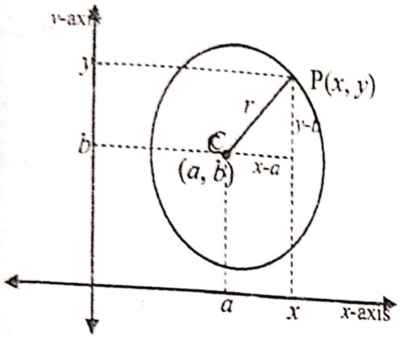
CIRCLES

A circle is a 2-dimensional shape in Euclidean geometry made by drawing a curve that is always the same distance from the center

A circle can also be defined as a locus of all points P(x, y) which are equidistant from the same given point fixed point C(a, b) [center]

Suppose that the distance of the points P from the given point C (a, b) is r



$$(x-a)^2+(y-b)^2=r^2$$

$(x-a)^2+(y-b)^2=r^2$ is the equation of the circle wit center (a, b) and radius r

If the center C is (0, 0) then the equation of the circle is $x^2+y^2=r^2$

For $(x-a)^2+(y-b)^2=r^2$

$$x^2-2ax+a^2+y^2-2by+b^2=r^2$$

$$x^2+y^2-2ax-2by+a^2+b^2-r^2=0$$

Suppose $-a=g, -b=f, C=a^2+b^2-r^2$

$$\Rightarrow C=g^2+f^2-r^2$$

The equation of the circle becomes

$$x^2+y^2+2gx+2fy+C=0$$

$x^2+y^2+2gx+2fy+C=0$ is the standard equation of a circle with center (-g, -f) and radius $r=\sqrt{g^2+f^2-C}$

Example I

Find the center and the radius of the circles below

(a) $(x-1)^2+(y-2)^2=9$

(b) $(x+1)^2+(y-3)^2=25$

(c) $x^2+y^2-4x-2y=4$

(d) $2x^2+2y^2-2x+2y=1$

Solution

(a) $(x-1)^2+(y-2)^2=9$

Comparing $(x-1)^2+(y-2)^2=9$ with $(x-a)^2+(y-b)^2=r^2$

$$a=1, b=2, r^2=9$$

$\Rightarrow$ The center is C (1, 2) and $r=3$

$(x-1)^2+(y-2)^2=9$ is a circle with radius 3 units and center (1, 2)

(b) $(x+1)^2+(y-3)^2=25$

Compare $(x+1)^2+(y-3)^2=25$ with

$$(x-a)^2+(y-b)^2=r^2$$

$$a=-1, b=3, r^2=25$$

$$r=5$$

The center is (-1, 3)

$\therefore (x+1)^2+(y-3)^2=25$ is the equation of the circle with center (-1, 3) and radius 5.

(c) $\boldsymbol{x^2+y^2-4x-2y=4}$

$x^2+y^2-4x-2y-4=0$

Comparing $x^2+y^2-4x-2y-4$ with

$x^2+y^2+2gx+2fy+C=0$

$$2g=-4, 2f=-2$$

$$C=-4$$

$$g=-2, f=-1$$

$$C=-4$$

Since the center is (-g, -f),

The center is (2, 1)

Radius $=\sqrt{g^2+f^2-C}$

$$r=\sqrt{(-2)^2+(-1)^2-(-4)}$$

$$r=\sqrt{4+1+4}$$

$$r=3$$

$x^2+y^2-4x-2y=4$ is a circle with radius 3 units and center (2, 1)

(d) $\boldsymbol{2x^2+2y^2-2x+2y=1}$

$$\frac{2x^2}{2}+\frac{2y^2}{2}-\frac{2x}{2}+\frac{2y}{2}=\frac{1}{2}$$

$$x^2+y^2-x+y-\frac{1}{2}=0$$

Comparing $x^2+y^2-x+y-\frac{1}{2}$ with

$x^2+y^2+2gx+2fy+C=0$

$2gx=-x, 2fy=y, c=\frac{-1}{2}$

$g=\frac{-1}{2}, f=\frac{1}{2}$

Center (-g, -f)

Centre $(\frac{1}{2}, -\frac{1}{2})$

Radius $=\sqrt{g^2+f^2-C}$

Radius $=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{-1}{2}\right)^2-\frac{-1}{2}}$

Radius $=\sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{2}}$

$= \sqrt{1}$

$= 1$

$2x^2 + 2y^2 - 2x + 2y = 1$ is the equation of the circle with center $\left(\frac{1}{2}, -\frac{1}{2}\right)$ and radius 1.

Example III

Find the equation of the circle with the following centers and radii

(a) Center (2, 3) radius 1

(b) Center (3, -4) radius 5

(c) Center $\left(\frac{-3}{2}, 2\right)$ and radius $\frac{1}{2}$

(d) Center $\left(\frac{-1}{4}, \frac{1}{2}\right)$ and radius $\frac{1}{2}\sqrt{2}$

(e) Center (0, -5) and radius 5

Solution

(a) Center (2, 3) radius 1

Given a circle of centre (a, b) and radius r. The equation of the circle is $(x-a)^2 + (y-b)^2 = r^2$.

Consider the equation of the circle

$(x-a)^2 + (y-b)^2 = r^2$ with center (a, b) and radius r

$$(x-2)^2 + (y-3)^2 = 1^2$$
$$(x-2)^2 + (y-3)^2 = 1$$
$$x^2 - 4x + 4 + y^2 - 6y + 9 = 1$$
$$x^2 + y^2 - 4x - 6y + 13 - 1 = 0$$
$$x^2 + y^2 - 4x - 6y + 12 = 0$$

The equation of the circle with center (2, 3) and radius 1 is $x^2 + y^2 - 4x - 6y + 12 = 0$

(b) Center (3, -4) radius 5

$$(x-a)^2 + (y-b)^2 = r^2$$
$$(x-3)^2 + (y - -4)^2 = 5^2$$
$$(x-3)^2 + (y+4)^2 = 5^2$$
$$x^2 - 6x + 9 + y^2 + 8y + 16 = 25$$
$$x^2 + y^2 - 6x + 8y = 0$$

The equation of the circle with center (3, -4) and radius 5 is $x^2 + y^2 - 6x + 8y = 0$

(c) Center $\left(\frac{-3}{2}, 2\right)$ and radius $\frac{1}{2}$

$$\left(x - \frac{-3}{2}\right)^2 + (y-2)^2 = \left(\frac{1}{2}\right)^2$$
$$\left(x + \frac{3}{2}\right)^2 + (y-2)^2 = \frac{1}{4}$$
$$x^2 + 3x + \frac{9}{4} + y^2 - 4y + 4 = \frac{1}{4}$$
$$x^2 + y^2 + 3x - 4y + 6 = 0$$

Equation of the circle with center $\left(\frac{-3}{2}, \ldots\right.$ radius $r = \frac{1}{2}$ is $x^2 + y^2 + 3x - 4y + 6 = 0$

(d) Center $\left(\frac{-1}{4}, \frac{1}{2}\right)$ and radius $\frac{1}{2}\sqrt{2}$

$$\left(x - \frac{-1}{4}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\sqrt{\ldots}\right.$$
$$\left(x + \frac{1}{4}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}(2)$$
$$\left(x + \frac{1}{4}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$$
$$x^2 + \frac{1}{2}x + \frac{1}{16} + y^2 - y + \frac{1}{4} = \ldots$$
$$x^2 + y^2 + \frac{1}{2}x - y + \frac{1}{16} + \frac{1}{4} - \frac{1}{2}\ldots$$
$$x^2 + y^2 + \frac{1}{2}x - y - \frac{3}{16} = 0$$
$$16x^2 + 16y^2 + 8x - 16y - 3 = 0$$

(e) Center (0, -5) and radius 5

$$(x-0)^2 + (y - -5)^2 = 5^2$$
$$x^2 + (y+5)^2 = 5^2$$
$$x^2 + y^2 + 10y + 25 = 25$$
$$x^2 + y^2 + 10y = 0$$

Example III

State which of the following are equations of the ...

(a) $x^2 + y^2 - 5 = 0$

(b) $x^2 + y^2 + 10 = 0$

(c) $x^2 + y^2 + c = 0$

(d) $x^2 + y^2 + bxy = 1$

(e) $9x^2 + 9y^2 = 1$

(f) $7x^2 + 3x - y^2 + 2y = 16$

(g) $x^2 + 3x - y^2 = 7$

(h) $x^2 + y^2 + 2x - 8y = 1$

(i) $x^2 + 2xy + y^2 = 4$

Solution

(a) $x^2 + y^2 - 5 = 0$

$x^2 + y^2 = 5$

$(x-0)^2 + (y-0)^2 = \left(\sqrt{5}\right)^2$

$x^2 + y^2 - 5 = 0$ is an equation of a circl...

(b) $x^2 + y^2 + 10 = 0$

$x^2 + y^2 = -10$

$(x-0)^2 + (y-0)^2 = \left(\sqrt{-10}\right)^2$

$x^2 + y^2 + 10 = 0$ is not an equation of a c...

since $r = \sqrt{-10}$ is not real.

(c) $x^2+y^2+c=0$

$x^2+y^2=-c$

$(x-0)^2+(y-0)^2=(\sqrt{-c})^2$

$x^2+y^2+c=0$ is an equation of the circle when $c<0$.

(d) $x^2+y^2+bxy=1$

$$x^2+y^2+bxy=1$$

Comparing $x^2+y^2+bxy=1$ with $x^2+y^2+2gx+2fy+C=0$

$\Rightarrow x^2+y^2+bxy=1$ is not an equation of a circle because of the component of bxy

(e) $9x^2+9y^2=1$

$\frac{9x^2}{9}+\frac{9y^2}{9}=\frac{1}{9}$

$x^2+y^2=\frac{1}{9}$

$(x-0)^2+(y-0)^2=\left(\frac{1}{3}\right)^2$

$9x^2+9y^2=1$ is a circle

(f) $7x^2+3x-y^2+2y=16$

Is not a circle because the co-efficient of $x^2 and\ y^2$ are not the same

(g) $x^2+3x-y^2=7$

Is not a circle because the co-efficient of x^2 and y^2 are not the same.

(h) $x^2+y^2+2x-8y=1$

Is a circle

(i) $x^2+2xy+y^2=4$

Is not a circle

Example IV (UNEB Question)

The equation of the circle with center O is given by $x^2+y^2+Ax+By+C=0$ where A, B and C are constants. Given that $4A=3B$, $3A=2C$ and $C=9$

Determine

(a) The coordinates of the center of the circle

(b) The radius of the circle

Solution

4A = 3B .... (1)

3A = 2C .... (2)

C = 9 .... (3)

Substituting eqn. (3) in eqn. (2)

$$3A=2(9)$$

$$3A=18$$

$$A=6$$

Substituting $A=6$ in Eqn (1)

$4\times 6=3B$

$B=8$

$x^2+y^2+Ax+By+C=0$

$x^2+y^2+6x+8y+9=0$

Comparing $x^2+y^2+6x+8y+9=0$ with $x^2+y^2+2gx+2fy+C=0$

$2g=6 \Rightarrow g=3$

$2f=8, \Rightarrow f=4$

$C=9$

Centre (-3, -4)

Radius $=\sqrt{g^2+f^2-C}$

Radius $=\sqrt{(-3)^2+(-4)^2-9}$

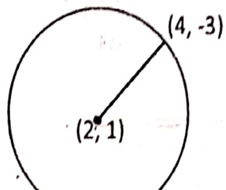
$=\sqrt{9+16-9}$

$=4$

Example V

Find the equation of a circle whose center is (2, 1) and passes through (4, -3)

Solution



$r=\sqrt{(2-4)^2+(1--3)^2}$

$r=\sqrt{4+16}$

$r=\sqrt{20}$

$(x-a)^2+(y-b)^2=r^2$

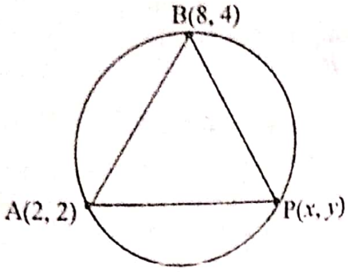
$(x-2)^2+(y-1)^2=(\sqrt{20})^2$

$x^2-4x+4+y^2-2y+1=20$

$x^2+y^2-4x-2y-15=0$

Example VI

Given the points A(2, 2) and B(8, 4) are end points of the diameter of circle. Find the centre, radius and the equation of the circle.



The diameter subtends 90° at the circumference of the circle.

(Gradient of P) × (Gradient of P) = -1

$$\left(\frac{y-4}{x-8}\right)\times\left(\frac{y-2}{x-2}\right)=-1$$

$(y-4)(y-2) = -1\,(x-8)(x-2)$

$(y-4)(y-2) = -1\,(8-x)(x-2)$

$y^2-2y-4y+8 = 8x-16-x^2+2x.$

$x^2+y^2-10x-6y+24=0.$

Comparing $x^2+y^2+2gx+2fy+c=0$ with $x^2+y^2-10x-6y+24=0.$

$\Rightarrow g = -5, f = -3.$

Centre $(-g, -f)$

$\Rightarrow c(5, 3)$

Radius $= \sqrt{g^2+f^2-c}$

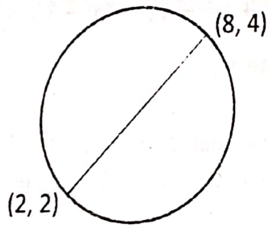
$$R=\sqrt{(-5)^2+(-3)^2-24}$$

$$R=\sqrt{25+9-24}$$

$$R=\sqrt{10}$$

$$R=3.1623$$

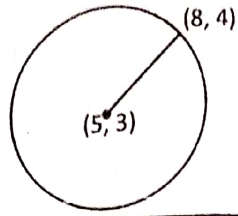
Alternatively



$$C\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

$$C\left(\frac{2+8}{2},\frac{2+4}{2}\right)$$

$$C(5,3)$$



$$r=\sqrt{(5-8)^2+(3-4)^2}$$

$$r=\sqrt{9+1}$$

$$r=\sqrt{10}$$

$$(x-a)^2+(y-b)^2=r^2$$

$$(x-5)^2+(y-3)^2=10$$

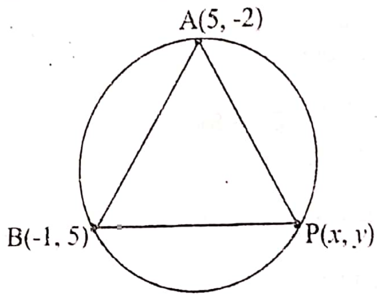
$$x^2-10x+25+y^2-6y+9=$$

$$x^2+y^2-10x-6y+34-10$$

$$x^2+y^2-10x-6y+24=($$

Example VII

Given the points A(5, -2) and B(-1, 5) are e points of the diameter of circle. Find the ce radius and the equation of the circle.



The diameter subtends 90° at the circumfer

$$\left(\frac{y-^-2}{x-5}\right)\times\left(\frac{y-5}{x-^-1}\right)=-1$$

$$\left(\frac{y+2}{x-5}\right)\times\left(\frac{y-5}{x+1}\right)=-1$$

$(y+2)(y-5) = -1\,(x-5)(x+1)$

$(y+2)(y-5) = (5-x)(x+1)$

$y^2-5y+2y-10 = 5x+5-x^2-x$

$x^2+y^2-4x-3y-15=0$

Comparing $x^2+y^2-4x-3y-15$ with x^2+

$2gx+2gf+c=0.$

$2g = -4$

$g = -2$

$2f = -3$

$f = \frac{-3}{2}$

Centre $(-g, -f)$

$\Rightarrow$ centre $\left(2, \frac{-3}{2}\right)$.

Radius $= \sqrt{g^2 + f^2 - c}$

$= \sqrt{4 + \frac{9}{4} - -15}$

$= \sqrt{\frac{16 + 9 + 60}{4}}$

$= \sqrt{\frac{85}{4}}$

$= \frac{\sqrt{85}}{2}$

Example VIII

Find the equation of a circle passing through points (2, 3) and (4, 5) having its center on the line $y = 4x + 3$

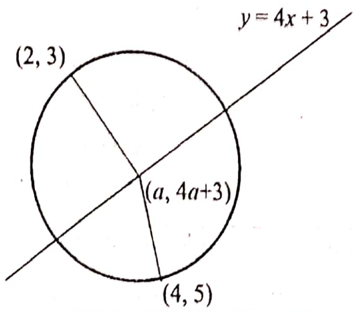
Solution

$$y = 4x + 3$$

Let the center be (x, y). Since it lies on the line $y = 4x + 3 = 0$, let the x-co-ordinate of the center be a.

Then the y-co-ordinate

$$y = 4a + 3$$



$r_1 = \sqrt{(a-2)^2 + (4a+3-3)^2}$

$r_1 = \sqrt{(a-2)^2 + (4a)^2}$

$r_1 = \sqrt{a^2 - 4a + 4 + 16a^2}$

$r_1 = \sqrt{17a^2 - 4a + 4}$

$r_2 = \sqrt{(a-4)^2 + (4a+3-5)^2}$

$r_2 = \sqrt{(a-4)^2 + (4a-2)^2}$

$r_2 = \sqrt{a^2 - 8a + 16 + 16a^2 - 16a + 4}$

$r_2 = \sqrt{17a^2 - 24a + 20}$

$r_1 = r_2 = r$

$\sqrt{17a^2 - 4a + 4} = \sqrt{17a^2 - 24a + 20}$

$17a^2 - 4a + 4 = 17a^2 - 24a + 20$

$20a = 16$

$a = \frac{16}{20} = \frac{4}{5}$

$r_1 = r = \sqrt{17 \times (\frac{4}{5})^2 - 4(\frac{4}{5}) + 4}$

$r = \sqrt{\frac{17 \times 16}{25} - \frac{16}{5} + 4}$

$r = \sqrt{\frac{292}{25}}$

Centre$(a, 4a+3)$

centre $\left(\frac{4}{5}, \frac{4 \times 4}{5} + 3\right)$

centre $\left(\frac{4}{5}, \frac{31}{5}\right)$

$(x - \frac{4}{5})^2 + (y - \frac{31}{5})^2 = \left(\sqrt{\frac{292}{25}}\right)^2$

$(x - \frac{4}{5})^2 + (y - \frac{31}{5})^2 = \frac{292}{25}$

$x^2 - \frac{8x}{5} + \frac{16}{25} + y^2 - \frac{62y}{5} + \frac{961}{25} = \frac{292}{25}$

$x^2 + y^2 - \frac{8x}{5} - \frac{62y}{5} + \frac{977}{25} - \frac{292}{25} = 0$

$x^2 + y^2 - \frac{8x}{5} - \frac{62y}{5} + \frac{685}{25} = 0$

$\Rightarrow 5x^2 + 5y^2 - 8x - 62y + 137 = 0$

Example

What is the equation of the circle whose center lies on the $x - 2y + 2 = 0$ which touches the positive axes.

Solution

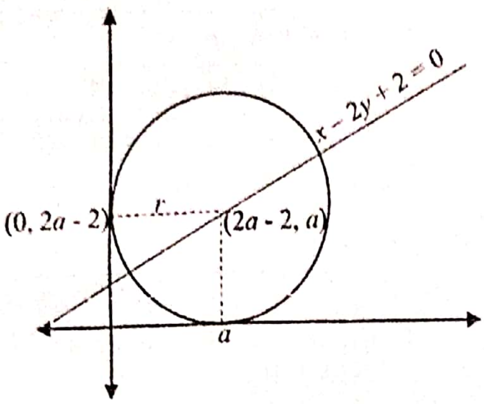
Let the y-coordinate of the centre be a

$x - 2y + 2 = 0$

$x - 2a + 2 = 0$

$x = 2a - 2$

$(2a - 2, a)$



$$2a - 2 = a$$
$$a = 2$$

The center is $(2, 2)$; radius $r = 2$

$$(x-2)^2 + (y-2)^2 = 2^2$$
$$x^2 - 4x + 4 + y^2 - 4y + 4 = 4$$
$$x^2 + y^2 - 4x - 4y + 4 = 0$$

Equation of circle passing through th points

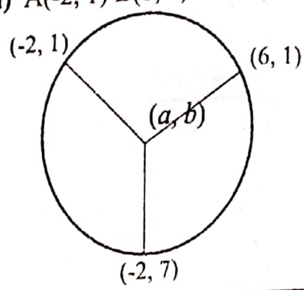
Example I

Find the equation of the circle passing through points

(a) A(-2, 1) B(6, 1) and C(-2, 7)
(b) A(-1, 4) B(2, 5) and C(0, 1)
(c) A(3, 1) B(8, 2) and C(2, 6)
(d) A(5, 7) B(1, 6) and C(2, 2)

Solution

(a) A(-2, 1) B(6, 1) and C(-2, 7)



$$r_1 = \sqrt{(a - {}^-2)^2 + (b-1)^2}$$
$$r_2 = \sqrt{(a - 6)^2 + (b-1)^2}$$
$$r_3 = \sqrt{(a - {}^-2)^2 + (b-7)^2}$$

Equating the radii; $r_1 = r_2 = r$

$$\sqrt{(a - {}^-2)^2 + (b-1)^2} = \sqrt{(a-6)^2 + (b-1)^2}$$
$$(a+2)^2 + (b-1)^2 = (a-6)^2 + (b-1)^2$$
$$a^2 + 4a + 4 + b^2 - 2b + 1$$
$$= a^2 - 12a + 36 + b^2 - 2b + 1$$
$$a^2 + b^2 + 4a - 2b + 5 = a^2 + b^2 - 12a - 2b + 37$$
$$4a - 2b + 5 = -12a - 2b + 37$$
$$16a = 32$$
$$a = 2$$

Also $r_1 = r_3 = r$

$$\sqrt{(a - -2)^2 + (b-1)^2} = \sqrt{(a - -2)^2 + (b-7)^2}$$
$$(a+2)^2 + (b-1)^2 = (a+2)^2 + (b-7)^2$$
$$a^2 + 4a + 4 + b^2 - 2b + 1$$
$$= a^2 + 4a + 4 + b^2 - 14b + 49$$
$$b^2 - 2b + 1 = b^2 - 14b + 49$$
$$12b = 48$$
$$b = 4$$

Center $(a, b) = (2, 4)$

$$radius = \sqrt{(a - -2)^2 + (b-1)^2}$$
$$= \sqrt{(2 - -2)^2 + (4-1)^2}$$
$$= \sqrt{16+9}$$
$$= 5$$

$$(x-a)^2 + (y-b) = r^2$$
$$(x-2)^2 + (y-4)^2 = 5^2$$
$$x^2 - 4x + 4 + y^2 - 8y + 16 = 25$$
$$x^2 + y^2 - 4x - 8y + 20 = 25$$
$$x^2 + y^2 - 4x - 8y - 5 = 0$$

Alternatively; Consider the general equation of the circle $x^2 + y^2 + 2gx + 2fy + C = 0$

At (-2, 1)

$$-2^2 + 1^2 + 2g(-2) + 2f(1) + c = 0$$
$$-4g + 2f + c = -5 \ldots\ldots\ldots\ldots (1)$$

At (6, 1), $6^2 + 1^2 + 2g(6) + 2f(1) + c = 0$

$$36 + 1 + 12g + 2f + c = 0$$
$$12g + 2f + c = -37 \ldots\ldots\ldots\ldots (2)$$

At (-2, 7), $-2^2 + 7^2 + 2g(-2) + 2f(7) + c = 0$

$$4 + 49 - 4g + 14f + c = 0$$
$$-4g + 14f + c = -53 \ldots\ldots\ldots\ldots (3)$$

Solving equation (1), 2 and 3 simultaneously

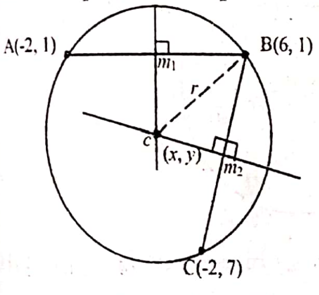
$$g = -2, f = -4, c = -5$$

Substituting $g = -2, f = -4, c = -5$ in the general equation of the circle $x^2 + y^2 + 2gx + 2fy + C = 0$

$$x^2 + y^2 + 2x(-2) + 2y(-4) + -5 = 0$$
$$x^2 + y^2 - 4x - 8y - 5 = 0$$

(As before)

For example, in the equation of the circle.



The perpendicular bisectors of the chord AB and BC meet at the centre.

$$m_1\left(\frac{-2+6}{2}, \frac{1+1}{2}\right)$$

$m_1(2, 1)$

Gradient of AB $= \dfrac{1-1}{6-^{-}2}$

$= 0$

The gradient of the perpendicular bisector of AB is a negative reciprocal of the gradient of AB.

$$= \frac{-1}{0}$$

$$\Rightarrow \frac{y-1}{x-2} = \frac{-1}{0}$$

$$x - 2 = 0$$

$$x = 2 \ldots\ldots\ldots\ldots (1)$$

$x = 2$ is the equation of the perpendicular bisector.

$$m_2\left(\frac{6+^{-}2}{2}, \frac{7+1}{2}\right)$$

$m_2(2, 4)$

gradient of BC

$$\Rightarrow \frac{1-7}{6-^{-}2}$$

$$= \frac{-3}{4}$$

Gradient of the perpendicular bisector of BC

$$= -\left(\frac{4}{-3}\right)$$

$$= \frac{4}{3}$$

The equation of the perpendicular bisector of BC

$$\Rightarrow \frac{y-4}{x-2} = \frac{4}{3}$$

$3(y-4) = 4(x-2)$

$3y - 12 = 4x - 8$

$3y = 4x + 4 \ldots\ldots\ldots\ldots (2)$

Substituting equation (1) in (2)

$\Rightarrow 3y = 4 \times 2 + 4$

$3y = 12$

$y = 4$

The centre of the circle C(2, 4).

$$r = \sqrt{(2-6)^2 + (4-1)^2}$$

$$r = \sqrt{16+9}$$

$= \sqrt{25}$

$= 5$

Equation of the circle

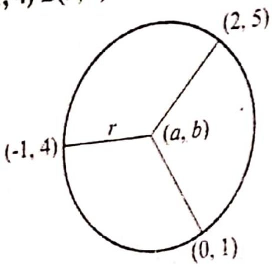
$(x-a)^2 + (y-b)^2 = r^2$

$\Rightarrow (x-2)^2 + (y-4)^2 = 5^2$

$x^2 - 4x + 4 + y^2 - 8y + 16 = 25$

$x^2 + y^2 - 4x - 8y - 5 = 0.$

(b) A(-1, 4) B(2, 5) and C(0, 1)



$$r_1 = \sqrt{(a - -1)^2 + (b-4)^2}$$

$$r_2 = \sqrt{(a-2)^2 + (b-5)^2}$$

$$r_3 = \sqrt{(a-0)^2 + (b-1)^2}$$

$r_1 = r_2 = r$

$$\sqrt{(a+1)^2 + (b-4)^2} = \sqrt{(a-2)^2 + (b-5)^2}$$

$$(a+1)^2 + (b-4)^2 = (a-2)^2 + (b-5)^2$$

$$a^2 + 2a + 1 + b^2 - 8b + 16$$
$$= a^2 - 4a + 4 + b^2 - 10b + 25$$

$$2a - 8b + 17 = -4a - 10b + 29$$

$$6a + 2b = 12$$

$\Rightarrow 3a + b = 6$............................ (1)

Similarly, $r_1 = r_3 = r$

$$\sqrt{(a+1)^2 + (b-4)^2} = \sqrt{(a-0)^2 + (b-1)^2}$$

$$(a+1)^2 + (b-4)^2 = (a-0)^2 + (b-1)^2$$

$$a^2 + 2a + 1 + b^2 - 8b + 16 = a^2 + b^2 - 2b + 1$$

$$2a - 8b + 17 = -2b + 1$$

$$2a - 6b = -16$$

$a - 3b = -8$............................ (2)

From eqn. (1)

$b = 6 - 3a$

Substituting $b = 6 - 3a$ in eqn. (2)

$a - 3(6 - 3a) = -8$

$a - 18 + 9a = -8$

$10a = 10$

$a = 1$

$\Rightarrow b = 6 - 3 \times 1$

$b = 3$

Center (1, 3)

$$r = \sqrt{(a+1)^2 + (b-4)^2}$$

$$r = \sqrt{(1 - -1)^2 + (3-4)^2}$$

$$r = \sqrt{4+1}$$

$$r = \sqrt{5}$$

$(x-a)^2 + (y-b)^2 = r^2$

$(x-1)^2 + (y-3)^2 = (\sqrt{5})^2$

$x^2 - 2x + 1 + y^2 - 6y + 9 = 5$

$x^2 + y^2 - 2x - 6y + 5 = 0$

Alternatively

$x^2 + y^2 + 2gx + 2fy + C = 0$

At (-1, 4): $(-1)^2 + 4^2 + 2g(-1) + 2f(4) + c = 0$

$1 + 16 - 2g + 8f + c = 0$

$-2g + 8f + c = -17$

At (2, 5): $(2)^2 + 5^2 + 2g(2) + 2f(5) + c = 0$

$4 + 25 + 4g + 10f + c = 0$

$4g + 10f + c = -29$.................... (2)

At (0, 1): $0^2 + 1^2 + 2g(0) + 2f(1) + c = 0$

$2f + c = -1$............................ (3)

Solving eqn. 1, 2 and 3 simultaneously

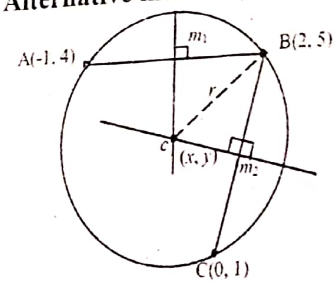
$g = 1, f = -3, c = 5$

Substituting $g = -1, f = -3, c = 5$ in the ge equation of the circle $x^2 + y^2 + 2gx + 2fy + C$ =

$x^2 + y^2 + 2x(1) + 2y(-3) + 5 = 0$

$x^2 + y^2 - 2x - 6y + 5 = 0$ (as before)

Alternative method (3)



The perpendicular bisector of the chords AB an BC meet at the centre.

$$m_1\left(\frac{-1+2}{2}, \frac{4+5}{2}\right)$$

$$m_1\left(\frac{1}{2}, \frac{9}{2}\right)$$

Gradient of AB

$\frac{5-4}{2-1}$

$= \frac{1}{3}$

The gradient of the perpendicular bisector of AB

$= -\left(\frac{3}{1}\right)$

$= -3$

Equation of the perpendicular bisector of AB

$\Rightarrow \frac{y-\frac{9}{2}}{x-\frac{1}{2}} = -3$

$y-\frac{9}{2} = -3\left(x-\frac{1}{2}\right)$

$y = -3x + 6$ .... (1)

Similarly

$m_2\left(\frac{2+0}{2}, \frac{1+5}{2}\right)$

$m_2(1, 3)$

gradient of BC

$\frac{5-1}{2-0} = \frac{4}{2}$

$= 2$

Gradient of the perpendicular bisector of BC = $\frac{-1}{2}$.

$2(y-3) = -1(x-1)$

$2y - 6 = -x + 1$

$2y = -x + 7$ .... (2)

Substituting equation (1) into (2)

$\Rightarrow 2(-3x+6) = -x + 7$

$-6x + 12 = -x + 7$

$-5x = -5$

$x = 1$

Substituting $x = 1$ in equation 2.

$\Rightarrow 2y = -1 + 7$

$y = 3$

$(1, 3)$

Radius $(r) = \sqrt{(1-2)^2+(3-5)^2}$

$= \sqrt{5}$

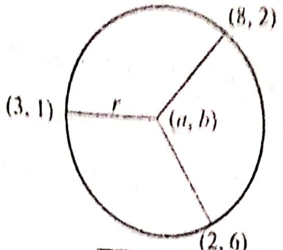
The equation of the circle

$(x-1)^2 + (y-3)^2 = \left(\sqrt{5}\right)^2$

$x^2 - 2x + 1 + y^2 - 6y + 9 = 5$

$x^2 + y^2 - 2x - 6y + 5 = 0$

(c) A(3, 1) B(8, 2) and C(2, 6)



$r_1 = \sqrt{(a-3)^2+(b-1)^2}$

$r_2 = \sqrt{(a-2)^2+(b-6)^2}$

$r_2 = \sqrt{(a-8)^2+(b-2)^2}$

$r_1 = r_2 = r$

$\sqrt{(a-3)^2+(b-1)^2} = \sqrt{(a-2)^2+(b-6)^2}$

$(a-3)^2+(b-1)^2 = (a-2)^2+(b-6)^2$

$a^2-6a+9+b^2-2b+1 = a^2-4a+4+b^2-12b+36$

$-6a-2b+10 = -4a-12b+40$

$-2a+10b = 30$

$-a+5b = 15$

$a = 5b - 15$.............................. (1)

Similarly; $r_1 = r_3 = r$

$\sqrt{(a-3)^2+(b-1)^2} = \sqrt{(a-8)^2+(b-2)^2}$

$(a-3)^2+(b-1)^2 = (a-8)^2+(b-2)^2$

$a^2-6a+9+b^2-2b+1 = a^2-16a+64+b^2-4b+4$

$-6a-2b+10 = -16a-4b+68$

$10a+2b = 58$

$\Rightarrow 5a + b = 29$.............................. (2)

Substituting equation (1) in (2)

$5(5b-15)+b = 29$

$25b - 75 + b = 29$

$26b = 104$

$b = 4$

$a = 5b - 15$

$a = 5 \times 4 - 15$

$a = 5$

Center (5, 4)

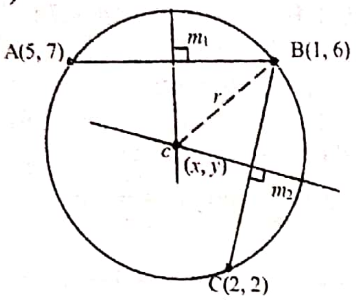
$r = \sqrt{(5-3)^2+(4-1)^2}$

$$x^2+y^2+2gx+2fy+C=0.$$

$$x^2+y^2+2\left(\frac{-7}{2}\right)x+2\left(\frac{-9}{2}\right)y+C=0$$

$$x^2+y^2-7x-9y+24=0$$

(d)



$$m_1\left(\frac{5+1}{2},\frac{7+6}{2}\right)$$

$$\left(3,\frac{13}{2}\right)$$

Gradient of AB

$$=\frac{6-7}{1-5}$$

$$=\frac{-1}{-4}$$

$$=\frac{1}{4}$$

The gradient of the perpendicular bisector of AB

$$\Rightarrow -\left(\frac{4}{1}\right)$$

$$\frac{y-\frac{13}{2}}{x-3}=-4$$

$$y-\frac{13}{2}=-4(x-3)$$

$$y-\frac{13}{2}=-4x+12$$

$$y=-4x+12+\frac{13}{2}$$

$$y=-4x+\frac{37}{2} \quad \text{.............. (1)}$$

Similarly

$$m_2\left(\frac{3}{2},4\right)$$

Gradient of BC

$$=\frac{6-2}{1-2}$$

$$=-4$$

The gradient of the perpendicular bisector of

$$=-\left(\frac{1}{-4}\right)$$

$$=\frac{1}{4}$$

The equation of the perpendicular bisector of

$$\frac{y-4}{x-\frac{3}{2}}=\frac{1}{4}$$

$$4(y-4)=x-\frac{3}{2}$$

$$4y-16=x-\frac{3}{2}$$

$$4y=x-\frac{3}{2}+16$$

$$4y=x-\frac{29}{2}$$

Substituting equation 2 in (1)

$$4\left(-4x+\frac{37}{2}\right)=x+\frac{29}{2}$$

$$-16x+74=x+\frac{29}{2}$$

$$-17x=\frac{-119}{2}$$

$$-34x=-119$$

$$x=\frac{7}{2}$$

Substituting $x=\frac{7}{2}$ in equation (1)

$$\Rightarrow y=-4\left(\frac{7}{2}\right)+\frac{37}{2}$$

$$y=\frac{-28}{2}+\frac{37}{2}$$

$$y=\frac{9}{2}$$

$$\Rightarrow \text{Centre}\left(\frac{7}{2},\frac{9}{2}\right)$$

$$\text{Radius} = \sqrt{\left(\frac{7}{2}-1\right)^2+\left(\frac{9}{2}-6\right)^2}$$

$$= \frac{\sqrt{34}}{2}$$

The equation of the circle

$$\Rightarrow \left(x-\frac{7}{2}\right)^2+\left(y-\frac{9}{2}\right)^2=\left(\frac{\sqrt{34}}{2}\right)^2$$

$$x^2-7x+\frac{49}{4}+y^2-9y+\frac{81}{4}=\frac{34}{4}$$

$$x^2+y^2-7x-9y+24=0$$

Parametric Equations of circle

Consider a circle $(x-a)^2+(y-b)^2=r^2$ the parametric equations of the above circles are $x-a=r\cos\theta$ and $y-b=r\sin\theta$

$$\therefore x=a+r\cos\theta \ and \ y=b+r\sin\theta$$

Example I

Find the parametric equation of the circle $(x-4)^2+(y-3)^2=4$

Solution

$$(x-4)^2+(y-3)^2=2^2$$
$$x-4=r\cos\theta$$
$$y-3=r\sin\theta$$
$$r=2$$
$$x-4=2\cos\theta$$
$$y-3=2\sin\theta$$
$$x=4+2\cos\theta$$
$$y=3+2\sin\theta$$

The parametric equations of the circle $(x-4)^2+(y-3)^2=4$ are

$$x=4+2\cos\theta$$

$$y=3+2\sin\theta$$

Example II

Find the parametric equations of the circle

$$(x+1)^2+(y-2)^2=9$$

Solution

Comparing $(x+1)^2+(y-2)^2=9$ with the equation of the circle $(x-a)^2+(y-b)^2=r^2$

$$a=-1, b=2, r=3$$
$$x+1=r\cos\theta$$
$$y-2=r\sin\theta$$
$$x+1=3\cos\theta$$
$$y-2=3\sin\theta$$
$$x=3\cos\theta-1$$
$$y=2+3\sin\theta$$

Example III

Find the parametric equations of the circle

$$x^2+y^2-4x-2y+1=0$$

Solution

$$x^2+y^2-4x-2y+1=0$$

By completing squares;s

$$(x^2-4x+4)-4+y^2-2y+1=0$$
$$(x-2)^2+(y-1)^2=4$$
$$x-2=r\cos\theta$$
$$y-1=r\sin\theta$$
$$x-2=2\cos\theta$$
$$y-1=2\sin\theta$$
$$x=2+2\cos\theta$$
$$y=1+2\sin\theta$$

Example IV

Find the parametric equation of a circle $x^2+y^2-6x+4y-12=0$

Solution

$$x^2+y^2-6x+4y-12=0$$
$$x^2-6x+y^2+4y-12=0$$

By completing squares;

$$x^2-6x+9-9+y^2+4y^2+4-4-12=0$$
$$(x-3)^2+(y+2)^2=25$$
$$(x-a)^2+(y-b)^2=5^2$$
$$a=3, b=-2, r=5$$
$$x-3=r\cos\theta$$
$$y+2=r\sin\theta$$
$$x-3=5\cos\theta$$
$$y+2=5\sin\theta$$
$$x=3+5\cos\theta$$
$$y=-2+5\sin\theta$$

Example V

Find the Cartesian equation of the circle with parametric equations

$x=-2+3\cos\theta$

$y=3+3\sin\theta$

Solution

$$\frac{x+2}{3}=\cos\theta$$
$$\frac{y-3}{3}=\sin\theta$$
$$\cos^2\theta+\sin^2\theta=1$$
$$\frac{(x+2)^2}{9}+\frac{(y-3)^2}{9}=1$$

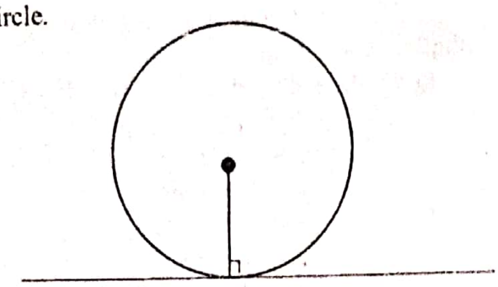
But $\frac{(x+2)^2}{9}+\frac{(y-3)^2}{9}=1$

$$(x+2)^2+(y-3)^2=9$$

$$x^2 + 4x + 4 + y^2 - 6y + 9 = 9$$
$$x^2 + 4x + y^2 - 6y + 4 = 0$$
$$x^2 + y^2 + 4x - 6y + 4 = 0$$

Tangents to the Circle

A tangent to the circle is a line which touches the circle at only one point and makes 90^0 with the radius of the circle.

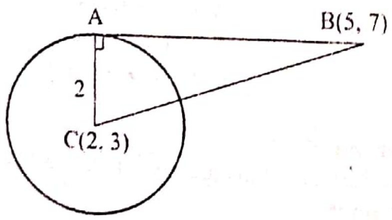


Length of the tangent to a circle

Example
Find the length of the tangent from (5, 7) to the circle
$$x^2 + y^2 - 4x - 6y + 9 = 0$$

Solution
Comparing $x^2 + y^2 + 4x - 6y - 9 = 0$ with
$x^2 + y^2 + 2gx + 2fy + C = 0.$
$$g = -2, f = -3, c = 9$$
Center (*-g*, *-f*)
Center (2, 3)

$$r = \sqrt{g^2 + f^2 - c}$$
$$r = \sqrt{4 + 9 - 9}$$
$$r = 2$$



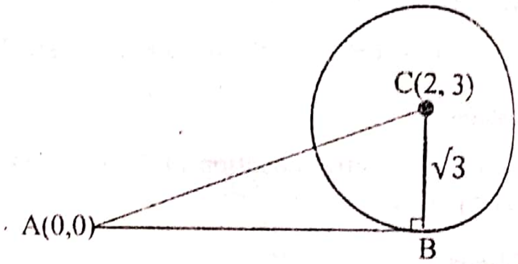
$$CB = \sqrt{(2-5)^2 + (3-7)^2}$$
$$CB = \sqrt{9 + 16}$$
$$CB = 5$$
$$AB^2 + AC^2 = CB^2$$
$$AB^2 + 2^2 = 5^2$$
$$AB^2 = 5^2 - 2^2$$
$$AB^2 = 21$$
$$AB = \sqrt{21}\ units$$
The length of the tangent is $\sqrt{21}$ units

Example II
Find the lengths of the tangents from the given p the following circles

(a) $x^2 + y^2 + 4x - 6y + 10 = 0, (0, 0)$
(b) $x^2 + y^2 + 6x + 10y - 2 = 0, (-2, 3)$

Solution
(a) $x^2 + y^2 + 4x - 6y + 10 = 0, (0, 0)$
Comparing $x^2 + y^2 + 4x - 6y + 10 = 0$
$x^2 + y^2 + 2gx + 2fy + C = 0.$
$$2gx = -4x$$
$$g = -2$$
$$2fy = -6y$$
$$f = -3$$

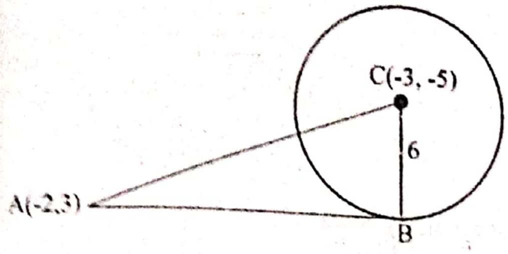
Center (*-g*, *-f*)
Center (2, 3)
$$r = \sqrt{g^2 + f^2 - c}$$
$$r = \sqrt{4 + 9 - 10}$$
$$r = \sqrt{3}$$



$$AC = \sqrt{(0-2)^2 + (0-3)^2}$$
$$AC = \sqrt{4 + 9}$$
$$AC = \sqrt{13}$$
$$AB^2 + CB^2 = AC^2$$
$$AB^2 + (\sqrt{3})^2 = (\sqrt{13})^2$$
$$AB^2 + 3 = 13$$
$$AB^2 = 10$$
$$AB = \sqrt{10}$$

(b) $\mathbf{x^2 + y^2 + 6x + 10y - 2 = 0, (-2, 3)}$
Comparing $x^2 + y^2 + 6x + 10y - 2 = 0$ with
$y^2 + 2gx + 2fy + C = 0.$
$$g = 3, f = 5, c = -2$$
Center (-3, -5)

$$r = \sqrt{g^2 + f^2 - c}$$
$$r = \sqrt{9 + 25 - -2}$$
$$r = 6$$



$$AC = \sqrt{(-2 - -3)^2 + (3 - -5)^2}$$
$$AC = \sqrt{1 + 64}$$
$$AC = \sqrt{65}$$
$$AB^2 + 6^2 = (\sqrt{65})^2$$
$$AB^2 + 36 = 65$$
$$AB^2 = 65 - 36$$
$$AB^2 = 29$$
$$AB = \sqrt{29}$$

Alternative method of finding length of the tangent to a circle

The length of a tangent drawn from a point (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + C = 0$ is given by

$$L = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

$$= \sqrt{S_1} \text{ where } L = \text{length of the tangent}$$

The square of the length of the tangent from the point P is called a power point with respect to the circle.

Example I
Find the length of the tangent drawn from the point (5, 1) to the circle $x^2 + y^2 + 6x - 4y - 3 = 0$

Solution
Comparing $x^2 + y^2 + 6x - 4y - 3 = 0$ with $x^2 + y^2 + 2gx + 2fy + C = 0$

$$g = 3, f = -2, c = 3$$

$$(x_1, y_1) = (5, 1)$$

$$L = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

$$L = \sqrt{5^2 + 1^2 + 2g(5) + 2f(1) + c}$$

$$L = \sqrt{5^2 + 1^2 + 2 \times 3(5) + 2(-2)(1) - 3}$$

$$L = \sqrt{25 + 1 + 30 - 4 - 3}$$

$$L = 7 \text{ Units}$$

Example II

If the length of the tangent from the point (f, g) to the circle $x^2 + y^2 = 4$ is four times the length of the tangent from (f_1, g_1) it to the circle $x^2 + y^2 = 4x$, show that $15f_1^2 + 15g_1^2 - 64f_1 + 4 = 0$

Solution

$$x^2 + y^2 - 4 = 0$$
$$g = 0, f = 0, c = -4$$
$$L_1 = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$
$$L_1 = \sqrt{g_1^2 + f_1^2 + 0 + 0 + -4}$$
$$L_1 = \sqrt{g_1^2 + f_1^2 - 4}$$

For $x^2 + y^2 - 4x = 0$, $g = -2$, $f = 0$ and $c = 0$

$$L_2 = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$
$$L_2 = \sqrt{g_1^2 + f_1^2 + 2(-2)f_1 + 0 + 0}$$
$$L_2 = \sqrt{g_1^2 + f_1^2 - 4g_1}$$

But $L_1 = 4L_2$

$$\sqrt{g_1^2 + f_1^2 - 4} = 4\sqrt{g_1^2 + f_1^2 - 4f_1}$$
$$g_1^2 + f_1^2 - 4 = 16\left(g_1^2 + f_1^2 - 4f_1\right)$$
$$g_1^2 + f_1^2 - 4 = 16g_1^2 + 16f_1^2 - 64f_1$$
$$15g_1^2 + 15f_1^2 - 64f_1 + 4 = 0 \text{ (as required)}$$

Equation of a Tangent

Example I
Find the equation of the tangent to the circle $x^2 + y^2 + 2x - 2y - 8 = 0$ at $(2, 2)$
Solution

$$\frac{d}{dx}(x^2 + y^2 + 2x - 2y - 8) = \frac{d}{dx}(0)$$
$$2xdx + 2ydy + 2dx - 2dy = 0$$
$$2x + 2y\frac{dy}{dx} + 2 - 2\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(2y - 2) = -2 - 2x$$
$$\frac{dy}{dx} = \frac{-2 - 2x}{2y - 2}$$
$$\frac{dy}{dx} = \frac{-2(1 + x)}{2(y - 1)}$$
$$\frac{dy}{dx} = \frac{-1(1 + x)}{y - 1}$$
$$\frac{dy}{dx} = \frac{-1 - x}{y - 1}$$

$$\left.\frac{dy}{dx}\right|_{(2,2)} = \frac{-1-2}{2-1}$$

$$\frac{dy}{dx} = -3$$

$$\frac{y-2}{x-2} = -3$$

$$y - 2 = -3(x-2)$$

$$y - 2 = -3x + 6$$

$$y = -3x + 8$$

Alternatively

Note: The equation of the tangent to the circle $x^2 + y^2 = a^2$ at (x_1, y_1) is $xx_1 + yy_1 = a^2$

The equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + C = 0$ at x_1, y_1 is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

We can now find the equation of the tangent to the $x^2 + y^2 + 2x - 2y - 8 = 0$ at $(2,2)$

Comparing $x^2 + y^2 + 2x - 2y - 8 = 0$ with $x^2 + y^2 + 2gx + 2fy + C = 0$

$$g = 1, f = -1, c = -8$$

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

$$x(2) + y(2) + g(x+2) + f(y+2) - 8 = 0$$

$$2x + 2y + 1(x+2) - 1(y+2) - 8 = 0$$

$$2x + 2y + x + 2 - y - 2 - 8 = 0$$

$$3x + y = 8$$

$$y = -3x + 8 \ (as\ before)$$

Example II

Find the equation of the tangent to the circle $2x^2 + 2y^2 - 8x - 5y - 1 = 0$ at C (1, -1)

Solution

$$2x^2 + 2y^2 - 8x - 5y - 1 = 0$$

$$4xdx + 4ydy - 8dx - 5dy = 0$$

$$4x + 4y\frac{dy}{dx} - 8 - 5\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(4y - 5) = 8 - 4x$$

$$\frac{dy}{dx} = \frac{8-4x}{4y-5}$$

$$\left.\frac{dy}{dx}\right|_{(1,-1)} = \frac{8-4(1)}{4\times{}^{-}1-5} = \frac{4}{-9}$$

$$\frac{y - -1}{x-1} = \frac{-4}{9}$$

$$9(y+1) = -4(x-1)$$

$$9y + 9 = -4x + 4$$

$$9y = -4x - 5$$

Alternative method

From $2x^2 + 2y^2 - 8x - 5y - 1 = 0$,

$$x^2 + y^2 - 4x - \frac{5y}{2} - \frac{1}{2} = 0$$

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$\Rightarrow g = -2,\ f = \frac{-5}{4},\ c = \frac{-1}{2}$$

$$x_1 = 1, y_1 = -1$$

The equation of the tangent is given by

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c =$$

$$x(1) + y(-1) + -2(x+1) + \frac{-5}{4}(y-1) +$$

$$x - y - 2x - 2 + \frac{-5y}{4} + \frac{5}{4} - \frac{1}{2} = 0$$

$$-x - \frac{9y}{4} - \frac{5}{4} = 0$$

$$-4x - 9y - 5 = 0$$

$$4x + 9y + 5 = 0$$

Example III

The tangent to the circle $x^2 + y^2 - 4x + 6y - 77 = 0$ point (5, 6) meets the axes at A and B. find A and

Solution

$$x^2 + y^2 - 4x + 6y - 77 = 0$$

$$2xdx + 2ydy - 4dx + 6dy = 0$$

$$2x + 2y\frac{dy}{dx} - 4 + 6\frac{dy}{dx} = 0$$

$$(2y + 6)\frac{dy}{dx} = 4 - 2x$$

$$\frac{dy}{dx} = \frac{4-2x}{2y+6}$$

$$\left.\frac{dy}{dx}\right|_{(5,6)} = \frac{4-2(5)}{2(6)+6}$$

$$\frac{dy}{dx} = \frac{-6}{18}$$

$$= \frac{-1}{3}$$

$$\frac{y-6}{x-5} = \frac{-1}{3}$$

$$3(y-6) = -1(x-5)$$

$$3y - 18 = -x + 5$$

$$3y = -x + 23$$

$$x + 3y = 23$$

Alternative method

Comparing $x^2 + y^2 - 4x + 6y - 77 = 0$ with $x^2 + y^2 + 2gx + 2fy + c = 0$

$$g = -2, f = 3, c = -77$$
$$x_1 = 5,\ y_1 = 6$$

The equation of the tangent is

$$x_1x + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$
$$\Rightarrow 5x + 6y + -2(x + 5) + 3(y + 6) - 77 = 0$$
$$5x + 6y - 2x - 10 + 3y + 18 - 77 = 0$$
$$3x + 9y = 69$$
$$\Rightarrow x + 3y = 23, \text{ as before.}$$

At the x- axis (A), $y = 0$

$$0 = -x + 23$$
$$x = 23$$

The tangent meets the x- axis at (23, 0)

At the y- axis (B), $x = 0$

$$3y = 23$$
$$y = \frac{23}{3}$$

The curve cuts the y- axis at $(0, \frac{23}{3})$

Example VII

Find the equation of the tangent to the circle $x^2 + y^2 - 30x + 6y + 109 = 0\ at\ (4, -1)$

Solution

$$x^2 + y^2 - 30x + 6y + 109 = 0$$
$$\frac{d}{dx}(x^2 + y^2 - 30x + 6y + 109) = \frac{d}{dx}(0)$$
$$2xdx + 2ydy - 30dx + 6dy = 0$$
$$2x + 2y\frac{dy}{dx} - 30 + 6\frac{dy}{dx} = 0$$
$$(2y + 6)\frac{dy}{dx} = 30 - 2x$$
$$\frac{dy}{dx} = \frac{30 - 2x}{2y + 6}$$
$$\frac{dy}{dx} = \frac{15 - x}{y + 3}$$
$$\left.\frac{dy}{dx}\right|_{(4,-1)} = \frac{15 - 4}{-1 + 3} = \frac{11}{2}$$
$$\frac{y - -1}{x - 4} = \frac{11}{2}$$
$$2y + 2 = 11x - 44$$
$$2y = 11x - 46$$
$$0 = 11x - 2y - 46$$

Alternatively

Given a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ the equation of the tangent at (x_1, y_1) is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

Comparing $x^2 + y^2 - 30x + 6y + 109 = 0$ with $x^2 + y^2 + 2gx + 2fy + c = 0$

$$g = -15, f = 3, c = 109, x_1 = 4, y_1 = -1$$
$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$
$$\Rightarrow x(4) + y(-1) + -15(x + 4) + 3(y + -1) + 109 = 0$$
$$4x - y - 15x - 60 + 3y - 3 + 109 = 0$$
$$-11x + 2y + 46 = 0$$
$$11x - 2y - 46 = 0\ (as\ before)$$

Example IV

Show that $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$

Solution

$$y = mx + c$$
$$x^2 + y^2 = a^2$$
$$c + (mx + c)^2 = a^2$$
$$x^2 + m^2x^2 + 2mcx + c^2 = a^2$$
$$x^2 + m^2x^2 + 2mcx + c^2 - a^2 = 0$$
$$(1 + m^2)x^2 + (2mc)x + c^2 - a^2 = 0$$
$$B^2 = 4AC\ (for\ tangency)$$
$$(2mc)^2 = 4(1 + m^2)[c^2 - a^2]$$
$$4m^2c^2 = 4(1 + m^2)[c^2 - a^2]$$
$$m^2c^2 = (1 + m^2)[c^2 - a^2]$$
$$m^2c^2 = c^2 - a^2 + m^2c^2 - m^2a^2$$
$$c^2 = a^2 + m^2a^2$$
$$c^2 = a^2(1 + m^2)$$

Example V

Show that the line $y = x + 1$ touches the circle $x^2 + y^2 - 8x - 2y + 9 = 0$.

Solution

$$x^2 + y^2 - 8x - 2y + 9 = 0$$
$$y = x + 1$$
$$x^2 + (x + 1)^2 - 8x - 2(x + 1) + 9 = 0$$
$$x^2 + x^2 + 2x + 1 - 8x - 2x - 2 + 9 = 0$$
$$2x^2 - 8x + 8 = 0$$
$$x^2 - 4x + 4 = 0$$

For the line to touch the circle

$$B^2 = 4AC$$
$$(-4)^2 = 4(4)(1)$$
$$16 = 16$$

The line $y = x + 1$ touches the circle $x^2 + y^2 - 8x - 2y + 9 = 0$

Note:

If $y = mx + c$ is a line and $x^2 + y^2 = a^2$ is a circle then
(i) $C^2 > a^2(1 + m^2)$ the line is a secant to the circle
(ii) If $C^2 = a^2(1 + m^2)$ the line touches the circle
(iii) If $C^2 < a^2(1 + m^2)$ the line doesn't meet the circle

Example VI

For what values of c will the line $y = 2x + c$ be tangent to the circle $x^2 + y^2 = 5^2$

Solution

$$y = 2x + c$$
$$x^2 + y^2 = 5^2$$
$$x^2 + (2x + c)^2 = 5$$
$$x^2 + 4x^2 + 4xc + c^2 = 5$$
$$5x^2 + 4xc + c^2 - 5 = 0$$

For tangency $B^2 = 4AC$

$$(4c)^2 = 4(5)(c^2 - 5)$$
$$16c^2 = 20c^2 - 100$$
$$100 = 4c^2$$
$$25 = c^2$$
$$c = 5, c = -5$$

Example VII

For what values of α, does the line $3x + 4y = \alpha$ touch the circle $x^2 + y^2 - 10x = 0$?

Solution

$$3x + 4y = \alpha \text{ .... (i)}$$
$$x^2 + y^2 - 10x = 10 \text{ .... (ii)}$$

Substituting $y = \frac{\alpha - 3x}{4}$ in Eqn (ii)

$$\Rightarrow x^2 + \left(\frac{\alpha - 3x}{4}\right)^2 - 10x = 0$$
$$x^2 + \frac{\alpha^2 - 6\alpha x + 9x^2}{16} - 10x = 0$$
$$16x^2 + \alpha^2 - 6x\alpha + 9x^2 - 160x = 0$$
$$25x^2 + (-6\alpha - 160)x + \alpha^2 = 0$$

For tangency $B^2 = 4AC$

$$(-6\alpha - 160)^2 = 4 \times 25\,(\alpha^2)$$
$$36\alpha^2 + 1920\alpha + 25600 = 100\alpha^2$$
$$64\alpha^2 - 1920\alpha - 25600 = 0$$
$$\alpha^2 - 30\alpha - 400 = 0$$
$$(\alpha - 40)(\alpha + 10) = 0$$
$$\alpha = 40, \alpha = -10$$

Example VIII

Find the equation of the tangents to the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ which are paral line $4x + 3y + 5 = 0$

Solution

Let the tangent be $y = mx + c$

Since the tangent is parallel to $4x + 3y + 5 =$ ($-\frac{4x}{3} - \frac{5}{3}$)

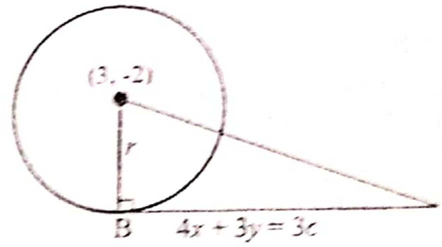
$$m = \frac{-4}{3}$$
$$y = -\frac{4x}{3} + c$$

$3y + 4x = 3c$ is equation of the tangent

Comparing $x^2 + y^2 - 6x + 4y - 12 = 0$ with $x^2 + y^2 + 2gx + 2fy + c = 0$

$$g = -3, f = 2, c = -12$$

Center (+3, -2)



$$r = \sqrt{g^2 + f^2 - c}$$
$$r = \sqrt{9 + 4 - -12}$$
$$r = 5$$

But we can obtain r using the formula for perpendi distance of a point from a line

$$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$
$$r = \left|\frac{4(3) + 3(-2) + -3c}{\sqrt{4^2 + 3^2}}\right|$$
$$5 = \left|\frac{12 - 6 - 3c}{5}\right|$$
$$5 = \pm\left(\frac{6 - 3c}{5}\right)$$
$$25 = 6 - 3c$$
$$3c = 6 - 25$$
$$3c = -19$$
$$5 = -\left(\frac{6 - 3c}{5}\right)$$
$$25 = -6 + 3c$$

$$31 = 3c$$

Since the tangents to the circle are given by

$$4x + 3y = 3c$$

$\Rightarrow$ The equations of the tangents are $4x + 3y = -19$ and

$$4x + 3y = +31$$

Example IX

(i) Find the equation of the tangents to the circle $x^2 + y^2 - 2x - 4y - 4 = 0$ which are parallel to line $3x - 4y - 1 = 0$

(ii) Which are perpendicular to the line $3x - 4y - 1 = 0$

Solution

Comparing $x^2 + y^2 + 2gx + 2fy + c = 0$ with $x^2 + y^2 - 2x - 4y - 4 = 0$

$$g = -1, f = -2, c = -4$$

Center (1, 2)

$$r = \sqrt{g^2 + f^2 - c}$$

$$r = \sqrt{1 + 4 - -4}$$

$$r = 3$$

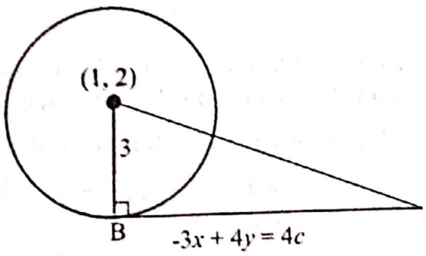
$$3x - 4y - 1 = 0$$

$$\frac{3x}{4} - \frac{1}{4} = y$$

Since the tangents are parallel to the line

$\Rightarrow m = \frac{3}{4}$ for the tangent $y = mx + c$

$y = \frac{3x}{4} + C$, $(4y - 3x) = 4C$ are the equations of the tangents



$$r = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

$$3 = \left|\frac{-3(1) + 4(2) - 4c}{\sqrt{-3^2 + 4^2}}\right|$$

$$3 = \left|\frac{5 - 4c}{5}\right|$$

$$3 = \pm\left(\frac{5 - 4c}{5}\right)$$

$$3 = \frac{5 - 4c}{5}$$

$$15 = 5 - 4c$$

$$4c = -10$$

$$3 = -\left(\frac{5 - 4c}{5}\right)$$

$$15 = -5 + 4c$$

$$4c = 20$$

Since the equations of the tangent that are parallel to the line $3x - 4y - 1 = 0$ are $-3x + 4y = 4c$

$\Rightarrow$ The required tangents are:

$$-3x + 4y = -10$$

$$-3x + 4y = 20$$

(ii) Let the tangents that are perpendicular to the line $3x - 4y - 1 = 0$ be $y = mx + c$

$$3x - 4y - 1 = 0$$

$$4y = 3x - 1$$

$$y = \frac{3x}{4} - \frac{1}{4}$$

$$m = \frac{3}{4}$$

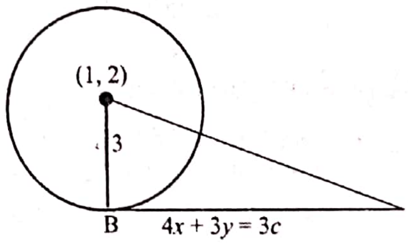
$$\Rightarrow m_1 = \frac{-4}{3}$$

$$y = \frac{-4x}{3} + c$$

$$3y + 4x = 3c$$

Center (1, 2)

$$r = 3$$



$$r = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

$$3 = \left|\frac{4(1) + 3(2) - 3c}{\sqrt{4^2 + 3^2}}\right|$$

$$3 = \left|\frac{4 + 6 - 3c}{5}\right|$$

$$3 = \frac{10 - 3c}{5}$$

$$15 = 10 - 3c$$
$$3c = -5$$
$$3 = \frac{-(10 - 3c)}{5}$$
$$15 = -10 + 3c$$
$$3c = 25$$

Since the tangent to the circle is given by;

$$3x + 4y = 3c$$

The equations of the tangents are:

$$3x + 4y = 25$$
$$3x + 4y = -5$$

Director Circle

The locus of the point of intersection of two perpendicular tangents is called the Director circle of a given circle. The Director circle of a circle is a concentric circle having radius equal to $\sqrt{2}$ times the original radius.

Example

Find the equation of the director circle of the circle $(x - 2)^2 + (y + 1)^2 = 2$

Solution

$$(x - 2)^2 + (y + 1)^2 = 2$$

Center (2, -1)

Radius $r = \sqrt{2}$

The center of the director circle is (2, -1) and the radius of the director circle is

$$\sqrt{2} \times r$$
$$= \sqrt{2}\, X\, \sqrt{2}$$
$$= 2$$

The equation of the director circle is

$$(x - 2)^2 + (y + 1)^2 = 2^2$$
$$x^2 - 4x + 4 + y^2 + 2y + 1 = 4$$
$$x^2 + y^2 - 4x + 2y + 1 = 0$$

Example II

Find the equation of a director circle of the circle whose diameters are $2x - 3y + 12 = 0$ and $x + 4y - 5 = 0$ and has an area of 154.

Solution

$$2x - 3y + 12 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$
$$x + 4y - 5 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Solving eqn. (1) and (2) simultaneously

$$x = -3, y = 2$$

The center of a circle is (-3, 2)

$$\pi r^2 = 154$$
$$\frac{22}{7}\, X\, r^2 = 154$$
$$r = 7$$

Radius of the director circle is $7\sqrt{2}$

The equation of the director circle is

$$(x - 3)^2 + (y - 2)^2 = (7\sqrt{2})^2$$
$$(x - 3)^2 + (y - 2)^2 = 98$$
$$x^2 + 6x + 9 + y^2 - 4y + 4 = 98$$

Therefore, $\quad x^2 + y^2 + 6x - 4y - 85 = 0$ is equation of the director circle.

INTERSECTING CIRCLES

Equation of a common chord of two circ

Let the equations of two intersecting circles be

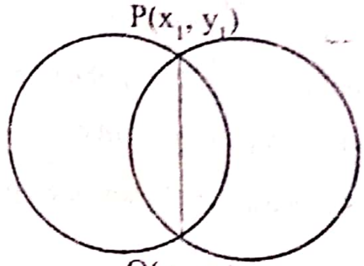
$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \ldots\ldots (1$$

And

$$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \ldots\ldots (2$$

Intersect at $P(x_1, y_1)$ and $Q(x_2, y_2)$

$P(x_1, y_1)$



$Q(x_2, y_2)$

Now we observe from the figure that $P(x_1, y_1)$ both given equations therefore, we get

$$x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1 = 0 \ldots\ldots (3)$$
$$x_1^2 + y_1^2 + 2g_2x_1 + 2f_2y_1 + c_2 = 0 \ldots\ldots (4)$$

Eqn. (3) – Eqn. (4)

$$2(g_1 - g_2)x_1 + 2(f_1 - f_2)y_1 + c_1 - c_2 = 0 \ldots$$

Again we observe from the above figure that $Q(x_2, y_2)$ lies on both circles

$$x_2^2 + y_2^2 + 2g_1x_2 + 2f_1y_2 + c_1 = 0 \ldots\ldots (6$$
$$x_2^2 + y_2^2 + 2g_2x_2 + 2f_2y_2 + c_2 = 0 \ldots\ldots (7$$

Eqn. 6 – eqn. 7

$$2(g_1 - g_2)x_2 + 2(f_1 - f_2)y_2 + c_1 - c_2 = 0 \ldots\ldots (8$$

From eqn. 5 and 8, it's evident that the $P(x_1, y_1)$and $Q(x_2, y_2)$ lie on $2(g_1 - g_2)x + \ldots f_2)y + c_1 - c_2 = 0$ which is a linear equation in y.

Note: While finding the equation of the common chord of two given intersecting circle, we fast express each equation in the form

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Example I

Determine the equation of the chord of the two intersecting circles $x^2 + y^2 - 4x - 2y - 31 = 0$ and $2x^2 + 2y^2 - 6x + 8y - 35 = 0$ and prove that the common chord is perpendicular to the line joining the two centres of the circles.

Solution

$x^2 + y^2 - 4x - 2y - 31 = 0$.......... (1)

$$2x^2 + 2y^2 - 6x + 8y - 35 = 0$$

$x^2 + y^2 - 3x + 4y - \frac{35}{2} = 0$........... (2)

Eqn. (1) – Eqn. (2)

$$-x - 6y - \frac{27}{2} = 0$$

$$-2x - 12y - 27 = 0$$

$$y = \frac{-2x}{12} - \frac{27}{12}$$

The equation of the chord:

The gradient of the chord is $\frac{-1}{6}$

Comparing $x^2 + y^2 - 4x - 2y - 31 = 0$ with $x^2 + y^2 + 2gx + 2fy + c = 0$

$$g = -2, f = -1$$

Center $(2, 1) = C_1$

Comparing $x^2 + y^2 - 3x + 4y - \frac{35}{2} = 0$ with $x^2 + y^2 + 2gx + 2fy + c = 0$

$$g = -\frac{3}{2}, f = 2$$

Center $(\frac{3}{2}, -2) = C_2$

The gradient joining the two centers

$$= \frac{-2 - 1}{\frac{3}{2} - 2}$$

$$= \frac{-3}{-\frac{1}{2}} = 6$$

Gradient of chord × gradient of line joining the two centres

$$6 \times \frac{-1}{6} = -1$$

The chord is perpendicular to the line joining the two centers

Example II

Show that the common chord of the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 4x - 2y - 4 = 0$ passes through the origin

Solution

$x^2 + y^2 = 4$

$x^2 + y^2 - 4 = 0$ .... (1)

$x^2 + y^2 - 4x - 2y - 4 = 0$ .... (2)

Eqn. (2) – eqn. (1)

$$4x + 2y = 0$$

$y = -2x$ is the equation of the common chord

At (0, 0), $x = 0, y = 0$

$$0 = -2 \times 0$$

$$0 = 0$$

The common chord passes through the origin.

Example III

Find the equation of the common chord of the circles

$x^2 + y^2 - 4x - 2y + 1 = 0$

$x^2 + y^2 + 4x - 16y - 10 = 0$

Solution

$x^2 + y^2 - 4x - 2y + 1 = 0$ .... (1)

$x^2 + y^2 + 4x - 16y - 10 = 0$ .... (2)

Eqn. (2) – eqn. (1)

$+8x - 14y - 11 = 0$

$$14y = 11 - 8x$$

Example IV

Find the point of intersection of the two circles

$x^2 + y^2 - 2x - 6y + 6 = 0$ and

$x^2 + y^2 - 6x - 6y + 14 = 0$

Solution

When we are finding the point of intersection, we first find the equation of the common chord and then we solve it simultaneously with one of the equations of the circles

$x^2 + y^2 - 2x - 6y + 6 = 0$ .... (1)

$x^2 + y^2 - 6x - 6y + 14 = 0$ .... (2)

Eqn. (1) – eqn. (2)

$$4x - 8 = 0$$

$$x = 2$$

$x = 2$ is the equation of the common chord

Substituting $x = 2$, in eqn. (1)

$$2^2 + y^2 - 2 \times 2 - 6y + 6 = 0$$

$$y^2 - 6y + 6 = 0$$

$$y = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 6}}{2 X 1}$$

$$y = \frac{6 \pm \sqrt{12}}{2}$$

$$y = 3 \pm \sqrt{3}$$

$$(2, 3 - \sqrt{3}) and\ (2, 3 + \sqrt{3})$$

The point of intersection of both circles is $(2, 3 - \sqrt{3}) and\ (2, 3 + \sqrt{3})$

Example V

Find the point of intersection of the circles

$x^2 + y^2 + 2x + 2y - 23 = 0$ and

$x^2 + y^2 - 10x - 7y + 31 = 0$

Solution

$x^2 + y^2 + 2x + 2y - 23 = 0$............ (1)

$x^2 + y^2 - 10x - 7y + 31 = 0$ .... (2)

Eqn (1) – Eqn (2)

$$12x + 9y - 54 = 0$$

$$4x + 3y = 18$$

$$y = \frac{18 - 4x}{3}$$

$$x^2 + \left(\frac{18 - 4x}{3}\right)^2 + 2x + 2\left(\frac{18 - 4x}{3}\right) - 23 = 0$$

$$x^2 + \frac{324 - 144x + 16x^2}{9} + 2x + \frac{36 - 8x}{3} - 23 = 0$$

$$9x^2 + 16x^2 - 144x + 18x + 108 - 24x + 324 - 207 = 0$$

$$25x^2 - 150x + 225 = 0$$

$$x^2 - 6x + 9 = 0$$

$$(x - 3)^2 = 0$$

$$x = 3$$

$$y = \frac{18 - 4 \times 3}{3}$$

$$y = 2$$

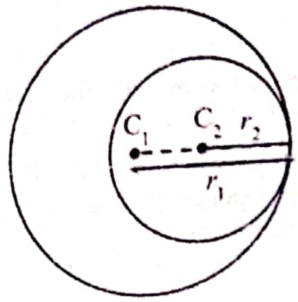
The point of intersection is (3, 2)

Types of intersecting circles

(1) Touching each other internally

Two circles touch each other internally distance between their centers is equal distance between their radii

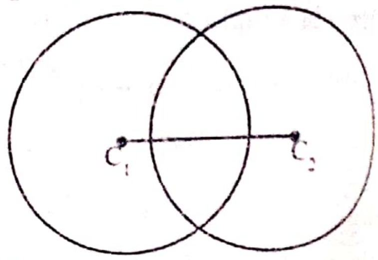
$$C_1C_2 = r_1 - r_2$$



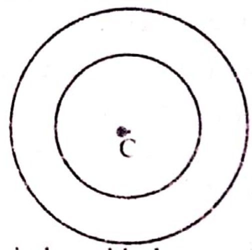
$$C_1C_2 = r_1 - r_2$$

(2) Circle intersect at two distinct points when

$$r_1 - r_2 < \overline{C_1C_2} < r_1 + r_2$$

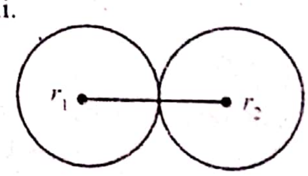


(3) Concentric circles



These are circles with the same center.

(4) Circle which touches each other externally distance between their centers is equal to the of their radii.



Example I

Prove that the circles $x^2 + y^2 - 10x - 7y + 31 = 0$ $x^2 + y^2 + 2x + 2y - 23 = 0$ touch each other exter

Solution

$$x^2 + y^2 - 10x - 7y + 31 = 0$$

$$x^2 + y^2 + 2x + 2y - 23 = 0$$

Comparing $x^2 + y^2 - 10x - 7y + 31 = 0$

$x^2 + y^2 + 2gx + 2fy + c = 0$

$$g = -5, f = -\frac{7}{2}, c = 31$$

Center $\left(5, \frac{7}{2}\right)$

Radius = $\sqrt{g^2+f^2-c}$

$$r=\sqrt{(-5)^2+\left(\frac{-7}{2}\right)^2-31}$$

$$r=\frac{5}{2}$$

Comparing $x^2+y^2+2x+2y-23=0$ with $x^2+y^2+2gx+2fy+c=0$

$$g=1, f=1, c=-23$$

Center (-1, -1)

$$radius=\sqrt{(-1)^2+(1)^2--23}$$

$$r=5$$

$$C_1\left(5,\frac{7}{2}\right) \text{ and } C_2(-1,-1)$$

$$C_1C_2=\sqrt{(5-{}^-1)^2+\left(\tfrac{7}{2}--1\right)^2}$$

$$=\sqrt{36+\frac{81}{4}}$$

$$C_1C_2=\frac{15}{2}$$

$$C_1C_2=7.5$$

$$r_1+r_2=7.5$$

Since $C_1C_2=r_1+r_2$

The two circles touch each other externally

Example II

Show that the circles

$x^2+y^2-6x-2y+1=0$ and

$x^2+y^2+2x-8y+13=0$ touch each other externally and find the equation of the tangent at the point of contact.

Solution

Considering the circle

$x^2+y^2-6x-2y+1=0$

comparing $x^2+y^2+2gx+2fy+c=0$ with

$x^2+y^2-6x-2y+1=0$

$\Rightarrow g=-3, f=-1$

Centre (C_1) (3, 1)

Radius (r_1) = $\sqrt{g^2+f^2-c}$

$$=\sqrt{(-3)^2+(-1)^2-1}$$

$$=3$$

Comparing $x^2+y^2+2x-8y+13$ with

$x^2+y^2+2gx+2fy+c=0$

$g=+4, f=-4$

Centre C_2 (-1, 4)

Radius $r_2=\sqrt{g^2+f^2-c}$

$$=\sqrt{1^2+(-4)^2-13}$$

$$=\sqrt{17-13}$$

$$=2$$

$$r_2=2$$

$r_2+r_2=5$

$C_1C_2=\sqrt{(3-{}^-1)^2+(1-4)^2}$

$=5$

Since $r_1+r_2=\overline{C_1C_2}$

$\Rightarrow$ The circles touch each other externally.

Example III

Show that the circles

$x^2+y^2-2x-2y-2=0$ and

$x^2+y^2-8x-10y+32=0$ touch externally and find the coordinates of the point.

Solution

Comparing $x^2+y^2-2x-2y-2=0$ with

$x^2+y^2+2gx+2fy+c=0$

$g=-1, f=-1, c=-2$

Centre $(-g, -f)$

$\Rightarrow$ Centre (1, 1)

C_1(1, 1)

Radius = $\sqrt{g^2+f^2-c}$

$$=\sqrt{(-1)^2+(-1)^2-{}^-2}$$

$$=2$$

Comparing $x^2+y^2-8x-10y+32=0$ with

$x^2+y^2+2gx+2fy+c=0$

$g=-4, f=-5, c=32$

Centre $(-g, -f)$

$\Rightarrow C_2$ (4, 5)

Radius = $\sqrt{g^2+f^2-c}$

$$=\sqrt{4^2+5^2-32}$$

$$=3$$

$$r_1+r_2=5$$

$\overline{C_1C_2}=\sqrt{(1-4)^2+(1-5)^2}$

$=5$

$\overline{C_1C_2} = 5$

Since $\overline{C_1C_2} = r_1 + r_2$, it implies the circles cut each other externally.

$x^2 + y^2 - 2x - 2y - 2 = 0$ .... (1)

$x^2 + y^2 - 8x - 10y + 32 = 0$ .... (2)

eqn (1) – eqn (2)

$\Rightarrow -6x + 8y - 34 = 0$

$3x + 4y - 12 = 0$

$y = \dfrac{17-3x}{4}$ .... (3)

Substituting eqn (3) in (1).

$$x^2 + \left(\frac{17-3x}{4}\right)^2 - 2x - 2\left(\frac{17-3x}{4}\right) - 2 = 0$$

$$x^2 + \frac{(17-3x)^2}{16} - 2x - \frac{(17-3x)}{2} - 2 = 0$$

$16x^2 + 289 - 102x + 9x^2 - 32x - 136 + 24x - 32 = 0$

$25x^2 - 110x + 121 = 0$

$$x = \frac{110 \pm \sqrt{(-110)^2 - 4 \times 25 \times 121}}{2 \times 25}$$

$$x = \frac{110 \pm 0}{50}$$

$$x = \frac{11}{5}$$

Substituting $x = \dfrac{11}{5}$ in eqn 3.

$$y = \frac{17 - \frac{33}{5}}{4}$$

$$y = \frac{85-33}{20}$$

$$y = \frac{52}{20}$$

$$y = \frac{13}{5}$$

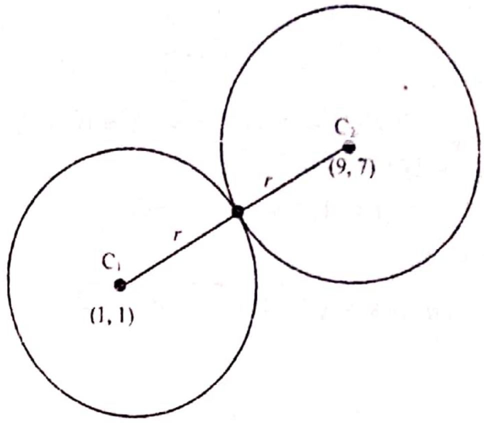
$$\left(\frac{11}{5}, \frac{13}{5}\right)$$

$\Rightarrow$ The point of contact is $\left(\dfrac{11}{5}, \dfrac{13}{5}\right)$

Example IV

Find the equation of two circles A(1, 1) and B(
which have equal radii and touch each other ex
Find the equations of the common chord of the
circles.

Solution.



For the two circles to touch each externally.

$C_1C_2 = r_2 + r_2$

$\Rightarrow \sqrt{(1-9)^2 + (1-7)^2} = r + r$

$\sqrt{100} = 2r$

$r = 5$

From $(x-a)^2 + (y-b)^2$

$\Rightarrow (x-1)^2 + (y-1)^2 = 5^2$

$x^2 - 2x + 1 + y^2 - 2y + 1 = 25$

$x^2 + y^2 - 2x - 2y - 23 = 0$

similarly,

$(x-9)^2 + (y-7)^2 = 5^2$

$x^2 - 18x + 81 + y^2 - 14y + 49 = 25$

$x^2 + y^2 - 18x - 14y + 105 = 0$

$\Rightarrow$ The equations of the two circles are

$x^2 - y^2 - 2x - 2y - 23 = 0$ .... (1)

$x^2 - y^2 - 18x - 14y - 105 = 0$ .... (2)

eqn (2) – eqn (1)

$\Rightarrow -16x - 12y + 128 = 0$

$4x + 3y = 32$

$\Rightarrow$ The equation of the common chord is $4x + 3y = 32$

Example

Two circles $x^2 + y^2 - 2x - 6y - 54 = 0$ and $x^2 + y^2 - 8x + 2y + 13 = 0$

Show that the one circle lies entirely inside th
other.

Solution

Comparing $x^2 + y^2 - 2x - 6y - 54 = 0$ with $x^2 + y^2$
$+ 2fy + c = 0$

$g = -1$ $\quad f = -3$

Centre (1, 3)

Radius $= \sqrt{g^2 + f^2 - c}$

$$= \sqrt{(-1)^2 + (-3)^2 - ^-54}$$

$$r_1 = 8$$

Comparing $x^2 + y^2 - 8x + 2y + 13 = 0$ with
$x^2 + y^2\ 2gx + 2fy + c = 0$

$\Rightarrow g = -4, \quad f = 1, \quad C = 13$

Centre (4, -1)

Radius $r_2 = \sqrt{(-4)^2 + 1^2 - 13}$

$$= 2$$

$$C_1C_2 = \sqrt{(1-4)^2 + (3 - ^-1)^2}$$

$$= \sqrt{25}$$

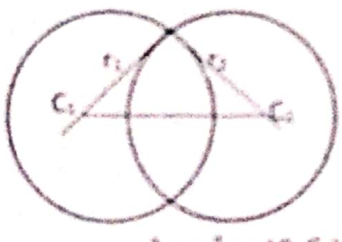
$$= 5$$

$$\overline{C_1C_2} < r_1 - r_2$$

$\Rightarrow$ One circle lies completely in another circle.

Orthogonal Circle

Two circles are said to be orthogonal if the tangents at their point of intersection cut at right angles as illustrated below.



$$r_1^2 + r_2^2 = (C_1C_2)^2$$

Example I

Prove that the circles $x^2 + y^2 + 4x - 2y - 11 = 0$ and $x^2 + y^2 - 4x - 8y + 11 = 0$ are orthogonal

Solution

Comparing $x^2 + y^2 + 2gx + 2fy + c = 0$ with $x^2 + y^2 + 4x - 2y - 11 = 0$

$$g = 2, f = -1, c = -11$$

Center $C_1(-2, 1)$

$$r_1 = \sqrt{2^2 + -1^2 - -11}$$

$$r_1 = 4$$

Similarly

Comparing $x^2 + y^2 - 4x - 8y + 11 = 0$ with

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

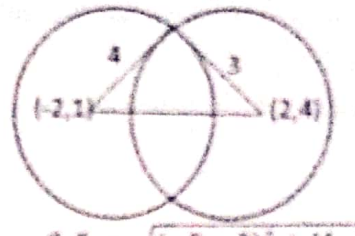
$$g = -2, f = -4, c = 11$$

Center $C_2(2, 4)$

$$r_2 = \sqrt{(-2)^2 + (-4)^2 - 11}$$

$$r_2 = \sqrt{4 + 16 - 11}$$

$$r_2 = 3$$



$$C_1C_2 = \sqrt{(-2-2)^2 + (1-4)^2}$$

$$\overline{C_1C_2} = 5$$

Since $r_1^2 + r_2^2 = \overline{C_1C_2}^2$

The two circles are orthogonal

Example II

Prove that the circles whose equations are
$x^2 + y^2 - 4x - 5 = 0$ and $x^2 + y^2 - 8x + 2y + 1 = 0$.
Cut orthogonally and find the equation of the common chord.

Solution

Comparing $x^2 + y^2 - 4y - 5 = 0$ with $x^2 + y^2 + 2gx + 2fy + c = 0$.

$\Rightarrow g = 0, \quad f = 2$

Centre (0, 2)

Radius $r_1 = \sqrt{g^2 + f^2 - c}$

$$= \sqrt{0^2 + (-2)^2 - ^-5}$$

$$= \sqrt{9}$$

$$= 3$$

Comparing $x^2 + y^2 - 8x + 2y + 1 = 0$ with $x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow g = -4, f = 1, c = 1$

Centre (4, -1)

Radius $r_2 = \sqrt{(-4)^2 + 1^2 - 1}$

$$= 4$$

$$C_1C_2 = \sqrt{(0-4)^2 + (2 - ^-1)^2}$$

$$= \sqrt{16 + 9}$$

$$= \sqrt{25}$$

$$= 5$$

$$\overline{C_1C_2}^2 = 25$$

$$r_1^2 + r_2^2 = 3^2 + 4^2$$
$$= 25$$

$$\overline{C_1C_2}^2 = 25$$

For two circles to be orthogonal to each other,

$$r_1^2 + r_2^2 = \overline{C_1C_2}^2$$

$\Rightarrow$ The circles are orthogonal.

Example (UNEB Question)

13. a) Form the equation of a circle that passes through the points $A(-1, 4)$, $B(2, 5)$ and $C(0, 1)$

b) The line $x + y = c$ is a tangent to the circle $x^2 + y^2 - 4y + 2 = 0$. Find the coordinates of the point of contact of the tangent for each value of c.

Solution

General equation of the circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

At $A(-1, 4)$,

$$-2g + 8f + c = -17 \text{.............................. (i)}$$

At $B(2, 5)$:

$$4g + 10f + c = -29 \text{.............................. (ii)}$$

At $C(0, 1)$:

$$2f + c = -1 \text{.............................. (iii)}$$

2 Eqn (i) + Eqn (ii)

$$10c = 50$$
$$c = 5$$

From Eqn (iii):

$$2f + 5 = -1$$
$$2f = -6$$
$$f = -3$$

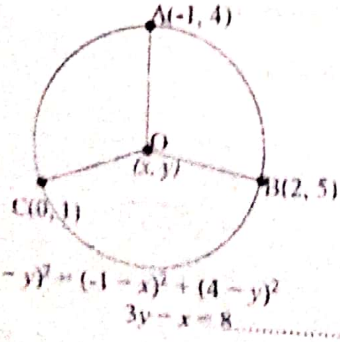
From Eqn (i)

$$-2g + 8(-3) + 5 = -17$$
$$-2g = 24 - 17 - 5$$
$$g = -1$$

Hence the equation of the circle is

$$\mathbf{x^2 + y^2 - 2x - 3y + 5 = 0}$$

Alternatively



$$(0 - x)^2 + (1 - y)^2 = (-1 - x)^2 + (4 - y)^2$$
$$3y - x = 8 \text{.............................. (i)}$$

Also

$$(0 - x)^2 + (1 - y)^2 = (2 - x)^2 + (5 - y)^2$$
$$2y + x = 7 \text{.......................}$$

Eqn (i) + Eqn (ii)

$$5y = 15$$
$$y = 3$$
$$3(3) - x = 8$$
$$x = 1$$

Centre of the circle = (1, 3) and the radius is

$$\sqrt{(0-1)^2 + (1-3)^2} = \sqrt{5}$$

Equation of the circle is $\mathbf{x^2 + y^2 - 2x - 6y + 5 = 0}$

b) $x^2 + y^2 - 4y + 2 = 0$,

And $y = c - x$

At the point of contact,

$$x^2 + (c - x)^2 - 4(c - x) + 2 = 0$$
$$2x^2 + (4 - 2c)x + (c^2 - 4c + 2) = 0$$

For tangency, $b^2 = 4ac$

$$(4 - 2c)^2 = 4 \times 2 \times (c^2 - 4c + 2)$$
$$4(2 - c)^2 = 8(c^2 - 4c + 2)$$
$$(2 - c)^2 = 2(c^2 - 4c + 2)$$
$$4 - 4c + c^2 = 2c^2 - 8c + 4$$
$$c^2 - 4c = c(c - 4) = 0$$

Either $c = 0$ or $c = 4$

If $c = 0$, $y = -x$

$$\Rightarrow \quad x^2 + x^2 + 4x + 2 = 0$$
$$2x^2 + 4x + 2 = 0$$
$$x^2 + 2x + 1 = (x + 1)^2 = 0$$
$$\Rightarrow \quad x = -1$$

Therefore $y = 1$

The point is $(-1, 1)$

If $c = 4$, $y = 4 - x$

$$(4 - x)^2 + x^2 - 4(4 - x) + 2 = 0$$
$$16 - 8x + x^2 + x^2 - 16 + 4x + 2 = 0$$
$$2x^2 - 4x + 2 = 0$$
$$x^2 - 2x + 1 = (x - 1)^2 = 0$$
$$x = 1, y = 3$$

The point is $(1, 3)$

MORE EXAMPLES ON CIRCLES

Example I (UNEB Question)

a) Find the equation of a circle which passes thr[ough the] points (5, 7), (1, 3) and (2, 2).

b) i) If $x = 0$ and $y = 0$ are tangents to the circle [$x^2 + y^2 +$] $2gx + 2fy + c = 0$, show that $c = g^2 = f^2$.

ii) Given that the line $3x - 4y + 6 = 0$ is also a ta[ngent to] the circle in (b) (i) above, determine the equa[tion of the] circle lying in the first quadrant. (06 m[arks])

Solution

(a) The equation of the circle is given by:

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Substituting for (5, 7),

$$25 + 49 + 10g + 14f + c = 0$$
$$74 + 10g + 14f + c = 0$$
$$10g + 14f + c = -74 \text{ ....(i)}$$

Substituting for (1, 3)

$1 + 9 + 2g + 4f + c = 0$

$2g + 6f + c = -10$ ....(ii)

Substituting for (2, 2)

$4 + 4 + 4g + 4f + c = 0$

$4g + 4f + c = -8$.......................... (iii)

Eqn (i) – Eqn (ii)

$6g + 10f = -64$

$g + 6f = -8$................................ (iv)

Eqn (i) – Eqn (iii)

$6g + 10f = -66$

$3g + 5f = -33$................................ (v)

3 Eqn (iv) – Eqn (v)

$3g + 3f = -24$

$3g + 5f = -33$

$-2f = 9$

$$f = \frac{-9}{2}$$

Substituting for f in Eqn (iv)

$$g = \frac{9}{2} - 8 = {}^{-7}\!/\!_{2}$$

Substituting for f and g in Eqn (iii)

$$4\left({}^{-7}\!/\!_{2}\right) + 4\left(\frac{-9}{2}\right) + c = -8$$

$-28 - 36 + 2c = -16$

$2c = 64 - 16$

$$c = \frac{48}{2} = 24$$

The equation of the circle is $x^2 + y^2 - 7x - 9y + 24 = 0$

b) Given $x^2 + y^2 + 2gx + 2fy + c = 0$

When $y = 0$, $\quad x^2 + 2gx + c = 0$

For tangency, $b^2 = 4ac$

$(2g)^2 = 4c$

$4g^2 = 4c$

$g^2 = c$

When $x = 0$, $y^2 + 2fy + c = 0$

For tangency, $b^2 = 4ac$

$(2f)^2 = 4c$

$4f^2 = 4c$

$f^2 = c$

Hence $c = g^2 = f^2$

ii) From the line $3x - 4y + 6 = 0$

$4y = 3x + 6$

$$y = \frac{3x+6}{4}$$

$$y^2 = \frac{(3x+6)^2}{16}$$

Substituting for y and y^2 in the equation

$x^2 + y^2 + 2gx + 2fy + c = 0$

$$x^2 + \frac{(3x+6)^2}{16} + 2fx + 2f\left(\frac{3x+6}{4}\right) + f^2 = 0$$

$16x^2 + (3x + 6)^2 + 32fx + 8f(3x + 6) + 16f^2 = 0$

$16x^2 + 9x^2 + 36x + 36 + 32fx + 24fx + 48f + 16f^2 = 0$

$25x^2 + (36 + 56f)x + (36 + 48f + 16f^2) = 0$

For tangency, $\quad b^2 = 4ac$

$(36 + 56f)^2 = 4 \times 24(36 + 48f + 16f^2)$

$(36 + 56f)^2 = 100(36 + 48f + 16f^2)$

By opening brackets and simplifying we obtain

$1536f^2 - 768f - 2304 = 0$

$2f^2 - f - 3 = 0$

$2f^2 - 3f + 2f - 3 = 0$

$f(2f - 3) + 1(2f - 3) = 0$

$(2f - 3)(f + 1) = 0$

Either $\quad 2f - 3 = 0$

$2f = 3$

$f = 3/2$

Or $\quad f + 1 = 0$

$f = -1$

Now $\quad f = g$

$\Rightarrow \quad g = 3/2$ or -1

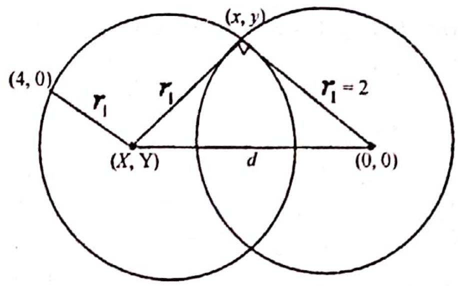
Centre of the circle is $(-g, -f)$. Since it is in the first quadrant, then the centre is (1, 1)

But $c = g^2 = f^2 = 1$

The equation of the circle is $x^2 + y^2 - 2x - 2y + 1 = 0$

Example II

A circle passing through the point (4, 0) is orthogonal to the circle $x^2 + y^2 = 4$. Find the locus of the variable circle.



For the two circles to be orthogonal.

$$r_1^2 + r_2^2 = d^2$$

$$r_1 = \sqrt{(X-x)^2 + (Y-y)^2}$$

$$= \sqrt{(X-4)^2 + (Y-0)^2}$$

$$r_1^2 = (X-4)^2 + Y^2$$

$$r_2^2 = 4$$

From $r_1^2 + r_2^2 = d^2$

$(X - 4)^2 + Y^2 + 4 = (X - 0)^2 + (Y - 0)^2$

$X^2 - 8X + 16 + Y^2 + 4 = X^2 + Y^2.$

$-8X + 20 = 0$

$20 = 8X$

$X = \frac{5}{2}$

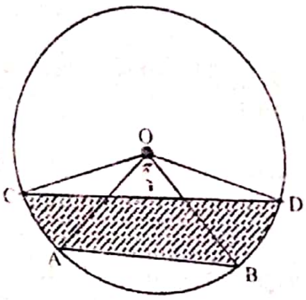
Example III

A circle has centre O and radius r. Two parallel chords AB and CD are on the same side of O. The angle AOB is $\frac{\pi}{3}$ and angle COD is $\left(\frac{\pi}{3}+2\theta\right)$.

Show that the area of the part of the circle between AB and CD is

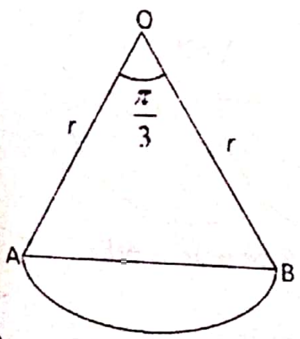
$$\frac{1}{4}r^2\left[\left(4\theta+\sqrt{3}\right)-2\sin\left(\frac{\pi}{3}+2\theta\right)\right]$$

Solution



$\angle COD = \frac{\pi}{3}+2\theta$

Extracting the sector AOB we have



Area of the sector

$$= \frac{\theta}{360}\times\pi r^2$$

$$360° = 2\pi r^2$$

$\Rightarrow$ Area of the sector $= \frac{\theta}{2\pi}\times\pi r^2$

Area of the sector OAB $= \frac{\frac{\pi}{3}}{2\pi}\times\pi r^2$

$$= \frac{1}{6}\pi r^2$$

(Area of the triangle OAB) $= \frac{1}{2}r.r\sin\left(\frac{\pi}{3}\right)$

$$= \frac{1}{2}r^2\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{4}\sqrt{3}r^2$$

Area of the segment formed by the chord AB the circle

$$= \frac{1}{6}\pi r^2 - \frac{1}{4}\sqrt{3}r^2$$

Area of the sector ODC

$$= \frac{\frac{\pi}{3}+2\theta}{2\pi}\times\pi r^2$$

$$= \left(\frac{\pi}{6}+\theta\right)r^2$$

$$= \frac{\pi}{6}r^2+\theta r^2$$

Area of the triangle ODC

$$\frac{1}{2}r.r\sin\left(\frac{\pi}{3}+2\theta\right)$$

$$\frac{1}{2}r^2\sin\left(\frac{\pi}{3}+2\theta\right)$$

Area of segment formed by chord CD and the circle

$$\frac{\pi r^2}{6}+\theta r^2-\frac{1}{2}r^2\sin\left(\frac{\pi}{3}+2\theta\right)$$

The area of the part of circle formed between and CD.

$$\left[\frac{\pi r^2}{6}+\theta r^2-\frac{1}{2}r^2\sin\left(\frac{\pi}{3}+2\theta\right)\right]-\left[\frac{\pi r^2}{6}-\frac{1}{4}\right.$$

$$= \theta r^2-\frac{1}{2}r^2\sin\left(\frac{\pi}{3}+2\theta\right)+\frac{1}{4}\sqrt{3}r^2$$

$$= \frac{4\theta r^2-2r^2\sin\left(\frac{\pi}{3}+2\theta\right)+\sqrt{3}r^2}{4}$$

$$= \frac{1}{4}r^2\left[\left(4\theta+\sqrt{3}\right)-2\sin\left(\frac{\pi}{3}+2\theta\right)\right]$$

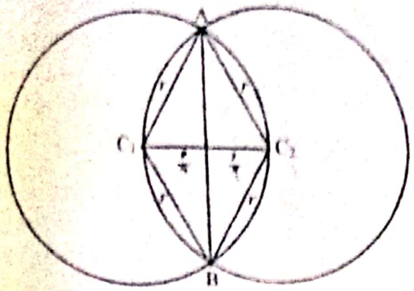
Example IV

A circle of radius r is drawn with it's centre circumference of another circle of radius r.

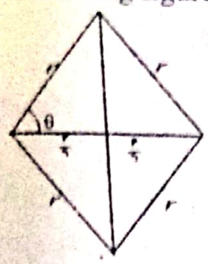
that the area common to both circles is,

$$2r^2\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)$$

Solution



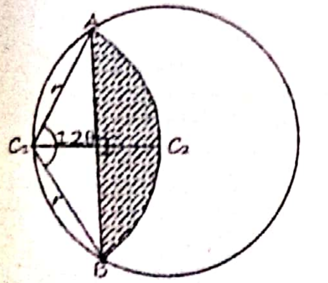
Extracting figure AC_1BC_2 we have



$$\cos\theta = \frac{\frac{r}{2}}{r} = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ = \frac{\pi}{3}$$

Area of the shaded part formed by AB and the circle with centre C_1.



Area of the shaded part formed by AB and the circle with centre C_1.
(Area of sector C_1AC_2B) – (Area of triangle C_1AB)

$$\frac{120}{360}\times\pi r^2 - \frac{1}{2}r^2\left(\frac{\sqrt{3}}{2}\right)$$

$$\frac{1}{3}\pi r^2 - \frac{\sqrt{3}}{4}r^2$$

The area common to both circles

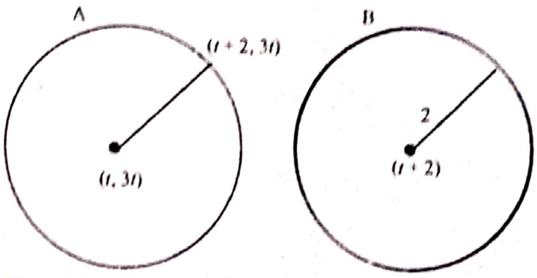
$$2\left(\frac{1}{3}\pi r^2 - \frac{1}{4}r^2\sqrt{3}\right)$$

$$= 2r^2\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)$$

Example V

A circle A passing through the point $(t + 2, 3t)$ has it's centre at $(t, 3t)$. Another circle B of radius 2 units has it's centre at $(t + 2, 3t)$.
Determine the equations of the circles A and B in terms of t.
If $t = 1$. Find the point of intersection of the two circles.
Show that the common area of intersection of two circles A and B is $8\left(\frac{\pi}{4}-\frac{\sqrt{3}}{4}\right)$

Solution



Consider the equation of the circle
$(x - a)^2 + (y - b)^2 = r^2$
With centre (a, b) and radius r.
The equation of the circle A is
$(x - t)^2 + (y - 3t)^2 = r^2$
Where

$$r = \sqrt{((t+2)-t)^2 + (3t-3t)^2}$$

$$r = \sqrt{4+0}$$

$r = 2$
$\Rightarrow (x - t)^2 + (y - 3t)^2 = r^2$
$x^2 - 2xt + t^2 + y^2 - 6yt + 9t^2 = 4$
$x^2 + y^2 - 2xt - 6yt + 10t^2 - 4 = 0$
The equation of circle B is
$\Rightarrow [x - (t + 2)]^2 + (y - 3t)^2 = 2^2$
$x^2 - 2x(t + 2) + (t + 2)^2 + y^2 - 6yt + 9t^2 = 4$
$x^2 + y^2 - 2(t + 2)x - 6yt + (t + 2)^2 + 9t^2 = 4$
the equation of the circle A at $t = 1$

$x^2 + y^2 - 2x - 6y + 6 = 0$ .... (1)

The equation of the circle B at $r = 1$

$x^2 + y^2 - 6x - 6y + 14 = 0$ .... (2)

eqn(1) − eqn (2)

$$4x - 8 = 0$$

$$4x = 8$$

$$x = 2$$

Substituting $x = 2$ in eqn (1)

$$\Rightarrow (2)^2 + y^2 - 2(2) - 6y + 6 = 0$$

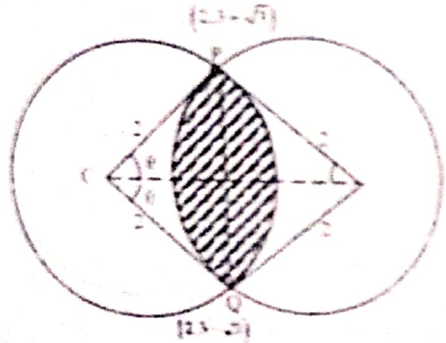
$$y^2 - 6y + 6 = 0$$

$$y = \frac{6 \pm \sqrt{6^2 - 4 \times 1 \times 6}}{2 \times 1}$$

$$y = \frac{6 \pm \sqrt{36 - 24}}{2}$$

$$y = \frac{6 \pm 2\sqrt{3}}{2}$$

$$y = 3 \pm \sqrt{3}$$



Length of the common chord

$$PQ = \sqrt{(2-2)^2 + \left[\left(3+\sqrt{3}\right) - \left(3-\sqrt{3}\right)\right]^2}$$

$$= \sqrt{(2\sqrt{3})^2}$$

$$= 2\sqrt{3}$$

$$PO = \frac{1}{2} RQ$$

$$= \frac{1}{2}\left(2\sqrt{3}\right)$$

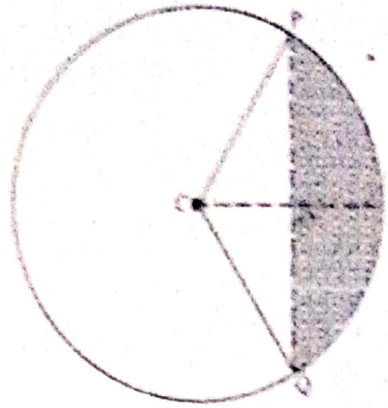
$$= \sqrt{3}$$

$$\sin\theta = \frac{\sqrt{3}}{2}$$

$$\theta = \left(\sin^{-1}\frac{\sqrt{3}}{2}\right)$$

$$\theta = 60^\circ$$

$$\angle PCQ = 2\theta = 120^\circ$$



Area of the shaded part

= (Area of the sector) − (Area of triangle

$$= \frac{120}{360} \times \pi\left(2^2\right) - \frac{1}{2} \times 2 \times 2\sin\theta$$

$$= \frac{4}{3}\pi - \frac{2\sqrt{3}}{2}$$

$$= 4\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$$

But the common area of the two circles i

the shaded area in eqn (1).

$$A = 2 \times 4\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$$

$$A = 8\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$$