A LEVEL

PURE MATHEMATICS

SELF HELP REVISION BOOK

WITH ANSWERS

N. S. Kirirabwa

Introduction

This question and answer booklet is designed to aid learners offering mathematics at Advanced level become familiar with various question settings , practice and learn to follow instructions as they workout to get suitable solutions.

The book contains a few worked examples but does not contain content on various exercises, this is to task students to discuss, consult and get in-depth content from their teachers and then apply by practicing to master concepts.

This book is purposely meant to be used in the 5th and 6th year of secondary mathematics course. The exercises are on topics in pure mathematics designed for advanced level particularly for students offering mathematics as a principle subject. However, it can also be used by students offering mathematics as a subsidiary especially in the pure mathematics section.

The book has been developed to help improve the learners' spirit of self reliance as they practice solving the exercises and getting correct answers. With practice, the subject is made easier, thus the book will be a useful instrument for A-level mathematics students and teachers.

Special thanks go to the almighty GOD who has given me grace to develop this book, my family members, fellow teachers in the Mathematics Department LUBIRI SECONDARY SCHOOL, Rebecca Nagawa, Luyimbazi Godfrey, Shamidah Zalwango, Aiden Magumba, Moses Sengendo, Fred Kijjambu, Viola Mayengo and HIGHER RISING INVESTMENTS LTD for their invaluable suggestions and work towards producing this book.

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EXERCISE 1

1. Simplify $\sqrt{1000} - \sqrt{40} - \sqrt{90}$ in the form $a\sqrt{b}$. Hence, find the product of a and b.
2. Find the roots of: $2t^2 - 2t - 12 = 0$.
3. Solve: (i) $2^m = 7$. (ii) Simplify: $\log_6 46656 + 3 \log_6 216 - 4 \log_6 36$.
4. Given that $3m^2 + 2m - 7 = 0$ has roots $m = \alpha$ and $m = \beta$, find an equation with roots: $1/\alpha$ and $1/\beta$.
5. Find the gradient of the cure $y = 3x^2 + 4x - 1$ at $x = 3$.
6. Simplify: $\dfrac{(1+x)^{1/3} - \frac{x}{3}(1+x)^{-2/3}}{x}$.
7. When $(x^5 + 4x + ax + b)$ is divided by $(x^2 - x)$, the remainder is $(2x+3)$. Find a and b.
8. (a) Given that $t = \tan\theta$, simplify $\dfrac{t}{\sqrt{1+t^2}}$.

   (b) Solve $\tan^2\theta + \tan\theta = 0$, given that $0^0 < \theta < 360^0$.

Prove your weight 1

EXERCISE 2

1. Solve the equation: $2\sqrt{x} + \sqrt{2x+1)} = 7$
2. Find $\dfrac{dy}{dx}$ in terms of x, y when $3(x-y)^2 = 2xy + 1$
3. Show that $$\log_a\left(a^k bc\right) = \frac{k}{\log_{bc} a}\left(\frac{1}{k} + \log_{bc} a\right)$$
4. If the equation $a^2x^2 + 6abx + ac + 8b^2 = 0$ has equal roots, prove that the roots of the Equation $ac(x+1)^2 = 4b^2x$ are also equal.

5. In triangle ABC, prove that; $\dfrac{bc}{ab+ac}=\dfrac{\operatorname{cosec}(B+C)}{\operatorname{cosec}B+\operatorname{cosec}C}$

6. A curve is represented parametrically by $x=(t^2-1)^2, y=t^3$

Find $\dfrac{d^2y}{dx^2}$ in terms of t

7. Solve for x in $\tan^{-1}\left(\dfrac{1}{x}\right)+\tan^{-1}\left(\dfrac{1}{3}\right)=\dfrac{\pi}{4}$

8. Given that y = f(x) is the equation of a curve, such that $f(1)=2$ and

$\dfrac{f(x+h)-f(x)}{h}=3(x+h)^2-2(x+h)$, find the equation of the curve.

9. (a) Show that

$$3^n=\binom{n}{0}+2\binom{n}{1}+4\binom{n}{2}+\ldots\ldots\ldots\ldots+2^n\binom{n}{n}$$

(b) The sum to infinity of a convergent geometric progression is m and the sum to infinity of the sum of squares of its terms is n. Show that the first term of the progression is given by $\dfrac{2mn}{m^2+n}$

(c) Prove that $\dfrac{\tan A+\sec A-1}{\tan A-\sec A+1}=\dfrac{1+\sin A}{\cos A}$

10. (a) Prove that $\log_6^x=\dfrac{\log_3^x}{1+\log_3^2}$. Hence given that $\log_3^2=0.631$. find without using tables or calculators $\log_6^4$. Correct to 3 decimal places.

(b) If $2\log_y x = 8$ and $x^2 + 2y^8 = 12$ Find for x . Find the values of K for which the equation $\dfrac{x^2-x+1}{x-1}=k$ has repeated roots. What are the repeated roots?

11.(a) Find the expansion $\sqrt{\frac{(1+2x)}{(1-2x)}}$ up to the term in x^3;Hence estimate $\sqrt{\frac{1.02}{0.98}}$

to 4 significate figures

(b) Find the sum of the series $1^3 + 2^3 + \ldots\ldots\ldots\ldots\ldots + (2n-1)^3$

12. (a) One stationary point of the curve $y = \frac{ax+b}{x^2+1}$ is $(2\ ,1)$. Find the values of a and b. Hence find the nature of the stationary points.

(b) Find the value of a and b if $ax^4 + bx^3 - 8x^2 + b$ has reminder $2x + 1$ when divided by $x^2 - 1$ and state the quotient.

EXERCISE 3

1. Prove by induction $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$

2. Find the value of **n** if $2\binom{n}{4} = {}^{n}P_{2}$

3. Determine the binomial expansion of $(y+h)^4$ Hence find $(1.996)^4$

4. Given that $b^q = a^p = (ab)^{qp}$. Show that $q = 1 - p$

5. The position vectors of the points A and B are $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix}$ respectively.

Determine the length of $\overline{AB}$.

6. Given that $\log_2^x + 2\log_4^y = 4$ show that $xy = 16$

7. If the n^{th} term of a series is $8n-2$ Show that the series is an arithmetic progression and write down the sum of the n terms

8. Given the complex numbers: $z_1 = 7+i$ and $z_2 = 1-i$. Determine the modulus and argument of $z_2 z_1$.

9. (a) Sketch the locus defined by the equation $\arg(z-6+i) = \frac{\pi}{3}$

   where z is complex.

   (b) Given that $2+i$ is a root of the equation $Z^3 - 6Z^2 + aZ + b = 0$

   Write down the quadratic factor of the polynomial $Z^3 - 6Z^2 + aZ + b$

   where a, b are real numbers.

10. (a)Given that $\frac{\log_{10} 3}{\log_{10} 2} = \frac{8}{5}$, Solve for x and y the simultaneous equations

   $3^x = 2^{3y+1}$ and $4^{x-1} = 12^{2y+1}$

   (b) Solve: $3(3^{2x}) + 2(3^x) - 1 = 0$

   (c) Express as equivalent fraction with a rational denominator. $\frac{\sqrt{2}}{\sqrt{2}-\sqrt{3}}$

11. (a) Distinguish the nature of the stationary points of the curve

   $y = x^3 - 2x^2 - x + 7$

   (b) Given that α and β are the roots of the equation $x^2 + px + q = 0$. Express $(\alpha-\beta^2)(\beta-\alpha^2)$ in terms of p and q. Deduce that for one to be the square of another. $p^3 - 3pq + q^2 + q = 0$ must hold.

12. Express $\sin\beta + \sin 3\beta$ in the form $m\cos k\beta \sin n\beta$ where m, k and n are constants.

13. A particle starting from rest at O moves along straight line OA so that its

 acceleration after seconds is $(24t - 12t^2)ms^{-2}$.

 (a) Find when it again returns to O and its velocity then.
 (b) Find its maximum displacement from O during this interval.
 (c) What is its maximum velocity and its greatest speed during this interval.

14. (a) Evaluate $\left[2x^3 - 4x^2 + x\right]_{-4}^{-1}$

 (b) Find the approximation for $\sqrt{102}$ by δx and δy Method.

 (c) Find the equation of a curve $y = f(x)$, given that $f(0) = 4$ and

 $f(x + \partial x) = f(x) + (3x^2 + 10x - 1)\partial x$.

EXERCISE 4

1. Solve the equation $3^{2y} - 3^{y+1} - 3^y + 3 = 0$

2. Rationalize the denominator $\dfrac{1+\sqrt{3}}{(\sqrt{2}-1)^2}$

3. Show that $\cos 15^o = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

4. Factorize completely $x^4 - 3x^3 + 4x^2 - 8$

5. Without using calculators or tables prove that

$$\cos 165^o + \sin 165^0 = \cos 135^o$$

6. Simplify $\dfrac{(1+x)^{1/3} - \frac{1}{3}x(1+x)^{-2/3}}{(1+x)^{2/3}}$

7. Express $5\sin\theta + 12\cos\theta$ in the form $\beta\sin(\theta+\alpha)$ state the values of β and α

8. If the roots of the equation $x^2 - 5x - 7 = 0$ are β and α . Find the equation whose roots are $\alpha^2\beta, \alpha\beta^2$

9. Show that $\cos\theta - \cos3\theta - \cos5\theta + \cos7\theta = -4\cos4\theta\sin2\theta\sin\theta$

10. Solve $\sqrt{x-5} + 2 = \sqrt{x+7}$

11. The quadratic polynomial P(x) leaves a reminder of 3 on division by (x-1), a remainder of 12 on division by (x-2) and no remainder on division by (x+2). Find P(x) and hence solve P(x) = 0

12. Solve the equations for all possible values of x and y

$$\log_3 x \log_3 y = -6 \quad , \quad \log_9 xy = \frac{5}{2}$$

EXERCISE 5

1. Find in terms of q the least value of the function

$f(x) \equiv x^2 + 3qx + q$

2. Simplify $\dfrac{27^{n+2} - 6(3^{3n+3})}{3^n(9^{n+2})}$

3. The sum of n terms of a series, s_n , is given by $s_n = n(n+3)$. Find the fourth term of series.

4. Find the equation of the tangent to $y = x^3$ at the point (t, t^3) .

5. One root of the equation $x^2 + px + q = 0$ is $2-3i$. Find the values of p and q.

6. Find the maximum and minimum ordinates of the curve $y = x^2(x+1)$.

7. Given the following pair of equations

$x = \operatorname{cosec}\theta - \cot\theta$, and $y = \operatorname{cosec}\theta + \cot\theta$, Eliminate θ .

8. Solve the equation $3+\log_3 x = 4\log_x 3$.

9. The perimeter of a triangle is 42cm; one side is of length 14cm and the area $21\sqrt{15}cm^2$. Find the lengths of the other two sides and show that the cosine of the largest angle is $\frac{1}{4}$

10. Prove that $\dfrac{\sin 3A \sin 6A + \sin A \sin 2A}{\sin 3A \cos 6A + \sin A \cos 2A} = \tan 5A$

Worked examples

1. i) Express $\frac{3x-2}{x^2-1}-\frac{2}{x+1}$ as a single fraction.

Find the LCM .

$$\frac{3x-2}{x^2-1}-\frac{2}{x+1}=\frac{3x-2-(x-1)}{(x+1)(x-1)}$$

$$=\frac{3x-2-2x+2}{(x+1)(x-1)}$$

$$\frac{3x-2}{x^2-1}-\frac{2}{x+1}=\frac{x}{x^2-1}$$

ii) Find the constants A and B such that

$$x^2+3x-2\equiv Ax(x-3)+B(x-1)(x+1)$$

Method I

If x = 0

-2 = -B

B = 2

If x = 1

$$1^2+3-2=A(1-3)+0$$

2 = -2A

A = -1

Method II

$$x^2+3x-2\equiv Ax(x-3)+B(x-1)(x+1)$$

$$x^2+3x-2=Ax^2-3Ax+Bx^2-B$$

Compare coefficients of x^2

1 = A+B - - - - - (i)

Compare coefficients of x

3 = -3A

A = -1

Substitute for A in equation (i)

B = 2

2. If the roots of the quadratic equation $2x^2 - 5x + k = 0$ *are such that one root is 4 times the other, find the*

i) Two roots *ii) values of constant k*

Let the roots be α *and* 4α

i) Sum of the roots is $-\frac{b}{a}$

$$\alpha + 4\alpha = \frac{5}{2}$$

$$5\alpha = \frac{5}{2}$$

$$\alpha = \frac{1}{2}$$

$$4\alpha = 2$$

The roots are $\frac{1}{2}$ *and 2*

ii) product of the root is $\frac{c}{a}$

$$\alpha(4\alpha) = \frac{k}{2}$$

$$4\alpha^2 = \frac{k}{2}$$

$$k = 8\alpha^2$$

But $\alpha = \frac{1}{2}$

Therefore $k = 2$

3. *Solve for x and y* $\frac{2y-4i}{2x+y} - \frac{y}{x-i} = 0$

$$\frac{2y-4i}{2x+y} = \frac{y}{x-i}$$

$$\frac{2y-4i}{2x+y} = \frac{y(x+i)}{(x-i)(x+i)}$$

$$\frac{2y-4i}{2x+y} = \frac{y(x+i)}{(x^2+1)}$$

$$\frac{2y}{2x+y} + \frac{4}{2x+y}i = \frac{xy}{x^2+1} + \frac{y}{x^2+1}i$$

Compare real parts

$$\frac{2y}{2x+y} = \frac{xy}{x^2+1}$$

$$2 = xy-------------i)$$

Compare Imaginary parts

$$\frac{4}{2x+y} = \frac{y}{x^2+1}$$

$$4x^2+4 = 2xy+y^2 -----ii)$$

Solving i) and ii) simultaneously

$$4x^2+4=2(2)+\left(\frac{2}{x}\right)^2$$

$$x^2=\frac{1}{x^2}$$

$$x^4=1$$

$x=1,-1$ *and* $y=2,-2$

4.*Given that* (x^2+r^2) *is a factor of* $(5x^4+px^3+qx^2+rx+4)$, *show that* $5r^2+4p^2-pqr=0$

for x^2+r^2 *a factor*

then $x^2+r^2=0$

$$x^2=-r^2$$

$$x=ir$$

$$5x^4+px^3+qx^2+rx+4=0$$

If x is a root

$$5(ir)^4+p(ir)^3-r^2q+rir+4=0$$

compare real parts

$$5r^4-r^2q+4=0-------(1)$$

compare imaginary parts

$$-pr^3+r^2=0$$

$$r^2(1-pr)=0$$

$r^2=0$ *or* $pr=1$

$$r = \frac{1}{p} - - - - - - - (2)$$

Substitute in equation (1)

$$5\frac{1}{p^4} - \frac{1}{p^2}q + 4 = 0$$

$$\frac{5}{p^2} - q + 4p^2 = 0$$

$$\frac{5}{p^2} + 4p^2 - q = 0$$

But $pr = 1$ *and* $r = \frac{1}{p}$

$$\therefore 5r^2 + 4p^2 - qpr = 0$$

5. Find the gradient of the curve $x^2 + 2xy - 2y^2 + x = 2$ *at point (1, -2)*

$$x^2 + 2xy - 2y^2 + x = 2$$

$$2x + 2x\frac{dy}{dx} + 2y - 4y\frac{dy}{dx} + 1 = 0$$

$$(2x - 4y)\frac{dy}{dx} = -1 - 2x - 2y$$

$$\frac{dy}{dx} = \frac{-1 - 2x - 2y}{(2x - 4y)}$$

At $(1, -2)$

$$\frac{dy}{dx} = \frac{-(1 + (2\times 1) + (2\times -2))}{(2\times 1) - (4\times -2)}$$

$$= -\left(\frac{1 + 2 - 4}{2 + 8}\right)$$

$$\frac{dy}{dx} = \frac{1}{10}$$

6. *Determine the expansion of* $\dfrac{x+4}{x^2-1}$ *in ascending powers of x up to the term containing* x^r *, for* $|x|>1$

$$\frac{x+4}{x^2-1} = (x+4)(x^2-1)^{-1}$$

Binomial Theorem

$$(a+b)^n = a + na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + --- + b^n$$

$$(x^2-1)^{-1} = (x^2)^{-1} + (-1)(x^2)^{-2}(-1) + (-1)(-2)(x^2)^{-3}(-1) + ---$$

$$(x^2-1)^{-1} = x^{-2} + x^{-4} + x^{-6} + ---$$

$$\left(\frac{x+4}{x^2-1}\right) = (x+4)(x^{-2} + x^{-4} + x^{-6} + ---)$$

$$= x^{-1} + x^{-3} + x^{-5} + 4x^{-2} + 4x^{-4} + 4x^{-6} + ---$$

$$\left(\frac{x+4}{x^2-1}\right) = \frac{1}{x} + \frac{4}{x^2} + \frac{1}{x^3} + \frac{4}{x^4} + \frac{1}{x^5} + \frac{4}{x^6} + ---$$

7. *If* $y^2 - 2y\sqrt{(x^2+1)} + x^2 = 0$ *show that* $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2+1}}$

$$y^2 - 2y\sqrt{(x^2+1\ )} + x^2 = 0$$

$$2y\frac{dy}{dx} - 2\frac{dy}{dx}(x^2+1)^{\frac{1}{2}} - 2y \times \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x + 2x = 0$$

$$2y\frac{dy}{dx} - 2\sqrt{(x^2+1)}\frac{dy}{dx} = \frac{2xy}{\sqrt{(x^2+1)}} - 2x$$

$$2\frac{dy}{dx}\left(y-\sqrt{(x^2+1)}\right)=2\left[\frac{xy-x\sqrt{x^2+1}}{\sqrt{x^2+1}}\right]$$

$$\frac{dy}{dx}=\frac{x\left(y-\sqrt{(x^2+1)}\right)}{\sqrt{(x^2+1)}\times\left[y-\sqrt{(x^2+1)}\right]}$$

$$\frac{dy}{dx}=\frac{x}{\sqrt{x^2+1}}$$

8. Find the sum of the series $3+6+11+---+(n^2+2)$

Sum

$$3+6+11+---+(n^2+2)=(1^2+2)+(2^2+2)+(3^2+2)+---+(n^2+2)$$

$$=(1^2+2^2+3^2+---+n^2)+(2+2+2+---+2)$$

$$=\frac{1}{6}n(n+1)(2n+1)+2n$$

$$3+6+11+---+(n^2+2)=\frac{n}{6}[(n+1)(2n+1)+12]$$

9a) Differentiate with respect to x:

$i)(1+x^2)$

$$\frac{d(1+x^2)}{dx}=2x$$

$ii)xy^2$

$$\frac{d(xy^2)}{dx}=x\left[\frac{d(y^2)}{dx}\right]+y^2\left[\frac{d(x)}{dx}\right]$$

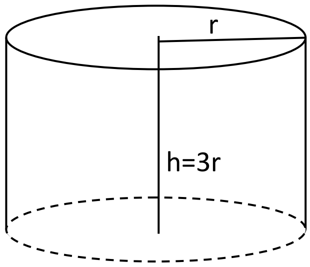
$$=x\times 2y\frac{dy}{dx}+y^2\times 1$$

$$\frac{d(xy^2)}{dx} = 2xy\frac{dy}{dx} + y^2$$

b) A closed right circular cylinder has base radius r cm and height 3r cm. if r is increased at the rate of 1mm/s, find the expression (in terms of r) for the rate of increase of the: i) total external surface area.

ii) Volume of cylinder.

$$\frac{dr}{dt} = 1mm/s$$
$$= 0.1cm/s$$
$$\text{h} = (3r)cm$$



i) total external surface area.

$$A = (2\pi r^2 + 2\pi rh) = 2\pi r^2 + 6\pi r^2 = 8\pi r^2$$

$$\frac{dA}{dr} = 16\pi r$$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = 1.6\pi r cm^2/s$$

ii) Volume of cylinder.

$$V = \pi r^2 h$$
$$= (\pi r^2)(3r)$$
$$= 3\pi r^3$$

$$\frac{dV}{dr} = \frac{d(3\pi r^3)}{dr}$$

$$\frac{dV}{dr} = 9\pi r^2$$

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = (9\pi r^2)(0.1)$$

$$\therefore \frac{dV}{dt} = \frac{9}{10}\pi r^2 \left(or 0.9\pi r^2\right) cm^3 / s$$

10. Given that $Z = (x + iy)$ *where x and y are real numbers;*

a) Show that when $\operatorname{Im}\left(\frac{z+i}{z+2}\right) = 0,$ *the point* (x, y) *lies on a straight line.*

$$z = x + iy$$

$$\operatorname{Im}\left(\frac{z+i}{z+2}\right) = 0$$

$$\operatorname{Im}\left(\frac{x+i(y+1)}{(x+2)+iy}\right) = 0$$

$$\operatorname{Im}\left[\frac{x+i(y+1)[(x+2)-iy]}{(x+2)^2+y^2}\right] = 0$$

$$\operatorname{Im}\left[\frac{x(x+2)-ixy+i(y+1)(x+2)+y(y+1)}{(x+2)^2+y^2}\right] = 0$$

Comparing imaginary parts

$$(y+1)(x+2) - yx = 0\left[\left(x^2+2\right)+y^2\right]$$

$$yx + 2y + x - yx + 2 = 0$$

$x + 2y + 2 = 0$ equation of the line

b) show that when $\text{Re}\left(\dfrac{z+i}{z+2}\right)=0,$ *the point* (x,y) *lies on a circle with centre* $\left(-1,-\frac{1}{2}\right)$ *and radius* $\dfrac{1}{2}\sqrt{5}$ *.*

$$\text{Re}\left(\frac{z+i}{z=2}\right)=0$$

$$\text{Re}\left(\frac{x+i(y+1)}{(x+2)+iy}\right)=0$$

$$\text{Re}\left(\frac{x(x+2)-iyx+i(y+1)(x+2)+y(y+1)}{(x+2)^2+y}\right)=0$$

Comparing real parts

$$\frac{x(x+2)+(y+1)}{(x+2)^2+y^2}=0$$

$$x^2+2x+y^2+y=0$$

Complete squares

$$x^2+2x+1+y^2+y+\frac{1}{4}=1+\frac{1}{4}$$

$$(x+1)^2+\left(y+\frac{1}{2}\right)^2=\frac{5}{4}$$

Circle centre $\left(-1,-\dfrac{1}{2}\right)$ and $r=\dfrac{1}{2}\sqrt{5}$ units

11a) use De Moivre's theorem to express $(\tan 5\theta)$ *in terms of* $\tan\theta$*.*

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin\theta$$

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$$

$$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin 5\theta$$

Compare real and imaginary parts

Re

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

Im

$$\sin 5\theta = 5c\cos^4\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

$$\frac{\sin 5\theta}{\cos 5\theta} = \frac{5\cos^4\theta\sin\ \theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\ \theta\sin^4\theta}$$

Divide both numerator and denominator by $\cos^5\theta$

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

11 b)Prove that: $$\frac{\sin\theta\sin\phi}{\cos\theta + \cos\phi} = \frac{2\tan\left(\frac{\theta}{2}\right)\tan\left(\frac{\phi}{2}\right)}{1 - \tan^2\left(\frac{\theta}{2}\right)\tan^2\left(\frac{\phi}{2}\right)}$$

From L.H.S

$$\frac{\sin\theta\sin\phi}{\cos\theta\cos\phi} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}{\left(2\cos^2\frac{\theta}{2} - 1\right) + \left(2\cos^2\frac{\phi}{2} - 1\right)}$$

$$= \frac{4\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\frac{\phi}{2}\cos\frac{\phi}{2}}{2\left(\cos^2\frac{\theta}{2} + \cos^2\frac{\phi}{2} - 1\right)}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\frac{\phi}{2}\cos\frac{\phi}{2}}{\cos\frac{\theta}{2} + \cos\frac{\phi}{2} - 1}$$

Divide through by $\cos^2\dfrac{\phi}{2}\cos^2\dfrac{\theta}{2}$

$$=\frac{2\tan\dfrac{\theta}{2}\tan\dfrac{\phi}{2}}{\dfrac{1}{\cos^2\dfrac{\phi}{2}}+\dfrac{1}{\cos^2\dfrac{\theta}{2}}-\dfrac{1}{\cos^2\dfrac{\theta}{2}\cos^2\dfrac{\phi}{2}}}$$

$$=\frac{2\tan\dfrac{\theta}{2}\tan\dfrac{\phi}{2}}{\sec^2\dfrac{\phi}{2}+\sec^2\dfrac{\theta}{2}-\sec^2\dfrac{\theta}{2}\sec^2\dfrac{\phi}{2}}$$

$$=\frac{2\tan\dfrac{\theta}{2}\tan\dfrac{\phi}{2}}{\left(1+\tan^2\dfrac{\phi}{2}\right)+\left(1+\tan^2\dfrac{\theta}{2}\right)-\left(1+\tan^2\dfrac{\theta}{2}\right)\left(1+\tan^2\dfrac{\phi}{2}\right)}$$

$$=\frac{2\tan\dfrac{\theta}{2}\tan\dfrac{\phi}{2}}{1+\tan^2\dfrac{\phi}{2}+1+\tan^2\dfrac{\theta}{2}-1-\tan^2\dfrac{\phi}{2}-\tan^2\dfrac{\theta}{2}-\tan^2\dfrac{\theta}{2}\tan^2\dfrac{\phi}{2}}$$

$$\frac{\sin\theta\sin\phi}{\cos\theta+\cos\phi}=\frac{2\tan\left(\dfrac{\theta}{2}\right)\tan\left(\dfrac{\phi}{2}\right)}{1-\tan^2\left(\dfrac{\theta}{2}\right)\tan^2\left(\dfrac{\phi}{2}\right)}$$

12a) A circle with centre A and radius r, externally touches both circle

$x^2+y^2=4$ *and* $x^2+y^2-6x+8=0$ *.show that the abscissa of point A is given*

by: $\left(\dfrac{r}{3}+2\right)$.

b) if $y = \frac{\sin x}{x}$, *prove that:* $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + xy = 0$

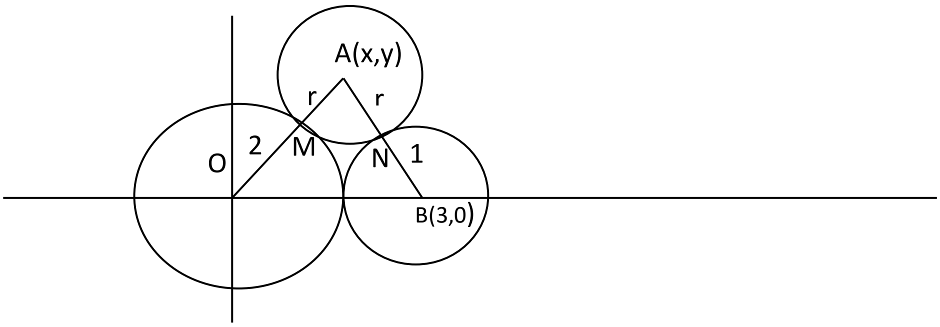
12a)

$$x^2 - 6x + y^2 = -8$$

$$x^2 - 6x + 9 + y^2 = -8 + 9$$

$$(x-3)^2 + (y+0)^2 = 1$$

Centre (3,0) or radius=1



$$\overline{OA} = \sqrt{x^2 + y^2}$$

$$r = \overline{OA} - \overline{OM}$$

$$r = \sqrt{x^2 + y^2} - 2$$

$$(r+2) = \sqrt{x^2 + y^2}$$

$$r^2 + 4r + 4 = x^2 + y^2 \text{ ------------}i)$$

$$\overline{AB} = \sqrt{(x-3)^2 + (y-0)^2}$$

$$AB = \sqrt{(x-3)^2 + y^2}$$

$$r = \overline{AB} - \overline{NB}$$

$$r = \sqrt{(x-3)^2 + y^2} - 1$$

$$(r+1)=\sqrt{(x-3)^2+y^2}$$

$$(r+1)^2=(x-3)^2+y^2$$

$$y^2=(r+1)^2-(x-3)^2 \text{ ----------}ii)$$

Substituting equation ii) in equation i)

$$r^2+4r+4=x^2+(r+1)^2-(x-3)^2$$

$$r^2+4r+4=x^2+r^2+2r+1-x^2+6x-9$$

$$2r+12=6x$$

$$x=\frac{2r+12}{6}$$

$$x=\left(\frac{r}{3}+2\right)$$

12 b) $y=\dfrac{\sin x}{x}$

$$xy=\sin x$$

$$\frac{d(xy)}{dx}=\frac{d(\sin x)}{dx}$$

$$y+x\frac{dx}{dy}=\cos x$$

$$y+x\frac{dy}{dx}=\cos x$$

$$\frac{dy}{dx}+\frac{d\left(x\frac{dy}{dx}\right)}{dx}=\frac{d(\cos x)}{dx}$$

$$\frac{dy}{dx}+x\frac{d^2y}{dx^2}+\frac{dy}{dx}=-\sin x$$

$$2\frac{dy}{dx}+x\frac{d^2y}{dx^2}+\sin x=0$$

But $xy = \sin x$

$$2\frac{dy}{dx} + x\frac{d^2y}{dx^2} + xy = 0$$

13a) $\int_0^{\frac{\pi}{2}} (\cos 2x \sin x)\, dx$

b)if $x = \left(\frac{1}{t^{-2}} - t\right)$ *and* $y = (t^2 - 4)$; *show that:* $\frac{d^2y}{dx^2} = -2(2t-1)^{-3}$

13 a) $\int_0^{\frac{\pi}{2}} \cos 2x \sin x dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} 2\cos 2x \sin x dx$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(2x + x) - \sin(2x - x) dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 3x dx - \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin x dx$$

$$= \frac{1}{2}\left[\frac{-\cos 3x}{3}\right]_0^{\frac{\pi}{2}} - \left[-\frac{1}{2}\cos x\right]_0^{\frac{\pi}{2}}$$

$$\therefore \int_0^{\frac{\pi}{2}} \cos 2x \sin x dx = \frac{1}{3}$$

13 b) $x = t^2 - t$ $\qquad y = t^2 - 4$

$$\frac{dx}{dt} = 2t - 1 \qquad \frac{dy}{dt} = 2t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$\frac{dy}{dx} = \frac{2t}{2t-1}$$

$$u = 2t, \qquad v = 2t - 1$$

$$\frac{du}{dt} = 2 \qquad \frac{dv}{dt} = 2$$

$$\frac{d\left(\frac{dy}{dx}\right)}{dt} = \frac{2(2t-1) - 2(2t)}{(2t-1)^2}$$

$$\frac{d\left(\frac{dy}{dx}\right)}{dt} = \frac{4t - 2 - 4t}{(2t-1)^2} = \frac{-2}{(2t-1)^2}$$

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dt} \cdot \frac{dt}{dx}$$

Thus $\frac{d^2y}{dx^2} = -2(2t-1)^{-2} \times (2t-1)^{-1}$

$$\frac{d^2y}{dx^2} = -2(2t-1)^{-3}$$

13 c) $a = 10 - 2t$

$$\frac{dv}{dt} = 10 - 2t$$

$$v = \int 10 - 2t dt$$

$$v = 10t - t^2 + c$$

At t=0 v=u=0

$$0 = c$$

$$v_t = 10t - t^2$$

When v=0

$$0 = 10t - t^2$$

$$0 = (10 - t)t$$

$$t = 10s$$

$$\frac{ds}{st} = 10t - t^2$$

$$s = \int 10t - t^2 dt$$

$$s = 5t^2 - \frac{t^3}{3} + c$$

At $t = 0, s = 0$ *then c=0*

At t=10s

$$s = 5 \times 10^2 - \frac{10^3}{3}$$

$$= 500 - \frac{1000}{3}$$

$$s = 166.7m$$

14 a) Given that $r_1 = (i + j + k) + \lambda(-4i + j + k)$ *and*

$r_2 = (2k + j) + \mu(2i - j + k)$ *are two lines, find*

i) their point of intersection.

ii) the angle between them.

*b) Find the resolved part of vector **a**=7i +2j in the direction of **b**= 2i – 3j.*

14a) i) $$r_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}$$

$$r_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$r_1 = r_2$$

$$1-4\lambda = 2\mu ----i$$
$$1+\lambda = 1-\mu ----ii$$
$$1+\lambda = 2+\mu ----iii$$

$$ii)+iii)$$

$$2+2\lambda = 3$$
$$2\lambda = 1$$
$$\lambda = \frac{1}{2}$$

Point of intersection $\left(-1, \frac{3}{2}, \frac{3}{2}\right)$

ii)

$$r_1 \text{ is parallel to } \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} = u$$

$$r_2 \text{ is parallel to } \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = v$$

From $u.v = |u||v|\cos\theta$

$$\cos\theta = \frac{(-8-1+1)}{\sqrt{4^2+1^2+1^2} \cdot \sqrt{2^2+1^2+1^2}}$$

$$\theta = \cos^{-1}\left(\frac{-8}{\sqrt{18}\times\sqrt{6}}\right)$$

$$\theta = 140.34^o$$

14b)

Resolved part of ***a*** in the direction of ***b***
$$= \frac{\underline{a} \cdot \underline{b}}{|\underline{b}|}$$

$$= \frac{(7i + 2j)(2i - 3j)}{\sqrt{2^2 + 3^2}}$$

$$= \frac{(7 \times 2) + (2 \times -3)}{\sqrt{13}}$$

$$= \frac{14 - 6}{\sqrt{13}}$$

$$= \frac{8}{\sqrt{13}}$$

15 a) Suppose that an error of 5% is made in measuring the radius of a sphere.

Find the corresponding percentage error in the surface area.

b) Using the technique of small changes, find the approximate value of $(10.02)^3$

15a) Let dr = change in radius and dA = change in S.A

$$\frac{dr}{r} = \frac{5}{100}$$

$$dr = 0.05r ----i$$

$$A = 4\pi r^2$$

$$\frac{dA}{dr} = 8\pi r ----ii$$

$$dA = \frac{dA}{dr}.dr = (8\pi r)(0.05r)$$

$$dA = 0.4\pi r^2$$

$$\text{But \% error in S.A} = \frac{dA}{A} = \frac{0.4\pi r^2}{4\pi r^2}$$

$$= \frac{1}{10} = 10\%$$

15b) Let $y = x^3$ *, where x=10,* $\delta x = 0.02$

$$\frac{dy}{dx} = 3x^2$$

$$\delta y \approx \frac{dy}{dx}.\delta x = 3x^2(0.02)\Big]_{x=10} = 6$$

$$(10.02)^3 = y + \delta y = x^3 + 6\Big]_{x=10}$$

$$\approx 10^3 + 6$$
$$\approx 1006$$

EXERCISE 6

1. The 3rd term of a convergent geometric progression (G.P) is the arithmetic mean of the 1st and 2nd terms. Find the common ratio and if the first term is l, find the sum to infinity.

2. Prove that $\displaystyle\sum_{r=1}^{n} r(3^r) = \frac{(2n-1)3^{n+1}+3}{4}$

3. Show that the circle $x^2 + y^2 = 36$ and $x^2 + y^2 - 6x - 8y + 24 = 0$ touch each other internally. Hence find their point of contact.

4. Write $4 + 2x^2 + 5x$ in the form $a(x + b)^2 + c$. Hence find its minimum value.

5. Given that $y = (3x-2)e^{2x}$, Show that $(3x-2)\frac{dy}{dx} = (6x-1)y$

6. Find the equation of a circle with centre on the y-axis, which cuts orthogonally each of the circles $x^2 + y^2 + 6x + 2y - 9 = 0$ and

   $x^2 + y^2 - 2x - 2y + 1 = 0$.

7. Three unequal numbers a, b, c are such that $^1/a$, $^1/b$, $^1/c$ are in arithmetical progression and a, c, b are in geometrical progression.

   Prove that b, a, c are in arithmetical progression.

8. Find the relation between q and r so that $x^3 + 3px^2 + qx = r$, should be a perfect cube for all values of x.

9. i) Find $\int \frac{x^3 - a^2x + a^3}{(x^2 - a^2)} dx$

(ii) Given that $y = x\sin(y + \varphi\pi)$

Show that $\frac{dy}{dx} = \frac{y + y(x^2 - y^2)^{1/2}}{x(1 + y^2 - x^2)}$

10. (a) Prove that in a triangle ABC.

$$\frac{1}{a}\cos^2\frac{A}{2} + \frac{1}{b}\cos^2\frac{B}{2} + \frac{1}{c}\cos^2\frac{C}{2} = \frac{(a+b+c)^2}{4abc}$$

(b) A line through the vertex M of a triangle MNQ meets the base at L making an angle MLQ = β, Show that

NL:LQ = (cot N – cot β):(cotQ + Cotβ).

11. (a) Solve the equation $Z^4 + 6Z^2 + 25 = 0$, expressing each of the roots of the equation in the form x + iy.

(b) If n is an integer and $Z = \cos\theta + i\sin\theta$, show that

$2\cos n\theta = z^n + \frac{1}{z^n}$

hence establish the formula $8\cos^4\theta = 6 + 2\cos 4\theta + 4\cos 2\theta$

12. (a) By solving $\frac{1}{x}\frac{dy}{dx} = \sqrt{(x-1)}$

Show that the general solution can be expressed as

$$y = \frac{2}{15}(x-1)(3x+2)\sqrt{(x-1)} + A$$

(b) A curve is such that at any point twice the gradient multiplied by $(1 + x^2)$ is equal to the x coordinate multiplied by $(4 - y^2)$. If the curve passes through the point (O,1) express y as a function of x.

13. Given that the curve $xy(x+2) = px + q$ has gradient zero at the point (1, -2). Find the value of p and q. Hence sketch the curve $xy(x+2) = px + q$.

14. (a) Find the point of intersection of a line and the plane r. (2i – j + k) = 4. If the vector equation of this line passes through the point (3,1,2) and is perpendicular to this plane.

    (b) Find the perpendicular distance of the point (4, -3, 10) to the line

$$\frac{x-1}{3} = \frac{y-2}{-1} = \frac{z-3}{2}$$

15. (a) Find the equation of the tangents to the hyperbola x = 4t, ty = 4 which passes through the point (4, 3).

    (b) Find the equation of the parabola whose focus is (-1, 1) and directrix x = y.

EXERCISE 7

1. Solve the equation $\log_2(x+y)=0$ and $\log_2 x - \frac{2}{\log_y 2} = 1$

2. By use of the *t-formula*, solve $\sin\theta + 2\cos\theta = 1$ for angles between 0^o and 360^o inclusive.

3. By a method of progressions, express 2.816816... in the form $\frac{y}{x}$ where x and y are integers and $x \neq 0$

4. Solve the equation $x^2 + 3x - 2 = \frac{8}{x^2+3x}$

5. Prove that $\log_a N = \frac{\log_b N}{\log_b a}$ hence deduce that $\log_9 4 = \frac{\log_3 4}{2}$

6. Solve the equation $\cos x + \sin x = \sec x$ for $0^o \le x \le 360^o$

7. i) Rationalize $\frac{1}{\sqrt{6}+\sqrt{2}+\sqrt{3}}$

ii) show that $\frac{1+\tan 60^o}{1-\tan 60^o} = -(2+\sqrt{3})$

8. Given that $x = \sin\theta + \cos\theta$ and $y = \tan\theta$; Show that $(y+1)^2 = x^2(1+y^2)$

9. i) Find the values of p and q so that *(x+1) and (x-2)* shall be factors of $x^3 + px^2 + 2x + q$, Determine the third factor.

ii) Solve for x correct to 2 significant figures

$$(4^{2x+1})(5^{x-2}) = 6^{1-x}$$

10. (a) If A, B and C are angles of a triangle, prove that

$$\sin 2A + \sin 2B + \sin 2C = 4 SinA \sin B \sin C$$

(b) By substituting $t = \tan\theta$ show that $\left(\frac{1-\sin 2\theta}{1+\sin 2\theta}\right)^{½} = \frac{1-\tan\theta}{1+\tan\theta}$

EXERCISE 8

1. Solve the equations for values of θ between 0^o and 360^o

a) $\cos(\theta - 30^o) = 2\sin\theta$ b) $5 + 3\tan^2\theta = 7\sec\theta$

2. Prove that $\frac{1}{8}(3 + 4\cos 2A + \cos 4A) = \cos^4 A$

3. Express 0.5270270... in the form $\frac{y}{x}$ where x and y are integers and $x \neq 0$

4. Solve the equation $9^{x+1} - 3^{x+3} - 3^x + 3 = 0$

5. Sketch the curve for $\tan\theta$, $0^0 \le \theta \le 360^o$

6. a) Prove that $\log_b a = \frac{\log_c a}{\log_c b}$

b) Solve the equation $1 + 2\log_4 x = \log_x 4$

7. Rationalize the denominator $\frac{1+2\sqrt{2}}{5-3\sqrt{2}}$

8. Given that $x=\sin\theta+\cos\theta$ and $y=\tan\theta$; eliminate θ

9. Prove that $\frac{\sec\theta-\operatorname{cosec}\theta}{\tan\theta-\cot\theta}=\frac{\tan\theta+\cot\theta}{\sec\theta+\operatorname{cosec}\theta}$

10. Simplify $\frac{1}{(1-\sin 45^\circ)^2}$ leaving surds in your answers.

EXERCISE 9

1. Find value of m for which the equation $x^2+(m+3)x+4m=0$ has equal roots. For what value of m is the sum of the roots zero?

2. Find the values of x between 0^0 and 360^0 satisfying the equation $10\sin^2 x+10\sin x\cos x-\cos^2 x=2$

3. If $\log_a n=x$ and $\log_c n=y$, $n\neq 1$, prove that

   $$\frac{x-y}{x+y}=\frac{\log_b c-\log_b a}{\log_b c+\log_b a}$$

4. Show that $\sin^3\theta-\cos^3\theta=(\sin\theta-\cos\theta)(1+\sin\theta\cos\theta)$

5. Prove that $\cos\theta=\frac{1-t^2}{1+t^2}$ where ($t=\tan\frac{1}{2}\theta$)

6. Prove that, if the difference between the roots and the equation $ax^2+bx+c=0$ is 1, then $a^2=b^2-4ac$

7. Solve the equation $\log_{49} 7W + \log_7 49W = 7$.

8. Given that the roots of the equation $2x^2-4x+12=0$ are m and n,

   Find the equation whose roots are: $(2+m)^{-2}$ and $(2+n)^{-2}$

9. Prove that (Cosec A – Sin A)(Sec A – Cos A) = $\dfrac{1}{\tan A+\cot A}$

10. (a) Prove that $2\cot 2A\cot\frac{1}{2}A=\dfrac{\cos A}{1-\cos A}-\dfrac{1+\cos A}{\cos A}$

(b) Find the maximum and minimum values of $2\sin\theta-5\cos\theta$ by expressing it in the form $R\sin(\theta-\alpha)$,and find the corresponding values of θ between 0^o and 360^0

EXERCISE 10

1. Solve for x, $2(2^x)+3(2^{-x})=5$

2. The curve $y=f(x)$ is such that $f(1)=2$ and $f(x+\delta x)=f(x)+3x^2\delta x-\dfrac{6\delta x}{x}+x(\delta x)^2$ where δx is a small increment in *x*. Find the equation of the curve.

3. Prove that: $\dfrac{\sin 4\theta+\sin 2\theta}{\cos 4\theta+\cos 2\theta}=\tan 3\theta$

4. If *y* is a product of two functions *u* and *v* of a variable *x* such that $y=uv$ then using first principles show that $\dfrac{dy}{dx}=v\dfrac{du}{dx}+u\dfrac{dv}{dx}$

5. Simplify the expression $\dfrac{5}{4-3i}+\dfrac{3}{3-2i}$ giving your answer in the form $a+bi$, hence find $|a+bi|$

6. Show that if the equations $x^2+bx+c=0, x^2+px+q=0$ have a common root then $(c-q)^2=(b-p)(cp-bq)$.

7. If the series $(1-x)^{-1}$ can be expanded as far as the term in x^3 in ascending powers of x, find the condition that the series is convergent.

8. Prove that, if the sum of the radii of two circles remains constant, the sum of the areas of the circles is least when the circles are equal.

9. i) Prove that if x is so small that the cube and higher powers can be neglected then

$$\sqrt{\frac{1+x}{1-x}} = 1 + x + \tfrac{1}{2}x^2$$

By taking $x = \frac{1}{9}$, prove that $\sqrt{5}$ is approximately $\frac{181}{81}$

ii) Obtain the first four terms of the expansion $\frac{(1+x)}{(1-x)^3}$ in a series of ascending powers of x

10.(a) In a triangle ABC prove that if the internal bisector of A meets BC at D the $AD = \dfrac{2bc}{b+c}\cos\frac{A}{2}$

(b) Prove that $\cos\theta = \dfrac{1-t^2}{1+t^2}$, where $(t = \tan\frac{1}{2}\theta)$

11. (i) Show that $3x^2 + 10x + 9 = f(x)$ cannot be negative and find the least value of *f(x)*.

(ii) By using the substitution $y = \dfrac{1}{x} + x$, solve the equation

$$2x^4 - 9x^3 + 14x^2 - 9x + 2 = 0$$

(iii) Solve $n^3 + 8 = 0$

12. (a) If $y = \sqrt{(5x^2+3)}$, Show that $y\dfrac{d^2y}{d^2x} + \left(\dfrac{dy}{dx}\right)^2 = 5$

(b) Find three numbers in arithmetical progression such that their sum is 27 and their product is 504.

13. (a) A spherical balloon is blown up so that its volume increases at a constant rate of $2cm^3/s$. Find the rate of increase of the radius when the volume of the balloon is $50\ cm^3$.

(b) Find the points of intersection of the x-axis and the curve $y = x^3 - 3x^2 + 2x$, find also the equation of the tangent and normal to the curve at each of these points.

EXERCISE 11

1. Show that $\dfrac{x^{\frac{3}{2}} + xy}{xy - y^3} - \dfrac{\sqrt{x}}{\sqrt{x} - y} = \dfrac{\sqrt{x}}{y}$

2. Using Heron's formula, find the area of a triangle with sides 6, 14, 16 units

3. Prove by induction that $8^n - 7n + 6$ is divisible by 7 for all positive integral values of n.

4. Find numbers a and b such that $3 + \sqrt{2} = (a + b)(6 - \sqrt{2})^2$

5. Find by completing squares the range of the function $f(x) = 10(2x + 1) - 5x^2$

6. Indicate on an argand diagram the region in which Z lies if $Arg(Z - w) > \dfrac{\pi}{4}$

If z is complex and w is real constant.

7. The first 3 terms of a G.P are (K-3), (2K-4) and (4K-3) respectively. Find the sum of the first eight terms of the progression.

8. If $\tan\alpha = \dfrac{a}{a+1}$ and $\tan\beta = \dfrac{1}{2a+1}$, Find the smallest values of the angle $\alpha + \beta$.

9. A particle starting from rest at O moves along a straight line OA so that its acceleration after t seconds is $(24t - 12t^2)m/s^2$.

a) Find when it again returns to O and its velocity then.

b) Find its maximum displacement from O during this interval.

c) What is its maximum velocity?

10. a) Expand $(1-x)^{1/2}$ as far as the term containing x^3. Use your expansion to evaluate $\sqrt{2}$ correct to four decimal places.

b) Express the recurring decimal 0.1576576.. as fraction in its lowest form.

11. i) Prove that $\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = \operatorname{cosec}\theta - \cot\theta$

ii) Sketch $y = |\sin x|$, in the interval, $0 \le x \le 2\pi$

iii) Express $4\sin\theta - 3\cos\theta$ in the form $R\sin(\theta - \alpha)$, where α is acute.

iv) Find the greatest and least value of $\dfrac{1}{6+4\sin\theta - 3\cos\theta}$

12. Sketch the graph of the function $f(x) = \dfrac{1}{3}x^3 - 2x^2 + 3x$

EXERCISE 12

1. Find the acute angle between the lines whose equations are $r = 2i + 3j - k + \beta(-4i + 3j - k)$ and $r = 3i + j - k + \lambda(2i + 6j - 5k)$.

2. Given that $y = \tan^{-1}\left(\dfrac{1-x}{1+x}\right)$, show that $\dfrac{dy}{dx} = \dfrac{-1}{1+x^2}$.

3. Given the lines $3y = 4x$, $4x + 3y = 0$ and $y = 8$ are tangents of a circle find the radius of the circle.

4. Using De Moivre theorem, solve $x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$.

5. Show that the maximum value of the volume of a right cylinder of height *h*, situated symmetrically inside a sphere of radius, *a is* $\dfrac{4\pi a^3}{3\sqrt{3}}$.

6. Determine the Cartesian equation of the plane containing the point $(4,-1,3)$ and the line $\dfrac{x-1}{1} = \dfrac{y-3}{1} = \dfrac{z+2}{5}$.

7. The 2nd, 3rd, and 9th terms of an AP form a geometric progression. Find the common ratio of the GP given that the common difference is none-zero.

8. The tangents from the curves $x^2 + y^2 = 8$ and $x^2 = 2y$ meet at a point (2,2) find the angle between the tangents.

9. i) y is defined by the formula $y = a + bx^n$, for different values of x corresponding values of y are given as below

x	1	2	4
y	7	10	12

Find the values of a and b. Show that $n = \log_2\left(\frac{2}{3}\right)$

ii) If $y_n = a + bx^n$ where $n = 1,2,3..$ Show that $(y_2 - y_1),(y_3 - y_2),(y_4 - y_3)..$ is a GP and state its common ratio

iii) Find the Cartesian equation of the curve given by the parametric equations

$$x = 20t$$
$$y = 90 - 5t^2$$

10. (a) Solve the equation $5\cos^2 3\theta = 3(1+\sin 3\theta), 0^o \le \theta \le 180^o$

(b) Evaluate $\int_0^2 3\cos\frac{1}{2}x dx$ and find $\int (2x+3)^2 dx$

11. (i) Solve the pair of simultaneous equations $\log(y - x) = 0$ and

$2\log y = \log(21 + x)$

(ii) The roots of the equation $2x^2 + px + q = 0$ are $2\alpha + \beta$ and $\alpha + 2\beta$.

Calculate the value of p and q.

(iii) Find the values of *a, b* and *c* for which $f(x) = ax^2 + bx + c$ leaves

remainders 1, 25, 1 on division by $x - 1, x + 1$ and $x - 2$ respectively.

12. (a) From first principles, find the derivative of $x^2 + \sec x$

(b) If $\sin y = 2\sin x$ show that $\left(\frac{dy}{dx}\right)^2 = 1 + 3\sec^2 y$

13. (a) Write down and simplify the binomial expansion of $(1+2x)^{-\frac{1}{2}}$ up to and including the term in x^3. By putting $x = \frac{1}{8}$, Use your expansion to show that $\sqrt{5} = \frac{2048}{915}$

(b) Evaluate $\sum_{i=1}^{n}(4i+2)$

EXERCISE 13

1. Given the vectors **a** and **b**, define their scalar product of **a** and **b**, thus show that their scalar product is commutative.

2. Given that $Z_2 + 5i = 2$ and $3Z_1 - 11 = (2 + 4Z_1)i$. Find the constants a and b such that $aZ_1 + bZ_2 + 4 = i$

3. Prove that if $\log_5 n = p$ and $\log_{75} n = q$, then $3^q = 5^{p-2q}$

4. Obtain the term independent of t in the expansion of $\left(4t^2 - \frac{1}{3t^3}\right)^{20}$

5. $\int_0^{\pi/4} Cos3xSin5xdx$

6. Find the vector equation of a straight line that passes through a point with position vector $2\boldsymbol{i} + 3\boldsymbol{j}$ and which is perpendicular to the line $\boldsymbol{r} = 3i + 2j + \lambda(i - 2j)$

7. Solve the equation; $Sin3x - Cos2x = Cos3x - Sin2x$; $0 \le x \le \pi$

8. Show that $\frac{d}{dx}\left(\frac{1+\sin x+\cos x}{1-\sin x+\cos x}\right) = \frac{1}{(1-\sin x)}$

9. a) Given that $y = x\sqrt{(3+2x)}$, show that $y^2\frac{d^2y}{dx^2} + 3x(1+x)\frac{dy}{dx} - 3y(1+2x) = 0$

   b) Differentiate from first principles $\sec x = y^2$

10. a) The sum of the first four terms of an A.P is 48 and the sum of the first seven is 63. Find the tenth term and the sum of the first 10 terms.

    b) Determine the square root of a complex number i

11. a) Determine the coordinates of the point of intersection of the lines with vector equations $r = \begin{pmatrix} 1-t \\ 2+4t \\ -3+3t \end{pmatrix}$ and $r = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \beta\begin{pmatrix} 3 \\ -11 \\ 1 \end{pmatrix}$ where t and β are scalars.

    b) Find the Cartesian equation of the plane containing the two lines in 11 a) above.

EXERCISE 14

1. Find the derivative of $\frac{1}{\sqrt{x}}$ from first principles

2. If $\sec A - \tan A = q$, prove that $\tan\frac{1}{2}A = \frac{1-q}{1+q}$

3. Find the sum of the following series

$$2 + 10 + 20 + \ldots + (n^3 + n)$$

4. Solve $\sin x + \sin 2x + \sin 3x = 0$, for $0^0 \le x \le 360^o$

5. Given that $U_n = A\left(\frac{3}{B}\right)^n$ where n=1, 2... with second and third term equal to $\frac{9}{32}$ and $\frac{27}{128}$ respectively. Find $\sum_1^\infty U_n$

6. In the expansion of $(1+ax+2x^2)^6$ in powers of x, the coefficients of x^2 and x^{11} are 27 and -192 respectively. Find the valve of a and coefficients of x^3.

7. Prove that $9^n - 1$ is always a multiple of 8, $n \geq 1$

8. Prove that $\dfrac{\cos A + \sin A}{\cos A - \sin A} = \sec A + \tan 2A$

9. a) Prove that $\sin^3 A + \sin^3(120^o + A) + \sin^3(240^o + A) + \dfrac{3}{4}\sin 3A = 0$

   b) If any triangle ABC $\sin\theta = \dfrac{2\sqrt{bc}}{b+c}\cos\dfrac{1}{2}A$, prove that $(b+c) = \dfrac{a}{\cos\theta}$

10. a) If the equation $a^2x^2 + 6abx + 8b^2 = -ac$ has equal roots, prove that the roots of the equation $ac(x+1)^2 = 4b^2x$

    b) The first, second, third and n^{th} terms of a series are 4, -3, -16 and $(an^2 + bn + c)$ respectively. Find *a, b, c.*

11. a) Expand $\left(\dfrac{1-x}{x+1}\right)^{1/2}$ in ascending powers of x as far as the term in x^2 and hence show that $\sqrt{7} = 2\frac{83}{128}$.

    b) Find three numbers in a G.P such that their sum is 39 and their product is 729.

12. a) If in a triangle ABC, $ab = c^2$, prove that

$$\cos(A-B) = \cos C + \cos 2C = 1$$

    b) If $x = \log_a bc$, $y = \log_b ca$, $z = \log_c ba$. Prove that $x + y + z + 2 = xyz$.

13. a) Show that $a^2 + b^2 + c^2 - bc - ca - ab$ cannot be negative and state the condition under which it is zero.

b) Find the rational numbers a and b such that

$$3+\sqrt{2}=\left(6-\sqrt{2}\right)^2\left(a+b\sqrt{2}\right)$$

EXERCISE 15

1. Represent the complex numbers below on the same argand diagram and find the modulus in each case.

   a) $1+i$ b) $-3+2i$ c) $8+3i$

2. Solve the equation $\dfrac{16^x-4^x}{4^x+2^x}=5(2^x)-8$

3. Express $\sqrt{(3+4i)}$ in the form a+ bi

4. State the zero product property and using its knowledge, solve $2x^2+7x-15=0$

5. a) Show that $\dfrac{n!}{(n-r)!r!}+\dfrac{n!}{(n-r+1)!(r-1)!}=\dfrac{(n+1)!}{r!(n-r+1)!}$

   b) Solve the equation $1+2\log_4 x=\log_x 4$

6. Find the values of p and q which make $x^4+6x^3+13x^2+px+q$ a perfect square.

7. Find from first principles, the derivatives of the function $f(x)=tx^4$, t is a constant.

8. Prove that $\dfrac{\sin^2 5A-\sin^2 A}{\cos^2 A-\cos^2 3A}=1+2\cos 4A$

9. Prove by induction $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$

Worked examples

1. *If the velocity of a body varies inversely as the square root of the distance, prove that the acceleration varies as the fourth power of the velocity.*

Denote the distance travelled by s, the velocity by v and acceleration by a

$$v \propto \frac{1}{\sqrt{s}}$$

$$v = \frac{k}{\sqrt{s}} ------(i) \qquad k \text{ is a constant}$$

$$a = \frac{dv}{dt}$$

But $v = \dfrac{ds}{dt}$

$$\therefore dt = \frac{ds}{v}$$

$$a = v\frac{dv}{ds}$$

From (i) $\dfrac{dv}{ds} = \dfrac{-k}{2s^{\frac{3}{2}}}$

$$\therefore a = \frac{k}{\sqrt{s}}\left(\frac{-k}{2s^{\frac{3}{2}}}\right)$$

$$= \frac{-k^2}{2s^2} \qquad \text{but } \sqrt{s} = \frac{k}{v} \qquad s = \frac{k^2}{v^2}$$

$$= \frac{-k^2}{2}\frac{v^4}{k^4}$$

$$a = \frac{-v^4}{2k^2}$$

2. *Express the complex number* $\dfrac{7-3i}{(2-4i)^2}$ *in the form* $x+iy$ *and find* $|x+iy|$

$$\frac{7\_3i}{(2-4i)^2}=\frac{7-3i}{4-16i+16i^2}$$

$$=\frac{7-3i}{-16i-12}$$

$$=\frac{7-3i}{-4(3+4i)}$$

$$=\frac{(7-3i)(3-4i)}{-4(3+4i)(3-4i)}$$

$$=\frac{9-37i}{-100}$$

$$\frac{7-3i}{(2-4i)^2}=0.09+0.037i$$

$$\left|\frac{-9}{100}+\frac{37}{100}i\right|=\sqrt{\left(\frac{-9}{100}\right)^2+\left(\frac{37}{100}\right)^2}$$

= 0.38078

3. *Find* $\int Sin^2xCos^2xdx$

$$\int \sin^2 x\cos^2 xdx=\int (1-\cos^2 x)\cos^2 xdx$$

$$=\int \cos^2 xdx-\int \cos^4 xdx$$

$$=\int \frac{1+\cos 2x}{2}dx-\int\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$=\frac{x}{2}+\frac{\sin 2x}{2}-\frac{1}{4}\int (1+2\cos 2x+\cos^2 2x)dx$$

$$=\frac{x}{4}+\frac{\sin 2x}{4}-\frac{1}{2}\int \cos 2x dx-\frac{1}{4}\int \frac{1+\cos 4x}{2}dx$$

$$=\frac{x+\sin 2x}{4}-\frac{\sin 2x}{4}-\frac{x}{8}-\frac{\sin 4x}{32}+c$$

$$=\frac{x}{8}-\frac{\sin 4x}{32}+c$$

$$\int Sin^2 xCos^2 xdx = \frac{1}{8}\left(x-\frac{\sin 4x}{4}\right)+c$$

4. Find the position vector of the point of intersection of the lines

$$r = 2i-3j+4k+\mu(6i+7j-k) \text{ and } r = 2i-12j-k+\lambda(-3i+j+3k)$$

$$. \ r_1=\begin{pmatrix} 2+6\mu \\ -3+7\mu \\ 4-\mu \end{pmatrix} \quad \text{and} \quad r_2=\begin{pmatrix} 2-3\lambda \\ -12+\lambda \\ -1+3\lambda \end{pmatrix}$$

$$r_1 = r_2$$

$$2+6\mu = 2-3\lambda ---------(i)$$
$$-3+7\mu = -12+\lambda -------(ii)$$
$$4-\mu = -1+3\lambda ---------(iii)$$

(i) + *(ii)*

$$6+5\mu = 1$$
$$5\mu = -5$$
$$\mu = -1$$

$$\text{Position vector is } \begin{pmatrix} 2-6 \\ -3-7 \\ 4+1 \end{pmatrix} = \begin{pmatrix} -4 \\ -10 \\ 5 \end{pmatrix}$$

5. *Find* $\int \sqrt{(1+\cos x)}dx$

Method I

$$\int \sqrt{(1+\cos x)}dx = \int \left(1+1-2\sin^2 \frac{x}{2}\right)^{\frac{1}{2}} dx$$

$$= \int \left(2-2\sin^2 \frac{x}{2}\right)^{\frac{1}{2}} dx$$

$$= \sqrt{2}\int \left(1-\sin^2 \frac{x}{2}\right)^{\frac{1}{2}} dx$$

$$= \sqrt{2}\int \sqrt{\cos^2 \frac{x}{2}}dx$$

$$= \sqrt{2}\int \cos\frac{x}{2}\, dx$$

$$= \sqrt{2}\left(\frac{\sin\frac{x}{2}}{\frac{1}{2}}\right)$$

$$\int \sqrt{1+\cos x}dx = 2\sqrt{2}\sin\frac{x}{2} + c$$

Method II

$$\int \sqrt{(1+\cos x)}dx = \int \sqrt{(1+\cos x)\left(\frac{1-\cos x}{1-\cos x}\right)}dx.$$

$$= \int \sqrt{\left(\frac{1-\cos^2 x}{1-\cos x}\right)}$$

$$= \int \frac{\sin x}{\sqrt{1-\cos x}}\, dx$$

Let $u = 1 - \cos x$ $\qquad dx = \dfrac{du}{\sin x}$

$$= \int \frac{\sin x}{\sqrt{u}} \cdot \frac{du}{\sin x}$$

$$= \int \frac{1}{\sqrt{u}} du$$

$$= 2u^{\frac{1}{2}} + c$$

$$= 2(1 - \cos x)^{\frac{1}{2}} + c$$

$$= 2\left(2\sin^2 \frac{x}{2}\right)^{½} + c$$

$$\int \sqrt{1 + \cos x}\, dx = 2\sqrt{2} \sin \frac{x}{2} + c$$

6. Sketch the curve $y = \dfrac{(x-1)^2}{x+1}$

Intercept when $x = 0, y = 1$

When $y = 0, x = 1$

Turning point for $y = \dfrac{(x^2 - 2x + 1)}{x+1}$

$$\frac{dy}{dx} = \frac{(x+1)(2x-2) - (x^2 - 2x + 1)(1)}{(x+1)^2} = 0$$

$$(x+1)(2x-2) - x^2 + 2x + 1 = 0$$

$$2x^2 - 2x + 2x - x + 2x - 1 = 0$$

$$x^2 + 2x - 3 = 0$$

$$x^2 - x + 3x - 3 = 0$$

$$x(x-1) + 3(x-1), x = -3, x = 1$$

When $x = -3$, $y = \dfrac{(-3-1)^2}{-3+1} = -8$, $(-3, -8)$

$$x = 1 \quad , \quad y = \frac{(1-1)^2}{-1+1} = 0$$

Nature of turning points:

For: $\dfrac{dy}{dx} = \dfrac{x^2+2x-3}{(x+1)^2}$

$$\frac{d^2y}{dx^2} = \frac{(x+1)^2[2x+2] - 2(x^2+2x-3)(x+1)}{(x+1)^4}$$

$$= \frac{(x+1)}{(x+1)^4}\left[(x+1)(2x+2) - 2(x^2+2x-3)\right]$$

$$= \frac{(x+1)(2x+1) - 2(x^2+2x-3)}{(x+1)^3}$$

$$\left.\frac{dy}{dx}\right|_{x=-3} < 0 \ , (-3,-8) \ \textit{maximum point}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} > 0 \ , (1,0) \ \textit{minimum point}$$

Asymptotes.

Vertical asymptote, $, x+1=0$ *then* $x=-1$

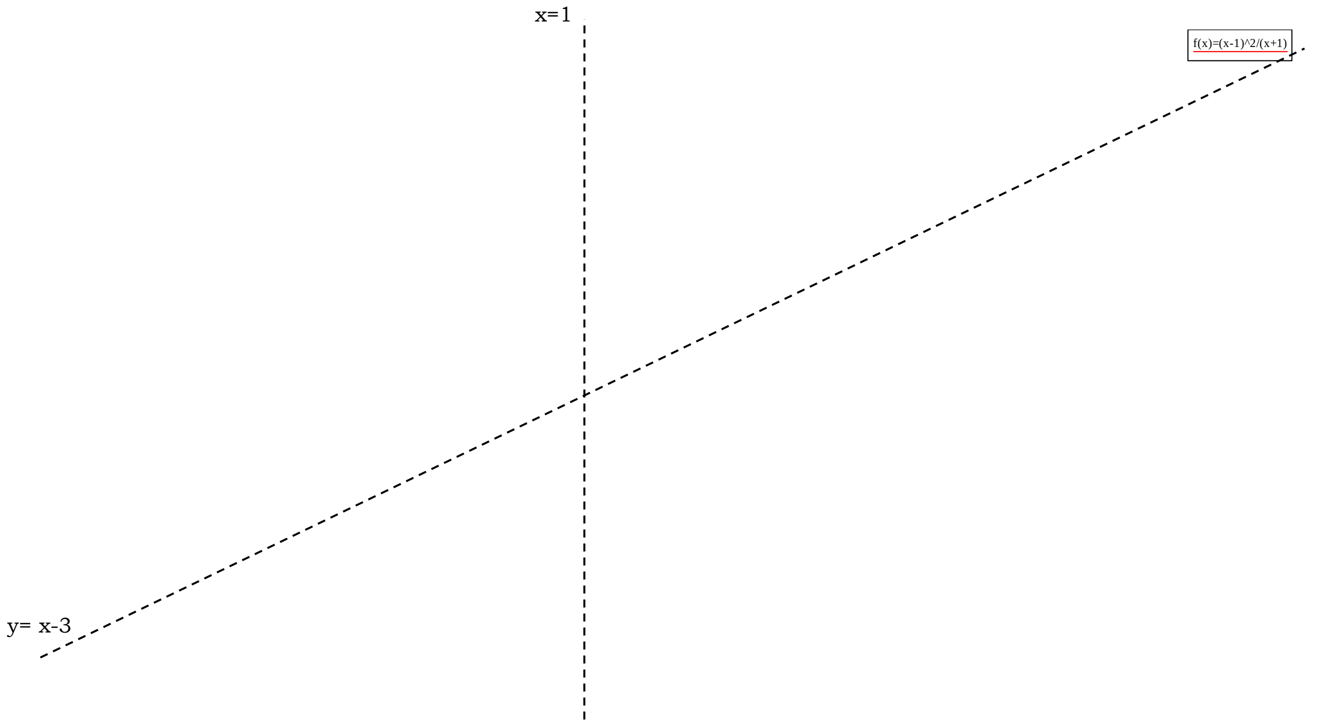
Slanting asymptote for: $y = \dfrac{(x-1)^2}{x+1} = \dfrac{x^2-2x+1}{x+1} = (x-3) + \dfrac{4}{x+1}$

As $x \to \infty, y = x-3$

Critical values are $x = -1, 1$

Table of existence:

	-1>x	-1<x>1	x>1
$(x-1)^2$	+	+	+
$(x+1)$	-	+	+
y	-	+	+



7. Evaluate $\int_{1}^{\sqrt{3}} \frac{1}{1+x^2}\,dx$; leave π in your answer.

Let $x = \tan\theta$

$$\frac{dx}{d\theta} = \sec^2\theta$$

$$\int_{1}^{\sqrt{3}} \frac{1}{1+x^2}\,dx = \int \frac{1}{1+\tan^2\theta}.\sec^2\theta d\theta$$

$$\int \frac{1}{\sec^2\theta}.\sec^2\theta d\theta = \int d\theta$$

$$= \theta + c$$

$$= \left[\tan^{-1} x\right]_{1}^{\sqrt{3}}$$

$$= \tan^{-1}\sqrt{3} - \tan^{-1}1$$

$$=\frac{\pi}{3}-\frac{\pi}{4}$$

$$=\frac{\pi}{12}$$

8. Expand $(1-\frac{1}{2}x-x^2)^5$ *in ascending powers of x as far as the term in* x^4

From $(1+x)^n = 1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^{\frac{3}{4}}}{3!}+---$

$$=\left[1-\left(\frac{1}{2}x+x^2\right)\right]^5 = 1+5\left(\frac{1}{2}x+x^2\right)+\frac{5\times4}{2}\left(\frac{1}{2}x+x^2\right)^2+\frac{5\times4\times3}{3\times2}\left(\frac{1}{2}x+x^2\right)+\frac{5\times4}{4\times3\times2}\left(\frac{1}{2}x+x^2\right)^4+---$$

$$=1-\frac{5}{2}x-5x^2+10\left(\frac{1}{4}x^2+x^3+x^4\right)+^{-}10\left(\frac{1}{8}x^3+\frac{3}{4}x^4+---\right)+5\left(\frac{1}{6}x^4+---\right)$$

$$=1-\frac{5}{2}x-5x^2+\frac{10}{4}x^2+10x^3-\frac{10}{8}x^3+10x^4-\frac{3}{4}x^4+\frac{5}{16}x^4+---$$

$$(1-\frac{1}{2}x-x^2)^5 = 1-\frac{5}{2}x-\frac{5}{2}x^2+\frac{35}{4}x^3+\frac{153}{16}x^4+---$$

9. If $y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, *show that* $\frac{dy}{dx}=\frac{2}{1+x^2}$

let $u=\frac{1-x^2}{1+x^2}$, $y=\cos^{-1}u$

$$u=\cos y$$

$$\frac{dy}{dx}=-\sin y$$

$$=-\sqrt{1-\cos^2 y}$$

$$=\sqrt{1-u^2}$$

$$= \sqrt{\left\{1 - \left(\frac{1-x^2}{1+x^2}\right)^2\right\}}$$

$$\frac{du}{dy} = -\frac{2x}{1+x^2}$$

$$\frac{du}{dx} = \frac{d\left(\frac{1-x^2}{1+x^2}\right)}{dx}$$

$$= \frac{(1+x^2)(-2x) - (1-x^2)(2x)}{(1+x^2)^2}$$

$$= \frac{-4x}{(1+x^2)^2}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \left(\frac{1}{du/dy}\right) \times \frac{du}{dx}$$

$$= \left(\frac{1+x^2}{2x}\right) \cdot \frac{-4x}{(1+x^2)^2}$$

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

10.Find) $\int \frac{x^2 - 1}{\sqrt{(x^3 - 3x)}}\, dx$

$$\int \frac{x^2 - 1}{\sqrt{(x^3 - 3x)}}\, dx = \int (x^2 - 1)(x^3 - 3x)^{\frac{-1}{2}}\, dx$$

Let $t = x^3 - 3x$

$$\frac{dt}{dx} = 3x^2 - 3$$

$$\frac{dt}{dx} = 3(x^2 - 1)$$

$$\int (x^2 - 1).(x^3 - 3x)^{\frac{-1}{2}} dx = \int (x^2 - 1).t^{\frac{-1}{2}} \frac{dt}{3(x^2 - 1)}$$

$$= \frac{1}{3}\int t^{\frac{-1}{2}} dt$$

$$= \frac{1}{3} t^{\frac{1}{2}} \times \frac{2}{1} + c$$

$$\int \frac{x^2 - 1}{\sqrt{(x^3 - 3x)}} dx = \frac{2}{3}\sqrt{(x^3 - 3x)} + c$$

EXERCISE 16

1. Solve the equations $x^2+4xy+y^2=13$ and $2x^2+3xy=8$ using $y=mx$
2. If $x=a(\theta-\sin\theta)$ and $y=a(1-\cos\theta)$. Show that $\frac{dy}{dx}=\cot\frac{1}{2}\theta$
3. Solve for x, given that $\log_{14}x=\log_7 4x$
4. A curve is such that at all points the gradient is $x-\frac{2y}{x}$. Given that the point (2, 4) lies on the curve, Find the equation of the curve.
5. Find the value of x $\sqrt{3(x-2)(x-3)}$ - $\sqrt{(x-2)(x-5)}=(x-2)$
6. A polynomial expression *P(x)* when divided by *(x-1)* leaves a remainder of 3 and when divided by *(x-2)* leaves a remainder of 1. find the remainder when *P(x)* is divided by *(x-1)(x-2)*
7. By row reduction to echelon form, solve the simultaneous equations

$$x+3y-z=6$$
$$2x-y+z=2$$
$$3x+2y-3z=11$$

8. Show that $\cos 18^0=\frac{1}{4}\sqrt{10+2\sqrt{5}}$

9. a) Show that $y=x+K(1+xy)$ is the solution to the differential equation $\frac{dy}{dx}=\frac{1+y^2}{1+x^2}$, where K is an arbitrary constant.

b) A clinical thermometer whose reading is 25° is placed in the mouth of a patient. The temperature *T*° C indicated by the thermometer rises at a rate proportional to $(T-\theta)$. Given that $\alpha,\beta,$and θ_1 are successive readings of θ at equal intervals of time t, show that $T=\frac{\alpha\theta_1-\beta^2}{\alpha+\theta_1-2\beta}$

10. a) The points A, B and C have position vectors $2i+3j-K, 3i+j+k$ and $5i-2j+3k$ respectively. Find the equation of the plane containing the points A, B and C in its i) scalar product form ii) Cartesian form

b) Find the position vector of a point where the line $r = 5i+3j-k+\lambda(i-4j+2k)$ meets the plane $r.(2i+j+3k)=12$

EXERCISE 17

1. The number of terms of an AP is even. The sum of the odd terms is 24 and the sum of the even terms is 30. If the last term exceeds the first term by 10.5, find the number of terms in the progression.

2. Differentiate with respect to x $i)\ y = Sin(\cos x)$

   $ii)\ y = (\cos x)^{\sin x}$.

3. Solve the equation for all possible values of x: $\dfrac{12}{x(x+1)} + x^2 + x - 8 = 0$.

4. If Z1 = 3 + 2i and Z_2 = 4 – 3i are two complex numbers, find the modulus and argument of $Z_1Z_2 + \frac{z_2}{z_1}$.

5. Show that the length of the tangent to the circle $x^2+y^2-4x-6y+12=0$ from the point $P(x, y)$ is $\left(x^2+y^2-4x-6y+12\right)^{\frac{1}{2}}$ units.

6. Find the value of n for which the co-efficient of x, x^2 and x^3 in the binomial expansion of $(1 + x)^n$ are in an Arithmetical progression.

7. Solve for n given that ${}^nC_{12} = {}^nC_8$

8. The coordinates of the point A and B are (0, 2, 5) and (-1, 3, 1) respectively and the equation of the line L is $\dfrac{x-3}{2} = \dfrac{y-2}{-2} = \dfrac{z-2}{-1}$. Find the:

(i) Equation of the plane P_1 which contains A and is perpendicular to the line L. Hence verify that point B lies in the plane P_1.

(ii) Coordinates of point C in which the line L meets the plane P_1

(iii) angle between CA and CB.

9. (i) From first principles find the derivative of $\frac{3}{4\sqrt{x}}$.

(ii) If $y = a\cos^2\theta + b\sin^2\theta$, Prove that $\frac{d^2y}{d\theta^2} + 4y = 2(a+b)$.

10. (a) Given that Z = 1 + i find the value of p and q such that:

$$\frac{p}{1+z} + \frac{q}{1+z^3} = 2i$$

(b) (i) Determine the stationary points of the curve $y = x^2(x-4)$

(ii) Sketch the curve above for $-2 \le x \le 5$

(iii) Find the area enclosed by the curve and the x-axis.

EXERCISE 18

1. Show that $\frac{dy}{dx} = \frac{-my}{nx}$ if $x^m y^n = k$ where k is a constant.

2. Find the length and direction cosines of the line PM where P is the midpoint of AB, M is the midpoint of BC and A,B,C are the points (3,-1,5), (7,1,3) and (-5,9,-1) respectively.

3. Find $\int \frac{x-1}{2-x}dx$

4. Prove that (Cosec A – Sin A)(Sec A – Cos A) = $\frac{1}{\tan A + \cot A}$

5. A circle has centre (2,0) and radius 2, and P is any point on this circle. OP is produced to Q so that OQ = 3 OP. Find the equation of the locus of Q.

6. Find the following Integral $\int \frac{4x^3+16x^2-15x+13}{(x+2)(2x-1)^2}dx$

7. In how many ways can the letters of the word i) **MATHS** ii) **MIME** be arranged?

8. Given $y=\frac{(1-x^2)e^{-x}}{1+x^2}$, $|x|<1$ find $\frac{dy}{dx}$ when x = 0

9. Expand $\frac{7+x}{(1+x)(1+x^2)}$ in ascending powers of x as far as the term x^4

EXERCISE 19

1. Factorize $(n+2)!+n^2(n-1)!$

2. Find the angle between the planes *2x – y + z = 6 and x + y + 2z = 3*

3. If $x^y = e^{x-y}$, Prove that $\frac{dy}{dx}=\frac{\ln x}{(1+\ln x)^2}$

4. Find the area enclosed between the two curves $y^2=4ax$ and $x^2=4ay$

5. If the first, third and thirteenth terms of an AP are in a GP, and the sum of the fourth and seventh terms of this AP is 40. Find the (non zero) common difference.

6. $\int \frac{x^2+1}{(x+1)(x-1)}dx$

7. Using a suitable formula, find $z^{\frac{1}{2}}$, given that $z=15+8i$

8. Establish the first derivative of $\sqrt{\frac{y}{x}}=6-\sqrt{\frac{x}{y}}$

9. Sketch i) $y=|2x+3|$

ii) $f(\theta)=\tan\theta,\ 0<\theta<2\pi$

10. Solve $\cos\theta + \cos 2\theta - \sin 3\theta = 0$ for $0^o \le \theta \le 360^o$

11. Kyankuddu makes his way to school at a speed which is proportional to the distance he still has to cover. He leaves home, $2km$ from school running at $10kmh^{-1}$. How long will it be before he has gone nine tenths of the way?

EXERCISE 20

1. a) Define a permutation.

   b) Find the number of permutations of the letters **PARALLEL**

2. Using the substitution $x^3u + 1 = 0$, Show that $\int_2^3 \frac{dx}{x(x^3-1)} = \frac{1}{3}Log_e\left(\frac{208}{189}\right)$

3. Prove that the vectors ***3i+j-2k, -i+3j+4k, 4i-2j-6k*** form sides of a triangle, thus find the area of the triangle

4. Solve for x $\tan^{-1}x + \tan^{-1}2x = \tan^{-1}\sqrt{2}$

5. Given that $e^{x-y} = x^y$ show that $\frac{dy}{dx} = \frac{\ln x}{(1+\ln x)^2}$

6. Find the general solution of $\sin\theta - \sin 3\theta + \sin 5\theta = 0$

7. $\int_0^{\pi/4} x\cos 2x dx$

8. Evaluate $(1+i\sqrt{2})^{2/3}$

9. Suppose Kaberege drives a nail using a hammer into a piece of wood, with the first impact, the nail enters the wood 20mm, with the second impact, it enters furthermore 18mm, Assuming the length it enters each time forms a geometric progression. Find the length it would have entered after the fourth impact.

10. Express $\frac{(x-2)^2}{x^3+1}$ as a partial fraction.

Worked examples

1. Given that α *and* β *are roots of the equation* $x^2 + px + q = 0$*, express* $(\alpha^2 - \beta^2)$ *and* $(\alpha^3 + \beta^3)$ *in terms of p and q.*

$$x^2 + px + q = 0$$

$$\alpha + \beta = -p, \qquad \alpha\beta = q$$

$$\left(\alpha^2 - \beta^2\right) = (\alpha + \beta)(\alpha - \beta)$$

But $$(\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta$$

$$(\alpha - \beta)^2 + 2\alpha\beta = \alpha^2 + \beta^2 \text{---------i)}$$

From

$$(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$$

$$(\alpha + \beta)^2 - 2\alpha\beta = \alpha^2 + \beta^2 \text{-------------ii)}$$

Equating i) and ii)

$$(\alpha - \beta)^2 + 2\alpha\beta = (\alpha + \beta)^2 - 2\alpha\beta$$

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$$

$$(\alpha - \beta) = \sqrt{\left[(\alpha + \beta)^2 - 4\alpha\beta\right]}$$

$$(\alpha - \beta) = \sqrt{\left[\left(p^2 - 4q\right)\right]}$$

$$\left(\alpha^2 - \beta^2\right) = (\alpha + \beta)(\alpha - \beta)$$

$$\left(\alpha^2 - \beta^2\right) = -p\sqrt{\left(p^2 + 4q\right)}$$

then

$$\left(\alpha^3 + \beta^3\right) = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

$$= (-p)^3 - (3q - p)$$

$$\left(\alpha^3 + \beta^3\right) = 3pq - p^3$$

2.Solve the equation $(3x + 5)^3 - (3x - 5)^3 = 730.$

Using $a^3 - b^3 = (a - b)\left(a^2 + ab + b^2\right)$

$$[(3x + 5) - (3x - 5)]\left[(3x + 5)^2 + (3x + 5)(3x - 5) + (3x - 5)^2\right] = 730$$

$$10\left[9^2+30x+25+9x^2-25+9x^2-30x+25\right]=730$$

$$27x^2+25=73$$

$$27x^2=48$$

$$x=\pm 1.33 \quad \text{or} \quad x=\pm\frac{4}{3}$$

3.If $(x+iy)^3=(a+ib)$, *show that* $\left(a^2+b^2\right)=\left(x^2+y^2\right)^3$.

expanding

$$(x+iy)^3=x^3+3x^2yi+3xy^2i^2+i^3y^3$$

$$=\left(x^3+3x^2yi-3xy^2-y^3i\right)$$

$$\therefore (x+iy)^3=\left(x^3-3xy^2\right)+\left(3x^2y-y^3\right)i=a+ib$$

So compare real parts

$$a=x^3-3xy^2$$

$$a=x\left(x^2-3y^2\right)$$

$$a^2=x^2\left(x^2-3y^2\right)^2$$

Compare imaginary parts

$$b=3x^2y-y^3$$

$$b=y\left(3x^2-y^2\right)^2$$

$$b^2=y^2\left(3x^2-y^2\right)^2$$

$$a^2+b^2=x^2\left(x^2-3y^2\right)^2+y^2\left(3x^2-y^2\right)^2$$

$$=x^2\left(x^4-6x^2y^2+9y^4\right)+y^2\left(9x^4-6x^2y^2+y^4\right)$$

$$=x^6-6x^4y^2+9x^2y^4+9x^4y^2-6x^2y^4+y^6$$

$$=x^6+3x^4y^2+3x^2y^4+y^6$$

$$a^2+b^2=\left(x^2+y^2\right)^3$$

4. Find the value of A,B and C for which:
$\left[Ax(x-2)(x+3)+Bx(x-2)+Cx(x+3)+(x-2)(x+3)\right]$ *has a constant value s of x.*

Suppose x=1 and the constant value is K

$$Ax(x-2)(x+3)+Bx(x-2)+Cx(x+3)+(x-2)(x+3)$$
$$=A(-1)(4)+B(-1)+C(4)+(-1)(4)=K$$

$$-4A-B+4C-4=K------i)$$

For $x=2$

$$2C(5)=K$$

$$10C=K$$

$$C=\frac{K}{10}-------------ii)$$

For *x=-3*

$$-3B(-5)=K$$

$$15B=K------------iii)$$

For *x=0*

$$(-2)(3)=0$$

K=-6---------------------------iv)

From iii)

$$B=\frac{-6}{15}$$

$$C=\frac{-6}{10}$$

From equation i)

$$-4A+\frac{6}{15}+\frac{-24}{10}-4=-6$$

$$-4A=\frac{180}{30}-6$$

$A = 0$

$\therefore A = 0, \quad B = \frac{6}{15}, \quad C = \frac{-6}{10}, \quad K = -6$

5.Determine and plot the locus represented by: $1 \le |z + 2i| \le 2$.

$1 \le |z + 2i| \le 2$

Case 1

Consider $|z + 2i| \ge 1$

Let $z = x + iy$

$|x + iy + 2i| \ge 1$

$|x + (y+2)i| \ge 1$

$x^2 + (y+2)^2 \ge 1$, which is a circle Centre $C_1(0,-2)$, Radius $r_1 = 1$

Case 2

$|Z + 2i| \le 2$

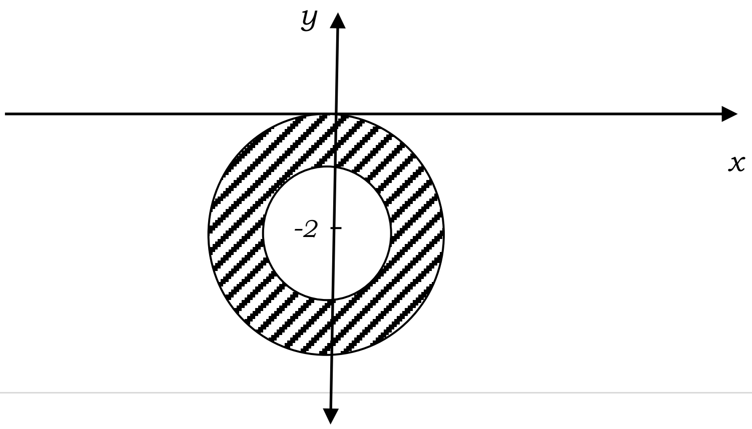
$|x + i(y+2)| \le 2$

$x^2 + (y+2)^2 \le 4$

$x^2 + (y+2)^2 \le 4$

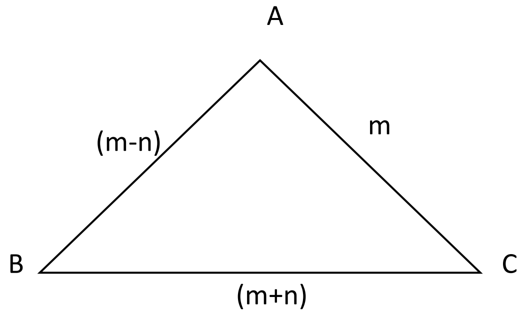
which is a circle *Centre* $C_1(0,-2)$, *radius*, $r_2 = 2$

Sketch



So $1 \leq |Z + 2i| \leq 2$ *lies in the shaded region.*

6.The sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ *of the triangle ABC are of length (m+n) ,m and (m-n), respectively. Show that:* $\cos A = \dfrac{m-4n}{2(m-n)}$.



From cosine rule

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$(m+n)^2 = m^2 + (m-n)^2 - 2m(m-n)\cos A$$

$$m^2 + 2mn + n^2 - m^2 - m^2 + 2mn - n^2 = -2m(m-n)\cos A$$

$$4mn - m^2 = -2m(m-n)\cos A$$

$$4n - m = -2(m-n)\cos A$$

$$\therefore \cos A = \frac{m-4n}{2(m-n)}$$

7.Solve the equation : $\sin\theta + \sin(60+\theta) = \dfrac{1}{2},\ for 0^o \leq \theta \leq 360^o$

$$\sin\theta + \sin(60+\theta) = \frac{1}{2}$$

$$2\sin\left(\frac{2\theta+60}{2}\right)\cos 30 = \frac{1}{2}$$

$$2\sin(\theta+30)\frac{\sqrt{3}}{2}=\frac{1}{2}$$

$$\sin(\theta+30)=\frac{1}{2\sqrt{3}}$$

$$(\theta+30)=\sin^{-1}\left(\frac{1}{2\sqrt{3}}\right)$$

$$(\theta+30)=16.8^o,376.8^o,163.2^o$$

$$\theta=133.2^o,\quad 346.8^o$$

8a).Given the sides of a water tank in terms of a cube was measured by measured by a student using a tape measure and was recorded as 25cm given that tape measure gives a reading which is higher by 0.05cm,use binomial to find the approximate percentage error in the volume of the tank.

Let x be the side

$$\therefore x=25cm\ ,\ \Delta x=0.05cm$$

Approximate volume $V_a=x^3$

$$=25^3cm^3$$

$$V_a=15{,}625cm^3$$

From binomial

$$(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)a^{n-2}b^2}{2!}+---+b^n$$

Actual volume $V_A=(x-\Delta x)^3$

$$=x^3-3x^2\Delta x+3x(\Delta x)^2-(\Delta x)^3$$

$$=25^2-3\times25^2\times0.05+3\times25\times(0.05)^2+(0.05)^3$$

$$V_A = 15{,}531.27$$

$$\%\ error = \left|\frac{V_A - V_a}{V_A}\right| \times 100\%$$

$$= \left|\frac{15531.27 - 15625}{15531.27}\right| \times 100\%$$

$$= 0.6\%$$

b) If $x^p = (xy)^q = (xy^2)^r$ *for values of x and y, prove that:* $2pr = q(p + r)$.

Consider

$$x^p = (xy)^q$$

Introduce log to base 10

$$\log x^p = \log (xy)^q$$

$$p\log x = q\log x + q\log y$$

$$(p - q)\log x = q\log y$$

$$\frac{p - q}{q} = \frac{\log y}{\log x} ------------- i)$$

also

$$x^p = (xy^2)^r$$

$$p\log x = r\log (xy^2)$$

$$p\log x = r\log x + 2r\log y$$

$$(p - r)\log x = 2r\log y$$

$$\frac{p-r}{2r}=\frac{\log y}{\log x}------------ii)$$

Equating i) and ii)

$$2r(p-q)=q(p-r)$$

$$2rp-2rq=pq-qr$$

$$2rq=pq+rq$$

$$2pr=q(p+r)$$

9i) Use the substitution $z=(x+x^{-1})$ *to solve the equation:* $2x^4-9x^3+14x^2-9x+2=0$.

$$2x^4-9x^3+14x^2-9x+2=0$$

Divide through by x^2

$$2\left(x^2+\frac{1}{x^2}\right)-9\left(x+\frac{1}{x}\right)+14=0------\#$$

But

$$z=(x+x^{-1})$$

$$z=\left(x+\frac{1}{x}\right)---------i)$$

$$z^2=x^2+\frac{1}{x^2}+2$$

$$(z^2-2)=x^2+\frac{1}{x^2}----ii)$$

Substitution i) and ii) in #

$$2(z^2\_2)-9z+14=0$$

$$2z^2-9z+10=0$$

$$(2z-4)(2z-5)=0$$

$$z = 2 \quad or \quad z = \frac{5}{2}$$

For $z = 2$

$$x + \frac{1}{x} = 2$$

$$x^2 - 2x + 1 = 0$$

$$x = +1 \text{ repeated}$$

For $z = \frac{5}{2}$

$$x + \frac{1}{x} = \frac{5}{2}$$

$$2x^2 - 5x + 2 = 0$$

$$(2x - 1)(x - 2) = 0$$

$$x = \frac{1}{2} \qquad x = 2$$

$$\therefore x = \frac{1}{2},\ 1,\ 2$$

ii)Given that $\tan^2 x = y$ *,prove that:* $\frac{d^2y}{dx^2} = 2(1+3y)(1+y)$.

$$\tan^2 x = y$$

$$y = \tan^2 x$$

$$\frac{dy}{dx} = 2\tan x \sec^2 x$$

$$\frac{dy}{dx} = 2\tan x\left(1+\tan^2 x\right)$$

$$\frac{dx}{dy} = 2\tan x + 2\tan^3 x$$

$$\frac{d^2y}{dx^2} = 2\sec^2 x + 2\times 3\tan^2 x\left(1+\tan^2 x\right)$$

$$= 2[(1+y)+3y(1+y)]$$

$$\frac{d^2y}{dx^2} = 2(1+y)(1+3y)$$

$$\frac{d^2y}{dx^2} = 2(1+3y)(1+y)$$

10a) Find the scope of the tangent to the curve:

$$xy^3 - 2x^2y^2 + x^4 - 1 = 0 at\ (1,2).$$

$$xy^3 - 2x^2y^2 + x^4 - 1 = 0 \qquad at\ (1,2)$$

$$y^3 + 3xy^2\frac{dy}{dx} - 4y^2x - 4x^2y\frac{dy}{dx} + 4x^3 = 0$$

$$\left(3xy^2 - 4x^2y\right)\frac{dy}{dx} = 4y^2x - 4x^3 - y^3$$

$$\frac{dy}{dx} = \frac{4y^2x - 4x^3 - y^3}{3xy^2 - 4x^2y}$$

$$At\ (1,2)\quad \frac{dy}{dx} = \frac{16-4-8}{12-8}$$

$$\frac{dy}{dx} = 1$$

b) Given L_1 *and* L_2 *have Cartesian equation.*

$$\frac{x-1}{3} = \frac{y+1}{2} = z-2 \quad and \quad x-5 = y-1 = \frac{z}{2}$$

i)Write down the vector equation of L_1 and L_2.

$$(l_1)\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-2}{1} \qquad , \frac{x-5}{1}=\frac{y-1}{1}=\frac{z}{2}(l_2)$$

$$L_1 \therefore r_1=(i-j+2k)+\lambda(3i+2j+k)$$

$$L_2 \therefore r_2=(5i+j)+\mu(i+j+2k)$$

ii) Find the acute angle between L_1 and L_2.

$$L_1 is \uparrow\uparrow to\, vector(3i+2j+k)=u$$

$$L_2 is \uparrow\uparrow to\, vector(i+j+2k)=v$$

Let θ be the angle between l_1 *and* l_2

$$\therefore u.v=|u||v|\cos\theta$$

$$(3i+2j+k)(i+j+2k)=\sqrt{3^2+2^2+1^2}\sqrt{1^2+1^2+2^2}\cos\theta$$

$$3+2+2=\sqrt{14}\sqrt{6}\cos\theta$$

$$\theta=40^o$$

c) Find the point of intersection of the line $x=\frac{y+2}{2}=\frac{z-5}{-1}$ and $\frac{x-1}{-1}=\frac{y+3}{-3}=z-4$.

let the point of intersection be (x_1, y_1, z_1)

$$\frac{x_1}{1}=\frac{y_1+2}{-3}=\frac{z_1-5}{-1}------(i)$$

At (x_1, y_1, z_1)

$$\frac{x_1-1}{-1}=\frac{y_1+3}{-3}=\frac{z_1-4}{1}------(ii)$$

From equation (i)

$$\frac{x_1}{1}=\frac{y_1+2}{2}$$

$$2x_1=y_1+2------(iii)$$

From equation (ii)

$$\frac{x_1 - 1}{-1} = \frac{y_1 + 3}{-3}$$

$$-3x_1 = y_1 - 6 - - - - - - - (iv)$$

Solving (iii) and (iv) simultaneously

$$x_1 = 4 \quad and \quad y_1 = 6$$

Using x_1 *and* y_1

Equation (i) and (ii) both give $z_1 = 1$ *the line intersects at a point* $(4,6,1)$

Prove your weight 2

EXERCISE 21

1. If $\frac{a}{b} = \frac{c}{d}$ prove that $\frac{a-c}{a+c} = \frac{b-d}{b+d}$

2. $\int_0^{\frac{\pi}{4}} \sec^4 x dx$

3. Solve $2\frac{dy}{dx}+\cos(x+y)=\cos(x-y)$

4. Find the turning points of the graph $y=x^2e^{-x}$ and hence sketch the graph.

5. Solve the equations $\frac{x-y}{4}=\frac{z-y}{3}=\frac{2z-x}{1}, x+3y+2z=4$

6. The area included between the parts of the two curves $x^2+y^2=1$ and $4x^2+y^2=4$ for which y is positive is rotated about the x-axis. Find the volume of the solid thus formed.

7. Find the sum to infinity of $\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{9}-\frac{1}{16}\right)+\left(\frac{1}{27}-\frac{1}{64}\right)+\ldots$

8. Given that the locus $\left|\frac{z-4}{z+3}\right|=2$ is a circle, find the radius of the circle where

   $z = x+iy$

9. The points $P(p^2,2p)$ is on the parabola $y^2=4x$ and S is the point *(1, 0).* The normal to the parabola at P intersects the *x-axis* at N and PQ produced is parallel to the *x-axis.* Given that the angle $PSO=\alpha$ and that the angle $PNO=\beta$, where O is the origin; $\alpha>\beta$

   i) Find $\tan\alpha$ in terms of *p.*

   ii) Show that $\tan\beta=p$

   iii) Deduce that $\alpha=2\beta$, and hence prove that PN bisects the angle *SPQ.*

10. a) Given the function $y=\frac{3x-2}{x^2-3x+2}$, state:

    i. the horizontal asymptotes
    ii. the vertical asymptotes
    iii. the range of values the function can take for real x.

    b) Hence sketch the graph of the function.

11. a) i) A vector $r = \begin{pmatrix} 1 \\ -\lambda \\ 4 \end{pmatrix}$ is perpendicular to the line $r = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix}$

Determine the value of λ.

ii) Find the equation of a plane which contains the point M with position vector 5**i** – 2**j** + 3**k** and perpendicular to the line in a(i) above.

b) **p** and **q** represent the position vectors of the points P and Q respectively. Show that the position vector of a point R which divides $\overline{PQ}$ in the ratio m : n is represented by $\dfrac{n\mathbf{p}+m\mathbf{q}}{m+n}$.

12. Use synthetic method to find the:

a) Remainder when $x^4 - 2x^2 + x - 3$ is divided by x – ½.

b) The roots of the equation $x^2 + 2x - 3$ are α and β. Form an equation whose roots are $\alpha^2 - \beta$ and $\beta^2 - \alpha$.

c) Without using the method of differentiation, find the maximum value of the function: $f(x) = 2x + 8 - 3x^2$

State the value of x for which $f(x)$ is maximum.

13. a) Evaluate $(1 + i\sqrt{3})^{2/3}$

b) Find the locus of $|z - 3| = 5$ where z is complex number, sketch the locus.

c) Find the modulus and argument of $Z = \dfrac{Z_1}{Z_2^2}$ where $Z_1 = 1 + 2i$, $Z_2 = 1 - i$

EXERCISE 22

1. Solve the equation: $16\sin\theta\cos\theta = \tan\theta + \cot\theta$, for $0^o \le x \le 180^o$

2. Factorise completely $x^4 + x^3 - 3x^2 - 4x - 4$

3. Evaluate $\int_0^1 \frac{dx}{(1+x^2)^2}$

4. Given a curve of the form $qr^n = c$,where c is a constant. If $q = 90$ when $r = 4$ and $q = 40$ when $r = 6.2$, find the values of n and c .

5. Given that $a\cos^2\theta + b\sin^2\theta = c,$ prove that $\tan^2\theta = \frac{c-a}{b-c}$,

6. Find the sketch the locus Arg $\left(\frac{z}{z-6}\right) = \frac{\pi}{4}$

7. Find the values of p such that the equation $\log(py) = 2\log(y+1)$ has exactly one root.

8. i) The roots of a quadratic equation are a and b; if $a^2 + b^2 = 63$ and $2ab = 1$, write the equation

   ii) The roots of the quadratic equation $x^2 - bx + 1 = 0$ are $\alpha\ and \beta$
   Given that $\alpha = \sqrt{3} + \sqrt{2}$. Show that b=$2\sqrt{3}$

9. Given the curve $y = \frac{(x-1)(x-4)}{(x-5)}$ Find the range of values of y for which the curve does not lie and hence sketch the curve

10. (a) Prove by induction that $(\cos x + i\sin x)^n = \cos nx + i\sin nx$ hence solve

    $Z^3 - 1 = 0$

    (b) Determine the possible values of Z if; $Z\overline{Z} + 2iZ = 12 + 6i$

    (c) Describe and hence sketch the locus defined by $\left|\frac{Z-1}{Z+1}\right| \geq 2$

11 (i) Find the equation of the plane through $\underset{\sim}{a} = 5\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$ and parallel to the line $r = 4i + (2\eta - 1)j + (5 - \eta)k$

(ii) If the line in (i) above is produced to meet the plain $2x + 6y - 7z + 10 = 0$ at point H. Find the coordinates of H.

(iii) The vectors $\underset{\sim}{i} - \lambda\underset{\sim}{j} + 4\underset{\sim}{k}$ and $3\underset{\sim}{i} - 4\underset{\sim}{j} + 6\underset{\sim}{k}$ are perpendicular, find the value of λ.

12 (a) Find $\int x^2\sqrt{(e^{-x^3} - 4)}dx$

(b) Using the expansion of $(1+x)^{1/3}$ up to the term in x^3, Find the value of $\sqrt[3]{1.27}$ to 5 decimal

(c)Determine the value of a given that $(x-1)$ is a factor of $2x^3 + (2-a)x^2 - (a-2)$ Hence solve $2x^3 + (2-a)x^2 - (a-2) = 0$

13 The roots of the equation ax² + bx + c = 0 are α *and* β

(i) Show that if $\alpha - \beta = 1$ then $a^2 = b^2 - 4ac$
(ii) Find from first principles the derivative of $y = \sqrt{p}$

(iii) The curve C is given by $y = ax^2 + b\sqrt{x}$ where a and b are constants. Give that the gradient of C at the point $(1, 1)$ is 5, find a.

Prove your weight 3

EXERCISE 23

1. Solve the equation $2\log_4 x + 1 - \log_x 4 = 0$
2. A GP has first term 10 and common ratio 1.5. How many terms of the series are needed to reach a sum greater than 200?

3. If $Z=\cos\theta+i\sin\theta$, simplify $\frac{2Z}{1-Z^2}$
4. Prove that if $I_n=\int x^n e^x dx$ then for $n\geq 1$, $I_n=x^n e^x-nI_{n-1}$
5. Find the maximum and minimum ordinates of the curve $y=x^2e^{-x}$
6. Prove that $x-2-x^2$ is negative for all real values of x.
7. Solve the simultaneous equations

$$x+3y-z=6$$
$$2x-y+z=2$$
$$3x+2y-3z=11$$

8. Show that $e^{-2\log_e x}=\frac{1}{x^2}$
9. a) Find the Cartesian Equation of the locus defined by $|Z+1|=|Z-2-i|$

   b) Find the distance from the point (3, 4, 5) to the point where the line

   $\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{2}$ meets the plane $x+y+z=2$

10. a) $\int \frac{1}{x\sqrt{x^2-9}}dx$

    b) Prove that $\int u\tan^{-1}u\,du=\frac{1}{2}(1+u^2)\tan^{-1}u-\frac{1}{2}u+c$

11. Sketch the curve $y=x^3-2x^2-5x+6$
12. a) Given that $y=Ae^{Bx}$, by differentiating & eliminating constants A, B,

    form a differential equation.

    b) Find the volume of revolution when $y^2=x$ (for $0\leq x\leq 9$) is rotated through 2π radians about the x-axis.

EXERCISE 24

1. If $y=a\cos(\log_e x)+b\sin(\log_e x)$ Show that $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$
2. Solve the inequality $\frac{1}{x-1}\rangle\frac{x}{3-x}$

3. Given that $\lg x+\lg 2+\lg x^2+\lg 4+\lg x^3+\lg 8+..$ is an arithmetic series, show that the sum of the first ten terms is $55\lg 2x$

4. Find $\int \frac{1}{x^2+3x+9}\,dx$

5. Prove that in a triangle ABC the length of the perpendicular from A to BC is $\frac{bc}{a}\sin A$

6. The total cost of producing x pocket radio per day is sh. $(\frac{1}{4}x^2+35x+25)$ and the price per hand set at which they may be sold is sh. $(50-\frac{x}{2})$. What should be the daily output to obtain a maximum total profit?

7. Find the square root of $14+6\sqrt{5}$

8. Find the equation of the circle which passes through the point (-3, 1) and the point of intersection of $x^2+y^2-y-5=0$ and $x^2+y^2+2x+5y-1=0$

9. a) By putting, $y=vx$ Solve $y^2dx+(xy+x^2)dy=0$

   b) A murder victim was discovered by the police at 6:00 a.m. The body

   temperature of the victim was measured and found to be 25°C. A doctor arrived on the scene of the crime 30 minutes later and measured the body temperature again. It was found to be 22°C. The temperature of the room had remained constant at 15°C. The doctor, knowing normal body temperature to be 37°C, was able to estimate the time of death of the victim. What would be your estimate for the time of death?

10. a) Sketch the curve $y=\dfrac{3(x-2)}{x(x+6)}$

11 . a) Obtain the expansion in ascending powers of x up to and including the

   term in x^3 of $\ln\left[\dfrac{(1+2)^{\frac{1}{2}}}{(1-3x)^2}\right]$

   b) Given the sum of the first n terms of an arithmetic series, S_n is

$s_n = n(n+3)$. Find the common difference of this series.

12. a) Evaluate $\int_2^3 \frac{3x^2-x}{x^2-1}\,dx$

b) Find $\int x\cos(3x^2+5)dx$

13. a) Simplify

$$\tan^{-1} x + \arctan\left(\tfrac{1-x}{1+x}\right)$$

b) Find the general solution to the equation

$$2\cos\theta = \sin(\theta + 30^o)$$

c) In the triangle ABC prove that

$$a^2 = (b-c)^2 + 4bc\sin^2 \tfrac{1}{2}A$$

14. a) Find the equation to the plane through (1, 2, 3) and parallel to

$3x + 4y - 5z = 0$

b) Find the locus of a point whose distance from the origin is 7 times its distance from the plane $2x + 3y - 6z = 2$

c) Find where the line $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{4}$ meets the plane $2x + 4y - z + 1 = 0$

Prove your weight 4

EXERCISE 25

1. Differentiate with respect to x

$(\sin x)^{\cos x} + (\cos x)^{\sin x}$.

2. Find $\int \sqrt{x(4-x)}dx$

3. Solve the differential equation $x\frac{dy}{dx}-2y=(x-2)e^x$

4. Show that the circles $x^2+y^2=36$ and $x^2+y^2-6x-8y+24=0$ touch each other internally. Hence find their point of contact.

5. Find the following Integral $\int \frac{4x^3+16x^2-15x+13}{(x+2)(2x-1)^2}dx$

6. a) Evaluate $\left(1+i\sqrt{3}\right)^{2/3}$

b) Find the locus of $|z-3|=5$ where z is complex number, sketch the locus.

c) Find the modulus and argument of $Z=\frac{Z_1}{Z_2^2}$ where $Z_1=1+2i$, $Z_2=1-i$

7. a) Given the point A(1, -4, 1) and B(5, 0, 1)

i) Find the equation of the line through C (2, 0, 1) which is parallel to AB.

ii) Determine the distance of the line AB from the plane $3x-6y+2z=8$

EXERCISE 26

1. Find m and n from the equations

$$\frac{1}{m+n}+\frac{2}{m-n}=8, \qquad m^2-n^2=\frac{1}{6}$$

2. Solve the differential equation

$$(1-x^2)\frac{dy}{dx}+9+y^2=0$$

3. Show that the equation of the common chord of the two circle $x^2+y^2+10x+8y+32=0$ and $x^2+y^2-4x-6y+12=0$ is perpendicular to the line of centres of the circles.
4. Prove that if $\frac{z-2i}{2z-1}$ is purely imaginary, the locus of the point representing z in the Argand diagram is a circle and find its centre and radius.
5. Given that $a=c^x$ and $b=c^y$, deduce that $\log_c a+\log_c b=\log_c ab$ and that $\log_c a-\log_c b=\log_c\left(\frac{a}{b}\right)$
6. Find range of the function $f(x)=3x^2+10x+9$ and show that it cannot be negative.
7. Sketch the $y=|2x+1|$ and find the values of x for which $y<3$
8. a) Find the length of the latus rectum of the ellipse $\frac{x^2}{16}+\frac{y^2}{7}=1$

   b) PQ is a chord of a rectangular hyperbola $xy=c^2$ and R is its mid-point. If PQ has a constant length k, Show that the locus of R is given by

$$4(xy-c^2)(x^2+y^2)=k^2xy$$

10. The number of terms of an AP is even. The sum of the odd terms is 24 and the sum of the even terms is 30. If the last term exceeds the first term by 10.5, find the number of terms in the progression.

11. Given the curve $y=\frac{x-1}{2x^2+x-1}$ Sketch the graph

12. a) Find the area of the region bounded by the curve $y=\frac{1}{2x^2+x}$, the x - axis, the lines $x=1$ and $x=2$.

    b) Solve $\cos x\frac{dy}{dx}-2y\sin x=1$

13 a) Show that the length of the line joining the common points of the line

$y = mx + c$ and the curve $y^2 = 4x$ is $\frac{4}{m^2}\sqrt{(1+m^2)(1-mc)}$

b) Find the derivative of $y = \sqrt{x}$ from first principles.

EXERCISE 27

1. Find the sum to n terms of a series whose general term is $1+3r+4r^2$

2. Prove that $\tan^{-1}(\frac{1}{3}) + \sin^{-1}(\frac{1}{\sqrt{5}}) = \frac{\pi}{4}$

3. Evaluate $\int_0^1 \frac{dx}{(1+x^2)^2}$

4. Differentiate $4x^5 + \cos 4x$ from first principles.

5. Three consecutive terms of geometric series have product 343 and sum $\frac{49}{2}$.

   Find the numbers.

6. Show that the square root of $a + b + \sqrt{(2ab + b^2)}$ is $\pm(\sqrt{a + \frac{1}{2}b} + \sqrt{\frac{1}{2}b})$

7. Solve the inequality $(0.8)^{-3x} \geq 4$ correct to two decimal places.

8. i) Show that $e^{\ln x} = x$

   ii) If $y = e^{-2x}\cos 4x$ establish that $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 20y = 0$

9. i) Prove that $\int \sec x dx = \ln k(\sec x + \tan x)$

   ii) Find $\int \frac{4x+5}{x^2+2x+2} dx$

10. (a) Solve the differential equation $x(4-y^2)=2(x^2+1)\frac{dy}{dx}$

given that y =1 when x = 0

(b) Form a differential equation from $x=\tan(By)$ given that B is an arbitrary constant.

(c) Find the general solution of the equation $e^x\frac{dy}{dx}+y^2+4=0$

11 (i) Prove that $\frac{\cos 3\theta}{\cos\theta}-\frac{\cos 6\theta}{\cos 2\theta}=2(\cos 2\theta-\cos 4\theta)$

(ii) Find the general solution for $\cos\theta-\sqrt{3}\sin\theta=1$ (use half angle formulae)

12 (a) By expressing as partial fractions, expand $\frac{3x+5}{(1-x)(1+3x)}$ as a series of ascending powers of x up to and including the term x^2

(c) The limit of the sum of a convergent G.P is ω and the limit of the sum of the squares of its terms is ρ .Find the first term and common ratio of the progression.

(d) Use binomial expansion to evaluate $\sqrt{25.1}$

13 Given the curve $x^3-9x^2+23x-15=f(x)$.

(i) Find the sets of values of x for which $x^3-9x^2+23x-15\le 0$

(ii) Establish the nature of the turning points

(iii) State the x-intercepts

(iv) Sketch the graph

EXERCISE 28

1. Solve the equation $x^{2/3}-5x^{1/3}+6=0$

2. Find the Cartesian form of the equation of the plane

$$r = (1+3\lambda+2\mu)i + (1+\lambda+4\mu)j + (\mu-\lambda)k$$

3. Solve the inequality $x > \frac{2}{x+1}$

4. By using the substitution $v = \frac{y}{x}$, show that the solution to the equation $(x-y)\frac{dy}{dx} = x+y$ can be written as $\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(\frac{x^2+y^2}{x^2}\right) = \ln kx$

5. An inverted right circular cone of semi vertical angle 45^0 is collecting water from a trough at a steady speed of $18\pi cm^3 s^{-1}$. When the depth of the water is hcm, the rate at which the depth is rising is $1cms^{-1}$. Determine the value of h.

6. Solve for x $5\sin 2x - 5\sin^2 x + 4 = 0$; $-180^0 \le x \le 180^0$

7. Find the equation of the circle with the chord $\overline{AB}$ where A(3, 4) and B(6, 1) and the tangent to the circle at the point A is the line $2y - 5 = x$

8. On a Celsius scale the distance from degree '1' to 'n' degrees is proportional to $\lg n$. If the distance from degree '1' to 10 degrees is 25cm, calculate the distances a) from degree '1' to 2 degrees

   b) from degree '2' to 3 degrees mark

9. i) Prove that

$$\int \frac{1-2x}{\sqrt{\{9-(x+2)^2\}}}dx = 5\sin^{-1}\left(\frac{x+2}{3}\right) + 2\sqrt{(5-4x-x^2)} + c$$

   ii) Integrate $\int e^{3x}\cos 2x dx$

10. (a) Find the equation to the plane through (1, 2, 3) and parallel to

$3x + 4y - 5z = 0$

(b) Find the locus of a point whose distance from the origin is 7 times its distance from the plane $2x + 3y - 6z = 2$

(c) Find where the line $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{4}$ meets the plane $2x + 4y - z + 1 = 0$

11 Sketch the curve $y = \frac{(x-2)^2}{(x-1)(x-4)}$ clearing showing the region where the curve does not lie.

12 (a) Find the general solution of the equation $(2\tan x - 1)^2 = 3(\sec^2 x - 2)$

(b) If A, B and C are angles of a triangle ABC; show that:

$$Sin^2 A + Sin^2 B + Sin^2 C = 2 + 2CosACosBCosC$$

13 The store of Sekandi girls' hostel can accommodate a maximum of 500 bags of rice. The rate of consumption of the rice is directly proportional to the product of the bags consumed and those not yet consumed by the girls in Sekandi. If the consumption of the rice increases from 100 to 280 bags in at least 3 days, then determine the;

i) number of bags of rice consumed by the girls in Sekandi at the end of the week.

ii) time in hours required for the rice bags to be consumed up to 430.

EXERCISE 29

1. Find the position vector of the points **P, Q** which divides **AB** in the ratio **1:2** and **5:-2 respectively**.

2. If α, β are roots of the $4x^2 - 6x + 1 = 0$, Find $\alpha^3 - \beta^3$

3. Express $\frac{4x-3}{x^4 + x^3}$ in terms of its partial fraction.

4. Given $x = \cos^{-1}\dfrac{1}{\sqrt{1+t^2}}$ and $y = \sin^{-1}\dfrac{1}{\sqrt{1+t^2}}$ Find $\dfrac{dy}{dx}$

5. Solve the equations i) $3^{x+1} = 4^{2x-1}$ ii) $z^3 = -1$

6. Calculate the term independent of x in the expansion of $\left(\dfrac{2x}{3} - \dfrac{3}{2x}\right)^6$

7. Given the coordinates of the points **C**, **E** and **D** are (-4, 0), (2, -3) and (3, 2) respectively. Find, i) the midpoint of **CE**

   ii) the angle $E\hat{D}C$

8. Solve $\cos x = \sin 4x + \cos 7x$ for $0^0 \le x \le 180^0$.

9. Determine the equations of the two tangents to the circle $x^2 + y^2 = 25$ which are parallel to the line $4x - 3y - 2 = 0$

10. Evaluate a) $\displaystyle\int_3^4 (x+1)(2x-5)^4\,dx$

    b) Integrate $\displaystyle\int \frac{1}{(1+x^2)^2}dx$

EXERCISE 30

1. Define a Geometric Series and State the condition necessary for such a series to be convergent.

2. If $2\log_8 m = x$, $\log_2 2m = y$ and $y - x = 4$, find the value of *m*.

3. Top scorer has to fence his rectangular field to reduce competition. One side is already fenced. What is the maximum area he can enclose on the other sides, if he has 288m of wire mesh?

4. Find the line of intersection of two planes $r.(i+j-3k)=6$ and

   $r.(2i-j+k)=4$

5. In the expansion of $\dfrac{1}{ax+1}-\dfrac{2}{2-x}$ in ascending powers of *x*, the term in the second power of x vanishes. Find the possible values of a.

6. Prove that $\dfrac{Sin\theta Sin\phi}{Cos\theta+Cos\phi}=\dfrac{2\tan\left(\frac{\theta}{2}\right)\tan\left(\frac{\phi}{2}\right)}{1-\tan^2\left(\frac{\theta}{2}\right)\tan^2\left(\frac{\phi}{2}\right)}$

7. The area bounded between the function $y^2=4ax$ and the lines $x=a$, $x=3a$ and $y=0$ in the first quadrant is rotated about the line $y=0$. Find the volume of the solid generated.

8. Two circles have their centres on the line $y=-3$ and touch the line $3y=2x$. If the radii of the circles are $\sqrt{13}$, find the coordinates of their centres and their equations.

9. Find the value of x and y, given that $\dfrac{x}{2+3i}-\dfrac{y}{3-2i}=\dfrac{6+2i}{1+8i}$

10. Find a) $\int Cos6xSin2xCos2xdx$

Prove your weight 5

EXERCISE 31

1. Given that $x^2+y^2=23xy$, show that $\log x+\log y=2\log\left(\dfrac{x+y}{5}\right)$

2. Solve the differential equation $x\frac{dy}{dx} - y = x^3 e^{x^2}$

3. Prove that $\cot\alpha + \operatorname{cosec}\alpha = \cot\frac{\alpha}{2}$

4. Draw Argand diagrams to illustrate i) $\arg(z) = \frac{\pi}{6}$ ii) $|z+1| = 3|z-1|$

5. Find the binomial expansion for $(1+x)^{1/2}$ as far as the term containing x^3.

   Hence find $\sqrt{1.03}$ to 4 decimal places.

6. Show that the circles $x^2 + y^2 = 36$ and $x^2 + y^2 - 6x - 8y + 24 = 0$ touch each other internally. Hence find their point of contact.

7. a) $\int_{\pi/4}^{\pi/2} \frac{\cos^3 x}{\sin x} dx$ b) $\int \frac{v^2}{v^2 - 1} dv$

8. Differentiate from first principles $x^2 + \cot x$

9. Show that for any isosceles triangle ABC, with the base AB = c, the area is given by $\Delta = \frac{1}{2}c\sqrt{s(s-c)}$, where s is half the perimeter of the triangle.

   Given that $\Delta = \sqrt{3}$ and $s = 4$, determine the sides of the triangle.

EXERCISE 32

1. (a) Sketch the locus defined by the equation $\arg(z - 6 + i) = \frac{\pi}{3}$

   (b) Given that $2+i$ is a root of the equation $Z^3 - 6Z^2 + aZ + b = 0$ Write down the quadratic factor of the polynomial $Z^3 - 6Z^2 + aZ + b$,where a, b are real numbers.

2. Show that $\dfrac{\sqrt{5^{2009}}}{\sqrt{5^{2011}}-\sqrt{5^{2007}}}=\dfrac{5}{24}$

3. Given that; $2x=p+\dfrac{1}{p}$, $2y=p-\dfrac{1}{p}$. Show that $\dfrac{d^2y}{dx^2}=\dfrac{-8p^3}{\left(p^2-1\right)^3}$

4. Given that $b^q=a^p=(ab)^{qp}$ Show that $q=1-p$

5. Given that the first three terms of the expansion of $(1+x+x^2)^n$ in ascending powers of x are the same as the first three terms in the expansion of $\left(\dfrac{1+ax}{1-3ax}\right)^3$

   Find the value of a and n.

6. If the n^{th} term of a series is $8n-2$ Show that the series is an arithmetic Progression and write down the sum of the n terms

7. Given a triangle ABC, where angle ACB is a right angle, prove that

   $$cCos\left(\frac{A-B}{2}\right)=(a+b)\sin\left(\frac{A+B}{2}\right)$$

8. a) Find the equation of a chord joining the points $\left(ct_1,\dfrac{c}{t_1}\right)$ to $\left(ct_2,\dfrac{c}{t_2}\right)$ on a hyperbola. Hence deduce the equation of the tangent at $\left(ct\ ,\dfrac{c}{t}\right)$

   b) Find the equation of the tangents to the hyperbola $x=4t,\quad y=\dfrac{4}{t}$ which passes through the point (4, 3).

EXERCISE 33

1. Find the value of *x*, given that $\log_{10}(19x^2+4)-2\log_{10}x=2$

2. A plane ω is 4 units from the origin and perpendicular to the vector $2i-\frac{7}{2}j+2k$. Find the closest distance of approach of this plane from the point (3, 1, 5).

3. Differentiate $\log_e\left(\frac{1+\ln 2x}{x+\ln 3x}\right)^{1/3}$ with respect to x.

4. Integrate $\int \frac{e^{\cot x}}{\sin^2 x}dx$

5. Prove that if $x+\frac{1}{x}=y+1$, then $\frac{(x^2-x+1)^2}{x(x-1)^2}=\frac{y^2}{y-1}$

   and hence solve the equation $(x^2-x+1)^2-4x(x-1)^2=0$

6. Find $\int \frac{x^3}{x^2-3x+2}dx$

7. If $y=Ae^{-x}\cos(x+\alpha)$, where A and α are constants, prove that

   $$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y=0$$

8. Find $\int Cos^4 x dx$

9. Find the solution set of $\frac{3x}{x-8}<\frac{2x-1}{5}$

10. Evaluate $\int_0^3 \frac{x^3}{\sqrt{(1+x^2)}}dx$

EXERCISE 34

1. Find the cube root of $1+i\sqrt{3}$

2. Prove that $\sum_{i}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

3. Given that $y = e^{-x}\cos 2x$, use the method of implicit differentiation to prove that $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$

4. Solve the simultaneous equations

$$\log_2 x - \log_4 y = 4$$
$$\log_2 (x-2y) = 5$$

5. Show that the equation of the tangent at the point (x_1, y_1) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $b^2xx_1 + a^2yy_1 = a^2b^2$

6. Find the points of intersection of the line $3x - 2y = 2$ with the curve $x = t-1,\ y = 1/t$

7. If $4x^3 + kx^2 + px + 2$ is divisible by $x^2 + q^2$, prove that $kp = 8$

8. Use Maclaurin's series to expand $\ln(1-x)$ as far as the term in x^4, hence find the value of $\ln 0.8$

9. Given that $\frac{a}{b} = \frac{c}{d} = k$. Show that $k = \frac{a+c}{b+d}$.

hence solve the equations: $\frac{x+4z}{4} = \frac{y+z}{6} = \frac{3x+y}{5}$, $4x+2y+5z = 30$

EXERCISE 35

1. If $2\cos\theta = x + \frac{1}{x}$, show that $2\cos 3\theta = x^3 + \frac{1}{x^3}$

2. Differentiate $\frac{1}{x^2}$ from first principles

3. Express in modulus argument form $\frac{(4+3i)\sqrt{(3+4i)}}{3+4i}$

4. Factorize completely $x^4-4x^3-7x^2+34x-24$ and hence solve

   $x^4-4x^3-7x^2+34x-24=0$

5. Solve $|x^2+4x-1|<2$.

6. Find $\int \frac{x^3}{x^2+x-20}dx$

7. a) Find the foci of the ellipse $x^2 + 4y^2 = 9$.

   b) A normal to the ellipse $x^2+4y^2=9$ at P $P(a\cos\theta, b\sin\theta)$ meets the y-axis at R. show that the locus of the midpoint of PR is another ellipse and hence find its focus and the length of the latus rectum.

8. Find the length and direction cosines of the line $\overline{PM}$ where P is the midpoint of $\overline{AB}$, M is the midpoint of $\overline{BC}$ and A,B,C are the points (3,-1,5), (7,1,3) and

   (-5,9,-1) respectively.

9. Prove that the circles $x^2+y^2+10x-4y-3=0$ and $x^2+y^2-2x-6y+5=0$ are

   orthogonal.

10. The police arrives at a scene of a murder at 8.00am. On arrival, the temperature of the body and its surroundings are measured at 34^o C and 17^o C respectively. At 9.00am the body temperature was measured as 33^0 C and the room temperature was still constant. Use Newton's law of cooling to estimate the time of death.

EXERCISE 36

1. The population of Uganda increases by 2.75% per annum. How long will

it take for the population of the school to treble?

2. Find the area enclosed between the two curves $y^2 = 4ax$ and $x^2 = 4ay$

3. Find the line of intersection of two planes $r.(i + j - 3k) = 6$ and $r.(2i - j + k) = 4$

4. Given that $z = x + iy$ show that $\left|z^2\right| = \left|z\right|^2$

5. The roots of the equation $2x^3 + 3x^2 + px - 6 = 0$ form an AP. Find the values of P and hence solve the equation

6. Show that $\int \cos x\sqrt{\cos 2x}dx = \frac{1}{2}\sin x\sqrt{(1-2\sin^2 x)} + \frac{1}{2\sqrt{2}}\sin^{-1}(\sqrt{2}\sin x) + c$

EXERCISE 37

1. Find the sum of the following series

$$2 + 12 + 36 + \ldots + (n^3 + n^2)$$

2. A water tank in the shape of a right circular cone of height 10cm and base radius 1 cm is catching the drips of water from a tap leaking at a rate of $0.1cm^3/s$. Find the rate at which the surface area of the water is increasing when the water is half way up the cone.

3. Express $\frac{16x}{x^4-16}$ as a partial fraction

4. If the normals from a point *(h,k)* meet the parabola $y^2 = 4ax$ at the three points t_1, t_2, t_3, show that $t_1 + t_2 + t_3 = 0$.

5. If $y = \cos\ln x$, implicitly, prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

6. Find the equation to the circle through the origin and through the points of intersection of the circles $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 + 8x - 4y + 6 = 0$

7. When the expression $x^5 + 4x^2 + ax + b$ is divided by $x^2 - 1$, the reminder is $2x + 3$. Find the values of a and b.

8. a) Given the function $y = \frac{3x-2}{x^2-3x+2}$, state:

   i) the horizontal asymptotes
   ii) the vertical asymptotes
   iii) the range of values the function can take for real x.

   b) Hence sketch the graph of the function

9. i) One root of the equation $z^2 + az + b = 0$ where a and b are real constants, is $2 + 3i$. Find the values of a and b.

   ii) Given that $2 + i$ is a root of the equation, $z^3 - 11z + 20 = 0$, find the remaining roots.

   iii) Verify that $2 + 3i$ is one of the square roots of $-5 + 12i$. Write down the other square root.

10. a) Prove that if $y = mx + c$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then

$c^2 = a^2m^2 - b^2$.

EXERCISE 38

1. Given that $i - 2$ is a root of the equation $Z^3 + pZ^2 + qZ - 15 = 0$. Find the values of the real constant p and q and hence determine the other roots.

2. Find the Cartesian equation of the plane containing the two lines below

   $r = i + j + k + \lambda(j - 4i + k)$ and $r = j + 2k + \mu(2i - j + k)$

3. Show that the line $y = mx$ is a tangent to $y = 2\sqrt{(ax)}$ if $y = \frac{a}{m}$

4. Solve the simultaneous equations for x and y

   $2\log_2 x + 3\log_3 y = 5$, and $xy = 6$

5. Show that $\int x\sin^{-1} x dx = \frac{1}{4}(2x^2 - 1)\sin^{-1} x + \frac{1}{4} x\sqrt{(1 - x^2)} + c$

6. The sum of the first four terms of an A.P is 48 and the sum of the first seven is 63. Find the tenth term and the sum of the first 10 terms.

7. If an error of 5% is made in measuring the radius a sphere, Find the corresponding percentage error in its surface area.

8. It can be proved by induction that for all positive values of n

   $$1^3 + 2^3 + 3^3 + ... + n^3 = \frac{1}{4}n^2(n+1)^2$$

   From the result deduce that

$$(n+1)^3+(n+2)^3+(n+3)^3+...+(2n)^3=\frac{1}{4}n^2(3n+1)(5n+3)$$

9. Factories in Uganda increase at a rate proportional to the number of bacteria present. If the number increases from 1000 to 2000 in one year, find how

   a) many factories will be in Uganda after $1\frac{1}{2}$ years?

   b) long will it take for the number of factories in Uganda increase to 4000?

EXERCISE 39

1. A particle P moves in a straight line, at time t, seconds, the velocity V m/s of P is given by $V\ =5-2t+t^2$. Find the acceleration of P when t = 3s and the distance traveled by P in the interval $0\le t\le 4$ seconds.

2. The perimeter of a triangle is 42cm; one side is of length 14cm and the area is $21\sqrt{15}cm^2$. Find the lengths of the other two sides and show that the cosine of the largest angle is $\frac{1}{4}$

3. The tangents from the curves $x^2+y^2=8$ and $x^2=2y$ meet at a point (2,2) find the angle between the tangents.

4. Given that $2+3i$ is a root of $z^3-6z^2+21z-26=0$ Find the other roots.

5. The 2nd, 3rd, and 9th terms of an AP form a geometric progression. Find the common ratio of the GP given that the common difference is none-zero.

6. If x, y and a. b are real numbers and $a+bi=\dfrac{x}{y+\cos\alpha+i\sin\alpha}$ show that $(y^2-1)(a^2+b^2)+x^2=2axy$

7. Prove that $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)$ for $-\pi\le(z_1z_2)\le\pi$ where z_1 and z_2 are complex numbers.

8. (i) Express the square root of $5+\sqrt{24}$ in the form $\pm(\sqrt{a}+\sqrt{b})$

(ii) Prove that in any triangle ABC

$$\sin\tfrac{1}{2}(B-C)=\frac{b-c}{a}\cos\tfrac{1}{2}A$$

9. A circle A passing through the point *(t+2, 3t)* has centre at *(t, 3t)* , another circle B of radius 2 units has centre at *(t+2, 3t).*

   i) Determine the equations of the circles A and B in terms of *t.*

   ii) If *t=1,* Find the points of intersection of the two circles.

   iii) Show that the common area of intersection of the circles A and B is $8(\pi/3-\sqrt{3}/4)$ Square units.

EXERCISE 40

1. Using the method of small increments, calculate the value of $\sqrt{9.01}$

2. Express the equations $y=m_1x+c_1$ and $y=m_2x+c_2$ in their vector form.

3. Find the value of p and q which make $x^4+6x^3+13x^2+px+q$ a perfect square.

4. In how many ways can the letters of the word BESIEGE and ANOTHERRAP be

arranged?

5. Solve $x^{\frac{1}{3}}-3=28x^{-\frac{1}{3}}$

6. Express $5x^2-30x+47$ in the form $a(x+p)^2+q$, hence show that the equation $5x^2-30x+47=0$ has no real roots

7. Solve the equation $3\sec^2\theta-4\tan\theta=2$

8. If $\sin y=2\sin x$ show that $\dfrac{d^2y}{dx^2}=3\sec^2 y\tan y$

Worked Examples

1. Given that α and β are roots of the equation $x^2+px+q=0$, express $(\alpha-\beta^2)(\beta-\alpha^2)$ in terms of p and q.

$$x^2+px+q=0$$

$$\alpha+\beta=-p,\ \alpha\beta=q$$

$\left(\alpha-\beta^2\right)\left(\beta-\alpha^2\right) = \alpha\beta-\alpha^3-\beta^3+\alpha^2\beta^2$

$= \alpha\beta+(\alpha\beta)^2-\left(\alpha^3+\beta^3\right)$

$= \alpha\beta+(\alpha\beta)^2-\left[(\alpha+\beta)\left(\alpha^2-\alpha\beta+\beta^2\right)\right]$

$= \alpha\beta+(\alpha\beta)^2-\left[(\alpha+\beta)\left(\alpha^2+\beta^2-\alpha\beta\right)\right]$

$= \alpha\beta+(\alpha\beta)^2-\left[(\alpha+\beta)\left\{(\alpha+\beta)^2-2\alpha-\alpha\beta\right\}\right]$

$= \alpha\beta+(\alpha\beta)^2-\left[(\alpha+\beta)\left\{(\alpha+\beta)^2-3\alpha\beta\right\}\right]$

$= q+q^2-\left(-p\left\{(-p)^2-3q\right\}\right)$

$\left(\alpha-\beta^2\right)\left(\beta-\alpha^2\right) = q+q^2+p^3-3pq$

2. Solve for n: $\log_n(4)+\log_4\left(n^2\right)=3$

$\log_n 4+\log_4 n^2=3$

$\log_n 4+\dfrac{2}{\log_n 4}=3$

Let $\log_n 4=m$

$m+\dfrac{2}{m}=3$

$m^2-3m+2=0$

$(m-2)(m-1)=0$

$\log_n 4=2$ *or* $\log_n 4=1$

$n^2=4$, $n=4$

$n=\pm2$, $n=4$

3.If $x\sin\theta=m-y\cos\theta$ *and* $y\sin\theta=n+x\cos\theta$, *show that:*
$m^2-y^2=x^2-n^2$.

$$x\sin\theta = m - y\cos\theta$$

$$y\sin\theta = n + x\cos\theta$$

$$(x\sin\theta + y\cos\theta)^2 = m^2$$

$$(y\sin\theta - x\cos\theta)^2 = n^2$$

$$x^2\sin^2\theta + 2xy\sin\theta\cos\theta + y^2\cos^2\theta = m^2 --------*$$

$$y^2\sin^2\theta - 2xy\sin\theta\cos\theta + x^2\cos^2\theta = n^2 ---------\#$$

*Add * and #*

$$(x^2 + y^2)\sin^2\theta + (x^2 + y^2)\cos^2\theta = m^2 + n^2$$

$$(x^2 + y^2) = m^2 + n^2$$

$$x^2 - n^2 = m^2 - y^2$$

$$m^2 - y^2 = x^2 - n^2$$

4. Solve the simultaneous equations: $xy = 8$ *and* $\log_x y = 2$

$$\log_x y = 2$$

$$x^2 = y ----i)$$

$$xy = 8$$

$$y = \frac{8}{x} ----ii)$$

Equate i) and ii)

$$x^2 = \frac{8}{x}$$

$$x^3 = 8$$

$$x^3 - 2^3 = 0$$

$$(x-2)(x^2 + 2x + 4) = 0$$

$(x-2) = 0$ *or* $x^2 + 2x + 4 = 0$

$x = 2$, $-1 \pm i\sqrt{3}$

Using equation ii)

$y = 4$, $\dfrac{8}{-1 \pm i\sqrt{3}}$

5. *Show that equation of the tangent to the rectangular hyperbola* $xy = c^2$ *at the point* (d, p) *may be written as:* $xp + yd - 2c^2 = 0$

$$xy = c^2$$
$$y = c^2 x^{-1}$$
$$\frac{dy}{dx} = \frac{-c^2}{x^2}$$

At (d , p) , $\dfrac{dy}{dx} = \dfrac{-c^2}{d^2}$

Equation of tangent

$$\frac{y-p}{x-d} = \frac{-c^2}{d^2}$$
$$yd - dp = c^2 - \frac{c^2 x}{d}$$

If $xy = c^2$ *then* $dp = c^2$

$$yd - c^2 = c^2 - px$$
$$xp + yd - 2c^2 = 0$$

6. *a) Prove that:*

$$\log a + \log(ax) + \log(ax^2) + ...(to\ n \text{ terms}) = n\log a + \tfrac{1}{2}n(n-1)\log x$$

The n^{th} *term is* $\log ax^{n-1}$

$$\log a + \log(ax) + \log(ax^2) + ... + \log ax^{n-1} = n\log a + \tfrac{1}{2}n(n-1)\log x$$

Suppose n = 1

$\log a = \log a$

Suppose n = 2

$\log a + \log ax = 2\log a + \log x$

$$= \log a + \log a + \log x$$
$$= \log a + \log ax$$

Suppose it holds for n = k

$$\log a + \log(ax) + \log(ax^2) + ... + \log ax^{k-1} = k\log a + \tfrac{1}{2}k(k-1)\log x$$

Does it hold for n = k+1

$$\log a + \log(ax) + \log(ax^2) + ... + \log ax^{k-1} + \log ax^k = (k+1)\log a + \tfrac{1}{2}(k+1)k\log x$$

$$= k\log a + \log a + \tfrac{1}{2}(k^2+k)\log x$$

From LHS

$$\log a + \log(ax) + \log(ax^2) + ... + \log ax^{k-1} + \log ax^k = k\log a + \tfrac{1}{2}k(k-1)\log x + \log ax^k$$

$$= k\log a + \tfrac{1}{2}k(k-1)\log x + \log a + \log x^k$$

$$= k\log a + \log a + \left[\tfrac{1}{2}k(k-1) + k\right]\log x$$

$$= k\log a + \log a + \tfrac{1}{2}(k^2+k)\log x$$

LHS =RHS

hence

$$\log a + \log(ax) + \log(ax^2) + ...(to\, n \text{ terms}) = n\log a + \tfrac{1}{2}n(n-1)\log x$$

b).The 2^{nd}, 5^{th} *and* 11^{th} *terms of an A.P are in G.P and the* 7^{th} *term is 4. Find the term* 1^{st} *and the common difference.*

For A.P, let the first term be a and the common difference be d.

2^{nd} term = a+d

5^{th} term = a+4d

11^{th} term = a+10d

For the GP

$$\frac{a+10d}{a+4d}=\frac{a+4d}{a+d}$$
$$(a+10d)(a+d)=(a+4d)^2$$

$$a^2+ad+10ad+10d^2=a^2+8ad+16d^2$$
$$11ad+10d^2=8ad+16d^2$$
$$3d(a-2d)=0$$
$$a=2d$$

7^{th} term

$$a+6d=4$$
$$2d+6d=4$$
$$d=\frac{1}{2}$$

$$a=4-6d$$

$$=4-\frac{6}{2}$$

$$a=1$$

7. *Given that* $\frac{(2+3i)(1-i)}{(3+4i)}=z$ *,determine:*

i) $\bar{z}$ *in the form* $a+ib$

ii) the locus represented by the a complex number such that $\left|\frac{z-1-i}{z-1+i}\right|=2.$

i)

$$z=\frac{(2+3i)(1-i)}{(3+4i)}$$

$$=\frac{2-2i+3i-3i^2}{(3+4i)}$$

$$= \frac{5+i}{(3+4i)}$$

$$= \frac{(5+i)(3-4i)}{(3+4i)(3-4i)}$$

$$= \frac{15-20i+3i-4i^2}{3^2+4^2}$$

$$z = \frac{19-17i}{25}$$

$$\bar{z} = \frac{19}{25} + \frac{17}{25}i$$

ii) $\left|\frac{z-1-i}{z-1+i}\right|$ *=2* *but z=x+iy*

$$\left|\frac{(x-1)+i(y-1)}{(x-1)+i(y+1)}\right| = 2$$

$$|(x-1)+i(y-1)| = 2|(x-1)+i(y-1)|$$

$$\sqrt{(x-1)^2+(y-1)^2} = 2\sqrt{(x-1)^2+(y+1)^2}$$

$$x^2-2x+1+y^2-2y+1 = 4(x^2-2x+1+y^2+2y+1)$$

$$x^2-2x+y^2-2y+2 = 4x^2-8x+4y^2+8y+8$$

$$3x^2+3y^2-6x+10y+6=0$$

8. Expand $\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}$ *up to the term in* x^2 *hence, find the value of* $\sqrt{\left(\frac{0.875}{1.125}\right)}$.

Deduce the value of $\sqrt{7}$.

$$\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}} = (1-x)^{\frac{1}{2}}(1+x)^{\frac{-1}{2}}$$

Binomial

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \dots$$

$$(1-x)^{\frac{1}{2}} = 1 + \frac{1}{2}(-x) + \frac{\frac{1}{2}\times\frac{-1}{2}(-x)^2}{2!} + ..$$

$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2$$

$$(1-x)^{\frac{1}{2}} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 + \dots$$

But also $(1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}(x) + \frac{-\frac{1}{2}\times\frac{-3}{2}(x)^2}{2!} + ..$

$$= 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \dots$$

$$(1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}} = \left[1 - \frac{1}{2}(x) - \frac{1}{8}x^2 + \dots\right]\left[1 - \frac{1}{2}x + \frac{3}{8}x^2 + \dots\right]$$

$$(1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}(x) + \frac{3}{8}x^2 - \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^2 + \dots$$

$$\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}} = 1 - x + \frac{x^2}{2}$$

ii) $1 - x = 0.875$

$$x = 0.125$$

And $1 + x = 1.125$

$$x = 0.125$$

So $\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}} = \left(\frac{1-0.125}{1+0.125}\right)^{\frac{1}{2}}$

but put $x = 0.125$ *Or* $x = \frac{1}{8}$

$$\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}} = 1 - x + \frac{x^2}{2}$$

$$\left(\frac{0.875}{1.125}\right)^{\frac{1}{2}} = 1 - \frac{1}{8} + \frac{\left(\frac{1}{8}\right)^2}{2}$$

$$= \frac{7}{8} + \frac{1}{128}$$

$$\sqrt{\left(\frac{0.875}{1.125}\right)} = \frac{113}{128}$$

$$\approx 0.8828$$

iii) From $\left(\frac{0.875}{1.125}\right)^{\frac{1}{2}} = \left(\frac{\left(\frac{7}{8}\right)}{\left(\frac{9}{8}\right)}\right)^{\frac{1}{2}}$

$$\left(\frac{7}{9}\right)^{\frac{1}{2}} = \frac{113}{128}$$

$$\frac{\sqrt{7}}{\sqrt{9}} = \frac{113}{128}$$

$$\sqrt{7} = \sqrt{9}\left(\frac{113}{128}\right)$$

$$= 3 \times \frac{113}{128}$$

$$\sqrt{7} \approx \frac{339}{128}$$

$$\sqrt{7} \approx 2.6484$$

ANSWERS

EXERCISE 1

1. $5\sqrt{10}$
2. $t = -2, 3$
3. (i) 2.807 (ii) 7
4. $7x^2 - 2x - 3 = 0$
5. 22
6. $\dfrac{2x+3}{3x(1+x)^{2/3}}$
7. $a = -18, b = 3$
8. (a) $\sin\theta$ (b) $135^o, 180^o, 315^0$

EXERCISE 3

2. $n = -1,6$

3. $y^4 + 4hy^3 + 6h^2y^2 + 4h^3y + h^4$, 15.87

5. $\sqrt{54}$ 8. 10 ,-43.87^0 9b. $z^2 - 4z + 5$

10 a $y = -\dfrac{29}{23},\ x = -\dfrac{40}{23}$ 10 b -1 10 c $-(2+\sqrt{6})$

11 a $\left(\dfrac{2+\sqrt{7}}{3}, 4.43\right)$,minima, $\left(\dfrac{2-\sqrt{7}}{3}, 7.11\right)$, maxima

11b $q + p^3 - 3qp + q^2$ 13 a 4s, -64ms^{-1}

13 b 27m, 13 c 16m/s, 64m/s

14 a 189 14b 10.1 14 c $y = x^3 + 5x^2 - x + 4$

EXERCISE 4

1. $y = 0,1$

2. $3 + 2\sqrt{2} + 3\sqrt{3} + 2\sqrt{6}$

4. $(x+1)(x-2)(x^2 - 2x + 4)$

6. $\dfrac{3+2x}{3(1+x)^{4/3}}$

7. $13\sin(\theta + 67.38)$

8. $x^2 + 35x - 343 = 0$

10. $x = 9$

11. $p(x) = 2x^2 + 3x - 2; x = \dfrac{1}{2}, 2$

12. $x = 3^6, y = \dfrac{1}{3}, and, x = \dfrac{1}{3}, y = 3^6$

EXERCISE 5

1. $\dfrac{1}{4}q(4-9q)$

2. $\dfrac{1}{4}q(4-9q)$

3. 10

4. $3t^2x - y - 2t^3 = 0$

5. $p = 4, q = 13$

6. $\min, 0, \max, \frac{4}{27}$

7. $xy = 1$

8. $x = \frac{1}{81}, 3$

9. 12,16

EXERCISE 6

1. $\frac{-1}{2}$, $s_\infty = \frac{2}{3}$

3. Point of contact (3.6, 4.8)

4. $2\left(x + \frac{5}{4}\right) + \frac{7}{8}$ min value is $\frac{7}{8}$

8. $q^3 = 27r^2$

9. $\frac{1}{2}\left[x + a^2 \ln\left(\frac{x-a}{x+a}\right)\right] + c$

11a) $\pm(1+2i), \pm(1-2i)$

12b) $y = \frac{6x^2+4}{3x^2+4}$

14a) $\left(2, \frac{3}{2}, \frac{3}{2}\right)$

14b) $3\sqrt{3}$ units

15a) $x + 4y - 16 = 0$, $9x + 4y - 48 = 0$

15b) $(x+y)^2 + 4(x - y + 1) = 0$

EXERCISE 7

1. $x = \frac{1}{2}, y = \frac{1}{2}, and, x = 2, y = -1$

2. $90^o, 323.14^o$

3. $\frac{938}{333}$

4. $-4, -2, -1, 1$

6. $0^o, 45^o, 180^o, 225^o, 360^o$

7i) $\frac{12+3\sqrt{6}-7\sqrt{2}-5\sqrt{3}}{-23}$

9i) $p=-5, q=8, (x-4)$

ii) 0.59

EXERCISE 8

1. $30^o, 210^o$

3. $\frac{39}{74}$

4. $-2, 1$

5.

6b). $2, \frac{1}{4}$

7. $17+\frac{13\sqrt{2}}{7}$

8. $x^2(1+y^2)=(y+1)^2$

10. $2+\sqrt{2}$

EXERCISE 9

1. $m=9,1, and, m=-3$

2. find $\tan^{-1}\frac{3}{2}$, $x=\tan^{-1}\frac{3}{2}$

2.find $\tan^{-1}\frac{3}{2}$, $x=\tan^{-1}\frac{3}{2}$ range $0^o \le \theta \le 360^o$

7. 7^3

8. $196x^2-8x+1$

10b) $\max, \sqrt{29}, \min, -\sqrt{29}$

EXERCISE 10

1. 0

2. $y = x^3 + 1 - 6\ln x$

5. $\frac{97}{65} + \frac{69i}{65}$ $|a+bi| = 1.83units$

7. $-1 < x < 1$

9b) $1 + 4x + ax^2 + 16x^3 + \ldots$

11i) $\frac{2}{3}$

11ii) $x = \frac{1}{2}, 1, 1, 2$

11iii) -2, $1 - i\sqrt{3}$, $1 + i\sqrt{3}$

12b) $4, 9, 14$

13a) $0.03cm/s$

13b) $(0,0)$, $(1,0)$, $(2,0)$

At $(0,0)$ tangent is $y = 2x$ normal $2y + x = 0$

At (1, 0) tangent is $y = 1 - x$ normal $y = x - 1$

At $(2,0)$ tangent is $-y = 2x - 4$ normal $2y = 2 - 2x$

EXERCISE 11

2. 41.57 Sq units

4. $a = \frac{69}{578}$, $b = \frac{37}{578}\sqrt{2}$

5. $f(x) = 30$

7. 4066.34

8. 45^o

9a) $4s$, $-64m/s$

9b) $27m$

9c) $16m/s$

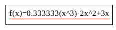
10a) $1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$

10b) $\frac{35}{222}$

11iii) $5\sin(\theta - 36.87)$

iv) greatest, 1, least = $\dfrac{1}{11}$

12.



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EXERCISE 12

1. 68.6^o

2.

3. $3units$

4. $1, \dfrac{1}{2} + or\_i\dfrac{\sqrt{3}}{2}$

5. $\dfrac{4\pi a^3}{3\sqrt{3}}$

6. $25x + 10y - 7z = 69$

7. 6

8. 18.43^o

9iii) $80y = 7200 - x^2$

10a) $7.86^o, 90^o, 127.86^o$

10b) 0.1047, $\dfrac{4}{3}x^3 + 6x^2 + 9x + c$

11i) $y = 5, x = 4$ and $y = -4, x = -5$

EXERCISE 13

2. $a = -2, b = -1$

4. ${}^{20}c_8 \cdot \dfrac{4^{12}}{3^8}$

5. $\frac{1}{4}$

6. $r = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + b\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

7. $\frac{\pi}{10}, \frac{\pi}{2}, \frac{9\pi}{10}, \pi$

9b) $2y\frac{dy}{dx} = \tan x \sec x$

10a) 60^o

10b) $+or-\left(\frac{1}{\sqrt{2}}\right) + i\frac{1}{\sqrt{2}}$

11a) $(-1,10,3)$

11b) $11x + 30y - 3z$

EXERCISE 14

1. $\frac{-1}{2\sqrt{x^3}}$

5. $S_\infty = \frac{3}{2}$

3. $\frac{1}{4}n(n+1)(n^2+n+2)$

4. $0^o, 90^o, 120^o, 180^o, 240^o, 270^o, 360^o$

6. $a = -1, -80$

10b) $a = -3, b = 2, c = 5$

11a) $1 - x + \frac{x^2}{2}$

11b) 3, 9, 27

13a) $a = b = c$

13b) $a = \frac{69}{578}, b = \frac{37}{578}$

EXERSCISE 15

1a) $\sqrt{2}$

1b) $\sqrt{3}$

1c) $\sqrt{73}$

2. 2, 1

3. $\pm(2+i)$

4a) zero property states that the product of two non zero

4b) $\frac{3}{2}, -5$

6. $p=12$, $q=4$

7. $4kx^3$

EXERCISE 16

1. $x=+or-1$, $y=\pm 2$

3. $\frac{1}{196}$

4. $4x^2y = x^4 + 48$

5. $6, 2$

6. $5-2x$

7. $x=2,\ y=1,\ z=-1$

10ai) $r.(2i+2j+k)=9$ scalar product

10aii) $2x+2y+z=9$ Cartesian

10b) $\begin{pmatrix} 5\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$

EXERCISE 17

1. 8

2i) $\frac{dy}{dx} = -\sin x \cos(\cos x)$

2ii) $\frac{dy}{dx} = (\cos x)^{\sin x}[\cos x \ln \cos x - \sin x \tan x]$

3. $-3, -2, 1, 2$

4. $\frac{240}{13} - \frac{30}{13}i$, 7.125

6. $7,2$

7. 20

8i) $2x - 2y - z = 9 = 0$

8ii) $C(1,4,3)$

8iii) 90^{o}

9i) $\frac{-3}{8x\sqrt{x}}$

10a) $p = -2$, $q = -4$

10bi) $(0,0)$, $\left(\frac{8}{3}, \frac{-257}{27}\right)$

10bii

f(x)=(x^2)(x-4)

10biii) $\frac{64}{3}$ sq units

EXERCISE 18

2. $5\sqrt{2}$ units, direction cosines are $\frac{-2\sqrt{2}}{5}$, $\frac{\sqrt{2}}{2}$, $\frac{-3\sqrt{2}}{10}$

3. $-x - i = \ln k(2 - x)$

5. $x^2 + y^2 - 12x = 0$

6. $x+3\ln(x+2)-\dfrac{2}{2x-1}+c$

7i). $120ways$

7ii) $12ways$

8. $\dfrac{dy}{dx}=-1$

9. $7-6x-x^2+7x^4+..........$

EXERCISE 19

1. $n!\left(n^2+4n+2\right)$

2. 60^o

4. $\dfrac{16a^2}{3}$ sq units

5. 4

6. $x+\ln\left(\dfrac{x-1}{x+1}\right)+c$

7. $\pm(4+i)$

8. $\dfrac{dy}{dx}=\dfrac{17y-x}{y-17x}$

10. $60^o, 180^o, 300^o$

EXERCISE 20

1a) is an arrangement of items.

1b) $\dfrac{8!}{2!3!}$

3. 8.6 sq units 8.6

4. 0.25

6. $\dfrac{1}{3}n\pi, n\pi, \dfrac{1}{6}\pi$

7. $\dfrac{\pi}{8}-\dfrac{1}{4}$

8. $1.22+1.02i$ $\quad -1.49+0.54i$ $\quad 0.27-1.56i$

9. 267.38

10. $\dfrac{3}{x+1}+\dfrac{1-2x}{x^2-x+1}$

EXERCISE 22

1. $\theta = 15^o, 75^o, 105^o, 165^o$

2. $f(x) = (x+2)(x-2)\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)$

3. 0.6427

4. $n = 1.8504 \qquad c = 1170.29$

6.

7. P= (0, 4)

8i). $2x^2 - 16x + 1$

OR

$2x + 16x + 1 = 0$

9.

10b) $z = 3 + 2i$ *OR* $z = 3 - i$

11i) $x + 4y + 8z = 25$

ii) $H = (x, y, z) = \left(4, \frac{27}{19}, \frac{72}{19}\right)$

iii) $\lambda = -6.75$

12a) $-\dfrac{2}{3}\left(\sqrt{e^{-x^3}-4} - 2\tan^{-1}\left(\dfrac{\sqrt{e^{-x^3}-4}}{2}\right)\right) + A$

b) 1.1

c) a=3 , $x = \left\{1, \frac{1}{4} \pm 0.66i\right\}$

13iii) a=3

EXERCISE 24

2. $-\sqrt{3}$, 1, $\sqrt{3}$, 3

3. $55\log 2x$

4. $\frac{2}{3\sqrt{3}}\tan^{-1}\left(\frac{2x+3}{3\sqrt{3}}\right)+c$

6. $10 radios$

7. $\pm\left(3+\sqrt{5}\right)$

8. $x^2+y^2-2x-7y-9=0$

9. a) $\sqrt{x}y = k(2y+x)^{\frac{1}{2}}$

b) $4:54am$

10.

11. a) $7x+8x^2+\frac{58}{3}x^3+...$

b) 2

12. a) 3.1178

b) $\frac{1}{6}\sin(3x^2+5)+c$

13. a) $\frac{\pi}{4}$

b) $n\pi+\frac{\pi}{3}$

14. a) $3x+4y-5z+4=0$

EXERCISE 27

1. $\frac{1}{6}n(8^2+21n+19)$

3. $\frac{\pi+1}{4}$

4. $20x^4-4\sin 4x$

5. $\frac{7}{2}, 7, 14$

7. $x \geq 2.07$

9ii). $2\ln(x^2+2x+2)+\tan^{-1}(x+1)+c$

10. a) $y+3y(x^2+1)=6(x^2+1)-2$

b) x

c) $y = 2\tan(2e^{-x} + A)$

11.ii) $2n\pi$

iii) $2n\pi - \dfrac{2\pi}{3}$

12.a) $5 - 7x + 29x^2 + \ldots$

b). $a = \dfrac{2w\rho}{w^2 + \rho}$

c) 5.0100

13) i). $x \le 1$ and $3 \le x \le 5$

ii). At $x = 1.845$ max At $x = 4.2$ min

iii). $x = 1,3,5$

13(iv)

f(x)=x^3-9x^2+23x-15

EXERCISE 28

1. 27,8

2. $x - y + 2z = 0$

3. $-2 < x < -1$ and $x > 1$

5. $\sqrt{18}$

6. $-21.07^0, 84.5^0, 95.5^0, 158.93^0$

7. $(x-4)^2 + (y-2)^2 = 5$

8 .a) $7.525cm$

b). $11.93cm$

9. b) 9. i) $\dfrac{1}{3}e^{3x}(2\sin 2x + 3\cos 2x) + c$

10. a) $3x + 4y - 5z + 4 = 0$

b) $3x^2+8y^2+35z^2-36yz-24zx+12xy-8x-12y+24z+4=0$

c) $\frac{10}{3},\frac{-3}{2},\frac{5}{3}$

11. y Can not lie between 0 and $\frac{8}{9}$

12.a) $n\pi+\tan^{-1}2$

13.i) 459 bags

ii) ≈ 6 days

EXERCISE 29

1. $p=\frac{2}{3}a+\frac{1}{3}b$ $\quad q=\frac{-2}{3}a+\frac{5}{3}b$

2. $\sqrt{5}$

3. $\frac{-7}{x}+\frac{7}{x^2}-\frac{3}{x^3}+\frac{7}{x+1}$

4. $4\frac{dy}{dx}=-1$

5i) 1.484

5ii) $-1,\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$

6. -20

7i) $(-1,-1.5)$

7ii) 62.74

8. $0^o,10^o,45^o,50^o,90^o,130^o,135^o,170^o,180^o$

9. $y=\frac{4}{3}x+\frac{25}{3}$, $xy=\frac{4}{3}x-\frac{25}{3}$

10a) 115.03

10b) $\frac{1}{2}\tan^{-1}x+\frac{1}{2}\left(\frac{x}{x^2+1}\right)+c$

EXERCISE 30

1. Geometric series is none with a common ratio.

Condition: if r is a common ratio then $|r|<0$ or $-1<r<1$

2. 512

3. $10,368m^2$

4. $\frac{x}{2}=\frac{y+9}{7}=\frac{z+5}{3}$

5. $a=\pm\frac{1}{2}$

7. $16a^3\pi$ Cubic units

8. $C_1(2,-3),(x-2)^2+(y+3)^2=13$

$C_2(-11,-3)$, $(x+11)^2+(y+3)^2=13$

9. $x=\frac{14}{5}, y=\frac{2}{5}$

10. $\frac{-1}{40}(\cos 10x-5\cos 2x)+c$