CONIC SECTION

Definition:

A conic section/a conic is the locus of a point that moves in such a way that its distance from a fixed point (FOCUS) bears a constant ratio to its distance from a fixed line (DIRECTRIX) not passing through the focus.

Terms used

A Focus:

Is a fixed point that lies on the axis of a conic.

A Directrix:

Is a line perpendicular to the axis of the conic not passing through the focus.

Eccentricity (e):

Is the constant ratio of distance of a point from a fixed point (focus) to the distance of a same point from the fixed line (Directrix).

Vertex of a conic:

A point (s) where the axis of the conic meets the conic.

Axis of the conic:

Is a line joining the focus and vertices of the conic and is perpendicular to the fixed line (Directrix).

Types of conic sections

- Circle
- Parabola
- Ellipse
- Hyperbola

The nature of locus of any point depends on the value of e, the eccentricity .i.e.

- If $e = 1$, the conic is called parabola
- If $0 < e < 1$, the conic is called ellipse
- If $e > 1$, the conic is called hyperbola

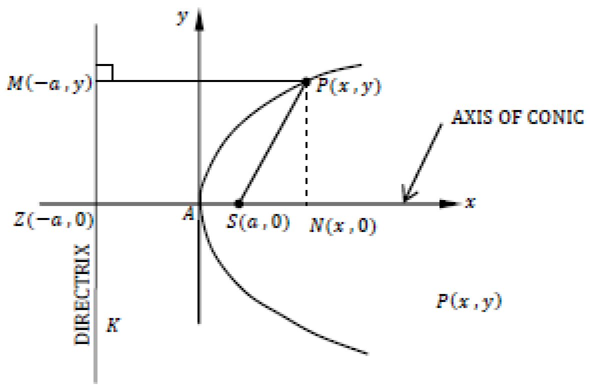
THE PARABOLA

Definition:

Is the locus of a point that moves in such away that its distance from a fixed point (Focus) is equal to its distance from a fixed line (Directrix) not passing through the focus.

Equation of a Parabola in Standard Form

Consider the figure below



Let $S(a\,,0)$ be the focus and ZK be the Directrix.

By definition, $\overline{PS} = \overline{PM}$ or $\overline{AS} = \overline{AZ}$

$\Rightarrow \overline{PS} = \sqrt{(x-a)^2 + (y-0)^2}$, $\overline{PM} = \sqrt{(x+a)^2 + (y-y)^2}$

$\Rightarrow \sqrt{(x-a)^2 + y^2} = \sqrt{(x+a)^2}$

Squaring both sides

$(x-a)^2 + y^2 = (x+a)^2$

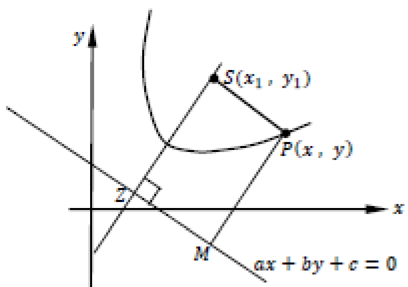
$\Rightarrow x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$

$\boxed{y^2 = 4ax}$, is the standard equation of a parabola

NOTE

1. The parabola in standard form .i.e. $y^2 = 4ax$ has:
   - Point $A(0\,,0)$ as the vertex
   - Line ZAS as the axis of conic with the equation $y = 0$
   - Line MZK as the Directrix with the equation $x = -a$
2. From $y^2 = 4ax$, if $a > 0$, the parabola opens to the right and if $a < 0$, the parabola opens to the left

To find the Equation of a Parabola when the Coordinates of the Focus and Equation of the Directrix are given



Let the coordinates of focus be $S(x_1, y_1)$, and the equation of the directrix be $ax + by + c = 0$.
Let $P(x, y)$ be a point tracing the locus.
By definition
$\overline{PS} = \overline{PM}$ or $\overline{AS} = \overline{AZ}$
$\overline{PS} = \sqrt{(x-x_1)^2 + (y-y_1)^2}$
$\overline{PM} = \perp$ Distance of P from directrix

$\therefore \overline{PM} = \frac{ax+by+c}{\sqrt{a^2+b^2}}$

$\Rightarrow \sqrt{(x-x_1)^2 + (y-y_1)^2} = \frac{ax+by+c}{\sqrt{a^2+b^2}}$

Squaring both sides gives

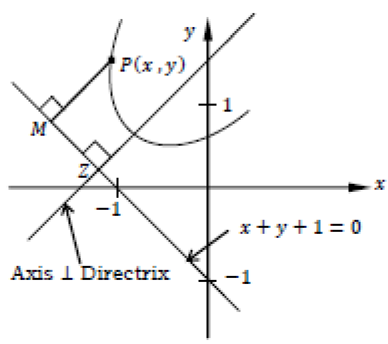
$$\boxed{(a^2+b^2)[(x-x_1)^2 + (y-y_1)^2] = (ax+by+c)^2}$$

, which on simplification can be written as:

$$\boxed{(bx-ay)^2 + 2gx + 2fy + k = 0}$$

NOTE
The 2^0 terms of the above equation form a perfect square.
Examples
1) *Find the equation of a parabola with;*
 a) *Focus* $(-1, 1)$ *and directrix* $x + y + 1 = 0$
 b) *Focus* $(0, 2)$ *and vertex as the origin*
 c) *Focus* $(-7, 3)$ *and vertex* $(-2, 3)$

Solution
a) Focus: $S(-1, 1)$, Directrix: $x + y + 1 = 0$

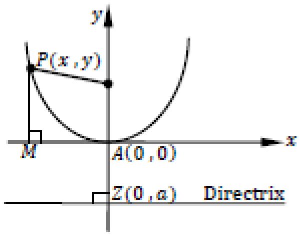


- Parabola opens to focus
- Directrix ⊥ Axis of the conic
- A, the vertex is the midpoint of $\overline{SZ}$

By definition, $\overline{PS} = \overline{PM}$ or $\overline{AS} = \overline{AZ}$
$\overline{PS} = \sqrt{(x+1)^2 + (y-1)^2}$
$\overline{PM} = \perp$ Distance of P from directrix
$\overline{PM} = \frac{x+y+1}{\sqrt{1+1}} = \frac{x+y+1}{\sqrt{2}}$
$\Rightarrow \sqrt{(x+1)^2 + (y-1)^2} = \frac{x+y+1}{\sqrt{2}}$, squaring
$\Rightarrow (x+1)^2 + (y-1)^2 = \frac{[(x+1)+y]^2}{2}$

$\therefore 2(x+1)^2 + 2(y-1)^2 = (x+1)^2 + 2y(x+1) + y^2$
$\Rightarrow (x+1)^2 + 2y^2 - 4y + 2 - y^2 - 2y(x+1) = 0$
$\Rightarrow x^2 + 2x + 1 + y^2 - 4y - 2xy - 2y + 2 = 0$
$\Rightarrow x^2 - 2xy + y^2 + 2x - 6y + 3 = 0$
$\therefore (x-y)^2 + 2x - 6y + 3 = 0$

b) $S(0,2)$, $A(0,0)$

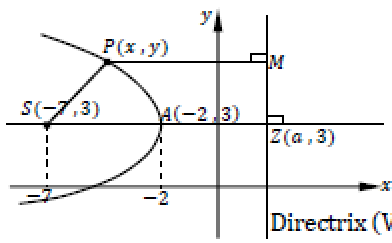


- Directrix ⊥ Axis of the conic
- Parabola opens to focus
- A, the vertex is the midpoint of $\overline{SZ}$

First finding the equation of the directrix
$A\left(\frac{0+0}{2}, \frac{2+a}{2}\right) \equiv A(0,0)$
$\Rightarrow \frac{2+a}{2} = 0$, $a = -2$
Equation of the directrix: $y + 2 = 0$

Now equation of parabola:
By definition, $\overline{PS} = \overline{PM}$ or $\overline{AS} = \overline{AZ}$
$\overline{PS} = \sqrt{(x-0)^2 + (y-2)^2} = \sqrt{x^2 + (y-2)^2}$
$\overline{PM}$ =⊥ Distance of P from directrix $y + 2 = 0 \Rightarrow 0x + y + 2 = 0$
$\overline{PM} = \frac{y+2}{\sqrt{0+1}} = y + 2$
$\therefore \sqrt{x^2 + (y-2)^2} = (y+2)$, squaring
$\Rightarrow x^2 + y^2 - 4y + 4 = y^2 + 4y + 4$
$\therefore x^2 = 8y$

c) $S(-7,3)$, $A(-2,3)$



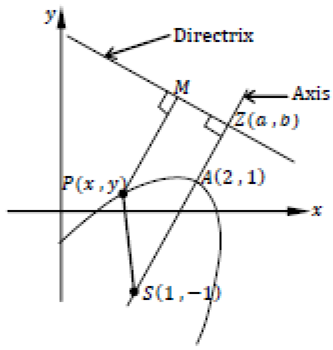
- Axis passes through SAZ
- Parabola opens to focus
- A, the vertex is the midpoint of $\overline{SZ}$

Finding the equation of the directrix
$A(-2,3) = A\left(\frac{-7+a}{2}, \frac{3+3}{2}\right) \Rightarrow -2 = \frac{-7+a}{2}$ $\therefore a = 3$
$\therefore Z(a,3) = Z(3,3)$
Equation of directrix is $x = 3$ or $x + 0y - 3 = 0$
By definition
$\overline{PS} = \overline{PM}$ or $\overline{AS} = \overline{AZ}$
$\Rightarrow \sqrt{(x+7)^2 + (y-3)^2} = \frac{x-3}{\sqrt{0+1}} = x - 3$, squaring both sides
$\Rightarrow (x+7)^2 + (y-3)^2 = (x-3)^2$

2) *Find the equation of a parabola with vertex* $(2,1)$ *and focus* $(1,-1)$.

Solution

Given: *Vertex* $A(2,1)$ *and Focus* $S(1,-1)$



Finding the equation of the directrix:

$A(2,1)$ =Midpoint of $\overline{SZ}$

$A(2,1) = A\left(\frac{a+1}{2}, \frac{-1+b}{2}\right) \Rightarrow 2 = \frac{a+1}{2} \quad \therefore a = 3$

$\Rightarrow \frac{-1+b}{2} = 1 \quad \therefore b = 3$

$\therefore Z(3,3)$

But the directrix is $\perp$ to axis of conic

From coordinate geometry, $m_1 \times m_2 = -1$

m_1 = Slope of axis of conic = Grad $\overline{AS}$

$= \frac{1+1}{2-1} = 2 \quad \therefore m_2 = -\frac{1}{2}$

Now finding equation of a line (Directrix) passing through a known point $Z(3,3)$ and perpendicular to a given line AS is:

$\frac{y-3}{x-3} = -\frac{1}{2} \Rightarrow y - 3 = -\frac{1}{2}(x-3)$ or $x + 2y - 9 = 0$

Now equation of parabola:

By definition, $\overline{PS} = \overline{PM}$ or $\overline{AS} = \overline{AZ}$

$\overline{PS} = \sqrt{(x-1)^2 + (y+1)^2}$

$\overline{PM} = \perp$ Distance of P from directrix $x + 2y - 9 = 0$

$\overline{PM} = \frac{x+2y-9}{\sqrt{1+4}} = \frac{x+2y-9}{\sqrt{5}}$

$\therefore \sqrt{(x-1)^2 + (y+1)^2} = \frac{x+2y-9}{\sqrt{5}}$, squaring

$\Rightarrow x^2 - 2x + 1 + y^2 + 2y + 1 = \frac{[(x+2y)-9]^2}{5}$

$\Rightarrow 5x^2 - 10x + 5 + 5y^2 + 10y + 5 = (x+2y)^2 - 18(x+2y) + 81$

$\Rightarrow 5x^2 + 5y^2 - 10x + 10y + 10 = x^2 + 4xy + 4y^2 - 18x - 36y + 81$

$\Rightarrow 4x^2 - 4xy + y^2 + 8x + 46y - 71 = 0$

$\Rightarrow (2x - y)^2 + 8x + 46y - 71 = 0$

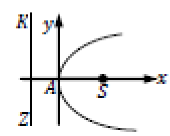
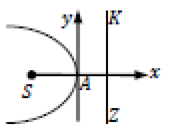
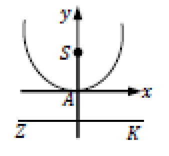
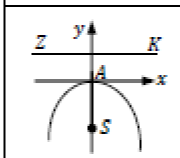
Task

Find the equation of a parabola with:

a) Vertex $(2,0)$, directrix $x = 0$ **Ans:** $y^2 = 8x - 16$

b) Vertex $(3,1)$, Focus at $(3,3)$ **Ans:** $(x-3)^2 = 8(y-1)$

Summary of Standard Equation of a Parabola

Curve	Eqn. of curve	Focus	Directrix	Axis	Vertex	Tangent at vertex
	$y^2 = 4ax$	$(a, 0)$	$x = -a$	$y = 0$	$A(0, 0)$	$x = 0$
	$y^2 = -4ax$	$-(a, 0)$	$x = a$	$y = 0$	$A(0, 0)$	$x = 0$
	$x^2 = 4ay$	$(0, a)$	$y = -a$	$x = 0$	$A(0, 0)$	$y = 0$
	$x^2 = -4ay$	$(0, -a)$	$y = a$	$x = 0$	$A(0, 0)$	$y = 0$

The above table s used in sketching the conic, determining the vertex, equation of axis, focus and directrix of a parabola from a given equation.

Example

1. *Find the vertex, axis, focus and directrix of a parabola*
   a) $y^2 = 5x - 4y - 9$
   b) $x^2 = 9x - y - 14$ c) $y^2 - 6y + 20x + 49 = 0$

Solution

a) Expressing the equation in one of the above standard forms of a parabola .i.e. $Y^2 = 4AX$. This form is taken because the perfect square variable is y.
$y^2 = 5x - 4y - 9 \Rightarrow y^2 - 4y = 5x - 9$
Completing squares
$\Rightarrow y^2 - 4y + 4 = 5x - 9 + 4$
$\Rightarrow (y-2)^2 = 5x - 5 = 5(x-1)$, which is of the form $Y^2 = 4AX$
By comparison

$\boxed{Y = y + 2,\ X = x - 1,\ 4A = 5\ \therefore A = 5/4}$

But for a parabola $Y^2 = 4AX$,

$\boxed{Y = y + 2,\ X = x - 1,\ 4A = 5\ \therefore A = 5/4}$

But for a parabola $Y^2 = 4AX$,

Vertex $(0, 0)$, Axis $= 0$, Focus $S(A, 0)$, Directrix $X = -A$

For vertex, $(0,0) = (X, Y)$
$\Rightarrow y + 2 = 0 \;\therefore y = -2,\; x - 1 = 0 \quad \therefore x = 1$
$\therefore$ Vertex is $(1, -2)$
Axis, $Y = 0 = y + 2 \quad \therefore y = -2$
Focus, $S(A, 0) = S(X, Y)$
$\Rightarrow Y = y + 2 = 0 \;\therefore y = -2,\; x - 1 = A = 5/4 \quad \therefore x = 9/4$
Focus $S\left(\frac{9}{4}, -2\right)$
Directrix, $X = -A \Rightarrow x - 1 = -\frac{5}{4} \quad \therefore x = -1/4$

b) $x^2 = 9x - y - 14 \Rightarrow x^2 - 9x = -y - 14$
Completing squares
$\Rightarrow x^2 - 9x + (-9/2)^2 = -y - 14 + (-9/2)^2$
$\Rightarrow (x - 9/2)^2 = -y - 14 + \frac{81}{4} = -y + \frac{25}{4}$
$\therefore (x - 9/2)^2 = -y + 25/4$, which is of the form $X^2 = 4AY$
For this parabola,
Focus is $S(0, A) = S(X, Y)$, Axis is $X = 0$, Vertex $A(0,0) = A(X, Y)$ and
Directrix $Y = -A$
Comparing $(x - 9/2)^2 = -y + 25/4$ with $X^2 = 4AY$

$$\boxed{X = x - 9/2\,,\; Y = -y + 25/4,\; 4A = 1 \;\therefore A = 1/4}$$

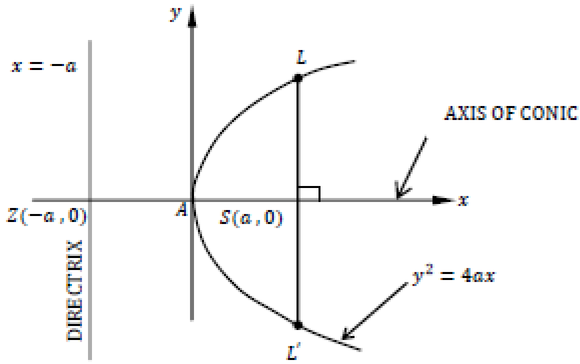
Focus $X = 0 = x - \frac{9}{2} \;\therefore x = \frac{9}{2};\;\; Y = -y + \frac{25}{4} = A = \frac{1}{4} \;\therefore y = 6$
Focus $S\left(\frac{9}{2}, 6\right)$
Axis, $X = x - \frac{9}{2} = 0 \;\therefore x = \frac{9}{2}$
Vertex, $X = 0 = x - \frac{9}{2} = 0 \;\therefore x = \frac{9}{2};\;\; Y = 0 = -y + \frac{25}{4} \;\therefore y = \frac{25}{4}$
Vertex $A\left(\frac{9}{2}, \frac{25}{4}\right)$
Directrix, $Y = -A = -y + \frac{25}{4} = -\frac{1}{4} \;\therefore y = \frac{13}{2}$ is the directrix

c) $y^2 - 6y + 20x + 49 = 0 \Rightarrow y^2 - 6y = -20x - 49$
Completing squares
$\Rightarrow y^2 - 6y + 9 = -20x - 49 + 9$
$\Rightarrow (y - 3)^2 = -20(x + 2)$, which is of the form $Y^2 = 4AX$
For this parabola
Focus $S(A, 0) = S(X, Y) \quad \therefore X = A,\; Y = 0$
Vertex $A(0,0) = A(X, Y) \quad \therefore X = 0, Y = 0$
Axis $Y = 0$
Directrix $X = -A$
Comparing $(y - 3)^2 = -20(x + 2)$ with $Y^2 = 4AX$

$$\boxed{X = y - 3,\;\; Y = x + 2,\;\; 4A = -20 \;\therefore A = -5}$$

LATUS RECTUM

The chord of a parabola which passes through the focus and perpendicular to the axis of the parabola is called *Latus Rectum*.



From the figure, LSL' is the rectum.

$LSL' = 2LS = 2L'S$

Since the $x-$ coordinate of $S(a,0)$ is $x = a$, then the $y-$ coordinate of ends of latus rectum are: $y^2 = 4a.a \Rightarrow y = \pm 2a$

$\therefore L(a, 2a)$, $L'(a, -2a)$, length $\overline{LL'} = 4a$

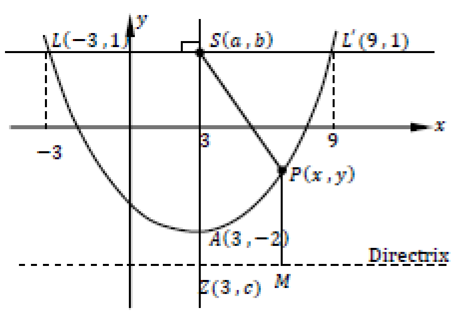
NOTE

Two parabolas are equal if latus rectum is the same.

Example

Find the equation of a parabola with vertex at $(3, -2)$, *end points of latus rectum at* $(-3, 1)$ *and* $(9, 1)$

Solution



Finding coordinates of focus $S(a, b)$

$S(a, b)$ is the midpoint of $\overline{LL'}$

$\Rightarrow (a, b) = \left(\frac{-3+9}{2}, \frac{1+1}{2}\right) = (3, 1)$

$\therefore S(3, 1)$

Also $\overline{AS} = \overline{AZ}$, $\overline{PS} = \overline{PM}$. From the definition A is the midpoint of $\overline{SZ}$

$\therefore (3, -2) = \left(\frac{a+3}{2}, \frac{b+c}{2}\right) = \left(\frac{3+3}{2}, \frac{1+c}{2}\right)$

$\Rightarrow \frac{1+c}{2} = -2 \quad \therefore c = -5$

Directrix passes through $Z(3, -5)$

Since axis of conic passes through S, V and Z and is vertical, the directrix equation is: $y = -5$ or $y + 5 = 0$

$\therefore \overline{PS}^2 = \overline{PM}^2$ becomes;

$(x-3)^2 + (y-1)^2 = (y+5)^2$

$(x-3)^2 = (y+5)^2 - (y-1)^2 = (y+5+y-1)(y+5-y+1)$

$\therefore (x-3)^2 = 12(y+2)$

A LINE AND A PARABOLA

The following questions must be understood in order to master this sub topic.

a) **To Find the Point of Intersection of the Line $y = mx + c$ and the Parabola $y^2 = 4ax$**

Solving the line and the parabola simultaneously

$y = mx + c$ .... (1)

$y^2 = 4ax$ .... (2)

Eqn. (1) in to eqn. (2)

$\Rightarrow (mx + c)^2 = 4ax$

$\Rightarrow m^2x^2 + 2mcx + c^2 - 4ax = 0$

$\Rightarrow m^2x^2 + (2mc - 4a)x + c^2 = 0$ .... (3)

Solving this equation gives *two* values of x. Obtain the corresponding values from equation (1), and hence two points of intersection in the form (x, y)

b) **To Show that a given Line $y = mx + c$ is a Tangent to the Parabola $y^2 = 4ax$.**

Here first solve the line and the parabola simultaneously to obtain eqn. (3) above

$m^2x^2 + (2mc - 4a)x + c^2 = 0$ .... (3)

Now proceed in two ways

Approach I

Applying condition for tangency .i.e. condition for equal roots of a quadratic equation

For tangency, $B^2 = 4AC$

$\Rightarrow (2mc - 4a)^2 = 4m^2c^2$

$\Rightarrow 4m^2c^2 - 16mca + 16a^2 = 4m^2c^2$

$\boxed{c = \frac{a}{m}}$, which is the condition for a line $y = mx + c$ to be a tangent to $y^2 = 4ax$

Approach II

Solving eqn. (3), two repeated/coincident values of x must be obtained.

Eqn. (3) can have coincident roots if the condition for tangency is used.

Put $c = a/m$ in to eqn. (3)

$\Rightarrow m^2x^2 + \left(2m\frac{a}{m} - 4a\right)x + \left(\frac{a}{m}\right)^2 = 0$

$\Rightarrow m^2x^2 - 2ax + \frac{a^2}{m^2} = 0 \quad \therefore \left(mx - \frac{a}{m}\right)^2 = 0$

$\Rightarrow mx - \frac{a}{m} = 0 \quad \therefore x = \frac{a}{m^2}, \frac{a}{m^2}$

Hence a line is a tangent to the parabola.

c) **To obtain the Point of Contact of a tangent line $y = mx + c$ to the parabola $y^2 = 4ax$**

Proceeding as in approach II above from eqn. (3), $x = \frac{a}{m^2}$, obtain the corresponding $y -$ value from the line.

$y = mx + c, \ x = \frac{a}{m^2}$

$\Rightarrow y = m \times \frac{a}{m^2} = \frac{a}{m}$

$\therefore$ Point of contact is $\boldsymbol{P\left(\frac{a}{m^2}, \frac{a}{m}\right)}$

d) **To state the General Point on the parabola $y^2 = 4ax$**

Using the line $y = mx + c$ as a tangent to the parabola $y^2 = 4ax$, obtain the point of contact $P\left(\frac{a}{m^2}, \frac{a}{m}\right)$

Let $t = \frac{1}{m} \Rightarrow P(at^2, 2at)$

$\therefore$ General point on the parabola is $\boldsymbol{T(at^2, 2at)}$

Example

1. *Find the point of intersection of the line $x + y + 1 = 0$ and the parabola $y^2 = 4(x+1)$*

Solution

$x + y + 1 = 0 \Rightarrow y = -x - 1$ .... (1)

$y^2 = 4(x+1)$ .... (2)

Eqn. (1) in to eqn. (2)

$(-x-1)^2 = 4(x+1) \Rightarrow (x+1)^2 = 4(x+1)$

$\Rightarrow (x+1)(x+1-4) = 0 \quad \therefore x = -1, 3$ .... (3)

Eqn. (3) in to eqn. (1)

For $x = -1,\ y = 0;\ x = 3, y = -4$

Points of intersection are: $(-1, 0)$, $(3, -4)$

2. *Prove that the line $3y - 6x - 2 = 0$ touches the parabola $3y^2 = 16x$. Hence find the coordinates of point of contact.*

Solution

For a line to be a tangent, it should touch the parabola at only one point, the point of contact.

$3y - 6x - 2 = 0 \Rightarrow y = \frac{6x+2}{3}$ .... (1)

$3y^2 = 16x$ .... (2)

Eqn. (2) in to eqn. (1)

$3\left(\frac{6x+2}{3}\right)^2 = 16x \quad \Rightarrow 3\left(\frac{36x^2+24x+4}{9}\right) = 16x$

$\Rightarrow 36x^2 + 24x + 4 - 48x = 0$ or $36x^2 - 24x + 4 = 0$

$\Rightarrow 9x^2 - 6x + 1 = 0$ or $(3x-1)^2 = 0 \quad \therefore x = \frac{1}{3}, \frac{1}{3}$

Hence line touches the parabola at one point since there is only one value of $x = \frac{1}{3}$

Point of contact:

For $x = \frac{1}{3}, y = ?$

Using eqn. (1), $y = \frac{6(1/3)+2}{3} = \frac{4}{3} \quad \therefore \left(\frac{1}{3}, \frac{4}{3}\right)$ is the point of contact

4. *A line $x+2y+k=0$ is a tangent to the parabola $y^2=8x$. Find the possible value(s) of k, hence deduce the point(s) of contact.*

Solution

$x+2y+k=0 \Rightarrow x=-(2y+k)$ .... (1)

$y^2=8x$ .... (2)

Eqn. (2) in to eqn. (1)

$y^2=8[-(2y+k)] \Rightarrow y^2+16y+8k=0$.. (3)

For tangency $B^2=4AC$

$\Rightarrow (16)^2=4(8k) \quad \therefore k=8$

$\therefore x+2y+8=0$ is a tangent.

Point of contact:

From eqn. (3), substituting the value of k,

$\Rightarrow y^2+16y+64=0$ or $(y+8)^2=0 \quad \therefore y=-8$...................................
(4)

From eqn. (1), putting $y=-8, k=8$

$\Rightarrow x=-(2\times 8+8)=-8$

$\therefore$ Point of contact is: $(-8,-8)$

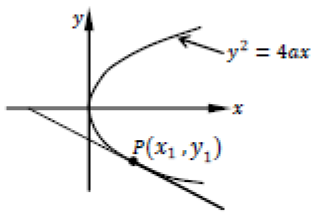
Task

1. *Prove that the line* $9x - 3y + 1 = 0$ *touches* $y^2 = 4x$
2. *Find the equation of a common tangent to* $y^2 = 4x$ *and* $x^2 = 4y$.

   Ans: $x + y + 1 = 0$
3. *Prove that the line* $6y = 18x + 1$ *touches* $y^2 = 2x$
4. *Find the equation of a straight line which touches both* $y^2 = 8ax$ *and* $x^2 + y^2 = 2a^2$ **Ans:** $y = x + 2a$
5. *Find the equation of the parabola whose focus is* $(0, 0)$ *and whose directrix is* $2x = y + 1$. *Show also that the line* $2y = 4x - 1$ *touches the parabola.* **Ans:** $(x + 2y)^2 = 2y - 4x + 1$

EQUATION OF A TANGENT AND NORMAL AT ANY POINT (x_1, y_1) ON A PARABOLA $y^2 = 4ax$

To Find an Equation of a Tangent at any point (x_1, y_1) on the Parabola $y^2 = 4ax$



Required line:

$y - y_1 = m(x - x_1)$ .... (1)

Now finding slope m of the line

From $y^2 = 4ax$, differentiating w.r.t x

$\Rightarrow 2y\frac{dy}{dx} = 4a \quad \therefore \frac{dy}{dx} = \frac{2a}{y}$

$\left.\frac{dy}{dx}\right|_{\text{at } P(x_1, y_1)} = \frac{2a}{y_1}$

$\therefore y - y_1 = \frac{2a}{y_1}(x - x_1)$ or $yy_1 - y_1^2 = 2a(x - x_1)$

$\therefore yy_1 = 2ax + y_1^2 - 2ax_1$ .... (2)

But (x_1, y_1) lies on the parabola

$\Rightarrow y_1^2 = 4ax_1$

Eqn. (2) becomes

$\therefore yy_1 = 2ax + 4ax_1 - 2ax_1 = 2a(x + x_1)$

$\boxed{yy_1 = 2a(x + x_1)}$, which is the required equation of the tangent at $P(x_1, y_1)$ on to the parabola $y^2 = 4ax$.

To Find an Equation of a Tangent at any point $(at^2, 2at)$ on the Parabola $y^2 = 4ax$

Following the above presentation, and using the equation of a tangent at any point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ which is $yy_1 = 2a(x + x_1)$, the required tangent at $(at^2, 2at)$ is:

Comparing $P(x_1, y_1) = (at^2, 2at) \Rightarrow x_1 = at^2, y_1 = 2at$

$\Rightarrow y(2at) = 2a(x + at^2)$ or $ty = x + at^2$

$$\boxed{ty = x + at^2}$$

To Find the Equation of a Tangent to the Parabola $y^2 = 4ax$ from a point $P(x_1, y_1)$ not on the parabola

Let the required tangent be: $y = mx + c$.. (1)

The task is to find the values of m and c

Obtaining the value(s) of m

Solving the equations of tangent and the parabola simultaneously gives:

$m^2x^2 + (2mc - 4a)x + c^2 = 0$

Now applying condition for tangency $B^2 = 4AC$ gives $c = \frac{a}{m}$............... (2)

Thus equation of tangent becomes:

$y = mx + \frac{a}{m}$ .... (3)

To find c, point $P(x_1, y_1)$ lies on the tangent

$\Rightarrow c = y_1 - mx_1$ .... (4)

Solving equation (2) and (4) simultaneously gives:

$\frac{a}{m} = y_1 - mx_1$ or $m^2x_1 - my_1 + a = 0$. Hence two values of m are obtained.

From eqn. (3), two tangents can be obtained as:

$y = m_1x + \frac{a}{m_1}$, $y = m_2x + \frac{a}{m_2}$

Alternatively

Considering the given point $P(x_1, y_1)$ to be the general point on the parabola $y^2 = 4ax$ as $T(at^2, 2at)$, the tangent equation at $T(at^2, 2at)$ is

$ty = x + at^2$... (1)

Now point $P(x_1, y_1)$ is on the tangent

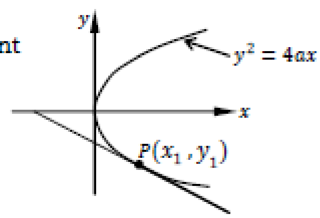
$\Rightarrow ty_1 = x_1 + at^2$ or $at^2 - ty_1 + x_1 = 0$, which is a quadratic in t.

Two values of t can be obtained. Using these values in eqn. (1), two tangents can be deduced.

$t_1y = x + at_1{}^2$ and $t_2y = x + at_2{}^2$

NORMAL AT ANY POINT $P(x_1, y_1)$ TO THE PARABOLA $y^2 = 4ax$

To Find the Equation of the Normal at any point $P(x_1, y_1)$ on the Parabola $y^2 = 4ax$



Using $m_1 \times m_2 = -1$, $m_1 =$ Gradient of tangent, and $m_2 =$ Gradient of the normal

Grad $PT = m_1 = ?$

From $y^2 = 4ax$, differentiating w.r.t x

$2y\frac{dy}{dx} = 4a$, $\frac{dy}{dx} = \frac{2a}{y}$

$\therefore \left.\frac{dy}{dx}\right|_{\text{at } P(x_1, y_1)} = \frac{2a}{y_1} = m_1$

$\therefore m_2 = \text{Grad NP} = -\frac{1}{m_1} = -\frac{y_1}{2a}$

Equation of the normal is:

$y - y_1 = m_2(x - x_1)$

To Find an Equation of a Normal at any point $T(at^2\ ,\ 2at)$ on the Parabola $y^2 = 4ax$

Comparing $P(x_1\ , y_1)$ with $T(at^2\ ,\ 2at)$, and using the equation of the normal above at $P(x_1\ , y_1)$ which is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$

The required equation of the normal at $T(at^2\ ,\ 2at)$ is:

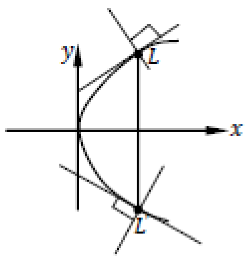
$\Rightarrow y - 2at = -\frac{2at}{2a}(x - at^2) = -tx + at^3$

$$\boxed{tx + y = at^3 + 2at}$$

General Examples

1. *Find the equation of the tangents and normals to the parabola* $y^2 = 4ax$ *at the ends of latus rectum.*

Solution



$L(a\ ,2a)\ ,\ L'(a\ ,2a)$ are end points of the latus rectum

<u>Equation of tangent at $L(a\ ,2a)$</u>

$y - 2a = m(x - a)$

<u>Obtaining slope m</u>

Differentiating $y^2 = 4ax$ w.r.t x

$\Rightarrow 2y\frac{dy}{dx} = 4a \quad \therefore \frac{dy}{dx} = \frac{2a}{y}$

$\frac{dy}{dx}$ at $P(a\ ,2a)$ $= \frac{2a}{2a} = 1$

$\Rightarrow y - 2a = x - a \quad \therefore y = x + a$

Equation of normal at $L(a\ ,2a)$

$y - 2a = m'(x - a)$

<u>Finding slope m'</u>

From $m \times m' = -1\ , m' = -\frac{1}{m} = -\frac{1}{1} = -1$ m =slope of tangent

$\Rightarrow y - 2a = -1(x - a)$

$\therefore y = -x + 3a$

<u>Equation of tangent at $L'(a\ ,-2a)$</u>

$y + 2a = m_1(x - a)$

<u>Obtaining slope m_1</u>

Differentiating $y^2 = 4ax$ w.r.t x

$\Rightarrow 2y\frac{dy}{dx} = 4a \quad \therefore \frac{dy}{dx} = \frac{2a}{y}$

$\frac{dy}{dx}$ at $P(a\ ,-2a)$ $= \frac{2a}{-2a} = -1$

$\Rightarrow y + 2a = -1(x - a)$

$\therefore y = -x - a$

Equation of normal at $L'(a\ ,-2a)$

$y + 2a = m_2(x - a)$

<u>Finding slope m_2</u>

2. *Show that the equation of the tangent to the curve $y^2 = 4ax$ at the ppoint $(at^2, 2at)$ is $ty = x + at^2$ and the equation of the tangent to the curve $x^2 = 4by$ at the point $(2bp, bp^2)$ is $y = px - bp^2$.*
The curves $y^2 = 32x$ and $x^2 = 4y$ intersect at the origin and at A. find the equation of the common tangent to these curves and the coordinates of the points of contact B and C between the tangent and the curves. Calculate area of triangle ABC.

Solution

$y^2 = 4ax$, required tangent: $ty = x + at^2$

<u>Tangent equation at $(at^2, 2at)$</u>

$y - 2at = m(x - at^2)$ .... (1)

<u>Obtaining slope, m</u>

For $y^2 = 4ax$, differentiating w.r.t x

$2y\frac{dy}{dx} = 4a$, $\frac{dy}{dx} = \frac{2a}{y}$

Now $\left.\frac{dy}{dx}\right|_{at\ (at^2, 2at)} = \frac{2a}{2at} = \frac{1}{t} = m$.. (2)

Eqn. (2) in to eqn. (1)

$\Rightarrow y - 2at = \frac{1}{t}.(x - at^2)$

$\Rightarrow ty - 2at^2 = x - at^2 \quad \therefore ty = x + at^2$

For $x^2 = 4by$, differentiating w.r.t x

$2x\frac{dy}{dx} = 4b$, $\frac{dy}{dx} = \frac{x}{2b}$

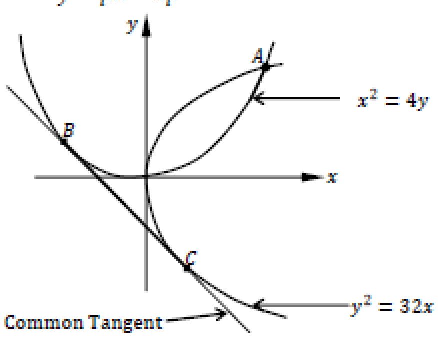
Now $\frac{dy}{dx}_{at(2bp, bp^2)} = \frac{2bp}{2b} = p = m'$

<u>Tangent equation at $(2bp, bp^2)$</u>

$y - bp^2 = m'(x - 2bp) \Rightarrow y - bp^2 = p(x - 2bp)$

$\Rightarrow y = px - 2bp^2 + bp^2$

$\therefore y = px - bp^2$



Comparing curves $y^2 = 32x$ and $y^2 = 4ax \Rightarrow 32 = 4a \quad \therefore a = 8$

Point on $y^2 = 4ax$ is $C(at^2, 2at)$

$\therefore C(8t^2, 16t)$

Now equation of tangent to $y^2 = 4ax$ at $(at^2, 2at)$ is $ty = x + at^2$

$\Rightarrow$ Equation of tangent at $C(8t^2, 16t)$ to the curve $y^2 = 32x$ is

$ty = x + 8t^2$............................ (1)

Eqn. (1) meets $x^2 = 4y$ at B
From curve $x^2 = 4by$ and $x^2 = 4y$, by comparison, $4b = 4 \quad \therefore b = 1$
Equation of the tangent to the curve $x^2 = 4y$ at $B(2bp, bp^2)$ is $y = px - bp^2$
Now equation of the tangent to the curve $x^2 = 4y$ at $B(2p, p^2)$ is:
$y = px - p^2$ .... (2)
Since the tangent at B and the tangent at C are the same, comparing the two equations,
$ty = x + 8t^2 \Rightarrow y = \frac{1}{t}x + 8t$; $y = px - p^2$
$\Rightarrow \frac{1}{t} = p$, and $8t = -p^2$, solving the two equations
$\Rightarrow 8t = -\frac{1}{t^2} \quad \therefore 8t^3 + 1 = 0$
$\Rightarrow (2t+1)(4t^2 - 2t + 1) = 0 \quad \therefore t = -\frac{1}{2}$
$\Rightarrow t = -\frac{1}{2}, p = -2$
Using eqn. (2) and $p = -2$
Equation of the common tangent is:
$y = -2x - 4$

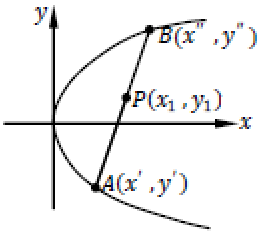
Alternatively
Tangent at B: $y = px - p^2$ is also a tangent to the curve $y^2 = 32x$
Solving the two equations simultaneously and applying the condition for tangency $B^2 = 4AC$
$(px - p^2)^2 = 32x$
$p^2x^2 - 2p^3x + p^4 - 32x = 0 \Rightarrow p^2x^2 - (2p^3 + 32)x + p^4 = 0$
For tangency $B^2 = 4AC$
$\Rightarrow [-(2p^3 + 32)]^2 = 4p^2.p^4$
$\Rightarrow 4p^6 + 128p^3 + 1024 = 4p^6$ or $p^3 + 8 = 0$
$\Rightarrow (p+2)(p^2 - 2p + 4) = 0 \quad \therefore p = -2$
From eqn. (2) $y = px - p^2$, $p = -2$
Equation of the common tangent is:
$y = -2x - 4$
Now obtaining the coordinates of B and C
$B(2p\,,\ p^2) \equiv B(-4\,,4)$
To find $C(8t^2\,,16t)$, lies on the tangent $y = -2x - 4$
$\Rightarrow 16t = -16t^2 - 4$ or $4t^2 + 4t + 1 = 0$, $t = -\frac{1}{2}$
$\therefore C(8t^2\,,16t) \equiv C(2\,,-8)$

3. *Find the equation of a tangent to the parabola from the given points:*
   a) *Parabola:* $y^2 = 8x$, *point* $(-10\,,-8)$
   b) *Parabola:* $y^2 = -32x$, *point* $(1\,,-12)$
   c) *Parabola:* $y^2 = 4ax$, *point* $(-4a\,,-3a)$

CHORD WITH GIVEN MIDPOINT

To find the equation of a chord of the parabola $y^2 = 4ax$ having $M(x_1, y_1)$ as its midpoint

Consider the figure below.



Equation of chord is

$y - y_1 = m(x - x_1)$ .... (1)

<u>Finding slope of the chord</u>

$P(x_1, y_1) = P\left[\frac{x' + x''}{2}, \frac{y' + y''}{2}\right]$

$\therefore 2x_1 = x' + x''$ .... (2)

$\therefore 2y_1 = y' + y''$ .... (3)

Also A and B lie on the parabola

$\Rightarrow y'^2 = 4ax'$ .... (3)

$\Rightarrow y''^2 = 4ax''$ .... (4)

Eqn. (3) – eqn. (4)

$y'^2 - y''^2 = 4a(x' - x'')$

$(y' + y'')(y' - y'') = 4a(x' - x'')$

$\Rightarrow$ Grad APB $= \frac{y' - y''}{x' - x''} = \frac{4a}{y' + y''}$.. (5)

Eqn. (3) into eqn. (5)

Grad APB $= \frac{4a}{2y_1} = \frac{2a}{y_1}$

From eqn. (1), $y - y_1 = m(x - x_1)$ becomes

$yy_1 - y_1^2 = 2a(x - x_1)$

$\boxed{yy_1 - y_1^2 = 2a(x - x_1)}$, which is the required equation

Rewriting this equation

$yy_1 = y_1^2 - 4ax_1 - 2ax + 2ax_1$

$\boxed{yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1}$

To Find the Equation of a Chord joining points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lying of the Parabola $y^2 = 4ax$

This is generally given as

$$\boxed{(y - y_1)(y - y_2) = y^2 - 4ax}$$

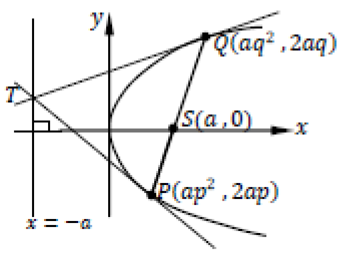
NOTE

A line joining two points on parabola and passing through the focus is called a *Focal Chord.*

Example

1. *Prove that the tangent at the extremities of the focal chord of a parabola $y^2 = 4ax$ intersects at right angles on the directrix. Take coordinates of extremities to be $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$.*

Solution



Slope of the tangent to the curve $y^2 = 4ax$

$2y\frac{dy}{dx} = 4a \quad \therefore \frac{dy}{dx} = \frac{2a}{y}$

$\frac{dy}{dx}_{\text{at } P} = \frac{2a}{2ap} = \frac{1}{p}$, $\frac{dy}{dx}_{\text{at } P} = \frac{2a}{2aq} = \frac{1}{q}$

<u>Equation of tangent at P</u>

$y - 2ap = \frac{1}{p}(x - ap^2)$ .... (1)

<u>Equation of tangent at Q</u>

$y - 2aq = \frac{1}{p}(x - aq^2)$..................... (2)

Solving for point of intersecti

Eqn. (1) − eqn. (2)

$2aq - 2ap = \frac{x - ap^2}{p} - \frac{x - aq^2}{q}$ .... $\times pq$

$pq(2aq - 2ap) = qx - ap^2q - px + apq^2$

$2apq(q - p) = (q - p)x - apq(q - p)$ or $2apq = x - apq \quad \therefore x = apq$

For the focal chord, Grad $\overline{PS}$ = Grad $\overline{SQ}$

$\Rightarrow \frac{2ap - 0}{ap^2 - a} = \frac{2aq - 0}{aq^2 - a}$ or $\frac{2p}{p^2 - 1} = \frac{2q}{p^2 - 1}$

$\Rightarrow pq^2 - p = p^2q - q$ or $pq(q - p) = p - q \quad \therefore pq = -1$

eqn. (3) − eqn. (4)

$x = -a$, which is the $x -$ coordinate of point of intersection, and is the equation of directrix.

Now for intersection at right angles,

Grad $\overline{PT} \times$ Grad $\overline{QT} = -1$

Grad $\overline{PT} = \frac{1}{p}$, Grad $\overline{QT} = \frac{1}{q}$

$\Rightarrow \frac{1}{p} \times \frac{1}{q} = \frac{1}{pq} = \frac{1}{-1} = -1 = \text{RHS}$

2. *Prove that the tangent to the parabola $y^2 = 4ax$ at points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ meet at $T(apq, a(p+q))$.*
If M is the midpoint of PQ, prove that TM is bisected by the parabola.
Solution
First case, see solution above in example (1)
For TM to be bisected by the parabola, midpoint of TM must lie on the parabola.

$\Rightarrow M\left(\frac{ap^2+aq^2}{2}, \frac{2ap+2aq}{2}\right) = M\left(\frac{a(p^2+q^2)}{2}, a(p+q)\right)$

$T(apq, a(p+q))$, $M\left(\frac{a(p^2+q^2)}{2}, a(p+q)\right)$

Let $N =$ midpoint of TM

$\Rightarrow N\left(\frac{apq+\frac{a(p^2+q^2)}{2}}{2}, \frac{a(p+q)+a(p+q)}{2}\right) = N\left(\frac{a(p^2+2pq+q^2)}{4}, a(p+q)\right)$

$\Rightarrow N\left(\frac{a(p+q)^2}{4}, a(p+q)\right)$

Let $t = p+q \Rightarrow N\left(\frac{at^2}{4}, at\right)$, $y^2 = 4ax$

$\Rightarrow (at)^2 = 4a\left(\frac{at^2}{4}\right) = a^2t^2$

Hence point N lies on parabola, which is the midpoint of TM. i.e. parabola bisects TM.

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a) *Find the equation of a tangent to the parabola $y^2 = \frac{x}{16}$ at the point $\left(t^2, \frac{t}{4}\right)$*
b) *If the tangents to the parabola in (a) above at points $P\left(p^2, \frac{p}{4}\right)$ and $Q\left(q^2, \frac{q}{4}\right)$ meet on the line $y = 2$*
   i) *Show that $p+q = 16$*
   ii) *Deduce the midpoint of PQ lies on the line $y = 2$*

Solution

a) Required tangent: $y - \frac{t}{4} = m(x - t^2)$... (1)
Finding the slope of the tangent at $\left(t^2, \frac{t}{4}\right)$
Differentiating $y^2 = \frac{x}{16}$ w.r.t x

$2y\frac{dy}{dx} = \frac{1}{16}$ $\quad \therefore \frac{dy}{dx} = \frac{1}{32y}$

$M = \frac{dy}{dx}_{\text{at } \left(t^2, \frac{t}{4}\right)} = \frac{1}{32 \times \frac{t}{4}} = \frac{1}{8t}$

$\Rightarrow y - \frac{t}{4} = \frac{1}{8t}(x - t^2)$ or $8ty - 2t^2 = x - t^2$

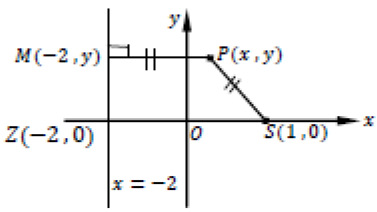
LOCUS PROBLEMS IN PARABOLA

1. *Find the locus of a point $P(x,y)$ which moves in a such a way that:*
   a) *It is equidistant from the point $S(1,0)$ and the line $x=-2$*
   b) *It is equidistant from point $S(-1,1)$ and the line $y=-3$*
   c) *It is equidistant from $S(2,-1)$ and the line $3x-4y=0$*

Hence describe and sketch the locus in each case above.

Solution

a) Using a very clear diagram,



Given: $\overline{PS}=\overline{PM}$

$\Rightarrow \sqrt{(x- -2)^2+(y-y)^2}=\sqrt{(x-1)^2+(y-0)^2}$

Squaring both sides

$\Rightarrow x^2+4x+4=x^2-2x+1+y^2$

$y^2=6x+3=3(2x+1)$

$\therefore y^2=3(2x+1)$, is the locus of P.

The locus is a parabola with focus $S(1,0)$ and equation of directrix $x=-2$

Sketching $y^2=3(2x+1)$

Obtaining the vertex and axis of conic

Axis of the conic is the line through the focus and perpendicular to the directrix, hence axis is $x-axis$. The parabola is symmetrical about the axis

Vertex is the midpoint of $\overline{SZ}$

$= A\left(\frac{-2+1}{2}, \frac{0+0}{2}\right) = A\left(-\frac{1}{2}, 0\right)$

Obtaining intercepts:

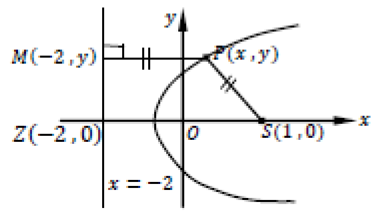
For $x = 0 \Rightarrow y^2 = 3(2(0)+1)$ $\therefore y = \pm\sqrt{3}$

$\therefore B\left(0, +\sqrt{3}\right)$, $B'\left(0, -\sqrt{3}\right)$, are $y-$intercepts.

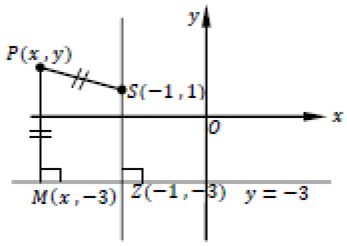
For $y = 0 \Rightarrow 0 = 3(2x+1)$ $\therefore x = -\frac{1}{2}$

$A\left(-\frac{1}{2}, 0\right)$, is an $x-$intercept.

Note: the $x-$intercept is the vertex



b) Representing the information on a clearer diagram.



Given: $\overline{PS} = \overline{PM}$

$\Rightarrow \sqrt{(x- -1)^2 + (y-1)^2} = \sqrt{(x-x)^2 + (y+3)^2}$

Squaring both sides

$\Rightarrow x^2 + 2x + 1 + y^2 - 2y + 1 = y^2 + 6y + 9$

$x^2 + 2x - 6y - 7 = 0$

$\therefore x^2 + 2x - 6y - 7 = 0$, is the locus of P.

The locus is a parabola with focus $S(-1, 1)$ and directrix $y = -3$

<u>Sketching $x^2 + 2x - 6y - 7 = 0$</u>

Obtaining the vertex and axis of conic

Axis of the conic is the line through the focus and perpendicular to the directrix, hence axis is $x = -1$. The parabola is symmetrical about the axis

Vertex is the midpoint of $\overline{SZ}$

$= A\left(\frac{-1-1}{2}, \frac{1+-3}{2}\right) = A(-1, -1)$

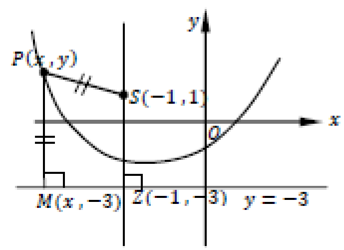
Obtaining intercepts:

For $x = 0$, $\frac{7}{6}$

$\therefore A(0, +7/6)$, is the $y-$intercepts.

For $y = 0 \Rightarrow x^2 + 2x - 7 = 0$

$\therefore B(1.828, 0)$, $B'(-1.828, 0)$ are $x-$intercepts



$\therefore pq + 1 = 0$

3. a) *Show that the equation of a chord joining points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ on a parabola $y^2 = 4ax$ is given as $y(p+q) = 2(x + apq)$*

b) *A variable focal chord PQ of a parabola $y^2 = 4ax$ is moving with its midpoint R. prove that the locus of point R defines a parabola of length of latus rectum $2a$. Find the focus and the equation of the directrix of locus of R.*

Solution

a) General equation of the chord joining any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the parabola is $(y - y_1)(y - y_2) = y^2 - 4ax$

Equation of the required chord is:

$(y - 2ap)(y - 2aq) = y^2 - 4ax$

$\Rightarrow y^2 - 2aqy - 2apy + 4a^2pq = y^2 - 4ax$

$\Rightarrow y(-2ap - 2aq) = -4ax - 4a^2pq$

$\Rightarrow y(p+q) = 2(x + apq)$

b) Since PQ is a variable focal chord, then PQ passes through the focus $S(a, 0)$

$y(p+q) = 2(x + apq) \Rightarrow 0 = 2(a + apq)$

$\Rightarrow 1 + pq = 0 \quad \therefore pq = -1$

$\therefore$ Equation of the focal chord is $y(p+q) = 2(x - a)$........................... (1)

But R is the midpoint of PQ

$\Rightarrow R\left[\frac{ap^2+aq^2}{2}, \frac{2ap+2aq}{2}\right] = R\left[\frac{a(p^2+q^2)}{2}, a(p+q)\right]$

Let the coordinates of R be $R(X, Y)$

$R(X, Y) = R\left[\frac{a(p^2+q^2)}{2}, a(p+q)\right]$

$\Rightarrow X = \frac{a(p^2+q^2)}{2}$.. (2)

$\Rightarrow Y = a(p+q)$.. (3)

Solving eqn. (2) and eqn. (3) simultaneously, eliminating the variables p and q in the coordinates,

From eqn. (2)

$\frac{2X}{a} = p^2 + q^2 = (p+q)^2 - 2pq$

$\Rightarrow \frac{2X}{a} = (p+q)^2 + 2 \quad [\because pq = -1]$ .... (4)

Eqn. (3) into eqn. (4) to eliminate p and q

$\Rightarrow \frac{2X}{a} = \left(\frac{Y}{a}\right)^2 + 2$ or $2aX = Y^2 + 2a^2$

$\Rightarrow Y^2 = 2a(X - a)$

$\therefore$ the locus of R is a parabola $y^2 = 2a(x - a)$

NOTE

When handling numbers of locus, the following points are worth noting.

(a) When mention of any point, line or any angle is made, you must first look for what has been mentioned in case it is not given in the question.

(b) The locus of a point needed is obtained by first letting the point to be (X, Y)

(c) Formulate two equations from step (b) above.

(d) Solve simultaneously the two formulated equations eliminating the variables/parameter involved in the coordinates given.

(e) Write the locus in terms of the coordinate variables on the curve. Usually write it in terms of variables x and y.

4. a) *Find the equation of a line parallel to the normal to the parabola $y^2 = 4ax$ at the point $(p^2 \ \ 2p)$ passing through $(1, 0)$.*

b) *The tangent to $y^2 = 4ax$ at the point $P(at^2, 2at)$ meets the y –axis at G and the normal at P meets the x – axis at H*

i) *Show that the midpoint M of HG has coordinates $\left(a + \frac{at^2}{2}, \frac{at}{2}\right)$*

ii) *Find the locus of M as P moves on the parabola.*

Solution

For parallel lines, gradients are the same $.i.e.$ $m_1 = m_2$

Let the required parallel line with slope m_1 be:

$y - 0 = m_1(x-1)$ or $y = m_1(x-1)$.. (1)

Obtaining the slope, m_2 of the normal at $(p^2 \ \ 2p)$

Given: $y^2 = 4ax$, differentiating w.r.t x to obtain the slope of the tangent

$\Rightarrow 2y\frac{dy}{dx} = 4a$ or $\frac{dy}{dx} = \frac{2a}{y}$

$\therefore m' = \left.\frac{dy}{dx}\right|_{\text{at }(p^2 \ \ 2p)} = \frac{2a}{2p} = \frac{a}{p} =$ Slope of the tangent at $(p^2 \ \ 2p)$

$\Rightarrow m_2 = -\frac{1}{m'} = -\frac{p}{a}$

$\therefore$ Slope of the required lie is $m_1 = m_2 = -\frac{p}{a}$

$\therefore$ Required equation of the line is: $y = -\frac{p}{a}(x-1)$or $\quad px + ay = p$

b) *Obtaining the tangent equation at $P(at^2, 2at)$*

$y - 2at = m(x - at^2)$ .... (1)

Obtaining slope m

$m = \left.\frac{dy}{dx}\right|_{\text{at } P(at^2, 2at)}$, differentiating $y^2 = 4ax$ w.r.t x

$\Rightarrow 2y\frac{dy}{dx} = 4a$ or $\frac{dy}{dx} = \frac{2a}{y}$

$\therefore m = \left.\frac{dy}{dx}\right|_{\text{at } P(at^2, 2at)} = \frac{2a}{2at} = \frac{1}{t}$

From eqn. (1), the tangent equation is:
$y-2at=\frac{1}{t}(x-at^2)\Rightarrow ty-2at^2=x-at^2$
$\therefore ty=x+at^2$, now we have the tangent.
This line meets $y-$ axis at G. This is the $y-$ intercept which occurs when $x=0$
$\Rightarrow ty=at^2$ or $y=at$
$\therefore G(0\,,at)$, now we have G
Obtaining the equation of the Normal at $P(at^2\,,2at)$
$y-2at=m'(x-at^2)$ .... (2)
Obtaining slope m
$m'=-\frac{1}{\text{Slope of the tangent at P}}=-\frac{1}{1/t}-t$
From eqn. (2)
$y-2at=-t(x-at^2)$ or $y=-tx+2at+at^3$
$\therefore y=-tx+2at+at^3$, now we have the normal
This line meets the x –axis at H. this is the x –intercept which occurs when $y=0$
$\Rightarrow 0=-tx+2at+at^3 \quad \therefore x=2a+at^2$
$\therefore H(2a+at^2\,,0)$, now we have also H
Midpoint of HG= $\left(\frac{0+2a+at^2}{2}\,,\frac{at+0}{2}\right)=\left(a+\frac{at^2}{2}\,,\frac{at}{2}\right)$
Obtaining the locus of M
Let $M(X\,,Y)=\left(a+\frac{at^2}{2}\,,\frac{at}{2}\right)$
$\Rightarrow X=a+\frac{at^2}{2}$ or $2X=2a+at^2$ .... (1)
$\Rightarrow Y=\frac{at}{2}$ or $t=\frac{2Y}{a}$ .... (2)
Eliminating t from eqn. (1) and eqn. (2), eqn. (2) into eqn. (1)
$2X=2a+a\left(\frac{2Y}{a}\right)^2\Rightarrow 2aX=2a^2+4Y^2$
$\therefore Y^2=\frac{a}{2}(X-a)$
$\therefore$ The locus of M is $\therefore y^2=\frac{a}{2}(x-a)$

5. *The normal to the parabola $y^2=4ax$ at $P(at^2\,,2at)$ meets the $x-$ axis at G and the midpoint of PG is N.*
   i) *Find the locus of N as P moves on the parabola*
   ii) *If the focus of the parabola is S, prove that $SN\perp PG$*
   iii) *If ΔSRG is equilateral, find the coordinates of R*
   iv) *If $P(at^2\,,2at)$, $Q\left(\frac{a}{t^2}\,,-\frac{2a}{t}\right)$ and $S(a\,,0)$ respectively are points in $x-y$ plane, show that $\frac{1}{SP}+\frac{1}{SQ}$ is independent of the parameter t*

the points meet the y −axis.

4. A tangent from the point $T(t^2, 2t)$ touches $y^2 = 4x$. Find:
   i) The equation of the tangent
   ii) The equation of the line L parallel to the normal at $T(t^2, 2t)$ and passing through $(1, 0)$
   iii) The point of intersection X of line L and tangent.
   (**Note:** point T lies on parabola)
5. Find the locus of the point of intersection of tangents to $y^2 = 4ax$ which are perpendicular to each other. **Ans:** $x + a = 0$
6. The normal at P to $y^2 = 4ax$ cuts the x −axis at G. A line through the midpoint of PG perpendicular to Ox cuts the curve at Q and Ox at N. prove that $QN = PG$
7. The normal to $y^2 = 4ax$ at P meets the x −axis in G. GP is produced so that $PQ = PG$. Find the locus of Q. **Ans:** $y^2 = 16a(x + 2a)$
8. P is any point on $y^2 = 4ax$. O is the origin and M is a point on Ox so that $MPO = 90^0$. If PN is the ordinate of P prove that $MN = 4a$
9. The focus of the parabola $y^2 = 4ax$ is S. Any tangent to the curve meets the directrix in M and the latus rectum in L. prove that $SM = SL$
10. The normal at the point $P(4a, 4a)$ to $y^2 = 4ax$ cuts the curve again at Q. if S is the focus prove that angle $PSQ = 90^0$
11. P is any point on the parabola $y^2 = 4ax$. The normal at P cuts the x −axis in Q. find the locus of the midpoint of PQ. **Ans:** $y^2 = a(x - a)$
12. The normal at P to the parabola $y^2 = 4ax$ cuts the curve again at Q. O is the origin and S is the focus. A line through P parallel to OQ meets the x −axis in R. prove that $OR = 2PS$
13. The vertex of the parabola $y^2 = 4ax$ is O. OP and OQ are two chords of the parabola and angle $POQ = 90^0$. Show that PQ cuts the x −axis in a fixed point.
14. Chords of the parabola $y^2 = 4ax$ are drawn so that the chords all subtend 90^0 at $O(0, 0)$. Prove that the midpoint of the chords all lie on the parabola $y^2 = 2a(x - 4a)$
15. The normal at P to the parabola $y^2 = 4ax$ cuts the x −axis in G. the normals at Q and R pass through the midpoint of PG. Prove that QR cuts the x −axis at $(-a, 0)$
16. PQ is a focal chord of $y^2 = 4ax$. O is the origin and PO cuts the directrix at M. prove that MQ is parallel to $y = 0$
17. P is the point $(at^2, 2at)$ on $y^2 = 4ax$. P is joined to point $S(a, 0)$ and the line PS cuts the curve again at Q. the tangents at P and Q cut at R. Prove that $RP.RQ = a^2\left(\frac{t^2+1}{t}\right)^3$

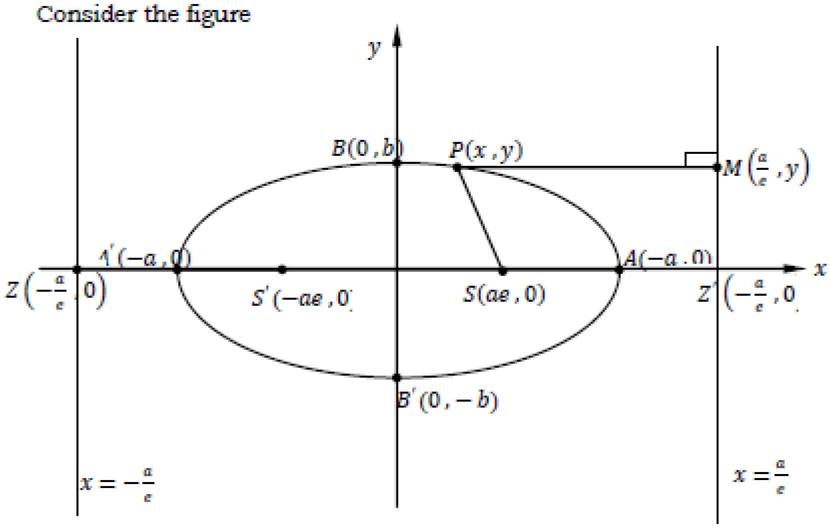
ELLIPSE

Definition:

An ellipse is the locus of a point which moves in such a way that its distance from a fixed point (Focus) bears a constant ratio e, $(0 < e < 1)$ to its distance from a fixed line (Directrix) not passing through the focus.

Equation of an Ellipse in Standard Form

Consider the figure



Point A divides ZS internally and externally in the ratio $1{:}\,e$

From the definition

$\overline{PS} = e\overline{PM}$ or $\overline{AS} = e\overline{AZ}$

$\overline{PS} = \sqrt{(x-ae)^2+(y-0)^2}$, $\overline{PM} = \sqrt{\left(x-\frac{a}{e}\right)^2+(y-y)^2}$

$\Rightarrow \overline{PS}^2 = e\overline{PM}^2$

$\Rightarrow (x-ae)^2+(y-0)^2 = e^2\left(x-\frac{a}{e}\right)^2$

$\Rightarrow x^2-2aex+a^2e^2+y^2 = (xe-a)^2$

$\Rightarrow x^2-2aex+a^2e^2+y^2 = x^2e^2-2aex+a^2$

$\Rightarrow x^2(1-e^2)+y^2 = a^2(1-e^2)$ .... $\div a^2(1-e^2)$

$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)} = 1$, let $b^2 = a^2(1-e^2)$

$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1,\ b^2 = a^2(1-e^2)}$, which is the standard equation of an ellipse

Definitions

1) AA' is called the *Major Axis* of the ellipse and its length is $2a$

2) BB' is called the *Minor Axis* of the ellipse and its length is $2b$

3) The points A and A' are *Vertices* of the ellipse

4) *Eccentricity* of the ellipse is given by the equation $b^2 = a^2(1-e^2)$

5) *Intercepts* of the ellipse are: $A(a,0), A'(-a,0), B(0,b), B'(0,-b)$

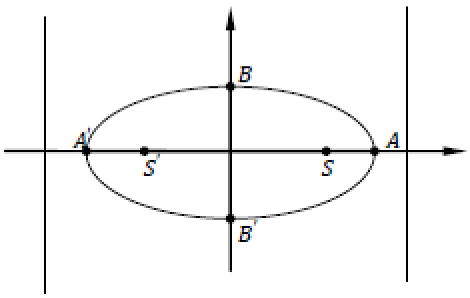
Note:

The points A and A' are $x-$intercepts and the points B and B' are $y-$intercepts.

The major axis is also called Transverse axis.

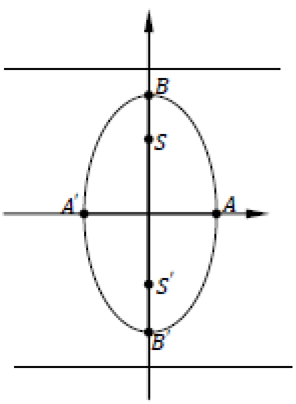
Summary of Standard Equation of an Ellipse

Case I: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$ **where** $b^2=a^2(1-e^2)$



Foci	$(\pm ae, 0)$
Directrices	$x=\pm\frac{a}{e}$
Vertices	$(\pm a,0),(0,\pm b)$
Major axis	$x-$axis or $y=0$
Minor axis	$y-$axis or $x=0$

Case II: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a<b$ **where** $a^2=b^2(1-e^2)$



Foci	$(0, \pm be)$
Directrices	$y=\pm\frac{b}{e}$
Vertices	$(\pm a,0),(0,\pm b)$
Major axis	$y-$axis or $y=0$
Minor axis	$x-$axis or $x=0$

NOTE:

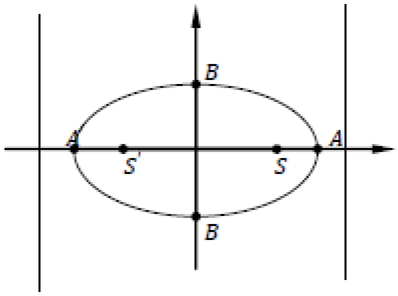
1. If the transverse axis (Major axis) is the y –axis rather than the x –axis, the equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a<b$ where $a^2=b^2(1-e^2)$
2. The table above is used in stating the focus, equation of Directrices, vertices, eccentricity and equation of the ellipse.

Example

1. *Find the equation of the ellipse with:*
   i) Foci at $(\pm3\,,0)$, vertices $(\pm5\,,0)$
   ii) Foci at $(0\,,\pm4)$, vertices $(0\,,\pm7)$
   iii) Foci at $(\pm2\,,0)$, minor axis of length $3\ units$
   iv) Foci at $(0\,,\pm5)$, major axis of length $24\ units$
   v) Vertices $(\pm8\,,0)$, contains $(4\,,\sqrt{3})$

Solution

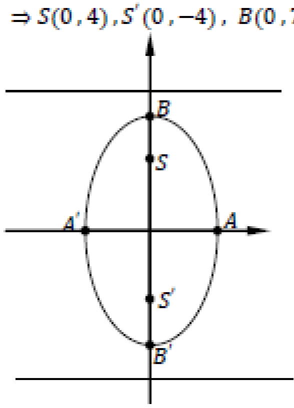
i) Foci at $(\pm3\,,0)$, vertices $(\pm5\,,0)$
$\Rightarrow S(3\,,0)\,,S'(-3\,,0)\,,\ A(5\,,0)\,,\ A'(5\,,0)$



Since the major axis is the x –axis, minor axis is the y –axis
The equation of the ellipse in of the form
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$
Vertices: $(\pm a\,,0)=(\pm5\,,0)\quad\therefore a=5$
Foci: $(\pm ae\,,0)=(\pm3\,,0)\quad\therefore ae=3$
$\Rightarrow e=\frac{3}{5}$
Also $b^2=a^2(1-e^2)=25\left(1-\frac{9}{25}\right)=16$

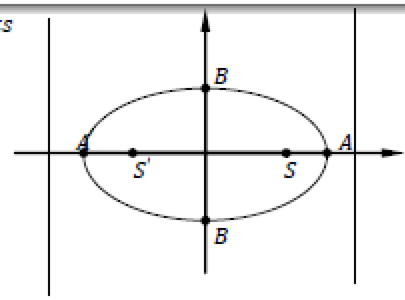
$\therefore\frac{x^2}{25}+\frac{y^2}{16}=1$, is the required ellipse.

ii) Foci at $(0\,,\pm4)$, vertices $(0\,,\pm7)$
$\Rightarrow S(0\,,4)\,,S'(0\,,-4)\,,\ B(0\,,7)\,,\ B'(0\,,-7)$



Since the major axis is the y –axis, minor axis is the x –axis
The equation of the ellipse in of the form
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a<b$
Vertices: $(0\,,\pm b)=(0\,,\pm7)\quad\therefore b=7$
Foci: $(0\,,\pm be)=(0\,,\pm4)\quad\therefore be=4$
$\Rightarrow e=\frac{4}{7}$
Also $a^2=b^2(1-e^2)=49\left(1-\frac{16}{49}\right)=33$
$\therefore\frac{x^2}{33}+\frac{y^2}{49}=1$

iii) Foci at $(\pm 2, 0)$, minor axis of length 3 *units*



Since the major axis is the x –axis, minor axis is the y –axis

The equation of the ellipse in of the form

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$

Foci: $(\pm ae, 0)=(\pm 2, 0)\quad \therefore ae=2$

Length of minor axis $=2b=3\quad \therefore b=\frac{3}{2}$

Since $a>b$, using $b^2=a^2(1-e^2)$

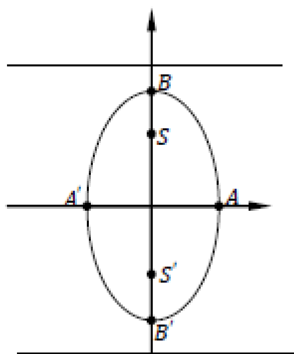
$\Rightarrow \frac{9}{4}=a^2(1-e^2)=a^2-a^2e^2$

$\Rightarrow \frac{9}{4}=a^2-4\quad [\because ae=2\Rightarrow a^2e^2=4]$

$\therefore a^2=\frac{25}{4}$

Required ellipse: $\frac{x^2}{25/4}+\frac{y^2}{9/4}=1$ or $\frac{4x^2}{25}+\frac{4y^2}{9}=1$

iv) Foci at $(0, \pm 5)$, major axis of length 24 *units*



Since the major axis is the y –axis, minor axis is the x –axis

The equation of the ellipse in of the form

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a<b$

Foci: $(0, \pm be)=(0, \pm 5)\quad \therefore be=5$

Length of Major axis $=24=2b\quad \therefore b=12$

$\Rightarrow e=5/12$

Also $a^2=b^2(1-e^2)=144\left(1-\frac{25}{144}\right)=119$

$\therefore \frac{x^2}{119}+\frac{y^2}{144}=1$

v) Vertices $(\pm 8, 0)$, contains $(4, \sqrt{3})$

Vertices $(\pm 8, 0)=(\pm a, 0)\ \Rightarrow a=8$, since the vertices are on the x –axis, it is the major axis.

Using $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$

$\Rightarrow \frac{x^2}{64}+\frac{y^2}{b^2}=1$, point $(4, \sqrt{3})$ is on the curve

$\Rightarrow \frac{4^2}{64}+\frac{(\sqrt{3})^2}{b^2}=1$ or $1-\frac{1}{4}=\frac{3}{b^2}\quad \therefore b^2=4$

$\therefore \frac{x^2}{64}+\frac{y^2}{4}=1$, is the ellipse

2. *Find the major axis, length of the major axis and semi minor axis and coordinates of the foci*

i) $\frac{x^2}{9}+\frac{y^2}{25}=1$ iii) $32x^2+16y^2=64$

ii) $9x^2+25y^2=225$ iv) $162x^2+128y^2=10368$

i) $\frac{x^2}{9}+\frac{y^2}{25}=1$, which is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a<b$, major axis is $y-$ axis

Foci: $(0,\pm be)$; $a^2=b^2(1-e^2)$

Now $a^2=9\Rightarrow a=\pm3$, $b^2=25\ \Rightarrow b=\pm5$

$\Rightarrow 9=25(1-e^2)\quad \therefore e=+4/5$

Major axis: $x=0\ \ (y-\text{axis})$

Length of major axis: $2b=10\ units$

Length of semi minor axis: $a=3\ units$

Foci: $\left(0,\pm5\times\frac{4}{5}\right)=(0,\pm4)$

ii) $9x^2+25y^2=225$……………………………….$(\div225)$

$\frac{x^2}{25}+\frac{y^2}{9}=1$, which is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$

$\Rightarrow a^2=25\ \ \therefore a=\pm5;\ \ b^2=9\ \ \therefore b=\pm3$

Using $b^2=a^2(1-e^2)$

$\Rightarrow 9=25(1-e^2)\ \ \therefore e=+4/5$

Transverse axis: $y=0\ \ (x-\text{axis})$

Length of major axis: $2a=10\ units$

Semi minor axis: $b=3\ units$

Foci: $\left(\pm5\times\frac{4}{5},0\right)=(\pm4,0)$

iii) Left as an exercise

iv) Left as an exercise

3. *Find the eccentricity, foci and Directrices of the ellipse;*

a) $\frac{x^2}{25}+\frac{y^2}{9}=1$ c) $4x^2+y^2=1$

b) $16x^2+25y^2=100$ d) $\frac{x^2}{4}+y^2=1$ e) $\frac{x^2}{2}+\frac{2y^2}{9}=2$

Solution

a) $\frac{x^2}{25}+\frac{y^2}{9}=1$ is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$

$a^2=25$, $b^2=9$, Foci: $(\pm ae,0)$, Directrices: $x=\pm\frac{a}{e}$, $b^2=a^2(1-e^2)$

$\Rightarrow 25=9(1-e^2)\ \ \therefore e=+4/5$

Foci: $\left(\pm5\times\frac{4}{5},0\right)=(\pm4,0)$

Directrices: $x=\pm\frac{5}{4/5}=\pm\frac{25}{4}$

b) $16x^2+25y^2=100\Rightarrow\frac{x^2}{25/4}+\frac{y^2}{4}=1$, which of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$

$a^2=25/4$, $b^2=4$, Foci: $(\pm ae,0)$, Directrices: $x=\pm\frac{a}{e}$, $b^2=$ $a^2(1-e^2)$

$\Rightarrow\frac{25}{4}=4(1-e^2)\ \ \therefore e=+3/5$

Foci: $\left(\pm\frac{5}{2}\times\frac{3}{5},0\right)=\left(\pm\frac{3}{2},0\right)$

Directrices: $x = \pm\frac{5/2}{3/5} = \pm\frac{25}{6}$

c) $4x^2 + y^2 = 1 \Rightarrow \frac{x^2}{1/4} + \frac{y^2}{1} = 1$ which is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a < b$

$a^2 = 1/4,\ b^2 = 1$, Foci: $(0, \pm be)$, Directrices: $y = \pm\frac{b}{e}$, $a^2 = b^2(1-e^2)$

$\Rightarrow \frac{1}{4} = 1(1-e^2)\ \ \therefore e = +\sqrt{3}/2$

Foci: $\left(0, \pm 1 \times \frac{\sqrt{3}}{2}\right) = \left(0, \pm\frac{\sqrt{3}}{2}\right)$

Directrices: $y = \pm\frac{1}{+\sqrt{3}/2} = \pm\frac{2}{\sqrt{3}}$

d) Left as an exercise. **Ans:** $e = \frac{\sqrt{3}}{2}, (\pm\sqrt{3},\ 0), x = \pm\frac{4\sqrt{3}}{3}$

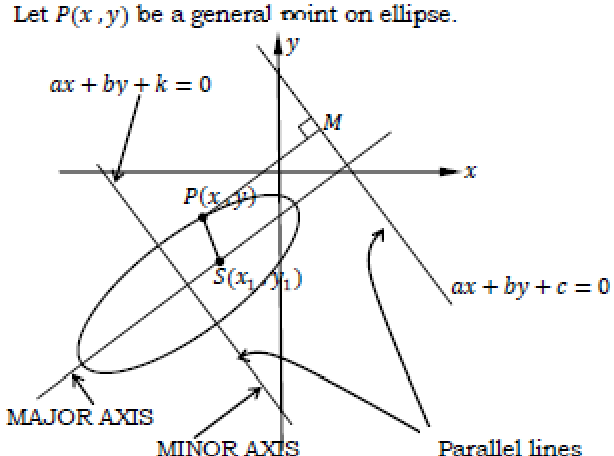
e) Left as an exercise. **Ans:** $e = \frac{\sqrt{5}}{3}, (0, \pm\sqrt{5}), y = \frac{9}{5}\sqrt{5}$

GENERAL EQUATION OF AN ELLIPSE

To Find the Equation of an Ellipse given coordinates of Focus and Equation of Directrix

Let the focus have coordinates $S(x_1, y_1)$ and the equation of the directrix $ax + by + c = 0$

Let $P(x, y)$ be a general point on ellipse.



By definition, $PS = ePM$ or $\overline{PS}^2 = e^2\overline{PM}$

$\overline{PM} = \perp$ Distance of P from the directrix, $ax + by + c = 0$

$\overline{PM} = \pm\frac{ax+by+c}{\sqrt{a^2+b^2}}$

$\overline{PS} = \sqrt{(x-x_1)^2 + (y-y_1)^2}$

$$\boxed{(x-x_1)^2 + (y-y_1)^2 = \frac{(ax+by+c)^2}{\sqrt{a^2+b^2}}}$$

USING STANDARD EQUATION OF AN ELIPSE TO DEDUCE THE CENTRE, LENGTH OF AXES, ECCENTRICITY, FOCI AND VERTICES OF A GIVEN ELLIPSE

Example

Find the centre, length of the axes, eccentricity, foci and vertices of the ellipse

i) $x^2 + 4y^2 - 4x + 24y + 31 = 0$
ii) $12x^2 + 4y^2 + 24x - 16y + 25 = 0$
iii) $4x^2 + 9y^2 - 48x + 72y + 144 = 0$
iv) $x^2 + 4y^2 - 10x - 8y + 13 = 0$

Solution

i) Rewriting the equation in the form $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$

Completing squares

$x^2 - 4x + 4 - 4 + 4(y^2 + 6y + 9 - 9) + 31 = 0$

$(x-2)^2 + 4(y+3)^2 - 4 - 36 + 31 = 0$

$(x-2)^2 + 4(y+3)^2 = 9$

$\frac{(x-2)^2}{9} + \frac{(y+3)^2}{9/4} = 9$, which is of the form $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1, a > b$

By comparison

$X = x - 2, Y = y + 3, a^2 = 9,\ b^2 = 9/4$

Centre: $(0, 0) = (X\ ,\ Y)$

$\Rightarrow x - 2 = 0\ \therefore x = 2$

$\Rightarrow y + 3 = 0\ \therefore y = -3$

$\therefore$ Centre: $(2, -3)$

Using $b^2 = a^2(1 - e^2) \Rightarrow \frac{9}{4} = 9(1 - e^2)\ \ \therefore e = +\frac{\sqrt{3}}{2}$

$\therefore$Eccentricity: $e = +\frac{\sqrt{3}}{2}$
Vertices: $(\pm a, 0) = (X, Y)$ and $(0, \pm b) = (X, Y)$
$\Rightarrow x - 2 = \pm 3 \;\therefore x = 5, -1$
$\Rightarrow y + 3 = 0 \;\therefore y = -3$
$\therefore$ Vertices: $(5, -3), (-1, -3)$
$\Rightarrow x - 2 = 0 \;\therefore x = 2$
$\Rightarrow y + 3 = \pm\frac{3}{2} \;\therefore y = -\frac{3}{2}, -\frac{9}{2}$
$\therefore$ Vertices: $\left(2, -\frac{3}{2}\right), \left(2, -\frac{9}{2}\right)$
Foci: $(\pm ae, 0) = (X, Y)$
$\Rightarrow x - 2 = \pm 3 \times \frac{\sqrt{3}}{2} \quad \therefore x = 2 \pm \frac{3\sqrt{3}}{2}$
$\Rightarrow y + 3 = 0 \;\therefore y = -3$
$\therefore$ Foci: $\left(2 \pm \frac{3\sqrt{3}}{2}, -3\right)$
Length of major axis: $2a = 2 \times 3 = 6\, units$
Length of minor axis: $2b = 2 \times \frac{3}{2} = 3\, units$

ii) Rewriting the equation in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Completing squares
$12(x^2 + 2x + 1 - 1) + 4(y^2 - 4y + 4 - 4) + 25 = 0$
$12(x+1)^2 + 4(y-2)^2 - 12 - 16 + 25 = 0$
$$12(x+1)^2 + 4(y-2)^2 = 3$$
$\frac{(x+1)^2}{1/4} + \frac{(y-2)^2}{3/4} = 1$, which is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a < b$
By comparison
$X = x + 1, Y = y - 2, a^2 = 1/4,\; b^2 = 3/4$
Centre: $(0, 0) = (X, Y)$
$\Rightarrow x + 1 = 0 \;\therefore x = -1$
$\Rightarrow y - 2 = 0 \;\therefore y = 2$
$\therefore$ Centre: $(-1, 2)$
Using $a^2 = b^2(1 - e^2) \Rightarrow \frac{1}{4} = \frac{3}{4}(1 - e^2) \;\; \therefore e = +\frac{\sqrt{6}}{3}$
$\therefore$Eccentricity: $e = +\frac{\sqrt{6}}{3}$
Vertices: $(\pm a, 0) = (X, Y)$ and $(0, \pm b) = (X, Y)$
$\Rightarrow x + 1 = \pm\frac{1}{2} \;\therefore x = -\frac{1}{2}, -\frac{3}{2}$
$\Rightarrow y - 2 = 0 \quad \therefore y = 2$
$\therefore$ Vertices: $\left(-\frac{1}{2}, 2\right), \left(-\frac{3}{2}, 2\right)$
$\Rightarrow x + 1 = 0 \;\therefore x = -1$
$\Rightarrow y - 2 = \pm\frac{\sqrt{3}}{2} \;\therefore y = 2 \pm \frac{\sqrt{3}}{2}$
$\therefore$ Vertices: $\left(-1, 2 \pm \frac{\sqrt{3}}{2}\right)$

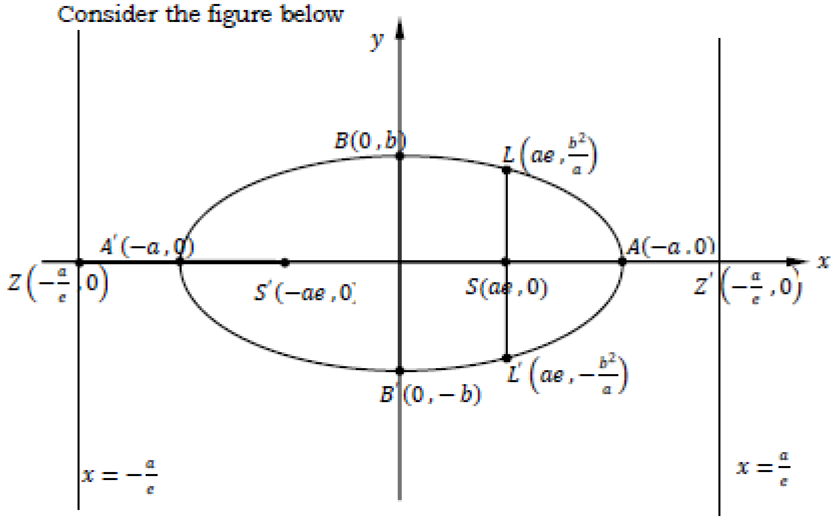
A CHORD, FOCAL CHORD AND LATUS RECTUM OF $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Chord: is any segment whose endpoints are on the ellipse

Focal Chord: is a chord containing the focus

Latus Rectum: the focal chord which is perpendicular to the transverse axis (Major axis)

Consider the figure below



The line LSL' is latus rectum (latus recta in plural).

Finding the coordinates of L' and L

The x −coordinate of L' and L is $x = +ae$

$\frac{(ae)^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{y^2}{b^2}=1-\frac{a^2e^2}{a^2}=1-e^2$

But from $b^2 = a^2(1-e^2) \Rightarrow 1-e^2 = \frac{b^2}{a^2}$

$\Rightarrow \frac{y^2}{b^2}=\frac{b^2}{a^2} \quad \therefore y = \pm\frac{b^2}{a}$

$\therefore L\left(ae, \frac{b^2}{a}\right)$, $L'\left(ae, -\frac{b^2}{a}\right)$

Length of latus rectum is $\frac{2b^2}{a}$

Example

Find the equation of the ellipse in the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a > b$ given that:

i) *Distance between the foci is 6 and distance between the Directrices is 24*

ii) *Latus rectum is $10\frac{2}{3}$ and distance between the Directrices is 36*

Solution

i) Foci: $(\pm ae, 0)$ $\therefore$ Distance$= 2ae = 6$

$\therefore ae = 3$ .... (1)

Distance between Directrices $= \frac{2a}{e} = 24 \quad \therefore a = 12e$.................... (2)

Eqn. (1) into eqn. (2)

$\Rightarrow 12e^2 = 3 \quad \therefore e = +\frac{1}{2}$

From eqn. (2), $a = 12 \times \frac{1}{2} = 6 \quad \Rightarrow a^2 = 36$

Using $b^2 = a^2(1-e^2)$

$\Rightarrow b^2 = 36(1-(1/2)^2) \quad \therefore b^2 = 27$

$\therefore$ Required ellipse is $\frac{x^2}{36}+\frac{y^2}{27}=1$

ii) Length of latus rectum is $\frac{2b^2}{a} = 10\frac{2}{3} \quad \therefore b^2 = \frac{16}{3}a$............................ (1)

Distance between Directrices $= \frac{2a}{e} = 36 \quad \therefore a = 18e$...................... (2)

Using $b^2 = a^2(1-e^2)$

$\Rightarrow \frac{16}{3}a = a^2\left(1-\frac{a^2}{324}\right)$ or $1728 = 324a - a^3$

$\Rightarrow a^3 - 324a + 1728 = 0 \quad \therefore a = 14.2337, -20.2337, 6$

Taking the most accurate value, the whole number, $a = 6$

$\Rightarrow a^2 = 36$, $b^2 = \frac{16}{3} \times 36 = 32$

$\therefore$ Required ellipse: $\frac{x^2}{36}+\frac{y^2}{32}=1$

A LINE AND AN ELLIPSE

Just like in the parabola, the following questions must be well understood in mastering this sub topic.

f) **To Find the Point of Intersection of the Line $y = mx + c$ and the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$**

Solving the equations simultaneously

$y = mx + c$ .... (1)

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .... (2)

Eqn. (1) into eqn. (2)

$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$

$\Rightarrow b^2x^2 + a^2(m^2x^2 + 2mcx + c^2) = a^2b^2$

$\Rightarrow (b^2 + a^2m^2)x^2 + (2mca^2)x + a^2c^2 - a^2b^2 = 0$ .... (3)

Solving this equation gives *two* values of x. Obtain the corresponding values from equation (1), and hence two points of intersection in the form (x, y)

NOTE

Points of intersection can be deduced from the roots of the equation above, which either can be

- Real and distinct
- Coincident/Repeated or
- Imaginary

, depending on the nature of the discriminant $B^2 - 4AC$

For real and distinct roots, $B^2 - 4AC > 0$

For Repeated roots, $B^2 - 4AC = 0$

For Imaginary roots, $B^2 - 4AC < 0$

g) **To Show that a given Line $y = mx + c$ is a Tangent to the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$**

Solving the equations of the line and the ellipse simultaneously, obtain eqn. (3) above as

$(b^2 + a^2m^2)x^2 + (2mca^2)x + (a^2c^2 - a^2b^2) = 0$ .... (3)

Proceed in two ways

Approach I:

Applying condition for tangency .i.e. condition for equal roots of a quadratic equation

For tangency, $B^2 = 4AC$

$\Rightarrow (2mca^2)^2 = 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2)$

$\Rightarrow 4m^2c^2a^4 = 4a^2\ (b^2 + a^2m^2)(c^2 - b^2)$.. $(\div 4a^2)$

$\Rightarrow m^2c^2a^2 = b^2c^2 - b^4 + a^2m^2c^2 - a^2m^2b^2$

$\Rightarrow m^2c^2a^2 = b^2c^2 - b^4 + a^2m^2c^2 - a^2m^2b^2$

$\Rightarrow b^2c^2 - b^4 - a^2m^2b^2 = 0$ .... $(\div b^2)$

$\Rightarrow c^2 - b^2 - a^2m^2 = 0 \quad \therefore c^2 = b^2 + a^2m^2$

$$\boxed{c^2 = a^2m^2 + b^2 \text{ or } c = \pm\sqrt{a^2m^2 + b^2}}$$

, which is the condition for a line $y = mx + c$ to be a tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Example

1. *Find the point of intersection of the line and the ellipse*
   Line: $y = x$ *and the ellipse* $4x^2 + 5y^2 = 49$

   Solution

   Solving the equations simultaneously
   $4x^2 + 5y^2 = 49$, $y = x$
   $\Rightarrow 4x^2 + 5x^2 = 49 \quad \therefore x = \pm 7$
   From the equation of the line, $y = x = \pm 7$
   $\therefore (7, 7)$, $(-7, -7)$ are the points of intersection

2. *Prove that* $y = 2x + 3$ *touches the ellipse* $\frac{x^2}{2} + y^2 = 1$ *and that the point of contact is* $\left(-\frac{4}{3}, \frac{1}{3}\right)$

   Solution

   $y = 2x + 3$ .... (1)
   $\frac{x^2}{2} + y^2 = 1 \Rightarrow x^2 + 2y^2 = 2$ .... (2)
   Solving the equations simultaneously
   Eqn. (1) into eqn. (2)
   $\Rightarrow x^2 + 2(2x + 3)^2 = 2$
   $\Rightarrow x^2 + 2(4x^2 + 12x + 9) = 2$
   $\Rightarrow 9x^2 + 24x + 16 = 0$ or $(3x + 4)^2 = 0$
   $\therefore x = -\frac{4}{3}, -\frac{4}{3}$, which is a repeated value. Since there is only one value of $x = -\frac{4}{3}$, then there is only one point of contact. Hence the line is a tangent to the ellipse.
   Point of contact
   For $x = -\frac{4}{3}$, $y = ?$
   From eqn. (1), $y = 2 \times -\frac{4}{3} + 3 = \frac{1}{3}$
   $\therefore$ Point of contact: $\left(-\frac{4}{3}, \frac{1}{3}\right)$

Task

1) *Prove that* $y = 2x + 5$ *touches* $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$ *and find the point of contact.*
   Ans: $\left(-\frac{8}{5}, \frac{9}{5}\right)$
2) *Prove that* $y = x - 5$ *touches* $\frac{1}{16}x^2 + \frac{1}{9}y^2 = 1$ *and find the point of contact.*
   Ans: $\left(\frac{16}{5}, -\frac{9}{5}\right)$
3) *Show that the line* $x - 2y + 10 = 0$ *is a tangent to the ellipse* $\frac{x^2}{64} + \frac{y^2}{9} = 1$.
   Hence deduce the point of contact. **(Uneb 2000)** **Ans:** $\left(-\frac{32}{4}, \frac{9}{5}\right)$

4. *Show that if the line $y = mx + c$ touches the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the $c^2 = a^2m^2 + b^2$. Hence find the equation of tangents to the ellipse;*

a) $4x^2 + 9y^2 = 1$ *which are perpendicular to* $y = 2x + 3$

b) $\frac{x^2}{64} + \frac{y^2}{9} = 1$, *that pass through* $(-2, 4)$

c) $\frac{x^2}{4} + \frac{y^2}{9} = 1$ *wwhich are inclined to* $x -$ *axis at an angle* $\sin^{-1}\left(\frac{2}{5}\sqrt{5}\right)$

d) $\frac{x^2}{3} + \frac{y^2}{4} = 1$, *that pass through* $(0, \sqrt{31})$

e) $x^2 + 2y^2 = 8$ *which are parallel to* $y = 2x$

Solution

The proof of a line being a tangent to the ellipse is left as an exercise.

a) Required tangent: $y = mx \pm \sqrt{a^2m^2 + b^2}$

Comparing $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $4x^2 + 9y^2 = 1$ or $\frac{x^2}{1/4} + \frac{y^2}{1/9} = 1$

$a^2 = \frac{1}{4}$, $b^2 = \frac{1}{9}$

$\Rightarrow y = mx \pm \sqrt{\frac{m^2}{4} + \frac{1}{9}}$, $m =$ slope of the tangent which must be

$-\frac{1}{\text{Slope of the given line}}$ $\quad \therefore m = -\frac{1}{2}$

$\therefore y = -\frac{1}{2}x \pm \sqrt{\frac{1}{16} + \frac{1}{9}}$ or $y = -\frac{1}{2}x \pm \frac{5}{12}$

b) Required tangent: $y = mx \pm \sqrt{a^2m^2 + b^2}$

Comparing $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $\frac{x^2}{64} + \frac{y^2}{9} = 1$

$a^2 = 64$, $b^2 = 9$

$\Rightarrow y = mx \pm \sqrt{64m^2 + 9}$, $m =$ slope of the tangent which must be obtained

Point $(-2, 4)$ lies on this line

$\Rightarrow 4 = -2m \pm \sqrt{64m^2 + 9}$

$\Rightarrow (4 + 2m) = \pm\sqrt{64m^2 + 9}$, squaring both sides

$\Rightarrow 16 + 16m + 4m^2 = 64m^2 + 9$ or $60m^2 - 16m - 7 = 0$ $\quad \therefore m = \frac{1}{2}, -\frac{7}{30}$

$\therefore$ Tangents are obtained from: $y = mx \pm \sqrt{64m^2 + 9}$

For $m = \frac{1}{2}$, $y = \frac{1}{2}x \pm 5$

For $m = -\frac{7}{30}$, $y = -\frac{7}{30}x \pm \frac{53}{15}$

c) Left as an exercise. **Ans:** $y = 2x \pm 5$

d) Left as an exercise. **Ans:** $y = 3x - \sqrt{31}$, $y = -3x + \sqrt{31}$

EQUATION OF A TANGENT AND NORMAL AT ANY POINT (x_1, y_1) ON AN ELLIPSE $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

To Find an Equation of a Tangent at any point $P(x_1, y_1)$ on the Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Required tangent: $y - y_1 = m(x - x_1)$.. (1)

Finding slope of the Tangent, m

From $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, differentiating w.r.t x

$\left.\frac{dy}{dx}\right|_{\text{at } P(x_1, y_1)} = m$

$\Rightarrow \frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$, multiplying through by $\frac{a^2b^2}{2}$

$\Rightarrow b^2x + a^2y\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{b^2x}{a^2y}$

$\therefore \left.\frac{dy}{dx}\right|_{\text{at } P(x_1, y_1)} = m = -\frac{b^2x_1}{a^2y_1}$ .... (2)

Eqn. (2) into eqn. (1)

$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1)$

$a^2y_1y - a^2{y_1}^2 = -b^2x_1x + b^2{x_1}^2$

$\Rightarrow a^2y_1y + b^2x_1x = a^2{y_1}^2 + b^2{x_1}^2$ .... $(\div a^2b^2)$

$\Rightarrow \frac{xx_1}{a^2}+\frac{yy_1}{b^2} = \frac{{x_1}^2}{a^2}+\frac{{y_1}^2}{b^2} = 1$

Since point $P(x_1, y_1)$ lies on the ellipse, $\frac{{x_1}^2}{a^2}+\frac{{y_1}^2}{b^2}=1$

$\boxed{\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1}$, which is the equation of the tangent at $P(x_1, y_1)$

To Find an Equation of the Normal at any point $P(x_1, y_1)$ on the Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Required tangent: $y - y_1 = m'(x - x_1)$.. (1)

Finding slope of the Normal, m'

Slope of tangent, $m = -\frac{b^2x_1}{a^2y_1}$

$\therefore$ Slope of Normal, $m' = \frac{a^2y_1}{b^2x_1} \quad [\because m \times m' = -1]$

$\Rightarrow y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1)$

$$\boxed{\frac{y-y_1}{\frac{y_1}{b^2}} = \frac{y-x_1}{\frac{x_1}{b^2}} \text{ or } \frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2}$$

Task

1. *Show that the x −coordinates of any point of intersection of the line $y = mx + c$ and the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ are given by the solutions of the quadratic equation* $(4+9m^2)x^2 + 18mc\,x + (9c^2-36) = 0$
   If the line $y = mx + c$ is a tangent of the ellipse, prove that $c^2 = 4 + 9m^2$.
   The line $y = mx + c$ passes through the point $(2, 3)$ write down a second equation connecting m and c, hence prove that m must satisfy the equation $5m^2 + 12m - 5 = 0$.
   Prove that the two tangents drawn from the point $(2, 3)$ to the ellipse are perpendicular to each other.
2. *Find the equation of the tangent and normal to $ax^2 + by^2 = c$ at the point (x_1, y_1).* **Ans:** $axx_1 + byy_1 = c$; $bxy_1 - ax_1y = (b-a)x_1y_1$

PARAMETRIC COORDINATES

The general coordinates on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are $(a\cos\theta\ ,b\sin\theta)$ or generally denoted as θ.

The parametric equations of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are:

$$\boxed{\boldsymbol{x=a\cos\theta\ \ ,\ y=b\sin\theta}}$$

To Find the Equation of a Tangent and Normal to an Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $(a\cos\theta\ ,b\sin\theta)$

Tangent Equation

$y-\ b\sin\theta=m_1(x-a\cos\theta)$ .... (1)

Obtaining slope, m_1

Method I:

$m_1=\frac{dy}{dx}\Big|_{at\ (a\cos\theta\ ,b\sin\theta)}$

Differentiating $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ w.r.t x

$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0\quad \therefore\frac{dy}{dx}=-\frac{b^2x}{a^2y}$

$\Rightarrow m_1=\frac{dy}{dx}\Big|_{at\ (a\cos\theta\ ,b\sin\theta)}=-\frac{b^2a\cos\theta}{a^2b\sin\theta}=-\frac{b\cos\theta}{a\sin\theta}$.

$\therefore m_1=-\frac{b\cos\theta}{a\sin\theta}$

Method II:

Formulating parametric equations

$x=a\cos\theta\ \ ,\ \ y=b\sin\theta$

$m_1=\frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}$

$\frac{dx}{d\theta}=-a\sin\theta\ ,\frac{dy}{d\theta}=b\cos\theta$

$\therefore m_1=b\cos\theta\times\frac{1}{-a\sin\theta}=-\frac{b\cos\theta}{a\sin\theta}$

Tangent equation becomes:

$y-\ b\sin\theta=-\frac{b\cos\theta}{a\sin\theta}(x-a\cos\theta)$

$\Rightarrow ya\sin\theta-ab\sin^2\theta=-b\cos\theta\, x+ab\cos^2\theta$

$\Rightarrow ya\sin\theta+b\cos\theta\, x=ab(\cos^2\theta+\sin^2\theta)$

$\therefore ay\sin\theta+bx\cos\theta=ab\qquad [\because\cos^2\theta+\sin^2\theta]$ .... $(\div ab)$

$\therefore\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$

Equation of the Normal

$y-\ b\sin\theta=m_2(x-a\cos\theta)$ .... (2)

Obtaining slope, m_2

$m_2=-\frac{1}{\text{Slope of the tangent at }(a\cos\theta\ ,b\sin\theta)}=-\frac{1}{m_1}=\frac{a\sin\theta}{b\cos\theta}$

$yb\cos\theta-\ b^2\cos\theta\sin\theta=a\sin\theta\, x-a^2\cos\theta\sin\theta$

$\therefore a\sin\theta\, x-by\cos\theta=(a^2-b^2)\cos\theta\sin\theta$ .... $(\div\cos\theta\sin\theta)$

$\Rightarrow a\frac{\sin\theta}{\cos\theta\sin\theta}x-by\frac{\cos\theta}{\cos\theta\sin\theta}=a^2-b^2$

$\therefore ax\sec\theta-by\,\mathrm{cosec}\,\theta=a^2-b^2$

To show that the parametric equations $x=a\cos\theta\ ,\ y=b\sin\theta$ represent an equation of an ellipse

$x=a\cos\theta\quad\Rightarrow\cos\theta=\frac{x}{a}$

$y=b\sin\theta\quad\Rightarrow\sin\theta=\frac{y}{b}$

Using the trigonometric identity $\cos^2\theta+\sin^2\theta=1$ to eliminate parameter θ

$\Rightarrow\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1\quad\therefore\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which is an ellipse.

the equation of the tangent at this point.

Ans: $\left(2\cos\theta\ ,\frac{4}{5}\sin\theta\right)$, $2x\cos\theta + 3y\sin\theta = 4$

2. *Show that the equation of the tangent to* $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$ *at* $(2\cos\theta\ ,3\sin\theta)$ *is* $3x\cos\theta + 2y\sin\theta = 6$

3. *The normal at any point P to the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *meets the axes in A and B. prove AB is divided externally at P in a constant ratio.*

4. S *and* S_1 *are foci of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$*.P is a point on the ellipse so that PS is parallel to the* y *−axis. The tangent at P meets the* y *−axis in M. prove* MS_1 *is parallel to the normal at P*

5. *The tangent at any point on the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *cuts the minor axis at Q and the tangent at one end of the major axis in R. if S is the focus prove* $QR = QS$

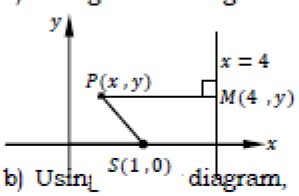
LOCUS PROBLEMS IN ELLIPSE

1. a) *Find the locus of a point* $P(x\ ,y)$ *such that twice its distance from* $S(1\ ,0)$ *is equal to its distance from the line* $x = 4$*. Identify the locus.*

b) *Find the locus of a point* $P(x\ ,y)$ *such that* $3PS = PM$ *where* $S(-3\ ,0)$ *and M is a point on the line* $x = -27$ *such that PM is parallel to x −a xis. Identify the locus.*

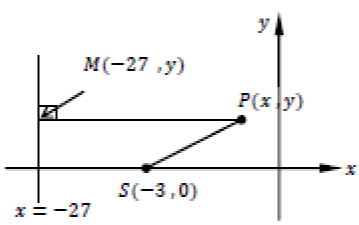
Solution

a) Using a clear diagram



$$\text{Given: } 2\overline{PS} = \overline{PM} \text{ or } 4\overline{PS}^2 = \overline{PM}^2$$
$$\Rightarrow 4[(x-1)^2 + (y-0)^2] = (x-4)^2 + (y-y)^2$$
$$\Rightarrow 4x^2 - 8x + 4 + 4y^2 = x^2 - 8x + 16$$
$$\Rightarrow 3x^2 + 4y^2 = 12 \text{ or } \frac{x^2}{4} + \frac{y^2}{3} = 1, \text{ which is an ellipse}$$

b) Usin diagram,



$$\text{Given: } 3\overline{PS} = \overline{PM} \text{ or } 9\overline{PS}^2 = \overline{PM}^2$$
$$\Rightarrow 9[(x+3)^2 + (y-0)^2] = (x+27)^2 + (y-y)^2$$
$$\Rightarrow 9x^2 + 54x + 81 + 9y^2 = x^2 + 54x + 729$$
$$\Rightarrow 8x^2 + 9y^2 = 648 \text{ or } \frac{x^2}{81} + \frac{y^2}{72} = 1, \text{ which is an ellipse}$$

2. *The normal at* $P(2\cos\theta\ ,\sin\theta)$ *of an ellipse meets x −axis in Q and y −axis in R. A point S is taken on QR produced such that* $QS = RQ$*, prove that the locus is a circle whose equation is* $x^2 + y^2 = 9$

$P(2\cos\theta, \sin\theta)$ lies on ellipse

$x = 2\cos\theta \Rightarrow \cos\theta = \frac{x}{2}$, $y = \sin\theta$

Using $\cos^2\theta + \sin^2\theta = 1$

The Cartesian equation of the curve is: $\frac{x^2}{4} + \frac{y^2}{1} = 1$, which is an ellipse

Obtaining the equation of the normal.

$y - \sin\theta = m_1(x - 2\cos\theta)$ .... (1)

$m_1 = -\frac{1}{\left.\frac{dy}{dx}\right|_{\text{at } P(2\cos\theta, \sin\theta)}}$

By chain rule, $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

For $x = 2\cos\theta \Rightarrow \frac{dx}{d\theta} = -2\sin\theta$

For $y = \sin\theta \Rightarrow \frac{dy}{d\theta} = \cos\theta$

$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \cos\theta \times \frac{1}{-2\sin\theta}$

$\Rightarrow m_1 = -\frac{1}{\left.\frac{dy}{dx}\right|_{\text{at } P(2\cos\theta, \sin\theta)}} = \frac{2\sin\theta}{\cos\theta}$

$\Rightarrow y - \sin\theta = \frac{2\sin\theta}{\cos\theta}(x - 2\cos\theta)$

$\Rightarrow y\cos\theta - \sin\theta\cos\theta = 2x\sin\theta - 2\sin\theta\cos\theta$

$\therefore 2x\sin\theta - y\cos\theta = 3\sin\theta\cos\theta$, which is the normal at P

For x –intercept, $y = 0$

$2x\sin\theta = 3\sin\theta\cos\theta \Rightarrow x = \frac{3}{2}\cos\theta$

$\therefore Q\left(\frac{3}{2}\cos\theta, 0\right)$

For y –intercept, $x = 0$

$-y\cos\theta = 3\sin\theta\cos\theta \Rightarrow y = -3\sin\theta$

$\therefore R(0, -3\sin\theta)$

$R(0, -3\sin\theta)$ $Q\left(\frac{3}{2}\cos\theta, 0\right)$ $S(X, Y)$

Q is the midpoint of SR

$\Rightarrow \left(\frac{3}{2}\cos\theta, 0\right) = \left(\frac{X+0}{2}, \frac{-3\sin\theta + Y}{2}\right)$

$\Rightarrow \frac{X}{2} = \frac{3}{2}\cos\theta \quad \therefore \cos\theta = \frac{X}{3}$ .... (1)

$\Rightarrow \frac{-3\sin\theta + Y}{2} = 0 \quad \therefore \sin\theta = \frac{Y}{3}$ .... (2)

Using $\cos^2\theta + \sin^2\theta = 1$

$\Rightarrow \frac{x^2}{9} + \frac{y^2}{9} = 1 \quad \therefore x^2 + y^2 = 9$

3. *The normal at the point $P(5\cos\theta, 4\sin\theta)$ on the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ meets the major and minor axes at L and M respectively. Show that the locus of R, the midpoint of LM is an ellipse having the same eccentricity as the given ellipse.*

Solution

The equation of the normal *at the point* $P(5\cos\theta\,,4\sin\theta)$ *is:*

$5x\sec\theta - 4y\,\mathrm{cosec}\,\theta = 9$ .... (1)

Let the student look for this equation.

The major axis is the x –axis. So L is the x – intercept

For $y = 0\,,\ 5x\sec\theta = 9 \quad \therefore x = \frac{9\cos\theta}{5}$

$\therefore L\left(\frac{9\cos\theta}{5}\,,0\right)$

The minor axis is the y –axis. So M is the y – intercept

For $x = 0\,,\ -4y\,\mathrm{cosec}\,\theta = 9 \quad \therefore y = -\frac{9\sin\theta}{4}$

$\therefore M\left(0\,,-\frac{9\sin\theta}{4}\right)$

Let $R(X\,,Y)$ be the midpoint of LM

$R(X\,,Y) = R\left(\frac{\frac{9\cos\theta}{5}+0}{2}\,,\frac{0-\frac{9\sin\theta}{4}}{2}\right)$

$\therefore X = \frac{9\cos\theta}{10} \quad \Rightarrow \cos\theta = \frac{10X}{9}$ .... (2)

$\therefore Y = -\frac{9\sin\theta}{8} \quad \Rightarrow \sin\theta = -\frac{8Y}{9}$ .... (3)

Using $\cos^2\theta + \sin^2\theta = 1$

$\left(\frac{10X}{9}\right)^2 + \left(-\frac{8Y}{9}\right)^2 = 1$ or $\frac{X^2}{81/100} + \frac{Y^2}{81/64} = 1$

Required locus is: $\frac{x^2}{81/100} + \frac{y^2}{81/64} = 1$

The locus is an ellipse with its major axis as the y – axis

Calculating Eccentricity for $\frac{x^2}{81/100} + \frac{y^2}{81/64} = 1$

Using $a^2 = b^2(1-e^2)$

$\Rightarrow \frac{81}{100} = \frac{81}{64}(1-e^2)$

$\Rightarrow e^2 = \frac{36}{100} \qquad \therefore e = +\frac{3}{5}$

Calculating Eccentricity for $\frac{x^2}{25} + \frac{y^2}{16} = 1$

Using $b^2 = a^2(1-e^2)$

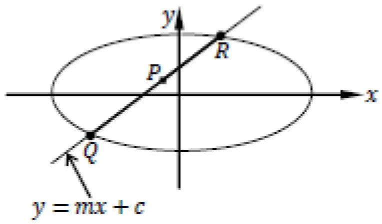
$\Rightarrow 16 = 25(1-e^2)$

$\Rightarrow e^2 = \frac{9}{16} \qquad \therefore e = +\frac{3}{5}$

The eccentricity for the required locus is the same as that of the given locus. i.e. $e = +\frac{3}{5}$

3. *A variable straight line with constant gradient m meets the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *at Q, R. find the locus of P the midpoint of QR.*

Solution



Let $Q(x_1, y_1)$ and $R(x_2, y_2)$ be the points of intersection of the line and the ellipse.
Let $y = mx + c$ be the line.
(c is unkown, m is known)
Solving the equations simultaneously,

$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$, simplifying

$x^2(a^2m^2 + b^2) + (2mca^2)x + a^2(c^2 - b^2) = 0$ .... (1)

This equation is quadratic in x, which gives the $x-$ coordinates of points of intersection for Q and R

Using sum of roots

$x_1 + x_2 = -\frac{2mc\,a^2}{a^2m^2+b^2}$ .... (2)

Let the midpoint of QR be $P(X, Y)$

$P(X, Y) = P\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$\therefore X = -\frac{2mc\,a^2}{a^2m^2+b^2} \times \frac{1}{2} = -\frac{mc\,a^2}{a^2m^2+b^2}$ .... (3)

In a similar way, solving $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $y = mx + c$ simultaneously we obtain

$\frac{\left(\frac{y-c}{m}\right)^2}{a^2} + \frac{y^2}{b^2} = 1$, simplifying

$y^2(a^2m^2 + b^2) - (2cb^2)y + b^2(c^2 - a^2m^2) = 0$ .... (4)

Using sum of roots

$y_1 + y_2 = \frac{cb^2}{a^2m^2+b^2}$ .... (5)

$\therefore Y = \frac{2cb^2}{a^2m^2+b^2} \times \frac{1}{2} = \frac{cb^2}{a^2m^2+b^2}$ .... (6)

Since c is unknown eliminating it form equations (3) and (6) to obtain the locus of P

$\Rightarrow \frac{X}{Y} = -\frac{mc\,a^2}{a^2m^2+b^2} \div \frac{cb^2}{a^2m^2+b^2} = \frac{-ma^2}{b^2}$

$\Rightarrow b^2X = -ma^2Y$ or $b^2X + ma^2Y = 0$

$\therefore$ Required locus is: $b^2x + ma^2y = 0$

4. *A variable tangent to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ at $T(4\cos\theta, 2\sin\theta)$ meets the parabola $y^2 = 4ax$ at L and M. Find the locus of the midpoint of LM.*

2) *A perpendicular OY is drawn from the origin to any tangent to* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. *Prove the locus of Y is* $x^2+y^2=a^2x^2+b^2y^2$

3) *The normal at point P on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *cuts the major axis at G. show that the locus of the midpoint of PG is an ellipse.*

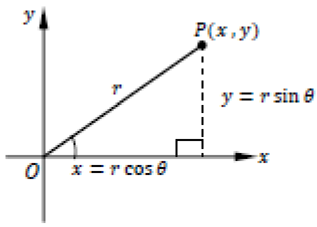
 Ans: $\frac{4a^2x^2}{(2a^2-b^2)^2}+\frac{4y^2}{b^2}=1$

4) *The tangent at any point P on* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *cuts Ox in T. if PN is the ordinate of P prove that* $ON.OT=a^2$

5) *The normal at the point* $P(a\cos\theta\ ,b\sin\theta)$ *on* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *meets the* $y-axis$ *at Q. if S is either focus of the ellipse, prove* $\frac{SQ}{PQ}$ *is independent of* θ

POLAR COORDINATES

These specify the position of a point in a plane.
Consider a point $P(x\ ,y)$ having coordinates in vector form. i.e.
Rectangular Coordinates



$OP=r$
The ordered pair of numbers $(r\ ,\theta)$ is called the *Polar Coordinate of P*
The equations $x=r\cos\theta$ and $y=r\sin\theta$ help in conversion of polar coordinates to rectangular coordinates.
Given $r\ ,\theta$ the $x\ ,y$ can be found using

$$\boxed{x=r\cos\theta\ \ ,y=r\sin\theta}$$

Conversely if x,y are given, then

$$\boxed{r^2=x^2+y^2,\ \tan\theta=\frac{y}{x}}$$

Angle θ is determined for $r=+\sqrt{x^2+y^2}$ and should lie in the interval $-\pi<\theta<\pi$

NOTE

If $r=+\sqrt{x^2+y^2}$, then
$\theta=\tan^{-1}\left(\frac{y}{x}\right)\ ;x>0$
$\theta=\tan^{-1}\left(\frac{y}{x}\right)+\pi\ ;\ x<0$

Examples

1) a) *Find the rectangular coordinates of* $(3\,,150^0)$

b) Find the polar coordinates of $(2\,,-2)$ *for* $r > 0$

Solution

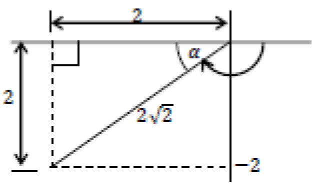
a) Polar coordinates: $(r\,,\theta) = (3\,,150^0)$

$\Rightarrow r = 3\,,\ \theta = 150^0$

Using $x = r\cos\theta\ ,\ y = r\sin\theta$

$x = 3\cos 150^0 = \left(-\frac{3\sqrt{3}}{2}\,,\frac{3}{2}\right)$

b) Polar coordinates: $(r\,,\theta) = ?$



$\tan\alpha = \frac{2}{2} = 1 \quad \therefore \alpha = 45^0$

$\therefore \theta = -135^0$

$r = \sqrt{x^2 + y^2} = \sqrt{2^2 + 2^2}$

$= 2\sqrt{2}$

$\therefore (r\,,\theta) = \left(2\sqrt{2}\,,-135^0\right)$

2. Find

i) The Cartesian coordinates of the point whose polar coordinates are $\left(5\,,\frac{\pi}{4}\right)$

ii) The polar coordinates whose Cartesian coordinates are $(-1\,,1)$

Solution

a) Cartesian coordinates: $x = r\cos\theta\ ,y = r\sin\theta$

Polar coordinates $\left(5\,,\frac{\pi}{4}\right) = (r\,,\theta)$

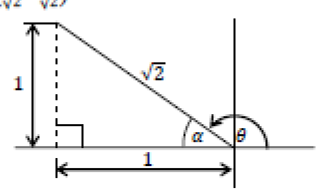
$r = 5\,,\ \theta = \frac{\pi}{4}$

$\therefore$ Cartesian coordinates:

$x = 5\cos\frac{\pi}{4} = \frac{5}{\sqrt{2}}\ ,\ y = 5\sin\frac{\pi}{4} = \frac{5}{\sqrt{2}}$

$\therefore \left(\frac{5}{\sqrt{2}}\,,\frac{5}{\sqrt{2}}\right)$

b)



$\tan\alpha = \frac{1}{1} = 1 \quad \therefore \alpha = 45^0 = \frac{\pi}{4}$

$\therefore \theta = 180^0 - 45^0 = 135^0 = \frac{3\pi}{4}$

θ

Polar coordinates:

$(r\,,\theta) = \left(\sqrt{2}\,,\frac{3\pi}{4}\right)$

Expressing Equations in terms of Polar Coordinates

Expressing the equations in polar coordinates

a) $x = 3$ c) $x^2 + y^2 = 16$ e) $y^2 = 4(x+3)$

b) $2x - y = 3$ d) $\frac{x^2}{4} + y^2 = 1$

Solution

Using $x = r\cos\theta$, $y = r\sin\theta$

a) $x = 3 \quad \Rightarrow r\cos\theta = 3$

b) $2x - y = 3 \quad \Rightarrow 2r\cos\theta - r\sin\theta = 3$

$\therefore r(2\cos\theta - \sin\theta) = 3$

c) $x^2 + y^2 = 16 \quad \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = 16$

$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = 16 \quad \therefore r = 4$

d) $\frac{x^2}{4} + y^2 = 1 \quad \Rightarrow \frac{r^2\cos^2\theta}{4} + r^2\sin^2\theta = 1$

$\therefore r^2(\cos^2\theta + 4\sin^2\theta) = 4$

e) $y^2 = 4(x+3) \quad \Rightarrow r^2\sin^2\theta = 4(r\cos\theta + 3)$

$r^2\sin^2\theta = 4(r\cos\theta + 3)$

Task

1. (I) *Find the coordinates of the points where the ellipse* $\frac{x^2}{4} + \frac{y^2}{9} = 1$ *cuts the axes*

i) *Express the equation in poplar form*

ii) *If the line* $y = mx + c$ *is a tangent to the ellipse, show that* $c^2 = 4m^2 + 9$

2. *Find the Cartesian equations corresponding to*

(i) $r = a\sec\theta$ (ii) $r^2\sin 2\theta = a^2$ **Ans:** (i) $x = a$ (ii) $2xy = a^2$

3. *Find the polar equations corresponding to:*

(i) $(x^2 + y^2)^2 = 2a^2xy$ (ii) $(x^2 + y^2)^2 = a^2(x^2 + y^2)$

(iii) $x(x^2 + y^2)ay^2$ (iv) $y^2(2a - x) = x^3$

Ans: (i) $r^2 = a^2\cos 2\theta$ (ii) $r = a\sin\theta\tan\theta$ (iii) $r^2 = a^2\sin 2\theta$

(iv) $r = 2a\sin^2\theta\sec\theta$

THE HYPERBOLA

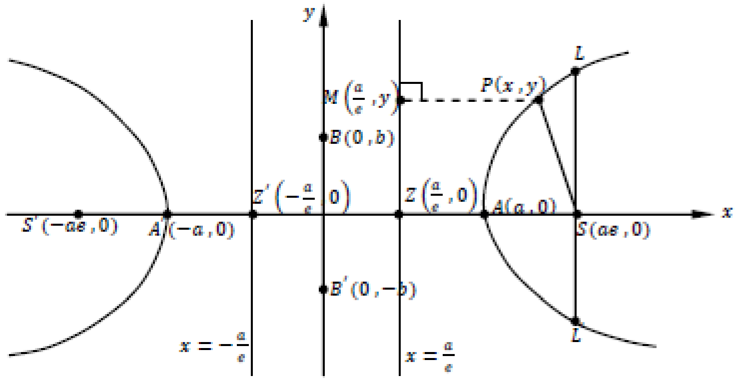
Definition:

Is the locus of a point which moves in such a way that its distance from a fixed point (Focus) bears a constant ratio, $e > 1$ to its distance from a fixed line (Directrix)

Standard Equation of a Hyperbola

To Find the equation of a Hyperbola in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Consider the figure below.



A and A' divide $\overline{SZ}$ internally and externally in the ratio $\overline{SA}:\overline{AZ} = e:1$

From the definition,

$\overline{PS}:\overline{PM} = e:1 \quad \therefore \overline{PS} = e\,\overline{PM}$ or $\overline{PS}^2 = e^2\,\overline{PM}^2$ or $\overline{AS}:\overline{AZ} = e:1$

$\overline{PS} = \sqrt{(x-ae)^2 + (y-0)^2} = \sqrt{x^2 - 2ae\,x + a^2e^2e^2}$

$\overline{PM} = \sqrt{\left(x - \frac{a}{e}\right)^2 + (y-y)^2} = \sqrt{x^2 - \frac{2a}{e}\,x + \frac{a^2}{e^2}}$

$\Rightarrow x^2 - 2ae\,x + a^2e^2 = e^2\left(x^2 - \frac{2a}{e}\,x + \frac{a^2}{e^2} + y^2\right)$

$\Rightarrow x^2 - 2ae\,x + a^2e^2 + y^2 = e^2x^2 - 2ae\,x + a^2$

$\Rightarrow x^2(e^2-1) - y^2 = a^2(e^2-1)$, dividing through by $a^2(e^2-1)$

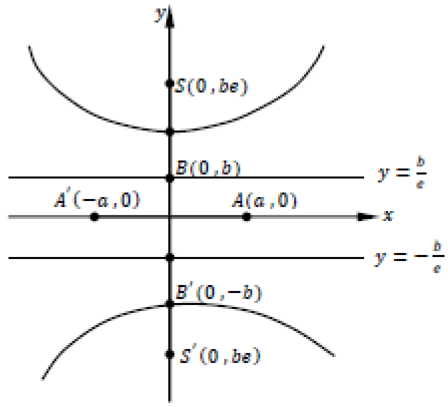
$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$

Let $b^2 = a^2(e^2-1)$

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\ ,\ b^2 = a^2(e^2-1)}$$

CONJUGATE HYPERBOLA

The equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is another standard equation of a hyperbola



Observations

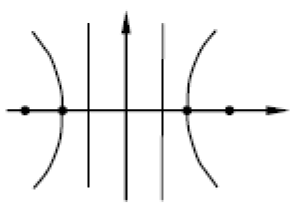
1. $\overline{AA'}$ is the conjugate axis, $\overline{BB'}$ is the transverse axis
2. Vertices are: $(0, \pm b)$
3. Directrices are: $y = \pm\frac{b}{e}$
4. Foci are: $(0, \pm be)$
5. Length of Latus rectum= $\frac{2a^2}{b}$
6. Equations of latus recta: $y = \pm be$
7. For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, $a^2 = b^2(e^2 - 1)$

Illustrative Examples

1. *Find the equation of the hyperbola with:*
   i) Foci $(\pm 5, 0)$ and equation of Directrices $x = \pm\frac{9}{5}$
   ii) Foci $(0, \pm 5)$ and vertices are $(0, \pm 4)$
   iii) Foci $(\pm 6, 0)$ and vertices are $(\pm 3, 0)$

Solution

i)



Since the foci are on the x −axis, then $y = 0$ is a transverse axis of hyperbola. Hence using $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Foci: $(\pm ae, 0) = (\pm 5, 0)$ $\therefore ae = 5$............... (1)

Directrices: $x = \pm\frac{a}{e} = \pm\frac{9}{5}$ $\therefore \frac{a}{e} = \frac{9}{5}$............... (2)

Solving simultaneously eqn. (1) × eqn. (2)

$ae \times \frac{a}{e} = \frac{9}{5} \times 5$ $\quad \therefore a^2 = 9$

Also using $b^2 = a^2(e^2 - 1)$

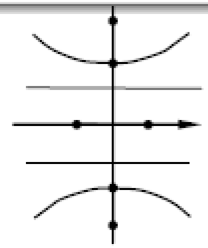
$\Rightarrow b^2 = 9(e^2 - 1)$

From eqn. (1), $e = \frac{5}{a} = \frac{5}{3}$

$\therefore b^2 = 9\left(\frac{25}{9} - 1\right) = 16$

$\therefore$Required equation of hyperbola is: $\frac{x^2}{9} - \frac{y^2}{16} = 1$

ii)



points, the equation of the required hyperbola is

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$

Given: Foci: $(0,\pm5)=(0,\pm be)$ $\therefore be=5$.......... (1)

Vertices: $(0,\pm4)=(0,\pm b)$ $\therefore b=4$.................. (2)

Solving eqn. (1) and eqn. (2) simultaneously $e=\frac{5}{4}$

Using $a^2=b^2(e^2-1)$

$a^2=b^2(e^2-1)=16\left(\frac{25}{16}-1\right)=9$

$\therefore$Required equation of hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=-1$

iii) Left as an exercise. **Ans:** $\frac{x^2}{9}-\frac{y^2}{27}=-1$

2. *Find the eccentricity, foci, equation of Directrices, centre of the hyperbola*

i) $16x^2-9y^2=144$ iii) $9x^2-16y^2+18x+32y-151=0$

ii) $x^2-y^2=32$ iv) $9x^2-16y^2+144=0$

Solution

i) $16x^2-9y^2=144$, rewriting the equation in standard form $(\div 144)$

$\Rightarrow \frac{x^2}{9}-\frac{y^2}{16}=1$ $\therefore a^2=9,\ b^2=16,\ b^2=a^2(e^2-1)$

$\Rightarrow 16=9(e^2-1)$ $\therefore e=5/3$

Foci: $(\pm ae,0)=\left(\pm3\times\frac{5}{3},0\right)=(\pm5,0)$

Directrices: $x=\pm\frac{a}{e}=\pm\frac{3}{5/3}=\pm\frac{9}{5}$ $\therefore x=\pm\frac{9}{5}$

Centre: $(0,0)$

ii) Left as an exercise.

Ans: Foci: $(\pm8,0)$, Directrices: $x=\pm4$, centre: $(0,0)$

iii) $9x^2-16y^2+18x+32y-151=0$

$(9x^2+18x)+(-16y^2+32y)=151$, completing squares,

$9(x^2+2x+1)-16(y^2-2y+1)=151+9-16$

$9(x+1)^2-16(y-1)^2=144$, dividing through by 144

$\frac{(x+1)^2}{16}-\frac{(y-1)^2}{9}=1$, which is of the form $\frac{X^2}{a^2}-\frac{Y^2}{b^2}=1$

$\therefore X=x+1, Y=y-1,\ a^2=16,\ b^2=9$

But for hyperbola $\frac{X^2}{a^2}-\frac{Y^2}{b^2}=1$

Foci: $(\pm Ae,\ 0)=(X,Y)$

$\Rightarrow Ae=X=x+1,\ Y=0=y-1$ $\therefore y=1$

Also $b^2=a^2(e^2-1)$

$9=16(e^2-1)$ $\therefore e=\frac{5}{4}$

$\therefore Ae=4\times\frac{5}{4}=x+1 \Rightarrow x=4$

A LINE AND A HYPERBOLA

j) **To Find the Point of Intersection of the Line $y = mx + c$ and the Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$**

Solving the two equations simultaneously, obtain

$(b^2 - a^2m^2)x^2 - (2mca^2)x - (a^2c^2 - a^2b^2) = 0$.................................... (1)

This equation gives two values of x, hence two points of intersection.

k) **To Show that a given Line $y = mx + c$ is a Tangent to the Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$**

Approach I:

First solve the two equations simultaneously to obtain

$(b^2 - a^2m^2)x^2 - (2mca^2)x - (a^2c^2 - a^2b^2) = 0$...................................... (1)

Applying the condition for tangency $B^2 = 4AC$ to obtain

$c^2 = a^2m^2 - b^2$ or $c = \pm\sqrt{a^2m^2 - b^2}$.. (2)

Approach II:

Solving eqn. (1) for point of contact

Eqn. (2) into eqn. (1) to obtain

$(a^2m^2 - b^2)x^2 \pm \left(2ma^2\sqrt{a^2m^2 - b^2}\right)x + a^4m^2 = 0$

$\left(x\sqrt{a^2m^2 - b^2} \pm ma^2\right)^2 = 0 \quad \therefore x = \pm\frac{ma^2}{\sqrt{a^2m^2 - b^2}}$, which will be a repeated value. Hence the line is a tangent to the ellipse.

EQUATION OF A TANGENT AND NORMAL AT ANY POINT (x_1, y_1) ON A HYPERBOLA $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

To Find an Equation of a Tangent at any point $P(x_1, y_1)$ on the Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Proceeding as in ellipse, the tangent equation is:

$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

To Find an Equation of the Normal at any point $P(x_1, y_1)$ on the Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Proceeding as in ellipse obtain

$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$

To Find an Equation of a Chord with the Midpoint $P(x_1, y_1)$ on the Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Follow the steps in ellipse for this question to obtain

$\frac{x^2x_1}{a^2} - \frac{y^2y_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$

General Examples

Find the equation of the tangent and the normal to the

a) $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at $(0, 3)$

b) $4x^2 - 9y^2 = 36$ at $\left(2, \frac{4\sqrt{5}}{3}\right)$

ASSYMPTOTES

Definition:

Are lines which meet the curve in two points at infinity but does not wholly lie at infinity.

To Find the Asymptotes of Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Two methods can be used.

Method I:

Let $y=mx+c$ be the asymptote.

Solving $y=mx+c$ and the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ obtain:

$(b^2-a^2m^2)x^2-(2mca^2)x-(a^2c^2-a^2b^2)=0$

Equating coefficients of higher powers of x to zero

For x^2: $b^2-a^2m^2=0 \;\Rightarrow m=\pm\frac{b}{a}$

For x: $2mca^2=0 \quad \Rightarrow c=0$

Equation of asymptotes: $\boxed{y=\pm\frac{b}{a}x}$

Method II:

From $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.. $(\times\, b^2x^2)$

$\frac{b^2}{a^2}-\frac{y^2}{x^2}=\frac{b^2}{x^2}$ or $\frac{y^2}{x^2}=\frac{b^2}{a^2}-\frac{b^2}{x^2}$

As $x\to\pm\infty$, $\frac{y^2}{x^2}\to\frac{b^2}{a^2}$ or $\frac{y}{x}\to\pm\frac{b}{a}$

$\boxed{\therefore y=\pm\frac{b}{a}x}$

General Equation of Asymptotes of Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

From $y=\pm\frac{b}{a}x$ or $\frac{y}{b}=\pm\frac{x}{a}$

$\Rightarrow\frac{x}{a}-\frac{y}{b}=0$, $\frac{x}{a}+\frac{y}{b}=0$

Multiplying he equations

$\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{a}+\frac{y}{b}\right)=0$

$$\boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=0}$$

Relationship between Equation of Hyperbola and its Asymptotes

The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ differs from its asymptotes $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$ by a constant. i.e. 1, and also the equation of a conjugate hyperbola differ from its asymptotes by the same constant.

Generally if $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, then the equation of asymptotes combined is:

$(a_1x+b_1y+c_1)(a_2x+b_2y+c_2)=0$

Then the equation of the hyperbola which differs from that of asymptotes by a constant k is:

$(a_1x+b_1y+c_1)(a_2x+b_2y+c_2)=k$

Example

1. *Find the equation of the hyperbola passing through the point* $(4,6)$ *and whose asymptotes are* $y=\pm\sqrt{3}x$

Solution

$y=\pm\sqrt{3}x$ or $\sqrt{3}x-y=0$ and $\sqrt{3}x+y=0$

Combined equation of asymptotes:

$(\sqrt{3}x+y)(\sqrt{3}x-y)=0 \;\Rightarrow 3x^2-y^2=0$

Equation of hyperbola differs from that of asymptotes by a constant

$\Rightarrow 3x^2-y^2=k$

Now $(4,6)$ lies on the hyperbola

$\Rightarrow 48-36=k \quad \therefore k=12$

$\therefore 3x^2-y^2=12$

PARAMETRIC EQUATIONS OF THE HYPERBOLA $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

The general point $(a\sec\theta\ ,b\tan\theta)$ lies on the hyperbola. Thus parametric equations are:

$$\boxed{x = a\sec\theta\ ,\ y = b\tan\theta}$$

To Show that the Parametric Equations represent a Hyperbola

For $x = a\sec\theta \quad \Rightarrow \sec\theta = \frac{x}{a}$

$y = b\tan\theta \qquad \Rightarrow \tan\theta = \frac{y}{b}$

Using $\sec^2\theta = 1+\tan^2\theta \ \Rightarrow \sec^2\theta - \tan^2\theta = 1$

$\therefore \ \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

To Find the Equation of a Tangent and Normal to a Hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ **at** $(a\sec\theta\ ,b\tan\theta)$

Tangent Equation

$y - b\tan\theta = m_1(x - a\sec\theta)$ .... (1)

Obtaining slope, m_1

Formulating parametric equations

$x = a\sec\theta\ \ ,\ y = b\tan\theta$

$m_1 = \frac{dy}{dx} = \frac{dy}{d\theta}\times\frac{d\theta}{dx}$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta \ \ ,\frac{dy}{d\theta} = b\sec^2\theta$

$\therefore m_1 = b\sec^2\theta \times \frac{1}{a\sec\theta\tan\theta} = \frac{b\sec\theta}{a\tan\theta}$

Tangent equation becomes:

$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$

$\Rightarrow ya\tan\theta - ab\tan^2\theta = b\sec\theta\, x - ab\sec^2\theta$

$\Rightarrow b\sec\theta\, x - ya\tan\theta = ab(\sec^2\theta - \tan^2\theta)$

$\therefore ay\tan\theta - bx\sec\theta = ab \qquad [\because \sec^2\theta - \tan^2\theta = 1]$ .... $(\div ab)$

$\therefore \frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$

Equation of the Normal

$y - b\tan\theta = m_2(x - a\sec\theta)$ .... (2)

Obtaining slope, m_2

$m_2 = -\frac{1}{\text{Slope of the tangent at }(a\cos\theta\ ,b\sin\theta)} = -\frac{1}{m_1} = -\frac{a\tan\theta}{b\sec\theta}$

$y - b\tan\theta = -\frac{a\tan\theta}{b\sec\theta}(x - a\sec\theta)$

$yb\sec\theta - b^2\tan\theta\sec\theta = -a\tan\theta\, x + a^2\tan\theta\sec\theta$

$\therefore a\tan\theta\, x + by\sec\theta = (a^2+b^2)\tan\theta\sec\theta$ .... $(\div\cos\theta\sin\theta)$

$\Rightarrow a\frac{\tan\theta}{\tan\theta\sec\theta}x + by\frac{\sec\theta}{\tan\theta\sec\theta} = a^2+b^2$

$\therefore \frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$

Examples

1. (i) *Show that the equation of the tangent to the hyperbola* $(a\sec\theta\ ,b\tan\theta)$ *is* $bx - ay\sin\theta - ab\cos\theta = 0$

ii) *Find the equations of tangents to* $\frac{x^2}{4}-\frac{y^2}{9}=1$ *at the points where* $\theta = 45^0$ *and where* $\theta = -135^0$

LOCUS PROBLEMS IN HYPERBOLA

1. *P is a variable point given by the parametric equations* $x=\frac{a}{2}\left(t+\frac{1}{t}\right)$, $y=\frac{b}{2}\left(t-\frac{1}{t}\right)$. *Show that the locus of P is* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. *State the asymptotes of the curve.*

Solution

$x=\frac{a}{2}\left(t+\frac{1}{t}\right) \quad \Rightarrow t+\frac{1}{t}=\frac{2x}{a}$ .... (1)

$y=\frac{b}{2}\left(t-\frac{1}{t}\right) \quad \Rightarrow t-\frac{1}{t}=\frac{2y}{b}$ .... (2)

Squaring eqn. (1) and eqn. (2) to create the squares on x and y

$\Rightarrow \left(t+\frac{1}{t}\right)^2=\frac{4x^2}{a^2}$ or $\frac{4x^2}{a^2}=t^2+\frac{1}{t^2}+2$................................ (3)

$\Rightarrow \left(t-\frac{1}{t}\right)^2=\frac{4y^2}{b^2}$ or $\frac{4y^2}{b^2}=t^2+\frac{1}{t^2}-2$................................ (4)

Eqn. (3)− eqn. (4)

$\frac{4x^2}{a^2}-\frac{4y^2}{b^2}=4 \quad \therefore \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, is the required locus.

Asymptotes: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$

2. *P is any point on the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ *and Q is the point* (a,b). *Find the locus of the point dividing PQ in the ratio* $2:1$.

Solution

Let $M(X,Y)$ be the point dividing PQ in the ratio $2:1$

Ratio of division; $PM:MQ=2:1$

Let $P(a\sec\theta, b\tan\theta)$, obtaining the coordinates of M in terms of a and b

From coordinate theorem, applying the ratio theorem

$\Rightarrow M(X,Y)=\left(\frac{1\times a\sec\theta+2.a}{2+1},\frac{1\times b\tan\theta+2.b}{2+1}\right)$

$\Rightarrow X=\frac{a}{3}(\sec\theta+2) \quad \therefore \sec\theta=\frac{3X}{a}-2$

$\Rightarrow Y=\frac{b}{3}(\tan\theta+2) \quad \therefore \tan\theta=\frac{3Y}{b}-2$

Using $1+\tan^2\theta=\sec^2\theta$

$\Rightarrow 1+\left(\frac{3Y}{b}-2\right)^2=\left(\frac{3X}{a}-2\right)^2$

$\Rightarrow 1+\frac{9Y^2}{b^2}-\frac{12Y}{b}+4=\frac{9X^2}{a^2}-\frac{12Y}{a}+4$ .... $(\times a^2b^2)$

$\therefore 9b^2X^2-9a^2Y^2=12ab^2X+12a^2bY-a^2b^2=0$

Thus the locus is:

$9b^2x^2-9a^2y^2=12ab^2x+12a^2by-a^2b^2=0$

3. *The normal at* $P(a\sec\theta, b\tan\theta)$ *to the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ *meets the axes at E and F. find the locus of the midpoint of EF.*

Solution

Equation of the normal: $ax\sin\theta + by - (a^2 + b^2)\tan\theta = 0$

The student should derive the equation

Finding the x and $y-$ intercepts

When $x = 0$, $by = (a^2 + b^2)\tan\theta$ $\therefore E\left(0, \frac{(a^2+b^2)\tan\theta}{b}\right)$

When $y = 0$, $ax\sin\theta - (a^2 + b^2)\tan\theta = 0$ $\therefore F\left(\frac{(a^2+b^2)\tan\theta}{a\sin\theta}, 0\right)$

Let $M(X, Y)$ be the midpoint

$\Rightarrow X = \frac{(a^2+b^2)\tan\theta}{2a\sin\theta} = \frac{(a^2+b^2)}{2a\cos\theta}$ $\therefore X = \frac{(a^2+b^2)\sec\theta}{2a}$ .... (1)

$\Rightarrow Y = \frac{(a^2+b^2)\tan\theta}{2b}$ .... (2)

From eqn. (1)

$\sec\theta = \frac{2aX}{(a^2+b^2)}$ $\Rightarrow \sec^2\theta = \frac{2a^2X^2}{(a^2+b^2)^2}$

From eqn. (2)

$\tan\theta = \frac{2bY}{(a^2+b^2)}$ $\Rightarrow \tan^2\theta = \frac{4b^2Y^2}{(a^2+b^2)^2}$

From $1 + \tan^2\theta = \sec^2\theta$

$\Rightarrow 1 + \frac{4b^2Y^2}{(a^2+b^2)^2} = \frac{2a^2X^2}{(a^2+b^2)^2}$ or $(a^2 + b^2)^2 = 4(a^2X^2 - b^2Y^2)$

$\therefore (a^2 + b^2)^2 = 4(a^2x^2 - b^2y^2)$

Task

1. The normal at any point P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the $x-$axis at G. Q is a point on either asymptote such that PQ is parallel to the $y-$axis. Show that GQ is perpendicular to the asymptote on which Q lies.
2. Find the eccentricity, the foci and the directrices of:
   a) $\frac{1}{9}x^2 - \frac{1}{16}y^2 = 1$ **Ans:** $\frac{5}{3}, (\pm 5, 0), 5x \pm 9 = 0$
   b) $5x^2 - 4y^2 = 20$ **Ans:** $\frac{3}{2}, (\pm 3, 0), 3x \pm 4 = 0$
   c) $\frac{1}{4}x^2 - \frac{1}{9}y^2 = 1$ **Ans:** $\frac{1}{2}\sqrt{13}, (\pm\sqrt{13}, 0), \sqrt{13}x \pm 4 = 0$
3. Prove that the locus of a point which moves so that the difference of its distances from the points $(\pm ae, 0)$ is $2a$ is $\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$
4. Find the equation of the tangent and normal to $\frac{1}{25}x^2 - \frac{1}{16}y^2 = 1$ at the point $\left(6\frac{1}{4}, 3\right)$. **Ans:** $4x - 3y = 16, 16y + 12x - 123 = 0$
5. Find the equation of the tangent to $x = a\sec\varphi, y = b\tan\varphi$ at the point $\varphi = 1/4\pi$ **Ans:** $\frac{x\sqrt{2}}{a} - \frac{y}{b} = 1$

THE RECTANGULAR HYPERBOLA

Definition: If the asymptotes of the hyperbola are perpendicular to each other, it is called a *rectangular hyperbola*.

i) Condition for $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ to be a Rectangular Hyperbola

Asymptotes: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$

$\Rightarrow \frac{x}{a}+\frac{y}{b}=0$ or $y=-\frac{b}{a}x$ $\therefore m_1=-\frac{b}{a}$

$\Rightarrow \frac{x}{a}-\frac{y}{b}=0$ or $y=\frac{b}{a}x$ $\therefore m_2=\frac{b}{a}$

For lines to be perpendicular to each other, $m_1 \times m_2=-1$

$\Rightarrow -\frac{b}{a}\times\frac{b}{a}=-1$ or $a^2=b^2$

$$\boxed{\therefore a=b}$$

Thus the equation of a rectangular hyperbola is:

$$\boxed{x^2-y^2=a^2}$$

ii) Eccentricity of a Rectangular Hyperbola $x^2-y^2=0$

From $b^2=a^2(e^2-1)$, $a=b$

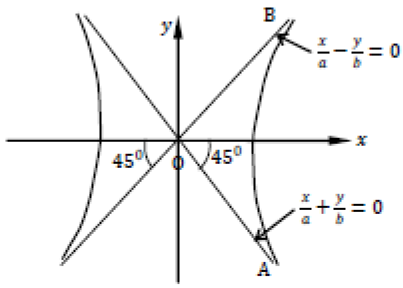
$\Rightarrow a^2=a^2(e^2-1)$ $\therefore e=\sqrt{2}$

Eccentricity of a Rectangular Hyperbola $x^2-y^2=a^2$ is $\boldsymbol{e=\sqrt{2}}$

iii) Equation of a Rectangular Hyperbola referred to its Asymptotes as Coordinate Axes is $xy=c^2$

Equation of the rectangular hyperbola is $x^2-y^2=a^2$

Asymptotes: $x^2-y^2=0$ or $(x+y)(x-y)=0$



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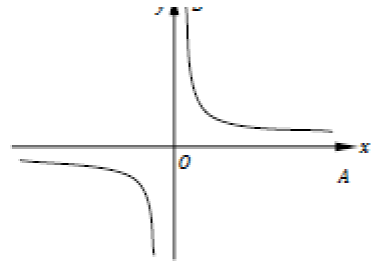


Rotation od aymptotes about O through $+45^0$

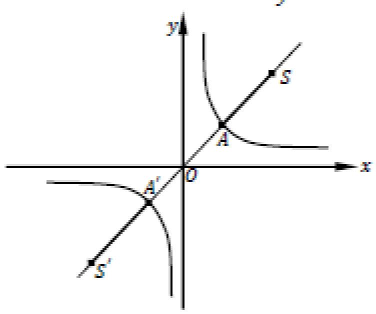
Asymptotes: $x^2-y^2=0$ 　　　 Asymptotes: $x.y=0$

Joint equation of asymptotes after rotation through $+45^0$ about the origin is

$x.y=0$

Since a hyperbola and its asymptotes differ by a constant, say c^2, the required equation of a rectangular hyperbola referred to its asymptotes as axes is: $\boxed{x.y=c^2}$

iv) **Staing the Vertices, coordinates of the Foci and Equations of Directrices of $x.y=c^2$**



Comparing $x.y=c^2$ with $xy=\frac{1}{2}a^2$ (Standard equation of rectangular hyperbola)

$\Rightarrow c^2=\frac{1}{2}a^2 \quad \therefore a=\pm c\sqrt{2}$

From the *figure*, $\overline{OA}=\overline{OA'}=c\sqrt{2}$; $\overline{OS}=\overline{OS'}=ae$

Since the hyperbola is rectangular $e=\sqrt{2}$

$\therefore ae=2c=\overline{OS}=\overline{OS'}$

Now the axis $SAOA'S'$ is inclined to the axes at 45^0

$x-$ coordinate of point A $=\overline{OA}\cos 45^0=c\sqrt{2}\times\frac{1}{\sqrt{2}}=c=\overline{OA'}$

$y-$ coordinate of point A $=\overline{OA}\sin 45^0=c\sqrt{2}\times\frac{1}{\sqrt{2}}=c$

$\therefore$ Vertices are: $(\pm \boldsymbol{c},\pm \boldsymbol{c})$

$x-$ coordinate of point S $=\overline{OS}\cos 45^0=2c\times\frac{\sqrt{2}}{2}=c\sqrt{2}$

$y-$ coordinate of point S $=\overline{OS}\sin 45^0=2c\times\frac{\sqrt{2}}{2}=c\sqrt{2}$

$\therefore$ Coordinates of foci are: $(\pm \boldsymbol{c}\sqrt{\boldsymbol{2}},\pm \boldsymbol{c}\sqrt{\boldsymbol{2}})$

Since directrices are perpendicular to the axis, let their distance from the origin be $\overline{OZ}=\frac{a}{e}=\frac{c\sqrt{2}}{\sqrt{2}}=c$

Task

Find the coordinates of the vertices, foci and equations of directrices of the hyperbola $xy = 18$

Ans: $(\pm 3\sqrt{2}, \pm 3\sqrt{2}), (\pm 6, \pm 6), x + y = \pm 6$

v) Parametric Equations of Rectangular Hyperbola $x.y = c^2$

The general point on the rectangular hyperbola $x.y = c^2$ is $\boldsymbol{P}\left(ct, \frac{c}{t}\right)$. Thus the parametric equations are:

$$\boxed{x = ct \quad , \quad y = \frac{c}{t}}$$

vi) Equation of the Tangent to Hyperbola $x.y = c^2$ at $P\left(ct, \frac{c}{t}\right)$

Required Tangent:

$y - \frac{c}{t} = m(x - ct)$.. (1)

Obtaining slope, m

Approach I:

$$m = \left.\frac{dy}{dx}\right|_{at\ P\left(ct, \frac{c}{t}\right)}$$

Differentiating the hyperbola $x.y = c^2$

$$\Rightarrow x\frac{dy}{dx} + y = 0 \quad \therefore \frac{dy}{dx} = -\frac{y}{x}$$

$$\Rightarrow m = \left.\frac{dy}{dx}\right|_{at\ P\left(ct, \frac{c}{t}\right)} = -\frac{c/t}{ct} = -\frac{1}{t^2}$$

Approach II:

$$m = \frac{dy}{dt} \times \frac{dt}{dx}$$

From $x = ct \quad , \quad y = \frac{c}{t}$

$$\frac{dx}{dt} = c \ , \frac{dy}{dt} = -\frac{c}{t^2}$$

$$\Rightarrow m = \frac{dy}{dt} \times \frac{dt}{dx} = -\frac{c}{t^2} \times \frac{1}{c} = -\frac{1}{t^2}$$

Eqn. (1) becomes

$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$ .... $(\times t^2)$

$t^2 y - ct = -x + ct$

$$\boxed{x + t^2 y = 2\,ct}$$

vii) Equation of the Normal to Hyperbola $x.y = c^2$ at $P\left(ct, \frac{c}{t}\right)$

Required Tangent:

$y - \frac{c}{t} = m'(x - ct)$ .... (1)

Obtaining slope, m'

$m' = -\frac{1}{\text{Slope of the Tangent at P}} = -\frac{1}{-1/t^2} = t^2$

$\therefore y - \frac{c}{t} = t^2(x - ct)$ or $t^3 x - ty = c(t^4 - 1)$

$$\boxed{t^3 x - ty = c(1 + t^4)}$$

viii) Equation of a Tangent and Normal to $x.y = c^2$ at $P(x_1, y_1)$

This point is on the curve $x.y = c^2$

Tangent equation:

From equation of tangent at $P\left(ct, \frac{c}{t}\right)$

$x + t^2 y = 2\,ct$ .... $(\div ct)$

$\frac{x}{ct} + \frac{t^2 y}{ct} = 2$, $P\left(ct, \frac{c}{t}\right) = P(x_1, y_1)$

$\Rightarrow \frac{x}{x_1} + \frac{y}{y_1} = 2$

$\therefore$ Equation of the tangent to $x.y = c^2$ at $P(x_1, y_1)$ is:

$$\boxed{\frac{x}{x_1} + \frac{y}{y_1} = 2}$$

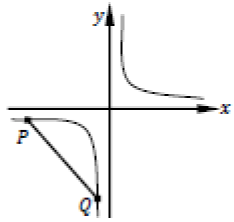
From equation of the Normal at $P\left(ct, \frac{c}{t}\right)$

$y - \frac{c}{t} = t^2(x - ct)$, but $P\left(ct, \frac{c}{t}\right) = P(x_1, y_1)$ and $t^2 = \frac{ct}{\frac{c}{t}} = \frac{x_1}{y_1}$

$\Rightarrow y - y_1 = \frac{x_1}{y_1}(x - x_1)$

$$\boxed{x.x_1 - yy_1 = {x_1}^2 - {y_1}^2}$$

x) To Find the Equation of a Chord joining $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ on the rectangular hyperbola $x.y = c^2$



Required equation of the chord is:

$$y - \frac{c}{p} = \frac{\frac{c}{p} - \frac{c}{q}}{cp - cq}(x - cp)$$

i.e. $y - \frac{c}{p} = -\frac{1}{pq}(x - cp)$

i.e. $y + \frac{x}{pq} = c\left(\frac{1}{p} + \frac{1}{q}\right)$

i.e. $\boldsymbol{pq\,y + x = c(p + q)}$

To Find the Equation of a Chord of the Rectangular Hyperbola $x.y = c^2$ having $M(x_1, y_1)$ as its midpoint

Let the chord have end points as $P(x', y')$ and $Q(x'', y'')$

Required chord:

$y - y_1 = m(x - x_1)$ .... (1)

Obtaining the slope of the chord

$M(x_1, y_1) = P\left[\frac{x' + x''}{2}, \frac{y' + y''}{2}\right]$

$2x_1 = x' + x''$ .... (2)

$\therefore 2y_1 = y' + y''$ .... (3)

Also P and Q lie on the hyperbola

$\Rightarrow x'.y' = c^2$ .... (4)

$\Rightarrow x''.y'' = c^2$ .... (5)

Eqn. (3) + eqn. (4)

$\Rightarrow x' + x'' = \frac{c^2}{y'} + \frac{c^2}{y''}$

$\Rightarrow x' + x'' = c^2\left(\frac{y'' + y'}{y'y''}\right) \quad \therefore y'y'' = \frac{c^2.2y_1}{2x_1} = \frac{c^2.y_1}{x_1}$ .... (6)

$m = \frac{y'' - y'}{x'' - x'} = \frac{y'' - y'}{\frac{c^2}{y''} - \frac{c^2}{y'}}$

$= \frac{y'' - y'}{-c^2\left(\frac{y'' - y'}{y'y''}\right)} = -\frac{y'y''}{c^2}$

$= -\frac{y_1}{x_1}$

Eqn. (1) becomes

$y - y_1 = -\frac{y_1}{x_1}(x - x_1) \Rightarrow x_1 y - x_1 y_1 = -xy_1 + x_1 y_1$

$\Rightarrow x_1 y + xy_1 = x_1 y_1 + x_1 y_1 = 2x_1 y_1$ .... $(\div x_1 y_1)$

$\therefore \frac{x}{x_1} + \frac{y}{y_1} = 2$

10. Prove that the straight line $lx + my = n$ touches the rectangular hyperbola $xy = c^2$ if $n^2 = 4lmc^2$. Find the co-ordinates of the point of contact. **Ans:** $\left(\frac{n}{2l}, \frac{2c^2l}{n}\right)$

11. Show that if the line $y = mx + c$ touches the hyperbola $x^2 - 3y^2 = 1$, then $3m^2 = 3c^2 + 1$.
Obtain an equation for the gradients of the two tangents to the hyperbola from the point $P(x_0, y_0)$. Show that if these tangents are perpendicular then P lies on the circle $x^2 + y^2 = \frac{2}{3}$
Ans: $m^2(x_0^2 - 1) - 2x_0y_0m + y_0^2 + \frac{1}{3} = 0$

12. Show that if the line $y = mx + c$ is a tangent to the rectangular hyperbola $x^2 - y^2 = a^2$ then $c^2 = a^2(m^2 - 1)$ and the co-ordinates of the point of contact T are $\left(-\frac{ma^2}{c}, -\frac{a^2}{c}\right)$
If the line meets the asymptotes at P, Q, show that T is the midpoint of PQ.

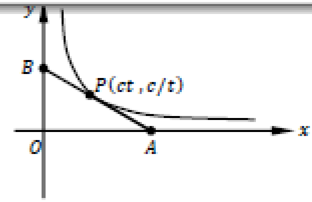
11. Find the equation of the tangents to the hyperbola $x^2 - y^2 = 7$ which are parallel to $3y = 4x$ and find their points of contact. Find the area of the triangle which one of these tangents makes with the asymptotes.
Ans: $4x - 3y = \pm 7, (4, 3)$ and $(-4, -3)$. 7 square units

LOCUS PROBLEMS ON RECTANGULAR HYPERBOLA $x.y = c^2$

1. *The tangent at any point $P(ct, c/t)$ on the hyperbola $x.y = c^2$ meets x and $y -$ axis at A and B respectively.* **O** *is the origin. Prove that:*
   i) $\overline{AP} = \overline{PB}$ ii) *The area of ΔAOB is a constant*

Solution

Find the tangent equation as:
$x + t^2y = 2\,ct$ .... (1)
Obtaining the $x, y -$ intercept
For $y -$ intercept, $x = 0$
$\Rightarrow t^2y = 2\,ct$ or $y = \frac{2c}{t}$ $\therefore B(0, 2c/t)$
For $x -$ intercept, $y = 0$
$\Rightarrow x = 2\,ct$ or $\therefore A(2ct, 0)$
Obtaining $\overline{AP} = \sqrt{(2ct - ct)^2 + (0 - c/t)^2} = c\sqrt{t^2 + 1/t^2}$
Obtaining $\overline{PB} = \sqrt{(0 - ct)^2 + \left(\frac{c}{t} - \frac{2c}{t}\right)^2} = c\sqrt{t^2 + 1/t^2}$
$\therefore \overline{AP} = \overline{PB} = c\sqrt{t^2 + 1/t^2}$



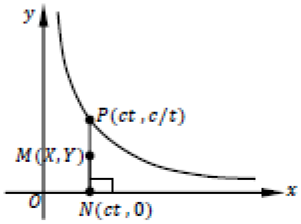
Area of $\Delta AOB = \frac{1}{2} \times \overline{OA} \times \overline{OB}$

$= \frac{1}{2} \times 2ct \times \frac{2c}{t} = 2c^2$

Hence there is no variable t in the expression of area.

2. *PN is the perpendicular to an asymptote from a point on a rectangular hyperbola $x.y = c^2$. Prove that the locus of the midpoint of PN is a rectangular hyperbola with the same axes.*

Solution



Midpoint of $PN = M(X, Y) = M\left(\frac{ct+ct}{2}, \frac{\frac{c}{t}+0}{2}\right)$

$\therefore X = ct\,,\; Y = \frac{c}{2t}$

Multiplying the equations

$X.Y = ct \times \frac{c}{2t} = \frac{c^2}{2}$

$\therefore xy = \frac{c^2}{2}$, which is a rectangular hyperbola with $x, y -$ axes

3. *PQ is a chord of the rectangular hyperbola $x.y = c^2$ and R is the midpoint of PQ. If PQ has a constant length k, find the locus of R.*

Solution

Let $P(cp, c/p), Q(cq, c/q)$ and $R(X, Y)$ be the given points

$\Rightarrow X = \frac{1}{2}c(p+q)$... (1)

$\Rightarrow Y = \frac{1}{2}c\left(\frac{p+q}{pq}\right)$... (2)

But $\overline{PQ} = k$ or $\overline{PQ}^2 = k^2$

$\Rightarrow \overline{PQ}^2 = k^2 = (cp - cq)^2 + \left(\frac{c}{p} - \frac{c}{q}\right)^2$

$\Rightarrow k^2 = c^2(p-q)^2 + c^2\left(\frac{1}{p} - \frac{1}{q}\right)^2$

$\Rightarrow k^2 = c^2(p-q)^2 + c^2\left(\frac{q-p}{pq}\right)^2$

$\Rightarrow k^2 = c^2(p-q)^2\left[1 + \frac{1}{p^2q^2}\right]$

But $(p-q)^2 = p^2 + q^2 - 2pq = (p+q)^2 - 4pq$

$\therefore k^2 = c^2[(p+q)^2 - 4pq]\left[1 + \frac{1}{p^2q^2}\right]$.. (3)

The task is to eliminate variables p, q from the formulated equations

From eqn. (1)

$p + q = \frac{2X}{c}$... (4)

Eqn. (1) ÷ eqn. (2)

$\frac{X}{Y} = pq$.. (5)

Eqn. (4), eqn. (5) into eqn. (3)

$\Rightarrow k^2 = c^2\left[\left(\frac{2X}{c}\right)^2 - 4\left(\frac{X}{Y}\right)\right]\left[1 + \frac{1}{(X/Y)^2}\right]$

$\Rightarrow k^2 = c^2\left[\frac{4X^2}{c^2} - \frac{4X}{Y}\right]\left[\frac{Y^2+X^2}{X^2}\right]$

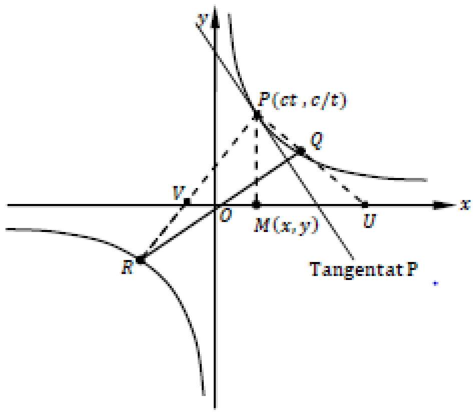
$\Rightarrow X^2Yk^2 = (4X^2Y - 4Xc^2)(X^2+Y^2)$, reducing both sides by X

$\Rightarrow XYk^2 = (4XY - 4c^2)(X^2+Y^2)$

$\therefore 4(X^2+Y^2)(xy - c^2) = xyk^2$, is the required locus

4. *The perpendicular from the origin to the tangent at a point P on the rectangular hyperbola $x.y = c^2$ meets the curve at Q and R. the chords PQ and PR meet the x –axis at u and v. prove that the midpoint of uv is the foot of the perpendicular from P to the x – axis*

Solution



To prove: $M(x, y)$ is the midpoint of uv.

Let $P(ct\ , c/t)$ be any point on $x.y = c^2$

$x = ct\ ,\ y = c/t$

Obtaining the tangent at P

$x + t^2y = 2\,ct$.................... (1)

Obtaining the equation of a line through origin and perpendicular to the tangent

$y - 0 = m'(x - ct) \Rightarrow y = m'x$

But $m' = -\frac{1}{\left.\frac{dy}{dx}\right|_{at\ P}} = -\frac{1}{-1/t^2} = t^2$

$\therefore y = t^2x$

Obtaining the points of intersection of $xy = c^2$ and $y = t^2x$

$\Rightarrow t^2x^2 = c^2 \qquad \therefore x = \pm\frac{c}{t}$

For $x = \frac{c}{t}, y = t^2 \times \frac{c}{t} = ct \qquad \therefore Q\left(\frac{c}{t}, ct\right)$

For $x = -\frac{c}{t}, y = t^2 \times -\frac{c}{t} = -ct \qquad \therefore Q\left(-\frac{c}{t}, -\ ct\right)$

Obtaining the equation of the chord PQ

$y - \frac{c}{t} = m_1(x - ct)$, but $m_1 = \frac{\frac{c}{t} - ct}{ct - c/t} = -1$

$\Rightarrow y - \frac{c}{t} = -1(x - ct)$ or $y = -x + ct + c/t$

Finding coordinates of u , $y = 0$

$0 = -x + ct + \frac{c}{t} \qquad \therefore x = ct + c/t$

Task

1. A chord PQ of the rectangular hyperbola $x.y = c^2$ subtends a right angle at another point R on the curve. Prove PQ parallel to the normal at R.
2. The normal at a point P on $x.y = c^2$ meets the asymptotes in Q and R and the transverse axis in G. prove that $PG^2 = PQ.PR$
3. N is the foot of the perpendicular from the point $(0,0)$ on to the tangent at point P on $x.y = c^2$. Prove that the locus of N is $(x^2 + y^2)^2 = 4c^2xy$
4. If chords of a constant length $2d$ are drawn in the curve $x.y = c^2$ prove the locus of their midpoint is $(x^2 + y^2)(xy - c^2) = d^2xy$
5. Show that the equation of the line joining points $(cp, c/p), (cq, c/q)$ on the rectangular hyperbola $xy = c^2$ is $x + pqy = c(p + q)$. Deduce the equation of the tangent at the point P. **Ans:** $x + p^2y = 2cp$
6. Show that the normal to the rectangular hyperbola $xy = c^2$ at the point $P(ct, c/t)$ meets the curve again at $Q(ct_1, c/t_1)$ such that $t^3t_1 = -1$
7. Show that the locus of the midpoint of the normal chords of a rectangular hyperbola $x^2 - y^2 = a^2$ is $(y^2 - x^2)^3 = 4a^2x^2y^2$

Conclusion

When handling numbers involving conic section,

- Knowledge of Coordinate Geometry I is relevant
- Try to interpret the question with an aid of a clear and illustrative diagram
- Knowledge of Algebra when handling equations involving parameters is also necessary
- Your ability to handle and interpret the question carefully determines how quick can you reach to the required answer
- DO NOT ASSUME THAT CONIC SECTION IS HARD.