. Median of a data, arranged in ascending order 7, 10, 15, x, y, 27, 30 is 17 and when one more observation 50 is added to the data, the median has become 18 Find x and y.

9.4.2.1 Median of a frequency distribution

Let us now discuss the method of finding the median for a data of weighted observations consider the monthly wages of 100 employees of a company.

Wages (in ₹)	7500	8000	8500	9000	9500	10000	11000
No. of employees	4	18	30	20	15	8	5

How to find the median of the given data? First arrange the observations given either in ascending or descending order. Then write the corresponding frequencies in the table and calculate less than cumulative frequencies. The cumulative frequency upto a particular observation is the progressive sum of frequencies upto that particular observation.

Wages (x)	No.of employees (f)	Cumulative frequency (cf)
7500	4	4
8000	18	22
8500	30	52
9000	20	72
9500	15	87
10000	8	95
11000	5	100
	100	

Find $\frac{N}{2}$ and identify the median class, whose cumulative frequencies just exceeds $\frac{N}{2}$, where N is sum of the frequencies.

Here N= 100 even so find $\left(\frac{N}{2}\right)^{th}$ and $\left(\frac{N}{2}+1\right)^{th}$ observations which are 50 and 51 respectively.

From the table corresponding values of 50^{th} and 51^{st} observations is the same, falls in the wages of 8500. So the median class of this distribution is 8500.

TRY THESE

1. Find the median marks in the data.

Marks	15	20	10	25	5
No of students	10	8	6	4	1

2. In finding the median, the given data must be written in order. Why?

9.4.3 Mode

Mode is the value of the observation which occurs most frequently, i.e., an observation with the maximum frequency is called mode.

Example-9. The following numbers are the sizes of shoes sold by a shop in a particular day. Find the mode.

6, 7, 8, 9, 10, 6, 7, 10, 7, 6, 7, 9, 7, 6.

Solution : First we have to arrange the observations in order 6, 6, 6, 6, 7, 7, 7, 7, 7, 8, 9, 9, 10, 10 to make frequency distribution table

Size	6	7	8	9	10
No. of shoes sold	4	5	1	2	1

Here No. 7 occurred most frequently. i.e, 5 times.

$\therefore$ Mode (size of the shoes) of the given data is 7. This indicates the shoes of size No. '7' is a fast selling item.

THINK AND DISCUSS

1. Classify your class mates according to their heights and find the mode of it.
2. If shopkeeper has to place a order for shoes, which number shoes should he order more?

Example-10. Test scores out of 100 for a class of 20 students are as follows:

93, 84, 97, 98, 100, 78, 86, 100, 85, 92, 55, 91, 90, 75, 94, 83, 60, 81, 95

(a) Make a frequency table taking class interval as 91-100, 81-90, ...

(b) Find the modal class. (The "Modal class" is the class containing the greatest frequency).

(c) find the interval that contains the median.

Solution :

(a)

Test Scores	Frequency	Greater than Cumulative frequency
91-100	9	20
81-90	6	11
71-80	3	5
61-70	0	2
51-60	2	2
Total	**20**	

(b) 91-100 is the modal class. This class has the maximum frequency.

(c) The middle of 20 is 10. If I count from the top, 10 will fall in the class interval 81-90. If I count from the bottom and go up, 10 will fall in the class interval 81-90. The class interval that contains the median is 81-90.

9.5 Deviation in values of central tendency

What will happen to the measures of central tendency if we add the same amount to all data values, or multiply each data value by the same amount.

Let us observe the following table

Particular	Data	Mean	Mode	Median
Original Data Set	6, 7, 8, 10, 12, 14, 14, 15, 16, 20	12.2	14	13
Add 3 to each data value	9, 10, 11, 13, 15, 17, 17, 18, 19, 23	15.2	17	16
Multiply 2 times each data value	12, 14, 16, 20, 24, 28, 28, 30, 32, 40	24.4	28	26

After observing the table, we see that

When added : Since all values are shifted by the same amount, the measures of central tendency are all shifted by the same amount. If 3 is added to each data value, the mean, mode and median will also increase by 3.

When multiplied : Since all values are affected by the same multiplicative values, the measures of central tendency will also be affected similarly. If each observation is multiplied by 2, the mean, mode and median will also be multiplied by 2.

EXERCISE - 9.2

1. Weights of parcels in a transport office are given below.

Weight (kg)	50	65	75	90	110	120
No of parcels	25	34	38	40	47	16

Find the mean weight of the parcels.

2. Number of families in a village in correspondence with the number of children are given below:

No of children	0	1	2	3	4	5
No of families	11	25	32	10	5	1

Find the mean number of children per family.

3. If the mean of the following frequency distribution is 7.2 find value of 'K'.

x	2	4	6	8	10	12
f	4	7	10	16	K	3

4. Number of villages with respect to their population as per India census 2011 are given below.

Population (in thousands)	12	5	30	20	15	8
Villages	20	15	32	35	36	7

Find the average population in each village.

5. AFLATOUN social and financial educational program intiated savings program among the high school children in Hyderabad district. Mandal wise savings in a month are given in the following table.

Mandal	No. of schools	Total amount saved (in rupees)
Amberpet	6	2154
Thirumalgiri	6	2478
Saidabad	5	975
Khairathabad	4	912
Secundrabad	3	600
Bahadurpura	9	7533

Find arithmetic mean of school wise savings in each mandal. Also find the arithmetic mean of saving of all schools.

6. The heights of boys and girls of IX class of a school are given below.

Height (cm)	135	140	147	152	155	160
Boys	2	5	12	10	7	1
Girls	1	2	10	5	6	5

Compare the heights of the boys and girls

[**Hint** : Find median heights of boys and girls]

7. Centuries scored and number of cricketers in the world are given below.

No. of centuries	5	10	15	20	25
No. of cricketers	56	23	39	13	8

Find the mean, median and mode of the given data.

8. On the occasion of New year's day a sweet stall prepared sweet packets. Number of sweet packets and cost of each packet are given as follows.

Cost of packet (in ₹)	₹25	₹50	₹75	₹100	₹125	₹ 150
No of packets	20	36	32	29	22	11

Find the mean, median and mode of the data.

9. The mean (average) weight of three students is 40 kg. One of the students Ranga weighs 46 kg. The other two students, Rahim and Reshma have the same weight. Find Rahims weight.

10. The donations given to an orphanage home by the students of different classes of a secondary school are given below.

Class	Donation by each student (in ₹)	No. of students donated
VI	5	15
VII	7	15
VIII	10	20
IX	15	16
X	20	14

Find the mean, median and mode of the data.

11. There are four unknown numbers. The mean of the first two numbers is 4 and the mean of the first three is 9. The mean of all four number is 15, if one of the four number is 2 find the other numbers.

WHAT WE HAVE DISCUSSED

- Representation of the data with actual observations with frequencies in a table is called '**Ungrouped Frequency Distribution Table**' or '**Table of Weighted Observations**'
- Representation of a large data in the form of a frequency distribution table enables us to view the data at a glance, to find the range easily, to find which observation is repeating for how many times, to analyse and to interpret the data easily.
- A measure of central tendency is a typical value of the data around which other observations congregate.
- Types of measure of central tendency : Mean, Mode, Median.
- Mean is the sum observation of a data divided by the number of observations.

$$\text{Mean} = \frac{\text{Sum of observations}}{\text{Number of observations}} \quad \text{or} \quad \bar{x} = \frac{\Sigma x_i}{n}$$

- For a ungrouped frequency distribution arithmetic mean $\bar{x} = \dfrac{\Sigma f_i x_i}{\Sigma f_i}$.

- By deviation method, arithmetic mean $= A + \frac{\Sigma f_i d_i}{\Sigma f_i}$ where A is assumed mean and where Σf_i is the sum of frequencies and $\Sigma f_i d_i$ is the sum of product of frequency and deviations.
- Median is the middle observation of a data, when arranged in order (ascending / descending).
- When number of observations 'n' is odd, median is $\left(\frac{n+1}{2}\right)^{th}$ observation.
- When number of observations 'n' is even, median is the average of $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ observations
- Median divides the data into two groups of equal number, one part comprising all values greater and the other comprising all values less than median.
- Mode is the value of the observation which occurs most frequently, i.e., an observation with the maximum frequency is called mode.

Brain teaser

In a row of students, Gopi is the 7th boy from left and Shankar is the 5th boy from the right. If they exchange their seats, Gopi is the 8th boy from the right. How many students are there in the row?

A boy chaitanya carved his name on the bark of a tree at a height of 1.5 m tall. The tree attains a height of a 6.75 m from the ground at what height from the ground will chaitanya's name can be located now?

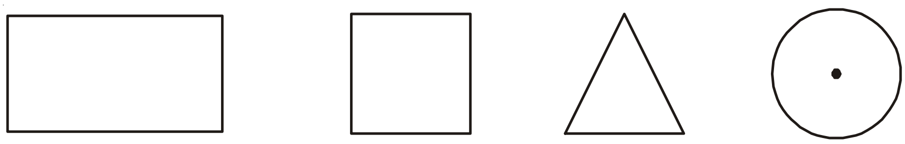
Give reason to your answer.

10 Surface Areas and Volumes

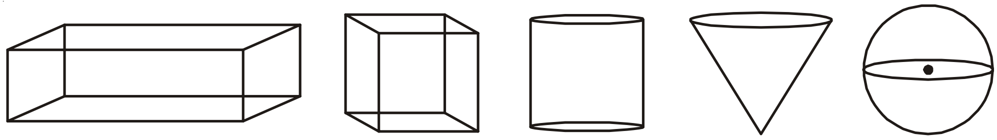
10.1 Introduction

Observe the following figures

(a)



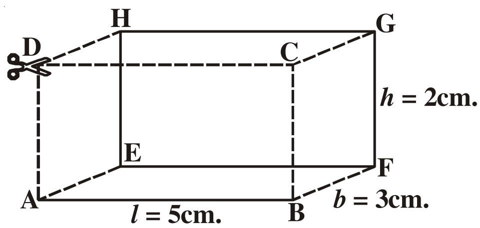
(b)



Have you noticed any differences between the figures of group (a) and (b)?

From the above, figures of group (a) can be drawn easily on our note books. These figures have length and breadth only and are named as two dimensional figures or 2-D objects. In group (b) the figures, which have length, breadth and height are called as three dimensional figures or 3-D objects. These are called solid figures. Usually we see solid figures in our surroundings. You have learned about plane figures and their areas. We shall now learn to find the surface areas and volumes of 3-dimensional objects such as cylinders, cones and spheres.

10.2 Surface area of Cuboid

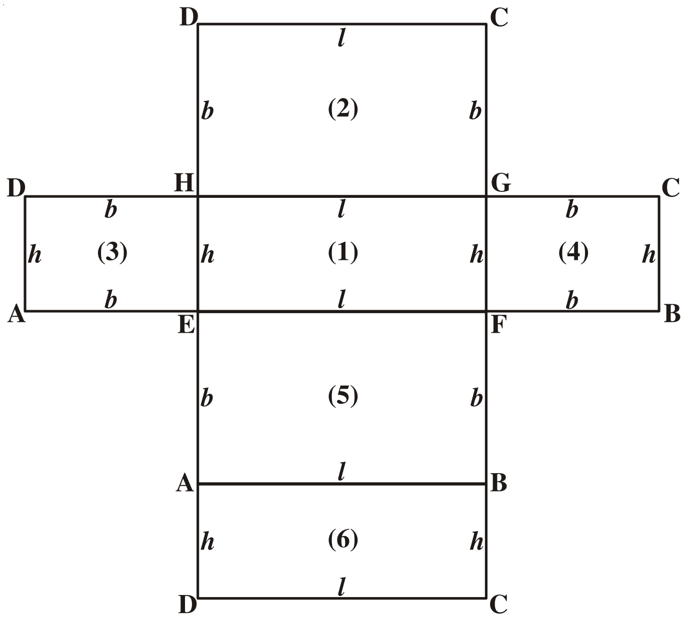


Observe the cuboid and find how many faces it has? How many corners and how many edges it has? Which pair of faces are equal in size? Do you get any idea to find the surface area of the cuboid?

Now let us find the surface area of a cuboid.

In the above figure length (l) = 5 cm; breadth (b) = 3 cm; height (h) = 2 cm

If we cut and open the given cuboid along CD, ADHE and BCGF. The figure we obtained is shown below:



This shows that the surface area of a cuboid is made up of six rectangles of three identical pairs of rectangles. To get the total surface area of cuboid, we have to add the areas of all six rectangular faces. The sum of these areas gives the total surface area of a cuboid.

Area of the rectangle EFGH = $l \times h = lh$ .....(1)

Area of the rectangle HGCD = $l \times b = lb$ .....(2)

Area of the rectangle AEHD = $b \times h = bh$ .....(3)

Area of the rectangle FBCG = $b \times h = bh$ .....(4)

Area of the rectangle ABFE = $l \times b = lb$ .....(5)

Area of the rectangle DCBA = $l \times h = lh$ .....(6)

On adding the above areas, we get the surface area of cuboid.

Surface Area of a cuboid = Areas of (1) + (2) + (3) + (4) + (5) + (6)

$= lh + lb + bh + bh + lb + lh$

$= 2\,lb + 2lh + 2bh$

$= 2(lb + bh + lh)$

(1), (3), (4), (6) are lateral surfaces of the cuboid

Lateral Surface Area of a cuboid = Area of (1) + (3) + (4) + (6)

$= lh + bh + bh + lh$

$= 2lh + 2bh$

$= 2h\,(l + b)$

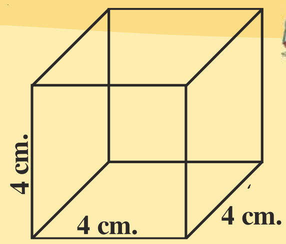
Now let us find the surface areas of cuboid for the above figure. Thus total surface area is 62 cm.^{2} and lateral surface area is 32 cm.^{2}.

TRY THIS

Take a cube of edge 'l' cm. and cut it as we did in the previous activity and find total surface area and lateral surface area of cube.

DO THIS

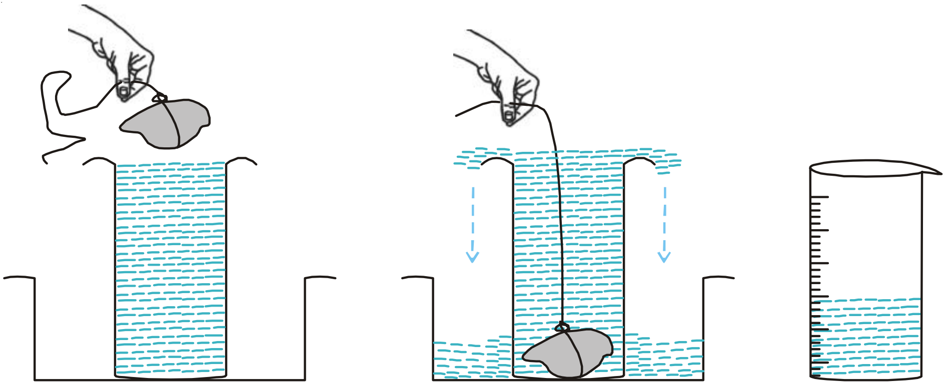
1. Find the total Surface area and lateral surface area of the Cube with side 4 cm. (By using the formulae deduced above)
2. Each edge of a cube is increased by 50%. Find the percentage increase in the surface area.



10.2.1 Volume

To recall the concept of volume, Let us do the following activity.

Take a glass jar, place it in a container. Fill the glass jar with water up to the its brim. Slowly drop a solid object (a stone) in it. Some of the water from the jar will overflow into the container. Take the overflowed water into measuring jar. It gives an idea of space occupied by a solid object called volume.

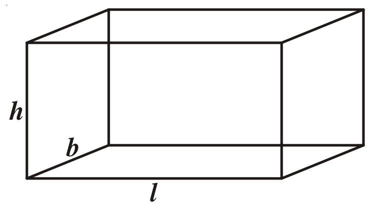


Every object occupies some space, the space occupied by an object is called its volume. Volume is measured in cubic units.

10.2.2 Capacity of the container

If the object is hollow, then interior is empty and it can be filled with air or any other liquid, that will take the shape of its container. Volume of the substance that can fill the interior is called the capacity of the container.

Volume of a Cuboid : Cut some rectangles from a cardboard of same dimensions and arrange them one over other. What do you say about the shape so formed?



The shape is a cuboid.

Now let us find volume of a cuboid.

Its length is equal to the length of the rectangle, and breadth is equal to the breadth of the rectangle.

The height up to which the rectangles are stacked is the height of the cuboid is 'h'

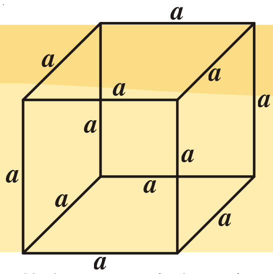
Space occupied by the cuboid = Area of plane region occupied by rectangle × height

Volume of the cuboid = $l\,b \times h = l\,bh$

$\therefore$ Volume of the cuboid = $l\,bh$

Where l, b, h are length, breadth and height of the cuboid.

TRY THESE



(a) Find the volume of a cube whose edge is 'a' units.

(b) Find the edge of a cube whose volume is 1000 cm^3.

We know that cuboid and cube are the solids. Do we call them as right prisms? You have observed that cuboid and cube are also called right prisms as their lateral faces are rectangle and perpendicular to base.

We know that the volume of a cuboid is the product of the area of its base and height.

Remember that volume of the cuboid = Area of base × height

$= lb \times h$

$= lbh$

In cube $= l = b = h = \mathrm{s}$ (All the dimensions are same)

volume of the cube $= \mathrm{s}^2 \times \mathrm{s}$

$= \mathrm{s}^3$

Hence volume of a cuboid should hold good for all right prisms.

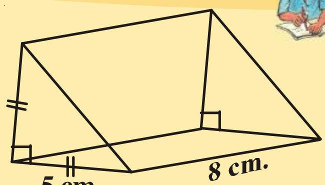
Hence volume of right prism = Area of the base × height

In particular, if the base of a right prism is an equilateral triangle its volume is $\frac{\sqrt{3}}{4}\,a^2 \times \mathrm{h}$ cu.units.

Where, 'a' is the length of each side of the base and 'h' is the height of the prim.

DO THESE

1. Find the volume of cuboid if $l = 12$ cm., $b = 10$ cm. and $h = 8$ cm.
2. Find the volume of cube, if its edge is 10 cm.
3. Find the volume of isosceles right angled triangular prism in (fig. 1).



(Fig 1)

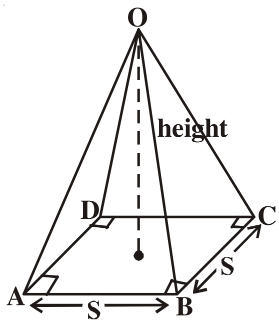
Like the prism, the pyramid is also a three dimentional solid figure. This figure has fascinated human beings from the ancient times. You might have read about pyramids of Egypt, which are, one of the seven wonders of the world. They are the remarkable examples of pyramids on square bases. How are they built? It is a mystery. No one really knows that how these massive structures were built.

Can you draw the shape of a pyramid?

What is the difference you have observed between the prism and pyramid?

What do we call a pyramid of square base?

Here OABCD is a square pyramid of side 'S' units and height 'h' units.

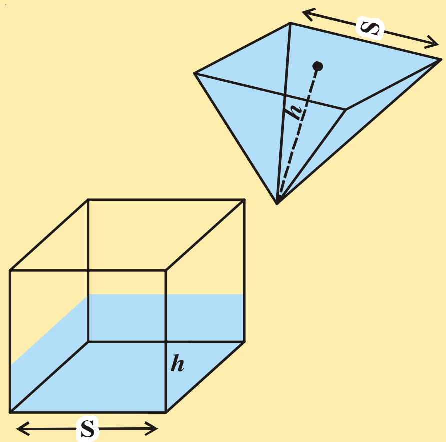


Can you guess the volume of a square pyramid in terms of volume of cube if their bases and height are equal?

ACTIVITY

Take the square pyramid and cube containers of same base and with equal heights.

Fill the pyramid with a liquid and pour into the cube (prism) completely. How many times it takes to fill the cube? From this, what inference can you make?



Thus volume of pyramid

$= \frac{1}{3}$ of the volume of right prism.

$= \frac{1}{3} \times$ Area of the base $\times$ height.

Note : A Right prism has bases perpendicular to the lateral edges and all lateral faces are rectangles.

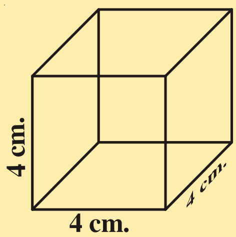
Do These

1. Find the volume of a pyramid whose square base is 10 cm. and height 8 cm.
2. The volume of cube is 1728 cubic cm. Find the volume of square pyramid of the same height.

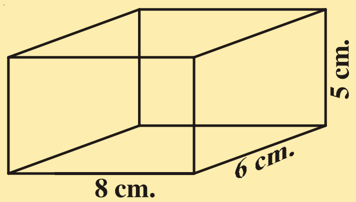
Exercise - 10.1

1. Find the later surface area and total surface area of the following right prisms.

(i)



(ii)



2. The total surface area of a cube is 1350 sq.m. Find its volume.
3. Find the area of four walls of a room (Assume that there are no doors or windows) if its length 12 m., breadth 10 m. and height 7.5 m.
4. The volume of a cuboid is 1200 cm^3. The length is 15 cm. and breadth is 10 cm. Find its height.
5. How does the total surface area of a box change if

(i) Each dimension is doubled? (ii) Each dimension is tripled?

Express in words. Can you find the total surface area of the box if each dimension is raised to *n* times?

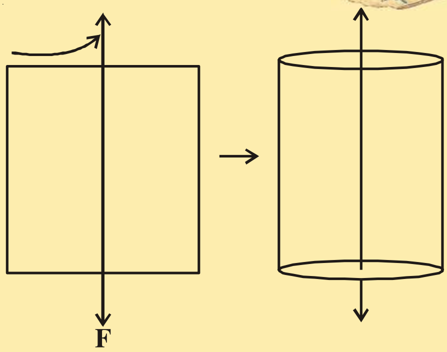
6. The base of a prism is triangular in shape with sides 3 cm., 4 cm. and 5 cm. Find the volume of the prism if its height is 10 cm.
7. A regular square pyramid is 3 m. height and the perimeter of its base is 16 m. Find the volume of the pyramid.
8. An Olympic swimming pool is in the shape of a cuboid of dimensions 50 m. long and 25 m. wide. If it is 3 m. deep throughout, how many liters of water does it hold? (1 cu.m = 1000 liters)

ACTIVITY

Cut out a rectangular sheet of paper. Paste a thick string along the line as shown in the figure. Hold the string with your hands on either sides of the rectangle and rotate the rectangle sheet about the string as fast as you can.

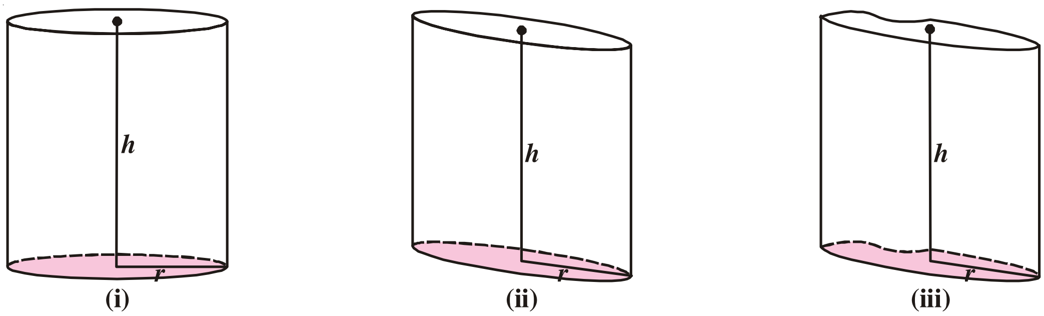
Do you recognize the shape that the rotating rectangle is forming ?

Does it remind you the shape of a cylinder ?



10.3 RIGHT CIRCULAR CYLINDER

Observe the following cylinders:



(i) What similiarties you have observed in figure (i), (ii) and (iii)?

(ii) What differences you have observed between fig. (i), (ii) and (iii)?

(iii) In which figure, the line segment is perpendicular to the base?

Every cylinder is made up of one curved surface and with two congruent circular faces on both ends. If the line segment joining the centre of circular faces, is perpendicular to its base, such a cylinder is called right circular cylinder.

Find out which is right circular cylinder in the above figures? Which are not? Give reasons.

Let us do an activity to generate a cylinder

10.3.1 Curved Surface area of a cylinder

Take a right circular cylinder made up of cardboard. Cut the curved face vertically and unfold it. While unfolding cylinder, observe its transformation of its height and the circular base. After unfolding the cylinder what shape do you find?

You will find it is in rectangular shape. The area of rectangle is equal to the area of curved surface area of cylinder. Its height is equal to the breadth of the rectangle, and the circumference of the base is equal to the length of the rectangle.

Height of cylinder = breadth of rectangle ($h = b$)

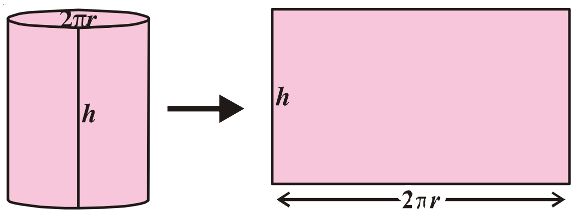
Circumferance of base of cylinder with radius 'r' = length of the rectangle ($2\pi r = l$)

Curved surface area of the cylinder = Area of the rectangle

$= \text{length} \times \text{breadth}$

$= 2\,\pi r \times h$

$= 2\pi rh$



Therefore, Curved surface area of a cylinder = 2πrh

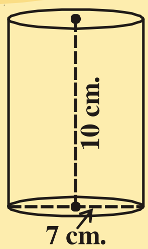
Do This

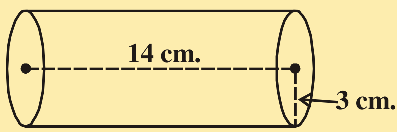
Find CSA of each of following cylinders

(i) $r = x$ cm., $h = y$ cm.

(ii) $d = 7$ cm., $h = 10$ cm.

(iii) $r = 3$ cm., $h = 14$ cm.





10.3.2 Total Surface area of a Cylinder

Observe the adjacent figure.

Do you find that it is a right circular cylinder? What surfaces you have to add to get its total surface area? They are the curved surface area and area of two circular faces.

Now the total surface area of a cylinder

= Curved surface area + Area of top + Area of base

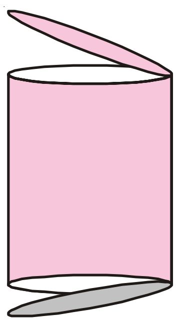
$= 2\pi rh + \pi r^2 + \pi r^2$

$= 2\pi rh + 2\pi r^2$

$= 2\pi r\,(h + r)$

$= 2\pi r\,(r + h)$

$\therefore$ The total surface area of a cylinder $= 2\pi r\,(r + h)$

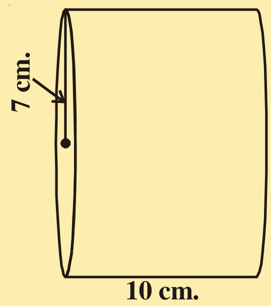


Where 'r' is the radius of the cylinder and 'h' is its height.

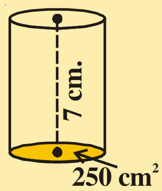
Do These

Find the Total surface area of each of the following cylinders.

(i)



(ii)



10.3.3 Volume of a cylinder

Take circles with equal radii and arrange one over the other.

Do this activity and find whether it form a cylinder or not.

In the figure 'r' is the radius of the circle, and the 'h' is the height up to which the circles are stacked.

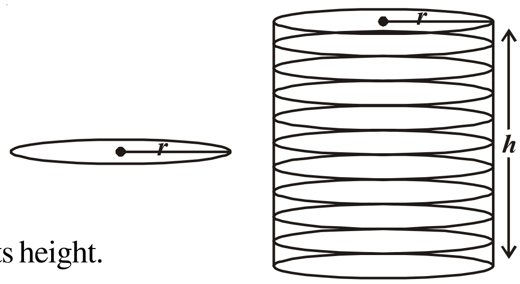
Volume of a cylinder $= \pi r^2 \times$ height

$= \pi r^2 \times h$

$= \pi r^2 h$

So volume of a cylinder = $\pi r^2 h$

Where 'r' is the radius of cylinder and 'h' is its height.



Example-1. A Rectangular paper of width 14 cm is folded along its width and a cylinder of radius 20 cm is formed. Find the volume of the cylinder (Fig 1) ? $\left(\text{Take } \pi = \frac{22}{7}\right)$

Solution : A cylinder is formed by rolling a rectangle about its width. Hence the width of the paper becomes height of cylinder and radius of the cylinder is 20 cm.

Height of the cylinder = h = 14 cm.

radius (r) = 20 cm.

Volume of the cylinder V = $\pi r^2 h$

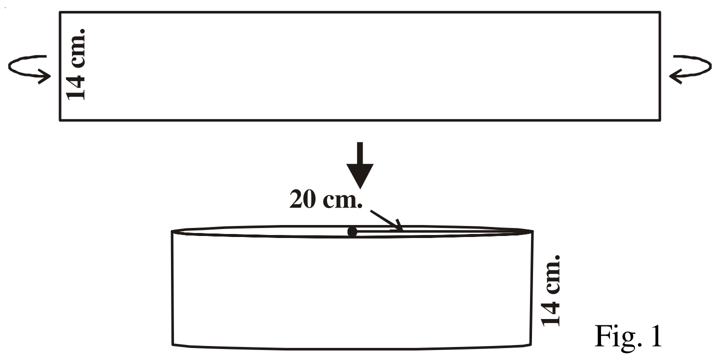


Fig. 1

$$= \frac{22}{7} \times 20 \times 20 \times 14$$

$$= 17600 \text{ cm}^3.$$

Hence the volume of the cylinder is 17600 cm^3.

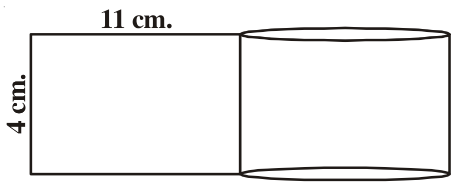
Example-2. A Rectangular piece of paper 11 cm × 4 cm is folded without overlapping to make a cylinder of height 4 cm. Find the volume of the cylinder.

Solution : Length of the paper becomes the circumference of the base of the cylinder and width becomes height.

Let radius of the cylinder = r and height = h

Circumference of the base of the cylinder = $2\pi r$ = 11 cm.

$$2 \times \frac{22}{7} \times r = 11$$



$$\therefore \quad r = \frac{7}{4} \text{ cm.}$$

$$h = 4 \text{ cm}$$

Volume of the cylinder (V) = $\pi r^2 h$

$$= \frac{22}{7} \times \frac{7}{4} \times \frac{7}{4} \times 4 \text{ cm}^3$$

$$= 38.5 \text{ cm}^3.$$

Example-3. A rectangular sheet of paper 44 cm × 18 cm is rolled along the length to form a cylinder. Assuming that the cylinder is solid (Completely filled), find its radius and the total surface area.

Solution : Height of the cylinder = 18 cm

Circumference of base of cylinder = 44 cm

$$2\pi r = 44 \text{ cm}$$

$$r = \frac{44}{2 \times \pi} = \frac{44 \times 7}{2 \times 22} = 7\text{cm.}$$

Total surface area $= 2\pi r\,(r + h)$

$$= 2\times\frac{22}{7}\times 7(7+18)\text{cm}^2$$

$= 1100\ \text{cm}^2$.

Example-4. Circular discs 5 mm thickness, are placed one above the other to form a cylinder of curved surface area 462 cm^2. Find the number of discs, if the radius is 3.5 cm.

Solution : Thickness of disc $= 5\text{ mm} = \frac{5}{10}\text{cm} = 0.5\,\text{cm}$

Radius of disc = 3.5 cm.

Curved surface area of cylinder = 462 cm^2.

$\therefore\ 2\pi rh = 462$ ..... (i)

Let the no of discs be x

$\therefore$ Height of cylinder = h = Thickness of disc $\times$ no of discs

$= 0.5\ x$

$\therefore\ 2\ \pi rh = 2\times\frac{22}{7}\times 3.5\times 0.5x$ ..... (ii)

From (i) are (ii) we get

$$2\times\frac{22}{7}\times 3.5\times 0.5x = 462$$

$$\therefore\ x = \frac{462\times 7}{2\times 22\times 3.5\times 0.5} = 42\ \text{discs}$$

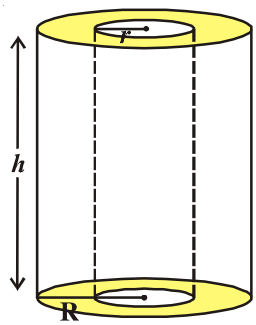
Example-5. A hollow cylinder having external radius 8 cm and height 10 cm has a total surface area of $338\,\pi\,\text{cm}^2$. Find the thickness of the hollow metallic cylinder.

Solution : External radius = R = 8 cm

Internal radius = r

Height = 10 cm

TSA = 338π cm^2.



But TSA = Area of external cylinder (CSA)

+ Area of internal cylinder (CSA)

+ Twice Area of base (ring)

$= 2\pi Rh + 2\pi rh + 2\pi (R^2 - r^2)$

$= 2\pi (Rh + rh + R^2 - r^2)$

$\therefore\ 2\pi (Rh + rh + R^2 - r^2) = 338\,\pi$

$Rh + rh + R^2 - r^2 = 169$

$\Rightarrow\ (10 \times 8) + (r \times 10) + 8^2 - r^2 = 169$

$\Rightarrow\ r^2 - 10r + 25 = 0$

$\Rightarrow\ (r - 5)^2 = 0$

$\therefore\ r = 5$

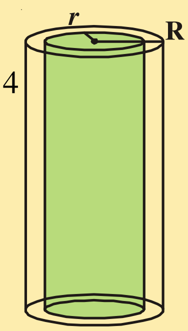
$\therefore$ Thickness of metal = $R - r = (8 - 5)$ cm = 3 cm.

TRY THESE

1. If the radius of a cylinder is doubled keeping its lateral surface area the same, then what is its height ?
2. A hot water system (Geyser) consists of a cylindrical pipe of length 14 m and diameter 5 cm. Find the total radiating surface of hot water system.

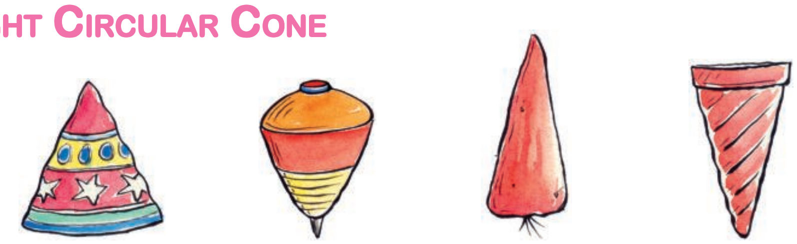
EXERCISE - 10.2

1. A closed cylindrical tank of height 1.4 m. and radius of the base is 56 cm. is made up of a thick metal sheet. How much metal sheet is required (Express in square meters)
2. The volume of a cylinder is 308 cm.^{3}. Its height is 8 cm. Find its lateral surface area and total surface area.
3. A metal cuboid of dimension 22 cm. × 15 cm. × 7.5 cm. was melted and cast into a cylinder of height 14 cm. What is its radius?
4. An overhead water tanker is in the shape of a cylinder has capacity of 61.6 cu.mts. The diameter of the tank is 5.6 m. Find the height of the tank.
5. A metal pipe is 77 cm. long. The inner diameter of a cross section is 4 cm., the outer diameter being 4.4 cm. (see figure) Find its
   (i) inner curved surface area
   (ii) outer curved surface area
   (iii) Total surface area.



6. A cylindrical piller has a diameter of 56 cm and is of 35 m high. There are 16 pillars around the building. Find the cost of painting the curved surface area of all the pillars at the rate of ₹5.50 per 1 m^2.
7. The diameter of a roller is 84 cm and its length is 120 cm. It takes 500 complete revolutions to roll once over the play ground to level. Find the area of the play ground in m^2.
8. The inner diameter of a circular well is 3.5 m. It is 10 m deep. Find
   (i) its inner curved surface area
   (ii) The cost of plastering this curved surface at the rate of Rs. 40 per m^2.
9. Find
   (i) The total surface area of a closed cylindrical petrol storage tank whose diameter 4.2 m. and height 4.5 m.
   (ii) How much steel sheet was actually used, if $\frac{1}{12}$ of the steel was wasted in making the tank.
10. A one side open cylinderical drum has inner radius 28 cm. and height 2.1 m. How much water you can store in the drum. Express in litres. (1 litre = 1000 cc.)
11. The curved surface area of the cylinder is 1760 $cm.^2$ and its volume is 12320 cm^3. Find its height.

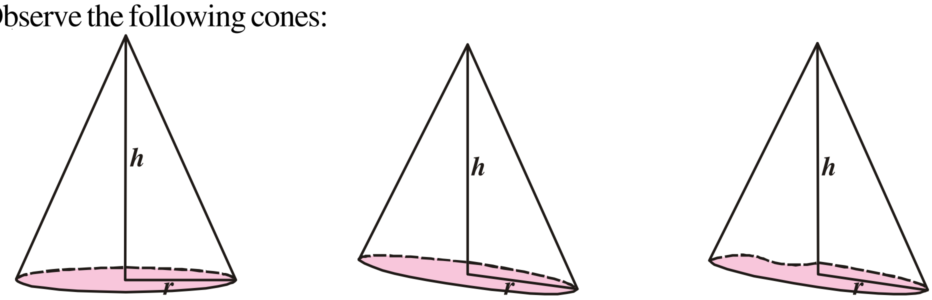
10.4 Right Circular Cone



Observe the above figures and which solid shape they resemble?

These are in the shape of a cone.

Observe the following cones:



(i) (ii) (iii)

(i) What common properties do you find among these cones?

(ii) What difference do you notice among them?

In fig.(i), lateral surface is curved and base is circle. The line segment joining the vertex of the cone and the centre of the circular base (vertical height) is perpendicular to the radius of the base. This type of cone is called Right Circular Cone.

In fig.(ii) although it has circular base, but its vertical height is not perpendicular to the radius of the cone.

Such type of cones are not right circular cones.

In the fig. (iii) although the vertical height is perpendicular to the base, but the base is not in circular shape.

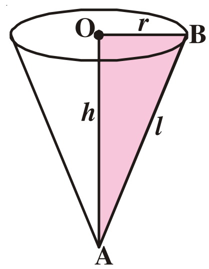
Therefore, this cone is not a right circular cone.

10.4.1 Slant Height of the Cone

In the adjacent figure (cone), $\overline{AO}$ is perpendicular to $\overline{OB}$

ΔAOB is a right angled triangle.

$\overline{AO}$ is the height of the cone (h) and $\overline{OB}$ is equal to the radius of the cone (r)



From ΔAOB

$AB^2 = AO^2 + OB^2$

$AB^2 = h^2 + r^2$ (AB is called slant height $= l$)

$l^2 = h^2 + r^2$

$l = \sqrt{h^2 + r^2}$

ACTIVITY

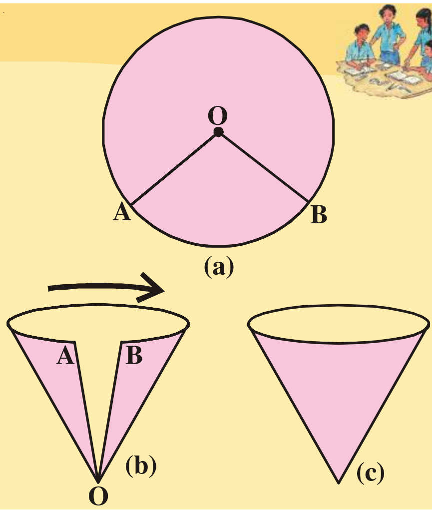
Making a cone from a sector

Follow the instructions and do as shown in the figure.

(i) Draw a circle on a thick paper Fig(a)

(ii) Cut a sector AOB from it Fig(b).

(iii) Fold the ends A, B nearer to each other slowly and join AB. Remember A, B should not overlap on each other. After joining A, B attach them with cello tape Fig(c).

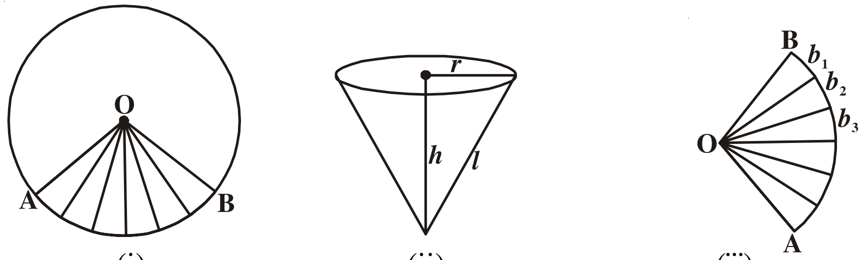


(iv) What kind of shape you have obtained?

Is it a right cone?

While making a cone observe what happened to the edges 'OA' and 'OB' and length of arc AB of the sector?

10.4.2 Curved Surface area of a cone



(i) (ii) (iii)

Let us find the surface area of a right circular cone that we made out of the paper as discussed in the activity.

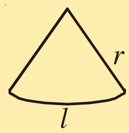
While folding the sector into cone you have noticed that OA, OB of sector coincides and becomes the slant height of the cone, whereas the length of $\overset{\frown}{AB}$ becomes the circumference of the base of the cone.

Now unfold the cone and cut the sector AOB as shown in the figure as many as you can, then you can see each cut portion is almost a small triangle with base b_1, b_2, b_3 ..... etc. and height 'l' i.e. equal to the slant height of the cone.

If we find the area of these triangles and adding these, it gives area of the sector. We know that sector forms a cone, so the area of a sector is equal to curved the surface area of the cone formed with it.

Try This

A sector with radius 'r' and length of its arc 'l' is cut from a circular sheet of paper. Fold it as a cone. How can you derive the formula of its curved surface area $A = \pi rl$



Area of the cone = Sum of the areas of triangles.

$$= \frac{1}{2}b_1 l + \frac{1}{2}b_2 l + \frac{1}{2}b_3 l + \frac{1}{2}b_4 l + ...$$

$$= \frac{1}{2}l(b_1 + b_2 + b_3 + b_4 + ...)$$

$$= \frac{1}{2}l \text{ (length of the curved part from A to B or circumference of the base of the cone)}$$

$$= \frac{1}{2}l(2\pi r) \quad (\because b_1 + b_2 + b_3 + ... = 2\pi r, \text{ where 'r' is the radius of the cone)}$$

as $\overset{\frown}{AB}$ forms a circle.

Thus, lateral surface area or curved surface area of the cone $= \pi rl$

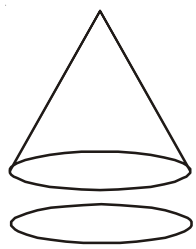
Where 'l' is the slant height of the cone and 'r' is its radius

10.4.3 Total surface area of the cone

If the base of the cone is to be covered, we need a circle whose radius is equal to the radius of the cone.

How to obtain the total surface area of cone? How many surfaces you have to add to get total surface area?

The area of the circle $= \pi r^2$



Total surface area of a cone = lateral surface area + area of its base

$$= \pi rl + \pi r^2$$

$$= \pi r\,(l + r)$$

Total surface area of the cone = $\pi r\,(l + r)$

Where 'r' is the radius of the cone and 'l' is its slant height.

DO THIS

1. Cut a right angled triangle, stick a string along its perpendicular side, as shown in fig. (i) hold the both the sides of a string with your hands and rotate it with constant speed.

   What do you observe ?

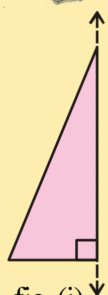
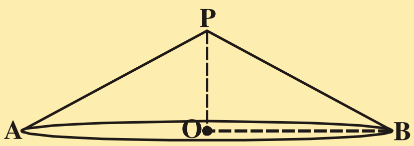
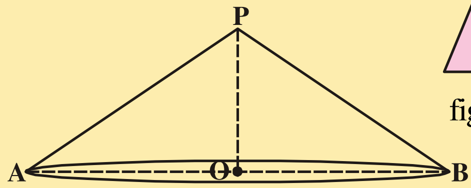


fig. (i)

2. Find the curved surface area and total surface area of the each following Right Circular Cones.

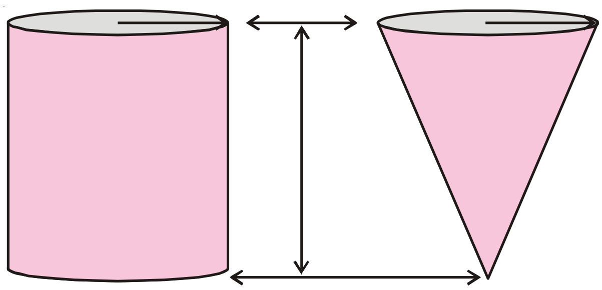


OP = 2cm.; OB = 3.5cm.

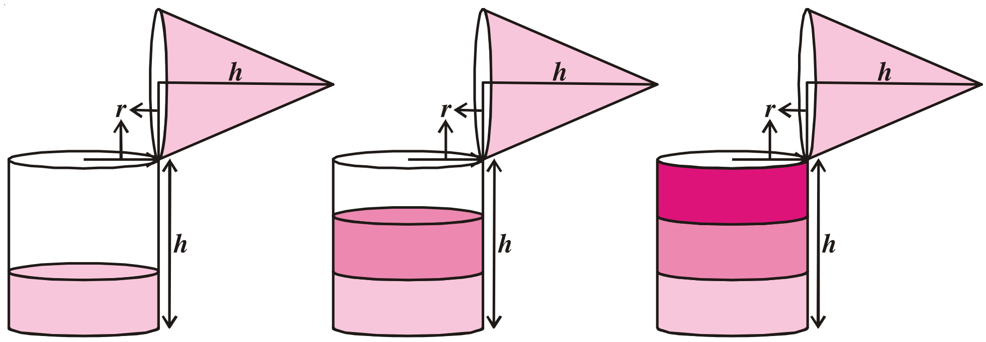


OP = 3.5cm.; AB = 10cm.

10.4.4 Volume of a right circular cone



Make a hollow cylinder and a hollow cone with the equal radius and equal height and do the following experiment, that will help us to find the volume of a cone.



i. Fill water in the cone up to the brim and pour into the hollow cylinder, it will fill up only some part of the cylinder.

ii. Again fill up the cone up to the brim and pour into the cylinder, we see the cylinder is still not full.

iii. When the cone is filled up for the third time and emptied into the cylinder, observe whether the cylinder is filled completely or not.

With the above experiment do you find any relation between the volume of the cone and the volume of the cylinder?

We can say that three times the volume of a cone makes up the volume of cylinder, which both have the same base and same height.

So the volume of a cone is one third of the volume of the cylinder.

$\therefore$ Volume of a cone $= \frac{1}{3}\pi r^2 h$

where 'r' is the radius of the base of cone and 'h' is its height.

Example-6. A corn cob (see fig), shaped like a cone, has the radius of its broadest end as 1.4 cm and length (height) as 12 cm. If each 1cm^2 of the surface of the cob carries an average of four grains, find how many grains approximately you would find on the entire cob.

Solution : Here $l = \sqrt{r^2 + h^2} = \sqrt{(1.4)^2 + (12)^2}\, cm.$

$= \sqrt{145.96} = 12.08$ cm. (approx.)

Therefore the curved surface area of the corn cob = πrl

$= \frac{22}{7} \times 1.4 \times 12.08\ cm^2$

$= 53.15 \text{ cm}^2$

$= 53.2 \text{ cm}^2$ (approx)

Number of grains of corn on 1 cm^2 of the surface of the corn cob = 4.

Therefore, number of grains on the entire curved surface of the cob.

$= 53.2 \times 4 = 212.8 = 213$ (approx)

So, there would be approximately 213 grain of corn on the cob.

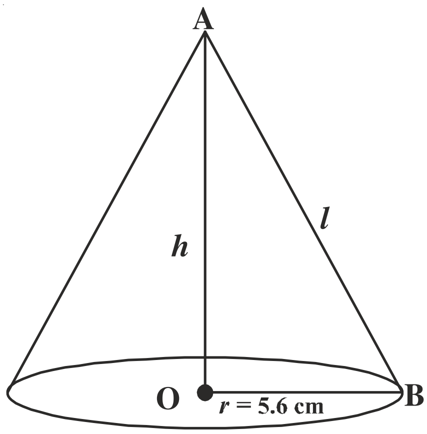
Example-7. Find the slant height and vertical height of a Cone with radius 5.6 cm and curved surface area 158.4 cm^2.

Solution : Radius = 5.6 cm, vertical height = h, slant height = l

CSA of cone $= \pi r l = 158.4 \text{ cm}^2$

$$\Rightarrow \frac{22}{7} \times 5.6 \times l = 158.4$$

$$\Rightarrow l = \frac{158.4 \times 7}{22 \times 5.6} = \frac{18}{2} = 9\text{cm}$$



we know $l^2 = r^2 + h^2$

$$h^2 = l^2 - r^2 = 9^2 - (5.6)^2$$

$$= 81 - 31.36$$

$$= 49.64$$

$$h = \sqrt{49.64}$$

$$h = 7.05 \text{ cm (approx)}$$

Example-8. A tent is in the form of a cylinder surmounted by a cone having its diameter of the base equal to 24 m. The height of cylinder is 11 m and the vertex of the cone is 5m above the cylinder. Find the cost of making the tent, if the rate of canvas is ₹10 per m^2.

Solution : Diametre of base of cylinder = diametre of cone = 24m

$\therefore$ Radius of base = 12 m

Height of cylinder = 11 m = h_1

Height of Cone = 5m = h_2

Let slant height of cone be l

$$l = GD = \sqrt{r^2 + h^2} = \sqrt{12^2 + 5^2} = 13\text{m}$$

Area of canvas required $= \text{CSA of cylinder} + \text{CSA of cone}$

$= 2\pi r h_1 + \pi r l$

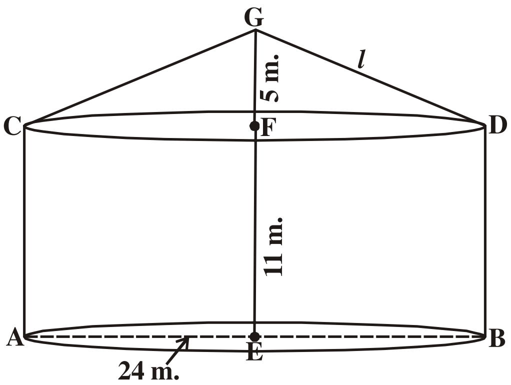
$= \pi r\,(2h_1 + l)$

$= \dfrac{22}{7} \times 12(2\times 11 + 13)\text{m}^2$

$= \dfrac{22\times 12}{7} \times 35\text{m}^2$

$= 22 \times 60\ \text{m}^2$

$= 1320\ \text{m}^2$



Rate of canvas $=$ ₹10 per m^2

$\therefore$ Cost of canvas $=$ Rate $\times$ area of canvas

$=$ ₹10 $\times$ 1320

$=$ ₹13,200.

Example-9. A conical tent was erected by army at a base camp with height 3m. and base diameter 8m. Find;

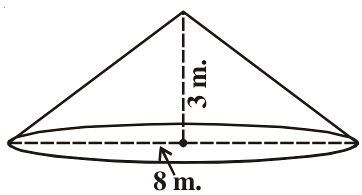
(i) The cost of canvas required for making the tent, if the canvas cost ₹ 70 per 1 sq.m.

(ii) If every person requires 3.5 m.^{3} air, how many can be seated in that tent.

Solution : Diameter of the tent $= 8$ m.

$$r = \frac{d}{2} = \frac{8}{2} = 4 \text{ m.}$$

height $= 3$ m.



Slant height $(l) = \sqrt{h^2 + r^2}$

$= \sqrt{3^2 + 4^2}$

$= \sqrt{25} = 5$ m.

$\therefore$ Curved surface area of tent $= \pi r l$

$$= \frac{22}{7} \times 4 \times 5 = \frac{440}{7}\ \text{m}^2$$

Volume of the cone $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times \frac{22}{7} \times 4 \times 4 \times 3$

$= \frac{352}{7}$ m^3

(i) Cost of canvas required for the tent

= CSA × Unit cost

$= \frac{440}{7} \times 70$

= ₹4400

(ii) No. of persons can be seated in the tent

$= \frac{\text{Volume of conical tent}}{\text{air required for each}}$

$= \frac{352}{7} \div 3.5$

$= \frac{352}{7} \times \frac{1}{3.5} = 14.36$

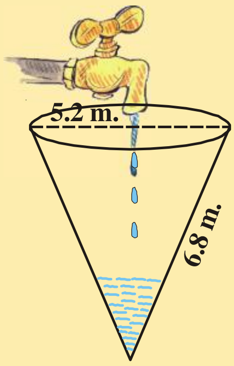
= 14 men (approx.)

EXERCISE - 10.3

1. The base area of a cone is 38.5 cm^2. Its volume is 77 cm^3. Find its height.
2. The volume of a cone is 462 m^3. Its base radius is 7 m. Find its height.
3. Curved surface area of a cone is 308 cm^2 and its slant height is 14 cm Find.

   (i) radius of the base (ii) Total surface area of the cone.
4. The cost of painting the total surface area of a cone at 25 paise per cm^2 is ₹176. Find the volume of the cone, if its slant height is 25 cm.
5. From a circle of radius 15 cm., a sector with angle 216° is cut out and its bounding radii are bent so as to form a cone. Find its volume.
6. The height of a tent is 9 m. Its base diameter is 24 m. What is its slant height? Find the cost of canvas cloth required if it costs ₹14 per sq.m.

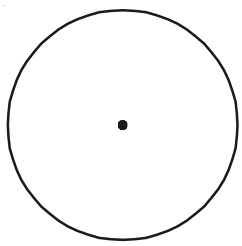
7. The curved surface area of a cone is $1159\frac{5}{7}$ cm^2. Area of its base is $254\frac{4}{7}$ cm^2. Find its volume.
8. A tent is cylindrical to a height of 4.8 m. and conical above it. The radius of the base is 4.5m. and total height of the tent is 10.8 m. Find the canvas required for the tent in square meters.
9. What length of tarpaulin 3 m wide will be required to make a conical tent of height 8m and base radius 6m ? Assume that extra length of material that will be required for stitching margins and wastage in cutting is approximately 20 cm (use $\pi = 3.14$)
10. A Joker's cap is in the form of a right circular cone of base radius 7 cm and height 27 cm. Find the area of the sheet required to make 10 such caps.
11. Water is pouring into a conical vessel of diameter 5.2m and slant height 6.8m (as shown in the adjoining figure), at the rate of 1.8 m^3 per minute. How long will it take to fill the vessel?



12. Two similiar cones have volumes 12π cu. units and 96π cu. units. If the curved surface area of the smaller cone is 15π sq. units, what is the curved surface area of the larger one?

Hint : For similar cones $\left(\frac{A_1}{A_2}\right)^3 = \left(\frac{V_1}{V_2}\right)^2$

10.5 Sphere



(i)

(ii)

(iii)

All the above figures are well known to you. Can you identify the difference among them.

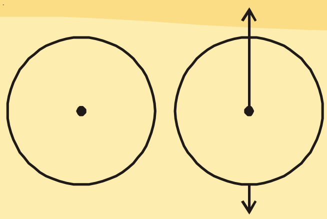
Figure (i) is a circle. You can easily draw it on a plane paper. Because it is a plane figure. A circle is plane closed figure whose every point lies at a constant distance (radius) from a fixed point (centre)

The remaining above figures are solids. These solids are circular in shape and are called spheres.

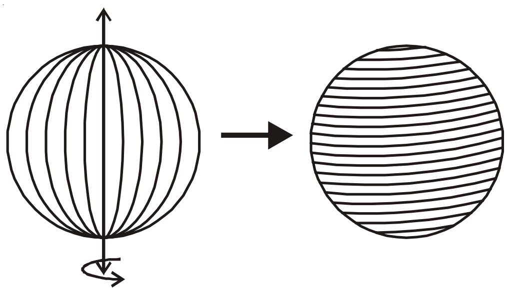
A sphere is a three dimensional figure, which is made up of all points in the space, which is at a constant distance from a fixed point. This fixed point is called centre of the sphere. The distance from the centre to any point on the surface of the sphere is its radius.

ACTIVITY

Draw a circle on a thick paper and cut it neatly. Stick a string along its diameter. Hold the both the ends of the string with hands and rotate with constant speed and observe the figure so formed.



10.5.1 Surface area of a sphere



Let us find the surface area of the figure with the following activity.

Take a tennis ball as shown in the figure and wind a string around the ball, use pins to keep the string in place. Mark the starting and ending points of the string. Slowly remove the string from the surface of the sphere.

Find the radius of the sphere and draw four circles of radius equal to the radius of the ball as shown in the pictures. Start filling the circles one after one with the string you had wound around the ball.

What do you observe?

The string, which had completely covered the surface area of the sphere (ball), has been used to completely fill the area of four circles, all have same radius as of the sphere.

With this we can understand that the surface area of a sphere of radius (r) is equal to the four times of the area of a circle of radius (r).

$\therefore$ Surface area of a sphere = 4 $\times$ the area of circle

$$= 4\pi r^2$$

Surface area of a sphere = $4\pi r^2$

Where 'r' is the radius of the sphere

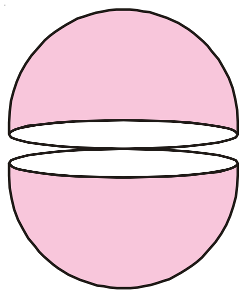
TRY THIS

Can you find the surface area of sphere in any other way?

10.5.2 Hemisphere

Take a solid sphere and cut it through the middle with a plane that passes through its centre.

Then it gets divided into two equal parts as shown in the figure



Each equal part is called a hemisphere.

A sphere has only one curved face. If it is divided into two equal parts, then its curved face is also divided into two equal curved faces.

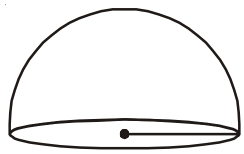
What do you think about the surface area of a hemisphere ?

Obviously,

Curved surface area of a hemisphere is equal to half the surface area of the sphere

So, surface area of a hemisphere = $\frac{1}{2}$ surface area of sphere

$$= \frac{1}{2} \times 4\pi r^2$$



$$= 2\ \pi r^2$$

$\therefore$ **surface area of a hemisphere = $2\ \pi r^2$**

The base of hemisphere is a circular region.

Its area is equal to = πr^2

Let us add both the curved surface area and area of the base, we get total surface area of hemisphere.

Total surface area of hemisphere = Its curved surface area + area of its base

$$= 2\ \pi r^2 + \pi r^2$$

$$= 3\ \pi r^2.$$

Total surface area of hemisphere = $3\pi r^2$.

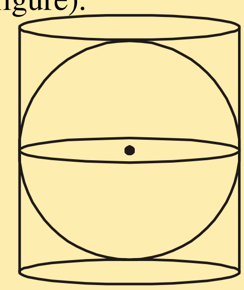
DO THESE

1. A right circular cylinder just encloses a sphere of radius r (see figure).

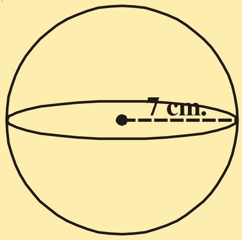
   Find : (i) surface area of the sphere

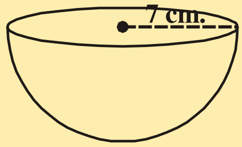
   (ii) curved surface area of the cylinder

   (iii) ratio of the areas obtained in (i) and (ii)



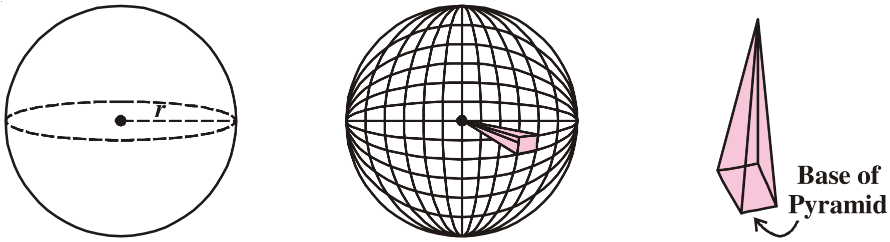
2. Find the surface area of each the following figure.

(i) 

(ii) 

10.5.3 Volume of Sphere

To find the volume of a sphere, imagine that a sphere is composed of a great number of congruent pyramids with all their vertices join at the centre of the sphere, as shown in the figure.



(i) (ii) (iii)

Let us follow the steps:

1. Let 'r' be the radius of the solid sphere as in fig. (i).
2. Assume that a sphere with radius 'r' is made of 'n' number of pyramids of equal sizes as shown in the fig. (ii).
3. Consider a part (pyramid) among them. Each pyramid has a base and let the area of the base of pyramids are A_1, A_2, A_3.....

   The height of the pyramid is equal to the radius of sphere, then the

   Volume of one pyramid $= \frac{1}{3} \times$ Area of the base $\times$ height

   $= \frac{1}{3} A_1 r$

   you can take any polygon as base of a pyramid

4. As there are 'n' number of pyramids, then

   Volume of 'n' pyramids $= \frac{1}{3} A_1 r + \frac{1}{3} A_2 r + \frac{1}{3} A_3 r + \ldots$ n times

   $= \frac{1}{3} r \, [A_1 + A_2 + A_3 + \ldots$ n times$]$

$$= \frac{1}{3} \times A\,r$$

$A = A_1 + A_2 + A_3 + \ldots$ n times
$=$ Surface areas of 'n' pyramids

5. As the sum of volumes of all these pyramids is equal to the volume of sphere and the sum of the areas of all the bases of the pyramids is very close to the surface area of the sphere, (i.e. $4\pi r^2$).

So, volume of sphere $= \frac{1}{3}(4\pi r^2)\,r$

$= \frac{4}{3}\pi r^3$ cub. units

Volume of a sphere = $\frac{4}{3}\pi r^3$

Where 'r' is the radius of the sphere

How can you find volume of hemisphere? It is half the volume of sphere.

$\therefore$ Volume of hemisphere $= \frac{1}{2}$ of volume of a sphere

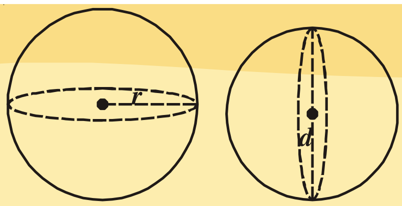
$$= \frac{1}{2} \times \frac{4}{3}\pi r^3$$

$$= \frac{2}{3}\pi r^3$$

[**Hint :** You can try to derive these formulae using water melon or any other like that]

Do This

1. Find the volume of the sphere given in the adjacent figures.
2. Find the volume of sphere of radius 6.3 cm.



r = 3cm. d = 5.4cm.

Example-10. If the surface area of a sphere is $154\ cm^2$, find its radius.

Solution : Surface area of sphere $= 4\pi r^2$

$$4\pi r^2 = 154 \Rightarrow 4 \times \frac{22}{7} \times r^2 = 154$$

$$\Rightarrow r^2 = \frac{154 \times 7}{4 \times 22} = \frac{7^2}{2^2}$$

$$\Rightarrow r = \frac{7}{2} = 3.5\,cm$$

Example-11. A hemispherical bowl is made up of stone whose thickness is 5 cm. If the inner radius is 35 cm, find the total surface area of the bowl.

Solution : Let R be outer radius and 'r' be inner radius Thickness of ring = 5 cm

$\therefore$ R = (r + 5) cm = (35 + 5) cm = 40 cm

Total Surface Area = CSA of outer hemisphere + CSA of inner hemisphere + area of the ring.

$$= 2\pi R^2 + 2\pi r^2 + \pi(R^2 - r^2)$$

$$= \pi(2R^2 + 2r^2 + R^2 - r^2)$$

$$= \frac{22}{7}(3R^2 + r^2) = \frac{22}{7}(3\times 40^2 + 35^2)\,cm^2$$

$$= \frac{6025 \times 22}{7} cm^2$$

$$= 18935.71 \text{ cm}^2 \text{ (approx)}.$$

Example-12. The hemispherical dome of a building needs to be painted (see fig 1). If the circumference of the base of dome is 17.6 m, find the cost of painting it, given the cost of painting is Rs.5 per 100 cm^2.

Solution : Since only the rounded surface of the dome is to be painted we need to find the curved surface area of the hemisphere to know the extent of painting that needs to be done. Now, circumference of base of the dome = 17.6 m Therefore 17.6 = $2\pi r$

So, The radius of the dome $= 17.6 \times \frac{7}{2\times 22}$ m

$= 2.8$ m

The curved surface area of the dome $= 2\pi r^2$

$$= 2 \times \frac{22}{7} \times 2.8 \times 2.8 m^2$$

$$= 49.28 \text{ m}^2.$$

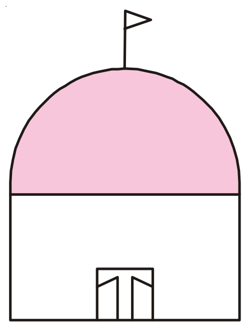


fig 1

Now, cost of painting 100 cm^2 is Rs.5

So, cost of painting 1m^2 = Rs. 500

Therefore, cost of painting the whole dome

= Rs.500 × 49.28

= Rs. 24640.

Example-13. The hollow sphere, in which the circus motor cyclist performs his stunts, has a diameter of 7m. Find the area available to the motor cyclist for riding.

Solution : Diameter of the sphere = 7 m. Therefore, radius is 3.5 m. So, the riding space available for the motorcyclist is the surface area of the 'sphere' which is given by

$$4\pi r^2 = 4 \times \frac{22}{7} \times 3.5 \times 3.5 m^2$$

$$= 154 \text{ m}^2.$$

Example-14. A shotput is a metallic sphere of radius 4.9 cm. If the density of the metal is 7.8 g. per cm^3, find the mass of the shotput.

Solution : Since the shot-put is a solid sphere made of metal and its mass is equal to the product of its volume and density, we need to find the volume of the sphere.

Now, volume of the sphere $= \frac{4}{3}\pi r^3$

$$= \frac{4}{3} \times \frac{22}{7} \times 4.9 \times 4.9 \times 4.9 \text{ cm}^3$$

$$= 493 \text{ cm}^3 \text{ (nearly)}$$

Further, mass of $1 cm^3$ of metal is 7.8 g

Therefore, mass of the shot-put $= 7.8 \times 493$ g

$= 3845.44\text{g} = 3.85$ kg (nearly)

Example-15. A hemispherical bowl has a radius of 3.5 cm. What would be the volume of water it would contain ?

Solution : The volume of water the bowl can contains = Volume of hemisphere

$$= \frac{2}{3}\pi r^3$$

$$= \frac{2}{3} \times \frac{22}{7} \times 3.5 \times 3.5 \times 3.5 \text{ cm}^3$$

$= 89.8 \text{ cm}^3$. (approx).

EXERCISE - 10.4

1. The radius of a sphere is 3.5 cm. Find its surface area and volume.
2. The surface area of a sphere is $1018\frac{2}{7}$ sq.cm. What is its volume?
3. The length of equator of the globe is 44 cm. Find its surface area.
4. The diameter of a spherical ball is 21 cm. How much leather is required to prepare 5 such balls.
5. The ratio of radii of two spheres is 2 : 3. Find the ratio of their surface areas and volumes.
6. Find the total surface area of a hemisphere of radius 10 cm. (use $\pi = 3.14$)
7. The diameter of a spherical balloon increases from 14 cm. to 28 cm. as air is being pumped into it. Find the ratio of surface areas of the balloons in the two cases.
8. A hemispherical bowl is made of brass, 0.25 cm. thickness. The inner radius of the bowl is 5 cm. Find the ratio of outer surface area to inner surface area.
9. The diameter of a lead ball is 2.1 cm. The density of the lead used is 11.34 g/c^3. What is the weight of the ball?
10. A metallic cylinder of diameter 5 cm. and height $3\frac{1}{3}$ cm. is melted and cast into a sphere. What is its diameter.
11. How many litres of milk can a hemispherical bowl of diameter 10.5 cm. hold?
12. A hemispherical bowl has diameter 9 cm. The liquid is poured into cylindrical bottles of diameter 3 cm. and height 3 cm. If a full bowl of liquid is filled in the bottles, find how many bottles are required.

WHAT WE HAVE DISCUSSED

1. Cuboid and cube are regular prisms having six faces and of which four are lateral faces and the base and top.
2. If length of cuboid is l, breadth is 'b' and height is 'h' then,

   Total surface area of a cuboid $= 2\,(lb + bh + lh)$

   Lateral surface area of a cuboid$= 2\,h\,(l + b)$

   Volume of a cuboid $= lbh$

3. If the length of the edge of a cube is 'l' units, then

   Total surface area of a cube $= 6l^2$

   Lateral surface area of a cube $= 4l^2$

   Volume of a cube $= l^3$

4. The volume of a pyramid is $\frac{1}{3}$rd volume of a right prism if both have the same base and same height.

5. A cylinder is a solid having two circular ends with a curved surface area. If the line segment joining the centres of base and top is perpendicular to the base, it is called right circular cylinder.

6. If the radius of right circular cylinder is 'r' and height is 'h' then;

   - Curved surface area of a cylinder = $2\pi rh$
   - Total surface area of a cylinder = $2\pi r\,(r + h)$
   - Volume of a cylinder = $\pi r^2 h$

7. Cone is a geometrical shaped object with circle as base, having a vertext at the top. If the line segment joining the vertex to the centre of the base is perpendicular to the base, it is called right circular cone.

8. The length joining the vertex to any point on the circular base of the cone is called slant height (l)

   $$l^2 = h^2 + r^2$$

9. If 'r' is the radius, 'h' is the height, 'l' is the slant height of a cone, then

   - Curved surface area of a cone = $\pi r l$
   - Total surface area of a cone = $\pi r\,(r + l)$

10. The volume of a cone is $\frac{1}{3}$rd the volume of a cylinder of the same base and same height

    i.e. volume of a cone = $\frac{1}{3}\pi r^2 h$.

11. A sphere is an geometrical object formed where the set of points are equidistant from the fixed point in the space. The fixed point is called centre of the sphere and the fixed distance is called radius of the sphere.

12. If the radius of sphere is 'r' then,

- Surface area of a sphere = $4\pi r^2$
- Volume of a sphere = $\frac{4}{3}\pi r^3$

13. A plane through the centre of a sphere divides it into two equal parts, each of which is called a hemisphere.

- Curved surface area of a hemisphere = $2\pi r^2$
- Total surface area of a hemisphere = $3\pi r^2$
- Volume of a hemisphere = $\frac{2}{3}\pi r^3$

Do You Know?

Making an 8 × 8 Magic Square

Simply place the numbers from 1 to 64 sequentially in the square grids, as illustrated on the left. Sketch in the dashed diagonals as indicated. To obtain the magic square below, replace any number which lands on a dashed line with its compliment (two numbers of a magic square are compliments if they total the same value as the sum of the magic's square smallest and largest numbers).

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

64	2	3	61	60	6	7	57
9	55	54	12	13	51	50	16
17	47	46	20	21	43	42	24
40	26	27	37	36	30	31	33
32	34	35	29	28	38	39	25
41	23	22	44	45	19	18	48
49	15	14	52	53	11	10	56
8	58	59	5	4	62	63	1

* A magic square is an array of numbers arrange in a square shape in which any row, column total the same amount. You can try more such magic squares.

11 Areas

11.1 Introduction

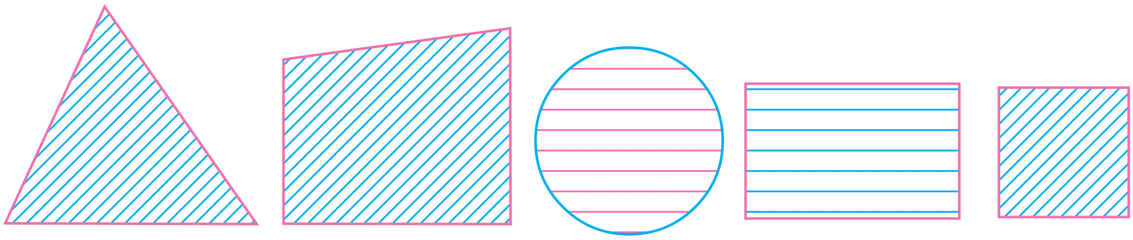
Have you seen agricultural fields around your village or town? The land is divided amongst various farmers and there are many fields. Are all the fields of the same shape and same size? Do they have the same area? If a field has to be further divided among some persons, how will they divide it? If they want equal area, what can they do?

How does a farmer estimate the amount of fertilizer or seed needed for field? Does the size of the field have anything to do with this number?

The earliest and the most important reason for the initiation of the study of geometry is agricultural organisation. This includes measuring the land, dividing it into appropriate parts and recasting boundaries of the fields for the sake of convenience. In history you may have discussed the floods of river Nile (Egypt) and the land markings generated later . Some of these fields resemble the basic shapes such as square, rectangle trapezium, parallelograms etc., and some are in irregular shapes. For the basic shapes, we follow the rules to find areas from given measurements. We would study some of them in this chapter. We will learn how to calculate areas of triangles, squares, rectangles and quadrilaterals by using formulae. We will also explore the basis of those formulae. We will discuss how are they derived ? What do we mean by 'area'?

11.2 Area of Planar regions

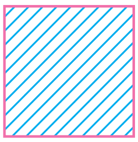
You may recall that the part of the plane enclosed by a simple closed figure is called a planar region corresponding to that figure. The magnitude or measure of this planar region is its area.



Triangular region | Quadrilateral region | Circular region | Rectangular region | Square region

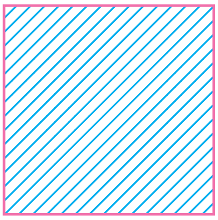
A planar region consists of a boundary and an interior region. How can we measure the area of this? The magnitude of measure of these regions (i.e. areas) is always expressed with a positive real number (in some unit of area) such as 10 cm^2, 215 m^2, 2 km^2, 3 hectares etc. So, we can say that area of a figure is a number (in some unit of area) associated with the part of the plane enclosed by the figure.

The unit area is the area of a square of a side of unit length. Hence **square centimeter** (or 1cm^2) is the area of a square drawn on a side one centimeter in length.



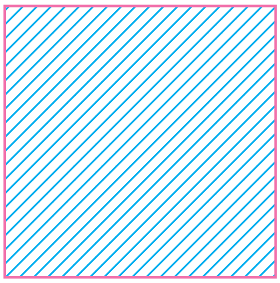
1 cm

1 cm
Area = 1 sq. cm



1 m

1 m
Area = 1 sq. m



1 km

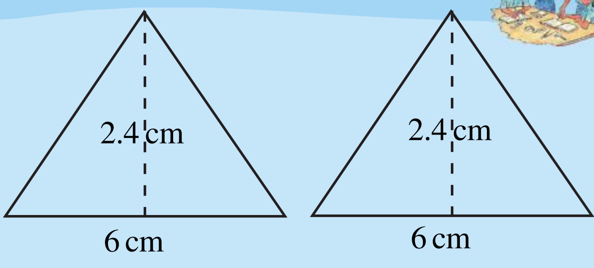
1 km
Area = 1 sq. km

The terms square meter (1m^2), square kilometer (1km^2), square millimeter (1mm^2) are to be understood in the same sense. We are familiar with the concept of congruent figures from earlier classes. Two figures are congruent if they have the same shape and the same size.

ACTIVITY

Observe Figure I and II. Find the area of both figures. Are the areas equal?

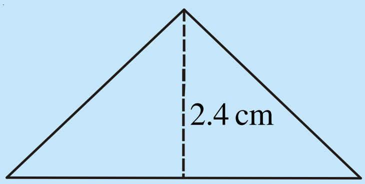
Trace these figures on a sheet of paper, cut them. Cover fig. I with fig. II. Do they cover each other completely? Are they congruent?



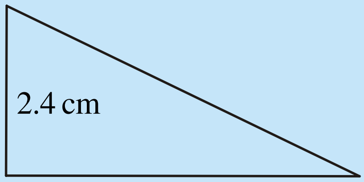
(i)

Observe fig. III and IV. Find the areas of both. What do you notice?

Are they congruent?

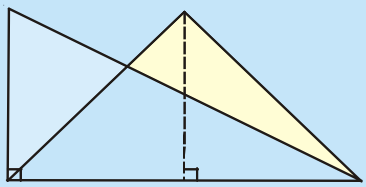


III



IV

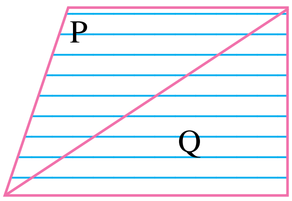
Now, trace these figures on sheet of paper. Cut them let us cover fig. III by fig. IV by conciding their bases (length of same side).


V

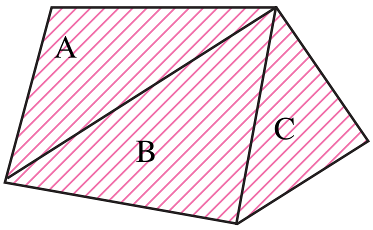
As shown in figure V are they covered completely?

We conclude that Figures I and II are congruent and equal in area. But figures III and IV are equal in area but they are not congruent.

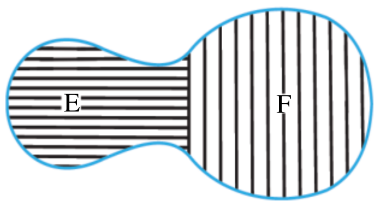
Now consider the figures given below:



X



Y



Z

You may observe that planar region of figures X, Y, Z is made up of two or more planar regions. We can easily see that

Area of figure X = Area of figure P + Area of figure Q.

Similarly area of (Y) = area of (A) + area of (B) + area of (C)

area of (Z) = area of (E) + area of (F).

Thus the area of a figure is a number (in some units) associated with the part of the plane enclosed by the figure with the following properties.

(**Note :** We use area of a figure (X) briefly as ar(X) from now onwards)

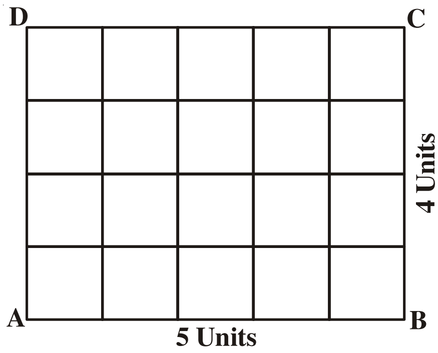
(i) The areas of two congruent figures are equal.

If A and B are two congruent figures, then ar(A) = ar(B)

(ii) The area of a figure is equal to the sum of the areas of finite number of parts of it.
If a planar region formed by a figure X is made up of two non-overlapping planar regions formed by figures P and Q then ar(X) = ar(P) + ar(Q).

11.3 Area of Rectangle

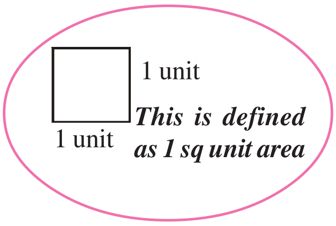
If the number of units in the length of a rectangle is multiplied by the number of units in its breadth, the product gives the number of square units in the area of rectangle



Let ABCD represent a rectangle whose length $\overline{AB}$ is 5 units and breadth $\overline{BC}$ is 4 units.

Divide $\overline{AB}$ into 5 equal parts and $\overline{BC}$ into 4 equal parts and through the points of division of each line draw parallels to the other. Each compartment in the rectangle represents one square unit (why ?)

$\therefore$ The rectangle contains (5 units $\times$ 4 units). That is 20 square units.



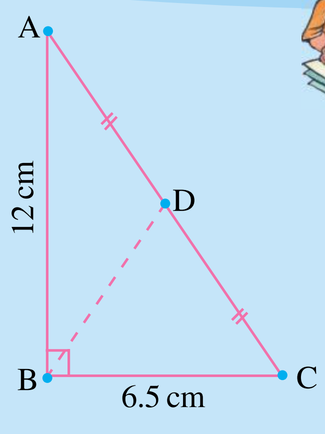
Similarly, if the length is 'a' units and breadth is 'b' units then the area of rectangle is 'ab' square units. That is "length $\times$ breadth" square units gives the area of a rectangle.

Think, Discuss and Write

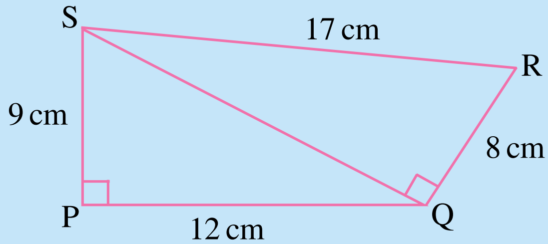
1. If 1cm represents 5m, what would be an area of 6 square cm. represent ?
2. Rajni says 1 sq.m = 100^2 sq.cm. Do you agree? Explain.

Exercise - 11.1

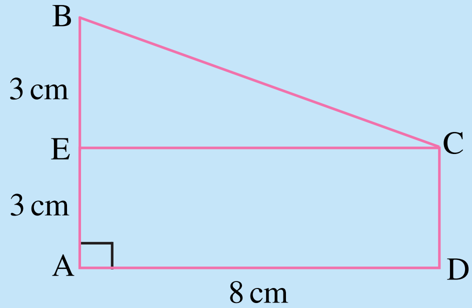
1. In ΔABC, $\angle ABC = 90^\circ$, $AD = DC$, $AB = 12cm$ and $BC = 6.5$ cm. Find the area of ΔADB.



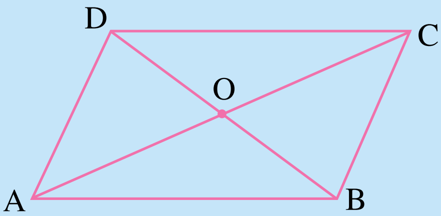
2. Find the area of a quadrilateral PQRS in which $\angle QPS = \angle SQR = 90°$, PQ = 12 cm, PS = 9 cm, QR = 8 cm and SR = 17 cm (**Hint:** PQRS has two parts)



3. Find the area of trapezium ABCD as given in the figure in which ADCE is a rectangle. (**Hint:** ABCD has two parts)



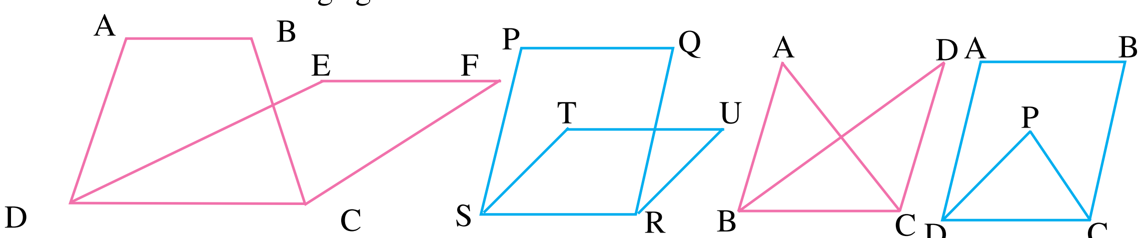
4. ABCD is a parallelogram. The diagonals AC and BD intersect each other at 'O'. Prove that ar(ΔAOD) = ar(ΔBOC). (**Hint:** Congruent figures have equal area)



11.4 FIGURES ON THE SAME BASE AND BETWEEN THE SAME PARALLELS

We shall now study some relationships between the areas of some geometrical figures under the condition that they lie on the same base and between the same parallels. This study will also be useful in understanding of some results on similarity of triangles.

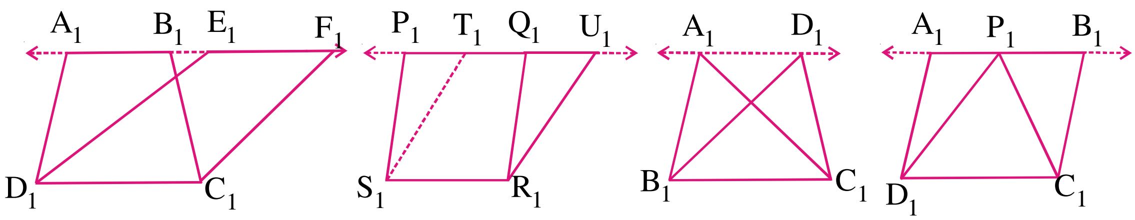
Look at the following figures.



(i) (ii) (iii) (iv)

In Fig(i) a trapezium ABCD and parallelogram EFCD have a common side CD. We say that trapezium ABCD and parallelogram EFCD are on the same base CD. Similarly in fig(ii) the base of parallelogram PQRS and parallelogram TURS is the same. In fig(iii) Triangles ABC and DBC have the same base BC. In Fig(iv) parallelogram ABCD and triangle PCD lie on DC so, all these figures are of geometrical shapes are therefore on the same base. They are however not between the same parallels as AB does not overlap EF and PQ does not overlap TU etc. Neither the points A, B, E, F are collinear nor the points P, Q, T, U. What can you say about Fig(iii) and Fig (iv)?

Now observe the following figures.



(v) (vi) (vii) (viii)

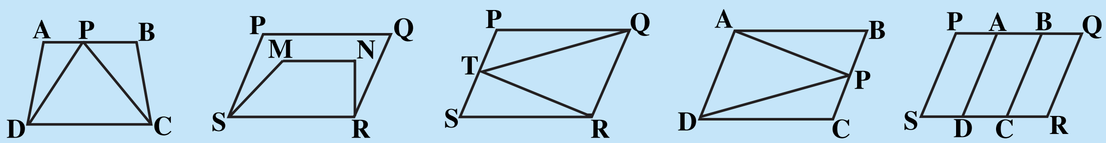
What difference have you observed among the figures? In Fig(v), We say that trapezium $A_1B_1C_1D_1$ and parallelogram $E_1F_1C_1D_1$ are on the same base and between the same parallels A_1F_1 and D_1C_1. The points A_1, B_1, E_1, F_1 are collinear and $A_1F_1 \parallel D_1C_1$. Similarly in fig. (vi) parallelograms $P_1Q_1R_1S_1$ and $T_1U_1R_1S_1$ are on the same base S_1R_1 and between the same parallels P_1U_1 and S_1R_1. Name the other figures on the same base and the parallels between which they lie in fig. (vii) and (viii).

So, two figures are said to be on the same base and between the same parallels, if they have a common base (side) and the vertices (or the vertex) opposite to the common base of each figure lie on a line parallel to the base.

THINK, DISCUSS AND WRITE

Which of the following figures lie on the same base and between the same parallels?

In such a cases, write the common base and the two parallels.



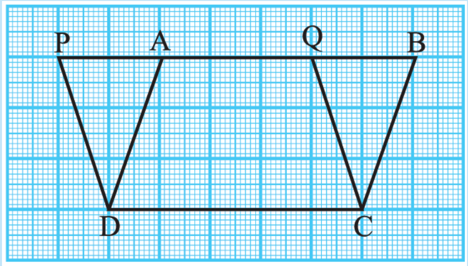
(a) (b) (c) (d) (e)

11.5 PARALLELOGRAMS ON THE SAME BASE AND BETWEEN THE SAME PARALLELS

Now let us try to find a relation, if any, between the areas of two parallelograms on the same base and between the same parallels. For this, let us perform the following activity.

ACTIVITY

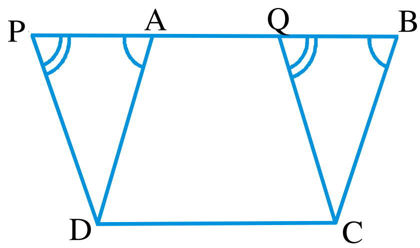
Take a graph sheet and draw two parallelograms ABCD and PQCD on it as show in the Figure-



The parallelograms are on the same base DC and between the same parallels PB and DC Clearly the part DCQA is common between the two parallelograms. So if we can show that ΔDAP and ΔCBQ have the same area then we can say ar(PQCD) = ar(ABCD).

Theorem-11.1 : Parallelograms on the same base and between the same parallels are equal in area.

Proof: Let ABCD and PQCD are two parallelograms on the samebase DC and between the parallel lines DC and PB.



In ΔDAP and ΔCBQ

$PD \parallel CQ$ and PB is transversal $\angle DPA = \angle CQB$

and $AD \parallel CB$ and PB is transversal $\angle DAP = \angle CBQ$

also $PD = QC$ as PQCD is a parallelogram.

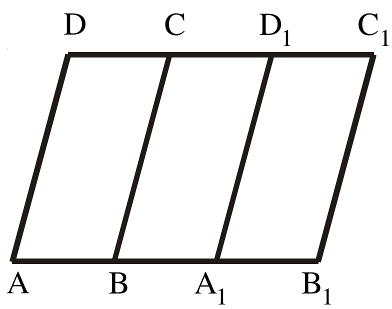
Hence ΔDAP and ΔCBQ are congruent and have equal areas.

So we can say ar(PQCD) = ar (AQCD) + ar(DAP)

= ar(AQCD) + ar(CBQ) = ar(ABCD)

You can verify by counting the squares of these parallelogram as drawn in the graph sheet.

Can you explain how to count full squares below half a square, above half a square on graph sheet.



Reshma argues that the parallelograms between same parallels need not have a common base for equal area. They only need to have an equal base. To understand her statement look at the adjacent figure.

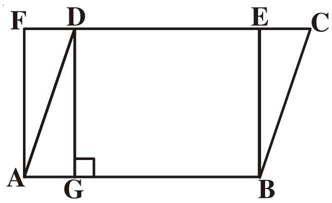
If $AB = A_1B_1$ When we cut out parallelogram $A_1B_1C_1D_1$ and place it over parallelogram ABCD, A would concide in with A_1 and B with B_1 and C_1, D_1 coincide with C, D. Thus these are equal in area. Thus the parallelogram with the equal base can be considered to be on the same base for the purposes of studying their geometrical properties.

Let us now take some examples to illustrate the use of the above Theorem.

Example-1. ABCD is parallelogram and ABEF is a rectangle and DG is perpendicular on AB.

Prove that (i) ar (ABCD) = ar(ABEF)

(ii) ar (ABCD) = AB × DG



Solution : (i) A rectangle is also a parallelogram

$\therefore$ ar(ABCD) = ar(ABEF) ..... (1)

(Parallelograms lie on the same base and between the same parallels)

(ii) ar(ABCD) = ar(ABEF) ($\because$ from (1))

= AB × BE ($\because$ ABEF is a rectangle)

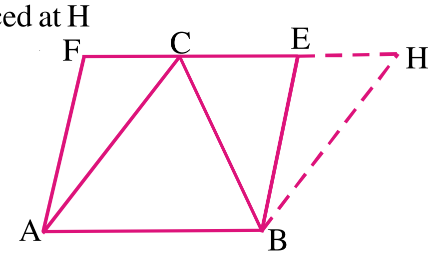
= AB × DG ($\because$ DG $\perp$ AB and DG = BE)

Therefore ar(ABCD) = AB × DG

From the result, we can say that "area of a parallelogram is the product of its any side and the corresponding altitude".

Example-2. Triangle ABC and parallelogram ABEF are on the same base, AB as in between the same parallels AB and EF. Prove that ar(ΔABC) = $\frac{1}{2}$ar($\parallel$ gm ABEF)

Solution : Through B draw BH $\parallel$ AC to meet FE produced at H



$\therefore$ ABHC is a parallelogram

Diagonal BC divides it into two congruent triangles

$\therefore$ ar(ΔABC) = ar(ΔBCH)

$= \frac{1}{2}$ar ($\parallel$ gm ABHC)

But $\parallel$ gm ABHC and $\parallel$ gm ABEF are on the same base AB and between same parallels AB and EF

$\therefore$ ar($\parallel$ gm ABHC) = ar($\parallel$ gm ABEF)

Hence ar(ΔABC) = $\frac{1}{2}$ar ($\parallel$ gm ABEF)

From the result, we say that "the area of a triangle is equal to half the area of the parallelogram on the same base and between the same parallels".

Example-3. Find the area of a figure formed by joining the mid-points of the adjacent sides of a rhombus with diagonals 12 cm. and 16 cm.

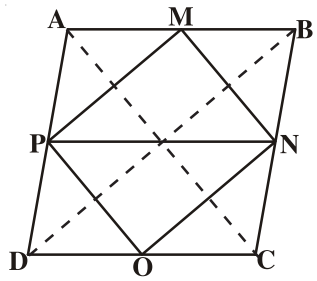
Solution : Join the mid points of AB, BC, CD, DA of a rhombus ABCD and name them M, N, O and P respectively to form a figure MNOP.

What is the shape of MNOP thus formed? Give reasons?

Join the line PN, then PN || AB and PN || DC (How?)

We know that if a triangle and a parallelogram are on the same base and between the same parallels, the area of the triangle is equal to one-half area of the parallelogram.

From the above result parallelogram ABNP and triangle MNP are on the same base PN and in between same parallel lines PN and AB.



$\therefore$ ar $\Delta MNP = \frac{1}{2}$ ar ABPN .....(i)

Similarly ar $\Delta PON = \frac{1}{2}$ ar PNCD .....(ii)

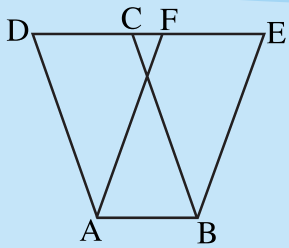
and Area of rhombus $= \frac{1}{2} \times d_1 d_2$ .....(iii)

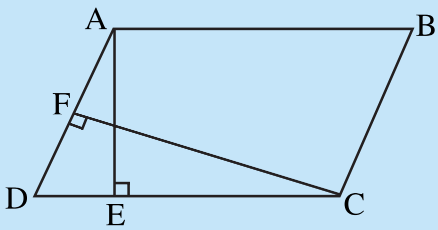
From (1), (ii) and (iii) we get

$$\begin{aligned} \text{ar}(MNOP) &= \text{ar}(\Delta MNP) + \text{ar}(\Delta PON) \\ &= \frac{1}{2}\text{ar}(ABNP) + \frac{1}{2}\text{ar}(PDCN) \\ &= \frac{1}{2}\text{ar(rhombus } ABCD) \\ &= \frac{1}{2}\left(\frac{1}{2} \times 12 \times 16\right) = 48 \text{ cm.}^2 \end{aligned}$$

Exercise - 11.2

1. The area of parallelogram ABCD is 36 cm^2. Calculate the height of parallelogram ABEF if AB = 4.2 cm.

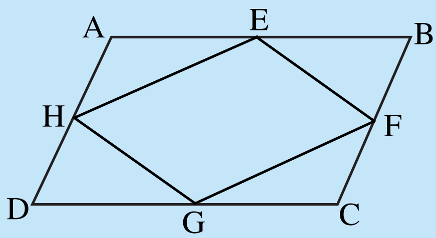




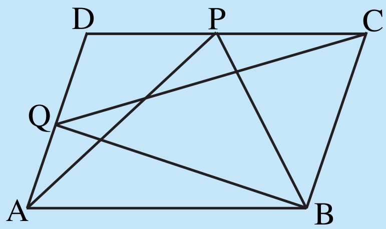
2. ABCD is a parallelogram. AE is perpendicular on DC and CF is perpendicular on AD.

   If AB = 10 cm, AE = 8 cm and CF = 12 cm. Find AD.

3. If E, F G and H are respectively the midpoints of the sides AB, BC, CD and AD of a parallelogram ABCD, show that ar(EFGH) $= \frac{1}{2}$ar(ABCD).



4. What type of quadrilateral do you get, if you join ΔAPM, ΔDPO, ΔOCN and ΔMNB in the example 3.



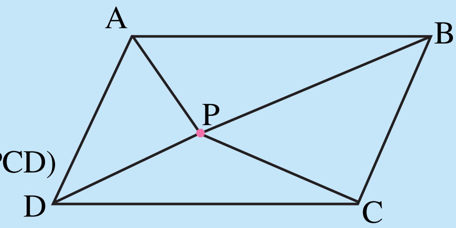
5. P and Q are any two points lying on the sides DC and AD respectively of a parallelogram ABCD show that ar(ΔAPB) = ar Δ(BQC).

6. P is a point in the interior of a parallelogram ABCD. Show that

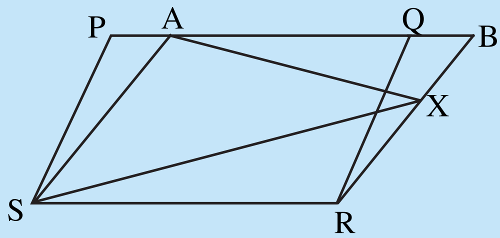
   (i) ar(ΔAPB) + ar(ΔPCD) $= \frac{1}{2}$ar(ABCD)

   (ii) ar(ΔAPD) + ar(ΔPBC) = ar(ΔAPB) + ar(ΔPCD)

   (Hint : Through P, draw a line parallel to AB)



7. Prove that the area of a trapezium is half the sum of the parallel sides multiplied by the distance between them.



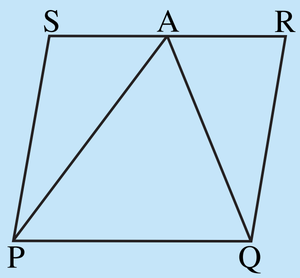
8. PQRS and ABRS are parallelograms and X is any point on the side BR. Show that

   (i) ar(PQRS) = ar(ABRS)

   (ii) ar(ΔAXS) $= \frac{1}{2}$ar(PQRS)

9. A farmer has a field in the form of a parallelogram PQRS as shown in the figure. He took the mid- point A on RS and joined it to points P and Q. In how many parts of field is divided? What are the shapes of these parts ?

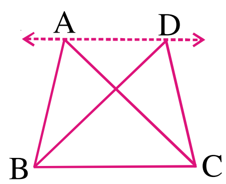
   The farmer wants to sow groundnuts which are equal to the sum of pulses and paddy. How should he sow? State reasons?



10. Prove that the area of a rhombus is equal to half of the product of the diagonals.

11.6 Triangles on the same base and between the same parallels

We are looking at figures that lie on the same base and between the same parallels. Let us have two triangles ABC and DBC on the same base BC and between the same parallels, AD and BC.



What can we say about the areas of such triangles? Clearly there can be infinite number of ways in which such pairs of triangle between the same parallels and on the same base can be drawn.

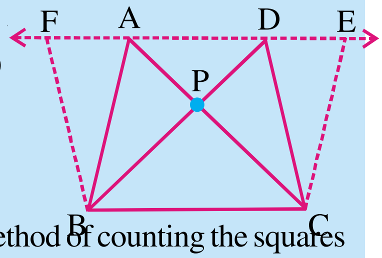
Let us perform an activity.

ACTIVITY

Draw pairs of triangles one the same base or (equal bases) and between the same parallels on the graph sheet as shown in the Figure.

Let ΔABC and ΔDBC be the two triangles lying on the same base BC and between parallels BC and AD. Extend AD on either sides and draw CE || AB and BF || CD. Parallelograms AECB and FDCB are on the same base BC and are between the same parallels BC and EF.

Thus ar(AECB) = ar(FDCB). (How ?)

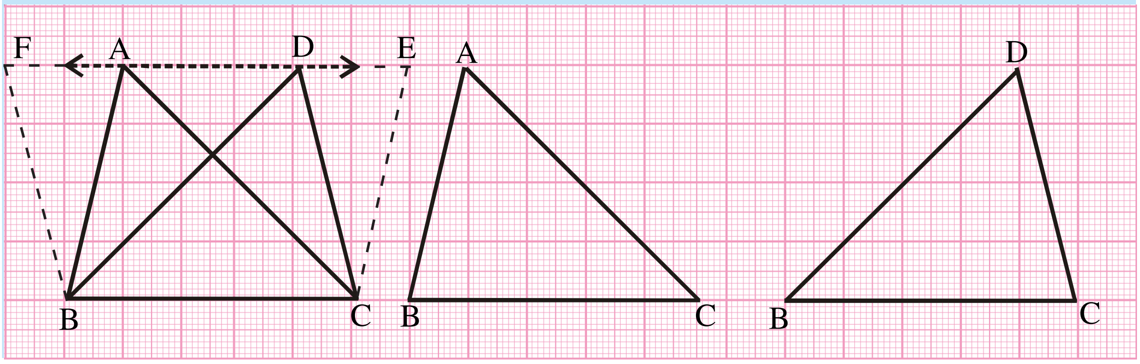


We can see ar(ΔABC) = $\frac{1}{2}$ ar(Parallelogram AECB) ...(i)

and ar(ΔDBC) = $\frac{1}{2}$ ar(Parallelogram FDCB) .... (ii)(

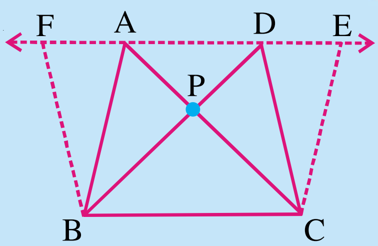
From (i) and (ii), we get ar(ΔABC) = ar(ΔDBC).

You can also find the areas of ΔABC and ΔDBC by the method of counting the squares in graph sheet as we have done in the earlier activity and check the areas are whether same.



THINK, DISCUSS AND WRITE

Draw two triangles ABC and DBC on the same base and between the same parallels as shown in the figure with P as the point of intersection of AC and BD. Draw CE || BA and BF || CD such that E and F lie on line AD.



Can you show ar(ΔPAB) = ar(ΔPDC)

Hint : These triangles are not congruent but have equal areas.

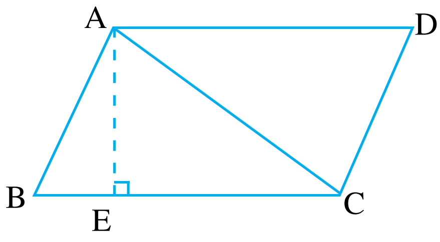
Corollary-1 : Show that the area of a triangle is half the product of its base (or any side) and the corresponding attitude (height).

Proof : Let ABC be a triangle. Draw AD || BC such that CD = BA.

Now ABCD is a parallelogram one of whose diagonals is AC.

We know $\Delta ABC \cong \Delta ACD$

So $ar\Delta ABC = ar\Delta ACD$ (Congruent triangles have equal area)



Therefore, $ar\Delta ABC = \frac{1}{2} ar(ABCD)$

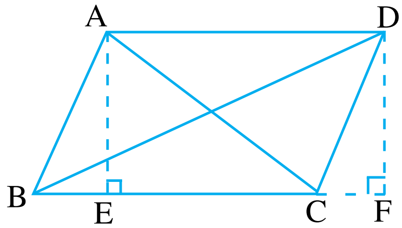
Draw $AE \perp BC$

We know $ar(ABCD) = BC \times AE$

We have $ar(\Delta ABC) = \frac{1}{2} ar(ABCD) = \frac{1}{2} \times BC \times AE$

So $ar\Delta ABC = \frac{1}{2} \times$ base BC $\times$ Corresponding attitude AE.

Theorem-11.2 : Two triangles having the same base (or equal bases) and equal areas will lie between the same parallels.



Observe the figure. Name the triangles lying on the same base BC. What are the heights of ΔABC and ΔDBC?

If two triangles have equal area and equal base, what will be their heights? Are A and D collinear?

Let us now take some examples to illustrate the use of the above results.

Example 4. Show that the median of a triangle divides it into two triangles of equal areas.

Solution : Let ABC be a triangle and Let AD be one of its medians.

In ΔABD and ΔADC the vertex is common and these bases BD and DC are equal.

Draw $AE \perp BC$.

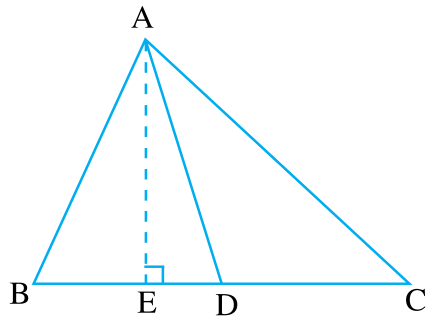
Now ar $(\Delta ABD) = \frac{1}{2} \times$ base $BD \times$ altitude of ΔADB

$= \frac{1}{2} \times BD \times AE$

$= \frac{1}{2} \times DC \times AE$ $\quad (\because BD = DC)$

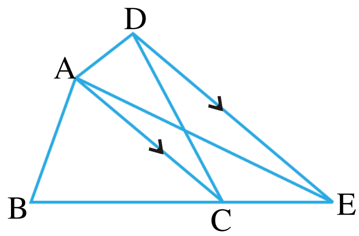
$= \frac{1}{2} \times$ base $DC \times$ altitude of ΔACD

$=$ ar ΔACD



Hence ar $(\Delta ABD) =$ ar (ΔACD)

Example-5. In the figure, ABCD is a quadrilateral. AC is the diagonal and $DE \parallel AC$ and also DE meets BC produced at E. Show that ar(ABCD) = ar(ΔABE).



Solution : ar(ABCD) = ar(ΔABC) + ar(ΔDAC)

ΔDAC and ΔEAC lie on the same base $\overline{AC}$

and between the parallels $DE \parallel AC$

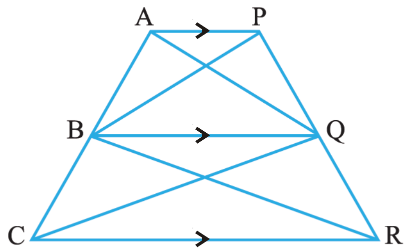
ar(ΔDAC) = ar(ΔEAC) (Why?)

Adding areas of same figures on both sides.

ar(ΔDAC) + ar(ΔABC) = ar(ΔEAC) + ar(ΔABC)

Hence ar(ABCD) = ar(ΔABE)

Example 6. In the figure, $AP \parallel BQ \parallel CR$. Prove that ar($\Delta$AQC) = ar($\Delta$PBR).



Solution : ΔABQ and Δ PBQ lie on the same base BQ and between the same parallels $AP \parallel BQ$.

ar(ΔABQ) = ar(ΔPBQ) ...(1)

Similarly

ar (ΔCQB) = ar (ΔRQB) (same base BQ and $BQ \parallel CR$) ...(2)

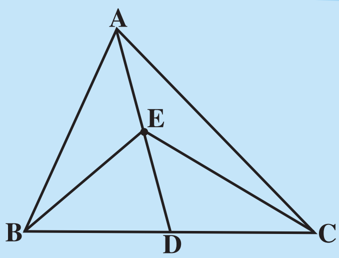
Adding results (1) and (2)

ar(ΔABQ) + ar(ΔCQB) = ar(ΔPBQ) + ar(ΔRQB)

Hence ar ΔAQC = ar ΔPBR

EXERCISE - 11.3

1. In a triangle ABC (see figure), E is the midpoint of median AD, show that



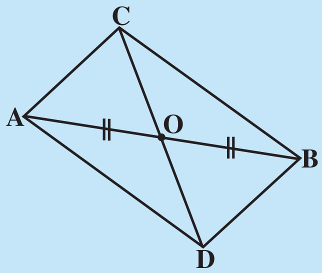
   (i) ar ΔABE = ar ΔACE

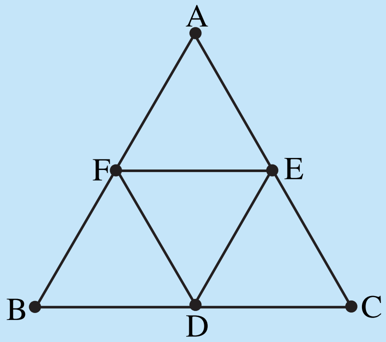
   (ii) arΔABE = $\frac{1}{4}$ar(ΔABC)

2. Show that the diagonals of a parallelogram divide it into four triangles of equal area.

3. In the figure, ΔABC and ΔABD are two triangles on the same base AB. If line segment CD is bisected by $\overline{AB}$ at O, show that

ar (ΔABC) = ar (ΔABD).





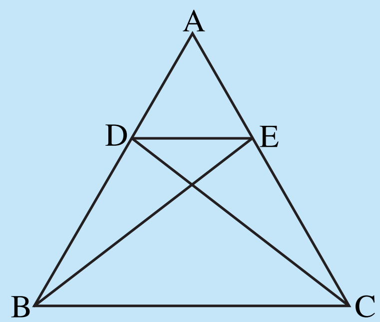
4. In the figure, ΔABC, D, E, F are the midpoints of sides BC, CA and AB respectively. Show that

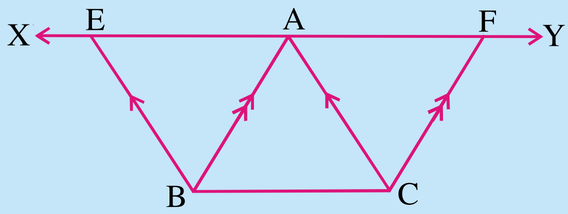
(i) BDEF is a parallelogram

(ii) $ar(\Delta DEF) = \frac{1}{4} ar(\Delta ABC)$

(iii) $ar(BDEF) = \frac{1}{2} ar(\Delta ABC)$

5. In the figure D, E are points on the sides AB and AC respectively of ΔABC such that $ar(\Delta DBC) = ar(\Delta EBC)$. Prove that $DE \parallel BC$.

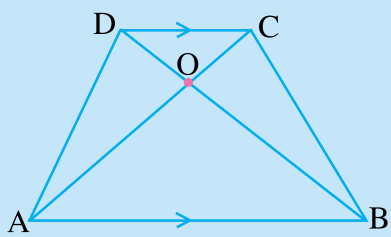


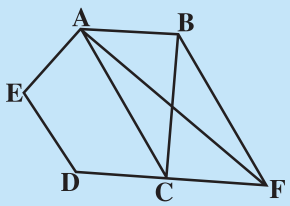


6. In the figure, XY is a line parallel to BC is drawn through A. If $BE \parallel CA$ and $CF \parallel BA$ are drawn to meet XY at E and F respectively. Show that $ar(\Delta ABE) = ar(\Delta ACF)$.

7. In the figure, diagonals AC and BD of a trapezium ABCD with $AB \parallel DC$ intersect each other at O. Prove that

$ar(\Delta AOD) = ar(\Delta BOC)$.

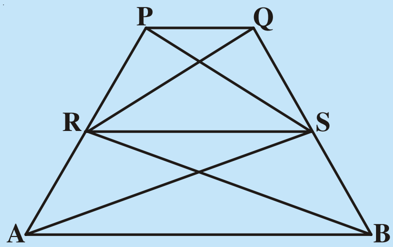




8. In the figure, ABCDE is a pentagon. A line through B parallel to AC meets DC produced at F. Show that

(i) ar (ΔACB) = ar (ΔACF)

(ii) ar (AEDF) = ar (ABCDE)



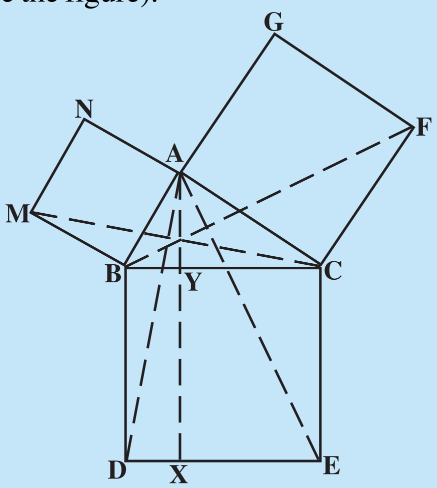
9. In the figure, if ar ΔRAS = ar ΔRBS and [ar (ΔQRB) = ar(ΔPAS) then show that both the quadrilaterals PQSR and RSBA are trapeziums.

10. A villager Ramayya has a plot of land in the shape of a quadrilateral. The grampanchayat of the village decided to take over some portion of his plot from one of the corners to construct a school. Ramayya agrees to the above proposal with the condition that he should be given equal amount of land in exchange of his land adjoining his plot so as to form a triangular plot. Explain how this proposal will be implemented. (Draw a rough sketch of plot).

THINK, DISCUSS AND WRITE

ABC is a right triangle right angled at A. BCED, ACFG and ABMN are squares on the sides BC, CA and AB respectively. Line segments AX $\perp$ DE meets BC at Y. and DE at X. Join AD, AE also BF and CM (See the figure).

Show that



(i) ΔMBC $\cong$ ΔABD

(ii) ar(BYXD) = 2ar (ΔMBC)

(iii) ar(BYXD) = ar(ABMN)

(iv) ΔFCB $\cong$ ΔACE

(v) ar(CYXE) = 2 ar(FCB)

(vi) ar(CYXE) = ar (ACFG)

(vii) ar(BCED) = ar(ABMN) + ar(ACFG)

Can you write the result (vii) in words ? This is a famous theorem of Pythagoras. You shall learn a simpler proof in this theorem in class X.

What we have discussed

In this chapter we have discussed the following.

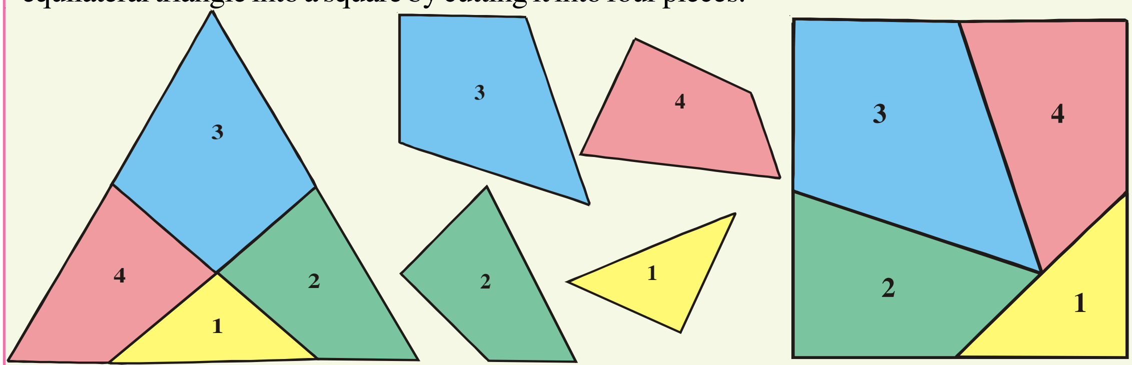
1. Area of a figure is a number (in some unit) associated with the part of the plane enclosed by that figure.
2. Two congruent figures have equal areas but the converse need not be true.
3. If X is a planer region formed by two non-overlapping planer regions of figures P and Q, then ar(X) = ar(P) + ar(Q)
4. Two figures are said to be on the same base and between the same parallels, if they have a common base (side) and the vertices (on the vertex) opposite to the common base of each figure lie on a line parallel to the base.
5. Parallelograms on the same base (or equal bases) and between the same parallels are equal in area.
6. Area of a parallelogram is the product of its base and the corresponding altitude.
7. Parallelogram on the same base (or equal bases) and having equal areas lie between the same parallels.
8. If a parallelogram and a triangle are on the same base and between the same parallels, then area of the triangle is half the area of the parallelogram.
9. Triangles on the same base (or equal bases) and between the same parallels are equal in area.
10. Triangles on the same base (or equal bases) and having equal areas lie between the same parallels.

Do You Know?

A Puzzle (Areas)

German mathematician David Hilbert (1862-1943) first proved that any polygon can be transformed into any other polygon of equal area by cutting it into a finite number of pieces.

Let us see how an English puzzlist, Henry Ernest Dudency (1847 - 1930) transforms an equilateral triangle into a square by cutting it into four pieces.



Try to make some more puzzles using his ideas and enjoy.

12 Circles

12.1 Introduction

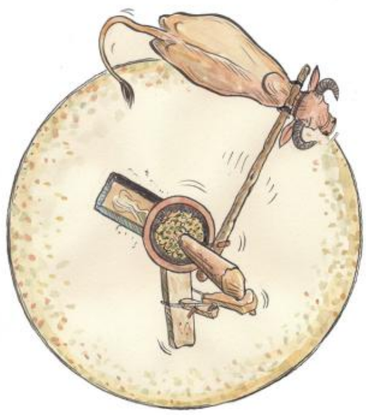
We come across many round shaped objects in our surroundings such as coins, bangles, clocks, wheels, buttons etc. All these are circular in shape.

You might have drawn an outline along the edges of a coin, a bangle, a button in your childhood to form a circle.

So, can you tell, the difference between the circular objects and the circles you have drawn with the help of these objects?

All the circular objects we have observed above have thickness and are 3-dimensional objects, where as, a circle is a 2-dimensional figure, with no thickness.

Let us take another example of a circle. You might have seen the oil press called oil mill (Spanish wheel - in Telugu known as ganuga). In the figure, a bullock is tied to fulcrum fixed at a point. Can you identify the shape of the path in which the bullock is moving? It is circular in shape.

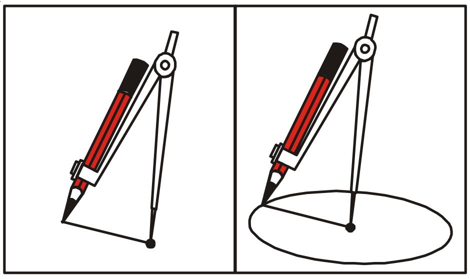


A line along the boundary made by the bullock is a circle. The oil press is attached to the ground at a fixed point, which is the centre of the circle. The length of the fulcrum with reference to the circle is radius of the circle. Think of some other examples from your daily life about circles.

In this chapter we will study circles, related terms and properties of the circle. Before this, you must know how to draw a circle with the help of a compass.

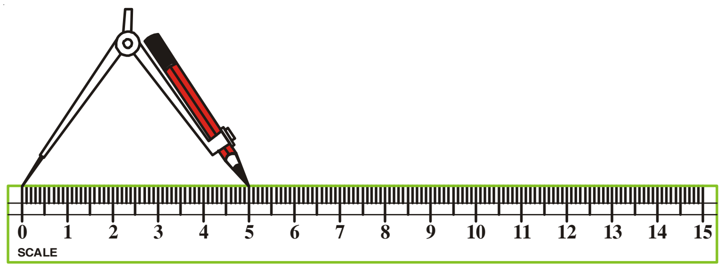
Let us do this.

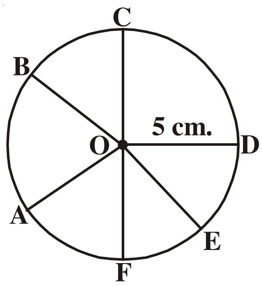
Insert a pencil in the pencil holder of the compass and tighten the screw. Mark a point 'O' on the drawing paper. Fix the sharp point of the compass on 'O'. Keeping the point of the compass firmly move the pencil round on the paper to draw the circle as shown in the figure.



If we need to draw a circle of given radius, we do this with the help of a scale.

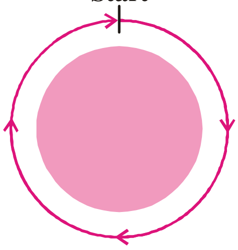
Adjust the distance between the sharp point of the compass and tip of the pencil equal to the length of the given radius, mark a point 'O'(radius of the circle in the figure is 5 cm.) and draw circle as described above.





Mark any 6 points A, B, C, D E and F on the circle. You can see that the length of each line segment OA, OB, OC, OD, OE and OF is 5 cm., which is equal to the given radius. Mark some other points on the circle and measure their distances from the point 'O'. What have you observed? We can say that a circle is a collection of all the points in a plane which are at a fixed distance from a fixed point on the plane.

The fixed point 'O' is called the centre of the circle and the fixed distance OA, is called the radius of the circle.



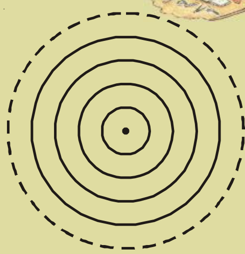
In a circular park Narsimha started walking from a point around the park and completed one round. What do you call the distance covered by Narsimha? It is the total length of the boundary of the circular park, and is called the circumference of the park.

So, the complete length of a circle is called its circumference.

ACTIVITY

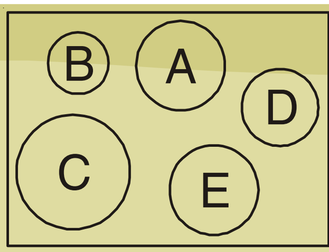
Let us now do the following activity. Mark a point on a sheet of paper. Taking this point as centre draw a circle with any radius. Now increase or decrease the radius and again draw some more circles with the same centre. What do you call the circles obtained in this activity?

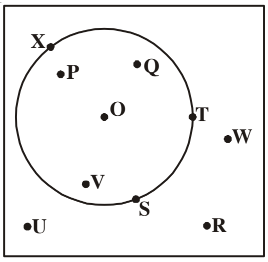
Circles having same centre with different radii are called **concentric circles**.



Do This

1. In the figure, which circles are congruent to the circle A?
2. What measure of the circles make them congruent?





A circle divides the plane on which it lies into three parts. They are (i) inside the circle, which is also called interior of the circle; (ii) on the circle, this is also called the circumference and (iii) outside the circle, which is also called the exterior of the circle. From the above figure, find the points which are inside, outside and on the circle.

The circle and its interior make up the circular region.

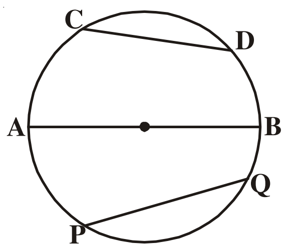
Activity

Take a thin circular sheet and fold it to half and open. Again fold it along any other half and open. Repeat this activity for several times. Finally when you open it, what do you observe?

You observe that all creases (traces of the folds) are intersecting at one point. Do you remember what do we call this point? This is the centre of the circle.

Measure the length of each crease of a circle with a divider. What do you notice ? They are all equal and each crease is dividing the circle into two equal halves. That crease is called diameter of circle. Diameter of a circle is twice its radius. A line segment joining any two points on the circle that passes through the centre is called the **diameter**.

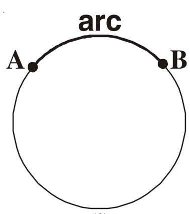
In the above activity if we fold the paper in any manner not only in half, we see that creases joining two points on circle. These creases are called chords of the circle.



So, a line segment joining any two points on the circle is called a **chord.**

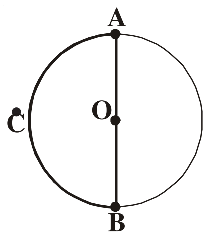
What do you call the longest chord? Is it passes through the centre?

See in the figure, CD, AB and PQ are chords of the circle.

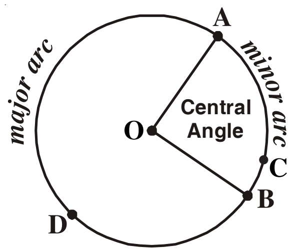


(i)

In the fig.(i), two points A and B are on the circle and they are dividing the circumference of the circle into two parts. The part of the circle between any two points on it is called an arc. In the fig.(i) AB is called an 'arc' and it is denoted by $\overset{\frown}{AB}$. If the end points of an arc become the end points of a diameter then such an arc is called a semicircular arc or a semicircle. In the fig.(ii) $\overset{\frown}{ACB}$ is a semicircle

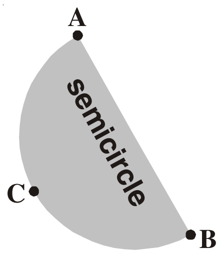


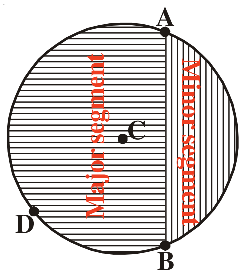
(ii)



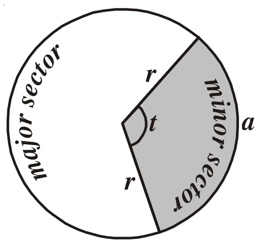
(iii)

If the arc is smaller than a semicircle, then the arc is called a minor arc and if the arc is longer than a semicircle, then the arc is called a major arc. In the fig.(iii) $\overset{\frown}{ACB}$ is a minor arc and $\overset{\frown}{ADB}$ is a major arc.





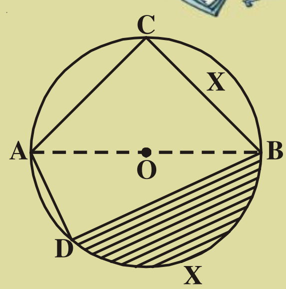
If we join the end points of an arc by a chord, the chord divides the circle into two parts. The region between the chord and the minor arc is called the **minor segment** and the region between the chord and the major arc is called the **major segment**. If the chord happens to be a diameter, then the diameter divides the circle into two equal segments.



The region enclosed by an arc and the two radii joining the centre to the end points of an arc is called a ***sector***. One is minor sector and another is major sector (see adjacent figure).

EXERCISE -12.1

1. Name the following parts from the adjacent figure where 'O' is the centre of the circle.



(i) $\overline{AO}$ (ii) $\overline{AB}$ (iii) $\overset{\frown}{BC}$

(iv) $\overline{AC}$ (v) $\overset{\frown}{DCB}$ (vi) $\overset{\frown}{ACB}$

(vii) $\overline{AD}$ (viii) shaded region

2. State true or false.

i. A circle divides the plane on which it lies into three parts. ( )

ii. The region enclosed by a chord and the minor arc is minor segment. ( )

iii. The region enclosed by a chord and the major arc is major segment. ( )

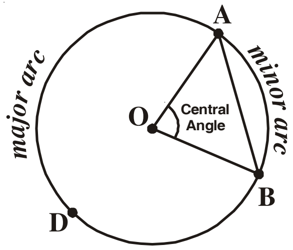
iv. A diameter divides the circle into two unequal parts. ( )

v. A sector is the area enclosed by two radii and a chord ( )

vi. The longest of all chords of a circle is called a diameter. ( )

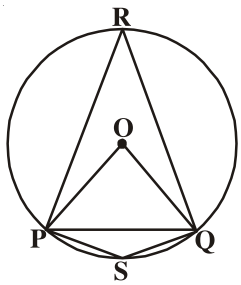
vii. The mid point of any diameter of a circle is the centre. ( )

12.2 ANGLE SUBTENDED BY A CHORD AT A POINT ON THE CIRCLE



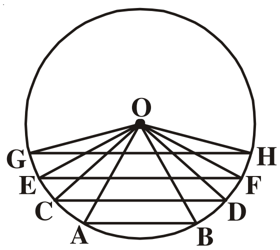
Let A, B be any two points on a circle with centre 'O'. Join AO and BO. Angle is made at centre 'O' by $\overline{AO}$, $\overline{BO}$ i.e. $\angle AOB$ is called the angle subtended by the chord $\overline{AB}$ at the centre 'O'.

What do you call the angles $\angle POQ$, $\angle PSQ$ and $\angle PRQ$ in the figure?



i. $\angle POQ$ is the angle subtended by the chord PQ at the centre 'O'

ii. $\angle PSQ$ and $\angle PRQ$ are respectively the angles subtended by the chord PQ at point S and point R on the minor and major arc.



In the figure, O is the centre of the circle and AB, CD, EF and GH are the chords of the circle.

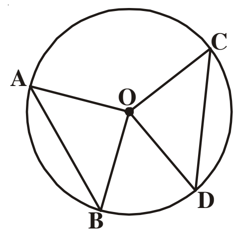
We can observe from the figure that GH > EF > CD > AB.

Now what do you say about the angles subtended by these chords at the centre?

After observing the angles, you will find that the angles subtended by the chords at the centre of the circle increases with increase in the length of chords.

So, now imagine what will happen to the angle subtended at the centre of the circle, if we take two equal chords of a circle?

Construct a circle with centre 'O' and draw equal chords AB and CD using the compass and ruler.



Join the centre 'O' with A, B and with C, D. Now measure the angles $\angle AOB$ and $\angle COD$. Are they equal to each other? Draw two or more equal chords of a circle and measure the angles subtended by them at the centre.

You will find that the angles subtended by them at the centre are equal.

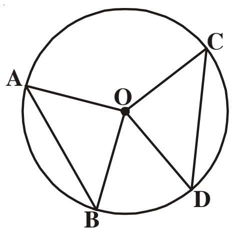
Let us try to prove this fact.

Theorem-12.1 : Equal chords of a circle subtend equal angles at the centre.

Given : Let 'O' be the centre of the circle. $\overline{AB}$ and $\overline{CD}$ are two equal chords and $\angle AOB$ and $\angle COD$ are the angles subtended by the chords at the centre.

R.T.P. : $\angle AOB \cong \angle COD$

Construction : Join the centre to the end points of each chord and you get two triangles ΔAOB and ΔCOD.



Proof: In triangles AOB and COD

$AB = CD$ (given)

$OA = OC$ (radii of same circle)

$OB = OD$ (radii of same circle)

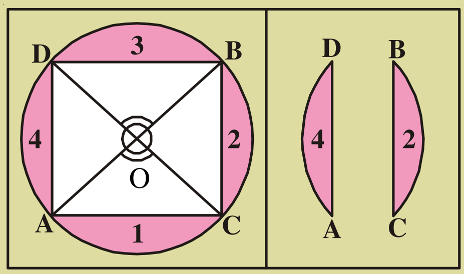
Therefore $\Delta AOB \cong \Delta COD$ (SSS rule)

Thus $\angle AOB \cong \angle COD$ (corresponding parts of congruent triangles)

In the above theorem, if in a circle, two chords subtend equal angles at the centre, what can you say about the chords? Let us investigate this by the following activity.

ACTIVITY

Take a circular paper. Fold it along any diameter such that the two edges coincide with each other. Now open it and again fold it into half along another diameter. On opening, we find two diameters meet at the centre 'O'. There forms two pairs of vertically opposite angles which are equal. Name the end points of the diameter as A, B, C and D



Draw the chords $\overline{AC}$, $\overline{BC}$, $\overline{BD}$ and $\overline{AD}$.

Now take cut-out of the four segments namely 1, 2, 3 and 4

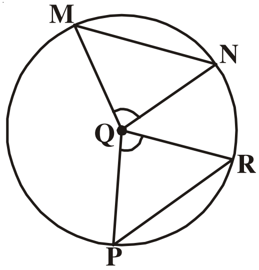
If you place these segments pair wise one above the other the edges of the pairs (1,3) and (2,4) coincide with each other.

Is $\overline{AD} = \overline{BC}$ and $\overline{AC} = \overline{BD}$?

Though you have seen it in this particular case, try it out for other equal angles too. The chords will all turn out to be equal because of the following theorem.

Can you state converse of the above theorem (12.1)?

Theorem-12.2 : If the angle subtended by the chords of a circle at the centre are equal, then the chords are equal.



This is the converse of the theorem 12.1.

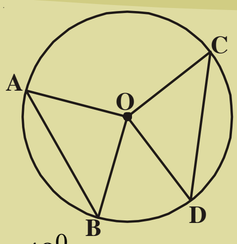
Note that in adjacent figure $\angle PQR = \angle MQN$, then

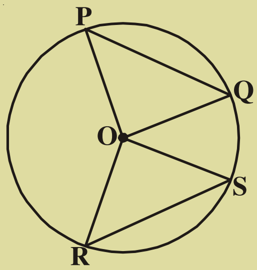
$\Delta PQR \cong \Delta MQN$ (Why?)

Is $PR = MN$? (Verify)

EXERCISE - 12.2

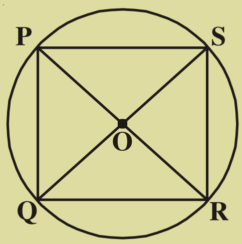
1. In the figure, if $AB = CD$ and $\angle AOB = 90^0$ find $\angle COD$





2. In the figure, $PQ = RS$ and $\angle ORS = 48^0$.

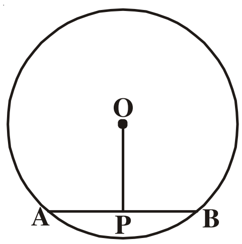
   Find $\angle OPQ$ and $\angle ROS$.



3. In the figure PR and QS are two diameters. Is $PQ = RS$?

12.3 PERPENDICULAR FROM THE CENTRE TO A CHORD

- Construct a circle with centre O. Draw a chord $\overline{AB}$ and a perpendicular to the chord $\overline{AB}$ from the centre 'O'.
- Let the point of intersection of the perpendicular on $\overline{AB}$ be P.
- After measuring PA and PB, we will find $PA = PB$.



Theorem-12.3 : The perpendicular from the centre of a circle to a chord bisects the chord.

Write a proof by yourself by joining O to A and B and prove that $\Delta OPA \cong \Delta OPB$.

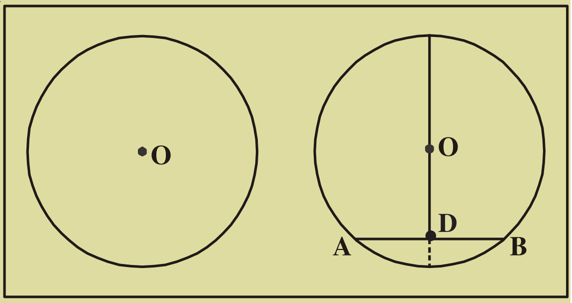
What is the converse of the theorem 12.3?

"If a line drawn from the centre of a circle bisects the chord then the line is perpendicular to that chord"

ACTIVITY

Take a circle shaped paper and mark centre 'O'

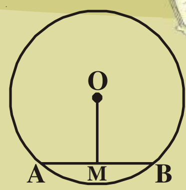
Fold it into two unequal parts and open it. Let the crease represent a chord AB, and then make a fold such that 'A' coincides with B. Mark the point of intersection of the two folds as D. Is AD = DB? ODA = ? 2∠ODB = ? Measure the angles between the creases. They are right angles. So, we can make a hypothesis "the line drawn through the centre of a circle to bisect a chord is perpendicular to the chord".



TRY THIS

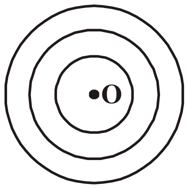
In a circle with centre 'O'. $\overline{AB}$ is a chord and 'M' is its midpoint. Now prove that $\overline{OM}$ is perpendicular to AB.

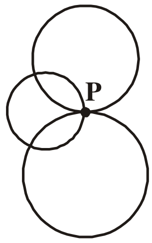
(**Hint** : Join OA and OB consider triangles OAM and OBM)



12.3.1 The three points that describe a circle

Let 'O' be a point on a plane. How many circles we can draw with centre 'O'? As many circles as we wish. We have already learnt that these circles are called concentric circles. If 'P' is a point other than the centre of the circle, then also we can draw many circles through P.

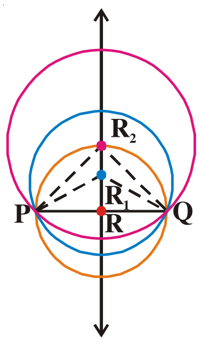


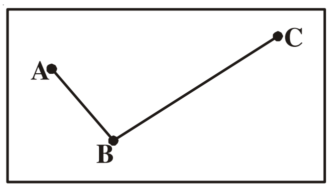


Suppose that there are two distinct points P and Q

How many circles can be drawn passing through given two points? We see that we can draw many circles passing through P and Q.

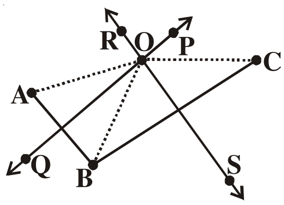
Let us join P and Q, draw the perpendicular bisector to PQ. Take any three points R, R_1 and R_2 on the perpendicular bisector and draw circles with centre R, R_1, R_2 and radii RP, R_1P and R_2P respectively. Does these circles also passes through Q (Why?)





If three non-collinear points are given, then how many circles can be drawn through them? Let us examine it. Take any three non-collinear points A, B, C and join AB and BC.

Draw $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ the perpendicular bisectors to $\overline{AB}$ and $\overline{BC}$. respectively. Both of them intersect at a point 'O'(since two lines cannot have more than one point in common)



Now O lies on the perpendicular bisector of $\overline{AB}$, so OA = OB. .....(i)

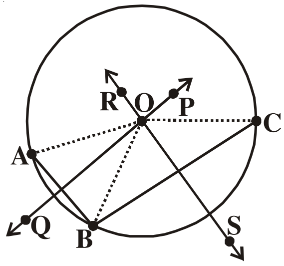
As every point on $\overleftrightarrow{PQ}$ is at equidistant from A and B

Also, 'O' lies on the perpendicular bisectors of $\overline{BC}$

Therefore OB = OC ..... (ii)

From equation (i) and (ii)

We can say that OA = OB = OC (transitive law)



Therefore, 'O' is the only point which is equidistant from the points A, B and C so if we draw a circle with centre O and radius OA, it will also pass through B and C i.e. we have only one circle that passes through A, B and C.

The hypothesis based on above observation is "there is one and only one circle that passes through three non-collinear points"

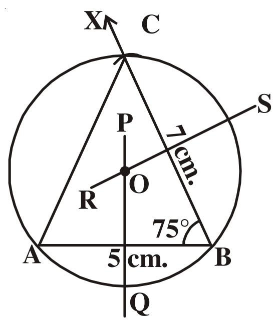
Note : If we join AC, the triangle ABC is formed. All its vertices lie on the circle. This circle is called circum circle of the triangle, the centre of the circle 'O' is circumcentre and the radius OA or OB or OC i.e. is circumradius.

TRY THIS

If three points are collinear, how many circles can be drawn through these points? Now, try to draw a circle passing through these three points.

Example-1. Construct a circumcircle of the triangle ABC where AB = 5cm; $\angle B = 75^0$ and BC = 7cm

Solution : Draw a line segment AB= 5 cm. Draw BX at B such that $\angle B = 75^0$. Draw an arc of radius 7cm with centre B to cut $\overrightarrow{BX}$ at C join CA to form ΔABC, Draw perpendicular bisectors $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ to $\overline{AB}$ and $\overline{BC}$ respectively. $\overleftrightarrow{PQ}$, $\overleftrightarrow{RS}$ intersect at 'O'. Keeping 'O' as a centre, draw a circle with OA as radius. The circle also passes through B and C and this is the required circumcircle.

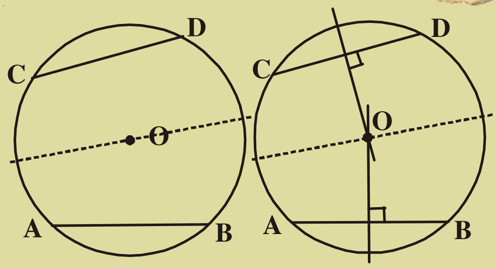


12.3.2 Chords and their distance from the centre of the circle

A circle can have infinite chords. Suppose we make many chords of equal length in a circle, then what would be the distance of these chords of equal length from the centre? Let us examine it through this activity.

ACTIVITY

Draw a big circle on a paper and take a cut-out of it. Mark its centre as 'O'. Fold it in half. Now make another fold near semi-circular edge. Now unfold it. You will get two congruent folds of chords. Name them as AB and CD. Now make perpendicular folds passing through centre 'O' for them. Using divider compare the perpendicular distances of these chords from the centre.

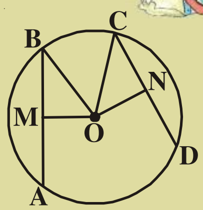


Repeat the above activity by folding congruent chords. State your observations as a hypothesis.

"The congruent chords in a circle are at equal distance from the centre of the circle"

TRY THIS

In the figure, O is the centre of the circle and AB = CD. OM is perpendicular on $\overline{AB}$ and $\overline{ON}$ is perpendicular on $\overline{CD}$. Then prove that OM = ON.



As the above hypothesis has been proved logically, it becomes a theorem 'chords of equal length are at equal distance from the centre of the circle.'

Example-2. In the figure, O is the centre of the circle. Find the length of CD, if AB = 5 cm.

Solution : In Δ AOB and Δ COD,

OA = OC (why?)

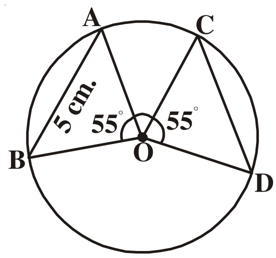
OB = OD (why?)

$\angle AOB = \angle COD$

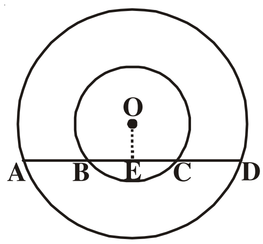
$\therefore$ $\Delta AOB \cong \Delta COD$

$\because$ AB = CD (Congruent parts of congruent triangles)

$\therefore$ AB = 5 cm. then CD = 5 cm.



Example-3. In the adjacent figure, there are two concentric circles with centre 'O'. Chord AD of the bigger circle intersects the smaller circle at B and C. Show that AB = CD.



Given : In two concentric circles with centre 'O'. $\overline{AD}$ is the chord of the bigger circle. $\overline{AD}$ intersect the smaller circle at B and C.

R.T.P. : AB = CD

Construction : Draw $\overline{OE}$ perpendicular to $\overline{AD}$

Proof : AD is the chord of the bigger circle with centre 'O' and $\overline{OE}$ is perpendicular to $\overline{AD}$.

$\because$ $\overline{OE}$ bisects $\overline{AD}$ (The perpendicular from the centre of a circle to a chord bisect it)

$\therefore$ AE = ED ..... (i)

$\overline{BC}$ is the chord of the smaller circle with centre 'O' and $\overline{OE}$ is perpendicular to AD.

$\because$ $\overline{OE}$ bisects $\overline{BC}$ (from the same theorem)

$\therefore$ BE = CE ..... (ii)

Subtracting the equation (ii) from (i), we get

AE - BE = ED - EC

AB = CD

Exercise - 12.3

1. Draw the following triangles and construct circumcircles for them.

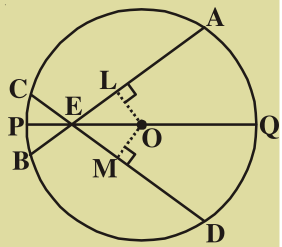
   (i) In Δ ABC, AB = 6cm, BC = 7cm and $\angle A = 60^o$

   (ii) In Δ PQR, PQ = 5cm, QR = 6cm and RP = 8.2cm

   (iii) In Δ XYZ, XY = 4.8cm, $\angle X = 60^o$ and $\angle Y = 70^o$

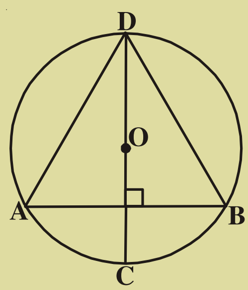
2. Draw two circles passing through A, B where AB = 5.4cm

3. If two circles intersect at two points, then prove that their centres lie on the perpendicular bisector of the common chord.

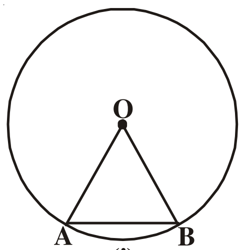


4. If two intersecting chords of a circle make equal angles with diameter passing through their point of intersection, prove that the chords are equal.

5. In the adjacent figure, AB is a chord of circle with centre O. CD is the diameter perpendicualr to AB. Show that AD = BD.



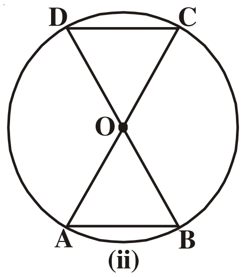
12.4 ANGLE SUBTENDED BY AN ARC OF A CIRCLE



(i)

In the fig.(i), $\overline{AB}$ is a chord and $\overset{\frown}{AB}$ is an arc (minor arc). The end points of the chord and arc are the same i.e. A and B.

Therefore angle subtended by the chord at the centre 'O' is the same as the angle subtended by the arc at the centre 'O'.



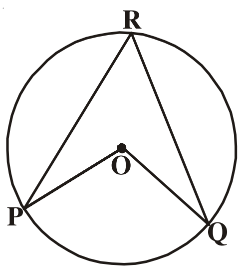
(ii)

In fig.(ii) $\overline{AB}$ and $\overline{CD}$ are two chords of a circle with centre 'O'. If AB = CD, then $\angle AOB = \angle COD$

Therefore we can say that the angle subtended by an arc $\overset{\frown}{AB}$ is equal to the angle subtended by the arc $\overset{\frown}{CD}$ at the centre 'O'. (Prove $\Delta AOB \cong \Delta DOC$)

From the above observations we can conclude that *"Arcs of equal length subtend equal angles at the centre"*

12.4.1 Angle subtended by an arc at a point on remaining part of circle

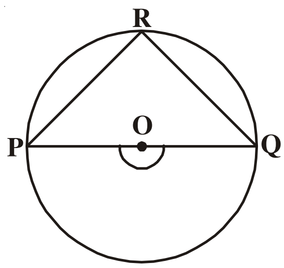


(i)

Consider the circle with centre 'O'.

Let $\overset{\frown}{PQ}$ in fig. (i) the minor arc, in fig. (ii) semicircle and in fig. (iii) major arc.

Take any point R on the circumference. Join R with P and Q.

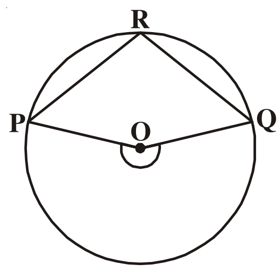


(ii)

$\angle PRQ$ is the angle subtended by the arc PQ at the point R on the circle while $\angle POQ$ is subtended at the centre.

Complete the following table for the given figures.

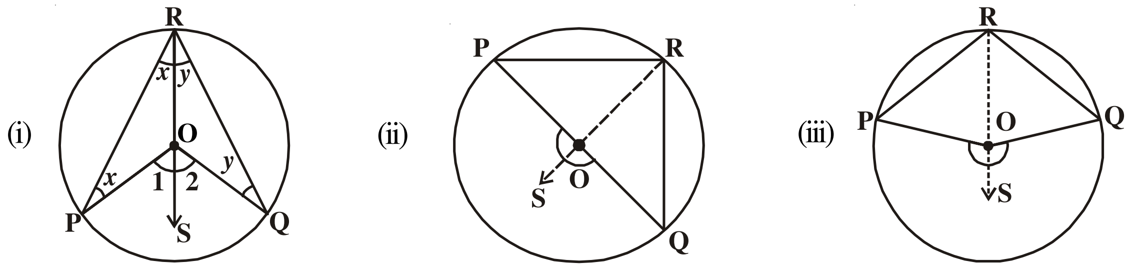
Angle	Fig. (i)	Fig. (ii)	Fig. (iii)
$\angle PRQ$			
$\angle POQ$			



(iii)

Similarly draw some circles and subtended angles on the circumference and centre of the circle by their arcs. What do you notice? Can you make a conjecture about the angle made by an arc at the centre and a point on the circle? So from the above observations, we can say that *"The angle subtended by an arc at the centre 'O' is twice the angle subtended by it on the remaining arc of the circle"*.

Theorem: The angle subtended by an arc at the centre of a circle is double the angle subtended by it at any point on the remaining circle.



Given : Let O be the centre of the circle.

$\overset{\frown}{PQ}$ is an arc subtending $\angle POQ$ at the centre.

Let R be a point on the remaining part of the circle (not on $\overset{\frown}{PQ}$)

Proof: Here we have three different cases in which (i) $\overset{\frown}{PQ}$ is minor arc, (ii) $\overset{\frown}{PQ}$ is semi-circle and (iii) $\overset{\frown}{PQ}$ is a major arc

Let us begin by joining the point R with the centre 'O' and extend it to a point S (in all cases)

For all the cases in Δ ROP

RO = OP (radii of the same circle)

Therefore $\angle ORP = \angle OPR$ (Angles opposite to equal sides of an isosceles triangle are equal).

$\angle POS$ is an exterior angle of Δ ROP (construction)

$\angle POS = \angle ORP + \angle OPR$ or $2\angle ORP$ ..... (1)

($\because$ exterior angle = sum of opp. interior angles)

Similarly for ΔROQ

$\angle SOQ = \angle ORQ + \angle OQR$ or $2\angle ORQ$ ... (2)

($\because$ exterior angle is equal to sum of the opposite interior angles)

From (1) and (2)

$\angle POS + \angle SOQ = 2(\angle ORP + \angle ORQ)$

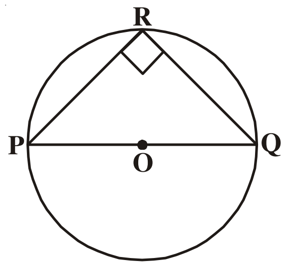
This is same as $\angle POQ = 2\angle QRP$ ..... (3)

> For convenience
> Let $\angle ORP = \angle OPR = x$
> $\angle POS = \angle 1$
> $\angle 1 = x + x = 2x$
> Let $\angle ORQ = \angle OQR = y$
> $\angle SOQ = \angle 2$
> $\angle 2 = y + y = 2y$
> Now $\angle POQ = \angle 1 + \angle 2 = 2x + 2y$
> $= 2(x+y) = 2(\angle PRO + \angle ORQ)$
> (i.e.) $\angle POQ = 2\angle PRQ$

Hence the theorem is "the angle subtended by an arc at the centre is twice the angle subtended by it at any point on the remaining part of the circle.

Example-4. Let 'O' be the centre of a circle, PQ is a diameter, then prove that $\angle PRQ = 90^o$

(OR) Prove that angle in a semi-circle is right angle.



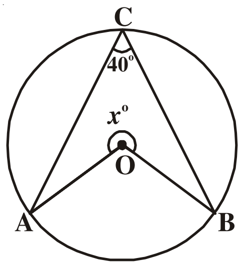
Solution : It is given that PQ is a diameter and 'O' is the centre of the circle.

$\therefore \angle POQ = 180^o$ [Angle on a straight line]

and $\angle POQ = 2\angle PRQ$ [Angle subtended by an arc at the centre is twice the angle subtended by it at any other point on circle]

$$\therefore \angle PRQ = \frac{180^o}{2} = 90^o$$

Example-5. Find the value of x° in the adjacent figure



Solution : Given $\angle ACB = 40^\circ$

By the theorem angle made by the arc AB at the centre

$\angle AOB = 2\ \angle ACB = 2 \times 40^\circ = 80^\circ$

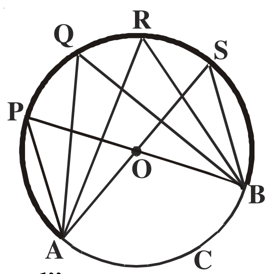
$\because \quad x^\circ + \angle AOB = 360^\circ$

Therefore $x^\circ = 360^\circ - 80^\circ = 280^\circ$

12.4.2 Angles in the same segment

Let us now discuss the measures of angles made by an arc in the same segment of a circle.

Consider a circle with centre 'O' and a minor arc AB (See figure). Let P, Q, R and S be points on the major arc AB i.e. on the remaining part of the circle. Now join the end points of the arc AB with points P, Q, R and S to form angles $\angle APB$, $\angle AQB$, $\angle ARB$ and $\angle ASB$.



$\because \quad \angle AOB = 2\angle APB$ (why?)

$\angle AOB = 2\angle AQB$ (why?)

$\angle AOB = 2\angle ARB$ (why?)

$\angle AOB = 2\angle ASB$ (why?)

Therefore $\angle APB = \angle AQB = \angle ARB = \angle ASB$

Observe that "angles subtended by an arc in the same segment are equal".

Note : In the above discussion we have seen that the point P, Q, R, S and A, B lie on the same circle. What do you call them? "Points lying on the same circle are called concyclic".

The converse of the above theorem can be stated as follows-

Theorem-12.4 : If a line segment joining two points, subtends equal angles at two other points lying on the same side of the line then these, the four points lie on a circle (i.e. they are concyclic)

Given : Two angles $\angle ACB$ and $\angle ADB$ are on the same side of a line segment $\overline{AB}$ joining two points A and B are equal.

R.T.P : A, B, C and D are concyclic (i.e.) they lie on the same circle.

Construction : Draw a circle passing through the three non colinear point A, B and C.

Proof: Suppose the point 'D' does not lie on the Circle.

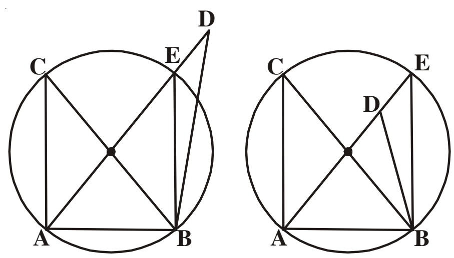
Then there may be other point 'E' such that it will intersect AD (or extension of AD)

If points A, B, C and E lie on the circle then

$\angle ACB = \angle AEB$ (Why?)

But it is given that $\angle ACB = \angle ADB$.

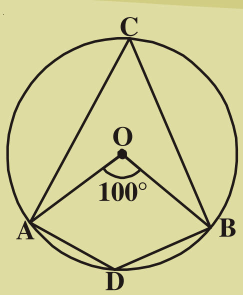
Therefore $\angle AEB = \angle ADB$

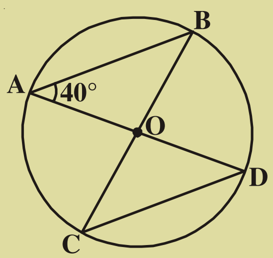


This is not possible unless E coincides with D (Why?)

EXERCISE – 12.4

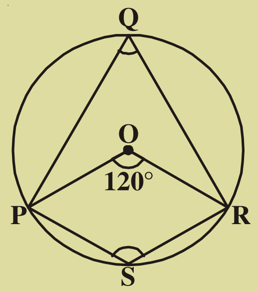
1. In the figure, 'O' is the centre of the circle. $\angle AOB = 100^\circ$ find $\angle ADB$.



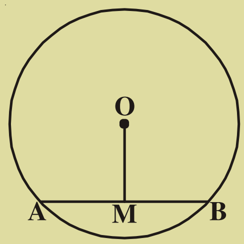


2. In the figure, $\angle BAD = 40^\circ$ then find $\angle BCD$.

3. In the figure, O is the centre of the circle and $\angle POR = 120^\circ$. Find $\angle PQR$ and $\angle PSR$

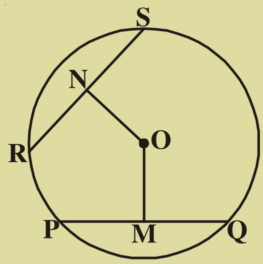


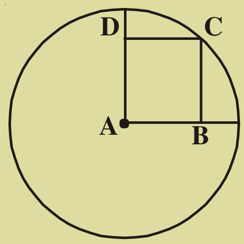
4. If a parallelogram is cyclic, then it is a rectangle. Justify.



5. In the figure, 'O' is the centre of the circle. $OM = 3cm$ and $AB = 8cm$. Find the radius of the circle

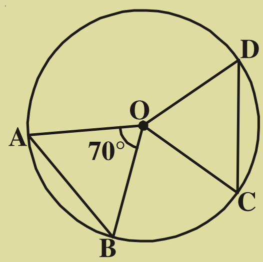
6. In the figure, 'O' is the centre of the circle and OM, ON are the perpendiculars from the centre to the chords PQ and RS. If $OM = ON$ and $PQ = 6cm$. Find RS.





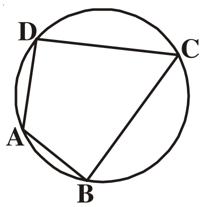
7. A is the centre of the circle and ABCD is a square. If $BD = 4cm$ then find the radius of the circle.

8. Draw a circle with any radius and then draw two chords equidistant from the centre.



9. In the given figure 'O' is the centre of the circle and AB, CD are equal chords. If $\angle AOB = 70^\circ$. Find the angles of the ΔOCD.

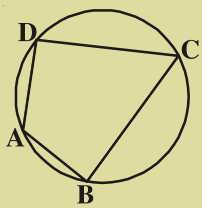
12.5 Cyclic Quadrilateral



In the figure, the vertices of the quadrilateral A, B, C and D lie on the same circle, this type of quadrilateral ABCD is called cyclic quadrilateral.

Activity

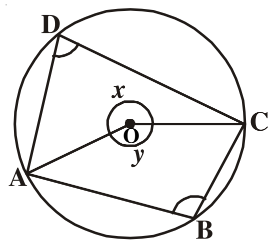
Draw a circle. Mark four points A, B, C and D on it. Draw quadrilateral ABCD. Measure its angles. Record them in the table. Repeat this activity for three more times.



S.No	$\angle A$	$\angle B$	$\angle C$	$\angle D$	$\angle A+\angle C$	$\angle B+\angle D$
1						
2						
3						
4						

What do you infer from the table?

Theorem-12.5 : The opposite angles of a cyclic quadrilateral are supplementary.



Given : ABCD is a cyclic quadrilateral .

To Prove : $\angle A + \angle C = 180°$

$\angle B + \angle D = 180°$

Construction : Join OA, OC

Proof: $\angle D = \frac{1}{2}\angle y$ (Why?) ..... (i)

$\angle B = \frac{1}{2}\angle x$ (Why?) ..... (ii)

By adding of (i) and (ii)

$$\angle D + \angle B = \frac{1}{2}\angle y + \frac{1}{2}\angle x$$

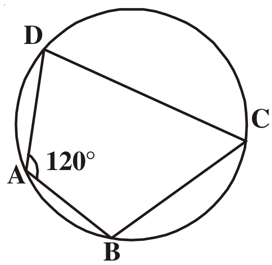
$$\angle D + \angle B = \frac{1}{2}(\angle y + \angle x)$$

$$\angle B + \angle D = \frac{1}{2} \times 360°$$

$$\angle B + \angle D = 180°$$

Similarly $\angle A + \angle C = 180°$

Example-6. In the figure, $\angle A = 120^{\circ}$ then find $\angle C$?



Solution: ABCD is a cyclic quadrilateral

Therefore $\angle A + \angle C = 180^{\circ}$

$120^0 + \angle C = 180^{\circ}$

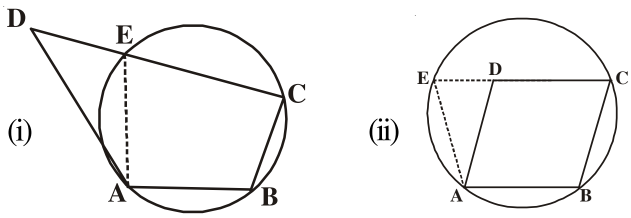
Therefore $\angle C = 180^{\circ} - 120^{\circ} = 60^{\circ}$

What is the converse of the above theorem?

"If the sum of a pair of opposite angles of a quadrilateral is 180^0, then the quadrilateral is cyclic".

The converse is also true.

Theorem-12.6 : If the sum of any pair of opposite angles in a quadrilateral is 180^o, then it is cyclic.



Given : Let ABCD be a quadrilateral such that

$\angle ABC + \angle ADC = 180^o$

$\angle DAB + \angle BCD = 180^o$

R.T.P. : ABCD is a cyclic quadrilateral.

Construction : Draw a circle through three non-collinear points A, B, and C.

If it passes through D, the theorem is proved since A, B, C and D are concyclic. If the circle does not pass through D, it intersects $\overline{CD}$ [fig (i) or $\overline{CD}$ produced [fig (ii)] at E.

Draw $\overline{AE}$

Proof : ABCE is a cyclic quadrilateral (construction)

$\angle AEC + \angle ABC = 180^o$ [sum of the opposite angles of a cyclic quadrilateral]

But $\angle ABC + \angle ADC = 180^o$ Given

$\therefore \angle AEC + \angle ABC = \angle ABC + \angle ADC \Rightarrow \angle AEC = \angle ADC$

But one of these is an exterior angle of ΔADE and the other is an interior opposite angle.

We know that the exterior angle of a triangle is always greater than either of the opposite interior angles.

$\therefore \angle AEC = \angle ADC$ is a contradiction.

So our assumption that the circle passing through A, B and C does not pass through D is false.

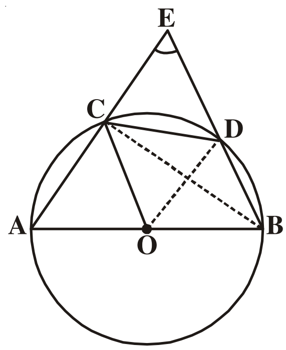
$\therefore$ The circle passing through A, B, C also passes through D.

$\therefore$ A, B, C and D are concyclic. Hence ABCD is a cyclic quadrilateral.

Example-7. In figure, $\overline{AB}$ is a diameter of the circle, $\overline{CD}$ is a chord equal to the radius of the circle. $\overrightarrow{AC}$ and $\overrightarrow{BD}$ when extended intersect at a point E. Prove that $\angle AEB = 60°$.

Solution : Join OC, OD and BC.

Triangle ODC is equilateral (Why?)



Therefore, $\angle COD = 60°$

Now, $\angle CBD = \frac{1}{2} \angle COD$ (Why?)

This gives $\angle CBD = 30°$

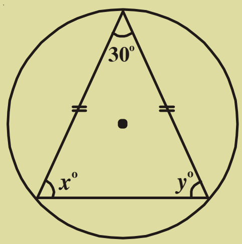
Again, $\angle ACB = 90°$ (Why?)

So, $\angle BCE = 180° - \angle ACB = 90°$

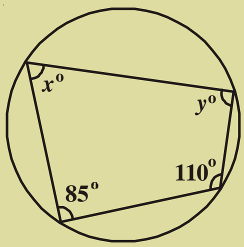
Which gives $\angle CEB = 90° - 30° = 60°$, i.e. $\angle AEB = 60°$

EXERCISE 12.5

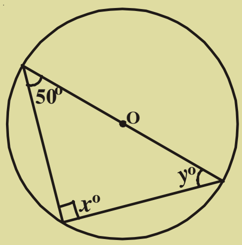
1. Find the values of x and y in the figures given below.



(i)



(ii)



(iii)

2. Given that the vertices A, B, C of a quadrilateral ABCD lie on a circle.

   Also $\angle A + \angle C = 180°$, then prove that the vertex D also lie on the same circle.

3. Prove that a cyclic rhombus is a square.

4. For each of the following, draw a circle and inscribe the figure given. If a polygon of the given type can't be inscribed, write not possible.
   (a) Rectangle
   (b) Trapezium
   (c) Obtuse triangle
   (d) Non-rectangular parallelogram
   (e) Accute isosceles triangle
   (f) A quadrilateral PQRS with $\overline{PR}$ as diameter.

WHAT WE HAVE DISCUSSED

- A collection of all points in a plane which are at a fixed distance from a fixed point in the same plane is called a circle. The fixed point is called the centre and the fixed distance is called the radius of the circle
- A line segment joining any points on the circle is called a chord
- The longest of all chords which also passes through the centre is called a diameter
- Circles with same radii are called congruent circles
- Circles with same centre and different radii are called concentric circles
- Diameter of a circle divides it into two semi-circles
- The part between any two points on the circle is called an arc
- The area enclosed by a chord and an arc is called a segment. If the arc is a minor arc then it is called the minor segment and if the arc is major arc then it is called the major segment
- The area enclosed by an arc and the two radii joining the end points of the arc with centre is called a sector
- Equal chords of a circle subtend equal angles at the centre
- Angles in the same segment are equal
- An angle in a semi circle is a right angle.
- If the angles subtended by two chords at the centre are equal, then the chords are congruent
- The perpendicular from the centre of a circle to a chord bisects the chords. The converse is also true
- There is exactly one circle passes through three non-collinear points
- The circle passing through the vertices of a triangle is called a circumcircle
- Equal chords are at equal distance from the centre of the circle, conversely chords at equidistant from the centre of the circle are equal in length
- Angle subtended by an arc at the centre of the circle is twice the angle subtended by it at any other point on the circle.
- If the angle subtended by an arc at a point on the remaining part of the circle is 90^o, then the arc is a semi circle.
- If a line segment joining two points subtends same angles at two other points lying on the same side of the line segment, the four points lie on the circle.
- The pairs of opposite angles of a cyclic quadrilateral are supplementary.

13 Geometrical Constructions

13.1 Introduction

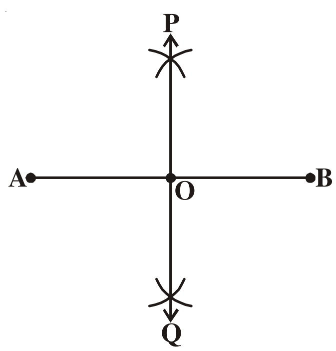
To construct geometrical figures, such as a line segment, an angle, a triangle, a quadrilateral etc., some basic geometrical instruments are needed. You must be having a geometry box which contains a graduated ruler (Scale) a pair of set squares, a divider, a compass and a protractor.

Generally, all these instruments are needed in drawing. A geometrical construction is the process of drawing a geometrical figure using only two instruments - an ungraduated ruler and a compass. We have mostly used ruler and compass in the construction of triangles and quadrilaterals in the earlier classes. In construction where some other instruments are also required, you may use a graduated scale and protractor as well. There are some constructions that cannot be done straight away. For example, when there are 3 measures available for the triangle, they may not be used directly. We will see in this chapter, how to extract the needed values and complete the required shape.

13.2 Basic Constructions

You have learnt how to construct (i) the perpendicular bisector of a line segment, (ii) angle bisector of 30°, 45°, 60°, 90° and 120° or of a given angle, in the lower classes. However the reason for these constructions were not discussed. The objective of this chapter is to give the process of necessary logical proofs to all those constructions.

13.2.1 To Construct the perpendicular bisector of a given line segment.



Example-1. Draw the perpendicular bisector of a given line segment AB and write justification.

Solution : Steps of construction.

Steps 1 : Draw the line segment AB

Step 2 : Taking A centre and with radius more than $\frac{1}{2}\overline{AB}$, draw an arc on either side of the line segment AB.

Step 3 : Taking 'B' as centre, with the same radius as above, draw arcs so that they intersect the previously drawn arcs.

Step 4 : Mark these points of intersection as P and Q.

Join P and Q.

Step 5 : Let PQ intersect $\overline{AB}$ at the point O

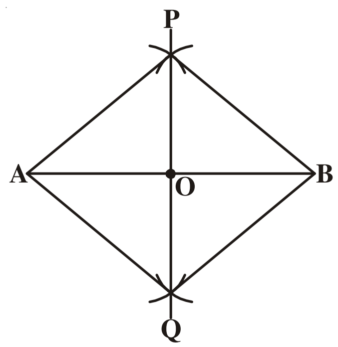
Thus the line POQ is the required perpendicular bisector of AB.

How can you prove the above construction i.e. "PQ is the perpendicular bisector of AB", logically?

Draw diagram of construction and join A to P and Q; also B to P and Q.

We use the congruency of triangle properties to prove the required.

Proof :



Steps	**Reasons**
In Δ^s PAQ and ΔPBQ	Selected
AP = BP ; AQ = BQ	Equal radii
PQ = PQ	Common side
$\therefore \Delta PAQ \cong \Delta PBQ$	SSS rule
So $\angle APO = \angle BPO$	CPCT (corresponding parts of congruent triangles)
Now In Δ^s APO and BPO	Selected
AP = BP	Equal radii as before
$\angle APO = \angle BPO$	Proved above
OP = OP	Common
$\therefore \Delta APO \cong \Delta BPO$	SAS rule
So OA = OB and $\angle APO = \angle BPO$	CPCT
As $\angle AOP + \angle BOP = 180^\circ$	Linear pair
We get $\angle AOP = \angle BOP = \frac{180^\circ}{2} = 90^\circ$	From the above result
Thus PO, i.e. POQ is the perpendicular bisector of AB	Required to prove.

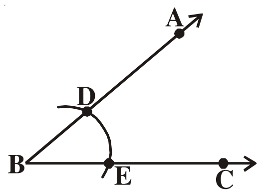
13.2.2 To construct the bisector of a given angle

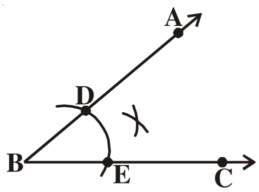
Example-2. Construct the bisector of a given angle ABC.

Solution : Steps of construction.

Step 1 : Draw the given angle ABC

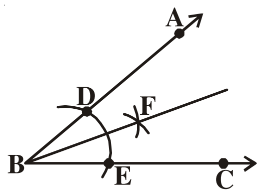
Step 2 : Taking B as centre and with any radius, draw an arc to intersect the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$, at D and E respectively, as shown in the figure.





Step 3 : Taking E and D as centres draw two arcs with equal radii to intersect each other at F.

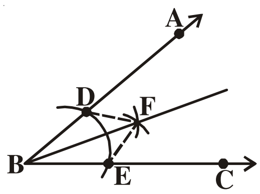
Step 4 : Draw the ray BF. It is the required bisector of $\angle$ABC.



Let us see the logical proof of above construction. Join D, F and E, F. We use congruency rule of triangles to prove the required.

Proof :

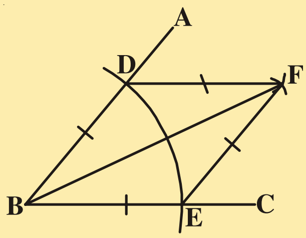
Steps	Reasons
In Δ^s BDF and ΔBEF	Selected triangles
BD = BE	radii of same arc
DF = EF	Arcs of equal radii
BF = BF	Common
$\therefore \Delta BDF \cong \Delta BEF$	SSS rule
So $\angle DBF = \angle EBF$	CPCT
Thus BF is the bisector of $\angle$ABC	Required to prove



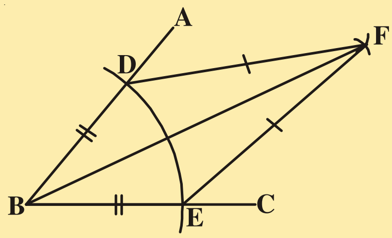
TRY THESE

Observe the sides, angles and diagonals of quadrilateral BEFD. Name the figures given below and write properties of figures.

1.



2.

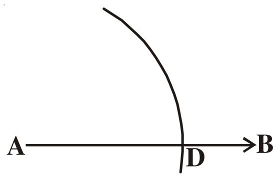


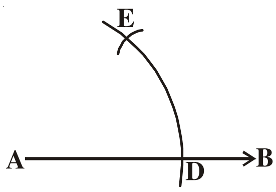
13.2.3 To construct an angle of 60° at the initial point of a given ray.

Example-3. Draw a ray AB with initial point A and construct a ray AC such that $\angle BAC = 60°$.

Solution : Steps of Construction

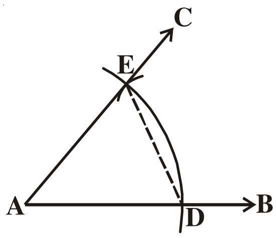
Step 1 : Draw the given ray AB and taking A as centre and some radius, draw an arc which intersects AB, say at a point D.





Step 2 : Taking D as centre and with the same radius taken before, draw an arc intersecting the previously drawn arc, say at a point E.

Step 3 : Draw a ray AC Passing through E then $\angle BAC$ is the required angle of $60°$.



Let us see how the construction is justified. Draw the figure again and join DE and prove it as follows .

Steps	Reasons
In Δ ADE	Selected
AE = AD	radii of same arc
AD = DE	Arcs of equal radius
Then AE = AD = DE	Same arc with same radii
$\therefore$ ΔADE is equilateral triangle	All sides are equal.
$\angle EAD = 60°$	each angle of equilateral triangle.
$\angle BAC$ is same as $\angle EAD$	$\angle EAD$ is a part of $\angle BAC$.
$\angle BAC = 60°$.	Required to prove.

TRY THIS

Draw a circle, Identify a point on it. Cut arcs on the circle with the length of the radius in succession. How many parts can the circle be divided into? Give reason.

EXERCISE - 13.1

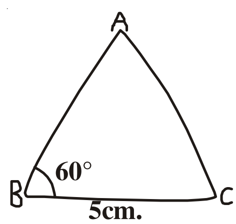
1. Construct the following angles at the initial point of a given ray and justify the construction.
 (a) 90^o (b) 45^o
2. Construct the following angles using ruler and compass and verify by measuring them by a protractor.
 (a) 30^o (b) $22\frac{1}{2}^o$ (c) 15^o
 (d) 75^o (e) 105^o f) 135^o
3. Construct an equilateral triangle, given its side of length of 4.5 cm and justify the construction.
4. Construct an isosceles triangle, given its base and base angle and justify the construction.
 [Hint : You can take any measure of side and angle]

13.3 CONSTRUCTION OF TRIANGLES (SPECIAL CASES)

We have so far, constructed some basic constructions and justified with proofs. Now we will construct some triangles when special type of measures are given. Recall the congruency properties of triangles such as SAS, SSS, ASA and RHS rules. You have already learnt how to construct triangles in class VII using the above rules.

You may have learnt that atleast three parts of a triangle have to be given for constructing it but not any combinations of three measures are sufficient for the purpose. For example, if two sides and an angle (not the included angle) are given, then it is not always possible to construct such a triangle uniquely. We can give several illustrations for such constructions. In such cases we have to use the given measures with desired combinations such as SAS, SSS, ASA and RHS rules.

13.3.1 Construction : To construct a triangle, given its base, a base angle and sum of other two sides.

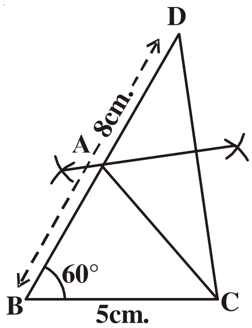


Example-4. Construct a ΔABC given BC = 5 cm., AB + AC = 8 cm. and $\angle ABC = 60^\circ$.

Solution : Steps of construction

Step 1 : Draw a rough sketch of ΔABC and mark the given measurements as usual. (How can you mark $AB + AC = 8cm$?)
How can you locate third vertex A in the construction ?

Analysis : As we have $AB + AC = 8$ cm., extend BA up to D so that BD = 8 cm.



$\therefore$ BD = BA + AD = 8 cm

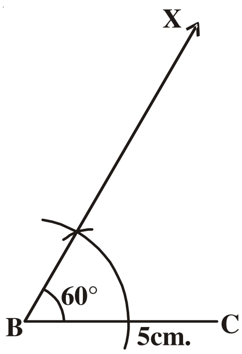
but AB + AC = 8 cm. (given)

$\because$ AD = AC

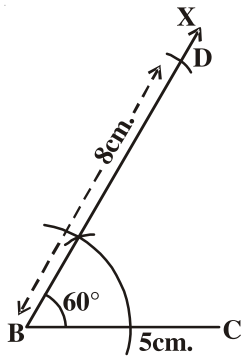
To locate A on BD what will you do ?

As A is equidistant from C and D, draw a perpendicular bisector of $\overline{CD}$ to locate A on BD.

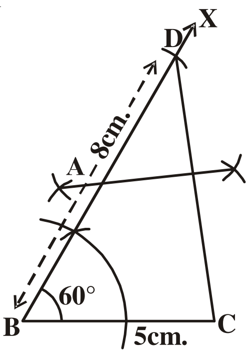
How can you prove AB + AC = BD ?



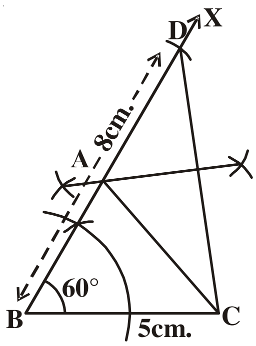
Step 2 : Draw the base $\overline{BC} = 5$ cm and construct $\angle CBX = 60^\circ$ at B



Step 3: With centre B and radius 8 cm (AB + AC = 8 cm) draw an arc on $\overrightarrow{BX}$ to intersect (meet) at D.



Step 4 : Join CD and draw a perpendicular bisector of CD to meet BD at A



Step 5 : Join AC to get the required triangle ABC.

Now, we will justify the construction.

Proof : A lies on the perpendicular bisector of $\overline{CD}$

$\because$ AC = AD

AB + AC = AB + AD

= BD

= 8 cm.

Hence ΔABC is the required triangle.

THINK, DISCUSS AND WRITE

Can you construct a triangle ABC with BC = 6 cm, $\angle B = 60^{o}$ and $AB + AC = 5cm.$? If not, give reasons.

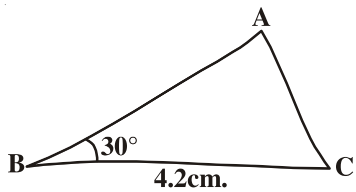
13.3.2 Construction : To Construct a triangle given its base, a base angle and the difference of the other two sides.

Given the base BC of a triangle ABC, a base angle say $\angle B$ and the difference of other two sides AB – AC in case AB>AC or AC-AB, in case AB<AC, you have to construct the triangle ABC. Thus we have two cases of constructions discussed in the following examples.

Case (i) Let AB > AC

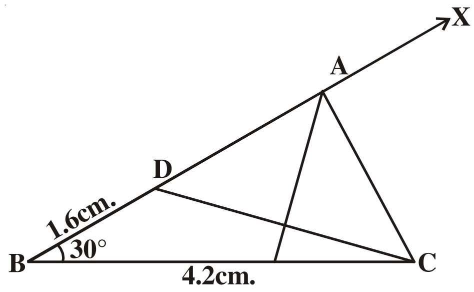
Example-5. Construct ΔABC in which BC = 4.2 cm, $\angle B = 30^{0}$ and AB – AC = 1.6 cm

Solution : Steps of Construction



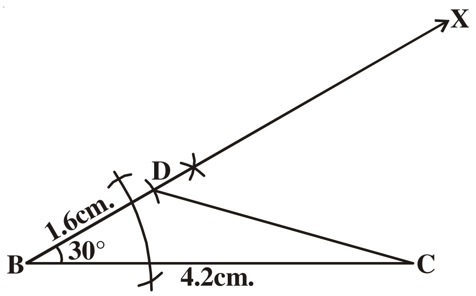
Step 1: Draw a rough sketch of ΔABC and mark the given measurements

(How can you mark AB – AC = 1.6 cm ?)

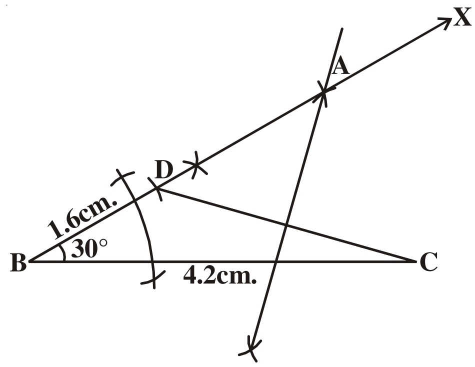


Analysis : Since AB – AC = 1.6 cm and AB > AC, mark D on AB such that AD = AC Now BD = AB – AC = 1.6 cm. Join CD and draw a perpendicular bisector of CD to find the vertex A on BD produced.

Join AC to get the required triangle ABC.

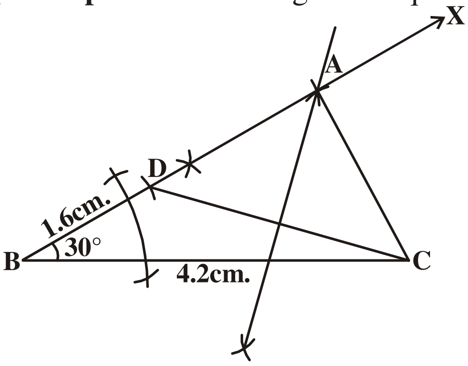


Step 2: Construct ΔBCD using S.A.S rule with measures BC = 4.2 cm $\angle B = 30^{0}$ and BD = 1.6 cm. (i.e. AB - AC)



Step 3 : Draw the perpendicular bisector of CD. Let it meet ray BDX at a point A.

Step 4: Join AC to get the required triangle ABC.



THINK, DISCUSS AND WRITE

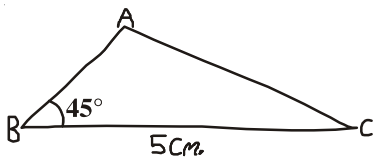
Can you construct the triangle ABC with the same measures by changing the base angle $\angle C$ instead of $\angle B$? Draw a rough sketch and construct it.

Case (ii) Let AB < AC

Example-6. Construct ΔABC in which $BC = 5cm$, $\angle B = 45^o$ and $AC - AB = 1.8$ cm.

Solution : Steps of Construction.

Step 1: Draw a rough sketch of ΔABC and mark the given measurements.



Analyse how $AC - AB = 1.8$ cm can be marked?

Analysis : Since $AC - AB = 1.8$ cm i.e. $AB < AC$ we have to find D on AB produced such that $AD = AC$

Now $BD = AC - AB = 1.8$ cm ($\because$ $BD = AD - AB$ and $AD = AC$)

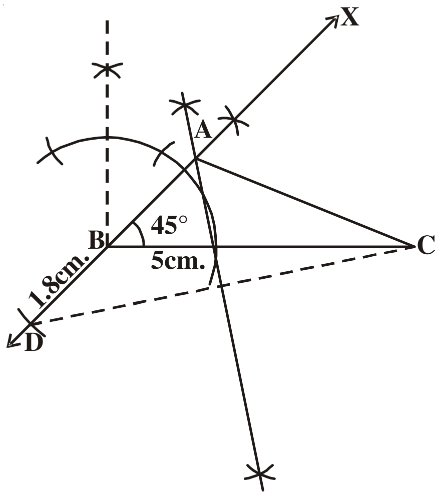
Join CD to find A on the perpendicular bisector of DC

Step 2 : Draw $BC = 5$ cm and construct $\angle CBX = 45°$

With centre B and radius 1.8 cm ($BD = AC - AB$) draw an arc to intersect the line XB extended at a point D.

Step 3 : Join DC and draw the perpendicular bisector of DC.

Step 4 : Let it meet $\overrightarrow{BX}$ at A and join AC ΔABC is the required triangle.



Now, you can justify the construction.

Proof: In ΔABC, the point A lies on the perpendicular bisector of DC.

$\therefore$ $AD = AC$

$AB + BD = AC$

So $BD = AC - AB$

$= 1.8$ cm

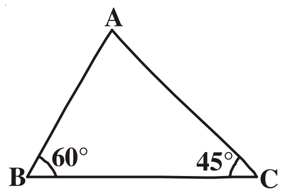
Hence ΔABC is the required that triangle.

13.3.3 Construction : To construct a triangle, given its perimeter and its two base angles.

Given the base angles, say $\angle B$ and $\angle C$ and perimeter AB + BC + CA, you have to construct the triangle ABC.

Example-7. Construct a triangle ABC, in which $\angle B = 60^\circ$, $\angle C = 45^\circ$ and

AB + BC + CA = 11 cm.



Solution : Steps of construction.

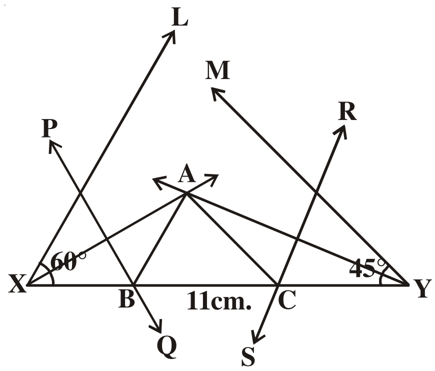
Step 1 : Draw a rough sketch of a triangle ABC and mark the given measures

(Can you mark the perimeter of triangle ?)

Analysis : Draw a line segment, say XY equal to perimeter of ΔABC i.e., AB + BC + CA. Make angles $\angle YXL$ equal to $\angle B$ and $\angle XYM$ equal to $\angle C$ and bisect them.

Let these bisectors intersect at a point A.

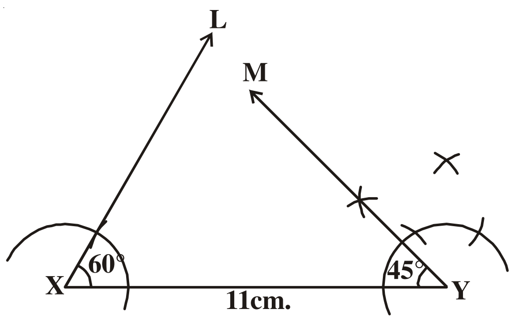
Draw perpendicular bisectors of AX to intersect XY at B and the perpendicular bisector of AY to intersect it at C. Then by joining AB and AC, we get required triangle ABC.



Step 2: Draw a line segment XY = 11 cm

(As XY = AB + BC + CA)

Step 3 : Construct $\angle YXL = 60^\circ$ and $\angle XYM = 45^\circ$ and draw bisectors of these angles.

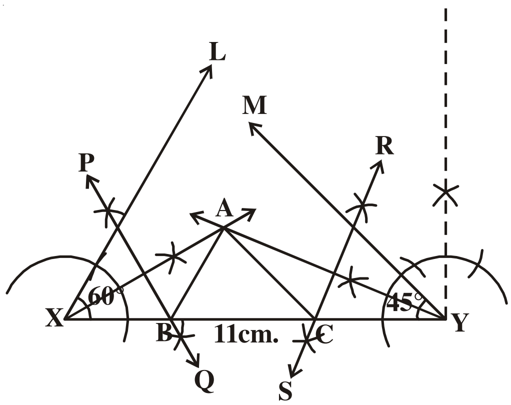


Step 4 : Let the bisectors of these angles intersect at a point A and join AX or AY.

Step 5 : Draw perpendicular bisectors of AX and AY to intersect $\overrightarrow{XY}$ at B and C respectively

Join AB and AC.

Then, ABC is the required triangle.



You can justify the construction as follows

Proof: B lies on the perpendicular bisector PQ of AX

$\therefore$ XB = AB and similarly CY = AC

This gives AB + BC + CA = XB + BC + CY

$= XY$

Again $\angle BAX = \angle AXB$ ($\because$ XB = AB in ΔAXB) and

$\angle ABC = \angle BAX + \angle AXB$

(Exterior angle of ΔABC).

$= 2\angle AXB$

$= \angle YXL$

$= 60^{o}$.

Similarly $\angle ACB = \angle XYM = 45^{o}$ as required

$\therefore$ $\angle B = 60°$ and $\angle C = 45°$ as given are constructed.

TRY THESE

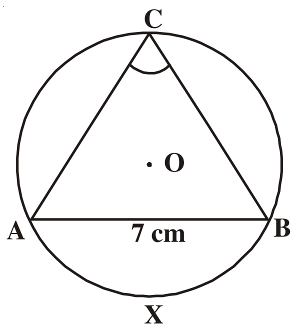
Can you draw the triangle with the same measurements in alternate way?

(**Hint**: Take $\angle YXL = \dfrac{60°}{2} = 30°$

and $\angle XYM = \dfrac{45°}{2} = 22\frac{1}{2}°$)

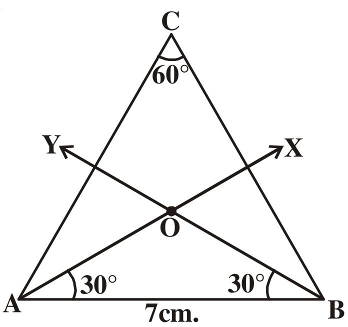
13.3.4 Construction : To construct a circle segment given a chord and a given an angle.

Example-8. Construct a segment of a circle on a chord of length 7cm. and containing an angle of 60°.



Solution : Steps of construction.

Step-1: Draw a rough sketch of a circle and a segment contains an angle 60°. (Draw major segment Why?) Can you draw a circle without a centre?



Analysis: Let 'O' be the centre of the circle. Let AB be the given chord and ACB be the required segment of the circle containing an angle C = 60°.

Let $\overparen{AXB}$ be the arc subtending the angle at C.

Since $\angle ACB = 60°$, $\angle AOB = 60° \times 2 = 120°$

In ΔOAB, OA=OB (radii of same circle)

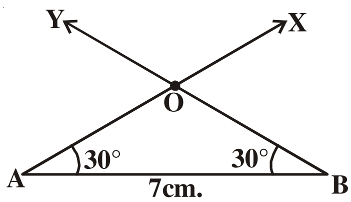
$\therefore \angle OAB = \angle OBA = \dfrac{180° - 120°}{2} = \dfrac{60°}{2} = 30°$

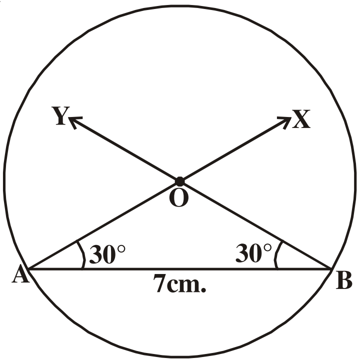
So we can draw ΔOAB then draw a circle with radius equal to OA or OB.

Step-2 : Draw a line segment AB = 7cm.

Step-3 : Draw $\overrightarrow{AX}$ such that $\angle BAX = 30°$ and draw $\overrightarrow{BY}$ such that $\angle YBA = 30°$ to intersect $\overrightarrow{AX}$ at O .

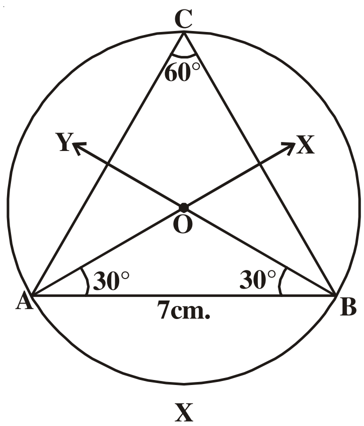
[Hint : Construct 30^o angle by bisecting 60^o angle]





Step-4 : With centre 'O' and radius OA or OB, draw the circle.

Step-5 : Mark a point 'C' on the arc of the circle. Join AC and BC. We get $\angle ACB = 60°$



Thus ACB is the required circle segment.

Let us justify the construction

Proof : OA = OB (radii of circle).

$\therefore \quad \angle OAB + \angle OBA = 30° + 30° = 60°$

$\therefore \quad \angle AOB = 180° - 60° = 120°$

$\overset{\frown}{AXB}$ Subtends an angle of 120° at the centre of the circle.

$$\therefore \quad \angle ACB = \frac{120°}{2} = 60°$$

$\therefore$ ACB is the required segment of a circle.

TRY THESE

What happen if the angle in the circle segment is right angle? What kind of segment do you obtain? Draw the figure and give reason.

EXERCISE - 13.2

1. Construct ΔABC in which BC = 7 cm, $\angle B = 75^\circ$ and AB + AC = 12 cm.
2. Construct ΔPQR in which QR = 8 cm, $\angle Q = 60°$ and PQ – PR = 3.5 cm
3. Construct $\Delta\, XYZ$ in which $\angle Y = 30^\circ$, $\angle Z = 60^\circ$ and XY + YZ + ZX = 10 cm.

4. Construct a right triangle whose base is 7.5cm. and sum of its hypotenuse and other side is 15cm.
5. Construct a segment of a circle on a chord of length 5cm. containing the following angles.
   i. 90° ii. 45° iii. 120°

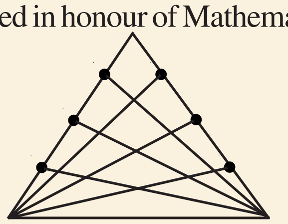
What we have discussed?

1. A geometrical construction is the process of drawing geometrical figures using only two instruments - an ungraduated ruler and a compass.
2. Construction of geometrical figures of the following with justifications (Logical proofs)
   - Perpendicular bisector of a given line segment.
   - bisector of a given angle.
   - Construction of 60° angle at the initial point of a given ray.
3. To construct a triangle, given its base, a base angle and the sum of other two sides.
4. To construct a triangle given its base, a base angle and the difference of the other two sides.
5. To construct a triangle, given its perimeter and its two base angle.
6. To construct a circle segment given a chord and an angle.

Brain Teaser

How many triangles are threre in the figure ?

(It is a 'Cevian' write formula of a triangle - named in honour of Mathematician Ceva)



(**Hint :** Let the number of lines drawn from each vertex to the opposite side be 'n')

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14 Probability

Probability theory is nothing but common sense reduced to calculation.
- Pierre-Simon Laplace

14.1 Introduction

Siddu and Vivek are classmates. One day during their lunch they are talking to each other.

Observe their conversation

Siddu : Hello Vivek , What are you going to do in the evening today?

Vivek : Most likely, I will watch India v/s Australia cricket match.

Siddu : Whom do you think will win the toss ?

Vivek : Both teams have equal chance to win the toss.
Do you watch the cricket match at home?

Siddu : There is no chance for me to watch the cricket at my home. Because my T.V. is under repair.

Vivek : Oh! then come to my home, we will watch the match together.

Siddu : I will come after doing my home work.

Vivek : Tomorrow is 2nd october. We have a holiday on the occasion of Gandhiji's birthday. So why don't you do your home work tomorrow?

Siddu : No, first I will finish the homework then I will come to your home.

Vivek : Ok.

Consider the following statements from the above conversation:

Most likely, I will watch India v/s Australia cricket match

There is ***no chance*** for me to watch the cricket match.

Both teams have ***equal chance*** to win the toss.

Here Vivek and Siddu are making judgements about the chances of the particular occurrence.

In many situations we make such statements and use our past experience and logic to take decisions. For example

It is a bright and pleasant sunny day. I need not carry my umbrella and will take a chance to go.

However, the decisions may not always favour us. Consider the situation. "Mary took her umbrella to school regularly during the rainy season. She carried the umbrella to school for many days but it did not rain during her walk to the school. However, by chance, one day she forgot to take the umbrella and it rained heavily on that day".

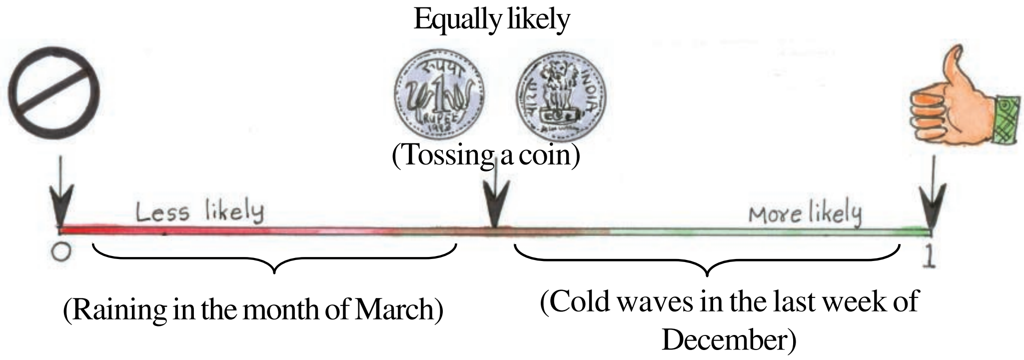
Usually the summer begins from the month of March, but one day in that month there was a heavy rainfall in the evening. Luckily Mary escaped becoming wet, because she carried umbrella on that day as she does daily.

Thus we take a decision by guessing the future happening that is whether an event occurs or not. In the above two cases, Mary guessed the occurrence and non-occurrence of the event of raining on that day. Our decision may favour us and sometimes may not. (Why?)

We try to measure numerically the chance of occurrence or non-occurrence of some events just as we measure many other things in our daily life. This kind of measurement helps us to take decision in a more systematic manner. Therefore we study probability to figure out the chance of something happening.

Before measuring numerically the chance of happening that we have discussed in the above situations, we grade it using the following terms given in the table. Let us observe the following table.

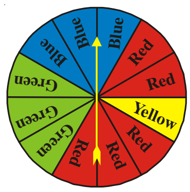
Term	Chance	Examples from conversation
certain	something that must occur	Gandhiji's birthday is on 2nd October.
more likely	something that would occur with great chance	Vivek watching the cricket match
equally likely	somethings that have the same chance of occurring	Both teams winning the toss.
less likely	Something that would occur with less chance	Vivek doing homework on the day of cricket match.
impossible	Something that cannot happen.	Sidhu watching the circket match at his home.



Do This

1. Observe the table given in the previous page and give some other example for each term.
2. Classify the following statements into the categories ***less likely, equally likely, more likely.***
   a) Rolling a die* and getting a number 5 on the top face.
   b) Cold waves in your village in the month of November.
   c) India winning the next soccer(foot ball)world cup
   d) Getting a tail or head when a coin is tossed.
   e) Winning the jackpot for your lottery ticket.

14.2 Probability



spinner

14.2.1 Random experiment and outcomes

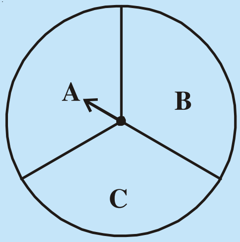
To understand and measure the chance, we perform the experiments like tossing a coin, rolling a die and spining the spinner etc.

When we toss a coin we have only two possible results, head or tail. Suppose you are the captain of a cricket team and your friend is the captain of the other cricket team. You toss the coin and ask your friend to choose head or tail. Can you control the result of the toss? Can you get a head or tail that you want? In an ordinary coin that is not possible. The chance of getting either is same and you cannot say what you would get. Such an experiment known as 'random experiment'. In such experiments though we know the possible outcomes before conducting the experiment, we cannot predict the exact outcome that occurs at a particular time, in advance. The outcomes of random experiments may be equally likely or may not be. In the coin tossing experiment head or tail are two possible outcomes.

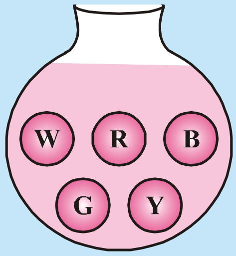
* A die (plural dice) is a well balanced cube with its six faces marked with numbers from 1 to 6, one number on each face. Sometimes dots appear in place of numbers.

TRY THESE

1. If you try to start a scooter , What are the possible outcomes?
2. When you roll a die, What are the six possible outcomes?
3. When you spin the wheel shown, What are the possible outcomes?
   (Out comes here means the possible sector where the pointer stops)



4. You have a jar with five identical balls of different colours (White, Red, Blue, Grey and Yellow) and you have to pickup (draw) a ball without looking at it. List the possible outcomes you get.



THINK, DISCUSS AND WRITE

In rolling a die.

- Does the first player have a greater chance of getting a six on the top face?
- Would the player who played after him have a lesser chance of getting a six on the top face?
- Suppose the second player got a six on the top face. Does it mean that the third player would not have a chance of getting a six on the top face?

14.2.2 Equally likely outcomes

When we toss a coin or roll a die , we assume that the coin and the die are fair and unbiased i.e. for each toss or roll the chance of all possibilities is equal. We conduct the experiment many times and collect the observations. Using the collected data, we find the measure of chance of occurrence of a particular happening.

A coin is tossed several times and the result is noted. Let us look at the result sheet where we keep on increasing the tosses.

Number of tosses	Tally marks (Heads)	Number of heads	Tally mark (Tails)	Number of tails
50	𝍸 𝍸 𝍸 𝍸 ||	22	𝍸 𝍸 𝍸 𝍸 𝍸 |||	28
60	𝍸 𝍸 𝍸 𝍸 𝍸 |	26	𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 ||||	34
70	......	30	......	40
80	......	36	......	44
90	......	42	......	48
100	......	48	......	52

We can observe from the above table as you increase the number of tosses, the number of heads and the number of tails come closer to each other.

Do This

Toss a coin for number of times as shown in the table. And record your findings in the table.

No. of Tosses	Number of heads	No. of tails
10		
20		
30		
40		
50		

What happens if you keep on increasing the number of tosses.

This could also be done with a die, roll it for large number of times and observe.

No. of times Die rolled	Number of times each outcome occured (i.e. each number appearing on the top face)					
	1	2	3	4	5	6
25	4	3	9	3	3	3
50	9	5	12	9	8	7
75	14	10	16	12	10	13
100	17	19	19	16	13	16
125	25	20	24	18	16	22
150	28	24	28	23	21	26
175	31	30	33	27	26	28
200	34	34	36	30	32	34
225	37	38	40	34	38	38
250	40	40	43	40	43	44
275	44	41	47	47	47	49
300	48	47	49	52	52	52

From the above table, it is evident that rolling a die for a larger number of times, the each of six outcomes, becomes almost equal to each other.

From the above two experiments, we may say that the different outcomes of the experiment are equally likely. This means each of the outcome has equal chance of occurring.

14.2.3 Trails and Events

In the above experiments each toss of a coin or each roll of a die is a **Trial** or **Random experiment**.

Consider a trial of rolling a die,

How many possible outcomes are there to get a number more than 5 on the top face?

It is only one (i.e., 6)

How many possible outcomes are there to get an even number on the top face?

They are 3 outcomes (2,4, and 6).

Thus each specific outcome or the collection of specific outcomes make an **Event**.

In the above trail getting a number more than 5 and getting an even number on the top face are two events. Note that event need not necessarily a single outcome. But, every outcome of a random experiment is an event.

Here we understand the basic idea of the event, more could be learnt on event in higher classes.

14.2.4 Linking the chance to Probability

Consider the experiment of tossing a coin once. What are the outcomes? There are only two outcomes Head or Tail and both outcomes are equally likely.

What is the chance of getting a head?

It is one out of two possible outcomes i.e. $\frac{1}{2}$. In other words it is expressed as the probability of getting a head when a coin is tossed is $\frac{1}{2}$, which is represented by

$$P(H) = \frac{1}{2} = 0.5 \text{ or } 50\%$$

What is the probability of getting a tail?

Now take the example of rolling a die. What are the possible outcomes in one roll? There are six equally likely outcomes 1,2,3,4,5,or 6.

What is the probability of getting an odd number on the top face?

1, 3 or 5 are the three favourable outcomes out of six total possible outcomes. It is $\frac{3}{6}$ or $\frac{1}{2}$

We can write the formula for Probability of an event 'A'

$$P(A) = \frac{\text{Number of favourable outcomes for event 'A'}}{\text{Number of total possible outcomes}}$$

Now let us see some examples :

Example 1: If two identical coins are tossed simultaneously. Find (a) the possible outcomes, (b) the number of total outcomes, (c) the probability of getting two heads, (d) probability of getting atleast one head, (e) probability of getting no heads and (f) probability of getting only one head.

Solution : (a) The possible outcomes are

Coin 1	Coin 2
Head	Head
Head	Tail
Tail	Head
Tail	Tail

b) Number of total possible outcomes is 4

c) Probability of getting two heads

$$= \frac{\text{Number of favourable outcomes of getting two heads}}{\text{Number of total possible outcomes}} = \frac{1}{4}$$

d) Probability of getting atleast one head $= \frac{3}{4}$

[At least one head means getting a head one or more number of times]

e) Probability of getting no heads $= \frac{1}{4}$.

e) Probability of getting only one head $= \frac{2}{4} = \frac{1}{2}$.

Do This

1. If three coins are tossed simultaneously then write their outcomes.
   a) All possible outcomes
   b) Number of possible outcomes
   c) Find the probability of getting at least one head
   (getting one or more than one head)
   d) Find the Probability of getting at most two heads
   (getting Two or less than two heads)
   e) Find the Probability of getting no tails

Example 2 : (a) Write the probability of getting each number on the top face when a die was rolled in the following table. (b) Find the sum of the probabilities of all outcomes.

Solution : (a) Out of six possibilities the number 4 occurs once hence probability is 1/6. Similarly we can fill the table for the remaining values.

Outcome	1	2	3	4	5	6
Probability (P)				1/6		

(b) The sum of all probabilities

$$P(1) + P(2) + P(3) + P(4) + P(5) + P(6)$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$$

We can generalize that

Sum of the probabilities of all the outcomes of a random experiment is always 1

TRY THIS

Find the probability of each event when a die is rolled once

Event	Favourable outcome(s)	Number of favourable outcome(s)	Total possible outcomes	Number of total possible outcomes	Probability = $\frac{\text{Number of favourable outcomes}}{\text{Number of total possible outcomes}}$
Getting a number 5 on the top face	5	1	1, 2, 3, 4, 5 and 6	6	1/6
Getting a number greater than 3 on the top face					
Getting a prime number on the top face					
Getting a number less than 5 on the top face					
Getting a number that is a factor of 6 on the top face					
Getting a number greater than 7 on the top face					
Getting a number that is a Multiple of 3 on the top face					
Getting a number 6 or less than 6 on the top face					

You can observe that

The probability of an event always lies between 0 and 1 (0 and 1 inclusive)

$$0 \leq \text{probability of an event } \leq 1$$

a) The probability of an event which is certain = 1

b) The probability of an event which is impossible = 0

14.2.5 Conduct your own experiments

1. We would work here in groups of 3-4 students each. Each group would take a coin of the same denomination and of the same type. In each group one student of the group would toss the coin 20 times and record the data. The data of all the groups would be placed in the table below (Examples are shown in the table).

Group No.	No. of tosses	Cumulative tosses of groups	Number of heads	Cumulative No. of heads	$\frac{\text{Cumulative heads}}{\text{total times tossed}}$	$\frac{\text{Cumulative tails}}{\text{total times tossed}}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	20	20	7	7	$\frac{7}{20}$	$\frac{20-7}{20} = \frac{13}{20}$
2	20	40	14	21	$\frac{21}{40}$	$\frac{40-21}{40} = \frac{19}{40}$
3	20	60				
4	20	80				
5	20	100				
6	.....	....				
7	....	....				

What happens to the value of the fractions in (6) and (7) when the total number of tosses of the coin increases? Could you see that the values are moving close to the probability of getting a head and tail respectively.

2. In this activity also we would work in groups of 3-4. One student from each group would roll a die for 30 times. Other students would record the data in the following table. All the groups should have the same kind of die so that all the throws will be treated as the throws of the same die.

No. of times Die rolled	Number of times the following outcomes turn up					
	1	2	3	4	5	6
30						

Complete the following table, using the data obtained from all the groups :

Group(s)	Number of times 1 turned up	Total number of times a die is rolled	$\frac{\text{Number of times 1 turned up}}{\text{Total number of times a die is rolled}}$
(1)	(2)	(3)	(4)
1^{st}			
$1^{st} + 2^{nd}$			
$1^{st}+2^{nd}+3^{rd}$			
$1^{st} + 2^{nd} + 3^{rd} + 4^{th}$			
$1^{st} + 2^{nd} + 3^{rd} + 4^{th} + 5^{th}$			

What do you observe as the number of rolls increases; the fractions in cloumn (4) move closer to $\frac{1}{6}$. We did the above experiment for the outcome 1. Check the same for the outcome 2 and the outcome 5.

What can you conclude about the values you get in column (4) and compare these with the probabilities of getting 1, 2, and 5 on rolling a die.

3. What would happen if we toss two coins simultaneously? We could have either both coins showing head, both showing tail or one showing head and one showing tail. Would the possibility of occurrence of these three be the same? Think about this while you do this group activity.

Divide class into small groups of 4 each. Let each group take two coins. Note that all the coins used in the class should be of the same denomination and of the same type. Each group would throw the two coins simultaneously 20 times and record the observations in a table.

No. of times two coins tossed	No. of times no head turns up	Number of times one head turns up	Number of times two heads turns up
20			

All the groups should now make a cummulative table:

Group(s)	Number of times two coins are tossed	Number of times no head turns up	Number of times one head turns up	Number of times two heads turns up
1^{st}				
$1^{st} + 2^{nd}$				
$1^{st} + 2^{nd} + 3^{rd}$				
$1^{st} + 2^{nd} + 3^{rd} + 4^{th}$				
....				

Now we find the ratio of the number of times no head turns up to the total number of times two coins are tossed. Do the same for the remaining events.

Fill the following table:

Group(s)	$\frac{\text{No. of times no head}}{\text{Total tosses}}$	$\frac{\text{No. of times one head}}{\text{Total tosses}}$	$\frac{\text{No. of times two heads}}{\text{Total tosses}}$
(1)	(2)	(3)	(4)
Group 1^{st}			
Group $1 + 2^{nd}$			
Group $1 + 2 + 3^{rd}$			
Group $1 + 2 + 3 + 4^{th}$			
....			

As the number of tosses increases, the values of the columns (2), (3) and (4) get closer to 0.25, 0.5 and 0.25 respectively.

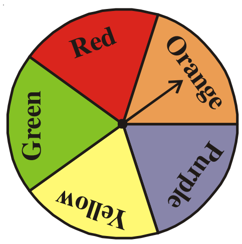
Example-3: A spinner was spun 1000 times and the frequency of outcomes was recorded as in given table:

Out come	Red	Orange	Purple	Yellow	Green
Frequency	185	195	210	206	204

Find (a) List the possible outcomes that you can see in the spinner (b) Compute the probability of each outcome. (c) Find the ratio of each outcome to the total number of times that the spinner spun (use the table)

Solution :

(a) The possible outcomes are 5. They are red, orange, purple, yellow and green. Here all the five colours occupy equal areas in the spinner. So, they are all equally likely.



spinner

(b) Compute the probability of each event.

$$P(\text{Red}) = \frac{\text{Favourable outcomes of red}}{\text{Total number of possible outcomes}}$$

$$= \frac{1}{5} = 0.2.$$

Similarly

P(Orange), P(Purple), P(Yellow) and P(Green) is also $\frac{1}{5}$ or 0.2.

(c) From the experiment the frequency was recorded in the table

$$\text{Ratio for red} = \frac{\text{No. of outcomes of red in the above experiment}}{\text{Number of times the spinner was spun}}$$

$$= \frac{185}{1000} = 0.185$$

Similarly, we can find the corresponding ratios for orange, purple, yellow and green are 0.195, 0.210, 0.206 and 0. 204 respectively.

Can you see that each of the ratio is approximately equal to the probability which we have obtained in (b) [i.e. before conducting the experiment]

Example-4. The following table gives the ages of audience in a theatre. Each person was given a serial number and a person was selected randomly for the bumper prize by choosing a serial number. Now find the probability of each event.

Age	Male	Female
Under 2	3	5
3 - 10 years	24	35
11 - 16 years	42	53
17 - 40 years	121	97
41- 60 years	51	43
Over 60	18	13

Total number of audience : 505

Find the probability of each event given below.

Solution :

a) The probability of audience of age less than or equal to 10 years

The audience of age less than or equal to 10 years = 24 + 35 + 5 + 3 = 67

Total number of people = 505

$$P(\text{audience of age} \leq 10 \text{ years}) = \frac{67}{505}$$

b) The probability of female audience of age 16 years or younger

The female audience with age less than or equal 16 years = 53 + 35 + 5 = 93

$P(\text{female audience of age} \leq 16 \text{ years}) = 93/505$

c) The probability of male audience of age 17 years or above

$$= 121 + 51 + 18 = 190$$

$$P(\text{male audience of age} \geq 17 \text{ years}) = \frac{190}{505} = \frac{38}{101}$$

d) The probability of audience of age above 40 years

$$= 51+43+18+13 = 125$$

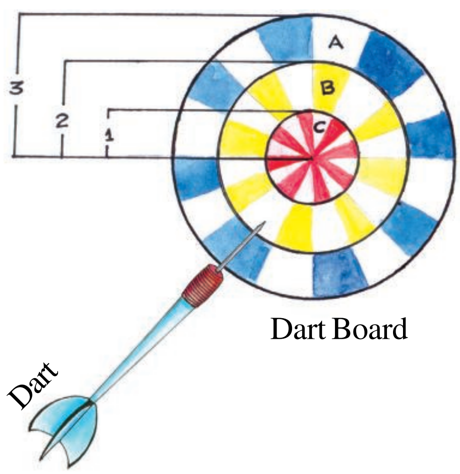
$$P(\text{audience of age} > 40 \text{ years}) = \frac{125}{505} = \frac{25}{101}$$

e) The probability of the person watching the movie is not a male

$$= 5 + 35 + 53 + 97 + 43 + 13 = 246$$

$$P(\text{A person watching movie is not a male}) = \frac{246}{505}$$

Example-5 : Assume that a dart will hit the dart board and each point on the dart board is equally likely to be hit in all the three concentric circles where radii of concetric circles are 3 cm, 2 cm and 1 cm as shown in the figure below.



Find the probability of a dart hitting the board in the region A. (The outer ring)

Solution : Here the event is hitting in region A.

The Total area of the circular region with radius 3 cm

$$= \pi(3)^2$$

Area of circular region A (i.e. ring A) = $\pi(3)^2 - \pi(2)^2$

Probability of the dart hitting the board in region A is

$$P(A) = \frac{\text{Area of circular region A}}{\text{Total Area}}$$

$$= \frac{\pi(3)^2 - \pi(2)^2}{\pi(3)^2}$$

$$= \frac{9\pi - 4\pi}{9\pi}$$

$$\frac{5}{9} = 0.556$$

> Remember
>
> Area of a circle = πr^2
>
> Area of a ring = $\pi R^2 - \pi r^2$

Try These

From the figure given in example 5.

1. Find the probability of the dart hitting the board in the circular region B (i.e. ring B).
2. Without calculating, write the percentage of probability of the dart hitting the board in circular region C (i.e. ring C).

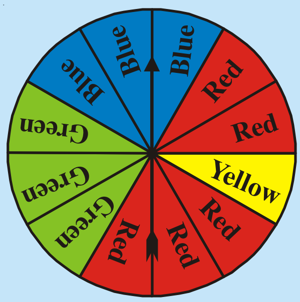
14.3 Uses of Probability in real life

- Meteorological department predicts the weather by observing trends from the data collected over many years in the past.
- Insurance companies calculate the probability of happening of an accident or casuality to determine insurance premiums.
- "An exit poll" is taken after the election . It is surveying the people to which party they have voted. This gives an idea of winning chances of each candidate and predictions are made accordingly.

EXERCISE - 14.1

1. A die has six faces numbered from 1 to 6. It is rolled and the number on the top face is noted. When this is treated as a random trial.
   a) What are the possible outcomes ?
   b) Are they equally likely? Why?
   c) Find the probability of a composite number turning up on the top face.
2. A coin is tossed 100 times and the following outcomes are recorded

   Head:45 times Tails:55 times from the experiment
   a) Compute the probability of each outcomes.
   b) Find the sum of probabilities of all outcomes.
3. A spinner has four colours as shown in the figure. When we spin it once, find
   a) At which colour, is the pointer more likely to stop?
   b) At which colour, is the pointer less likely to stop?
   c) At which colours, is the pointer equally likely to stop?
   d) What is the chance the pointer will stop on white?
   e) Is there any colour at which the pointer certainly stops?



4. A bag contains five green marbles, three blue marbles, two red marbles, and two yellow marbles. One marble is drawn out randomly.
   a) Are the four different colour outcomes equally likely? Explain.
   b) Find the probability of drawing each colour marble

      i.e. , P(green), P(blue), P(red) and P(yellow)
   c) Find the sum of their probabilities.
5. A letter is chosen from English alphabet. Find the probability of the letters being
   a) A vowel b) a letter that comes after P
   c) A vowel or a consonant d) Not a vowel

6. Eleven bags of wheat flour, each marked 5 kg, actually contained the following weights of flour (in kg):

   4.97, 5.05, 5.08, 5.03, 5.00, 5.06, 5.08, 4.98, 5.04, 5.07, 5.00

   Find the probability that any of these bags chosen at random contains more than 5 kg of flour.

7. An insurance company selected 2000 drivers at random (i.e., without any preference of one driver over another) in a particular city to find a relationship between age and accidents. The data obtained is given in the following table:

Age of Drivers (in years)	**Accidents in one year**				**More than 3 accidents**
	0	**1**	**2**	**3**	
18-29	440	160	110	61	35
30- 50	505	125	60	22	18
Over 50	360	45	35	15	9

   Find the probabilities of the following events for a driver chosen at random from the city:

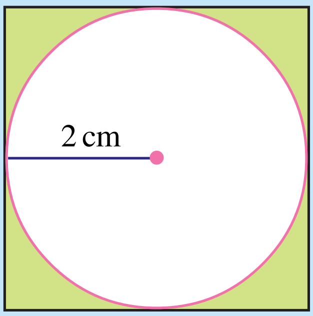
   (i) The driver being in the age group 18-29 years and having exactly 3 accidents in one year.

   (ii) The driver being in the age group of 30-50 years and having one or more accidents in a year.

   (iii) Having no accidents in the year.

8. What is the probability that a randomly thrown dart hits the square board in shaded region

   (Take $\pi = \frac{22}{7}$ and express in percentage)



WHAT WE HAVE DISCUSSED

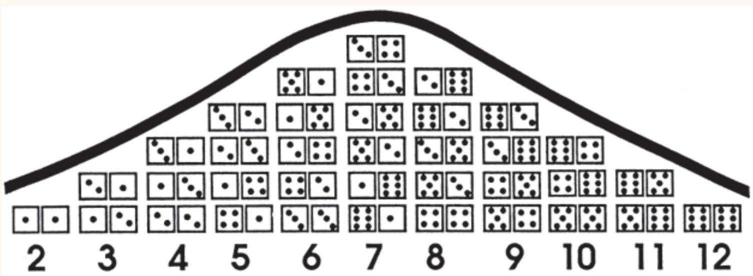
- There is use of words like most likely, no chance, equally likely in daily life, are showing the manner of chance and judgement.
- There are certain experiments whose outcomes have equal chance of occurring. Outcomes of such experiments are known as **equally likely** outcomes.
- An event is a collection of a specific outcome or some of the specific outcomes of the experiment.
- In some random experiments all outcomes have equal chance of occurring.
- As the number of trials increases, the probability of all equally likely outcomes come very close to each other.
- The probability of an event A

$$P(A) = \frac{\text{Number of favourable outcomes of event A}}{\text{Number of total possible outcomes}}$$

- The probability of an event which is certain = 1.
- The probability of an event which is impossible = 0
- The probability of an event always lies between 0 and 1 (0 and 1 inclusive).

Do you Know?

The diagram below shows the 36 possible outcomes when a pair of dice are thrown. It is interesting to notice how the frequency of the outcomes of different possible numbers (2 through 12) illustrate the Gaussian curve.



This curve illustrate the Gaussian curve, name after 19th century famous mathematician Carl Friedrich Gauss.

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15 Proofs in Mathematics

15.1 Introduction

We come across many statements in our daily life. We gauge the worth of each statement. Some statements we consider to be appropriate and true and some we dismiss. There are some we are not sure of. How do we make these judgements? In case there is a statement of conflict about loans or debts. You want to claim that bank owes your money then you need to present documents as evidence of the monetary transaction. Without that, people would not believe you. If we think carefully we can see that in our daily life we need to prove if a statement is true or false. In our conversations in daily life we sometimes do not consider to prove or check statements and accept them without serious examination. That however will not be accepted in mathematics. Consider the following:

1. The sun rises in the east.
2. $3 + 2 = 5$
3. New York is the capital of USA.
4. $4 > 8$
5. How many siblings do you have?
6. Goa has better football team than Bengal.
7. Rectangle has 4 lines of symmetry.
8. $x + 2 = 7$
9. Please come in.
10. What is the probability of getting two consecutive 6's on throws of a 6 sided dice?
11. How are you?
12. The sun is not stationary but moving at high speed all the time.
13. $x < y$
14. Where do you live?

We know, out of these some sentences are false. For example, 4>8 and present New York is not the capital of USA. Some are correct. These include "sun rises in the east." The probability............"

The Sun is not stationary......................

Besides those there are some other sentences that are true for some known cases but not true for other cases, for example $x + 2 = 7$ is true only when $x = 5$ and $x < y$ is only true for those values of x and y where x is less than y.

Look at the other sentences which of them are clearly false or clearly true. These are statements. We say these sentences that can be judged on some criteria, no matter by what process for their being true or false are statements.

Think about these:

1. Please ignore this notice.....
2. The statement I am making is false.
3. This sentence has some words.
4. You may find water on the moon.

Can you say whether these sentences are true or false? Is there any way to check them being true or false?

Look at the first sentence, if you ignore the notice, you do that because it tells you to do so. If you do not ignore the notice, then you have paid some attention to it. So you can never follow it and being an instruction it cannot be judged on a true/false scale. 2^{nd} and 3^{rd} sentences are talking about themselves. 4th sentence have words that show only likely or possibility and hence ambiguity of being on both sides.

The sentences which are talking about themselves and the sentences with possibility are not statements.

Do This

Make 5 more sentences and check whether they are statements or not. Give reasons.

15.2 Mathematical Statements

We can write infinetely large number of sentences. You can think the kind of sentences you use and can you count the number of sentences you speak? Not all these however, they can be judged on the criteria of false and true. For example, consider, please come in. Where do you live? Such sentences can also be very large in number.

All these the sentences are not statements. Only those that can be judged to be true or false but not both are statements. The same is true for mathematical statements. A mathemtical statement can not be ambiguous. In mathematics a statement is only acceptable if it is either true or false. Consider the following sentences:

1. 3 is a prime number.
2. Product of two odd integers is even.
3. For any real number x; $4x + x = 5x$
4. The earth has one moon.
5. Ramu is a good driver.
6. Bhaskara has written a book "Leelavathi".
7. All even numbers are composite.
8. A rhombus is a square.
9. $x > 7$.
10. 4 and 5 are relative primes.

11. Silver fish is made of silver.
12. Humans are meant to rule the earth.
13. Fo any real number x, $2x > x$.
14. Havana is the capital of Cuba.

Which of these are mathematical and which are not mathematical statements?

15.3 Verifying the Statements

Let us consider some of the above sentences and discuss them as follows:

Example-1. We can show that (1) is true from the definition of a prime number.

Which of the sentences from the above list are of this kind of statements that we can prove mathematically? (Try to prove).

Example-2. "Product of two odd integers is even". Consider 3 and 5 as the odd integers. Their product is 15, which is not even.

Thus it is a statement which is false. So with one example we have showed this. Here we are able to verify the statement using an example that runs counter to the statement. Such an example, that counters a statement is called a counter example.

Try This

Which of the above statements can be tested by giving a counter example ?

Example-3. Among the sentences there are some like **"Humans are meant to rule the earth"** or **"Ramu is a good driver**."

These sentences are ambiguous sentences as the meaning of ruling the earth is not specific. Similarly, the definition of a good driver is not specified.

We therefore recognize that a 'mathematical statement' must comprise of terms that are understood in the same way by everyone.

Example-4. Consider some of the other sentences like

The earth has one Moon.

Bhaskara has written the book "Leelavathi"

Think about how would you verify these to consider as statements?

These are not ambiguous statements but needs to be tested. They require some observations or evidences. Besides, checking this statement cannot be based on using previously known results. The first sentence require observations of the solar system and more closely of the earth. The second sentence require other documents, references or some other records.

Mathematical statements are of a distinct nature from these. They cannot be proved or justified by getting evidence while as we have seen, they can be disproved by finding an example counter to the statement. In the statement for any real number $2x > x$, we can take $x = -1$ or $-\frac{1}{2}$ .... and disprove the statement by giving counter example. You might have also noticed that $2x > x$ is true with a condition on x i.e. x belong to set N.

Example-5. Restate the following statements with appropriate conditioins, so that they become true statements.

i. For every real number x, $3x > x$.

ii. For every real number x, $x^2 \geq x$.

iii. If you divide a number by two, you will always get half of that number.

iv. The angle subtended by a chord of a circle at a point on the circle is 90°.

v. If a quadrilateral has all its sides equal, then it is a square.

Solution :

i. If $x > 0$, then $3x > x$.

ii. If $x \leq 0$ or $x \geq 1$, then $x^2 \geq x$.

iii. If you divide a number other than 0 by 2, then you will always get half of that number.

iv. The angle subtended by a diameter of a circle at a point on the circle is 90°.

v. If a quadrilateral has all its sides and interior angles equal, then it is a square.

EXERCISE - 15.1

1. State whether the following sentences are always true, always false or ambiguous. Justify your answer.

   i. There are 27 days in a month.
   ii. Makarasankranthi falls on a Friday.
   iii. The temperature in Hyderabad is 2°C.
   iv. The earth is the only planet where life exist.
   v. Dogs can fly.
   vi. February has only 28 days.

2. State whether the following statements are true or false. Give reasons for your answers.

   i. The sum of the interior angles of a quadrilateral is 350°.
   ii. For any real number x, $x^2 \geq 0$.
   iii. A rhombus is a parallelogram.
   iv. The sum of two even numbers is even.
   v. Square numbers can be written as the sum of two odd numbers.

3. Restate the following statements with appropriate conditions, so that they become true statements.

   i. All numbers can be represented as the product of prime factors.
   ii. Two times a real number is always even.
   iii. For any x, $3x + 1 > 4$.
   iv. For any x, $x^3 \geq 0$.
   v. In every triangle, a median is also an angle bisector.

4. Disprove, by finding a suitable counter example, the statement $x^2 > y^2$ for all $x > y$.

15.4 REASONING IN MATHEMATICS

We human beings are naturally curious. This curiosity makes us to interact with the world. What happens if we push this? What happens if we stuck our finger in that? What happens if we make various gestures and expressions? From this experimentation, we begin to form a more or less consistant picture of the way that the physical world behaves. Gradually, in all situations, we make a shift from

'What happens if.....?' to *'this will happen if'*

The experimentation moves on to the exploration of new ideas and the refinement of our world view of previously understood situations. This description of the playtime pattern very nicely models the concept of 'making and testing hypothesis.' It follows this pattern:

- Make some observations, Collect data based on the observations.
- Draw conclusion (called a 'hypothesis') which will explain the pattern of the observations.
- Test out hypothesis by making some more targeted observations.

So, we have

- A **hypothesis** is a statement or idea which gives an explanation to a series of observations.

Sometimes, following observation, a hypothesis will clearly need to be refined or rejected. This happens if a *single* contradictory observation occurs. In general we use word conjecture in mathematics instead of hypothesis. You will learn the similarities and difference between these two in the higher classes.

15.4.1 Using deductive reasoning in hypothesis testing

There is often confusion between the ideas surrounding proof, making and testing an experimental hypothesis which is mathematics, which is science. The difference is rather simple:

- Mathematics is based on *deductive reasoning* : a proof is a logical deduction from a set of clear inputs.
- Science is based on *inductive reasoning* : hypotheses are strengthened or rejected based on an accumulation of experimental evidence.

Of course, to be good at science, you need to be good at deductive reasoning, although experts at deductive reasoning need not be mathematicians.

Detectives, such as Sherlock Holmes and Hercule Poirot, are such experts : they collect evidence from a crime scene and then draw logical conclusions from the evidence to support the hypothesis that, for example, person M. committed the crime. They use this evidence to create sufficiently compelling deductions to support their hypothesis *beyond reasonable doubt*. The key word here is 'reasonable'.

15.4.2 Deductive Reasoning

The main logical tool used in establishing the truth of an **unambiguous** statement is *deductive reasoning*. To understand what deductive reasoning is all about, let us begin with a puzzle for you to solve.

You are given four cards. Each card has a number printed on one side and a letter on the other side.

$\boxed{A}$ $\boxed{V}$ $\boxed{8}$ $\boxed{5}$

Suppose you are told that these cards follow the rule:

"If a card has an odd number on one side, then it has a vowel on the other side."

What is the **smallest number** of cards you need to turn over to check if the rule is true?

Of course, you have the option of turning over all the cards and checking. But can you manage with turning over a fewer number of cards?

Notice that the statement mentions that a card with an odd number on one side has a vowel on the other. It does not state that a card with a vowel on one side must have an odd number on the other side. That may or may not be so. The rule also does not state that a card with an even number on one side must have a consonant on the other side. It may or may not.

So, do we need to turn over $\boxed{A}$? No! Whether there is an even number or an odd number on the other side, the rule still holds.

What about $\boxed{8}$? Again we do not need to turn it over, because whether there is a vowel or a consonant on the other side, the rule still holds.

But you do need to turn over $\boxed{V}$ and 5. if $\boxed{V}$ has an odd number on the other side, then the rule has been broken. Similarly, if $\boxed{5}$ has a consonant on the other side, then the rule has been broken.

The kind of reasoning we have used to solve the puzzle is called **deductive reasoning**. It is called 'deductive' because we arrive at (i.e., deduce or infer) a result or a statement from a previously established statement using logic. For example, in the puzzle by a series of logical arguments we deduced that we need to turn over only $\boxed{V}$ and $\boxed{5}$.

Deductive reasoning also helps us to conclude that a particular statement is true, because it is a special case of a more general statement that is known to be true. For example, once we prove that the product of two even numbers is always even, we can immediately conclude (without computation) that 56702 × 19992 is even simply because 56702 and 19992 are even.

Consider some other examples of deductive reasoning:

i. If a number ends in '0' it is divisible by 5. 30 ends in 0.

From the above two statements we can deduce that 30 is divisible by 5 because it is given that the number ends in 0 is divisible by 5.

ii. Some singers are poets. All lyrists are Poets.

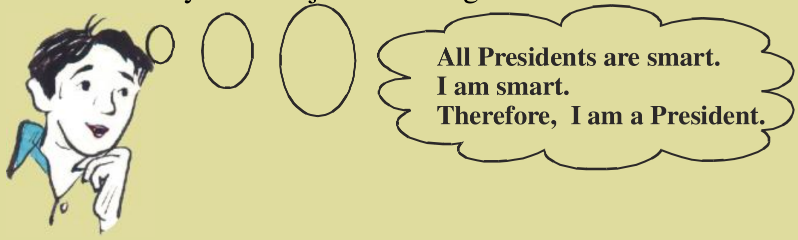
Here the deduction based on two statement is wrong. (Why?) All lyricist are poets (wrong). Because we are not sure about it. There are three posibilities (i) all lyricists could be poets, (ii) few could be poets or (iii) none of the lyricists is a poet.

You may come to a conclusion that if - then conditional statement comes into deductive reasoning. In mathematics we use this reasoning a lot like if linear pair of angles are 180°. Then only the sum of angles in a triangle is equal to 180°. Like wise if we are using decimal number system to write a number 5. If we use the binary system we represent the quantity by 101.

Unfortunately we do not always use correct reasoning in our daily life. We often come to many conclusions based on faulty reasoning. For example, if your friend does not talk to you one day, then you may conclude that she is angry with you. While it may be true that "if she is angry at me she will not talk to me", it may also be true that "if she is busy, she will not talk to me. Why don't you examine some conclusions that you have arrived at in your day-to-day existence, and see if they are based on valid or faulty reasoning?

EXERCISE - 15.2

1. Use deductive reasoning to answer the following:
   i. Human beings are mortal. Jeevan is a human being. Based on these two statements, what can you conclude about Jeevan ?
   ii. All Telugu people are Indians. X is an Indian. Can you conclude that X belongs to Telugu people.
   iii. Martians have red tongues. Gulag is a Martian. Based on these two statements, what can you conclude about Gulag?
   iv. What is the fallacy in the Raju's reasoning in the cartoon below?



2. Once again you are given four cards. Each card has a number printed on one side and a letter on the other side. Which are the only two cards you need to turn over to check whether the following rule holds?

   "If a card has a consonant on one side, then it has an odd number on the other side."

B	3	U	8

3. Think of this puzzle What do you need to find a chosen number from this square?

   Four of the clues below are true but do nothing to help in finding the number.

   Four of the clues are necessary for finding it.

   Here are eight clues to use:

   a. The number is greater than 9.
   b. The number is not a multiple of 10.
   c. The number is a multiple of 7.
   d. The number is odd.
   e. The number is not a multiple of 11.
   f. The number is less than 200.
   g. Its ones digit is larger than its tens digit.
   h. Its tens digit is odd.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

   What is the number?

   Can you sort out the four clues that help and the four clues that do not help in finding it?

   First follow the clues and strike off the number which comes out from it.

   Like - from the first clue we come to know that the number is not from 1 to 9. (strike off numbers from 1 to 9).

   After completing the puzzle, see which clue is important and which is not?

15.5 Theorems, Conjectures and Axioms

So far we have discussed statements and how to check their validity. In this section, you will study how to distinguish between the three different kinds of statements, Mathematics is built up from, namely, a theorem, a conjecture and an axiom.

You have already come across many theorems before. So, what is a theorem? A mathematical statement whose truth has been established (proved) is called a *theorem*. For example, the following statements are theorems.

Theorem-15.1 : The sum of the interior angles of a triangle is 180°.

Theorem-15.2 : The product of two odd natural numbers is odd.

Theorem-15.3 : The product of any two consecutive even natural numbers is divisible by 4.

A *conjecture* is a statement which we believe as true, based on our mathematical understanding and experience, i.e., our mathematical intuition. The conjecture may turn out to be true or false. If we can prove it, then it becomes a theorem. Mathematicians often come up with conjectures by looking for patterns and making intelligent mathematical guesses. Let us look at some patterns and see what kind of intelligent guesses we can make.

While studying some cube numbers Raju noticed that "if you take three consecutive whole numbers and multiply them together and then add the middle number of the three, you get the middle number cubed"; *e.g.*, 3, 4, 5, gives $3 \times 4 \times 5 + 4 = 64$, which is a perfect cube. Does this always work? Take some more consecutive numbers and check it.

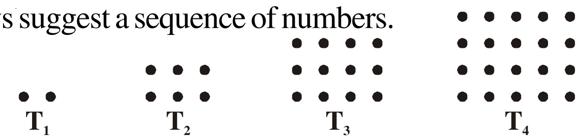
Rafi took 6, 7, 8 and checked this conjecture. Here 7 is the middle term so according to the rule $6 \times 7 \times 8 + 7 = 343$, which is also a perfect cube. Try to generalize it by taking numbers as n, $n + 1$, $n + 2$. See other example:

Example-6. The following geometric arrays suggest a sequence of numbers.

(a) Find the next three terms.

(b) Find the 100th term.

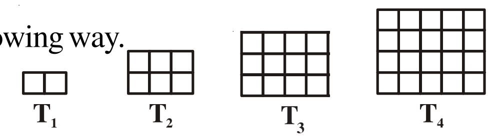
(c) Find the n^{th} term.



The dots here arranged in such a way that they form a rectangle. Here $T_1 = 2$, $T_2 = 6$, $T_3 = 12$, $T_4 = 20$ and so on. Can you guess what T_5 is? What about T_6? What about T_n?

Make a conjecture about T_n.

It might help if you redraw them in the following way.



Solution :

$$\begin{array}{cccccc} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 \\ 2 & 6 & 12 & 20 & ? & \ldots\ldots \\ & +4 & +6 & +8 & +10 & \end{array}$$

So, $T_5 = T_4 + 10 = 20 + 10 = 30 = 5 \times 6$

$T_6 = T_5 + 12 = 30 + 12 = 42 = 6 \times 7$ ..... Try for T_7?

$T_{100} = 100 \times 101 = 10,100$

$T_n = n \times (n + 1) = n^2 + n$

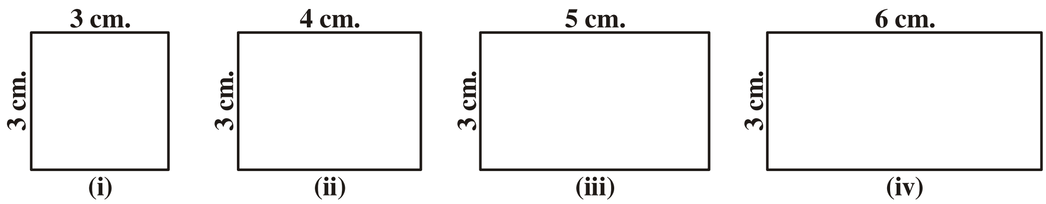
This type of reasoning which is based on examining a variety of cases or sets of data, discovering patterns and forming conclusions is called ***inductive reasoning***. Inductive reasoning is very helpful technique for making conjecture.

Gold bach the renounced mathematician, observed a pattern:

$6 = 3 + 3$	$8 = 3 + 5$	$10 = 3 + 7$
$12 = 5 + 7$	$14 = 11 + 3$	$16 = 13 + 3 = 11 + 5$

From the pattern Gold bach in 1743 reasoned that every even number greater than 4 can be written as the sum of two primes (not necessarily distinct primes). His conjecture has not been proved to be true or false so far. Perhaps you will prove that this result is true or false and will become famous.

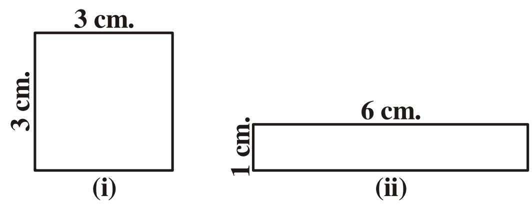
But just by looking few patterns some time lead us to a wrong conjecture like: in class 8th Janvi and Kartik while studying Area and Perimeter chapter..... observed a pattern



	(i)	(ii)	(iii)	(iv)
Perimeter :	**12 cm.**	**14 cm.**	**16 cm.**	**18 cm.**
Area :	**9 cm^2**	**12 cm^2**	**15 cm^2**	**18 cm^2**

and stated a conjecture that when the perimeter of the rectangle increases the area will also increase. What do you think? Are they right?

While working on this pattern. Inder drew some rectangles and disproved the conjecture stated by Janvi and Kartik.

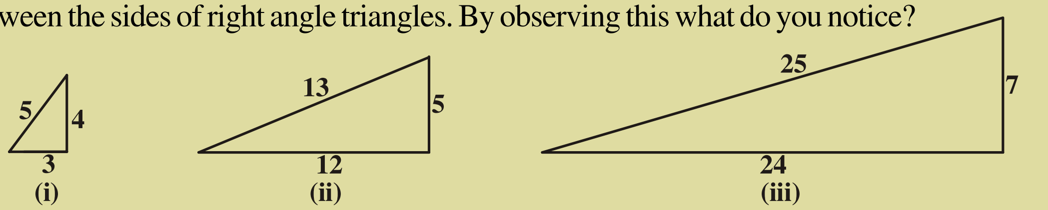


	(i)	(ii)
Perimeter :	**12 cm.**	**14 cm.**
Area :	**9 cm^2**	**6 cm^2**

Do you understand that while making a conjecture we have to look all the possibilities.

Try This

Envied by the popularity of Pythagoras his disciple claimed a different relation between the sides of right angle triangles. By observing this what do you notice?



Liethagoras Theorem : In any right angle triangle the square of the smallest side equals the sum of the other sides.

Check this conjucture, whether it is right or wrong.

You might have wondered - do we need to prove every thing we encounter in mathematics and if not, why not?

In mathematics some statements are assumed to be true and are not proved, these are self-evident truths' which we take to be true without proof. These statements are called *axioms*. In chapter 3, you would have studied the axioms and postulates of Euclid. (We do not distinguish between axioms and postulates these days generally we use word postulate in geometry).

For example, the first postulate of Euclid states:

A straight line may be drawn from any point to any other point.

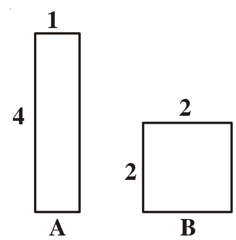
And the third postulate states:

A circle may be drawn with any centre and any radius.

These statements appear to be perfectly true and Euclid assumed them to be true. Why? This is because we cannot prove everything and we need to start somewhere, we need some statements which we accept as true and then we can build up our knowledge using the rules of logic based on these axioms.

You might then wonder why don't we just accept all statements to be true when they appear self evident. There are many reasons for this. Very often our intuition can be wrong, pictures or patterns can deceive and the only way to be sure that something is true is to prove it. For example, many of us believe that if a number is added to another number, the result will be large than the numbers. But we know that this is not always true : for example 5 + (-5) = 0, which is smaller than 5.

Also, look at the figures. Which has bigger area ?



It turns out that both are of exactly the same area, even though B appears bigger.

You might then wonder, about the validity of axioms. Axioms have been chosen based on our intuition and what appears to be self-evident. Therefore, we expect them to be true. However, it is possible that later on we discover that a particular axiom is not true. What is a safeguard against this possibility? We take the following steps:

i. Keep the axioms to the bare minimum. For instance, based only on axioms and five postulates of Euclid, we can derive hundreds of theorems.

ii. Make sure that the axioms are consistent.

We say a collection of axioms is *inconsistent*, if we can use one axiom to show that another axiom is not true. For example, consider the following two statements. We will show that they are inconsistent.

Statement-1 : No whole number is equal to its successor.

Statement-2 : A whole number divided by zero is a whole number.

(Remember, **division by zero is not defined**. But just for the moment, we assume that it is possible, and see what happens.)

From Statement-2, we get $\frac{1}{0} = a$, where a is some whole number. This implies that, 1=0. But this disproves Statement-1, which states that no whole number is equal to its successor.

iii. A false axiom will, sooner or later, result into contradiction. We say that *there is a contradiction, when we find a statement such that, both the statement and its negation are true.* For example, consider Statement-1 and Statement-2 above once again.

From Statement-1, we can derive the result that $2 \neq 1$.

Let $x = y$

$x \times x = xy$

$x^2 = xy$

$x^2 - y^2 = xy - y^2$

$(x+y)(x-y) = y(x-y)$ From Statement-2, we can cancel $(x - y)$ from both the sides.

$x + y = y$

But $x = y$

so $x + x = x$

or $2x = x$

$2 = 1$

So we have both the statements $2 \neq 1$ and its negation, $2 = 1$ are true. This is a contradiction. The contradiction arose because of the false axiom, that a whole number divided by zero is a whole number.

So, the statement we choose as axioms require a lot of thought and insight. We must make sure they do not lead to inconsistencies or logical contradictions. Moreover, the choice of axioms themselves, sometimes leads us to new discoveries.

We end the section by recalling the differences between an axiom, a theorem and a conjecture. An **axiom** is a mathematical statement which is true without proof; a **conjecture** is a mathematical statement whose truth or falsity is yet to be established; and a **theorem** is a mathematical statement whose truth has been logically established.

EXERCISE - 15.3

1. (i) Take any three consecutive odd numbers and find their product;

   for example, $1 \times 3 \times 5 = 15$, $3 \times 5 \times 7 = 105$, $5 \times 7 \times 9$ - .....

   (ii) Take any three consecutive even numbers and add them, say,

   $2 + 4 + 6 = 12$, $4 + 6 + 8 = 18$, $6 + 8 + 10 = 24$, $8 + 10 + 12 = 30$ and so on.

   Is there any pattern can you guess in these sums? What can you conjecture about them?

2. Go back to Pascal's triangle.

   Line-1 : $1 = 11^0$

   Line-2 : $11 = 11^1$

   Line-3 : $121 = 11^2$

   Make a conjecture about Line-4 and Line-5.

   Does your conjecture hold? Does your conjecture hold for Line-6 too?

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

3. Look at the following pattern:

   i) $28 = 2^2 \times 7^1$, Total number of factors $(2+1)(1+1) = 3 \times 2 = 6$
   28 is divisible by 6 factors i.e. 1, 2, 4, 7, 14, 28

   ii) $30 = 2^1 \times 3^1 \times 5^1$, Total number of factors $(1+1)(1+1)(1+1) = 2 \times 2 \times 2 = 8$
   30 is divisible by 8 factors i.e. 1, 2, 3, 5, 6, 10, 15, 30

   Find the pattern.
   (**Hint :** Product of every prime base exponent +1)

4. Look at the following pattern:

   $1^2 = 1$

   $11^2 = 121$

   $111^2 = 12321$

   $1111^2 = 1234321$

   $11111^2 = 123454321$

Make a conjecture about each of the following:

$111111^2 =$

$1111111^2 =$

Check if your conjecture is true.

5. List five axioms (postulates) used in this book.
6. In a polynomial $p(x) = x^2 + x + 41$ put different values of x and find $p(x)$. Can you conclude after putting different values of x that $p(x)$ is prime for all. Is x an element of N? Put $x = 41$ in $p(x)$. Now what do you find?

15.6 What is a Mathematical Proof?

Before you study proofs in mathematics, you are mainly asked to verify statements.

For example, you might have been asked to verify with examples that "the product of two odd numbers is odd". So you might have picked up two random odd numbers, say 15 and 2005 and checked that $15 \times 2005 = 30075$ is odd. You might have done so for many more examples.

Also, you might have been asked as an activity to draw several triangles in the class and compute the sum of their interior angles. Apart from errors due to measurement, you would have found that the interior angles of a triangle add up to 180°.

What is the flaw in this method? There are several problems with the process of verification. While it may help you to make a statement you believe is true, you cannot be *sure* that it is true in *all* cases. For example, the multiplication of several pairs of even numbers may lead us to guess that the product of two even numbers is even. However, it does not ensure that the product of all pairs of even numbers is even. You cannot physically check the products of all possible pairs of even numbers because they are endless. Similarly, there may be some triangles which you have not yet drawn whose interior angles do not add up to 180°.

Moreover, verification can often be misleading. For example, we might be tempted to conclude from Pascal's triangle (Q.2 of Exercise), based on earlier verification, that $11^5 = 15101051$. But in fact $11^5 = 161051$.

So, you need another approach that does not depend upon verification for some cases only. There is another approach, namely 'proving a statement'. A process which can establish the truth of a mathematical statement based purely on logical arguments is called a *mathematical proof*.

To make a mathematical statement false, we just have to produce a single counter-example. So while it is not enough to establish the validity of a mathematical statement by checking or verifying it for thousands of cases, it is enough to produce one counter example to *disprove* a statement.

Let us look what should be our procedure to prove.

i. First we must understand clearly, what is required to prove, then we should have a rough idea how to proceed.

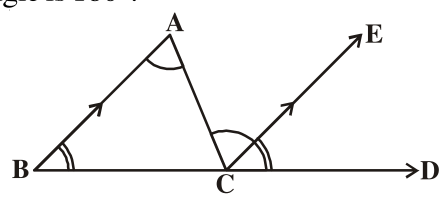
ii. A proof is made up of a successive sequence of mathematical statements. Each statement is a proof logically deduced from a previous statement in the proof or from a theorem proved earlier or an axiom or our hypothesis and what is given.

iii. The conclusion of a sequence of mathematically true statements laid out in a logically correct order should be what we wanted to prove, that is, what the theorem claims.

To understand that, we will analyse the theorem and its proof. You have already studied this theorem in chapter-4. We often resort to diagrams to help us to prove theorems, and this is very important. However, each statement in proof has to be established using only logic. Very often we hear or said statement like those two angles must be 90°, because the two lines look as if they are perpendicular to each other. Beware of being deceived by this type of reasoning.

Theorem-15.4 : The sum of three interior angles of a triangle is 180°.

Proof : Consider a triangle ABC.



We have to prove that

$\angle ABC + \angle BCA + \angle CAB = 180°$

Construct a line CE parallel to BA through C and produce line BC to D.

CE is parallel to BA and AC is transversal.

So, $\angle CAB = \angle ACE$, which are alternate angles. ..... (1)

Similarly, $\angle ABC = \angle DCE$ which are corresponding angles. ..... (2)

adding eq. (1) and (2) we get

$$\angle CAB + \angle ABC = \angle ACE + \angle DCE \quad ..... (3)$$

add $\angle BCA$ on both the sides.

We get, $\angle ABC + \angle BCA + \angle CAB = \angle DCE + \angle BCA + \angle ACE$ ..... (4)

But $\angle DCE + \angle BCA + \angle ACE = 180°$, since they form a straight angle...... (5)

Hence, $\angle ABC + \angle BCA + \angle CAB = 180°$

Now, we see how each step has been logically connected in the proof.

Step-1: Our theorem is concerned with a property of triangles. So we begin with a triangle ABC.

Step-2: The construction of a line CE parallel to BA and producing BC to D is a vital step to proceed so that to be able to prove the theorem.

Step-3: Here we conclude that $\angle CAB = \angle ACE$ and $\angle ABC = \angle DCE$, by using the fact that CE is parallel to BA (construction), and previously known theorems, which states that if two parallel lines are intersected by a transversal, then the alternate angles and corresponding angles are equal.

Step-4: Here we use Euclid's axiom which states that "if equals are added to equals, the wholes are equal" to deduce $\angle ABC + \angle BCA + \angle CAB = \angle DCE + \angle BCA + \angle ACE$.

That is, the sum of three interior angles of a triangle is equal to the sum of angles on a straight line.

Step-5: Here in concluding the statement we use Euclid's axiom which states that "things which are equal to the same thing are equal to each other" to conclude that

$$\angle ABC + \angle BCA + \angle CAB = \angle DCE + \angle BCA + \angle ACE = 180°$$

This is the claim made in the theorem we set to prove.

You now prove theorem-15.2 and 15.3 without analysing them.

Theorem-15.5 : The product of two odd natural numbers is odd.

Proof : Let x and y be any two odd natural numbers.

We want to prove that xy is odd.

Since x and y are odd, they can be expressed in the form $x = (2m - 1)$, for some natural number m and $y = 2n - 1$, for some natural number n.

Then, $xy = (2m - 1)(2n - 1)$

$= 4mn - 2m - 2n + 1$

$= 4mn - 2m - 2n + 2 - 1$

$= 2(2mn - m - n + 1) - 1$

Let $2mn - m - n + 1 = l$, any natural number, replace it in the above equation.

$= 2l - 1, l \in N$

This is definitely an odd number.

Theorem-15.6 : The product of any two consecutive even natural numbers is divisible by 4.

Any two consecutive even number will be of the form $2m$, $2m + 2$, for some natural number n. We have to prove that their product $2m\,(2m + 2)$ is divisible by 4. (Now try to prove this yourself).

We conclude this chapter with a few remarks on the difference between how mathematicians discover results and how formal rigorous proofs are written down. As mentioned above, each proof has a key intiative idea. Intution is central to a mathematicians' way of thinking and discovering results. A mathematician will often experiment with several routes of thought, logic and examples, before she/he can hit upon the correct solution or proof. It is only after the creative phase subsides that all the arguments are gathered together to form a proper proof.

We have discussed both inductive reasoning and deductive reasoning with some examples.

It is worth mentioning here that the great Indian mathematician Srinivasa Ramanujan used very high levels of intuition to arrive at many of his statements, whch he claimed were true. Many of these have turned out to be true and as well as known theorems.

EXERCISE - 15.4

1. State which of the following are mathematical statements and which are not? Give reason.
   - i. She has blue eyes
   - ii. $x + 7 = 18$
   - iii. Today is not Sunday.
   - iv. For each counting number x, $x + 0 = x$
   - v. What time is it?
2. Find counter examples to disprove the following statements:
   - i. Every rectangle is a square.
   - ii. For any integers x and y, $\sqrt{x^2 + y^2} = x + y$
   - iii. If n is a whole number then $2n^2 + 11$ is a prime.
   - iv. Two triangles are congruent if all their corresponding angles are equal.
   - v. A quadrilateral with all sides are equal is a square.
3. Prove that the sum of two odd numbers is even.
4. Prove that the product of two even numbers is an even number.

5. Prove that if x is odd, then x^2 is also odd.

6. Examine why they work ?

   i. Choose a number. Double it. Add nine. Add your original number. Divide by three. Add four. Subtract your original number. Your result is seven.

   ii. Write down any three-digit number (for example, 425). Make a six-digit number by repeating these digits in the same order (425425). Your new number is divisible by 7, 11, and 13.

WHAT WE HAVE DISCUSSED

1. The sentences that can be judged on some criteria, no matter by what process for their being true or false are statements.
2. Mathematical statements are of a distinct nature from general statements. They can not be proved or justified by getting evidence while they can be disproved by finding a counter example.
3. Making mathematical statements through observing patterns and thinking of the rules that may define such patterns.

   A hypothesis is a statement of idea which gives an explanation to a sense of observation.
4. A process which can establish the truth of a mathematical statement based purely on logical arguments is called a mathematical proof.
5. Axioms are statements which are assumed to be true without proof.
6. A conjecture is a statement we believe is true based on our mathematical intution, but which we are yet to prove.
7. A mathematical statement whose truth has been established or proved is called a theorem.
8. The prime logical method in proving a mathematical statement is deductive reasoning.
9. A proof is made up of a successive sequence of mathematical statements.
10. Begining with given (Hypothesis) of the theorem and arrive at the conclusion by means of a chain of logical steps is mostly followed to prove theorems.
11. The proof in which, we start with the assumption contrary to the conclusion and arriving at a contradiction to the hypothesis is another way that we establish the original conclusion is true is another type of deductive reasoning.
12. The logical tool used in establishment the truth of an unambiguious statements to deductive reasoning.
13. The resoning which is based on examining of variety of cases or sets of data discovering pattern and forming conclusion is called Inductive reasoning.