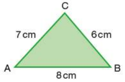
21 Geometrical constructions and scale drawings

Constructing triangles

Triangles can be drawn accurately by using a ruler and a pair of compasses. This is called **constructing** a triangle.

Worked example The sketch shows the triangle ABC.

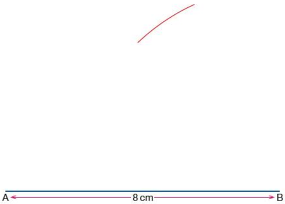


Construct the triangle ABC given that:

AB = 8 cm, BC = 6 cm and AC = 7 cm

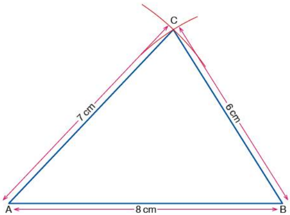
- Draw the line AB using a ruler:

- Open up a pair of compasses to 6 cm. Place the compass point on B and draw an arc:



Note that every point on the arc is 6 cm away from B.

- Open up the pair of compasses to 7 cm. Place the compass point on A and draw another arc, with centre A and radius 7 cm, ensuring that it intersects with the first arc. Every point on the second arc is 7 cm from A. Where the two arcs intersect is point C, as it is both 6 cm from B and 7 cm from A.
- Join C to A and C to B:



Exercise 21.1

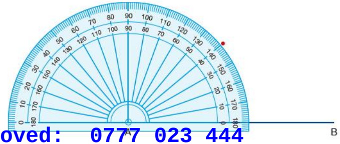
Using only a ruler and a pair of compasses, construct the following triangles:

1. $\triangle ABC$ where $AB = 10$ cm, $AC = 7$ cm and $BC = 9$ cm
2. $\triangle LMN$ where $LM = 4$ cm, $LN = 8$ cm and $MN = 5$ cm
3. $\triangle PQR$, an equilateral triangle of side length 7 cm
4. a) $\triangle ABC$ where $AB = 8$ cm, $AC = 4$ cm and $BC = 3$ cm
   b) Is this triangle possible? Explain your answer.

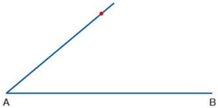
Constructing simple geometric figures

Worked example Using a ruler and a protractor only, construct the parallelogram ABCD in which AB = 6 cm, AD = 3 cm and angle DAB = 40°.

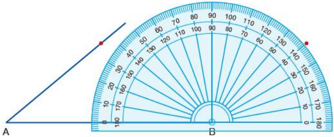
- Draw a line 6 cm long and label it AB.
- Place the protractor on A and mark an angle of 40°.



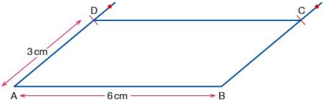
- Draw a line from A through the marked point.



- Place the protractor on B and mark an angle of 40° reading from the inner scale. Draw a line from B through the marked point.



- Measure 3 cm from A and mark the point D.
- Measure 3 cm from B and mark the point C.
- Join D to C.



Exercise 21.2

1. Using only a ruler and a protractor, construct the triangle PQR in which PQ = 4 cm, angle RPQ = 115° and angle RQP = 30°.

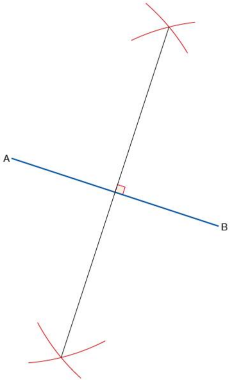
2. Using only a ruler and a protractor, construct the trapezium ABCD in which AB = 8 cm, AD = 4 cm, angle DAB = 60° and angle ABC = 90°.

Bisecting lines and angles

The word **bisect** means 'to divide in half'. Therefore, to bisect an angle means to divide an angle in half. Similarly, to bisect a line means to divide a line in half. A **perpendicular bisector** to a line is another line which divides it in half and meets the original line at right angles. To bisect either a line or an angle involves the use of a pair of compasses.

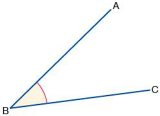
Worked examples **a)** A line AB is drawn below. Construct the perpendicular bisector to AB.

- Open a pair of compasses to more than half the distance AB.
- Place the compass point on A and draw arcs above and below AB.
- With the same radius, place the compass point on B and draw arcs above and below AB. Note that the two pairs of arcs should intersect (see diagram below).
- Draw a line through the two points where the arcs intersect:

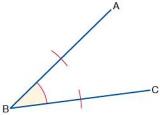


The line drawn is known as the perpendicular bisector of AB, as it divides AB in half and also meets it at right angles.

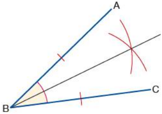
b) Using a pair of compasses, bisect the angle ABC below:



- Open a pair of compasses and place the point on B. Draw two arcs such that they intersect the arms of the angle:

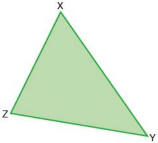


- Place the compasses in turn on the points of intersection, and draw another pair of arcs of the same radius. Ensure that they intersect.
- Draw a line through B and the point of intersection of the two arcs. This line bisects angle ABC.



Exercise 21.3

1. Draw a triangle similar to the one shown below:

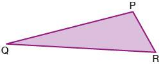


Construct the perpendicular bisector of each of the sides of your triangle.

Use a pair of compasses to draw a circle using the point where the three perpendicular bisectors cross as the centre and passing through the points X, Y and Z.

This is called the **circumcircle** of the triangle.

2. Draw a triangle similar to the one shown below:



By construction, draw a circle to pass through points P, Q and R.

Scale drawings

Scale drawings are used when an accurate diagram, drawn in proportion, is needed. Common uses of scale drawings include maps and plans. The use of scale drawings involves understanding how to scale measurements.

Worked examples

a) A map is drawn to a scale of 1 : 10 000. If two objects are 1 cm apart on the map, how far apart are they in real life? Give your answer in metres.

A scale of 1 : 10 000 means that 1 cm on the map represents 10 000 cm in real life.

Therefore the distance = 10 000 cm
= 100 m

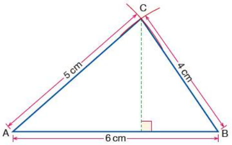
b) A model boat is built to a scale of 1 : 50. If the length of the real boat is 12 m, calculate the length of the model boat in cm.

A scale of 1 : 50 means that 50 cm on the real boat is 1 cm on the model boat.

12 m = 1200 cm

Therefore the length of the model boat = 1200 ÷ 50 cm
= 24 cm

c) i) Construct, to a scale of 1 : 1, a triangle ABC such that AB = 6 cm, AC = 5 cm and BC = 4 cm.



ii) Measure the perpendicular length of C from AB.
Perpendicular length is 3.3 cm.

iii) Calculate the area of the triangle.

$$\text{Area} = \frac{\text{base length} \times \text{perpendicular height}}{2}$$

$$\text{Area} = \frac{6 \times 3.3}{2}\ \text{cm} = 9.9\ \text{cm}^2$$

Exercise 21.4

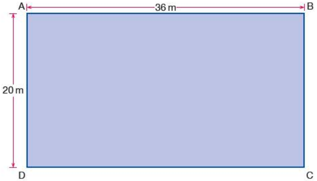
1. In the following questions, both the scale to which a map is drawn and the distance between two objects on the map are given.
   Find the real distance between the two objects, giving your answer in metres.

a)	1 : 10 000	3 cm	b)	1 : 10 000	2.5 cm
c)	1 : 20 000	1.5 cm	d)	1 : 8000	5.2 cm

2. In the following questions, both the scale to which a map is drawn and the true distance between two objects are given.
   Find the distance between the two objects on the map, giving your answer in cm.

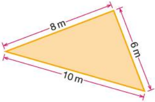
a)	1 : 15 000	1.5 km	b)	1 : 50 000	4 km
c)	1 : 10 000	600 m	d)	1 : 25 000	1.7 km

3. A rectangular pool measures 20 m by 36 m as shown below:



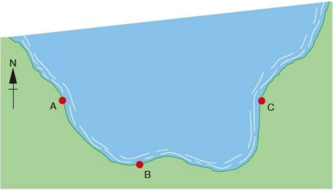
   a) Construct a scale drawing of the pool, using 1 cm for every 4 m.
   b) A boy swims across the pool in such a way that his path is the perpendicular bisector of BD. Show, by construction, the path that he takes.
   c) Work out the distance the boy swam.

4. A triangular enclosure is shown in the diagram below:



   a) Using a scale of 1 cm for each metre, construct a scale drawing of the enclosure.
   b) Calculate the true area of the enclosure.

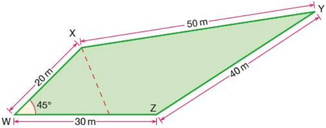
5. Three radar stations A, B and C pick up a distress signal from a boat at sea.



C is 24 km due East of A, AB = 12 km and BC = 18 km. The signal indicates that the boat is equidistant from all three radar stations.

a) By construction and using a scale of 1 cm for every 3 km, locate the position of the boat.
b) What is the boat's true distance from each radar station?

6. A plan view of a field is shown below:



a) Using a scale of 1 cm for every 5 m, construct a scale drawing of the field.
b) A farmer divides the field by running a fence from X in such a way that it bisects angle WXY. By construction, show the position of the fence on your diagram.
c) Work out the length of fencing used.

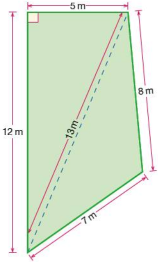
Student assessment 1

1. Construct △ABC such that AB = 8 cm, AC = 6 cm and BC = 12 cm.

2. Three players, P, Q and R, are approaching a football. Their positions relative to each other are shown below:

The ball is equidistant from all three players. Copy the diagram and show, by construction, the position of the ball.

3. A plan of a living room is shown below:



a) Using a pair of compasses, construct a scale drawing of the room using 1 cm for every metre.
b) Using a set square if necessary, calculate the total area of the actual living room.

Student assessment 2

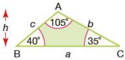
1. a) Draw an angle of 320°.
   b) Using a pair of compasses, bisect the angle.

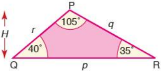
2. In the following questions, both the scale to which a map is drawn and the true distance between two objects are given. Find the distance between the two objects on the map, giving your answer in cm.
   a) 1 : 20 000 4.4 km b) 1 : 50 000 12.2 km

3. a) Construct a regular hexagon with sides of length 3 cm.
   b) Calculate its area, showing your method clearly.

22 Similarity

NB: All diagrams are not drawn to scale.





Similar shapes

Two polygons are said to be **similar** if a) they are equi-angular and b) corresponding sides are in proportion.

For triangles, being equi-angular implies that corresponding sides are in proportion. The converse is also true.

In the diagrams (left) $\triangle ABC$ and $\triangle PQR$ are similar.

For similar figures the ratios of the lengths of the sides are the same and represent the **scale factor**, i.e.

$$\frac{p}{a} = \frac{q}{b} = \frac{r}{c} = k \text{ (where } k \text{ is the scale factor of enlargement)}$$

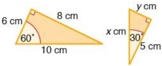
The heights of similar triangles are proportional also:

$$\frac{H}{h} = \frac{p}{a} = \frac{q}{b} = \frac{r}{c} = k$$

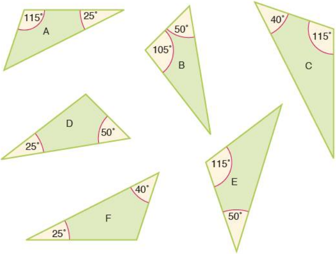
The ratio of the areas of similar triangles (the **area factor**) is equal to the square of the scale factor.

$$\frac{\text{Area of } \triangle PQR}{\text{Area of } \triangle ABC} = \frac{\frac{1}{2}H \times p}{\frac{1}{2}h \times a} = \frac{H}{h} \times \frac{p}{a} = k \times k = k^2$$

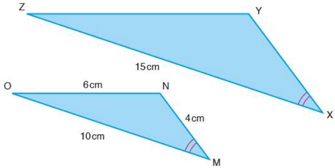
Exercise 22.1



1. a) Explain why the two triangles (left) are similar.
   b) Calculate the scale factor which reduces the larger triangle to the smaller one.
   c) Calculate the value of x and the value of y.
2. Which of the triangles below are similar?

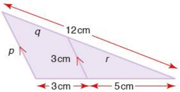


3. The triangles below are similar.

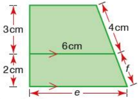


a) Calculate the length XY.
b) Calculate the length YZ.

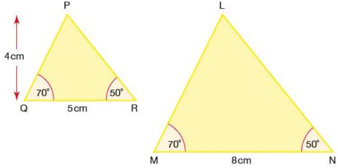
4. In the triangle (right) calculate the lengths of sides p, q and r.



5. In the trapezium (right) calculate the lengths of the sides e and f.

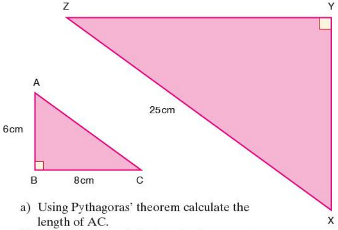


6. The triangles PQR and LMN are similar.

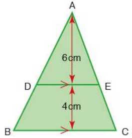


Calculate:
a) the area of $\triangle$PQR
b) the scale factor of enlargement
c) the area of $\triangle$LMN.

7. The triangles ABC and XYZ below are similar.

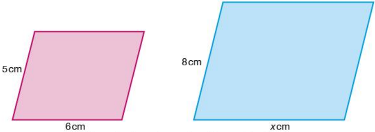


a) Using Pythagoras' theorem calculate the length of AC.
b) Calculate the scale factor of enlargement.
c) Calculate the area of $\triangle XYZ$.



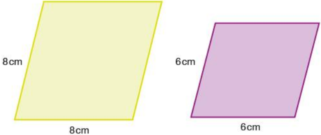
8. The triangle ADE shown (left) has an area of 12cm^2.
a) Calculate the area of $\triangle ABC$.
b) Calculate the length BC.

9. The parallelograms below are similar.



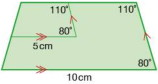
Calculate the length of the side marked x.

10. The diagram below shows two rhombuses.



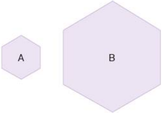
Explain, giving reasons, whether the two rhombuses are definitely similar.

11. The diagram (right) shows a trapezium within a trapezium. Explain, giving reasons, whether the two trapezia are definitely similar.



Exercise 22.2

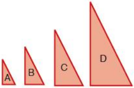
1. In the hexagons below, hexagon B is an enlargement of hexagon A by a scale factor of 2.5.



If the area of A is $8\,\text{cm}^2$, calculate the area of B.

2. P and Q are two regular pentagons. Q is an enlargement of P by a scale factor of 3. If the area of pentagon Q is $90\,\text{cm}^2$, calculate the area of P.

3. The diagram below shows four triangles A, B, C and D. Each is an enlargement of the previous one by a scale factor of 1.5



a) If the area of C is $202.5\,\text{cm}^2$, calculate the area of:
i) triangle D ii) triangle B iii) triangle A.
b) If the triangles were to continue in this sequence, which letter triangle would be the first to have an area greater than $15\,000\,\text{cm}^2$?

4. A square is enlarged by increasing the length of its sides by 10%. If the length of its sides was originally 6 cm, calculate the area of the enlarged square.

5. A square of side length 4 cm is enlarged by increasing the lengths of its sides by 25% and then increasing them by a further 50%. Calculate the area of the final square.

6. An equilateral triangle has an area of $25\,\text{cm}^2$. If the lengths of its sides are reduced by 15%, calculate the area of the reduced triangle.

Area and volume of similar shapes

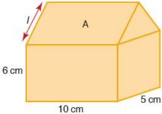
Earlier in the chapter we found the following relationship between the scale factor and the area factor of enlargement:

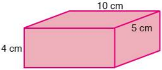
Area factor = (scale factor)^2

A similar relationship can be stated for volumes of similar shapes:

i.e. Volume factor = (scale factor)^3

Exercise 22.3





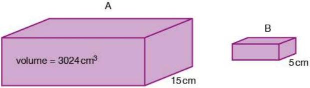
1. The diagram (left) shows a scale model of a garage. Its width is 5 cm, its length 10 cm and the height of its walls 6 cm.
   a) If the width of the real garage is 4 m, calculate:
      i) the length of the real garage
      ii) the real height of the garage wall.
   b) If the apex of the roof of the real garage is 2 m above the top of the walls, use Pythagoras' theorem to find the real slant length l.
   c) What is the area of the roof section A on the model?

2. A cuboid has dimensions as shown in the diagram (left): If the cuboid is enlarged by a scale factor of 2.5, calculate:
   a) the total surface area of the original cuboid
   b) the total surface area of the enlarged cuboid
   c) the volume of the original cuboid
   d) the volume of the enlarged cuboid.

3. A cube has side length 3 cm.
   a) Calculate its total surface area.
   b) If the cube is enlarged and has a total surface area of 486 cm^2, calculate the scale factor of enlargement.
   c) Calculate the volume of the enlarged cube.

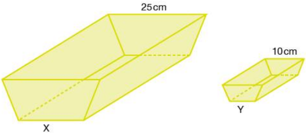
4. Two cubes P and Q are of different sizes. If n is the ratio of their corresponding sides, express in terms of n:
   a) the ratio of their surface areas
   b) the ratio of their volumes.

5. The cuboids A and B shown below are similar.



Calculate the volume of cuboid B.

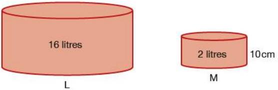
6. Two similar troughs X and Y are shown below.



If the capacity of X is 10 litres, calculate the capacity of Y.

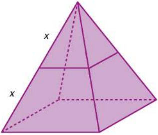
Exercise 22.4

1. The two cylinders L and M shown below are similar.

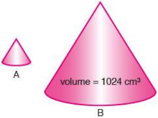


If the height of cylinder M is 10 cm, calculate the height of cylinder L.

2. A square-based pyramid (below) is cut into two shapes by a cut running parallel to the base and made half-way up.

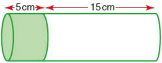


a) Calculate the ratio of the volume of the smaller pyramid to that of the original one.
b) Calculate the ratio of the volume of the small pyramid to that of the truncated base.

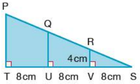


3. The two cones A and B (left) are similar. Cone B is an enlargement of A by a scale factor of 4.

If the volume of cone B is 1024 cm^3, calculate the volume of cone A.

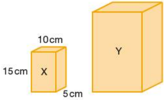


4. a) Stating your reasons clearly, decide whether the two cylinders shown (left) are similar or not.
   b) What is the ratio of the curved surface area of the shaded cylinder to that of the unshaded cylinder?



5. The diagram (left) shows a triangle.
   a) Calculate the area of $\triangle$RSV.
   b) Calculate the area of $\triangle$QSU.
   c) Calculate the area of $\triangle$PST.

6. The area of an island on a map is $30\,\text{cm}^2$. The scale used on the map is 1 : 100 000.
   a) Calculate the area in square kilometres of the real island.
   b) An airport on the island is on a rectangular piece of land measuring 3 km by 2 km. Calculate the area of the airport on the map in cm^2.



7. The two packs of cheese X and Y (left) are similar.

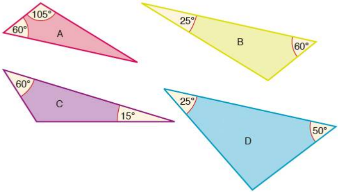
   The total surface area of pack Y is four times that of pack X.

   Calculate:

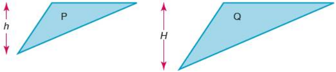
   a) the dimensions of pack Y
   b) the mass of pack X if pack Y has a mass of 800g.

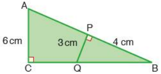
Student assessment 1

1. Which of the triangles below are similar?



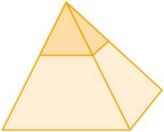
2. Triangles P and Q (below) are similar. Express the ratio of their areas in the form, area of P : area of Q.

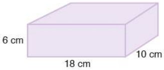




3. Using the triangle (left),
   a) explain whether △ABC and △PBQ are similar,
   b) calculate the length QB,
   c) calculate the length BC,
   d) calculate the length AP.

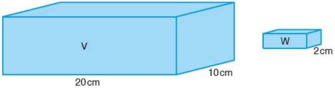
4. The vertical height of the large square-based solid pyramid (below) is 30 m. Its mass is 16 000 tonnes. If the mass of the smaller (shaded) pyramid is 2000 tonnes, calculate its vertical height.





5. The cuboid (left) undergoes a reduction by a scale factor of 0.6.
   a) Draw a sketch of the reduced cuboid labelling its dimensions clearly.
   b) What is the volume of the new cuboid?
   c) What is the total surface area of the new cuboid?

6. Cuboids V and W (below) are similar.



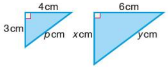
If the volume of cuboid V is 1600 cm^3, calculate:
   a) the volume of cuboid W,
   b) the total surface area of cuboid V,
   c) the total surface area of cuboid W.

7. An island has an area of 50 km^2. What would be its area on a map of scale 1 : 20 000?

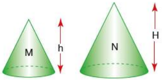
8. A box in the shape of a cube has a surface area of 2400 cm². What would be the volume of a similar box enlarged by a scale factor of 1.5?

Student assessment 2

1. The two triangles (below) are similar.

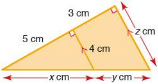


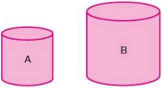
a) Using Pythagoras' theorem, calculate the value of p.
b) Calculate the values of x and y.



2. Cones M and N (left) are similar.
a) Express the ratio of their surface areas in the form, area of M : area of N.
b) Express the ratio of their volumes in the form, volume of M : volume of N.

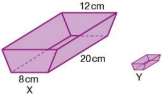
3. Calculate the values of x, y and z in the triangle below.





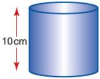
4. The tins A and B (left) are similar. The capacity of tin B is three times that of tin A. If the label on tin A has an area of 75 cm², calculate the area of the label on tin B.

5. A cube of side 4 cm is enlarged by a scale factor of 2.5.
a) Calculate the volume of the enlarged cube.
b) Calculate the surface area of the enlarged cube.



6. The two troughs X and Y (left) are similar. The scale factor of enlargement from Y to X is 4. If the capacity of trough X is 1200 cm³, calculate the capacity of trough Y.

7. The rectangular floor plan of a house measures 8 cm by 6 cm. If the scale of the plan is 1 : 50, calculate:
a) the dimensions of the actual floor,
b) the area of the actual floor in m².



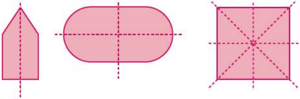
8. The volume of the cylinder (left) is 400 cm³. Calculate the volume of a similar cylinder formed by enlarging the one shown by a scale factor 2.

23 Symmetry

NB: All diagrams are not drawn to scale.

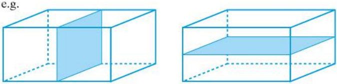
Symmetry and three-dimensional shapes

A **line of symmetry** divides a two-dimensional (flat) shape into two congruent (identical) shapes.
e.g.



1 line of symmetry 2 lines of symmetry 4 lines of symmetry

A **plane of symmetry** divides a three-dimensional (solid) shape into two congruent solid shapes.
e.g.



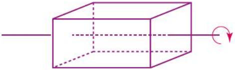
A cuboid has at least three planes of symmetry, two of which are shown above.

A shape has **reflective symmetry** if it has one or more lines or planes of symmetry.

A two-dimensional shape has **rotational symmetry** if, when rotated about a central point, it fits its outline. The number of times it fits its outline during a complete revolution is called the **order of rotational symmetry**.
e.g.

rotational symmetry of order 2 rotational symmetry of order 4

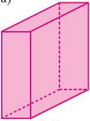
A three-dimensional shape has **rotational symmetry** if, when rotated about a central axis, it looks the same at certain intervals.
e.g.

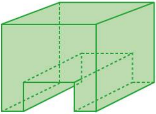


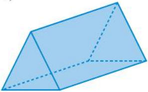
This cuboid has rotational symmetry of order 2 about the axis shown.

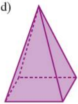
Exercise 23.1

1. Draw each of the solid shapes below twice, then:
   i) on each drawing of the shape, draw a different plane of symmetry,
   ii) state how many planes of symmetry the shape has in total.

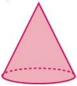
a) 
cuboid

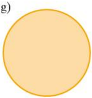
b) 
prism

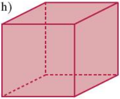
c) 
equilateral triangular prism

d) 
square-based pyramid

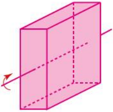
e)
cylinder

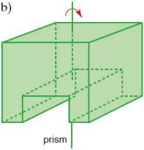
f) 
cone

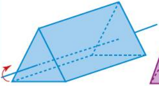
g) 
sphere

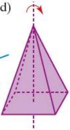
h) 
cube

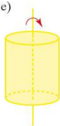
2. For each of the solid shapes shown below determine the order of rotational symmetry about the axis shown.

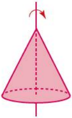
a) 
cuboid

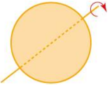
b) 
prism

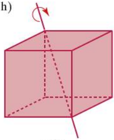
c) 
equilateral triangular prism

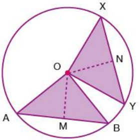
d) 
square-based pyramid

e) 
cylinder

f) 
cone

g) 
sphere

h) 
cube



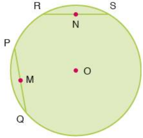
Circle properties

Equal chords and perpendicular bisectors

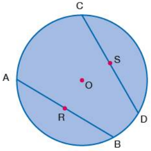
If chords AB and XY are of equal length, then, since OA, OB, OX and OY are radii, the triangles OAB and OXY are congruent isosceles triangles. It follows that:

- the section of a line of symmetry OM through $\triangle OAB$ is the same length as the section of a line of symmetry ON through $\triangle OXY$,
- OM and ON are perpendicular bisectors of AB and XY respectively.

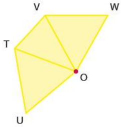
Exercise 23.2



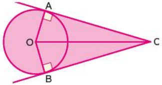
1. In the diagram (left) O is the centre of the circle, PQ and RS are chords of equal length and M and N are their respective midpoints.
   a) What kind of triangle is $\triangle POQ$?
   b) Describe the line ON in relation to RS.
   c) If $\angle POQ$ is 80°, calculate $\angle OQP$.
   d) Calculate $\angle ORS$.
   e) If PQ is 6 cm calculate the length OM.
   f) Calculate the diameter of the circle.



2. In the diagram (left) O is the centre of the circle. AB and CD are equal chords and the points R and S are their midpoints respectively.
   State whether the statements below are true or false, giving reasons for your answers.
   a) $\angle COD = 2 \times \angle AOR$
   b) $OR = OS$
   c) If $\angle ROB$ is 60° then $\triangle AOB$ is equilateral.
   d) OR and OS are perpendicular bisectors of AB and CD respectively.



3. Using the diagram (left) state whether the following statements are true or false, giving reasons for your answer.
   a) If $\triangle VOW$ and $\triangle TOU$ are isosceles triangles, then T, U, V and W would all lie on the circumference of a circle with its centre at O.
   b) If $\triangle VOW$ and $\triangle TOU$ are congruent isosceles triangles, then T, U, V and W would all lie on the circumference of a circle with its centre at O.



Tangents from an external point

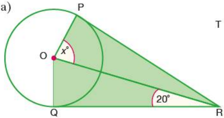
Triangles OAC and OBC are congruent since ∠OAC and ∠OBC are right angles, OA = OB because they are both radii, and OC is common to both triangles. Hence AC = BC.

In general, therefore, tangents being drawn to the same circle from an external point are equal in length.

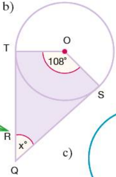
Exercise 23.3

1. Copy each of the diagrams below and calculate the size of the angle marked $x°$ in each case. Assume that the lines drawn from points on the circumference are tangents.

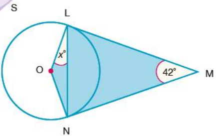
a)



b)

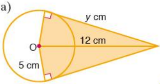


c)

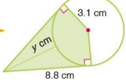


2. Copy each of the diagrams below and calculate the length of the side marked y cm in each case. Assume that the lines drawn from points on the circumference are tangents.

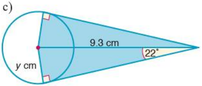
a)



b)

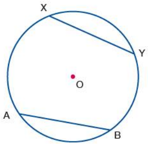


c)



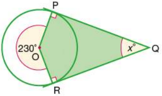
Student assessment 1

1. Draw a shape with exactly:
   a) one line of symmetry,
   b) two lines of symmetry,
   c) three lines of symmetry.
2. Draw and name a shape with:
   a) two planes of symmetry,
   b) four planes of symmetry.

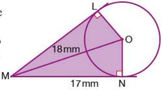


3. In the diagram (left) O is the centre of the circle and the lengths AB and XY are equal. Prove that $\triangle AOB$ and $\triangle XOY$ are congruent.

4. In the diagram (below) that PQ and QR are both tangents to the circle. Calculate the size of the angle marked $x°$.



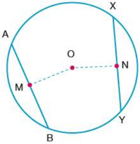
5. Calculate the diameter of the circle (right) given that LM and MN are both tangents to the circle, O is its centre and OM = 18 mm.



Student assessment 2

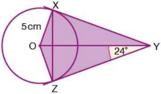
1. Draw a two-dimensional shape with exactly:
   a) rotational symmetry of order 2,
   b) rotational symmetry of order 4,
   c) rotational symmetry of order 6.

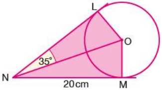
2. Draw and name a three-dimensional shape with the following orders of rotational symmetry. Mark the position of the axis of symmetry clearly.
   a) Order 2
   b) Order 3
   c) Order 8



3. In the diagram (left), OM and ON are perpendicular bisectors of AB and XY respectively. OM = ON. Prove that AB and XY are chords of equal length.

4. In the diagram (right), XY and YZ are both tangents to the circle with centre O.
   a) Calculate $\angle OZX$.
   b) Calculate the length XZ.





5. In the diagram (left), LN and MN are both tangents to the circle centre O. If $\angle LNO$ is 35°, calculate the circumference of the circle.

24 Angle properties

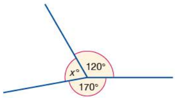
NB: All diagrams are not drawn to scale.

Angles at a point and on a line

One complete revolution is equivalent to a rotation of 360° about a point. Similarly, half a complete revolution is equivalent to a rotation of 180° about a point. These facts can be seen clearly by looking at either a circular angle measurer or a semi-circular protractor.

Worked examples

a) Calculate the size of the angle x in the diagram below:

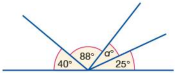


The sum of all the angles around a point is 360°. Therefore:

$$120 + 170 + x = 360$$
$$x = 360 - 120 - 170$$
$$x = 70$$

Therefore angle x is 70°.
Note that the size of the angle x is **calculated** and **not measured**.

b) Calculate the size of angle a in the diagram below:



The sum of all the angles at a point on a straight line is 180°. Therefore:

$$40 + 88 + a + 25 = 180$$
$$a = 180 - 40 - 88 - 25$$
$$a = 27$$

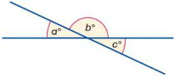
Therefore angle a is 27°.

Angles formed within parallel lines

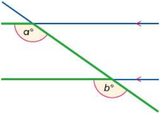
When two straight lines cross, it is found that the angles opposite each other are the same size. They are known as **vertically opposite angles**. By using the fact that angles at a point on a straight line add up to 180°, it can be shown why vertically opposite angles must always be equal in size.

$$a + b = 180°$$
$$c + b = 180°$$

Therefore, a is equal to c.

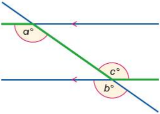


When a line intersects two parallel lines, as in the diagram below, it is found that certain angles are the same size.



The angles a and b are equal and are known as **corresponding angles**. Corresponding angles can be found by looking for an 'F' formation in a diagram.

A line intersecting two parallel lines also produces another pair of equal angles, known as **alternate angles**. These can be shown to be equal by using the fact that both vertically opposite and corresponding angles are equal.



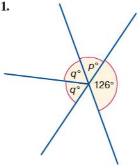
In the diagram above, $a = b$ (corresponding angles). But $b = c$ (vertically opposite). It can therefore be deduced that $a = c$.

Angles a and c are alternate angles. These can be found by looking for a 'Z' formation in a diagram.

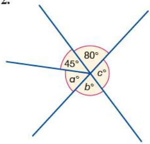
Exercise 24.1

In each of the following questions, some of the angles are given. Deduce, giving your reasons, the size of the other labelled angles.

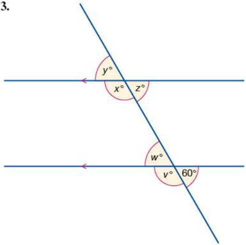
1.



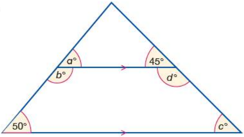
2.



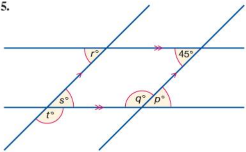
3.



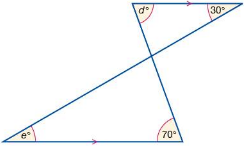
4.



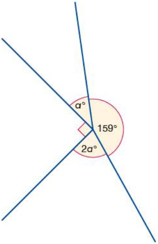
5.



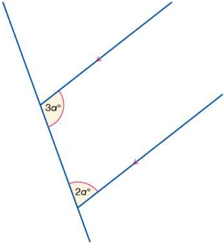
6.



7.



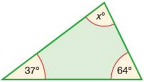
8.



Angles in a triangle

The sum of the angles in a triangle is 180°.

Worked example Calculate the size of the angle x in the triangle below:



$37 + 64 + x = 180$

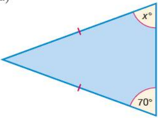
$x = 180 - 37 - 64$

Therefore angle x is 79°.

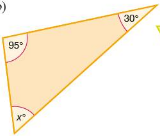
Exercise 24.2

1. For each of the triangles below, use the information given to calculate the size of angle x.

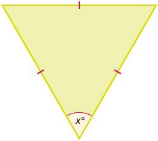
a)



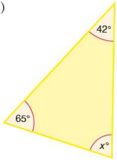
b)



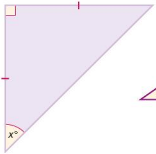
c)



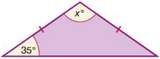
d)



e)

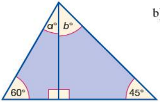


f)

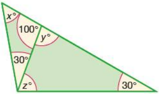


2. In each of the diagrams below, calculate the size of the labelled angles.

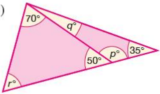
a)



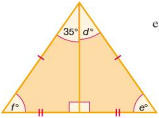
b)



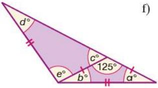
c)



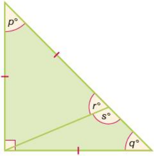
d)



e)

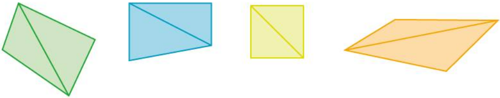


f)



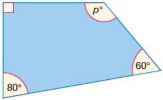
Angles in a quadrilateral

In the quadrilaterals below, a straight line is drawn from one of the corners (vertices) to the opposite corner. The result is to split the quadrilaterals into two triangles.



The sum of the angles of a triangle is 180°. Therefore, as a quadrilateral can be drawn as two triangles, the sum of the four angles of any quadrilateral must be 360°.

Worked example Calculate the size of angle p in the quadrilateral below:



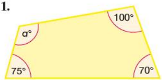
$$90 + 80 + 60 + p = 360$$
$$p = 360 - 90 - 80 - 60$$

Therefore angle p is 130°.

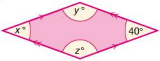
Exercise 24.3

In each of the diagrams below, calculate the size of the labelled angles.

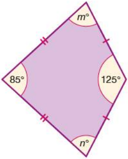
1.



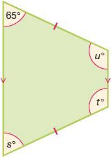
2.



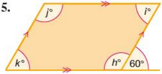
3.



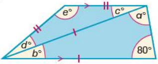
4.



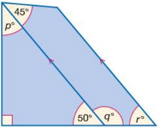
5.



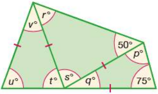
6.



7.

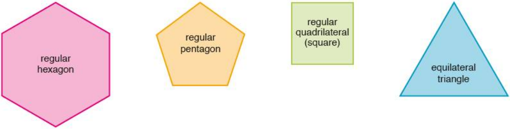


8.



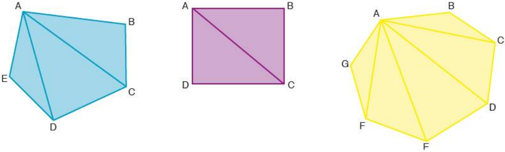
Polygons

A **regular polygon** is distinctive in that all its sides are of equal length and all its angles are of equal size. Below are some examples of regular polygons.



The sum of the interior angles of a polygon

In the polygons below a straight line is drawn from each vertex to vertex A.



As can be seen, the number of triangles is always two less than the number of sides the polygon has, i.e. if there are n sides, there will be $(n - 2)$ triangles.

Since the angles of a triangle add up to 180°, the sum of the interior angles of a polygon is therefore $180(n - 2)$ degrees.

Worked example Find the sum of the interior angles of a regular pentagon and hence the size of each interior angle.

For a pentagon, $n = 5$.

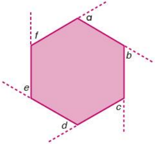
Therefore the sum of the interior angles $= 180(5 - 2)°$
$= 180 \times 3°$
$= 540°$

For a regular pentagon the interior angles are of equal size.

Therefore each angle $= \dfrac{540°}{5} = 108°$.

The sum of the exterior angles of a polygon

The angles marked a, b, c, d, e and f in the diagram below represent the exterior angles of a regular hexagon.



For any convex polygon the sum of the exterior angles is 360°.

If the polygon is regular and has n sides, then each exterior angle $= \dfrac{360°}{n}$.

Worked examples **a)** Find the size of an exterior angle of a regular nonagon.

$$\frac{360°}{9} = 40°$$

b) Calculate the number of sides a regular polygon has if each exterior angle is 15°.

$$n = \frac{360}{15}$$
$$= 24$$

The polygon has 24 sides.

Exercise 24.4

1. Find the sum of the interior angles of the following polygons:
   a) a hexagon b) a nonagon c) a heptagon

2. Find the value of each interior angle of the following regular polygons:
   a) an octagon b) a square
   c) a decagon d) a dodecagon

3. Find the size of each exterior angle of the following regular polygons:
   a) a pentagon b) a dodecagon c) a heptagon

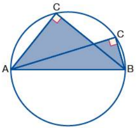
4. The exterior angles of regular polygons are given below. In each case calculate the number of sides the polygon has.
   a) 20° b) 36° c) 10°
   d) 45° e) 18° f) 3°

5. The interior angles of regular polygons are given below. In each case calculate the number of sides the polygon has.
   a) 108° b) 150° c) 162°
   d) 156° e) 171° f) 179°

6. Calculate the number of sides a regular polygon has if an interior angle is five times the size of an exterior angle.

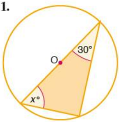
The angle in a semi-circle

In the diagram below, if AB represents the diameter of the circle, then the angle at C is 90°.

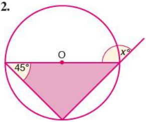


Exercise 24.5 In each of the following diagrams, O marks the centre of the circle. Calculate the value of x in each case.

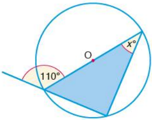
1.



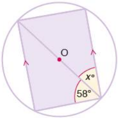
2.



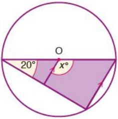
3.



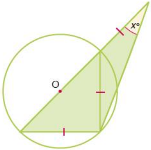
4.



5.



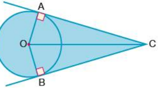
6.



The angle between a tangent and a radius of a circle

The angle between a tangent at a point and the radius to the same point on the circle is a right angle.

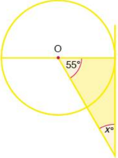
In the diagram below, triangles OAC and OBC are congruent as ∠OAC and ∠OBC are right angles, OA = OB because they are both radii and OC is common to both triangles.



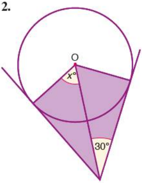
Exercise 24.6

In each of the following diagrams, O marks the centre of the circle. Calculate the value of x in each case.

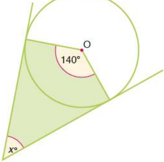
1.



2.



3.



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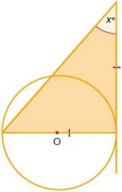
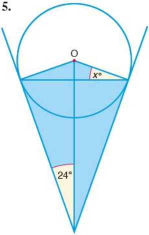
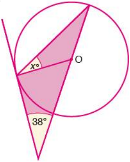


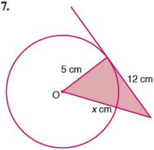
5.



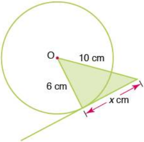
6.



7.



8.



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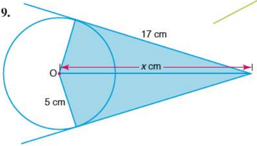


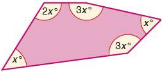
Angle properties of irregular polygons

As explained earlier in this chapter, the sum of the interior angles of a polygon is given by $180(n - 2)°$, where n represents the number of sides of the polygon. The sum of the exterior angles of any polygon is $360°$.

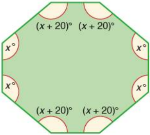
Both of these rules also apply to irregular polygons, i.e. those where the lengths of the sides and the sizes of the interior angles are not all equal.

Exercise 24.7

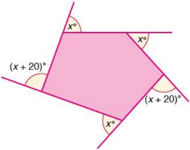
1. For the pentagon (right):
a) calculate the value of x,
b) calculate the size of each of the angles.



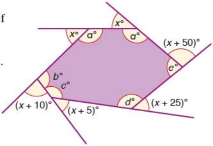
2. Find the size of each angle in the octagon (below).



3. Calculate the value of x for the pentagon shown.



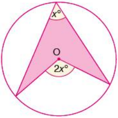
4. Calculate the size of each of the angles a, b, c, d and e in the hexagon (right).

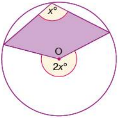


Angle at the centre of a circle

The angle subtended at the centre of a circle by an arc is twice the size of the angle on the circumference subtended by the same arc.

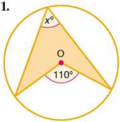
Both diagrams below illustrate this theorem.



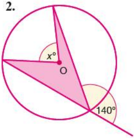


Exercise 24.8 In each of the following diagrams, O marks the centre of the circle. Calculate the size of the marked angles:

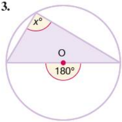
1.



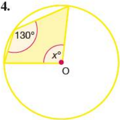
2.



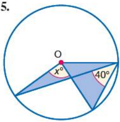
3.



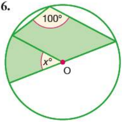
4.



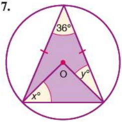
5.



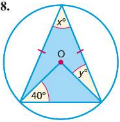
6.

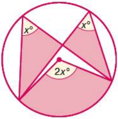


7.



8.





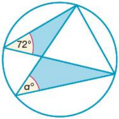
Angles in the same segment

Angles in the same segment of a circle are equal.

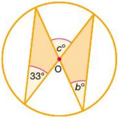
This can be explained simply by using the theorem that the angle subtended at the centre is twice the angle on the circumference. Looking at the diagram (left), if the angle at the centre is $2x°$, then each of the angles at the circumference must be equal to $x°$.

Exercise 24.9 Calculate the marked angles in the following diagrams:

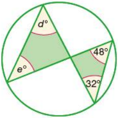
1.



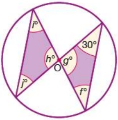
2.



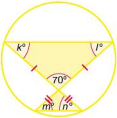
3.



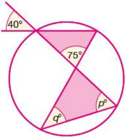
4.



5.



6.

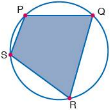


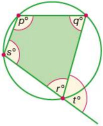
Angles in opposite segments

Points P, Q, R and S all lie on the circumference of the circle (below). They are called concyclic points. Joining the points P, Q, R and S produces a **cyclic quadrilateral**.

The opposite angles are **supplementary**, i.e. they add up to 180°. Since $p° + r° = 180°$ (supplementary angles) and $r° + t° = 180°$ (angles on a straight line) it follows that $p° = t°$.

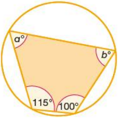
Therefore the exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.



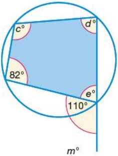


Exercise 24.10 Calculate the size of the marked angles in each of the following:

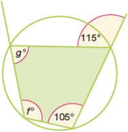
1.



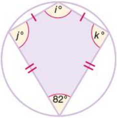
2.



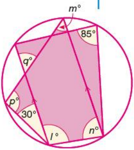
3.



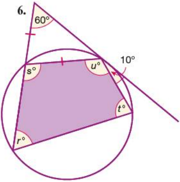
4.



5.



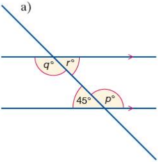
6.



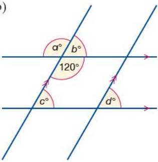
Student assessment 1

1. For the diagrams below, calculate the size of the labelled angles.

a)

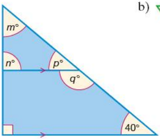


b)

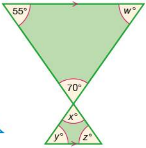


2. For the diagrams below, calculate the size of the labelled angles.

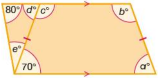
a)



b)



c)

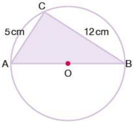


3. Find the size of each interior angle of a twenty-sided regular polygon.

4. What is the sum of the interior angles of a nonagon?

5. What is the sum of the exterior angles of a polygon?

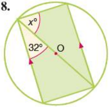
6. What is the size of each exterior angle of a regular pentagon?



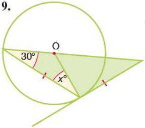
7. If AB is the diameter of the circle (left) and AC = 5 cm and BC = 12 cm, calculate:
   a) the size of angle ACB,
   b) the length of the radius of the circle.

In questions 8–11, O marks the centre of the circle. Calculate the size of the angle marked x in each case.

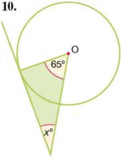
8.



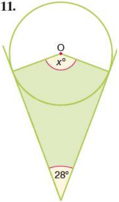
9.



10.



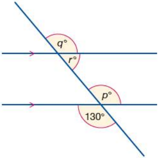
11.



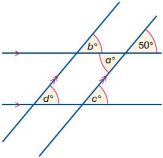
Student assessment 2

1. For the diagrams below, calculate the size of the labelled angles.

   a)

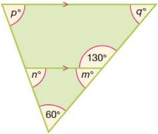


   b)

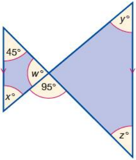


2. For the diagrams below, calculate the size of the labelled angles.

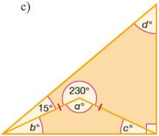
   a)



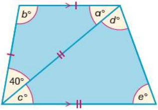
   b)



   c)



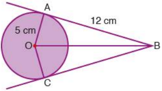
   d)



3. Find the value of each interior angle of a regular polygon with 24 sides.

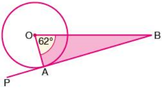
4. What is the sum of the interior angles of a regular dodecagon?

5. What is the size of an exterior angle of a regular dodecagon?



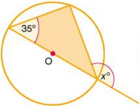
6. AB and BC are both tangents to the circle centre O (left). If OA = 5 cm and AB = 12 cm calculate:
a) the size of angle OAB,
b) the length OB.

7. If OA is a radius of the circle (below) and PB the tangent to the circle at A, calculate angle ABO.

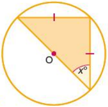


In question 8–11, O marks the centre of the circle. Calculate the size of the angle marked x in each case.

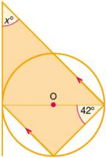
8.



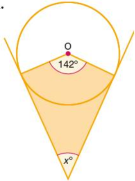
9.



10.



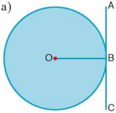
11.



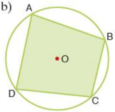
Student assessment 3

1. In the following diagrams, O marks the centre of the circle. Identify which angles are:
i) supplementary angles,
ii) right angles,
iii) equal.

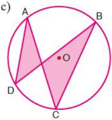
a)



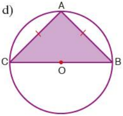
b)

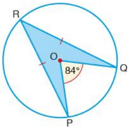


c)



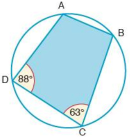
d)



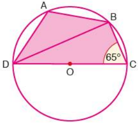


2. If $\angle POQ = 84°$ and O marks the centre of the circle in the diagram (left), calculate the following:
a) $\angle PRQ$ b) $\angle OQR$

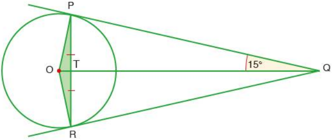
3. Calculate $\angle DAB$ and $\angle ABC$ in the diagram below.



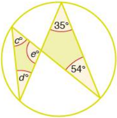
4. If DC is a diameter, and O marks the centre of the circle, calculate angles BDC and DAB.



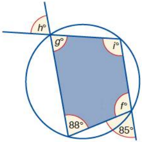
5. Calculate as many angles as possible in the diagram below. O marks the centre of the circle.



6. Calculate the values of *c*, *d* and *e*.

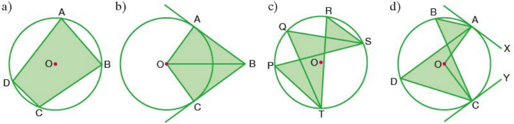


7. Calculate the values of *f*, *g*, *h* and *i*.

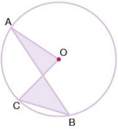


Student assessment 4

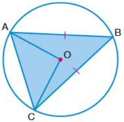
1. In the following diagrams, O is the centre of the circle. Identify which angles are:
i) supplementary angles,
ii) right angles,
iii) equal.



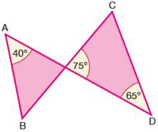
2. If ∠AOC is 72°, calculate ∠ABC.



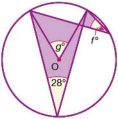
3. If ∠AOB = 130°, calculate∠ABC, ∠OAB and ∠CAO.



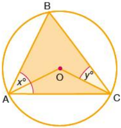
4. Show that ABCD is a cyclic quadrilateral.



5. Calculate *f* and *g*.



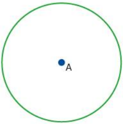
6. If $y = 22.5°$ calculate the value of x.



25 Loci

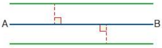
NB: All diagrams are not drawn to scale.

A **locus** (plural **loci)** refers to all the points which fit a particular description. These points can belong to either a region or a line, or both. The principal types of loci are explained below.



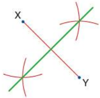
- **The locus of the points which are at a given distance from a given point**

In the diagram (left) it can be seen that the locus of all the points equidistant from a point A is the circumference of a circle centre A. This is due to the fact that all points on the circumference of a circle are equidistant from the centre.



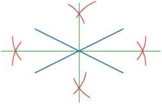
- **The locus of the points which are at a given distance from a given straight line**

In the diagram (left) it can be seen that the locus of the points equidistant from a straight line AB runs parallel to that straight line. It is important to note that the distance of the locus from the straight line is measured at right angles to the line. This diagram, however, excludes the ends of the line. If these two points are taken into consideration then the locus takes the form shown in the next diagram (left).



- **The locus of the points which are equidistant from two given points**

The locus of the points equidistant from points X and Y lies on the perpendicular bisector of the line XY.

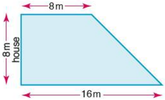


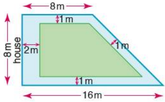
- **The locus of the points which are equidistant from two given intersecting straight lines**

The locus in this case lies on the bisectors of both pairs of opposite angles as shown left.

The application of the above cases will enable you to tackle problems involving loci at this level.

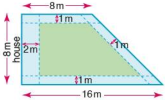
Worked example The diagram shows a trapezoidal garden. Three of its sides are enclosed by a fence, and the fourth is adjacent to a house.





i) Grass is to be planted in the garden. However, it must be at least 2 m away from the house and at least 1 m away from the fence. Shade the region in which the grass can be planted.

The shaded region is therefore the locus of all the points which are both at least 2 m away from the house and at least 1 m away from the surrounding fence. Note that the boundary of the region also forms part of the locus of the points.



ii) Using the same garden as before, grass must now be planted according to the following conditions: it must be **more than** 2 m away from the house and **more than** 1 m away from the fence. Shade the region in which the grass can be planted.

The shape of the region is the same as in the first case; however, in this instance the boundary is not included in the locus of the points as the grass cannot be exactly 2 m away from the house or exactly 1 m away from the fence.

Note: If the boundary is included in the locus points, it is represented by a **solid** line. If it is not included then it is represented by a **dashed** (broken) line.

Exercise 25.1

Questions 1–4 are about a rectangular garden measuring 8 m by 6 m. For each question draw a scale diagram of the garden and identify the locus of the points which fit the criteria.

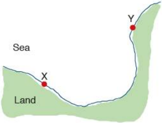
1. Draw the locus of all the points at least 1 m from the edge of the garden.

2. Draw the locus of all the points at least 2 m from each corner of the garden.

3. Draw the locus of all the points more than 3 m from the centre of the garden.

4. Draw the locus of all the points equidistant from the longer sides of the garden.

5. A port has two radar stations at P and Q which are 20 km apart. The radar at P is set to a range of 20 km, whilst the radar at Q is set to a range of 15 km.
a) Draw a scale diagram to show the above information.
b) Shade the region in which a ship must be sailing if it is only picked up by radar P. Label this region 'a'.
c) Shade the region in which a ship must be sailing if it is only picked up by radar Q. Label this region 'b'.
d) Identify the region in which a ship must be sailing if it is picked up by both radars. Label this region 'c'.

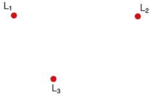


6. X and Y are two ship-to-shore radio receivers (left). They are 25 km apart.
A ship sends out a distress signal. The signal is picked up by both X and Y. The radio receiver at X indicates that the ship is within a 30 km radius of X, whilst the radio receiver at Y indicates that the ship is within 20 km of Y. Draw a scale diagram and identify the region in which the ship must lie.

7. a) Mark three points L, M and N not in a straight line. By construction find the point which is equidistant from L, M and N.
b) What would happen if L, M and N were on the same straight line?

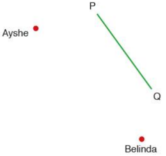
8. Draw a line AB 8 cm long. What is the locus of a point C such that the angle ACB is always a right angle?

9. Draw a circle by drawing round a circular object (do not use a pair of compasses). By construction determine the position of the centre of the circle.



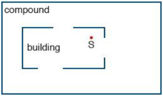
10. Three lionesses L_1, L_2 and L_3 have surrounded a gazelle. The three lionesses are equidistant from the gazelle. Draw a diagram with the lionesses in similar positions to those shown (left) and by construction determine the position (g) of the gazelle.

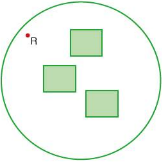
Exercise 25.2



1. Three girls are playing hide and seek. Ayshe and Belinda are at the positions shown (left) and are trying to find Cristina. Cristina is on the opposite side of a wall PQ to her two friends.
Assuming Ayshe and Belinda cannot see over the wall identify, by copying the diagram, the locus of points where Cristina could be if:
a) Cristina can only be seen by Ayshe,
b) Cristina can only be seen by Belinda,
c) Cristina can not be seen by either of her two friends,
d) Cristina can be seen by both of her friends.

2. A security guard S is inside a building in the position shown. The building is inside a rectangular compound. If the building has three windows as shown, identify the locus of points in the compound which can be seen by the security guard.



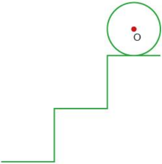


3. The circular cage shown (left) houses a snake. Inside the cage are three obstacles.
A rodent is placed inside the cage at R. From where it is lying, the snake can see the rodent.
Trace the diagram and identify the regions in which the snake could be lying.

Exercise 25.3

1. A coin is rolled in a straight line on a flat surface as shown below.

Draw the locus of the centre of the coin O as the coin rolls along the surface.



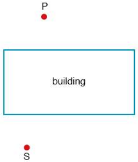
2. The diameter of the disc (left) is the same as the width and height of each of the steps shown.
Copy the diagram and draw the locus of the centre of the disc as it rolls down the steps.

3. A stone is thrown vertically upwards. Draw the locus of its trajectory from the moment it leaves the person's hand to the moment it is caught again.

4. A stone is thrown at an angle of elevation of 45°. Sketch the locus of its trajectory.

5. X and Y are two fixed posts in the ground. The ends of a rope are tied to X and Y. A goat is attached to the rope by a ring on its collar which enables it to move freely along the rope's length.
Copy the diagram (left) and sketch the locus of points in which the goat is able to graze.

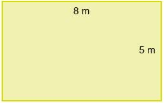
Student assessment 1



1. Pedro and Sara are on opposite sides of a building as shown (left). Their friend Raul is standing in a place such that he cannot be seen by either Pedro or Sara. Copy the above diagram and identify the locus of points at which Raul could be standing.

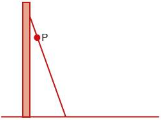
2. A rectangular garden measures 10 m by 6 m. A tree stands in the centre of the garden. Grass is to be planted according to the following conditions:
   - it must be at least 1 m from the edge of the garden,
   - it must be more than 2 m away from the centre of the tree.

   a) Make a scale drawing of the garden.
   b) Draw the locus of points in which the grass can be planted.



3. A rectangular rose bed in a park measures 8 m by 5 m as shown (left).
   The park keeper puts a low fence around the rose bed. The fence is at a constant distance of 2 m from the rose bed.
   a) Make a scale drawing of the rose bed.
   b) Draw the position of the fence.

4. A and B (left) are two radio beacons 80 km apart, either side of a shipping channel. A ship sails in such a way that it is always equidistant from A and B.
   Showing your method of construction clearly, draw the path of the ship.

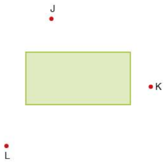


5. A ladder 10 m long is propped up against a wall as shown (left). A point P on the ladder is 2 m from the top.
   Make a scale drawing to show the locus of point P if the ladder were to slide down the wall. Note: several positions of the ladder will need to be shown.

6. The equilateral triangle PQR is rolled along the line shown. At first, corner Q acts as the pivot point until P reaches the line, then P acts as the pivot point until R reaches the line, and so on.

   Showing your method clearly, draw the locus of point P as the triangle makes one full rotation, assuming there is no slipping.

Student assessment 2



1. Jose, Katrina and Luis are standing at different points around a building as shown (left).
Trace the diagram and show whether any of the three friends can see each other or not.

2. A rectangular courtyard measures 20 m by 12 m. A horse is tethered in the centre with a rope 7 m long. Another horse is tethered, by a rope 5 m long, to a rail which runs along the whole of the left-hand (shorter) side of the courtyard. This rope is able to run freely along the length of the rail. Draw a scale diagram of the courtyard and draw the locus of points which can be reached by both horses.

3. The view in the diagram (left) is of two walls which form part of an obstacle course. A girl decides to ride her bicycle in between the two walls in such a way that she is always equidistant from them.
Copy the diagram and, showing your construction clearly, draw the locus of her path.

4. A ball is rolling along the line shown in the diagram (below). Copy the diagram and draw the locus of the centre, O, of the ball as it rolls.

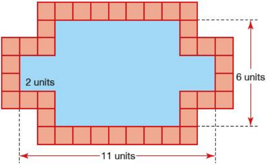
5. A square ABCD is 'rolled' along the flat surface shown below. Initially corner C acts as a pivot point until B touches the surface, then B acts as a pivot point until A touches the surface, and so on.

Assuming there is no slipping, draw the locus of point A as the square makes one complete rotation. Show your method clearly.

Topic 3 Mathematical investigations and ICT

Fountain borders

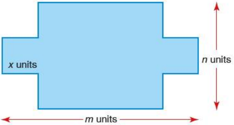
The Alhambra Palace in Granada, Spain has many fountains which pour water into pools. Many of the pools are surrounded by beautiful ceramic tiles. This investigation looks at the number of square tiles needed to surround a particular shape of pool.



The diagram above shows a rectangular pool 11×6 units, in which a square of dimension 2×2 units is taken from each corner.

The total number of unit square tiles needed to surround the pool is 38.

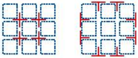
The shape of the pools can be generalised as shown below:



1. Investigate the number of unit square tiles needed for different sized pools. Record your results in an ordered table.
2. From your results write an algebraic rule in terms of m, n and x (if necessary) for the number of tiles T needed to surround a pool.
3. Justify, in words and using diagrams, why your rule works.

Tiled walls

Many cultures have used tiles to decorate buildings. Putting tiles on a wall takes skill. These days, to make sure that each tile is in the correct position, 'spacers' are used between the tiles.



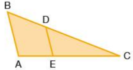
You can see from the diagrams that there are + shaped spacers (used where four tiles meet) and ⊤ shaped spacers (used at the edges of a pattern).

1. Draw other sized squares and rectangles, and investigate the relationship between the dimensions of the shape (length and width) and the number of + shaped and ⊤ shaped spacers.
2. Record your results in an ordered table.
3. Write an algebraic rule for the number of + shaped spacers c in a rectangle l tiles long by w tiles wide.
4. Write an algebraic rule for the number of ⊤ shaped spacers t in a rectangle l tiles long by w tiles wide.

ICT activity 1

In this activity you will be using a dynamic geometry package such as Cabri or Geogebra to demonstrate that for the triangle (below):

$$\frac{AB}{ED} = \frac{AC}{EC} = \frac{BC}{DC}$$



1. a) Using the geometry package construct the triangle ABC.
   b) Construct the line segment ED such that it is parallel to AB. (You will need to construct a line parallel to AB first and then attach the line segment ED to it.)
   c) Using a 'measurement' tool, measure each of the lengths AB, AC, BC, ED, EC and DC.
   d) Using a 'calculator' tool, calculate the ratios $\frac{AB}{ED}$, $\frac{AC}{EC}$, $\frac{BC}{DC}$.
2. Comment on your answers to question 1(d).

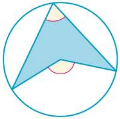
3. a) Grab vertex B and move it to a new position. What happens to the ratios you calculated in question 1(d)?
   b) Grab the vertices A and C in turn and move them to new positions. What happens to the ratios? Explain why this happens.

4. Grab point D and move it to a new position along the side BC. Explain, giving reasons, what happens to the ratios.

ICT activity 2

Using a geometry package, such as Cabri or Geogebra, demonstrate the following angle properties of a circle:

1. The angle subtended at the centre of a circle by an arc is twice the size of the angle on the circumference subtended by the same arc.
   The diagram below demonstrates the construction that needs to be formed:



2. The angles in the same segment of a circle are equal.

3. The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

Topic 4 Mensuration

Syllabus

E4.1

Use current units of mass, length, area, volume and capacity in practical situations and express quantities in terms of larger or smaller units.

E4.2

Carry out calculations involving the perimeter and area of a rectangle, triangle, parallelogram and trapezium and compound shapes derived from these.

E4.3

Carry out calculations involving the circumference and area of a circle.

Solve problems involving the arc length and sector area as fractions of the circumference and area of a circle.

E4.4

Carry out calculations involving the volume of a cuboid, prism and cylinder and the surface area of a cuboid and a cylinder.

Carry out calculations involving the surface area and volume of a sphere, pyramid and cone.

E4.5

Carry out calculations involving the areas and volumes of compound shapes.

Contents

The Egyptians

The Egyptians must have been very talented architects and surveyors to have planned the many large buildings and monuments built in that country thousands of years ago.

Evidence of the use of mathematics in Egypt in the Old Kingdom (about 2500BCE) is found on a wall near Meidum (south of Cairo) which gives guidelines for the slope of the stepped pyramid built there. The lines in the diagram are spaced at a distance of one cubit. A cubit is the distance from the tip of the finger to the elbow (about 50 cm). They show the use of that unit of measurement.

The earliest true mathematical documents date from about 1800BCE. The Moscow Mathematical Papyrus, the Egyptian Mathematical Leather Roll, the Kahun Papyri and the Berlin Papyrus all date to this period.

The Rhind Mathematical Papyrus, which was written in about 1650BCE, is said to be based on an older mathematical text. The Moscow Mathematical Papyrus and Rhind Mathematical Papyrus are so-called mathematical problem texts. They consist of a collection of mainly mensuration problems with solutions. These could have been written by a teacher for students to solve similar problems to the ones you will work on in this topic.

During the New Kingdom (about 1500–100BCE) papyri record land measurements. In the worker's village of Deir el-Medina several records have been found that record volumes of dirt removed while digging the underground tombs.

The Rhind Mathematical Payrus

26 Measures

Metric units

The metric system uses a variety of units for length, mass and capacity.

- The common units of length are: kilometre (km), metre (m), centimetre (cm) and millimetre (mm).
- The common units of mass are: tonne (t), kilogram (kg), gram (g) and milligram (mg).
- The common units of capacity are: litre (L or l) and millilitre (ml).

Note: 'centi' comes from the Latin *centum* meaning hundred (a centimetre is one hundredth of a metre);
'milli' comes from the Latin *mille* meaning thousand (a millimetre is one thousandth of a metre);
'kilo' comes from the Greek *khilloi* meaning thousand (a kilometre is one thousand metres).

It may be useful to have some practical experience of estimating lengths, volumes and capacities before starting the following exercises.

Exercise 26.1

Copy and complete the sentences below:

1. a) There are . . . centimetres in one metre.
b) There are . . . millimetres in one metre.
c) One metre is one . . . of a kilometre.
d) One milligram is one . . . of a gram.
e) One thousandth of a litre is one . . . .

2. Which of the units below would be used to measure the following?

mm, cm, m, km, mg, g, kg, t, ml, litres

a) your height
b) the length of your finger
c) the mass of a shoe
d) the amount of liquid in a cup
e) the height of a van
f) the mass of a ship
g) the capacity of a swimming pool
h) the length of a highway
i) the mass of an elephant
j) the capacity of the petrol tank of a car

Converting from one unit to another

Length

1 km = 1000 m

Therefore 1 m = $\frac{1}{1000}$ km

1 m = 1000 mm

Therefore 1 mm = $\frac{1}{1000}$ m

1 m = 100 cm

Therefore 1 cm = $\frac{1}{100}$ m

1 cm = 10 mm

Therefore 1 mm = $\frac{1}{10}$ cm

Worked examples

a) Change 5.8 km into m.
Since 1 km = 1000 m,
5.8 km is 5.8 × 1000 m

5.8 km = 5800 m

b) Change 4700 mm to m.
Since 1 m is 1000 mm,
4700 mm is 4700 ÷ 1000 m

4700 mm = 4.7 m

c) Convert 2.3 km into cm.
2.3 km is 2.3 × 1000 m = 2300 m
2300 m is 2300 × 100 cm

2.3 km = 230 000 cm

Exercise 26.2

1. Put in the missing unit to make the following statements correct:
a) 300 . . . = 30 cm
b) 6000 mm = 6 . . .
c) 3.2 m = 3200 . . .
d) 4.2 . . . = 4200 mm
e) 2.5 km = 2500 . . .

2. Convert the following to millimetres:
a) 8.5 cm
b) 23 cm
c) 0.83 m
d) 0.05 m
e) 0.004 m

3. Convert the following to metres:
a) 560 cm
b) 6.4 km
c) 96 cm
d) 0.004 km
e) 12 mm

4. Convert the following to kilometres:
a) 1150 m
b) 250 000 m
c) 500 m
d) 70 m
e) 8 m

Mass

1 tonne is 1000 kg

Therefore 1 kg $= \frac{1}{1000}$ tonne

1 kilogram is 1000 g

Therefore 1 g $= \frac{1}{1000}$ kg

1 g is 1000 mg

Therefore 1 mg $= \frac{1}{1000}$ g

Worked examples

a) Convert 8300 kg to tonnes.
Since 1000 kg = 1 tonne, 8300 kg is 8300 ÷ 1000 tonnes

8300 kg = 8.3 tonnes

b) Convert 2.5 g to mg.

Since 1 g is 1000 mg, 2.5 g is 2.5 × 1000 mg

2.5 g = 2500 mg

Exercise 26.3

1. Convert the following:
a) 3.8 g to mg
b) 28 500 kg to tonnes
c) 4.28 tonnes to kg
d) 320 mg to g
e) 0.5 tonnes to kg

Capacity

1 litre is 1000 millilitres

Therefore 1 ml $= \frac{1}{1000}$ litre

Exercise 26.4

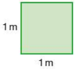
1. Calculate the following and give the totals in ml:
a) 3 litres + 1500 ml
b) 0.88 litre + 650 ml
c) 0.75 litre + 6300 ml
d) 450 ml + 0.55 litre

2. Calculate the following and give the total in litres:
a) 0.75 litre + 450 ml
b) 850 ml + 490 ml
c) 0.6 litre + 0.8 litre
d) 80 ml + 620 ml + 0.7 litre

Area and volume conversions

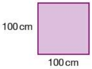
Converting between units for area and volume is not as straightforward as converting between units for length.

The diagram below shows a square of side length 1 m.



Area of the square = $1\ m^2$

However, if the lengths of the sides are written in cm, each of the sides are 100 cm.

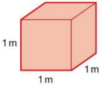


Area of the square = $100 \times 100 = 10\ 000\ cm^2$

Therefore an area of $1\ m^2 = 10\ 000\ cm^2$.

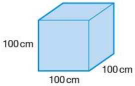
Similarly a square of side length 1 cm is the same as a square of side length 10 mm. Therefore an area of $1\ cm^2$ is equivalent to an area of $100\ mm^2$.

The diagram below shows a cube of side length 1 m.



Volume of the cube = $1\ m^3$

Once again, if the lengths of the sides are written in cm, each of the sides are 100 cm.



Volume of the cube = $100 \times 100 \times 100 = 1\ 000\ 000\ cm^3$

Therefore a volume of $1\ m^3 = 1\ 000\ 000\ cm^3$.

Similarly a cube of side length 1 cm is the same as a cube of side length 10 mm.

Therefore a volume of $1\ cm^3$ is equivalent to a volume of $1000\ mm^3$.

Exercise 26.5

1. Convert the following areas:

a) $10\ m^2$ to cm^2 b) $2\ m^2$ to mm^2

c) $5\ km^2$ to m^2 d) $3.2\ km^2$ to m^2

e) $8.3\ cm^2$ to mm^2

2. Convert the following areas:

a) $500\ cm^2$ to m^2 b) $15\ 000\ mm^2$ to cm^2

c) $1000\ m^2$ to km^2 d) $40\ 000\ mm^2$ to m^2

e) $2\ 500\ 000\ cm^2$ to km^2

3. Convert the following volumes:
   a) 2.5 m^3 to cm^3
   b) 3.4 cm^3 to mm^3
   c) 2 km^3 to m^3
   d) 0.2 m^3 to cm^3
   e) 0.03 m^3 to mm^3

4. Convert the following volumes:
   a) 150 000 cm^3 to m^3
   b) 24 000 mm^3 to cm^3
   c) 850 000 m^3 to km^3
   d) 300 mm^3 to cm^3
   e) 15 cm^3 to m^3

Student assessment 1

1. Convert the following lengths into the units indicated:
   a) 2.6 cm to mm
   b) 0.88 m to cm
   c) 6800 m to km
   d) 0.875 km to m

2. Convert the following masses into the units indicated:
   a) 4.2 g to mg
   b) 3940 g to kg
   c) 4.1 kg to g
   d) 0.72 tonnes to kg

3. Convert the following liquid measures into the units indicated:
   a) 1800 ml to litres
   b) 3.2 litres to ml
   c) 0.083 litre to ml
   d) 250 000 ml to litres

4. Convert the following areas:
   a) 56 cm^2 to mm^2
   b) 2.05 m^3 to cm^3

5. Convert the following volumes:
   a) 8670 cm^3 to m^3
   b) 444 000 cm^3 to m^3

Student assessment 2

1. Convert the following lengths into the units indicated:
   a) 3100 mm to cm
   b) 6.4 km to m
   c) 0.4 cm to mm
   d) 460 mm to cm

2. Convert the following masses into the units indicated:
   a) 3.6 mg to g
   b) 550 mg to g
   c) 6500 g to kg
   d) 1510 kg to tonnes

3. Convert the following measures of capacity to the units indicated:
   a) 3400 ml to litres
   b) 6.7 litres to ml
   c) 0.73 litre to ml
   d) 300 000 ml to litres.

4. Convert the following areas:
   a) 0.03 m^2 to mm^2
   b) 0.005 km^2 to m^2

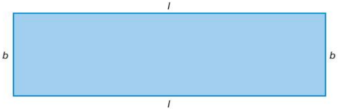
5. Convert the following volumes:
   a) 100 400 cm^3 to m^3
   b) 5005 m^3 to km^3

27 Perimeter, area and volume

NB: All diagrams are not drawn to scale.

The perimeter and area of a rectangle

The **perimeter** of a shape is the distance around the outside of the shape. Perimeter can be measured in mm, cm, m, km, etc.



The perimeter of the rectangle above of length l and breadth b is therefore:

Perimeter $= l + b + l + b$

This can be rearranged to give:

Perimeter $= 2l + 2b$

This in turn can be factorised to give:

Perimeter $= 2(l + b)$

The **area** of a shape is the amount of surface that it covers. Area is measured in mm^2, cm^2, m^2, km^2, etc.

The area A of the rectangle above is given by the formula:

$A = lb$

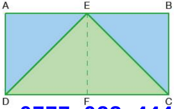
Worked example Calculate the breadth of a rectangle of area 200 cm^2 and length 25 cm.

$$A = lb$$
$$200 = 25b$$
$$b = 8$$

So the breadth is 8 cm.

The area of a triangle

Rectangle ABCD has a triangle CDE drawn inside it.

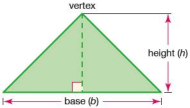


Point E is said to be a **vertex** of the triangle.
EF is the **height** or **altitude** of the triangle.
CD is the **length** of the rectangle, but is called the **base** of the triangle.

It can be seen from the diagram that triangle DEF is half the area of the rectangle AEFD.

Also triangle CFE is half the area of rectangle EBCF.

It follows that **triangle CDE is half the area of rectangle ABCD**.

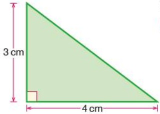


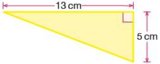
Area of a triangle $A = \frac{1}{2}bh$, where b is the base and h is the height.

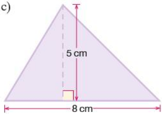
Note: it does not matter which side is called the base, but the height must be measured at right angles from the base to the opposite vertex.

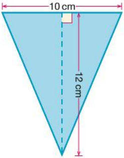
Exercise 27.1

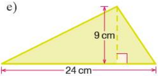
1. Calculate the areas of the triangles below:

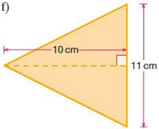
a)


b)


c)


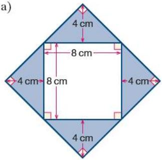
d)


e)


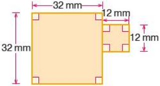
f)


2. Calculate the areas of the shapes below:

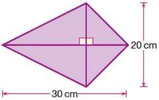
a)



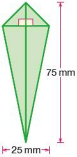
b)



c)



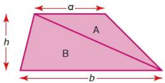
d)



The area of a parallelogram and a trapezium

A **parallelogram** can be rearranged to form a rectangle as shown below:

Therefore: area of parallelogram
= base length × perpendicular height.



A **trapezium** can be visualised as being split into two triangles as shown on the left:

Area of triangle $A = \frac{1}{2} \times a \times h$

Area of triangle $B = \frac{1}{2} \times b \times h$

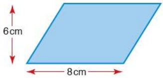
Area of the trapezium

$= \text{area of triangle } A + \text{area of triangle } B$

$= \frac{1}{2}ah + \frac{1}{2}bh$

$= \frac{1}{2}h(a + b)$

Worked examples

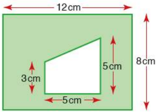


a) Calculate the area of the parallelogram (left):

Area = base length × perpendicular height

$= 8 \times 6$

$= 48\ \text{cm}^2$



b) Calculate the shaded area in the shape (left):

Area of rectangle $= 12 \times 8$

$= 96\ \text{cm}^2$

Area of trapezium $= \frac{1}{2} \times 5(3 + 5)$

$= 2.5 \times 8$

$= 20\ \text{cm}^2$

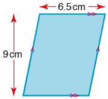
Shaded area $= 96 - 20$

$= 76\ \text{cm}^2$

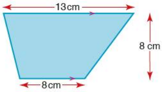
Exercise 27.2

Find the area of each of the following shapes:

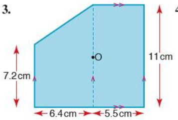
1.



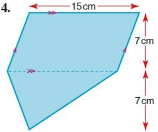
2.



3.

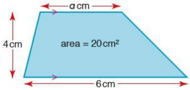


4.

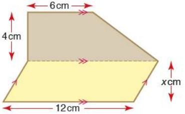


Exercise 27.3

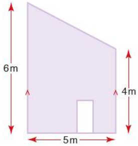
1. Calculate a.



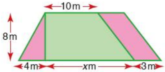
2. If the areas of this trapezium and parallelogram are equal, calculate x.



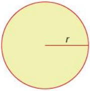
3. The end view of a house is as shown in the diagram (below). If the door has a width and height of 0.75 m and 2 m respectively. Calculate the area of brickwork.

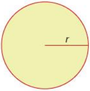


4. A garden in the shape of a trapezium is split into three parts: flower beds in the shape of a triangle and a parallelogram; and a section of grass in the shape of a trapezium, as shown below. The area of the grass is two and a half times the total area of flower beds. Calculate:
   a) the area of each flower bed,
   b) the area of grass,
   c) the value of x.



The circumference and area of a circle





The circumference is $2\pi r$.

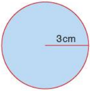
$C = 2\pi r$

The area is πr^2.

$A = \pi r^2$

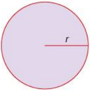
Worked examples

a) Calculate the circumference of this circle, giving your answer to 3 s.f.



$$C = 2\pi r$$
$$= 2\pi \times 3 = 18.8$$

The circumference is 18.8 cm.



b) If the circumference of this circle is 12 cm, calculate the radius, giving your answer to 3 s.f.

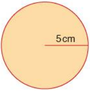
$$C = 2\pi r$$

$$r = \frac{C}{2\pi}$$

$$r = \frac{12}{2\pi} = 1.91$$

The radius is 1.91 cm.

c) Calculate the area of this circle, giving your answer to 3 s.f.



$$A = \pi r^2$$
$$= \pi \times 5^2 = 78.5$$

The area is 78.5 cm^2.

d) The area of a circle is 34 cm^2, calculate the radius, giving your answer to 3 s.f.

$$A = \pi r^2$$

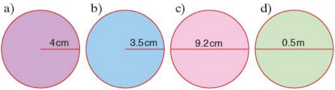
$$r = \sqrt{\frac{A}{\pi}}$$

$$r = \sqrt{\frac{34}{\pi}} = 3.29$$

The radius is 3.29 cm.

Exercise 27.4

1. Calculate the circumference of each circle, giving your answer to 3 s.f.



2. Calculate the area of each of the circles in question 1. Give your answers to 3 s.f.

3. Calculate the radius of a circle when the circumference is:
   a) 15 cm b) π cm
   c) 4 m d) 8 mm

4. Calculate the diameter of a circle when the area is:
   a) 16 cm^2 b) 9π cm^2
   c) 8.2 m^2 d) 14.6 mm^2

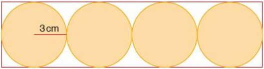
Exercise 27.5

1. The wheel of a car has an outer radius of 25 cm. Calculate:
   a) how far the car has travelled after one complete turn of the wheel,
   b) how many times the wheel turns for a journey of 1 km.

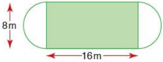
2. If the wheel of a bicycle has a diameter of 60 cm, calculate how far a cyclist will have travelled after the wheel has rotated 100 times.

3. A circular ring has a cross-section as shown (left). If the outer radius is 22 mm and the inner radius 20 mm, calculate the cross-sectional area of the ring.

4.



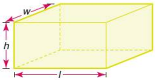
Four circles are drawn in a line and enclosed by a rectangle as shown. If the radius of each circle is 3 cm, calculate:
   a) the area of the rectangle,
   b) the area of each circle,
   c) the unshaded area within the rectangle.



5. A garden is made up of a rectangular patch of grass and two semi-circular vegetable patches. If the dimensions of the rectangular patch are 16 m (length) and 8 m (width) respectively, calculate:
   a) the perimeter of the garden,
   b) the total area of the garden.

The surface area of a cuboid and a cylinder

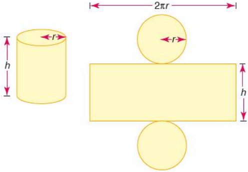
To calculate the surface area of a **cuboid** start by looking at its individual faces. These are either squares or rectangles. The surface area of a cuboid is the sum of the areas of its faces.



Area of top $= wl$ Area of bottom $= wl$
Area of front $= lh$ Area of back $= lh$
Area of one side $= wh$ Area of other side $= wh$

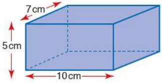
Total surface area
$$= 2wl + 2lh + 2wh$$
$$= 2(wl + lh + wh)$$

For the surface area of a **cylinder** it is best to visualise the net of the solid: it is made up of one rectangular piece and two circular pieces.



Area of circular pieces $= 2 \times \pi r^2$
Area of rectangular piece $= 2\pi r \times h$
Total surface area $= 2\pi r^2 + 2\pi rh$
$= 2\pi r(r + h)$

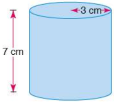
Worked examples



a) Calculate the surface area of the cuboid shown (left).

Total area of top and bottom $= 2 \times 7 \times 10 = 140\text{ cm}^2$
Total area of front and back $= 2 \times 5 \times 10 = 100\text{ cm}^2$
Total area of both sides $= 2 \times 5 \times 7 = 70\text{ cm}^2$

Total surface area $= 310\text{ cm}^2$



b) If the height of a cylinder is 7 cm and the radius of its circular top is 3 cm, calculate its surface area.

Total surface area $= 2\pi r(r + h)$
$= 2\pi \times 3 \times (3 + 7)$
$= 6\pi \times 10$
$= 60\pi$
$= 188\text{ cm}^2$ (3 s.f.)

The total surface area is 188 cm².

Exercise 27.6

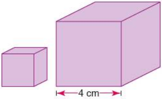
1. Calculate the surface area of each of the following cuboids:
   a) $l = 12$ cm, $w = 10$ cm, $h = 5$ cm
   b) $l = 4$ cm, $w = 6$ cm, $h = 8$ cm
   c) $l = 4.2$ cm, $w = 7.1$ cm, $h = 3.9$ cm
   d) $l = 5.2$ cm, $w = 2.1$ cm, $h = 0.8$ cm

2. Calculate the height of each of the following cuboids:
   a) $l = 5$ cm, $w = 6$ cm, surface area $= 104\ \text{cm}^2$
   b) $l = 2$ cm, $w = 8$ cm, surface area $= 112\ \text{cm}^2$
   c) $l = 3.5$ cm, $w = 4$ cm, surface area $= 118\ \text{cm}^2$
   d) $l = 4.2$ cm, $w = 10$ cm, surface area $= 226\ \text{cm}^2$

3. Calculate the surface area of each of the following cylinders:
   a) $r = 2$ cm, $h = 6$ cm b) $r = 4$ cm, $h = 7$ cm
   c) $r = 3.5$ cm, $h = 9.2$ cm d) $r = 0.8$ cm, $h = 4.3$ cm

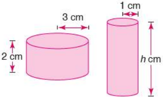
4. Calculate the height of each of the following cylinders. Give your answers to 1 d.p.
   a) $r = 2.0$ cm, surface area $= 40\ \text{cm}^2$
   b) $r = 3.5$ cm, surface area $= 88\ \text{cm}^2$
   c) $r = 5.5$ cm, surface area $= 250\ \text{cm}^2$
   d) $r = 3.0$ cm, surface area $= 189\ \text{cm}^2$

Exercise 27.7



1. Two cubes (left) are placed next to each other. The length of each of the edges of the larger cube is 4 cm. If the ratio of their surface areas is 1 : 4, calculate:
   a) the surface area of the small cube,
   b) the length of an edge of the small cube.

2. A cube and a cylinder have the same surface area. If the cube has an edge length of 6 cm and the cylinder a radius of 2 cm calculate:
   a) the surface area of the cube,
   b) the height of the cylinder.



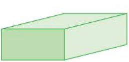
3. Two cylinders (left) have the same surface area. The shorter of the two has a radius of 3 cm and a height of 2 cm, and the taller cylinder has a radius of 1 cm. Calculate:
   a) the surface area of one of the cylinders,
   b) the height of the taller cylinder.

4. Two cuboids have the same surface area. The dimensions of one of them are: length = 3 cm, width = 4 cm and height = 2 cm.
   Calculate the height of the other cuboid if its length is 1 cm and its width is 4 cm.

The volume of a prism

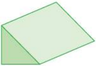
A prism is any three-dimensional object which has a constant cross-sectional area.

Below are a few examples of some of the more common types of prism.



Rectangular prism (cuboid)

Circular prism (cylinder)

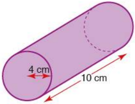


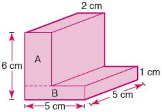
Triangular prism

When each of the shapes is cut parallel to the shaded face, the cross-section is constant and the shape is therefore classified as a prism.

Volume of a prism = area of cross-section × length

Worked examples





a) Calculate the volume of the cylinder in the diagram (left):

Volume = cross-sectional area × length

$= \pi \times 4^2 \times 10$

Volume = 503 cm^3 (3 s.f.)

b) Calculate the volume of the 'L' shaped prism shown in the diagram (below left):

The cross-sectional area can be split into two rectangles:

Area of rectangle A $= 5 \times 2$

$= 10$ cm^2

Area of rectangle B $= 5 \times 1$

$= 5$ cm^2

Total cross-sectional area = (10 cm^2 + 5 cm^2) = 15 cm^2

Volume of prism $= 15 \times 5$

$= 75$ cm^3

Exercise 27.8

1. Calculate the volume of each of the following cuboids, where w, l and h represent the width, length and height respectively.

a) $w = 2$ cm, $l = 3$ cm, $h = 4$ cm
b) $w = 6$ cm, $l = 1$ cm, $h = 3$ cm
c) $w = 6$ cm, $l = 23$ mm, $h = 2$ cm
d) $w = 42$ mm, $l = 3$ cm, $h = 0.007$ m

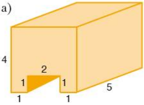
2. Calculate the volume of each of the following cylinders, where r represents the radius of the circular face and h the height of the cylinder.

a) $r = 4$ cm, $h = 9$ cm
b) $r = 3.5$ cm, $h = 7.2$ cm
c) $r = 25$ mm, $h = 10$ cm
d) $r = 0.3$ cm, $h = 17$ mm

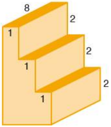
3. Calculate the volume of each of the following triangular prisms, where b represents the base length of the triangular face, h its perpendicular height and l the length of the prism.
   a) $b = 6$ cm, $h = 3$ cm, $l = 12$ cm
   b) $b = 4$ cm, $h = 7$ cm, $l = 10$ cm
   c) $b = 5$ cm, $h = 24$ mm, $l = 7$ cm
   d) $b = 62$ mm, $h = 2$ cm, $l = 0.01$ m

4. Calculate the volume of each of the following prisms. All dimensions are given in centimetres.

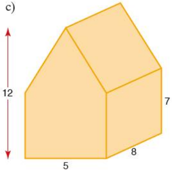
a)



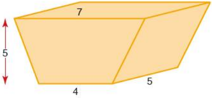
b)



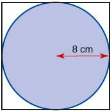
c)



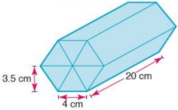
d)



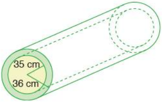
Exercise 27.9



1. The diagram shows a plan view of a cylinder inside a box the shape of a cube. If the radius of the cylinder is 8 cm, calculate:
   a) the height of the cube,
   b) the volume of the cube,
   c) the volume of the cylinder,
   d) the percentage volume of the cube not occupied by the cylinder.



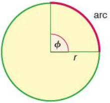
2. A chocolate bar is made in the shape of a triangular prism. The triangular face of the prism is equilateral and has an edge length of 4 cm and a perpendicular height of 3.5 cm. The manufacturer also sells these in special packs of six bars arranged as a hexagonal prism.
   If the prisms are 20 cm long, calculate:
   a) the cross-sectional area of the pack,
   b) the volume of the pack.



3. A cuboid and a cylinder have the same volume. The radius and height of the cylinder are 2.5 cm and 8 cm respectively. If the length and width of the cuboid are each 5 cm, calculate its height to 1 d.p.

4. A section of steel pipe is shown in the diagram. The inner radius is 35 cm and the outer radius 36 cm. Calculate the volume of steel used in making the pipe if it has a length of 130 m.

Arc length



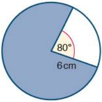
An **arc** is part of the circumference of a circle between two radii.

Its length is proportional to the size of the angle ϕ between the two radii. The length of the arc as a fraction of the circumference of the whole circle is therefore equal to the fraction that ϕ is of 360°.

$$\text{Arc length} = \frac{\phi}{360} \times 2\pi r$$

Worked examples

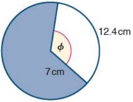
a) Find the length of the minor arc in the circle (right). Give your answer to 3 s.f.



$$\text{Arc length} = \frac{80}{360} \times 2 \times \pi \times 6$$

$$= 8.38 \text{ cm}$$

b) In the circle (left), the length of the minor arc is 12.4 cm and the radius is 7 cm.



i) Calculate the angle ϕ.

$$\text{Arc length} = \frac{\phi}{360} \times 2\pi r$$

$$12.4 = \frac{\phi}{360} \times 2 \times \pi \times 7$$

$$\frac{12.4 \times 360}{2 \times \pi \times 7} = \phi$$

$$\phi = 101.5° \text{ (1 d.p.)}$$

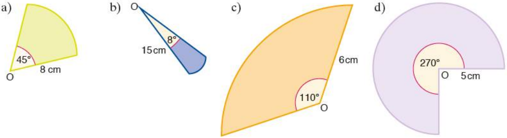
ii) Calculate the length of the major arc.

$$C = 2\pi r$$
$$= 2 \times \pi \times 7 = 44.0 \text{ cm (3 s.f.)}$$

$$\text{Major arc} = \text{circumference} - \text{minor arc}$$
$$= (44.0 - 12.4) = 31.6 \text{ cm}$$

Exercise 27.10

1. For each of the following, give the length of the arc to 3 s.f. O is the centre of the circle.

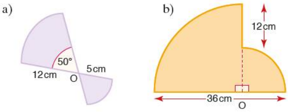


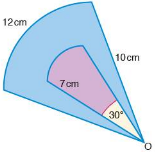
2. A sector is the region of a circle enclosed by two radii and an arc. Calculate the angle ϕ for each of the following sectors. The radius r and arc length a are given in each case.
   a) $r = 14$ cm, $a = 8$ cm
   b) $r = 4$ cm, $a = 16$ cm
   c) $r = 7.5$ cm, $a = 7.5$ cm
   d) $r = 6.8$ cm, $a = 13.6$ cm

3. Calculate the radius r for each of the following sectors. The angle ϕ and arc length a are given in each case.
   a) $\phi = 75°$, $a = 16$ cm
   b) $\phi = 300°$, $a = 24$ cm
   c) $\phi = 20°$, $a = 6.5$ cm
   d) $\phi = 243°$, $a = 17$ cm

Exercise 27.11

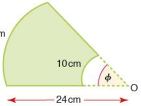
1. Calculate the perimeter of each of these shapes.

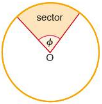




2. A shape (left) is made from two sectors arranged in such a way that they share the same centre. The radius of the smaller sector is 7 cm and the radius of the larger sector is 10 cm. If the angle at the centre of the smaller sector is 30° and the arc length of the larger sector is 12 cm, calculate:
   a) the arc length of the smaller sector,
   b) the total perimeter of the two sectors,
   c) the angle at the centre of the larger sector.

3. For the diagram (right), calculate:
   a) the radius of the smaller sector,
   b) the perimeter of the shape,
   c) the angle ϕ.



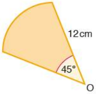


The area of a sector

A **sector** is the region of a circle enclosed by two radii and an arc. Its area is proportional to the size of the angle ϕ between the two radii. The area of the sector as a fraction of the area of the whole circle is therefore equal to the fraction that ϕ is of 360°.

$$\text{Area of sector} = \frac{\phi}{360} \times \pi r^2$$

Worked examples

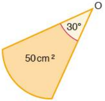


a) Calculate the area of the sector (left), giving your answer to 3 s.f.

$$\text{Area} = \frac{\phi}{360} \times \pi r^2$$

$$= \frac{45}{360} \times \pi \times 12^2$$

$$= 56.5 \text{ cm}^2$$



b) Calculate the radius of the sector (left), giving your answer to 3 s.f.

$$\text{Area} = \frac{\phi}{360} \times \pi r^2$$

$$50 = \frac{30}{360} \times \pi \times r^2$$

$$\frac{50 \times 360}{30\pi} = r^2$$

$$r = 13.8$$

The radius is 13.8 cm.

Exercise 27.12

1. Calculate the area of each of the following sectors, using the values of the angles ϕ and radius r in each case.
   a) $\phi = 60°$, $r = 8$ cm
   b) $\phi = 120°$, $r = 14$ cm
   c) $\phi = 2°$, $r = 18$ cm
   d) $\phi = 320°$, $r = 4$ cm

2. Calculate the radius for each of the following sectors, using the values of the angle ϕ and the area A in each case.

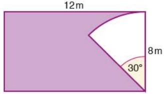
a) $\phi = 40°$, $A = 120\ \text{cm}^2$
b) $\phi = 12°$, $A = 42\ \text{cm}^2$
c) $\phi = 150°$, $A = 4\ \text{cm}^2$
d) $\phi = 300°$, $A = 400\ \text{cm}^2$

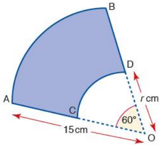
3. Calculate the value of the angle ϕ, to the nearest degree, for each of the following sectors, using the values of A and r in each case.

a) $r = 12$ cm, $A = 60\ \text{cm}^2$
b) $r = 26$ cm, $A = 0.02\ \text{m}^2$
c) $r = 0.32$ m, $A = 180\ \text{cm}^2$
d) $r = 38$ mm, $A = 16\ \text{cm}^2$

Exercise 27.13

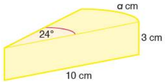
1. A rotating sprinkler is placed in one corner of a garden (below). If it has a reach of 8 m and rotates through an angle of 30°, calculate the area of garden not being watered.





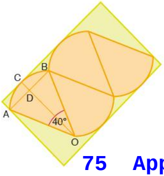
2. Two sectors AOB and COD share the same centre O. The area of AOB is three times the area of COD. Calculate:

a) the area of sector AOB,
b) the area of sector COD,
c) the radius *r* cm of sector COD.



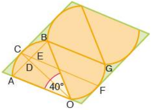
3. A circular cake is cut. One of the slices is shown. Calculate:

a) the length *a* cm of the arc,
b) the total surface area of all the sides of the slice,
c) the volume of the slice.



4. The diagram shows a plan view of four tiles in the shape of sectors placed in the bottom of a box. C is the midpoint of the arc AB and intersects the chord AB at point D. If the length OB is 10 cm, calculate:

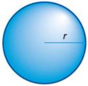
a) the length OD,
b) the length CD,
c) the area of the sector AOB,
d) the length and width of the box,
e) the area of the base of the box not covered by the tiles.



5. The tiles in question 4 are repackaged and are now placed in a box, the base of which is a parallelogram. Given that C and F are the midpoints of arcs AB and OG respectively, calculate:
   a) the angle OCF,
   b) the length CE,
   c) the length of the sides of the box,
   d) the area of the base of the box not covered by the tiles.

The volume of a sphere

Volume of sphere $= \frac{4}{3}\pi r^3$



Worked examples

a) Calculate the volume of the sphere (left), giving your answer to 3 s.f.

$$\text{Volume of sphere} = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3} \times \pi \times 3^3$$
$$= 113.1$$

The volume is 113 cm³.

b) Given that the volume of a sphere is 150 cm³, calculate its radius to 3 s.f.

$$V = \frac{4}{3}\pi r^3$$
$$r^3 = \frac{3V}{4\pi}$$
$$r^3 = \frac{3 \times 150}{4 \times \pi}$$
$$r = \sqrt[3]{35.8} = 3.30$$

The radius is 3.30 cm.

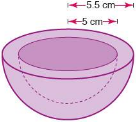
Exercise 27.14

1. Calculate the volume of each of the following spheres. The radius r is given in each case.
   a) $r = 6$ cm b) $r = 9.5$ cm
   c) $r = 8.2$ cm d) $r = 0.7$ cm

2. Calculate the radius of each of the following spheres. Give your answers in centimetres and to 1 d.p. The volume V is given in each case.
   a) $V = 130$ cm³ b) $V = 720$ cm³
   c) $V = 0.2$ m³ d) $V = 1000$ mm³

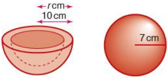
Exercise 27.15

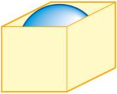
1. Given that sphere B has twice the volume of sphere A, calculate the radius of sphere B. Give your answer to 1 d.p.



2. Calculate the volume of material used to make the hemispherical bowl on the left, if the inner radius of the bowl is 5 cm and its outer radius 5.5 cm.

3. The volume of the material used to make the sphere and hemispherical bowl (right) are the same. Given that the radius of the sphere is 7 cm and the inner radius of the bowl is 10 cm, calculate, to 1 d.p., the outer radius *r* cm of the bowl.

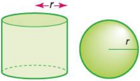




4. A ball is placed inside a box into which it will fit tightly. If the radius of the ball is 10 cm, calculate:
   a) the volume of the ball,
   b) the volume of the box,
   c) the percentage volume of the box not occupied by the ball.

5. A steel ball is melted down to make eight smaller identical balls. If the radius of the original steel ball was 20 cm, calculate to the nearest millimetre the radius of each of the smaller balls.

6. A steel ball of volume 600 cm^3 is melted down and made into three smaller balls A, B and C. If the volumes of A, B and C are in the ratio 7 : 5 : 3, calculate to 1 d.p. the radius of each of A, B and C.



7. The cylinder and sphere shown (left) have the same radius and the same height. Calculate the ratio of their volumes, giving your answer in the form, volume of cylinder : volume of sphere.

The surface area of a sphere

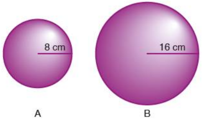
Surface area of sphere $= 4\pi r^2$

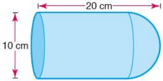
Exercise 27.16

1. Calculate the surface area of each of the following spheres when:
   a) $r = 6$ cm b) $r = 4.5$ cm
   c) $r = 12.25$ cm d) $r = \frac{1}{\sqrt{\pi}}$ cm

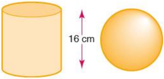
2. Calculate the radius of each of the following spheres, given the surface area in each case.
   a) $A = 50\ \text{cm}^2$ b) $A = 16.5\ \text{cm}^2$
   c) $A = 120\ \text{mm}^2$ d) $A = \pi\ \text{cm}^2$

3. Sphere A has a radius of 8 cm and sphere B has a radius of 16 cm. Calculate the ratio of their surface areas in the form $1 : n$.





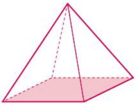
4. A hemisphere of diameter 10 cm is attached to a cylinder of equal diameter as shown.
   If the total length of the shape is 20 cm, calculate:
   a) the surface area of the hemisphere,
   b) the length of the cylinder,
   c) the surface area of the whole shape.



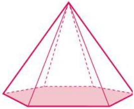
5. A sphere and a cylinder both have the same surface area and the same height of 16 cm.
   Calculate:
   a) the surface area of the sphere,
   b) the radius of the cylinder.

The volume of a pyramid

A pyramid is a three-dimensional shape in which each of its faces must be plane. A pyramid has a polygon for its base and the other faces are triangles with a common vertex, known as the **apex**. Its individual name is taken from the shape of the base.



Square-based pyramid

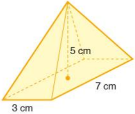


Hexagonal-based pyramid

Volume of any pyramid

$= \frac{1}{3} \times$ area of base $\times$ perpendicular height

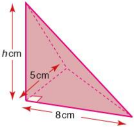
Worked examples



a) A rectangular-based pyramid has a perpendicular height of 5 cm and base dimensions as shown. Calculate the volume of the pyramid.

$$\text{Volume} = \tfrac{1}{3} \times \text{base area} \times \text{height}$$
$$= \tfrac{1}{3} \times 3 \times 7 \times 5$$
$$= 35$$

The volume is 35 cm^3.



b) The pyramid shown has a volume of 60 cm^3. Calculate its perpendicular height h cm.

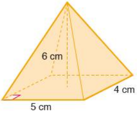
$$\text{Volume} = \tfrac{1}{3} \times \text{base area} \times \text{height}$$
$$\text{Height} = \frac{3 \times \text{volume}}{\text{base area}}$$
$$h = \frac{3 \times 60}{\frac{1}{2} \times 8 \times 5}$$
$$h = 9$$

The height is 9 cm.

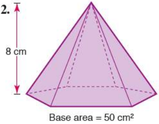
Exercise 27.17

Find the volume of each of the following pyramids:

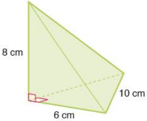
1.



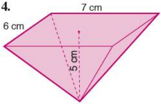
2.



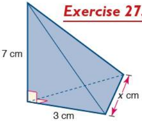
3.



4.

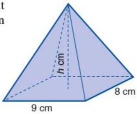


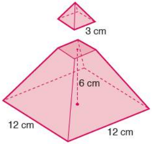
Exercise 27.18



1. Calculate the perpendicular height h cm for the pyramid (right), given that it has a volume of 168 cm^3.

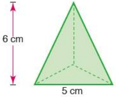
2. Calculate the length of the edge marked x cm, given that the volume of the pyramid (left) is 14 cm^3.

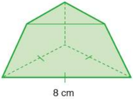




3. The top of a square-based pyramid (left) is cut off. The cut is made parallel to the base. If the base of the smaller pyramid has a side length of 3 cm and the vertical height of the truncated pyramid is 6 cm, calculate:
   a) the height of the original pyramid,
   b) the volume of the original pyramid,
   c) the volume of the truncated pyramid.

4. The top of a triangular-based pyramid (tetrahedron) is cut off. The cut is made parallel to the base. If the vertical height of the top is 6 cm, calculate:
   a) the height of the truncated piece,
   b) the volume of the small pyramid,
   c) the volume of the original pyramid.



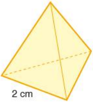


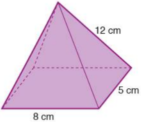
The surface area of a pyramid

The surface area of a pyramid is found simply by adding together the areas of all of its faces.

Exercise 27.19

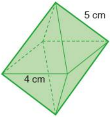
1. Calculate the surface area of a regular tetrahedron with edge length 2 cm.

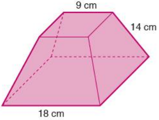




2. The rectangular-based pyramid shown (left) has a sloping edge length of 12 cm. Calculate its surface area.

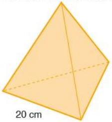
3. Two square-based pyramids are glued together as shown (right). Given that all the triangular faces are identical, calculate the surface area of the whole shape.

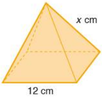




4. Calculate the surface area of the truncated square-based pyramid shown (left). Assume that all the sloping faces are identical.

5. The two pyramids shown below have the same surface area.



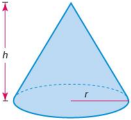


Calculate:
a) the surface area of the regular tetrahedron,
b) the area of one of the triangular faces on the square-based pyramid,
c) the value of x.

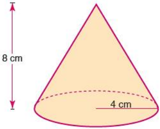
The volume of a cone

A cone is a pyramid with a circular base. The formula for its volume is therefore the same as for any other pyramid.

$$\text{Volume} = \tfrac{1}{3} \times \text{base area} \times \text{height}$$
$$= \tfrac{1}{3}\pi r^2 h$$



Worked examples

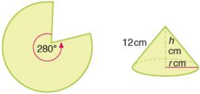


a) Calculate the volume of the cone (left).

$$\text{Volume} = \tfrac{1}{3}\pi r^2 h$$
$$= \tfrac{1}{3} \times \pi \times 4^2 \times 8$$
$$= 134.0 \text{ (1 d.p.)}$$

The volume is 134 cm^3 (3 s.f.).

b) The sector below is assembled to form a cone as shown.



i) Calculate the base circumference of the cone.
The base circumference of the cone is equal to the arc length of the sector.

$$\text{Sector arc length} = \frac{\phi}{360} \times 2\pi r$$
$$= \frac{280}{360} \times 2\pi \times 12 = 58.6 \text{ (3 s.f.)}$$

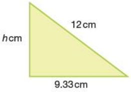
So the base circumference is 58.6 cm.

ii) Calculate the base radius of the cone.
The base of a cone is circular, therefore:

$$C = 2\pi r$$
$$r = \frac{C}{2\pi} = \frac{58.6}{2\pi}$$
$$= 9.33 \text{ (3 s.f.)}$$

So the radius is 9.33 cm.

iii) Calculate the vertical height of the cone.
The vertical height of the cone can be calculated using Pythagoras' theorem on the right-angled triangle enclosed by the base radius, vertical height and the sloping face, as shown below.
Note that the length of the sloping face is equal to the radius of the sector.



$$12^2 = h^2 + 9.33^2$$
$$h^2 = 12^2 - 9.33^2$$
$$h^2 = 56.9$$
$$h = 7.54 \text{ (3 s.f.)}$$

So the height is 7.54 cm.

iv) Calculate the volume of the cone.

$$\text{Volume} = \tfrac{1}{3} \times \pi r^2 h$$
$$= \tfrac{1}{3} \times \pi \times 9.33^2 \times 7.54$$
$$= 688 \text{ (3 s.f.)}$$

So the volume is 688 cm^3.

It is important to note that, although answers were given to 3 s.f. in each case, where the answer was needed in a subsequent calculation the exact value was used and not the rounded one. By doing this we avoid introducing rounding errors into the calculations.

Exercise 27.20

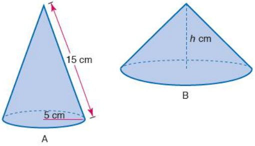
1. Calculate the volume of each of the following cones. Use the values for the base radius r and the vertical height h given in each case.
   a) $r = 3$ cm, $h = 6$ cm
   b) $r = 6$ cm, $h = 7$ cm
   c) $r = 8$ mm, $h = 2$ cm
   d) $r = 6$ cm, $h = 44$ mm

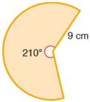
2. Calculate the base radius of each of the following cones. Use the values for the volume V and the vertical height h given in each case.
   a) $V = 600\ \text{cm}^3$, $h = 12$ cm
   b) $V = 225\ \text{cm}^3$, $h = 18$ mm
   c) $V = 1400\ \text{mm}^3$, $h = 2$ cm
   d) $V = 0.04\ \text{m}^3$, $h = 145$ mm

3. The base circumference C and the length of the sloping face l is given for each of the following cones. Calculate
   i) the base radius,
   ii) the vertical height,
   iii) the volume in each case.
   Give all answers to 3 s.f.
   a) $C = 50$ cm, $l = 15$ cm
   b) $C = 100$ cm, $l = 18$ cm
   c) $C = 0.4$ m, $l = 75$ mm
   d) $C = 240$ mm, $l = 6$ cm

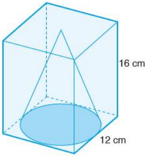
Exercise 27.21

1. The two cones A and B shown below have the same volume. Using the dimensions shown and given that the base circumference of cone B is 60 cm, calculate the height h cm.



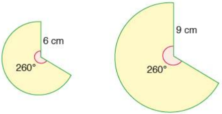


2. The sector shown is assembled to form a cone. Calculate:
   a) the base circumference of the cone,
   b) the base radius of the cone,
   c) the vertical height of the cone,
   d) the volume of the cone.



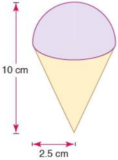
3. A cone is placed inside a cuboid as shown (left). If the base diameter of the cone is 12 cm and the height of the cuboid is 16 cm, calculate:
   a) the volume of the cuboid,
   b) the volume of the cone,
   c) the volume of the cuboid not occupied by the cone.

4. Two similar sectors are assembled into cones (below). Calculate:
   a) the volume of the smaller cone,
   b) the volume of the larger cone,
   c) the ratio of their volumes.

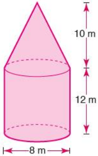


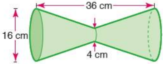
Exercise 27.22

1. An ice cream consists of a hemisphere and a cone (right). Calculate its total volume.

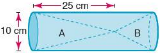


2. A cone is placed on top of a cylinder. Using the dimensions given (right), calculate the total volume of the shape.





3. Two identical truncated cones are placed end to end as shown.
   Calculate the total volume of the shape.

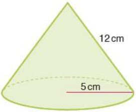


4. Two cones A and B are placed either end of a cylindrical tube as shown.
Given that the volumes of A and B are in the ratio 2 : 1, calculate:
a) the volume of cone A,
b) the height of cone B,
c) the volume of the cylinder.

The surface area of a cone

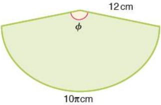
The surface area of a cone comprises the area of the circular base and the area of the curved face. The area of the curved face is equal to the area of the sector from which it is formed.

Worked example Calculate the total surface area of the cone shown (left).



$$\text{Surface area of base} = \pi r^2 = 25\pi \text{ cm}^2$$

The curved surface area can best be visualised if drawn as a sector as shown in the diagram below left:



The radius of the sector is equivalent to the slant height of the cone. The curved perimeter of the sector is equivalent to the base circumference of the cone.

$$\frac{\phi}{360} = \frac{10\pi}{24\pi}$$

$$\text{Therefore } \phi = 150°$$

$$\text{Area of sector} = \frac{150}{360} \times \pi \times 12^2 = 60\pi \text{ cm}^2$$

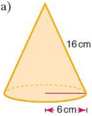
$$\begin{aligned}\text{Total surface area} &= 60\pi + 25\pi \\ &= 85\pi \\ &= 267 \text{ (3 s.f.)}\end{aligned}$$

The total surface area is 267 cm^2.

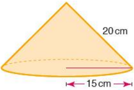
Exercise 27.23

1. Calculate the surface area of each of the following cones:

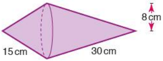
a)



b)

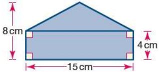


2. Two cones with the same base radius are stuck together as shown. Calculate the surface area of the shape.

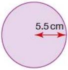


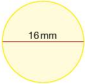
Student assessment 1

1. Calculate the area of the shape below.

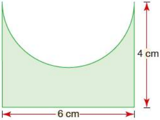


2. Calculate the circumference and area of each of the following circles. Give your answers to 3 s.f.

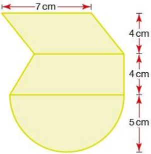
a) 

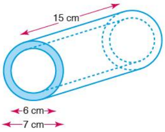
b) 

3. A semi-circular shape is cut out of the side of a rectangle as shown. Calculate the shaded area to 3 s.f.



4. For the diagram (right), calculate the area of:
   a) the semi-circle,
   b) the trapezium,
   c) the whole shape.

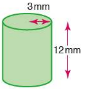




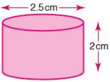
5. A cylindrical tube has an inner diameter of 6 cm, an outer diameter of 7 cm and a length of 15 cm. Calculate the following to 3 s.f.:
   a) the surface area of the shaded end,
   b) the inside surface area of the tube,
   c) the total surface area of the tube.

6. Calculate the volume of each of the following cylinders:

   a)

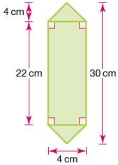


   b)



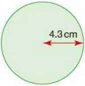
Student assessment 2

1. Calculate the area of this shape:

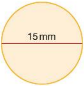


2. Calculate the circumference and area of each of the following circles. Give your answers to 3 s.f.

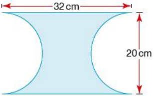
   a)



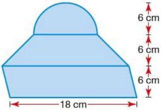
   b)



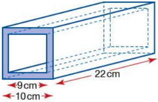
3. A rectangle of length 32 cm and width 20 cm has a semi-circle cut out of two of its sides as shown (below). Calculate the shaded area to 3 s.f.



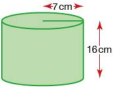
4. Calculate the area of:
   a) the semi-circle,
   b) the parallelogram,
   c) the whole shape.



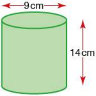
5. A prism in the shape of a hollowed-out cuboid has dimensions as shown. If the end is square, calculate the volume of the prism.



6. Calculate the surface area of each of the following cylinders:
   a)



   b)

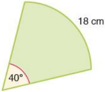


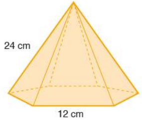
Student assessment 3

1. Calculate the arc length of each of the following sectors. The angle ϕ and radius r are given in each case.
   a) $\phi = 45°$
   $r = 15$ cm
   b) $\phi = 150°$
   $r = 13.5$ cm

2. Calculate the angle ϕ in each of the following sectors. The radius r and arc length a are given in each case.
   a) $r = 20$ mm
   $a = 95$ mm
   b) $r = 9$ cm
   $a = 9$ mm

3. Calculate the area of the sector shown below:





4. A sphere has a radius of 6.5 cm. Calculate to 3 s.f:
   a) its total surface area,
   b) its volume.

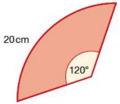
5. A pyramid with a base the shape of a regular hexagon is shown (left). If the length of each of its sloping edges is 24 cm, calculate:
   a) its total surface area,
   b) its volume.

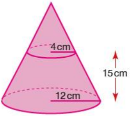
Student assessment 4

1. Calculate the arc length of the following sectors. The angle ϕ and radius r are given in each case.
   a) $\phi = 255°$, $r = 40$ cm
   b) $\phi = 240°$, $r = 16.3$ mm

2. Calculate the angle ϕ in each of the following sectors. The radius r and arc length a are given in each case.
   a) $r = 40$ cm, $a = 100$ cm
   b) $r = 20$ cm, $a = 10$ mm

3. Calculate the area of the sector shown below:

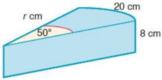




4. A hemisphere has a radius of 8 cm. Calculate to 1 d.p.:
   a) its total surface area,
   b) its volume.

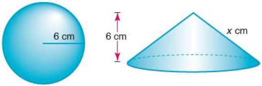
5. A cone has its top cut as shown (left). Calculate:
   a) the height of the large cone,
   b) the volume of the small cone,
   c) the volume of the truncated cone.

Student assessment 5

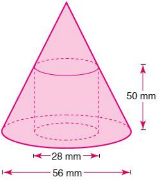


1. The prism (left) has a cross-sectional area in the shape of a sector.
   Calculate:
   a) the radius r cm,
   b) the cross-sectional area of the prism,
   c) the total surface area of the prism,
   d) the volume of the prism.

2. The cone and sphere shown (below) have the same volume.

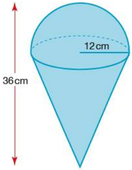


If the radius of the sphere and the height of the cone are both 6 cm, calculate:
a) the volume of the sphere,
b) the base radius of the cone,
c) the slant height x cm,
d) the surface area of the cone.



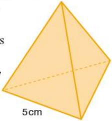
3. The top of a cone is cut off and a cylindrical hole is drilled out of the remaining truncated cone as shown (left). Calculate:
a) the height of the original cone,
b) the volume of the original cone,
c) the volume of the solid truncated cone,
d) the volume of the cylindrical hole,
e) the volume of the remaining truncated cone.

Student assessment 6



1. A metal object (left) is made from a hemisphere and a cone, both of base radius 12 cm. The height of the object when upright is 36 cm. Calculate:
a) the volume of the hemisphere,
b) the volume of the cone,
c) the curved surface area of the hemisphere,
d) the total surface area of the object.

2. A regular tetrahedron (right) has edges of length 5 cm. Calculate:
a) the surface area of the tetrahedron,
b) the surface area of a tetrahedron with edge lengths of 10 cm.



3. A regular tetrahedron and a sphere have the same surface area. If the radius of the sphere is 10 cm, calculate:
a) the area of one face of the tetrahedron,
b) the length of each edge of the tetrahedron.

(Hint: Use the trigonometric formula for the area of a triangle.)