INTERGRATION:

Integration is a process of obtaining a function from its derivative.

Techniques of Integration

Recognizing the presence of a function of its derivative:

Example I

$$\int x(x^2-3)^5 dx$$

$Let\ u = x^2-3$

$du = 2xdx$

$dx = \frac{du}{2x}$

$$\int x(x^2-3)^2.dx = \int xu^5.\frac{du}{2x}$$

$$= \frac{1}{2}\int u^5 du$$

$$= \frac{1}{2}\left[\frac{u^6}{6}\right] + c$$

$$= \frac{1}{12}u^6 + C$$

$$= \frac{1}{12}(x^2-3)^6 + C$$

Example II

$$\int (3x-1)^7 dx$$

Solution

$$\int (3x-1)^7 dx$$

Let $u = 3x-1$

$du = 3dx$

$dx = \frac{du}{3}$

$$\int (3x-1)^7 dx = \int u^7.\frac{du}{3}$$

$$= \frac{1}{3}\int u^7\, du$$

$$= \frac{1}{3}\left[\frac{u^8}{8}\right] + C$$

$$= \frac{1}{24}u^8 + C$$

$$\int (3x-1)^7 dx = \frac{1}{24}(3x-1)^8 + C$$

Example III

$$\int (2x-3)(x^2-3x+7)^4\, dx$$

Let $u = x^2-3x+7$

$du = (2x-3)dx$

$dx = \frac{du}{2x-3}$

$$\Rightarrow \int (2x-3)u^4.\frac{du}{2x-3}$$

$$= \int u^4 du$$

$$= \frac{u^5}{5} + C$$

$$\frac{1}{5}(x^2-3x+7)^5 + C$$

Example III

$$\int \frac{2x}{(4x^2-7)^2}dx$$

$let\ u = 4x^2-7$

$du = 8xdx$

$dx = \frac{du}{8x}$

$$\int \frac{2x}{u^2}.\frac{du}{8x}$$

$$\frac{1}{4}\int \frac{1}{u^2}.du$$

$$\frac{1}{4}\int u^{-2}du$$

$$\frac{1}{4}\left[\frac{u^{-2+1}}{-1}\right] + C$$

$$= \frac{-1}{4}\left[\frac{1}{u}\right] + C$$

$$= \frac{-1}{4(4x^2-7} + C$$

Example IV

$$\int \frac{x^2-1}{\sqrt{x^3-3x}}dx$$

Solution

$$\int \frac{x^2-1}{\sqrt{x^3-3x}}dx$$

Let $\sqrt{x^3-3x} = u$

$x^3-3x = u^2$

$(3x^2-3)dx = 2udu$

$3(x^2-1)dx = 2udu$

$$dx = \frac{2udu}{3(x^2-1)}$$

$$\int \frac{x^2-1}{\sqrt{x^3-3x}}dx = \int \frac{x^2-1}{u}.\frac{2udu}{3(x^2-1)}$$

$$= \frac{2}{3}\int du$$

$$= \frac{2}{3}u + C$$

$$= \frac{2}{3}\left(\sqrt{x^3-3x}\right) + C$$

Example VI

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$$

$$let\ \sqrt{x} = u$$

$$x = u^2$$

$$dx = 2udu$$

$$\int \frac{\cos u}{u}.2udu$$

$$2\int \cos u\, du$$

$$2\sin u + C$$

$$2\sin\sqrt{x} + C$$

Example VII

$\int \cos x \sin x\, dx.$

Let $u = \sin x$

$du = \cos x\, dx$

$$dx = \frac{du}{\cos x}$$

$$\Rightarrow \int \cos x \sin x\, dx. = \int \cos x\, u. \frac{du}{\cos x}$$

$$= \int u\, du$$

$$= \frac{u^2}{2} + C$$

$$= \frac{1}{2}\sin^2 x + C$$

Example VIII

$$\int sec^2 x tan^2 x\, dx$$

$$let\ u = \tan x$$

$$du = sec^2 x\, dx$$

$$dx = \frac{du}{sec^2 x}$$

$$\int sec^2 x\, u^2 . \frac{du}{sec^2 x}$$

$$= \frac{u^3}{3} + C$$

$$= \frac{1}{3}tan^3 x + C$$

Example IX

$$\int \cos x\sqrt{\sin x}\, dx$$

$$let\ u = \sqrt{\sin x}$$

$$u^2 = \sin x$$

$$2udu = \cos x\, dx$$

$$dx = \frac{2udu}{\cos x}$$

$$\int \cos x\sqrt{\sin x}\, dx = \int \cos x . \frac{2udu}{\cos x}$$

$$= 2\int u^2\, du$$

$$= 2\left[\frac{u^3}{3}\right] + C$$

$$= \frac{2}{3}u^3 + C$$

$$= \frac{2}{3}\left(\sqrt{\sin x}\right)^3 + C$$

Example X

$$\int sec^5 x \tan x\, dx$$

Solution

$$\int sec^5 x \tan x\, dx = \int sec^4 x\, (\sec x \tan x)\, dx$$

Let $u = \sec x$

$du = \sec x \tan x\, dx$

$$dx = \frac{du}{\sec x \tan x}$$

$$\int u^4 \sec x \tan x \frac{du}{\sec x \tan x}$$

$$= \frac{u^5}{5} + C$$

$$= \frac{1}{5}sec^5 x + C$$

Example XI

$$\int cosec\ x\ cot^3 x\, dx$$

Solution

$$\int cosec\ x\ cot^3 x\, dx$$

$$\int cosec\ x \cot x . cot^2 x\, dx$$

$$\int cosec\ x \cot x\, (cosec^2 x - 1) dx$$

Let $u = \text{cosec}\, x$

$du = -cosec\, x \cot x\, dx$

$$dx = -\frac{du}{cosec\, x \cot x}$$

$$\int cosec\, x\, cot^3 x\, dx$$

$$= \int cosec\, x \cot x\, (u^2 - 1)\frac{-du}{cosec\, x \cot x}$$

$$\int (-u^2 + 1)du = \frac{-u^3}{3} + u + C$$

$$\int cosec\, x\, cot^3 x\, dx = \frac{-1}{3} cosec^3 x + cosec\, x + C$$

Students Exercise

1) $\int x\sqrt{4 - 3x^2}\, dx$

2) $\int (1 - 2x)\,(x^2 - x - 3)^3 dx$

3) $\int \frac{(1 + \sqrt{x})^5}{\sqrt{x}} dx$

4) $\int x(x^2 + 4)^4 dx$

5) $\int_0^1 16x(x^2 + 5)^3$

6) $\int_{-1}^1 (3 + 2x)^5\, dx$

7) $\int (x - 1)(x^2 - 2x + 4)^7\, dx$

8) $\int (1 + 2x)(4 + x + x^2)^5 dx$

9) $\int 2(3 + 4x)^4 dx$

10) $\int x^3(x^4 + 6)^3 dx$

11) $\int x^2\sqrt{4x^3 + 1}\, dx$

12) $\int x(2x^2 + 3)^5 dx$

13) $\int \sin(4x - 8) dx$

14) $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx$

15) $\int x\, cosec^2 x^2 dx$

16) $\int_0^{\frac{\pi}{2}} sec^2 x \sqrt{\tan x}\, dx$

17) $\int \frac{x}{\sqrt{2x^2 - 5}} dx$

18) $\int \frac{3x^2 - 1}{(x^3 - x + 4)^3} dx$

19) calculate the area enclosed by the curve $y = \frac{x}{x^2 - 1}$

20) Find the area enclosed between the curve $y = \sin x + 3\cos x$ and the x − axis from $x = 0\ and\ x = \frac{\pi}{2}$

21) $\int tan^6 x\, sec^2 x\ dx$

22) $\int_0^{\pi} \cos\left(3x + \frac{\pi}{2}\right) dx$

Integrating trigonometric functions

Considering integration as the reverse process of differentiation. The following examples illustrate the way in which trigonometric functions can be integrated.

$f(x)$	$\int \boldsymbol{f(x)dx}$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\sec^2 x$	$tan\ x$
$\sin ax$	$\frac{-1}{a}\cos ax$
$\text{cosec}^2 x$	$-cot\ x$
$\cot x\ \text{cosec}\ x$	$-cosec\ x$

Technique II of integration

(Integration of product of two cosines) - two sines or a sine and a cosine

The product of two sines, two cosines or a sine and a cosine can be integrated by first expressing the product as a sum or difference of trigonometric functions by use of factor formulae.

Example I

$$\int 2\cos 3x \cos x\ \ dx$$

Solution

$$\int 2\cos 3x \cos x\ \ dx$$

Consider $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

Now compare $2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$ with $2\cos 3x \cos x$

$$\frac{A+B}{2} = \boldsymbol{3x}$$

$A + B = 6x$ ……………………………(i)

$$\frac{A-B}{2} = x$$

A – $B = 2x$ ……………………………….(ii)

Adding Eqn (i) and Eqn (ii);

$$\Longrightarrow 2A = 8x$$
$$A = 4x$$

Eqn (i) – Eqn (ii);

$2B = 4x$

$B = 2x$

$\Longrightarrow \cos 4x + \cos 2x = 2\cos 3x \cos x$

$$\int 2\cos 3x \cos x\, dx = \int (\cos 4x + \cos 2x)dx$$

$$= \frac{1}{4}\sin 4x + \frac{1}{2}\sin 2x + C$$

Example II

$$\int \cos 3x \cos 5x\, dx$$

Solution

$$\int \cos 3x \cos 5x\, dx = \int \cos 5x \cos 3x\, dx$$

From the factor formulae,

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\Rightarrow \cos\frac{A+B}{2}\cos\frac{A-B}{2} = \frac{1}{2}(\cos A + \cos B)$$

Comparing $\cos\frac{A+B}{2}\cos\frac{A-B}{2}$ with $\cos 5x \cos 3x$

$$\Longrightarrow \frac{A+B}{2} = 5x$$

$A + B = 10x$ ……………………….(i)

$$\frac{A-B}{2} = 3x$$

$A - B = 6x$ ……………………….(ii)

Eqn (i) + Eqn (ii)

$$2A = 16x$$
$$A = 8x$$

Eqn (i) – Eqn (ii)

$$2B = 4x$$
$$B = 2x$$

$$\frac{1}{2}(\cos 8x + \cos 2x) = \cos 5x \cos 3x$$

$$\Longrightarrow \int (\cos 3x \cos 5x\, dx = \int \frac{1}{2}(\cos 8x + \cos 2x)dx$$

$$= \int \frac{1}{2}\cos 8x\, dx + \int \frac{1}{2}\cos 2x\, dx$$

$$= \frac{1}{2}\int \cos 8x\, dx + \frac{1}{2}\int \cos 2x\, dx$$

$$= \frac{1}{16}\sin 8x + \frac{1}{4}\sin 2x + C$$

Example III

$$\int_0^{\pi/3} 2\sin 3x \cos x\, dx$$

Solution

$$\int_0^{\pi/3} 2\sin 3x \cos x\, dx$$

From $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$$\Longrightarrow \frac{A+B}{2} = 3x$$

$A + B = 6x$ … … … … … … … … (1)

$$\frac{A-B}{2} = x$$

$A - B = 2x$ … … … … … … … (2)

Eqn (i) + Eqn (ii)

$$\Longrightarrow 2A = 8x$$
$$A = 4x$$

Eqn (i) – Eqn (ii)

$\Longrightarrow 2B = 4x$

$B = 2x$

$\sin 4x + \sin 2x = 2\sin 3x \cos x$

$$\Longrightarrow \int_0^{\pi/3} (2\sin 3x\, cosx)dx = \int_0^{\pi/3} (\sin 4x + \sin 2x)\, dx$$

$$= \frac{-1}{4}\cos 4x - \frac{1}{2}\cos 2x \Bigg|_0^{\pi/3}$$

$$= \left(-\frac{1}{4}\cos\frac{4\pi}{3} - \frac{1}{2}\cos\frac{2\pi}{3}\right) - \left(-\frac{1}{4}\cos o - \frac{1}{2}\cos 0\right)$$

$$= -\frac{1}{4}\left(-\frac{1}{2}\right) - \frac{1}{2}\left(-\frac{1}{2}\right) - \left(-\frac{1}{4} - \frac{1}{2}\right)$$

$$= \frac{1}{8} + \frac{1}{4} + \frac{1}{4} + \frac{1}{2}$$

$$= \frac{1+2+2+4}{8}$$

$$= \frac{9}{8}$$

Example IV (UNEB 2001)

$$\int \sin x \sin 3x\, dx$$

Solution

$$\int \sin x \sin 3x \, dx = \int \sin 3x \sin x \, dx$$

Consider $\cos A - \cos A = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

$$\int \sin x \sin 3x \, dx = \int sin 3x \sin x \, dx$$

$$\sin\frac{A+B}{2}\sin\frac{A-B}{2} = -\frac{1}{2}(\cos A - \cos B)$$

Comparing $(\sin 3x \sin x)$ with $\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

$$\frac{A+B}{2} = 3x$$

$A + B = 6x$ ………………….. (i)

$$\frac{A-B}{2} = 2x$$

$A - B = 2x$ ………………….. (i)

Equation (1) +(2)

$2A = 8x$

$A = 4x$

Eqn (i) – Eqn (ii);

$2B = 4x$

$B = 2x$

$$\Longrightarrow \sin 3x \sin x = -\frac{1}{2}(\cos 4x - \cos 2x)$$

$$\int \sin x \sin 3x \, dx = \int -\frac{1}{2}(\cos 4x - \cos 2x)$$

$$= -\frac{1}{2}\int \cos 4x \;\; dx - \frac{1}{2}\int \cos 2x \, dx$$

$$= \frac{-1}{2}\left[\frac{1}{4}\sin 4x\right] - \frac{1}{2}\left[\frac{1}{2}\sin 2x\right] + C$$

$$= \frac{-1}{8}\sin 4x - \frac{1}{4}\sin 2x + C$$

Integration of odd and even powers of trigonometric functions

Odd powers of trigonometric functions

Under this we use the following trigonometric identities

1) $cos^2 x + sin^2 x = 1$
2) $1 + tan^2 x = sec^2 x$
3) $1 + cot^2 x = cosec^2 x$

Example I

$$\int cos^3 x \, dx$$

$$\int \cos x \, cos^2 x \, dx$$

$$\int \cos x (1 - sin^2 x) \, dx$$

Let $u = \sin x$

$du = \cos x \, dx$

$$dx = \frac{du}{\cos x}$$

$$\int \cos x (1 - u^2).\frac{du}{\cos x}$$

$$\int (1 - u^2) \, du$$

$$u - \frac{u^3}{3} + C$$

$$\sin x - \frac{1}{3} sin^3 x + C$$

Example II

$$\int sin^3 2x \, dx$$

Solution

$$\int sin^3 2x \, dx = \int (\sin 2x)(sin^2 2x) dx$$

$$= \int \sin 2x \, (1 - cos^2 2x) dx$$

$let \; u = \cos 2x$

$du = -2\sin 2x \, dx$

$$dx = -\frac{du}{2\sin 2x}$$

$$\int \sin 2x \, (1 - u^2).\frac{-du}{2\sin 2x}$$

$$-\frac{1}{2}\int (1 - u^2) du$$

$$-\frac{1}{2}\left(u - \frac{u^3}{3}\right) + C$$

$$= -\frac{1}{2}u + \frac{1}{6}u^3 + C$$

$$= -\frac{1}{2}\cos 2x + \frac{1}{6} cos^{\,3} 2x + C$$

$$\Longrightarrow \int sin^3 2x \, dx = -\frac{1}{2}\cos 2x + \frac{1}{6} cos^3 2x + C$$

Example III

$$\int cos^5 3x \, dx$$

Solution

$$\int cos^5 3x \, dx$$

$$\int \cos 3x \, cos^4 3x \, dx$$

$$\int \cos 3x(cos^2 3x)^2\, dx$$
$$\int \cos 3x(1-sin^2 3x)^2\, dx$$
$let\ u = \sin 3x$
$du = 3\cos 3x\, dx$
$$dx = \frac{du}{3\cos 3x}$$
$$\int \cos 3x\,(1-u^2)^2 . \frac{du}{3\cos 3x}$$
$$\frac{1}{3}\int (1-u^2)^2\, du$$
$$\frac{1}{3}\int (1-2u^2+u^4)du$$
$$\frac{1}{3}\left(u-\frac{2u^3}{3}+\frac{u^5}{5}\right)+C$$
$$\frac{1}{3}u-\frac{2u^3}{9}+\frac{u^5}{15}+C$$
$$=\frac{1}{3}\sin 3x-\frac{2}{9}sin^3 3x+\frac{1}{15}sin^5 3x+C$$

Example IV

$$\int cos^2\frac{x}{2}sin^3\frac{x}{2}dx$$
Solution
$\int cos^2\frac{x}{2}sin^3\frac{x}{2}dx = \int cos^2\frac{x}{2}\sin\frac{x}{2}sin^2\frac{x}{2}dx$
$$\int cos^2\frac{x}{2}\sin\frac{x}{2}\left(1-cos^2\frac{x}{2}\right)dx$$
Let $u = \cos\frac{x}{2}$
$$du = -\frac{1}{2}\sin\frac{x}{2}dx$$
$$dx = \frac{-2du}{\sin\frac{x}{2}}$$
$$\int cos^2\frac{x}{2}sin^3\frac{x}{2}dx = \int u^2\sin\frac{x}{2}(1-u^2)\frac{-2du}{\sin\frac{x}{2}}$$
$$-2\int (u^2-u^4)\,du$$
$$=-\frac{2u^3}{3}+\frac{2u^5}{5}+C$$
$$=\frac{2\,cos^3\frac{x}{2}}{3}+\frac{2}{5}\left(cos^5\frac{x}{2}\right)+C$$
$$\Rightarrow \int cos^2\frac{x}{2}sin^3\frac{x}{2}dx = \frac{2}{3}(\cos^3\tfrac{x}{2})+\frac{2}{5}\left(\cos^5\tfrac{x}{2}\right)+C$$

Example V

$$\int \sec x\, tan^3 x\, dx$$
Solution
$$\int tan^3 x\sec x\, dx$$
$$\int tan^2 x\,(\tan x\sec x)dx$$
$$\int (sec^2-1)\sec x\tan x\, dx$$
$let\ u = \sec x$
$du = \sec x\tan x\, dx$
$$dx = \frac{du}{\sec x\tan x}$$
$$\int (u^2-1)\sec x\tan x . \frac{du}{\sec x\tan x}$$
$$\int (u^2-1)du$$
$$\frac{u^3}{3}-u+C$$
$$\Rightarrow \int \sec x\, tan^3 x\, dx = \frac{1}{3}sec^3 x-\sec x+C$$

TAN & SIN SUBSTITUTION

Show that :

(ii) $\int \frac{1}{a^2+b^2x^2}dx = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right)+C$

(ii) $\int \frac{1}{\sqrt{a^2-b^2x^2}}dx = \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right)+C$

(iii) $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

(iv) $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}(\frac{x}{a})+C$

Solution

$$\int \frac{1}{a^2+b^2x^2}dx = \int \frac{1}{a^2(1+\frac{b^2x^2}{a^2}}dx$$
$$let\ \sqrt{\frac{b^2x^2}{a^2}} = \tan\theta$$
$$\frac{bx}{a} = \tan\theta$$
$$\frac{b}{a}dx = sec^2\theta d\theta$$

$$dx = \frac{asec^2\theta}{b}d\theta$$

$$\int \frac{1}{a^2\left(1+\frac{b^2x^2}{a^2}\right)}dx = \int \frac{1}{a^2(1+tan^2\theta)}.\frac{asec^2\theta d\theta}{b}$$

$$= \int \frac{1}{ab}d\theta = \frac{1}{ab}\theta + C$$

$$\int \frac{1}{a^2+b^2x^2}dx = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right) + C$$

(ii) $\int \frac{1}{\sqrt{a^2-b^2x^2}}dx = \int \frac{1}{\sqrt{a^2\left(1-\frac{b^2x^2}{a^2}\right)}}dx$

Let $\sqrt{\frac{b^2x^2}{a^2}} = \sin\theta$

$$\frac{bx}{a} = \sin\theta$$

$$\frac{b}{a}dx = \cos\theta d\theta$$

$$dx = \frac{a\cos\theta}{b}d\theta$$

$$\int \frac{1}{\sqrt{a^2(1-\frac{b^2x^2}{a^2})}}dx = \int \frac{1}{\sqrt{a^2(1-\sin^2\theta)}}\cdot\frac{a\cos\theta}{b}d\theta$$

$$= \int \frac{1}{b}\,d\theta$$

$$= \frac{1}{b}\theta + C$$

$$\int \frac{1}{\sqrt{a^2-b^2x^2}}dx = \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right) + C$$

(iii) $\int \frac{1}{a^2+x^2}dx$

Solution

$$\int \frac{1}{a^2+x^2}dx$$

$$\int \frac{1}{a^2(1+\frac{x^2}{a^2})}dx$$

Let $\frac{x}{a} = \tan\theta$

$$\frac{1}{a}dx = \sec^2\theta d\theta$$

$$dx = a\sec^2\theta\, d\theta$$

$$\int \frac{1}{a^2+x^2}dx = \int \frac{1}{a^2(1+\tan^2\theta)}a\sec^2\theta\, d\theta$$

$$= \int \frac{1}{a}d\theta$$

$$= \frac{1}{a}\theta + c$$

$$= \frac{1}{a}\tan^{-1}(\tfrac{x}{a}) + c$$

(iv) $\int \frac{1}{\sqrt{a^2-x^2}}dx$

$$\int \frac{1}{\sqrt{a^2(1-\frac{x^2}{a^2})}}dx$$

Let $\frac{x}{a} = \sin\theta$

$$\frac{1}{a}dx = \cos\theta\, d\theta$$

$$dx = a\cos\theta\, d\theta$$

$$\int \frac{1}{\sqrt{a^2(1-\frac{x^2}{a^2})}}dx = \int \frac{1}{\sqrt{a^2(1-\sin^2\theta)}}\cdot a\cos\theta\, d\theta$$

$$= \int \theta d\theta = \theta + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}(\tfrac{x}{a}) + c$$

Tan Substitution

Example

Find the following integrals

a) $\int \frac{1}{4+x^2}dx$, **b**) $\int \frac{1}{1+16x^2}dx$

c) $\int_{\frac{\sqrt{3}}{2}}^{\frac{3}{2}} \frac{1}{3+4x^2}dx$

Solution

$a)$ $\int \frac{1}{4+x^2}dx = \int \frac{1}{4\left(1+\frac{x^2}{4}\right)}dx$

$$let \sqrt{\frac{x^2}{4}} = tan\theta$$

$$\frac{x}{2} = \tan\theta$$

$$\frac{1}{2}dx = sec^2\theta d\theta$$

$$dx = 2sec^2\theta d\theta$$

$$\int \frac{1}{4\left(1+\frac{x^2}{4}\right)}dx = \int \frac{1}{4(1+tan^2\theta)}.2\sec^2\theta d\theta$$
$$= \int \frac{1}{2}d\theta = \frac{1}{2}\theta + C$$
$$\int \frac{1}{4+x^2}dx = \frac{1}{2}tan^{-1}\left(\frac{x}{2}\right) + C$$

b) $\int \frac{1}{1+16x^2}dx$

Let $\sqrt{16x^2} = \tan\theta$

$4x = \tan\theta$

$4dx = sec^2\theta d\theta$

$dx = \frac{sec^2\theta}{4}d\theta$

$$\int \frac{1}{1+tan^2\theta}.\frac{sec^2\theta}{4}d\theta$$

$\int \frac{1}{4}d\theta = \frac{1}{4}\theta + C$

$$\int \frac{1}{1+16x^2}dx = \frac{1}{4}tan^{-1}(4x) + C$$

c) $\int_{\frac{\sqrt{3}}{2}}^{\frac{3}{2}} \frac{1}{3+4x^2}dx$

Solution

Consider $\int \frac{1}{3+4x^2}dx$

$$= \int \frac{1}{3\left(1+\frac{4x^2}{3}\right)}dx$$

$let \sqrt{\frac{4x^2}{3}} = \tan\theta$

$= \frac{2x}{\sqrt{3}} = \tan\theta$

$\frac{2}{\sqrt{3}}dx = \sec^2\theta\, dx$

$dx = \frac{\sqrt{3}\sec^2\theta d\theta}{2}$

$$= \int \frac{1}{3\left(1+\frac{4x^2}{3}\right)}dx = \int \frac{1}{3(1+\tan^2\theta)}.\frac{\sqrt{3}\sec^2\theta\, d\theta}{2}$$
$$= \int \frac{\sqrt{3}}{6}d\theta = \frac{\sqrt{3}}{6}\theta + c$$
$$= \int \frac{1}{3+4x^2}dx = \frac{\sqrt{3}}{6}\tan^{-1}\frac{2x}{\sqrt{3}} + c$$
$$\int_{\frac{\sqrt{3}}{2}}^{\frac{3}{2}} \frac{1}{3+4x^2}dx = \frac{\sqrt{3}}{6}\tan^{-1}\left(\frac{2x}{\sqrt{3}}\right)\Bigg|_{\frac{\sqrt{3}}{2}}^{\frac{3}{2}}$$
$$= \frac{\sqrt{3}}{6}\tan^{-1}\frac{2\left(\frac{3}{2}\right)}{\sqrt{3}} - \frac{\sqrt{3}}{6}\tan^{-1}\frac{2\left(\frac{\sqrt{3}}{2}\right)}{\sqrt{3}}$$
$$= \frac{\sqrt{3}}{6}\left(\frac{\pi}{3}\right) - \frac{\sqrt{3}}{6}\left(\frac{\pi}{4}\right)$$
$$= \frac{\sqrt{3}}{72}\pi$$

Example II

Find the integral of the following.

$a) \int \frac{x}{1+x^4}dx$

$b) \int \frac{2x^3}{16+x^8}dx$

$c) \int \frac{1}{(x^2+9)^2}dx$

$d) \int_0^1 x\sqrt{4+x^2}\,dx$

Solution

$$\int \frac{x}{1+x^4}dx$$

$let \sqrt{x^4} = \tan\theta, \Rightarrow x^2 = \tan\theta.$

$2xdx = \sec^2\theta d\theta$

$dx = \frac{\sec^2\theta}{2x}d\theta$

$$= \int \frac{x}{1+x^4}dx = \int \frac{x}{1+\tan^2\theta}.\frac{\sec^2\theta}{2x}d\theta$$
$$= \int \frac{1}{2}d\theta = \frac{1}{2}\theta + c$$
$$\Rightarrow \int \frac{x}{x+x^4}dx = \frac{1}{2}\tan^{-1}(x^2) + c.$$

b) $\int \frac{2x^3}{16+x^8}dx$

$$= \int \frac{2x^3}{16\left(1+\frac{x^8}{16}\right)}dx$$

$let \sqrt{\frac{x^8}{16}} = \tan\theta$

$\frac{x^4}{4} = \tan\theta$

$\Longrightarrow \frac{4x^3}{4}dx = \sec^2\theta d\theta.$

$$dx = \frac{\sec^2\theta}{x^3} d\theta$$

$$\Rightarrow \int \frac{2x^3}{16\left(1+\frac{x^8}{16}\right)} dx = \int \frac{2x^3}{16(1+\tan^2\theta)} \cdot \frac{\sec^2\theta}{x^3} d\theta$$

$$\Longrightarrow \int \tfrac{1}{8} d\theta = \tfrac{1}{8}\theta + c$$

$$\Rightarrow \int \frac{2x^3}{16+x^8} = \frac{1}{8}\tan^{-1}\left(\frac{x^4}{4}\right) + c.$$

c) $\int \frac{1}{(x^2+9)^2} dx$

Solution

$$\int \frac{1}{(x^2+9)^2} = \int \frac{1}{(9+x^2)^2} dx$$

$$= \int \frac{1}{\left(9\left(1+\frac{x^2}{9}\right)\right)^2} dx$$

$$= \int \frac{1}{9^2\left(1+\frac{x^2}{9}\right)^2} dx$$

$$= \int \frac{1}{81\left(1+\frac{x^2}{9}\right)^2} dx$$

$$\sqrt{\frac{x^2}{9}} = \tan\theta$$

Let $\frac{x}{3} = \tan\theta$.

$$\frac{1}{3} dx = \sec^2\theta d\theta.$$

$$dx = 3\sec^2\theta d\theta.$$

$$\int \frac{1}{\left(9\left(1+\frac{x^2}{9}\right)\right)^2} dx = \int \frac{1}{81(1+\tan^2\theta)^2} . 3\sec^2\theta d\theta$$

$$= \int \frac{1}{81(\sec^2\theta)^2} . 3\sec^2\theta d\theta$$

$$= \frac{1}{27}\int \frac{1}{\sec^2\theta} d\theta$$

$$= \frac{1}{27}\int \cos^2\theta d\theta.$$

But $\cos 2\theta = 2\cos^2\theta - 1)$

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$= \frac{1}{27}\int \frac{1}{2}(1+\cos 2\theta) d\theta.$$

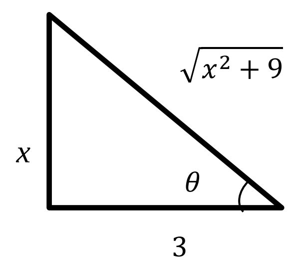
$$= \frac{1}{54}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{54}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$

$$= \frac{1}{54}\theta + \frac{1}{108}. 2\sin\theta\cos\theta] + C$$

$$= \frac{1}{54}(\theta + \sin\theta\cos\theta) + C$$

$$\tan\theta = \frac{x}{3}$$



$$\Rightarrow \cos\theta = \frac{3}{\sqrt{x^2+9}}, \quad \sin\theta = \frac{x}{\sqrt{x^2+9}}$$

$$\int \frac{1}{(x^2+9)^2} dx = \frac{1}{54}\left(\tan^{-1}\left(\frac{x}{3}\right) + \frac{x}{\sqrt{x^2+9}} . \frac{3}{\sqrt{x^2+9}}\right)$$

$$= \frac{1}{54}\left(\tan^{-1}\left(\frac{x}{3}\right) + \frac{3x}{x^2+9}\right) + C$$

Example III

Find the integral of the following:

a) $\int \frac{1}{x^2-2x+5} dx$

b) $\int \frac{1}{2x^2+4x+11} dx$

c) $\int \frac{1}{4x^2-8x+7} dx$

Solution

$$\int \frac{1}{x^2-2x+5} dx$$

Note: For the tan substitution to be used the denominator should not be factorized.

$x^2 - 2x + 5$

By completing squares;

$$= x^2 - 2x + \left(\frac{1}{2}(-2)\right)^2 - \left(\frac{1}{2}(-2)\right)^2 + 5$$
$$= x^2 - 2x + 1 - 1 + 5$$
$$= x^2 - 2x + 1 + 4.$$
$$= 4 + x^2 - 2x + 1$$
$$= 4 + (x-1)^2$$

$$\Rightarrow \int \frac{1}{x^2 - 2x + 5}dx = \int \frac{1}{4 + (x-1)^2}dx$$
$$= \int \frac{1}{4\left(1 + \frac{(x-1)^2}{4}\right)}dx$$

Let $\sqrt{\frac{(x-1)^2}{4}} = \tan\theta$

$$\frac{x-1}{2} = \tan\theta.$$
$$\frac{1}{2}dx = \sec^2\theta d\theta$$
$$dx = 2\sec^2\theta d\theta$$

$$\int \frac{1}{4\left(1 + \frac{(x-1)^2}{4}\right)}dx = \int \frac{1}{4(1 + \tan^2\theta)}.2\sec^2\theta d\theta$$

$$\frac{1}{2}\int d\theta$$

$$\frac{1}{2}\theta + C$$

$\frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right)$+C

$$\int \frac{1}{4\left(1 + \frac{(x-1)^2}{4}\right)}dx = \frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C$$

$$\int \frac{1}{x^2 - 2x + 5}dx = \frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C$$

b) $\int \frac{1}{2x^2 + 4x + 11}dx$

Consider $2x^2 + 4x + 11$

$2(x^2 + 2x) + 11$

By completing squares;

$2(x^2 + 2x + 1) - 2 + 11$

$2(x+1)^2 + 9$

$\Rightarrow 2(x+1)^2 + 9$

$9 + 2(x+1)^2.$

$$\int \frac{1}{9 + 2(x+1)^2}dx.$$

$$\int \frac{1}{2x^2 + 4x + 11}dx = \int \frac{1}{9\left(1 + \frac{2(x+1)^2}{9}\right)}dx.$$

Let $\sqrt{\frac{2(x+1)^2}{9}} = \tan\theta$

$$\frac{\sqrt{2}}{3}(x+1) = \tan\theta$$
$$\frac{\sqrt{2}}{3}dx = \sec^2\theta d\theta$$
$$dx = \frac{3\sec^2\theta d\,\theta}{\sqrt{2}}$$

$$\int \frac{1}{2x^2 + 4x + 11}dx$$
$$= \int \frac{1}{9(1 + \tan^2\theta)}.\frac{3\sec^2 d\theta}{\sqrt{2}}$$
$$\int \frac{1}{3\sqrt{2}}d\theta.$$
$$= \frac{1}{3\sqrt{2}}\theta + C.$$
$$= \frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}(x+1)}{3}\right) + C$$
$$= \frac{\sqrt{2}}{6}\tan^{-1}\frac{\sqrt{2}(x+1)}{3} + C$$

c) $\int \frac{1}{4x^2 - 8x + 7}dx.$

$4x^2 - 8x + 7.$

$= 4(x^2 - 2x) + 7$

$4(x^2 - 2x + 1) - 4 + 7.$

$= 4(x-1)^2 + 3$

$$\int \frac{1}{4x^2 - 8x + 7}dx = \int \frac{1}{3 + 4(x-1)^2}dx$$
$$= \int \frac{1}{3\left(1 + \frac{4(x-1)^2}{3}\right)}dx$$

Let $\frac{2}{\sqrt{3}}(x-1) = \tan\theta$

$$\frac{2}{\sqrt{3}}dx = \sec^2\theta d\theta$$

$$dx = \frac{\sqrt{3}\sec^2\theta}{2}d\theta$$

$$\int \frac{1}{3\left(1+\frac{4(x-1)^2}{3}\right)}dx = \int \frac{1}{3(1+\tan^2\theta)}\frac{\sqrt{3}\sec^2\theta d\theta}{2}$$

$$= \int \frac{\sqrt{3}}{6}d\theta$$

$$= \frac{\sqrt{3}}{6}\theta + C.$$

$$= \frac{\sqrt{3}}{6}\tan^{-1}\left(\frac{2(x-1)}{\sqrt{3}}\right) + C$$

$$\Rightarrow \int \frac{1}{4x^2-8x+7}dx = \frac{\sqrt{3}}{6}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

Sine Substitution

Find the following integrals

a) $\int \frac{1}{\sqrt{9-4x^2}}dx$

b) $\int_0^{\sqrt{3}} \frac{1}{\sqrt{3-x^2}}dx$

c) $\int \frac{1}{\sqrt{4-(x-1)^2}}dx$

d) $\int \frac{x^2}{\sqrt{1-x^2}}dx$

f) $\int \frac{4}{\sqrt{16-5x^2}}dx$

g) $\int \frac{1}{(1-9x^2)\sqrt{(1-9x^2)}}dx$

Solution

$$\int \frac{1}{\sqrt{9-4x^2}}dx = \int \frac{1}{\sqrt{9\left(1-\frac{4x^2}{9}\right)}}dx$$

Let $\sqrt{\frac{4x^2}{9}} = \sin\theta$

$$\frac{2x}{3} = \sin\theta.$$

$$\frac{2}{3}dx = \cos\theta d\theta$$

$$dx = \frac{3\cos\theta d\theta}{2}$$

$$\int \frac{1}{\sqrt{9\left(1-\frac{4x^2}{9}\right)}}dx = \int \frac{1}{\sqrt{9(1-\sin^2\theta)}}.\frac{3\cos\theta}{2}d\theta.$$

$$\int \frac{1}{2}d\theta = \frac{1}{2}\theta + C$$

$$\Rightarrow \int \frac{1}{\sqrt{9-4x^2}}dx = \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + C$$

b) $\int_0^{\sqrt{3}} \frac{1}{\sqrt{3-x^2}}dx$

Solution

Consider $\int \frac{1}{\sqrt{3-x^2}}dx = \int \frac{1}{\sqrt{3\left(1-\frac{x^2}{3}\right)}}dx$

$$\frac{x}{\sqrt{3}} = \sin\theta$$

$$\frac{1}{\sqrt{3}}dx = \cos\theta d\theta$$

$$dx = \sqrt{3}\cos\theta d\theta$$

$$\int \frac{1}{\sqrt{3\left(1-\frac{x^2}{3}\right)}}dx = \int \frac{1}{\sqrt{3(1-\sin^2\theta)}}\sqrt{3}\cos\theta d\theta$$

$$= \int d\theta = \theta + C$$

$$\Longrightarrow \int \frac{1}{\sqrt{3-x^2}}dx = \sin^{-1}\frac{x}{\sqrt{3}} + C$$

$$\Rightarrow \int_0^{\sqrt{3}} \frac{1}{\sqrt{3-x^2}}dx = \sin^{-1}\frac{x}{\sqrt{3}}\Big|_0^{\sqrt{3}}$$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) - \sin^{-1}\frac{0}{\sqrt{3}}$$

$$= \frac{\pi}{2} - 0$$

$$\Rightarrow \int_0^{\sqrt{3}} \frac{1}{\sqrt{3-x^2}}dx = \frac{\pi}{2}$$

c) $\int \frac{1}{\sqrt{4-(x-1)^2}}dx.$

$$\int \frac{1}{\sqrt{4\left(1-\frac{(x-1)^2}{4}\right)}}dx.$$

let $\frac{x-2}{2} = \sin\theta.$

$$\frac{1}{2}dx = \cos\theta d\theta$$

$$dx = 2\cos\theta d\theta.$$

$$\int \frac{1}{\sqrt{4\left(1-\frac{(x-1)^2}{4}\right)}}dx. = \int \frac{1}{\sqrt{4(1-\sin^2\theta)}}.2\cos\theta d\theta$$

$$= \int d\theta$$

$$= \theta + C$$

$$\sin^{-1}\left(\frac{x-1}{2}\right)+C.$$

$$\Rightarrow \int \frac{1}{\sqrt{4-(x-1)^2}}\,dx = \sin^{-1}\left(\frac{x-1}{2}\right)+C$$

d) $\int \frac{x^2}{\sqrt{1-x^2}}\,dx.$

Let $\sqrt{x^2}=\sin\theta$

$x=\sin\theta$

$dx=\cos\theta d\theta$

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta d\theta.$$

$$\int \sin^2\theta d\theta = \int \frac{1}{2}(1-\cos 2\theta)d\theta$$

$$\frac{1}{2}\int (1-\cos 2\theta)d\theta$$

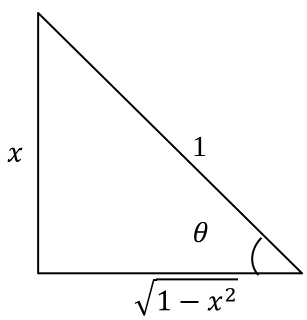
$$\frac{1}{2}\left[\theta-\frac{1}{2}\sin 2\theta\right]+C$$

$$=\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta + C$$

$$=\frac{1}{2}\theta-\frac{1}{4}.2\sin\theta\cos\theta + C.$$

$$=\frac{1}{2}\theta-\frac{1}{4}\sin\theta\cos\theta + C$$

$\sin\theta = x$



$\cos\theta=\sqrt{1-x^2}$

$$\Rightarrow \frac{1}{2}\theta-\frac{1}{2}\sin\theta\cos\theta + C$$

$$\Rightarrow \frac{1}{2}\sin^{-1}x-\frac{1}{2}x\sqrt{1-x^2}+C$$

$$\Rightarrow \int \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\left(\sin^{-1}x - x\sqrt{1-x^2}\right)+C$$

f) $\int \frac{4}{\sqrt{16-5x^2}}\,dx.$

$$\int \frac{4}{\sqrt{16\left(1-\frac{5x^2}{16}\right)}}\,dx$$

$$\frac{(\sqrt{5})dx}{4}=\sin\theta$$

$$\frac{\sqrt{5}x}{4}=\cos\theta d\theta$$

$$dx=\frac{4\cos\theta d\theta}{\sqrt{5}}$$

$$\int \frac{4}{\sqrt{16(1-\sin^2\theta)}}\,.\frac{4\cos\theta}{\sqrt{5}}\,d\theta.$$

$$\frac{4}{\sqrt{5}}\int d\theta.$$

$$\frac{4}{\sqrt{5}}\theta + C.$$

$$\frac{4}{\sqrt{5}}\sin^{-1}\left(\frac{\sqrt{5}x}{4}\right)+C.$$

$$\Rightarrow \int \frac{4}{\sqrt{16-5x^2}}\,dx = \frac{4}{\sqrt{5}}\sin^{-1}\left(\frac{\sqrt{5}x}{4}\right)+C$$

g) $\int \frac{1}{(1-9x^2)\sqrt{1-9x^2}}$

Solution

$$\int \frac{1}{(1-9x^2)\sqrt{1-9x^2}} = \int \frac{1}{(1-9x^2)^{\frac{3}{2}}}\,dx.$$

$$= \int \frac{1}{\left(\sqrt{1-9x^2}\right)^3}\,dx.$$

Let $\sqrt{9x^2}=\sin\theta.$

$3x=\sin\theta$

$3dx=\cos\theta d\theta.$

$$dx=\frac{\cos\theta}{3}d\theta$$

$$dx=\frac{\cos\theta d\theta}{3}.$$

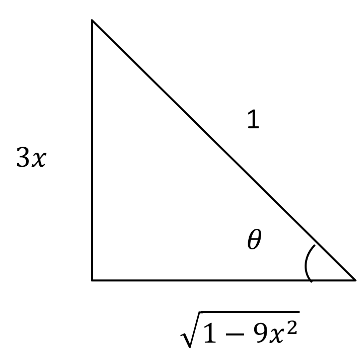
$$\int \frac{1}{\left(\sqrt{1-9x^2}\right)^3}\,dx = \int \frac{1}{\left(\sqrt{1-\sin^2\theta}\right)^3}\,\frac{\cos\theta d\theta}{3}$$

$$=\frac{1}{3}\int \frac{1}{\cos^2\theta}\,d\theta.$$

$$=\frac{1}{3}\int \sec^2\theta d\theta$$

$$=\frac{1}{3}\tan\theta + C$$

$\sin\theta = 3x$



$$\Rightarrow \tan\theta = \frac{3x}{\sqrt{1-9x^2}}$$

$$\Rightarrow \int \frac{1}{\left(\sqrt{1-9x^2}\right)^3}dx = \frac{1}{3}\left(\frac{3x}{\sqrt{1-9x^2}}\right) + C$$

$$= \frac{x}{\sqrt{1-9x^2}} + C$$

$$\Rightarrow \int \frac{1}{(1-9x^2)\left(\sqrt{1-9x^2}\right)}dx = \left(\frac{x}{\sqrt{1-9x^2}}\right) + C$$

Example II

Find the following integrals

a) $\int \frac{x}{\sqrt{4-x^4}}dx$

b) $\int \frac{4x^2}{\sqrt{1-x^6}}dx.$

c) $\int \frac{2+x}{\sqrt{9-x^2}}dx$

Solution

a) $\int \frac{x}{\sqrt{4-x^2}}dx$

$$\int \frac{x}{\sqrt{4-x^4}}dx = \int \frac{x}{\sqrt{4\left(1-\frac{x^4}{4}\right)}}$$

Let $\sqrt{\frac{x^4}{4}} = \sin\theta$

$$\frac{x^2}{2} = \sin\theta$$

$$\frac{2x}{2}dx = \cos\theta d\theta.$$

$$dx = \frac{\cos\theta}{x}d\theta$$

$$\int \frac{x}{\sqrt{4\left(1-\frac{x^4}{4}\right)}}dx = \int \frac{x}{\sqrt{4(1-\sin^2\theta)}}.\frac{\cos\theta d\theta}{x}$$

$$= \int \frac{1}{2}\,d\theta$$

$$= \frac{1}{2}\theta + C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{x^2}{2}\right) + C.$$

$$\int \frac{x}{\sqrt{4-x^4}}dx = \frac{1}{2}\sin^{-1}\left(\frac{x^2}{2}\right) + C$$

b) $\int \frac{4x^2}{\sqrt{1-x^6}}dx$

Let $\sqrt{x^6} = \sin\theta,$

$$\Rightarrow x^3 = \sin\theta$$

$$3x^2dx = \cos\theta d\theta$$

$$dx = \frac{\cos\theta d\theta}{3x^2}$$

$$\int \frac{4x^2}{\sqrt{1-x^6}}dx$$

$$= \int \frac{4x^2}{\sqrt{1-\sin^2\theta}}.\frac{\cos\theta}{3x^2}d\theta$$

$$= \frac{4}{3}\int d\theta.$$

$$= \frac{4}{3}\theta + C$$

$$= \frac{4}{3}\sin^{-1}(x^3) + C.$$

$$\Rightarrow \int \frac{4x^3}{\sqrt{1-x^6}}dx = \frac{4}{3}\sin^{-1}(x^3) + C$$

c) $\int \frac{2+x}{\sqrt{9-x^2}}$

$$= \int \frac{2+x}{\sqrt{9\left(1-\frac{x^2}{9}\right)}}dx$$

Let $\frac{x}{3} = \sin\theta$

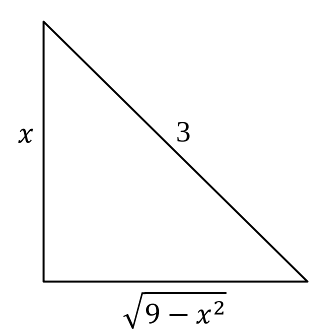
$$\frac{1}{3}dx = \cos\theta d\theta.$$

$$dx = 3\cos\theta d\theta$$

$$\int \frac{2+x}{\sqrt{9+\left(1-\frac{x^2}{9}\right)}}dx. = \int \frac{2+3\sin\theta}{\sqrt{9(1-\sin^2\theta)}}.3\cos\theta d\theta.$$

$$\int(2+3\sin\theta)d\theta. = 2\theta - 3\cos\theta + C$$

But $\sin\theta = \frac{x}{3}.$



$\cos\theta = \dfrac{\sqrt{9-x^2}}{3}$

$$= 2\sin^{-1}\left(\frac{x}{3}\right) - 3\left(\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$\Rightarrow \int \frac{2+x}{\sqrt{9-x^2}}\,dx = 2\sin^{-1}\left(\frac{x}{3}\right) - \sqrt{9-x^2} + C$$

Find the following integrals

$a) \displaystyle\int \frac{1}{\sqrt{3-2x-x^2}}\,dx$

$b) \displaystyle\int \frac{1}{\sqrt{12+4x+x^2}}\,dx$

$c) \displaystyle\int \frac{1}{\sqrt{-2x^2+12x-9}}\,dx$

$d) \displaystyle\int \frac{1}{\sqrt{1+8x-4x^2}}\,dx$

$f) \displaystyle\int \frac{x+3}{\sqrt{7-6x-x^2}}\,dx$

$g) \displaystyle\int \frac{3-7x}{4x-x^2}\,dx.$

Solution

$a) \displaystyle\int \frac{1}{\sqrt{3-2x-x^2}}\,dx.$

Consider $3-2x-x^2$

$3-(x^2+2x)$

By completing squares;

$3-(x^2+2x+1)--1$

$4-(x+1)^2$

$$\Rightarrow \int \frac{1}{\sqrt{3-2x-x^2}}\,dx = \int \frac{1}{\sqrt{4-(x+1)^2}}\,dx$$

$$= \int \frac{1}{\sqrt{4\left(1-\frac{(x+1)^2}{4}\right)}}\,dx.$$

$\dfrac{x+1}{2} = \sin\theta.$

$\dfrac{1}{2}dx = \cos\theta d\theta.$

$dx = 2\cos\theta d\theta.$

$$\int \frac{1}{\sqrt{4\left(1-\frac{(x+1)^2}{4}\right)}}\,dx = \int \frac{1}{\sqrt{4(1-\sin^2\theta)}}2\cos\theta d\theta$$

$$= \theta + C.$$

$$\int \frac{1}{\sqrt{3+2x-x^2}}\,dx = \sin^{-1}\left(\frac{x+1}{2}\right) + C$$

b) $\displaystyle\int \frac{1}{\sqrt{12+4x-x^2}}$

$12+4x-x^2 = 12-(x^2-4x)$

$= 12-(x^2-4x+4)--4$

$= 16-(x-2)^2$

$$\Rightarrow \int \frac{1}{\sqrt{12+4x-x^2}}\,dx = \int \frac{1}{\sqrt{16-(x-2)^2}}\,dx$$

$$= \int \frac{1}{\sqrt{16\left(1-\frac{(x-2)^2}{16}\right)}}\,dx$$

Let $\frac{x-2}{4} = \sin\theta.$

$\dfrac{1}{4}dx = \cos\theta d\theta$

$dx = 4\cos\theta d\theta$

$$\int \frac{1}{\sqrt{16\left(1-\frac{(x-2)^2}{16}\right)}}\,dx$$

$$= \int \frac{1}{\sqrt{16(1-\sin^2\theta)}}4\cos\theta d\theta$$

$$= \int d\theta$$

$$= \theta + C$$

$$= \sin^{-1}\left(\frac{x-2}{4}\right)$$

$$\Rightarrow \int \frac{1}{\sqrt{12+4x-x^2}}\,dx = \sin^{-1}\left(\frac{x-2}{4}\right) + C$$

c) $\displaystyle\int \frac{1}{\sqrt{-2x^2+12x-9}}\,dx.$

$-2x^2 + 12x - 9 = -2(x^2 - 6x) - 9$

By completing squares;

$-2(x^2 - 6x + 9) - -18 - 9$

$-2(x-3)^2 + 9$

$9 - 2(x-3)^2$

$$\Rightarrow \int \frac{1}{\sqrt{-2x^2+12x-9}}\,dx = \int \frac{1}{\sqrt{9-2(x-3)^2}}$$

$$= \int \frac{1}{\sqrt{9\left(1-\frac{2(x-3)^2}{9}\right)}}\,dx$$

$$= \int \frac{1}{\sqrt{9\left(1-\frac{2(x-3)^2}{9}\right)}}\,dx$$

Let $\frac{\sqrt{2}(x-3)}{3} = \sin\theta$

$$\frac{\sqrt{2}}{3}dx = \cos\theta d\theta$$
$$dx = \frac{3\cos\theta}{\sqrt{2}}d\theta$$
$$\int \frac{1}{\sqrt{9\left(1 - \frac{2(x-3)^2}{9}\right)}}dx$$
$$= \int \frac{1}{\sqrt{9(1-\sin^2\theta)}}\frac{3\cos\theta}{\sqrt{2}}d\theta$$
$$= \int \frac{1}{\sqrt{2}}d\theta$$
$$= \frac{\sqrt{2}}{2}\theta + C.$$
$$= \frac{\sqrt{2}}{2}\sin^{-1}\left(\frac{\sqrt{2}(x-3)}{3}\right) + C$$
$$\Rightarrow \int \frac{1}{\sqrt{-2x^2+12x-9}}dx$$
$$= \frac{\sqrt{2}}{2}\sin^{-1}\left(\frac{\sqrt{2}(x-3)}{3}\right) + C$$

d) $\int \frac{1}{\sqrt{1+8x-4x^2}}dx$

$$1 + 8x - 4x^2 = 1 - 4(x^2 - 2x).$$
$$= 1 - 4(x^2 - 2x).$$
$$= 1 - 4(x^2 - 2x + 1) - -4$$
$$= 5 - 4(x-1)^2$$
$$\int \frac{1}{\sqrt{1+8x-4x^2}}dx = \int \frac{1}{\sqrt{5-4(x-1)^2}}dx.$$
$$= \int \frac{1}{\sqrt{5\left(1 - \frac{4(x-1)^2}{5}\right)}}dx$$

Let $\frac{2(x-1)}{\sqrt{5}} = \sin\theta$

$$\frac{2}{\sqrt{5}}dx = \cos\theta d\theta.$$
$$dx = \frac{\sqrt{5}(\cos\theta)}{2}d\theta$$
$$\int \frac{1}{\sqrt{5\left(1 - \frac{4(x-1)^2}{5}\right)}}dx$$
$$= \int \frac{1}{\sqrt{5(1-\sin^2\theta)}}\frac{\sqrt{5}\cos\theta d\theta}{2}$$
$$\int \frac{1}{2}d\theta = \frac{1}{2}\theta + C$$
$$\Rightarrow \int \frac{1}{\sqrt{1+8x-4x^2}}dx = \frac{1}{2}\sin^{-1}\left(\frac{2(x-1)}{\sqrt{5}}\right) + C$$
$$= \frac{1}{2}\sin^{-1}\frac{2\sqrt{5}(x-1)}{5} + C.$$

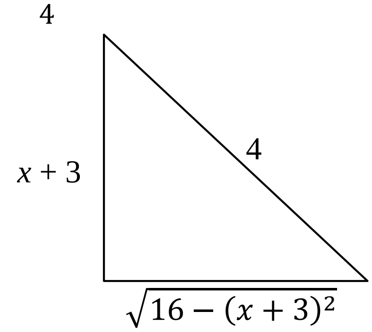
f) $\int \frac{x+3}{\sqrt{7-6x-x^2}}dx.$

$$7 - 6x - x^2 = 7 - (x^2 + 6x)$$
$$= 7 - (x^2 + 6x + 9) - -9$$
$$= 16 - (x+3)^2$$
$$\int \frac{x+3}{\sqrt{7-6x-x^2}}dx = \int \frac{x+3}{\sqrt{16-(x+3)^2}}dx.$$
$$= \int \frac{x+3}{\sqrt{16\left(1 - \frac{(x+3)^2}{16}\right)}}dx$$

Let $\frac{x+3}{4} = \sin\theta.$

$$x = 4\sin\theta - 3$$
$$dx = 4\cos\theta d\theta$$
$$\int \frac{x+3}{\sqrt{16\left(1 - \frac{(x+3)^2}{16}\right)}}dx$$
$$= \int \frac{(4\sin\theta - 3) + 3}{\sqrt{16(1-\sin^2\theta)}}.4\cos\theta d\theta$$
$$= \int 4\sin\theta d\theta$$
$$= -4\cos\theta + C.$$

But $\sin\theta = \frac{x+3}{4}$



$$\cos\theta = \frac{\sqrt{16-(x+3)^2}}{4}.$$
$$\Rightarrow \int \frac{x+3}{\sqrt{7-6x-x^2}}dx = -4\left(\frac{\sqrt{16-(x+3)^2}}{4}\right) + C$$
$$= -\sqrt{16-(x+3)^2} + C$$
$$= -\sqrt{7-6x-x^2} + C$$

Sec Substitution

Note: When we are integrating integrand in the form. $\frac{K}{\sqrt{a^2x^2-b^2}}$, we use the **sec** substitution.

Example

Find the following integrals.

$a)\int \frac{1}{x\sqrt{x^2-1}}dx$

$b)\int \frac{1}{x\sqrt{x^2-9}}dx.$

$c)\int_1^2 \frac{dx}{x^2\sqrt{5x^2-1}}dx$

Solution

$$\int \frac{1}{x\sqrt{x^2-1}}dx.$$

$$let\ \sqrt{x^2} = \sec\theta.$$

$$x = \sec\theta.$$

$$dx = \sec\theta\tan\theta d\theta.$$

$$\int \frac{1}{x\sqrt{x^2-1}}dx = \int \frac{\sec\theta\tan\theta}{\sec\theta\sqrt{\sec^2\theta-1}}d\theta$$

$$= \theta + C$$

$$= \sec^{-1}(x) + C$$

$$\Rightarrow \int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}(x) + C$$

b) $\int \frac{1}{x\sqrt{x^2-9}}dx$

$$\int \frac{1}{x\sqrt{9\left(\frac{x^2}{9}-1\right)}}dx$$

$$\sqrt{\frac{x^2}{9}} = \sec\theta.$$

$$\frac{x}{3} = \sec\theta.$$

$$\frac{1}{3}dx = \sec\theta\tan\theta\, d\theta.$$

$$dx = 3\sec\theta\tan\theta d\theta$$

$$\int \frac{1}{x\sqrt{9\left(\frac{x^2}{9}-1\right)}}dx = \int \frac{1}{3\sec\theta\sqrt{9(\sec^2\theta-1)}}3\sec\theta\tan\theta d\theta$$

$$\int \frac{1}{3}d\theta = \frac{1}{3}\theta + C$$

$$= \frac{1}{3}\sec^{-1}\left(\frac{x}{3}\right) + C$$

c) $\int_1^2 \frac{dx}{x^2\sqrt{5x^2-1}}dx.$

Let $\left(\sqrt{5x^2}\right) = \sec\theta.$

$$x\sqrt{5} = \sec\theta$$

$$\sqrt{5}dx = \sec\theta\tan\theta d\theta$$

$$dx = \frac{\sec\theta\tan\theta}{\sqrt{5}}d\theta$$

$$\int \frac{1}{\frac{\sec\theta}{5}\sqrt{\sec^2\theta-1}}.\frac{\sec\theta\tan\theta}{\sqrt{5}}d\theta$$

$$\int \frac{5}{\sec^2\theta\tan\theta}\frac{\sec\theta\tan\theta}{(\sqrt{5})}d\theta.$$

$$\frac{5}{\sqrt{5}}\int \frac{1}{\sec\theta}d\theta$$

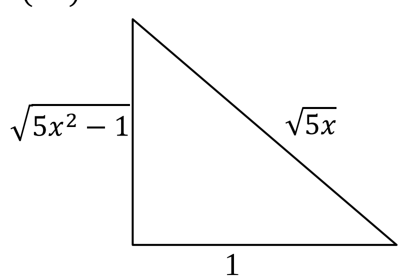
$$\frac{5}{\sqrt{5}}\int \cos\theta d\theta.$$

$$\frac{5}{\sqrt{5}}(\sin\theta) + C$$

But $\sec\theta = (\sqrt{5})x$

$$\frac{1}{\cos\theta} = \sqrt{5}x.$$

$$\cos\theta = \frac{1}{(\sqrt{5})x}$$



$$\frac{5}{\sqrt{5}}\sin\theta + C = \frac{5}{\sqrt{5}}\left(\frac{\sqrt{5x^2-1}}{\sqrt{5}-x}\right) + C$$

$$= \frac{\sqrt{5x^2-1}}{x} + C$$

$$\int_1^2 \frac{dx}{x\sqrt{5x^2-1}} = \left[\frac{\sqrt{5x^2-1}}{x}\right]_1^2$$

$$= \frac{\sqrt{20-1}}{2} - 2$$

$$= \frac{\sqrt{19}}{2} - 2$$

Partial Fractions

Content:

- Revision of addition and subtraction of rational expressions.
- Expressing rational expressions as a sum of it's partial fractions
- Rational expression where the denominator has a quadratic term (quadratic factor) which is not factorisable
- Rational expression where the denominator has repeated factors.
- Dealing with improper functions.

Partial Fractions

It is a process of expressing a rational expression into simpler rational expression that we can add or subtract to get the original rational expression. Given a rational expression where the numerators are polynomials.

If the degree of the numerator is less than the degree of the denominator the fraction is said to be a proper fractional. If the degree of the numerator is greater or equal to degree of the denominator, the fraction is said to be improper.

Consider $\frac{2}{x-3}-\frac{1}{2x+1}$

$$=\frac{2(2x+1)-(x-3)}{(x-3)(2x+1)}$$

$$=\frac{4x+2-x+3}{(3-x)(2x+1)}$$

$$=\frac{3x+5}{(x-3)(2x+1)}$$

$$\Rightarrow \frac{3x+5}{(x-3)(2x+1)}=\frac{2}{x-3}-\frac{1}{2x+1}$$

$\Rightarrow \frac{2}{x-3}-\frac{1}{2x+1}$ can be expressed as a single fraction $\frac{3x+5}{(x-3)(2x+1)}$

The process of getting back to $\frac{2}{x-3}-\frac{1}{2x+1}$ from $\frac{3x+5}{(x-3)(2x+1)}$ is called expressing $\frac{3x+5}{(x-3)(2x+1)}$ as a partial fraction.

Methods of Partial Fractions

1. Denominator with only linear factors

Example 1

Express $\frac{3x}{(x-1)(x+2)}$ as a partial fraction

Solution

$$\frac{3x}{(x-1)(x+2)}=\frac{A}{x-1}+\frac{B}{x+2}$$

$$\Rightarrow \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$

$$A(x+2)+B(x-1)=3x$$

If $x=-2$

$$-3B=-6$$

$$B=2$$

If $x = 1$,

$3A = 3 \Rightarrow A=1$

$$\Rightarrow \frac{3x}{(x-1)(x+2)}=\frac{1}{x-1}+\frac{2}{x+2}$$

Example 2

Express $\frac{3x+5}{(x-3)(2x+1)}$ as a partial fraction.

Solution:

$$\frac{3x}{(x-3)(2x+1)}=\frac{A}{x-3}+\frac{B}{2x+1}$$

$$\frac{3x+5}{(x-3)(2x+1)}=\frac{A(2x+1)+B(x-3)}{(x-3)(2x+1)}$$

$$3x+5=A(2x+1)+B(x-3)$$

If $x=-\frac{1}{2}$

$$3\left(-\frac{1}{2}\right)+5=B\left(-\frac{1}{2}-3\right)$$

$$-\frac{3}{2}+5=-\frac{7}{2}B$$

$$\frac{7}{2}=-\frac{7}{2}B$$

$$B = -1$$

If $x = 3$,

$3(3) + 5 = A(2 \times 3+1)+0$

$14 = 7A$

$A = 2$

$$\Rightarrow \frac{3x+5}{(x-3)(2x+1)}=\frac{2}{x-3}-\frac{1}{2x+1}$$

Example III

Express $\frac{x-1}{3x^2-11x+10}$ as partial fraction.

Solution

$$\frac{x-1}{3x^2-11x+10}$$

Consider $3x^2-11x+10$

$$3x^2-6x-5x+10$$
$$3x(x-2)-5(x-2)$$
$$(3x-5)(x-2)$$
$$\frac{x-1}{(3x-5)(x-2)}=\frac{A}{3x-5}+\frac{B}{x-2}$$
$$A(x-2)+B(3x-5)=x-1$$

If $x = 2$,

$B(1) = 1 \Rightarrow B = 1$

$If\ x=\frac{5}{3}$,

$$A\left(\frac{-1}{3}\right)=\frac{2}{3}$$
$$-A=2$$
$$A=-2$$
$$\Rightarrow \frac{x-1}{3x^2-11x+10}=\frac{-2}{3x-5}+\frac{1}{x-2}$$

Example IV

Express $\frac{3x^2-21x+24}{(x+1)(x-2)(x-3)}$ as partial fraction.

Solution

$$\frac{3x^2-21x+24}{(x+1)(x-2)(x-3)}=\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{x-3}$$

$$\frac{3x^2-21x+24}{(x+1)(x-2)(x-3)}=$$
$$\frac{A(x-2)(x-3)+B(x+1)(x-3)+C(x+1)(x-2)}{(x+1)(x-2)(x-3)}$$

$$A(x-2)(x-3)+B(x+1)(x-3)+C(x+1)(x-2)$$
$$=3x^2-21x+24$$

If $x = 2$, $B(3)(-1)=12-42+24$

$$-3B=-6$$
$$B=2$$

If $x = 3$, $C(4)(1)=3(3^2)-21\times 3+24$

$$4C=-12$$
$$C=-3$$

If $x=-1$, $A(-3)(-4)=3+21+24$

$12A = 48 \Rightarrow A = 4$

$$\frac{3x^2-21x+24}{(x+1)(x-2)(x-3)}=\frac{4}{x+1}+\frac{2}{x-2}-\frac{3}{x-3}$$

Example V

Express $\frac{32}{x^3-16x}$ as a partial fraction

Solution

$$\frac{32}{x^3-16x}=\frac{32}{x(x^2-16)}=\frac{32}{x(x+4)(x-4)}$$
$$\frac{32}{x(x+4)(x-4)}=\frac{A}{x}+\frac{B}{x+4}+\frac{C}{x-4}$$
$$A(x-4)(x+4)+Bx(x-4)+Cx(x+4)=32$$

If $x= 4$,

$32C=32$, $\Rightarrow C = 1$

If $x=-4$,

$32B=32 \Rightarrow B=1$

If $x = 0$,

$-16A = 32 \Rightarrow A = -2$

$$\frac{32}{x^3-16x}=\frac{-2}{x}+\frac{1}{x+4}+\frac{1}{x-4}$$

Example VI

Express $\frac{68+11x}{(3+x)(16-x^2)}$ in partial fractions.

Solution

$$\frac{68+11x}{(3+x)(16-x^2)}=\frac{68+11x}{(3+x)(4+x)(4-x)}$$
$$\frac{68+11x}{(3+x)(4+x)(4-x)}=\frac{A}{(3+x)}+\frac{B}{(4+x)}+\frac{C}{(4-x)}$$
$$A(4+x)(4-x)+B(3+x)(4-x)+$$
$$C(3+x)(4+x)=68+11x$$

If $x = 4$, $56C = 68 + 44$

$56C = 112 \Rightarrow C = 2$

If $x = -4$, $-8B = 68 - 44$

$-8B = 24 \Rightarrow B = -3$

If $x=-3$, $A(1)(7)=68-33$

$$7A=35,\ A=5$$
$$\frac{68+11x}{(3+x)(4+x)(4-x)}=\frac{5}{3+x}-\frac{3}{4+x}+\frac{2}{4-x}$$

Example VII

Express $\frac{x-9}{x(x^2+2x-3)}$ as a partial fraction

Solution

$$\frac{x-9}{x(x^2+2x-3)}=\frac{x-9}{x(x+3)(x-1)}$$
$$\frac{x-9}{x(x^2+2x-3)}=\frac{A}{x}+\frac{B}{x+3}+\frac{C}{x-1}$$

$$A(x+3)(x-1)+Bx(x-1)+Cx(x+3)=x-9$$
$$If\ x=-3$$
$$B(-3)(-4)=-12$$
$$12B=-12,\ \ B=-1$$
If $x=0, -3A=-9, \Rightarrow A=3$

If $x = 1$, $4C = -8$

$$C=-2$$
$$\Rightarrow \frac{x-9}{x^3+2x^2-3x}=\frac{3}{x}+\frac{-1}{x-3}+\frac{-2}{x-1}$$
$$=\frac{3}{x}-\frac{1}{x-3}-\frac{2}{x-1}$$

Denominator with quadratic factor not factorisable

Example I

Express $\frac{3x^2-2x+5}{(x-1)(x^2+5)}$ in partial fractions

Solution

$$\frac{3x^2-2x+5}{(x-1)(x^2+5)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+5}$$
$$A(x^2+5)+(Bx+C)(x-1)=3x^2-2x+5$$
$If\ x=1,$
$$6A=6,\ \ A=1$$
$$Ax^2+5A+Bx^2-Bx+Cx-C=3x^2-2x+5$$
$$(A+B)x^2+(C-B)x+5A-C=3x^2-2x+5$$
$$A+B=3;\ \text{But } A=1$$
$$1+B=3$$
$$B=2$$
$$C-B=-2$$
$$C-2=-2$$
$$C=0$$

$$\frac{3x^2-2x+5}{(x-1)(x^2+5)}=\frac{1}{x-1}+\frac{2x}{x^2+5}$$

Example II

Express $\frac{11x}{(2x-3)(2x^2+1)}$ in partial fractions

Solution

$$\frac{11x}{(2x-3)(2x^2+1)}=\frac{A}{2x-3}+\frac{Bx+C}{2x^2+1}$$
$$A(2x^2+1)+(2x-3)(Bx+C)=11x$$
If $x=\frac{3}{2}, A\left(2\times\frac{9}{4}+1\right)=\frac{33}{2}$

$A\left(\frac{11}{2}\right)=\frac{33}{2},\ \ A=3$

$$2Ax^2+A+2Bx^2-3Bx+2Cx-3C=11x$$
$$(2A+2B)x^2+(2C-3B)x+A-3c=11x$$
Equating the corresponding co-efficients

$$2A+2B=0;\ \text{But } A=3$$
$$2(3)+2B=0$$
$$B=-3$$
$$2C-3B=11$$
$$2C-3(-3)=11$$
$$2C+9=11$$
$$2C=2, C=1$$
$$A-3C=0$$
$$A=3C$$
$$A=3$$
$$\Rightarrow \frac{11x}{(2x-3)(2x^2+1)}=\frac{3}{2x-3}+\frac{-3x+1}{2x^2+1}$$

Example III

Express $\frac{6-3x}{(x+1)(x^2+3)}$ in partial fraction

Solution

$$\frac{6-3x}{(x+1)(x^2+3)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+3}$$
$$A(x^2+3)+(Bx+C)(x+1)=6-3x$$
$$If\ x=-1,\qquad 4A=9$$
$$A=\frac{9}{4}$$
$$Ax^2+3A+Bx^2+Bx+Cx+C=6-3x$$
$$(A+B)x^2+(B-C)x+3A+C=6-3x$$
$$A+B=0$$
$$\frac{9}{4}+B=0$$
$$B=\frac{-9}{4}$$
$$B+C=-3,\qquad -\frac{9}{4}+C=-3$$
$$C=-3+\frac{9}{4}$$
$$C=\frac{-3}{4}$$
$$\frac{6-3x}{(x+1)(x^2+3)}=\frac{9}{4(x+1)}+\frac{\frac{-9x}{4}+\frac{-3}{4}}{x^2+3}$$
$$=\frac{9}{4(x+1)}-\frac{3(3x+1)}{4(x^2+3)}$$

Example IV

Express $\frac{1}{x^4+5x^2+6}$ in partial fractions

Solution

$$\frac{1}{x^4+5x^2+6}$$
$$x^4+5x^2+6=(x^2)^2+5x^2+6$$
Let $y=x^2$
$$y^2+5y+6$$
$$=(y+2)(y+3)$$
$$\frac{1}{x^4+5x^2+6}=\frac{1}{(x^2+2)(x^2+3)}$$
$$\frac{1}{(x^2+2)(x^2+3)}=\frac{Ax+B}{x^2+2}+\frac{Cx+D}{x^2+3}$$
$$(Ax+B)(x^2+3)+(Cx+D)(x^2+2)=1$$
$$Ax^3+3Ax+Bx^2+3B+Cx^3+2Cx+Dx^2+2D=1$$
$$(A+C)x^3+(B+D)x^2+(3A+2C)x+3B+2D=1$$
$A+C=0$ ……………………………..(i)
$B+D=0$……………………………..(ii)
$3A+2C=0$ ………………………… (iii)
$3B+2D=1$ …………………………. (iv)
From Eqn (i), $A=-C$
Substitute in Eqn (iii)
$$-3C+2C=0,\quad C=0$$
From Eqn (ii), $B=-D$
Substitute in Eqn (iv);
$$3(-D)+2D=1$$
$$-D=1,\Rightarrow D=-1$$
$$\therefore B=1$$
$$\Rightarrow \frac{1}{x^4+5x^2+6}=\frac{1}{x^2+2}-\frac{1}{x^2+3}$$

Example IV

Express $\frac{2x+1}{x^3+1}$ in partial fraction

Solution

$$\frac{2x+1}{x^3+1}$$
Consider $x^3-1=x^3-1^3$
$$a^3-b^3=(a-b)(a^2+ab+b^2)$$
$$\Rightarrow x^3-1^3=(x-1)(x^2+x+1)$$
$$\frac{2x+1}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$$
$$A(x^2+x+1)+(Bx+C)(x-1)=2x+1$$
$$If\ x=1,\ \ 3A=3,\ \ A=1$$
$$Ax^2+Ax+A+Bx^2-Bx+Cx-C=2x+1$$
Equating coefficients of the same monomial;
$A+B=0$; But $A=1$
$1+B=0 \Rightarrow B=-1$
$A-B+C=2$
$1-{}^{-}1+C=2 \Rightarrow C=0$
$$\frac{2x+1}{x^3-1}=\frac{1}{x-1}-\frac{x}{x^2+x+1}$$

Example V

Express $\frac{13x+7}{(x-4)(3x^2+2x+3)}$ in partial fractions

Solution

$$\frac{13x+7}{(x-4)(3x^2+2x+3)}=\frac{A}{x-4}+\frac{Bx+C}{3x^2+2x+3}$$
$$A(3x^2+2x+3)+(Bx+C)(x-4)=13x+7$$
If $x=4$, $A(12+8+3)=59$
$$A(23)=59$$
$$A=\frac{59}{23}$$
$3Ax^2+2Ax+3A+Bx^2-4Bx+Cx-4C=13x+7$
$3A+B=0$
$B=-3A$
$$B=-3\left(\frac{59}{23}\right)=\frac{-177}{23}$$
$$3A-4C=7$$
$$\frac{177}{23}-4C=7$$
$$\frac{177}{23}-7=4C$$
$$\frac{16}{23}=4C \quad \Rightarrow C=\frac{4}{23}$$
$$\frac{13x+7}{(x-4)(3x^2+2x+3)}=\frac{59}{23(x-4)}+\frac{\frac{-177x}{23}+\frac{4}{23}}{3x^2+2x+3}$$
$$=\frac{59}{23(x-4)}-\frac{(177x-4)}{23(3x^2+2x+3)}$$

Example VI

Express $\frac{5x}{(x^2+x+1)(x-2)}$ in partial fractions

Solution

$$\frac{5x}{(x^2+x+1)(x-2)}=\frac{Ax+B}{x^2+x+1}+\frac{C}{x-2}$$
$(Ax+B)(x-2)+C(x^2+x+1)=5x.$
If $x=2$, $C(4+2+1)=10$
$$7C=10$$
$$C=\frac{10}{7}$$
$$Ax^2-2Ax+Bx-2B+Cx^2+Cx+C=5x$$
$A+C=0$ ……………………………. (i)
$(C-2A+B)=5$ …………………….. (ii)

$C - 2B = 0$ …………………………… (iii)

$A = -C$

$\Rightarrow A = \frac{-10}{7}$

From Eqn (iii); $C = 2B$

$\frac{10}{7} = 2B.$

$B = \frac{5}{7}.$

$$\Rightarrow \frac{5x}{(x^2+x+1)(x-2)} = \frac{\frac{-10}{7}x + \frac{5}{7}}{x^2+x+1} + \frac{\frac{10}{7}}{x-2}$$

$$\frac{5x}{(x^2+x+1)(x-2)} = \frac{-10x+5}{7(x^2+x+1)} + \frac{10}{7(x-2)}$$

Denominator with Repeated Factors

Express the following in partial fractions.

Example (Hints)

Express the following in partial fractions

a) $\frac{1}{(x+2)(x-3)^3(x+5)^2}$

b) $\frac{1}{x^2(x+5)}$

c) $\frac{1}{(x+9)^4(x+1)^2}$

d) $\frac{1}{(x^2+3)(x+1)^3}$

Solution

(a) $\frac{1}{(x+2)(x-3)^3(x+5)^2}$

$$= \frac{A}{(x+2)} + \frac{B}{(x-3)} + \frac{C}{(x-3)^2} + \frac{D}{(x-3)^3} + \frac{E}{x+5} + \frac{F}{(x+5)^2}$$

(b) $\frac{1}{x^2(x+5)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+5}$

(c) $\frac{1}{(x+9)^4(x+1)^2}$

$$= \frac{A}{x+9} + \frac{B}{(x+9)^2} + \frac{C}{(x+9)^3} + \frac{D}{(x+9)^4} + \frac{E}{x+1} + \frac{F}{(x+1)^2}$$

(d) $\frac{1}{(x^2+3)(x+1)^3} = \frac{Ax+B}{x^2+3} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}$

The above hint will help us to express rational expressions with denominators of repeated factors into partial functions

Example I

Express $\frac{x-3-2x^2}{x^2(x-1)}$ in partial fractions

Solution

$$\frac{x-3-2x^2}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$Ax(x-1) + B(x-1) + Cx^2 = x - 3 - 2x^2$

If $x = 1$, $C = 1 - 3 - 2$

$C = -4.$

If $x = 0$, $-B = -3 \Rightarrow B = 3$

If $x = 2$, $2A + B + 4C = 2 - 3 - 8$

$2A + 3 - 16 = -9$

$2A - 13 = -9$

$2A = -9 + 13$

$2A = 4$

$A = 2$

$$\Rightarrow \frac{x-3-2x^2}{x^2(x-1)} = \frac{2}{x} + \frac{3}{x^2} - \frac{4}{x-1}$$

Example II

Express $\frac{x+4}{(x+1)(x-2)^2}$ in partial fractions

Solution

$$\frac{x+4}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$\Rightarrow A(x-2)^2 + B(x-2)(x+1) + C(x+1) = x + 4$

If $= 2$, $C(3) = 6$

$C = 2$

If $x = -1$, $9A = 3$

$A = \frac{1}{3}$

If $x = 0$, $4A + -2B + C = 4$

$\frac{4}{3} - 2B + 2 = 4$

$2B = \frac{4}{3} + 2 - 4$

$2B = \frac{4}{3} - \frac{2}{1}$

$2B = -\frac{2}{3}$

$B = \frac{-1}{3.}$

$$\frac{x+4}{(x+1)(x-2)^2} = \frac{1}{3(x+1)} + \frac{-1}{3(x-2)} + \frac{2}{(x-2)^2}$$

Example III

Express $\frac{4x+3}{(x-1)^2}$ in partial fraction.

Solution

$$\frac{4x+3}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$\Rightarrow A(x-1) + B = 4x + 3$

If $x = 1,\ B = 7$

If $x = 0,\ -A + B = 3$

$$-A + 7 = 3$$
$$-A = -4$$
$$A = 4$$

$\Rightarrow \frac{4x+3}{(x-1)^2} = \frac{4}{x-1} + \frac{7}{(x-1)^2}.$

Example IV

Express $\frac{10+6x-3x^2}{(2x-1)(x+3)^2}$ in partial fractions

Solution

$$\frac{10+6x-3x^2}{(2x-1)(x+3)^2} = \frac{A}{2x-1} + \frac{B}{(x+3)} + \frac{C}{(x+3)^2}$$

$A(x+3)^2 + B(x+3)(2x-1) + C(2x-1)$
$= 10 + 6x - 3x^2$

If $x = \frac{1}{2},\ A\left(\frac{49}{4}\right) = 10 + 3 - 3\left(\frac{1}{4}\right)$

$$A\left(\frac{49}{4}\right) = 13 - \frac{3}{4}$$
$$A\left(\frac{49}{4}\right) = \frac{49}{4}$$
$$A = 1$$

If x = -3, $-7C = 10 - 18 - 27$

$$-7C = -35$$
$$C = 5.$$

If x = 0, $9A - 3B - C = 10$

$$9 - 3B - 5 = 10$$
$$-3B = 6$$
$$B = -2.$$

$$\frac{10+6x-3x^2}{(2x-1)(x+3)^2} = \frac{1}{2x-1} + \frac{2}{(x+3)} + \frac{5}{(x+3)^2}$$

Example V

Express $\frac{3x+1}{(x-1)^2(x+2)}$ in partial fraction.

Solution

$$\frac{3x+1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$$

$A(x-1)(x+2) + B(x+2) + C(x-1)^2 = 3x + 1$

If x = 1, $3B = 4$

$$B = \frac{4}{3}$$

If x = -2, $9C = -6 + 1$

$$C = -\frac{5}{9}$$

If x = 0, $-2A + 2B + C = 1$

$$-2A + \left(\frac{8}{3}\right) - \left(\frac{5}{9}\right) = 1.$$
$$2A = \left(\frac{8}{3}\right) - \left(\frac{5}{9}\right) - 1$$
$$2A = \frac{24-5-9}{9}$$
$$2A = \frac{10}{9}$$
$$A = \frac{5}{9}.$$

$$\Rightarrow \frac{3x+1}{(x-1)^2(x+2)} = \frac{5}{9(x-1)} + \frac{4}{3(x-1)^2} - \frac{5}{9(x+2)}$$

Example VI

Express $\frac{5x^2-6x-21}{(x-4)^2(2x-3)}$ in partial fractions.

Solution

$$\frac{5x^2-6x-21}{(x-4)^2(2x-3)} = \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{C}{2x-3}$$

$\Rightarrow A(2x-3)(x-4) + B(2x-3) + C(x-4)^2 = 5x^2 - 6x - 21$

If x = 4, B(5) = 80 – 24 – 21

$$5B = 35$$
$$B = 7$$

If $x = \frac{3}{2}$, $C\left(\frac{25}{4}\right) = 5\left(\frac{9}{4}\right) - 9 - 21$

$$\frac{25}{4}C = \frac{45}{4} - 30$$
$$\frac{25C}{4} = \frac{-75}{4} \quad \Rightarrow C = -3$$

If x = 0, $12A - 3B + 16C = -21$

$$12A - 21 - 48 = -21$$
$$12A = -21 + 69$$
$$12A = 48$$
$$A = 4$$

$$\Rightarrow \frac{5x^2-6x-21}{(x-4)^2(2x-3)} = \frac{4}{x-4} + \frac{7}{(x-4)^2} + \frac{-3}{2x-3}$$

Improper Fractions

So far we have only dealt with proper fractions for which the numerator is of lower degree than the denominator. We can now look at how to deal with improper fractions where the degree of the numerator is greater or equal to the degree of the denominator.

Examples of improper fraction are:

$$\frac{x}{x+1},\ \frac{x^2+1}{x},\ \frac{x^4+x+1}{x(x-1)(x+1)(x+2)}$$

$$\frac{x^3+1}{x^2+1},\ \frac{2x^4+3x^2+1}{x^2+3x+2},\ \frac{7x^2-1}{x^2+3}$$

Example I

Express $\frac{4x^3+10x+4}{x(2x+1)}$ in partial fractions.

Solution

$$\frac{4x^3+10x+4}{x(2x+1)} = \frac{4x^3+10x+4}{2x^2+x}$$

$$\begin{array}{r|l} & \quad 2x-1 \\ \hline 2x^2+x & 4x^3+10x+4 \\ - & 4x^3+2x^2 \\ \hline & -2x^2+10x+4 \\ - & -2x^2-x \\ \hline & \quad 11x+4 \end{array}$$

$$\Rightarrow \frac{4x^3+10x+4}{x(2x+1)} = (2x-1) + \frac{11x+4}{x(2x+1)}$$

$$\frac{11x+4}{x(2x+1)} = \frac{A}{x} + \frac{B}{2x+1}$$

$A(2x+1) + Bx = 11x + 4.$

If $x = 0$, $A = 4$

If $x = -\frac{1}{2}$, $-\frac{1}{2}B = \frac{-11}{2} + 4$

$$-\frac{1}{2}B = -\frac{3}{2}$$

$$B = 3$$

$$\Rightarrow \frac{4x^2+10x+4}{x(2x+1)} = (2x-1) + \frac{4}{x} + \frac{3}{2x+1}$$

Example II

Express $\frac{2x^3-x-1}{(x-3)(x^2+1)}$ in partial fraction

Solution

$$\frac{2x^3-x-1}{(x-3)(x^2+1)}$$

Consider $(x-3)(x^2+1)$

$$= x^3 + x - 3x^2 - 3.$$

$$\frac{2x^3-x-1}{x^3-3x^2+x-3.}$$

$$\begin{array}{r|l} & \quad 2 \\ \hline x^3-3x^2+x-3 & 2x^3-x-1 \\ - & 2x^3-6x^2+2x-6 \\ \hline & \quad 6x^2-3x+5 \end{array}$$

$$\Rightarrow \frac{2x^3-x-1}{(x-3)(x^2+1)} = 2 + \frac{6x^2-3x+5}{(x-3)(x^2+1)}$$

$$\frac{6x^2-3x+5}{(x-3)(x^2+1)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+1}$$

$A(x^2+1) + (Bx+C)(x-3) = 6x^2 - 3x + 5$ ………. (i)

If $x = 3$, $10A = 54 - 9 + 5$

$10A = 50$

$A = 5$

From Eqn (i);

$Ax^2 + A + Bx^2 - 3Bx + Cx - 3C = 6x^2 - 3x + 5$

$\Rightarrow A + B = 6$

$5 + B = 6$

$B = 1$

$C - 3B = -3$

$C - 3 = -3$

$C = 0.$

$$\Rightarrow \frac{2x^3-x-1}{(x-3)(x^2+1)} = 2 + \frac{5}{x-3} + \frac{x}{x^2+1}$$

Example III

Express $\frac{x^3-3}{(x-2)(x^2+1)}$ in partial fraction

Solution

$$\frac{x^3-3}{(x-2)(x^2+1)} = \frac{x^3-3}{x^3-2x^2+x-2}$$

$$\begin{array}{r|l} & \quad 1 \\ \hline x^3-2x^2+x-2 & x^3-3 \\ - & x^3-2x^2+x-2 \\ \hline & \quad 2x^2-x-1 \end{array}$$

$$\Rightarrow \frac{x^3-3}{x^3-2x^2+x-2} = 1 + \frac{2x^2-x-1}{x^3-2x^2+x-2}$$

$$= 1 + \frac{2x^2-x-1}{(x-2)(x^2+1)}$$

But $\frac{2x^2-x-1}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$

$$A(x^2+1) + (Bx+C)(x-2) = 2x^2-x-1$$

If $x = 2$, $5A = 8 - 2 - 1 \Rightarrow A = 1$

$Ax^2 + A + Bx^2 - 2Bx + Cx - 2C = 2x^2 - x - 1$

$A + B = 2$

$1 + B = 2$

$B = 1$

$-2B + C = -1$

$-2 + C = -1$

$C = 1.$

$$\Rightarrow \frac{2x^2-x-1}{(x-1)(x^2+1)} = \frac{1}{x-2} + \frac{x+1}{x^2+1}$$

$$\Rightarrow \frac{x^3-3}{(x-2)(x^2+1)} = 1 + \frac{1}{x-2} + \frac{x+1}{x^2+1}$$

Example IV

Express $\frac{x^4+3x-1}{(x+2)(x-1)^2}$ in partial fractions

Solution

$$\frac{x^4+3x-1}{(x+2)(x-1)^2} = \frac{x^4+3x-1}{x^3-3x+2}$$

$$\begin{array}{r|l} & x \\ \hline x^3-3x+2 & x^4+3x-1 \\ - & x^4-3x^2+2x \\ \hline & 3x^2+x-1 \end{array}$$

$$\frac{x^4+3x-1}{x^3-3x+2} = x + \frac{3x^2+x-1}{(x+2)(x-1)^2}$$

$$\frac{3x^2+x-1}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$A(x-1)^2 + B(x-1)(x+2) + C(x+2) = 3x^2 + x - 1$

If $x = 1$, $C(3) = 3 + 1 - 1$

$C = 1$

If $x = -2$, $9A = 12 - 2 - 1$

$9A = 9$

$A = 1$

If $x = 0$, $A - 2B + 2C = -1$

$1 - 2B + 2 = -1$

$-2B = -1 - 3$

$-2B = -4$

$B = 2$

$$\Rightarrow \frac{x^4+3x-1}{(x+2)(x-1)^2} = x + \frac{1}{x+2} + \frac{2}{x-1} + \frac{1}{(x-1)^2}$$

Example V

Express $\frac{2x^3+7x^2+2x-10}{(x+3)(2x-1)}$ in partial fractions

Solution

$$\frac{2x^3+7x^2+2x-10}{(x+3)(2x-1)} = \frac{2x^3+7x^2+2x-10}{2x^2+5x-3}$$

$$\begin{array}{r|l} & x+1 \\ \hline 2x^2+5x-3 & 2x^3+7x^2+2x-10 \\ - & 2x^3+5x^2-3x \\ \hline & 2x^2+5x-10 \\ & 2x^2+5x-3 \\ \hline & -7 \end{array}$$

$$\frac{2x^3+7x^2+2x-10}{2x^2+5x-3} = (x+1) - \left(\frac{7}{(x+3)(2x-1)}\right)$$

But $\frac{7}{(x+3)(2x-1)} = \frac{A}{x+3} + \frac{B}{2x-1}$

$A(2x-1) + B(x+3) = 7$

If $x = -3$, $-7A = 7$

$A = -1$

If $x = \frac{1}{2}$, $B\left(\frac{7}{2}\right) = 7$

$B = 2$

$$\Rightarrow \frac{2x^3+7x^2+2x-10}{(x+3)(2x-1)}$$

$$= x + 1 - \left(\frac{-1}{x+3} + \frac{2}{2x-1}\right)$$

$$= x + 1 + \frac{1}{x+3} - \frac{2}{2x-1}$$

Example VI

Express $\frac{x^3}{(x+4)(x-1)}$ in partial fractions.

Solution

$$
\begin{array}{r|l}
 & x - 3 \\
\hline
x^2 + 3x - 4 & x^3 \\
 & -\underline{\left(x^3 + 3x^2 - 4x\right)} \\
 & \quad -3x^2 + 4x \\
 & \quad \underline{-3x^2 - 9x + 12} \\
 & \qquad 13x - 12
\end{array}
$$

$$\frac{x^3}{(x+4)(x-1)} = (x-3) + \frac{3x-12}{x^2+3x-4}$$

$$\frac{13x+12}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$$

$A(x-1) + B(x+4) = 13x - 12.$

If $x = 1$, $5B = 1 \Rightarrow B = \frac{1}{5}$

If $x = -4$, $-5A = -13 \times 4 - 12$

$$A = \frac{-64}{5}$$

$$\frac{x^3}{(x+4)(x-1)} = (x-3) + \frac{64}{5(x+4)} + \frac{1}{5(x-1)}$$

Integration of Partial Fraction

In this section, we are going to look at how we can integrate some algebraic fraction. We will be using partial fractions to express the integrand as a sum of simpler fractions which can be integrated separately. We will also need to call upon wide variety of other techniques including completing squares, integration by substitution, integration using stands results and so on. In order to understand the integration of partial fractions, it's vital that we undertake a plenty of practice exercise so that they become second nature.

Note: It's important to recognize certain standard integrals and method here.

(1) $\textit{the use of } \int \frac{f^1(x)}{f(x)} dx = \ln f(x) + C$

(2) $\int \frac{x}{2x^2+3} dx = \frac{1}{4}\ln(2x^2+3) + C$

(3) Splitting up the expression

$$\int \frac{2x+1}{x^2+1} dx = \int \frac{2x}{x^2} + \frac{1}{x^2+1} dx$$
$$= \ln(x^2+1) + \tan^{-1} x + C$$

Example 1

$$\int \frac{3x+1}{(x-1)(2x+1)} dx.$$

Consider $\frac{3x+1}{(x-1)(2x+1)} = \frac{A}{x-1} + \frac{B}{2x+1}$

$A(2x+1) + B(x-1) = 3x + 1$

If $x = 1$, $3A = 4$

$$A = \frac{4}{3}$$

If $x = \frac{-1}{2}$, $B\left(\frac{-3}{2}\right) = \frac{-1}{2}$

$$B = \frac{1}{3}$$

$$\Rightarrow \frac{3x+1}{(x-1)(2x+1)} = \frac{4}{3(x-1)} + \frac{1}{3(2x+1)}$$

$$\Rightarrow \int \frac{3x+1}{(x-1)(2x+1)} dx = \int \frac{4}{3(x-1)} + \frac{1}{3(2x+1)} dx$$

$$\frac{4}{3}\int \frac{1}{x-1} dx + \frac{1}{3}\int \frac{1}{2x+1} dx.$$

$$= \frac{4}{3}\ln(x-1) + \frac{1}{3}\left(\frac{1}{2}\ln(2x+1)\right) + C$$

$$= \frac{4}{3}\ln(x-1) + \frac{1}{6}\ln(2x+1) + C$$

Example II

$$\int \frac{3x+1}{(x-1)(x^2+1)} dx.$$

Solution

Consider,

$$\frac{3x+1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$A(x^2+1) + (Bx+C)(x-1) = 3x + 1$

If $x = 1$, $2A = 4$

$A = 2$

$Ax^2 + A + Bx^2 - Bx + Cx - C = 3x + 1$

$A + B = 0$

$2 + B = 0$

$B = -2.$

$C - B = 3.$

$C + 2 = 3$

$c = 1$

$$\Rightarrow \frac{3x+1}{(x-1)(x^2+1)} = \frac{2}{x-1} + \frac{-2x+1}{x^2+1}$$

$$\int \frac{3x+1}{(x-1)(x^2+1)} dx = \int \frac{2}{x-1} + \frac{-2x+1}{x^2+1} dx$$

$$= \int \frac{2}{x-1}\,dx + \int \frac{-2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$2\ln(x-1) - \ln(x^2+1) + \tan^{-1} x + C$$

$$\ln\frac{(x-1)^2}{x^2+1} + \tan^{-1}(x) + C$$

Example III

$$\int_2^4 \frac{36}{(x-1)^2(x+5)}\,dx.$$

Consider:

$$\frac{36}{(x-1)^2(x+5)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+5}$$

$$A(x-1)(x+5) + B(x+5) + C(x-1)^2 = 36$$

If $x = 1$, $6B = 36$

$B = 6$

If $x = -5$, $36C = 36$

$C = 1$

If $x = 0$, $-5A + 5B + C = 36$

$-5A + 30 + 1 = 36$

$-5A = 5$

$A = -1$

$$\Rightarrow \frac{36}{(x-1)^2(x+5)} = \frac{-1}{(x-1)} + \frac{6}{(x-1)^2} + \frac{1}{(x+5)}$$

$$\int_2^4 \frac{36}{(x-1)^2(x+5)}\,dx$$

$$= \int_2^4 \left(\frac{-1}{x-1} + \frac{6}{(x-1)^2} + \frac{1}{x+5}\right)dx$$

$$= {}^{-}\ln(x-1) - \frac{6}{x-1} + \ln(x+5)\Big|_2^4$$

$$= [-\ln(3) + \ln 1] - \left(\frac{6}{3} - 6\right) + (\ln 9 - \ln 7)$$

$$-\ln 3 + 4 + \ln\left(\frac{9}{7}\right)$$

$$= 4 + \ln\left(\frac{9}{7}\right) - \ln 3$$

$$= 4 + \ln\left(\tfrac{3}{7}\right)$$

$= 3.1527$

Example IV

$$\int_1^2 \frac{1+x}{x^2(x^2+1)}\,dx.$$

Solution

$$\frac{1+x}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{(Cx+D)}{x^2+1}$$

$$A(x)(x^2+1) + B(x^2+1) + (Cx+D)x^2 = 1 + x$$

If $x = 0$, $B = 1$

$$Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2 = 1 + x$$

$\Rightarrow A + C = 0 \ldots\ldots\ldots.(1)$

$B + D = 0 \ldots\ldots\ldots\ldots..(2)$

$A = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3)$

$B = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$

From Eqn (2)

$B + D = 0$

$1 + D = 0.$

$D = -1.$

$A + C = 0$

$\Rightarrow 1 + C = 0$

$c = -1$

$$\Rightarrow \frac{1+x}{x^2(x^2+1)} = \frac{1}{x} + \frac{1}{x^2} + \frac{-(x+1)}{x^2+1}$$

$$\Rightarrow \int_1^2 \frac{1+x}{x^2(x^2+1)}\,dx = \int_1^2 \frac{1}{x} + \frac{1}{x^2} - \frac{(x+1)}{x^2+1}\,dx$$

$$\int_1^2 \frac{1}{x}\,dx + \int_1^2 \frac{1}{x^2}\,dx - \int_1^2 \frac{x}{x^2+1} - \int_1^2 \frac{1}{x^2+1}$$

$$\ln x\Big|_1^2 + \frac{-1}{x}\Big|_1^2 - \ln(x^2+1)\Big|_1^2 + \tan^{-1} x\Big|_1^2$$

$$(\ln 2 - \ln 1) + \left(\frac{-1}{2}\right) - (-1)$$

$$-\left(\frac{1}{2}\ln 5 - \frac{1}{2}\ln 2\right) - (\tan^{-1} 2 - \tan^{-1} 1)$$

$$\ln 2 + \frac{1}{2} + \frac{1}{2}\left(\ln\frac{2}{5}\right) - (\tan^{-1} 2 - \tan^{-1} 1)$$

Example V

$$\int \frac{x^3}{x^2-4}\,dx.$$

Solution

Consider $\frac{x^3}{x^2-4}$

$$\begin{array}{r|l} & x \\ \hline x^2-4 & x^3 \\ - & x^3 - 4x \\ \hline & 4x \end{array}$$

$$\frac{x^3}{x^2-4} = x + \frac{4x}{(x^2-4)}$$

But $\frac{4x}{x^2-4} = \frac{4x}{(x+2)(x-2)}$

$$\frac{A}{x+2}+\frac{B}{x-2}$$

$A(x-2)+B(x+2)=4x$

If $x = 2$, $4B = 8$

$B = 2$

If $x = -2$, $-4A = -8$

$A = 2$

$$\frac{x^3}{x^2-4}=x+\frac{2}{x+2}+\frac{2}{x-2}$$

$$\int\frac{x^3}{x^2-4}dx=\int xdx+\int\frac{2}{x+2}dx+\int\frac{2}{x-2}dx$$

$$\int\frac{x^3}{x^2-4}dx=\frac{x^2}{2}+2\ln(x+2)+2\ln(x-2)+C$$

$$\Rightarrow\int\frac{x^3}{x^2-4}dx=\frac{x^2}{2}+\ln\left(\frac{x+2}{x-2}\right)^2+C$$

Example VI

$$\int_4^5\frac{24x^3(x-3)}{(x-1)(2x+1)}dx$$

Solution

$$\frac{24x^3(x-3)}{(x-1)(2x+1)}=\frac{24x^4-72x^3}{2x^2-x-1}$$

$$\begin{array}{r|l} & 12x^2-30x-9 \\ \hline 2x^2-x-1 & 24x^4-72x^3 \\ - & 24x^4-12x^3-12x^2 \\ \hline & -60x^3+12x^2 \\ & -60x^3+30x^2+30x \\ \hline & -18x^2-30x \\ & -18x^2+9x+9 \\ \hline & -39x-9 \end{array}$$

$$\frac{24x^3(x-1)}{(x-1)(2x+1)}=(12x^2-30x-9)-\frac{(39x+9)}{(x-1)(2x+1)}$$

But $\frac{39x+9}{(x-1)(2x+1)}=\frac{A}{x-1}+\frac{B}{2x+1}$

$A(2x+1)+B(x-1)=39x+9.$

If $x=1$, $3A=48$

$A=16.$

If $x=\frac{-1}{2}$, $\frac{-3}{2}B=\frac{-21}{2}$

$B=7$

$$\frac{24x^3(x-3)}{(x-1)(2x+1)}=12x^2-30x-9+\frac{16}{x-1}+\frac{7}{2x+1}$$

$$\int_4^5\frac{24x^3(x-3)}{(x-1)(2x+1)}dx$$

$$4x^3-15x^2-9x\Big|_4^5+16\ln(x-1)\Big|_4^5+\tfrac{7}{2}\ln(2x+1)\Big|_4^5$$

$= 500 - 375 - 45) - (256 - 240 - 36) + (16\ln 4 - 16\ln 3)$

$+\frac{7}{2}(\ln 11-\ln 9)$

$= 100 + 23\ln 3 - \frac{7}{2}\ln(11) - 32\ln(2)$

Example VII

$$\int\frac{6x}{(x-2)(x+4)^2}dx.$$

Solution

Consider.

$$\frac{6x}{(x-2)(x+4)^2}=\frac{A}{x-2}+\frac{B}{(x+4)}+\frac{C}{(x+4)^2}$$

$A(x+4)^2+B(x-2)(x+4)+C(x-2)=6x$

$If\ x=2,\ 36A=12$

$36A=12$

$A=\frac{1}{3}$

$If\ x=-4,\ C(-6)=-24$

$C=4$

$If\ x=0,\ 16A-8B-2C=0$

$16\left(\frac{1}{3}\right)-8B-8=0$

$8B=8-\frac{16}{3}$

$8B=\frac{8}{3}$

$B=\frac{1}{3}$

$$\Rightarrow\frac{6x}{(x-1)(x+4)^2}$$

$$=\frac{1}{3(x-2)}+\frac{1}{3(x+4)}+\frac{4}{(x+4)^2}$$

$$\Rightarrow\int\frac{6x}{(x-2)(x+4)}dx$$

$$=\frac{1}{3}\int\frac{1}{x-2}dx+\frac{1}{3}\int\frac{1}{x+4}dx+4\int\frac{1}{(x+4)^2}dx$$

$$=\frac{1}{3}\ln(x-2)+\frac{1}{3}\ln(x+4)-\frac{4}{x+4}+C$$

Example

$$\int\frac{x^2}{x^4-1}dx.$$

$$\frac{x^2}{x^4-1}=\frac{x^2}{(x^2)^2-1^2}=\frac{x^2}{(x^2+1)(x^2-1)}$$
$$=\frac{x^2}{(x^2+1)(x+1)(x-1)}$$
$$\frac{x^2}{(x^2+1)(x+1)(x-1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x+1}+\frac{D}{x-1}$$
$$(Ax+B)(x+1)(x-1)+C(x-1)(x^2+1)$$
$$+D(x+1)(x^2+1)=x^2$$
If $x = 1$, $D(2)(2)=1$.
$$4D=1$$
$$D=\frac{1}{4}$$
$$Ax^3-Ax+Bx^2-B+Cx^3-Cx^2+Cx-C+Dx^3$$
$$+Dx^2+Dx+D=x^2$$
$A+C+D=0$ ……………………..(1)
$B-C+D=1$ ………………………(2)
$-A+C+D=0$ ………….…….(3)
$D-C-B=0$ ………………………(4)

Eqn(1) − Eqn(3)
$$2A=0 \Rightarrow A=0$$
Eqn (2) − Eqn(4)
$$2B=1 \Rightarrow B=\frac{1}{2}.$$
From Eqn (2).
$$B-C+D=1$$
$$\frac{1}{2}-C+\frac{1}{4}=1$$
$$\frac{3}{4}-C=1$$
$$C=\frac{-1}{4}$$
$$\Rightarrow \frac{x^2}{x^4-1}=\frac{1}{2(x^2+1)}-\frac{1}{4(x+1)}+\frac{1}{4(x-1)}$$
$$\Rightarrow \int\frac{x^2}{x^4-1}dx$$
$$=\int\frac{1}{2(x^2+1)}-\frac{1}{4(x+1)}+\frac{1}{4(x-1)}dx$$
$$\frac{1}{2}\tan^{-1}x-\frac{1}{4}\ln(x+1)+\frac{1}{4}\ln(x-1)+C$$
$$\frac{1}{2}\tan^{-1}x+\frac{1}{4}\ln\left(\frac{x-1}{x+1}\right)+C$$

Example IX

Show that $\int_0^1\frac{x^2+6}{(x^2+4)(x^2+9)}=\frac{\pi}{20}$

Solution
$$\frac{x^2+6}{(x^2+4)(x^2+9)}=\frac{Ax+B}{x^2+4}+\frac{Cx+D.}{x^2+9}$$
$$(Ax+B)(x^2+9)+(Cx+D)(x^2+4)=x^2+6$$
$$Ax^3+9Ax+Bx^2+9B+Cx^3+4Cx+Dx^2+4D$$
$$=x^2+6$$
$A+C=0$ ………………………….(1)
$B+D=1$ ………………………….(2)
$9A+4C=0$ ……………………….(3)
$9B+4D=6$ ……………………….(4)

From Eqn (2), $B = 1 - D$

Substitute $B = 1 - D$ in Eqn (4)
$$9(1-D)+4D=6$$
$$9-9D+4D=6$$
$$3=5D, \qquad D=\frac{3}{5.}$$
$$B+D=1$$
$$B+\frac{3}{5}=1$$
$$B=\frac{2}{5}$$
From Eqn (1); $A = -C$

Substitute in Eqn (3);
$$9(-C)+4C=0$$
$$-5C=0$$
$$C=0$$
$$A=0$$
$$\frac{x^2+6}{(x^2+4)(x^2+9)}=\frac{2}{5(x^2+4)}+\frac{3}{5(x^2+9)}$$
$$\int_0^1\frac{x^2+6}{(x^2+4)(x^2+9)}dx$$
$$=\frac{2}{5}\int_0^1\frac{1}{(x^2+4)}dx+\frac{3}{5}\int_0^1\frac{1}{(x^2+9)}dx$$
$$=\frac{2}{5}\left(\frac{1}{2}\tan^{-1}\frac{1}{2}x\right)\Big|_0^1+\frac{3}{5}\left(\frac{1}{3}\tan^{-1}\frac{1}{3}x\right)\Big|_0^1$$
$$=\frac{1}{5}\left(\tan^{-1}\frac{1}{2}-0\right)+\frac{1}{5}\left(\tan^{-1}\frac{1}{3}-0\right)$$
$$=\frac{1}{5}\left(\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}\right)$$
Let $\tan^{-1}\frac{1}{2}=A$, $\tan^{-1}\frac{1}{3}=B$
$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$
$$\tan(A+B)=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}$$

$$A+B=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{5/6}\right)$$

$$A+B=\tan^{-1}\left(\frac{\frac{3+2}{6}}{5/6}\right)$$

$$A+B=\tan^{-1}(1)$$

$$\tan^{-1}\tfrac{1}{2}+\tan^{-1}\tfrac{1}{3}=\frac{\pi}{4}$$

$$\Rightarrow \frac{1}{5}\left(\tan^{-1}(\tfrac{1}{2})+\tan^{-1}(\tfrac{1}{3})\right)$$

$$=\frac{1}{5}\left(\frac{\pi}{4}\right)=\frac{\pi}{20}$$

$$=\frac{1}{5}\left(\frac{\pi}{4}\right)$$

$$=\frac{\pi}{20}.$$

Example

Express $\dfrac{3x^2+x+1}{(x-2)(x+1)^3}$ into partial fractions. Hence evaluate $\int_3^4 \frac{3x^2+x+1}{(x-2)(x+1)^3}dx$.

Consider $\dfrac{3x^2+x+1}{(x-2)(x+1)^3}=\dfrac{A}{x-2}+\dfrac{B}{x+1}+\dfrac{C}{(x+1)^2}+\dfrac{D}{(x+1)^3}$

$$\Rightarrow A(x+1)^3+B(x+1)^2(x-2)+C(x+1)(x-2)+D(x-2)=3x^2+x+1$$

If x = -1, $D(-3)=3-1+1$

$$D=-1$$

If x = 2

$$27A=12+2+1$$

$$27A=15$$

$$A=\frac{15}{27}=\frac{5}{9}$$

If x = 0, $A-2B-2C-2D=1$

$$\Rightarrow \frac{5}{9}-2B-2C+2=1$$

$$2B+2C=+1+\frac{5}{9}$$

$$2B+2C=\frac{14}{9}$$

$$B+C=\frac{7}{9}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

If x = 1, $8A$ +B(4)(-1) + C(2)(-1) – D = 3 + 1 + 1

$$8A-4B-2C-D=5$$

$$\frac{40}{9}-4B-2C+1=5$$

$$-4B-2C=5-1-\frac{40}{9}$$

$$-4B-2C=4-\frac{40}{9}$$

$$-4B-2C=\frac{-4}{9}$$

$$2B+C=\frac{2}{9}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2)$$

Eqn (2) – Eqn (1)

$$\Rightarrow B=\frac{-5}{9}$$

$$\frac{-5}{9}+C=\frac{7}{9}$$

$$C=\frac{12}{9}$$

$$C=\frac{4}{3}$$

$$\Rightarrow \frac{3x^2+x+1}{(x-2)(x+1)^3}$$

$$=\frac{5}{9(x-2)}+\frac{-5}{9(x+1)}+\frac{4}{3(x+1)^2}-\frac{1}{(x+1)^3}$$

$$\Rightarrow \int_3^4 \frac{3x^3+x+1}{(x-2)(x+1)^3}$$

$$=\frac{5}{9}\int_3^4\frac{1}{x-2}dx-\frac{5}{9}\int_3^4\frac{1}{x+1}dx+\frac{4}{3}\int_3^4\frac{1}{(x+1)^2}-\int_3^4\frac{1}{(x+1)^3}$$

$$=\frac{5}{9}\ln(x-2)\Big|_3^4-\frac{5}{9}\ln(x+1)\Big|_3^4+\frac{-4}{3(x+1)}\Big|_3^4+\frac{1}{2(x+1)^2}\Big|_3^4$$

$$=0.317.$$

Example **(UNEB Question)**

$$\int\frac{x^4-x^3+x^2+1}{x^3+x}dx$$

Solution

Consider $\frac{x^4-x^3+x^2+1}{x^3+x}$

$$\Rightarrow \frac{x^4-x^3+x^2+1}{x^3+x}=(x-1)+\frac{x+1}{x(x^2+1)}$$

But $\dfrac{x+1}{x(x^2+1)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+1}$

$$A(x^2+1)+(Bx+C)x=x+1.$$

If $x = 0$, $A = 1$

$Ax^2 + A + Bx^2 + Cx = x + 1$

$A + B = 0$

$\Rightarrow 1 + B = 0$

$B = -1$

$C = 1$

$$\frac{x+1}{x(x^2+1)} = \frac{1}{x} + \frac{-x+1}{x^2+1}$$

$$\frac{x^4 - x^3 + x^2 + 1}{x^3 + x} = (x-1) + \frac{1}{x} + \frac{-x+1}{x^2+1}$$

$$\int \frac{x^4 - x^3 + x^2 + 1}{x^3 + x} dx$$

$$= \int (x-1)dx + \int \frac{1}{x} dx + \int \frac{1-x}{x^2+1} dx.$$

$$\frac{x^2}{2} - x + \ln x + \int \frac{1}{x^2+1} dx - \int \frac{x}{x^2+1} dx$$

$$\frac{x^2}{2} - x + \ln(x) + \tan^{-1} x - \frac{1}{2}\ln(x^2+1) + C$$

Example II

$$\int_2^3 \frac{3x^2 + 4x - 1}{x^3 + 2x^2 + x} dx$$

Solution

$$\frac{3x^2 + 4x - 1}{x^3 + 2x^2 + x} = \frac{3x^2 + 4x - 1}{x(x^2 + 2x + 1)}$$

$$= \frac{3x^2 + 4x + 1}{x(x+1)^2}$$

$$\frac{3x^2 + 4x - 1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$A(x+1)^2 + Bx(x+1) + Cx = 3x^2 + 4x - 1$$

If $x = 0$, $A = -1$

If $x = -1$, $-C = 3 - 4 - 1$

$C = 2$

If $x = 1$, $4A + 2B + C = 8$

$-4 + 2B + 2 = 8$

$2B = 10$

$B = 5$

$$\int_1^3 \frac{3x^2+4x-1}{x^3+2x^2+x} dx = \int_1^3 \frac{-1}{x} dx + \int_1^3 \frac{5}{x+1} + \int_1^3 \frac{2}{(x+1)^2} dx$$

$$-\ln x\Big|_1^3 + 5\ln(x+1)\Big|_1^3 - \frac{2}{x+1}\Big|_1^3$$

$$= \frac{1}{6} + 9(\ln 2) - 5(\ln 3)$$

Example (UNEB Question)

$$\int_1^3 \frac{x^2+1}{x^3 + 4x^2 + 3x} dx.$$

Solution

$$\frac{x^2+1}{x^3 + 4x^2 + 3x} = \frac{x^2+1}{x(x^2 + 4x + 3)} = \frac{x^2+1}{x(x+1)(x+3)}$$

$$\frac{x^2+1}{x(x+1)(x+3)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+3.}$$

$$A(x+3)(x+1) + Bx(x+3) + Cx(x+1) = x^2 + 1$$

If $x = -1$, $-2B = 2$

$B = -1$

If $x = -3$, $6c = 10$

$C = \frac{5}{3}$

If $x = 0$, $3A = 1$

$A = \frac{1}{3.}$

$$\frac{x^2+1}{x(x+1)(x+3)} = \frac{1}{3(x)} - \frac{1}{x+1} + \frac{5}{3(x+3)}$$

$$\int_2^3 \frac{x^2+1}{x(x+1)(x+3)} dx$$

$$= \frac{1}{3}\ln x\Big|_1^3 - \ln\left(x+1\right)\Big|_1^3 - \frac{5}{3}\ln\left(x+3\right)\Big|_1^3$$

$$\frac{1}{3}\ln 3 + \ln 2 + \frac{5}{3}\ln\frac{2}{3}$$

$$\frac{1}{3}\ln 3 - \frac{1}{3}\ln 1 - (\ln 4 - \ln 2) - \frac{5}{2}(\ln 6 - \ln 4)$$

$$\frac{1}{3}\ln 3 - \frac{1}{3}\ln 2 - \frac{5}{2}\left(\tfrac{3}{2}\right)$$

DIFFERENTIATION II

Differentiation is a process of finding derivatives

The derivative is the instantaneous rate of change of a function with respect to one of its variables

Objectives of the topic:

- To know the derivatives of exponential functions of any base.
- To know the derivatives of logarithmic functions.
- Use the techniques of logarithmic differentiation to find derivatives of functions involving products and quotients.

Differentiation of exponential functions

1. Differentiate the following

$a)\ 4e^x$ $b)\ e^{-2x}$ $c)e^{ax^2+b}$

$d)\ e^{\sqrt{cosx}}$ $e)\ e^{xe^x}$ $f)\ e^{\tan x^2}$

$g)\ e^{\sqrt{x^2}+1}$ $h)\ e^{-cotx}$

Solution

Note: $\frac{d}{dx}\left(e^{ax}\right)=ae^{ax}$

When we are differentiating an exponential function, we first differentiate the power of the expression multiplied by the same expression.

$a)\ \frac{d}{dx}(4e^x) = 4e^x$

$b)\ \frac{d}{dx}(e^{-2x}) = -2e^{-2x}$

c) $\frac{d}{dx}\left(e^{ax^2+b}\right) = 2axe^{ax^2+b}$

Alternatively;

$y= e^{ax^2}$ using chain rule

Let $ax^2 = u$

$y = e^u$

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

d) $\frac{d}{dx}\left(e^{\sqrt{\cos x}}\right)$

$= \frac{1}{2}(\cos x\ )^{\ -\frac{1}{2}}(-\sin x)(e^{\sqrt{cosx}})$

$= \frac{-sinx}{2\sqrt{\cos x}}e^{\sqrt{cosx}}$

e) $\frac{d}{dx}(e^{xe^x}) = (xe^x + e^x)e^{xe^x}$

$= e^x[x+1]e^{xe^x}$

$=(x+1)e^{x(1+e^x)} = (x+1)e^{x(1+e^x)}$

$e^u \times\ 2ax$

$2axe^{ax^2}$

Example II

Differentiate the following:

a) $e^{\tan x^2}$

b) $e^{\sqrt{x^2}+1}$

c) e^{-cotx}

d) $e^{\tan(x^2+4x+1)}$

Solution

a) $y = e^{\tan x^2}$

$\frac{dy}{dx}=2x\sec^2 x^2e^{(\tan x^2)}$

b) $y=e^{\sqrt{x^2}+1}$

$\frac{dy}{dx}=\frac{1}{2}(x^2+1)^{-1/2}e^{\sqrt{x^2+1}}\times 2x$

$\frac{dy}{dx}=\frac{x}{\sqrt{x^2+1}}e^{\sqrt{x^2+1}}$

c) $y = e^{-cotx}$

$\frac{dy}{dx}=\text{cosec}^2 x e^{-cotx}$

d) $y=e^{\tan(x^2+4x+1)}$

$\frac{dy}{dx}=(2x+4)e^{\tan(x^2+4x+1)}\sec^2(x^2+4x+1)$

Differentiation of logarithmic functions

Note i) $\frac{d}{dx}(\ln ax) = \frac{a}{ax} = \frac{1}{x}$

ii) $\frac{d}{dx}[\ln(x-1)] = \frac{1}{x-1}$

Example III

Differentiate the following

a) $\ln(2x^3)$ $b)\ \ln(x^3+1)$ $c) \ln\sec x$

d) $\ln\left(\frac{1+cosx}{1-sinx}\right)$ $e)\frac{\ln x}{\sqrt{1+x^2}}$ $f)3x\ln x^2$

Solution

$a)\ y = \ln(2x^3)$

$$\frac{dy}{dx} = \frac{6x^2}{2x^3} = \frac{3}{x}$$

$b)$ $\ln(x^3 + 1)$

$$\frac{dy}{dx} = \frac{3x^2}{x^3 + 1}$$

c) $y = \ln secx$

$$\frac{dy}{dx} = \frac{secxtanx}{secx} = \tan x$$

$d)$ $y = \ln\frac{(1+cosx)}{(1-sinx)}$

$$= \ln(1 + cosx) - \ln(1 - sinx)$$

$$\frac{dy}{dx} = \frac{-sinx}{(1 + cosx)} + \frac{cosx}{(1 - sinx)}$$

$$\frac{(1 - sinx)(-sinx) + cosx(1 + cosx)}{(1 + cosx)(1 - sinx)}$$

$$\frac{-sinx + sin^2x + cosx + cos^2x}{(1 + cosx)(1 - sinx)}$$

$$\frac{dy}{dx} = \frac{cosx - sinx + 1}{(1 + cosx)(1 - sinx)}$$

e) $\ln\frac{x}{\sqrt{1 + x^2}}$

$$y = \ln x - \ln\sqrt{(1 + x^2)}$$

$$y = \text{lnx} - \ln(1 + x^2)^{\frac{1}{2}}$$

$$\text{lnx} - \frac{1}{2}\ln(1 + x^2)$$

$$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{2}\left(\frac{2x}{1 + x^2}\right)$$

$$\frac{dy}{dx} = \frac{1}{x} - \frac{x}{1 + x^2}$$

f) y =3x $\ln x^2$ (*use product rule*)

$$\frac{dy}{dx} = 3x.\frac{2x}{x^2} + \ln(x^2).3$$

$$\frac{\text{dy}}{\text{dx}} = 3x.\frac{2}{x} + 3\ln x^2$$

$$\frac{dy}{dx} = 6 + 3\ln x^2$$

Example I

Differentiate the following:

a) $\ln\cos x$ b) $\ln(\sec x + \tan x)$

c) $\ln\frac{(x + 1)^2}{\sqrt{x - 1}}$ $c)$ $\frac{dy}{dx}(\ln x\sqrt{x^2 - 1})$

Solution

a) $y = \ln(\cos x)$

$$\frac{dy}{dx} = \frac{-sinx}{cosx} = -tanx$$

b) $\ln(\sec x + \tan x)$

$$\frac{dy}{dx} = \frac{secxtanx + sec^2x}{secx + tanx}$$

$$\frac{dy}{dx} = \frac{secx(tanx + secx)}{(tanx + secx)}$$

$$\frac{dy}{dx} = secx$$

$$c)\ \ln\frac{(x + 1)^2}{\sqrt{x - 1}} = \ln(x + 1)^2 - \ln\sqrt{(x - 1)}$$

$$\ln(x + 1)^2 - \ln(x - 1)^{\frac{1}{2}}$$

$$y = 2\ln(x+1) - \frac{1}{2}(x-1)$$

$$\frac{dy}{dx} = \frac{2}{x+1} - \frac{1}{2}\left(\frac{1}{x-1}\right)$$

$$\frac{dy}{dx} = \frac{2(x-1)-(x+1)}{2(x^2-1)}$$

$$= \frac{x-3}{2(x^2-1)}$$

Examples

Differentiate the following:

(**a**) $\ln sin^2x$ (**b**) $\ln\tan(3x)$

(**c**) $\ln 3cos^2x$ (**d**) $\ln\left(\frac{(x + 1)^2}{x - 1}\right)$ **e**) $\ln(x + \sqrt{(x^2 - 1)}$

f $\sqrt[3]{\frac{x + 1}{x - 1}}$ (**g**) $\frac{e^{x^2}\sqrt{sinx}}{(2x + 1)^3}$

Solution

(a) ***y* = ln sin²x**

$$\frac{dy}{dx} = \frac{2sinxcosx}{sin^2x} = 2\cot x$$

(b) ***y* = ln tan(3*x*)**

$$c)\ \frac{dy}{dx} = \frac{3sec^2 3x}{tan3x}$$

(c) ***y* = ln 3cos²x**

$d)\ \frac{dy}{dx} = \frac{6cosx(-sinx)}{3cos^2x} = \frac{-6sinx}{3cosx}$
$= -2\tan x$

f) $\ln(x+1)^2 - \ln(x-1)$

$y = \ln(x+1)^2 - \ln(x-1)$
$y = 2\ln(x+1) - \ln(x-1)$
$\frac{dy}{dx} = \frac{2}{(x+1)} - \frac{1}{(x-1)}$
$\frac{dy}{dx} = \frac{2(x-1)-(x+1)}{(x+1)(x-1)} = \frac{2x-2-x-1}{(x^2-1)}$
$\frac{dy}{dx} = \frac{x-3}{x^2-1}$

g) $\ln(x + \sqrt{(x^2-1)}$

$\frac{dy}{dx} = 1 + \frac{\frac{1}{2}(x^2-1)^{\frac{-1}{2}} \times 2x}{\sqrt{x^2-1}}$

$\frac{dy}{dx} = \frac{1 + \frac{1}{2}(x^2-1)^{\frac{-1}{2}} \times 2x}{x + \sqrt{x^2-1}}$

$= \frac{1 + \left(\frac{x}{\sqrt{x^2-1}}\right)}{x + \sqrt{(x^2+1)}}$

$= \frac{\left(\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right)}{x + (x^2+1)}$

$\frac{dy}{dx} = \frac{\sqrt{x^2-1}+x}{\sqrt{(x^2-1)}(x+\sqrt{x^2-1})}$

i) $\sqrt[3]{\frac{x+1}{x-1}}$

let $y = \sqrt[3]{\frac{x+1}{x-1}}$

$\text{lny} = \ln\left(\frac{(x+1)^{\frac{1}{3}}}{(x-1)^{\frac{1}{3}}}\right)$

$\text{lny} = \ln(x+1)^{\frac{1}{3}} - \ln(x-1)^{\frac{1}{3}}$

$\text{lny} = \frac{1}{3}\ln(x+1) - \frac{1}{3}\ln(x-1)$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{3}\left(\frac{1}{x+1}\right) - \frac{1}{3}\left(\frac{1}{x-1}\right)$

$\frac{dy}{dx} = y\left[\frac{1}{3(x+1)} - \frac{1}{3(x-1)}\right]$

$\frac{dy}{dx} = \sqrt[3]{\frac{x+1}{x-1}}\left[\frac{1}{3(x+1)} - \frac{1}{3(x-1)}\right]$

h) $y = \frac{e^{x^2}\sqrt{sinx}}{(2x+1)^3}$

$\text{lny} = \text{ln}e^{x^2} + \ln(\text{sinx})^{\frac{1}{2}} - \ln(2x+1)^3$

$\text{lny} = x^2 + \frac{1}{2}\text{lnsinx} - 3\ln(2x+1)$

$\frac{1}{y}\frac{dy}{dx} = 2x + \frac{1cosx}{2sinx} - 3\left(\frac{2}{2x+1}\right)$

$\frac{1}{y}\frac{dy}{dx} = 2x + \frac{1}{2}cotx - \frac{6}{2x+1}$

$\frac{\text{dy}}{\text{dx}} = y\left[2x + \frac{1}{2}cotx - \frac{6}{2x+1}\right]$

$\frac{dy}{dx} = \frac{e^{x^2}\sqrt{sinx}}{(2x+1)^3}\left[2x + \frac{1}{2}cotx - \frac{6}{2x+1}\right]$

Example

Differentiate $\log_{10} cos3x$

Solution

Let $y = \log_{10} cos3x$

$cos3x = 10^y$
$\text{lncos3x} = \ln 10^y$
$\text{lncos3x} = \text{yln10}$
$\frac{-3sin3x}{cos3x}dx = (\ln 10)\text{dy}$
$-3tan3xdx = (\ln 10)\text{dy}$
$\frac{dy}{dx} = \frac{-3tan3x}{\ln 10}$

Examples

Differentiate the following:

a) x^x b) $(sinx)^x$ c) 2^x d) $x10^{sinx}$ e) $\ln(x)^x$ f) $\frac{\ln x}{x^2}$ g) x^{sinx}

Solution

a) $y = x^x$

$\ln y = \ln x^x$
$\ln y = \text{xlnx}$
$\frac{1}{y}dy = [(x.\frac{1}{x} + \text{lnx} \times 1)]\ dx$
$\frac{dy}{dx} = y[1 + \ln x]$
$\frac{dy}{dx} = x^x(1 + \text{lnx})$

b) $y = (\sin x)^x$

$$\ln y = \ln(\sin x)^x$$
$$\ln y = x\ln\sin x$$
$$\frac{1}{y}dy = \left(x\frac{cosx}{sinx} + \ln\sin x\right)dx$$
$$\frac{1}{y}dy = (xcotx + \ln(\sin x))\mathrm{dx}$$
$$\frac{dy}{dx} = y(xcotx + \ln\sin x)$$
$$\frac{dy}{dx} = (\sin x)^x.(xcotx + \ln\sin x)$$

c) $y = 2^x$
$$\ln y = \ln 2^x$$
$$\ln y = x(\ln 2)$$
$$\frac{1}{y}dy = \ln 2\, dx$$
$$\frac{dy}{dx} = y\ln 2$$
$$\frac{dy}{dx} = (2^x)\ln 2$$

d) $x10^{sinx}$
$$y = x10^{sinx}$$
$$\ln y = \ln(x10^{sinx})$$
$$\ln y = \ln x + \ln 10^{sinx}$$
$$\ln y = \ln x + \sin x.\ln 10$$
$$\frac{1}{y}dy = \left(\frac{1}{x} + cosx(\ln 10)\right)dx$$
$$\frac{dy}{dx} = y\left(\frac{1}{x} + cosx(\ln 10)\right)$$
$$\frac{dy}{dx} = x10^{sinx}\left(\frac{1}{x} + cosx(\ln 10)\right)$$

e) $(\ln x)^3$

Let $y = (\ln x)^3$
$$\frac{dy}{dx} = 3(\ln x)^2\frac{1}{x}$$
$$\frac{dy}{dx} = \frac{3}{x}(\ln x)^2$$

f) $\frac{\ln x}{x^2}$
$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{x^2.\frac{1}{x} - (\ln x)2x}{x^4}$$
$$\frac{dy}{dx} = \frac{x - 2x(\ln x)}{x^4}$$

g) x^{sinx}
$$y = x^{sinx}$$
$$\ln y = \ln x^{sinx}$$
$$\ln y = \sin x(\ln x)$$
$$\frac{1}{y}dy = \left(\sin x.\frac{1}{x} + (\ln x)\cos x\right)dx$$
$$\frac{dy}{dx} = y\left(\sin x.\frac{1}{x} + (\ln x)\cos x\right)$$
$$\frac{dy}{dx} = x^{\sin x}\left(\sin x.\frac{1}{x} + (\ln x)\cos x\right)$$

Differentiation of inverse trigonometric functions

Example

$a)$ $\cos^{-1}x$ $b)$ $\sin^{-1}x$ $c)$ $\tan^{-1}x$

d) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\sqrt{1} - x^2$ $f)$ $\tan^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

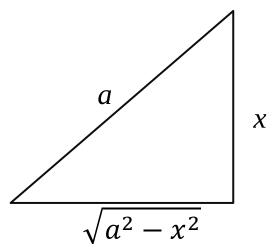
Solution

a) $y = \cos^{-1}x$
$$\cos y = x$$
$$-\sin y\, dy = dx$$
$$\frac{dy}{dx} = -\frac{1}{\sin y}$$
$$\sin^2 y + \cos^2 y = 1$$
$$\sin y = \sqrt{1-\cos^2 y}$$
$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

b) $y = \sin^{-1}x$
$$\sin y = x$$
$$\cos y\, dy = dx$$
$$\frac{dy}{dx} = \frac{1}{\cos y}$$
$$\cos y = \sqrt{1-\sin^2 y}$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

c) $y = \sin^{-1}\left(\frac{x}{a}\right)$
$$\sin y = \frac{x}{a}$$
$$\cos y\, dy = \frac{1}{a}dx$$
$$\frac{dy}{dx} = \frac{1}{a\cos y}$$

$$\frac{dy}{dx} = \frac{1}{a\sqrt{1-\sin^2 y}}$$



$$\frac{dy}{dx} = \frac{1}{a\left(\frac{\sqrt{a^2-x^2}}{a}\right)}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{a^2-x^2}}$$

e) $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$\cos y = \left(\frac{1-x^2}{1+x^2}\right)$$

$$-\text{sin}y\frac{dy}{dx} = \frac{(1+x^2)(-2x) - (1-x^2) \times 2x}{(1+x^2)^2}$$

$$\frac{dy}{dx} = \frac{-2x - 2x^3 - 2x + 2x^3}{-\sin y(1+x^2)^2}$$

$$\frac{dy}{dx} = \frac{-4x}{-siny(1+x^2)^2}$$

$$\frac{dy}{dx} = \frac{4x}{\sin y(1+x^2)^2}$$

$$\frac{dy}{dx} = \frac{4x}{\frac{2x}{(1+x^2)}(1+x^2)^2}$$

$$\frac{dy}{dx} = \frac{2}{(1+x^2)^2}$$

f) $\tan^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$y = tan^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$tany = \left(\frac{1-x^2}{1+x^2}\right)$$

$$sec^2y\frac{dy}{dx} = \frac{(1+x^2).-2x - (1-x^2).2x}{(1+x^2)^2}$$

$$sec^2y\frac{dy}{dx} = \frac{-2x - 2x^3 - 2x + 2x^3}{(1+x^2)^2}$$

$$sec^2y\frac{dy}{dx} = \frac{-4x}{(1+x^2)^2}$$

$$\frac{dy}{dx} = \frac{-4x}{sec^2y(1+x^2)^2}$$

$$\frac{dy}{dx} = \frac{-4x}{(1+tan^2y)(1+x^2)^2}$$

$$\frac{dy}{dx} = \frac{-4x}{\left(1+\left(\frac{1-x^2}{1+x^2}\right)^2\right)(1+x^2)^2}$$

$$\frac{dy}{dx} = \frac{-4x}{\left(\frac{(1+x^2)^2+(1-x^2)^2}{(1+x^2)^2}\right)(1+x^2)^2}$$

$$\frac{dy}{dx} = \frac{-4x}{(1+x^2)^2+(1-x^2)^2}$$

$$\frac{dy}{dx} = \frac{-4x}{1+2x^2+x^4+1-2x^2+x^4}$$

$$\frac{dy}{dx} = \frac{-4x}{2+2x^4}$$

$$\frac{dy}{dx} = \frac{-4x}{2(1+x^4)}$$

$$\frac{dy}{dx} = \frac{-2x}{(1+x^4)}$$

Differentiate the following

$a)\ \frac{e^{x/2}\sin x}{x^4} \qquad b)\frac{1}{xe^x\cos x}$

c) $\log 10\sin(9x^2+4x+3)$

d) if $\sin e^{xy} = x,$ show that;

$$\frac{dy}{dx} = \frac{x - \sqrt{1-x^2}(\ln\sin^{-1}x)\sin^{-1}x}{x^2(\sqrt{1-x^2})sin^{-1}x}$$

Solution

a) $y = \frac{e^{x/2}\sin x}{x^4}$

$$\ln y = ln\left(\frac{e^{x/2}\sin x}{x^4}\right)$$

$$lny = \ln e^{x/2} + \ln\sin x - \ln x^4$$

$$\ln y = \frac{x}{2} + \ln\,(\text{sin}x) - 4\ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2} + \frac{\cos x}{sinx} - 4.\frac{1}{x}$$

$$\frac{dy}{dx} = y\left(\frac{1}{2} + \cot x - \frac{4}{x}\right)$$

$$\frac{dy}{dx} = \frac{e^{x/2}\sin x}{x^4}\left(\frac{1}{2} + \cot x - \frac{4}{x}\right)$$

b) $y = \frac{1}{xe^x \cos x}$

$\ln y = \ln\left(\frac{1}{xe^x \cos x}\right)$

$\ln y = \ln 1 - (lnx + \ln e^x + \ln \cos x)$

$\ln y = \ln 1 - (lnx + x \ln e + \ln \cos x)$

$\frac{1}{y} dy = \left(0 - \left(\frac{1}{x} + 1 - \frac{\sin x}{\cos x}\right)\right) dx$

$\frac{dy}{dx} = y\left(-\frac{1}{x} - 1 + tanx\right)$

$\frac{dy}{dx} = \frac{1}{xe^x \cos x}\left(-\frac{1}{x} - 1 + tanx\right)$

c) $y = \log_{10} \sin(9x^2 + 4x + 3)$

$10^y = \sin(9x^2 + 4x + 3)$

$\sin(9x^2 + 4x + 3) = 10^y$

$\ln\sin(9x^2 + 4x + 3) = \ln 10^y$

$\ln\sin(9x^2 + 4x + 3) = y \ln 10$

$\frac{(18x + 4)(cos(9x^2 + 4x + 3))}{\sin(9x^2 + 4x + 3)} = (\ln 10)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{18x+4}{\ln 10}\cot(9x^2 + 4x + 3)$

d) $e^{xy} = \sin^{-1} x$……………………..(1)

$\left(x\frac{dy}{dx} + y\right)e^{xy} = \frac{1}{\sqrt{1 - x^2}}$

$\left(x\frac{dy}{dx} + y\right)\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$

$\left(x\frac{dy}{dx} + y\right) = \frac{1}{(\sin^{-1}x)\sqrt{1 - x^2}}$

$\left(x\frac{dy}{dx}\right) = \frac{1}{\sin^{-1} x\sqrt{1 - x^2}} - y \ldots\ldots (2)$

From Eqn (1); $\ln e^{xy} = \ln \sin^{-1} x$

$y = \frac{ln \sin^{-1} x}{x} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (3)$

Substituting Eqn (3) in (2)

$\left(x\frac{dy}{dx}\right) = \frac{1}{\sin^{-1} x\sqrt{1 - x^2}} - \frac{ln \sin^{-1} x}{x}$

$x\frac{dy}{dx} = \frac{x - \left(\sqrt{1 - x^2}\right)\sin^{-1} x \, ln(\sin^{-1} x)}{x \sin^{-1}(x\sqrt{1 - x^2})}$

$\frac{dy}{dx} = \frac{x - \left(\sqrt{1 - x^2}\right)\sin^{-1} x \, ln(\sin^{-1} x)}{x^2 \sin^{-1} x\sqrt{1 - x^2}}$ as required

Example (UNEB Questions)

Determine $\frac{d}{dx}\left\{\ln\left(\frac{x}{\sqrt{1+x^2}}\right)\right\}$, when $x = 2$

(05 *marks*)

Solution

Let $y = \ln\left(\frac{x}{\sqrt{1+x^2}}\right)$

$y = \ln x - \ln\left(1+x^2\right)^{1/2}$

$y = \ln x - \frac{1}{2}\ln\left(1+x^2\right)$

$\frac{dy}{dx} = \frac{d}{dx}\ln x - \frac{1}{2}\frac{d}{dx}\ln\left(1+x^2\right)$

$= \frac{1}{x} - \frac{x}{1+x^2}$

$= \frac{1+x^2-x^2}{x\left(1+x^2\right)}$

$= \frac{1}{x(1+x^2)}$

When $x = 2$

$\frac{d}{dx}\ln y = \frac{1}{2(5)} = \frac{1}{10}$

Example (UNEB Question)

Given that: $y = \sqrt{\frac{1+\sin x}{1-\sin x}}$, show that $\frac{dy}{dx} = \frac{1}{1-\sin x}$.

Solution

$y = \sqrt{\frac{1+\sin x}{1-\sin x}}$

$y^2 = \frac{1+\sin x}{1-\sin x}$

Introducing $\log_e$ on both sides,

$2\ln y = \ln(1 + \sin x) - \ln(1 - \sin x)$

$2\frac{1}{dx}(\ln y) = \frac{d}{dx}[\ln(1+\sin x) - \ln(1-\sin x)]$

$\frac{2}{y}\frac{dy}{dx} = \frac{\cos x}{1+\sin x} - \frac{-\cos x}{1-\sin x}$

$\frac{dy}{dx} = \frac{y}{2}\left[\frac{\cos x - \sin x\cos x + \cos x + \sin x\cos x}{(1+\sin x)(1-\sin x)}\right]$

$\frac{dy}{dx} = \frac{y}{2}\left[\frac{2\cos x}{1-\sin^2 x}\right]$

$= \frac{y}{2}\left[\frac{2\cos x}{\cos^2 x}\right] \qquad = \frac{y}{\cos x}$

Substitute for y,

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{1+\sin x}{1-\sin x}} \bullet \frac{1}{\cos x}$$

$$= \sqrt{\frac{(1+\sin x)(1-\sin x)}{(1-\sin x)(1-\sin x)}} \bullet \frac{1}{\cos x}$$

$$= \frac{\sqrt{1-\sin^2 x}}{1-\sin x} \bullet \frac{1}{\cos x}$$

$$= \frac{\sqrt{\cos^2 x}}{1-\sin x} \bullet \frac{1}{\cos x}$$

$$= \frac{\cos x}{1-\sin x} \bullet \frac{1}{\cos x} = \frac{1}{1-\sin x}$$

Hence $\frac{dy}{dx} = \frac{1}{1-\sin x}$ as required

Example (UNEB Question)

a) Differentiate the following with respect to x

i) $(\sin x)^x$

ii) $\frac{(x+1)^2}{(x+4)^3}$ Giving your answers in their simplest forms.

b) The distance of a particle moving in a straight line from a fixed point after time t is given by

$x = e^{-t}\sin t$.

Show that the particle is instantaneously at rest at time

$t = \frac{\pi}{4}$ seconds. Find its acceleration at $t = \frac{\pi}{4}$ seconds.

Solution

i) Let $y = (\sin x)^x$

Introducing $\log_e$ to both sides,

$\ln y = x\ln \sin x$

$$\frac{1}{y}\frac{dy}{dx} = x\frac{\cos x}{\sin x} + \ln\sin x$$

$$\frac{dy}{dx} = y[x\cot x + \ln\sin x]$$

$$= (\sin x)^x[x\cot x + \ln\sin x]$$

ii) $$y = \frac{(x+1)^2}{(x+4)^3}$$

Introducing $\log_e$ to both sides,

$$\ln y = 2\ln(x+1) - 3\ln(x+4)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x+1} - \frac{3}{x+3}$$

Example (UNEB Question)

Given that $y = e^{\tan x}$, show that

$$\frac{d^2y}{dx^2} - (2\tan x + \sec^2 x)\frac{dy}{dx} = 0.$$

Solution

Given $x = e^{-t}\sin t$.

$$V = \frac{dx}{dt} = e^{-t}\cos t - e^{-t}\sin t$$

for instantaneous rest, $\frac{dx}{dt} = 0$

=> $e^{-t}(\cos t - \sin t) = 0$

$\cos t - \sin t = 0$

$\cos t = \sin t$

$\tan t = 1$

$t = \tan^{-1}(1)$

$t = \frac{\pi}{4}$ seconds

Acceleration = $\frac{dv}{dt}$

$$= \frac{d}{dt}\left(e^{-t}\cos t - e^{-t}\sin t\right)$$

$$= e^{-t}\sin t - e^{-t}\cos t - e^{-t}\cos t - e^{-t}\sin t$$

$$= -2e^{-t}\cos t$$

When $t = \frac{\pi}{4}$,

$$\frac{dv}{dt} = -2e^{\frac{-\pi}{4}}\cos\frac{\pi}{4}$$

$$= -\sqrt{2}e^{\frac{-\pi}{4}}$$

$$= -0.6447$$

Example (UNEB Question)

a) i) If $x^2\sec x - xy + 2y^2 = 15$, find $\frac{dy}{dx}$.

ii) Given that $y = \theta - \cos\theta$; $x = \sin\theta$; show that

$$\frac{d^2y}{dx^2} = \frac{1+\sin\theta}{\cos^3\theta}.$$

b) Determine the maximum and minimum values of x^2e^{-x}

Solution

a) $x^2\sec x - xy + 2y^2 = 15$

$$\frac{d}{dx}\left(x^2\sec x\right) - \frac{d}{dx}(xy) + \frac{d}{dx}\left(2y^2\right) = \frac{d}{dx}(15)$$

$$x^2\sec x\tan x + 2x\sec x - \left(x\frac{dy}{dx} + y\right) + 4y\frac{dy}{dx} = 0$$

$$x^2 \sec x \tan x + 2x \sec x + (4y - x)\frac{dy}{dx} - y = 0$$

$$(4y - x)\frac{dy}{dx} = y - x^2 \sec x \tan x - 2x \sec x$$

$$\frac{dy}{dx} = \frac{y - x^2 \sec x \tan x - 2x \sec x}{4y - x}$$

ii) $y = \theta - \cos\theta$ and $x = \sin\theta$

$$\frac{dy}{d\theta} = 1 + \sin\theta$$

$$\frac{dx}{d\theta} = \cos\theta$$

Now $\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$ (by the chain rule)

$$= (1 + \sin\theta) \cdot \frac{1}{\cos\theta} = \frac{1 + \sin\theta}{\cos\theta}$$

Again by using the chain rule,

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta} \cdot \frac{dy}{dx} \cdot \frac{d\theta}{dx}$$

$$= \frac{\cos\theta \cdot \cos\theta - (1 + \sin\theta)(-\sin\theta)}{\cos^2\theta} \cdot \frac{1}{\cos\theta}$$

$$= \frac{\cos^2\theta + \sin\theta + \sin^2\theta)}{\cos^3\theta}$$

$$= \frac{1 + \sin\theta}{\cos^3\theta}$$

b) Let $y = x^2e^{-x}$

By introducing $\log_e$ on both sides

$\ln y = \ln (x^2e^{-x})$

$\ln y = \ln x^2 + \ln e^{-x}$

$\ln y = 2 \ln x - x$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} - 1$$

$$\frac{dy}{dx} = y\left(\frac{2}{x} - 1\right)$$

$$\frac{dy}{dx} = x^2e^{-x}\left(\frac{2}{x} - 1\right)$$

$$= 2xe^{-x} - x^2e^{-x}$$

For maximum or minimum values of y;

$$\frac{dy}{dx} = 0$$

$\Rightarrow \quad 2xe^{-x} - x^2e^{-x} = 0$

$$= x^2e^{-x}(2 - x) = 0$$

Either $x^2e^{-x} = 0$

$x = 0$

Or $\quad 2 - x = 0$

$x = 2$

When $x = 0$, $\Rightarrow \quad y = 0$

The turning point is (0, 0)

When $x = 2 \Rightarrow y = 4e^{-2} = 0.5413$ (4 dps)

The turning point is (2, 0.5413)

Finding the nature of the turning points

$$\frac{dy}{dx} = 2xe^{-x} - x^2e^{-x}$$

$$\frac{d^2y}{dx^2} = \left(2x \cdot -e^{-x} + e^{-x} \cdot 2\right) - 2xe^{-x} - x^2e^{-x}$$

$$= -2xe^{-x} + 2e^{-x} - 2xe^{-x} + x^2e^{-x}$$

$$= x^2e^{-x} - 4xe^{-x} + 2e^{-x}$$

At $x = 0$, $\frac{d^2y}{dx^2} = 2$ (*positive*)

Hence the turning point at (0, 0) is a minimum.

Therefore the minimum value of x^2e^{-x} is 0

At $x = 2$, $\frac{d^2y}{dx^2} = 4e^{-2} - 8e^{-2} + 2e^{-2}$

$= -2e^{-2}$ (*negative*)

Hence the turning point at (2, 0.5413) is a maximum

Therefore the maximum value of x^2e^{-x} is 0.5413

MACLAURIN'S EXPANSION

Maclaurin's theorem states that:

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \frac{x^4 f^{IV}(0)}{4!} + \ldots$$

Example I

1) Use Maclaurin's theorem to expand $\ln(1 + x)$ in ascending powers of x as far as the term x^5

$$f(x) = \ln(1 + x)$$

$$f(0) = \ln(1) = 0$$

$$f'(x) = \frac{1}{(1 + x)} = (1 + x)^{-1}$$

$$f'(0) = 1$$

$$f''(x) = -1(1 + x)^{-2}.1 = \frac{-1}{(1 + x)^2}$$

$$f''(0) = \frac{-1}{1} = -1$$

$$f'''(x) = 2(1 + x)^{-3}.1 = \frac{2}{(1 + x)^3}$$

$$f'''(0) = 2$$

$$f^{IV}(x) = -6(1 + x)^{-4} = \frac{-6}{(1 + x)^4}$$

$$f^{IV}(0) = \frac{-6}{(1 + 0)^4} = -6$$

$$f^{V}(x) = 24(1+x)^{-5} = \frac{24}{(1+x)^5}$$

$f^{V}(0) = 24$

$$f(0) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \frac{x^4 f^{IV}(0)}{4!} + \ldots$$

$$\ln(1+x) = 0 + x(1) + \frac{x^2(-1)}{2} + x^3\frac{(2)}{6} + \frac{x^4(-6)}{24} + \frac{x^5(24)}{120} + \ldots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{6x^4}{24} + \frac{1}{5}x^5 + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{1}{3}x^3 + \frac{x^4}{4} + \frac{1}{5}x^5 + \cdots$$

2) Use Mauclaurin's theorem to expand sec x in ascending powers of x as far as the term ln x^3

Solution

$f(x) = \sec x$

$f(0) = 1$

$f'(x) = \sec x \tan x$

$f'(0) = 0$

$f''(x) = \sec x\,(sec^2 x) + \tan x(\sec x \tan x)$

$f''(x) = sec^3 x + tan^2 x\ secx$

$f''(0) = \sec 0 + 0 = 1$

$$f'''(x) = [3sec^2 x(\sec x \tan x) + tan^2 x\ (\sec x \tan x) + \sec x.(2\tan x\, sec^2 x)]$$

$f'''(0) = 0$

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \cdots$$

$$f(x) = 1 + 0 + \frac{x^2(1)}{2} + \cdots$$

$$\sec x = 1 + \frac{1}{2}x^2 + \cdots$$

Example III

3(a) Find the first three terms of the expansion of $\frac{1}{1+x}$ using Maclaurin's theorem.

(b) Use Maclaurin's theorem to expand $\tan x$ in ascending powers of x up to the term in x^3

Solution

a) $f(x) = \frac{1}{1+x}$

$f(0) = 1$

$$f'(x) = \frac{-1}{(1+x)^2}$$

$f'(0) = -1$

$$f''(x) = \frac{2}{(1+x)^3}$$

$f''(0) = 2$

$f'''(x) = -6(1+x)^{-4}$

$$f'''(x) = \frac{-6}{(1+x)^4}$$

$f'''(0) = -6$

$$f(x) = 1 + x(-1) + \frac{x^2(2)}{2!} + \frac{-6x^3}{3!} + \ldots$$

$$f(x) = 1 - x + x^2 + -x^3 + \cdots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

b) $f(x) = \tan x$

$f(0) = 0$

$f'(x) = \sec^2 x$

$f'(0) = sec^2 0 = 1$

$f''(x) = 2\sec x[\sec x \tan x)$

$f''(x) = 2sec^2 x\ tanx$

$f''(0) = 0$

$f'''(x) = 2sec^2 x(sec^2 x) + \tan x(2\sec x(\sec x \tan x)$

$f'''(0) = 2(1+0) = 2$

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \cdots$$

$$\tan x = 0 + x(1) + \frac{x^2(0)}{2} + \frac{x^3(2)}{6} + \cdots$$

$$\tan x = x + \frac{x^3}{3} + \cdots$$

Use Maclaurin's theorem to expand $\ln(1+ax)$, where a is a constant hence or otherwise expand $\ln\frac{(1+x)}{\sqrt{(1-2x)}}$ up to the term ln x^3

Solution

$f(x) = \ln(1+ax)$

$f(0) = \ln(1+0) = 0$

$$f'(x) = \frac{a}{(1+ax)}$$

$f'(0) = a$

$f''(x) = a(1+ax)^{-1}$

$f''(x) = -a(1+ax)^{-2}.a$

$$f''(x) = \frac{-a^2}{(1+ax)^2}$$

$f''0 = -a^2$

$$f'''(x) = 2a^2(1+ax)^{-3}.a = \frac{2a^3}{(1+ax)^3}$$

$f'''(0) = 2a^3$

$$\ln(1+ax) = ax - \frac{a^2x^2}{2!} + \frac{2a^3x^3}{3!} + \cdots$$

$$\ln(1+ax) = ax - \frac{a^2x^2}{2} - \frac{a^3x^3}{3} + \ldots$$

$$\ln\left(\frac{(1+x)}{\sqrt{1-2x}}\right) = \ln(1+x) - \ln\sqrt{(1-2x)}$$

$$= \ln(1+x) - \ln(1-2x)^{1/2}$$

$$= \ln(1+x) - \frac{1}{2}\ln(1-2x)$$

$$\ln(1+ax) = ax - \frac{a^2x^2}{2} - \frac{a^3x^3}{3} + \ldots$$

Comparing $\ln(1 + x)$ with $\ln(1 + ax)$;

$\Rightarrow a = 1$

$$\Rightarrow \ln(1+x) = x - \frac{x^2}{2} - \frac{x^3}{3} + \ldots$$

Comparing $\ln(1 - 2x)$ with $\ln(1 + ax)$;

$\Rightarrow a = -2$

$$\ln(1 - 2x) = -2x - 2x^2 - \frac{8x^3}{3}$$

$$\Rightarrow \ln(1+x) - \frac{1}{2}\ln(1-2x)$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right) - \frac{1}{2}\left(-2x - 2x^2 - \frac{8x^3}{3} + \ldots\right)$$

$$= 2x + \frac{1}{2}x^2 + \frac{5}{2}x^3 + \ldots$$

$$\ln\left(\frac{(1+x)}{\sqrt{1-2x}}\right) = 2x + \frac{1}{2}x^2 + \frac{5}{2}x^3 + \ldots$$

Example IV

Use Maclaurin's theorem to show that $e^{-x}\sin x$ up to the term in x^3 is $\frac{x}{3}(x^2 - 3x + 3)$.

Hence evaluate $e^{\pi/3}\sin\frac{\pi}{3}$.

Solution

$f(x) = e^{-x}sinx$

$f(0) = e^{-0}\sin 0 = 0$

$f'(x) = e^{-x}\cos x + \sin x(-e^{-x})$

$f'(0) = 1$

$f'' = e^{-x}(-\sin x) + \cos x\,(-e^{-x}) - (e^{-x}\cos x - e^{-x}\sin x)$

$f'' = -2e^{-x}\cos x$

$f''(0) = -2$

$f'''(x) = -2(-e^{-x}\,sinx - e^{x}\cos x)$

$f'''(0) = 2$

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \frac{x^4 f^{IV}(0)}{4!} + \ldots$$

$$e^{-x}\,sinx = \frac{3x - 3x^2 + x^3}{3}$$

$$e^{-x}\,sinx = \frac{x}{3}(3 - 3x + x^2)$$

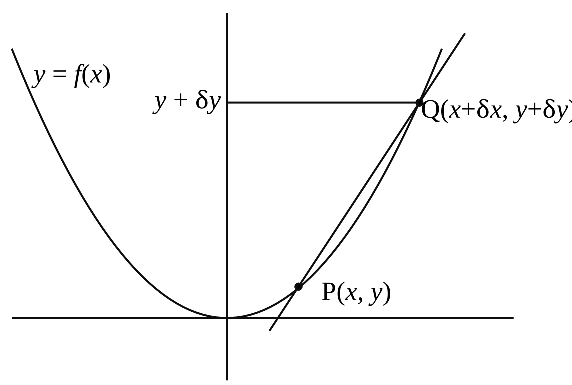
$$e^{-x}\sin x = \frac{x}{3}(x^2 - 3x + 3)$$

$$e^{\pi/3}\sin\frac{\pi}{3} = \frac{\pi}{9}\left(\frac{\pi^2}{9} - \pi + 3\right)$$

$$= \frac{\pi^3}{81} - \frac{\pi^2}{9} + 3 \times \frac{\pi}{9}$$

Differentiation from first principle

Suppose we have a smooth function $f(x)$ which is represented graphically by a curve $y = f(x)$ then we can draw the tangent to the curve at any point P. It is important to be able to calculate the slope of the tangent of the curve a graphical method can be used but this is rather imprecise so we use the following analytical method. We choose a second point Q on the curve which is near P and join the two points with a straight line PQ called a secant and calculate the slope of the line. Then, we allow Q to approach P so that the secant swings around until it just touches the curve and becomes a tangent. The limit of the slope of a secant is required to find the slope of a tangent.



The Greek letter δ(delta) is used to denote small change (very small change).

In the diagram above figure P(x, y) and Q(x+δx, y+δy) are two points on the curve $y = f(x)$. If the increase in x in moving from P to Q is δx then the corresponding increase in y is δy. The coordinates of Q are (x+ δx), $(y + \delta y)$. The gradient of the chord

$$\text{PQ} = \frac{\delta y}{\delta x}$$

As Q approaches P along the curve $(\delta x) \longrightarrow 0$ then δx becomes zero, PQ coincides with the tangent PT.

Hence, the gradient of the curve at P is the limiting value of $\frac{\delta y}{\delta x}$

The limiting value of $\frac{\delta y}{\delta x}$ as $\delta x \longrightarrow 0$, which is written as $\frac{dy}{dx}$ and is called derived function. It is not a fraction but a symbol meaning derivative of y with respect to x.

Example 1

Differentiate $y = x^2$ from the first principles.

$$y + \delta y = (x + \delta x)^2$$
$$y + \delta y = x^2 + 2x(\delta x) + (\delta x)^2$$
$$y + \delta y = x^2 + 2x(\delta x) + (\delta x)^2$$
$$\delta y = x^2 + 2x(\delta x) + (\delta x)^2 - y$$
$$\delta y = x^2 + 2x(\delta x) + (\delta x)^2 - x^2$$
$$\delta y = 2x(\delta x) + (\delta x)^2$$

Dividing through by δx

$$\frac{\delta y}{\delta x} = 2x + \delta x$$

As $\delta x \longrightarrow 0$

$$\frac{\delta y}{\delta x} \longrightarrow \frac{dy}{dx}$$

Hence $\frac{dy}{dx} = 2x$

Example 2

Differentiate: $y = \frac{4}{3\sqrt{x}}$ from the first principles

Solution

$$y + \delta y = \frac{4}{3\sqrt{x + \delta x}}$$
$$\delta y = \frac{4}{3\sqrt{x + \delta x}} - y$$
$$\delta y = \frac{4}{3\sqrt{x + \delta x}} - \frac{4}{3\sqrt{x}}$$
$$\delta y = \frac{4\sqrt{x} - 4\sqrt{x + \delta x}}{3\sqrt{x(x + \delta x)}}$$
$$\delta y = \frac{4\left(\sqrt{x} - \sqrt{x + \delta x)}\right)\left(\sqrt{x} + \sqrt{x + \delta x)}\right)}{3\sqrt{x(x + \delta x)}\left(\sqrt{x} + \sqrt{x + \delta x)}\right)}$$
$$\delta y = \frac{4\left(\left(\sqrt{x}\right)^2 - \left(\sqrt{x + \delta x}\right)^2\right)}{3\sqrt{(x^2 + x\delta x)}\left(\sqrt{x} + \sqrt{x + \delta x)}\right)}$$
$$\delta y = \frac{-4\delta x}{3\sqrt{(x^2 + x\delta x)}\left(\sqrt{x} + \sqrt{x + \delta x)}\right)}$$

Divide through by δx

$$\frac{\delta y}{\delta x} = \frac{-4}{3\sqrt{(x^2 + x\delta x)}\left(\sqrt{x} + \sqrt{x + \delta x)}\right)}$$

As $\delta x \longrightarrow 0$

$$\frac{\delta y}{\delta x} \longrightarrow \frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{-4}{3x(2\sqrt{x})}$$
$$\Longrightarrow \frac{dy}{dx} = \frac{-2}{3x^{3/2}}$$

Example 3

Differentiate $y = \frac{x}{x^2+1}$ from the first principle

Solution

$$y + \delta y = \frac{x + \delta x}{(x + \delta x)^2 + 1}$$
$$\delta y = \frac{x + \delta x}{(x + \delta x)^2 + 1} - y$$
$$\delta y = \frac{x + \delta x}{(x + \delta x)^2 + 1} - \frac{x}{x^2 + 1}$$
$$\delta y = \frac{(x + \delta x)(x^2 + 1) - x((x + \delta x)^2 + 1)}{((x + \delta x)^2 + 1)(x^2 + 1)}$$
$$\delta y = \frac{x^3 + x + x^2\delta x + \delta x - x^3 - 2x^2\delta x - x(\delta x)^2 - x}{((x + \delta x)^2 + 1)(x^2 + 1)}$$
$$\delta y = \frac{\delta x - x^2\delta x}{((x + \delta x)^2 + 1)(x^2 + 1)}$$

Dividing through by δx

$$\frac{\delta y}{\delta x} = \frac{(1 - x^2)}{((x + \delta x)^2 + 1)(x^2 + 1)\delta x}$$
$$\frac{\delta y}{\delta x} = \frac{1 - x^2}{((x + \delta x)^2 + 1)(x^2 + 1)}$$

As $\delta x \longrightarrow 0$

$$\frac{\delta y}{\delta x} \longrightarrow \frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1 - x^2}{(x^2 + 1)(x^2 + 1)}$$
$$\frac{dy}{dx} = \frac{1 - x^2}{(x^2 + 1)^2}$$

Example 4

Differentiate $y = \sin x$ from the first principle.

Solution

$$y + \delta y = \sin(x + \delta x)$$
$$\delta y = \sin(x + \delta x) - y$$
$$\delta y = \sin(x + \delta x) - \sin x$$
$$\delta y = 2\cos\frac{(\mathrm{x} + \delta x + x)}{2}\sin\frac{(x + \delta x - x)}{2}$$

$$\delta y = 2\cos\left(x + \frac{\delta x}{2}\right)\sin\frac{\delta x}{2}$$
$as\ \delta x \longrightarrow 0$
$\sin\frac{\delta x}{2} \longrightarrow \frac{\delta x}{2}$
$$\delta y = 2\cos\left(\mathrm{x} + \frac{\delta x}{2}\right).\frac{\delta x}{2}$$
$$\frac{\delta y}{\delta x} = \cos\left(x + \frac{\delta x}{2}\right)$$
As $\delta x \longrightarrow 0$
$$\frac{\delta y}{\delta x} \longrightarrow \frac{dy}{dx}$$
$$\frac{dy}{dx} = \cos x$$

Example 5

Differentiate $y = \cos x$ from the first principle

Solution

$$y = \cos x$$
$$y + \delta y = \cos(x + \delta x)$$
$$\delta y = \cos(x + \delta x) - y$$
$$\delta y = \cos(x + \delta x) - \cos x$$
$$\delta y = -2\sin\frac{(x + \delta x + x)}{2}\sin\frac{(x + \delta x - x)}{2}$$
$$\delta y = -2\sin\left(x + \frac{\delta x}{2}\right)\sin\frac{\delta x}{2}$$
As $\delta x \longrightarrow 0$
$$sin\frac{\delta x}{2} \longrightarrow \frac{\delta x}{2}$$
$$\delta y = -2\sin\left(x + \frac{\delta x}{2}\right)\frac{\delta x}{2}$$
$$\delta y = -\,\delta x\sin\left(x + \frac{\delta x}{2}\right)$$
$$\frac{\delta y}{\delta x} = -\sin\left(x + \frac{\delta x}{2}\right)$$
As $\delta x \longrightarrow 0$, $\frac{\delta y}{\delta x} \rightarrow \frac{dy}{dx}$
$$\frac{dy}{dx} = -\sin x$$

Example 6

Show that $\frac{d}{dx}(tanx) = \sec^2 x$ from first principles

Solution

$$y = \tan x$$
$$y + \delta y = \tan(x + \delta x)$$
$$\delta y = \tan(x + \delta x) - y$$
$$\delta y = \tan(x + \delta x) - \tan x$$
$$\delta y = \frac{\tan \mathrm{x} + \tan\delta x}{1 - \tan x\ tan\delta x} - \tan x$$
$$\delta y = \frac{\tan \mathrm{x} + \tan\delta x - \tan x + \tan^2 x\ tan\delta x}{1 - \tan x\ tan\delta x}$$
$$\delta y = \frac{\tan\delta x(1 + \tan^2 x)}{1 - \tan x\ tan\delta x}$$
As $\delta x \longrightarrow 0$, $\tan\delta x \approx \delta x$
$$\delta y = \frac{(1 + \tan^2 x)\delta x}{1 - \tan x\ (\delta x)}$$
As $\delta x \longrightarrow 0$,
$$\frac{\delta y}{\delta x} = \frac{(1 + \tan^2 x)}{1}$$
$$\frac{\delta y}{\delta x} \longrightarrow \frac{dy}{dx}$$
$$\frac{dy}{dx} = 1 + \tan^2 x$$
$$\frac{dy}{dx} = \sec^2 x$$

Example 7

Differentiate $y = x^2 + \cos 2x$ from the first principle.

$$y + \delta y = (x + \delta x)^2 + \cos 2(x + \delta x)$$
$$\delta y = (x + \delta x)^2 + \cos 2(x + \delta x) - y$$
$$\delta y = x^2 + 2x\delta x + (\delta x)^2 + \cos 2(x + \delta x) - y$$
$$\delta y = x^2 + 2x\delta x + (\delta x)^2 + \cos 2(x + \delta x) - x^2 - \cos 2x$$
As $\delta x \longrightarrow 0$, $(\delta x)^2 \approx 0$
$$\delta y = 2x\delta x + \cos 2(x + \delta x) - \cos 2x$$
$$\delta y = 2x\delta x - 2\sin(2x + \delta x)\sin\delta x$$
For small angles, $\sin\delta x \rightarrow \delta x$
$$\delta y = 2x\delta x - 2\delta x sin\ (2x + \delta x)$$
$$\frac{\delta y}{\delta x} = 2x - 2sin\ (2x + \delta x)$$
As $\delta x \longrightarrow 0$
$$\frac{\delta y}{\delta x} \longrightarrow \frac{dy}{dx}$$
$$\frac{dy}{dx} = 2x - 2\sin 2x$$

Example 8

Differentiate: $y = \sec 3x$ from first principle

$$y = \frac{1}{\cos 3x}$$
$$y + \delta y = \frac{1}{\cos 3(x + \delta x)}$$
$$\delta y = \frac{1}{\cos 3(x + \delta x)} - y$$
$$\delta y = \frac{\cos 3x - cos 3(x + \delta x)}{\cos 3x\cos 3(x + \delta x)}$$

$$\delta y=\frac{-2\sin\left(3x+\frac{3\delta x}{2}\right)\sin\left(\frac{-3\delta x}{2}\right)}{\cos 3x\cos 3(x+\delta x)}$$

As $\delta x\to 0, \sin\left(\frac{-3\delta x}{2}\right)\longrightarrow \frac{-3\delta x}{2}$

$$\delta y=\frac{-2\left(\frac{-3\delta x}{2}\right)\sin\left(3x+\frac{-3\delta x}{2}\right)}{\cos 3x\cos 3x}$$

As $\delta x\to 0,\quad \delta y=\dfrac{3\delta x\sin 3x}{\cos^2 3x}$

$$\frac{\delta y}{\delta x}=\frac{3\sin 3x}{\cos^2 3x}$$

$$\frac{\delta y}{\delta x}=3\tan 3x\sec 3x$$

As $\delta x\to 0,\ \frac{\delta y}{\delta x}\longrightarrow\frac{dy}{dx}$

$$\frac{dy}{dx}=3\tan 3x\sec 3x$$

Example 9

Differentiate y=sin^{2}x from the first principles

Solution:

$$y=\sin^2 x$$
$$y+\delta y=\sin^2(x+\delta x)$$
$$\delta y=\sin^2(x+\delta x)-y$$
$$\delta y=\sin^2(x+\delta x)-\sin^2 x$$
$$\delta y=(\sin(x+\delta x)+\sin x)(\sin(x+\delta x)-\sin x)$$
$$\delta y=\left[2\sin\left(x+\frac{\delta x}{2}\right)\cos\frac{\delta x}{2}\right]\left(2\cos\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}\right)$$

As $\longrightarrow 0$, $\sin\frac{\delta x}{2}\longrightarrow\frac{\delta x}{2}$, $\cos\frac{\delta x}{2}\longrightarrow 1$ and

$$\frac{\delta y}{\delta x}\longrightarrow\frac{dy}{dx}$$
$$\frac{dy}{dx}=2\sin x\cos x$$

Example 10

Differentiate $y=\tan^{-1}x$ from the first principle

Solution

$$y=\tan^{-1}x$$
$$y+\delta y=\tan^{-1}(x+\delta x)$$
$$y+\delta y=\tan^{-1}(x+\delta x)$$
$$\delta y=\tan^{-1}(x+\delta x)-y$$
$$\delta y=\tan^{-1}(x+\delta x)-\tan^{-1}x$$
$$\delta y=\tan^{-1}(x+\delta x)-\tan^{-1}x$$

Let A=$\tan^{-1}(x+\delta x)$ and B=$\tan^{-1}x$

$$\tan A=x+\delta x$$
$$\tan B=x$$
$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$
$$A-B=\tan^{-1}\left(\frac{\tan A-\tan B}{1+\tan A\tan B}\right)$$
$$\delta y=\tan^{-1}\left(\frac{x+\delta x-x}{1+x^2+x\delta x}\right)$$
$$\delta y=\tan^{-1}\left(\frac{\delta x}{1+x^2+x\delta x}\right)$$
$$\tan\delta y=\left(\frac{\delta x}{1+x^2+x\delta x}\right)$$

As $\delta x\longrightarrow 0$

$\tan\delta y\to\delta y$

$$\delta y=\frac{\delta x}{1+x^2}$$
$$\frac{\delta y}{\delta x}=\frac{1}{1+x^2}$$

As $\delta x\longrightarrow 0$

$$\frac{\delta y}{\delta x}\longrightarrow\frac{dy}{dx}$$
$$\frac{dy}{dx}=\frac{1}{1+x^2}$$

Example 11

Differentiate $y=e^{ax}$ from the first principles

Solution

$$y=e^{ax}$$
$$y+\delta y=e^{a(x+\delta x)}$$
$$\delta y=e^{a(x+\delta x)}-y$$
$$\delta y=e^{a(x+\delta x)}-e^{ax}$$
$$\delta y=e^{ax}.e^{a\delta x}-e^{ax}$$
$$\delta y=e^{ax}\left(e^{a\delta x}-1\right)$$

But $e^x=1+x+\frac{x^2}{2}+\cdots$ (*from the tables*)

$$e^{a\delta x}=1+(a\delta x)+\frac{(a\delta x)^2}{2}+\cdots$$
$$\delta y=e^{ax}\left(1+a\delta x+\frac{a^2(\delta x)^2}{2}+\cdots-1\right)$$
$$\delta y=e^{ax}(a\delta x)$$
$$\frac{\delta y}{\delta x}=ae^{ax}$$

As $\delta x\longrightarrow 0$

$$\frac{\delta y}{\delta x}\longrightarrow\frac{dy}{dx}$$
$$\frac{dy}{dx}=ae^{ax}$$

More examples on differentiation

Example I

Given that $y=\sin\sqrt{x}$, prove that

$$2\frac{dy}{dx}+y+4x\frac{d^2y}{dx^2}=0$$

Solution

$y=\sin\sqrt{x}$

$$\frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}\cos\sqrt{x}$$

$$\frac{dy}{dx}=\frac{\cos\sqrt{x}}{2\sqrt{x}}$$

If $y=\frac{u}{v}$ then, $\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

$$\frac{d^2y}{dx^2}=\frac{2\sqrt{x}\left(\frac{-1}{2\sqrt{x}}\sin\sqrt{x}\right)-\cos\sqrt{x}\left(\frac{1}{2}2x^{\frac{-1}{2}}\right)}{4x}$$

$$\frac{d^2y}{dx^2}=\frac{\sin\sqrt{x}-\frac{\cos\sqrt{x}}{\sqrt{x}}}{4x}$$

$$\frac{d^2y}{dx^2}=\frac{-\sin\sqrt{x}-2\frac{dy}{dx}}{4x}$$

$$4x\frac{d^2y}{dx^2}=-\sin\sqrt{x}-2\frac{dy}{dx}$$

$$4x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=-\sin\sqrt{x}$$

$$4x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=-y$$

$$4x\frac{d^2y}{dx^2}+y+2\frac{dy}{dx}=0$$

Example II

If $y=e^{2x}\cos 3x$ show that

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}-13y=0$$

Solution

$y=e^{2x}\cos 3x$

$$\frac{dy}{dx}=e^{2x}(-3\sin 3x)+(\cos 3x)2e^{2x}$$

$$\frac{dy}{dx}=-3e^{2x}\sin 3\text{x}+2y\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{d^2y}{dx^2}=-3(e^{2x}3\cos 3\text{x}+(\sin 3x)2e^{2x})+2\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2}=-3(3y+2e^{2x}\sin 3x)+2\frac{dy}{dx}$$

From equation (1)

$$e^{2x}\sin 3x=\frac{\frac{dy}{dx}-2y}{-3}$$

$$\frac{d^2y}{dx^2}=-3\left(3y+\frac{2\left(\frac{dy}{dx}-2y\right)}{-3}\right)+2\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2}=-9y+2\frac{dy}{dx}-4y+2\frac{dy}{dx}$$

$\frac{d^2y}{dx^2}-4\frac{dy}{dx}-13y=0$ As required

Example III

$y=xe^{-x}$ Show that $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$

Solution

$y=xe^{-x}$

$$\frac{dy}{dx}=x.(-e^{-x})+e^{-x}(1)$$

$$\frac{dy}{dx}=-y+e^{-x}\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

From Eqn (1);

$\frac{d^2y}{dx^2}=-\frac{dy}{dx}-e^{-x}$……………. (2)

From Eqn (1)

$\frac{dy}{dx}+y=e^{-x}$ ………. (3)

Substituting Eqn (3) in Eqn (2);

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}=-\left(\frac{dy}{dx}+y\right)$$

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$$

INTEGRATION BY PARTS

Integration by parts is often used when one has an integral where the integrand can be made to take the form of a product.

Consider $\frac{d}{dx}(UV) = U\frac{dV}{dx} + V\frac{dU}{dx}$

$$\int \frac{d}{dx}(UV)dx = \int U\frac{dV}{dx}dx + \int V\frac{dU}{dx}dx$$

$$UV = \int U\frac{dV}{dx} + \int V\frac{dU}{dx}dx$$

$$\Rightarrow \int U\frac{dV}{dx}dx = UV - \int V\frac{dU}{dx}dx$$

When we are integrating by parts, we let the easily differentiable function be U and the easily integrable function to be $\frac{dV}{dx}$. However, there are some exceptions.

LIATE: Choose U to be a function that comes first in this list (LIATE)

L – logarithm function
I – Inverse trigonometric functions
A – Algebraic function
T – Trigonometric function
E – exponential functions

Example I

$$\int x\cos x\,dx$$

Since x is an algebraic function (A) and cos x is a trigonometric function A comes before T in LIATE.

$\therefore let\ U = x$

$\frac{dv}{dx} = \cos x$

$U = x, \frac{du}{dx} = 1$

$\frac{dv}{dx} = \cos x,$

$dv = \cos x\,dx$

$v = \sin x$

$$\Rightarrow \int U\frac{dv}{dx}dx = UV - \int V\frac{du}{dx}dx$$

$$\int x\cos x\,dx = x\sin x - \int \sin x\ dx$$

$$\int x\cos x\,dx = x\sin x - -\cos x + C$$

$$\int x\cos x\,dx = x\sin x + \cos x + C$$

Example II

$$\int x^2e^xdx$$

x^2 = Algebraic function (A)

e^x = exponential function (E)

A – comes before E in the word LIATE

$U = x^2, \quad \frac{dv}{dx} = e^x$

$$U = x^2 \Longrightarrow \frac{du}{dx} = 2x.$$

$$\frac{dv}{dx} = e^x \Longrightarrow \int dv = \int e^x\,dx$$

$$V = e^x$$

$$\int U\frac{dv}{dx}dx = UV - \int V\frac{du}{dx}dx$$

$$\int x^2e^xdx = x^2e^x - \int e^x.2x\,dx$$

$$\int x^2\,e^xdx = x^2e^x - 2\int x\,e^xdx$$

Consider $\int x\,e^x$

Let $u = x \ \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^x$

$dv = e^xdx$

$v = e^x$

$$\int xe^x\,dx \ = xe^x - \int e^x\,dx$$

$$= xe^x - e^x + C$$

$$\Rightarrow \int x^2e^xdx = x^2e^x - 2(xe^x - e^x)$$

$$= x^2e^x - 2xe^x + 2e^x + A$$

$$\Rightarrow \int x^2e^xdx = x^2e^x - 2xe^x + 2e^x + A$$

Example III

$$\int x^2sin^2x\,dx$$

x^2=algebraic function (A)

sin^2x=trigonometric function (T)

A comes first before **T** in the LIATE

$$u = x^2 \ \Rightarrow \frac{du}{dx} = 2x$$

$$\frac{dv}{dx} = sin^2x$$

$\Longrightarrow\ dv = sin^2xdx$

$$dv = \frac{1}{2}(1 - \cos 2x)\, dx$$
$$v = \int (\frac{1}{2} - \frac{1}{2}\cos 2x)\, dx$$
$$v = \frac{1}{2}x - \frac{1}{4}\sin 2x$$
$$\Rightarrow \int x^2\, sin^2 x dx$$
$$\int U \frac{dv}{dx} dx = UV - \int V \frac{du}{dx} dx$$
$$= \frac{1}{2}x^3 - \frac{1}{4}x^2 \sin 2x - \int (\frac{1}{2}x - \frac{1}{4}\sin 2x)\,.\, 2x$$
$$= \frac{1}{2}x^3 - \frac{1}{4}x^2 \sin 2x - \int x^2 + \int \frac{1}{2} x \sin 2x\, dx$$
$$= \frac{1}{2}x^3 - \frac{1}{4}x^2 \sin 2x - \frac{x^3}{3} + \frac{1}{2}\int x \sin 2x\ \ dx$$

Consider $\int x \sin 2x\, dx$

$$let\ u = x \ \Rightarrow \frac{du}{dx} = 1$$
$$\frac{dv}{dx} = \sin 2x \ \Rightarrow dv = \sin 2x\, dx$$
$$v = -\frac{1}{2}\cos 2x$$
$$\int x \sin 2x\, dx = -\frac{1}{2}x \cos 2x - \int -\frac{1}{2}\cos 2x dx$$
$$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$$
$$\int x^2 \sin^2 x\, dx = \frac{1}{2}x^3 - \frac{1}{4}x^2 \sin 2x - \frac{x^3}{3}$$
$$= \frac{1}{2}\left[\frac{1}{2}x\ \cos 2x + \frac{1}{4}\sin 2x\right] + C$$

Example IV

$$\int \sin x\, (\ln \cos x)\, dx$$

$\ln \cos x\ (logarithmic\ function)$

$\sin x\ (trigonometric\ function)$

$L\ comes\ first\ before\ T\ in\ the\ LIATE$

$$\Rightarrow U = \ln \cos x,\ \ \frac{du}{dx} = -\frac{\sin x}{\cos x}$$
$$\frac{dv}{dx} = \sin x$$
$$dv = (\sin x)dx$$
$$v = -\cos x$$
$$\int (\sin x)(\ln \cos x)\, dx$$
$$= -(\ln \cos x)\cos x - \int -\cos x\,.\, -\frac{\sin x}{\cos x} dx$$
$$= (-\ln \cos x)\cos x + \int \sin x\, dx$$
$$= -(\ln \cos x)\cos x + \cos x + C$$

Example V

$$\int x^3 (\ln x) dx$$

x^3 = algebraic function (A)

$(\ln x)$ = logarithmic function (L)

L come before A in LIATE

$$u = \ln x \quad \Rightarrow\ \frac{du}{dx} = \frac{1}{x}$$
$$\frac{dv}{dx} = x^3$$
$$\Rightarrow dv = x^3 dx$$
$$v = \frac{x^4}{4}$$
$$\int x^3\, (\ln x) dx = \frac{(\ln x)x^4}{4} - \frac{1}{4}\int x^3 dx$$
$$= \frac{(\ln x)\, x^4}{4} - \frac{1}{4}\left(\frac{x^4}{4}\right) + C$$
$$\Rightarrow \int x^3 (\ln x)\, dx = \frac{(\ln x)x^4}{4} - \frac{1}{16}x^4 + \text{C}$$

Example VI UNEB 2012

$$\int_0^{\frac{\pi}{2}} x^2 \sin x\, dx$$

Solution

Consider $\int x^2 \sin x\, dx$

x^2 = algebraic function (A)

$\sin x$ =trigonometric function (T)

A comes before T in the word LIATE

$$\Rightarrow u = x^2$$
$$\frac{du}{dx} = 2x$$
$$\frac{dv}{dx} = \sin x$$
$$v = -\cos x$$
$$\int x^2 \sin x dx = -x^2 \cos x - \int (-\cos x) 2x dx$$
$$= -x^2 \cos x + 2\int x \cos x\, dx$$

consider $\int x \cos x dx$

$$u = x, \frac{dv}{dx} = \cos x$$

$$\Rightarrow \frac{du}{dx} = 1 \qquad v = \sin x$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int x\cos x\, dx = x\sin x - \int \sin x\, dx$$

$$= x\sin x + \cos x + C$$

$$\Longrightarrow \int x^2\, sinx\, dx = -x^2\cos x + 2x\sin x + 2\cos x$$

$$\int_0^{\frac{\pi}{2}} x^2\sin x\, dx = [-x^2\cos x + 2x\sin x + 2\cos x]_0^{\pi/2}$$

$$= -\frac{\pi^2}{4}(0) + \pi - (0+2)$$

$$= \pi - 2$$

Example VII

$$\int x^2\sin^2 x\, dx$$

$$\int x^2\frac{1}{2}(1 - \cos 2x)dx$$

$$= \frac{1}{2}\int x^2 - x^2\cos 2x\, dx$$

$$= \frac{1}{2}\int x^2 dx - \int \frac{1}{2}x^2\cos 2x\, dx$$

$= \frac{x^3}{6} - \frac{1}{2}\int x^2\cos 2x\, dx$………….. (i)

Consider $\int x^2\cos 2x\, dx$

Let $u = x^2$, $\frac{dv}{dx} = \cos 2x$

$$\frac{du}{dx} = 2x, \quad v = \frac{1}{2}\sin 2x$$

$$\int x^2\cos 2x\, dx = \frac{1}{2}x^2(\sin 2x) - \int \frac{1}{2}(sin2x)2x\, dx$$

$\int x^2\cos 2x\, dx = \frac{1}{2}x^2(\sin 2x) - \int x\sin 2x\, dx$ ……..(ii)

Consider $\int x\sin 2x\, dx$

Let $u = x \ \Rightarrow \frac{du}{dx} = 1$

$$\frac{dv}{dx} = \sin 2x, \quad \Rightarrow v = \frac{-1}{2}\cos 2x$$

$$\int x\sin 2x\, dx = \frac{-1}{2}x\cos 2x - \int \frac{-1}{2}\cos 2x\, dx$$

$$= \frac{-1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$$

$\int x^2\cos 2x\, dx =$
$\frac{1}{2}x^2\sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x + C$

$\Rightarrow \int x^2\sin^2 x\, dx$

$$= \frac{x^3}{6} - \frac{1}{4}x^2 sin2x - \frac{1}{4}x^2 cos2x + \frac{1}{8}sin2x + C$$

Example VIII **UNEB 2002**

$$\int x^2\sin 2x\, dx$$

Let $u = x^2 \quad \Rightarrow \frac{du}{dx} = 2x$

$$\frac{dv}{dx} = \sin 2x \qquad \Rightarrow dv = \sin 2x\, dx$$

$$\Longrightarrow v = -\frac{1}{2}\cos 2x$$

$$\int x^2\sin 2x\, dx = -\frac{1}{2}x^2\cos 2x - \int -\frac{1}{2}(\cos 2x)\, 2xdx$$

$$\int x^2\sin 2xdx$$

$$= -\frac{1}{2}x^2\cos 2x + \int x\cos 2x\, dx$$

$$consider \int x\cos 2x\, dx$$

$$u = x \quad \Rightarrow \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = \cos 2x$$

$$dv = \cos 2x\, dx \ \Rightarrow \ v = \frac{1}{2}\sin 2x$$

$$\int x\cos 2x\, dx = \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2xdx$$

$$\int x\cos 2x\, dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$$

$$\int x^2\sin 2x\, dx$$

$$= -\frac{1}{2}x^2\cos 2x + +\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$$

Example VIII UNEB 2003

$$\int x(\ln x)dx$$

$x = algebraic\ function$
$(\ln x) = logarithimic\ function$
L comes before **A** in the word LIATE

$$\Longrightarrow u = \ln x \quad \Rightarrow \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = x \quad \Rightarrow v = \frac{x^2}{2}$$

$$\int x(\ln x)\,dx = \frac{\ln x\, x^2}{2} - \frac{1}{2}\int x\,dx$$
$$= \frac{(\ln x)\,x^2}{2} - \frac{x^2}{4} + C$$

Example IX UNEB 2003

$$\int \ln(x^2-4)dx$$

Solution

$$\int \ln(x^2-4)dx$$

Let $u = \ln(x^2-4) \;\Rightarrow \dfrac{du}{dx} = \dfrac{2x}{x^2-4}$

$$\frac{dv}{dx} = 1 \;\Rightarrow v = x$$

$$\int \ln(x^2-4)dx = x\ln(x^2-4) - \int x.\frac{2x}{x^2-4}dx$$
$$= x\ln(x^2-4) - \int \frac{2x^2}{x^2-4}dx$$

$$\begin{array}{r|l} & 2 \\ \hline x^2-4 & 2x^2 \\ & \underline{2x^2-8} \\ & \quad 8 \end{array}$$

$$\frac{2x^2}{x^2-4} = 2 + \frac{8}{x^2-4}$$
$$\Rightarrow \frac{2x^2}{x^2-4} = 2 + \frac{8}{(x+2)(x-2)}$$

(By partial fractions)

$$= x\ln(x^2-4) - \int 2 + \frac{2}{x-2} - \frac{2}{x+2}dx$$
$$= x\ln(x^2-4) - 2x - 2\ln(x-2) + 2\ln(x+2)$$
$$= x\ln(x^2-4) - 2x + \ln\frac{(x+2)^2}{(x-2)^2} + C$$

Alternative method of integration by parts

If an expression can be broken down into two parts one differentiatable up to zero and the other can be integrated each time the former is differentiated

Example 1

$$\int x^2\cos 2x\,dx$$

Sign change	Differentiate	integrate
+	x^2	$\cos 2x$
−	$2x$	$\frac{1}{2}\sin 2x$
+	2	$\frac{-1}{4}\cos 2x$
−	0	$\frac{-1}{8}\sin 2x$

$$\Rightarrow \int x^2\cos 2x\,dx$$
$$= \frac{1}{2}x^2\sin 2x + \frac{1}{2}x\cos x - \frac{1}{4}\sin 2x + C$$

Example II

$$\int x^3 e^{2x}dx$$

Sign change	Differentiate	integrate
+	x^3	e^{2x}
−	$3x^2$	$\frac{1}{2}e^{2x}$
+	$6x$	$\frac{1}{4}e^{2x}$
−	6	$\frac{1}{8}e^{2x}$
+	0	$\frac{1}{16}e^{2x}$

$$\int x^3e^{2x}dx = \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}e^{2x} + C$$

Example III

Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} xcos^2 3x\;\,dx$

Solution

Consider $\displaystyle\int x\,cos^2 3xdx$

$cos^2 3x = \dfrac{1}{2}(1+\cos 6x)$

Sign change	Differentiate	Integrate
+	x	½ (1+cos 6x)
-	1	$\frac{1}{2}x + \frac{1}{12}\sin 6x$
+	0	$\frac{x^2}{4} + -\frac{1}{72}\cos 6x$

$$\Rightarrow \int xcos^2 3x\,dx$$
$$= \frac{1}{2}x^2 + \frac{1}{12}x\sin 6x - \frac{x^2}{4} + \frac{1}{72}\cos 6x + C$$
$$= \frac{x^2}{4} + \frac{1}{12}x\sin 6x + \frac{1}{72}\cos 6x + C$$

$$\int_0^{\frac{\pi}{2}} x\,cos^2 3x\,dx = \frac{x^2}{4} + \frac{1}{12}x\sin 6x + \frac{1}{72}\cos 6x\Big|_0^{\pi/2}$$
$$= \left(\frac{\pi^2}{16} + 0 - \frac{1}{72}\right) - \left(0 + 0 + \frac{1}{72}\right)$$
$$= \frac{\pi^2}{16} - \frac{1}{36}$$

More examples on integration by parts

Example I

$$\int \frac{(\ln x)}{x^2}\,dx$$

Solution

Since (ln x) is a logarithmic function **L** and $\frac{1}{x^2}$ is an algebraic function (A)

L comes before A in LIATES

$$\Rightarrow u = (\ln x)\ and \frac{dv}{dx} = \frac{1}{x^2}$$

$$\int \frac{udv}{dx}dx = uv - \int \frac{vdu}{dx}dx$$

$$u = \ln x$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = \frac{1}{x^2}$$

$$\int dv = \int \frac{1}{x^2}dx$$

$$v = -\frac{1}{x}$$

$$\Rightarrow \int \frac{\ln x}{x^2}dx = -\frac{\ln x}{x} - \int -\frac{1}{x}.\frac{1}{x}dx$$

$$= \frac{\ln x}{x} + \int x^{-2}dx$$

$$= \frac{\ln x}{x} - \frac{1}{x} + C$$

$$\int \frac{\ln x}{x^2}dx = \frac{\ln x}{x} - \frac{1}{x} + C$$

Example II

$$\int x10^x\,dx$$

Let $u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$

$$\frac{dv}{dx} = 10^x$$

$$dv = 10^x dx$$

$$v = \frac{10^x}{\ln 10} \quad since \int a^x dx = \frac{a^x}{\ln a}$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{dv}{dx}dx$$

$$\int x10^x\,dx = \frac{x10^x}{\ln 10} - \int \frac{10^x}{\ln 10}dx$$

$$= \frac{x10^x}{\ln 10} - \frac{1}{\ln 10}\int 10^x\,dx$$

$$= \frac{x10^x}{\ln 10} - \frac{1}{\ln 10}\left(\frac{10^x}{\ln 10}\right) + C$$

$$\Rightarrow \int x10^x\,dx = \frac{x10^x}{\ln 10} - \frac{10^x}{(\ln 10)^2} + C$$

Example III

$$\int_1^{10} x\log_{10} x\,dx$$

Solution

$$u = \log_{10} x, \quad \frac{dv}{dx} = x$$

$$\Rightarrow dv = x\,dx$$

$$v = \frac{x^2}{2}$$

$$10^u = x$$

$$u\ln 10 = \ln x$$

$$\ln 10\,du = \frac{1}{x}dx$$

$$\frac{du}{dx} = \frac{1}{x\ln 10}$$

$$\int x\log_{10} x\,dx = \frac{x^2}{2}\log_{10} x - \int \frac{x^2}{2}.\frac{1}{x\ln 10}dx$$

$$\int x\log_{10} x\,dx = \frac{x^2}{2}\log_{10} x - \frac{1}{2(\ln 10)}\left(\frac{x^2}{2}\right)$$

$$\Rightarrow \int x\log_{10} x\,dx = \frac{x^2}{2}\log_{10} x - \frac{1}{4\ln 10}(x^2) + C$$

$$\int_1^{10} x\log_{10}x = \left|\frac{x^2}{2}\log_{10}x - \frac{1}{4(\ln 10)}x^2\right|_1^{10}$$

$$= 50 - \frac{99}{4\ln 10}$$

Example IV

$$\int 3^{\sqrt{2x-1}}\,dx$$

Solution

$$let\ \sqrt{2x} - 1 = m$$

$$2x - 1 = m^2$$

$$2dx = 2m\,dm$$

$$dx = dm$$

$$\int 3^m\,mdm = \int m3^m dm$$

$$let\ u = m$$

$$\frac{du}{dm} = 1$$

$$\frac{dv}{dm} = 3^m$$

$v = \frac{3^m}{\ln 3}$ (Since $\int a^m dm = \frac{a^m}{\ln a} + c$)

$$\int u\frac{dv}{dm}dm = uv - \int v\frac{du}{dm}dm$$

$$\int m3^m dm = \frac{m3^m}{\ln 3} - \int \frac{3^m}{\ln 3}\, dm$$

$$\int m3^m dm = \frac{m3^m}{\ln 3} - \frac{1}{\ln 3}\int 3^m dm$$

$$= \frac{m3^m}{\ln 3} - \frac{1}{\ln 3}\left(\frac{3^m}{\ln 3}\right) + C$$

$$\int 3^{\sqrt{2x-1}}\, dx$$

$$= \frac{\sqrt{2x-1}\left(3^{\sqrt{2x-1}}\right)}{\ln 3} - \frac{1}{\ln 3}\left(\frac{3^{\sqrt{2x-1}}}{\ln 3}\right) + C$$

$$\int 3^{\sqrt{2x-1}}\, dx = \frac{\sqrt{2x-1}\left(3^{\sqrt{2x-1}}\right)}{\ln 3} - \frac{3^{\sqrt{2x-1}}}{(\ln 3)^2} + C$$

Example IV

$$\int x^3 e^{x^2} dx$$

Solution

$$\int x^3 e^{x^2} dx$$

$$\int x.x^2 e^{x^2} dx$$

$let\ u = x^2$

$$\frac{du}{dx} = 2x$$

$$dx = \frac{du}{2x}$$

$$\int x^3 e^{x^2}\, dx$$

$$= \int x.x^2 e^{x^2}\, dx = \int x.ue^u.\frac{du}{2x}$$

$$= \frac{1}{2}\int ue^u du$$

Sign change	Differentiate	Integrate
+	u	e^u
−	1	e^u
+	0	e^u

$$\int ue^u du = ue^u - e^u$$

$$\frac{1}{2}\int ue^u\, du = \frac{1}{2}ue^u - \frac{1}{2}e^u + C$$

$$\int x^3 e^{x^2} dx = \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C$$

Example V

$$\int \theta^3 \sin(\theta^2) d\theta$$

$$\int \theta\,.\,\theta^2 \sin(\theta^2) d\theta$$

Let $p = \theta^2$

$$dp = 2\theta d\theta$$

$$d\theta = \frac{dp}{2\theta}$$

$$\int \theta.\theta^2 \sin\theta^2\, d\theta = \int \theta.p\sin p\frac{dp}{2\theta}$$

$$= \frac{1}{2}\int p\sin p\, dp$$

Sign change	Differentiate	Integrate
+	p	$\sin p$
−	1	$-\cos p$
+	0	$-\sin p$

$$\int p\sin p\, dp = -p\cos p + \sin p + C$$

$$\frac{1}{2}\int p\sin p\, dp = -\frac{p\cos p}{2} + \frac{1}{2}\sin p$$

$$\int \theta^3 \sin\theta^2\, d\theta = -\frac{\theta^2\cos\theta^2}{2} + \frac{1}{2}\sin\theta^2 + C$$

Example VI

$$\int x\, sec^2 x\, dx$$

Solution

Let $u = x \ \Rightarrow \ \frac{du}{dx} = 1$

$$\frac{dv}{dx} = sec^2 x$$

$dv = sec^2 x\, dx \ \Rightarrow \ v = \tan x$

$$\int xsec^2 x\, dx = x\tan x - \int \tan x\, dx$$

$$= x\tan x - \int \frac{\sin x}{\cos x}dx$$

$$= x\tan x - -\ln\cos x + C$$

$$= x\tan x + (\ln\cos x) + C$$

Example VII

$$\int x^n \ln x\, dx$$

Let $u = \ln x$

$$\frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = x^n \quad \Rightarrow v = \frac{x^{n+1}}{n+1}$$

$$\int x^n(\ln x)dx = \frac{(\ln x)x^{n+1}}{n+1} - \int \frac{x^{n+1}}{n+1}.\frac{1}{x}dx$$

$$= (\ln x)\frac{x^{n+1}}{n+1} - \frac{1}{n+1}\int x^n dx$$

$$= \frac{(\ln x)x^{n+1}}{n+1} - \frac{1}{(n+1)^2}(x^{n+1}) + C$$

Example VIII

$$\int x\, cosec^2 x\, dx$$

Solution

$$u = x \Rightarrow \tfrac{du}{dx} = 1$$

$$\frac{dv}{dx} = cosec^2 x$$

$$dv = cosec^2 x\, dx \;\Rightarrow\; v = -\cot x$$

$$\int x\, cosec^2 x\, dx = -x\cot x - \int -\cot x\, dx$$

$$= -x\cot x + \int \frac{\cos x}{\sin x}dx$$

$$= -x\cot x + \ln(\sin x) + C$$

$$\int x\, cosec^2 x\, dx = -x\cot x + \ln\sin x + C$$

Example IX

$$\int x\sin 2x\cos 2x\, dx$$

Solution:

$$let\ u = x, \quad \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = \sin 2x\cos 2x$$

$$dv = \sin 2x\cos 2x\, dx$$

$$\int dv = \int \frac{1}{2}\sin 4x\, dx$$

$$v = -\frac{1}{8}\cos 4x$$

$$\int x\sin 2x\cos 2x\, dx = -\frac{x}{8}\cos 4x - \int -\frac{1}{8}\cos 4x\, dx$$

$$= -\frac{x}{8}\cos 4x + \frac{1}{8}\int \cos 4x\, dx$$

$$= -\frac{x}{8}\cos 4x + \frac{1}{32}\sin 4x + C$$

Cases where the original integral re-appears

When integrating functions with the original integral re-appearing we use integration by parts. This common with integrals consisting of periodic functions like sin x and cos x

Example I

$$\int e^x\cos x\, dx$$

$$let\ u = e^x$$

$$\frac{du}{dx} = e^x$$

$$\frac{dv}{dx} = \cos x$$

$$\int dv = \int \cos x\, dx$$

$$v = \sin x$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int e^x\cos x\, dx = e^x\sin x - \int \sin x\,(e^x)dx$$

$$= e^x\sin x - \int e^x\sin x\, dx$$

But the integral on R.H.S is still a product so we can repeat the process

Consider $\int e^x\sin x\, dx$

$$let\ u = e^x$$

$$\frac{du}{dx} = e^x$$

$$\frac{dv}{dx} = \sin x$$

$$v = -\cos x$$

$$\int e^x\sin x\, dx = -e^x\cos x - \int -\cos x\,(e^x)dx$$

$$\int e^x\sin x\, dx = -e^x\cos x + \int e^x\cos x\, dx$$

$$\Rightarrow \int e^x\cos x\, dx = e^{-x}\sin x - \left[-e^x\cos x + \int e^x\cos x\right]$$

$$\int e^x\cos x\, dx$$

$$= e^x\sin x + e^x\cos x - \int e^x\cos x\, dx$$

$$2\int e^x\cos x\, dx = e^x\sin x + e^x\cos x$$

$$\int e^x\cos x\, dx = \frac{1}{2}e^x\sin x + \frac{1}{2}e^x\cos x + A$$

Example II

$$\int e^{2x}\sin 3x\, dx$$

Let $u = e^{2x}$

$$\frac{du}{dx} = 2e^{2x}$$

$$\frac{dv}{dx} = \sin 3x$$

$$v = -\frac{1}{3}\cos 3x$$

$$\int e^{2x}\sin 3x\,dx = -\frac{1}{3}e^{2x}\cos 3x - \int -\frac{2}{3}e^{2x}\cos 3x\,dx$$

$$= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int e^{2x}\cos 3x\,dx$$

But the integral on RHS still a product so we can repeat the process

$$\int e^{2x}\cos 3x\,dx$$

$$u = e^{2x}$$

$$\frac{du}{dx} = 2e^{2x}$$

$$\frac{dv}{dx} = \cos 3x$$

$$v = \frac{1}{3}\sin 3x$$

$$\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x - \int \frac{2}{3}e^{2x}\sin 3x\,dx$$

$$= \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\int e^{2x}\sin 3x\,dx$$

$$\Rightarrow \int e^{2x}\sin 3x\,dx$$

$$= \frac{-1}{3}e^{2x}\cos 3x + \frac{2}{3}\left[\frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\int e^{2x}\sin 3x\,dx\right]$$

$$= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x - \frac{4}{9}\int e^{2x}\sin 3x\,dx$$

Let $I = \int e^{2x}\sin 3x\,dx$

$$I = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x - \frac{4}{9}I$$

$$I + \frac{4}{9}I = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x$$

$$\frac{13}{9}I = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x$$

$$I = \frac{9}{13}\left(-\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x\right)$$

$$\Rightarrow \int e^{2x}\sin 3x\,dx$$

$$= \frac{9}{13}\left(-\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x\right) + C$$

$$\int e^{2x}\sin 3x\,dx = \frac{2}{13}e^{2x}\sin 3x - \frac{3}{13}e^{2x}\cos 3x + C$$

Example III

$$\int e^{-x}\cos\frac{x}{2}dx$$

Let $u = e^{-x} \Rightarrow \dfrac{du}{dx} = -e^{-x}$

$$\frac{dv}{dx} = \cos\frac{x}{2} \quad \Rightarrow\ v = 2\sin\frac{x}{2}$$

$$\int e^{-x}\cos\frac{x}{2}\,dx = 2e^{-x}\sin\frac{x}{2} - \int -2e^{-x}\sin\frac{x}{2}dx$$

$$= 2e^{-x}\sin\frac{x}{2} + 2\int e^{-x}\sin\frac{x}{2}dx$$

But the integral on RHS is still a product so we can repeat the process

Consider $\int e^{-x}\sin\frac{x}{2}dx$

$$u = e^{-x} \ \Rightarrow\ \frac{du}{dx} = -e^{-x}$$

$$\frac{dv}{dx} = \sin\frac{x}{2}$$

$$v = -2\cos\frac{x}{2}$$

$$\int e^{-x}\sin\frac{x}{2}dx = 2e^{-x}\cos\frac{x}{2} - \int 2e^{-x}\cos\frac{x}{2}dx$$

$$\int e^{-x}\cos\frac{x}{2}dx$$

$$= -2e^{-x}\sin\frac{x}{2} + 4e^{-x}\cos\frac{x}{2} - 4\int e^{-x}\cos\frac{x}{2}\,dx$$

Let $\int e^{-x}\cos\frac{x}{2}dx = I$

$$I = -2e^{-x}\sin\frac{x}{2} + 4e^{-x}\cos\frac{x}{2} - 4I$$

$$5I = -2e^{-x}\sin\frac{x}{2} + 4e^{-x}\cos\frac{x}{2}$$

$$I = \frac{1}{5}(2e^{-x}\sin\frac{x}{2} - 4e^{-x}\cos\frac{x}{2})$$

$$\int e^{-x}\cos\frac{x}{2}dx = \frac{2}{5}\ (2e^{-x}\cos\frac{x}{2} - e^{x}\sin\frac{x}{2})\ + C$$

However, we can also use the alternative method to integration by parts to evaluate the following integrals

$$\int e^{x}\cos x\,dx$$

Sign change	Differentiate	Integrate
+	e^{x}	$\cos x$
-	e^{x}	-$\sin x$
+	e^{x}	-$\cos x$

$$\int e^{x}\cos x\,dx = -e^{x}\sin x + e^{x}\cos x$$

$$+ \int e^{x}(-\cos x)dx$$

$$\int e^{x}\cos x\,dx = -e^{x}\sin x + e^{x}\cos x - \int e^{x}\cos x\,dx$$

$$\int e^{x}\cos x\,dx = I$$

$$I = -e^{x}\sin x + e^{x}\cos x - I$$

$$2I = -e^{x}\sin x + e^{x}\cos x$$

$$I = \frac{1}{2}(-e^{-x}\sin x + e^{x}\cos x) + C$$

$$\Longrightarrow \int e^{x}\cos x\,dx = \frac{1}{2}e^{x}(\cos x - \sin x) + C$$

As before

Example III

$$\int e^{2x}\sin 3x\,dx$$

Sign change	Differentiate	Integrate
+	e^{2x}	$\sin 3x$
-	$2e^{2x}$	$\frac{-1}{3}\cos 3x$
+	$4e^{2x}$	$\frac{-1}{9}\sin 3x$

$$\int e^{2x}\sin 3x\,dx$$

$$= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x - \frac{4}{9}\int e^{2x}\sin 3x$$

Let $\int e^{2x}\sin 3x\,dx = I$

$$I = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x - \frac{4}{9}I$$

$$\frac{13}{9}I = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x$$

$$I = \frac{9}{13}\left(\frac{2}{9}e^{2x}\sin 3x - \frac{1}{3}e^{2x}\cos 3x\right)$$

$$I = \frac{1}{13}(2e^{2x}\sin 3x - 3e^{2x}\cos 3x)$$

$$\Longrightarrow \int e^{2x}\sin 3x\,dx$$

$$= \frac{1}{13}(e^{2x}\sin 3x - 3e^{2x}\cos 3x) + C$$

As before

Example IV

$$\int e^{-x}\cos\frac{x}{2}dx$$

Sign change	Differentiate	Integrate
+	e^{-x}	$\cos\frac{x}{2}$
−	$-e^{-x}$	$2\sin\frac{x}{2}$
+	e^{-x}	$-4\cos\frac{x}{2}$

$$\int e^{-x}\cos\frac{x}{2}dx$$

$$= 2e^{-x}\sin\frac{x}{2} - 4e^{-x}\cos\frac{x}{2} - 4\int e^{-x}\cos\frac{x}{2}dx$$

Let $\int e^{-x}\cos\frac{x}{2}dx = I$

$$I = 2e^{-x}\sin\frac{x}{2} - 4\,e^{-x}\cos\frac{x}{2} - 4I$$

$$5I = 2e^{-x}\sin\frac{x}{2} - 4e^{-x}\cos\frac{x}{2}$$

$$I = \frac{2}{5}\left(e^{-x}\sin\frac{x}{2} - 2e^{-x}\cos\frac{x}{2}\right) + C$$

$$\int e^{-x}\cos\frac{x}{2}dx = \frac{2}{5}\left(e^{-x}\sin\frac{x}{2} - 2e^{-x}\cos\frac{x}{2}\right) + C$$

Integration of inverse trigonometric functions

Example I

$$\int \tan^{-1}x\,dx$$

Solution

$$\int (tan^{-1}x)dx = \int (tan^{-1}x)(1)dx$$

x^0 = 1 = algebraic function (A)

$tan^{-1}x = inverse\ algebraic\ function\ (I)$

'I' comes before 'A' in the word ***LIATE***

$\Longrightarrow u = tan^{-1}x$

$\tan u = x$

$sec^2u\,du = dx$

$(1 + tan^2u)du = dx$

$$\frac{du}{dx} = \frac{1}{1 + tan^2u}$$

$$\frac{du}{dx} = \frac{1}{1 + x^2}$$

$$\frac{dv}{dx} = 1$$

$v = x$

$$\int tan^{-1}x\,dx = xtan^{-1}x - \int \frac{x}{1 + x^2}dx$$

$$\Longrightarrow \int tan^{-1}x\,dx = xtan^{-1}x - \frac{1}{2}\ln(1 + x^2) + C$$

Example II

$$\int sin^{-1}x\,dx = \int (sin^{-1}x)\,1dx$$

$u = sin^{-1}x$

$\sin u = x$

$\cos u\,du = dx$

$$\frac{du}{dx} = \frac{1}{\cos u}$$

$cos^2u + sin^2u = 1$

$\cos u = \sqrt{1 - sin^2u}$

$\cos u = \sqrt{1 - x^2}$

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{dv}{dx} = 1$$
$$v = x$$
$$\int (sin^{-1}x)dx = x(sin^{-1}x) - \int \frac{x}{\sqrt{1-x^2}}dx$$

Consider
$$\int \frac{x}{\sqrt{1-x^2}}dx$$
Let $\sqrt{1-x^2} = p$
$$1 - x^2 = p^2$$
$$-2x\,dx = 2p\,dp$$
$$dx = -\frac{pdp}{x}$$
$$\int \frac{x}{\sqrt{1-x^2}}dx = \int \frac{x}{p} \times \frac{-pdp}{x}$$
$$= -p + C$$
$$= -\sqrt{1-x^2}$$
$$\int (\,sin^{-1}x)dx = xsin^{-1}x + \sqrt{1-x^2} + C$$

Example III
$$\int cos^{-1}x\,dx$$
$$u = \cos^{-1}x$$
$$\cos u = x$$
$$-\sin u\,du = dx$$
$$\frac{du}{dx} = -\frac{1}{\sin u}$$
$$\frac{du}{dx} = -\frac{1}{\sqrt{1-cos^2u}}$$
$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{dv}{dx} = 1$$
$$v = x$$
$$\int cos^{-1}x\,dx = xcos^{-1}x - \int x\,.-\frac{1}{\sqrt{1-x^2}}dx$$
$$\int cos^{-1}x\,dx = xcos^{-1}x - \int \frac{x}{\sqrt{1-x^2}}dx$$
Consider $\int \frac{x}{\sqrt{1-x^2}}dx$
$$let\sqrt{1-x^2} = p$$
$$1 - x^2 = p^2$$
$$-2x\,dx = 2pdp$$
$$dx = \frac{-dp}{x}$$
$$\int \frac{x}{\sqrt{1-x^2}}dx = \int \frac{x}{p} \times \frac{-pdp}{x} = -p + C$$
$$\int \frac{x}{\sqrt{1-x^2}}dx = -\sqrt{1-x^2} + C$$
$$\int \cos^{-1}x\,dx = xcos^{-1}x + \sqrt{1-x^2} + C$$

Example IV
$$\int x^2 \tan^{-1}x\,dx$$
$$let\; u = \tan^{-1}x$$
$$\tan u = x$$
$$sec^2u\,du = dx$$
$$\frac{du}{dx} = \frac{1}{sec^2u}$$
$$\frac{du}{dx} = \frac{1}{1+tan^2u}$$
$$\frac{du}{dx} = \frac{1}{1+x^2}$$
$$\frac{dv}{dx} = x^2$$
$$v = \frac{x^3}{3}$$
$$\int x^2 \tan^{-1}x\,dx = \frac{x^3}{3}\tan^{-1}x - \int \frac{x^3}{3(1+x^2)}dx$$
$$\int x^2 \tan^{-1}x\,dx = \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int \frac{x^3}{1+x^2}dx$$
$$\begin{array}{r|l} & x \\ \hline x^2+1 & x^3 \\ & -(x^3+x) \\ \hline & -x \end{array}$$
$$\Rightarrow \frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$$
$$\int \frac{x^3}{x^2+1}dx = \int x - \frac{x}{x^2+1}dx$$
$$= \frac{x^2}{2} - \frac{1}{2}\ln(x^2+1) + C$$
$$\Rightarrow \int x^2 \tan^{-1}x\,dx$$
$$= \frac{x^3}{3}\tan^{-1}x - \frac{x^2}{6} - \frac{1}{6}\ln(x^2+1) + C$$

Change of Variable

(1) $t = \tan\frac{x}{2}$

(2) $t = \tan x$

The above substitution can be applied to integration of certain trigonometric functions

> Case I
> Where the denominator of the variable being integrated is a linear function of the trigonometric function.
> e.g. $\boldsymbol{C + \cos x}$
> $\boldsymbol{C + \sin x}$
> $\boldsymbol{C + \sec x}$
> Where C is a constant
> We use the substitution $\mathbf{t = \tan\frac{x}{2}}$

> **Case II**
> When the expression being integrated is a linear function of the second under trigonometric function
> e.g.
> $(\boldsymbol{i})$ $\boldsymbol{C + cos^2 x}$
> $(\boldsymbol{ii})$ $\boldsymbol{C + sin^2 x}$
> $(\boldsymbol{iii})$ $\boldsymbol{C + sec^2 x}$
> $(\boldsymbol{iv})$ $\boldsymbol{C + \sin 2x}$
> $(\boldsymbol{v})$ $\boldsymbol{C + \cos 2x}$ **Etc.**
> We use substitution $\boldsymbol{t = \tan x}$

Note $when\ t = \tan\frac{x}{2}$

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$when\ t = \tan x$

$when\ t = \tan x$

$$\sin 2x = \frac{2t}{1+t^2}$$

$$\cos 2x = \frac{1-t^2}{1+t^2}$$

Proof (students' exercise).

Example I

Integrate the following:

$(a) \int \frac{1}{(1+\cos\theta)} d\theta$ $\quad (b) \int \sec 2\theta\, d\theta$

$(c) \int cosec\frac{x}{2} dx$ $\quad (d) \int \frac{1}{1+\sin 2x} dx$

$(e) \int \frac{1}{5+3\cos\frac{1}{2}\theta} d\theta$ $\quad (f) \int \frac{1}{1+2sin^2 x} dx$

$(g) \int \frac{sin^2 x}{1+cos^2 x} dx$ $\quad (h) \int \frac{1}{1-10sin^2 x} dx$

$(i) \int \frac{1}{\cos 2x - 3sin^2 x} dx$

Solution

(a) $\int \frac{1}{1+\cos\theta} d\theta$

$$let\ t = \tan\frac{\theta}{2}$$

$$dt = \frac{1}{2} sec^2\frac{\theta}{2} d\theta$$

$$\frac{2dt}{sec^2\frac{\theta}{2}} = d\theta \quad \Rightarrow d\theta = \frac{2dt}{1+t^2}$$

$$\int \frac{1}{1+cos\theta} d\theta = \int \frac{1}{1+\frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$= \int \left(\frac{1}{\frac{1+t^2+1-t^2}{1+t^2}}\right) . \frac{2dt}{1+t^2}$$

$$= \int dt$$

$$= t + C$$

$$= \tan\frac{\theta}{2} + C$$

$$\Rightarrow \int \frac{1}{1+\cos\theta} d\theta = \tan\frac{\theta}{2} + C$$

(b) $\int \sec 2\theta\, d\theta$

$$\int \sec 2\theta\, d\theta = \int \frac{1}{\cos 2\theta} d\theta$$

$$let\ t = \tan\theta$$

$$dt = sec^2\theta\, d\theta$$

$$d\theta = \frac{dt}{sec^2\theta} = \frac{dt}{(1+t^2)}$$

$$\int \sec 2\theta\, d\theta = \int \left(\frac{1}{\frac{1-t^2}{1+t^2}}\right) \times \frac{dt}{1+t^2}$$

$$= \int \frac{1}{1-t^2} dt$$

$$\frac{1}{1-t^2} = \frac{1}{(1+t)(1-t)} = \frac{A}{1+t} + \frac{B}{1-t}$$

$$A(1-t) + B(1+t) = 1$$

If $t = 1$, $2B = 1 \ \Rightarrow\ B = \frac{1}{2}$

If t = -1, $2A = 1 \Rightarrow A = \frac{1}{2}$

$$\int \frac{1}{2(1+t)} + \frac{1}{2(1-t)} dt$$

$$\frac{1}{2}\ln(1+t) - \frac{1}{2}\ln(1-t) + C$$

$$\frac{1}{2}\ln\left(\frac{1+t}{1-t}\right) + C$$

$$\frac{1}{2}\ln\left(\frac{1+\tan\theta(1+\tan\theta)}{1-\tan\theta(1+\tan\theta)}\right) + C$$

$$\frac{1}{2}\ln\left(\frac{1+2\tan\theta + tan^2\theta}{1-tan^2\theta}\right) + C$$

$$\frac{1}{2}\ln\left(\frac{2\tan\theta}{1-tan^2\theta} + \frac{1+tan^2\theta}{1-tan^2\theta}\right) + C$$

$$\frac{1}{2}\ln(\tan 2\theta + \sec 2\theta) + C$$

(C) $\int cosec\frac{x}{2}dx = \int \frac{1}{\sin\frac{x}{2}dx}$

Let $t = \tan\frac{x}{4}$

$$dt = \frac{1}{4}sec^2\frac{x}{4}dx$$

$$dx = \frac{4dt}{sec^2\frac{x}{2}}$$

$$\int \frac{1}{\sin\frac{x}{2}dx} = \int\left(\frac{\frac{1}{2t}}{1+t^2}\right).\frac{4dt}{1+t^2}$$

$$= \int \frac{2}{t}dt$$

$$= 2\ln t + C$$

$$= 2\ln\tan\frac{x}{4} + C$$

$$\int cosec\frac{x}{2}dx = \ln tan^2\frac{x}{4} + C$$

(d) $\int \frac{1}{(1+\sin 2x)}\ dx$

$let\ t = \tan x$

$dt = sec^2x\ dx$

$$\int \frac{1}{1+\sin 2x}dx = \int \frac{1}{1+\frac{2t}{1+t^2}}.\frac{dt}{1+t^2}$$

$$= \int \frac{\frac{1}{1+2t+t^2}}{1+t^2} \times \frac{dt}{1+t^2}$$

$$\int \frac{1}{(1+t^2)}dt = \frac{-1}{1+t} + C$$

$$= \frac{-1}{(1+\tan x)}$$

$$\Rightarrow \int \frac{1}{1+\sin 2x}dx = \frac{-1}{1+\tan x} + C$$

(e) $\int \frac{1}{5+3\cos\frac{1}{2}\theta}d\theta$

$let\ t = \tan\frac{\theta}{4}$

$$dt = \frac{1}{4}sec^2\frac{\theta}{4}d\theta$$

$$\frac{4dt}{sec^2\frac{\theta}{4}} = d\theta$$

$$\int \frac{1}{5+\frac{3(1-t^2)}{1+t^2}}.\frac{4dt}{1+t^2}$$

$$\int \frac{\frac{1}{5+5t^2+3-3t^2}}{1+t^2}.\frac{4dt}{1+t^2}$$

$$\int \frac{4}{8-2t^2}dt = \int \frac{2}{4-t^2}dt$$

$$\frac{2}{4-t^2} = \frac{2}{(2+t)(2-t)} = \frac{A}{2+t} + \frac{B}{2-t}$$

$$A(2-t) + B(2+t) = 2$$

If t = 2, $4B = 2 \Rightarrow B = \frac{1}{2}$

If t = -2, $4A = 2 \Rightarrow A = \frac{1}{2}$

$$\int \frac{2}{4-t^2}dt = \int \frac{1}{2(2+t)} - \frac{1}{2(2-t)}dt$$

$$= \frac{1}{2}\ln(2+t) - \frac{1}{2}\ln(2-t) + C$$

$$= \frac{1}{2}\ln\left(\frac{2+t}{2-t}\right) + C$$

$$= \ln\left(\frac{2+\tan\frac{\theta}{4}}{2-\tan\frac{\theta}{4}}\right) + C$$

(f) $\int \frac{1}{1+2sin^2x}dx$

Solution

$$\int \frac{1}{1+2sin^2x}dx = \int \frac{\frac{1}{cos^2x}}{\frac{1}{cos^2x} + \frac{2sin^2x}{cos^2x}}dx$$

$$= \int \frac{sec^2x}{sec^2x + 2tan^2x} dx$$
$$= \int \frac{1 + tan^2x}{1 + tan^2x + 2\,tan^2x} dx$$
$$= \int \frac{1 + tan^2x}{1 + 3tan^2x} dx$$

Let $t = \tan x$

$$dt = sec^2x\, dx$$
$$dx = \frac{dt}{sec^2x}$$
$$\int \frac{1 + tan^2x}{1 + 3tan^2x} dx = \int \frac{1 + t^2}{1 + 3t^3} \times \frac{dt}{1 + t^2}$$
$$= \int \frac{1}{1 + 3t^2} dt$$
$$let\ (\sqrt{3})t = \tan\theta$$
$$\sqrt{3}dt = sec^2\theta\, d\theta$$
$$dt = \frac{sec^2\theta\, d\theta}{\sqrt{3}}$$
$$\int \frac{1}{1 + 3t^2} dt = \int \frac{1}{1 + tan^2\theta} . \frac{sec^2\theta}{\sqrt{3}} d\theta$$
$$\frac{1}{\sqrt{3}} \int d\theta$$
$$\frac{1}{\sqrt{3}}\theta + C$$
$$= \frac{1}{\sqrt{3}} tan^{-1}(\sqrt{3}t) + C$$
$$= \frac{1}{\sqrt{3}} tan^{-1}(\sqrt{3}\tan x) + C$$
$$\int \frac{1}{1 + 2sin^2x} dx = \frac{1}{\sqrt{3}} tan^{-1}(\sqrt{3}\tan x) + C$$

(g) $\int \frac{sin^2x}{1 + cos^2x} dx$

$$\int \frac{sin^2x}{1 + cos^2x} dx = \int \frac{\frac{sin^2x}{cos^2x}}{\frac{1}{cos^2x} + \frac{cos^2x}{cos^2x}} dx$$
$$= \int \frac{tan^2x}{sec^2x + 1} dx$$
$$let\ t = \tan x$$
$$dt = sec^2x\, dx$$
$$dx = \frac{dt}{sec^2x}$$
$$\int \frac{sin^2x}{1 + cos^2x} dx = \int \frac{\tan^2x}{sec^2x + 1} dx$$
$$= \int \frac{t^2}{1 + t^2 + 1} . \frac{dt}{1+t^2}$$
$$= \int \frac{t^2}{(2 + t^2)(1 + t^2)} dt$$
$$\frac{t^2}{(2 + t^2)(1 + t^2)} = \frac{At + B}{2 + t^2} + \frac{Ct + D}{1 + t^2}$$
$$(At + B)(1 + t^2) + (Ct + D)(2 + t^2) = t^2$$
$$At + At^3 + B + Bt^2 + 2Ct + Ct^3 + 2D + Dt^2 = t^2$$
$$A + C = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$
$$B + D = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$
$$A + 2C = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$
$$B + 2D = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$
$$from\ eqn\ (1). C = -A$$

Substituting *C* = -*A*, in Eqn (3)

$$A - 2A = 0$$
$$-A = 0$$
$$A = 0$$
$$C = 0$$

From Eqn (4), *B* = -2*D*

Substituting *B* = -2*D* in Eqn 2

$$-2D + D = 1$$
$$-D = 1$$
$$D = -1$$
$$B + D = 1$$
$$B - 1 = 1$$
$$B = 2$$
$$\int \frac{t^2}{4 - t^2} dt = \int \frac{2}{2 + t^2} - \frac{1}{1 + t^2} dt$$
$$\left[\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} tan^{-1} \frac{1}{a} x + C\right]$$
$$\int \frac{2}{2 + t^2} - \frac{1}{1 + t^2} dt =$$
$$\left[2\left(\frac{1}{\sqrt{2}} tan^{-1}\left(\frac{1}{\sqrt{2}} t\right)\right) - tan^{-1}t + C\right]$$
$$\int \frac{sin^2x}{1 + cos^2x} dx = \frac{2\sqrt{2}}{2} tan^{-1}\left(\frac{\sqrt{2}t}{2}\right) - tan^{-1}t + C$$
$$= \sqrt{2}\, tan^{-1}\left(\frac{\sqrt{2}}{2}\tan x\right) - tan^{-1}(\tan x) + C$$
$$\int \frac{1}{1 - 10sin^2x} dx = \int \frac{\frac{1}{cos^2x}}{\frac{1}{cos^2x} - \frac{10sin^2x}{cos^2x}} dx$$
$$= \int \frac{sec^2x}{sec^2x - 10tan^2x} dx$$

$$let\ t = \tan x$$
$$dt = sec^2x\,dx$$
$$dx = \frac{dt}{sec^2x}$$
$$\int \frac{sec^2x}{sec^2x - 10tan^2x}.\frac{dt}{sec^2x}$$
$$\int \frac{1}{1 + t^2 - 10t^2}dt$$
$$\int \frac{1}{1 - 9t^2}dt$$
$$\frac{1}{1 - 9t^2} = \frac{1}{1 - 3^2t^2} = \frac{1}{(1 + 3t)(1 - 3t)}$$
$$\frac{A}{1 + 3t} + \frac{B}{1 - 3t}$$
$$A(1 - 3t) + B(1 + 3t) = 1$$
If $t = \frac{1}{3}$, $2B = 1$
$$B = \frac{1}{2}$$
If $t = -\frac{1}{3}$, $2A = 1$
$$A = \frac{1}{2}$$
$$\int \frac{1}{2(1 + 3t)} + \frac{1}{2(1 - 3t)}dt$$
$$\frac{1}{6}\ln(1 + 3t) - \frac{1}{6}\ln(1 - 3t)$$
$$= \frac{1}{6}\ln\left(\frac{1 + 3t}{1 - 3t}\right) + C$$
$$= \frac{1}{6}\ln\left(\frac{1 + 3\tan x}{1 - 3\tan x}\right) + C$$

(**i**) $\int \frac{1}{(\cos 2x - 3sin^2x)}\ dx$
$$= \int \frac{1}{cos^2x - sin^2x - 3sin^2x}dx$$
$$= \int \frac{1}{cos^2x - 4sin^2x}dx$$
$$\int \frac{\frac{1}{cos^2x}}{\frac{cos^2x}{cos^2x} - \frac{4sin^2x}{cos^2x}}dx = \int \frac{sec^2x}{1 - 4tan^2x}dx$$
$$let\ t = \tan x$$
$$dt = sec^2x\,dx$$
$$dx = \frac{dt}{sec^2x}$$
$$\int \frac{sec^2x}{1 - 4tan^2x}dx = \int \frac{sec^2x}{1 - 4t^2}.\frac{dt}{sec^2x}$$
$$= \int \frac{1}{1 - 4t^2}dt$$
$$\frac{1}{1 - 4t^2} = \frac{1}{(1 + 2t)(1 - 2t)}$$
$$\frac{A}{1 + 2t} + \frac{B}{1 - 2t}$$
$$A(1 - 2t) + B(1 + 2t) = 1$$
If $t = \frac{1}{2}$, $2B = 1$
$$B = \frac{1}{2}$$
If $t = -\frac{1}{2}$, $2A = 1$
$$A = \frac{1}{2}$$
$$\int \frac{1}{1 - 4t^2}dt = \int \frac{1}{2(1 + 2t)} + \frac{1}{2(1 - 2t)}dt$$
$$\frac{1}{4}\ln\left(\frac{1 + 2t}{1 - 2t}\right) + C$$
$$\frac{1}{4}\ln\left(\frac{1 + 2\tan x}{1 - 2\tan x}\right) + C$$

Splitting the Numerator

When a fractional integrand with a quadratic denominator cannot be written in simple partial fractions, it is often to express it as a sum of two fractions by splitting the numerator.
$$\int \frac{1 + x}{1 + x^2}dx = \int \frac{1}{1 + x^2} + \frac{x}{1 + x^2}dx$$
$$= \tan^{-1}x + \frac{1}{2}\ln(1 + x^2) + C$$
The key to a more general application of this method is to express the numerator in two parts, one of which is a multiple of the derivative of the denominator.

Numerator = A(Derivative of denominator) + B

Example
$$\int \frac{5x + 7}{x^2 + 4x + 8}dx$$

Formula
Numerator = A(derivative of denominator) + B

$$5x + 7 = A(2x + 4) + B$$
$$5x + 7 = 2Ax + 4A + B$$
Equating coefficients of the same monomial;
$$5 = 2A,\ 4A + B = 7$$
$$A = \frac{5}{2}$$
$$4A + B = 7$$
$$4\left(\frac{5}{2}\right) + B = 7$$

$$10 + B = 7$$
$$B = -3$$
$$\int \frac{5x+7}{x^2+4x+8}dx = \int \frac{\frac{5}{2}(2x+4) + \ -3}{x^2+4x+8}dx$$
$$\int \frac{\frac{5}{2}(2x+4)}{x^2+4x+8} - \frac{3}{x^2+4x+8}dx$$
$$= \frac{5}{2}\ln(x^2+4x+8) - 3\int \frac{1}{x^2+4x+8}dx$$
Consider
$$x^2+4x+8 = x^2+4x+4-4+8$$
$$= (x+2)^2+4$$
$$\Rightarrow \int \frac{1}{x^2+4x+8}dx = \int \frac{1}{(4+(x+2)^2)}dx$$
$$= \int \frac{1}{4\left(1+\frac{(x+2)^2}{4}\right)}dx$$
Let $\frac{x+2}{2} = \tan\theta$
$$\frac{1}{2}dx = sec^2\theta\ d\theta$$
$$dx = 2sec^2\theta\ d\theta$$
$$\Rightarrow \int \frac{1}{4\left(1+\frac{(x+2)^2}{4}\right)}dx = \int \frac{1}{4(1+tan^2\theta)}2sec^2\theta\ d\theta$$
$$= \frac{1}{2}\theta + C$$
$$= \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + C$$
$$\int \frac{5x+7}{x^2+4x+8}dx$$
$$= \frac{5}{2}\ln(x^2+4x+8) - \frac{3}{2}tan^{-1}\left(\frac{x+2}{2}\right) + A$$

Example II
$$\int \frac{3x+4}{9x^2+6x+5}dx$$
Solution

Numerator = A(Derivative of denominator) + B
$$((3x)+4) = A(18x+6)+B$$
$$3x+4 = 18Ax+6A+B$$
$$A = \frac{1}{6}$$
$$6A+B = 4$$
$$B = 3$$
$$\int \frac{3x+4}{(9x^2+6x+5)}dx = \int \frac{\left(\frac{1}{6}(18x+6)+3\right)}{9x^2+6x+5}dx$$
$$= \frac{1}{6}\int \frac{18x+6}{9x^2+6+5}dx + \int \frac{3}{9x^2+6x+5}dx$$
$$\frac{1}{6}\ln(9x^2+6x+5) + 3\int \frac{1}{9x^2+6x+5}dx$$
But $9x^2+6x+5$
$$= 9\left(x^2+\frac{6x}{9}\right)+5$$
$$= 9\left(x^2+\frac{2}{3}x\right)+5$$
$$= 9\left(x^2+\frac{2}{3}x\right)+5$$
$$= 9\left(x^2+\frac{2}{3}x+\frac{1}{9}\right)+5-1$$
$$= 4+9\left(x+\frac{1}{3}\right)^2$$
$$= 4\left(1+\frac{9}{4}\left(x+\frac{1}{3}\right)^2\right)$$
$$\int \frac{1}{9x^2+6x+5}dx = \int \frac{1}{4\left(1+\frac{9}{4}\left(x+\frac{1}{3}\right)^2\right)}dx$$
$$let\, \frac{3}{2}\left(x+\frac{1}{3}\right) = \tan\theta$$
$$\frac{3}{2}dx = sec^2\theta\ d\theta$$
$$dx = \frac{2\ sec^2\theta}{3}d\theta$$
$$\int \frac{1}{4(1+tan^2\theta)}.\frac{2\ sec^2\theta}{3}d\theta = \frac{1}{6}\theta + C$$
$$\int \frac{1}{9x^2+6x+5}dx = \frac{1}{6}tan^{-1}\left[\frac{3}{2}\left(\frac{3x+1}{3}\right)\right] + C$$
$$= \frac{1}{6}tan^{-1}\left(\frac{3x+1}{2}\right) + C$$
$$\Rightarrow \int \frac{3x+4}{(9x^2+6x+5)}dx$$
$$= \frac{1}{6}\ln(9x^2+6x+5) + \frac{1}{2}tan^{-1}\left(\frac{3x+1}{2}\right) + C$$
Example III
$$\int \frac{x}{2x^2-x+1}dx$$
Solution
$$x = A(4x-1)+B$$
$$4A = 1$$
$$A = \frac{1}{4}$$
$$-A+B = 0$$
$$A = B$$

$$B = \frac{1}{4}$$

$$\Rightarrow \int \frac{x}{2x^2 - x + 1} dx = \int \frac{\frac{1}{4}(4x-1) + \frac{1}{4}}{2x^2 - x + 1} dx$$

$$= \frac{1}{4}\int \frac{4x-1}{2x^2 - x + 1} dx + \frac{1}{4}\int \frac{1}{2x^2 - x + 1} dx$$

$$= \frac{1}{4}\ln(2x^2 - x + 1) + \frac{1}{4}\int \frac{1}{2x^2 - x + 1} dx$$

Consider

$$2x^2 - x + 1 = 2\left(x^2 - \frac{x}{2}\right) + 1$$

$$2\left(x^2 - \frac{x}{2} + \frac{1}{16}\right) + 1 - \frac{1}{8}$$

$$2\left(x - \frac{1}{4}\right)^2 + \frac{7}{8}$$

$$\Rightarrow \int \frac{1}{2x^2 - x + 1} dx = \int \frac{1}{\frac{7}{8}\left(1 + \frac{16}{7}\left(x - \frac{1}{4}\right)^2\right)} dx$$

$$let \frac{4}{\sqrt{7}}\left(x - \frac{1}{4}\right) = \tan\theta$$

$$\frac{4}{\sqrt{7}} dx = sec^2\theta \, d\theta$$

$$dx = \frac{\sqrt{7} sec^2\theta d\theta}{4}$$

$$\int \frac{8}{7(1 + tan^2\theta)} . \frac{\sqrt{7} sec^2\theta}{4} d\theta$$

$$\frac{2\sqrt{7}}{7}\theta + C$$

$$= \frac{2\sqrt{7}}{7}\tan^{-1}\frac{4}{\sqrt{7}}\left(x - \frac{1}{4}\right) + C$$

$$\Rightarrow \int \frac{x}{2x^2 - x + 1} dx =$$

$$\frac{1}{4}\ln(2x^2 - x + 1) + \frac{\sqrt{7}}{14} tan^{-1}\left(\frac{4x-1}{\sqrt{7}}\right) + C$$

Splitting the numerator for trigonometric functions

The above method is appropriate to integrals of the form

$$\frac{a\cos x + b\sin x}{a\cos x + b\sin x}$$

When splitting the numerator for the trigonometric functions

Numerator = A (derivative of the denominator) + B (Denominator)

Example

$$\int \frac{2\cos x + 9\sin x}{3\cos x + \sin x} dx$$

$2\cos x + 9\sin x$

$$= A(-3\sin x + \cos x) + B(3\cos x + \sin x)$$

$$= (-3A + B)\sin x + (A + 3B)\cos x$$

$$= (A + 3B)\cos x + (B - 3A)\sin x$$

$$A + 3B = 2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

$$B - 3A = 9 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

From Eqn (i);

$$A = 2 - 3B$$

$$B - 3(2 - 3B) = 9$$

$$B - 6 + 9B = 9$$

$$10B = 15$$

$$B = \frac{3}{2}$$

$$A = 2 - 3\left(\frac{3}{2}\right)$$

$$A = -\frac{5}{2}$$

$$\int \frac{2\cos x + 9\sin x}{3\cos x + \sin x} dx$$

$$= \int \frac{-\frac{5}{2}(-3\,sin\,x + cos\,x)}{3\,cos\,x + sin\,x} dx + \int \frac{\frac{3}{2}(3\,cos\,x + sinx)}{3\,cos\,x + sin\,x} dx$$

$$= -\frac{5}{2}\int \frac{-3\sin x + \cos x}{3\cos x + \sin x} dx + \int \frac{3}{2} dx$$

$$= -\frac{5}{2}\ln(3\cos x + \sin x) + \frac{3}{2}x + C$$

Example II

$$\int \frac{\sin x}{\cos x + \sin x} dx$$

Solution

Numerator=A(Derivative of the Denominator) + B(Denominator)

$\sin x = A(-\sin x + \cos x) + B(\cos x + \sin x)$

$\sin x = -A\sin x + A\cos x + B\cos x + B\sin x$

$\sin x = (B - A)\sin x + (A + B)\cos x$

$$B - A = 1 \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$A + B = 0 \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Solving Eqn (i) and Eqn (ii) simultaneously,

$$\Rightarrow A = \frac{-1}{2}, \quad B = \frac{1}{2}$$

$$\int \frac{\sin x}{\cos x + \sin x} dx =$$

$$\int \frac{-\frac{1}{2}(-\sin x + \cos x)}{\cos x + \sin x} + \frac{\frac{1}{2}(\cos x + \sin x)}{\cos x + \sin x} dx$$

$$-\frac{1}{2}\int \frac{-\sin x + \cos x}{\cos x + \sin x} dx + \frac{1}{2}\int \frac{\cos x + \sin x}{\cos x + \sin x} dx$$

$$= -\frac{1}{2}\ln(\cos x + \sin x) + \frac{1}{2}x$$

Example III

$$\int \frac{2\cos x + 3\sin x}{\cos x + \sin x} dx$$

Solution

$2\cos x + 3\sin x =$

$A(-\sin x + \cos x) + B(\cos x + \sin x)$

$$A + B = 2 \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$-A + B = 3 \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Solving Eqn (i) and Eqn (ii) simultaneously

$$A = -\frac{1}{2}, \quad B = \frac{5}{2}$$

$$\Rightarrow \int \frac{2\cos x + 3\sin x}{\cos x + \sin x} dx =$$

$$\int \frac{\frac{-1}{2}(-\sin x + \cos x)}{\cos x + \sin x} + \frac{\frac{5}{2}(\cos x + \sin x)}{\cos x + \sin x} dx$$

$$= -\frac{1}{2}\ln(\cos x + \sin x) + \frac{5}{2}x + C$$

Example IV

Show that $\int_0^{\frac{\pi}{2}} \frac{\sin x}{3\sin x + 4\cos x} dx = \frac{3\pi}{50} + \frac{4}{25}\ln\left(\frac{4}{3}\right)$

Solution

Consider $\int \frac{\sin x}{3\sin x + 4\cos x} dx$

$\sin x = A(3\cos x - 4\sin x) + B(3\sin x + 4\cos x)$

$3A + 4B = 0$ ... ... (1)

$3B - 4A = 1$ ... ... (2)

From Eqn (1) $A = -\frac{4B}{3}$

$$3B - 4\left(-\frac{4B}{3}\right) = 1$$

$$3B + \frac{16B}{3} = 1$$

$$\frac{25B}{3} = 1$$

$$B = \frac{3}{25}$$

$$A = -\frac{4}{3}\left(\frac{3}{25}\right) = -\frac{4}{25}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin x}{3\sin x + 4\cos x} dx$$

$$.= \int_0^{\pi/2} \frac{-\frac{4}{25}(3\cos x - 4\sin x)}{3\sin x + 4\cos x} dx + \frac{3}{25}\int_0^{\pi/2} \frac{(3\sin x + 4\sin x)}{3\sin x + 4\cos x} dx$$

$$= -\frac{4}{25}\ln(3\sin x + 4\cos x)\Big|_0^{\pi/2} + \frac{3x}{25}\Big|_0^{25}$$

$$= -\frac{4}{25}\ln 3 + \frac{3\pi}{50} + \frac{4}{25}(\ln 4)$$

$$= \frac{3\pi}{50} + \frac{4}{25}\ln\left(\frac{4}{3}\right)$$

Revision Exercise

1. Express in partial fractions.

   (a) $\frac{x-11}{(x+3)(x-4)}$ (b) $\frac{x}{25-x^2}$

   (c) $\frac{3x^2-21x+24}{(x+1)(x-2)(x-3)}$ (d) $\frac{4x^2+x+1}{x(x^2-1)}$

   (e) $\frac{8x^2+15x^2-15x-5}{(x+3)(x-2)}$

   (f) $\frac{2x^3+x^2-15-5}{(x+3)(x-2)}$

2. Express the following in partial fractions:

   (a) $\frac{5x^2-10x+11}{(x-3)(x^2+4)}$ (b) $\frac{2x^2-x+3}{(x+1)(x^2+2)}$

   (c) $\frac{3x^2-2x+5}{(x-1)(x^2+5)}$ (d) $\frac{11x}{(2x-3)(2x^3+1)}$

   (e) $\frac{20x+84}{(x+5)(x^2-9)}$ (f) $\frac{2x^3-x-1}{(x-3)(x^2+1)}$

3. Express the following in partial fractions:

   (a) $\frac{x-5}{(x-2)^2}$ (b) $\frac{5x+4}{(x-1)(x+2)}$

   (c) $\frac{5x^2+2}{(3x+1)(x+1)^2}$ (d) $\frac{x^4+3x-1}{(x+2)(x-1)^2}$

4. Express in partial fractions

   (a) $\frac{3x+7}{x(x+2)(x-1)}$ (b) $\frac{3}{x^2(x+2)}$

   (c) $\frac{2x^4-17x-1}{(x-2)(x^2+5)}$ (d) $\frac{68+11x}{(3+x)(16-x^2)}$

   (e) $\frac{2x+1}{(x^3-1)}$ (f) $\frac{2x^2+39x+12}{(2x+1)^2(x-3)}$

5. Find the following integrals

(a) $\int \frac{1}{x(x-2)} dx$ (b) $\int \frac{1}{(x+3)(5x-2)} dx$

(c) $\int \frac{7x+2}{3x^3+x^2} dx$ (d) $\int \frac{x}{16-x^2} dx$

(e) $\int \frac{1}{x^2-4x-5} dx$ (f) $\int \frac{x-2}{x^2-4x-5} dx$

(g) $\int \frac{2x^2+2x+3}{(x+2)(x^2+3)} dx$ (h) $\int \frac{22-16x}{(3+x)(2-x)(4-x)} dx$

(i) $\int \frac{4x-33}{(2x+1)(x^2-9)} dx$

6. Evaluate the following, correct to 3 significant figures.

(a) $\int_3^5 \frac{2}{x^2-1} dx$ (b) $\int_{-1}^0 \frac{2}{(1-x)(1+x^2)} dx$

(c) $\int_2^3 \frac{x-9}{x(x-1)(x+3)} dx$ (d) $\int_0^3 \frac{13x+7}{(x-4)(3x^2+2x+3)} dx$

7. Find the following indefinite integrals

(a) $\int \frac{3}{(x-1)(x+2)} dx$ (b) $\int \frac{1}{1-x^2} dx$

(c) $\int \frac{1}{x(x-3)} dx$ (d) $\int \frac{x}{x^2-4} dx$

(e) $\int \frac{4x}{x^2-2x-3} dx$ (f) $\int \frac{2x-5}{(x-2)(x-3)} dx$

8. Evaluate the following definite integrals

(a) $\int_1^2 \frac{x}{(x+1)(x+2)} dx$ (b) $\int_3^4 \frac{5}{x^2+x-6} dx$

(c) $\int_4^5 \frac{2x}{x^2-4x+3} dx$ (d) $\int_0^{1/3} \frac{3x}{1-x^2} dx$

(e) $\int_0^{1/3} \frac{3+x}{(1-x)(1+3x)} dx$ (f) $\int_2^3 \frac{1}{x(x-1)} dx$

9. Find the following indefinite integrals

(a) $\int x \sin x \, dx$ (b) $\int x \cos \tfrac{1}{2} x \, dx$

(c) $\int x e^{-x} dx$ (d) $\int x \ln 2x \, dx$

(e) $\int x e^{2x} dx$ (f) $\int x \sec^2 x \, dx$

(g) $\int x \sin x \cos x \, dx$ (h) $\int x \tan^2 x \, dx$

(i) $\int x \cos^2 x \, dx$ (j) $\int x^2 \cos x \, dx$

10. Find the following definite integrals

(a) $\int_0^{\pi} x \cos x dx$ (b) $\int_{-1}^{1} x e^x dx$

(c) $\int_0^{\pi/3} x \sin 3x \, dx$ (d) $\int_1^e \ln x \, dx$

(e) $\int_0^{\pi} x \sin^2 x \, dx$ (f) $\int_1^2 x^2 \ln x \, dx$

(g) $\int_2^3 x^2 e^{-x} \, dx$ (h) $\int_1^e \frac{1}{x^3} \ln x \, dx$

(i) $\int_0^{\pi/2} x^2 \sin x \, dx$ (j) $\int_0^{\pi} e^x \sin x \, dx$

11. Find $\int x(x+1)^4 dx$ using integration by parts

12. Find the integral of the following

(a) $\int x\sqrt{(x-1)} \, dx$ (b) $\int \frac{x}{(x+2)^3} dx$

13. Show that:

(a) $2\int \cos^2 x \, dx = \cos x \sin x + x + c$

(b) $3\int \cos^3 x \, dx = \cos^2 x \sin x + 2\sin x + c$

14. Find the following integrals, using the the given change of variable.

(a) $\int 3x\sqrt{14x-1} \, dx$, $\sqrt{14x-1} = u$

(b) $\int x\sqrt{5x+2} \, dx$, $\sqrt{5x+2} = u$

(c) $\int x(2x-1)^6 \, dx$, $2x - 1 = u$

(d) $\int \frac{x}{\sqrt{x-2}} \, dx$, $\sqrt{x-2} = u$

(e) $\int (x+2)(x-1)^4 \, dx$, $x - 1 = u$

(f) $\int (x-2)^5(x+3)^2 \, dx$, $x - 2 = u$

(g) $\int \frac{x(x-4)}{(x-2)^2} \, dx$, $x - 2 = u$

(e) $\int \frac{(x-1)}{\sqrt{(2x+3}} \, dx$, $\sqrt{(2x+3} = u$

15. Find the following integrals using a suitable change of variables only when necessary.

(a) $\int x\sqrt{2x^2+1} \, dx$ (b) $\int 2x\sqrt{2x-1} \, dx$

(c) $\int \frac{3x^2-1}{(x^3-x+4)^3} \, dx$ (d) $\int \cos^3 2x \, dx$

(e) $\int \sin x\sqrt{\cos x} \, dx$ (f) $\int \cot^2 x \operatorname{cosec}^2 x \, dx$

(g) $\int 2x(4x^2-1)^3 \, dx$ (h) $\int \frac{x}{\sqrt{(2x^2-5)}} \, dx$

(i) $\int \frac{3x}{\sqrt{(4-x)}} \, dx$ (j) $\int \frac{\sin\sqrt{x}}{\sqrt{x}} \, dx$

16. Evaluate the following definite integrals by changing the variable and the limits.

(a) $\int_2^3 x\sqrt{(x-2)} \, dx$ (b) $\int_0^1 x(x-1)^4 \, dx$

(c) $\int_1^2 \frac{x}{\sqrt{2x-1}}\,dx$ (d) $\int_1^2 (2x-1)(x-2)^3\,dx$

(e) $\int_{-3/8}^{0} \frac{x+3}{\sqrt{2x+1}}\,dx$

17. Evaluate the following definite integrals either by writing down the integral as a function of x or by using the given change of variable.

(a) $\int_0^{\pi/6} \sec^4 x \tan x\,dx$ (sec $x = u$)

(b) $\int_0^{\pi/2} \sin^5 x\,dx$ (cos $x = u$)

(c) $\int_{\pi/6}^{\pi/2} \frac{\cot x}{\sqrt{\operatorname{cosec}^3 x}}\,dx$ (cosec $x = u$)

18. Evaluate

(a) $\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx$ (b) $\int_0^4 2x\sqrt{4-x}\,dx$

(c) $\int_{-1}^{0} x(x^2-1)^4\,dx$ (d) $\int_0^{\pi/4} \sec^4 x\,dx$

(e) $\int_{1/2}^{1} \frac{x-2}{(x+2)^3(x-6)^3}\,dx$ (f) $\int_{-1}^{2} (x+1)(2-x)^4\,dx$

(g) $\int_{-\pi/2}^{\pi/2} \cos^3 x\,dx$ (h) $\int_{5/3}^{8/3} \frac{x+2}{\sqrt{3x-4}}\,dx$

(i) $\int_0^{\pi/2} \sin x\sqrt{\cos x}\,dx$

19. Calculate the area enclosed by the curve $y = \frac{x}{\sqrt{x^2-1}}$, the x-axis, $x = 2$ and $x = 3$.

20. Calculate the area under $y = \sin^3 x$ from $x = 0$ to $x = \frac{2\pi}{3}$.

21. Calculate the volume of the solid generated when the area under $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$ is rotated through four right angles about the x-axis.

22. The area of a uniform lamina is that enclosed by the curve $y = \sin x$, the x-axis, and the line $x = \frac{\pi}{2}$. Find the distance from the x-axis of the centre of gravity of the lamina.

Answers

1. (a) $\frac{2}{x+3} - \frac{1}{x-4}$

(b) $\frac{1}{2(5-x)} - \frac{1}{2(5+x)}$ (c) $\frac{4}{x+1} + \frac{2}{x-2} - \frac{3}{x-3}$

(d) $\frac{3}{x-1} - \frac{1}{x} + \frac{2}{x+1}$ (e) $\frac{1}{x+2} + \frac{2}{2x+1} - \frac{2}{3x+2}$

(f) $2x-1+\frac{1}{x+3} - \frac{3}{x-2}$

2. (a) $\frac{2}{x-3} + \frac{3x-1}{x^2+4}$ (b) $\frac{2}{x+1} - \frac{1}{x^2+4}$

(c) $\frac{1}{x-1} + \frac{2x}{x^2+5}$ (d) $\frac{3}{2x-3} + \frac{1-3x}{2x^2+1}$

(e) $\frac{3}{x-3} - \frac{2}{x+3} - \frac{1}{x+5}$ (f) $2+\frac{5}{x-3} + \frac{x}{x^2+1}$

3. (a) $\frac{1}{x-2} - \frac{3}{(x-2)^2}$

(b) $\frac{1}{x-1} - \frac{1}{x+2} + \frac{2}{(x+2)^2}$

(c) $\frac{23}{4(3x+1)} - \frac{1}{4(x+1)} - \frac{7}{2(x+1)^2}$

(d) $x + \frac{1}{x+2} + \frac{2}{x-1} + \frac{1}{(x-1)^2}$

4. (a) $\frac{1}{6(x+2)} - \frac{7}{2x} + \frac{10}{3(x-1)}$

(b) $\frac{3}{2x^2} - \frac{3}{4x} + \frac{3}{4(x+2)}$

(c) $2x+4-\frac{1}{3(x-2)} - \frac{5x+61}{3(x^2+5)}$

(d) $\frac{5}{3+x} + \frac{2}{4-x} - \frac{3}{4+x}$

(e) $\frac{1}{x-1} - \frac{x}{x^2+x+1}$

(f) $\frac{2}{(2x+1)^2} - \frac{5}{2x+1} + \frac{3}{x-3}$

5. (a) $\frac{1}{2}\ln\frac{k(x-2)}{x}$ (b) $\frac{1}{17}\ln\frac{k(5x-2)}{x+3}$

(c) $\ln\frac{kx}{3x+1} - \frac{2}{x}$ (d) $\ln\frac{k}{\sqrt{16-x^2}}$

(e) $\frac{1}{6}\ln\frac{k(x-5)}{(x+1)}$ (f) $\ln\left[k(x^2-4x-5)^{1/2}\right]$

(g) $\ln\left[k(x+2)\sqrt{(x^2+3)}\right]$

(h) $\ln\frac{k(3+x)^2(2-x)}{(4-x)^3}$

(i) $2\ln\left[k(2x+1)\right] - \frac{1}{2}\ln\left[(x-3)(x+3)^3\right]$

6. (a) $\ln\frac{4}{3}\approx 0.288$ (b) $\frac{1}{2}\ln 2+\frac{1}{4}\pi=1.13$

(c) $\ln\frac{45}{64}\approx -0.352$ (d) $-3\ln 2-\frac{1}{2}\ln 3\approx -2.63$

7. (a) $\ln\left[\frac{x-1}{x+2}\right]+c$ (b) $\frac{1}{2}\ln\left[\frac{1+x}{1-x}\right]+c$

(c) $\frac{1}{3}\ln\left[\frac{x-3}{x}\right]$ (d) $\frac{1}{2}\ln(x^2-4)+c$

(e) $\ln((x-3)^3(x+1))+c$

(f) $\ln\big((x-2)(x-3)\big)+c$

8 (a) $\ln\frac{32}{27}$ (b) $\ln\frac{12}{7}$ (c) ln 6

(d) $\frac{3}{2}\ln\frac{9}{8}$ (e) $\frac{1}{3}\ln\frac{27}{2}$ (f) $\ln\frac{4}{3}$

9. (a) $\sin x - x\cos x + c$

(b) $2x\sin\frac{1}{2}x+4\cos\frac{1}{2}x+c$

(c) $-e^{-x}(x+1)+c$ (d) $\frac{1}{4}x^2(2\ln 2x-1)+c$

(e) $\frac{1}{4}e^{2x}(2\ln 2x-1)+c$ (f) $x\tan x+\ln(\cos x)+c$

(g) $\frac{1}{4}\sin 2x-\frac{1}{4}x\cos 2x+c$

(h) $x\tan x+\ln(\cos x)-\frac{1}{2}x^2+c$

(i) $\frac{1}{4}x^2+\frac{1}{4}x\sin 2x+\frac{1}{8}\cos 2x+c$

(j) $(x^2-2)\sin x+2x\cos x+c$

10. (a) $-2x$ (b) $\frac{2}{e}$

(c) $\frac{\pi}{9}$ (d) 1

(e) $\frac{1}{4}\pi^2$ (f) $\frac{8}{3}\ln 2-\frac{7}{9}$

(g) $10e^{-2}-17e^{-3}$ (h) $\frac{1}{4}-\frac{3}{4}e^{-2}$

(i) $\pi-2$ (j) $\frac{1}{2}(e^{\pi}+1)$

11. $\frac{1}{30}(x+1)^5(5x-1)+c$

12. (a) $\frac{2}{15}(x-1)^{3/2}(3x+2)+c$

(b) $\frac{-(x+1)}{(x+2)^2}+c$

14. (a) $\frac{1}{20}(4x-1)^{3/2}(6x+1)+C$

(b) $\frac{2}{375}(5x+1)^{3/2}(15x-4)+c$

(c) $\frac{1}{224}(2x-1)^7(14x+1)+c$

(d) $\frac{2}{3}(x+4)\sqrt{x-2}+c$

(e) $\frac{1}{30}(x-1)^5(5x+13)+c$

(f) $\frac{1}{168}(x-2)^6(21x^2+156x+304)+c$

(g) $\frac{x^2-4x+8}{x-2}+c$ (h) $\frac{1}{3}(x-6)\sqrt{2x+3}+c$

15. (a) $\frac{1}{6}(2x^2+1)^{3/2+c}$ (b) $\frac{2}{15}(2x-1)^{3/2}(3x+1)+c$

(c) $\frac{-1}{2}(x^3-x+4)^{-2}+c$ (d) $\frac{1}{2}\sin 2x-\frac{1}{6}\sin^3 2x+c$

(e) $\frac{-2}{3}(\cos x)^{3/2}+c$ (f) $\frac{-1}{3}\cot^3 x+c$

(g) $\frac{1}{16}(4x^2-1)^4+c$ (h) $\frac{1}{2}\sqrt{2x^2-5}+c$

(i) $-2(8+x)\sqrt{4-x}+c$ (j) $-2\cos\sqrt{x}+c$

16. (a) $\frac{26}{15}$ (b) $\frac{1}{30}$ (c) $\sqrt{3}-\frac{2}{3}$

(d) $\frac{-7}{20}$ (e) $\frac{67}{48}$

17. (a) $\frac{7}{36}$ (b) $\frac{8}{15}$ (c) $\frac{1}{6}(4-\sqrt{2}$

18. (a) $1-\frac{1}{2}\sqrt{3}$ (b) $\frac{256}{15}$ (c) $\frac{-1}{10}$

(d) $\frac{4}{3}$ (e) $\frac{23}{108900}$ (f) 24.3

(g) $\frac{4}{3}$ (h) $\frac{74}{27}$ (i) $\frac{2}{3}$

19. $2\sqrt{3}-\sqrt{3}$ 20. $\frac{9}{8}$ 21. $\frac{1}{4}\pi^2$ 22. $\frac{1}{8}\pi$