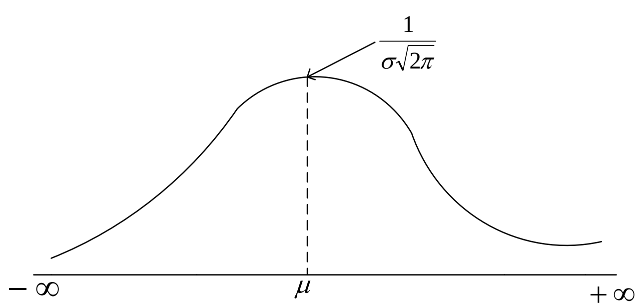
TOPIC 6: NORMAL DISTRIBUTION

A normal distribution is one of the most important distributions in statistics. Many measured quantities in natural Sciences follow a normal distribution. It is a special kind of continuous random variable with a pdf given as $f(x) =$

$$\begin{cases} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} & ; -\infty < x < +\infty \\ 0 & ; elsewhere \end{cases}$$

If X is normally distributed then we write $X \sim N(\mu, \sigma^2)$ where $\mu = mean\ and\ \sigma$ is the standard deviation. The distribution above gives the variance σ^2 rather than the standard deviationσ. The general curve of the normal distribution



The normal curve has the following features.

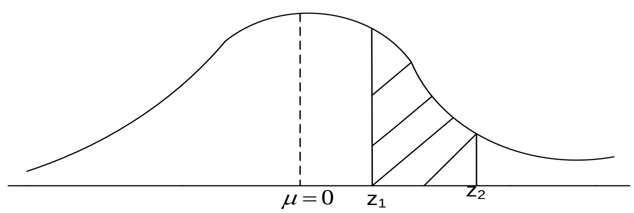
- It is bell shaped
- It is symmetric about the mean, μ
- It extends from $-\infty$ to $+\infty$
- The maximum value of f(x) is $\frac{1}{\sigma\sqrt{2\pi}}$
- The total area under the curve is 1
- The x-axis is an asymptotic to the curve

Finding probabilities

In order to determine probability $P(x_1 \le X \le x_2)$, It would be required to compute the integral, $\int_{x_1}^{x_2} f(x)dx$. But because of the difficulty experienced in evaluating this we can reduce this amount of work by using a standard normal variable Z where Z=$\frac{x-\mu}{\sigma}$. This curve will still follow the same shape but now symmetric about the y-axis so that its mean is 0 and the standard deviation is 1

Therefore if X~$N(\mu,\sigma^2)$ then Z~$N(0,1^2)$. For us to determine, $P(x_1 \le X \le x_2)$, we shall have to standardize x_1 and x_2. Here we have

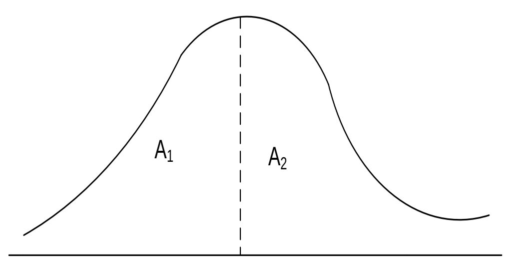
$z_1 = \frac{x_1-\mu}{\sigma}$ and $z_2 = \frac{x_2-\mu}{\sigma}$ such that $P(x_1 \le X \le x_2) = P(z_1 \le z \le z_2)$



Such that $P(x_1 \le x \le x_2) = P(z_1 \le z \le z_2)$

<u>**Reading the normal distribution tables (How to determine probability)**</u>

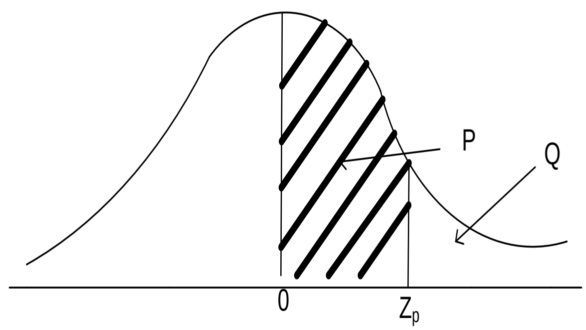
The value of the normal distribution is first standardized using $z = \frac{x-\mu}{\sigma}$



From the curve above the area under the curve is 1

$A_1 + A_2 = 1$ OR $A_1 = A_2 = 0.5$

Note: The area on the left of the line of symmetry is 0.5 and that on the right is also 0.5. We have to use strictly UNEB tables



Note:

i) **The shaded area shows** $P = P(0 < z < z_p)$

ii) $Q = P(z > z_p) = 0.5 - P(0 < z < z_p)$

Example 1:

Given that the standard normal variable z, find:

i) $P(z < 1.09)$

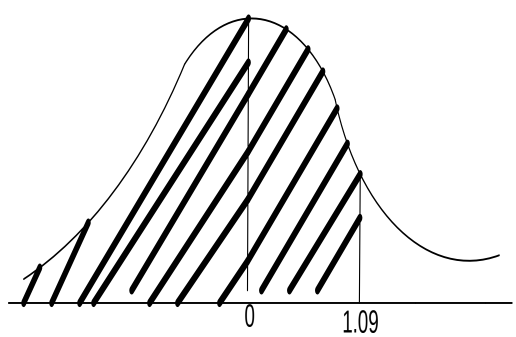
ii) $P(z < -0.22)$

iii) $P(z > -1.54)$

iv) $P(-2.34 < z < 2.34)$

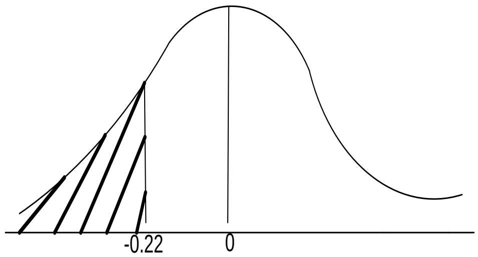
Soln:

i) Since the variable is already standardized, no need to standardize again.

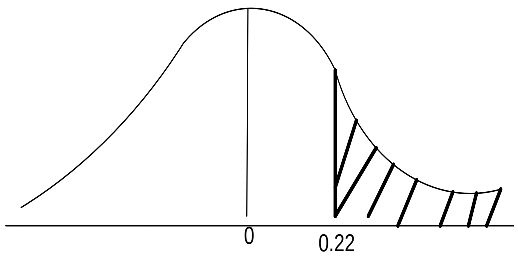


$$P(z < 1.09) = 0.5 + P(0 < z < 1.09) = 0.5 + 0.3621 = 0.8621$$

ii) $P(z < -0.22)$

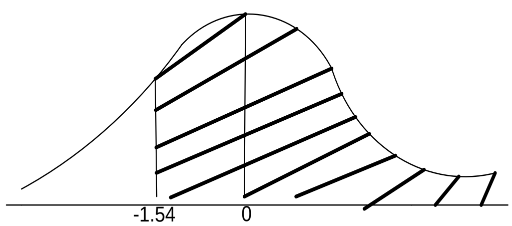


By symmetry $P(z<-0.22)=P(z>0.22)$



$$P(z>0.22)=0.5-P(0<z<0.22)=0.5-0.0871=0.4129$$

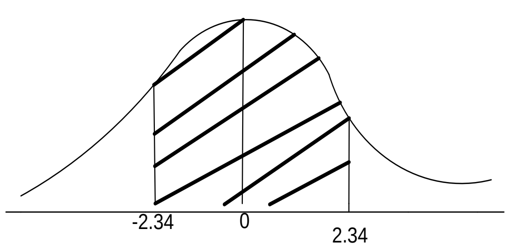
iii) $P(z>-1.54)$



$P(z>-1.54)=0.5+P(-1.54<z<0)=0.5+P(0<z<1.54)$, by symmetry

$=0.5+0.4382=0.9382$

iv) $P(-2.34<z<2.34)$



$$P(-2.34<z<2.34)=P(-2.34<z<0)+P(0<z<2.34)$$

$= P(0 < z < 2.34) + P(0 < z < 2.34)$ By symmetry

$= 2P(0 < z < 2.34) = 2(0.4904) = 0.9808$

Example 2:

A random variable X is normally distributed with mean 3 and variance 9.

Determine:

i) $P(x \geq 0)$

ii) $P(2 < x < 5)$

Soln:

i) $X \sim N(\mu, \sigma^2) \Rightarrow X \sim N(3,9), \mu = 3, and, \sigma = \sqrt{9}$

$$P(x > 0) = P\left(z > \frac{0-3}{\sqrt{9}}\right) = P(Z > -1) = 0.5 + 0.3413 = 0.8413$$

ii) $$P(2 < X < 5) = P\left(\frac{2-3}{\sqrt{9}} < Z < \frac{5-3}{\sqrt{9}}\right) = P(-0.333 < Z < 0.667)$$

$= 0.1304 + 0.2477 = 0.3781$

Example 3:

The life time of batteries produced by a certain Company is approximately normally distributed with mean time of 160 hours and Variance of 900 $hours^2$.

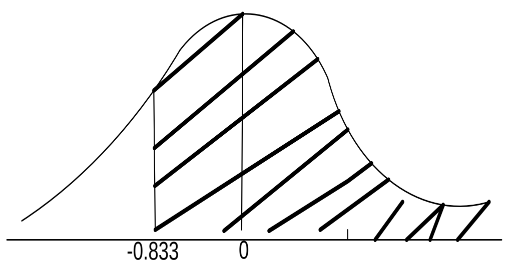
a) Calculate the:
   i) Probability that the battery chosen will run for not less than 135 hours.
   ii) Percentage of batteries with in a life time between 150 and 180 hours.

b) If a radio takes three of these batteries and requires all of them to be working , what is the probability that the radio will run for at least 135 hours.

Soln:

a) $\mu = 160, \sigma^2 = 900$ Let $X \sim N(160,900)$

i) $P(X \geq 135) = P\left(Z \geq \frac{135-160}{\sqrt{900}}\right) = P(Z \geq -0.833)$



$= 0.5 + 0.2975 = 0.7975$

ii) $P(150 < X < 180) = P\left(\frac{150-160}{30} < Z < \frac{180-160}{30}\right)$

$P(-0.333 < Z < 0.667) = 0.1304 + 0.2477 = 0.3781$

The percentage required $= 0.3781 \times 100 = 37.81\%$

b) It gives a binomial distribution

$n = 3, p = P(X \geq 135) = 0.7975, q = 0.2025, x = 3$

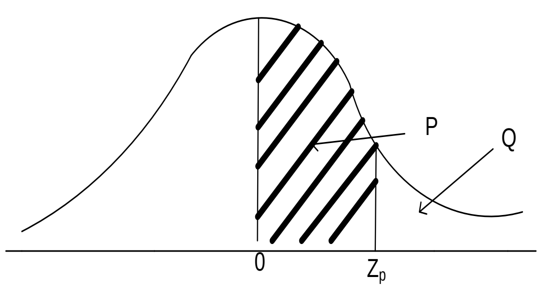
Then $P(X = 3) = \binom{3}{3}(0.7975)^3(0.2025) = 0.50737$

Use of tables to find Z value when given probability

We may be interested to find the Z-value when given probability. It is required to note that the Z-value may be positive or negative.

The value Z can be read off from the normal in two ways.

a) Using critical points from the normal distribution
Given the distribution below

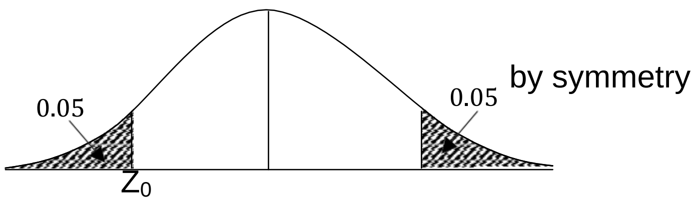


Below is an extract from the table for critical points of the normal distribution;

P	Q	Z
0.00	0.50	0.000
0.05	0.45	0.126
0.10	0.40	0.252

For $P(0<Z<Z_0)=0.05$, here P=0.05 and Q=0.45 and therefore the Z-value is 0.126

For $P(Z<Z_0)=0.05$, here from the tables, P=0.05 and Q=0.45 by symmetry. But since 0.05<0.5, the Z-value is negative



The Z-value, Z_0=-0126. But since 0.05<0.5, therefore the value is negative

b) Using cumulative normal distribution, P(Z). Here we find the anti-Z of probability

Example:

Suppose $P(0<Z<Z_0)=0.05$. From the tables, 0.05 we look for exactly this value but if it does not exist we the immediate less. From the tables 0.05 lies between 0.0478 and 0.0517, the immediate less is 0.0478 and it corresponds to 0.120. So when we consider the row for ADD, the exact value we could add to 0.0478 to get 0.05 is 22, but it is not there so we can roughly consider adding 24 which corresponds to 6

Hence the Z-value corresponding to 0.05 is 0.0502

Worked examples:

Example 4:

A random Variable X is normally distributed where $P(X>9)=0.9192$ and $P(X<11)=0.7580$. Find the;

i) The values of the mean and standard deviation

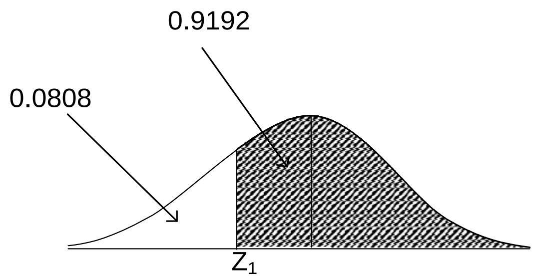
ii) $P(X>10)$

Solution:

i) P(X>9)=0.9192

$$P\left(Z > \frac{9-\mu}{\sigma}\right) = 0.9192 \text{ Let } Z_1 = \frac{9-\mu}{\sigma}$$

$$P(Z > Z_1) = 0.9192$$



$$0.5 + P(Z_1 < Z < 0) = 0.9192$$

$$P(Z_1 < Z < 0) = 0.4192$$

$$Z_1 = -1.400$$

$$\frac{9-\mu}{\sigma} = -1.4$$

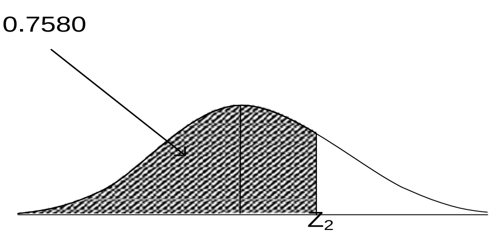
$$9 = -1.4\sigma + \mu \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

$$P(X < 11) = 0.7580$$

$$P(Z < \frac{11-\mu}{\sigma}) = P(Z < Z_2) = 0.7580$$

$$0.5 + P(0 < Z < Z_2) = 0.7580$$

$$P(0 < Z < Z_2) = 0.2580$$



$$Z_2 = 0.700$$

$$\frac{11-\mu}{\sigma} = 0.7$$

$$11 = 0.7\sigma + \mu \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

From (ii)-(i) we get $\sigma = 0.9524$

From equation (i) $\mu = 9 + 1.4\sigma = 10.3334$

ii) $$P(X > 10) = P\left(Z > \frac{10 - 10.3334}{0.9524}\right) = P(Z > 0.350)$$
$$= 0.5 - P(0 < Z < 0.350) = 0.3632$$

Example 5:

The speeds of cars passing a certain point on a motor way can be taken to be normally distributed. Observations show that of cars passing the point 95% are travelling at less than 85km/hr and 10% are travelling at at less than 55km/hr. Find the mean and standard deviation of the speeds of the cars.

Solution:

Let X be the speeds of cars

$$P(X < 85) = 0.95,\ P\left(Z < \frac{85 - \mu}{\sigma}\right) = P(Z < Z_1) = 0.95$$

$$0.5 + P(0 < Z < Z_1) = 0.95$$
$$P(0 < Z < Z_1) = 0.45$$
$$Z_1 = 1.645$$

$$\frac{85 - \mu}{\sigma} = 1.645$$
$$85 - \mu = 1.645\sigma \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

$$P(X < 55) = P\left(Z < \frac{55 - \mu}{\sigma}\right) = P(Z < Z_2) = 0.1$$
$$0.5 - P(Z_2 < Z < 0) = 0.1$$
$$P(Z_2 < Z < 0) = 0.4$$
$$Z_2 = -1.282$$
$$\frac{55 - \mu}{\sigma} = -1.282$$
$$55 = -1.282\sigma + \mu \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

Equation (i) – (ii) gives;

$$\sigma = 10.249$$

From equation 1

$\mu = 85 - 1.645\sigma = 68.140$

Normal approximation to binomial distribution

The normal approximation is used to approximate a binomial distribution when;

i) The number of trials of a binomial experiment is large i.e n>20
ii) The probability of a success is constant and close to 0.5

If x is a random variable following a binomial distribution, then $E(X) = np$ and Var(X) =npq

The Z- value corresponding to such a value of X is

Z= $\frac{(X \pm 0.5) - np}{\sqrt{npq}}$ where 0.5 is a continuity correction value. i.e It is a value that transforms a distinct variable into a continuous variable.

Example:

Between two values inclusive $\Rightarrow P(X_1 \le X \le X_2) \Rightarrow P(X_1 - 0.5 \le X \le X_2 + 0.5)$

More than $\Rightarrow P(X > X_1) \Rightarrow P(X > X_1 + 0.5)$

At most $\Rightarrow P(X \le X_1) \Rightarrow P(X \le X_1 + 0.5)$

At least $\Rightarrow P(X \ge X_1) \Rightarrow P(X - 0.5)$

Between $\Rightarrow P(X_1 < X < X_2) \Rightarrow P(X_1 + 0.5 < X < X_2 - 0.5)$

Example 6:

A fair coin is tossed 100 times. Determine the probability of obtaining

i) At most 53 heads
ii) At least 45 heads
iii) Between 46 and 54 heads

Solution

Let X be the variable for obtaining a head

$p=\frac{1}{2}$ and $q=\frac{1}{2}$ $\mu=np=100\times\frac{1}{2}=50, \sigma^2=100\times\frac{1}{2}\times\frac{1}{2}=25$

i) $P(X\leq 53)\Rightarrow P(X\leq 53.5)=P(Z\leq\frac{53.5-50}{5})=P(Z\leq 0.7)=0.5+0.2580=0.7580$

ii) $P(X\geq 45)\Rightarrow P(X\geq 44.5)=P\left(Z\geq\frac{44.5-50}{5}\right)=P(Z\geq -1.1)$

$=0.5+0.3643=0.8643$

iii) $P(46<X<54)\Rightarrow P(46.5<X<53.5)=P\left(\frac{46.5-50}{5}<Z<\frac{53.5-50}{5}\right)$

$P(-0.7<Z<0.7)=2\times 0.258=0.516$

SAMPLING AND ESTIMATION

In statistical inquiry, we often need information about a particular group. This group is known as target population. It could be small large or even infinite. Population does not necessarily mean people, but can mean cans of soft drinks, pages in book, etc

Sample statistics:

When you are trying to find out information about a population, it seems sensible to take random samples and then consider values obtained from them to carry out some analysis.

Distribution of sample mean of a normal distribution:

Suppose $X_1, X_2,\ldots\ldots, X_n$ of a sample size, n, is taken from a large population whose mean is μ and variance σ^2, then the distribution of the sample mean $\overline{X}$ is said to be normally distributed with mean μ and variance $\frac{\sigma^2}{n}$, hence the sample standard deviation is $\frac{\sigma}{\sqrt{n}}$

If X~ $N(\mu, \sigma^2)$, then $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$, the Z-value, Z= $\frac{\bar{X} - \mu}{\sigma}$

Example 7:

The drying time of newly manufactured pain is normally distributed with mean 110.5 minutes and standard deviation of 12 minutes.

a) Find the probability that the paint dries between 104 and 109 minutes
b) If a random sample of 20 tins of paint was taken, find the probability that the mean drying time of the sample is between 108 and 112 minutes

Solution:

a) $P(104 < X < 109) = P\left(\frac{104 - 110.5}{12} < Z < \frac{109 - 110.5}{12}\right)$

$= P(-0.542 < Z < -0.125) = P(0 < Z < 0.542) - P(0 < Z < 0.125)$

=0.2061-0.0498=0.1563

b) $\bar{X} \sim N\left(110.5, \frac{12^2}{20}\right)$

$$P(100 < \bar{X} < 112) = P\left(\frac{108 - 110.5}{\frac{12}{\sqrt{20}}} < Z < \frac{112 - 110.5}{\frac{12}{\sqrt{20}}}\right) = P(-0.932 < Z < 0.559)$$

=0.3243+0.2119=0.5362

UNBIASED ESTIMATES OF POPULATION PARAMETERS

Suppose we don't know the value of a particular parameter of the distribution e.g mean, variance. It seems sensible that you will take a random sample from the distribution and use it to estimate the value of the unknown parameter of the population. This estimate is unbiased if the average of a large number of values taken is the same way the true value.

Ways of estimating unbiased estimates

a) **Point estimate of the population parameter.**

The best unbiased estimate of the population mean is $\hat{\mu} = \bar{X} = \frac{\sum X}{n}$

The best unbiased estimate of the population variance is $\hat{\sigma}^2$ where $\hat{\sigma}^2 = \frac{n}{n-1} S^2$ where S^2 is the sample variance.

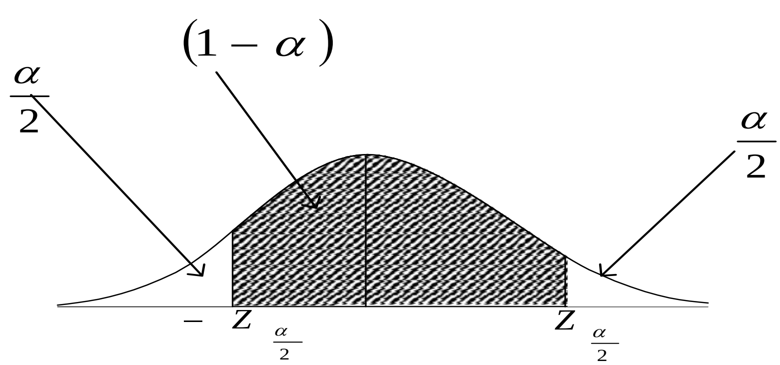
$$\hat{\sigma}^2 = \frac{n}{n-1}\left(\frac{\sum\left(X-\bar{X}^2\right)}{n}\right) = \frac{n}{n-1}\left(\frac{\sum X^2}{n} - \left(\frac{\sum X}{n}\right)^2\right) = \frac{n}{n-1}\left(\frac{\sum fX^2}{\sum f} - \left(\frac{\sum fX}{\sum f}\right)^2\right)$$

b) **Interval estimates**

Another way of using a sample value to give a good idea of an unknown population parameter is to construct an interval known as the confidence interval. A confidence interval is an interval that has a specified probability of including the unknown parameter.

Note: The wider the confidence interval the more we are confident that the given interval contains the unknown parameter.

Recall that for a random sample of size, n, such that $X \sim N(\mu, \sigma^2)$, $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$. For 95% confidence interval, you need to find the values of Z between which the 95% distribution lies where Z= $\frac{\bar{X}-\mu}{\sigma}$



Where α, is the degree of confidence interval.

The $(1-\alpha)\%$ confidence interval for mean μ is given by, Confidence interval $=\left[\overline{X}-Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \overline{X}+Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right]$,

Where $\overline{X}-Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}$ =lower limit

$\overline{X}+Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}$ =upper limit

Note:

- The width of the confidence interval is given by $2\left(Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$ and $\left(\frac{\sigma}{\sqrt{n}}\right)$ is the standard error of the mean.
- If the variance of the population is not given, but we have that of the sample we can use the unbiased estimate of the population variance, $\hat{\sigma}^2$.

<u>Assignment 5.1.8:</u>

1. The marks scored in an examination were found to be normally distributed with mean of 53.9 and standard deviation of 16.2. 20% of the candidates who sat the examinations failed. Find the pass mark for the examination.
2. Given that X~N$\left(\mu,\sigma^2\right)$ such that $P(X>200)=0.01$ and $P(165<X<200)=0.76$. Determine the values of μ and σ.
3. A biased die with faces labeled 1,2,2,3,5 and 6 is tossed 45 times. Calculate the probability that 2 will appear:
   i) More than 18 times
   ii) Exactly 11 times
4. On average 15% of all boiled eggs sold in a restaurant have cracks. Find the probability that a sample of 300 boiled eggs will have more than 50 cracked eggs.

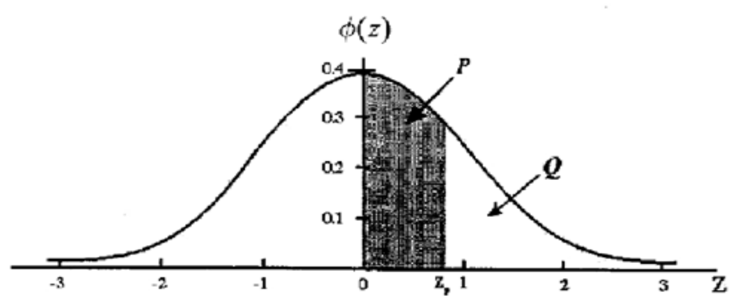
CUMULATIVE NORMAL DISTRIBUTION $P(z)$

Z	0	1	2	3	4	5	6	7	8	9	ADD								
											1	2	3	4	5	6	7	8	9
0.0	0.0000	0040	0080	0120	0160	0199	0239	0279	0319	0359	4	8	12	16	20	24	28	32	36
0.1	0.0398	0438	0478	0517	0557	0596	0636	0675	0714	0753	4	8	12	16	20	24	28	32	36
0.2	0.0793	0832	0871	0910	0948	0987	1026	1064	1103	1141	4	8	12	15	19	22	27	31	35
0.3	0.1179	1217	1255	1293	1331	1368	1406	1443	1480	1517	4	8	11	15	19	22	26	30	34
0.4	0.1554	1591	1628	1664	1700	1736	1772	1808	1844	1879	4	7	11	14	18	22	25	29	32
0.5	0.1915	1950	1985	2019	2054	2088	2123	2157	2190	2224	3	7	10	14	17	21	24	27	31
0.6	0.2257	2291	2324	2357	2389	2422	2454	2486	2517	2549	3	6	10	13	16	19	23	26	29
0.7	0.2580	2611	2642	2673							3	6	9	12	15	19	22	25	28
					2704	2734	2764	2794	2823	2852	3	6	9	12	15	18	21	24	27
0.8	0.2881	2910	2939	2967	2995	3023					3	6	8	11	14	17	20	22	25
							3051	3078	3106	3133	3	5	8	11	13	16	19	22	24
0.9	0.3159	3186	3212	3238	3264	3289					3	5	8	10	13	16	18	21	23
							3315	3340	3365	3389	2	5	7	10	12	15	17	20	22
1.0	0.3413	3438	3461	3485	3508						2	5	7	10	12	14	17	19	22
						3531	3554	3577	3599	3621	2	4	7	9	11	13	15	18	20
1.1	0.3643	3665	3686	3708							2	4	6	8	11	13	15	17	19
					3729	3749	3770	3790	3810	3830	2	4	6	8	10	12	14	16	18
1.2	0.3849	3869	3888	3907	3925						2	4	6	8	10	11	13	15	17
						3944	3962	3980	3997	4015	2	4	5	7	9	11	13	14	16
1.3	0.4032	4049	4066	4082	4099	4115	4131	4147	4162	4177	2	3	5	6	8	10	11	13	14
1.4	0.4192	4207	4222	4236	4251	4265	4279	4292	4306	4319	1	3	4	6	7	8	10	11	13
1.5	0.4332	4345	4357	4370	4382	4394	4406	4418	4429	4441	1	2	4	5	6	7	8	10	11
1.6	0.4452	4463	4474	4484	4495	4505	4515	4525	4535	4545	1	2	3	4	5	6	7	8	9
1.7	0.4554	4564	4573	4582	4591	4599	4608	4616	4625	4633	1	2	3	3	4	5	6	7	8
1.8	0.4641	4649	4656	4664	4671	4678	4686	4693	4699	4706	1	1	2	3	4	4	5	6	6
1.9	0.4713	4719	4726	4732	4738	4744	4750	4756	4761	4767	1	1	2	2	3	4	4	5	5
2.0	0.4772	4778	4783	4788	4793	4798	4803	4808	4812	4817	0	1	1	2	2	3	3	4	4
2.1	0.4821	4826	4830	4834	4838	4842	4846	4850	4854	4957	0	1	1	2	2	2	3	3	4
2.2	0.4861	4864	4868	4871	4875	4878	4881	4884	4887	4890	0	1	1	1	2	2	2	3	3
2.3	0.4893	4896	4898	4901	4904	4906	4909	4911	4913	4916	0	0	1	1	1	2	2	2	2
2.4	0.4918	4920	4922	4925	4927	4929	4931	4932	4934	4936	0	0	1	1	1	1	1	2	2
2.5	0.4938	4940	4941	4943	4945	4946	4948	4949	4951	4952									
2.6	0.4953	4955	4956	4957	4959	4960	4961	4962	4963	4964									
2.7	0.4965	4966	4967	4968	4969	4970	4971	4972	4973	4974									
2.8	0.4974	4975	4976	4977	4977	4978	4979	4979	4980	4981									
2.9	0.4981	4982	4982	4983	4984	4984	4985	4985	4986	4986									
3.0	0.4987	4990	4993	4995	4997	4998	4998	4999	4999	5000									

The table gives $P(z) = \int_0^z \phi(z)dz$

If the random variable Z is distributed as the standard normal distribution N(0,1) then:

1. $P(o < Z < z_p) = P(\text{Shaded Area})$
2. $P(Z > Z_p) = Q = ½ - P$
3. $P(Z > |Z_p|) = 1 - 2P = 2Q$



NORMAL DISTRIBUTION N(0,1) ϕ (Z)

Z	0	1	2	3	4	5	6	7	8	9	SUBTRACT 1	2	3	4	5	6	7	8	9
0.0	0.3989	3989	3989	3988	3986														
						3984	3982	3980	3977	3973	0	1	1	1	1	2	2	2	3
0.1	0.3970	3965	3961	3956	3951						0	1	1	2	2	3	3	4	4
						3945	3939	3932	3925	3918	1	1	2	3	3	4	5	6	6
0.2	0.3910	3902	3894	3885	3876						1	2	3	4	4	5	6	7	8
						3867	3857	3847	3836	3825	1	2	3	4	5	7	8	9	10
0.3	0.3814	3802	3790	3778	3765						1	2	4	5	6	7	8	10	11
						3752	3739	3725	3712	3697	1	3	4	6	7	8	10	11	13
0.4	0.3683	3668	3653	3637	3621	3605	3589	3572	3555	3538	2	3	5	6	8	10	11	13	14
0.5	0.3521	3503	3485	3467	3448	3429	3410	3391	3372	3352	2	4	6	8	10	11	13	15	17
0.6	0.3332	3312	3292	3271	3251	3230	3209	3187	3166	3144	2	4	6	8	10	13	15	17	19
0.7	0.3123	3101	3079	3056	3034	3011	2989	2966	2943	2920	2	5	7	9	11	14	16	18	21
0.8	0.2897	2874	2850	2827	2803	2780	2756	2732	2709	2685	2	5	7	10	12	14	17	19	22
0.9	0.2661	2637	2613	2589	2565	2541	2516	2492	2468	2444	2	5	7	10	12	14	17	19	22
1.0	0.2420	2396	2371	2347	2323	2299	2275	2251	2227	2203	2	5	7	10	12	14	17	19	22
1.1	0.2179	2155	2131	2107	2083	2059	2036	2012	1989	1965	2	5	7	10	12	14	17	19	22
1.2	0.1942	1919	1895	1872	1849	1826	1804	1781	1758	1736	2	5	7	9	12	14	16	18	21
1.3	0.1714	1691	1669	1647	1626	1604	1582	1561	1539	1518	2	4	7	9	11	13	15	18	20
1.4	0.1497	1476	1456	1435	1415	1394	1374	1354	1334	1315	2	4	6	8	10	12	14	16	18
1.5	0.1295	1276	1257	1238	1219	1200	1182	1163	1145	1127	2	4	6	8	9	11	13	15	17
1.6	0.1109	1092	1074	1057	1040	1023	1006	0989	0973	0957	2	3	5	7	8	10	12	14	15
1.7	0.0940	0925	0909	0893	0878	0863	0848	0833	0818	0804	2	3	5	6	8	9	11	12	14
1.8	0.0790	0775	0761	0748	0734						1	3	4	6	7	8	10	11	13
						0721	0707	0694	0681	0669	1	3	4	5	7	8	9	10	12
1.9	0.0656	0644	0632	0620	0608	0596	0584	0573	0562	0551	1	2	4	5	6	7	8	10	11
2.0	0.0540	0529	0519	0508	0498	0488	0478	0468	0459	0449	1	2	3	4	5	6	7	8	9
2.1	0.0440	0431	0422	0413	0404	0396	0387	0379	0371	0363	1	2	3	4	4	5	6	7	8
2.2	0.0355	0347	0339	0332	0325	0317	0310	0303	0297	0290	1	1	2	3	4	4	5	6	6
2.3	0.0283	0277	0270	0264	0258	0252	0246	0241	0235	0229	1	1	2	2	3	4	4	5	5
2.4	0.0224	0219	0231	0208	0203	0198	0194	0189	0184	0180	0	1	1	2	2	3	3	4	4
2.5	0.0175	0171	0167	0163	0158	0154	0151	0147	0143	0139	0	1	1	2	2	2	3	3	4
2.6	0.0136	0132	0129	0126	0122	0119	0116	0113	0110	0107	0	1	1	1	2	2	2	2	3
2.7	0.0104	0101	0099	0096	0093	0091	0088	0086	0084	0081	0	1	1	1	1	2	2	2	3
2.8	0.0079	0077	0075	0073	0071	0069	0067	0065	0063	0061									
2.9	0.0060	0058	0056	0055	0053	0051	0050	0048	0047	0046									
3.	0.0044	0033									1	2	3	4	5	6	7	8	9
			0024	0017							1	1	2	2	3	4	4	5	5
					0012	0009	0006	0004	0003	0002									

The functions tabled are:

$\phi(Z) = \sqrt{\left(\frac{1}{2\pi}\right)} \exp\left(-\frac{1}{2} Z^2\right)$, where $\phi(Z)$ is the probability density of the standardized normal distribution N(0,1)

CRITICAL POINTS OF THE NORMAL DISTRIBUTION Z_p

P	Q	z	P	Q	z	P	Q	z
.00	.50	0.000	.460	.040	1.751	.490	.010	2.326
.05	.45	0.126	.462	.038	1.774	.491	.009	2.366
.10	.40	0.253	.464	.036	1.799	.492	.008	2.409
.15	.35	0.385	.466	.034	1.825	.493	.007	2.457
.20	.30	0.524	.468	.032	1.852	.494	.006	2.512
.25	.25	0.674	.470	.030	1.881	.495	.005	2.576
.30	.20	0.842	.472	.028	1.911	.496	.004	2.652
.35	.15	1.036	.474	.026	1.943	.497	.003	2.748
.40	.10	1.282	.476	.024	1.977	.498	.002	2.878
.45	.05	1.645	.478	.022	2.014	.499	.001	3.090
.450	.050	1.645	.480	.020	2.054	.4995	.0005	3.291
.452	.048	1.665	.482	.018	2.097	.4999	.0001	3.719
.454	.046	1.685	.484	.016	2.144	.49995	.00005	3.891
.456	.044	1.706	.486	.014	2.197	.49999	.00001	4.265
.458	.042	1.728	.488	.012	2.257	.499995	.000005	4.417