

# DIFFERENTIATION PRACTICE

# THE OPERATION OF DIFFERENTIATION

**Question 1**

Evaluate the following.

a)  $\frac{d}{dx}(5x^6)$

$$\frac{d}{dx}(5x^6) = 30x^5$$

b)  $\frac{d}{dx}\left(2x^{\frac{3}{2}}\right)$

$$\frac{d}{dx}\left(2x^{\frac{3}{2}}\right) = 3x^{\frac{1}{2}}$$

c)  $\frac{d}{dx}(6x^4 - x^3)$

$$\frac{d}{dx}(6x^4 - x^3) = 24x^3 - 3x^2$$

d)  $\frac{d}{dx}(3x^2 + 5x + 1)$

$$\frac{d}{dx}(3x^2 + 5x + 1) = 6x + 5$$

e)  $\frac{d}{dx}\left(4x^{\frac{1}{2}} - 2x - 7\right)$

$$\frac{d}{dx}\left(4x^{\frac{1}{2}} - 2x - 7\right) = 2x^{-\frac{1}{2}} - 2$$

$\text{(1)} \frac{d}{dx}(5x^6) = 6 \times 5x^{6-1} = 30x^5$
$\text{(2)} \frac{d}{dx}(2x^{\frac{3}{2}}) = \frac{3}{2} \times 2x^{\frac{3}{2}-1} = 3x^{\frac{1}{2}}$
$\text{(3)} \frac{d}{dx}(6x^4 - x^3) = 4 \times 6x^{4-1} - 3x^{3-1} = 24x^3 - 3x^2$
$\text{(4)} \frac{d}{dx}(3x^2 + 5x + 1) = 2x \cdot 3x^{2-1} + \textcircled{S} + \textcircled{C} = 6x + 5$
$\text{(5)} \frac{d}{dx}(4x^{\frac{1}{2}} - 2x - 7) = \frac{1}{2} \times 4x^{\frac{1}{2}-1} - 2 + \textcircled{C} = 2x^{-\frac{1}{2}} - 2$

**Question 2**

Evaluate the following.

a)  $\frac{d}{dx}(4x^3)$

$$\boxed{\frac{d}{dx}(4x^3) = 12x^2}$$

b)  $\frac{d}{dx}(7x^5)$

$$\boxed{\frac{d}{dx}(7x^5) = 35x^4}$$

c)  $\frac{d}{dx}(4x^2 + 3x^4)$

$$\boxed{\frac{d}{dx}(4x^2 + 3x^4) = 8x + 12x^3}$$

d)  $\frac{d}{dx}(x^2 + 7x + 5)$

$$\boxed{\frac{d}{dx}(x^2 + 7x + 5) = 2x + 7}$$

e)  $\frac{d}{dx}\left(8x^{\frac{1}{2}} + 2x^{-2}\right)$

$$\boxed{\frac{d}{dx}\left(8x^{\frac{1}{2}} + 2x^{-2}\right) = 4x^{-\frac{1}{2}} - 4x^{-3}}$$

(a)  $\frac{d}{dx}(4x^3) = 12x^2$

(b)  $\frac{d}{dx}(7x^5) = 35x^4$

(c)  $\frac{d}{dx}(4x^2 + 3x^4) = 8x + 12x^3$

(d)  $\frac{d}{dx}(x^2 + 7x + 5) = 2x + 7 + 0 = 2x + 7$

(e)  $\frac{d}{dx}(8x^{\frac{1}{2}} + 2x^{-2}) = 4x^{-\frac{1}{2}} - 4x^{-3}$

**Question 3**Differentiate the following expressions with respect to  $x$ 

a)  $y = x^2 - 4x^6$

$$\frac{dy}{dx} = 2x - 24x^5$$

b)  $y = 5x^3 - 6x^{\frac{3}{2}}$

$$\frac{dy}{dx} = 15x^2 - 9x^{\frac{1}{2}}$$

c)  $y = 9x^{-3} + 7x^{-2}$

$$\frac{dy}{dx} = -27x^{-4} - 14x^{-3}$$

d)  $y = 5 - 5x^{-1}$

$$\frac{dy}{dx} = 5x^{-2}$$

e)  $y = 7x + \sqrt{x}$

$$\frac{dy}{dx} = 7 + \frac{1}{2}x^{-\frac{1}{2}}$$

(a)  $y = x^2 - 4x^6$   
 $\frac{d}{dx}(x^2 - 4x^6) = \frac{d}{dx}(y) = \boxed{\frac{dy}{dx} = 2x - 24x^5}$

(b)  $y = 5x^3 - 6x^{\frac{3}{2}}$   
 $\frac{d}{dx}(5x^3 - 6x^{\frac{3}{2}}) = \frac{d}{dx}(y) = \boxed{\frac{dy}{dx} = 15x^2 - 9x^{\frac{1}{2}}}$

(c)  $y = 9x^{-3} + 7x^{-2}$   
 $\frac{d}{dx}(9x^{-3} + 7x^{-2}) = \frac{d}{dx}(y) = \boxed{\frac{dy}{dx} = -27x^{-4} - 14x^{-3}}$

(d)  $y = 5 - 5x^{-1}$   
 $\frac{d}{dx}(5 - 5x^{-1}) = \frac{d}{dx}(y) = \boxed{\frac{dy}{dx} = 0 + 5x^{-2} = 5x^{-2}}$

(e)  $y = 7x + \sqrt{x}$   
 $\frac{d}{dx}(7x + \sqrt{x}) = \frac{d}{dx}(7x + x^{\frac{1}{2}}) = \frac{d}{dx}(y) = \boxed{\frac{dy}{dx} = 7 + \frac{1}{2}x^{-\frac{1}{2}}}$

**Question 4**Differentiate the following expressions with respect to  $x$ 

a)  $y = x^6 - 7x^2$

$$\boxed{\frac{dy}{dx} = 6x^5 - 14x}$$

b)  $y = 1 - 6x^{\frac{5}{2}}$

$$\boxed{\frac{dy}{dx} = 15x^{\frac{3}{2}}}$$

c)  $y = 2x + 8x^{-2}$

$$\boxed{\frac{dy}{dx} = 2 + 16x^{-3}}$$

d)  $y = (2x-1)(4x+3)$

$$\boxed{\frac{dy}{dx} = 16x + 2}$$

e)  $y = 4x^3(2-3x)$

$$\boxed{\frac{dy}{dx} = 24x^2 - 48x^3}$$

(a)  $y = x^6 - 7x^2$   
 $\frac{dy}{dx}(x^6 - 7x^2) = \frac{dy}{dx}(y) = \boxed{\frac{dy}{dx} = 6x^5 - 14x}$

(b)  $y = 1 - 6x^{\frac{5}{2}}$   
 $\frac{dy}{dx}(1 - 6x^{\frac{5}{2}}) = \frac{dy}{dx}(y) = \boxed{\frac{dy}{dx} = 0 - \frac{5}{2}x^{\frac{3}{2}}} \quad \therefore \boxed{\frac{dy}{dx} = 15x^{\frac{3}{2}}}$

(c)  $y = 2x + 8x^{-2}$   
 $\frac{dy}{dx}(2x + 8x^{-2}) = \frac{dy}{dx}(y) = \boxed{\frac{dy}{dx} = 2 + 16x^{-3}}$

(d)  $y = (2x-1)(4x+3)$   
 $\frac{dy}{dx}[(2x-1)(4x+3)] = \frac{dy}{dx}[8x^2 + 2x - 3] = \frac{dy}{dx}(y) = \boxed{\frac{dy}{dx} = 16x + 2}$

(e)  $y = 4x^3(2-3x) = 8x^3 - 12x^4$   
 $\frac{dy}{dx}(8x^3 - 12x^4) = \frac{dy}{dx}(y) = \boxed{\frac{dy}{dx} = 24x^2 - 48x^3}$

**Question 5**Find  $f'(x)$  for each of the following functions.

a)  $f(x) = 4x^3 - 9x + 2$

$$f'(x) = 12x^2 - 9$$

b)  $f(x) = 6x^{-\frac{1}{2}} + 2x$

$$f'(x) = -3x^{-\frac{3}{2}} + 2$$

c)  $f(x) = x^4 + 2x^{\frac{5}{2}}$

$$f'(x) = 4x^3 + 5x^{\frac{3}{2}}$$

d)  $f(x) = \frac{1}{2}x^2 - 4x^{-\frac{3}{2}}$

$$f'(x) = x + 6x^{-\frac{5}{2}}$$

e)  $f(x) = \frac{1}{2}x^{\frac{1}{3}} + 5x$

$$f'(x) = \frac{1}{6}x^{-\frac{2}{3}} + 5$$

(a)  $f(x) = 4x^3 - 9x + 2$   
 $\frac{d}{dx}(4x^3 - 9x + 2) = \frac{d}{dx}(f(x)) = \boxed{f'(x) = 12x^2 - 9}$

(b)  $f(x) = 6x^{-\frac{1}{2}} + 2x$   
 $\frac{d}{dx}(6x^{-\frac{1}{2}} + 2x) = \frac{d}{dx}(f(x)) = \boxed{f'(x) = -3x^{-\frac{3}{2}} + 2}$

(c)  $f(x) = x^4 + 2x^{\frac{5}{2}}$   
 $\frac{d}{dx}(x^4 + 2x^{\frac{5}{2}}) = \frac{d}{dx}(f(x)) = \boxed{f'(x) = 4x^3 + 5x^{\frac{3}{2}}}$

(d)  $f(x) = \frac{1}{2}x^2 - 4x^{-\frac{3}{2}}$   
 $\frac{d}{dx}\left(\frac{1}{2}x^2 - 4x^{-\frac{3}{2}}\right) = \frac{d}{dx}(f(x)) = \boxed{f'(x) = x + 6x^{-\frac{5}{2}}}$

(e)  $f(x) = \frac{1}{2}x^{\frac{1}{3}} + 5x$   
 $\frac{d}{dx}\left(\frac{1}{2}x^{\frac{1}{3}} + 5x\right) = \frac{d}{dx}(f(x)) = \boxed{f'(x) = \frac{1}{6}x^{-\frac{2}{3}} + 5}$

**Question 6**

Differentiate each of the following functions with respect to  $x$ .

a)  $f(x) = 6x^{-\frac{3}{2}} + 4x + 1$

$$f'(x) = -9x^{-\frac{5}{2}} + 4$$

b)  $g(x) = x^4 - x^{-1}$

$$g'(x) = 4x^3 + x^{-2}$$

c)  $h(x) = 9x^2 - \frac{1}{2}x^4$

$$h'(x) = 18x - 2x^3$$

d)  $p(x) = 4x^{\frac{1}{2}} - 6x^{\frac{1}{3}} + \frac{1}{2}x^{-\frac{1}{4}}$

$$p'(x) = 2x^{-\frac{1}{2}} - 2x^{-\frac{2}{3}} - \frac{1}{8}x^{-\frac{5}{4}}$$

e)  $v(x) = \left(8x + \frac{1}{2}\right)^2$

$$v'(x) = 128x + 8$$

$\text{(a)} \quad f(x) = 6x^{-\frac{3}{2}} + 4x + 1$ $\frac{d}{dx}(6x^{-\frac{3}{2}} + 4x + 1) = \frac{d}{dx}(f(x)) = f'(x) = -\frac{3}{2} \cdot 6x^{-\frac{5}{2}} + 4 + 0$ $\therefore f'(x) = -9x^{-\frac{5}{2}} + 4$
$\text{(b)} \quad g(x) = x^4 - x^{-1}$ $\frac{d}{dx}(x^4 - x^{-1}) = \frac{d}{dx}(g(x)) = g'(x) = 4x^3 + x^{-2}$
$\text{(c)} \quad h(x) = 9x^2 - \frac{1}{2}x^4$ $\frac{d}{dx}(9x^2 - \frac{1}{2}x^4) = \frac{d}{dx}(h(x)) = h'(x) = 18x - 2x^3$
$\text{(d)} \quad p(x) = 4x^{\frac{1}{2}} - 6x^{\frac{1}{3}} + \frac{1}{2}x^{-\frac{1}{4}}$ $\frac{d}{dx}(4x^{\frac{1}{2}} - 6x^{\frac{1}{3}} + \frac{1}{2}x^{-\frac{1}{4}}) = \frac{d}{dx}(p(x)) = p'(x) = 2x^{-\frac{1}{2}} - 2x^{-\frac{2}{3}} - \frac{1}{8}x^{-\frac{5}{4}}$
$\text{(e)} \quad v(x) = \left(8x + \frac{1}{2}\right)^2$ $\frac{d}{dx}\left(\left(8x + \frac{1}{2}\right)^2\right) = \frac{d}{dx}(v(x)) = v'(x) = \sqrt{v(x)} = \sqrt{128x + 8}$

**Question 7**

Carry out the following differentiations.

a)  $\frac{d}{dt}(4t^2 - 7t + 5)$

$$\boxed{\frac{d}{dt}(4t^2 - 7t + 5) = 8t - 7}$$

b)  $\frac{d}{dy}\left(y^{\frac{1}{2}} - \frac{2}{3}y^{-\frac{1}{2}}\right)$

$$\boxed{\frac{d}{dy}\left(y^{\frac{1}{2}} - \frac{2}{3}y^{-\frac{1}{2}}\right) = \frac{1}{2}y^{-\frac{1}{2}} + \frac{1}{3}y^{-\frac{3}{2}}}$$

c)  $\frac{d}{dz}(2z^2 - 3z^{-1} + z)$

$$\boxed{\frac{d}{dz}(2z^2 - 3z^{-1} + z) = 4z + 3z^{-2} + 1}$$

d)  $\frac{d}{dw}\left(w^2 - w^{-\frac{3}{2}}\right)$

$$\boxed{\frac{d}{dw}\left(w^2 - w^{-\frac{3}{2}}\right) = 2w + \frac{3}{2}w^{-\frac{5}{2}}}$$

e)  $\frac{d}{dx}(ax^2 - 3x^2)$

$$\boxed{\frac{d}{dx}(ax^2 - 3x^2) = 2ax - 6x}$$

(a) $\frac{d}{dt}(4t^2 - 7t + 5) = 8t - 7$
(b) $\frac{d}{dy}\left(y^{\frac{1}{2}} - \frac{2}{3}y^{-\frac{1}{2}}\right) = \frac{1}{2}y^{-\frac{1}{2}} + \frac{1}{3}y^{-\frac{3}{2}}$
(c) $\frac{d}{dz}(2z^2 - 3z^{-1} + z) = 4z + 3z^{-2} + 1$
(d) $\frac{d}{dw}\left(w^2 - w^{-\frac{3}{2}}\right) = 2w + \frac{3}{2}w^{-\frac{5}{2}}$
(e) $\frac{d}{dx}(ax^2 - 3x^2) = 2ax - 6x$
$\frac{d}{dx}[(a-3)x^2] = 2(a-3)x$

**Question 8**

Carry out the following differentiations.

a)  $\frac{d}{dy}(4y^3 + 6y + 2)$

$$\boxed{\frac{d}{dy}(4y^3 + 6y + 2) = 12y^2 + 6}$$

b)  $\frac{d}{dt}\left(7t^2 - 4t^{\frac{1}{2}}\right)$

$$\boxed{\frac{d}{dt}\left(7t^2 - 4t^{\frac{1}{2}}\right) = 14t - 2t^{-\frac{1}{2}}}$$

c)  $\frac{d}{dx}(ax^2 + bx + c)$

$$\boxed{\frac{d}{dx}(ax^2 + bx + c) = 2ax + b}$$

d)  $\frac{d}{dz}\left(\frac{1}{4}z^2 - \frac{1}{z}\right)$

$$\boxed{\frac{d}{dz}\left(\frac{1}{4}z^2 - \frac{1}{z}\right) = \frac{1}{2}z + \frac{1}{z^2}}$$

e)  $\frac{d}{dw}\left(\frac{1}{4}w^{\frac{4}{5}} + \frac{k}{w^2}\right)$

$$\boxed{\frac{d}{dw}\left(\frac{1}{4}w^{\frac{4}{5}} + \frac{k}{w^2}\right) = \frac{1}{5}w^{-\frac{1}{5}} - \frac{2k}{w^3}}$$

(a) $\frac{d}{dy}(4y^3 + 6y + 2) = 12y^2 + 6$
(b) $\frac{d}{dt}(7t^2 - 4t^{\frac{1}{2}}) = 14t - 2t^{-\frac{1}{2}}$
(c) $\frac{d}{dx}(ax^2 + bx + c) = 2ax + b$
(d) $\frac{d}{dz}\left(\frac{1}{4}z^2 - \frac{1}{z}\right) = \frac{d}{dz}\left(\frac{1}{4}z^2 - z^{-1}\right) = \frac{1}{2}z + z^{-2} = \frac{1}{2}z + \frac{1}{z^2}$
(e) $\frac{d}{dw}\left(\frac{1}{4}w^{\frac{4}{5}} + \frac{k}{w^2}\right) = \frac{d}{dw}\left(\frac{1}{4}w^{\frac{4}{5}} + kw^{-2}\right) = \frac{1}{5}w^{-\frac{1}{5}} - 2kw^{-3} = \frac{1}{5}w^{-\frac{1}{5}} - \frac{2k}{w^3}$

**Question 9**

- a) If  $A = \pi x^2 - 20x$ , find the rate of change of  $A$  with respect to  $x$ .
- b) If  $V = x - 2\pi x^3$ , find the rate of change of  $V$  with respect to  $x$ .
- c) If  $P = at^2 - bt$ , find the rate of change of  $P$  with respect to  $t$ .
- d) If  $W = 6kh^2 - h$ , find the rate of change of  $W$  with respect to  $h$ .
- e) If  $N = (at + b)^2$ , find the rate of change of  $N$  with respect to  $t$ .

$$\boxed{\frac{dA}{dx} = 2\pi x - 20}, \quad \boxed{\frac{dV}{dx} = 1 - 6\pi x^2}, \quad \boxed{\frac{dP}{dt} = 2at - b}, \quad \boxed{\frac{dW}{dh} = 3kh^{\frac{1}{2}} - 1},$$

$$\boxed{\frac{dN}{dt} = 2a^2t + 2ab}$$

(a) $A = \pi x^2 - 20x$ $\frac{dA}{dx} = 2\pi x - 20$	(d) $W = 6kh^{\frac{1}{2}} - h$ $\frac{dW}{dh} = 3kh^{\frac{1}{2}} - 1$
(b) $V = x - 2\pi x^3$ $\frac{dV}{dx} = 1 - 6\pi x^2$	(e) $N = (at + b)^2$ $N = a^2t^2 + 2abt + b^2$ $\frac{dN}{dt} = 2a^2t + 2ab$
(c) $P = at^2 - bt$ $\frac{dP}{dt} = 2at - b$	

# DIFFERENTIATING INDICES

**Question 1**

Differentiate the following expressions with respect to  $x$

a)  $y = 4\sqrt{x} - \sqrt[3]{x}$

$$\frac{dy}{dx} = 2x^{-\frac{1}{2}} - \frac{1}{3}x^{-\frac{2}{3}}$$

b)  $y = 2\sqrt{x} - 4\sqrt[3]{x^3}$

$$\frac{dy}{dx} = x^{-\frac{1}{2}} - 6x^{\frac{1}{2}}$$

c)  $y = \frac{1}{2\sqrt{x}} + \frac{4}{x^2}$

$$\frac{dy}{dx} = -\frac{1}{4}x^{-\frac{3}{2}} - 8x^{-3}$$

d)  $y = x\sqrt{x} - \frac{1}{x^2}$

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} + 2x^{-3}$$

e)  $y = 4\sqrt{x} + \frac{1}{4\sqrt{x}}$

$$\frac{dy}{dx} = 2x^{-\frac{1}{2}} - \frac{1}{8}x^{-\frac{3}{2}}$$

(a) $\frac{dy}{dx}(4\sqrt{x} - \sqrt[3]{x^3}) = \frac{d}{dx}(4x^{\frac{1}{2}} - x^{\frac{3}{3}}) = 2x^{-\frac{1}{2}} - \frac{1}{3}x^{-\frac{2}{3}}$
(b) $\frac{dy}{dx}(2\sqrt{x} - 4\sqrt[3]{x^3}) = \frac{d}{dx}(2x^{\frac{1}{2}} - 4x^{\frac{3}{3}}) = x^{-\frac{1}{2}} - 6x^{\frac{1}{2}}$
(c) $\frac{dy}{dx}\left(\frac{1}{2\sqrt{x}} + \frac{4}{x^2}\right) = \frac{d}{dx}\left(\frac{1}{2}x^{-\frac{1}{2}} + 4x^{-2}\right) = -\frac{1}{4}x^{-\frac{3}{2}} - 8x^{-3}$
(d) $\frac{dy}{dx}(x\sqrt{x} - \frac{1}{x^2}) = \frac{d}{dx}(x^{\frac{3}{2}} - x^{-2}) = \frac{d}{dx}(x^{\frac{1}{2}} - x^{-2}) = \frac{3}{2}x^{\frac{1}{2}} + 2x^{-3}$
(e) $\frac{dy}{dx}(4\sqrt{x} + \frac{1}{4\sqrt{x}}) = \frac{d}{dx}(4x^{\frac{1}{2}} + \frac{1}{4}x^{-\frac{1}{2}}) = 2x^{-\frac{1}{2}} - \frac{1}{8}x^{-\frac{3}{2}}$

**Question 2**Find  $f'(x)$  for each of the following functions.

a)  $f(x) = \frac{2}{x^3} + 5x^{\frac{2}{3}}$

$$f'(x) = -6x^{-4} + \frac{10}{3}x^{-\frac{1}{3}}$$

b)  $f(x) = 8x^{\frac{3}{4}} - \frac{2}{x^4}$

$$f'(x) = 6x^{-\frac{1}{4}} + 8x^{-5}$$

c)  $f(x) = 2x - \frac{3}{x^2} + 4\sqrt{x} + 2$

$$f'(x) = 2 + 6x^{-3} + 2x^{-\frac{1}{2}}$$

d)  $f(x) = \sqrt[3]{x^2} - \frac{3}{2x^3}$

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} + \frac{9}{2}x^{-4}$$

e)  $f(x) = \sqrt{x^3} - \frac{1}{2x^2}$

$$f'(x) = \frac{3}{2}x^{\frac{1}{2}} + x^{-3}$$

(a) $f(x) = \frac{2}{x^3} + 5x^{\frac{2}{3}}$ $f'(x) = 5x^{-3} + 5x^{\frac{1}{3}}$ $\therefore f'(x) = -5x^{-4} + \frac{5}{3}x^{\frac{1}{3}}$	(b) $f(x) = \sqrt[3]{x^2} - \frac{3}{2x^3}$ $f'(x) = x^{\frac{2}{3}} - \frac{3}{2}x^{-4}$ $\therefore f'(x) = \frac{2}{3}x^{-\frac{1}{3}} + \frac{9}{2}x^{-4}$
(c) $f(x) = 8x^{\frac{3}{4}} - \frac{2}{x^4}$ $f'(x) = 6x^{\frac{3}{4}} - 2x^{-5}$ $\therefore f'(x) = 6x^{-\frac{1}{4}} + 8x^{-5}$	(d) $f(x) = \sqrt{x^3} - \frac{1}{2x^2}$ $f'(x) = x^{\frac{1}{2}} - \frac{1}{2}x^{-3}$ $\therefore f'(x) = \frac{3}{2}x^{\frac{1}{2}} + x^{-3}$
(e) $f(x) = 2x - \frac{3}{x^2} + 4\sqrt{x} + 2$ $f'(x) = 2x^{\frac{1}{2}} - 2x^{-3}$ $\therefore f'(x) = 2 + 6x^{-3} + 2x^{-\frac{1}{2}}$	

**Question 3**

Differentiate the following expressions with respect to  $x$

a)  $y = \frac{4}{x^3} - \frac{4}{3x^2}$

$$\frac{dy}{dx} = \frac{8}{3}x^{-3} - 12x^{-4}$$

b)  $y = \frac{3}{4x^2} - \frac{12}{x^2\sqrt{x}}$

$$\frac{dy}{dx} = 30x^{-\frac{7}{2}} - \frac{3}{2}x^{-3}$$

c)  $y = \frac{1}{3x} + \frac{2x^3+1}{3\sqrt{x}}$

$$\frac{dy}{dx} = -\frac{1}{3}x^{-2} + \frac{5}{3}x^{\frac{3}{2}} - \frac{1}{6}x^{-\frac{3}{2}}$$

d)  $y = 2\sqrt{x}(7x - x^2)$

$$\frac{dy}{dx} = 21x^{\frac{1}{2}} - 5x^{\frac{3}{2}}$$

e)  $y = (3 + 2\sqrt{x})^2$

$$\frac{dy}{dx} = 6x^{-\frac{1}{2}} + 4$$

(a)  $y = \frac{4}{x^3} - \frac{4}{3x^2} = 4x^{-3} - \frac{4}{3}x^{-2}$   $\therefore \frac{dy}{dx} = -12x^{-4} + \frac{8}{3}x^{-3}$

(b)  $y = \frac{3}{4x^2} - \frac{12}{x^2\sqrt{x}} = \frac{3}{4}x^{-2} - \frac{12}{3x^{\frac{3}{2}}} = \frac{3}{4}x^{-2} - \frac{12}{3}x^{-\frac{3}{2}} = \frac{3}{4}x^{-2} - 4x^{-\frac{3}{2}}$   
 $\therefore \frac{dy}{dx} = -\frac{3}{2}x^{-3} + 3x^{-\frac{5}{2}}$

(c)  $y = \frac{1}{3x} + \frac{2x^{\frac{3}{2}}+1}{3\sqrt{x}} = \frac{1}{3}x^{-1} + \frac{2x^{\frac{3}{2}}+1}{3\sqrt{x}} = \frac{1}{3}x^{-1} + \frac{2x^{\frac{3}{2}}}{3\sqrt{x}} + \frac{1}{3\sqrt{x}}$   
 $= \frac{1}{3}x^{-1} + \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{3}x^{-\frac{1}{2}}$   
 $\therefore \frac{dy}{dx} = -\frac{1}{3}x^{-2} + \frac{2}{3}x^{\frac{1}{2}} - \frac{1}{3}x^{-\frac{3}{2}}$

(d)  $y = 2\sqrt{x}(7x - x^2) = 2x^{\frac{1}{2}}(7x - x^2) = 14x^{\frac{3}{2}} - 2x^{\frac{5}{2}}$   
 $\therefore \frac{dy}{dx} = 21x^{\frac{1}{2}} - 5x^{\frac{3}{2}}$

(e)  $y = (3 + 2\sqrt{x})^2 = \overset{?}{3} + 2 \times 3x \cdot 2\sqrt{x} + (2\sqrt{x})^2 = 9 + 12x^{\frac{1}{2}} + 4x$   
 $= 9 + 12x^{\frac{1}{2}} + 4x$   
 $\therefore \frac{dy}{dx} = 6x^{-\frac{1}{2}} + 4$

OP: MULTIPLY TWO BRACKETS  
 $= (3 + 2\sqrt{x})(3 + 2\sqrt{x}) = 9 + 6\sqrt{x} + 6\sqrt{x} + 4x$   
 $= 9 + 12\sqrt{x} + 4x = 9 + 12x^{\frac{1}{2}} + 4x$

**Question 4**

Evaluate the following.

a)  $\frac{d}{dx} \left( 6x^{\frac{4}{3}} - 2x^{\frac{5}{2}} \right)$

$$8x^{\frac{1}{3}} - 5x^{\frac{3}{2}}$$

b)  $\frac{d}{dx} \left( \frac{1}{x} - \frac{1}{\sqrt{x}} \right)$

$$-x^{-2} + \frac{1}{2}x^{-\frac{3}{2}}$$

c)  $\frac{d}{dx} \left( \sqrt[3]{x} - \frac{27}{x} \right)$

$$\frac{1}{3}x^{-\frac{2}{3}} + 27x^{-2}$$

d)  $\frac{d}{dx} \left( \frac{3\sqrt{x}-2}{x^{\frac{3}{2}}} \right)$

$$-3x^{-2} + 3x^{-\frac{5}{2}}$$

e)  $\frac{d}{dx} \left[ \frac{1}{3\sqrt{x}} \left( \frac{2}{x} - 3 \right) \right]$

$$-x^{-\frac{5}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$$

$$\begin{aligned}
 \text{(a)} \quad & \frac{d}{dx} \left[ 6x^{\frac{4}{3}} - 2x^{\frac{5}{2}} \right] = 8x^{\frac{1}{3}} - 5x^{\frac{3}{2}} // \\
 \text{(b)} \quad & \frac{d}{dx} \left[ \frac{1}{x} - \frac{1}{\sqrt{x}} \right] = \frac{d}{dx} \left[ x^{-1} - x^{-\frac{1}{2}} \right] = -x^{-2} + \frac{1}{2}x^{-\frac{3}{2}} // \\
 \text{(c)} \quad & \frac{d}{dx} \left[ \sqrt[3]{x} - \frac{27}{x} \right] = \frac{d}{dx} \left[ x^{\frac{1}{3}} - 27x^{-1} \right] = \frac{1}{3}x^{-\frac{2}{3}} + 27x^{-2} // \\
 \text{(d)} \quad & \frac{d}{dx} \left[ \frac{3\sqrt{x}-2}{x^{\frac{3}{2}}} \right] = \frac{d}{dx} \left[ \frac{3x^{\frac{1}{2}}-2}{x^{\frac{3}{2}}} \right] = \frac{d}{dx} \left[ \frac{3x^{\frac{1}{2}}}{x^{\frac{3}{2}}} - \frac{2}{x^{\frac{3}{2}}} \right] \\
 & = \frac{d}{dx} \left[ 3x^{-\frac{1}{2}} - 2x^{-\frac{3}{2}} \right] = -3x^{-2} + 3x^{-\frac{5}{2}} // \\
 \text{(e)} \quad & \frac{d}{dx} \left[ \frac{1}{3\sqrt{x}} \left( \frac{2}{x} - 3 \right) \right] = \frac{1}{3} \cdot \frac{d}{dx} \left[ \frac{2}{x^{\frac{3}{2}}} - 3x^{-\frac{1}{2}} \right] = \frac{1}{3} \cdot \left[ \frac{3}{2}x^{-\frac{5}{2}} - \frac{3}{2}x^{-\frac{3}{2}} \right] \\
 & = -x^{-\frac{5}{2}} + \frac{1}{2}x^{-\frac{3}{2}}
 \end{aligned}$$

**Question 5**

Evaluate the following.

a) 
$$\frac{d}{dx} \left( \frac{x+x^2}{\sqrt{x}} \right)$$

$$\boxed{\frac{1}{2}x^{-\frac{1}{2}} + \frac{3}{2}x^{\frac{1}{2}}}$$

b) 
$$\frac{d}{dx} \left( \frac{4x+\sqrt{x}}{2x^2} \right)$$

$$\boxed{-2x^{-2} - \frac{3}{4}x^{-\frac{5}{2}}}$$

c) 
$$\frac{d}{dx} \left( \frac{x^2+2}{x^3} \right)$$

$$\boxed{-x^{-2} - 6x^{-4}}$$

d) 
$$\frac{d}{dx} \left( \frac{1-\sqrt{x}}{4x^3} \right)$$

$$\boxed{-\frac{3}{4}x^{-4} + \frac{5}{8}x^{-\frac{7}{2}}}$$

e) 
$$\frac{d}{dx} \left[ \frac{\sqrt[3]{x^5} - 2x\sqrt{x}}{3x} \right]$$

$$\boxed{\frac{2}{9}x^{-\frac{1}{3}} - \frac{1}{3}x^{-\frac{1}{2}}}$$

$$\begin{aligned}
 \text{(a)} \quad & \frac{d}{dx} \left( \frac{x+x^2}{\sqrt{x}} \right) = \frac{d}{dx} \left( \frac{x}{\sqrt{x}} + \frac{x^2}{\sqrt{x}} \right) = \frac{d}{dx} \left( x^{\frac{1}{2}} + x^{\frac{3}{2}} \right) = \frac{1}{2}x^{-\frac{1}{2}} + \frac{3}{2}x^{\frac{1}{2}} \\
 \text{(b)} \quad & \frac{d}{dx} \left( \frac{4x+\sqrt{x}}{2x^2} \right) = \frac{d}{dx} \left( \frac{4x^{\frac{1}{2}} + x^{\frac{1}{2}}}{2x^2} \right) = \frac{d}{dx} \left( 2x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}} \right) = -2x^{-2} - \frac{3}{4}x^{-\frac{5}{2}} \\
 \text{(c)} \quad & \frac{d}{dx} \left( \frac{x^2+2}{x^3} \right) = \frac{d}{dx} \left( \frac{x^{\frac{2}{3}} + \frac{2}{3}}{x^3} \right) = \frac{d}{dx} \left( x^{\frac{2}{3}} + 2x^{-\frac{2}{3}} \right) = -2x^{-\frac{5}{3}} - 6x^{-4} \\
 \text{(d)} \quad & \frac{d}{dx} \left( \frac{1-\sqrt{x}}{4x^3} \right) = \frac{d}{dx} \left( \frac{1-x^{\frac{1}{2}}}{4x^3} \right) = \frac{d}{dx} \left( \frac{1}{4x^3} - \frac{x^{-\frac{1}{2}}}{4x^3} \right) = -\frac{3}{4}x^{-4} + \frac{5}{8}x^{-\frac{7}{2}} \\
 \text{(e)} \quad & \frac{d}{dx} \left[ \frac{\sqrt[3]{x^5} - 2x\sqrt{x}}{3x} \right] = \frac{d}{dx} \left( \frac{1}{3x} - \frac{x^{\frac{1}{2}}}{3x} \right) = \frac{d}{dx} \left( \frac{1}{3x^2} - \frac{1}{6}x^{-\frac{1}{2}} \right) = -\frac{2}{9}x^{-\frac{1}{3}} - \frac{1}{3}x^{-\frac{1}{2}}
 \end{aligned}$$

**Question 6**

Differentiate the following expressions with respect to  $x$

a)  $y = \frac{4+x}{2x^3}$

$$\frac{dy}{dx} = -6x^{-4} - x^{-3}$$

b)  $y = \frac{x^2 + 3x}{2\sqrt{x}}$

$$\frac{dy}{dx} = \frac{3}{4}x^{-\frac{1}{2}} + \frac{3}{4}x^{\frac{1}{2}}$$

c)  $y = \frac{x + 4\sqrt{x}}{2x^3}$

$$\frac{dy}{dx} = -5x^{-\frac{7}{2}} - x^{-3}$$

d)  $y = \frac{\sqrt{x}(2x-4)}{3x^2}$

$$\frac{dy}{dx} = -\frac{1}{3}x^{-\frac{3}{2}} + 2x^{-\frac{5}{2}}$$

e)  $y = \frac{(x+2)(2x-3)}{4x^5}$

$$\frac{dy}{dx} = -\frac{3}{2}x^{-4} - x^{-5} + \frac{15}{2}x^{-6}$$

$\text{(a)} \quad y = \frac{4+2x}{2x^3} + \frac{4}{2x^3} + \frac{2}{2x^3} = 2x^{-1} + \frac{1}{2}x^{-2}$ $\frac{dy}{dx} = -6x^{-4} - x^{-3}$
$\text{(b)} \quad y = \frac{x^2 + 3x}{2\sqrt{x}} = \frac{x^2 + 3x}{2x^{\frac{1}{2}}} = \frac{x^2}{2x^{\frac{1}{2}}} + \frac{3x}{2x^{\frac{1}{2}}} = \frac{1}{2}x^{\frac{3}{2}} + \frac{3}{2}x^{\frac{1}{2}}$ $\frac{dy}{dx} = \frac{3}{4}x^{-\frac{1}{2}} + \frac{3}{4}x^{\frac{1}{2}}$
$\text{(c)} \quad y = \frac{x + 4\sqrt{x}}{2x^3} = \frac{\frac{1}{2}x^{\frac{1}{2}} + \frac{4}{2}x^{\frac{1}{2}}}{2x^3} = \frac{1}{2}x^{-2} + 2x^{-\frac{5}{2}}$ $\frac{dy}{dx} = -2x^{-3} - 5x^{-\frac{7}{2}}$
$\text{(d)} \quad y = \frac{\sqrt{x}(2x-4)}{3x^2} = \frac{2x^{\frac{3}{2}}(2x-4)}{3x^2} = \frac{2x^{\frac{3}{2}} - 8x^{\frac{1}{2}}}{3x^2} = \frac{2x^{\frac{1}{2}}}{3x^2} - \frac{8x^{\frac{1}{2}}}{3x^2} = \frac{2}{3x^{\frac{3}{2}}} - \frac{8}{3x^{\frac{3}{2}}} = \frac{2}{3}x^{-\frac{5}{2}} - \frac{8}{3}x^{-\frac{5}{2}}$ $\frac{dy}{dx} = -\frac{1}{3}x^{-\frac{3}{2}} + 2x^{-\frac{5}{2}}$
$\text{(e)} \quad y = \frac{(x+2)(2x-3)}{4x^5} = \frac{2x^2 + 4x - 6x - 12}{4x^5} = \frac{2x^2 - 2x - 12}{4x^5} = \frac{2x^2}{4x^5} - \frac{2x}{4x^5} - \frac{12}{4x^5} = \frac{1}{2}x^{-3} - \frac{1}{2}x^{-4} - \frac{3}{x^4}$ $\frac{dy}{dx} = -\frac{3}{2}x^{-4} - x^{-5} + \frac{15}{2}x^{-6}$

**Question 7**Find  $f'(x)$  for each of the following functions.

a)  $f(x) = x(\sqrt{x} + x^{-4})$

$$f'(x) = \frac{3}{2}x^{\frac{1}{2}} - 3x^{-4}$$

b)  $f(x) = \frac{1}{\sqrt{x}}\left(\frac{2}{x} - \frac{3}{4x^2}\right)$

$$f'(x) = -3x^{-\frac{5}{2}} + \frac{15}{8}x^{-\frac{7}{2}}$$

c)  $f(x) = 4x^{\frac{7}{2}}\left(\frac{6}{x^2} - \frac{5}{\sqrt{x}}\right)$

$$f'(x) = 36x^{\frac{1}{2}} - 60x^2$$

d)  $f(x) = 2\sqrt{x}\left(\frac{5}{x} + x^2\right)$

$$f'(x) = -5x^{-\frac{3}{2}} + 5x^{\frac{3}{2}}$$

e)  $f(x) = \frac{2}{x^{\frac{3}{2}}}\left(\frac{7x^3 - 5x^2}{4x}\right)$

$$f'(x) = \frac{7}{4}x^{-\frac{1}{2}} + \frac{5}{4}x^{-\frac{3}{2}}$$

**(a)**  $f(x) = x(\sqrt{x} + x^{-4}) = x^{\frac{1}{2}}(x^{\frac{1}{2}} + x^{-4}) = x^{\frac{3}{2}} + x^{-3}$   
 $\therefore f'(x) = \frac{3}{2}x^{\frac{1}{2}} - 3x^{-4}$

**(b)**  $f(x) = \frac{1}{\sqrt{x}}\left(\frac{2}{x} - \frac{3}{4x^2}\right) = x^{-\frac{1}{2}}\left(2x^{-1} - \frac{3}{4}x^{-3}\right) = 2x^{-\frac{3}{2}} - \frac{3}{4}x^{-\frac{7}{2}}$   
 $\therefore f'(x) = -3x^{-\frac{5}{2}} + \frac{15}{8}x^{-\frac{7}{2}}$

**(c)**  $f(x) = 4x^{\frac{7}{2}}\left(\frac{6}{x^2} - \frac{5}{\sqrt{x}}\right) = 4x^{\frac{1}{2}}(6x^{-2} - 5x^{-\frac{3}{2}}) = 24x^{\frac{1}{2}} - 20x^{\frac{1}{2}}$   
 $\therefore f'(x) = 24x^{\frac{1}{2}} - 10x^{\frac{1}{2}}$

**(d)**  $f(x) = 2\sqrt{x}\left(\frac{5}{x} + x^2\right) = 2x^{\frac{1}{2}}(5x^{-1} + x^3) = 10x^{-\frac{1}{2}} + 2x^{\frac{5}{2}}$   
 $\therefore f'(x) = -5x^{-\frac{3}{2}} + 5x^{\frac{3}{2}}$

**(e)**  $f(x) = \frac{2}{x^{\frac{3}{2}}}\left(\frac{7x^3 - 5x^2}{4x}\right) = 2x^{-\frac{1}{2}}\left(\frac{7x^2 - 5x}{4x}\right) = 2x^{-\frac{1}{2}}\left(\frac{7}{4}x^2 - \frac{5}{4}x\right)$   
 $= \frac{7}{2}x^{\frac{1}{2}} - \frac{5}{2}x^{-\frac{1}{2}}$   
 $\therefore f'(x) = \frac{7}{4}x^{-\frac{1}{2}} + \frac{5}{4}x^{-\frac{3}{2}}$

**Question 8**

Differentiate the following expressions with respect to  $x$

a)  $y = \frac{(2x-1)(3x-2)}{2x^{\frac{3}{2}}}$

$$\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}} + \frac{7}{4}x^{-\frac{3}{2}} - \frac{3}{2}x^{-\frac{5}{2}}$$

b)  $y = \frac{(3+2\sqrt{x})^2}{4x}$

$$\frac{dy}{dx} = -\frac{3}{2}x^{-\frac{3}{2}} - \frac{9}{4}x^{-2}$$

c)  $y = \frac{4x^3 + \sqrt{x^5}}{4\sqrt{x}}$

$$\frac{dy}{dx} = \frac{1}{2}x + \frac{5}{2}x^{\frac{3}{2}}$$

d)  $y = \frac{(4x+\sqrt{x})(x^2-3)}{3\sqrt{x}}$

$$\frac{dy}{dx} = \frac{2}{3}x + \frac{10}{3}x^{\frac{3}{2}} - 2x^{-\frac{1}{2}}$$

e)  $y = \frac{\left(2x^{\frac{1}{2}} + 6x^{-\frac{1}{2}}\right)\left(6x^{\frac{3}{2}} - 2x^{-\frac{1}{2}}\right)}{3x}$

$$\frac{dy}{dx} = \frac{4}{3}x^{-2} + 8x^{-3} + 4$$

$$\begin{aligned}
 \text{(a)} \quad & y = \frac{(2x-1)(3x-2)}{2x^{\frac{3}{2}}} = \frac{6x^2-7x+2}{2x^{\frac{3}{2}}} = \frac{6x^{\frac{5}{2}}-7x^{\frac{3}{2}}+2}{2x^{\frac{3}{2}}} \\
 & \therefore y = 3x^{\frac{3}{2}} - \frac{7}{2}x^{\frac{1}{2}} + 2x^{-\frac{3}{2}} \quad \therefore \frac{dy}{dx} = \frac{9}{2}x^{\frac{1}{2}} + \frac{7}{4}x^{-\frac{3}{2}} - \frac{3}{2}x^{-\frac{5}{2}} \quad // \\
 \text{(b)} \quad & y = \frac{(3+2\sqrt{x})^2}{4x} = \frac{9+12\sqrt{x}+4x}{4x} = \frac{9}{4x} + \frac{12}{4} + \frac{1}{4}x^{\frac{1}{2}} \\
 & \therefore y = \frac{9}{4}x^{-\frac{1}{2}} + 3x^{\frac{1}{2}} + 1 \quad \therefore \frac{dy}{dx} = -\frac{9}{8}x^{-\frac{3}{2}} - \frac{3}{2}x^{-\frac{1}{2}} \quad // \\
 \text{(c)} \quad & y = \frac{4x^3 + \sqrt{x^5}}{4\sqrt{x}} = \frac{4x^{\frac{5}{2}}}{4\sqrt{x}} + \frac{\sqrt{x^5}}{4\sqrt{x}} = x^{\frac{3}{2}} + \frac{1}{4}x^{\frac{5}{2}} \\
 & \therefore \frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} + \frac{5}{4}x^{\frac{3}{2}} \quad // \\
 \text{(d)} \quad & y = \frac{(4x+\sqrt{x})(x^2-3)}{3\sqrt{x}} = \frac{(4x+\sqrt{x})(x^{\frac{3}{2}}-3)}{3x^{\frac{1}{2}}} = \frac{4x^3-12x^{\frac{5}{2}}+3x^2-3\sqrt{x}}{3x^{\frac{1}{2}}} \\
 & = \frac{4x^2}{3x^{\frac{1}{2}}} - \frac{12x^{\frac{5}{2}}}{3x^{\frac{1}{2}}} - \frac{3x^{\frac{3}{2}}}{3x^{\frac{1}{2}}} = \frac{4}{3}x^{\frac{3}{2}} - 4x^{\frac{7}{2}} - x^{\frac{5}{2}} - 3x^{\frac{1}{2}} \quad \therefore \frac{dy}{dx} = \frac{12}{3}x^{\frac{1}{2}} - \frac{4}{3}x^{\frac{5}{2}} - \frac{5}{3}x^{\frac{3}{2}} - \frac{3}{3}x^{\frac{1}{2}} \\
 \text{(e)} \quad & y = \frac{\left(2x^{\frac{1}{2}} + 6x^{-\frac{1}{2}}\right)\left(6x^{\frac{3}{2}} - 2x^{-\frac{1}{2}}\right)}{3x} = \frac{12x^2-4+x_{\frac{1}{2}}-12x^{-\frac{1}{2}}}{3x} \\
 & = \frac{12x^2}{3x} - \frac{4}{3x} + \frac{x_{\frac{1}{2}}}{3x} - \frac{12x^{-\frac{1}{2}}}{3x} = 4x - \frac{4}{9}x^{-2} + \frac{1}{6}x^{\frac{1}{2}} - 4x^{-\frac{3}{2}} \quad \therefore \frac{dy}{dx} = 4 + \frac{2}{9}x^{\frac{1}{2}} + \frac{1}{3}x^{-\frac{5}{2}}
 \end{aligned}$$

# **TANGENTS & NORMALS**

**Question 1 (non calculator)**

For each of the following curves find an equation of the tangent to the curve at the point whose  $x$  coordinate is given.

a)  $y = x^2 - 9x + 13$ , where  $x = 6$

$y = 3x - 23$

b)  $y = x^4 + x + 1$ , where  $x = 1$

$y = 5x - 2$

c)  $y = 2x^2 + 6x + 7$ , where  $x = -1$

$y = 2x + 5$

d)  $y = 2x^3 - 4x + 5$ , where  $x = 1$

$y = 2x + 1$

e)  $y = 2x^3 - 4x^2 - 3$ , where  $x = 2$

$y = 8x - 19$

f)  $y = 3x^3 - 17x^2 + 24x - 9$ , where  $x = 2$

$y = -8x + 11$

Q1)  $y = x^2 - 9x + 13$   
 $\frac{dy}{dx} = 2x - 9$   
• When  $x=6$   
 $y = 36 - 54 + 13 = -5$   
 $\frac{dy}{dx}|_{x=6} = 12 - 9 = 3$   
•  $y_1 = 3$ ,  $(6, 3)$   
 $y - y_1 = m(x - x_1)$   
 $y + 5 = 3(x - 6)$   
 $y = 3x - 23$

Q2)  $y = x^4 + x + 1$   
 $\frac{dy}{dx} = 4x^3 + 1$   
•  $y_1 = 1$ ,  $(1, 1)$   
 $y - y_1 = m(x - x_1)$   
 $y - 1 = 5(x - 1)$   
 $y = 5x - 4$

Q3)  $y = 2x^2 + 6x + 7$   
 $\frac{dy}{dx} = 4x + 6$   
•  $y_1 = 5$ ,  $(-1, 5)$   
 $y - y_1 = m(x - x_1)$   
 $5 - 5 = 2(-1 - x)$   
 $0 = 2x + 2$   
 $y = 2x + 5$

Q4)  $y = 2x^3 - 4x + 5$   
 $\frac{dy}{dx} = 6x^2 - 4$   
•  $y_1 = 5$ ,  $(1, 5)$   
 $y - y_1 = m(x - x_1)$   
 $5 - 5 = 2(1 - x)$   
 $0 = 2 - 2x$   
 $y = 2x + 3$

Q5)  $y = 2x^3 - 4x^2 - 3$   
 $\frac{dy}{dx} = 6x^2 - 8x$   
•  $y_1 = -3$ ,  $(2, -3)$   
 $y - y_1 = m(x - x_1)$   
 $-3 - (-3) = 2(2 - x)$   
 $0 = 4 - 2x$   
 $y = 2x - 4$

Q6)  $y = 3x^3 - 17x^2 + 24x - 9$   
 $\frac{dy}{dx} = 9x^2 - 34x + 24$   
•  $y_1 = 11$ ,  $(2, 11)$   
 $y - y_1 = m(x - x_1)$   
 $11 - 11 = 9(2 - x)$   
 $0 = 18 - 9x$   
 $y = -9x + 18$

**Question 2 (non calculator)**

For each of the following curves find an equation of the tangent to the curve at the point whose  $x$  coordinate is given.

a)  $f(x) = x^3 - 4x^2 + 2x - 1$ , where  $x = 2$

$y = -2x - 1$

b)  $f(x) = 3x^3 + x^2 - 8x - 5$ , where  $x = 1$

$y = 3x - 12$

c)  $f(x) = 2x^3 - 5x^2 + 2x - 1$ , where  $x = 2$

$y = 6x - 13$

d)  $f(x) = x^3 - x^2 - 3x - 2$ , where  $x = 1$

$y = -2x - 3$

e)  $f(x) = 2x^3 + x^2 - 2x - 2$ , where  $x = 1$

$y = 6x - 7$

$\text{(a)} \quad f(x) = 2x^3 - 4x^2 + 2x - 1$ $f'(x) = 3x^2 - 8x + 2$ $f'(2) = 3(2)^2 - 8(2) + 2$ $= 12 - 16 + 2$ $= -2$ $f(2) = 2(2)^3 - 4(2)^2 + 2(2) - 1$ $= 8 - 16 + 4 - 1$ $= -5$ $m = -2$ $y - y_1 = m(x - x_1)$ $y - 5 = -2(x - 2)$ $y + 5 = -2x + 4$ $y = -2x - 1$	$\text{(b)} \quad f(x) = 2x^3 + x^2 - 2x - 2$ $f'(x) = 6x^2 + 2x - 2$ $f'(1) = 6(1)^2 + 2(1) - 2$ $= 6 + 2 - 2$ $= 6$ $f(1) = 2(1)^3 + 1^2 - 2(1) - 2$ $= 2 + 1 - 2 - 2$ $= -1$ $m = 6$ $y - y_1 = m(x - x_1)$ $y - (-1) = 6(x - 1)$ $y + 1 = 6x - 6$ $y = 6x - 7$	$\text{(c)} \quad f(x) = 2x^3 - 2x^2 - 2x - 2$ $f'(x) = 6x^2 - 4x - 2$ $f'(2) = 6(2)^2 - 4(2) - 2$ $= 24 - 16 - 2$ $= 6$ $f(2) = 2(2)^3 - 2(2)^2 - 2(2) - 2$ $= 16 - 16 - 4 - 2$ $= -4$ $m = 6$ $y - y_1 = m(x - x_1)$ $y - (-4) = 6(x - 2)$ $y + 4 = 6x - 12$ $y = 6x - 13$
$\text{(d)} \quad f(x) = x^3 - x^2 - 3x - 2$ $f'(x) = 3x^2 - 2x - 3$ $f'(1) = 3(1)^2 - 2(1) - 3$ $= 3 - 2 - 3$ $= -2$ $f(1) = 1^3 - 1^2 - 3(1) - 2$ $= 1 - 1 - 3 - 2$ $= -5$ $m = 3$ $y - y_1 = m(x - x_1)$ $y - (-5) = 3(x - 1)$ $y + 5 = 3x - 3$ $y = 3x - 12$	$\text{(e)} \quad f(x) = 2x^3 - 2x^2 - 2x - 2$ $f'(x) = 6x^2 - 4x - 2$ $f'(1) = 6(1)^2 - 4(1) - 2$ $= 6 - 4 - 2$ $= 0$ $f(1) = 2(1)^3 - 2(1)^2 - 2(1) - 2$ $= 2 - 2 - 2 - 2$ $= -4$ $m = 0$ $y - y_1 = m(x - x_1)$ $y - (-4) = 0(x - 1)$ $y + 4 = -2x + 2$ $y = -2x + 2$	

**Question 3 (non calculator)**

For each of the following curves find an equation of the tangent to the curve at the point whose  $x$  coordinate is given.

a)  $y = x^2 - \frac{3}{x} - \frac{1}{2}$ , where  $x = -2$

$$13x + 4y + 6 = 0$$

b)  $y = x^3 - 6x + \frac{8}{x} + 1$ , where  $x = 2$

$$y = 4x - 7$$

c)  $y = 4x^2 + \frac{5}{x} - 1$ , where  $x = 1$

$$y = 3x + 5$$

d)  $y = 2\sqrt{x} - \frac{6}{\sqrt{x}}$ , where  $x = 4$

$$7x - 8y - 20 = 0$$

e)  $y = 3x^{\frac{3}{2}} - \frac{32}{x}$ , where  $x = 4$

$$y = 11x - 28$$

(a)  $y = x^2 - \frac{3}{x} - \frac{1}{2} = x^2 - 3x^{-1} - \frac{1}{2}$

$$\frac{dy}{dx} = 2x + 3x^{-2} = 2x + \frac{3}{x^2}$$

$$\frac{dy}{dx} \Big|_{x=2} = 2(2) + \frac{3}{2^2} = 4 + \frac{3}{4} = \frac{11}{4}$$

when  $x=2$ ,  $y = 2^2 - \frac{3}{2} - \frac{1}{2} = 5$

$$y + 4 = \frac{11}{4}x - \frac{1}{2}$$

$$4(y+4) = 11x - 2$$

$$4y + 16 = 11x - 2$$

$$4y = 11x - 18$$

$$4y = 11(2) - 18$$

$$4y = 22 - 18$$

$$4y = 4$$

$$y = 1$$

$$\therefore m = 3 \quad (\text{from } y = mx + c)$$

$$y - 1 = m(x - 2)$$

$$y - 1 = 3(x - 2)$$

$$y - 1 = 3x - 6$$

$$y = 3x - 5$$

(b)  $y = x^2 - 6x + \frac{8}{x} + 1$

$$y = x^2 - 6x + 8x^{-1} + 1$$

$$\frac{dy}{dx} = 2x - 6 - 8x^{-2} = 2x - 6 - \frac{8}{x^2}$$

$$\frac{dy}{dx} \Big|_{x=2} = 2(2) - 6 - \frac{8}{2^2} = 4 - 6 - 2 = -4$$

when  $x=2$ ,  $y = 2^2 - 6(2) + \frac{8}{2} + 1 = 1$

$$y - 1 = m(x - 2)$$

$$y - 1 = m(2 - 2)$$

$$y - 1 = 4(2 - 2)$$

$$y - 1 = 4(0)$$

$$y = 1$$

$$\therefore m = 4 \quad (\text{from } y = mx + c)$$

$$y - 1 = 4(x - 2)$$

$$y - 1 = 4x - 8$$

$$y = 4x - 7$$

(c)  $y = 3x^{\frac{3}{2}} - \frac{32}{x}$

$$y = 3x^{\frac{3}{2}} - 32x^{-1}$$

$$\frac{dy}{dx} = \frac{9}{2}x^{\frac{1}{2}} + 32x^{-2}$$

$$\frac{dy}{dx} \Big|_{x=4} = \frac{9}{2}(4)^{\frac{1}{2}} + \frac{32}{4^2} = \frac{9}{2}(2) + \frac{32}{16} = 9 + 2 = 11$$

when  $x=4$ ,  $y = 3(4)^{\frac{3}{2}} - \frac{32}{4} = 3(8) - 8 = 24 - 8 = 16$

$$y - 16 = m(x - 4)$$

$$y - 16 = 11(x - 4)$$

$$y - 16 = 11(4 - 4)$$

$$y - 16 = 0$$

$$y = 16$$

**Question 4 (non calculator)**

For each of the following curves find an equation of the normal to the curve at the point whose  $x$  coordinate is given.

a)  $f(x) = x^3 - 4x^2 + 1$ , where  $x = 2$

$$4y = x - 30$$

b)  $f(x) = x^3 - 7x^2 + 11x$ , where  $x = 3$

$$4y = x - 15$$

c)  $f(x) = 3x^4 - 7x^3 + 5$  where  $x = 2$

$$12y + x + 34 = 0$$

d)  $f(x) = \frac{1}{4}x^5 - 18x + 11$  where  $x = 2$

$$2y + x + 32 = 0$$

<p>(1) <math>f(x) = x^3 - 4x^2 + 1</math>  <math>f'(x) = 3x^2 - 8x</math>  <math>f'(2) = 3(2)^2 - 8(2) = 12 - 16 = -4</math>  <math>\therefore</math> NORMAL GRADIENT IS <math>\frac{1}{4}</math>  <math>f(x) = x^3 - 4x^2 + 1 = 8 - 16 + 1 = -7</math>  <math>\therefore</math> <math>m = \frac{1}{4}</math> (2, -7)</p> $\Rightarrow y - y_1 = m(x - x_1)$ $\Rightarrow y - 7 = \frac{1}{4}(x - 2)$ $\Rightarrow 4y + 28 = x - 2$ $\Rightarrow 4y = x - 30$	<p>(2) <math>f(x) = 3x^4 - 7x^3 + 5</math>  <math>f'(x) = 12x^3 - 21x^2</math>  <math>f'(2) = 12(2)^3 - 21(2)^2 = 96 - 84 = 12</math>  <math>\therefore</math> NORMAL FORMULA <math>= -\frac{1}{12}</math>  <math>f(x) = 3x^4 - 7x^3 + 5 = 48 - 56 + 5 = -3</math>  <math>\therefore m = -\frac{1}{12}</math> (2, -3)</p> $\Rightarrow y - y_1 = m(x - x_1)$ $\Rightarrow y + 3 = -\frac{1}{12}(x - 2)$ $\Rightarrow 12y + 36 = -x + 2$ $\Rightarrow 12y + 34 = 0$
<p>(3) <math>f(x) = x^3 - 7x^2 + 11x</math>  <math>f'(x) = 3x^2 - 14x + 11</math>  <math>f'(3) = 3(3)^2 - 14(3) + 11 = 27 - 42 + 11 = -4</math>  <math>\therefore</math> NORMAL GRADIENT IS <math>\frac{1}{4}</math> (3, -4)  <math>\Rightarrow y - y_1 = m(x - x_1)</math> <math>\Rightarrow y + 4 = \frac{1}{4}(x - 3)</math> <math>\Rightarrow 4y + 16 = x - 3</math> <math>\Rightarrow 4y = x - 15</math> </p>	<p>(4) <math>f(x) = \frac{1}{4}x^5 - 18x + 11</math>  <math>f'(x) = \frac{5}{4}x^4 - 18</math>  <math>f'(2) = \frac{5}{4}(2)^4 - 18 = 80 - 72 = 8</math>  <math>\therefore</math> NORMAL GRADIENT IS <math>-\frac{1}{8}</math>  <math>f(x) = \frac{1}{4}x^5 - 18x + 11 = 8 - 36 + 11 = -17</math>  <math>\therefore m = -\frac{1}{8}</math> (2, -17)  <math>\Rightarrow y - y_1 = m(x - x_1)</math> <math>\Rightarrow y + 17 = -\frac{1}{8}(x - 2)</math> <math>\Rightarrow 2y + 34 = -x + 2</math> <math>\Rightarrow 2y + 32 = 0</math> </p>

**Question 5 (non calculator)**

For each of the following curves find an equation of the normal to the curve at the point whose  $x$  coordinate is given.

a)  $f(x) = 2x^3 - 3x^2 - 10x + 18$ , where  $x = 2$

$$x + 2y = 6$$

b)  $f(x) = x^3 - 4x^2 + 6x + 1$ , where  $x = 1$

$$x + y = 5$$

c)  $f(x) = 4x^3 + 2x^2 - 18x - 10$  where  $x = -2$

$$22y + x = 42$$

d)  $f(x) = -2x^3 + 4x^2 - 1$ , where  $x = 2$

$$8y = x - 10$$

<p>(a) <math>f(x) = 2x^3 - 3x^2 - 10x + 18</math></p> $\begin{aligned} f'(x) &= (2x^2 - 6x - 10) \\ f'(2) &= 2(2^2) - 12 - 10 = 2 \\ f(2) &= 16 - 12 - 20 + 18 \\ &= 2 \end{aligned}$ <p>∴ NORMAL line: <math>y - y_0 = m(x - x_0)</math></p> $\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 2 &= -2(x - 2) \\ y - 2 &= -2x + 4 \\ y + 2x &= 6 \end{aligned}$	<p>(c) <math>f(x) = x^3 + 2x^2 - 18x - 10</math></p> $\begin{aligned} f'(x) &= (3x^2 + 4x - 18) \\ f'(-2) &= 3(-2)^2 + 4(-2) - 18 \\ &= -32 + 8 + 36 - 10 \\ &= 2 \end{aligned}$ <p>∴ NORMAL line: <math>y - y_0 = \frac{1}{2}(x - 2)</math></p> $\begin{aligned} y - y_0 &= \frac{1}{2}(x - 2) \\ y - 2 &= \frac{1}{2}(2x + 2) \\ 2y + 4x &= x - 2 \\ 2y + x &= 4 \end{aligned}$
<p>(b) <math>f(x) = x^3 - 4x^2 + 6x + 1</math></p> $\begin{aligned} f'(x) &= 3x^2 - 8x + 6 \\ f'(1) &= 3 - 8 + 6 = 1 \\ f'(1) &= 4 \\ \therefore \text{NORMAL line: } y - 1 &= 4(x - 1) \end{aligned}$ <p>∴ NORMAL line: <math>y - 1 = 4(x - 1)</math></p> $\begin{aligned} y - 1 &= m(x - x_0) \\ y - 1 &= 4(x - 1) \\ y - 4 &= -x + 1 \\ y + x &= 5 \end{aligned}$	<p>(d) <math>f(x) = -2x^3 + 4x^2 - 1</math></p> $\begin{aligned} f'(x) &= -6x^2 + 8x \\ f'(2) &= -24 + 16 = -8 \\ f'(2) &= -16 \\ \therefore \text{NORMAL line: } y - 1 &= \frac{1}{8}(x - 2) \end{aligned}$ <p>∴ NORMAL line: <math>y - 1 = \frac{1}{8}(x - 2)</math></p> $\begin{aligned} y - 1 &= \frac{1}{8}(2x - 2) \\ 8y + 8 &= 2x - 2 \\ 8y + 6 &= 2x - 2 \\ 8y &= 2x - 10 \end{aligned}$

**Question 6 (non calculator)**

For each of the following curves find an equation of the normal to the curve at the point whose  $x$  coordinate is given.

a)  $y = x^2(x-6) + \frac{5}{x} - 1$ , where  $x=1$

$$x - 14y - 15 = 0$$

b)  $y = 2x^{\frac{3}{2}} - \frac{16}{x}$ , where  $x=4$

$$x + 7y = 88$$

c)  $y = 4x^2 + x^{-\frac{3}{2}}$ , where  $x=1$

$$2x + 13y = 67$$

d)  $y = 2x^2 - 4x^{\frac{3}{2}} - \frac{8}{x} - 1$ , where  $x=4$

$$2x + 9y + 19 = 0$$

**(a)**  $y = x^2(x-6) + \frac{5}{x} - 1$   
 $y = x^3 - 6x^2 + \frac{5}{x^2} - 1$   
 $\frac{dy}{dx} = 3x^2 - 12x - \frac{10}{x^3}$   
 $\bullet$  when  $x=1$   
 $y = (1^2)(1-6) + \frac{5}{1^2} - 1 = -1$  (in (1, -1))  
 $\frac{dy}{dx}_{(1,-1)} = 3 - 12 - 5 = -14$

**(b)**  $y = 2x^{\frac{3}{2}} - \frac{16}{x}$   
 $y = 2x^{\frac{1}{2}} - 16x^{-1}$   
 $\frac{dy}{dx} = 3x^{\frac{1}{2}} + 16x^{-2}$   
 $\bullet$  when  $x=4$   
 $y = 2(4)^{\frac{3}{2}} - \frac{16}{4} = 16 - 4 = 12$  (in (4, 12))  
 $\frac{dy}{dx}_{(4,12)} = 3(4)^{\frac{1}{2}} + 16(4)^{-2} = 6 + 1 = 7$

**(c)**  $y = 4x^2 + x^{-\frac{3}{2}}$   
 $\frac{dy}{dx} = 8x - \frac{3}{2}x^{-\frac{5}{2}}$   
 $\bullet$  when  $x=1$ ,  $y+4=5$  (in (1, 1))  
 $\frac{dy}{dx}_{(1,1)} = 8 - \frac{3}{2} = \frac{13}{2}$

**(d)**  $y = 2x^2 - 4x^{\frac{3}{2}} - \frac{8}{x} - 1$   
 $y = 2x^2 - 4x^{\frac{1}{2}} - \frac{8}{x^2} - 1$   
 $\frac{dy}{dx} = 4x - 2x^{\frac{1}{2}} + 8x^{-3}$   
 $\bullet$  when  $x=4$ ,  $y-22=2(-1)$  (in (4, -23))  
 $\frac{dy}{dx}_{(4,-23)} = 16 - 6(2) + 8(4)^{-3} = 4 - \frac{1}{2} = \frac{7}{2}$

NORMAL GRADIENT IS  $\frac{1}{14}$   
 $\Rightarrow y - y_1 = m(x - x_1)$   
 $\Rightarrow y + 1 = \frac{1}{14}(x - 1)$   
 $\Rightarrow 14y + 14 = x - 1$   
 $\Rightarrow 14y + 15 = x$   
 $\Rightarrow x - 14y - 15 = 0$

NORMAL GRADIENT IS  $-\frac{1}{7}$   
 $\Rightarrow y - y_1 = m(x - x_1)$   
 $\Rightarrow y - 12 = -\frac{1}{7}(x - 4)$   
 $\Rightarrow 7y - 84 = -x + 4$   
 $\Rightarrow x + 7y = 88$

NORMAL GRADIENT IS  $-\frac{2}{3}$   
 $\Rightarrow y - y_1 = m(x - x_1)$   
 $\Rightarrow y - 5 = -\frac{2}{3}(x - 1)$   
 $\Rightarrow 3y - 15 = -2x + 2$   
 $\Rightarrow 2x + 3y = 17$

NORMAL GRADIENT IS  $-\frac{7}{2}$   
 $\Rightarrow y - y_1 = m(x - x_1)$   
 $\Rightarrow y + 22 = -\frac{7}{2}(x - 4)$   
 $\Rightarrow 7y + 154 = -7x + 28$   
 $\Rightarrow 2x + 7y = 126$   
 $\Rightarrow 2x + 9y + 19 = 0$

# STATIONARY POINTS

**Question 1** (non calculator)

For each of the following cubic equations find the coordinates of their stationary points and determine their nature.

a)  $y = x^3 - 3x^2 - 9x + 3$

b)  $y = x^3 + 12x^2 + 45x + 50$

c)  $y = 2x^3 - 6x^2 + 12$

d)  $y = 25 - 24x + 9x^2 - x^3$

$\min(3, -24), \max(-1, 8)$ ,  $\min(-3, -4), \max(-5, 0)$   $\min(2, 4), \max(0, 12)$ ,

$\min(2, 5), \max(4, 9)$

<p>(a) <math>y = x^3 - 3x^2 - 9x + 3</math></p> $\frac{dy}{dx} = 3x^2 - 6x - 9$ $\frac{d^2y}{dx^2} = 6x - 6$ <p>Eq. S.T.P.   <math>\frac{dy}{dx} = 0</math>  <math>\Rightarrow 3x^2 - 6x - 9 = 0</math>  <math>\Rightarrow x^2 - 2x - 3 = 0</math>  <math>\Rightarrow (x+1)(x-3) = 0</math>  <math>\Rightarrow x = -1, 3</math></p> <p><math>\frac{d^2y}{dx^2} _{x=-1} = 6(-1) - 6 = -12 &lt; 0</math>  <math>\therefore (-1, 8)</math> is a MAX</p> <p><math>\frac{d^2y}{dx^2} _{x=3} = 6(3) - 6 = 12 &gt; 0</math>  <math>\therefore (3, -24)</math> is a MIN</p>	<p>(b) <math>y = x^3 + 12x^2 + 45x + 50</math></p> $\frac{dy}{dx} = 3x^2 + 24x + 45$ $\frac{d^2y}{dx^2} = 6x + 24$ <p>Eq. S.T.P.   <math>\frac{dy}{dx} = 0</math>  <math>\Rightarrow 3x^2 + 24x + 45 = 0</math>  <math>\Rightarrow x^2 + 8x + 15 = 0</math>  <math>\Rightarrow (x+5)(x+3) = 0</math>  <math>\Rightarrow x = -5, -3</math></p> <p><math>\frac{d^2y}{dx^2} _{x=-5} = 6(-5) + 24 = 6 &gt; 0</math>  <math>\therefore (-5, 0)</math> is a MIN</p> <p><math>\frac{d^2y}{dx^2} _{x=-3} = 6(-3) + 24 = -6 &lt; 0</math>  <math>\therefore (-3, 0)</math> is a MAX</p>	<p>(c) <math>y = 2x^3 - 6x^2 + 12</math></p> $\frac{dy}{dx} = 6x^2 - 12x$ $\frac{d^2y}{dx^2} = 12x - 12$ <p>Eq. S.T.P.   <math>\frac{dy}{dx} = 0</math>  <math>\Rightarrow 6x^2 - 12x = 0</math>  <math>\Rightarrow 6x(x-2) = 0</math>  <math>\Rightarrow x = 0, 2</math></p> <p><math>\frac{d^2y}{dx^2} _{x=0} = 12(0) - 12 = -12 &lt; 0</math>  <math>\therefore (0, 12)</math> is MAX</p> <p><math>\frac{d^2y}{dx^2} _{x=2} = 12(2) - 12 = 12 &gt; 0</math>  <math>\therefore (2, 4)</math> is MIN</p>
<p>(d) <math>y = 25 - 24x + 9x^2 - x^3</math></p> $\frac{dy}{dx} = -24 + 18x - 3x^2$ $\frac{d^2y}{dx^2} = 18 - 6x$ <p>Eq. S.T.P.   <math>\frac{dy}{dx} = 0</math>  <math>\Rightarrow -24 + 18x - 3x^2 = 0</math>  <math>\Rightarrow 3x^2 - 18x + 24 = 0</math>  <math>\Rightarrow (x-2)(x-4) = 0</math>  <math>\Rightarrow x = 2, 4</math></p> <p><math>\frac{d^2y}{dx^2} _{x=2} = 18 - 6(2) = -6 &lt; 0</math>  <math>\therefore (2, 5)</math> is MAX</p> <p><math>\frac{d^2y}{dx^2} _{x=4} = 18 - 6(4) = -6 &lt; 0</math>  <math>\therefore (4, 9)</math> is MIN</p>		

**Question 2**

For each of the following equations find the coordinates of their stationary points and determine their nature.

a)  $y = x + \frac{4}{x}$ ,  $x \neq 0$

b)  $y = x^2 + \frac{16}{x}$ ,  $x \neq 0$

c)  $y = x - 4\sqrt{x}$ ,  $x > 0$

d)  $y = 4x^2 + \frac{1}{x}$ ,  $x \neq 0$

$\boxed{\min(2, 4), \max(-2, -4)}$ ,  $\boxed{\min(2, 12)}$ ,  $\boxed{\min(4, -4)}$ ,  $\boxed{\min(\frac{1}{2}, 3)}$

(a)  $y = x + \frac{4}{x} = x + 4x^{-1}$

$\frac{dy}{dx} = 1 - 4x^{-2} = 1 - \frac{4}{x^2}$

$\frac{d^2y}{dx^2} = 8x^{-3} = \frac{8}{x^3}$

EQN MAX/MIN  $\frac{dy}{dx} = 0$   
 $1 - \frac{4}{x^2} = 0$   
 $1 = \frac{4}{x^2}$   
 $x^2 = 4$   
 $x = \pm 2$ ,  $y = \pm 4$

$\frac{d^2y}{dx^2} \Big|_{x=2} = \frac{8}{2^3} = 1 > 0$   
 $\therefore (2, 4)$  is a MIN

$\frac{d^2y}{dx^2} \Big|_{x=-2} = \frac{8}{(-2)^3} = -1 < 0$   
 $\therefore (-2, -4)$  is a MAX

(b)  $y = x^2 + \frac{16}{x} = x^2 + 16x^{-1}$

$\frac{dy}{dx} = 2x - 16x^{-2} = 2x - \frac{16}{x^2}$

$\frac{d^2y}{dx^2} = 2 + 32x^{-3} = 2 + \frac{32}{x^3}$

EQN MIN/MAX  $\frac{dy}{dx} = 0$   
 $2x - \frac{16}{x^2} = 0$   
 $2x = \frac{16}{x^2}$   
 $x^3 = 8$   
 $x = 2$

$\frac{d^2y}{dx^2} \Big|_{x=2} = \frac{1}{2^3} = \frac{1}{8} > 0$   
 $\therefore (2, 12)$  is a MAX

(c)  $y = x - 4\sqrt{x} = x - 4x^{1/2}$

$\frac{dy}{dx} = 1 - 2x^{-1/2} = 1 - \frac{2}{\sqrt{x}}$

$\frac{d^2y}{dx^2} = x^{-3/2} = \frac{1}{x^{3/2}}$

EQN MIN/MAX  $\frac{dy}{dx} = 0$   
 $1 - \frac{2}{\sqrt{x}} = 0$   
 $1 = \frac{2}{\sqrt{x}}$   
 $\sqrt{x} = 2$   
 $x = 4$   
 $\therefore y = 4 - 4\sqrt{4} = 4 - 8 = -4$

$\frac{d^2y}{dx^2} \Big|_{x=4} = \frac{1}{4^{3/2}} = \frac{1}{8} > 0$   
 $\therefore (-4, -4)$  is a MIN

(d)  $y = 4x^2 + \frac{1}{x} = 4x^2 + x^{-1}$

$\frac{dy}{dx} = 8x - x^{-2} = 8x - \frac{1}{x^2}$

$\frac{d^2y}{dx^2} = 8 + 2x^{-3} = 8 + \frac{2}{x^3}$

EQN MIN/MAX  $\frac{dy}{dx} = 0$   
 $8x - \frac{1}{x^2} = 0$   
 $8x = \frac{1}{x^2}$   
 $8x^3 = 1$   
 $x^3 = \frac{1}{8}$   
 $x = \frac{1}{2}$   
 $\therefore y = 4(\frac{1}{2})^2 + \frac{1}{\frac{1}{2}} = 2 + 2 = 4$

$\frac{d^2y}{dx^2} \Big|_{x=\frac{1}{2}} = 8 + \frac{2}{(\frac{1}{2})^3} = 8 + 16 = 24 > 0$   
 $\therefore (\frac{1}{2}, 3)$  is a MIN

**Question 3**

For each of the following equations find the coordinates of their stationary points and determine their nature.

a)  $y = 12\sqrt{x} - x^{\frac{3}{2}}, \quad x > 0$

b)  $y = x^{\frac{3}{2}} - 6x^{\frac{1}{2}}, \quad x > 0$

c)  $y = 6x^{\frac{1}{2}} - 4x - 2, \quad x > 0$

d)  $y = x^{\frac{7}{2}} - 14x^{\frac{5}{2}} + 100, \quad x > 0$

$\boxed{\max(4, 16)}, \boxed{\min(2, -4\sqrt{2})}, \boxed{\max\left(\frac{9}{16}, \frac{1}{4}\right)}, \boxed{\min(4, 4)}$

(a)

$$y = 12\sqrt{x} - x^{\frac{3}{2}}$$

$$\frac{dy}{dx} = 12\cdot\frac{1}{2\sqrt{x}} - \frac{3}{2}x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 6\sqrt{x} - \frac{3}{2}x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{3}{2}\sqrt{x} - \frac{3}{2}\sqrt{x}$$

$$\frac{dy}{dx} = 0$$

• F.O.T.C.P.  $\frac{dy}{dx} = 0$

$$\Rightarrow 6\sqrt{x} - \frac{3}{2}x^{\frac{1}{2}} = 0$$

$$\Rightarrow 6x^{\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}} = 0$$

$$\Rightarrow (2x)^{\frac{1}{2}} = 3x^{\frac{1}{2}}$$

$$\Rightarrow \frac{2}{x} = 3x$$

$$\Rightarrow x = \frac{2}{3x}$$

$$\therefore \boxed{x = \frac{2}{3}}$$

• When  $x=4$

$$y = 12\sqrt{4} - 4^{\frac{3}{2}}$$

$$y = 12\sqrt{4} - 8$$

$$\boxed{y = 16}$$

• To find nature

$$\frac{d^2y}{dx^2} = 6\cdot\frac{1}{2\sqrt{x}} - \frac{3}{4}x^{\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = 3\sqrt{x} - \frac{3}{4}x^{\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = -\frac{3}{8}x^{\frac{1}{2}} < 0$$

$$\therefore (4, 16) \text{ is A MAX}$$

(b)

$$y = 2\sqrt{x} - x^{\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{2}{2\sqrt{x}} - \frac{3}{2}x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}} - \frac{3}{2}x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}} - \frac{3}{2}\sqrt{x}$$

$$\frac{dy}{dx} = 0$$

• F.O.T.C.P.  $\frac{dy}{dx} = 0$

$$\Rightarrow \frac{1}{\sqrt{x}} - \frac{3}{2}\sqrt{x} = 0$$

$$\Rightarrow \frac{2}{x} = 3x^{\frac{1}{2}}$$

$$\Rightarrow 2x^{\frac{1}{2}} = 3x^{\frac{1}{2}}$$

$$\Rightarrow 2x^{\frac{1}{2}} = \frac{6}{\sqrt{x}}$$

$$\Rightarrow x = \frac{6}{2\sqrt{x}}$$

$$\Rightarrow x = 3$$

$$\therefore \boxed{x = 3}$$

• When  $x=2$

$$y = 2\sqrt{2} - 2^{\frac{3}{2}}$$

$$y = 2\sqrt{2} - 4\sqrt{2}$$

$$\boxed{y = -2\sqrt{2}}$$

• To find nature

$$\frac{d^2y}{dx^2} = 2\sqrt{x} - \frac{3}{2}x^{\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = \frac{1}{\sqrt{x}} + \frac{3}{2}\times\frac{1}{\sqrt{x}}$$

$$\frac{d^2y}{dx^2} = \frac{1}{\sqrt{x}} + \frac{3}{2}\sqrt{x} > 0$$

$$\therefore (2, -2\sqrt{2}) \text{ is A MIN}$$

(c)

$$y = x^{\frac{1}{2}} - 4x - 2$$

$$\frac{dy}{dx} = 3x^{\frac{1}{2}} - 4$$

$$\frac{dy}{dx} = \frac{3}{2}\sqrt{x} - 4$$

$$\frac{dy}{dx} = -\frac{3}{2}\sqrt{x} - 4$$

$$\frac{dy}{dx} = -\frac{3}{8}x^{\frac{1}{2}} - 4$$

$$\frac{dy}{dx} = -\frac{3}{8}\sqrt{x} - 4$$

• When  $x=2$

$$y = 2^{\frac{1}{2}} - 4\cdot 2 - 2$$

$$y = \sqrt{2} - 8$$

$$\boxed{y = \frac{9}{2}}$$

• To find nature

$$\frac{d^2y}{dx^2} = -\frac{3}{2}\left(\frac{1}{2}\right)x^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = -\frac{3}{4}\left(\frac{1}{2}\right)x^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = -\frac{3}{8}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} < 0$$

$$\therefore \boxed{(2, \frac{9}{2}) \text{ is A MAX}}$$

(d)

$$y = x^{\frac{7}{2}} - 14x^{\frac{5}{2}} + 100$$

$$\frac{dy}{dx} = \frac{7}{2}x^{\frac{5}{2}} - 20x^{\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{35}{4}x^{\frac{3}{2}} - 20x^{\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{35}{4}x^{\frac{3}{2}} - 20x^{\frac{3}{2}}$$

$$\frac{dy}{dx} = 0$$

• F.O.T.C.P.  $\frac{dy}{dx} = 0$

$$\Rightarrow \frac{35}{4}x^{\frac{3}{2}} - 20x^{\frac{3}{2}} = 0$$

$$\Rightarrow \frac{35}{4}x^{\frac{3}{2}} = 20x^{\frac{3}{2}}$$

$$\Rightarrow \frac{35}{4}x^{\frac{3}{2}} = 5x^{\frac{3}{2}}$$

$$\Rightarrow \frac{35}{4} = 5$$

$$\Rightarrow x^{\frac{3}{2}} = 8$$

$$\Rightarrow (x^{\frac{1}{2}})^3 = 8^{\frac{1}{2}}$$

$$\therefore \boxed{(x^{\frac{1}{2}})^3 = 8^{\frac{1}{2}}}$$

• When  $x=4$

$$y = 4^{\frac{7}{2}} - 14\cdot 4^{\frac{5}{2}} + 100$$

$$y = 128 - 448 + 100$$

$$y = -280 + 100$$

$$\boxed{y = -180}$$

• To find nature

$$\frac{d^2y}{dx^2} = \frac{35}{4}x^{\frac{1}{2}} - 20$$

$$\frac{d^2y}{dx^2} = \frac{35}{4}\times 4^{\frac{1}{2}} - 20$$

$$\frac{d^2y}{dx^2} = 35 - 20$$

$$\frac{d^2y}{dx^2} = 15 > 0$$

$$\therefore (4, -180) \text{ is A MIN}$$

**Question 4**

For each of the following equations find the coordinates of their stationary points and determine their nature.

a)  $y = x^3 - 16x^{\frac{3}{2}} + 60, \quad x > 0$

b)  $y = 5x^2 - 6x^{\frac{5}{2}} + 10, \quad x > 0$

c)  $y = 6x^{\frac{4}{3}} - x^2 - 20, \quad x > 0$

d)  $y = 5x^2 - 2x^{\frac{5}{2}} - 10, \quad x > 0$

min (4,-4), min (1,9), max (8,12), max (4,6)

(a)  $y = x^3 - 16x^{\frac{3}{2}} + 60$

• For MIN/MAX  $\frac{dy}{dx} = 0$       • CHECK NATURE  
 $\frac{dy}{dx} = 3x^2 - 24x^{\frac{1}{2}} = 0$   
 $\Rightarrow 3x^2 = 24x^{\frac{1}{2}}$   
 $\Rightarrow x^2 = 8x^{\frac{1}{2}}$   
 $\Rightarrow x^{\frac{1}{2}} = 8$   
 $\Rightarrow x = 64$   
 $\Rightarrow (2\sqrt{16})^{\frac{3}{2}} = 8^{\frac{3}{2}}$   
 $\therefore (x_1, y_1) = (2\sqrt{16}, 8^{\frac{3}{2}})$   
 $\therefore (x_1, y_1) = (4, 4)$   
 $\therefore y = 4^3 - 16 \cdot 4^{\frac{3}{2}} + 60 = -4$   
 $\therefore y = -4$

(b)  $y = 5x^2 - 6x^{\frac{5}{2}} + 10$

• For MIN/MAX  $\frac{dy}{dx} = 0$       • CHECK NATURE  
 $\frac{dy}{dx} = 10x - 15x^{\frac{3}{2}} = 0$   
 $\Rightarrow 10x = 15x^{\frac{3}{2}}$   
 $\Rightarrow x = 2\sqrt{3}$   
 $\Rightarrow \frac{3}{2}x^{\frac{1}{2}} = 1$   
 $\Rightarrow x^{\frac{1}{2}} = \frac{2}{3}$   
 $\Rightarrow (x_1, y_1) = (\frac{4}{9}, \frac{50}{27})$   
 $\therefore (x_1, y_1) = (\frac{4}{9}, \frac{50}{27})$   
 $\therefore y = 5 \cdot \frac{4}{9}^2 - 6 \cdot \frac{4}{9}^{\frac{5}{2}} + 10 = 9$

(c)  $y = 6x^{\frac{4}{3}} - x^2 - 20$

• For MIN/MAX  $\frac{dy}{dx} = 0$       • CHECK NATURE  
 $\frac{dy}{dx} = 8x^{\frac{1}{3}} - 2x = 0$   
 $\Rightarrow 8x^{\frac{1}{3}} = 2x$   
 $\Rightarrow 4x^{\frac{1}{3}} = x$   
 $\Rightarrow 4 = x^{\frac{2}{3}}$   
 $\Rightarrow x = \sqrt[3]{16}$   
 $\Rightarrow (x_1, y_1) = (\sqrt[3]{16}, 0)$   
 $\therefore (x_1, y_1) = (2\sqrt{4}, 0)$   
 $\therefore (x_1, y_1) = (4, 0)$   
 $\therefore y = 6 \cdot 4^{\frac{4}{3}} - 4^2 - 20 = 12$   
 $\therefore y = 12$

(d)  $y = 5x^2 - 2x^{\frac{5}{2}} - 10$

• For MIN/MAX  $\frac{dy}{dx} = 0$       • CHECK NATURE  
 $\frac{dy}{dx} = 10x - 5x^{\frac{3}{2}} = 0$   
 $\Rightarrow 2x - x^{\frac{3}{2}} = 0$   
 $\Rightarrow 2x = x^{\frac{3}{2}}$   
 $\Rightarrow 2 = x^{\frac{1}{2}}$   
 $\Rightarrow x = 4$   
 $\Rightarrow (x_1, y_1) = (4, 0)$   
 $\therefore (x_1, y_1) = (4, 0)$   
 $\therefore y = 5 \cdot 4^2 - 2 \cdot 4^{\frac{5}{2}} - 10 = 0$   
 $\therefore y = 0$

**Question 5**

For each of the following equations find the coordinates of their stationary points and determine their nature.

a)  $y = \frac{1}{x} - \frac{1}{\sqrt{x}}, \quad x > 0$

b)  $y = \frac{3\sqrt{x}-2}{x^2}, \quad x > 0$

c)  $y = \sqrt[3]{x} + \frac{27}{x}, \quad x > 0$

d)  $y = \frac{1}{3\sqrt{x}} \left[ \frac{2}{x} - 3 \right], \quad x > 0$

$$\boxed{\min\left(4, -\frac{1}{4}\right)}, \boxed{\max(1, 1)}, \boxed{\min(27, 4)}, \boxed{\min\left(2, -\frac{\sqrt{2}}{3}\right)}$$

<p>(a) <math>y = \frac{1}{x} - \frac{1}{\sqrt{x}}</math></p> <p>• To find min/max <math>\frac{dy}{dx} = 0</math></p> <p><math>y = x^{-1} - x^{-\frac{1}{2}}</math>  <math>\Rightarrow -x^{-2} + \frac{1}{2}x^{-\frac{3}{2}} = 0</math>  <math>\Rightarrow -\frac{1}{x^2} + \frac{1}{2x^{\frac{1}{2}}} = 0</math>  <math>\Rightarrow \frac{1}{2x^{\frac{1}{2}}} = x^{-2}</math>  <math>\Rightarrow 2x^{-\frac{1}{2}} = x^{-4}</math>  <math>\Rightarrow 2 = x^{-\frac{3}{2}}</math>  <math>\Rightarrow 2 = \frac{1}{x^{\frac{3}{2}}}</math>  <math>\Rightarrow 2 = x^{\frac{-3}{2}}</math>  <math>\Rightarrow [4 = x]</math>  <math>y = \frac{1}{4} - \frac{1}{\sqrt{4}} = \frac{1}{4} - \frac{1}{2}</math>  <math>\therefore y = -\frac{1}{4}</math></p> <p>• To check nature  <math>\frac{d^2y}{dx^2} = \frac{2}{x^3} - \frac{3}{4x^{\frac{5}{2}}}</math>  <math>\frac{d^2y}{dx^2} = \frac{2}{x^3} - \frac{3}{4x^{\frac{5}{2}}} &lt; 0</math>  <math>\therefore (4, -\frac{1}{4})</math> is a max</p>	<p>(b) <math>y = \sqrt[3]{x} + \frac{27}{x}</math></p> <p>• To find min/max <math>\frac{dy}{dx} = 0</math></p> <p><math>y = x^{\frac{1}{3}} + 27x^{-1}</math>  <math>\Rightarrow \frac{1}{3}x^{-\frac{2}{3}} - 27x^{-2} = 0</math>  <math>\Rightarrow \frac{1}{3x^{\frac{2}{3}}} = 27x^{-2}</math>  <math>\Rightarrow x^{\frac{-4}{3}} = \frac{1}{81}</math>  <math>\Rightarrow x^{\frac{4}{3}} = 81</math>  <math>\Rightarrow x^{\frac{3}{2}} = 81^{\frac{1}{3}}</math>  <math>\Rightarrow x = 81^{\frac{1}{3}}</math>  <math>\Rightarrow x = 3\sqrt[3]{3}</math>  <math>y = \sqrt[3]{3} + \frac{27}{3} = 3 + 1</math>  <math>\therefore y = 4</math></p> <p>• To check nature  <math>\frac{d^2y}{dx^2} = -\frac{5}{9x^{\frac{5}{3}}} + \frac{54}{x^3}</math>  <math>\frac{d^2y}{dx^2} = -\frac{5}{9x^{\frac{5}{3}}} + \frac{54}{x^3} &lt; 0</math>  <math>\therefore (3\sqrt[3]{3}, 4)</math> is a min</p>
<p>(c) <math>y = \frac{3\sqrt{x}-2}{x^2}</math></p> <p>• To find min/max <math>\frac{dy}{dx} = 0</math></p> <p><math>y = \frac{3x^{\frac{1}{2}}-2}{x^2}</math>  <math>\Rightarrow -3x^{\frac{3}{2}} - 2x^{-2} = 0</math>  <math>\Rightarrow -\frac{3}{2}x^{\frac{5}{2}} - \frac{2}{x^2} = 0</math>  <math>\Rightarrow -\frac{3}{2}x^{\frac{5}{2}} = \frac{2}{x^2}</math>  <math>\Rightarrow x^{\frac{15}{2}} = \frac{4}{3}</math>  <math>\Rightarrow x^{\frac{3}{2}} = 2</math>  <math>\Rightarrow x^{\frac{3}{2}} = 2^{\frac{2}{3}}</math>  <math>\Rightarrow x^{\frac{3}{2}} = 1</math>  <math>\Rightarrow x = 1</math>  <math>\therefore y = 1</math></p> <p>• To check nature  <math>\frac{d^2y}{dx^2} = -\frac{3}{2}x^{\frac{3}{2}} - \frac{2}{x^3}</math>  <math>\frac{d^2y}{dx^2} = -\frac{3}{2}x^{\frac{3}{2}} - \frac{2}{x^3} &lt; 0</math>  <math>\therefore (1, 1)</math> is a max</p>	<p>(d) <math>y = \frac{1}{3\sqrt{x}} \left[ \frac{2}{x} - 3 \right]</math></p> <p>• To find min/max <math>\frac{dy}{dx} = 0</math></p> <p><math>y = \frac{1}{3x^{\frac{1}{2}}} \left[ \frac{2}{x} - 3 \right]</math>  <math>\Rightarrow -\frac{1}{3}x^{-\frac{3}{2}} + \frac{1}{3}x^{-\frac{5}{2}} = 0</math>  <math>\Rightarrow \frac{1}{3x^{\frac{5}{2}}} = \frac{1}{3x^{\frac{3}{2}}}</math>  <math>\Rightarrow x^{\frac{3}{2}} = 2</math>  <math>\Rightarrow x^{\frac{3}{2}} = 2^{\frac{2}{3}}</math>  <math>\Rightarrow x^{\frac{3}{2}} = 2</math>  <math>\Rightarrow x = 2</math>  <math>\therefore y = \frac{1}{6}</math></p> <p>• To check nature  <math>\frac{d^2y}{dx^2} = \frac{5}{9x^{\frac{7}{2}}} - \frac{3}{4x^{\frac{5}{2}}}</math>  <math>\frac{d^2y}{dx^2} = \frac{5}{9x^{\frac{7}{2}}} - \frac{3}{4x^{\frac{5}{2}}} &lt; 0</math>  <math>\therefore (2, \frac{1}{6})</math> is a min</p>

# **INCREASING and DECREASING FUNCTIONS**

**Question 1**

For each of the following equations find the range of the values of  $x$ , for which  $y$  is increasing or decreasing.

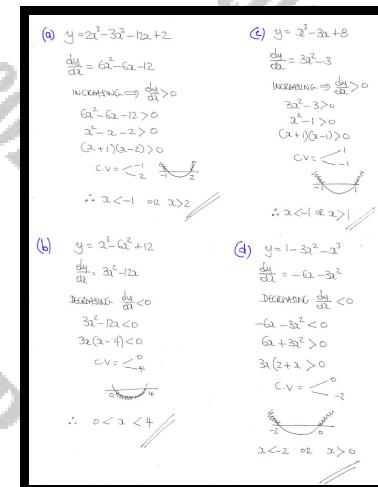
a)  $y = 2x^3 - 3x^2 - 12x + 2$ , increasing

b)  $y = x^3 - 6x^2 + 12$ , decreasing

c)  $y = x^3 - 3x + 8$ , increasing

d)  $y = 1 - 3x^2 - x^3$ , decreasing

$$\boxed{x < -1 \text{ or } x > 2}, \boxed{0 < x < 4}, \boxed{x < -1 \text{ or } x > 1}, \boxed{x < -2 \text{ or } x > 0}$$



**Question 2**

Find the range of the values of  $x$ , for which  $f(x)$  is increasing or decreasing.

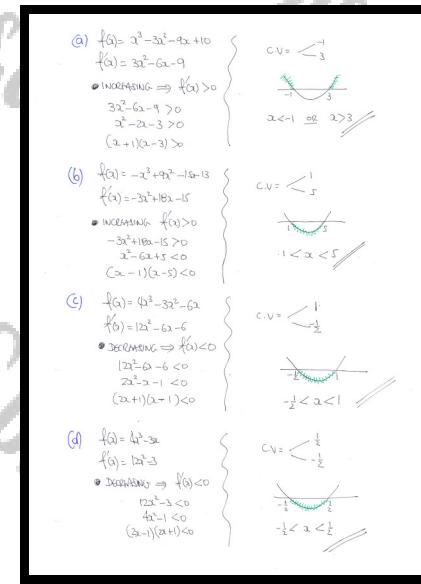
a)  $f(x) = x^3 - 3x^2 - 9x + 10$ , increasing

b)  $f(x) = -x^3 + 9x^2 - 15x - 13$ , increasing

c)  $f(x) = 4x^3 - 3x^2 - 6x$ , decreasing

d)  $f(x) = 4x^3 - 3x$ , decreasing

$$\boxed{x < -1 \text{ or } x > 3}, \boxed{1 < x < 5}, \boxed{-\frac{1}{2} < x < 1}, \boxed{-\frac{1}{2} < x < \frac{1}{2}}$$



# DIFFERENTIATION PRACTICE IN CONTEXT

**Question 1**

The curve  $C$  has equation

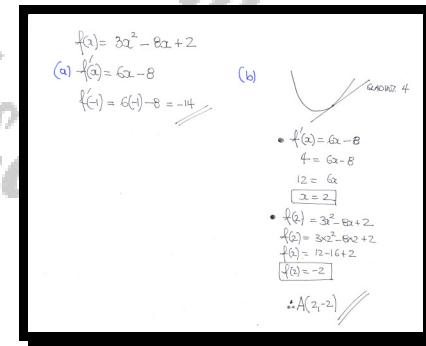
$$f(x) = 3x^2 - 8x + 2.$$

- a) Find the gradient at the point on  $C$ , where  $x = -1$ .

The point  $A$  lies on  $C$  and the gradient at that point is 4.

- b) Find the coordinates of  $A$ .

-14, A(2, -2)



**Question 2**

The curve  $C$  has equation

$$y = x^3 - 11x + 1.$$

- a) Find the gradient at the point on  $C$ , where  $x = 3$ .

The point  $P$  lies on  $C$  and the gradient at that point is 1.

- b) Find the possible coordinates of  $P$ .

[16], P(2, -13) or P(-2, 15)

(a)  $y = x^3 - 11x + 1$   
 $\Rightarrow \frac{dy}{dx} = 3x^2 - 11$   
 $\Rightarrow \left. \frac{dy}{dx} \right|_{x=3} = 3(3)^2 - 11 = 27 - 11 = 16$

(b)  $\frac{dy}{dx} = 1$   
 $3x^2 - 11 = 1$   
 $3x^2 = 12$   
 $x^2 = 4$   
 $x = \pm 2$   
 $x = 2 \rightarrow y = 2^3 - 11(2) + 1 = 8 - 22 + 1 = -13$   
 $x = -2 \rightarrow y = (-2)^3 - 11(-2) + 1 = -8 + 22 + 1 = 15$   
 $\therefore P(2, -13) \text{ or } P(-2, 15)$

**Question 3**

The curve  $C$  has equation

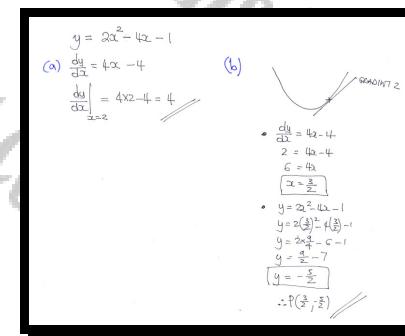
$$y = 2x^2 - 4x - 1.$$

- a) Find the gradient at the point on  $C$ , where  $x = 2$ .

The point  $P$  lies on  $C$  and the gradient at that point is 2.

- b) Find the coordinates of  $P$ .

4,  $\boxed{P\left(\frac{3}{2}, -\frac{5}{2}\right)}$



**Question 4**

The curve  $C$  has equation

$$f(x) = x + \frac{1}{x}, \quad x \neq 0.$$

- a) Find the gradient at the point on  $C$ , where  $x = \frac{1}{2}$ .

The point  $A$  lies on  $C$  and the gradient at that point is  $\frac{3}{4}$ .

- b) Find the possible coordinates of  $A$ .

$-3$	$A\left(2, \frac{5}{2}\right)$ or $A\left(-2, -\frac{5}{2}\right)$
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<p>(a) <math>f(x) = x + \frac{1}{x}</math>  <math>\rightarrow f'(x) = x + x^{-1}</math>  <math>\rightarrow f'(x) = 1 - x^{-2}</math>  <math>\rightarrow f'(x) = 1 - \frac{1}{x^2}</math>  <math>\rightarrow f'\left(\frac{1}{2}\right) = 1 - \frac{1}{\left(\frac{1}{2}\right)^2}</math>  <math>= 1 - \frac{1}{\frac{1}{4}}</math>  <math>= 1 - 4</math>  <math>= -3</math></p>	<p>(b) <math>\frac{dy}{dx} = \frac{3}{4} \Rightarrow f'(x) = \frac{3}{4}</math>  <math>\Rightarrow 1 - \frac{1}{x^2} = \frac{3}{4}</math>  <math>\Rightarrow \frac{1}{x^2} = \frac{1}{4}</math>  <math>\Rightarrow x^2 = 4</math>  <math>\Rightarrow x = \pm 2</math>          Since <math>f(x) = x + \frac{1}{x}</math>  <math>f(2) = 2 + \frac{1}{2} = \frac{5}{2}</math>  <math>f(-2) = -2 + \frac{1}{-2} = -\frac{5}{2}</math>  <math>\therefore A\left(2, \frac{5}{2}\right)</math> or <math>A\left(-2, -\frac{5}{2}\right)</math></p>
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**Question 5**

The curve  $C$  has equation

$$y = x^3 - x^2 - 5x + 2.$$

Find the  $x$  coordinates of the points on  $C$  with gradient 3.

$$x = -\frac{4}{3}, 2$$

$$\begin{aligned} y &= x^3 - x^2 - 5x + 2 \\ \frac{dy}{dx} &= 3x^2 - 2x - 5 \\ 3 &= 3x^2 - 2x - 5 \end{aligned} \quad \left. \begin{aligned} \Rightarrow 0 &= 3x^2 - 2x - 8 \\ \Rightarrow 0 &= (3x+4)(x-2) \\ \therefore x &= -\frac{4}{3}, 2 \end{aligned} \right\}$$

**Question 6**

The curve  $C$  has equation

$$y = x^5 - 6x^3 - 3x + 25.$$

Find an equation of the tangent to  $C$  at the point where  $x = 2$ .

$$y = 5x - 7$$

$$\begin{aligned} \bullet \quad y &= x^5 - 6x^3 - 3x + 25 \\ \text{when } x &= 2, \\ \Rightarrow y &= 2^5 - 6 \cdot 2^3 - 3 \cdot 2 + 25 \\ \Rightarrow y &= 32 - 48 - 6 + 25 \\ \Rightarrow y &= 57 - 54 \\ \Rightarrow y &= 3 \\ \therefore (2, 3) & \end{aligned} \quad \left. \begin{aligned} \frac{dy}{dx} &= 5x^4 - 18x^2 - 3 \\ \frac{dy}{dx} &= 5 \cdot 2^4 - 18 \cdot 2^2 - 3 \\ &= 80 - 72 - 3 \\ &= 5 \\ y_1 - y_2 &= m(x - x_0) \\ y - 3 &= 5(2 - 2) \\ y - 3 &= 5x - 10 \\ y &= 5x - 7 \end{aligned} \right\}$$

**Question 7**

The curve  $C$  has equation

$$y = -x^2(x+1), \quad x \in \mathbb{R}.$$

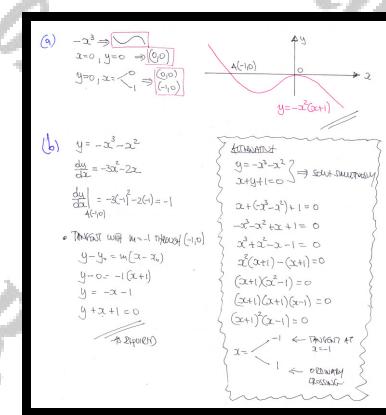
The curve meets the coordinate axes at the origin  $O$  and at the point  $A$ .

- a) Sketch the graph of  $C$ , indicating clearly the coordinates of  $A$ .
- b) Show that the straight line with equation

$$x + y + 1 = 0,$$

is a tangent to  $C$  at  $A$ .

A(-1, 0)



**Question 8**

The curve  $C$  has equation

$$y = \frac{6}{x^2} + \frac{5x}{4} - 4, \quad x \neq 0.$$

- a) Find an expression for  $\frac{dy}{dx}$ .
- b) Determine an equation of the normal to the curve at the point where  $x = 2$ .

$$\boxed{\frac{dy}{dx} = \frac{5}{4} - \frac{12}{x^3}}, \quad \boxed{y = 4x - 8}$$

<p>(a) <math>y = \frac{6}{x^2} + \frac{5x}{4} - 4</math></p> $y = 6x^{-2} + \frac{5}{4}x - 4$ $\frac{dy}{dx} = -12x^{-3} + \frac{5}{4}$	<p>(b) • when <math>x=2</math></p> $y = \frac{6}{2^2} + \frac{5 \cdot 2}{4} - 4$ $y = \frac{3}{2} + \frac{5}{2} - 4$ $y = 0$ $\therefore (2, 0)$
<p>(c) <math>\frac{dy}{dx} = \frac{5}{4} - \frac{12}{x^3}</math></p>	<p>• <math>\left. \frac{dy}{dx} \right _{x=2} = \frac{5}{4} - \frac{12}{2^3} = \frac{5}{4} - \frac{12}{8}</math>  <math>= \frac{5}{8} - \frac{12}{8} = -\frac{7}{8}</math></p> <p>• Normal (gradient) is <math>4</math>, <math>(2, 0)</math></p> $\Rightarrow y - y_1 = m(x - x_1)$ $\Rightarrow y - 0 = 4(x - 2)$ $\Rightarrow y = 4x - 8$

**Question 9**

The curve  $C$  has equation

$$f(x) = 4x\sqrt{x} - \frac{25x^2}{16}, \quad x \geq 0.$$

- a) Find a simplified expression for  $f'(x)$ .
- b) Determine an equation of the tangent to  $C$  at the point where  $x = 4$ , giving the answer in the form  $ax + by = c$ , where  $a$ ,  $b$  and  $c$  are integers.

$$f'(x) = 6x^{\frac{1}{2}} - \frac{25}{8}x, \quad x + 2y = 18$$

$\text{(a)} \quad f(x) = 4x\sqrt{x} - \frac{25x^2}{16}$ $\Rightarrow f(x) = 4x^{\frac{3}{2}} - \frac{25}{16}x^2$ $\Rightarrow f'(x) = 6x^{\frac{1}{2}} - \frac{25}{8}x$	$\text{with } x=4$ $y = f(4) = 4(4)\sqrt{4} - \frac{25}{16}(4)^2$ $= 32 - 25$ $= 7$ $\therefore (4, 7) \text{ is on the tangent line}$
$\text{(b)} \quad f'(x) = 6x^{\frac{1}{2}} - \frac{25}{8}x^{\frac{3}{2}}$ $= 6x^{\frac{1}{2}} - \frac{25}{8}x^{\frac{3}{2}}$ $= 6(2) - \frac{25}{8}(2)$ $= 12 - \frac{25}{4}$ $= \frac{28}{4} - \frac{25}{4}$ $= \frac{3}{4}$	$\Rightarrow y - y_0 = m(x - x_0)$ $\Rightarrow y - 7 = \frac{3}{4}(x - 4)$ $\Rightarrow 2y - 14 = -3x + 12$ $\Rightarrow 2y + 2 = 18$

**Question 10**

A curve has the following equation

$$f(x) = \frac{(2x-3)(x+2)}{\sqrt{x}}, \quad x > 0.$$

- a) Express  $f(x)$  in the form  $Ax^{\frac{3}{2}} + Bx^{\frac{1}{2}} + Cx^{-\frac{1}{2}}$ , where  $A$ ,  $B$  and  $C$  are constants to be found.
- b) Show that the tangent to the curve at the point where  $x=1$  is parallel to the line with equation

$$2y = 13x + 2.$$

$$\boxed{A = 2}, \boxed{B = 1}, \boxed{C = -6}$$

(a)  $\frac{d}{dx} f(x) = \frac{(2x-3)(x+2)}{\sqrt{x}} = \frac{2x^2 + 2x - 6}{x^{\frac{1}{2}}} = \frac{2x^2}{x^{\frac{1}{2}}} + \frac{2x}{x^{\frac{1}{2}}} - \frac{6}{x^{\frac{1}{2}}} = 2x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 6x^{-\frac{1}{2}}$

$\begin{matrix} A = 2 \\ B = 1 \\ C = -6 \end{matrix}$

(b)  $f'(x) = 3x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}} + 3x^{-\frac{3}{2}}$

$f'(1) = 3 + \frac{1}{2} + 3$

$f'(1) = \frac{13}{2}$

(IT GRADIENT OF TANGENT @  $x = 1$ )

Also  $2y = 13x + 2$

$y = \frac{13}{2}x + 1$

✓ SAME GRADIENT AS TANGENT  
∴ INDEED PARALLEL

**Question 11**

A cubic curve has equation

$$f(x) = 2x^3 - 7x^2 + 6x + 1.$$

The point  $P(2,1)$  lies on the curve.

- a) Find an equation of the tangent to the curve at  $P$ .

The point  $Q$  lies on the curve so that the tangent to the curve at  $Q$  is parallel to the tangent to the curve at  $P$ .

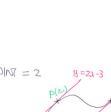
- b) Determine the  $x$  coordinate of  $Q$ .

$$y = 2x - 3, \quad x_Q = \frac{1}{3}$$

(a)  $f(x) = 2x^3 - 7x^2 + 6x + 1$   
 $f'(x) = 6x^2 - 14x + 6$        $f'(2) = 24 - 28 + 6 = 2$

$\text{Eqn } m=2$   
 $y - y_0 = m(x - x_0)$   
 $y - 1 = 2(x - 2)$   
 $y - 1 = 2x - 4$   
 $y = 2x - 3$

(b) PARALLEL TANGENTS  $\Rightarrow$  SAME GRADIENT = 2  
 $2 = 6x^2 - 14x + 6$   
 $0 = 6x^2 - 14x + 4$   
 $0 = 3x^2 - 7x + 2$   
 $0 = (x-2)(3x-1)$   
 $x = 2 \leftarrow \text{Point P (Clearly known)}$   
 $x = \frac{1}{3} \leftarrow \text{Point Q}$



**Question 12**

The curve  $C$  has equation

$$y = 2x^3 - 9x^2 + 12x - 10.$$

- a) Find the coordinates of the two points on the curve where the gradient is zero.

The point  $P$  lies on  $C$  and its  $x$  coordinate is  $-1$ .

- b) Determine the gradient of  $C$  at the point  $P$ .

The point  $Q$  lies on  $C$  so that the gradient at  $Q$  is the same as the gradient at  $P$ .

- c) Find the coordinates of  $Q$ .

$$\boxed{(1, -5), (2, -6)}, \boxed{36}, \boxed{Q(4, 22)}$$

$$y = 2x^3 - 9x^2 + 12x - 10$$

(a)  $\frac{dy}{dx} = 6x^2 - 18x + 12$

$$\frac{dy}{dx} = 0$$

$$0 = 6x^2 - 18x + 12$$

$$0 = 6(x^2 - 3x + 2)$$

$$0 = 6(x-1)(x-2)$$

$$x = \begin{cases} 1 \\ 2 \end{cases}$$

$$y = \begin{cases} 2(-1)^3 - 9(-1)^2 + 12(-1) - 10 = -16 \\ 16 - 36 + 24 - 10 = -6 \end{cases} \Rightarrow \begin{cases} (1, -5) \\ (2, -6) \end{cases}$$

(b)  $\frac{dy}{dx} = 6x^2 - 18x + 12$

$$\frac{dy}{dx} \Big|_{x=-1} = 6(-1)^2 - 18(-1) + 12 = 36$$

(c)  $\frac{dy}{dx} = 6x^2 - 18x + 12$

$$6x^2 - 18x + 12 = 0$$

$$6x^2 - 18x - 24 = 0$$

$$x^2 - 3x - 4 = 0$$

$$(x+1)(x-4) = 0$$

$$x = \begin{cases} -1 \\ 4 \end{cases}$$

$$y = \begin{cases} 2(-1)^3 - 9(-1)^2 + 12(-1) - 10 = -16 \\ 16 - 36 + 24 - 10 = -6 \\ 2(4)^3 - 9(4)^2 + 12(4) - 10 = 22 \\ 128 - 144 + 48 - 10 = 8 \end{cases} \Rightarrow \begin{cases} (-1, -6) \\ (4, 8) \end{cases}$$

ALREADY KNOWN

**Question 13**

The curve  $C$  has equation

$$y = ax^3 + bx^2 - 10,$$

where  $a$  and  $b$  are constants.

The point  $A(2, 2)$  lies on  $C$ .

Given that the gradient at  $A$  is 4, determine the value of  $a$  and the value of  $b$ .

$$\boxed{a = -2}, \boxed{b = 7}$$

Working for Question 13:

Given:  $y = ax^3 + bx^2 - 10$  and  $\frac{dy}{dx} = 3ax^2 + 2bx$ .  
Point  $A(2, 2)$  lies on the curve.  
 $2 = 8a + 4b - 10$   
 $12 = 8a + 4b$   
 $3 = 2a + b$

Gradient at  $x=2$ :  $\frac{dy}{dx} = 12a + 4b$   
 $4 = 12a + 4b$   
 $1 = 3a + b$

Solving the system of equations:  
 $b = 3 - 2a$   
 $b = 1 - 3a$   
 $3 - 2a = 1 - 3a$   
 $a = -2$   
 $b = 7$

**Question 14**

The curve  $C$  has equation

$$y = x^3 - 4x^2 + 6x - 3.$$

The point  $P(2,1)$  lies on  $C$  and the straight line  $L_1$  is the tangent to  $C$  at  $P$ .

- a) Find an equation of  $L_1$ .

The straight line  $L_2$  is a tangent to  $C$  at the point  $Q$ .

- b) Given that  $L_2$  is parallel to  $L_1$ , determine ...

- i. .... the exact coordinates of  $Q$ .

- ii. .... an equation of  $L_2$ .

$$\boxed{y = 2x - 3}, \quad \boxed{Q\left(\frac{2}{3}, -\frac{13}{27}\right)}, \quad \boxed{27y = 54x - 49}$$

(a)  $y = x^3 - 4x^2 + 6x - 3$

$$\frac{dy}{dx} = 3x^2 - 8x + 6$$

$$\left. \frac{dy}{dx} \right|_{x=2} = 3(2)^2 - 8(2) + 6$$

$$= 12 - 16 + 6$$

$$= 2.$$

$\bullet (2,1), m=2, \text{ TANGENT AT } (2,1)$

$$\Rightarrow y - 1 = m(x - 2)$$

$$\Rightarrow y - 1 = 2(x - 2)$$

$$\Rightarrow y - 1 = 2x - 4$$

$$\Rightarrow y = 2x - 3$$

(b) (i)  $P(2,1)$   $L_1$

$L_2$   $Q$

$\frac{dy}{dx} = 3x^2 - 8x + 6$

$$2 = 3x^2 - 8x + 6$$

$$0 = 3x^2 - 8x + 4$$

$$0 = (x-2)(3x-2)$$

$$x = 2 \leftarrow P$$

$$\frac{2}{3} \leftarrow Q$$

$y = \frac{2}{3}^3 - 4\left(\frac{2}{3}\right)^2 + 6\left(\frac{2}{3}\right) - 3 = \frac{8}{27} - 4 \times \frac{4}{9} + 4 - 3 = \frac{8}{27} - \frac{16}{9} + 1$

$$= \frac{8}{27} - \frac{48}{27} + \frac{27}{27} = -\frac{13}{27} \therefore Q\left(\frac{2}{3}, \frac{13}{27}\right)$$

(ii)  $y - y_1 = m(x - x_1)$

$$y + \frac{13}{27} = 2\left(x - \frac{2}{3}\right)$$

$$y + \frac{13}{27} = 2x - \frac{4}{3}$$

$$27y + 13 = 54x - 36$$

$$27y = 54x - 53$$

**Question 15**

A curve  $C$  and a straight line  $L$  have respective equations

$$y = 2x^2 - 6x + 5 \quad \text{and} \quad 2y + x = 4.$$

- a) Find the coordinates of the points of intersection between  $C$  and  $L$ .
- b) Show that  $L$  is a normal to  $C$ .

The tangent to  $C$  at the point  $P$  is parallel to  $L$ .

- c) Determine the  $x$  coordinate of  $P$ .

$$\boxed{(2,1), \left(\frac{3}{4}, \frac{13}{8}\right)}, \quad x_P = \frac{11}{8}$$

(a)

$$\begin{cases} y = 2x^2 - 6x + 5 \\ 2y + x = 4 \end{cases} \Rightarrow \begin{aligned} 2(2x^2 - 6x + 5) + x &= 4 \\ \Rightarrow 4x^2 - 12x + 10 + x &= 4 \\ \Rightarrow 4x^2 - 11x + 6 &= 0 \\ \Rightarrow (4x - 3)(x - 2) &= 0 \end{aligned}$$

$x = \frac{3}{4}, x = 2$

$$\Rightarrow x = \frac{3}{4}, y = \frac{4x^2 - 12x + 10 + x}{2} = 1$$

$$\therefore (2,1) \text{ or } \left(\frac{3}{4}, \frac{13}{8}\right)$$
  

(b)

$$\frac{dy}{dx} = 4x - 6$$

↙ Gradient

$$\begin{aligned} \text{at } (2,1) \quad &\frac{dy}{dx} = 2(2) - 6 = -2 \\ &2y + x = 4 \\ &2(-2) + x = 4 \\ &x = 8 \end{aligned}$$

↙ Line

$$\begin{aligned} \frac{dy}{dx} &= 4(2) - 6 = 2 \quad \leftarrow \text{Tangent at } (2,1) \text{ has gradient 2} \\ &\therefore \text{normal at } (2,1) \text{ has gradient } -\frac{1}{2} \\ &\therefore 2y + x + 4 \text{ is the normal to the curve at } (2,1) \end{aligned}$$
  

(c)

TANGENT (GRADIENT  $-\frac{1}{2}$ )

NORMAL (GRADIENT  $\frac{1}{2}$ )

$\frac{dy}{dx} = 4x - 6$

$$\begin{aligned} -\frac{1}{2} &= 4x - 6 \\ -1 &= 8x - 12 \\ 11 &= 8x \\ x &= \frac{11}{8} \end{aligned}$$

**Question 16**

The curve  $C$  has equation

$$y = 2x^3 - 6x^2 + 3x + 5.$$

The point  $P(2,3)$  lies on  $C$  and the straight line  $L_1$  is the tangent to  $C$  at  $P$ .

- a) Find an equation of  $L_1$ .

The straight lines  $L_2$  and  $L_3$  are parallel to  $L_1$ , and they are the respective normals to  $C$  at the points  $Q$  and  $R$ .

- b) Determine the  $x$  coordinate of  $Q$  and the  $x$  coordinate of  $R$ .

$$y = 3x - 3, \quad x = \frac{1}{3}, \frac{5}{3}$$

(a)

$$\begin{aligned} y &= 2x^3 - 6x^2 + 3x + 5 \\ \frac{dy}{dx} &= 6x^2 - 12x + 3 \\ \left. \frac{dy}{dx} \right|_{x=2} &= 6(2)^2 - 12(2) + 3 \\ &= 24 - 24 + 3 \\ &= 3 \end{aligned}$$

TANGENT AT  $(2,3)$ , EQUATION

$$\begin{aligned} \Rightarrow y - y_0 &= m(x - x_0) \\ \Rightarrow y - 3 &= 3(x - 2) \\ \Rightarrow y - 3 &= 3x - 6 \\ \Rightarrow y &= 3x - 3 \end{aligned}$$

(b)

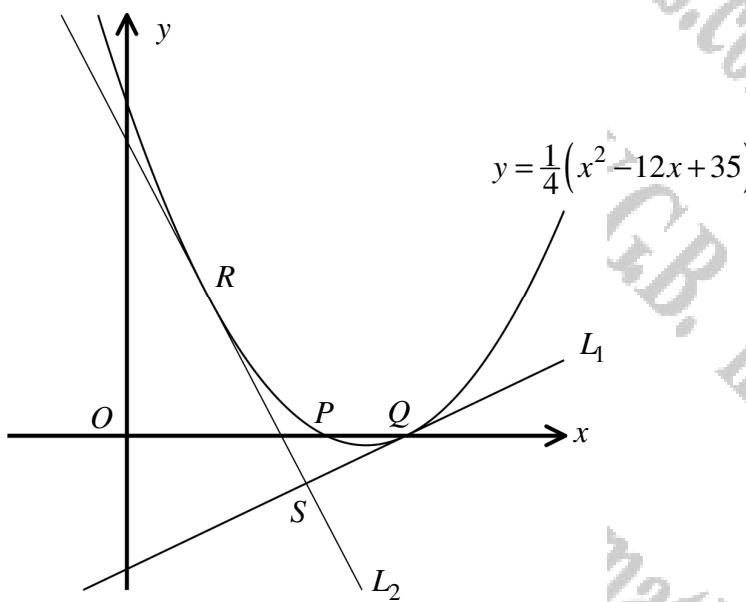
$$\begin{aligned} \frac{dy}{dx} &= 6x^2 - 12x + 3 \\ \Rightarrow -\frac{1}{3} &= 6x^2 - 12x + 3 \\ \Rightarrow -1 &= 18x^2 - 36x + 9 \\ \Rightarrow 0 &= 18x^2 - 36x + 10 \\ \Rightarrow 9x^2 - 18x + 5 &= 0 \end{aligned}$$

$$\Rightarrow (3x - 5)(3x - 1) = 0$$

$$\Rightarrow x = \frac{1}{3}, \frac{5}{3}$$

IN MATHS DEPARTMENT

## Question 17



The figure above shows the curve with equation

$$y = \frac{1}{4}(x^2 - 12x + 35).$$

The curve crosses the  $x$  axis at the points  $P(x_1, 0)$  and  $Q(x_2, 0)$ , where  $x_2 > x_1$ .

The tangent to the curve at  $Q$  is the straight line  $L_1$ .

- a) Find an equation of  $L_1$ .

The tangent to the curve at the point  $R$  is denoted by  $L_2$ . It is further given that  $L_2$  meets  $L_1$  at right angles, at the point  $S$ .

- b) Find an equation of  $L_2$ .

- c) Determine the exact coordinates of  $S$ .

,  $y = \frac{1}{2}x - \frac{7}{2}$  ,  $4y + 8x = 31$  ,  $S\left(\frac{9}{2}, -\frac{5}{4}\right)$

<p>(a) <math>y = \frac{1}{4}(x^2 - 12x + 35)</math>  <math>y = \frac{1}{4}(x-5)(x-7)</math>  <math>\text{when } y=0, x=5 &lt; x &lt; 7 \quad \leftarrow \text{PC}(x_1)</math>  <math>\frac{dy}{dx} = \frac{1}{2}(2x-12) = \frac{1}{2}x - 3</math>  <math>\frac{dy}{dx} _{Q} = \frac{1}{2}x_2 - 3 = \frac{7}{2} - 3 = \frac{1}{2}</math></p> <p><i>Therefore <math>\Delta x</math></i>  <math>y_2 - y_1 = \frac{1}{2}(x_2 - x_1)</math>  <math>y_2 = \frac{1}{2}(x_2 - 7)</math>  <math>y_2 = \frac{1}{2}x_2 - \frac{7}{2}</math></p>	<p>(b) <math>\bullet</math> since <math>CF</math> <math>L_2 = -2</math> (<math>\perp</math> to <math>L_1</math>)  <math>\bullet \frac{dy}{dx} = \frac{1}{2}x - 3</math>  <math>-2 = \frac{1}{2}x_2 - 3</math>  <math>1 = \frac{1}{2}x_2</math>  <math>x_2 = 2</math>  <math>\therefore y = \frac{1}{4}(2^2 - 12 \cdot 2 + 35) = \frac{1}{4} \cdot 15 = \frac{15}{4}</math>  <math>\therefore S\left(2, \frac{15}{4}\right)</math></p> <p><i>Therefore <math>\Delta x</math></i>  <math>y_2 - y_1 = \frac{1}{2}(2 - 7)</math>  <math>y_2 = \frac{1}{2}(2 - 7)</math>  <math>y_2 = \frac{1}{2} \cdot -5 = -\frac{5}{2}</math>  <math>\frac{dy}{dx} = 4(x-2)</math>  <math>4(x-2) = 4(2-2)</math>  <math>4x - 16 = 8(2-2)</math>  <math>4x - 16 = 0</math>  <math>4x = 16</math>  <math>x = 4</math></p>	<p>(c) <i>Second Stage Method</i>  <math>\begin{cases} y = \frac{1}{2}x - \frac{7}{2} \\ 4y + 8x = 31 \end{cases} \Rightarrow 4\left(\frac{1}{2}x - \frac{7}{2}\right) + 8x = 31</math>  <math>2x - 14 + 8x = 31</math>  <math>10x = 45</math>  <math>x = \frac{9}{2}</math>  <math>y = \frac{1}{2} \cdot \frac{9}{2} - \frac{7}{2} = \frac{9}{4} - \frac{7}{2} = \frac{9}{4} - \frac{14}{4} = -\frac{5}{4}</math>  <math>\therefore S\left(\frac{9}{2}, -\frac{5}{4}\right)</math></p>
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**Question 18**

The point  $P(1,0)$  lies on the curve  $C$  with equation

$$y = x^3 - x, \quad x \in \mathbb{R}.$$

- a) Find an equation of the tangent to  $C$  at  $P$ , giving the answer in the form  $y = mx + c$ , where  $m$  and  $c$  are constants.

The tangent to  $C$  at  $P$  meets  $C$  again at the point  $Q$ .

- b) Determine the coordinates of  $Q$ .

$$y = 2x - 2, \quad Q(-2, -6)$$

(a)  $y = x^3 - x$   
 $\frac{dy}{dx} = 3x^2 - 1$   
 $\left. \frac{dy}{dx} \right|_{x=1} = 3(1)^2 - 1 = 2$

From part (a):  
 $y - y_1 = m(x - x_1)$   
 $y - 0 = 2(x - 1)$   
 $y = 2x - 2$

(b)  $y = x^3 - x$   
 $y = 2x - 2$       Same simultaneously

$\Rightarrow x^3 - x = 2x - 2$   
 $\Rightarrow x^3 - 3x + 2 = 0$

Since  $P(1,0)$  must be solution to the problem  
 $\Rightarrow x=1$  is a root (given point)  
 $\Rightarrow$  further more it must be squared (tangent point)  
 $\Rightarrow (x-1)^2(x+2)=0$

Point P:  $x=1, y=0$   
Point Q:  $x=-2, y=-6$

Check:  
 $(x+2)(x-1)^2$   
 $= (x+2)(x^2-2x+1)$   
 $= x^3 - 2x^2 + x + 2x^2 - 4x + 2$   
 $= x^3 - 3x + 2$

$\therefore x = -2$   
 $y = 2(-2) - 2 = -4 - 2 = -6$   
So  $Q(-2, -6)$

**Question 19**

A curve  $C$  with equation

$$y = 4x^3 + 7x^2 + x + 11, \quad x \in \mathbb{R}.$$

The point  $P$  lies on  $C$ , where  $x = -1$ .

- a) Find an equation of the tangent to  $C$  at  $P$ .

The tangent to  $C$  at  $P$  meets  $C$  again at the point  $Q$ .

- b) Determine the  $x$  coordinate of  $Q$ .

$$y = 12 - x, \quad x_Q = \frac{1}{4}$$

(a)

$$\begin{aligned} y &= 4x^3 + 7x^2 + x + 11 \\ \frac{dy}{dx} &= 12x^2 + 14x + 1 \\ \frac{dy}{dx}|_{x=-1} &= 12 - 14 + 1 = -1 \end{aligned}$$

where  $x = -1$

$$\begin{aligned} y &= 4(-1)^3 + 7(-1)^2 + (-1) + 11 \\ &\in C_1(8) \end{aligned}$$

Tangent at  $P(-1, 8)$

$$\begin{aligned} \frac{dy}{dx} &= m(x - x_0) \\ \frac{dy}{dx} &= -1(x + 1) \\ y - 8 &= -1(x + 1) \\ y &= 12 - x \end{aligned}$$

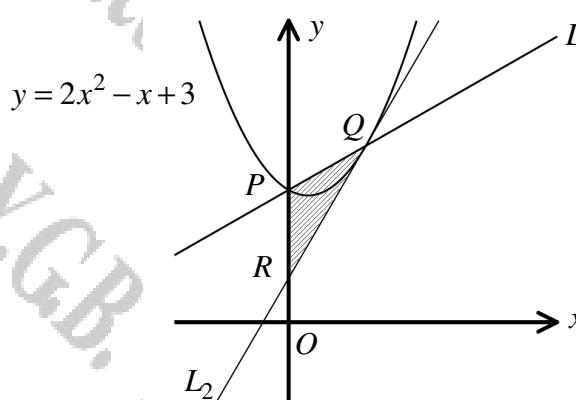
(b)

$$\begin{aligned} y &= 4x^3 + 7x^2 + x + 11 \\ y &= 12 - x \\ \Rightarrow 4x^3 + 7x^2 + x + 11 &= 12 - x \\ \Rightarrow 4x^3 + 7x^2 + 2x - 1 &= 0 \\ \Rightarrow (2x+1)(4x^2+2x-1) &= 0 \end{aligned}$$

Roots of  $2x+1=0$ :  $x = -\frac{1}{2}$

Roots of  $4x^2+2x-1=0$ :  $x = -1, \frac{1}{4}$

Point  $Q$  is the second root:  $x = \frac{1}{4}$

**Question 20**

The figure above shows the curve  $C$  with equation

$$y = 2x^2 - x + 3.$$

$C$  crosses the  $y$  axis at the point  $P$ . The normal to  $C$  at  $P$  is the straight line  $L_1$ .

- a) Find an equation of  $L_1$ .

$L_1$  meets the curve again at the point  $Q$ .

- b) Determine the coordinates of  $Q$ .

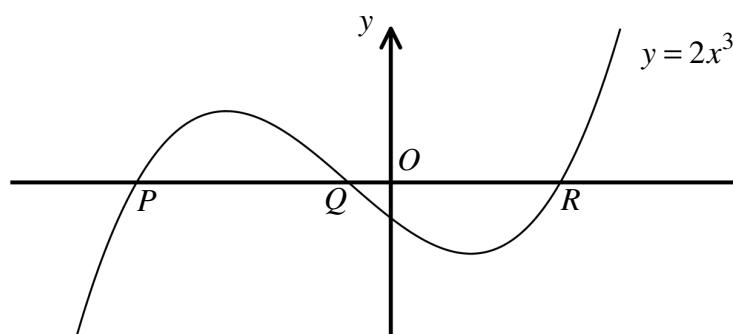
The tangent to  $C$  at  $Q$  is the straight line  $L_2$ .

$L_2$  meets the  $y$  axis at the point  $R$ .

- c) Show that the area of the triangle  $PQR$  is one square unit.

$$y = x + 3, \quad Q(1, 4)$$

<p>(a) <math>y = 2x^2 - x + 3</math> • BY DIFFERENTIATION <math>\frac{dy}{dx}(3, 3)</math> • <math>\frac{dy}{dx} = 4x - 1</math> <math>\frac{dy}{dx} _{(3, 3)} = 4(3) - 1 = 11</math> ← tangent gradient ∴ EQUATION OF <math>L_1</math>: <math>y - 3 = 11(x - 3)</math> <math>y - 3 = 11x - 33</math> <math>y = 11x - 30</math></p>	<p>(b) <math>\text{SOLVING } 2x^2 - x + 3 = 11x - 30</math> <math>2x^2 - 11x + 30 = 0</math> <math>x^2 - \frac{11}{2}x + 15 = 0</math> <math>x^2 - 5x - \frac{1}{2}x + 15 = 0</math> <math>x(x - 5) - \frac{1}{2}x(x - 5) = 0</math> <math>(x - 5)(x - \frac{1}{2}x) = 0</math> <math>x = 5, \frac{1}{2}</math></p>	<p>(c) <math>\frac{dy}{dx} _{(1, 4)} = 4(1) - 1 = 3</math> ∴ EQUATION <math>L_2</math>: <math>y - 4 = 3(x - 1)</math> <math>y - 4 = 3x - 3</math> <math>y = 3x + 1</math></p>	<p>BY DIFFERENTIATION <math>R(0, 1)</math> <math>y = 3x + 1</math> <math>y = 1</math></p>
		<p><math>\therefore \triangle PQR</math></p> <p><math>\text{AREA} = \frac{1}{2} \times \text{BASE} \times \text{HEIGHT}</math> <math>= \frac{1}{2} \times 2 \times  3 - 1 </math> <math>= \frac{1}{2} \times 2 \times 1</math> <math>= 1</math></p>	

**Question 21**

The figure above shows the curve  $C$  with equation

$$y = 2x^3 + 3x^2 - 11x - 6.$$

The curve crosses the  $x$ -axis at the points  $P$ ,  $Q$  and  $R(2,0)$ .

The tangent to  $C$  at  $R$  is the straight line  $L_1$ .

- a) Find an equation of  $L_1$ .

The normal to  $C$  at  $P$  is the straight line  $L_2$ .

The straight lines  $L_1$  and  $L_2$  meet at the point  $S$ .

- b) Show that  $\angle PSR = 90^\circ$ .

$$y = 25x - 50$$

**(a)**

$$y = 2x^3 + 3x^2 - 11x - 6$$

- $\frac{dy}{dx} = 6x^2 + 6x - 11$
- $\frac{dy}{dx} = 6x^2 + 6x - 11$
- $3x_1 = 24 + 12 - 11 = 25$

Then  $x_1 = 25/3$  (2,0)

$$y - y_1 = m(x - x_1)$$

$$y - 0 = 25(x - 2)$$

$$y = 25x - 50$$

**(b)**

$$y = 2x^3 + 3x^2 - 11x - 6$$

$$y = (x-2)(2x^2+4x+3)$$

From point  $R$ :

- Quadratic coefficients in 2:
- $3a - 24a = -11$
- $3 - 24 = -11$
- $14 = 24$
- $4 = 7$

$$y = (x-2)(2x^2+7x+3)$$

$$y = (x-2)(2x+1)(x+3)$$

Diagram of the curve  $y = 2x^3 + 3x^2 - 11x - 6$

$\frac{dy}{dx} = 6(x^2 + x - \frac{11}{6})$

At  $P(2,0)$

$$\frac{dy}{dx} = 6(2^2 + 2 - \frac{11}{6}) = 25$$

GRADIENT OF THE TANGENT AT P

GRADIENT OF THE NORMAL AT P

GRADIENT OF THE TANGENT AT R

GRADIENT OF THE NORMAL AT R

$\therefore L_1 \perp L_2$  MEET AT  $90^\circ$

**Question 22**

A curve has equation

$$y = 6\sqrt[3]{x^5} - 15\sqrt[3]{x^4} - 80x + 16, \quad x \in \mathbb{R}, \quad x \geq 0.$$

Find the coordinates of the stationary point of the curve and determine whether it is a local maximum, a local minimum or a point of inflexion.

local minimum at  $(16, -2800)$

WRITE THE EQUATION IN IMAGINARY FORM AND DIFFERENTIATE

$$\Rightarrow y = 6x^{\frac{5}{3}} - 15x^{\frac{4}{3}} - 80x + 16$$

$$\Rightarrow y = 6x^{\frac{5}{3}} - 15x^{\frac{4}{3}} - 80x + 16$$

$$\Rightarrow \frac{dy}{dx} = 10x^{\frac{2}{3}} - 20x^{\frac{1}{3}} - 80$$

FIND STATIONARY POINTS, SOLVING  $\frac{dy}{dx} = 0$

$$\Rightarrow 0 = 10x^{\frac{2}{3}} - 20x^{\frac{1}{3}} - 80$$

$$\Rightarrow 2x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 8 = 0$$

$$\Rightarrow (2x^{\frac{1}{3}})^2 - 2(2x^{\frac{1}{3}}) - 8 = 0$$

$$\Rightarrow 2x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 8 = 0 \quad \text{, where } a = x^{\frac{1}{3}}$$

$$\Rightarrow (a+2)(a-4) = 0$$

$$\Rightarrow a = -2$$

$$\Rightarrow x^{\frac{1}{3}} = -2$$

$$\Rightarrow x^{\frac{1}{3}} = \frac{-2}{4}$$

$$\Rightarrow x = \frac{-8}{64}$$

$$\Rightarrow x = -\frac{1}{8}$$

$$\Rightarrow y = 6(-\frac{1}{8})^{\frac{5}{3}} - 15(-\frac{1}{8})^{\frac{4}{3}} - 80(-\frac{1}{8}) + 16$$

$$= 6x(-\frac{1}{8})^{\frac{5}{3}} - 15x(-\frac{1}{8})^{\frac{4}{3}} - 80x + 16$$

$$= 6144 - 3840 - 5120 + 16$$

$$= -2800$$

DETERMINING THE NATURE BY THE SECOND DERIVATIVE TEST

$$\frac{dy}{dx} = 10x^{\frac{2}{3}} - 20x^{\frac{1}{3}} - 80$$

$$\frac{d^2y}{dx^2} = \frac{20}{3}x^{-\frac{1}{3}} - \frac{20}{3}x^{-\frac{2}{3}}$$

$$\frac{d^3y}{dx^3} = \frac{20}{3}\left[\frac{1}{3}x^{-\frac{2}{3}} - \frac{1}{3}x^{-\frac{5}{3}}\right]$$

$$\left.\frac{d^3y}{dx^3}\right|_{x=64} = \frac{20}{3}\left[\frac{1}{3}x^{-\frac{2}{3}} - \frac{1}{3}x^{-\frac{5}{3}}\right] = \frac{5}{4} > 0$$

$\therefore (16, -2800)$  is a  
LOCAL MINIMUM

**Question 23**

A curve has equation

$$y = x^2 - 6x \sqrt[3]{x} + 2, \quad x \in \mathbb{R}, \quad x \geq 0.$$

Find the coordinates of the stationary points of the curve and classify them as local maxima, local minima or points of inflexion.

local minimum at  $(8, -30)$ , local maximum at  $(0, 2)$

REWRITE THE EQUATION IN INDICES & DIFFERENTIATE.

$$\begin{aligned} \Rightarrow y &= x^2 - 6x^{1/3}x^2 + 2 \\ \Rightarrow y &= x^2 - 6x^{7/3} + 2 \\ \Rightarrow y &= x^2 - 6x^{7/3} + 2 \\ \Rightarrow \frac{dy}{dx} &= 2x - 6x^{4/3} \end{aligned}$$

SEARCH FOR ZERO, SEEING STATIONARY POINTS

$$\begin{aligned} \Rightarrow 2x - 6x^{4/3} &= 0 \\ \Rightarrow 2x &= 6x^{4/3} \\ \Rightarrow x &= 4x^{4/3} \\ \text{EITHER } x &= 0 \text{ (BY INSPECTION) OR, IF WE DIVIDE WE OBTAIN} \\ \Rightarrow x^{2/3} &= 4 \\ \Rightarrow (\sqrt[3]{x})^2 &= 4 \\ \Rightarrow \sqrt[3]{x} &= \sqrt[3]{-2} \\ \Rightarrow x &= -\sqrt[3]{-2} \quad x > 0 \end{aligned}$$

FIND FIRST THE CORRESPONDING Y COORDINATES

$$\begin{aligned} x=0, \quad y &= 2 \\ x=8, \quad y &= 8^2 - 6 \cdot 8^{4/3} + 2 = 64 - 6 \cdot 16 + 2 = 64 - 96 + 2 = -30 \\ \therefore (0, 2) \quad &\text{and} \quad (8, -30) \end{aligned}$$

DETERMINING THE NATURE OF THESE POINTS BY USING THE SECOND DERIVATIVE TEST

$$\begin{aligned} \Rightarrow \frac{d^2y}{dx^2} &= 2 - 8x^{1/3} \\ \Rightarrow \frac{d^2y}{dx^2} &= 2 - \frac{8}{x^{2/3}} \\ \Rightarrow \frac{d^2y}{dx^2} &= 2 - \frac{8}{3x^{2/3}} \end{aligned}$$

$$\left. \frac{d^2y}{dx^2} \right|_{x=8} = 2 - \frac{8}{3 \cdot 8^{2/3}} = 2 - \frac{8}{12} = \frac{4}{3} > 0$$

$\therefore (8, -30)$  is a local minimum

$$\left. \frac{d^2y}{dx^2} \right|_{x=0} = 2 - \frac{8}{3 \cdot 0^{2/3}} = 2 - \frac{8}{0} \leftarrow \text{WE CANNOT USE THIS TEST WITHOUT EXAMINING OF LIMTING BEHAVIOUR}$$

CHECKING WHETHER THE GRADIENT OR THE VALUE OF  $y$  TO THE RIGHT OF  $x=0$  (AS  $x \rightarrow 0$ )

$$\begin{aligned} \left. \frac{dy}{dx} \right|_{x=0+} &\approx -3.51... \\ \left. \frac{dy}{dx} \right|_{x=0+} &\approx 1.73... \end{aligned}$$

$\therefore (0, 2)$  is a local max

**Question 24**

A curve has equation

$$y = x(x^2 - 128\sqrt{x}), \quad x \in \mathbb{R}, \quad x > 0.$$

The curve has a single stationary point with coordinates  $(2^\alpha, -2^\beta)$ , where  $\alpha$  and  $\beta$  are positive integers.

Find the value of  $\beta$  and justify that the stationary point is a local minimum.

$$\boxed{\beta = 12}$$

<p><u>REWRITE THE EQUATION IN INDICIAL FORM &amp; DIFFERENTIATE</u></p> $\begin{aligned} \Rightarrow y &= x(x^2 - 128x^{\frac{1}{2}}) \\ \Rightarrow y &= x^3 - 128x^{\frac{3}{2}} \\ \Rightarrow \frac{dy}{dx} &= 3x^2 - 192x^{\frac{1}{2}} \\ \Rightarrow \text{FOR STATIONARY POINTS SET } \frac{dy}{dx} &= 0 \end{aligned}$ $\begin{aligned} \Rightarrow 3x^2 - 192x^{\frac{1}{2}} &= 0 \\ \Rightarrow x^2 - 64x^{\frac{1}{2}} &= 0 \\ \Rightarrow x^2 &= 64x^{\frac{1}{2}} \\ \Rightarrow \frac{x^2}{x^{\frac{1}{2}}} &= 64 \quad (\text{WE ARE NOT CONCERNED WITH } x=0) \\ \Rightarrow x^{\frac{3}{2}} &= 64 \\ \Rightarrow (x^{\frac{3}{2}})^{\frac{2}{3}} &= (64)^{\frac{2}{3}} \\ \Rightarrow x^2 &= (64)^{\frac{2}{3}} \\ \Rightarrow x &= 16 \end{aligned}$ <p><u>CHECK THE NATURE OF THE POINT BY THE SECOND DERIVATIVE TEST</u></p> $\begin{aligned} \Rightarrow \frac{d^2y}{dx^2} &= 6x - 96x^{\frac{1}{2}} \\ \Rightarrow \frac{d^2y}{dx^2}_{x=16} &= 6 \times 16 - 96 \times 16^{\frac{1}{2}} = 96 - 96 \times \frac{1}{4} = 96 - 24 = 72 > 0 \end{aligned}$ <p style="text-align: right;"><u>LOCAL MINIMUM</u></p>	<p><u>FINDING THE Y CO-ORDINATE IN THE REQUIRED FORM</u></p> $\begin{aligned} y &= x(x^2 - 128\sqrt{x}) \\ y &= 16(16^2 - 128\sqrt{16}) \\ y &= -4096 \\ y &= -2^{12} \quad (\text{TERM } 2 \text{ IS A POWER OF 2}) \end{aligned}$ <p style="text-align: center;"><math>\therefore \underline{\text{LOCAL MINIMUM AT } (16, -2^{12})}</math></p>
--	---

**Question 25**

The point  $P$ , whose  $x$  coordinate is  $\frac{1}{4}$ , lies on the curve with equation

$$y = \frac{k + 4x\sqrt{x}}{7x}, \quad x \in \mathbb{R}, \quad x > 0,$$

where  $k$  is a non zero constant.

- a) Determine, in terms of  $k$ , the gradient of the curve at  $P$ .

The tangent to the curve at  $P$  is parallel to the straight line with equation

$$44x + 7y - 5 = 0.$$

- b) Find an equation of the tangent to the curve at  $P$ .

$$\left. \frac{dy}{dx} \right|_{x=\frac{1}{4}} = \frac{4-16k}{7}, \quad [44x+7y=25]$$

a) FIND THE EQUATION INTO INEQUAL FORM AND DIFFERENTIATE

- $y = \frac{4x\sqrt{x} + k}{7x} = \frac{4x^{\frac{3}{2}} + k}{7x} = \frac{4x^{\frac{1}{2}}}{7} + \frac{k}{7x} = \frac{2}{7}x^{\frac{1}{2}} + \frac{k}{7x}$
- $\frac{dy}{dx} = \frac{2}{7}x^{-\frac{1}{2}} - \frac{k}{7x^2}$
- $\left. \frac{dy}{dx} \right|_{x=\frac{1}{4}} = \frac{2}{7}\left(\frac{1}{4}\right)^{\frac{1}{2}} - \frac{k}{7\left(\frac{1}{4}\right)^2} = \frac{2}{7} \times 2 - \frac{k}{7 \times 16} = \frac{4}{7} - \frac{k}{112}$

b) REARRANGE THE EQUATION OF THE LINE TO "READ" THE GRADIENT

$$\begin{aligned} \Rightarrow 7y + 44x - 5 &= 0 \\ \Rightarrow 7y &= -44x + 5 \\ \Rightarrow y &= -\frac{44}{7}x + \frac{5}{7} \end{aligned}$$

FIND THE GRADIENT AT  $P$  WHICH IS  $-\frac{44}{7}$  (PARALLEL)

$$\begin{aligned} \Rightarrow \frac{4}{7} - \frac{k}{112} &= -\frac{44}{7} \\ \Rightarrow 4 - 16k &= -44 \\ \Rightarrow 48 &= 16k \\ \Rightarrow k &= 3 \end{aligned}$$

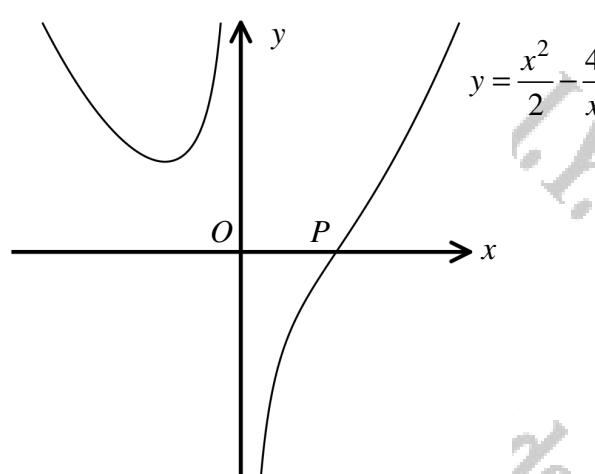
FIND THE  $y$ -COORDINATE OF  $P$

$$y = \frac{4x\sqrt{x} + 3}{7x} = \frac{4x^{\frac{3}{2}} + 3}{7x} = \frac{\left(\frac{1}{4} + 3\right)^{\frac{3}{2}}}{7 \times \frac{1}{4}} = \frac{27}{7} = 2$$

EQUATION OF TANGENT AT  $P(2, 2)$

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 2 &= -\frac{44}{7}(x - 2) \\ y - 2 &= -\frac{44}{7}x + \frac{88}{7} \\ y - 2 &= -6.2857x + 12.5714 \end{aligned} \quad \therefore 44x + 7y = 25$$

## Question 26



The figure above shows the curve  $C$  with equation

$$y = \frac{x^2 - 4}{2x}, \quad x \neq 0.$$

The curve crosses the  $x$  axis at the point  $P$ .

The straight line  $L$  is the normal to  $C$  at  $P$ .

a) Find ...

i. ... the coordinates of  $P$ .

ii. ... an equation of  $L$ .

b) Show that  $L$  does not meet  $C$  again.

$$P(2, 0), \quad x + 3y = 2$$

**(a)**  $y = 0$

$$0 = \frac{x^2 - 4}{2x}$$

$$\frac{2x}{2} = \frac{x^2 - 4}{2x}$$

$$2x^2 - 8 = 0$$

$$2(x^2 - 4) = 0$$

$$x^2 - 4 = 0$$

$$(x-2)(x+2) = 0$$

$$x = 2, -2$$

$$\therefore P(2, 0)$$

**(b)**  $y = \frac{x^2 - 4}{2x}$

$$\frac{dy}{dx} = \frac{2x \cdot 2x - (x^2 - 4) \cdot 2}{(2x)^2}$$

$$\frac{dy}{dx} = \frac{4x^2 - 2x^2 + 8}{4x^2}$$

$$\frac{dy}{dx} = \frac{2x^2 + 8}{4x^2}$$

$$\frac{dy}{dx} = \frac{2x^2}{4x^2}$$

$$\frac{dy}{dx} = \frac{1}{2}$$

$$\therefore \text{gradient of } C \text{ at } P(2, 0) = \frac{1}{2}$$

$$\therefore \text{gradient of } L = -2$$

$$y - 0 = -2(x - 2)$$

$$y = -2x + 4$$

$$2x + y = 4$$

$$x + 3y = 2$$

**(c)**  $\text{Solve } x + 3y = 2$

$$x + 3y = 2 \quad (1)$$

$$y = \frac{2-x}{3} \quad (2)$$

$$\text{Substitute } (2) \text{ into } (1)$$

$$x + 3\left(\frac{2-x}{3}\right) = 2$$

$$x + 2 - x = 2$$

$$2 = 2$$

$$\therefore \text{only solution is } (2, 0)$$

**Normal to**  $C$  **at**  $P(2, 0)$

$$\frac{dy}{dx} = \frac{1}{2}$$

$$\therefore \text{gradient of } L = -2$$

$$L: y = -2x + 4$$

$$y = -2x + 4$$

$$2x + y = 4$$

$$x + 3y = 2$$

$$2x + 3(-2x + 4) = 2$$

$$2x - 6x + 12 = 2$$

$$-4x + 12 = 2$$

$$-4x = 2 - 12$$

$$-4x = -10$$

$$x = \frac{-10}{-4}$$

$$x = 2.5$$

$$\therefore \text{only solution is } (2, 0)$$

**Normal to**  $C$  **at**  $P(2, 0)$

$$\frac{dy}{dx} = \frac{1}{2}$$

$$\therefore \text{gradient of } L = -2$$

$$L: y = -2x + 4$$

$$y = -2x + 4$$

$$2x + y = 4$$

$$x + 3y = 2$$

$$2x + 3(-2x + 4) = 2$$

$$2x - 6x + 12 = 2$$

$$-4x + 12 = 2$$

$$-4x = 2 - 12$$

$$-4x = -10$$

$$x = \frac{-10}{-4}$$

$$x = 2.5$$

$$\therefore \text{only solution is } (2, 0)$$

**Normal to**  $C$  **at**  $P(2, 0)$

$$\frac{dy}{dx} = \frac{1}{2}$$

$$\therefore \text{gradient of } L = -2$$

$$L: y = -2x + 4$$

$$y = -2x + 4$$

$$2x + y = 4$$

$$x + 3y = 2$$

$$2x + 3(-2x + 4) = 2$$

$$2x - 6x + 12 = 2$$

$$-4x + 12 = 2$$

$$-4x = 2 - 12$$

$$-4x = -10$$

$$x = \frac{-10}{-4}$$

$$x = 2.5$$

$$\therefore \text{only solution is } (2, 0)$$

**Question 27**

The curve  $C$  has equation

$$y = (x-1)(x^2+4x+5), \quad x \in \mathbb{R}.$$

- a) Show that  $C$  meets the  $x$  axis at only one point.

The point  $A$ , where  $x = -1$ , lies on  $C$ .

- b) Find an equation of the normal to  $C$  at  $A$ .

The normal to  $C$  at  $A$  meets the coordinate axes at the points  $P$  and  $Q$ .

- c) Show further that the area of the triangle  $OPQ$ , where  $O$  is the origin, is  $12\frac{1}{4}$  square units.

$$2y = x - 7$$

(a)  $y = (x-1)(x^2+4x+5)$

$$\begin{aligned} &\text{Factorise} \\ &= (x-1)(x^2+4x+5) \\ &= (x-1)(x^2+4x+4+1) \\ &= (x-1)(x+2)^2 \\ &= (x-1)(x+2)(x+2) \end{aligned}$$

$\therefore$  ONLY solution  $x=1$   
(It touches the  $x$ -axis at  $(1,0)$  only)

(c) Normal is  $2y = x - 7$

- when  $x=0, 2y = -7, y = -\frac{7}{2}$   
say  $P(0, -\frac{7}{2})$
- when  $y=0, x = 7, x = 7$   
say  $Q(7, 0)$

$$\begin{aligned} \text{Area } &= \frac{1}{2}|xy|/|o| \\ &= \frac{1}{2} \times \frac{7}{2} \times 7 \\ &= \frac{49}{4} \\ &= 12\frac{1}{4} \quad \text{Hence required} \end{aligned}$$

(b)  $y = x^3+4x^2+5x-5$

$$\begin{aligned} &\text{Find } \frac{dy}{dx} \\ &= 3x^2+8x+5 \\ &\text{when } x=-1 \\ &y = -1+3-1-5 = -4 \\ &\therefore A(-1, -4) \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= 3x^2+8x+1 \\ \frac{dy}{dx} &= 3-6+1 = -2 \\ \therefore \text{NORMAL SLOPES IS } &\frac{1}{2} \\ \therefore \text{NORMAL EQUATION IS } &2y = x + 1 \\ \therefore y - x = -x(2-x) & \\ y + x = x^2 + 1 & \\ 2y + 2x = x^2 + 2x + 1 & \\ 2y = x^2 + 1 & \\ 2y = x^2 + 1 & \end{aligned}$$

**Question 28**

A curve has equation

$$y = x - 8\sqrt{x}, \quad x \in \mathbb{R}, \quad x \geq 0.$$

The curve meets the coordinate axes at the origin and at the point  $P$ .

- a) Determine the coordinates of  $P$ .

The point  $Q$ , where  $x=4$ , lies on the curve.

- b) Find an equation of the normal to curve at  $Q$ .

- c) Show clearly that the normal to the curve at  $Q$  does not meet the curve again.

$$\boxed{P(64,0)}, \quad \boxed{y = x - 16}$$

(a) Given  $y = x - 8\sqrt{x}$ . When  $y=0$ ,  $0 = x - 8\sqrt{x}$ .  
 $8\sqrt{x} = x$   
 $8\sqrt{x}^2 = x^2$   
 $64 = x^2$   
 $x = \pm 8$   
 $\therefore x = 8$  (out of the required solution)  
 $\therefore P(64,0)$

(b) When  $x=4$ ,  $y = 4 - 8\sqrt{4} = 4 - 16 = -12$ .  $P(4,-12)$   
 $y = x - 8\sqrt{x}$   
 $\frac{dy}{dx} = 1 - 4\sqrt{\frac{1}{x}}$   
 $\frac{dy}{dx} = 1 - \frac{4}{\sqrt{x}}$   
 $\left.\frac{dy}{dx}\right|_{x=4} = 1 - \frac{4}{\sqrt{4}} = 1$   
 $\therefore$  NORMAL QUAINT IS 1  
 $\therefore$  EQUATION OF NORMAL  
 $y - y_1 = m(x - x_1)$   
 $y + 12 = 1(x - 4)$   
 $y = x - 16$

(c)  $y = x - 16$   
 $y = x - 8\sqrt{x}$  } SOLV-SIMULTANOUSLY  
 $\Rightarrow x - 16 = x - 8\sqrt{x}$   
 $\Rightarrow 8\sqrt{x} = 16$   
 $\Rightarrow \sqrt{x} = 2$   
 $\Rightarrow x = 4 \quad \therefore$  Point P  
 $\therefore$  NORMAL DOES NOT MEET THE CURVE AGAIN

**Question 29**

The curve  $C$  has equation

$$y = x^3 - 9x^2 + 24x - 19, \quad x \in \mathbb{R}.$$

- Show that the tangent to  $C$  at the point  $P$ , where  $x = 1$ , has gradient 9.
- Find the coordinates of another point  $Q$  on  $C$  at which the tangent also has gradient 9.

The normal to  $C$  at  $Q$  meets the coordinate axes at the points  $A$  and  $B$ .

- Show further that the **approximate** area of the triangle  $OAB$ , where  $O$  is the origin, is 11 square units.

Q(5,1)

**(a)**

$$y = x^3 - 9x^2 + 24x - 19$$

$$\frac{dy}{dx} = 3x^2 - 18x + 24$$

$$\left. \frac{dy}{dx} \right|_{x=1} = 3 - 18 + 24 = 9$$

At  $x=1$ ,  $y=1$

**(b)**

$$\frac{dy}{dx} = 3x^2 - 18x + 24$$

$$9 = 3x^2 - 18x + 24$$

$$0 = 3x^2 - 18x + 15$$

$$0 = 3(x^2 - 6x + 5)$$

$$(x-1)(x-5) = 0$$

$$x = 1 \quad \leftarrow \begin{matrix} 1 \\ 5 \end{matrix} \quad \leftarrow \begin{matrix} 1 \\ 5 \end{matrix}$$

$$\therefore y = x^3 - 9x^2 + 24x - 19$$

$$y = 15 - 225 + 120 - 19$$

$$y = 1$$

$\therefore Q(5,1)$

**(c)**

- GRADIENT AT  $Q$  IS 9
- NORMAL GRADIENT IS  $-\frac{1}{9}$
- EQUATION OF NORMAL
$$y - y_1 = m(x - x_1)$$

$$y - 1 = -\frac{1}{9}(x - 5)$$

$$9y - 9 = -x + 5$$

$$9y + x = 14$$

When  $x=0, 9y=14$   
 $y = \frac{14}{9}$   
 $(0, \frac{14}{9}) \leftarrow \text{DAY A}$

When  $y=0, x=14$   
 $x = 14$   
 $(14, 0) \leftarrow \text{DAY B}$

$$\text{Area} = \frac{1}{2} |OA||OB|$$

$$= \frac{1}{2} \times \frac{14}{9} \times 14$$

$$= \frac{98}{9} \approx 11$$

Required

**Question 30**

The point  $A(2,1)$  lies on the curve with equation

$$y = \frac{(x-1)(x+2)}{2x}, \quad x \in \mathbb{R}, \quad x \neq 0.$$

- a) Find the gradient of the curve at  $A$ .
- b) Show that the tangent to the curve at  $A$  has equation

$$3x - 4y - 2 = 0.$$

The tangent to the curve at the point  $B$  is parallel to the tangent to the curve at  $A$ .

- c) Determine the coordinates of  $B$ .

gradient at $A = \frac{3}{4}$	$B(-2,0)$
-------------------------------	-----------

(a)  $y = \frac{(x-1)(x+2)}{2x} = \frac{x^2+x-2}{2x} = \frac{x^2}{2x} + \frac{x}{2x} - \frac{2}{2x} = \frac{1}{2}x + \frac{1}{2} - \frac{1}{x}$

$$\therefore y = \frac{1}{2}x + \frac{1}{2} - \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{2} + \frac{1}{x^2}$$

$$\left. \frac{dy}{dx} \right|_{x=2} = -\frac{1}{2} + \frac{1}{2^2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

(b)  $A(2,1)$  RESTRICTION  $= \frac{3}{4} \Rightarrow y - y_0 = m(x - x_0)$

$$y - 1 = \frac{3}{4}(x - 2)$$

$$4y - 4 = 3x - 6$$

$$0 = 3x - 4y - 2$$

*As required*

(c) PARALLEL TANGENTS  $\Rightarrow$  SAME GRADIENT OF  $\frac{3}{4}$

$$\frac{dy}{dx} = \frac{1}{2} + \frac{1}{x^2}$$

$$\Rightarrow \frac{3}{4} = \frac{1}{2} + \frac{1}{x^2}$$

$$\Rightarrow \frac{1}{4} = \frac{1}{x^2}$$

$$\Rightarrow 4 = x^2$$

$$\Rightarrow x = \sqrt{2} \quad \sqrt{-4} \quad \sqrt{4}$$

$$\Rightarrow x = \sqrt{2} \quad \sqrt{-4} \quad \sqrt{3}$$

WORKING  $y = \frac{(x-1)(x+2)}{2x}$  when  $x = -2$

$$y = \frac{(-2-1)(-2+2)}{2(-2)} = 0$$

$$\therefore B(-2,0)$$

**Question 31**

The curve  $C$  has equation  $y = f(x)$  given by

$$f(x) = 2(x-2)^3, \quad x \in \mathbb{R}.$$

- a) Sketch the graph of  $f(x)$ .
- b) Find an expression for  $f'(x)$ .

The point  $P(3, 2)$  lies on  $C$  and the straight line  $l_1$  is the tangent to  $C$  at  $P$ .

- c) Find an equation of  $l_1$ .

The straight line  $l_2$  is another tangent at a different point  $Q$  on  $C$ .

- d) Given that  $l_1$  is parallel to  $l_2$  show that an equation of  $l_2$  is

$$y = 6x - 8.$$

$$f'(x) = 6x^2 - 24x + 24, \quad y = 6x - 16$$

(a)

(b)

$$\begin{aligned} f(2) &= 2(2-2)^3 \\ f'(2) &= 2(2-2)(2-2)^2 \\ f'(2) &= 2(2-2)(2^2-4) \\ f'(2) &= 2(2-4)(2^2-4) \\ f'(2) &= 2^3 \cdot 2^2 \cdot (-4) \\ f'(2) &= 2^5 \cdot 2^2 \cdot (-4) \\ f'(2) &= 2^7 \cdot 2^2 \cdot (-4) \\ f'(2) &= 2^{10} \cdot 2^2 \cdot (-4) \\ f'(2) &= 2^{12} \cdot 2^2 \cdot (-4) \\ f'(2) &= 2^{14} \end{aligned}$$

(c)

NEED THE GRADIENT AT  $Q(x_1, y_1)$

$$\begin{aligned} f'(x_1) &= 6x_1^2 - 24x_1 + 24 \\ f'(x_1) &= 54 - 72 + 24 \\ f'(x_1) &= 6 \end{aligned}$$

Thus  $y - y_1 = m(x - x_1)$

$$\begin{aligned} y - 2 &= 6(x - 3) \\ y - 2 &= 6x - 18 \\ y &= 6x - 16 \end{aligned}$$

(d)

- PARALLEL LINES  $\Rightarrow$  SAME GRADIENT  $\Rightarrow l_2$  HAS GRADIENT 6
- NEED ANOTHER POINT ON  $C$  WITH GRADIENT 6

$$\begin{aligned} f'(x) &= 6 \\ \Rightarrow 6x^2 - 24x + 24 &= 6 \\ \Rightarrow 6x^2 - 24x + 18 &= 0 \\ \Rightarrow x^2 - 4x + 3 &= 0 \\ \Rightarrow (x-1)(x-3) &= 0 \\ x &= 1 \leftarrow Q \quad 3 \rightarrow P \end{aligned}$$

• WHEN  $x=1$   $y = 2(-1-2)^3 = -2$

$$\therefore Q(1, -2)$$

Hence

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y + 2 &= 6(x - 1) \\ y + 2 &= 6x - 6 \\ y &= 6x - 8 \end{aligned}$$

As Required

**Question 32**

The point  $P(2,9)$  lies on the curve  $C$  with equation

$$y = x^3 - 3x^2 + 2x + 9, \quad x \in \mathbb{R}, \quad x \geq 1.$$

- a) Find an equation of the tangent to  $C$  at  $P$ , giving the answer in the form  $y = mx + c$ , where  $m$  and  $c$  are constants.

The point  $Q$  also lies on  $C$  so that the tangent to  $C$  at  $Q$  is perpendicular to the tangent to  $C$  at  $P$ .

- b) Show that the  $x$  coordinate of  $Q$  is

$$\frac{6 + \sqrt{6}}{6}.$$

$$y = 2x + 5$$

(a)  $y = x^3 - 3x^2 + 2x + 9$

$$\begin{aligned} \frac{dy}{dx} &= 3x^2 - 6x + 2 \\ \left. \frac{dy}{dx} \right|_{x=2} &= 3(2)^2 - 6(2) + 2 \\ &= 12 - 12 + 2 = 2 \end{aligned}$$

So  $\frac{dy}{dx}|_{x=2} = m$  (gradient of  $C$  at  $P$ )

$$\begin{aligned} \Rightarrow y - 9 &= 2(x - 2) \\ \Rightarrow y - 9 &= 2x - 4 \\ \Rightarrow y &= 2x + 5 \end{aligned}$$

(b) 

GRADIENT OF  $l_1$  IS 2. (GIVEN)  
∴ GRADIENT OF  $l_2$  MUST BE  $-\frac{1}{2}$

$$\begin{aligned} \therefore \frac{dy}{dx} &= -\frac{1}{2} \\ \Rightarrow 3x^2 - 6x + 2 &= -\frac{1}{2} \\ \Rightarrow 6x^2 - 12x + 4 &= -1 \\ \Rightarrow 6x^2 - 12x + 5 &= 0 \end{aligned}$$

BY FORMULAR METHOD OR  
BY SOLVING THE QUADRATIC EQUATION

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{(12 \pm \sqrt{144 - 4 \times 6 \times 5})}{12}$$

$$x = \frac{(12 \pm \sqrt{24})}{12}$$

$$x = \frac{6 \pm \sqrt{6^2}}{6}$$

$$x = \frac{6 \pm \sqrt{36}}{6} \quad (x > 1) \quad \checkmark$$

**Question 33**

The volume,  $V \text{ cm}^3$ , of a soap bubble is modelled by the formula

$$V = (p - qt)^2, t \geq 0,$$

where  $p$  and  $q$  are positive constants, and  $t$  is the time in seconds, measured after a certain instant.

When  $t = 1$  the volume of a soap bubble is  $9 \text{ cm}^3$  and at that instant its volume is decreasing at the rate of  $6 \text{ cm}^3$  per second.

Determine the value of  $p$  and the value of  $q$ .

$$p = 4, q = 1$$

Working for Question 33:

$V = (p - qt)^2$        $\Rightarrow V = p^2 - 2pqt + q^2t^2$   
 $t=1, V=9$        $\frac{dV}{dt} = -2pq + 2q^2t$   
 $q = (p - qt)^2$        $-6 = -2pq + 2q^2 \times 1$   
 $q = (p - q)^2$        $2pq - 6 = 2q^2$   
 $p - q = \boxed{< -3}$

•  $p - q = 3$       •  $p - q = -3$   
 $p = q+3$        $p = q-3$   
 $\downarrow$        $\downarrow$   
 $2q(q+3) - 6 = 2q^2$        $2q(q-3) - 6 = 2q^2$   
 $2q^2 + 6q - 6 = 2q^2$        $2q^2 - 6q - 6 = 2q^2$   
 $6q = 6$        $-6q = 6$   
 $q = 1$        $q > 0$   
 $p = 4$        $\boxed{q > 0}$

### **Question 34**

A curve  $C$  has equation

$$y = 2x^3 - 5x^2 + a, \quad x \in \mathbb{R}$$

where  $a$  is a constant.

The tangent to  $C$  at the point where  $x = 2$  and the normal to  $C$  at the point where  $x = 1$ , meet at the point  $O$ .

Given that  $O$  lies on the  $x$  axis, determine in any order ..

- a) ... the value of  $a$ .
  - b) ... the coordinates of  $O$

$$a = \frac{8}{3}, Q\left(\frac{7}{3}, 0\right)$$

$y = 2x^3 - 5x^2 + a$   
 •  $\frac{dy}{dx} = 6x^2 - 10x$   
 •  $\frac{dy}{dx} \Big|_{x=2} = 6(2)^2 - 10(2) = 4$   
 •  $\frac{dy}{dx} \Big|_{x=1} = 6(1)^2 - 10(1) = -4$   
 • NORMAL:  $y - y_1 = m(x - x_1)$   $\frac{1}{4}$   
 •  $y - 1 = \frac{1}{4}(x - 2)$   
 •  $y = \frac{1}{4}x + \frac{3}{2}$   
 •  $y - 4 = -4(x - 1)$   
 •  $y = -4x + 8$   
 •  $\text{INTERSECTION POINT}$   
 $\frac{1}{4}x + \frac{3}{2} = -4x + 8$   
 $\frac{1}{4}x + 4x = 8 - \frac{3}{2}$   
 $\frac{17}{4}x = \frac{13}{2}$   
 $x = \frac{13}{2} \times \frac{4}{17}$   
 $x = \frac{26}{17}$   
 $\text{SUBSTITUTE } x = \frac{26}{17} \text{ INTO } y = \frac{1}{4}x + \frac{3}{2}$   
 $y = \frac{1}{4} \times \frac{26}{17} + \frac{3}{2}$   
 $y = \frac{13}{17} + \frac{3}{2}$   
 $y = \frac{13}{17} + \frac{51}{34}$   
 $y = \frac{64}{34}$   
 $y = \frac{32}{17}$   
 $\text{MIDPOINT: } \left( \frac{26}{17}, \frac{32}{17} \right)$

**Question 35**

The curve  $C$  has equation

$$y = \frac{x^3(5x\sqrt{x} - 128)}{\sqrt{x}}, \quad x \in \mathbb{R}, \quad x > 0.$$

- a) Determine expressions for  $\frac{dy}{dx}$ ,  $\frac{d^2y}{dx^2}$  and  $\frac{d^3y}{dx^3}$ .
- b) Show that the  $y$  coordinate of the stationary point of  $C$  is  $-k\sqrt[3]{4}$ , where  $k$  is a positive integer.
- c) Evaluate  $\frac{d^2y}{dx^2}$  at the stationary point of  $C$ .

Give the answer in terms of  $\sqrt[3]{2}$ .

- d) Find the value of  $\frac{d^3y}{dx^3}$  at the point on  $C$ , where  $\frac{d^2y}{dx^2} = 0$ .

	$\frac{dy}{dx} = 20x^3 - 320x^{\frac{3}{2}}$	$\frac{d^2y}{dx^2} = 60x^2 - 480x^{\frac{1}{2}}$	$\frac{d^3y}{dx^3} = 120x - 240x^{-\frac{1}{2}}$
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$k = 3072$	$960\sqrt[3]{2}$	$360$
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a) START BY REWRITING THE EQUATION IN STANDARD FORM, THEN DIFFERENTIATE

$$y = \frac{x^3(5x\sqrt{x} - 128)}{\sqrt{x}} = \frac{5x^{\frac{7}{2}} - 128x^3}{\sqrt{x}} = \frac{5x^{\frac{5}{2}}}{\sqrt{x}} - \frac{128x^3}{\sqrt{x}}$$

- $y = 5x^{\frac{5}{2}} - 128x^{\frac{3}{2}}$
- $\frac{dy}{dx} = 25x^{\frac{3}{2}} - 384x^{\frac{1}{2}}$
- $\frac{d^2y}{dx^2} = 60x^{\frac{1}{2}} - 480x^{-\frac{1}{2}}$
- $\frac{d^3y}{dx^3} = 120x - 240x^{-\frac{3}{2}}$

b) FIND THE STATIONARY POINTS  $\frac{dy}{dx} = 0$

$$\begin{aligned} \Rightarrow 25x^{\frac{3}{2}} - 384x^{\frac{1}{2}} &= 0 \\ \Rightarrow x^{\frac{3}{2}} - 16x^{\frac{1}{2}} &= 0 \\ \Rightarrow x^{\frac{1}{2}}(x^{\frac{1}{2}} - 16) &= 0 \quad (\text{if } x \neq 0) \\ \Rightarrow x^{\frac{1}{2}} &= 16 \\ \Rightarrow x^{\frac{1}{2}} &= 16 \\ \Rightarrow (x^{\frac{1}{2}})^2 &= 16^2 \\ \Rightarrow x^1 &= 256 \\ \Rightarrow x &= 256 \end{aligned}$$

SUBSTITUTE INTO EQUATION TO FIND  $y$

$$\begin{aligned} \Rightarrow y &= 5x^{\frac{5}{2}} - 128x^{\frac{3}{2}} \\ \Rightarrow y &= 5(256)^{\frac{5}{2}} - 128(256)^{\frac{3}{2}} \\ \Rightarrow y &= 5(2^{10})^{\frac{5}{2}} - 128(2^{12})^{\frac{3}{2}} \\ \Rightarrow y &= 2^{25} - 128 \times 2^{18} \\ \Rightarrow y &= 2^{24}(-48) \\ \Rightarrow y &= -48 \times 2^{24} \\ \Rightarrow y &= -3072 \times 2^{\frac{1}{2}} \end{aligned}$$

c)  $\frac{d^2y}{dx^2} = 60x^{\frac{1}{2}} - 480x^{-\frac{1}{2}}$

$$\begin{aligned} \Rightarrow \frac{d^2y}{dx^2} &= 60(x^{\frac{1}{2}})^2 - 480(x^{-\frac{1}{2}})^2 \\ \Rightarrow \left. \frac{d^2y}{dx^2} \right|_{x=64} &= 60(2^{\frac{1}{2}})^2(64) - 480(2^{-\frac{1}{2}})^2(64) = 60 \times 2^{\frac{1}{2}} \times 64 - 480 \times 2^{\frac{1}{2}} \times 64 \\ \Rightarrow \left. \frac{d^2y}{dx^2} \right|_{x=64} &= 960\sqrt{2} \end{aligned}$$

d) FINALLY  $\frac{d^3y}{dx^3} = 0$

$$\begin{aligned} \Rightarrow 120x - 240x^{-\frac{3}{2}} &= 0 \\ \Rightarrow 2^{\frac{1}{2}} - 2x^{\frac{1}{2}} &= 0 \\ \Rightarrow x^{\frac{1}{2}} &= 2x^{\frac{1}{2}} \quad (\text{if } x \neq 0) \\ \Rightarrow \frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}}} &= 1 \\ \Rightarrow x^{\frac{1}{2}} &= 2^{\frac{1}{2}} \\ \Rightarrow x &= 2^1 \\ \Rightarrow (x^{\frac{1}{2}})^2 &= 2^{\frac{1}{2}} \\ \Rightarrow x^1 &= (\sqrt{2})^2 \\ \Rightarrow x &= 2 \end{aligned}$$

FINALLY

$$\begin{aligned} \Rightarrow \frac{d^3y}{dx^3} &= 120x - 240x^{-\frac{3}{2}} \\ \Rightarrow \left. \frac{d^3y}{dx^3} \right|_{x=2} &= 120 \times 2 - 240 \times 2^{-\frac{3}{2}} \\ &= 240 - 240 \times \frac{1}{2} \\ &= 240 - 120 = 120 \end{aligned}$$