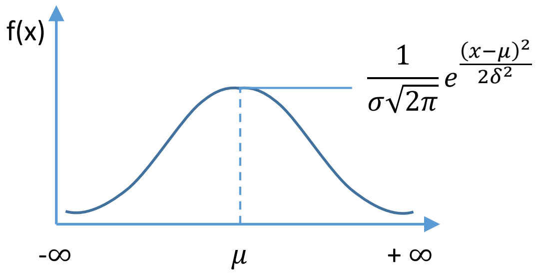
Normal distribution

A continuous random variable X follows a normal distribution with mean, μ and variance, σ^2 if

X~N(μ, σ^2) root

Its p.d.f is given by fx = $\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{(x-\mu)^2}{2\delta^2}}, -\infty < x < \infty$

A sketch of f(x) gives a normal curve



Properties of the curve

- It is bell shaped
- It is symmetrical about μ
- It extends from $-\infty < x < \infty$

The maximum value of f(x) =$\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{(x-\mu)^2}{2\delta^2}}$

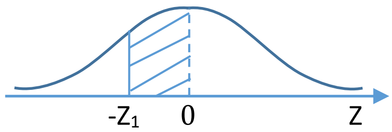
- The total area under the curve = 1

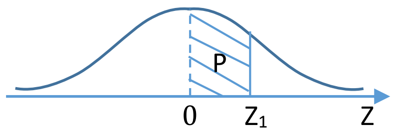
How to read the cumulative normal distribution table

(i) Between 0 and any z value

(a) P(0 ≤ Z ≤ Z_1) = ϕ(Z_1) = region P (b) P(-Z_1≤ Z ≤ 0)= P(0 ≤ Z ≤ Z_1) = ϕ(Z_1) = region P

By symmetrical



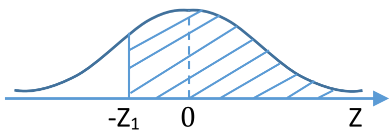


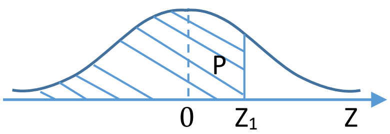
(ii) Less than any positive z value

(a) P(Z < Z_1) = 0.5 + P(0 ≤ Z ≤ Z_1) = ϕ(Z_1) = region P

(b) P(Z > -Z_1) = P(Z < Z_1) = 0.5 + P(0 ≤ Z ≤ Z_1) = 0.5 + ϕ(Z_1) = region P

By symmetrical



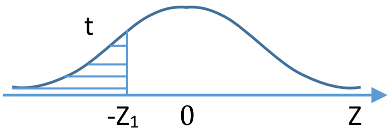


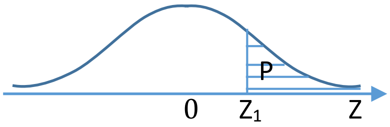
(iii) Greater than any positive z value

P(Z > Z_1) = 0.5 - P(0 ≤ Z ≤ Z_1) = 0.5 - ϕ(Z_1) = region P

P(Z < -Z_1) = P(Z > Z_1) = 0.5 - P(0 ≤ Z ≤ Z_1) = 0.5 - ϕ(Z_1) = region P

By symmetrical





Example 1

Find (i) $P(Z< 2)$ (ii) $P(Z > 0.85)$ (iii) $P(X< 0.345)$

Solution

(i) $P(X < 2) = 0.5 + \phi(2) = 0.5 + 0.4772 = 0.9772$

(ii) $P(Z > 0.85) = 0.5 - \phi(0.85) = 0.5- 0.3023 = 0.1977$

(iii) $P(X< 0.345) = 0.5 + - \phi(0.345) =0.5 + 0.1331 + 0.0019 = 0.6350$

Example 2

Find (i)$P(Z < -0.25)$ (ii) $P(Z > -1.377)$ (iii) $P(Z < -1.377)$

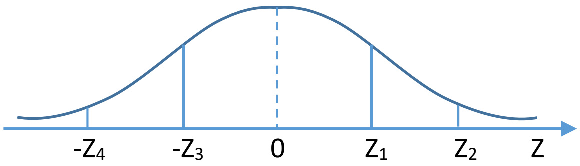
Solution

(i) $P(Z < -0.25) = P(Z > 0.25) = 0.5 - \phi(0.25)= 0.5 – 0.0987 = 0.4013$

(ii) $P(Z > -1.377) = P(Z < 1.377) = 0.5 + \phi(1.377)= 0.5 + 0.4147 + 0.0011 = 0.9158$

(iii) $P(Z < -1.377) = P(Z > 1.377 = 0.5 - \phi(1.377) = 0.5 – (0.4147 + 0.0011) = 0.0842$

Other important results



(i) Between two Z values on the same side of the mean

(a) $P(Z_1 < Z < Z_2) = P(0 < Z < Z_2) - P(0 < Z < Z_1) = \phi(Z_2) - \phi(Z_1)$

(b) $P(-Z_4 < Z < -Z_3) = P(0 < Z < Z_4) - P(0 < Z < Z_3) = \phi(Z_4) - \phi(Z_3)$

(ii) Between two Z values on the opposite side of the mean

(a) $P(-Z_3 < Z < Z_1) = P(0 < Z < Z_3) + P(0 < Z < Z_1) = \phi(Z_3) + \phi(Z_1)$

(b) $P(|Z|< Z_1) = P(-Z_1 < Z < Z_1) = 2 \times P(0 < Z < Z_1) = 2 \times \phi(Z_1)$

(c) $P(|Z|> Z_1) = 1 - P(-Z_1 < Z < Z_1) = 1 -2 \times \phi(Z_1)$

Example 3

Find (i) $P(1.5 < Z < 1.88)$ (iii) $P(-2.696 < Z < 1.865)$ (v) $P(|Z|< 1.75)$
(ii) $P(-2.5 < Z < 1)$ (iv) $P (-1.4 < Z < -0.6)$ (vi) $P(|Z|> 1.433)$

Solution

(i) $P(1.5 < Z < 1.88) = \phi(1.88) - \phi(1.5) = 0.4699 – 0.4332 = 0.0367$

(ii) $P(-2.5 < Z < 1) = \phi(1) + \phi(2.5) = 0.3413 + 0.4938 = 0.8351$

(iii) $P(-2.696 < Z < 1.865) = \phi(1.865) + \phi(2.696) = 0.469 + 0.4964 = 0.9654$

(iv) $P (-1.4 < Z < -0.6) = \phi(1.4) + \phi(0.6) = 0.4192 – o.2257 =0.1935$

(v) $P(|Z|< 1.75) = P(-1.75 < Z < 1.75) = 2 \times \phi(1.75) = 2 \times 0.4625= 0.925$

(vi) $P(|Z|> 1.433) = 1 - P(|Z|< 1.433) = 1 – 2 \times \phi(1.433) = 1 – 2 \times 0.424 = 0.152$

Standardizing a random variable X

If a random variable X follows a normal distribution with mean, μ and variance, σ^2, then X ~N(μ, σ^2) and can be standardized using the equation below and read from a cumulative normal table

Z = $\frac{X-\mu}{\sigma}$

Example 4

Given that the random variable X is X ~N(300, 25). Find

(i) P(X > 305) (ii) P(X < 291) (iii) P(X < 312) (iv) P(X > 286)

Solution

(i) P(X> 305) = $P\left(Z < \frac{305-300}{5}\right)$ = P(X > 1) = 0.5 – ϕ(1) = 0.5 – 0.3413 = 0.1587

(ii) P(X < 291) = $P\left(Z < \frac{291-300}{5}\right)$ = P(Z < -1.8)
= P(Z >1.8) = 0.5 - ϕ(1.8) = 0.5 – 0.4641 = 0.0359

(iii) P(X < 312) = $P\left(Z < \frac{312-300}{5}\right)$ = P(X < 2.4) = 0.5 + ϕ(2.4)
= 0.5 – 0.4918 = 0.9918

(iv) P(X > 286) = $P\left(Z < \frac{286-300}{5}\right)$ = P(Z < -2.8)
= P(Z < 2.8) = 0.5 + ϕ(2.8) = 0.5 + 0.4974= 0.9974

Example 5

Given that the random variable X is X ~N(10, 4). Find

Find (i) P(X < 7) (ii) P(X > 12) (iii) P(7 < X < 12) (iv) P(9 < X < 11)

Solution

(i) P(X < 7) = $P\left(Z < \frac{7-10}{2}\right)$ = P(Z < - 0.15) = P(Z > 1.5)
= 0.5 – ϕ(1.5) = 0.5 – 0.4332 0.0668

(ii) P(X > 12) = $P\left(Z > \frac{12-10}{2}\right)$ = P(Z > 1) = 0.5 – ϕ(1) = 0.5 – 0.3413 = 0.1587

(iii) P(7 < X < 12) = $P\left(\frac{7-10}{2} < Z > \frac{12-10}{2}\right)$
= P(-1.5 < Z < 1) = ϕ(1.5) + ϕ(1) = 0.4332 + 0.3413 = 0.7745

(iv) P(9 < X < 11) = $P\left(\frac{9-10}{2} < Z > \frac{11-10}{2}\right)$
= P(-0.5 < Z < 0.5) = ϕ(0.5) + ϕ(0.5) = 2 x 0.1915 = 0.3830

Example 6

Given that the random variable X is X ~N(50, 8). Find

(i) P(48 < X < 54) (ii) P(52 < X < 55) (iii) P(46 < X < 49) (iv) P(|X- 50|< $\sqrt{8}$)

Solution

(i) P(48 < X < 54) = $P\left(\frac{48-50}{\sqrt{8}} < Z > \frac{54-50}{\sqrt{8}}\right)$ = P(-0.707 < Z < 1.414)

= ϕ(1.414) + ϕ(0.707) = 0.4213 + 0.2601 = 0.6814

(ii) P(52 < X < 55) = $P\left(\frac{52-50}{\sqrt{8}} < Z > \frac{55-50}{\sqrt{8}}\right)$ = P(0.707 < Z < 1.768)

= ϕ(1.768) - ϕ(0.707) = 0.4615 -0.2601 = 0.2014

(iii) P(46 < X < 49) = $P\left(\frac{46-50}{\sqrt{8}} < Z > \frac{49-50}{\sqrt{8}}\right)$ = P(-1.414 < Z < -0.354)

=ϕ(1.414) – ϕ(0.354) = 0.4213 – 0.1383 -0.283

(iv) P(|X- 50|< $\sqrt{8}$) = $P\left(\frac{-\sqrt{8}+50-50}{\sqrt{8}} < Z > \frac{\sqrt{8}+50-50}{\sqrt{8}}\right)$ P(-1 < X < 1) = 2 x ϕ(1) = 2 x 0.3413 = 0.6826

Example 6

A random variable X is normally distributed with mean 65 and variance 100, find the probability that X assumes a value between 50 and 90.

P(50 < X < 90) = $P\left(\frac{50-65}{10} < Z < \frac{90-65}{10}\right)$= P(-1.5 < Z < 2.5) = ϕ(1.5) + ϕ(2.5) = 0.4332 + 0.4938 = 0.927

Example 7

Lengths of metal strips produced by a machine are normally distributed with mean length of 150cm and standard deviation of 10cm. find the probability that the length of a randomly selected strip is

(i)shorter than 165 (ii) within 5cm of the mean

Solution

P(X < 165) = $P\left(Z < \frac{165-150}{10}\right)$ =P(Z< 1.5) = 0.5 + ϕ(1.5) = 0.5 + 0.4332 = 0.9332

P(150 – 5 < X < 150 + 5) = $P\left(\frac{-5}{10} < Z < \frac{5}{10}\right)$= P(-0.5 < Z < 0.5) = 2 x ϕ(0.5) = 2 x 0.1915 = 0.383

Example 8

In end of year exams, the marks are normally distributed with a mean mark of 50 and standard deviation 5. If a mark 45 is required to pass the exam, what percentage of the students failed the exam.

P(X < 45) = $P\left(Z < \frac{45-50}{5}\right)$ = P(Z < -1) = P(Z > 1) = 0.5 – ϕ(1) = 0.5 – 0.3413 = 0.1587

Example 9

A bakery supplies bread to a shop every day. The time to deliver bread to the shop is normally distributed with mean 12 minutes and standard deviation of 2 minutes. Estimate the number of days the year when he takes

(i) longer than 17 minutes (ii) less than 10 minutes (iii) between 9 and 13 minutes

Solution

(i) P(X > 17) = $P\left(Z > \frac{17-12}{2}\right)$= P(Z > 2.5) = 0.5 – ϕ(2.5) = 0.5 – 0.4938 = 0.0062

The number of days = 0.0062 x 365 = 2 days

(ii) P(X < 10) = $P\left(Z > \frac{10-12}{2}\right)$= P(Z < -1) = P(Z> 1) = 0.5 – ϕ(1) + 0.5 – 0.3413 0.1587

The number of days = 0.1587 x 365 = 58 days.

(iii) P(9 < X <13) = $P\left(\frac{9-12}{2} < Z < \frac{13-12}{2}\right)$= P(-1.5 < Z < 0.5) = ϕ(1.5) + ϕ(0.5) 0.4332 + 0.1915 = 0.6247

Number of days = 0.6247 x 365 = 228 days.

Example 10

(a) In a certain athletics competition, points are awarded according to level of performance. The average grade was 82 points with standard deviation of 5 points. All competitors whose grades ranged between 88 to 94 points received certificates. If the grades are normally distributed and 8 competitors received certificates. How many participants took part in the competition?

P(88 < X < 94) = $P\left(\frac{88-82}{5} < Z < \frac{94-82}{5}\right)$= P(1.2 < Z < 2.4) =$\frac{8}{n}$

ϕ(2.4) – ϕ(1.2) = 0.4918 – 0.3849 = 0.1069 = $\frac{8}{n}$; n = 74.84

hence 75 participants took part.

(b) If certificates were to be awarded to only those having between 90 and 94 points. What proportion of the participants would acquire certificates.

P(90 < X < 94 = $P\left(\frac{90-82}{5} < Z < \frac{94-82}{5}\right)$=P(1.6 < Z < 2.4 = ϕ(2.4) – ϕ(1.6)

= 0.4918 – 0.4452 = 0.0466

= 0.0466 x 100% = 4.66%

Revision exercise 1

1. The amount of meat sold by a butcher is normally distributed with mean 43kg and standard deviation 4kg. Determine the probability that the amount of meat sold is between 40kg and 50kg. (0.7333)
2. Given that a random variable X is X~N(2, 2.89). Find P(X<0)(0.1198)
3. In a school of 800 students their average weight is 54.5kg and standard deviation 6.8kg. given that the weight of students are normally distributed, find
   (i) Probability that the weight of any student randomly selected is 52.8 kg or less = 0.4014
   (ii) Number of students who weigh over 75kg = 1
   (iii) Weight of the middle 56% of the students (49.251< X < 59.750)
4. A sugar factory sells sugar in bags of mean weight 50kg and standard deviation 2.5kg. given that the weight of the bags is normally distributed, find the
   (i) Probability that the weight of any bag of sugar randomly selected lies between 51.5kg and 53kg = 0.1592
   (ii) Percentage of bags whose weight exceeds 54kg = 5.48%
   (iii) Number of bags that will be rejected out of 1000 bags purchased for weighing below 45.0kg = 23
5. A certain maize firm sells maize in bags of mean weight 40kg and standard deviation 2kg,. given that the weight of the bags are normally distributed, find
   (i) Probability that the weight of any bag of maize randomly selected lies between 41.0 and 42.5kg = 0.2029
   (ii) Percentage of bags whose weight exceeds 43kg = 6.68%
   (iii) Number of bags that will be rejected out of 500 bags purchased for weighing below 38.5kg = 113
6. Given that the random variable X is X~N(300, 25) Find
   (i) P(X>308) = 0.0548   (ii) P(X> 311.5) = 0.0107   (iii) P(X < 294) = 0.8849

(iv) $P(X < 290.5) = 0.9713$ (v) $P(X > 302) = 0.6554$ (vi) $P(X > 312) = 0.9918$

7. If $X \sim N(50, 20)$. Find
   (i) $P(X > 60.3) = 0.0106$ (ii) $P(X < 47.3) = 0.273$ (iii) $P(X > 48.9) = 0.5972$
   (iv) $P(X > 53.5) = 0.2831$ (v) $P(X < 59.8) = 0.9857$ (vi) $P(X < 62.3) = 0.9970$
8. If $X \sim N(-8, 12)$. Find
   (i) $P(X < -9.8) = 0.1587$ (ii) $P(X > 0) = 0.8413$ (iii) $P(X < -3.4) =0.9079$
   (iv) $P(X > -5.7) =0.2533$ (v) $P(X < 10.8) = 0.2097$ (vi) $P(X > -1.6) = 0.0323$
9. If $X \sim N(\alpha, \alpha^2)$. Find
   (i) $P(X < 0) = 0.1587$ (ii) $P(X > 0) = 0.8413$ (iii) $P(X < 0.5\alpha) = 0.6915$ $P(X > 0.5\alpha) = 0.3085$
10. If $X \sim N(100, 80)$. Find
    (i) $P(85 < X < 112) = 0.8634$ (ii) $P(105 < X < 115) = 0.2413$
    (iii) $P(85 < X < 92) = 0.1388$ (iv) $P(|X|< \sqrt{80} = 0.6826$
11. If $X \sim N(84, 12)$. Find
    (i) $P(80 < X < 89)= 0.8014$ (ii) $P(X <79 \text{ or } X > 92) = 0.085$ (iii) $P(76 < X < 82) = 0.2714$
    (iv) $P(|X- 84|> 2.9) = 0.4028$ (v) $P(87 < X < 93) = 0.1886$
12. The masses of packages from a particular machine are normally distributed with a mean of 200g and standard deviation of 2g, find the probability that a randomly selected package from the machine weighs
    (i) less than 197g = 0.0668
    (ii) more than 200.5g = 0.4013
    (iii) between 198.5g and 199.5g = 0.1747
13. The heights of boys at a certain school follow a normal distribution with mean = 150.3cm and variance 25cm, find the probability that a boy picked at random from the group has a height;
    (i) less than 153cm = 0.7054
    (ii) more than 158cm = 0.018
    (iii) between 150 cm and 158 cm= 0.4621
    (iv) more than 10cm difference from the mean height = 0.0046
14. The masses of a certain type of cabbages are normally distributed with mean of 1000g and standard deviation of 0.15kg. In a batch of 800 cabbages, estimate how many have a mass between 750g and 1290g = 740
15. Cartons of milk from quality super market are advertised as containing 1 litre, but in fact the volume of the content is normally distributed with a mean of 1012ml and standard deviation of 15ml.
    (i) Find the probability that a randomly chosen carton contains more than 1010ml =0.6554
    (ii) In a batch of 1000 cartons, estimate the number of cartons containing less than the advertised volume of milk =8
16. A random variable X is such that $X \sim N(-5, 9)$. Find the probability that;
    (i) A randomly chosen item from the population will have positive value = 0.0478
    (ii) Out of 10 items chosen randomly, exactly 4 will have a positive value = 0.00082
17. The life of a laptop is normally distributed with a mean of 2000 hours and standard deviation of 120 hours. Estimate the probability that the life of such a laptop will be
    (i) greater than 2150 hours = 0.1056
    (ii) greater than 1910hours = 0.7734
    (iii) within a range 1850 hours to 2090 hours = 0.6678

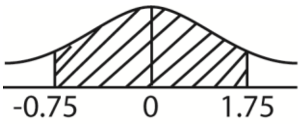
Solutions to revision questions 1

1. The amount of meat sold by a butcher is normally distributed with mean 43kg and standard deviation 4kg. Determine the probability that the amount of meat sold is between 40kg and 50kg.

   X ~N(43,4)

   P(40<x<50) = $P\left(\frac{40-43}{4} < Z < \frac{50-43}{4}\right)$

   = P(- 0.75 < Z < 1.75)



   P(40<x<50) = P(-0.75 < Z < 0) + P(0 < Z < 1.75)

   By property of symmetry

   P(40<x<50) = P(-0.75 < Z < 0) + P(0 < Z < 1.75)

   = 0.2735 + 0.4599

   = 0.733

2. Given that a random variable X is X~N(2, 2.89). Find P(X<0)

   μ = 2, σ = $\sqrt{2.89}$ = 1.7

   P(X < 0) = $P(Z < \frac{0-2}{1.7})$ = P(Z < - 1.176) = P(X> 1.176)

   = 0.5 – P(0 < Z < 1.176)

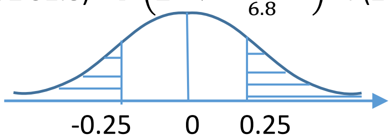
   = 0.5 – 0.3802 = 0.1198

3. In a school of 800 students their average weight is 54.5kg and standard deviation 6.8kg. given that the weight of students are normally distributed, find

   (i) Probability that the weight of any student randomly selected is 52.8 kg or less

   Let x be the weight of the student

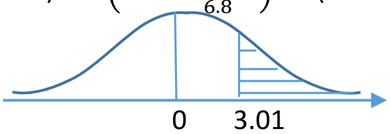
   P(x ≤ 52.8) = $P\left(Z < \frac{52.8-54.5}{6.8}\right)$= P(Z <-0.25)



   = P(Z> 0.25) = 0.5- P(0 < Z < 0.25) = 0.5 -0.0987 = 0.4013

   (ii) Number of students who weigh over 75kg = 1

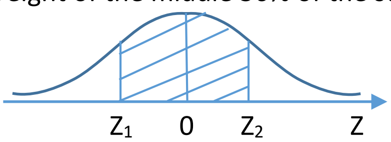
   P(Z > 75) = $P\left(Z > \frac{75-54.5}{6.8}\right)$ = P(Z > 3.01)



   P(Z > 3.01) = 0.5 – P(0 < Z < 3.01) = 0.5 – 0.4990= 0.001

   Number of students who weigh more than 75g = 800 x 0.001 =1

   (iii) Weight of the middle 56% of the students



   $P(X_1 < X < X_2) = P(Z_1 < Z < Z_2) = 0.56$

   But $P(0 < Z < Z_2) = 2.8$; $Z_2 = 0.772$ and $Z_1 = -0.772$

   $Z_1 = \frac{x_1-54.5}{6.8}$

   $-0.772 = \frac{x_1-54.5}{6.8}$; $x_1 = 49.251$

   $Z_2 = \frac{x_1-54.5}{6.8}$

   $0.772 = \frac{x_1-54.5}{6.8}$; $x_2 = 59.750$

Hence the weight range of the middle 56% of students of the school is 49.251< X < 59.750

4. A sugar factory sells sugar in bags of mean weight 50kg and standard deviation 2.5kg. given that the weight of the bags is normally distributed, find the
   (i) Probability that the weight of any bag of sugar randomly selected lies between 51.5kg and 53kg

   P(51.5 < X < 53) = $\frac{51.5-50}{2.5} < Z < \frac{53-50}{2.5}$ = P(0.6 < Z 1.2)

   = ϕ(1.2) – ϕ(0.6) = 0.3849 – 0.2257== 0.1592
   (ii) Percentage of bags whose weight exceeds 54kg

   P(X > 54) =P($Z > \frac{54-50}{2.5}) = P(Z > 1.6)$= 0.5 –ϕ(1.6) = 0.5 – 0.4452 = 0.0548

   = 0.0548 x 100 = 5.48%
   (iii) Number of bags that will be rejected out of 1000 bags purchased for weighing below 45.0kg

   P(X < 45) = P($Z < \frac{45-50}{2.5}) = P(Z < -2) = P(Z < 2)$ = 0.5 –ϕ(2)= 0.5- 0.4772 = 0.0228

   Number of bags rejected = 0.0228 x 1000 = 22.8 ≈23
5. A certain maize firm sells maize in bags of mean weight 40kg and standard deviation 2kg, given that the weight of the bags are normally distributed, find
   (i) Probability that the weight of any bag of maize randomly selected lies between 41.0 and 42.5kg

   P(41.0 < X < 42.5) = $\frac{41-40}{2} < Z < \frac{42.5-40}{2}$ = P(0.5 < Z 1.25)

   = ϕ(1.25) – ϕ(0.5) = 0.3944 - 0.1915 = 0.2029

   (ii) Percentage of bags whose weight exceeds 43kg

   P(X > 43) =P($Z > \frac{43-40}{2}) = P(Z > 1.5)$= 0.5 –ϕ(1.5) = 0.5 – 0.4332 = 0.0668

   = 0.0668 x 100 = 6.68%

   (iii) Number of bags that will be rejected out of 500 bags purchased for weighing below 38.5kg

   P(X < 38.5) = P($Z < \frac{38.5-40}{2}) = P(Z < -0.77) = P(Z < 0.75)$ = 0.5 –ϕ(0.75)

   = 0.5- 0.2734 = 0.2266

   Number of bags rejected = 0.2266 x 500 = 113
7. If X~N(50, 20). Find
   (i) P(X > 60.3)

   P(X > 60.3) = P(Z > $\frac{60.3-50}{\sqrt{20}}$) = P(Z > 2.303) = 05 –ϕ(2.303)

   = 0.5 –(0.4893 + 0.0001) = 0.0106
   (ii) P(X < 47.3)

   P(X < 47.3) P(Z > $\frac{47.3-50}{\sqrt{20}}$) = P(Z <-0.6037) =P(Z> 0.6037) = 05 –ϕ(0.6037)

   = 0.5 –(0.2257 + 0.0013) = 0.273
   (iii) P(X > 48.9)

   P(X < 48.9) = P(Z > $\frac{48.9-50}{\sqrt{20}}$) = P(Z <-0.246) = P(Z < 0.246) = 0.5 +ϕ(0.246)

   = 0.5 + 0.0948 + 0.0022 = 0.597
   (iv) P(X > 53.5)

   P(X > 53.5) = P(Z > $\frac{53.5-50}{\sqrt{20}}$) = P(Z > 0.783) =0.2823 + 0.0008 = 0.2831

(v) P(X < 59.8)

$P(X < 59.8) = P(Z < \frac{59.8-50}{\sqrt{20}}) = P(Z < 2.191) = 0.5 + \phi(2.191)$

$= 0.5 + 0.4826 + 0.0001 = 0.9857$

(vi) P(X < 62.3)

$P(X < 62.3) = P(Z < \frac{62.3-50}{\sqrt{20}}) = P(Z < 2.750) = 0.5 + \phi(2.730$

$= 0.5 + 0.4970 = 0.9970$

How to obtain Z-values from a given probabilities

If you are interested in finding the Z-values whose probabilities are given, it is important to note that the Z-value may be positive or negative.

Sign	Probability	Z-value
<	< 0.5	-
>	>0.5	-
<	>0.5	+
>	<0.5	+

Note: for the above table the probability given in the question always correspond to Q in the critical table

Example 11

$P(Z<Z_1) = 0.5$, find Z_1

$P(Z<Z_1) = 0.5$, find Z_1

Solution

$P(Z<Z_1) = 0.5(Q)$

$P(Z<Z_1) = 0.5$, find Z_1

$Z_1 = -0.674$ (negative since 0.25 < 0.5 read directly from a critical table)

Example 12

$P(Z < Z_1) = 0.0968$, find Z_1

Since 0.0968(Q) is not on critical table

P(Z < Z1) = 0.5 – 0.0968 = 0.403(P)

$Z_1 = -1.3$ (negative since 0.0968 < 0.5 read directly from a critical table)

Example 13

$P(Z < Z_1) = 0.5$, find Z_1

Solution

$P(Z < Z_1) = 0.05(Q)$

$Z_1 = -1.645$ (negative since 0.05 < 0.5 read directly from a critical table)

Example 14

P(Z< a) = 0.787, find a

Since 0.787 is not on critical table

P(Z < a) = 0.787 – 0.5 = 0.287

From the table0.287 lies between 0.2852 and 0.2881. Since the extra information to the right hand side is **add**, we consider the smallest value i.e. 0.2852 but 2852 corresponds to 0.79

to get the next

0.2870 – 0.285 = 0.0018

So we look for 0.0018 on the add column which gives 6

∴ a = 0.79 + 0.006 = 0.796

Example 15

P(Z > b) = 0.01, find b

P(Z > b) = 0.01(Q)

b = 2.326 read directly from critical table

Inverse process (De-standardizing Z)

It involves converting the Z-value to raw data (X) form

Example 16

If X~(100, 36) and P(X > α) = 0.8907, find the value of α

Since 0.8907 is not critical on a critical table

P(Z < $\frac{\alpha-100}{6}$) = 0.8907 – 0.5 = 0.390(P)

From the table Z =1.23

1.23 = $\frac{\alpha-100}{6}$

α = 100 + 1.23 x 6 =107.38

Example 17

If X~(24, 9) and P(X > b) = 0.974, find the value of b

P(Z < $\frac{b-24}{3}$) = 0.974 – 0.5 = 0.474(P)

From the table Z =-1.943

1.943 = $\frac{b-24}{3}$

α = 24 -1.943 x 3 =18.171

Example 18

The height of flowers in a farm is normally distributed with the mean 169 cm and standard deviation 9cm. if X stands for the height of flowers in cm, find X values for

(a) P(X < a) = 0.8

Solution

P(X < a) = 0.8(Q)

$P(Z < \frac{a-169}{9})$ = 0.8 – 0.5 = 0.3(P)

From the table Z = 0.842

a = 0.842 x 9 + 169 = 176.38

(b) P(X > b) = 0.6

Solution

P(X > b) = 0.6 (Q)

$P(Z < \frac{a-169}{9})$ = 0.6 – 0.5 = 0.1(P)

From the table Z = -0.253

b = -0.253 x 9 + 169 = 166.72

Example 19

The period of a certain machine approximately follows a normal distribution with mean of five years and standard deviation of 1 year. Given that the manufacturer of this machine replaces the machine that fails under guarantee, determine the

(i) Length of the guarantee required so that not more than 2% of the machine that fail are replaced.

$P(X < X_0) = 0.02(Q)$

$P\left(Z < \frac{X_0-5}{1}\right) = 0.02(Q)$

From the table Z = -2.054

$X_0 = -2.054 \times 1 + 5 = 2.946$

∴ the guarantee period is 2.946 years

(ii) The proportion of the machines that would be replaced if the guarantee period was four years

$P(X < 4) = P\left(Z < \frac{4-5}{1}\right) = P(Z < -1) = P(Z > 1) = 5 - \phi(1) = 5 - 0.3413 = 0.1587$

$P(Z < 4) = 0.1587 \times 100 = 15.87\%$

Example 20

The marks of 500 students in a mock examination for which the ass mar was 50%. Their marks are normally distributed with mean 45 marks and standard deviation 20 marks.

(a) Given that the pass mar is 41, estimate the number of candidates who passed the examination.

$P(X \geq 41) = P\left(Z < \frac{41-45}{20}\right) = P(Z \geq -0.2) = P(Z \leq 0.2) = 0.5 + \phi(0.2) = 0.5 + 0.0793 = 0.5793$

Number of candidates who assed =0.5793 x 500 = 290

(b) If 5% of the candidates obtain a distinction by scoring X marks or more, estimate the value of X.

$P(X > X_0) = 0.05(Q)$

$P\left(Z < \frac{X_0-45}{20}\right) = 0.5 - 0.05 = 0.45(P)$; from the table Z = 1.645

$X_0 = 1.645 \times 20 + 45 = 78$

∴the distinction starts at 78%

(c) Estimate the interquartile range of the distribution

Interquartile range = q3 – q1

$P\left(0 < Z < \frac{q_3-55}{20}\right) = 0.25(P)$; from the table Z = 0.674

$q_3 = 0.674 \, x \, 20 + 45 = 58.48$

$P\left(\frac{q_1-45}{20} < Z < 0\right) = 0.25(P)$; from the table Z = -0.674

$q_1 = -0.674 \, x \, 20 + 45 = 31.52$

∴ interquartile range = 58.48 – 31.52 = 26.96

Example 21

If X~N(70, 25) and P(|X - 70|< a) = 0.8, find the value of a and hence the limits within which the central 80% of the distribution lies.

$P(|X - 70| < a) = P(-a < X - 70 < a) = P(-a + 70 < X < a + 70) = 0.8$

$P\left(\frac{-a+70-70}{5} < X < \frac{a+70-70}{5}\right) = P\left(\frac{-a}{5} < X < \frac{a}{5}\right) = 0.8$

$2\,x\,P\left(0 < X < \frac{a}{5}\right)$= 0.8; $P\left(0 < X < \frac{a}{5}\right)$= 0.4(P)

From table Z = 1.282

1.282 = $\frac{a}{5}$; a = 6.41

But P(-a + 70 < X < a + 70) = 0.8

P(63.59 < X < 76.41) = 0.8

∴ Central 80% of the distribution lies between 63.59 and 76.41

Revision Exercise 2

1. Find the value of the following
   (i) P(Z < a) = 0.506 [a = 0.015]   (ii) P(Z < a) = 0.787 [a= 0.796]   (iii)P(Z< a) =0.0296[a= -0.1887]
   (iv) P(Z > a) = 0.713 [a = -0.562]   (iii)P(Z< a) = 0.325 [a=-0.454] (vi) P(|Z| > a) = 0.5 [a =0.674]
   (vii) P(|Z|> a) = 0.6 [a = 0.842]   (viii) P(Z< a) = 0.9738 [a =1.94] (ix) P(Z< a)= 0.2435[a = -0.695]
   (x) P(Z > a) = 0.82 [a = - 0.915] (xi) P(Z >a) = 0.2351 [a= 0.628] (xii) P(|Z|<a)= 0.6372[a=0.91]
   (xiii) P(Z>a) = 0.097 a=1.66]   (xv) P(|Z|> a) = 0.0404[a= 2.05]

2. Find the value of a if
   (i) P(Z< a) =0.9693 [a =1.87] (ii) P(Z > a) = 0.3802 [a = 0.305] (iii) P(Z > a)= 0.7367 [a = -0.633]
   (iv) P(Z< a) = 0.0793[a = -1.41] (v) P(|Z|< a) = 0.9 [a = 1.645]
3. If X~N(60, 25) find a if
   (i) P(X > a) = 0.2324 [a = 63.66]   (ii) P(X > a) = 0.0702 [a =67.37]
   (iii) P(X > a) = 0.837 [a = 55.09]   (iv) P(X > a) =0.7461 [a = 56.69]
4. If X~N(45, 16) find a if
   (i) P(X < a) = 0.0317 [a = 37.57]   (ii) P(X < a) = 0.895 [a = 50.01]
   (iii) P(X < a) = 0.0456 [a = 38.24]   (iv) P(X < a) = 0.996 [a = 55.6]
5. If X~N(400, 64) find a if
   (i) P(|X-400|< a) = 0.75 [9.2]   (ii) P(|X-400|< a) = 0.98 [18.61]
   (iii) P(|X-400|< a) = 0.95 [15.68]   (iv) P(|X-400|< a) = 0.975 [17.92]
   (v) The limits within which the central 95% of distribution lies. [384.32 < X < 415.68]
   (vi) Interquartile range of distribution [394.61, 405.39]
6. Bags of flour paced by a particular machine have masses which are normally distributed with mean 500g and standard deviation 20g. 2% of the bags are rejected for being overweight. Between what ranges of values should the mass of a bag of flour lie if it is to be accepted. [0458.92, 546.52]
7. The masses of mangoes sold at a market are normally distributed with mean mass 600g and standard deviation 20g.
   (i) If a mango is chosen at random, find the probability that its mass lies between 570g and 610g [0.6247]
   (ii) Find the mass exceeded by 7% of mangoes [629.52]
   (iii) In one day 1000 mangoes are sold. Estimate how many weigh less than 545g [3]
8. The length of metal strips are normally distributed with mean of 120cm and standard deviation of 10cm.
   (a) Find the probability that a strip selected at random has a length
      (i) greater than 105cm [0.9332] (ii) within 5cm of the mean =[0.383]

(b) Strips that are shorter than L cm are rejected. Estimate the value of L, if 9% or all the strips are rejected. [106.6cm]
(c) In a sample of 500 strips, estimate the number having a length over 126cm. [137]

9. The number of shirts sold in a week by a boutique are normally distributed with a mean 2080 and standard deviation of 50. Estimate
   (i) The probability that in a given week fewer than 2000 shirts are sold [0.0548]
   (ii) The number of weeks in a year that between 2060 and 2130 shirts are sold [26]
   (iii) The least number n of shirts such that the probability that more than n are sold in a given week is less than 0.02 [2183]
10. Batteries for a transistor radio have a mean life under normal usage of 160 hours, with standard deviation of 30 hours. Assuming the battery life follow normal distribution
    (i) Find the percentage of batteries which have a life between 150 hours and 180 hours. [37.8%]
    (ii) Calculate the range, symmetrical about the mean, within which 75% of the batteries lives lie. [125.5, 194.5]
    (iii) If the radio takes four of these batteries and require all of them to be working, find the probability that the radio will run for at least 135 hours. [0.405]
11. The length of type A rod is normally distributed with mean of 15cm and a standard deviation of 0.1cm. the length of another type B is also normally distributed with mean of 20cm and standard deviation 0.16cm. For type A rod to be acceptable, its length must be between 14.8cm and 15.2 cm and type B rod, the length must be between 19.8cm and 20.2cm.
    (i) What is the proportion of type a rod is of acceptable length? [95.44%]
    (ii) What is the probability that one of them is of acceptable length [0.7528, 0.2375]
12. The marks of 1000 students in an examination were normally distributed with mean 55 marks and standard deviation 8 marks.
    (i) If a mark of 71 or more is required for A-pass, estimate the number of a-passes awarded. [23]
    (ii) If 15% of the candidates failed, estimate the minimum mark required to pass. [47]
    (iii) Calculate the probability that two candidates chosen at random both passes examination [0.7225]
13. The burning life of a bulb approximately follows a normal distribution with mean of 1300hours and standard deviation of 125 hours
    (i) What is the probability that the bulb selected at random will burn for more than 1500 hours. [0.0548]
    (ii) Given that the manufacturer guarantees to replace any bulb that burns for less than 1050hours, what percentage of the bulbs will have to be replaced. [2.28%]
    (iii) If two bulbs are installed at the same time, what is the probability that bot will burn less than 1400 hours but more than 1200 hours [0.3320]
14. The marks in an examination were found to be normally distributed with mean 53.9 and standard deviation 16.5. 20% od the candidates who sat this examination failed. Find the pass mark [40.007]

Finding the value of mean, μ or standard deviation, σ or both

Hint: X = Zσ + μ

Example 22

If X~N(100, σ^2) and P(X < 106) = 0.8849, find the value of standard deviation, σ.

Solution

P(X < 106) = 0.8849 (Q –value)

$P\left(Z < \frac{106-100}{\sigma}\right)$ = 0.8849 – 0.5 = 0.3849 (P – value)

From table Z = 1.2

$\frac{106-100}{\sigma}$ = 1.2

σ = 5

Example 23

The length of a certain item follows a normal distribution with mean, μ cm and standard deviation of 6cm. it is known that 4.78% of the items have length greater than 82cm, find the mean, μ.

Solution

P(X > 82) = 0.0478 (Q – value)

$P\left(Z > \frac{82-\mu}{6}\right)$ = 0.5 – 0.0478 = 0.4522(P – value)

From table Z = 1.667

$\frac{82-\mu}{6}$ = 1.667

μ = 72 cm

Example 24

The masses of boxes of oranges are normally distributed such that 30% of them are greater than 4.00kg and 20% are greater than 4.53kg. Estimate the mean and standard deviation of the masses

Solution

P(X > 4) = 0.3 (Q – value)

$P\left(Z > \frac{4-\mu}{\sigma}\right)$ = 0.3 (Q – value)

$\frac{4-\mu}{\sigma} = 0.524$

4 = $\mu + 0.524\sigma$ ……………….(i)

$P\left(Z > \frac{4.53-\mu}{\sigma}\right)$ = 0.2 (Q – value)

$\frac{4.53-\mu}{\sigma} = 0.842$

4.53 = $\mu + 0.842\sigma$ ……………….(ii)

Eqn. (ii) – eqn. (i): 0.53 = 0.318σ; σ = 1.67kg

From eqn. (i)

μ = 4 – 0.524 x 1.67 = 3.13kg

Example 25

The speed of cars passing certain Entebbe high way can be taken to be normally distributed. 95% of the cars are travelling at less than 85m/s and 10% are travelling at less than 55m/s.

(i) Find the average speed of the cars passing through the high way
(ii) Find the proportion of the cars that travel at more than 70m/s

Solution

P(X < 85) = 0.95 (Q – value)

$P\left(Z < \frac{85-\mu}{\sigma}\right)$ = 0.45 (Q – value)

$\frac{85-\mu}{\sigma} = 1.645$

85 = $\mu + 1.645\sigma$ ……………….(i)

P(X < 55) = 0.1 (Q – value)

$P\left(Z < \frac{55-\mu}{\sigma}\right)$ = 0.1 (Q – value)

$\frac{55-\mu}{\sigma} = -1.282$

55 = μ – 1.282σ …………………………… (ii)

Eqn. (i) – eqn. (ii): 30 = 2.927σ; σ = 10.25m/s

From eqn. (i)

μ = 85 – 1.645x 10.25 = 68.14m/s

(ii) P(X > 70) = $P\left(Z < \frac{70-68.14}{10.25}\right)$= P(Z > 0.182)

= 0.5 – 0.0722

= 0.4278

Example 26

The masses of articles produced in a particular shop are normally distributed with mean μ and standard deviation σ. 5% of the articles have greater than 85g and 10% have masses less than 25g.

(i) Find the values of μ and σ
(ii) Find the symmetrical limits, about the mean, within which75% of the masses lie.

Solution

P(X > 85) = 0.05 (Q – value)

$P\left(Z > \frac{85-\mu}{\sigma}\right)$ = 0.05 (Q – value)

$\frac{85-\mu}{\sigma} = 1.645$

85 = $\mu + 1.645\sigma$ ……………….(i)

P(X < 25) = 0.1 (Q – value)

$P\left(Z < \frac{25-\mu}{\sigma}\right)$ = 0.1 (Q – value)

$\frac{25-\mu}{\sigma} = -1.282$

55 = μ – 1.282σ …………………………… (ii)

Eqn. (i) – eqn. (ii): 60 = 2.927σ; σ = 20.5m/s

From eqn. (i)

μ = 85 – 1.645x 20.5 = 513g

(ii) P(|X-51.3| < a) = 0.75

P(-a + 51.3 < X < a + 51.3) = 0.75

$$P\left(\frac{-a + 51.3 - 51.3}{20.5} < Z < \frac{a + 51.3 - 51.3}{20.5}\right) = 0.75$$

$P\left(\frac{-a}{20.5} < Z < \frac{a}{20.5}\right)$ = 0.75

2 x $P\left(0 < Z < \frac{a}{20.5}\right)$= 0.375 (P – value)

$\frac{a}{20.5} = 1.15$; a = 23.575

Lower limit = -23.575 + 51.3 = 27.73

Upper limit = 23.575 + 51.3 = 74.88

Example 27

A total population of 700 students sat a mock examination for which the pass mark was 50%. Their marks were normally distributed. 28 students scored below 40% while 35 students scored above 60%.

(a) Find the mean mark and standard deviation of the students' marks.
(b) What is the probability that a student chosen at random passed the exam?

(c) Suppose the pass mark is lowered by 2%, how many more students will pass.

Solution

P(X < 40) = $\frac{28}{700}$ =0.04 (Q – value)

$P\left(Z > \frac{40-\mu}{\sigma}\right)$ = 0.04 (Q – value)

$\frac{40-\mu}{\sigma} = -1.751$

40 = $\mu - 1.751\sigma$ ………………(i)

P(X > 60) = $\frac{35}{700}$ =0.05 (Q – value)

$P\left(Z > \frac{60-\mu}{\sigma}\right)$ = 0.05 (Q – value)

$\frac{60-\mu}{\sigma} = 1.645$

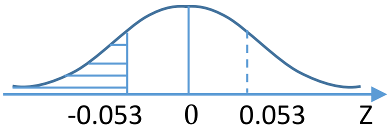
60 = μ + 1.645σ …………………………… (ii)

Eqn. (ii) – eqn. (ii): 20 = 3.396σ; σ = 5.889

From eqn. (i)

μ = 40 + 1.751 x 5.889 = 50.312

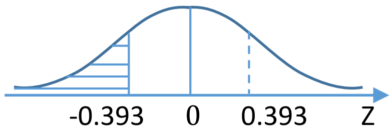
(ii) P(X ≥ 50) = $P\left(Z \geq \frac{50-50.312}{5.889}\right)$= P(Z ≥ - 0.053)



P(Z ≥ -0.053) = 0.5 + P(0 < Z < 0.053)

= 0.5 + 0.0211 = 0.5211

(iii) P(X ≥ 48) = $P\left(Z \geq \frac{48-50.312}{5.889}\right)$= P(Z ≥ - 0.053)



P(Z ≥ -0.392) = 0.5 + P(0 < Z < 0.392)

= 0.5 + 0.1528 = 0.6528

More proportion = 0.6528 – 0.5211 = 0.1317

More students = 0.1317 x 700 = 92

Example 28

A random variable X has a normal distribution with P(X > 55) = 0.2 and P(35 < X < 55) = 0.5. Find

(a) The value of the mean, μ and standard deviation, σ.
(b) The percentage of those with P(X > 45)

Solution

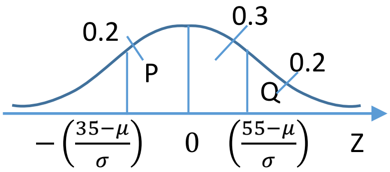
P(X > 55) = $\frac{28}{700}$ =0.2 (Q – value)

$P\left(Z > \frac{55-\mu}{\sigma}\right)$ = 0.2 (Q – value)

$\frac{40-\mu}{\sigma} = 0.842$

55 = $\mu + 0.842\sigma$ ………………(i)

P(35 < X < 55) = 0.5



$P\left(\frac{35-\mu}{\sigma} < Z < 0\right)$ = 0.2(P – value)

$\frac{35-\mu}{\sigma} = -0.842$

35 = $\mu - 0.842\sigma$ ………………(ii)

μ = 70.8797, σ = 42.5872

(b) P(X > 45) = $P\left(Z > \frac{45-70.8797}{42.5872}\right)$= P(Z > -0.608)

= 0.5 + P(0 < Z < 0.608)

= 0.5 + 0.2283 = 0.7283

Percentage = 0.7283 x 100 = 72.83

Revision exercise 3

1. $X \sim N(45, \sigma^2)$ and $P(X > 51) = 0.288$. find σ. [σ = 10.7]
2. $X \sim N(21, \sigma^2)$ and $P(X < 27) = 0.9332$. find σ. [σ = 4]
3. $X \sim N(\mu, 25)$ and $P(X < 27.5) = 0.3085$. find μ. [μ = 30]
4. $X \sim N(\mu, 12)$ and $P(X > 32) = 0.8438$. find μ. [μ = 35.5]
5. $X \sim N(\mu, \sigma^2)$ and $P(X > 80) = 0.0113$ and $P(X > 30) = 0.9713$. find σ and μ. [μ =52.73 and σ= 11.96]
6. $X \sim N(\mu, \sigma^2)$ and $P(X > 102) = 0.42$ and $P(X < 97) = 0.25$. find σ and μ. [μ =100.8 and σ= 5.71]
7. $X \sim N(\mu, \sigma^2)$ and $P(X < 57.84) = 0.90$ and $P(X < 50) = 0.5$. find σ and μ. [μ =50 and σ= 6.12]
8. $X \sim N(\mu, \sigma^2)$ and $P(X < 35) = 0.20$ and $P(35 < X < 45) = 0.65$. find σ and μ. [μ =39.5 and σ= 5.32]
9. The length of rods produced in a workshop follow a normal distribution with mean μ and variance 4. 10%of the rods are less than 17,4cm long. Find the probability that a rod chosen at random will be between 18cm and 23 cm. [0.7725]
10. The length of a stick follow a normal distribution. 10% are of length 250cm or more while 55% have a length over 240cm. Find the probability that a stick chosen at random is less than 235cm long. [0.203]
11. A certain make of car tyres can be safely used for 25000km on average before replaced. The makers guarantee to pay compensation to anyone whose tyre does not last for 22000km. they expect 7.5% of all the tyres sold to qualify for compensation. If the distance X travelled before a tyre is replaced has normal distribution.
    (i) Find the standard deviation [2080]
    (ii) Estimate the number of tyres per 1000 which will not have been replaced when they have covered 26500km. [236]
12. The continuous random variable X is normally distributed with mean μ and standard deviation σ. If $P(X < 53) = 0.04$ and $P(X < 65) = 0.97$, find the interquartile range [4.46]
13. Tea sold in packages marked 750g. The masses are normally distributed with mean 760g and standard deviation σ. What is the maximum value of σ, if less than 1% of the packages are underweight? [4.299]
14. In an examination 30% of the candidates fail and 10% achieve distinction. Last year the pass mark (out of 200) was 84 and the minimum mark required for a distinction was 154. Assuming that the marks of candidates are normally distributed, estimate the mean mark and standard deviation. [μ = 104.31, σ = 38.76]
15. AT St Noa junior, the heights of students are normally distributed. 10% are over 1.8m and 20% are below 1.6m.
    (i) Find the mean height μ and standard deviation σ. [μ =1.68, μ = 0.09]
    (ii) Find the interquartile range [0.13]
16. Observation of a very rage number of cars are certain point on a motor way established that the speeds are normally distributed. 90% of the cars have speed less than 77.7km/h and only 5% of cars have speed less than 63.1km/h. find the mean speed μ and standard deviation σ. [μ = 71.305, σ = 4.988]
17. A sample of 100 apples is taken from a load. The apple have the following distribution of size.

Diameter (cm)	6	7	8	9	10
Frequency	11	21	38	17	13

Assuming that the distribution is approximately normal with mean μ and standard deviation σ.

(i) Determine μ and σ [μ = 8, σ = 1.16]

(ii) Find the range of sizes of apples for packing, if 5% are to be rejected as too small and 5% are to be rejected as too large [6.10, 9.90]

18. The volumes of soda in bottles are normally distributed with mean of 333ml. Given that 20% of the bottles contain more than 340ml, find
    (i) Standard deviation of the volume of bottle. [8.31]
    (ii) Percentage of bottles that contain less than 330ml. [35.9%]
19. The heights of 500 students are normally distributed with a standard deviation of 0.080cm. If the heights of 129 of the students are greater than the mean height but less than 1.806m find the mean height. [1.75]
20. The masses of boxes of apples are normally distributed such that 20% of them are greater than 5.08kg and 15% are greater than 5.62kg; find the mean and standard deviation. [μ= 2.74, σ = 2.78]
21. The masses of sugar are normally distributed. If 5% of the packets have mass greater than 510g and 2% have masses greater than 515g. Find the mean and standard deviation. [μ = 490, σ = 12.2g]
22. Sugar packed in 500g packets is observed to be approximately normally distributed with standard deviation of 4. If only 2% of the packets contained less than 500g of sugar. Find the mean weight of sugar in the packets. [508.216g]
23. Sixty students sat for a mathematics contest whose pass mark was 40marks. Their scores in the contest were approximately normally distributed. 9 students scored less than 20 marks while 3 scored more than 70 marks. Find the
    (i) Mean scored and the standard deviation of the contest. [μ = 39.32, σ = 18.65]
    (ii) Find the probability that a student chosen at random passed the contest. [0.4856]
24. The number of cows owned by residents n a village is assumed to be normally distributed. 15% of the residents have less than 60 cows while 5% of residents have over 90 cows.
    (a) Determine the values of the mean and standard deviations of cows [μ =71.5926, σ= 11.1899]
    (b) If there are 200 residents, find how many have more than 80 cows. [45]
25. A random variable X has a normal distribution when P(X > 9) = 0.9192 and P (X < 11) = 0.7580. find
    (a) the value of the mean and standard deviation. [μ= 10.3333, σ = 0.9524]
    (b) P(X > 10) [0.6386

26. The marks in an examination were normally distributed with mean μ and standard deviation σ. 20% of the candidates scored less than 40marks and 10% more than 75 marks. Find the
    (a) values of μ and σ. (08marks)
    (b) percentage of the candidate who scored more than 50 marks.(04marks)

Solutions to revision exercise 3

20. The masses of boxes of apples are normally distributed such that 20% of them are greater than 5.08kg and 15% are greater than 5.62kg; find the mean and standard deviation.

Solution

P(X > 5.08) = 0.2 (Q – value)

$P\left(Z > \frac{5.08-\mu}{\sigma}\right)$ = 0.2 (Q – value)

$\frac{5.08-\mu}{\sigma} = 0.842$

5.08 = $\mu + 0.842\sigma$ ……………….(i)

P(X > 5.62) = 0.15 (Q – value)

$\frac{5.62-\mu}{\sigma} = 1.036$

5.62 = $\mu + 1.036\sigma$ ……………….(ii)

Eqn. (ii) – eqn. (i)

0.54 = 0.194σ; σ = 2.7835kg

From eqn. (i)

μ = 5.08 – 0.842 x 2.7835 = 2.7363kg

$P\left(Z > \frac{5.62 - \mu}{\sigma}\right)$ = 0.15 (Q – value)

21. The masses of sugar are normally distributed. If 5% of the packets have mass greater than 510g and 2% have masses greater than 515g. Find the mean and standard deviation.

Solution

P(X > 510) = 0.05 (Q – value)

$P\left(Z > \frac{510 - \mu}{\sigma}\right)$ = 0.05 (Q – value)

$\frac{510-\mu}{\sigma} = 1.645$

510 = $\mu + 1.645\sigma$ ……………….(i)

P(X > 515) = 0.02 (Q – value)

$P\left(Z > \frac{515 - \mu}{\sigma}\right)$ = 0.02 (Q – value)

$\frac{515-\mu}{\sigma} = 2.054$

515 = $\mu + 2.054\sigma$ ……………….(ii)

Eqn. (ii) – eqn. (i)

5 = 0.409σ; σ = 12.225kg

From eqn. (i)

μ = 510 – 1.645 x 12.225 = 489.89kg

22. Sugar packed in 500g packets is observed to be approximately normally distributed with standard deviation of 4. If only 2% of the packets contained less than 500g of sugar. Find the mean weight of sugar in the packets.

P(X < 500) = 0.02 (Q – value)

$P\left(Z < \frac{500 - \mu}{4}\right)$ = 0.02 (Q – value)

$\frac{500-\mu}{\sigma} = -2.054$

Mean weight, μ = 500 + 2.054 x 4 = 508.216g

23. Sixty students sat for a mathematics contest whose pass mark was 40marks. Their scores in the contest were approximately normally distributed. 9 students scored less than 20 marks while 3 scored more than 70 marks. Find the
    (i) Mean scored and the standard deviation of the contest.

Solution

P(X < 20) = $\frac{9}{60}$ =0.15 (Q – value)

$P\left(Z > \frac{20 - \mu}{\sigma}\right)$ = 0.15 (Q – value)

$\frac{20-\mu}{\sigma} = -1.036$

20 = $\mu - 1.036\sigma$ ……………….(i)

P(X > 70) = $\frac{3}{60}$ = 0.05 (Q – value)

$P\left(Z > \frac{70 - \mu}{\sigma}\right)$ = 0.05 (Q – value)

$\frac{70-\mu}{\sigma} = 1.645$

70 = $\mu + 1.645\sigma$ ……………….(ii)

Eqn. (ii) – eqn. (i)

50 = 2.681; σ = 18.65

From eqn. (i)

μ = 20 + 1.036 x 18.65 = 39.3214

(ii) Find the probability that a student chosen at random passed the contest.

P(X > 40) =$P\left(Z > \frac{40-39.3214}{18.65}\right)$= P(Z > 0.0364) = 0.5 – ϕ(0.036) = 0.5 – 0.0144 = 0.4856

24. The number of cows owned by residents n a village is assumed to be normally distributed. 15% of the residents have less than 60 cows while 5% of residents have over 90 cows.
   (a) Determine the values of the mean and standard deviations of cows
   (b) If there are 200 residents, find how many have more than 80 cows. [45]

Solution

(i) P(X < 60) =0.15 (Q – value)

$P\left(Z < \frac{60-\mu}{\sigma}\right)$ = 0.15 (Q – value)

$\frac{20-\mu}{\sigma} = -1.036$

60 = $\mu - 1.036\sigma$ ……………….(i)

P(X > 90) = 0.05 (Q – value)

$P\left(Z > \frac{70-\mu}{\sigma}\right)$ = 0.05 (Q – value)

$\frac{70-\mu}{\sigma} = 1.645$

90 = $\mu + 1.645\sigma$ ……………….(ii)

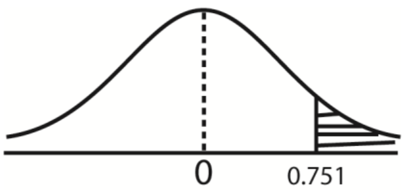
Eqn. (ii) – eqn. (i)

30 = 2.681; σ = 11.1899

From eqn. (i)

μ = 60 + 1.036 x 11.1899= 71.5927

(ii) P(X > 80) = $P\left(Z > \frac{80-71.5927}{11.1899}\right)$= P(Z > 0.751)



P(Z > 0.751) = 0.5 – (0 < Z < 0.751)

= 0.5 – 0.2737

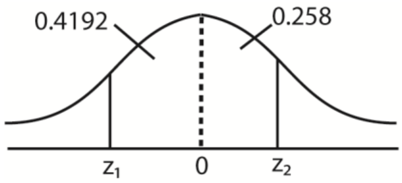
=0.2263

Number of residents =200 x 0.2263 = 45

25. A random variable X has a normal distribution when P(X > 9) = 0.9192 and P (X < 11) = 0.7580. find
   (a) The values of the mean and standard deviation (08marks)

   P(X, >9) = P$\left(z_1 > \frac{9-\mu}{\delta}\right)$ =0.9192

   P(X, < 11) = P$\left(z_2 > \frac{11-\mu}{\delta}\right)$ =0.0.7580



$z_1 = -\phi(0.4192) = -1.4$

$z_2 = \phi(0.258) = 0.7$

⇨ $\frac{9-\mu}{\delta} = -1.4$

$9 - \mu = -1.4\delta$ …………………… (i)

⇨ $\frac{11-\mu}{\delta} = 0.7$

$11 - \mu = 0.7\delta$ …………………… (ii)

$Eqn\ (i) - Eqn\ (ii)$

$-2 = -2.1\delta$

$\delta = \frac{-2}{-2.1} = 0.9524$

From (i)

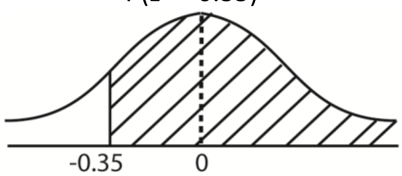
$9 - \mu = -1.4 \times 0.9524$

$\mu = 10.333$

(b) P(X>10) (04marks)

$P(X>10) = P\left(z > \frac{10-10.333}{0.9524}\right)$

$= P(z > -0.35)$



$P(X > 10) = P(0.5 + P(0 < z < 0.35)$

$= 0.5 + 0.1368$

$= 0.6368$

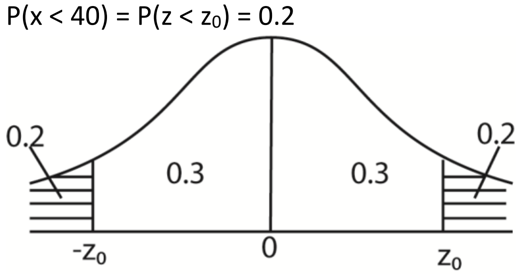
26. The marks in an examination were normally distributed with mean μ and standard deviation σ. 20% of the candidates scored less than 40marks and 10% more than 75 marks. Find the

(c) values of μ and σ. (08marks)

Let x = marks scored

$P(x < z < z_0) = 20\% = 0.2$

$P(x < 40) = P(z < z_0) = 0.2$



$P(0 < x < z_0) = 0.3$

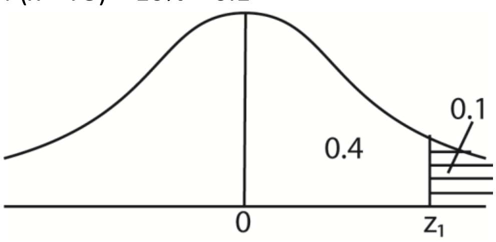
$z_0 = -0.842$

but $z = \frac{40-\mu}{\sigma}$

$-0.842 = \frac{40-\mu}{\sigma}$

$-0.842\sigma = 40 - \mu$ …………………… (i)

P(x > 75) = 10% = 0.1



$P(0 < z < z_1) = 0.4$

$z_1 = 1.282$

$1.282 = \frac{75 - \mu}{\sigma}$

$1.282\sigma = 75 - \mu$ ……………………. (ii)

Eqn. (ii) – eqn. (i)

$2.124\sigma = 35$

$\sigma = 16.478\ (3D)$

$substituting\ \sigma\ into\ eqn.\ (ii)$

$1.282 \times 16.478 = 75 - \mu$

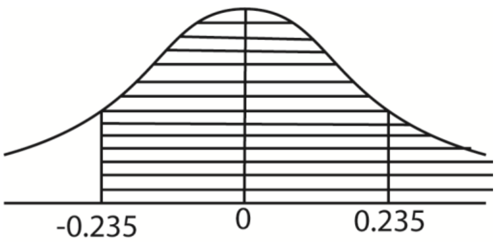
$\mu = 53.875$

Hence μ = 53.875 and σ = 16.478

(d) percentage of the candidate who scored more than 50 marks.(04marks)

$P(x > 50) = P\left(z - \frac{50-53.875}{16.478}\right)$

$= P(z > -0.235)$



= 0.5 + (0< z < 0.235)

= 0.5 + 0.0929

= 0.5929

= 59.29%

Binomial approximation to a normal distribution

Under the following conditions, the normal distribution is used to approximate binomial distribution

Conditions

(i) the number of trials of the binomial experiment should be large, n > 20.

(ii) The probability of success not to small or too large i.e. p constant and very close to 0.5

$X \sim N(np, npq)$

The z-value is obtained from $Z = \frac{X \pm 0.5 - np}{\sqrt{npq}}$

Where $\pm$0.5 is used to make the binomial distribution continuous.

Note; 0.5 must be subtracted from the minimum value and added to the maximum value

(i) P(X$\geq x_1$) = $P\left(Z \ \geq \frac{(X-0.5)-np}{\sqrt{npq}}\right)$

(ii) P(X$\leq x_1$) = $P\left(Z \ \leq \frac{(X+0.5)-np}{\sqrt{npq}}\right)$

(iii) P(x_1 $\leq$X$\leq$ x_2) = $P\left(\frac{(X-0.5)-np}{\sqrt{npq}} \leq Z \ \leq \frac{(X=0.5)-np}{\sqrt{npq}}\right)$

Example 29

In a box containing different pens, the probability that a pen is red is 0.35. Find the probability that in a random sample of 400 pens from the box

(i) Less than 120 are red pens
(ii) More than 160 are red pens
(iii) Between 120 and 150 inclusive are red pens.

Solution

N = 400, p = 0.35, q = 0.65

Mean, μ = np = 400 x 0.35 = 140

σ = $\sqrt{npq} = \ \sqrt{400 \ x \ 0.35 \ x \ 0.65} = \sqrt{91}$

(i) P(X < 120) = $P\left(Z \leq \frac{119.5 - 140}{\sqrt{91}}\right) = P(Z \leq \ 2.149)$

=$P(Z \ \geq 2.149)$ = 0.5 – ϕ(2.149)

=0.5 – 0.4842 = 0.0158

(ii) P(X > 160) = P(X $\geq$ 161)

= $P\left(Z \leq \frac{116.5 - 140}{\sqrt{91}}\right) = P(Z \geq \ 2.149)$

=0.5 – ϕ(2.149) = 0.5 – 0.4821 = 0.0158

(iii) P(120 ≤ X ≤ 150)

=$P\left(\frac{119.5 - 140}{\sqrt{91}} \ \leq Z \ \frac{150.5 - 140}{\sqrt{91}}\right)$

=P(-2.149 ≤ Z ≤ 1.101)

=0.4842 + 0.3645 = 0.8487

Example 30

In unbiased coin is tossed 100 times, what is the probability that

(i) There will be more than 60 heads (ii) there will be less than 43 head

(iii) there will be between 45 heads and 55 head

Solution

N = 100, p = 0.5, q = 0.5

Mean, μ = np = 100 x 0.5 = 50

σ = $\sqrt{npq} = \ \sqrt{100 \ x \ 0.5 \ x \ 0.5} = 5$

(i) P(X > 60) = P(X ≥ 61)

= $P\left(Z \geq \frac{60.5 - 50}{5}\right) = P(Z \geq \ 2.1)$

=0.5 – ϕ(2.149) =0.5 – 0.4821 = 0.0179

(ii) P(X < 43) = P(X ≤ 42)

= $P\left(Z \leq \frac{42.5 - 50}{5}\right) = P(Z \leq \ -1.5$

= $P(Z \geq \ 1.5)$ =0.5 – ϕ(1.5)

= 0.5 – 0.4332 = 0.0668

(iii) P(45 ≤ X ≤ 55)

=$P\left(\frac{44.5 - 50}{5} \ \leq Z \ \frac{55.5 - 50}{5}\right)$

=P(-1.1 ≤ Z ≤ 1.1) = 2 x ϕ(1.1)

=2 x 0.3643 = 0.7286

Example 31

It is known that 72% f NTV viewer watch news at 9 pm. What is the probability that a sample of 500 viewer chosen at random

(i) More than 350 watch news (ii) fewer than 340 watch news (ii) exactly 350 watch news

Solution

(i) P(X > 350) = P(X ≥ 351)

$= P\left(Z \geq \frac{350.5 - 500\ x\ 0.72}{\sqrt{500\ x\ 0.72\ x\ 0.28}}\right)$

$= P(Z \geq -0.946) = P(Z \leq 0.946$

=0.5 + ϕ(0.946) =0.5 + 0.328 = 0.8280

(ii) P(X < 340) = P(X ≤ 339)

$= P\left(Z \leq \frac{338.5 - 500\ x\ 0.72}{\sqrt{500\ x\ 0.72\ x\ 0.28}}\right)$

$= P(Z \leq -2.042) = P(Z \geq 2.042$

=0.5 - ϕ(2.042) =0.5 - 0.4794= 0.0206

(iii) P(X = 350) = P(349.5 ≤ X ≤ 350.5)

$=P\left(\frac{349.5 - 500\ x\ 0.72}{\sqrt{500\ x\ 0.72\ x\ 0.28}} \leq Z \leq \frac{350.5 - 500\ x\ 0.72}{\sqrt{500\ x\ 0.72\ x\ 0.28}}\right)$

= ϕ(-1.046 – ϕ0.946)

= 0.3522 – 0.3280 = 0.0242

∴ P(X = 350) = 0.0242

Example 32

A pair of balanced dice, each numbered 1 to 6 is tossed 150 times. Determine the probability that a sum of seven appear at least 26 times

P(sum of 7) = p= $\frac{6}{36} = \frac{1}{6}$

P(X ≥ 26) = $P\left(Z \geq \frac{(26-0.5) - 150\ x \frac{1}{6}}{\sqrt{150\ x \frac{1}{6}\ x \frac{5}{6}}}\right)$

= $P\left(Z \geq \frac{25.5 - 25}{4.56}\right)$ = P(Z ≥ 0.11)

= 0.5 – ϕ(0.11) = 0.5 – 0.0438

= 0.4562

Example 33

Two players play a game in which each of them tosses a balanced coin. The game ends in a draw if both get the same result. Determine the probability that in 100 trials, the game ends in a draw.

(i) At least 53 times (ii) at most 53 times

P(sum of 7) = p= $\frac{2}{4} = \frac{1}{2}$

P(X ≥ 53) = $P\left(Z \geq \frac{(53-0.5) - 100\ x \frac{1}{2}}{\sqrt{100\ x \frac{1}{2}\ x \frac{1}{2}}}\right)$

= $P\left(Z \geq \frac{52.5 - 50}{5}\right)$ = P(Z ≥ 0.5)

= 0.5 – ϕ(0.5) = 0.5 – 0.1915 = 0.3085

P(X ≤ 53) = $P\left(Z \leq \frac{(53+0.5) - 100\ x \frac{1}{5}}{\sqrt{100\ x \frac{1}{2}\ x \frac{1}{2}}}\right)$

= $P\left(Z \leq \frac{53.5 - 50}{5}\right)$ = P(Z ≤ 0.7)

= 0.5 + ϕ(0.5) = 0.5 + 0.2580 = 0.7580

∴ P(X ≤ 53) = 0.7580

Example 34

In a certain book of words per page follow normal distribution with mean 800 words and standard deviation 40 words. Three pages are chosen at random, what is the probability that

(i) None of them has between 830 and 845 words.
(ii) At least tw pages have between 830 and 845 words

Solution

(i) P(830 ≤ X ≤ 845) = $P\left(\frac{830-800}{40} \le Z \le \frac{845-800}{40}\right)$= P(0.75 ≤ Z ≤ 1.25)

= $\phi(1.25) - \phi(0.75)$ = 0.3522 – 0.3280 = 0.0962

P(X = 0) = $^3C_0(0.0962)^0(0.9038)^3$ = 0.7383

(ii) P(X ≥ 2) = P(X = 2) + P(X = 3)

=$^3C_2(0.0962)^2(0.9038)^1 + {}^3C_3(0.0962)^3(0.9038)^0$ = 0.02509 + 0.00089 = 0.02598

Revision exercise 4

1. A random variable X~B(200, 0.7). Find
   (i) P(X ≥ 130) [0.9474] (ii) P(136 ≤ X < 148) [0.6325] (iii) P(X < 142) [0.5914] (iv) P(X= 152) [0.0111]
2. An ordinary unbiased die is thrown 120 times. Find the probability of obtaining at least 24 sixes. [0.1958]
3. A pair f dice is tossed 144 times and the sum of the outcomes recorded. Find the probability that a sum of 7 occurs at least 26 times. [0.3688]
4. In a school 45% f the boys are circumcised. Find the probability that in a group of 200 boys 97 are circumcised. [0.1432]
5. 10% of phones imported to Uganda are I-phones, a random sample of 1000 phones is taken. Find the probability that
   (i) Less than 80 are I-phones [0.0154]
   (ii) Between 90 and 115 inclusive are I-phones [0.8145]
   (iii) 120 or more are I-phones [0.02]
6. During Christmas, the probability that message is sent on phone successfully is 0.85.
   (i) When 8 messages are sent, find the probability that at least 7 are successfully sent [0.657]
   (ii) When 50 messages are sent, find the probability that at least 45 are successfully sent [0.2142]
7. One –fifth f tourist have COVID 19. Find the probability that the number of tourist with COVID 19 is
   (i) More than 20 in a sample f 100 people [0.4502]
   (ii) Exactly 20 in a random sample of 100 people [0.0996]
   (iii) More than 200 in a random sample of 1000 people [0.484]
8. If a fair die is thrown 300 times, what is the probability that
   (i) There will be more than 60 sixes [0.0519]
   (ii) There will be fewer than 45 sixes [0.1971]
9. A coin is biased such that head is twice as likely to occur as a tail. The coin is tossed 120 times. find the probability that there will be
   (i) Between 42 and 51 tails inclusive [0.3729]
   (ii) 48 tails or less [0.9501]
   (iii) Less than 34 tails [0.1039]
   (iv) At least 72 and at most 90 heads [0.9290]
10. A lorry of potatoes has average one rotten potato in six. A green grocer tests a random sample of 100 potatoes and decides to turn away the lorry if he finds more than 18 rotten potatoes in the sample. Find the probability that he accepts the consignment. [0.6886]
11. On a certain farm, 20% of all the cows are infected by a tick disease. Find the probability that in a sample of 50 cows selected at random not more than 10% of the cows are infected. [0.0558]

12. A pair of balanced dice, each numbered from 1 to 6 is tossed 180 times. determine the probability that a sum of seven appears;
    (i) Exactly 40 times [0.0108]
    (ii) Between 25 and 35 inclusive times [0.7286]
13. On average 20% of all the eggs supplied by a farm have cracks. Find the probability that in a sample of 900 eggs supplied by a far will have more than 200 cracked eggs. [0.0439]
14. On average 15% of all boiled eggs sold in a restaurant have cracks. Find the probability that in a sample of 300 boiled eggs will have more than 50 cracked eggs [0.215]
15. Among spectators watching a football watch, 80% were the home supporters while 20% were the visiting team supporters. If 2500 of the spectators are selected randomly, what is the probability that there are at least 541 visitors in the sample? [0.0215]
16. A die is tossed 40 times and the probability of getting at any one toss is 0.122, estimate the probability of getting between 6 to 10 sixes. [0.2048]
17. In an examination which consists of 100 questions, a student has a probability of 0.6 of getting each answer correct. A student fails the examination if he obtains a mark less than 55, and obtains a distinction for a mark of 68 or more. Calculate
    (i) The probability that he fails the examination [0.1308]
    (ii) The probability that he obtains a distinction [0.0629]
18. A research station supplies three varieties of seeds s_1, S_2 and S_3 in the ratio 4:2:1. The probabilities of germination of S_1, S_2 and S_3 are 50%, 60% and 80% respectively
    (i) Find the probability that a selected seed will germinate [
    (ii) Given that 150 seeds are selected at random, find the probability that less than 90 seed will germinate.
19. A biased die with faces labelled 1, 2, 3, 4, 5, and 6 is tossed 45 times. calculate the probability that 2 appears
    (i) More than 18 times [0.1342]
    (ii) Exactly 11 times[0.0568]

Solutions to revision exercise 4

11. On a certain farm, 20% of all the cows are infected by a tick disease. Find the probability that in a sample of 50 cows selected at random not more than 10% of the cows are infected. [0.0558]

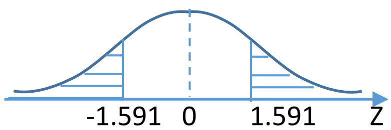
    Since n is large, we use the normal distribution to approximate binomial distribution

    Given: n = 50, p = 0.2, q = 1 – 0.2 = 0.8, μ = np = 50 x 0.2 = 10,

    $=\sqrt{npq} = \sqrt{50\ x\ 0.2\ x\ 0.8} = \sqrt{8}$

    10% of 50 cows = 0.1 x 50 = 5

    P(x < 5) = $P\left(Z < \frac{5.5}{\sqrt{8}}\right) = P(Z < -1.591)$



$P(Z < -1.591) = P(Z > 1.591)$

= 0.5 – P(0 < Z < 1.591)

= 0.5 – 0.4442 = 0.0558

12. A pair of balanced dice, each numbered from 1 to 6 is tossed 180 times. determine the probability that a sum of seven appears;
    (i) Exactly 40 times [0.0108]
    (ii) Between 25 and 35 inclusive times [0.7286]

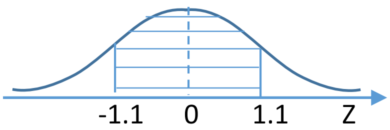
Solution

Dice	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Let E = event that the sum 7 is picked when a pair of dice is tossed

n(E) = 7 and P(E) = $\frac{6}{36} = \frac{1}{6}$

Since n is large, we use the normal distribution to approximate binomial distribution
Given: n = 180, p = $\frac{1}{6}$, q = 1 – $\frac{1}{6}$ = $\frac{5}{6}$, μ = np = 180 x $\frac{1}{6}$= 30,
=$\sqrt{npq} = \sqrt{180\ x\ \frac{1}{6}\ x\ \frac{5}{6}}$ = 5
Let x = number of times a sum of 7 appears
(i) P(x = 40) = P(39.5 < X < 40.5) = $P\left(\frac{39.5-30}{5} < Z < \frac{39.5-30}{5}\right) = P(1.9 < Z < 2.1)$
=P(0 < Z < 2.1) - (0 < Z < 1.9) = 0.4821 – 0.4713 = 0.0108
(ii) P(25 ≤ x ≤ 35) = $P\left(\frac{24.5-30}{5} < Z < \frac{35.5-30}{5}\right)$= P(-.1.1 < Z < 1.1)

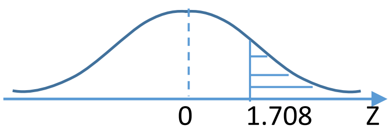


P(-.1.1 < Z < 1.1) = 2 x P() < Z < 1.1) = 2 x 0.3643 = 0.7286

Hence P(25 ≤ x ≤ 35) = 0.7286

13. On average 20% of all the eggs supplied by a farm have cracks. Find the probability that in a sample of 900 eggs supplied by a far will have more than 200 cracked eggs.

Since n is large, we use the normal distribution to approximate binomial distribution
Given: n = 900, p = 0.2, q = 1 – 0.2 = 0.8, μ = np = 900 x 0.2 = 180,
=$\sqrt{npq} = \sqrt{900\ x\ 0.2\ x\ 0.8}$ = 12
let x = number of eggs with cracks
P(x > 200) = P(Z > Z_1)
Where $Z_1 = \frac{200.5-180}{12} = 1.708$



$P(Z > 1.708) = 0.5 – P(0 < Z < 1.708)$

$= 0.5 – 0.4561 = 0.0439$

14. On average 15% of all boiled eggs sold in a restaurant have cracks. Find the probability that in a sample of 300 boiled eggs will have more than 50 cracked eggs

Since n is large, we use the normal distribution to approximate binomial distribution

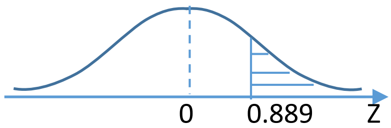
Given: n = 300, p = 0.15, q = 1 – 0.15 = 0.85, μ = np = 300 x 0.15 = 45,

$=\sqrt{npq} = \sqrt{300 \; x \; 0.15 \; x \; 0.85} = 6.1847$

let x = number of eggs with cracks

$P(x > 50) = P(Z > Z_1)$

Where $Z_1 = \frac{50.5-45}{6.1847} = 0.889$



$P(Z > 0.889) = 0.5 – P(0 < Z < 0.889)$

$= 0.5 – 0.2850 = 0.215$

15. Among spectators watching a football watch, 80% were the home supporters while 20% were the visiting team supporters. If 2500 of the spectators are selected randomly, what is the probability that there are at least 541 visitors in the sample?

Solution

Since n is large, we use the normal distribution to approximate binomial distribution

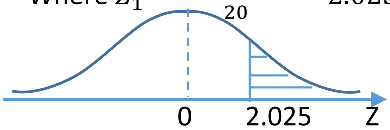
Given: n = 2500, p = 0.2, q = 1 – 0.2 = 0.8, μ = np = 2500 x 0.2 = 500,

$\sigma = \sqrt{npq} = \sqrt{2500 \; x \; 0.2 \; x \; 0.8} = 20$

let x = number of visitors to support their team

$P(x > 540) = P(Z > Z_1)$

Where $Z_1 = \frac{540.5-500}{20} = 2.025$



$P(Z > 2.025) = 0.5 – P(0 < Z < 2.025)$

$= 0.5 – 0.4785 = 0.0215$

16. A die is tossed 40 times and the probability of getting at any one toss is 0.122, estimate the probability of getting between 6 to 10 sixes.

Solution

Given: n = 40, p = 0.122, q = 1 – 0.122 = 0.878, μ = np = 40 x 0.122 = 4.88,

$\sigma = \sqrt{npq} = \sqrt{40 \; x \; 0.122 \; x \; 0.878} = 2.07$

Let x be the number of sixes

$P(6 < x < 10) = P(7 \leq x \leq 9)$

Using normal approximation to binomial

$Z = \frac{X \pm 0.5 - \mu}{\sigma}$

$P(7 \leq x \leq 9) = P\left(\frac{6.5-4.88}{2.07} \leq Z \leq \frac{9.5-4.88}{2.07}\right)$

$= P(0.78 < Z < 2.23$

$= P(0 < Z < 2.23) - P(0 < Z < 0.78)$

$= 0.4871 - 0.2823$

$= 0.2048$

17. In an examination which consists of 100 questions, a student has a probability of 0.6 of getting each answer correct. A student fails the examination if he obtains a mark less than 55, and obtains a distinction for a mark of 68 or more. Calculate
    (i) The probability that he fails the examination [0.1308]
    (ii) The probability that he obtains a distinction [0.0629]

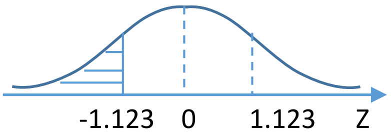
Solution

Given: n = 100, p = 0.6, q = 0.4, μ = np = 100 x 0.6 = 60 and σ = $\sqrt{npq} = \sqrt{100 \ x \ 0.6 \ x \ 0.4} = 4.899$

Let x = mark scored

(i) $P(x < 55) = P(x \leq 54) = P\left(Z \leq \frac{54.5-60}{4.899}\right)$

$= (Z \leq -1.123)$



$P(Z < -1.123) = P(Z > 1.123)$

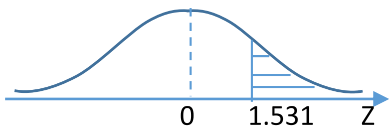
$= 0.5 - P(0 \leq Z \leq 1.123)$

$= 0.5 - 0.3692$

$= 0.1308$

(ii) $P(x \geq 68) = P\left(Z \leq \frac{67.5-60}{4.899}\right)$

$= P(Z \geq 1.531)$



$P(Z \geq 1.531) = 0.5 - P(0 \leq Z \leq 1.531)$

$= 0.5 - 0.4371$

$= 0.0629$

18. A research station supplies three varieties of seeds s_1, S_2 and S_3 in the ratio 4:2:1. The probabilities of germination of S_1, S_2 and S_3 are 50%, 60% and 80% respectively
    (i) Find the probability that a selected seed will germinate [
    (ii) Given that 150 seeds are selected at random, find the probability that less than 90 seed will germinate.

Solution

Given

4:2:1

4 + 2+ 1 = 7

$P(S_1) = \frac{4}{7}$; $P(S_2) = \frac{2}{7}$; $P(S_3) = \frac{1}{7}$

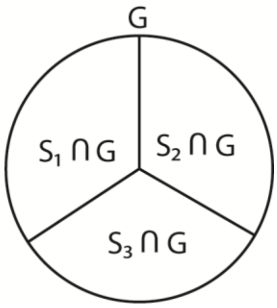
Let G = germination of seeds

$P\left({}^{G}/_{S_1}\right) = 50\% = 0.5$

$P\left({}^{G}/_{S_2}\right) = 60\% = 0.6$

$P\left({}^{G}/_{S_3}\right) = 80\% = 0.8$

(a) Find the probability that a seed selected at random will germinate.



$P(G) = P(s_1 \cap G) + P(s_2 \cap G) + P(s_3 \cap G)$

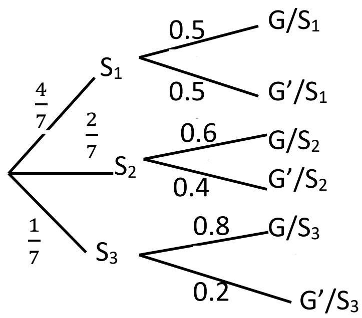
$= P(S_1).P\left({}^{G}/_{S_1}\right) + P(S_2).P\left({}^{G}/_{S_2}\right) + P(S_3).P\left({}^{G}/_{S_3}\right)$

$= \frac{4}{7}\ x\ 0.5 + \frac{2}{7}\ x\ 0.6 + \frac{1}{7}\ x\ 0.8$

$= \frac{2}{7} + \frac{1.2}{7} + \frac{0.8}{7}$

$= \frac{4}{7}$

Or Using factor tree diagram



P(G) $= \frac{4}{7}\ x\ 0.5 + \frac{2}{7}\ x\ 0.6 + \frac{1}{7}\ x\ 0.8$

$= \frac{2}{7} + \frac{1.2}{7} + \frac{0.8}{7}$

$= \frac{4}{7}$

(b) Given that 150 seeds are selected at random, find the probability that less than 90 of the seeds will germinate. Give your answer to two decimal places.

n = 150; $P = \frac{4}{7}$; $q = \frac{3}{7}$

since n = large (= 150), we use the normal approximate this binomial

$\mu = np = \frac{4}{7}\ x150 = \frac{600}{7}$

$\sigma = \sqrt{npq} = \sqrt{\frac{600}{7}\ x\ \frac{3}{7}} = \frac{30\sqrt{2}}{7}$

Let X = number of seeds that will germinate

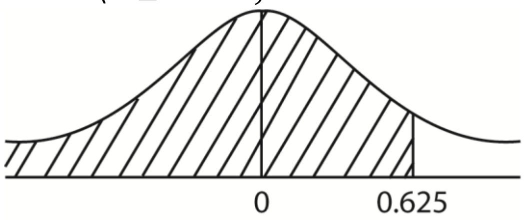
$P(x < 90) = P(x \leq 89)$

$= P\left(z \leq \frac{89.5 - \frac{600}{7}}{\frac{30\sqrt{2}}{7}}\right)$

$= P\left(z \leq \frac{7\left(89.5 - \frac{600}{7}\right)}{30\sqrt{2}}\right)$

$= P\left(z \le \frac{628.5-600}{30\sqrt{2}}\right)$

$= P(z \le 0.6250)$



$= 0.5 + (0 \le z \le 0.625)$

$= 0.5 + 0.2340$

$= 0.7340$

$= 0.73$ (2D)

19. A biased die with faces labelled 1, 2, 3, 4, 5, and 6 is tossed 45 times. calculate the probability that 2 appears

(i) More than 18 times (07marks)

$n = 45$, $p = \frac{2}{6} = \frac{1}{3}$, $q = \frac{2}{3}$

$\mu = np = 45 \times \frac{1}{3} = 15$

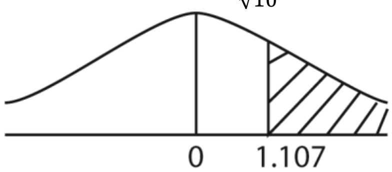
$\sigma = \sqrt{npq} = \sqrt{45 \, x \, \frac{1}{3} x \, \frac{2}{3}} = \sqrt{10}$

Changing binomial to normal distribution.

$P(X > x) = P(X > 18 + 0.5) = P(X > 18.5)$

Standardizing using $z = \frac{\overline{x - \mu}}{\sigma}$

$P(X > 18.5) = P(z > \frac{18.5-15}{\sqrt{10}} = P(z > 1.107)$



$P(z > 1.107) = 0.5 - P(0 < z < 1.107)$

$= 0.5 - 0.3658$

$= 0.1342$

$\therefore P(X > 18) = 0.1342$

(ii) Exactly 11 times (05marks)

$P(X = 11) = P(11 - 0.5 < X < 11 + 0.5)$

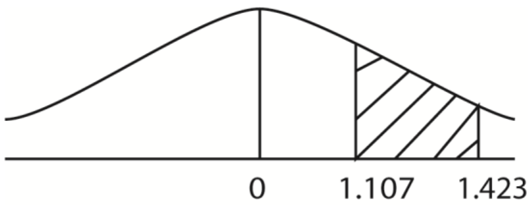
$= P(10.5 < X < 11.5)$

$= P\left(\frac{10.5-15}{\sqrt{10}} < z < \frac{11.5-15}{\sqrt{10}}\right)$

$= P(-1.423 < z < 1.107)$

By symmetry

$P(-1.423 < z < 1.107) = P(1.107 < z < 1423)$



P(1.107 < z < 1423) = P(0 < z < 1.423) – P(0 < z < 1.107)
= 0.4226 – 0.3658
= 0.0568

Distribution of sample mean of a normal distribution population

If a random variable X of a sample of size n from a normal distribution with mean μ and variance σ^2, then distribution of the sample mean $\overline{x}$ is also said to be normally distributed with mean μ and variance $\frac{\sigma^2}{n}$, such that $\overline{x} \approx \left(\mu,\ \frac{\sigma^2}{n}\right)$

Then $Z = \frac{\overline{x}-\mu}{\frac{\sigma^2}{n}}$

Example 35

At a certain school, the masses of students are normally distributed with mean 70kg and standard deviation 5kg. If 4 students are randomly selected, find the probability that their mean is less than 65.

P$(\overline{X} < 65) = P\left(Z < \frac{65-70}{\frac{5}{\sqrt{4}}}\right) = P(Z < \ -2)$

$P(Z < \ -2) = P(Z > 2) = 0.5 - P(0 < Z < 2)$ =0.5 – 0.4772 = 0.0228

Example 36

A random sample of size 15 is taken from a normal population with mean 60 and standard deviation 4. Find the probability that the mean of the sample is less than 58

P$(\overline{X} < 58) = P\left(Z < \frac{58-60}{\frac{4}{\sqrt{15}}}\right) = P(Z < \ -1.936)$

$P(Z < \ -1.936) = P(Z > 1.936) = 0.5 - P(0 < Z < 1.936)$ =0.5 – 0.4736 = 0.0264

Example 37

The height of students are normally distributed with mean 164cm and standard deviation 7.2cm. Calculate the probability that the mean height of a sample of 36 students will be between 162cm and 166cm.

P$(162 < \ \overline{X} < 166) = P\left(\frac{162-164}{\frac{7.2}{\sqrt{36}}} < Z < \ \frac{166-164}{\frac{7.2}{\sqrt{36}}}\right) = P(-1.667 < Z < \ 1.667)$

$P(-1.667 < Z < \ 1.667) = 2\ x\ P(0 < Z < 1.667$ = 2 x 0.4522 = 0.9044

Example 38

The height of a certain plant follows a normal distribution with mean 21cm and standard deviation $\sqrt{90}cm$. A random sample of 10 plants is taken and the mean height calculated. Find the probability that this sample mean lies between 18cm and 27 cm

$$P(18 < \overline{X} < 27) = P\left(\frac{18-21}{\frac{\sqrt{90}}{\sqrt{10}}} < Z < \frac{27-21}{\frac{\sqrt{90}}{\sqrt{10}}}\right) = P(-1 < Z < 2)$$

$$P(-1 < Z < 2) = P(0 < Z < 1) + P(0 < Z < 2) = 0.3413 + 0.4772 = 0.8185$$

Example 39

A large number of random sample of size n is taken from a distribution X where X ~N(74, 36) and the sample mean $\overline{x}$ for each sample is noted. If $P(\overline{x} > 72) = 0.854, find\ the\ value\ of\ n$.

$$P(\overline{X} > 72) = P\left(Z > \frac{72-74}{\frac{6}{\sqrt{n}}}\right) = 0.854$$

$$P\left(Z > \frac{-\sqrt{n}}{3}\right) = 0.854$$

From table Z = - 1.054

$$\frac{-\sqrt{n}}{3} = -1.054$$

n = 10

Example 40

The distribution of a random variable x is X ~N(25, 340) and the sample mean $\overline{x}$ for each sample is calculated. If $P(\overline{x} > 28) = 0.005, find\ the\ value\ of\ n$.

$$P(\overline{X} > 28) = P\left(Z > \frac{28-25}{\frac{\sqrt{340}}{\sqrt{n}}}\right) = 0.005$$

$$P\left(Z > \frac{3\sqrt{n}}{\sqrt{340}}\right) = 0.005$$

From table Z = 2.576

$$\frac{3\sqrt{n}}{\sqrt{340}} = 2.576$$

n = 250

Revision exercise 5

1. If $X \sim N(200, 80)$ and a random sample of size 5 is taken from the distribution, find the probability that the sample mean
   (i) is greater than 207 [0.0401]
   (ii) lies between 201 and 209 [0.3891]
2. If $X \sim N(200, 10)$ and a random sample of size 10 is taken from the distribution, find the probability that the sample mean lies outside the range 198 and 205 [0.3206]
3. If $X \sim N(50, 12)$ and a random sample of size 12 is taken from the distribution, find the probability that the sample mean

(i) Is less than48.5 [0.0668]
(ii) Is less than 52.3[0.9893]
(iii) Lie between 50.7 and 51.7 [0.1974]

4. Biscuits are produced with weight (W g) where W is N(10, 4) and are packed at random into boxes consisting of 25 biscuits. Find the probability that
   (i) a biscuit chosen at random weigh between 9.25g and 10.7g [0.2924]
   (ii) the content of a box weighs between 245g and 255g [0.0796]
   (iii) the average weight of the biscuit in the box lies between 9.7g and 10.3g [0.5468]
5. A normal distribution has a mean of 40 and standard deviation of 4. If 25 items are drawn at random, find the probability that their mean
   (i) 41.4 or more [0.0401]
   (ii) Between 38.7 and 40 .7 [0.7571]
   (iii) Less than 39.5 [0.2660]
6. A random sample of size 25 is taken from a normal population with mean 60 and standard deviation 4. Find the probability that the mean of the sample
   (i) Less than 58 [0.0062]
   (ii) Greater than 58 [0.9918]
   (iii) Between 58 and 62[0.9876]
7. At St. Noa Junior, the marks of the pupils can be modelled by a normal distribution with mean 70% and standard deviation 5%. If four pupils are chosen at random, find the probability that the mean mark is
   (i) Less than 65% [0.9772]
   (ii) Greater than 65% [0.0228]
   (iii) Greater than 75%[0.0228]
   (iv) Between 72% and 75% [0.1891]
8. The volume of soda in bottle are normally distributed with mean 758ml and standard deviation of 12ml. a random sample of 10 bottles is taken and mean volume is found. Find the probability that the sample mean is less than 750ml. [0.0176]
9. The height of cassava plants are normally distributed with mean of 2m and standard deviation of 40cm. a random sample of 50 cassava plants is taken and the mean height found. Find the probability that the sample mean lies between 195cm and 205cm. [0.6234]
10. In an examination, marks are normally distributed with mean 64.5 and variance 64. The mean ark in a random sample of 100 scripts is denoted by X. find
    (i) P(X> 65.5) [0.1056]
    (ii) P(63.8 < $\overline{X}$ < 64.5[0.3092]
11. The marks of an examination were normally distributed. 20% of the students scored below 40 marks while 10%of the students scored above 75 marks
    (i) Find the mean mark and standard deviation of the students [μ = 53.87, σ = 16.473]
    (ii) If 25 students were chosen at random from those who sat for the examination, what is the probability that their average mark exceeds 60. [0.0313]
    (iii) If a sample of 8 students were chosen, find the probability that not more than 3 scored between 45 and 65 marks. [0.5419]
12. The life time of batteries produced by a certain factory is normally distributed. Out of 10,000 batteries selected at random, 668have life tie less than 130 hours and 228 have life time more than 200 hours.
    (i) Find the mean mark and standard deviation of the battery life time[μ =160, σ = 20]
    (ii) Find the percentage of the batteries with life time between 150 and 180 hours.

(iii) If the sample of 25 batteries is selected at random, find the probability that the mean of the life time exceeds 165 hours [0.1056]

13. A normal distribution has a mean of 30 and a variance of 5. Find the probability that
    (i) The average of 10 observation exceeds 30.5 [0.2399]
    (ii) The average of 40 observation exceeds 30.5[0.0787]
    (iii) The average of 100 observation exceeds 30.5 [0.0127]
    (iv) Find n such that the probability that the average of observations exceed 30.5 is less than 1%, [n > 108]
14. The random variable is such that X$\sim N$(μ, 4). A random sample size n is taken from the population. Find the least n such that P($|\overline{X} - \mu| < 0.5) = 0.95$ [62]
15. Boxes made in a factory have weight which are normally distributed with a mean of 4.5kg and a standard deviation of 2.0kg. if a sample of 16 boxes is drawn at random, find the probability that their mean is
    (i) between 4.6 and 4.7 kg [0.0761]
    (ii) between 4.3 and 4.7g [0.3108]
16. the masses of soap powder in a certain packet are normally distributed with mean 842g and variance 225g. find the probability that a random sample of 120 packets has sample mean mass
    (i) between 844g and 846g [0.0702]
    (ii) less than 843g[0.7673]

Estimation of population parameters

Statistical estimation is used to describe the unknown characteristics of the population (population parameters) by using sample characteristics.

A sample is a representation of the population parameter such as population mean, μ and population variance, σ^2.

Types of parameter estimation

- point estimation
- interval estimation

(a) point estimates

(i) the unbiased estimate of the population mean, μ is

$\overline{x} = \frac{\sum x}{n}$ or $\overline{x} = \frac{\sum fx}{\sum f}$ where $\overline{x}$ is sample mean

(ii) the unbiased estimate of the population variance, σ^2 is $\hat{\sigma}^2$ where $\hat{\sigma}^2 = \frac{n}{n-1}s^2$ where s^2 is sample variance

OR $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2\right]$ or $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$

Example 41

Find the best unbiased estimate of mean μ and variance σ2 of the population from each of the following sample is drawn

(i) 46, 48, 50, 45, 53, 50, 48, 51

Solution

x	f	fx	fx^2
45	1	45	2025
46	1	46	2116
48	2	96	4608
50	2	100	5000
51	1	51	2601
53	1	53	2809
	$\sum f = 8$	$\sum fx = 391$	$\sum fx^2 = 19159$

Unbiased estimate for the mean $\overline{x} = \frac{\sum fx}{\sum f} = \frac{391}{8} = 48.875$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$

$= \frac{8}{8-1}\left[\frac{19159}{8} - \left(\frac{391}{8}\right)^2\right] = 6.982$

(ii) $\sum x = 100$, $\sum x^2 = 1028$, n = 10

Unbiased estimate for the mean $\overline{x} = \frac{\sum fx}{\sum f} = \frac{100}{10} = 10$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$

$= \frac{10}{10-1}\left[\frac{1028}{10} - \left(\frac{100}{10}\right)^2\right] = 3.11$

(iii) $\sum x = 120$, $\sum x^2 = 2102$, n = 8

Unbiased estimate for the mean $\overline{x} = \frac{\sum fx}{\sum f} = \frac{120}{8} = 15$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$

$= \frac{8}{8-1}\left[\frac{2102}{8} - \left(\frac{120}{8}\right)^2\right] = 43.14$

(iv) $\sum x = 330$, $\sum x^2 = 23700$, n = 34

Unbiased estimate for the mean $\overline{x} = \frac{\sum fx}{\sum f} = \frac{330}{34} = 9.71$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$

$= \frac{34}{34-1}\left[\frac{23700}{34} - \left(\frac{330}{34}\right)^2\right] = 621.12$

(v) $\sum x = 738$, $\sum x^2 = 16526$, n = 50

Unbiased estimate for the mean $\overline{x} = \frac{\sum fx}{\sum f} = \frac{738}{50} = 14.76$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$

$$= \frac{50}{50-1}\left[\frac{16526}{50} - \left(\frac{738}{50}\right)^2\right] = 114.96$$

Example 42

The fuel consumption of a new car model was being tested. In one trials 8 cars chosen at random were driven under identical conditions and distance x km covered on one litre of petro was recorded. The following results were obtained. $\sum x = 152.98, \sum x^2 = 2927.1$. Calculate the unbiased estimate of the mean and variance of the distance covered by the car.

Solution

Unbiased estimate for the mean $\overline{x} = \frac{\sum fx}{\sum f} = \frac{152.98}{8} = 19.1225$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$

$$= \frac{8}{8-1}\left[\frac{2927.1}{8} - \left(\frac{152.98}{8}\right)^2\right] = 0.25$$

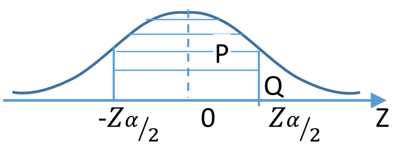
(b) Interval estimate

Here we are interested in obtaining the interval over which the true population mean lies (confidence interval)

The unbiased estimate of the population mean, μ is $\overline{x}$

Z = $\frac{\overline{x}-\mu}{\frac{\sigma}{\sqrt{n}}}$ where n is the sample size

Z is the area under the normal curve leaving an area of $\frac{\alpha}{2}$ on either side of the curve



$P\left(-Z\alpha/_2 < Z < Z\alpha/_2\right) = \frac{\alpha}{2}$

$P\left(-Z\alpha/_2 < \frac{\overline{x}-\mu}{\frac{\sigma}{\sqrt{n}}} < Z\alpha/_2\right) = \frac{\alpha}{2}$

$P\left(\overline{x} - Z\alpha/_2\frac{\sigma}{\sqrt{n}} < \mu < \overline{x} + Z\alpha/_2\frac{\sigma}{\sqrt{n}}\right) = \frac{\alpha}{2}$

Confidence interval $\left[\overline{x} - Z\alpha/_2\frac{\sigma}{\sqrt{n}} < \mu < \overline{x} + Z\alpha/_2\frac{\sigma}{\sqrt{n}}\right]$

Confidence Limits $\left[\overline{x} - Z\alpha/_2\frac{\sigma}{\sqrt{n}} < \mu < \overline{x} + Z\alpha/_2\frac{\sigma}{\sqrt{n}}\right]$

Or $\mu < \overline{x} \pm Z\alpha/_2\frac{\sigma}{\sqrt{n}}$

(i) Confidence interval for population mean μ

- of a normal or non-normal population
- with known population variance σ2 or standard deviation σ
- using any sample size

The confidence interval is obtained from $\mu < \overline{x} \pm Z\alpha/_2\frac{\sigma}{\sqrt{n}}$ where $\overline{x}$ is sample mean

Example 43

The length of a bar of a metal is normally distributed with mean of 115cm and standard deviation of 3cm. find the 95% confidence limits for the length of the bar

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$

$Z\alpha/_2 = 1.96$

$\mu < \overline{x} \pm Z\alpha/_2 \frac{\sigma}{\sqrt{n}}$

$\mu < 115 \pm 1.96 \frac{3}{\sqrt{1}}$

Lower limit =109.12

Upper limit = 120.88

Example 44

The mass of vitamin in a capsule is normally distributed with standard deviation 0.042mg. a random sample of 5 capsules was taken and the mean mass of vitamin e found to be 5.12. calculate a symmetric confidence interval for the population mean mass.

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$

$Z\alpha/_2 = 1.96$

$\mu < \overline{x} \pm Z\alpha/_2 \frac{\sigma}{\sqrt{n}}$

$\mu < 5.12 \pm 1.96 \frac{0.042}{\sqrt{5}}$

Lower limit =5.08

Upper limit = 5.16

Example 44

It is known that an examination paper is marked in such a way that the standard deviation of the marks is 15.1. in a certain school, 80 candidates took the examination and they had an average mark of 57.4. find

(i) 95% confidence limits for the mean mark in the exam.

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$

$Z\alpha/_2 = 1.96$

$\mu < \overline{x} \pm Z\alpha/_2 \frac{\sigma}{\sqrt{n}}$

$\mu < 5.12 \pm 1.96 \frac{15.1}{\sqrt{80}}$

Lower limit =54.091

Upper limit = 60.709

(ii) 99% confidence limits for the mean mark in the exam.

$\frac{\alpha}{2} = \frac{0.99}{2} = 0.495$

$Z\alpha/_2 = 2.575$

$\mu < \overline{x} \pm Z\alpha/_2 \frac{\sigma}{\sqrt{n}}$

$\mu < 5.12 \pm 2.595 \frac{15.1}{\sqrt{80}}$

Lower limit =53.053

Upper limit = 61.746

Example 45

After a particular rainy night, 12 worms were picked and their length in cm measured;
9.5, 9.5, 11.2, 10.6, 9.9, 11.1, 10.9, 9.8, 10.1, 10.2, 10.9, 11.0

Assuming that this sample came from a normal population with standard deviation 2, find the 95 confidence interval for the mean length of all the worms

$$\overline{x} = \frac{\sum x}{n} = \frac{9.5 + 9.5 + 11.2 + 10.6 + 9.9 + 11.1 + 10.9 + 9.8 + 10.1 + 10.2 + 10.9 + 11.0}{12}$$

= 10.39

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$

$Z\alpha/_2 = 1.96$

$\mu < \overline{x} \pm Z\alpha/_2 \frac{\sigma}{\sqrt{n}}$

$\mu < 10.39 \pm 1.96 \frac{2}{\sqrt{12}}$

Lower limit =9.81

Upper limit = 10.97

The height of students are normally distributed with mean μ and standard deviation σ. On the bsis of results obtained from a random sample of 100 students from school, the 95% confidence interval of the mean was calculated and found to be (177.22cm, 179.18cm). Calculate

(i) the value of the sample mean

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$

$Z\alpha/_2 = 1.96$

$\mu < \overline{x} \pm Z\alpha/_2 \frac{\sigma}{\sqrt{n}}$

$177.22 < \overline{x} - 1.96\frac{\sigma}{\sqrt{100}}$ …. (i)

$179.18 < \overline{x} + 1.96\frac{\sigma}{\sqrt{100}}$ …. (ii)

Eqn. (i) + eqn. (ii)

$2\overline{x}$ = 356.4; $\overline{x}$ = 178.2

(ii) the value of standard deviation

$177.22 < 178.2 - 1.96\frac{\sigma}{\sqrt{100}}$;

σ = 5

(iii) 90% confidence interval of the mean, μ

$\frac{\alpha}{2} = \frac{0.90}{2} = 0.45; Z\alpha/_2 = 1.645$

Example 46

A plant produces steel sheets whose weight are normally distributed with standard deviation of 2.4kg. A random sample of 36 sheets had a mean weight of 31.4kg.

(i) Find the 99% confidence limit for the population

$\frac{\alpha}{2} = \frac{0.99}{2} = 0.495$

$Z\alpha/_2 = 2.575$

$\mu < \overline{x} \pm Z\alpha/_2 \frac{\sigma}{\sqrt{n}}$

$\mu < 31.4 \pm 2.595\frac{2.4}{\sqrt{36}}$

Lower limit =30.37kg

Upper limit = 32.43kg

(ii) Find the width of the 99% confidence limit

=32.43kg - 30.37kg = 2.06kg

Example 47

The marks scored by students are normally distributed with mean μ and standard deviation1.3. it is required to have 95% confidence interval for μ with width less than 2. Find the least number of students that be sampled to achieve this.

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$

$Z\alpha/_2 = 1.96$

width = 2 $xZ\alpha/_2 \frac{\sigma}{\sqrt{n}}$ < 2

$2\ x\ 1.96\frac{\sigma}{\sqrt{n}}$ < 2

$2\ x\ 1.96\frac{\sigma}{2}$ < $\sqrt{n}$; n < 6.49

n > 6.49

the least number = 7

(ii) Confidence interval for population mean μ

- of a normal or non-normal population
- with unknown population variance σ2 or standard deviation σ
- using a large sample size (n ≥ 30)

If the population variance σ2 s not given or unknown, the confidence interval is obtained from $\mu < \overline{x} \pm Z\alpha/_2 \frac{\hat{\sigma}}{\sqrt{n}}$ where $\overline{x}$ is sample mean, $\hat{\sigma} = \frac{n}{n-1} s^2$ and s = sample variance

$$\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2\right]$$

Or

$$\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$$

Example 48

the fuel consumption of a new car was being tested. In one trials 50 cars chosen at random were driven under identical conditions and the distance x km covered on one litre of petrol was recorded. the following results were obtained. $\sum x = 525$, $\sum x^2 = 5625$. Calculate the 95%confidence interval for the mean petrol consumption, in km per litre of cars of this type..

Unbiased estimate for the mean $\overline{x} = \frac{\sum fx}{\sum f} = \frac{525}{50} = 10.5$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$

$= \frac{50}{50-1}\left[\frac{5625}{50} - \left(\frac{525}{50}\right)^2\right] = 2.2952$

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$

$Z\alpha/_2 = 1.96$

$\mu < \overline{x} \pm Z\alpha/_2 \frac{\sigma}{\sqrt{n}}$

$\mu < 10.5 \pm 1.96\frac{\sqrt{2.2952}}{\sqrt{50}}$

Lower limit =10.08km/litre

Upper limit = 10.92km/litre

Example 49

The height x cm of each man in a random sample of 200 men living in Nairobi was measured. The following results were obtained$\sum x = 35050$, $\sum x^2 = 6163109$.

(a) calculate the unbiased estimate of the mean and variance of the heights of men living Nairobi

Unbiased estimate for the mean $\overline{x} = \frac{\sum fx}{\sum f} = \frac{35050}{200} = 175.25$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$

$= \frac{200}{200-1}\left[\frac{6163109}{200} - \left(\frac{35050}{200}\right)^2\right] = 103.5$

(b) Determine the 90% confidence interval for the mean height of mean living in Nairobi.

$\frac{\alpha}{2} = \frac{0.90}{2} = 0.45$

$Z\alpha/_2 = 1.645$

$\mu < \overline{x} \pm Z\alpha/_2 \frac{\sigma}{\sqrt{n}}$

$\mu < 175.25 \pm 1.645\frac{\sqrt{103.5}}{\sqrt{200}}$

Lower limit =174.07cm

Upper limit = 176.43cm

Example 50

A random sample of 100 observations from a normal population with mean μ gave the following results $\sum x = 8200$, $\sum x^2 = 686000$.

(a) calculate the unbiased estimate of the mean and variance of the heights of men living Nairobi

Unbiased estimate for the mean $\overline{x} = \frac{\sum fx}{\sum f} = \frac{8200}{100} = 82$

The unbiased estimate of the population variance, $\hat{\sigma}^2 = \frac{n}{n-1}\left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$

$$= \frac{200}{200-1}\left[\frac{68600}{100} - \left(\frac{8200}{100}\right)^2\right] = 11.72$$

(b) Determine the 98% confidence interval for the mean

$\frac{\alpha}{2} = \frac{0.98}{2} = 0.49$

$Z\alpha/_2 = 2.326$

$\mu < \overline{x} \pm Z\alpha/_2 \frac{\sigma}{\sqrt{n}}$

$\mu < 82 \pm 2.326\frac{11.72}{\sqrt{100}}$

Lower limit =79.274

Upper limit = 84.726

(c) determine the 99% confidence interval for the mean

$\frac{\alpha}{2} = \frac{0.99}{2} = 0.495$

$Z\alpha/_2 = 2.575$

$\mu < \overline{x} \pm Z\alpha/_2 \frac{\sigma}{\sqrt{n}}$

$\mu < 82 \pm 2.595\frac{11.72}{\sqrt{100}}$

Lower limit =78.981

Upper limit = 85.726

Example 50

The mean and standard deviation of a random sample of size 100 is 900 and 60 respectively. Given that the population is normally distributed, find 96% confidence interval of the population mean.

$$\hat{\sigma} = \sqrt{\frac{n}{n-1}s^2} = \sqrt{\frac{100}{100-1}x\ 60^2} = 60.302$$

$\frac{\alpha}{2} = \frac{0.96}{2} = 0.48$

$Z\alpha/_2 = 2.054$

$\mu < \overline{x} \pm Z\alpha/_2 \frac{\sigma}{\sqrt{n}}$

$\mu < 900 \pm 2.054\frac{60.302}{\sqrt{100}}$

Lower limit =887.614

Upper limit = 912.386

Revision exercise 6

1. the concentration in mg per litre of a trace element in 7 randomly chosen samples of water from a spring were 240.8, 237.3, 236.7, 236.6, 234.2, 233.9, 232.5.
   Determine the unbiased mean and variance of the concentration of the trace element per litre from spring [236, 7.58]
2. Cartons of oranges are filled by a machine. A sample of 10 cartons selected at random from the population contained the following quantities in ml) 201.2, 205.0, 209.1, 202.3,204.6, 206.4,

210.1,201.9, 203.7, 207.3. Determine the unbiased mean and variance of the population from which the sample was taken. [203.16, 9.223]

3. A factory produces cans of meat whose masses are normally distributed with standard deviation 18g. A random sample of 25 cans is found to have the mean of 458g. find the 99% confidence interval for the population mean of a can of meat produced at the factory. [448.7, 467.3g]
4. The height of bounce of a tennis ball is normally distributed with standard deviation 2cm. A sample of 60 tennis balls is tested and the mean height of bounce is 140cm. Find
   (i) 95% [139.5, 140.51] (ii) 98% [139.4, 140.6] confidence interval for the mean height of bounce of the tennis ball
5. A random sample of 100 is taken from a population. The sample is found to have a mean of 76.0 and standard deviation of 120. Find
   (i) 90% [747.51, 748.49] (ii) 95% [747.42, 748.58] (iii) 98% [747.31, 748.69] confidence interval for the mean of the population
6. 150 bags of flour of a particular brand are weighed and the mean mass is found to be 748g with standard deviation 3.6g. Find
   (i) 90% [74.02, 77.98] (ii) 97% [73.38, 78.62] (iii) 99% [72.89, 79.11] confidence interval for the mean mass of bags of flour of this brand.
7. A random sample of 100 readings taken from a normal population gave the following data: $\overline{x}$ = 82, $\sum x^2 = 686800$. Find
   (i) 98% [79.19, 84.81] (ii) 99% [78.89, 85.11] confidence interval of the population mean
8. 80 people were asked to measure their pulse rates when they woke up in the morning. The mean was 69 beats and standard deviation 4 beats. find
   (i) 95% [68.12, 69.88] (ii) 99% [67.84, 70.16] (iii) 97% [68.0, 70.0] confidence interval of the population mean
9. The 95% confidence interval for the mean length of a particular brand of light bulbs is [1023.3h, 1101.7h]. This is interval is based on results from a sample of 36 light bulbs. Find the 99% confidence interval for the mean length of life of this brand of light bulbs assuming that the length of life is normally distributed. [1011, 1114]
10. A random sample of 6 items taken from a normal population with variance 4.5cm^2 gave the following data: 12.9cm, 13.2cm, 14.6cm, 12.6cm, 11.3cm, and 10.1 cm.
    (i) Find the 95% confidence interval for the population mean. [10.75, 14.15].
    (ii) What is the width of this confidence interval [3.4]
11. A random sample of 60 loaves is taken from a population whose mean masses are normally distributed with mean μ and standard deviation 10g.
    (i) calculate the width of 95% confidence interval for μ bases on the sample [5.06]
    (ii) Find the confidence level having the same width as in (i) but based on a random sample of 40 loaves. [89%]
12. The distribution of measurements of masses of a random sample of bags packed in a factory is shown below

Mass (kg)	72.5	77.5	82.5	87.5	92.5	97.5	102.5	107.5
frequency	6	18	32	57	102	51	25	9

    (i) Find the mean and standard deviation of the masses [μ =91.317, σ=7.41]
    (ii) find the 95% confidence limits[90.5, 92.2]

Thank you
Dr. Bbosa Science