PHY1111: Mathematical Methods in Physics I
Lecture Notes

Taddeo SSENYONGA (TJS)
email: taddeo.ssenyonga@mak.ac.ug

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Detailed Course Description

Course Name:	:	Mathematical Methods in Physics I
Course Code	:	PHY1111
Credit Units	:	3(3-0)

Course Description:

- This is an introductory course intended to give the student necessary mathematical tools for understanding the courses in Classical Mechanics, Electricity and Magnetism, Properties of Matter, and Heat and Thermodynamics.
- After a brief review of complex numbers, the student will be introduced to methods of solving linear equations, partial derivatives and their applications.
- Vector analysis will be introduced with special emphasis on applications. Line integrals, Green's theorem in a plane, the divergence theorem, Gauss' law and Stokes' theorem will be studied.

Detailed Course Description ctned and Objectives

- Vector analysis will be introduced with special emphasis on applications. Line integrals, Green's theorem in a plane, the divergence theorem, Gauss' law and Stokes' theorem will be studied.

Objectives

By the end of this course, students should be able to:-

- Use complex numbers to solve problems.
- Solve simple linear equations.
- Find partial derivatives.
- Apply vector algebra in physics problems.
- Solve line integrals

Course Outline

- **Complex Numbers:** the complex plane, polar form of a complex number, principal angle, complex conjugate; addition, subtraction, multiplication and division of complex numbers; powers and roots of complex numbers $(z^n, z^{1/n})$ and applications: complex amplitudes in theory of electric circuits, combination of light waves in optics, etc.
- **Linear Equations**: Writing of linear equations in matrix form; solving of linear equations using Crammer's rule; Example: circuit analysis.
- **Derivatives:** partial derivatives; definition, notation and examples; total differential of functions of several variables. Examples: Heat and wave equations in one dimension.
- **Vectors:** Vector components in a Cartesian coordinate system; the zero and unit vectors; addition and subtraction of vectors; multiplication of vectors; scalar product; vector product. Applications of vector multiplication – work, torque, angular velocity, triple scalar product, triple vector product. Differentiation of vectors – displacement, velocity, acceleration. Directional derivative, Gradient of a function, the divergence, the curl. Vector Identities.

Course Outline ctned

- **Differential Equations**: Linear; second order; homogeneous; and partial differential equations.
- **Line Integrals**: Work-done; conservative fields; potentials; exact differential; and integration by parts.
- **Theorems:** Green's theorem in a plane; divergence theorem; Gauss' law; and Stokes' theorem.
- **Curve Fitting:** Least squares lines; least squares polynomials; and nonlinear curve fitting.
- **Numerical Integration:** Trapezoidal rule for numerical integration; and Simpson's rule for numerical Integration.
- **Tutorials:** Several tutorials will be carried out

Mode of Delivery and Reference Materials

Mode of delivery The course will be conducted through lectures and tutorials. **References:**

1. Boas, M.L.: Mathematical Methods in the Physical Sciences. John Wiley and Sons (2006), Third edition. (Textbook)
2. Spiegel, M.R.: Schaum's Outline of Complex Variables. McGraw-Hill, (1972).
3. Spiegel, M.R.: Schaum's Vector Analysis. McGraw-Hill, (1967)
4. Internet

Differential Equations

Introduction: A great many applied problems involve rates, that is, derivatives. An equation containing derivatives is called a **differential equation**.
If it contains partial derivatives, it is called **a partial differential equation**; **otherwise it is called an ordinary differential equation**.
Differential equations are important in Physics e.g,

$$\vec{F} = m\vec{a} \qquad \text{Newton's second law in vector form}$$

$$\vec{F} = m\frac{d\vec{v}}{dt} = m\frac{d^2\vec{r}}{dt^2} \qquad \text{are differential equations}$$

The rate of heat escape

$$\frac{dQ}{dt} = kA\left(\frac{dT}{dx}\right),$$

where $\frac{dT}{dx}$ is temperature gradient and k is thermal conductivity and it depends on the material.

Differential Equations

Electronics

If $I(t)$ is the current flowing through the circuit at time t and $q(t)$ is the charge on the capacitor plates, then

$$I(t) = \frac{dq}{dt}, \text{ and}$$

$$L\frac{dI}{dt} + IR + \frac{q}{C} = V$$

Further differentiation with respect to t and putting $\frac{dq}{dt} = I$,

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}\frac{dq}{dt} = \frac{dV}{dt}$$

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dV}{dt}$$

is a differential equation in I

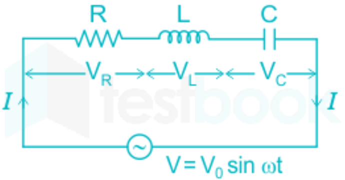


Figure 1:

Differential Equations

The **order of the differential equation** is the **order of the highest derivative**, e.g

$$\frac{dy}{dx} + xy^2 = 1 \qquad \text{or} \qquad y' + xy^2 = 1$$
$$x\frac{dy}{dx} + y = e^x \qquad \text{or} \qquad xy' + y = e^x$$
$$\frac{dv}{dt} = -g \qquad \text{or} \qquad v' = -g$$

e.t.c are first order differential equations, and

$$m\frac{d^2r}{dt^2} = -kr \qquad \text{is second order differential equation}$$

$$a_0y + a_1y' + a_2y'' + a_3y''' + ... = b,$$

is a linear differential equation, where a_i are constants or functions of x.

Non linear differential equation

- Equations below are non linear differential equations.

$$\begin{aligned} y' &= \cot y \\ yy' &= 1 \\ y'^2 &= xy, \end{aligned}$$

  A solution of a differential equation (in variables x and y) is a relation between x and y which, if substituted into the differential equation, gives an identity'

- **Example:** The relation $y = \sin x + C$ is a solution of the differential equation $y' = \cos x$, because a substitution gives

$$y = \sin x, \quad \Rightarrow \quad y' = \cos x$$

  Example: The differential equation $y'' = y$ has solutions $y = e^x$ and $y = e^{-x}$ or $y = Ae^x + Be^{-x}$ as can be verified by substitution.

Non linear differential equation

- **Example:** Find the distance which an object falls under gravity in t seconds if it starts from rest.

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = g$$

$$\text{on integrating} \quad \frac{dr}{dt} = gt + \text{constant} = gt + v_0$$

$$r(t) = \frac{1}{2}gt^2 + v_0 t + r_0$$

From rest $t = 0$, and free fall $v_0 = 0$ ms^{-1} $\Rightarrow r(t = 0) = r_0 = 0$ $\Rightarrow r(t) = \frac{1}{2}gt^2$

Separable Equations

- **Example:** The rate at which a radioactive substance decays is proportional to the remaining number of atoms. If there are N_0 atoms at time $t = 0$, find the number of atoms at any time t.
- **Solution:**

$$\begin{aligned} \frac{dN}{dt} &= -\lambda N, \qquad \text{where } \lambda \text{ is a decay constant} \\ \int \frac{dN}{N} &= \int -\lambda dt \\ \ln N &= -\lambda t + \text{constant} \\ N &= N_0 e^{-\lambda t}, \end{aligned}$$

where $N = N_0$ at $t = 0$

Separable Equations

- **Example:** Solve the differential equation

$$xy' = y + 1$$

- **Solution:**

$$\begin{aligned} \frac{y'}{y+1} &= \frac{1}{x} \\ \text{Or} \quad \frac{dy}{(y+1)} &= \frac{dx}{x} \\ \ln(y+1) &= \ln x + \text{constant} \\ &= \ln x + \ln a \\ &= \ln(ax) \\ \text{Or} \quad y+1 &= ax \end{aligned}$$

First-order Linear Equations

- Written in the form

$$y' + Py = Q, \tag{1}$$

where P and Q are functions of x.

Example: Decay equation

$$\frac{dN}{dt} = -\lambda N \qquad \Rightarrow \qquad N' + \lambda N = 0$$

To solve Eqn. (1), we set $Q = 0$

$$y' + Py = 0 \quad \text{Or} \quad \frac{dy}{dx} = -Py$$

Separating the variables, $\displaystyle \frac{dy}{y} = -Pdx$

$$\ln y = -\int Pdx + \text{constant}$$

First-order Linear Equations

- $$y = Ae^{-\int P dx},$$
- where $A = e^{\text{constant}}$. To simplify the notation for future use, put
$$I = \int P dx$$
Then
$$\frac{dI}{dx} = P$$
We can write,
$$y = Ae^{-I} \tag{2}$$
$$\text{Or} \quad ye^{I} = A \tag{3}$$

First-order Linear Equations

We can start from Eqn. (3) and try to get Eqn. (1)

$$\begin{aligned}\frac{d}{dx}\left(ye^I\right) &= y'e^I + ye^I\frac{dI}{dx}\\ &= y'e^I + ye^I P\\ &= e^I(y' + Py),\end{aligned}$$

which is the L.H.S of Eqn. (1) $\times e^I$. Thus we can write Eqn. (1) $\times e^I$

$$\frac{d}{dx}(ye^I) = e^I(y' + Py) = Qe^I$$

Since Q and e^I are functions of x only, we can now integrate both sides with respect to x;

$$\begin{aligned}ye^I &= \int Qe^I dx + \text{constant}\\ y &= e^{-I}\int Qe^I dx + Ce^{-I}\end{aligned}$$

$$\text{where} \qquad I = \int P dx$$

Example: Solve

$$x^2 y' - 2xy = \frac{1}{x}$$

Solution: Writing it in the form $y' + Py = Q$

$$y' - \frac{2}{x} y = \frac{1}{x^3}$$

$\Rightarrow \quad P = \frac{-2}{x}$ and $\quad Q = \frac{1}{x^3}$

$$I = \int P dx = \int \frac{-2}{x} dx = -2 \ln x$$

$$\text{Then} \quad e^I = e^{-2\ln x} = e^{\ln x^{-2}} = \frac{1}{x^2}$$

First-order Linear Equations

$$
\begin{aligned}
ye^{I} = y\left(\frac{1}{x^2}\right) &= \int Qe^{I}dx = \int \frac{1}{x^3}\cdot\frac{1}{x^2}dx = \int x^{-5}dx \\
&= \frac{x^{-4}}{-4} + \text{constant} \\
\Rightarrow \quad y &= \left(\frac{-1}{4x^4} + C\right)x^2 \\
y &= \frac{-1}{4x^2} + Cx^2
\end{aligned}
$$

First-order Linear Equations

Example:

$$
\begin{aligned}
N' + \lambda N &= 0 \\
\Rightarrow \quad P = \lambda, \quad Q &= 0 \\
\Rightarrow \quad I = \int P dt = \int \lambda dt &= \lambda t \\
e^{I} &= e^{\lambda t} \\
N e^{I} &= N e^{\lambda t} = \int Q e^{I} dt + C \\
&= \int 0 + C(N_0) \\
N e^{\lambda t} &= N_0 \\
N(t) &= N_0 e^{-\lambda t}
\end{aligned}
$$

Second-Order Linear Equations with Constant Coefficients

$$a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx}a_0y = 0,$$

where a_0, a_1 and a_2, are constants. The equation is homogeneous because every term contains y or a derivative of y.

Example: Solve the differential equation

$$y'' + 5y' + 4y = 0 \tag{4}$$

Solution: Let

$$Dy = \frac{dy}{dx} = y', \quad D^2y = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y''$$

Then we can write

$$\begin{aligned} D^2y + 5Dy + 4y &= 0 \\ \left(D^2 + 5D + 4\right)y &= 0 \\ (D+1)(D+4)y &= 0 \end{aligned}$$

Note: To find the roots you may use

$$\frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}.$$

Second-Order Linear Equations with Constant Coefficients

We now solve the simpler equations

$$(D+4)\,y = 0 \qquad (D+1)\,y = 0$$

$$\frac{dy}{dx} + 4y = 0 \qquad \frac{dy}{dx} + y = 0$$

$$\int \frac{dy}{y} = \int -4dx \qquad \int \frac{dy}{y} = \int -dx$$

$$y = c_1 e^{-4x} \qquad y = c_2 e^{-x}$$

Now if $(D+4)y = 0$, then

$$(D+1)(D+4)y = (D+1)\cdot 0 = 0$$

Therefore, any solution of $(D+4)y$ is a solution of the differential equation in Eqn. (4).

Similarly, any solution of $(D+1)y = 0$ is a solution of Eqn. (4). Since the two solutions above are linearly independent , a linear combination of them contains two arbitrary constants and so is the general solution. Then

$$y = c_1 e^{-4x} + c_2 e^{-x},$$

is the general solution of Eqn. (4).

Note: For a differential equation $(D-a)(D-b)y = 0$, $a \neq b$, the general solution is $y = c_1 e^{ax} + c_2 e^{bx}$.

Second-Order Linear Equations with Constant Coefficients

Example: Find the general solution of

$$(D^2 + 1)(D^2 - 1)y = 0$$

Solution:

$$\begin{aligned}(D^2 + 1)(D^2 - 1)y &= 0\\ (D + i)(D - i)(D + 1)(D - 1)y &= 0\end{aligned}$$

$$y = A_1 e^{-ix} + A_2 e^{ix} + A_3 e^{-x} + A_4 e^{x}$$

Equal Roots of the Auxiliary Equation

If the two roots of the auxiliary equation are equal, the differential equation can be written as

$$(D-a)(D-a)y = 0 \tag{5}$$

From the previous discussion,

$$y_1 = c_1 e^{ax} \qquad \text{and} \qquad y_2 = c_2 e^{ax}$$

here $a = b$

$$\Rightarrow \quad y_1 = y_2 = y = ce^{ax}$$

To find the second solution for this case, we let

$$u = (D-a)y \tag{6}$$

The Eqn. (5) becomes

$$(D-a)u = 0$$

From which

$$\left(\frac{d}{dx} - a\right)u = 0$$

$$\frac{du}{dx} - au = 0$$

or

Equal Roots of the Auxiliary Equation

or

$$\frac{du}{u} = adx$$

$$\ln u = ax + d$$

$$u = Ae^{ax} \qquad (7)$$

put Eqn. (7) into (6)

$$(D - a)y = Ae^{ax}$$

or

$$\frac{dy}{dx} - ay = Ae^{ax}$$

This is a first order linear equation

Here

$$I = \int -adx = -ax$$

$$e^{I} = e^{-ax}$$

then

$$ye^{I} = ye^{-ax} = \int e^{-ax} \cdot Ae^{ax} dx = \int Adx = Ax + B$$

$$\Rightarrow \qquad y = (Ax + B)e^{ax}$$

Example: Solve the differential equation

$$y'' - 6y' + 9y = 0$$

Solution: We can write the Eqn as

$$\begin{aligned}(D^2 - 6D + 9)y &= 0 \\ (D-3)(D-3)y &= 0\end{aligned}$$

Since the roots are equal, then the solution is

$$y = (Ax + B)e^{3x}$$

Second-Order Linear Equations with Constant coefficients and Right-hand side not Zero

$$a_2\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_0y=f(x), \tag{8}$$

or

$$\frac{d^2y}{dx^2}+\frac{a_1}{a_2}\frac{dy}{dx}+\frac{a_0}{a_2}y=F(x),$$

Example: Consider a differential equation

$$(D^2+5D+4)y=\cos 2x \tag{9}$$

Solution: The solution with R.H.S equal to zero

$$\begin{aligned}(D^2+5D+4)y &= 0\\(D+1)(D+4)y &= 0\end{aligned}$$

Solution of a complementary eqn is

Second-Order Linear Equations with Constant coefficients and Right-hand side not Zero

Solution of a complementary eqn is

$$y_c = Ae^{-x} + Be^{-4x} \tag{11}$$

Suppose we know any solution of Eqn. (9), we call that solution a particular solution and denote it by y_p.
It can be shown that

$$y_p = \frac{1}{10}\sin 2x$$

is a particular solution to Eqn. (9). Then

$$(D^2 + 5D + 4)y_p = \cos 2x \tag{12}$$

And

$$(D^2 + 5D + 4)y_c = 0 \tag{13}$$

Adding Eqn. (12) and (13)

$$(D^2 + 5D + 4)(y_p + y_c) = \cos 2x$$

Thus

$$y = y_p + y_c = Ae^{-x} + Be^{-4x} + \frac{1}{10}\sin 2x$$

The general solution of an equation of the form Eqn. (8) is

$$y = y_c + y_p \tag{14}$$

where the complementary function y_c is the general solution of the homogeneous equation and y_p is a particular solution of Eqn. (8).

Successive Integration of two first-order Equations

Example: Solve

$$y'' + y' - 2y = e^x$$

Solution:

$$\begin{aligned}(D^2 + D - 2)y &= e^x \\ (D+2)(D-1)y &= e^x\end{aligned}$$

Now let

$$u = (D+2)y \tag{15}$$

Then we have

$$\begin{aligned}(D-1)u &= e^x \\ Du - u &= e^x \\ u' - u &= e^x,\end{aligned}$$

which is a first-order linear differential equation. From which

$$
\begin{aligned}
I &= \int P dx = -\int dx = -x \\
ue^{-x} &= \int e^{-x} \cdot e^{x} dx = x + c_1 \\
u &= xe^{x} + c_1 e^{x}
\end{aligned}
$$

Then Eqn. (15) becomes

$$
\begin{aligned}
(D+2)y &= xe^{x} + c_1 e^{x} \\
Dy + 2y &= xe^{x} + c_1 e^{x} \\
y' + 2y &= xe^{x} + c_1 e^{x},
\end{aligned}
$$

which is a first-order differential equation

$$
\begin{aligned}
I &= \int 2dx = 2x \\
ye^{2x} &= \int e^{2x}\left(xe^{x} + c_1 e^{x}\right)dx \\
&= \int \left(xe^{3x} + c_1 e^{3x}\right)dx \\
&= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + \frac{1}{3}c_1 e^{3x} + c_2 \\
&= \frac{1}{3}xe^{3x} + d_1 e^{3x} + c_2 \\
\Rightarrow \ y &= \frac{1}{3}xe^{x} + d_1 e^{x} + c_2 e^{-2x}
\end{aligned}
$$

Note: We have obtained the general solution all in one process rather than getting the complementary and particular solutions in 2 separate processes.

$$(D-a)(D-b)y = f(x) = ke^{cx}, \tag{16}$$

where c may also be complex. Suppose $c \neq a$ and $c \neq b$
Solving Eqn. (16) by successive integration of 2 first-order equations gives the particular solution as a multiple of e^{cx}.
Example: Solve the differential equation

$$(D-1)(D+5)y = 7e^{2x} \tag{17}$$

Solution: Here $c \neq -5$ and $c \neq 1$, i.e, is not equal to either roots of the auxiliary equation.

$$\Rightarrow \quad y_p = Ae^{2x}$$

substitute y_p into Eqn. (17)

Exponential Right-Hand Side

$$
\begin{aligned}
(D^2 + 4D - 5)y &= 7e^{2x} \\
y'' + 4y' - 5y &= 7e^{2x} \\
\text{substituting} \quad y &= y_p = Ae^{2x} \\
y_p'' + 4y_p' - 5y_p &= A(4e^{2x} + 8e^{2x} - 5e^{2x}) = 7e^{2x} \\
&= A(7e^{2x}) = 7e^{2x} \\
\Rightarrow \; A &= 1,
\end{aligned}
$$

and the general solution is

$$y = a_1 e^{x} + a_2 e^{-5x} + e^{2x}$$

Summary for a particular solution

$$
\begin{aligned}
y_p &= Ae^{cx}, & &\text{if } c \neq a \text{ and } c \neq b \\
y_p &= Axe^{cx}, & &\text{if } c = a \text{ or } c = b \text{ but } \; a \neq b \\
y_p &= Ax^2e^{cx}, & &\text{if } c = a = b
\end{aligned}
$$

Example: Find a solution for the differential equation

$$(D-1)(D+2)y = e^x$$

Solution: Here $c = a$

$$y_p = Axe^x$$

Summary for a particular solution

$$
\begin{aligned}
y'' + y' - 2y &= e^x \\
y_p'' + y_p' - 2y_p &= \frac{d}{dx}\left(Axe^x + Ae^x\right) + \left(Axe^x + Ae^x\right) - 2\left(Axe^x\right) = e^x \\
\Rightarrow \quad e^x &= \left(Axe^x + Ae^x + Ae^x\right) + Axe^x + Ae^x - 2Axe^x \\
3A &= 1 \qquad \Rightarrow \qquad A = \frac{1}{3}
\end{aligned}
$$

Therefore, the general solution is

$$
y = a_1 e^x + a_2 e^x + \frac{1}{3} x e^x
$$

Use of Complex Exponentials

Example: Solve

$$y'' + y' - 2y = 4\sin 2x \tag{18}$$

We start by solving the equation

$$Y'' + Y' - 2Y = 4e^{2ix} \tag{19}$$

Since

$$e^{2ix} = \cos 2x + i\sin 2x$$

$$\begin{aligned} Y &= Y_R + iY_I \qquad \text{complex solution} \\ Y_R'' + Y_R' - 3Y_R &= Re4e^{2ix} = 4\cos 2x \\ Y_I'' + Y_I' - 3Y_I &= Im4e^{2ix} = 4\sin 2x \\ (D+2)(D-1)y &= 4e^{2ix} \end{aligned}$$

Use of Complex Exponentials

From the previous subsection,

$$Y_p = Ae^{2ix}$$

Substitution into Eqn. (19)

$$\begin{aligned} Y_R'' + Y_R' - 3Y_R &= 4e^{2ix} \\ -4Ae^{2ix} + 4Aie^{2ix} - 2Ae^{2ix} &= 4e^{2ix} \\ (-4 + 2i - 2)Ae^{2ix} &= 4e^{2ix} \\ \Rightarrow \quad A &= \frac{4}{(-6+2i)} = \frac{-1}{5}(i+3) \\ Y_p &= \frac{-1}{5}(i+3)e^{2ix} \end{aligned}$$

Taking the imaginary part of Y_p we find y_p for Eqn. (19)

$$y_p = -\frac{1}{5}\cos 2x - \frac{3}{5}\sin 2x$$

Summary for the method of complex exponentials

To find the particular solution of

$$(D-a)(D-b) = \begin{cases} k\cos\alpha x \\ k\cos\alpha x \end{cases}$$

First solve

$$(D-a)(D-b) = ke^{\alpha i x}$$

and then take the real or imaginary part

Method of Undetermined Coefficients

- The previous method is an example of the method of undetermined coefficients.
- If the right-hand side is an exponential times a polynomial:
- A particular solution y_p of $(D-a)(D-b)y = e^{cx}P_n(x)$, where $P_n(x)$ is a polynomial of degree n is

$$y_p = \begin{cases} e^{cx}Q_n(x) & \text{if c is not equal to either a or b,} \\ \\ xe^{cx}Q_n(x) & \text{if } c \text{ equals } a \text{ or } b, a \neq b, \\ \\ x^2e^{cx}Q_n(x) & \text{if } c = a = b, \end{cases}$$

- where $Q_n(x)$ is a polynomial of the same degree as $P_n(x)$ with undetermined coefficients to be found to satisfy the given differential equation.

Example: Solve $y'' + y' - 2y = x^2 - x$

Solution: $(D-2)(D+1)y = x^2 - x$. We assume a particular solution

$$\begin{aligned} y_p &= a_2x^2 + a_1x + a_0 \\ y_p' &= 2a_2x + a_1 \\ y_p'' &= 2a_2 \\ y_p'' + y_p' - 2y_p &= 2a_2 + 2a_2x + a_1 - 2(a_2x^2 + a_1x + a_0) = x^2 - x \end{aligned}$$

Equating the coefficients of x^2,

$$\Rightarrow \quad -2a_2 = 1 \quad \Rightarrow \quad a_2 = -\frac{1}{2}$$

Using the coefficients of x,

$$\Rightarrow \quad 2a_2 - 2a_1 = -1 \quad \Rightarrow \quad a_1 = 0$$

$$\Rightarrow \quad 2a_2 + a_1 - 2a_0 = 0 \quad \Rightarrow \quad a_0 = -\frac{1}{2}$$

$y_p = -\frac{1}{2}\left(x^2 + 1\right)$

For y_c

$$
\begin{aligned}
y'' + y' - 2y &= 0 \\
(D^2 + D - 2)y &= 0 \\
(D-2)(D+1)y &= 0 \\
y_c &= a_1 e^{2x} + a_2 e^{-x} \\
y &= y_c + y_p = a_1 e^{2x} + a_2 e^{-x} - \frac{1}{2}\left(x^2 + 1\right)
\end{aligned}
$$

Exr: Solve $(D-3)(D+1)y = 16x^2 e^{-x}$

Partial Differential Equation

1. Laplace's equation

$$\nabla^2 u = 0, \tag{20}$$

where u may be the gravitational potential in a region containing no matter, or electrostatic potential in a charge free region, or the steady state temperature ($\frac{dT}{dt}$) in the region containing no heat, or the velocity potential for an in-compressible fluid.

2. Poisson's equation

$$\nabla^2 u = f(x, y, z) \tag{21}$$

3. The diffusion or heat flow equation

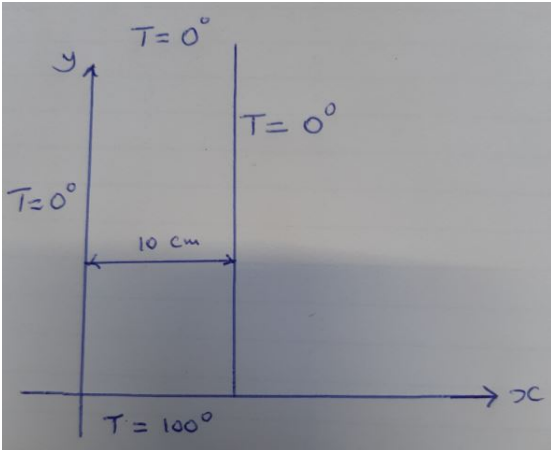
$$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial t^2}, \tag{22}$$

where u is temperature.

Partial Differential Equation

1. The wave equation
$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}, \tag{23}$$
where u is displacement from equilibrium of a vibrating string, etc
2. Helmholtz equation
$$\nabla^2 F + k^2 F = 0, \tag{24}$$
where F is space part (time independent part)

- Laplace's equation; steady-state temperature in a rectangular plate.



$$\nabla^2 T = 0 \quad \text{or} \quad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0,$$

Since the boundary of the plate is rectangular.
Trial solution

$$\begin{aligned} T(x,y) &= X(x)Y(y) \\ \Rightarrow \quad Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} &= 0 \\ \text{divide throught by } XY & \\ \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} &= 0 \\ \frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} &= -k^2 \\ \text{where} \qquad k &\geq 0 \end{aligned}$$

k^2 is called the separation constant.

$$
\begin{aligned}
X'' &= -k^2 X & \qquad Y'' &= k^2 Y \\
X'' + k^2 X &= 0 & \qquad Y'' - k^2 Y &= 0 \\
(D^2 + k^2) X &= 0 & \qquad (D^2 - k^2) Y &= 0 \\
(D + ik)(D - ik) X &= 0 & \qquad (D + k)(D - k) Y &= 0
\end{aligned}
$$

Solution

$$
X = \begin{cases} \sin kx \\ \cos kxx \end{cases}
$$

$$
Y = \begin{cases} e^{ky} \\ e^{-ky} x \end{cases}
$$

Thus

$$T = XY = \begin{cases} e^{ky} \sin kx \\ e^{-ky} \sin kx, \\ e^{ky} \cos kx, \\ e^{-ky} \cos kx \end{cases}$$

We put in the boundary conditions to find the solution

$T = 0$ at $y = \infty$

It eliminates e^{ky} for $k > 0$

$T = 0$ when $x = 0$

It eliminates $\cos kx$

We are left with $e^{-ky} \sin kx$

Now $T = 0$ at $x = 10$, this is true if $\sin(10k) = 0$

$$\Rightarrow \quad k = \frac{n\pi}{10}, \qquad n = 1, 2, 3, \ldots$$

Hence the solution is

$$T = e^{-n\pi y/10} \sin\left(\frac{n\pi x}{10}\right)$$

Example: The diffusion or heat flow equation;

$$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t} \tag{25}$$

Here we assume a solution of the form

$$u = F(x, y, z)T(t), \tag{26}$$

where u is temperature and T is the time dependent factor in u.

Put Eqn. (26) into (25)

$$T\nabla^2 F = \frac{1}{\alpha^2} F \frac{\partial T}{\partial t} \tag{27}$$

Divide through by FT

$$\frac{1}{F}\nabla^2 F = \frac{1}{\alpha^2}\frac{1}{T}\frac{\partial T}{\partial t} \tag{28}$$

The left-hand side of Eqn. (28) is a function of only space variables (x,y,z)
Now we can write

$$\begin{aligned} \frac{1}{F}\nabla^2 F &= -k^2 \\ \nabla^2 F + k^2 F &= 0 \end{aligned}$$

is Helmholtz equation, and

$$\begin{aligned} \frac{1}{\alpha^2}\frac{1}{T}\frac{\partial T}{\partial t} &= -k^2 \\ \frac{dT}{dt} &= -k^2\alpha^2 T \end{aligned}$$

The time equation gives

$$\int \frac{dT}{T} = \int -k^2\alpha^2 dt$$
$$\ln T = -k^2\alpha^2 t$$
$$T = e^{-k^2\alpha^2 t}$$

$-k^2$ is negative because as t increases, T goes to zero. And $+k^2$ means T may increase to infinity

Solution for F

$$F(x) = \begin{cases} \sin kx \\ \cos kxx \end{cases}$$

$$T = XY = \begin{cases} e^{-k^2\alpha^2 t} \sin kx \\ e^{-k^2\alpha^2 t} \cos kx \end{cases}$$

The wave equation: The vibrating string

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \tag{29}$$

to separate the variables we substitute

$$y = X(x)T(t) \tag{30}$$

Then

$$T\frac{d^2X}{dx^2} = \frac{1}{v^2}X\frac{d^2T}{dt^2} \tag{31}$$

$$\text{divide through by by } XT \tag{32}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2} = -k^2 \tag{33}$$

$$X'' + k^2X = 0, \qquad T'' + k^2v^2T = 0 \tag{34}$$

Recall

$$\begin{aligned}
&\nu && \text{is frequency } (\text{sec}^{-1})\\
&\lambda && \text{is wavelength (m)}\\
&v = \lambda\nu && \text{is velocity}\\
&\omega = 2\pi\nu && \text{angular frequency (in radians)}\\
&k = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{v} = \frac{\omega}{v} && \text{is the wave number}
\end{aligned}$$

Solutions of Eqns. (34)

$$X = \begin{cases} \sin kx \\ \cos kxx \end{cases}$$

$$T = \begin{cases} \sin kvt = \sin \omega t, \\ \cos kvt = \cos \omega t \end{cases}$$

General solution is

$$y = XT = \begin{cases} \sin kx \sin \omega t, \\ \sin kx \cos \omega t, \\ \cos kx \sin \omega t, \\ \cos kx \cos \omega t \end{cases}$$