LINEAR SYSTEMS

25/10/2021
Monday

System of Equations

A set of equations with the same Variables (two or more equations graphed in the same Coordinate plane.

Solution of the system.

An ordered pair that is a solution to all equations is a Solution to the equation.

terminologies.

- a - one solution
- b - no solution
- c - an infinite number of solutions.

- Consistent system.

A system that has at least one solution: a.

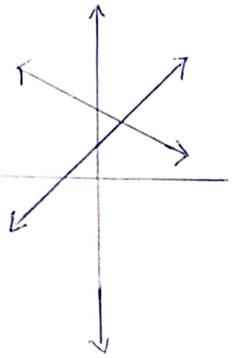
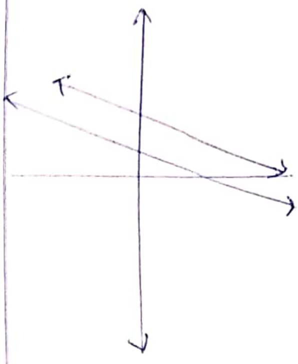
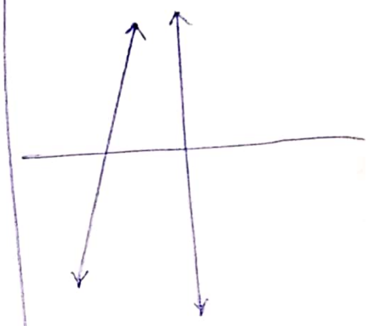
a) independent - has exactly one solution.

b) dependent - an infinite number of solutions.

- Inconsistent System.

a System that has no solution.

Illustrative display of the Solutions:

Number of Solutions. (Solutions are where they intersect)	one solution.	no solution	infinitely many solutions.
Definitions.	Consistent and independent	inconsistent	Consistent and dependent.
Graph.			

To solve a system of Equations by graphing simply graph both equations on the same coordinate plane and find where they intersect.

Solving Linear systems.

$\boxed{1}$ Graphing

a) Graph one equation.

b) Graph the other equation on the same plane

c) Find the point or points, or

$y = 2x - 7 \quad -(1)$

$y = \frac{2}{3}x + 1 \quad = (11)$.

(1) into (11).

$2x - 7 = \frac{2}{3}x + 1$

$\frac{2x}{1} - \frac{2}{3}x = 1 + 7$

$\frac{6x - 2x}{3} = 8$

$\frac{4x}{3} = \frac{8}{1}$

$4x = 24$

$x = 6$

$(6, 5)$

$y = 2x -$

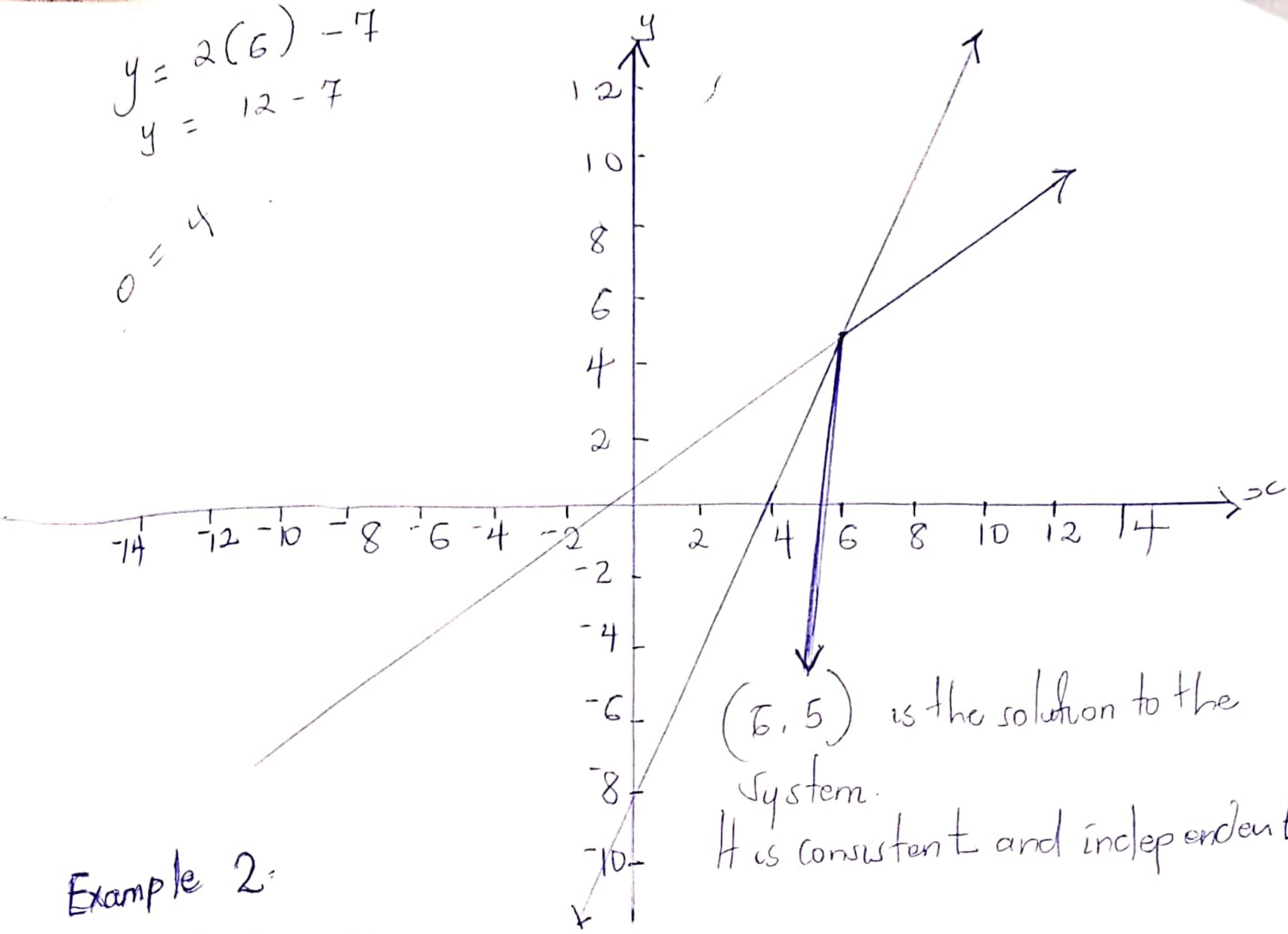
$y = 2(6$

$y =$

$y = 2(6) - 7$

$y = 12 - 7$

$0 = 4$



(6, 5) is the solution to the system.
It is consistent and independent.

Example 2:

$x - 27 = 9y$ — (i)

$18 = x - 9y$ — (ii)

Eqn (i)

$\frac{x}{9} - \frac{27}{9} = \frac{9y}{9}$

$\frac{1}{9}x - 3 = y$

Eqn (ii)

$\frac{18}{-9} = \frac{x}{-9} - \frac{9y}{-9}$

$\frac{1}{9}x - 2 = y$

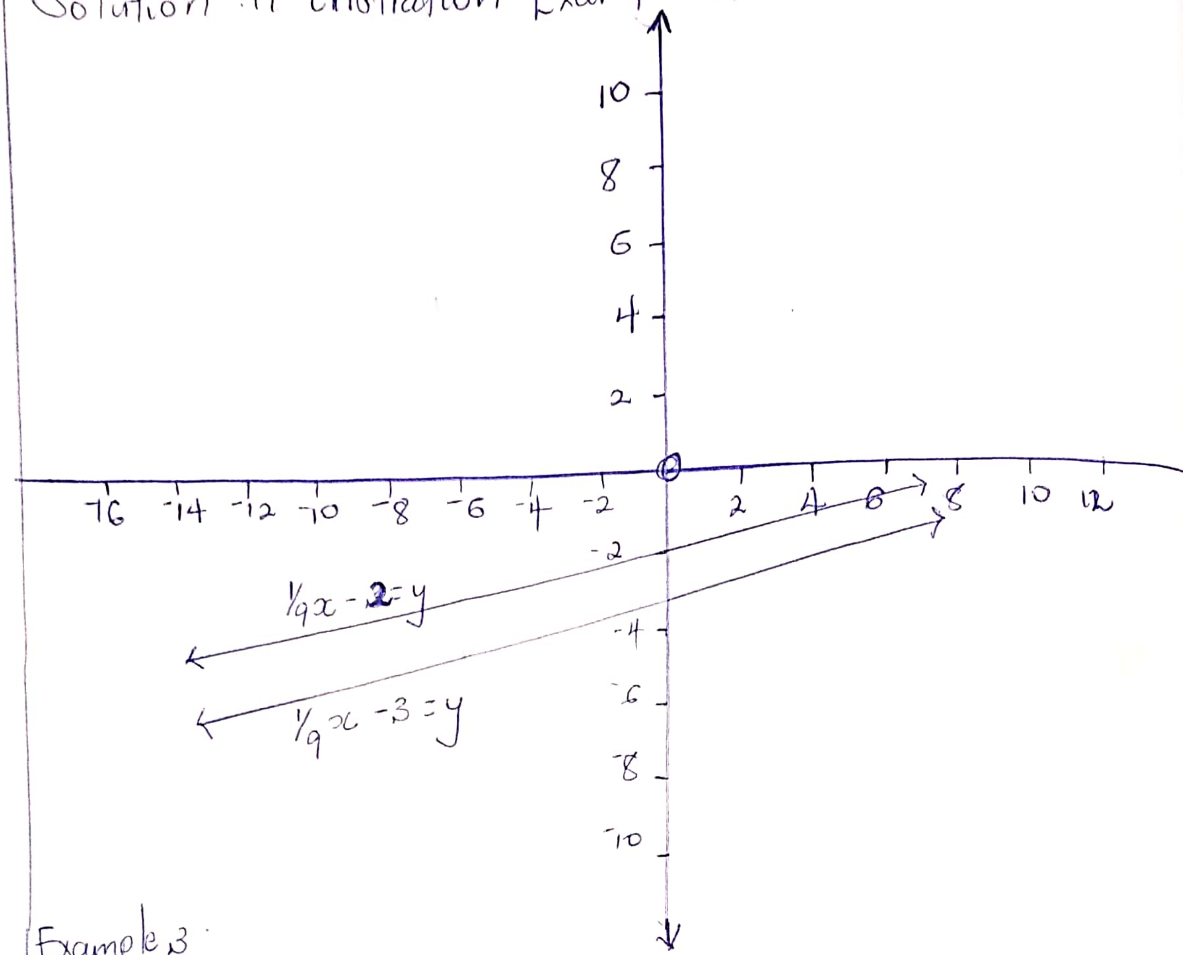
The lines are parallel to each other because they have the same gradient.

∴ There is no solution on this system

It is inconsistent.

3/3

Solution in illustration Example 2.



Example 3

$2x + 3y = 15$ — (i)

$y = -\frac{2}{3}x + 5$ — (ii)

Eqn (i).

$\frac{2x}{3} + \frac{3y}{3} = \frac{15}{3}$

$\frac{2}{3}x + y = 5$

$y = -\frac{2}{3}x + 5$

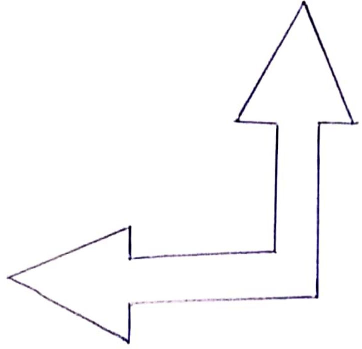
$\frac{3y}{3} = \frac{-2x}{3} + \frac{15}{3}$

$y = -\frac{2}{3}x + 5$

They are the same equation so they would graph into the same line

There are infinitely many solution. then the system is said to consistent and dependent.

illustration.



12 Substitution.

a) If possible, solve at least one equation for one Variable.

b) Substitute the result into the other equation to replace one of the Variables.

c) solve the equation.

d) Substitute the Value just found into the first equation.

e) solve for the other Variable.

f) Write the solution as an ordered pair.

Quick steps: Solve
Substitute
Solve,
Substitute
solve
Write the solution

Example 1.

$8x + 5y = 2$ — (i)

$-2x+y = 4$ — (ii)

$8x + 5y = 2$

$8x+5(4+2x) = 2$

$8x + 20 + 10x = 2$

$\frac{18x}{18} = \frac{-18}{18}$

$x = -1$

using Eqn (ii).

$y = 4+2x$ — (iii)

Substitute into eqn (iii).

$y = 4+2x$.

$y = 4+2x-1$

$y = 4-2$

$y = 2$

The solution is $(-1, 2)$
It is consistent and independent.

Example 2

$-2x + 2y = -4$ — (i)

$-x = -y - 4$ — (ii)

Solution for (ii)

$\frac{-x}{-1} = \frac{-y}{-1} - \frac{4}{-1}$

$x = y + 4$ — (iii)

$y = 4 - x$ — (iii).

Substitute Eqn (iii) into Eqn (i),

$-2x + 2y = -4$

$-2x + 2(4 - x) = -4$

$-2x + 8 - 2x = -4$

$\frac{-4x}{-4} = \frac{-12}{-4}$

~~$x = 3$~~

Substitute Eqn (iii) into Eqn (i).

$-2x + 2y = -4$

$-2(y + 4) + 2y = -4$

$-2y - 8 + 2y = -4$

$-8 = -4$

This is a false statement, therefore this system no solution. The lines are parallel and are inconsistent.

Example 3

$2x + y = 5 \quad -(i)$

$-6x - 3y = -15 \quad -(ii)$

Using Eqn (i)

$y = -2x + 5$

$-6x - 3(-2x + 5) = -15$

$-6x + 6x - 15 = -15$

$-15 = -15$

→ This is a true statement, therefore this System has infinitely many solutions.
It is consistent and dependent.

26/10/2021
Tuesday

E Elimination

a) Write both equations so that like terms are aligned Vertically
b) Multiply one or both equations by a Constant to get two equations that contain at least one set of exactly opposite terms.

c. Add the equations eliminating one Variable.

d) Solve for the remaining Variable.

e) Substitute the Value from (d) into one of the equations and Solve for the other Variable

F) Write the solution as an ordered pair.

Example I

$6x + 14y = 6$ — (I)
$-4x - 7y = -11$ — (II)

$$\begin{array}{r|l} & 6x + 14y = 6 \\ 2 & -4x - 7y = -11 \\ \hline \end{array}$$

$$6x + 14y = 6$$
$$-8x - 14y = -22$$
$$\overline{-2x = -16}$$
$$\frac{-2x}{-2} = \frac{-16}{-2}$$

$x = 8$ Substitute 8 into Eqn (I)

$$6x + 14y = 6$$
$$6(8) + 14y = 6$$
$$48 + 14y = 6$$
$$\frac{14y}{14} = \frac{-42}{14}$$
$$y = -3$$

The solution is $(8, -3)$

It is Consistent and independent.

Example II

$-8x + 2y = -10$ — (I)
$-4x + y = -2$ — (II)

$$\begin{array}{r|l} & -8x + 2y = -10 \\ -2 & -4x + y = -2 \\ \hline \end{array}$$

$$-8x + 2y = -10$$
$$8x - 2y = 4$$

$0 = -6.$

This is a fake | false statement and has no solution

The lines are parallel. It is inconsistent.

Example 3:

$6x + 8y = -28$ — (I)

$-3x - 4y = 14$ — (II)

$$\begin{array}{r|l} & 6x + 8y = -28 \\ \times 2 & -3x - 4y = 14 \\ \hline & 6x + 8y = -28 \\ & -6x - 8y = 28 \\ \hline & 0 = 0 \end{array}$$

This is a true statement and has infinitely many Solutions.

The equations are the exact same line. It is Consistent and dependent.

GAUSS' ELIMINATION METHOD.

A system of equation
Is a collection of two or more equations with the same set of unknown variables that are considered simultaneously.

Example.
The following set of equations is a system of equations.

$$x - 2y + 3z = 9$$
$$-x + 3y = -4$$
$$2x - 5y + 5z = 17.$$

An augmented matrix.
Is a rectangular array of numbers that represents a system of equations.

Ex. Turn the following system of equations into an augmented matrix.

Becomes:
$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

Gaussian Elimination
Is a method for solving systems of equations in matrix form.

Goal: turn matrix into row-echelon form.

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

Once in this form, we can say that $z = f$ and use back substitution to solve for y and x.

Use the elementary row operations and follow these steps:

1) Get a 1 in the 1st column, first row.
2) Use the 1 to get 0's in the remainder of the first column.
3) Get a 1 in the second column, second row
4) use the 1 to get 0's in the remainder of the second column
5) Get a 1 in the third column, third row.

3 Elementary Row Operations.

① Exchange two rows.
(Written $R_1 \leftrightarrow R_j$)

2 Multiply a row by a non-zero constant.
(Written $\# R_1 - R_i$ or $\# R_i \to NR_j$)

3 Add a multiple of a row to another row
(Written $\# R_i + R_j \to R_j$ or
$\# R_i + R_j \to NR_j$).

3rd

1st

$$\left[\begin{array}{ccc|c} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{array}\right]$$

2nd

4th 5th.

Points to note in Gaussian method

- Multiply a row with a zero is not allowed.
- Adding a multiple to itself is not allowed
- Swapping a row with itself is not allowed

Solve the following using Gaussian methods.

Eqn (1)

$x + y = 3 \quad -(i)$
$2x + y = 3 \quad -(ii)$

$x + y = 3 \ldots R_1$
$2x + y = 3 \ldots R_2$

$\left| x + y = 3 \right|$

$-2R_1 + R_2 \rightarrow R_2$

$\left| x + y = 3 \right|$
$\left| \quad -y = -3 \right|$

$\frac{-y}{-1} = \frac{-3}{-1}$

$y = 3$

$x + y = 3$
$x + 3 = 3$
$x = 0$

The solution is $(0, 3)$

It has only one solution.
It is independent and Consistent.

$x + y = 3 \quad \cdots R_1$

$2x + y = 3 \quad \cdots R_2$

$\left| x + y = 3 \right|$

$-1R_1 + R_2 \quad -R_2 \qquad \left| -2x \quad = 0 \right|$

$-2x = 0$

$x = 0$

$x + y = 3$

$0 + y = 3$

$y = 3$

The solution is $(0, 3)$.

It is has only one solution.

It is independent and Consistent.

Example 2.

$x - 3y = 1 \quad - R_1$

$-2x + 2y - 4z = 2 \quad - R_2$

$x + 3y - 2z = 3 \quad - R_3.$

$\left| x - 3y \quad = 1 \right|$

$2R_1 + R_2 = R_2 \longrightarrow \left| \quad -4y - 4z = 4 \right|$

$-1R_1 + R_3 = R_3 \longrightarrow \left| \quad 6y - 2z = 2 \right|$

$\cdot \frac{1}{4}R_2 = R_2 \qquad \left| -y - z = 1 \right|$

$\left| 6y - 2z = 2 \right|$

$6R_2 + R_3 \longrightarrow$

$-6z = 6$

$-2z = 2$

$-8z = 8$

$\overline{-8 \quad -8}$

$z = -1$

Scanned with CamScanner

$z = -1$

$-y - z = 1$

$-y - (-1) = 1$

$-y + 1 = 1$

$\frac{-y}{-1} = \frac{0}{-1}$

$y = 0$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$x \Rightarrow \quad x - 3y = 1$

$x - 3(0) = 1$

$x - 0 = 1$

$x = 1$

The solution set $(1, 0, -1)$

Example 3

Solve the following set of equations.

$$x - 2y + 3z = 9$$
$$-x + 3y = -4$$
$$2x - 5y + 5z = 17.$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

$R_1 + R_2 \rightarrow R_2$.

$-2R_1 + R_3 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right]$$

$R_2 + R_3 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$\frac{1}{2}R_3 \rightarrow R_3$ $\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

New equation.

$x + -2y + 3z = 9$ — (i)

$y + 3z = 5$ — (ii)

$z = 2$ — (iii)

$y = -1$

$x + -2(-1) + 3(2) = 9$

$x + 2 + 6 = 9$

$x = 1$

x

The solution set $(1, -1, 2)$

a) $3x + y - 6z = -10$ — (I) R_1

$2x + y - 5z = -8$ — (II) R_2

$6x - 3y + 3z = 0$ — (III) R_3

Use Gauss's method to solve the linear systems below

$\frac{1}{3}R_1 = R_1$

$$x + \frac{1}{3}y - 2z = -\frac{10}{3} \quad - R_1$$

$$2x + y - 5z = -8 \quad - R_2$$

$$6x - 3y + 3z = 0 \quad - R_3$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{3} & -2 & -\frac{10}{3} \\ 2 & 1 & -5 & -8 \\ 6 & -3 & 3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{3} & -2 & -\frac{10}{3} \\ 0 & \frac{1}{3} & -1 & -1\frac{1}{3} \\ 0 & -5 & 15 & 20 \end{array}\right] \begin{array}{l} \\ -2R_1 + R_2 = R_2 \\ -6R_1 + R_3 = R_3 \end{array}$$

$$\begin{array}{c} \\ 3R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & \frac{1}{3} & -2 & -\frac{10}{3} \\ 0 & 1 & -3 & -4 \\ 0 & -5 & 15 & 20 \end{array}\right]$$

$$\begin{array}{c} \\ \\ 5R_2 + R_3 \end{array} \left[\begin{array}{ccc|c} 1 & \frac{1}{3} & -2 & -\frac{10}{3} \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

GAUSS JORDAN REDUTION AND REDUCED ROW ECHELON FORM.

Gauss - Jordan elimination

Gauss - Jordan elimination is another method for solving s
of equations in matrix from.
It is really a continuation of Gaussian elimation.

Goal: turn matrix into reduced row - echelon form.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$$

Once it is in this form,
we can say $x = a$, $y = b$ and $z = c$ or
$(x, y, z) = (a, b, c)$

Use same row operations as before
The steps are slighty different because we need zeros above the diagonal line of 1's as well as below. We can either complete Gaussion elimination and then work on the 0's above the 1's, or work on the zeros above as we move through the rows, as demonstrated below.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$$

1st, Second, 3rd, 4th, 5th, 6th.

or

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$$

1st, 2nd, 3rd, 4th, 5th, 6th

Solve the following systems using Gauss-Jordan elimination method.

$$
\begin{aligned}
x + y + z &= 5 \\
2x + 3y + 5z &= 8 \\
4x \qquad + 5z &= 2
\end{aligned}
$$

$$
\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right]
\xrightarrow[-4R_1+R_3]{-2R_1,\ R_2}
\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right]
$$

$$
\begin{array}{c} \\ \\ 4R_2+R_3 \end{array}
\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right]
$$

$$
\begin{array}{c} \\ \\ \tfrac{1}{13}R_3 \end{array}
\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]
$$

$$
\begin{array}{c} -1R_3+R_1 \\ -3R_3+R_2 \\ \\ \end{array}
\left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & +1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]
$$

$$
\begin{array}{c} -1R_2+R_1 \\ \\ \\ \end{array}
\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]
$$

From this final matrix, we can read the solution of the system.

$x = 3,\ y = 4,\ z = -2$

Solution $= \{3, 4, -2\}$

2. $x + 2y - 3z = 2$

$6x + 3y - 9z = 6$

$7x + 14y - 21z = 13$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 7 & 14 & -21 & 13 \end{array}\right]$$

$$\begin{array}{c} \\ -6R_1+R_2 \\ -7R_1+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & -9 & 9 & -6 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

We obtain a row whose elements are all zeros except the last one on the right. Therefore, we conclude that the system of equations is inconsistent ie has no solution.

3. Example III

$A + B + 2C = 1$

$2A - B + D = -2$

$A - B - C - 2D = 4$

$2A - B + 2C - D = 0$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 2 & -1 & 0 & 1 & -2 \\ 1 & -1 & -1 & 2 & 4 \\ 2 & -1 & 2 & -1 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ -2R_1+R_2 \\ -1R_1+R_3 \\ -2R_1+R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & -3 & -2 & -1 & -2 \end{array}\right]$$

$$
\begin{array}{c}
\\
-\tfrac{1}{3}R_2 \\
\\
\\
\end{array}
\left[\begin{array}{cccc|c}
1 & 1 & 2 & 0 & 1 \\
0 & 1 & \tfrac{4}{3} & -\tfrac{1}{3} & \tfrac{4}{3} \\
0 & -2 & -3 & -2 & 3 \\
0 & -3 & -2 & -1 & -2
\end{array}\right]
$$

$$
\left[\begin{array}{cccc|c}
1 & 1 & 2 & 0 & 1 \\
0 & 1 & \tfrac{4}{3} & -\tfrac{1}{3} & \tfrac{4}{3} \\
0 & 0 & -\tfrac{1}{3} & -\tfrac{8}{3} & \tfrac{11}{3} \\
0 & 0 & +2 & -2 & 2
\end{array}\right]
\begin{array}{l}
\\
\\
2R_2 + R_3 \\
3R_2 + R_4
\end{array}
$$

$$
\begin{array}{c}
\\
\\
-3R_3 \\
\\
\end{array}
\left[\begin{array}{cccc|c}
1 & 1 & 2 & 0 & 1 \\
0 & 1 & \tfrac{4}{3} & -\tfrac{1}{3} & \tfrac{4}{3} \\
0 & 0 & 1 & 8 & -17 \\
0 & 0 & 2 & -2 & 2
\end{array}\right]
$$

$$
\begin{array}{c}
\\
\\
\\
-2R_3 + R_4
\end{array}
\left[\begin{array}{cccc|c}
1 & 1 & 2 & 0 & 1 \\
0 & 1 & \tfrac{4}{3} & -\tfrac{1}{3} & \tfrac{4}{3} \\
0 & 0 & 1 & 8 & -17 \\
0 & 0 & 0 & -18 & 36
\end{array}\right]
$$

$$
\begin{array}{c}
\\
\\
\\
-\tfrac{1}{18}
\end{array}
\left[\begin{array}{ccccc|c}
1 & 1 & 2 & 0 & & 1 \\
0 & 1 & 1 & \tfrac{4}{3} & -\tfrac{1}{3} & \tfrac{4}{3} \\
0 & 0 & 0 & 1 & 8 & -17 \\
0 & & 0 & 0 & 1 & -2
\end{array}\right]
$$

$$
\begin{array}{c}
\\
\tfrac{1}{3}R_4 + R_2 \\
-8R_4 + R_3 \\
\\
\end{array}
\left[\begin{array}{cccc|c}
1 & 1 & 2 & 0 & 1 \\
0 & 1 & \tfrac{4}{3} & 0 & \tfrac{2}{3} \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & -2
\end{array}\right]
$$

$$\begin{array}{l} -2R_3+R_1 \\ -\frac{4}{3}R_3+R_2 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

$$-R_2+R_1 \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

$A = 1 \quad B = 2 \quad C = -1 \quad D = -2$

Solution Set $= \{1, 2, -1, -2\}$

ECHELON FORM

In each row of linear system, the first Variable with a non-zero coefficient is the row's leading Variable.

. Specifically. amatrix is in row echelon form if:

All rows consisting of only zeros are at the botton.

The leading Coefficient (also called the pivot) of a nonzero row is always strictly to the right of the leading Coefficient of the row above it.

Row Echelon form.

Property 2.

F a column Contains aleading entry then all entries below that leading entry are zero.

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 & 7 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Show that the solution to the following linear system is.

$$\left\{ \frac{3-z}{2},\ \frac{1-3z}{2},\ z \ \middle|\ z \in \mathbb{R} \right\}$$ Ans:

$2x + z = 3$ ①
$x - y - z = 1$
$3x - y = 4$

Swapping $R_1 \leftrightarrow R_2$.

$x - y - z = 1$
$2x + z = 3$
$3x - y = 4$

$$\left|\begin{array}{cccc} x & -y & -z & = 1 \\ 2 & 0 & 1 & = 3 \\ 3 & -1 & 0 & = 4 \end{array}\right.$$

$$\begin{array}{l} \\ -2R_1+R_2 \\ -3R_1+R_3 \end{array} \left|\begin{array}{cccc} x & -y & -z & = 1 \\ 0 & +2y & +3z & = 1 \\ 0 & +2y & +3z & = 1 \end{array}\right.$$

$$\begin{array}{l} \\ \\ -R_2+R_3 \end{array} \left|\begin{array}{cccc} x & -y & -z & = 1 \\ 0 & +2y & +3z & = 1 \\ 0 & +0 & +0 & = 0 \end{array}\right.$$ - Redundancy

Redundancy

$$2y + 3z = 1$$

$$\frac{2y}{2} = \frac{1 - 3z}{2}$$

$$y = \frac{1+3z}{2}$$

$$\frac{3z}{3} = \frac{1-2y}{3}$$

$$z = \frac{1-2y}{3}$$

$$x - y - z = 1$$

$$\frac{x}{1} - \frac{1+3z}{2} - \frac{z}{1} = \frac{1}{1}$$

$$2x + 1 + 3z - 2z = 2$$

$$2x + z = 1$$

$$\frac{2x}{2} = \frac{1-z}{2}$$

$$x = \frac{1-z}{2} \quad \times\times$$

$$2x + z = 3$$

$$x = \frac{3-z}{2}$$

$$x - y - z = 1$$

$$\frac{3-z}{2} - \frac{(1+3z)}{2} - z = 1$$

$$\frac{3 - z - 1 - 3z}{2} - z = 1$$

$$\frac{-4z - 2}{2} - z = 1$$

$$-2z - 1 - z = 1$$

$$\frac{-3z}{3} = \frac{0}{3}$$

$$z = 0 \quad \times\times$$

Solution Set $= \{x, y, z\} = \left\{\frac{3-z}{2}, \frac{1-3z}{2}, z \,\middle|\, z \in R\right\}$

Example 2.

$$x + y + z + w = 1$$
$$y - z + w = -1$$
$$3x + 6z - 6w = 6$$
$$-y - + z - w = 1$$

$$\begin{array}{l|l} & x + y + z + w = 1 \\ & 0 + y - z + w = -1 \\ -3R_1 + R_3 & 0 - 3y + 3z - 9w = 3 \\ & 0 \ -y + z = w = 1 \end{array}$$

$$\begin{array}{l|llllll} & x + & y + & z + & w & = & 1 \\ & 0 + & y & - z + & w & = & -1 \\ 3R_2 + R_3 & 0 & 0 & 0 & -6w & = & 0 \\ +R_2 + R_4 & 0 & 0 & 0 & 0 & = & 0 \end{array}$$

$$-6w = 0$$
$$w = 0$$

$$y - z + w = -1$$
$$y - z = -1$$
$$y = z - 1$$

$$x + y + z + w = 1$$
$$x + z - 1 + z + w = 1$$
$$x + 2z = 1 + 1$$
$$x = 2 - 2z$$

Solution Set = $\left\{ 2-2z,\ z-1,\ z, 0 \mid z \in \mathbb{R} \right\}$

Inconsistent System Examples

Solve the following.

$$2x - 2z = 6$$
$$y + z = 1$$
$$2x + y - z = 7$$
$$3y + 3z = 0$$

$$\left(\begin{array}{ccc|c} 2 & 0 & -2 & 6 \\ 0 & 1 & 1 & 1 \\ 2 & 1 & -1 & 7 \\ 0 & 0 & 3 & 0 \end{array}\right)$$

$$\left[\begin{array}{ccc|c} 2 & 0 & -2 & 6 \\ 2 & 1 & -1 & 7 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

$$\tfrac{1}{2}R_1 \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 2 & 1 & -1 & 7 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \end{array}\right]$$

$$2R_1 + R_2 \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \end{array}\right]$$

Addendum-

- A linear System with a unique Solution has a Solution with one element.
A linear system with no Solution has a Solution set that is empty. However, describing a Solution set with many Solutions can be a challenge.

- Any linear System solution set can be described as

- $\vec{P} + \overrightarrow{C_1 B_1} + \overrightarrow{C_2 B_2} + \cdots + \overrightarrow{C_k B_k} : C \in R$

- Where $\vec{P}$ - any particular Solution.

- B_i - number of free Variables that a System has after a Gaussian reduction.

Homogeneous System

- A linear equation is said to be homogeneous if it has a Constant of zero i.e. it can be expressed in this form.

- $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$

- A linear system comprising of a homogeneous equations is referred to as a homogeneous System.

Solve the homogenous equivalent of the System.

$x + 2y = 0$

$2x + z = 0$

$3x + 2y + z - w = 0$

Swapping $R_3 \leftrightarrow R_1$

$3x + 2y + z - w = 0$

$x + 2y = 0$

$2x + z = 0$

$$\left(\begin{array}{cccc|c} 3 & 2 & 1 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \end{array}\right)$$

$$\begin{array}{l} \\ -R_1 + 3R_2 \\ -R_1 + 3R_3 \end{array} \left(\begin{array}{cccc|c} 3 & 2 & 1 & -1 & 0 \\ 0 & 4 & -1 & 1 & 0 \\ 0 & -4 & 1 & 2 & 0 \end{array}\right)$$

$$\begin{array}{l} \\ \\ R_2 + R_3 \end{array} \left[\begin{array}{cccc|c} 3 & 2 & 1 & -1 & 0 \\ 0 & 4 & -1 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right)$$

$3w = 0$

$w = 0$

$4y - 1z + w = 0$

$4y - z = 0$

$y = z/4$

$3x + 2y + z - k1 = 0$

$3x + 2\left(\frac{z}{4}\right) + z - 0 = 0$

$3x + \frac{z}{4} + z = 0$

$3x = \frac{-3z}{2}$

$x = -\frac{z}{2}$

$\left\{ -\frac{z}{2}, \frac{z}{4}, z, 0 \mid z \in R \right\}$

Example 2.

$3x + 2y + z - w = 4$

$x + 2y = 1$

$2x + z = 2$

$$\left(\begin{array}{cccc|c} 3 & 2 & 1 & -1 & 4 \\ 1 & 2 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 2 \end{array}\right)$$

$$\begin{array}{r} \\ -R_1+3R_2 \\ -2R_1+3R_3 \end{array} \left(\begin{array}{cccc|c} 3 & 2 & 1 & -1 & 4 \\ 0 & 4 & -1 & 1 & -1 \\ 0 & -4 & 1 & 2 & 2 \end{array}\right)$$

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$$R_2+R_3\left(\begin{array}{cccc|c} 3 & 2 & 1 & -1 & 4 \\ 0 & 4 & -1 & 1 & -1 \\ 0 & 0 & 0 & 3 & -3 \end{array}\right)$$

$$3w = -3$$
$$w = -1$$

$$4y - z + w = -1$$
$$4y - z - 1 = -1$$
$$4y = z$$
$$y = z/4$$

$$3x + 2y + z - w = 4$$
$$3x + 2\left(\frac{z}{4}\right) + z - -1 = 4$$
$$3x + z/2 + z + 1 = 4$$
$$3x + \frac{z}{2} + z = 3$$
$$6x + z + 2z = 6$$
$$6x + 3z = 6$$
$$\frac{6x}{6} = \frac{6-3z}{6}$$
$$x = \frac{2-z}{2}$$

Solution set is $\left\{\frac{2-z}{2}, \frac{z}{4}, z, -1\right\}$

$1 - z/2$

$1/4(z)$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} \frac{2 \cdot z}{2} \\ z/4 \\ z \\ -1 \end{pmatrix}$$

Solution in the form $= \begin{pmatrix} 1 - z/2 \\ \frac{1}{4}z \\ z \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} -1/2 \\ 1/4 \\ 1 \\ 0 \end{pmatrix} z$

$\vec{p} + \overrightarrow{c_1 B_1} + \overrightarrow{c_2 \beta_2} + \cdots + \overrightarrow{c_k \beta k}$

Solution $= \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + z \begin{pmatrix} -1/2 \\ 1/4 \\ 1 \\ 0 \end{pmatrix} \middle| z \in R \right\}$

Example 2.

$x + y + z + w = 1$

$y - z + w = -1$

$3x + 6z - 6w = 6$

$-y + z - w = 1$

Solution $\{ 2 - 2z, z - 1, z, 0 \mid z \in R \}$

In the form $\vec{P} + z^k \overrightarrow{c_i \beta_i}$

$$\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \middle| z \in R \right\}$$

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Row Equivalence.

Two matrices are said to be row equivalent if there is a sequence of elementary row operations that transforms one matrix into the other

A linear system is also said to be row equivalent if the Row Reduced Echelon form (RREF) of a matrix is unique if you can obtain an reduced echelon matrix from a system, then it is said to be row equivalent

No matter what sequence of row operations you use, each matrix is row equivalent to one and only one reduced echelon matrix.

Example.

Show matrices A and B row equivalent?

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 4 \\ 5 & 8 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\xrightarrow{-3R_1 + R_2} \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} \xrightarrow{R_2 + R_1} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = C$$

$$B = \begin{pmatrix} 2 & 4 \\ 5 & 8 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 1 & 2 \\ 5 & 8 \end{pmatrix} \xrightarrow{-5R_1+R_2} \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} \xrightarrow{R_1+R_2} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = D$$

$\therefore$ They are row equivalent.

Example 2.

$$\begin{pmatrix} 0 & 0 & 4 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 \end{pmatrix}$$

$R_2 \leftrightarrow R_1$

$R_3 \leftrightarrow R_4$

$$\begin{pmatrix} 0 & 3 & 0 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{3}R_1} \begin{pmatrix} 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 4 & 1 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-4R_1 + R_3 \quad \begin{pmatrix} 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 4 & 1 \\ 0 & 0 & +2 & -\frac{4}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{1}{4}R_2 \quad \begin{pmatrix} 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & +2 & -\frac{4}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-2R_2 + R_3 \quad \begin{pmatrix} 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & -\frac{11}{6} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-\tfrac{11}{6}R_3 \begin{pmatrix} 0 & 1 & 0 & \tfrac{1}{3} \\ 0 & 0 & 1 & \tfrac{1}{4} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{l} -\tfrac{1}{3}R_3 + R_1 \\ -\tfrac{1}{4}R_3 + R_2 \end{array} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is RREF

$\therefore$ It's row equivalent.

09/11/2021
Tuesday.

Dr. Johnson Ssekakubo.

VECTOR SPACES. / linear Space.

Is a system consisting of a set of generalized Vectors and a field of scalars, having the same rules for vector addition and scalar multiplication as physical Vectors and Scalar.

Let V be a non empty set of goods or objects on which operations of addition and multiplication by scalars are defined.

If the following axioms are satisfied by all objects u, v, w in V & all scalars k1, k2 then V is called a Vector space and the objects in V are called Vectors.

Addition Conditions:-

*1) If u and v are objects in V, then u + v is in V $u+v \in V$ (vector addition)

2 $u + v = v + u$ → (associative)

3 $u + (v + w) = (u + v) + w$ (distributive)

4 There is an object 0 in V, called a zero Vector for V, such that $0 + u = u + 0 = u$ for all u in V

5) For each u in V, there is an object −u in V, called a negative of u, such that $u + (-u) = (-u) + u = 0$ $\forall u \in V, u + (-u) = 0$ (additive inverse)

*6) If k is any scalar and u is any object in V then ku is in V {$ku \in V$} {Scalar multiplication}

7 $k(u + v) = ku + kv$

8 $(k + l)(u) = ku + lu$

9 $k(lu) = (kl)u$

10 $1u = u$ 1 ≡ unit Vector.

A Vector space (linear space), V, is a set that satisfies the following ∀ (for all) $u, v, w \in V$ and $c, d \in \mathbb{R}$

c, d are scalars.

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Example of Vector Space :

Determine whether the set of V of all pair of real numbers (x, y) with the operations $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$ and $k(x, y) = (kx, ky)$ is a vectorspace.

Solution

Let $u = (x_1, y_1)$,

$v = (x_2, y_2)$

$w = (x_3, y_3)$ are objects in V and k_1, k_2 are some Scalars.

① $u + v = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2 + 1, y_1 + y_2 + 1)$

~~Since $x_1 + x_2$~~

Since $x_1 + x_2 + 1$, $y_1 + y_2 + 1$ are also real numbers

Therefore, $u + v$ is also an object in V. $(u + v \in V)$

② $u + v = (x_1 + x_2 + 1, y_1 + y_2 + 1)$

$= (x_2 + x_1 + 1, y_2 + y_1 + 1)$

$= v + u$ $(u + v = v + u)$.

Therefore, Vector addition is commutative.

③ $u + (v + w)$

$(x_1, y_1) + [(x_2, y_2) + (x_3, y_3)]$

$(x_1, y_1) + [x_2 + x_3 + 1, y_2 + y_3 + 1]$

$[x_1 + (x_2 + x_3 + 1) + 1, y_1 + (y_2 + y_3 + 1) + 1]$

$[(x_1 + x_2 + 1) + x_3 + 1, (y_1 + y_2 + 1) + y_3 + 1)]$

$[x_1 + x_2 + 1, y_1 + y_2 + 1) + (x_3 + y_3)$

$= (u + v) + w$.

Hence the vector addiction is associative.

4 Let (a,b) be an object in V such that $(a,b)+u=u$

$(a,b)+(x1,y1)=(x1,y1)$

$(a+x1+1,\ b1+y1+1)=(x1,y1)$

$a=-1,\quad b=-1$

Hence, $(-1,-1)$ is zero vector in V.

5 let (a,b) be an object in V such that $(a,b)+u=(-1,-1)$

$(a,b)+(x1,y1)=(-1,-1)$

$(x1+a+1,\ y1+b+1)=(1,-1)$

$a=-x1-2,\quad b=-y1-2$

Hence, $(-x1-2,\ y1-2)$ is the negative of u in V.

6 $k1u=k1(x1,y1)$

$=k1x1,\ k1y1$

Since $k1x1$, $k1y1$ are real numbers.

Therefore, V is closed under scalar multiplication.

7 $k1(u+v)=k1(x1+x2+1,\ y1+y2+1)$

$=(k1x1+k1x2+k1,\ k1y1+k1y2+k1)$

$\neq k1u+k1v$

V is not distributive under scalar multiplication

Hence, V is not a vector space.

SUB SPACES:

For any Vectorspace, a subspace is a subset that is itself a vectorspace under the Vector addition and Scalar multiplication operations.

Definition:

$(V, +, \cdot)$: a Vectorspace

$\left.\begin{array}{l} W \neq \emptyset \\ W \subseteq V \end{array}\right\}$: a non empty subset

$(W, +, \cdot)$: a vectorspace (under the operations of addition and Scalar multiplication defined in V)

$\Rightarrow$ W is a subspace of V.

Prove that $w = \left\{ \begin{pmatrix} x \\ 2x+3z \\ z \end{pmatrix} : x, z \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$

Solution

$$\begin{pmatrix} x \\ 2x+3z \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}_{u} x + \underbrace{\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}}_{v} z \quad ; x, z \in \mathbb{R}$$

let $u = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $v \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$

$$u+v = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} \in W$$

$$xu, \ x \in \mathbb{R} = x \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} x \\ 2x \\ 0 \end{pmatrix} \in W$$

Since Vector addition and scalar multiplication rules are satisfied, w is a subspace of $\mathbb{R}^3$

Show that $V = \{(x, -3x) \mid x \in R\}$ is a subspace of $\mathbb{R}^2$

Solution

let $u, w \in V$

let $u = (a, -3a)$

$w = (b, -3b)$

Vector addition.

$u + w = [(a+b), (-3a + -3b)]$

$u + w \ [(a+b), -3(a+b)]$

V is a set of coordinates x, y such that $y = -3x$

If $x = a+b$, $y = -3(a+b)$

$u + w \in V$

Scalar multiplication

let $r \in R$, then $r.u \in V$

$\Rightarrow r(a, -3a) = (ra, r(-3a))$

$(ra, -3(ra))$

$\downarrow \quad \downarrow$

$x \quad y$

Since V is defined on Vector addition and Scalar multiplicat

then V is a subspace of $\mathbb{R}^2$.

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Span of Set of Vectors

A spanning set of the vectorspace is defined as a set of all the vectors in the set $S = \{v_1, v_2 \cdots v_r\}$ is called span of S and denoted by span $\{v_1, v_2, \cdots v_r\}$.

→ If $S = \{v_1, v_2 \cdots v_r\}$ is a set of vector in a vector space V then,

(1) The span is a subspace of V

(2) The span S is the smallest subspace of V that contains the set S.

– If S_1 and S_2 are two sets of vectors in V then, span S_1 = Span S_2

If and only if each Vector in S_1 is a linear Combination of those in S_1 and S_2 and Vice versa.

Method to check span of V

① choose an arbitrary Vector b in V

② Express b as a linear combination of $v_1, v_2, \cdots v_r$

$b = k_1 v_1 + k_2 v_2 + \cdots + k_r v_r$

③ if the system of the equation is Consistent for all choice of b then vectors $v_1, v_2 \cdots v_r$ span V. If it is inconsistent for some choice of b, vectors do not span V.

Consider a set of Vectors. $\{v_1, v_2 \cdots v_k\}$

The set of all linear Combinations of this Vector space is called a span.

Determine whether following Vector Span the R^3

$V_1\ (2, 2, 2)$

$V_2\ (0, 0, 3)$

$V_3\ (0, 1, 1)$

let $b = (b_1, b_2, b_3)$ be an arbitary vector in R^3 and can be expressed as a linear combination of given vectors.

$$b = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$(b_1, b_2, b_3) = (2u_1,\ 2u_1 + u_3,\ 2k_1 + 3u_2 + u_3)$$

$$2k_1 = b_1$$
$$2k_1 \quad k_3 = b_2$$
$$2k_1 + 3k_2 + k_3 = b_3$$

Coefficient matrix = $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix}$

let $A = -6 \neq 0$

Hence given vectors span R^3

$\boxed{2}$ Given $w = \left\{ \begin{pmatrix} x \\ 2x + z \\ z \end{pmatrix},\ x, z \in R \right\}$,

find $[w]$ or Span $[w]$

Solution

$$w = \begin{pmatrix} x \\ 2x + z \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\text{Span}(w) = \left\{ x \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},\ x, z \in R \right\}$$

Linear Combination

A vector v is a vectorspace V is called a linear combination of Vectors $u_1, u_2, \ldots, u_k$ in V if v can be written in the form

$$V = c_1 u_1 + c_2 u_2 + \ldots + c_k u_k$$

$c_1, c_2, \ldots, c_k$: Scalars

If $S = (w_1, w_2, w_3 \ldots w_r)$ is a nonempty set of vectors in a Vector space V, then:

a) The set W of all possible linear Combinations of the Vectors in S is a subspace of V

b) The set W in part (a) is the "smallest" subspace of V that contains all of the vectors in S in the Sense that any other subspace that contains those vectors contains W

Example:

Every vector $V = (a, b, c)$ in R^3 is expressible as a linear Combination of the standard basis vectors $i = (1,0,0)$, $j = (0,1,0)$, $k = (0,0,1)$ Since.

$$V = (a, b, c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$$

$$= ai + bj + ck$$

Example:

Consider the vectors $u = (1,2,-1)$ and $v(6,4,2)$ in R^3. Show that $w = (9,2,7)$ is a linear combination of u and V and that $w = (4,-1,8)$ is not a linear Combination of u and V.

$$w' = k_1 u + k_2 V$$

$$(4,-1,8) = k_1(1,2,-1) + k_2(6,4,2)$$

$$(4,-1,8) = (k_1 + 6k_2,\ 2k_1 + 4k_2,\ 2k_1 + 2k_2)$$

$$k_1 + 6k_2 = 4$$

$2k_1 + 4k_2 = -1$

$2k_1 + 2k_2 = 8$

finding a linear combination

$V_1 = (1,2,3)$ $V_2 = (0,1,2)$ $V_3 = (-1,0,1)$

Prove $W = (1,2,2)$ is not a linear combination of V_1, V_2, V_3

Solution

$W = C_1 V_1 + C_2 V_2 + C_3 V_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 2 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gauss Jordan}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

This system has no solution ($0 \neq 7$)

$\Rightarrow W \neq C_1 V_1 + C_2 V_2 + C_3 V_3$

Basis of a Vector Space

V a vector space.

$S = \{V_1, V_2, \dots V_n\} \subseteq V$

$\boxed{\begin{array}{l} S \text{ spans } V \text{ (i.e., } \operatorname{Span}(S) = V \\ S \text{ is linearly independent} \end{array}}$

$\Rightarrow$ S is called a basis for V

let V denote a Vector space and $S = \{u_1, u_2, \dots, u_n\}$ a Subset of V.

S is called a basis for V if the following is true:

i) S spans V

ii) S is linearly independent.

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A subspace is linearly independent of ~~more~~ none of its elements is a linear combination of the others

i.e given $V_i = \{V_1, V_2\}$, V_1 and V_2 are linearly independent if $V_1 \neq kV_2$; $k \in \mathbb{R}$.

A set of Vectors u_i is linearly independent iff (if and only if).

$\sum c_i u_i = 0$ has a trivial Solution. otherwise it is linearly dependent. i.e $c_i = 0$

$$\Rightarrow \underset{\underset{0}{\uparrow}}{C_1} u_1 + \underset{\underset{0}{\uparrow}}{C_2} u_2 + \cdots + \underset{\underset{0}{\uparrow}}{C_n} u_n = 0$$

In other words, a set of Vectors $V_1, V_2, \ldots V_n \in V$ are linearly dependent if there exists a set of scalars c_i not all zero Such that

$$\sum c_i v_i = 0$$

Examples.

Determine of the Vectors below are linearly dependent or independent.

1) $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $V_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

ii) $u_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $u_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $u_3 \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$

iii) $w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $w_2 = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$

4) Given $V_1 \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ and $V_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$,

linear independent.

Show that $V = \{V_1, V_2\}$ is a basis for $\mathbb{R}^2$

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Solution.

1) $v_1\begin{pmatrix}1\\1\end{pmatrix}$, $v_2\begin{pmatrix}1\\0\end{pmatrix}$

V_1 and V_2 are linearly independent if $c_1V_1+c_2V_2=0$ has a trivial Solution.

$$c_1\begin{pmatrix}1\\1\end{pmatrix}+c_2\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}c_1\\c_1\end{pmatrix}+\begin{pmatrix}c_2\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}c_1+c_2\\c_1+0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\left[\begin{array}{cc|c}1&1&0\\1&0&0\end{array}\right]$$

$$-R_1+R_2\quad\left[\begin{array}{cc|c}1&1&0\\0&-1&0\end{array}\right]$$

$$-R_2\quad\left[\begin{array}{cc|c}1&1&0\\0&1&0\end{array}\right]$$

$c_2=0$

$c_1+c_2=0$

$c_1+0=0$

$c_1=0$.

$\therefore$ V_1 and V_2 are linearly independent.

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$$

u_1, u_2, u_3 are linearly independent if $c_1u_1 + c_2u_2 + c_3u_3 = 0$ has a trivial Solution.

$$c_1\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ 0 \\ c_1 \end{pmatrix} + \begin{pmatrix} c_2 \\ c_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -2c_3 \\ 3c_3 \\ -c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 + c_2 + -2c_3 \\ 0 + c_2 + 3c_3 \\ c_1 + 0 - c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$$

$$-R_1 + R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right]$$

$$R_2 + R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

$$\tfrac{1}{4}R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$c_3 = 0$

$c_2 + 3c_3 = 0$

$c_2 = 0$

$c_1 + c_2 - 2c_3 = 0$

$c_1 = 0$

Vectors u_1, u_2 and u_3 are linearly independent.

$w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $w_2 \begin{pmatrix} -2 \\ -2 \end{pmatrix}$

w_1 and w_2 are linearly independent if $c_1 w_1 + c_2 w_2 = 0$ has a trivial solution

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} -2c_2 \\ -2c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 - 2c_2 \\ c_1 - 2c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\cancel{\begin{pmatrix} 1 & -2 \end{pmatrix}} \xrightarrow{-R_1 + R_2} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} c_1 - 2c_2 = 0 \\ \Rightarrow c_1 = 2c_2 \end{array}$$

w_1 and w_2 are linearly dependent.

Matrix Algebra.

Is a rectangular array of numbers.

A matrix is a simple mathematical structure that holds numeral information in a rectangular or tabular form.

→ rows
↓ columns.

Matrix are always in upper case ie A, B, C

They are always in [ ]

eg $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ 2X2 $B = \begin{bmatrix} 1 & 2 \\ 4 & 6 \\ 1 & -1 \end{bmatrix}$ 2X3 $C = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ 3 X 1

Terminologies used.

(i) Order of a matrix | Dimension | Size

This refering to the no of rows and columns in a matrix.

written as Rows X Columns.

m X n

m n

(ii) Column matrix.

This is a matrix with only one Column. it is also known as the column vector

eg $C = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ 3X1

(iii) Row matrix.

This is a matrix with only one Row.

eg [1 3 4] 1X3

iv Scalar matrix.

This is a matrix that contains only one entry

eg $D = [2]$

1X1

Equal matrices
These are matrices that are equal in size and have corresponding entrances eg

$$P = \begin{pmatrix} 0 & 2 \\ 5 & 1 \end{pmatrix} \quad Q = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix} \quad S\begin{pmatrix} 0 & 2 \\ 5 & 1 \end{pmatrix}$$

$$\therefore P = S$$

Square matrix
This is a matrix with equal number of rows and colums.

$$P = \begin{pmatrix} 0 & 2 \\ 5 & 1 \end{pmatrix} \quad Q = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix} \quad S\begin{pmatrix} 0 & 2 \\ 5 & 1 \end{pmatrix}$$

$2 \times 2 \qquad 2 \times 2 \qquad 2 \times 2$

Diagonal matrix
This is a square matrix whose elements ā in that are in ē major diagonal are non zero's and all others are Zero's.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$ major diagonal.

Identity matrix / unit matrix
This is a square matrix whose elements in ē major diagonal are ones (1) and the rest are zero's

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transpose of a matrix.
This is a matrix obtained after interchanging the rows of a matrix and the columns of the same matrix.

$$P^T \begin{bmatrix} 0 & 5 \\ 2 & 1 \end{bmatrix} \text{ using } P\begin{bmatrix} 0 & 2 \\ 5 & 1 \end{bmatrix}$$

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Zero / null matrix.

This is a matrix with only zero's eg.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Symmetric matric.

This is a square matric whose transpose is the same with the orginal matrix.

$$kl = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad w^T \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

in ~~dots~~ conlusion all diagonal matrix are the same of as symmetric matric.

Read about
- Triagular
- Sigular and non sigular.

$$\begin{matrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

det (0)

Properties of Matrices.

1. matrix $A + B = B + A$
2. $A + B + C = A + (B + C) = (A + B) + C$
3. $AB = BA$ if and only if one of the matrix is a Scalar matrix.
   And if one of the matrix is an indentity Rule.
4. $A \cdot B \cdot C = A(BC) = AB(C)$
   if and only if $A_{m \times n}$ $B_{n \times q}$ $C_{q \times p}$
5. $A(B + C) = AB + AC$
6. If we have a transpose of a product of matrixes it is the same as

$$(AB)^T = A^T \times B^T$$

7. If we have an matrix $A \cdot A^{-1} = I_n$ (Identity inverse)

$$A \cdot A^{-1} \times A \cdot A^{-1} = I_n$$

8. If we have an inverse of A

$$(A^{-1})^{-1} = A$$

9. $(A^{-1})^T = (A^T)^{-1}$

10. $(AB)^{-1} = A^{-1} \times B^{-1}$

11 $(A^T)^T = A$

12 $(A+B)^T = A^T + B^T$

13 $AB \neq BA$ if properties above donot hold.

eg $A_{m \times n}$ $B_{n \times q}$

14 $A + 0 = A$

15 $(A + (-A)) = 0$

16 $k(A+B) = kA + kB$ ie k is a scalar.

OPERATIONS.

- Addition. & substraction.
- Scalar multiplication
- Multiplication of matrices.

iii) $\frac{1}{2} B$.

$$= \begin{pmatrix} 3/2 & \cancel{1} & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

Matrix multiplication.

For multiplication to be possible it is essential. That the number of columns in the 1st matrix should be equal to number of rows in the 2st matrix

ie

Columns

Rows.

$A_{m \times n}$ $B_{n \times q}$ $AB_{m \times q}$

$n = n$

New matrix

$m \times q$

$A_{3 \times 2}$ $B_{2 \times 3}$

$2 = 2$

New matrix.

3×3 .

Answer $AB_{3 \times 3}$

Given $A = \begin{pmatrix} -2 & 4 \\ 5 & 3 \end{pmatrix}$

&

$B = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$

$$AB = \begin{pmatrix} -6+28 & -4+5 \\ 15+21 & 10+15 \end{pmatrix}$$

$$AB = \begin{pmatrix} 22 & 1 \\ 36 & 25 \end{pmatrix}$$

$$BA \; \cancel{\left((-6+28)\right)}$$

$$\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ 5 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} (-6+10) & (12+6) \\ (-14+25) & (28+15) \end{pmatrix}$$

$$BA = \begin{pmatrix} 4 & 18 \\ 11 & 43 \end{pmatrix}$$

$$AB \neq BA$$

$$A = \begin{pmatrix} -2 & 4 \\ 5 & 3 \\ 4 & -1 \end{pmatrix} \quad \& \quad B = \begin{pmatrix} -2 & 4 \\ 5 & 3 \end{pmatrix}$$

BA.

$$\begin{pmatrix} -2 & 4 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ 5 & 3 \\ 4 & -1 \end{pmatrix}$$

$2 \times 2 \qquad 3 \times 2$

BA Not possible.

$$AB = \begin{pmatrix} -2 & 4 \\ 5 & 3 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ 5 & 3 \end{pmatrix}$$

$$\begin{pmatrix} (4+20) & (-8+12) \\ (-10+25) & (20+9) \\ (-8-5) & (16-3) \end{pmatrix}$$

$$AB = \begin{pmatrix} 24 & 4 \\ 5 & 29 \\ -13 & 13 \end{pmatrix}$$

In any matrix product AB if the 1st matrix A has (m) rows and (n) colomns and Second matrix B has (n) rows and (p) colomns,

Then the product AB has m rows and P colomns.

If the elements lies in the ith row and jth colomn of AB is the same of the products of the Corresponding elements of the ith row of A and the j^{th} column of B

CB

$$C = \begin{bmatrix} 2 & 1 \\ 4 & 4 \\ 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

~~3×3~~

$3 \times 2 \quad 2 \times 2$

$$CB = \begin{bmatrix} (2+3) & (4+5) \\ (4+12) & (8+20) \\ (3+6) & (6+10) \end{bmatrix} \qquad CB = \begin{bmatrix} 5 & 9 \\ 16 & 28 \\ 9 & 16 \end{bmatrix}$$

Free Variable-

is a variable that can take on any Value

3x

$$\left[\begin{array}{ccc|c} 3 & -6 & 2 & -3 \\ 5 & 9 & 4 & 1 \\ 2 & 4 & 2 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -15/11 \end{array}\right]$$

$z = -15/11$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 5 & 9 & 4 & 1 \\ 3 & -6 & 2 & -3 \end{array}\right]$$

$y + z = -1$

$y + \frac{-15}{11} = -1$

$y = -1 + \frac{15}{11}$

$y = \frac{-11+15}{11}$

$= \frac{4}{11}$

$$\begin{array}{r} \\ -5R_1+R_2=R_2 \\ -3R_1+R_3=R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & -12 & -1 & -3 \end{array}\right]$$

$$\begin{array}{r} \\ \\ 12R_2+R_3=R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 11 & -15 \end{array}\right]$$

$x + 2y + z = 0$

$x + \frac{8}{11} - \frac{15}{11} = 0$

$x = 7/11$

the j^{th} Column of B

CB $\quad C = \begin{bmatrix} 2 & 1 \\ 4 & 4 \\ 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

~~3×3~~ $3 \times 2 \qquad 2 \times 2$

$$CB = \begin{bmatrix} (2+3) & (4+5) \\ (4+12) & (8+20) \\ (3+6) & (6+10) \end{bmatrix} \qquad CB = \begin{bmatrix} 5 & 9 \\ 16 & 28 \\ 9 & 16 \end{bmatrix}$$

Free Variable-

is a variable that can take on any Value

3x

$$\left[\begin{array}{ccc|c} 3 & -6 & 2 & -3 \\ 5 & 9 & 4 & 1 \\ 2 & 4 & 2 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -15/11 \end{array}\right]$$

$z = -15/11$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 5 & 9 & 4 & 1 \\ 3 & -6 & 2 & -3 \end{array}\right]$$

$$\begin{array}{r} \\ -5R_1+R_2=R_2 \\ -3R_1+R_3=R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & -12 & -1 & \cancel{0}\,-3 \end{array}\right]$$

$$\begin{array}{r} \\ \\ 12R_2+R_3=R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 11 & -15 \end{array}\right]$$

$y + z = -1$

$y + \frac{-15}{11} = -1$

$y = -1 + \frac{15}{11}$

$y = \frac{-11+15}{11}$

$= \frac{4}{11}$

$x + 2y + z = 0$

$x + \frac{8}{11} - \frac{15}{11} = 0$

$x = 7/11$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -15/11 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 15/11 \\ 0 & 1 & 0 & 4/11 \\ 0 & 0 & 1 & -15/11 \end{array}\right] \begin{array}{l} -R_3+R_1=R_1 \\ -R_3+R_2=R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7/11 \\ 0 & 1 & 0 & 4/11 \\ 0 & 0 & 1 & -15/11 \end{array}\right] \quad -2R_2+R_1=R_1$$

A1

b).
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 3 & 2 & 1 & 3 \\ 1 & -2 & -5 & 1 \end{array}\right]$$

$$A \quad \begin{array}{r} \\ -3R_1+R_2 \\ -R_1+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & -2 & -6 \\ 0 & -3 & -6 & -2 \end{array}\right]$$

$$-R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 6 \\ 0 & -3 & -6 & 2 \end{array}\right]$$

$$\begin{array}{r} \\ \\ 3R_2+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 20 \end{array}\right]$$

$$0 \neq 20$$

Determinant of a matrix.

Given A, the determinant is denoted as det A or |A| or ΔA and can only be obtained for square matrices.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

A square matrice is one wic has the same number of row and columns

$$|A| = ad - cb$$

given $A = [a]$

$$|A| = a$$

Higher order determinants.

To obtain the determinants of a 3×3, 4×4, 5×5, 6×6 matrix etc, we use the Laplace expression which involves obtaining minors and cofactors.

Minors.

A minor of a matrix is denoted as (M_{ij}) which is a determinant of a matrix obtained by deleting the ith row and jth column of a certain matrix

eg Given $\underset{m_i \quad j_n}{A_{3\times4}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

Given $B_{3\times3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

Sub matrix M_{11} $\left| \begin{bmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{bmatrix} \right|$

= Determinant of sub matrix $= (b_{22} \times b_{33}) - (b_{32} \times b_{23})$

Example ①

Given $A = \begin{bmatrix} 2 & 3 & 7 \\ -4 & 0 & 6 \\ 1 & 5 & 0 \end{bmatrix}$, find det A.

$M_{11} = \left|\begin{bmatrix} 0 & 6 \\ 5 & 0 \end{bmatrix}\right| = (0 \times 0) - (5 \times 6) = -30$

$M_{12} = \left|\begin{bmatrix} -4 & 6 \\ 1 & 0 \end{bmatrix}\right| = (0 \times -4) - (1 \times 6) = -6$

$M_{13} = \left|\begin{bmatrix} -4 & 0 \\ 1 & 5 \end{bmatrix}\right| = (-4 \times 5) - (0 \times 1) = -20$

$M_{21} = \left|\begin{pmatrix} 3 & 7 \\ 5 & 0 \end{pmatrix}\right| = (0 \times 3) - (5 \times 7) = -35$

$M_{22} = \left|\begin{pmatrix} 2 & 7 \\ 1 & 0 \end{pmatrix}\right| = (2 \times 0) - (7 \times 1) = -7$

$M_{23} = \left|\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}\right| = (5 \times 2) - (1 \times 3) = 7$

$M_{31} = \left|\begin{pmatrix} 3 & 7 \\ 0 & 6 \end{pmatrix}\right| = (3 \times 6) - (0 \times 7) = 18$

$M_{32} = \left|\begin{pmatrix} 2 & 7 \\ -4 & 6 \end{pmatrix}\right| = (2 \times 6) - (7 \times -4) = 40$

$M_{33} = \left|\begin{pmatrix} 2 & 3 \\ -4 & 0 \end{pmatrix}\right| = (0 \times 2) - (-4 \times 3) = 12$

$|A| = a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + a_{13} C_{13}$

or $a_{21} \cdot C_{21} + a_{22} C_{22} + a_{23}$

or $a_{31} C_{31} + a_{32} C_{32} + a_{33} C_{33}$

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Cofactors

A cofactor is a signed minor.
Denoted as C_{ij}
or is a signed determinant of a submatrix.

$$C_{ij} = (-1)^{i+j} \cdot M_{ij}$$

ij - even - $C_{ij} = m_{ij}$
ij - odd - $C_{ij} \neq = -m_{ij}$

Given our example:

$C_{11} = (-1)^{(1+1)} \cdot -30 = -30$

$C_{12} = (-1)^{(1+2)} \cdot -6 = 6$

$C_{13} = (-1)^{(1+3)} \cdot -20 = -20$

$C_{21} = (-1)^{(2+1)} \cdot -35 = 35$

$C_{22} = (-1)^{(2+2)} \cdot -7 = -7$

$C_{23} = (-1)^{(2+3)} \cdot 7 = -7$

$C_{31} = (-1)^{(2+1)} \cdot 18 = 18$

$C_{32} = (-1)^{(3+2)} \cdot 40 = -40$

$C_{33} = (-1)^{(3+3)} \cdot 12 = 12$

Cofactor matrix.

$$\begin{matrix} -30 & 6 & -20 \\ 35 & -7 & -7 \\ 18 & -40 & 12 \end{matrix}$$

$$\begin{matrix} -30 & 35 & 18 \\ 6 & -7 & -40 \\ -20 & -7 & 12 \end{matrix}$$

$\therefore \det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$= 2(-30) + 3(6) + 7(-20)$

$= -60 + 18 + -140$

$= -182$

or $\det A = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$

$1(18) + 5(-40) + 0(12)$

$18 - 200 + 0$

$= -182$

Inverse of a matrix : (A^{-1}).

Denoted as A^{-1}, can only be defined if A is a Square mat and it satisfies

$$A \cdot A^{-1} \times A^{-1} \cdot A = I_n$$

Procedures of determining A^{-1}

① Determine all the minor.
② Find the cofactors.
③ Find the determinant of the matrix
④ Put the cofactors together and form a cofactor matrix c.
⑤ Find C^T (the transpose of the cofactor matrix) Adjoint matrix
⑥ Determine the inverse by :-

$$A^{-1} = \frac{1}{|A|} \cdot \text{transpose of } \bar{e} \text{ Cofactor matrix } A$$

Using the previous example A.

$$C = \begin{bmatrix} -30 & 6 & -20 \\ 35 & -7 & -7 \\ 18 & -40 & 12 \end{bmatrix}$$

$$C^T = \begin{bmatrix} -30 & 35 & 18 \\ 6 & -7 & -40 \\ -20 & -7 & 12 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \cdot C^T$$

$$A^{-1} = \frac{1}{-182} \cdot \begin{bmatrix} -30 & 35 & 18 \\ 6 & -7 & -40 \\ -20 & -7 & 12 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 30/182 & -35/182 & -18/182 \\ -6/182 & 7/182 & 40/182 \\ 20/182 & 7/182 & -12/182 \end{bmatrix}$$

Prove for the inverse.

$AA^{-1} = I_n$

$$\begin{bmatrix} 2 & 3 & 7 \\ -4 & 0 & 6 \\ 1 & 5 & 0 \end{bmatrix} \begin{bmatrix} 30/182 & -35/182 & -18/182 \\ -6/182 & 7/182 & 40/182 \\ 20/182 & 7/182 & -12/182 \end{bmatrix}$$

$$\left(2 \times 30/182\right) + \left(3 \times -6/182\right) + \left(7 \times 20/182\right)$$

$$\frac{60 - 18 + 140}{182}$$

$$= \frac{182}{182}$$

$$= 1$$

$$\left(2 \times -35/182\right) + \left(3 \times 7/182\right) + \left(7 \times 7/182\right)$$

$$\frac{-70 + 21 + 49}{182}$$

$$= 0$$

$$\left(2 \times -18/182\right) + \left(3 \times 40/182\right) + \left(7 \times -12/182\right)$$

$$\frac{-36 + 120 + -84}{182}$$

$$0/182$$

$$= 0$$

$$\left(-4 \times 30/182\right) + \left(0 \times 6/182\right) + \left(6 \times 20/182\right)$$

$$\frac{-120 + 0 + 120}{182}$$

$$= 0$$

$$\left(-4 \times -35/182\right) + \left(0 \times 7/182\right) + \left(6 \times 7/182\right)$$

$$\frac{140 + 0 + 42}{182} = 1$$

$$\left(-4 \times -18/182\right) + \left(0 \times 40/182\right) + \left(6 \times -12/182\right)$$

$$\frac{72 + 0 + -72}{182}$$

$$= 0$$

$$\left(1 \times 30/182\right) + \left(5 \times -6/182\right) + \left(0 \times 20/182\right)$$

$$\frac{30 - 30 + 0}{182}$$

$$= 0$$

$$\left(1 \times -35/182\right) + \left(5 \times 7/182\right) + \left(0 \times 7/182\right)$$

$$\frac{-35 + 35 + 0}{182}$$

$$= 0$$

$$\left(1 \times -18/182\right) + \left(5 \times 40/182\right) + \left(0 \times -12/182\right)$$

$$\frac{-18 + 200 + 0}{182}$$

$$= 1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ identity matrix

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Example 3.

Find the inverse of $B = \begin{bmatrix} 3 & 5 \\ -2 & 1 \end{bmatrix}$

$|B| = (3 \times 1) - (5 \times -2)$

$3 + 10$

$|B| = 13$

Adjoint $B = \begin{bmatrix} 1 & -5 \\ 2 & 3 \end{bmatrix}$

$B^{-1} = \frac{1}{|B|} \cdot \begin{bmatrix} 1 & -5 \\ 2 & 3 \end{bmatrix}$

$\frac{1}{13} \begin{bmatrix} 1 & -5 \\ 2 & 3 \end{bmatrix}$

$\begin{bmatrix} 1/13 & -5/13 \\ 2/13 & 3/13 \end{bmatrix}$

Prove the inverse.

$B \cdot B^{-1} = I_n$.

$\begin{bmatrix} 3 & 5 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/13 & -5/13 \\ 2/13 & 3/13 \end{bmatrix}$

$(3 \times 1/13) + (5 \times 2/13)$

$\frac{3+10}{13}$

$= 1$

$(3 \times -5/13) + (5 \times 3/13)$

$\frac{-15+15}{13}$

$= 0$

$(-2 \times 1/13) + (1 \times 2/13)$

$\frac{-2+2}{13}$

$= 0$

Adjoint

Note: ① change the element in the main Diagonal

② change the signs in the minor

$(-2 \times -5/13) + (1 \times 3/13)$

$\frac{10+3}{13}$

$= 1$

$B \cdot B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_n$

$\boxed{2}$ $A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$ find A^{-1}.

$|A| = (4 \times 2) - (3 \times 3)$

$|A| = -1$

Adjoint $A = \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} \cdot \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}$

$\frac{1}{-1} \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} =$

$A^{-1} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$

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Example 4.

Given $A = \begin{bmatrix} 5 & 7 & 9 \\ 4 & 3 & 8 \\ 7 & 5 & 6 \end{bmatrix}$, find A^{-1}

~~Given~~ Given Solution.

minors:

$m_{11} \left|\begin{pmatrix} 3 & 8 \\ 5 & 6 \end{pmatrix}\right| = 18 - 40 = -22$

$m_{12} \left|\begin{pmatrix} 4 & 8 \\ 7 & 6 \end{pmatrix}\right| = 24 - 56 = -32$

$m_{13} \left|\begin{pmatrix} 4 & 3 \\ 7 & 5 \end{pmatrix}\right| = 20 - 21 = -1$

$m_{21} \left|\begin{pmatrix} 7 & 9 \\ 5 & 6 \end{pmatrix}\right| = 42 - 45 = -3$

$m_{22} \left|\begin{pmatrix} 5 & 9 \\ 7 & 6 \end{pmatrix}\right| \; 30 - 63 = -33$

$m_{23} \left|\begin{pmatrix} 5 & 7 \\ 7 & 5 \end{pmatrix}\right| \; 25 - 49 = -24$

$M_{31} \left|\begin{pmatrix} 7 & 9 \\ 3 & 8 \end{pmatrix}\right| \; 56 - 27 = 29$

$M_{32} \left|\begin{pmatrix} 5 & 9 \\ 4 & 8 \end{pmatrix}\right| \; 40 - 36 = 4$

$M_{33} \left|\begin{pmatrix} 5 & 7 \\ 4 & 3 \end{pmatrix}\right| = 15 - 28 = -13$

Cofactors.

$C_{ij} = (-1)^{i+j} \cdot M_{ij}$

$C_{11} = (-1)^{(1+1)} \cdot -22 = -22$

$C_{12} = (-1)^{(1+2)} \cdot (-32) = 32$

$C_{13} = (-1)^{(1+3)} \cdot (-1) = -1$

$C_{21} = (-1)^{(2+1)} (-3) = 3$

$C_{22} = (-1)^{(2+2)} \cdot (-33) = -33$

$C_{23} = (-1)^{(2+3)} \cdot (-24) = 24$

$C_{31} = (-1)^{(3+1)} \cdot (29) = 29$

$C_{32} = (-1)^{(3+2)} \cdot (4) = -4$

$C_{33} = (-1)^{(3+3)} \cdot (-13) = -13$

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$(5 \times -22) + (7 \times 32) + (9 \times -1)$

$-110 + 224 + -9$

$|A| = 105$

Cofactor matrix $\begin{bmatrix} \cancel{-22} \\ \cancel{32} \end{bmatrix}$

$$\begin{bmatrix} -22 & 32 & -1 \\ 3 & -33 & 24 \\ 29 & -4 & -13 \end{bmatrix}$$

$$C^T = \begin{bmatrix} -22 & 3 & 29 \\ 32 & -33 & -4 \\ -1 & 24 & -13 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \cdot C^T$$

$$\frac{1}{105} \cdot \begin{bmatrix} -22 & 3 & 29 \\ 32 & -33 & -4 \\ -1 & 24 & -13 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -22/105 & 3/105 & 29/105 \\ 32/105 & -33/105 & -4/105 \\ -1/106 & 24/105 & -13/105 \end{bmatrix}$$

Prove for the unverse.

$$A \cdot A^{-1} = I_n.$$

$$\begin{bmatrix} 5 & 7 & 9 \\ 4 & 3 & 8 \\ 7 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} -22/105 & 3/105 & 29/105 \\ 32/105 & -33/105 & -4/105 \\ -1/105 & 24/105 & -13/105 \end{bmatrix}$$

$$(5 \times -22/105) + (7 \times 32/105) + (9 \times -1/105)$$

$$\frac{-110 + 224 - 9}{105}$$

$$= 1$$

$$(5 \times 3/105) + (7 \times -33/105) + (9 \times 24/105)$$

$$\frac{15 + -231 + 216}{105}$$

$$= 0$$

$$(5 \times 29/105) + (7 \times -4/105) + (9 \times -13/105)$$

$$= 0$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

APPLICATION OF MATRIX ALGEBRA

Storing information.

Branch	A	B	C
Kla	20	30	10
Arua	15	16	10
Gulu	10	20	17

No of employees in an organisation

$$\begin{array}{cl} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} & \begin{array}{l} \text{Peon} \\ \text{Technicians} \\ \text{operations manager.} \end{array} \end{array}$$

Transport costs.

1. B'ness applications.
   Applications of matrix algebra.
2. Solving linear equations.
3. Input and ouput analysis / industry analysis.
4. Eigen vectors and given Values.

1. B'NESS APPLICATIONS / MANAGEMENT.

In b'ness mgt matrix algebra can be used to determine prices, profits, costs, revenue and any other b'ness situations.

Example 1

The quaterly sells of books pens and pencils for the year 2020 and 2021 are given below.

Year 2020 (A)

$$\begin{array}{l} \\ \text{Books} \\ \text{Pens} \\ \text{Pencils} \end{array} \begin{array}{c} \begin{array}{cccc} Q_1 & Q_2 & Q_3 & Q_4 \end{array} \\ \begin{bmatrix} 20 & 25 & 22 & 20 \\ 10 & 20 & 18 & 10 \\ 15 & 20 & 15 & 15 \end{bmatrix} \end{array}$$

Year 2021 (B)

$$\begin{array}{l} \text{Books} \\ \text{Pens} \\ \text{Pencils} \end{array} \begin{bmatrix} 10 & 15 & 20 & 20 \\ 5 & 20 & 18 & 10 \\ 8 & 30 & 15 & 10 \end{bmatrix}$$

Find the totaly quarterly sales of books, pens and pencils of the two years.

$$A+B = \begin{bmatrix} 30 & 40 & 42 & 40 \\ 15 & 40 & 36 & 20 \\ 23 & 50 & 30 & 25 \end{bmatrix}$$

Nelly
P Q R

Example (8)

Nelly, Marvin and Becky purchased biscuits of different brands ~~PQR~~ P, Q and R. Nelly purchase 10 packets of P, 7 packets of Q and 3 packets of R. Marvin purchased 4 packets of P, 8 packets of Q and 10 packets of R. Becky purchased 4 packets of P, 7 packets of Q and 8 packets of R.
If brand P costs Shs 600, Q sh 400 and R sh 500 each, using matrix operation, find the amount of money spent by these persons individually.

Solution

Let Q denote matrix for biscuit quantities and P denote matrix for biscuit prices.

$$Q = \begin{array}{c} \\ \text{Nelly} \\ \text{Marvin} \\ \text{Becky} \end{array} \begin{array}{c} \begin{array}{ccc} P & Q & R \end{array} \\ \begin{bmatrix} 10 & 7 & 3 \\ 4 & 8 & 10 \\ 4 & 7 & 8 \end{bmatrix} \end{array} \qquad P = \begin{array}{c} \text{Price of P} \\ \text{Price of Q} \\ \text{price of R} \end{array} \begin{bmatrix} 600 \\ 400 \\ 500 \end{bmatrix}$$

$$Q \times P = \begin{bmatrix} 10 & 7 & 3 \\ 4 & 8 & 10 \\ 4 & 7 & 8 \end{bmatrix} \begin{bmatrix} 600 \\ 400 \\ 500 \end{bmatrix}$$

$$= \begin{bmatrix} (10\times600)+(7\times400)+(3\times500) \\ (4\times600)+(8\times400)+(10\times500) \\ (4\times600)+(7\times400)+(8\times500) \end{bmatrix}$$

$$= \begin{bmatrix} 6000+2800+1500 \\ 2400+3200+5000 \\ 2400+2800+4000 \end{bmatrix}$$

$$= \begin{bmatrix} 10,300 \\ 10,600 \\ 9,200 \end{bmatrix} \begin{array}{l} \text{Nelly} \\ \text{Marvin} \\ \text{Becky} \end{array}$$

Example (4)

Ms Jane went to a market to purchase 3kg of Sugar, 10kg of wheat and 1kg of Salt. In a shop near Ms Jane's residence, these commodities are priced at 3400, 6000 and 1000 shillings respectively whereas in a local market, these commodities are priced at 3000, 5500 and 700 shillings respectively.
If the cost of travelling to the local market is shs 1000, find the net savings of Ms Jane using matrix multiplication method

Solution:

Let J denote the matrix of items to be purchased by Jane
P denote the price of these items at nearby shop
Q denote the price of these items in the market.

$$J \times P = \begin{array}{ccc} S & W & Sa \end{array} \begin{bmatrix} 3 & 10 & 1 \end{bmatrix} \times \begin{bmatrix} 3400 \\ 6000 \\ 1000 \end{bmatrix}$$

$= 3 \times 3400 + 10 \times 6000 + 1 \times 1000$

$= 10,200 + 60,000 + 1000$

$= 71,200$ at near by shop.

$$J \times Q = \begin{bmatrix} 3 & 10 & 1 \end{bmatrix} \times \begin{bmatrix} 3000 \\ 5500 \\ 700 \end{bmatrix}$$

$= 9000 + 55000 + 700$

$= 64,700$ at market

Add transport of 1000

$= 65,700$ at market.

$\begin{bmatrix} 71,200 & 64,700 \end{bmatrix}$

$= 71200 - 65,700$

$= 5500 =$ net savings.

Prices

Item

$$\begin{array}{c} S \\ W \\ Sa \end{array} \begin{bmatrix} 3400 & 6000 & 1000 \\ 3000 & 5500 & 700 \end{bmatrix} \begin{bmatrix} 3 \\ 10 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3400 \times 3 + 6000 \times 10 + 1000 \times 1 \\ 3000 \times 3 + 5500 \times 10 + 700 \times 1 \end{bmatrix}$$

$=$

coefficient matrix $\downarrow$ $AX = B$ $\checkmark$ matrix for constants.

$\uparrow$ matrix for unknowns

$X = A^{-1}B$

Solving systems of linear Equations:

For equations with more than one unknown, we use the matrix algebra to solve for the unknowns using the matrix inverse.

Given $AX = B$

where A = matrix for the coefficients

X = matrix for the unknowns

B = matrix for the constants.

multiply by A^{-1} all through

$$A \cdot A^{-1} X = A^{-1} \cdot B$$

but $A \cdot A^{-1} = I_n$.

$$\Rightarrow IX = A^{-1}B$$

$$\therefore \boxed{X = A^{-1}B}$$

$3x + 2y = 8$

$4x - 3y = 5$

$$X = \frac{Dx}{D}, \quad Y = \frac{Dy}{D}$$

$\boxed{\text{Crammer's rule}}$

D = Determinant of original matrix

Dx = Determinant of original matrix but with solution on RHS substituted in for X

Dy = Determinant of original matrix but with solution on RHS substituted in for Y.

Example ①

Given $3x + 2y = 8$, find the value of x and y.

$4x - 3y = 5$

Solution:

$$\begin{array}{ccc} A & X & B \\ \begin{bmatrix} 3 & 2 \\ 4 & -3 \end{bmatrix} & \begin{bmatrix} x \\ y \end{bmatrix} = & \begin{bmatrix} 8 \\ 5 \end{bmatrix} \end{array}$$

$\det A = -3\times3 - (4\times2)$

$= -9 - 8$

$= -17.$

Adjoint $A = \begin{bmatrix} -3 & -2 \\ -4 & 3 \end{bmatrix}$.

$A^{-1} = \frac{1}{|A|} \cdot \text{Adjoint } A$

$= \frac{1}{-17}\begin{bmatrix} -3 & -2 \\ -4 & 3 \end{bmatrix}$

$= \begin{bmatrix} 3/17 & 2/17 \\ 4/17 & -3/17 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3/17 & 2/17 \\ 4/17 & -3/17 \end{bmatrix}\begin{bmatrix} 8 \\ 5 \end{bmatrix}$

$x = (\frac{3}{17}\times8) + (\frac{2}{17}\times5) = \frac{24+10}{17}$

$x = \frac{24+10}{17}$

$= \frac{34}{17}$

$\underline{x = 2.}$

$y = (\frac{4}{17}\times8) + (\frac{-3}{17}\times5)$

$= \frac{32-15}{17}$

$= \frac{17}{17}$

$\underline{y = 1}$

(Using Crammer's rule)

$x = \frac{Dx}{D}$

$D = -17$

$Dx = \begin{vmatrix} 8 & 2 \\ 5 & -3 \end{vmatrix}$

$= (-3\times8) - (5\times2)$

$= -24 - 10$

$= -34$

$x = \frac{Dx}{D}$

$= \frac{-34}{-17}$

$= 2$

$y = \frac{Dy}{D}$.

$Dy = \begin{vmatrix} 3 & 8 \\ 4 & 5 \end{vmatrix}$

$= (5\times3) - (4\times8)$

$= 15 - 32$

$= -17$

$\therefore \cancel{x}\ y = \frac{Dy}{D}$

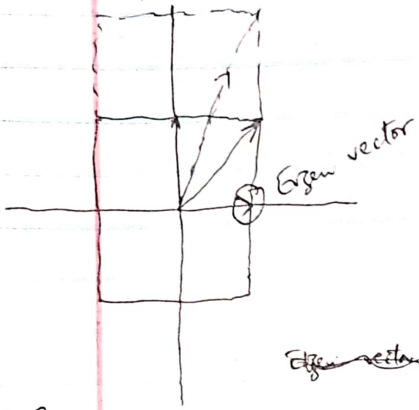
$= \frac{-17}{-17}$

$\underline{y = 1}$

EIGEN VECTORS

02/12/2021
Thursday.

An Eigen vector is a non zero vector whose direction doesn't change when linear transformation is applied to it.
Eigen values or scalars are a special type of scalars that are associated with the linear system of equations or matrix equations that are known as the x-tic roots, values, proper scalar or latent variables



~~Eigen vector for and~~

$\lambda = 2$

Mathematical expression:

$$AX = \lambda X$$

LHS
$A \rightarrow$ matrix of transformation
$X \rightarrow$ Eigen vector
RHS
λ = Eigen value.

Proof

Given $AX = \lambda X$

$\Rightarrow AX - \lambda X = 0$

$(A - \lambda)X = 0$

$\Rightarrow \boxed{(A - \lambda I)X = 0} \rightarrow$ produces Eigen vector

$\boxed{|A - \lambda I| = 0} \rightarrow$ produces Eigen values.

Example ①

Show that $x = \begin{pmatrix}2\\1\end{pmatrix}$ is the Eigen vector for $A = \begin{pmatrix}3 & 2\\2 & +2\end{pmatrix}$ corresponding to $\lambda = 4$.

Solution

$AX = \lambda X$

LHS: $AX = \begin{pmatrix}3 & 2\\2 & +2\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}$

$= \begin{pmatrix}(3\times2)+(2\times1)\\(2\times2)+(+2\times1)\end{pmatrix}$

$AX = \begin{pmatrix}8\\6\end{pmatrix}$.

RHS $= \lambda X = 4\begin{pmatrix}2\\1\end{pmatrix}$

$= \begin{pmatrix}8\\4\end{pmatrix}$

$\therefore$ Since LHS $\neq$ RHS ie $AX \neq \lambda X$, vector $X = \begin{pmatrix}2\\1\end{pmatrix}$ is not an eigen vector for $A = \begin{pmatrix}3 & 2\\2 & 2\end{pmatrix}$

Example ②

Given $A = \begin{pmatrix}1 & 1\\-2 & 4\end{pmatrix}$, find all the Eigen vectors and their Eigen values.

Eigen vector for matrix $A = \begin{pmatrix}1 & 1\\-2 & 4\end{pmatrix}$ corresponding to Eigen value $\lambda = 2$ is $\begin{pmatrix}1\\1\end{pmatrix}$.

Solution

$|A - \lambda I| = 0$

$\left|\begin{pmatrix}1 & 1\\-2 & 4\end{pmatrix} - \lambda\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right| = 0$

$\left|\begin{pmatrix}1-\lambda & 1\\-2 & 4-\lambda\end{pmatrix}\right| = 0$

$(4-\lambda)(1-\lambda) + 2 = 0$

$4 - 4\lambda - \lambda + \lambda^2 + 2 = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$\lambda^2 - 2\lambda - 3\lambda + 6 = 0$

$\lambda(\lambda - 2) - 3(\lambda - 2) = 0$

$(\lambda - 3)(\lambda - 2) = 0$

$\lambda - 3 = 0$

$\Rightarrow \lambda_1 = 3$

or $\lambda - 2 = 0$

$\lambda_2 = 2$.

Eigen values are $\lambda_1 = 3$ and $\lambda_2 = 2$.

Using $\lambda_1 = 3$

$(A - \lambda_1 I)X = 0$

$\left[\begin{pmatrix}1 & 1\\-2 & 4\end{pmatrix} - 3\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right]X = 0$

$\begin{pmatrix}-2 & 1\\-2 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$\begin{array}{c} R_1 \\ R_2 \end{array}\left[\begin{array}{cc|c}-2 & 1 & 0\\-2 & 1 & 0\end{array}\right]$

$C_{ij} = -1^{(i+j)} \cdot M_{ij}$

$-R_1 + R_2 = R_2 \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow x_1 - \frac{1}{2}x_2 = 0$

$\Rightarrow x_1 = \frac{1}{2}x_2$

$\Rightarrow 2x_1 = x_2$

let $x_2 = r$

$\Rightarrow 2x_1 = r$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix}$

let $x_1 = r$

$= \begin{pmatrix} r \\ 2r \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} r$

using $\lambda_2 = 2$,

$(A - \lambda_2 I)X = 0$

$\left[\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] X = 0$

$\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{array}{c} R_1 \\ R_2 \end{array}\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -2 & 2 & 0 \end{array}\right]$

$\begin{array}{c} -R_1 \to R_1 \\ -2R_1 + R_2 \to R_2 \end{array}\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow x_1 - x_2 = 0$

$x_1 = x_2$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}$

let $x_1 = v$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} v$

$\therefore \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is the eigen vector corresponding to eigen value $\lambda = 3$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the eigen vector corresponding to eigen value $\lambda = 2$.

Example (3) — $\lambda = 4$, E.V $= \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Given $B = \begin{pmatrix} 7 & 3 \\ -3 & 1 \end{pmatrix}$, find all the Eigen vectors & their corresponding Eigen values

Solution:

$B = \begin{pmatrix} 7 & 3 \\ 3 & -1 \end{pmatrix}$

$\lambda_1 = 8,\ \lambda_2 = -2$.

$\begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

E.V

$|A - \lambda I| = 0$

$\left|\begin{pmatrix} 7 & 3 \\ -3 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right| = 0$

$\left|\begin{pmatrix} 7-\lambda & 3 \\ -3 & 1-\lambda \end{pmatrix}\right| = 0$

$(7-\lambda)(1-\lambda) + 9 = 0$

$7 - 7\lambda - \lambda + \lambda^2 + 9 = 0$

$\lambda^2 - 8\lambda + 16 = 0$

$\lambda^2 - 4\lambda - 4\lambda + 16 = 0$

$\lambda(\lambda - 4) - 4(\lambda - 4) = 0$

$(\lambda - 4)(\lambda - 4) = 0$

$\lambda = 4$.

Eigen value $= 4$.

$(A - \lambda I)X = 0$

$\left[\begin{pmatrix} 7 & 3 \\ -3 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\right] X = 0$

$\begin{pmatrix} 3 & 3 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{array}{c} R_1 \\ R_2 \end{array}\left[\begin{array}{cc|c} 3 & 3 & 0 \\ -3 & -3 & 0 \end{array}\right]$

$\begin{array}{c} \frac{1}{3}R_1 \to R_1 \\ R_1 + R_2 \to R_2 \end{array}\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow x_1 = -x_2 \Rightarrow x_2 = -x_1$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix}$

let $x_1 = r$

$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} r \\ r \end{pmatrix} = r\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Example (4)

Given $A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$ find all the Eigen vectors and the corresponding Eigen values.

Solution:

$|A - \lambda I| = 0$

$$\left| \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{pmatrix} \right| = 0$$

$M_{11} = \begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = (2-\lambda)(5-\lambda) - 4$
$= 10 - 2\lambda - 5\lambda + \lambda^2 - 4$
$= \underline{\lambda^2 - 7\lambda + 6}$

$M_{12} = \begin{vmatrix} 4 & 2 \\ 2 & 2-\lambda \end{vmatrix} = 4(2-\lambda) - 4$
$8 - 4\lambda - 4$
$= \underline{-4\lambda + 4}$

$M_{13} = \begin{vmatrix} 4 & 5-\lambda \\ 2 & 2 \end{vmatrix} = 8 - 2(5-\lambda)$
$= 8 - 10 + 2\lambda$
$= \underline{2\lambda - 2}$

$M_{21} = \begin{vmatrix} 4 & 2 \\ 2 & 2-\lambda \end{vmatrix} = 4(2-\lambda) - 4$
$8 - 4\lambda - 4$
$= \underline{-4\lambda + 4}$

$M_{22} = \begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = (2-\lambda)(5-\lambda) - 4$
$= \underline{\lambda^2 - 7\lambda + 6}$

$M_{23} = \begin{vmatrix} 5-\lambda & 4 \\ 2 & 2 \end{vmatrix} = 2(5-\lambda) - 8$
$= 10 - 2\lambda - 8$
$= \underline{-2\lambda + 2}$

$M_{31} = \begin{vmatrix} 4 & 2 \\ 5-\lambda & 2 \end{vmatrix} = 8 - 2(5-\lambda)$
$= 8 - 10 + 2\lambda$
$= \underline{2\lambda - 2}$

$M_{32} = \begin{vmatrix} 5-\lambda & 2 \\ 4 & 2 \end{vmatrix} = 2(5-\lambda) - 8$
$= 10 - 2\lambda - 8$
$= \underline{-2\lambda + 2}$

$M_{33} = \begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = (5-\lambda)(5-\lambda) - 16$
$= 25 - 5\lambda - 5\lambda + \lambda^2 - 16$
$= \underline{\lambda^2 - 10\lambda + 9}$

$C_{11} = -1^{(1+1)} \cdot M_{11}$ ($i+j$ = even →
$= \lambda^2 - 7\lambda +$

$C_{11} = -1^{(1+1)} \cdot M_{11} = \lambda^2 - 7\lambda + 6$

$C_{12} = -1^{(1+2)} \cdot M_{12} = 4\lambda - 4$

$C_{13} = -1^{(1+3)} \cdot M_{13} = 2\lambda - 2$

$C_{21} = -1^{(2+1)} \cdot M_{21} = 4\lambda - 4$

$C_{22} = -1^{(2+2)} \cdot M_{22} = \lambda^2 - 7\lambda + 6$

$C_{23} = -1^{(2+3)} \cdot M_{23} = 2\lambda - 2$

$C_{31} = -1^{(3+1)} \cdot M_{31} = 2\lambda - 2$

$C_{32} = -1^{(3+2)} \cdot M_{32} = 2\lambda - 2$

$C_{33} = -1^{(3+3)} \cdot M_{33} = \lambda^2 - 10\lambda + 9$

Eq =

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$
$= (5-\lambda)(\lambda^2 - 7\lambda + 6) + 4(4\lambda - 4) + 2(2\lambda - 2)$
$= 5\lambda^2 - \lambda^3 - 35\lambda + 7\lambda^2 + 30 - 6\lambda + 16\lambda - 16 + 4\lambda - 4$
$= -\lambda^3 + 12\lambda^2 - 21\lambda + 10 = 0$
$\Rightarrow$ solve for λ to get $\lambda_1 = 10$ & $\lambda_2 = 1$.

for $\lambda_1 = 10$,

$(A - \lambda I)X = 0$

$$\begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix} - \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} -5 & 4 & 2 & 0 \\ 4 & -5 & 2 & 0 \\ 2 & 2 & -8 & 0 \end{array} \right]$$

$$\begin{array}{r} \rightarrow R_1 \\ 4R_1 + 5R_2 \rightarrow R_2 \\ 2R_1 + 5R_3 \rightarrow R_3 \end{array} \left[\begin{array}{ccc|c} -5 & 4 & 2 & 0 \\ 0 & -9 & 18 & 0 \\ 0 & 18 & -36 & 0 \end{array} \right]$$

$$\begin{array}{r} R_1 \\ R_2 \\ 2R_2 + R_3 \rightarrow R_3 \end{array} \left[\begin{array}{ccc|c} -5 & 4 & 2 & 0 \\ 0 & -9 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$\Rightarrow -9x_2 + 18x_3 = 0$

$$\frac{-9x_2}{-9} = \frac{-18x_3}{-9}$$

36 $\quad 9\times 4$ $\quad$ 38

back $\quad x_2 = 2x_3$.

Substitute for x_2

$\Rightarrow -5x_1 + 4x_2 + 2x_3 = 0$

$\Rightarrow -5x_1 + 4(2x_3) + 2x_3 = 0$

$-5x_1 + 8x_3 + 2x_3 = 0$

$-5x_1 + 10x_3 = 0$

$$\frac{-5x_1}{-5} = \frac{-10x_3}{-5}$$

$x_1 = 2x_3$.

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ 2x_3 \\ x_3 \end{pmatrix} = x_3\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$\therefore \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is the eigen vector for the matrix $\begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$ corresponding to the eigen vector $\lambda_1 = 10$.

For $\lambda_2 = 1$,

$(A - \lambda_2 I)x = 0$

$$\left[\begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]x = 0$$

$$\begin{pmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 4 & 4 & 2 & 0 \\ 4 & 4 & 2 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right]$$

$$\begin{array}{r} R_1 \\ -R_1 + R_2 \to R_2 \\ -R_1 + 2R_3 \to R_3 \end{array}\left[\begin{array}{ccc|c} 4 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow 4x_1 + 4x_2 + 2x_3 = 0$

$\Rightarrow 2x_3 = -4x_1 - 4x_2$

$\Rightarrow x_3 = -2x_1 - 2x_2$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ -2x_1 - 2x_2 \end{pmatrix}$$

$$= x_1\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}.$$

$\therefore \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ are the Eigen vectors for the matrix $\begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$ corresponding to the eigen value $\lambda_2 = 1$.

Algebraic multiplicity of Geometric

Diagonalization: $\Rightarrow$ Produces a diagonal matrix if not all the matrix is diagonalizable.

The application of Eigen vectors.

We use the formula

$D = P^{-1}AP$,

where P is a matrix of Eigen vectors $(x_1, x_2, x_3, \dots, x_n)$

and P^{-1} is the inverse of P.

Example ①

Diagonalize matrix $A = \begin{pmatrix} -3 & 4 \\ 5 & 6 \end{pmatrix}$.

Solution

$|A - \lambda I| = 0$

$$\left|\begin{pmatrix} -3 & 4 \\ 5 & 6 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right| = 0$$

$$\left|\begin{pmatrix} -3-\lambda & 4 \\ 5 & 6-\lambda \end{pmatrix}\right| = 0$$

$(6-\lambda)(-3-\lambda) - 20 = 0$

$-18 - 6\lambda + 3\lambda + \lambda^2 - 20 = 0$

$\lambda^2 - 3\lambda - 38 = 0$

$\lambda_1 = 2, \ \lambda_2 = 1$

$\begin{pmatrix} -4/5 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$P = \begin{pmatrix} -4/5 & -1 \\ 1 & 1 \end{pmatrix}$

$P^{-1} = \begin{pmatrix} 5 & 5 \\ -5 & -4 \end{pmatrix}$

Conditions for diagonalization

- ~~Have~~ Same number of eigen vectors as eigen values
- n eigen vectors that are linearly independent.

Example (4)

Given $A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$ find all the Eigen vectors and the corresponding Eigen values.

Solution

$|A - \lambda I| = 0$

$$\left| \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{pmatrix} \right| = 0$$

$M_{11} = \begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = (2-\lambda)(5-\lambda) - 4$
$= 10 - 2\lambda - 5\lambda + \lambda^2 - 4$
$= \lambda^2 - 7\lambda + 6$

$M_{12} = \begin{vmatrix} 4 & 2 \\ 2 & 2-\lambda \end{vmatrix} = 4(2-\lambda) - 4$
$8 - 4\lambda - 4$
$= -4\lambda + 4$

$M_{13} = \begin{vmatrix} 4 & 5-\lambda \\ 2 & 2 \end{vmatrix} = 8 - 2(5-\lambda)$
$= 8 - 10 + 2\lambda$
$= 2\lambda - 2$

$M_{21} = \begin{vmatrix} 4 & 2 \\ 2 & 2-\lambda \end{vmatrix} = 4(2-\lambda) - 4$
$= 8 - 4\lambda - 4$
$= -4\lambda + 4$

$M_{22} = \begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = (2-\lambda)(5-\lambda) - 4$
$= \lambda^2 - 7\lambda + 6$

$M_{23} = \begin{vmatrix} 5-\lambda & 4 \\ 2 & 2 \end{vmatrix} = 2(5-\lambda) - 8$
$= 10 - 2\lambda - 8$
$= -2\lambda + 2$

$M_{31} = \begin{vmatrix} 4 & 2 \\ 5-\lambda & 2 \end{vmatrix} = 8 - 2(5-\lambda)$
$= 8 - 10 + 2\lambda$
$= 2\lambda - 2$

$M_{32} = \begin{vmatrix} 5-\lambda & 2 \\ 4 & 2 \end{vmatrix} = 2(5-\lambda) - 8$
$= 10 - 2\lambda - 8$
$= -2\lambda + 2$

$M_{33} = \begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = (5-\lambda)(5-\lambda) - 16$
$= 25 - 5\lambda - 5\lambda + \lambda^2 - 16$
$= \lambda^2 - 10\lambda + 9$

$C_{11} = -1^{(1+1)} \cdot M_{11}$ $i+j$ = even +
$= \lambda^2 - 7\lambda +$

$C_{11} = -1^{(1+1)} \cdot M_{11} = \lambda^2 - 7\lambda + 6$
$C_{12} = -1^{(1+2)} \cdot M_{12} = 4\lambda - 4$
$C_{13} = -1^{(1+3)} \cdot M_{13} = 2\lambda - 2$
$C_{21} = -1^{(2+1)} \cdot M_{21} = 4\lambda - 4$
$C_{22} = -1^{(2+2)} \cdot M_{22} = \lambda^2 - 7\lambda + 6$
$C_{23} = -1^{(2+3)} \cdot M_{23} = 2\lambda - 2$
$C_{31} = -1^{(3+1)} \cdot M_{31} = 2\lambda - 2$
$C_{32} = -1^{(3+2)} \cdot M_{32} = 2\lambda - 2$
$C_{33} = -1^{(3+3)} \cdot M_{33} = \lambda^2 - 10\lambda + 9$

$C_{ij} =$

$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$
$= (5-\lambda)(\lambda^2 - 7\lambda + 6) + 4(4\lambda - 4) + 2(2\lambda - 2)$
$= 5\lambda^2 - \lambda^3 - 35\lambda + 7\lambda^2 + 30 - 6\lambda + 16\lambda - 16 + 4\lambda - 4$
$= -\lambda^3 + 12\lambda^2 - 21\lambda + 10 = 0$
$\Rightarrow$ solve for λ to get $\lambda_1 = 10$ \& $\lambda_2 = 1$.

for $\lambda_1 = 10$,
$(A - \lambda I)X = 0$

$$\left[\begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix} - \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} -5 & 4 & 2 & 0 \\ 4 & -5 & 2 & 0 \\ 2 & 2 & -8 & 0 \end{array} \right]$$

$$\begin{array}{r} \rightarrow R_1 \\ 4R_1 + 5R_2 \rightarrow R_2 \\ 2R_1 + 5R_3 \rightarrow R_3 \end{array} \left[\begin{array}{ccc|c} -5 & 4 & 2 & 0 \\ 0 & -9 & 18 & 0 \\ 0 & 18 & -36 & 0 \end{array} \right]$$

$$\begin{array}{r} R_1 \\ R_2 \\ 2R_2 + R_3 \rightarrow R_3 \end{array} \left[\begin{array}{ccc|c} -5 & 4 & 2 & 0 \\ 0 & -9 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$\Rightarrow -9x_2 + 18x_3 = 0$

$\frac{-9x_2}{-9} = \frac{-18x_3}{-9}$

36 9×4 38

$x_2 = 2x_3$

back Substitute for x_2

$\Rightarrow -5x_1 + 4x_2 + 2x_3 = 0$

$\Rightarrow -5x_1 + 4(2x_3) + 2x_3 = 0$

$-5x_1 + 8x_3 + 2x_3 = 0$

$-5x_1 + 10x_3 = 0$

$\frac{-5x_1}{-5} = \frac{-10x_3}{-5}$

$x_1 = 2x_3$.

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ 2x_3 \\ x_3 \end{pmatrix} = x_3\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$\therefore \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is the eigen vector for the matrix $\begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$ corresponding to the eigen vector $\lambda_1 = 10$.

For $\lambda_2 = 1$,

$(A - \lambda_2 I)x = 0$

$$\left[\begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] x = 0$$

$$\begin{pmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 4 & 4 & 2 & 0 \\ 4 & 4 & 2 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right]$$

$$\begin{array}{r} R_1 \\ -R_1 + R_2 \to R_2 \\ -R_1 + 2R_3 \to R_3 \end{array}\left[\begin{array}{ccc|c} 4 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow 4x_1 + 4x_2 + 2x_3 = 0$

$\Rightarrow 2x_3 = -4x_1 - 4x_2$

$\Rightarrow x_3 = -2x_1 - 2x_2$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ -2x_1 - 2x_2 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}.$$

$\therefore \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ are the Eigen vectors for the matrix $\begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$ corresponding to the eigen value $\lambda_2 = 1$.

Algebraic multiplicity ≠ Geometric

Diagonalization: ⇒ Produces a diagonal matrix if at all the matrix is diagonalizable.

The application of Eigen vectors.

We use the formula

$D = P^{-1}AP$,

where P is a matrix of Eigen vectors $(x_1, x_2, x_3, \ldots, x_n)$

and P^{-1} is the inverse of P.

Example ①

Diagonalize matrix $A = \begin{pmatrix} -3 & 4 \\ 5 & 6 \end{pmatrix}$.

Solution

$|A - \lambda I| = 0$

$$\left|\begin{pmatrix} -3 & 4 \\ 5 & 6 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right| = 0$$

$$\left|\begin{pmatrix} -3-\lambda & 4 \\ 5 & 6-\lambda \end{pmatrix}\right| = 0$$

$(6-\lambda)(-3-\lambda) - 20 = 0$

$-18 - 6\lambda + 3\lambda + \lambda^2 - 20 = 0$

$\lambda^2 - 3\lambda - 38 = 0$

$\lambda_1 = 2,\ \lambda_2 = 1$

$\begin{pmatrix} -4/5 \\ 1 \end{pmatrix}$ $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$P = \begin{pmatrix} -4/5 & -1 \\ 1 & 1 \end{pmatrix}$

$P^{-1} = \begin{pmatrix} 5 & 5 \\ -5 & -4 \end{pmatrix}$

Conditions for diagonalization

- ~~Have~~ Same number of eigen vectors as eigen values
- n eigen vectors that are linearly independent.

Example ① $\lambda_1 = \lambda_2 = \lambda_3 = 2$

Diagonalize $B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & 2 \end{pmatrix}$

Ans: $\lambda_1 = 2 \quad \lambda_2 = 1 \quad \lambda_3 = -1$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$D = P^{-1} B P$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

Qn: Show that $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

Application of diagonalization

– Used in getting higher powers of a given matrix.. $\Rightarrow A^K = P^{-1} D^K P$.

Example ①

Given $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$, find A^{13}.

Ans:

$\lambda_1 = 2 \quad \lambda_2 = 1$

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^{13} = P^{-1} D^{13} P$$

$$= \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2^{13} & 0 & 0 \\ 0 & 2^{13} & 0 \\ 0 & 0 & 1^{13} \end{pmatrix} \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$A^{13} = \begin{pmatrix} -8190 & 0 & -16382 \\ 8191 & 8192 & 8191 \\ 8191 & 0 & 16383 \end{pmatrix}$$

Solution

$$|A - \lambda I| = 0$$

$$(A - \lambda I)X = 0$$

$$|A - \lambda I| = \left| \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right|$$

$$= \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix}$$

$$M_{11} = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix}$$

$$= (2-\lambda)(3-\lambda) - 0$$

$$M_{11} = \lambda^2 - 5\lambda + 6$$

$$M_{12} = \begin{vmatrix} 1 & 1 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda) - 1$$

$$M_{12} = 2 - \lambda$$

$$M_{13} = \begin{vmatrix} 1 & 2-\lambda \\ 1 & 0 \end{vmatrix}$$

$$= 0 - (2-\lambda)$$

$$M_{13} = \lambda - 2$$

$$M_{21} = \begin{vmatrix} 0 & -2 \\ 0 & 3-\lambda \end{vmatrix}$$

$$= 0 - 0$$

$$M_{21} = 0$$

$$M_{22} = \begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= -\lambda(3-\lambda) - (-2)$$

$$M_{22} = \lambda^2 - 3\lambda + 2$$

$$M_{23} = \begin{vmatrix} -\lambda & 0 \\ 1 & 0 \end{vmatrix}$$

$$= 0 - 0$$

$$M_{23} = 0$$

$$M_{31} = \begin{vmatrix} 0 & -2 \\ 2-\lambda & 1 \end{vmatrix}$$

$$= 0 - [-2(2-\lambda)]$$

$$= 0 - (2\lambda - 4)$$

$$M_{31} = 4 - 2\lambda$$

$$M_{32} = \begin{vmatrix} -\lambda & -2 \\ 1 & 1 \end{vmatrix}$$

$$= -\lambda - -2$$

$$M_{32} = 2 - \lambda$$

$$M_{33} = \begin{vmatrix} -\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= -\lambda(2-\lambda) - 0$$

$$M_{33} = \lambda^2 - 2\lambda$$

$c_{11} = \lambda^2 - 5\lambda + 6 \qquad a_{11} = -\lambda$

$c_{12} = \lambda - 2 \qquad a_{12} = 0$

$c_{13} = \lambda - 2 \qquad a_{13} = -2$

$c_{21} = 0 \qquad a_{21} = 1$

$c_{22} = \lambda^2 - 3\lambda + 2 \qquad a_{22} = 2-\lambda$

$c_{23} = 0 \qquad a_{23} = 1$

$c_{31} = 4 - 2\lambda \qquad a_{31} = 1$

$c_{32} = \lambda - 2 \qquad a_{32} = 0$

$c_{33} = \lambda^2 - 2\lambda \qquad a_{33} = 3-\lambda$

$$|A-\lambda I| = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$$
$$= -\lambda(\lambda^2-5\lambda+6) + 0 + -2(\lambda-2)$$
$$= -\lambda^3 + 5\lambda^2 - 6\lambda - 2\lambda + 4$$
$$|A-\lambda I| = -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$
$$\lambda_1 = 1, \quad \lambda_2 = 2$$

$(A-\lambda I)X = 0$

when $\lambda = 1$,

$$\left[\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right]$$

$$\begin{array}{r} -R_1 \to R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right]$$

$$\begin{array}{r} R_1 \\ -R_1+R_2 \to R_2 \\ -R_1+R_3 \to R_3 \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_2 - x_3 = 0$

$x_2 = x_3$

$x_1 + 2x_3 = 0$

$x_1 = -2x_3$.

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_3 \\ x_3 \\ x_3 \end{pmatrix} = x_3\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ = eigen vector corresponding to eigen value $\lambda_1 = 1$.

when $\lambda = 2$,

$$\left[\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{r} -\tfrac{1}{2}R_1 \to R_1 \\ \to R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{r} R_1 \\ -R_1+R_2 \to R_2 \\ -R_1+R_3 \to R_3 \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_3 = 0$

$\Rightarrow x_1 = -x_3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
$$= x_3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\therefore \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are the eigen vectors corresponding to eigen value $\lambda_2 = 2$.

$P \equiv$ matrix of eigen vectors

$$\therefore P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$$

$$M_{12} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$$

$$M_{13} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 - 0 = 1$$

$$M_{21} = \begin{vmatrix} -1 & 0 \\ 1 & 0 \end{vmatrix} = 0 - 0 = 0$$

$$M_{22} = \begin{vmatrix} -2 & 0 \\ 1 & 0 \end{vmatrix} = 0 - 0 = 0$$

$$M_{23} = \begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} = -2 + 1 = -1$$

$M_{31} = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1-0 = -1$

$M_{32} = \begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix} = -2-0 = -2$

$M_{33} = \begin{vmatrix} -2 & -1 \\ 1 & 0 \end{vmatrix} = 0-1 = 1$

$$C = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ -1 & 2 & 1 \end{pmatrix}$$

$|P| = -2(-1) + -1(1) + 0 = 2-1 = 1$

$= 0 + 0 + 1(1) = 1$

$= 1(-1) + 1(2) + 0 = -1+2 = 1$

$$C^T = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$P^{-1} = \frac{1}{|P|} C^T$

$= \frac{1}{1}\begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$

$$P^{-1} = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$D = P^{-1}AP$

$= \begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}\begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 & -1 \\ 2 & 0 & 4 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$A^{13} = P^{-1}D^{13}P$

$= \begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 8192 & 0 \\ 0 & 0 & 8192 \end{pmatrix}\begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 & -8192 \\ 1 & 0 & 16384 \\ 1 & 8192 & 8192 \end{pmatrix}\begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$$A^{13} = \begin{pmatrix} -8190 & -8191 & 0 \\ 16382 & 16383 & 0 \\ 16382 & 8191 & 8192 \end{pmatrix}$$

change arrangement of P:

$$P = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$M_{11} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$

$M_{12} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$

$M_{13} = |$

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06/12/2021 - Monday-

LEONTIF INPUT-OUTPUT ANALYSIS

The main feature of the input-output model is that it incorporates the interactions between industries or sectors which make up an economy.

The aim of the Leontif model is to allow economists to forecast the future production levels of each industry in order to meet the future demands for the various products.

Such forecasting is complicated by linkages between the different industries because the change in the demand of one industry can induce a change in the production levels of another industry e.g an ↑se in the demand of automobiles leads not only to the increase in production of automobiles but also an increase in the levels of a variety of other industries within the economy such as steel, rubber, leather, etc.

Consider an economy with n industries or sectors, each industry/sector producing a homogeneous product. Part of the industry or sector is required/used by households as final goods.

In a modern economy, production of one good requires the input of many other goods in the production process. (intermediary goods)

Therefore, the demands of an individual product will need to find the output required to meet the final demand from within and outside the industry.

i.e X = summation of all intermediary goods + External demand.

Mathematically, the total demand, X is given by the summation of all intermediary products and the final demand.

i.e for example if an economy with 3 sectors, 1, 2 & 3,

$$X_1 = a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + D_1$$
$$X_2 = a_{21}X_1 + a_{22}X_2 + a_{33}X_3 + D_2$$
$$X_3 = a_{31}X_1 + a_{31}X_2 + a_{33}X_3 + D_3$$

$$\Rightarrow \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} + \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix}$$

$$\Rightarrow X = AX + D$$

X = Total output matrix

A = Matrix of technical coefficients

D = External demand.

$$\Rightarrow X - AX = D$$

$$X(1-A) = D$$ Introduce Identity matrix.

$$\Rightarrow X(I-A) = D$$

$$\boxed{\cancel{(I-A)^{-1}(IA)}}$$

$$(I-A)^{-1}(I-A)X = (I-A)^{-1}D$$

$$\Rightarrow \boxed{X = (I-A)^{-1}D} \equiv \text{Leontif matrix.}$$

Example ①:

Consider a hypothetical two industry economy with the matrix of technical coefficients given as C I and

$$\begin{array}{c c} & \begin{array}{cc} C & I \end{array} \\ \begin{array}{c} C \\ I \end{array} & \begin{pmatrix} 0.2 & 0.3 \\ 0.4 & 0.1 \end{pmatrix} \end{array}$$

external demand as $D = \begin{array}{c} C \\ I \end{array}\begin{pmatrix} 24 \\ 15 \end{pmatrix}$, determine the total output required to meet the internal and external demands.

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Solution:

$$X = (I-A)^{-1}D.$$

$$(I-A) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.2 & 0.3 \\ 0.4 & 0.1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.8 & -0.3 \\ -0.4 & 0.9 \end{pmatrix}$$

$$|(I-A)| = (0.9 \times 0.8) - (0.4 \times 0.3)$$

$$= 0.72 - 0.12$$

$$|(I-A)| = 0.60$$

$$(I-A)^T = \begin{pmatrix} 0.9 & +0.3 \\ +0.4 & 0.8 \end{pmatrix}$$

$$(I-A)^{-1} = \frac{1}{0.6}\begin{bmatrix} 0.9 & +0.3 \\ +0.4 & 0.8 \end{bmatrix}$$

$$= \begin{pmatrix} 1.5 & 0.5 \\ 0.67 & 1.33 \end{pmatrix}$$

$$X = (I-A)^{-1}D$$

$$= \begin{pmatrix} 1.5 & 0.5 \\ 0.67 & 1.33 \end{pmatrix}\begin{pmatrix} 24 \\ 15 \end{pmatrix}$$

$$= \begin{pmatrix} (1.5 \times 24) + (0.5 \times 15) \\ (0.67 \times 24) + (1.33 \times 15) \end{pmatrix}$$

$$= \begin{pmatrix} 36 + 7.5 \\ 16.08 + 19.95 \end{pmatrix}$$

$$X = \begin{pmatrix} 43.5 \\ 36.03 \end{pmatrix}$$

⇒ Industry C needs to produce 43.5 units in order to meet the internal & external d'd.

⇒ Industry I needs to produce 36.03 units in order to meet the internal & external d'd.

Example (2)

Suppose an economy is divided into 2 sectors, A & B with a matrix of technical coefficients given as

$$\begin{array}{c|cc} & A & B \\ A & 0.4 & 0.3 \\ B & 0.5 & 0.7 \end{array} \rightarrow \text{output}, \downarrow \text{inputs}$$

a) Interprete the coefficient of technical matrix and discuss the economic feasibility of the two industries.

b) If the input-output model is set up as $X = AX + D$, derive the final solution of the model using the Leontif approach.

c) Given the final demand as \$2000 and \$1500 for industry A and B respectively, determine the final output matrix.

Solution:

a) $\overset{0.4}{\text{0.4}}$ or 40% of the output of industry A is used as its input

$\overset{0.3}{\text{0.3}}$ or 30% → industry A uses 30% of the output of B as its input.

$\overset{0.5}{}$ industry B uses 50% of the output of A as its input.

$\overset{0.7}{}$ Industry B uses 70% of its output as input in production in the very industry.

There is a high level of interdependence between industries A and B just laying off 0.1 units to the external demand.

b) → covered.

c) $X = (I-A)^{-1}D$

$$X = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.4 & 0.3 \\ 0.5 & 0.7 \end{pmatrix}\right]^{-1}\begin{pmatrix} 2000 \\ 1500 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.4 & 0.3 \\ 0.5 & 0.7 \end{pmatrix}$$

$$(I-A) = \begin{pmatrix} 0.6 & -0.3 \\ -0.5 & 0.3 \end{pmatrix}$$

$$|(I-A)| = (0.3\times0.6)-(-0.3\times-0.5)$$
$$= 0.18-0.15$$
$$= 0.03.$$

$$(I-A)^T = \begin{pmatrix} 0.3 & 0.3 \\ 0.5 & 0.6 \end{pmatrix}$$

$$(I-A)^{-1} = \frac{1}{0.03}\begin{bmatrix} 0.3 & 0.3 \\ 0.5 & 0.6 \end{bmatrix}$$

$$= \begin{pmatrix} 10 & 10 \\ 16.67 & 20 \end{pmatrix}$$

$$\therefore X = (I-A)^{-1}D$$

$$= \begin{pmatrix} 10 & 10 \\ 16.67 & 20 \end{pmatrix}\begin{pmatrix} 2000 \\ 1500 \end{pmatrix}$$

$$= \begin{pmatrix} 10\times2000 + 10\times1500 \\ 16.67\times2000 + 20\times1500 \end{pmatrix}$$

$$= \begin{pmatrix} 20{,}000 + 15000 \\ 33{,}340 + 30000 \end{pmatrix}$$

$$X = \begin{pmatrix} 35{,}000 \\ 63{,}340 \end{pmatrix}$$

Final output matrix $= X = \begin{pmatrix} 35{,}000 \\ 48{,}340 \end{pmatrix}$.

Example (3)

	Industry I	Industry II	Final Demand	Gross output
Industry I	240	750	210	1200
Industry II	720	450	330	1500
Primary inputs	240	300		

a) Determine the input-output matrix A

b) Determine the output matrix if the final demand changes to 312 units for industry I and 299 units for industry II respectively.

Example (3)

c) what will be the new primary inputs for the two industries?

Solution

$$A = \begin{array}{c|cc} & I & II \\ \hline I & \frac{240}{1200} & \frac{750}{1500} \\ II & \frac{720}{1200} & \frac{450}{1500} \end{array}$$

$$A = \begin{pmatrix} 0.2 & 0.5 \\ 0.6 & 0.3 \end{pmatrix}$$

b) $D = \begin{array}{c} I \\ II \end{array}\begin{pmatrix} 312 \\ 299 \end{pmatrix}$

$$X = (I-A)^{-1}D.$$

$$(I-A) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.2 & 0.5 \\ 0.6 & 0.3 \end{pmatrix} = \begin{pmatrix} 0.8 & -0.5 \\ -0.6 & 0.7 \end{pmatrix}$$

$$|I-A| = (0.7\times0.8)-(-0.6\times-0.5)$$
$$= 0.56-0.30$$
$$= 0.26.$$

$$(I-A)^T = \begin{pmatrix} 0.7 & 0.5 \\ 0.6 & 0.8 \end{pmatrix}$$

$$(I-A)^{-1} = \frac{1}{|I-A|}\cdot(I-A)^T$$

$$= \frac{1}{0.26}\begin{pmatrix} 0.7 & 0.5 \\ 0.6 & 0.8 \end{pmatrix}$$

$$= \begin{pmatrix} 2.692 & 1.923 \\ 2.308 & 3.077 \end{pmatrix}$$

$$X = (I-A)^{-1}D$$

$$= \begin{pmatrix} 2.692 & 1.923 \\ 2.308 & 3.077 \end{pmatrix}\begin{pmatrix} 312 \\ 299 \end{pmatrix}$$

$$= \begin{pmatrix} (2.692\times312)+(1.923\times299) \\ (2.308\times312)+(3.077\times299) \end{pmatrix} \quad \begin{array}{l} 705.744+574.9 \\ 720.096+920.0 \end{array}$$

$$= \begin{array}{c} \cancel{1280.721} \\ \cancel{1640.119} \end{array} \quad X = \begin{pmatrix} 1415 \\ 1640.119 \end{pmatrix}$$

c) for industry I, initial fraction of inputs

$= \frac{240}{1200} = 0.2$ of the output

New primary input $= 0.2 \times 1415$

$= 283$

For industry II, initial fraction of inputs

$= \frac{300}{1500} = 0.2$ of the output

New primary input $= 0.2 \times 1640.119$

$= 328$

Example (4) ≡ Exercise

	Agriculture	Manufacturing	Final Demand	Gross output
Agriculture	240	270	90	600
Manufacturing	300	90	60	450
Labour	60	90		

a) Determine the input-output matrix A

b) Suppose that in 3 years, the demand for agricultural products decreases to 63 units and increases to 105 units for manufactured products, determine the new output vector to meet the new demand.

c) What will be the new labour requirement for each sector?

Example (5) ≡ Exercise

Given a 3 sector economy with the input-output matrix A given as

$$A = \begin{array}{c} \\ C \\ I \\ A \end{array} \begin{array}{c} \begin{array}{ccc} C & I & A \end{array} \\ \begin{pmatrix} 0.5 & 0.1 & 0.2 \\ 0.2 & 0.8 & 0.1 \\ 0.1 & 0 & 0.4 \end{pmatrix} \end{array}$$

with the external demand as $D = \begin{pmatrix} 100 \\ 150 \\ 200 \end{pmatrix} \begin{array}{c} C \\ I \\ A \end{array}$,

determine the total output to meet the new demand.

Accounting Principles.

Fundamental Accounting Principle
This is a way to figure out the number of out comes in a probability problem.

or

A way to figureout a given situation.

Addition principle
If an operation can be performed M ways and also if another operation can be performed N ways and only one operation can be done at a time then either of the ~~Note it was~~ two operations can be performed in M+N ways.

Example.
A student is shopping for a new computer he is decid[ing] a ~~monthly~~ three desk top computer and four laptop Computer what is the total number of Computer options.

Solution.

M+N

3+4

7 computer options.

Multiplication.
If an operation can be performed in M ways and if another operation can be performed in N ways independent of the 1st then the total number of ways of Performing the operations sumand is MxN ways.

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Examples

If a baby has two shirts and 3 skirts how many outfits can be put together.

2X3

6 outfits

Example 2:

You will take a survey with 5 (Yes, No) ans How many diff't ways would you complete the survey.

2X2X2X2X2

32 ways. $^{5}C_{2}$

You toss 3 coins how many possible outcomes are there.

2 X 2 X 2

8 outcomes.

Note:

This works when all chooses are independent of each other. If one choose depends on another then simple multiplication is not right.

Factorial notation.

This is a product of positive intergers from 1 to n it is always denoted by (!) written as n!

ie $n! = n \times (n-1) \times (n-2) \ldots 1$

$n! = 1\ 2\ 3\ \ldots\ (n-2)(n-1) \times n$

$0! = 1$

$2^0 = 1$

$5!$

$5\times4\times3\times2\times1$

$= 120$

$$\frac{10!}{8!} = \frac{10\times9\times8\times7\times6\times5\times4\times2\times1}{8\times7\times6\times5\times4\times2\times1}$$

$= 90$

Evalut

$$\frac{10!}{2!3!5!}$$

$$\frac{10\times9\times8\times7\times6\times\cancel{5\times4\times3\times2\times1}}{\cancel{2\times1}\times3\times2\times1\times\cancel{5\times4\times3}\times2\times1}$$

$$\frac{30240}{12}$$

2520

$$\frac{6!\,2!}{8!}$$

$$\frac{6\times\cancel{5\times4\times3}\times2\times1\times\cancel{2!}}{8\times7\times6\times\cancel{5\times4\times3}\times\cancel{2!}}$$

$$\frac{12}{336}$$

$\frac{1}{28}$

Exerc

$$\frac{(n+1)!}{2(n!)}$$

$$\frac{(n+1)(n+\cancel{1}-\cancel{1})!}{2\cancel{n!}}$$

$\frac{1}{2}(n+1)$

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PERMUTATIONS.

Arrangement of a number of objects in a particular order.

OR.

Diff^t arrangements that can be made out of the diff^t set of objects or items by taking some or all of them at a time.

When given a set of N objects and we are required and arrange them in a particular order the total number of arrangments is denoted as ^nP_r

It means that the number of possible arrangements of r items taken from n items.

$$^nP_r = \frac{n!}{(n-r)!}$$

Diff^t types of Permutation.

① Permutations in which objects are replaced.

n^r

② Permutations of objects not all diff^t

$$\frac{n!}{p!\,q!\,r!}$$

③ C

There we Fix

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From a class of 20 students we need to select 3 for a com
1 to be ē president, 2 as VP and the other as.
how many diff th Committee can be ~~sect~~ performed.

$n = 20$

$r = 3$

$$^nP_r = \frac{n!}{(n-r)!}$$

$$\frac{20!}{(20-3!)!}$$

$$\frac{20!}{17!}$$

$$\frac{20 \times 19 \times 18 \times 17!}{17!}$$

$= 6840$ ways.

Example 2.

How many unique ways are there to arrange in the word THAT.

$$\frac{n!}{P!}$$

$$\frac{4!}{2!}$$

$$\frac{4 \times 3 \times \cancel{2!}}{\cancel{2!}}$$

$= 12$ ways

How many diff't two digit numbers can be formed from the 1st natural numbers if repetition is allowed.

n^r

5^2

25 ways

A Father mother two boys and 3 girls are asked to line up for a pic. determine the number of ways if

(a) There are no restrictions.

(b) If the parents stand together.

c) If the parents do not stand together.

d) all females stand together.

Solution.

a) $7! = 5040$ ways.

b) $6! \times 2!$ $= 1440$ ways.

c) $7! - (6! \times 2!)$

$5040 - 1440$

$= 3600$

(d) $4! \times 4!$

$= 576$ ways.

In how many ways can 5 people to sitted on a round table.

$(n-1)!$

$(5-1)!$

$= 4!$

$= 24$ ways.

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MATHEMATICS OF FINANCE

The study of financial mathematics is Concerned with financial decisions we help in allocating, paying and receiving money at intervals. It helps us to take efficient financial decisions including how best to invest money.

Examples of finanical Calculations:

- Compounding
- discounting
- annuities
- Amortization.
- Net present Value (NPV
- Internal Rateof Return.

READ: Principal, Rate of interest & Accrued Amount.

Interest rate / Cost of Capital.

Simple Interest:

The Interest earned under this approach can be calculated as, $I = P \times R \times T$ where R is the rate of interest in %.

Example:

Suppose that 20000 was invested at 10% interest per annum. Compute the simple interest earned after 3 years and the cumulative amount accrued.

$$I = P \times R \times T$$

$$I = 200{,}000 \times \frac{10}{100} \times 3$$

$$I = 60{,}000$$

Amount accrued $= P + I$

$200{,}000 + 60{,}000$

$260{,}000$

Compound interest:

In this technique, the amount accrued can be Calculated as: $S = P(1+r)^n$ Where S is the sum Value or amount accrued, r is rate of interest in %. n is the time period.

Example 2.

$$S = P(1+r)^n$$

$$260{,}000 = 200{,}000\left(1 + \frac{10}{100}\right)^3$$

$260{,}000 = 200$

Compounding and Discounting.

Compounding is the process of obtaining Future Value from present Value of money. For example determing the money that will be realized after depositing money in a bank for a specific period of time into the future.

$$A = P(1+r)^n$$

Discounting.

Is the process of obtaining ~~Future~~ present Value of money from the Future Value of money. For example obtain amount to deposit today inorder to achieve a certain amount in the future.

other definitions

a) An annuity

Is a sequence of equal payments (or receipts) over uniform time intervals. Annuities can be ordinary (paid at end of the period) or due (paid at the beginning of the period). They normally begin and end on fixed dates (ordinary), but occasionally carry on indefinitely (Contingent).

b) An amortized debt.

Is one that is repaid with an annuity that takes account of both interest and repayment of principal amount borrowed.

Investment Appraisal.

Since investment involves risks, there is need to evalute the investments.

Traditional techniques that can be used include
Accounting Rate of Return (ARR)
payback Period.

Modern techniques (discounting techniques) Consider time
Net Present Value (NPV)
Internal Rate of Returns (IRR)

Capital investments.
Are projects that involve an initial outlay of capital and a set of estimated cash inflows and outflows over the life of a project.

Present Value and Investment appraisal
This is a technique that enables a future Cash flow to be represent in equivalent of today's money terms.
The present Value P of amount A payable in n-years' time subject to a discount rate of r%
is given by:

$$P = \frac{A}{(1+r)^n}$$

Recall: Formula for Compounding, $S = P(1+r)^n$

$P = \frac{A}{(1+r)^n}$ Can be rearranged as $P = A \times \frac{1}{(1+r)^n}$

where $\frac{1}{(1+r)^n}$ is Known as the discount factor or Presents Value factor.

Example

Find the present Value of \$1,500 in 6 years subject to a discount rate of 19%

$$PV = \frac{A}{(1+i)^n} = \frac{1,500}{(1+0.19)^6}$$

$$= \$\ 528.15$$

Net Present Value (NPV)

It involves calculating the sum of the present Values of all cash flows accociated with the project. The sum is known as the net present Value. The cash flows are normally tabulated per year.

It can be computed as $NPV = \sum_{n=0}^{t} PV_n$ or $NPV = \sum_{n=1}^{t} PV_n - I_0$

where I_0 is the initial outlay.

Interpretation of NPV:

NPV > 0; The project earns more than the discount rate hence it ~~brea~~is ~~the initial outlay~~ in profit (worthwhile)

NPV = 0 The project earns Same as the discount rate hence it is ~~in loss (not~~ breaks even (Revenue = Cost)

NPV < 0; The project earns less than the discount rate hence it is in loss (not worthwhile)

Example.

A business project is Considered to have $ 12000 initial Costs and the associated inflows (Revenues) and outflows (Costs) over the following 4 years are given below:

Number of Year	Cash inflow($)	cash outflows($)
1	8600	8500
2	12000	3000
3	10,000	1500
4	6500	1500

Evaluate the Projects NPV at the discount rate of 18.5% and Comment on your results.

Solution.

Note since this involves many years, we can use a DF (discount factor) table.

$DF = \frac{1}{(1+r)^n}$, and $PV = A \times DF$

Year	inflow $	outflows $	Net cash flow $	DF	PV
0	0	12.000	-(12.000)	1	
1	8000	8 500	500	0.84388	
2	12000	3,000	9000	0.712137	
3	10,000	1,500	8500	0.600959	
4	65000	1,500	5000	0.507139	