CHAPTER 23: DIFFERENTIAL EQUATIONS

A differential equation is an equation containing differential coefficients (derivatives)

TERMS USED

Linear differential equation

A differential equation is linear in y if its order decreases gradually by 1 (one). e.g

$$A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + Cy = D$$

Non-linear differential equation

A differential equation is said to be non-linear if its differential coefficient is raised to a power greater than 1 (one). E.g.

$$\left(\frac{dy}{dx}\right)^2, \left(\frac{d^2y}{dx^2}\right)^2, (y''')^4, e.t.c$$

Differential coefficients

A differential equation is said to have constant coefficients if all the coefficients of the derivatives are constant. If one of the coefficients is variable, then the differential equation is said to have variable coefficients. e.g. in the differential equation $y'' + (4x)y' + (5x^2)y = sinx$, the first derivative ($y'$) has a variable coefficient whereas the second derivative (y'') has a constant coefficient.

Solution of a differential equation

The solution of a differential equation, containing $\left(\frac{dy}{dx}\right)$, involves finding an equation connecting x *and* y without containing the derivatives. This solution is obtained by or through integration.

The general solution of a differential equation is a solution containing the arbitrary constant (constant of integration).

The particular solution of a differential equation is a solution without the arbitrary constant (constant of integration).

Order and degree of a differential equation

The order of a differential equation is determined by the highest derivative present in a differential equation.

The degree of a differential equation is determined by the power of the highest derivative present in a differential equation. e.g.

$$(y'')^2 + 4y' + 5y = 0 \text{ (second order, degree two)}$$
$$y'' + 4(y')^3 + 5y = 0 \text{ (second order, degree one)}$$
$$(y')^2 + 2xy = 4 \text{ (first order, degree two)}$$

FORMING DIFFERENTIAL EQUATIONS WORKED EXAMPLES

Qn 1: Form differential equations by eliminating the constants A and B from the following:
(a). $y = Ax + B$
(b). $y = (Ax + B)e^{4x}$
Sol:
(a). $y = Ax + B$
First derivative,

$$\frac{dy}{dx} = A$$

Second derivative,

$$\frac{d^2y}{dx^2} = 0$$

This is the differential equation since it does not contain constants A and B.
(b). $y = (Ax + B)e^{4x}$. Here we shall first divide both sides by e^{4x}

$$\frac{y}{e^{4x}} = \frac{(Ax + B)e^{4x}}{e^{4x}}$$

$$ye^{-4x} = Ax + B$$

First derivative,

$$y(-4e^{-4x}) + e^{-4x}\frac{dy}{dx} = A$$

$$e^{-4x}\left(\frac{dy}{dx} - 4y\right) = A$$

Second derivative,

$$e^{-4x}\left(\frac{d^2y}{dx^2} - 4\frac{dy}{dx}\right) - 4e^{-4x}\left(\frac{dy}{dx} - 4y\right) = 0$$

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$$

This is the differential equation since it does not contain constants A and B.
Qn 2: Given that $y = Ae^{3x} + Be^{-2x}$, show that: $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$.
Sol:

$$y = Ae^{3x} + Be^{-2x}$$

$$\frac{dy}{dx} = 3Ae^{3x} - 2Be^{-2x}$$

$$\frac{d^2y}{dx^2} = 9Ae^{3x} + 4Be^{-2x}$$

$$\frac{d^2y}{dx^2} = 3Ae^{3x} - 2Be^{-2x} + 6Ae^{3x} + 6Be^{-2x}$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + 6y$$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$$

Alternatively:

$$y = Ae^{3x} + Be^{-2x}$$

$$ye^{2x} = Ae^{5x} + B$$

$$2ye^{2x} + e^{2x}\frac{dy}{dx} = 5Ae^{5x}$$

$$2ye^{-3x} + e^{-3x}\frac{dy}{dx} = 5A$$

$$\left(2y + \frac{dy}{dx}\right)e^{-3x} = 5A$$

$$\left(2\frac{dy}{dx} + \frac{d^2y}{dx^2}\right)e^{-3x} - 3e^{-3x}\left(2y + \frac{dy}{dx}\right) = 0$$

$$\left(2\frac{dy}{dx} + \frac{d^2y}{dx^2}\right) - 3\left(2y + \frac{dy}{dx}\right) = 0$$

$$2\frac{dy}{dx} + \frac{d^2y}{dx^2} - 6y - 3\frac{dy}{dx} = 0$$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$$

FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS

These are generally of the form $y' + py = q$ where $p\ and\ q$ are functions of x.
They are grouped into two major categories:
Those which have separable variables (separable differential equations)
Those which have non-separable variables (non-separable differential equations)

SEPARABLE DIFFERENTIAL EQUATIONS

These are generally of the form

$$\frac{dy}{dx} = f(x)g(y)$$

Where $f(x)$ is a function of x and $g(y)$ is a function of y.

Solving a separable differential equation

To solve a separable differential equation, first separate the variables $f(x)\ and\ g(y)$ and then apply direct integration in order to obtain an equation connecting $x\ and\ y$ without containing the derivative $\left(\frac{dy}{dx}\right)$. i.e.

$$\int \frac{dy}{g(y)} = \int f(x)\ dx$$

WORKED EXAMPLES

Qn 1: Solve $y^2\frac{dy}{dx} = x(x-1)$

Sol:

$$y^2\frac{dy}{dx} = x(x-1)$$
$$y^2\frac{dy}{dx} = (x^2 - x)$$
$$y^2\ dy = (x^2 - x)\ dx$$
$$\int y^2\ dy = \int (x^2 - x)\, dx$$
$$\frac{y^3}{3} = \frac{x^3}{3} - \frac{x^2}{2} + c$$

This is the general solution of the given differential equation. It's called a general solution because the value of its constant of integration, c, can't be got. c can only be got if initial conditions are given, as we shall see in the next examples below.

Qn 2: Given that $\frac{dy}{dx} = 7$; and that $x = 1$ when $y = 10$, find:

(i). y in terms of x.

(ii). $x$when $y = 24$.

Sol:

(i).

$$\frac{dy}{dx} = 7$$
$$1\ dy = 7\ dx$$

$$\int 1\ dy = \int 7\,dx$$
$$y = \frac{7x^2}{2} + c$$
This is the general solution of the given differential equation. But since the initial conditions are given (i.e. $x = 1$ when $y = 10$), we shall use these conditions to obtain the value of c.
$$10 = \frac{7 \times 1^2}{2} + c, \qquad \Rightarrow c = 10 - \frac{7}{2} = \frac{13}{2}$$
$$\therefore y = \frac{7x^2}{2} + \frac{13}{2}$$
This is the particular solution of the given differential equation. It's called a particular solution because the value of its constant of integration, c, is known.
(ii). When $y = 24$,
$$24 = \frac{7x^2}{2} + \frac{13}{2}$$
$$17.5 = 3.5x^2$$
$$x^2 = 5, \qquad \Rightarrow x = \pm\sqrt{5} = \pm 2.236$$

Qn 3: Find $y$if$\frac{dy}{dx} = \frac{3x^2+2x-3}{3y^2-4y}$, given that $y(1) = 2$. Hence state the value of $y$when$x = -\frac{1}{2}$.

Sol:
$$\frac{dy}{dx} = \frac{3x^2 + 2x - 3}{3 - 4y}$$
$$(3 - 4y)\,dy = (3x^2 + 2x - 3)\,dx$$
$$\int (3 - 4y)\ dy = \int (3x^2 + 2x - 3)\,dx$$
$$3y - \frac{4y^2}{2} = \frac{3x^3}{3} + \frac{2x^2}{2} - 3x + c$$
$$3y - 2y^2 = x^3 + x^2 - 3x + c$$
This is the general solution of the given differential equation.The particular solution of the given differential equation got got by using the given initial conditions, $y(1) = 2$, which means that $y = 2$ when $x = 1$.
$$3(2) - 2(2)^2 = (1)^3 + (1)^2 - 3(1) + c$$
$$6 - 8 = 1 + 1 - 3 + c$$
$$c = -1$$

$$3y - 2y^2 = x^3 + x^2 - 3x - 1$$

This is the particular solution of the given differential equation.

When$x = -\frac{1}{2}$,

$$3y - 2y^2 = (-0.5)^3 + (-0.5)^2 - 3(-0.5) - 1$$
$$3y - 2y^2 = -0.125 + 0.25 + 1.5 - 1$$
$$3y - 2y^2 = 0.625$$
$$2y^2 - 3y + 0.625 = 0$$
$$y = \frac{3 \pm \sqrt{(-3)^2 - 4 \times 2 \times 0.625}}{2 \times 2} = \frac{3 \pm 2}{4}$$
$$y = \frac{3-2}{4} = 0.25, \qquad \text{or,} \qquad y = \frac{3+2}{4} = 1.25$$

NON-SEPARABLE DIFFERENTIAL EQUATION

These are differential equations which can not be solved by separating variables. They are of three forms: homogeneous, exact and inexact differential equations.

HOMOGENEOUS DIFFERENTIAL EQUATIONS

A homogeneous differential equation is one in which all the terms are of the same degree. That is, all the terms are of the same dimensions. E.g

$$for, \qquad xy\frac{dy}{dx} = x^2 + y^2$$
$$\text{dimensions are, } (L)(L)\left(\frac{L}{L}\right) = L^2 + L^2$$
$$L^2 = L^2 + L^2, \qquad \text{hence homogeneous}$$
$$\text{for,} \qquad x\frac{dy}{dx} = 2x + y$$
$$\text{dimensions are, } (L)\left(\frac{L}{L}\right) = L + L$$
$$L = L + L, \qquad \text{hence homogeneous}$$

Solving homogeneous differential equations

homogeneous differential equations can be solved by using an appropriate substitution, $y = ux \quad \rightarrow (i)$ where u is a function of x. Hence,

$$\frac{dy}{dx} = \left(u + x\frac{du}{dx}\right) \quad \rightarrow (ii)$$

This substitution helps to transform the homogeneous non-separable differential equation into a separable differential equation. At this stage, the method used to solve separable differential equations can be adopted.

WORKED EXAMPLES

Qn 1: Use the substitution $y = ux$, where u is a function of x, to solve the differential equation$x\frac{dy}{dx} = x + 2y$, given that $y(3) = 1.5$

Sol: From$y = ux \quad \rightarrow (i)$, $\frac{dy}{dx} = \left(u + x\frac{du}{dx}\right) \quad \rightarrow (ii)$. Given

$$x\frac{dy}{dx} = x + 2y$$

Substituting equation (i) and (ii) in the differential equation gives

$$x\left(u + x\frac{du}{dx}\right) = x + 2ux$$

$$ux + x^2\frac{du}{dx} = x + 2ux$$

$$x^2\frac{du}{dx} = x + ux = x(1 + u)$$

$$x\frac{du}{dx} = (1 + u)$$

This is now a separable differential equation. We shall therefore separate the variables and then integrate.

$$\int \frac{du}{(u + 1)} = \int \frac{dx}{x}$$

$$\ln(u + 1) = \ln x + c$$

Where c is the constant of integration. Taking $c = \ln k$

$$\ln(u + 1) = \ln x + \ln k$$

$$\ln(u + 1) = \ln kx$$

$$u + 1 = kx$$

$$u = kx - 1 \quad \rightarrow (iii)$$

From $= ux$, $u = \frac{y}{x}$. substituting into (iii) gives

$$\frac{y}{x} = kx - 1$$

$$y = x(kx - 1)$$

This is the general solution of the given differential equation. $y(3) = 1.5$ implies that when $x = 3, y = 1.5$.

$$k = \frac{1}{x}\left(\frac{y}{x} + 1\right) = \frac{1}{3}\left(\frac{1.5}{3} + 1\right) = 0.5$$

$$y = x(0.5x - 1)$$

This is the particular solution of the given differential equation

Qn 2: Solve $x^2 \frac{dy}{dx} = xy + y^2$

Sol: Let $y = ux \quad \to (i)$, hence $\frac{dy}{dx} = \left(u + x\frac{du}{dx}\right) \quad \to (ii)$. Given

$$x^2 \frac{dy}{dx} = xy + y^2$$

Substituting equation (i) and (ii) in the differential equation gives

$$x^2\left(u + x\frac{du}{dx}\right) = x(ux) + (ux)^2$$

$$ux^2 + x^3\frac{du}{dx} = ux^2 + u^2x^2$$

$$x\frac{du}{dx} = u^2$$

This is now a separable differential equation. We shall therefore separate the variables and then integrate.

$$\int \frac{du}{u^2} = \int \frac{dx}{x} \quad , \qquad \Rightarrow -\frac{1}{u} = \ln x + c$$

Where c is the constant of integration. Taking $c = \ln k$

$$-\frac{1}{u} = \ln x + \ln k \, , \qquad \Rightarrow -\frac{1}{u} = \ln kx \quad \to (iii)$$

From $= ux$, $u = \frac{y}{x}$. Substituting into (iii) gives

$$-\frac{x}{y} = \ln kx \, , \qquad \Rightarrow kx = e^{-x/y}$$

This is the general solution of the given differential equation.

Qn 3: Using the substitution $y = ux$, solve the differential equation $x^2 \frac{dy}{dx} = x^2 + xy + y^2$.

Sol:

$$x^2\frac{dy}{dx} = x^2 + xy + y^2$$

$$\text{but,} \qquad y = ux, \qquad \Rightarrow \frac{dy}{dx} = \left(u + x\frac{du}{dx}\right)$$

Substituting for y and $\frac{dy}{dx}$ gives:

$$x^2\left(u + x\frac{du}{dx}\right) = x^2 + ux^2 + u^2x^2$$

$$ux^2 + x^3\frac{du}{dx} = (1 + u + u^2)x^2$$

$$u + x\frac{du}{dx} = 1 + u + u^2$$

$$x\frac{du}{dx} = 1 + u^2$$
$$\int \frac{du}{1+u^2} = \int \frac{1}{x}dx$$
$$\tan^{-1} u = \ln x + c$$
$$\tan^{-1}\left(\frac{y}{x}\right) = \ln x + c$$

EXACT AND INEXACTDIFFERENTIAL EQUATIONS

If a non-separable differential equation is not homogeneous, then it's either exact or inexact. It's in the form $\frac{dy}{dx} + py = q$ where $p \; and \; q$ are functions of x.

e.g. consider the differential equation

$$\frac{dy}{dx} + 2xy = x$$

$$\text{dimensions,} \qquad \frac{L}{L} + (L)(L) = L,$$

$$1 + L^2 = L, \text{hence it's not homogeneous}$$

Since the dimensions are not the same, then it's not homogeneous. Also, since its variables can not be separated, it's not separable. Hence the equation is either exact or inexact. After discovering that the differential equation is exact or inexact, there's no need to further discover between exact and inexact differential equations since the same method is used to solve both of them.

SOLVING EXACT AND INEXACT DIFFERENTIAL EQUATIONS

Such equations can be solved with the help of the integrating factor $(I.F)$. consider the general form of exact and inexact equations below.

$$\frac{dy}{dx} + py = q$$

where $p \; and \; q$ are functions of x.

Note that the coefficient of $\left(\frac{dy}{dx}\right)$ is equal to 1 (one)

Step 1: Calculate the integrating factor,R, using the formula

$$R = e^{\int p dx}$$

Step 2: Multiply the integrating factor,R, throughout the given differential equation.

$$R\frac{dy}{dx} + Rpy = qR$$

Step 3: After step 2, the resulting equation will be an exact differential equation. Hence

$$\frac{d}{dx}[yR] = [qR]$$

Step 4: Put integrals on both sides and then integrate to obtain the solution of the given differential equation.

$$\int \frac{d}{dx}[Ry] = \int [Rq]\,dx$$
$$[yR] = \int [qR]\,dx$$

WORKED EXAMPLES

Qn 1: Solve $\frac{dy}{dx} + y + 3 = x$

Sol:

$$\frac{dy}{dx} + y = (x - 3)$$

Comparing with the general equation, $p = 1, q = (x - 3)$

Step 1:

Integrating factor, $R = e^{\int p dx} = e^{\int dx} = e^{x}$

Step 2:

$$e^{x}\frac{dy}{dx} + ye^{x} = (x - 3)e^{x}$$

Step 3:

$$\frac{d}{dx}(ye^{x}) = (x - 3)e^{x}$$

Step 4:

$$\int \frac{d}{dx}(ye^{x}) = \int (x - 3)e^{x}\,dx$$
$$ye^{x} = \int xe^{x}\,dx - \int 3e^{x}\,dx \quad \rightarrow (i)$$
$$for, \quad \int xe^{x}\,dx, \quad let\ u = x\ and\ \frac{dv}{dx} = e^{x}$$
$$\frac{du}{dx} = 1, \quad v = e^{x}$$
$$\int xe^{x}\,dx = uv - \int v\frac{du}{dx}dx = xe^{x} - \int e^{x}\,dx = xe^{x} - e^{x}$$

From (i),

$$ye^{x} = \int xe^{x}\,dx - \int 3e^{x}\,dx = xe^{x} - e^{x} - 3e^{x} + c$$

$$ye^x = (x-4)e^x + c$$

This is the general solution of the given differential equation.

Qn 2: Solve $\frac{dy}{dx} - y\tan x = x$

Sol: Comparing with the general equation, $p = -\tan x\,, q = x$

Step 1:

$$R = e^{\int p dx} = e^{\int -\tan x dx} = e^{\ln\cos x} = \cos x$$

Step 2:

$$\cos x\frac{dy}{dx} - y\tan x\cos x = x\cos x$$

Step 3:

$$\frac{d}{dx}(y\cos x) = x\cos x$$

Step 4:

$$\int\frac{d}{dx}(y\cos x) = \int x\cos x\,dx \quad \rightarrow (i)$$

$$\text{for,} \int x\cos x\,dx, \qquad let\ u = x\ and\ \ \frac{dv}{dx} = \cos x$$

$$\frac{du}{dx} = 1, \qquad v = \sin x$$

$$\int x\cos x\,dx = uv - \int v\frac{du}{dx}dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x$$

From (i),

$$y\cos x = x\sin x + \cos x + c$$

This is the general solution of the given differential equation.

Qn 3: Solve $\tan x\frac{dy}{dx} - y = \sin^2 x$

Sol:

Step 1:

$$\tan x\frac{dy}{dx} - y = \sin^2 x\,, \qquad \Rightarrow \frac{dy}{dx} - \frac{y}{\tan x} = \frac{\sin^2 x}{\tan x}$$

$$\frac{dy}{dx} - y\cot x = \sin^2 x\cot x$$

Comparing with the general equation, $p = -\cot x\,, q = \sin^2 x\cot x$

Step 1:

$$R = e^{\int p dx} = e^{\int -\cot x dx} = e^{-\ln\sin x} = e^{\ln(\sin x)^{-1}} = (\sin x)^{-1} = \frac{1}{\sin x}$$

Step 2:

$$\frac{1}{\sin x}\times\frac{dy}{dx} - y\cot x\times\frac{1}{\sin x} = \sin^2 x\cot x\times\frac{1}{\sin x}$$

Step 3:

$$\frac{d}{dx}\left(y \times \frac{1}{\sin x}\right) = \sin^2 x \cot x \times \frac{1}{\sin x}$$
$$\frac{d}{dx}\left(\frac{y}{\sin x}\right) = \sin^2 x \times \frac{\cos x}{\sin x} \times \frac{1}{\sin x}$$
$$\frac{d}{dx}\left(\frac{y}{\sin x}\right) = \cos x$$

Step 4:

$$\int \frac{d}{dx}\left(\frac{y}{\sin x}\right) = \int \cos x \, dx, \qquad \Rightarrow \frac{y}{\sin x} = \sin x + c$$

This is the general solution of the given differential equation.

APPLICATIONS OF DIFFERENTIAL EQUATIONS

The best concept of differential equations is the rates of change. In life, most systems work on rates of change. Examples include:

1) Rate of cooling of a liquid/dead body/specimen.
2) Rate of decay of a radioactive particle.
3) Rates of spread of a rumor.
4) Rate of growth of a specimen/population.
5) Rates of spread of a disease.
6) Rates of change of a chemical reaction.
7) Rate of change of distance, velocity, e.t.c.

RATE OF COOLING OF A SPECIMEN

To get the rate of cooling of a specimen, we use Newton's law of cooling which states that

$$\begin{pmatrix}\text{rate of cooling} \\ \text{of a specimen}\end{pmatrix} \propto \begin{pmatrix}\text{excess temperature above} \\ \text{room temperature}\end{pmatrix}$$
$$-\frac{d\theta}{dt} \propto (\theta - \theta_R) \quad \rightarrow (1)$$

$\frac{d\theta}{dt}$ is the rate of cooling of the body, θ is the temperature of the body at a time t, and θ_R is the room temperature. The negative sign is because the excess temperature decreases as time increases.

$$-\frac{d\theta}{dt} = k(\theta - \theta_R)$$

Where k is the proportionality constant.

$$\frac{d\theta}{(\theta - \theta_R)} = -kdt$$

Putting integrals on both sides

$$\int \frac{d\theta}{(\theta - \theta_R)} = \int -k\, dt$$

$$\ln(\theta - \theta_R) = -kt + c \quad \rightarrow (2)$$

Where c is the constant of integration.

Qn 1: A liquid cools in a room of constant temperature 22°C at a rate proportional to the excess temperature. Initially, the temperature of the liquid was 100°C and after one minute later, it was 92.2°C. Find the temperature of the liquid after 5 minutes.

Sol:

Let θ be the temperature of the body at a time t, and θ_R be the room temperature.

$$-\frac{d\theta}{dt} \propto (\theta - \theta_R)$$

$$-\frac{d\theta}{dt} = k(\theta - 22)$$

Where k is the proportionality constant.

$$\int \frac{d\theta}{(\theta - 22)} = \int -k\, dt$$

$$\ln(\theta - 22) = -kt + c$$

When $t = 0, \theta = 100$°C

$$\ln(100 - 22) = -k \times 0 + c$$

$$c = \ln 78$$

$$\therefore \ln(\theta - 22) = -kt + \ln 78$$

When $t = 1, \theta = 92.2$°C

$$\ln(92.2 - 22) = -k \times 1 + \ln 78$$

$$k = \ln 78 - \ln 70.2$$

$$= \ln\left(\frac{78}{70.2}\right)$$

$$\ln(\theta - 22) = -t\ln\left(\frac{78}{70.2}\right) + \ln 78$$

$$\ln(\theta - 22) = t\ln 0.9 + \ln 78$$

When $t = 5$,

$$\ln(\theta - 22) = 5\ln 0.9 + \ln 78$$

$$\ln(\theta - 22) = 3.8299$$
$$\theta - 22 = 46.05822$$
$$\theta = 68.05822°C$$

RATE OF DECAY OF A RADIOACTIVE PARTICLE

Qn 1: A sample of a radioactive rhodium loses mass at a rate proportional to the amount present. If m is the mass after t years,
(i). Form a differential equation connecting m, t and a constant k.
(ii). If initially the mass of rhodium was m_o, deduce that $m = m_o e^{-kt}$.
(iii). Given that its initial mass is halved in 1600 years, find the value of k and hence determine the number of years it takes 5 g of rhodium to reduce to 3.6 g.
Sol:
(i)
$$\begin{pmatrix}\text{rate of decay} \\ \text{of a rhodium}\end{pmatrix} \propto \begin{pmatrix}\text{ammount of rhodium} \\ \text{present after t years}\end{pmatrix}$$
$$-\frac{dm}{dt} \propto m$$
The negative sign is because the mass of rhodium decreases as time increases.
$$-\frac{dm}{dt} = km$$
This is the differential equation connecting m, t and a constant k.
(ii).
$$\int \frac{dm}{m} = \int -k\, dt$$
$$\ln m = -kt + c$$
When $t = 0, m = m_o$
$$\ln m_o = -k \times 0 + c$$
$$c = \ln m_o$$
Equation (1) becomes
$$\ln m = -kt + \ln m_o$$
$$\ln m - \ln m_o = -kt$$
$$\ln\left(\frac{m}{m_o}\right) = -kt$$
$$\frac{m}{m_o} = e^{-kt}$$

$$m = m_o e^{-kt}, \qquad \text{as required}$$

(iii).

When $t = 1600, m = \frac{1}{2}m_o$

$$\frac{1}{2}m_o = m_o e^{-1600k}$$

$$0.5 = e^{-1600k}$$

$$-1600k = \ln 0.5$$

$$k = -\frac{\ln 0.5}{1600}$$

When $m_o = 5$ g and $m = 3.6$ g

$$-kt = \ln\left(\frac{m}{m_o}\right)$$

$$\frac{t\ln 0.5}{1600} = \ln\left(\frac{3.6}{5}\right)$$

$$t = \frac{1600\ln 0.72}{\ln 0.5} = 758.2899 \text{ years}$$

RATES OF SPREAD OF A RUMOR

Qn 1: A rumor spreads through a town at a rate which is proportional to the product of the number of people who have heard it and those who have not heard it. Given that x is a fraction of the population of the town who have heard the rumor after time t,

(i). Form a differential equation connecting x, t and a constant k.

(ii). If initially a fraction D, had heard the rumor, deduce that

$$x = \frac{D}{D + (1 - D)e^{-kt}}$$

(iii). Given that 15%had heard the rumor at 9: 00 am and another 15% by noon, find what further fraction of the population would have heard the rumor by 3: 00 pm.

Sol:

(i).

$$\begin{pmatrix}\text{rate of spread}\\ \text{of a rumor}\end{pmatrix} \propto \begin{pmatrix}\text{those who have}\\ \text{heard it}\end{pmatrix} \times \begin{pmatrix}\text{those who have}\\ \text{not heard it}\end{pmatrix}$$

$$\frac{dx}{dt} \propto x(1 - x)$$

$$\frac{dx}{dt} = kx(1 - x)$$

The sign is positive because the number of people who have heard the rumor increases as time increases.

(ii).

$$\int \frac{dx}{x(1-x)} = \int k\,dt$$

$$\text{but,} \qquad \frac{1}{x(1-x)} \equiv \frac{A}{x} + \frac{B}{1-x}$$

$$1 \equiv A(1-x) + Bx$$

$$\text{when } x = 0, A = 1\,, \qquad \text{and when } x = 1, B = 1$$

$$\Longrightarrow \frac{1}{x(1-x)} \equiv \frac{1}{x} + \frac{1}{1-x}$$

$$\therefore \int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx = \int k\,dt$$

$$\ln x - \ln(1-x) = kt + c$$

When $t = 0, x = D$

$$\ln D - \ln(1-D) = k \times 0 + c$$

$$c = \ln\left(\frac{D}{1-D}\right)$$

$$\therefore \ln x - \ln(1-x) = kt + \ln\left(\frac{D}{1-D}\right)$$

$$-kt = \ln\left(\frac{D}{1-D}\right) + \ln(1-x) - \ln x$$

$$-kt = \ln\left[\frac{D(1-x)}{x(1-D)}\right]$$

$$e^{-kt} = \frac{D(1-x)}{x(1-D)}$$

$$x(1-D)e^{-kt} = D - Dx$$

$$(1-D)xe^{-kt} + Dx = D$$

$$x[(1-D)e^{-kt} + D] = D$$

$$x = \frac{D}{D + (1-D)e^{-kt}}\,, \qquad \text{as required}$$

(iii).

At 9:00 am, $t = 0$ and $D = 15\% = 0.15$

$$x = \frac{0.15}{0.15 + (1-0.15)e^{-kt}}$$

$$x = \frac{0.15}{0.15 + 0.85e^{-kt}}$$

At noon, $t = 3$ and $x = 15\% + 15\% = 30\% = 0.3$

$$0.3 = \frac{0.15}{0.15 + 0.85e^{-3k}}$$
$$0.045 + 0.255e^{-3k} = 0.15$$
$$0.255e^{-3k} = 0.105$$
$$e^{-3k} = \frac{0.105}{0.255}$$
$$-3k = \ln\left(\frac{0.105}{0.255}\right)$$
$$k = \frac{-1}{3}\ln\left(\frac{0.105}{0.255}\right) = 0.2958$$
$$\therefore x = \frac{0.15}{0.15 + 0.85e^{-0.2958t}}$$

At 3:00 pm, $t = 6$

$$x = \frac{0.15}{0.15 + 0.85e^{-0.2958\times 6}} = 0.510$$
$$\text{further fraction} = 0.51 - 0.3 = 0.21 = 21\%$$

RATE OF GROWTH OF A SPECIMEN/ POPULATION

Qn 1: The size S of a population at a time$t$, satisfies a differential equation $\frac{dS}{dt} = kS$. Show that $S = Be^{kt}$ where B is a constant.
If the population was 32,000 in 1910 and had increased to 48,000 by 1980, estimate the size of the population in the year 2010. Give your answer to the nearest 1000.

Sol:

$$\frac{dS}{dt} = kS$$
$$\int \frac{dS}{S} = \int k\,dt$$
$$\ln S = kt + c$$

When $t = 0, S = B$

$$\ln B = k \times 0 + c$$
$$c = \ln B$$
$$\therefore \ln S = kt + \ln B$$
$$\ln S - \ln B = kt$$
$$\ln\left(\frac{S}{B}\right) = kt$$

$$\frac{S}{B} = e^{-kt}$$
$$S = Be^{kt}$$
In 1910, $t = 0$ and $B = 32{,}000$
$$S = 32{,}000e^{kt}$$
In 1980, $t = 70$ and $S = 48{,}000$
$$48{,}000 = 32{,}000e^{70k}$$
$$e^{70k} = \frac{48{,}000}{32{,}000}$$
$$e^{70k} = 1.5$$
$$70k = \ln 1.5$$
$$k = \frac{\ln 1.5}{70} = 0.0058$$
$$S = 32{,}000e^{\left(\frac{t\ln 1.5}{70}\right)}$$
In 2010, $t = 100$
$$S = 32{,}000e^{\left(\frac{100\ln 1.5}{70}\right)} = 57{,}109.5739 \approx 57{,}000 \text{ (nearest thousand)}$$

RATES OF SPREAD OF A DISEASE

Qn 1: The rate of spread of AIDS disease in a community is proportional to the number of people already affected. If the number of infected people rose from 2,000 to 4,000 between 1980 and 1990, how many people shall be affected by the year 2000?

Sol: Let N be the number of people already affected at time t.
$$\begin{pmatrix}\text{rate of spread}\\ \text{of AIDS}\end{pmatrix} \propto \begin{pmatrix}\text{number of people}\\ \text{already affected}\end{pmatrix}$$
$$\frac{dN}{dt} \propto N$$
The positive sign is because the number of people already affected increases as time increases.
$$\frac{dN}{dt} = kN$$
$$\int \frac{dN}{N} = \int k\, dt$$
$$\ln N = kt + c$$
In 1980, $t = 0$ and $N = 2000$
$$\ln 2000 = k \times 0 + c$$
$$c = \ln 2000$$

$$\therefore \ln N = kt + \ln 2000$$

In 1990, $t = 10$ and $N = 4000$

$$\ln 4000 = k \times 10 + \ln 2000$$
$$10k = \ln 4000 - \ln 2000$$
$$10k = \ln\left(\frac{4000}{2000}\right)$$
$$10k = \ln 2$$
$$k = \frac{\ln 2}{10}$$
$$\therefore \ln N = \frac{t \ln 2}{10} + \ln 2000$$

In 2000, $t = 20$

$$\ln N = \frac{20 \ln 2}{10} + \ln 2000$$
$$\ln N = \ln 4 + \ln 2000$$
$$\ln N = \ln 8000$$
$$N = 80000$$

RATES OF CHANGE OF A CHEMICAL REACTION

Qn 1: In a certain type of chemical reaction, a substance P is continuously transformed into substance Q. throughout the reaction, the sum of the mass of P and Q remain constant and equal to m. the mass of Qpresent at time t after commencement of the reaction is denoted by x. at any instant, the rate of increase of the mass of Q is k times the mass of P where k is a positive constant.

Write down the differential equation relating $x\ and\ t$.

Given that $x = 0$ when $t = 0$, solve the differential equation.

Given also that $x = \frac{1}{2}m$, when $t = \ln 2$, determine the value of k and show that at time t, $x = m(1 - e^{-t})$. Hence find

(i). The value of x when $t = 3 \ln 2$.

(ii). The value t of when $x = \frac{3}{4}m$.

Sol: Let m_p be the mass of P.

$$m_p + x = m, \qquad \Rightarrow m_p = (m - x)$$
$$\begin{pmatrix}\text{rate of increase} \\ \text{of the mass of Q}\end{pmatrix} = \{k \times (\text{mass of P})\}$$

$$\frac{dx}{dt} = k(m - x)$$
$$\int \frac{dx}{m - x} = \int k\,dt$$
$$-\ln(m - x) = kt + c$$

When $t = 0, x = 0$

$$-\ln m \ = k \times 0 + c$$
$$c = -\ln m$$
$$-\ln(m - x) = kt - \ln m$$
$$-kt = \ln(m - x) - \ln m$$
$$-kt = \ln\left(\frac{m - x}{m}\right)$$
$$e^{-kt} = \left(\frac{m - x}{m}\right)$$
$$me^{-kt} = m - x$$
$$x = m - me^{-kt}$$
$$x = m(1 - e^{-kt})$$

When $t = \ln 2\,, x = \frac{1}{2}m$

$$\frac{1}{2}m = m\left(1 - e^{-k\ln 2}\right)$$
$$0.5 = \left(1 - e^{\ln 2^{-k}}\right)$$
$$0.5 = 1 - 2^{-k}$$
$$2^{-k} = 0.5$$
$$2^{-k} = 2^{-1}$$
$$k = 1$$
$$\therefore x = m(1 - e^{-t})$$

(i). When $t = 3\ln 2$

$$x = m\left(1 - e^{-3\ln 2}\right) = m\left(1 - e^{\ln 2^{-3}}\right) = m(1 - 2^{-3}) = \frac{7}{8}m$$

(ii). when $x = \frac{3}{4}m$

$$\frac{3}{4}m = m(1 - e^{-t})$$
$$0.75 = 1 - e^{-t}$$
$$e^{-t} = 0.25$$
$$t = -\ln 0.25 = 1.3863$$

RATE OF CHANGE OF SPEED

Qn 1: John walks towards a trading centre which is 1000 m away at a rate which is proportional to the distance he still has to cover. He starts by walking at a speed of 1 m s^{-1} from his home towards the trading centre. How many minutes does he take to cover 600 m from his home?

Sol:

$$v \propto (1000 - x)$$
$$v = -k(1000 - x)$$

When $t = 0, v = 1, x = 0$

$$1 = -k(1000 - 0)$$
$$k = \frac{-1}{1000}$$
$$\therefore v = \frac{1}{1000}(1000 - x)$$
$$\frac{dx}{dt} = \frac{1}{1000}(1000 - x)$$

Where k is the proportionality constant.

$$\int \frac{dx}{1000 - x} = \int \frac{1}{1000} dt,$$
$$-\ln(1000 - x) = \frac{t}{1000} + c$$

When $t = 0, x = 0$

$$-\ln(1000 - 0) = 0 + c$$
$$c = -\ln 1000$$

Equation (1) becomes,

$$-\ln(1000 - x) = \frac{t}{1000} - \ln 1000$$
$$\ln 1000 - \ln(1000 - x) = \frac{t}{1000}$$
$$t = 1000 \ln\left(\frac{1000}{1000 - x}\right)$$

When $x = 600$,

$$t = 1000 \ln\left(\frac{1000}{1000 - 600}\right) = 1000 \ln 2.5 = 916.2907 \text{ s}$$
$$t = \frac{916.2907}{60} = 15.2715 \text{ minutes}$$