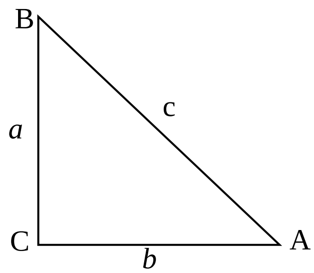
S5 TRIGONOMETRY

The Covid-19 pandemic has greatly affected us this year. I want to encourage everyone wherever you are to style up and get back to serious business. We can make it though time is running out.

KEEP SAFE BY AVOIDING CROWDS AND WASH HANDS REGULARLY

Trigonometry is the study of the properties of triangles, trigonometric functions and their applications.

Consider a right angled triangle ABC, the sides opposite to A, B and C being a, b and c respectively.



The sine of the angle A, written as $\sin A$ is defined as the ratio of the Opposite (BC) to the hypotenuse(AB). Thus $\sin A = \frac{BC}{AB} = \frac{a}{c}$.

The cosine of the angle A, written as $\cos A$ is defined as the ratio of the Adjacent (AC) to the hypotenuse(AB). Thus $\cos A = \frac{AC}{AB} = \frac{b}{c}$.

The tangent of the angle A, written as $\tan A$ is defined as the ratio of the Opposite (BC) to the adjacent(AC). Thus $\tan A = \frac{BC}{AC} = \frac{a}{b}$.

It should be clearly noted that $\tan A = \frac{BC}{AC} = \frac{a}{b} = \frac{\sin A}{\cos A}$.

From the common three trigonometric ratios above, we can derive their reciprocals.

The cosecant of angle A, written as $\operatorname{cosec} A$ is defined as $\operatorname{cosec} A = \frac{1}{\sin A} = \frac{c}{a}$.

The secant of angle A, written as $\sec A$ is defined by $\sec A = \frac{1}{\cos A} = \frac{c}{b}$.

The cotangent of angle A, written as $\cot A$ is defined by $\cot A = \dfrac{1}{\tan A} = \dfrac{\cos A}{\sin A} = \dfrac{b}{a}$.

The general angle

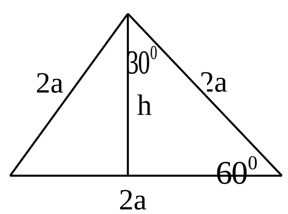
- An acute angle is one that lies between 0 and 90^0 i.e $0^0 < x < 90^0$ - first quadrant.
- An obtuse angle is one that lies between 90^0 and 180^0 i.e $90^0 < x < 180^0$ - second quadrant.
- A reflex angle is one that lies between 180^0 and 360^0 i.e $180^0 < x < 360^0$ - third and fourth quadrants.

We should remember that clockwise rotations are negative, so angles measured in a clockwise direction are always negative whereas the anticlockwise rotations are positive and so the angles measured in an anticlockwise direction are always positive.

It is also very important to remember the signs to the different trigonometric ratios in the different quadrants. The three ratios- sine, cosine and tangent are ALL positive in the first quadrant, the sine is the only one positive in the second quadrant, the tangent is the only one positive in the third quadrant and the cosine is the only one positive in the fourth quadrant.

Trigonometric ratios of 30^0 and 60^0

Consider an equilateral triangle of sides 2a.



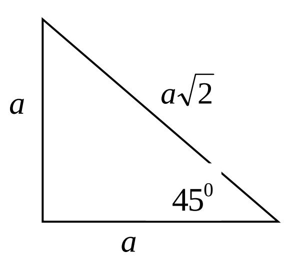
By Pythagoras theorem,

$$h = \sqrt{(2a)^2 - a^2} = a\sqrt{3}$$

Now; $\sin 30^0 = \cos 60^0 = \dfrac{a}{2a} = \dfrac{1}{2}$, $\cos 30^0 = \sin 60^0 = \dfrac{a\sqrt{3}}{2a} = \dfrac{\sqrt{3}}{2}$,

$\tan 30^0 = \dfrac{a}{a\sqrt{3}} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$ and $\tan 60^0 = \dfrac{a\sqrt{3}}{a} = \sqrt{3}$.

Consider an isosceles right angled triangle below.



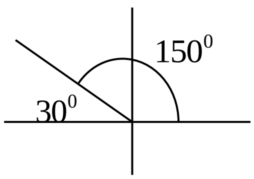
By Pythagoras' theorem

Hyp = $\sqrt{a^2 + a^2} = a\sqrt{2}$

Now, $\sin 45^0 = \cos 45^0 = \frac{a}{a\sqrt{2}} = \frac{\sqrt{2}}{2}$ and $\tan 45^0 = \frac{a}{a} = 1$.

Example

Find the sines, cosines and tangents of the angles 150^0, 210^0 and 315^0 by expressing them in terms of the trigonometric ratios of acute angles.



Since 150^0 lies in the second quadrant, there its only the sine which is positive. So

$$\sin 150^0 = +\sin(180-150) = +\sin 30^0 = 0.5$$

$$\cos 150^0 = -\cos(180-150) = -\cos 30^0 = -\frac{\sqrt{3}}{2}$$ and

$$\tan 150^0 = -\tan(180-150) = -\tan 30^0 = -\frac{\sqrt{3}}{3}.$$

Since 210^0 lies in the third quadrant, there it is only the tangent which is positive. So

$$\sin 210^0 = -\sin(210-180) = -\sin 30^0 = -0.5$$

$$\cos 210^0 = -\cos(210-180) = -\cos 30^0 = -\frac{\sqrt{3}}{2}$$ and

$$\tan 210^0 = +\tan(210-180) = +\tan 30^0 = \frac{\sqrt{3}}{3}.$$

Since 315^0 lies in the fourth quadrant, there it is only the cosine which is positive. So

$$\sin 315^0 = -\sin(360-315) = -\sin 45^0 = -\frac{\sqrt{2}}{2}$$

$$\cos 315^0 = +\cos(360-315) = +\cos 45^0 = \frac{\sqrt{2}}{2}$$

$$\tan 315^0 = -\tan(360-315) = -\tan 45^0 = -1$$

Suppose we wanted to find $\operatorname{cosec} 315^0$, use the definition $\operatorname{cosec} 315^0 = \frac{1}{\sin 315^0} = -\frac{2}{\sqrt{2}} = -2$.

Trial exercise

1. Evaluate the following without using mathematical tables.

   (a) $\sec 150^0$ (b) $\cot 135^0$ (c) $\sin 230^0$ (d) $\cos 156^0$

   (e) $\cos 225^0$ (f) $\sin 480^0$ (g) $\cos ec\, 600^0$ (h) $\tan 1020^0$

2. Given that $\sin x = \frac{5}{13}$ and $0^0 < x < 360^0$, find the possible values of $\tan x - \cot x$.

3. Given that $\cos\theta = -\frac{8}{17}$ and θ is a reflex angle, find the values of $4\sec^2\theta + \tan\theta$.

Pythagorean identity

It should be noted that for any value of the angle A, $\cos^2 A + \sin^2 A = 1$……. (i)

Proof

We saw already that $\sin A = \frac{a}{c}$, $\cos A = \frac{b}{c}$ in the right angled triangle ABC.

Now $\sin^2 A + \cos^2 A = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = \frac{a^2 + b^2}{c^2} = \frac{c^2}{c^2} = 1$

When we divide through equation (i) by $\cos^2 A$, we obtain $1 + \tan^2 A = \sec^2 A$….. (ii)

When we divide through equation (i) by $\sin^2 A$, we obtain $1 + \cot^2 A = \cos ec^2 A$….. (iii)

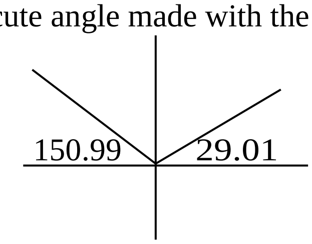
Solving trigonometric equations

Examples

Solve the equations below for $0^0 < x < 360^0$.

(a) $\sin x = 0.485$

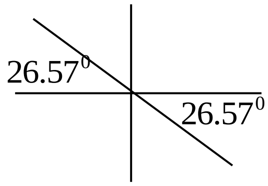
Here the sine is positive. So the angle must lie in the first and second quadrants. Thus to find *x*, first obtain the acute angle made with the horizontal.



$x = \sin^{-1} 0.485 = 29.01^0$, 150.99^0

(b) $\tan x = -0.5$

Here the tangent is negative and so the angle must lie in the second and third quadrants. However the acute angle made with the horizontal is obtained by $\tan^{-1} 0.5 = 26.57^0$.



The required angles are therefore

$x = 153.47^0$, 333.43^0

Trial question

Solve the equations below for $-360^0 < x < 360^0$

(i) $\cos x = -0.821$ (ii) $\tan x = 0.75$ (iii) $\cot x = -1.7254$

(iv) $\sec x = 1.432$ (v) $\cos x = 0.28$ (vi) $\operatorname{cosec} x = -2$

Example

1. *Solve the equations below for* $-180^0 \le \theta \le 180^0$.

(a) $6\cos^2\theta - \cos\theta - 1 = 0$

This equation is quadratic in $\cos\theta$. Solve it for the values of $\cos\theta$ using the quadratic formula.

$$\cos\theta = \frac{-(-1) \pm \sqrt{(-1)^2 - 4\times 6\times -1}}{2\times 6} = 0.5 \; or \; -\frac{1}{3}$$

Then obtain the required angles for each of the solutions in the given interval.

For $\cos\theta = 0.5$

$\theta = -60^0 , 60^0$

For $\cos\theta = -\frac{1}{3}$

$\theta = -109.47^0 , 109.47^0$

(b) $5\sin\theta + 6\operatorname{cosec}\theta = 17$

Re write this equation as $5\sin\theta + \dfrac{6}{\sin\theta} = 17 \Rightarrow 5\sin\theta - 17\sin\theta + 6 = 0$.

$$\sin\theta = \frac{-(-17) \pm \sqrt{(-17)^2 - 4\times5\times6}}{2\times5} = 3(discard)\ or\ 0.4$$

Thus for $\sin\theta = 0.4$, $\theta = 23.58^0$, 156.42^0 .

(c) $4\cos^3\theta = \cos\theta$.

For this question, do not divide through by $\cos\theta$ as this will lead to loss of some solutions. Therefore rearrange and factorise.

$$4\cos^3\theta = \cos\theta \Rightarrow 4\cos^3\theta - \cos\theta = 0,\ \cos\theta(4\cos^2\theta - 1) = 0$$

$$\cos\theta = 0 \Rightarrow \theta = -90^0\ ,\ 90^0$$

Or $\cos\theta = \sqrt{\dfrac{1}{4}} = \pm\,0.5 \Rightarrow \theta = -120^0\ ,\ -60^0\ ,\ 60^0\ ,\ 120^0$

(d) $2\sin^2\theta - 1 = \cos\theta$

For this question, first express $\sin^2\theta$ as $\sin^2\theta = 1 - \cos^2\theta$ (in terms of cosines) and the solve.

$$\cos\theta = \frac{-1 \pm \sqrt{1^2 - 4\times2\times-1}}{2\times2} = -1\ or\ 0.5$$

The angles are $-180^0\ ,\ -60^0\ ,\ 60^0\ ,\ 180^0$

(e) $4\cos\theta - 3\sec\theta = 2\tan\theta$

$$4\cos\theta - \frac{3}{\cos\theta} = \frac{2\sin\theta}{\cos\theta} \Rightarrow 4\cos^2\theta - 3 = 2\sin\theta$$

$$4(1 - \sin^2\theta) - 3 = 2\sin\theta \Rightarrow 4\sin^2\theta + 2\sin\theta - 1 = 0$$

$$\sin\theta = \frac{-2 \pm \sqrt{2^2 - 4\times4\times-1}}{2\times4} = 0.3090\ or\ -0.8090$$

The angles are $\theta = 18^0\ ,\ 162^0$ and $\theta = -18^0\ ,\ -162^0$

(f) $5\sin^2 2x - 3\sin 2x\cos 2x - 14\cos^2 2x = 0$.

First dividing through by $\cos^2 2x$, we have

$$5\tan^2 2x - 3\tan 2x - 14 = 0$$

$$\tan 2x = \frac{3 \pm \sqrt{9+280}}{10} = \frac{3 \pm 17}{10} = 2, -\frac{7}{5}$$

Thus for $\tan 2x = 2$, $2x = -116.57^0, 63.43^o, 243.43^0$; $x = -58.29^0, 31.72^o, 121.72^0$

Note that the interval is doubles for this question.

Also $\tan 2x = -1.4 \Rightarrow 2x = -54.46^0, 125.54^o$; $x = -27.23^0, 62.77^o$

2. *Solve the equation:* $5\sin^2 2x - 3\sin 2x\cos 2x - 14\cos^2 2x = 0$, *for* $0^o \le x \le 90^o$.

Dividing through by $\cos^2 2x$, we have $5\tan^2 2x - 3\tan 2x - 14 = 0$

$$\tan 2x = \frac{3 \pm \sqrt{9+280}}{10} = \frac{3 \pm 17}{10} = 2, -\frac{7}{5}$$

Thus for $\tan 2x = 2$, $2x = 63.43^o$, $x = 31.72^o$

$\tan 2x = -1.4$, $2x = 125.54^o$, $x = 62.77^o$

3. *Solve the equation* $(2\cot 2x - 1)^2 = 3(\operatorname{cosec}^2 2x - 2)$ *for* $0^o \le x \le 200^o$.

$$4\cot^2 2x - 4\cot 2x + 1 = 3(\cot^2 2x + 1) - 6$$

$$\cot^2 2x - 4\cot 2x + 4 = 0, \quad (\cot 2x - 2)(\cot 2x - 2) = 0$$

$$\cot 2x = 2, \quad \tan 2x = \frac{1}{2}$$

$$2x = 26.6^o, 206.6^o, 386.6^o \text{ , thus } x = 13.3^o, 103.3^o, 193.3^o$$

Trial exercise

1. Solve the equation $\sin^2\theta = -\sin\theta\cos\theta$ for $0^0 \le \theta \le 360^0$.

2. Solve the equations below for $0^0 \le \theta \le 360^0$:

(a) $\sec^2\theta = 3\tan\theta - 1$ (b) $2\cot^2\theta = 7\operatorname{cosec}\theta - 8$.

3. Solve the equation $5\cos^2 3x = 3(1+\sin 3x)$ for $-180^0 < x < 180^0$.

4. Solve the equation $\sin^2\theta + 5\cos^2\theta = 3$ for $-360^0 < x < 360^0$.

5. Solve the equation: $5\sin^2 3x - 3\sin 3x\cos 3x - 14\cos^2 3x = 0$, for $0^o \le x \le 90^o$.

6. Solve the equation $7\tan^2 x - 5\sec x\tan x + 1 = 0$ for values of x between 0^0 and 180^0.

7. Given that $\cot^2\theta + 3\operatorname{cosec}^2\theta = 7$, show that $\tan\theta = \pm 1$.

8. Solve the trigonometric equation $(2\cot 2x - 1)^2 = 3(\operatorname{cosec}^2 2x - 2)$ for $0 \le x \le 360^0$.

9. Solve the equation: $6\cos^2\theta - \cos\theta - 1 = 0$, for $0^o \le \theta \le 360^o$.

10. Solve the equation $4\sin x + 1 = 3\operatorname{cosec} x$ for $0^o \le x \le 360^o$.

11. Find all the values of x in the interval $180^0 \le x \le 540^0$ for which $\cot x + 5\operatorname{cosec}^2 x = 6$.

12. Solve the equation: $(1-\sin x)^2 + (1+\cos x)^2 = 1$ for $-180^o \le x \le 180^o$.

MORE EXAMPLES

1. *Eliminate θ from each of the following.*

In order to eliminate θ, we shall make use of the trigonometric identities already discussed.

(a) $x = a\cos\theta$, $y = b\sin\theta$

Here $\cos\theta = \dfrac{x}{a}$ and $\sin\theta = \dfrac{y}{b}$.

Now using $\cos^2\theta + \sin^2\theta = 1$,

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(b) $x = 3\cos\theta + 2$, $y = 3\sin\theta - 1$

Here $\cos\theta = \dfrac{x-2}{3}$ and $\sin\theta = \dfrac{y+1}{3}$.

Now using $\cos^2\theta + \sin^2\theta = 1$, $\Rightarrow \left(\frac{x-2}{3}\right)^2 + \left(\frac{y+1}{3}\right)^2 = 1$

So $(x-2)^2 + (y+1)^2 = 9$

(c) $x = a\sin\theta$, $y = b\tan\theta$

Here $\sin\theta = \frac{x}{a}$ and $\tan\theta = \frac{y}{b}$. It now implies that $\cot\theta = \frac{b}{y}$ and $\operatorname{cosec}\theta = \frac{a}{x}$.

So, using $1 + \cot^2\theta = \operatorname{cosec}^2\theta$, $1 + \frac{b^2}{y^2} = \frac{a^2}{x^2} \Rightarrow x^2y^2 + b^2x^2 = a^2y^2$.

2. *If* $x = a\operatorname{cosec}\theta$, *simplify* $\frac{1}{\sqrt{x^2 - a^2}}$.

$$\frac{1}{\sqrt{x^2-a^2}} = \frac{1}{\sqrt{a^2\operatorname{cosec}^2\theta - a^2}} = \frac{1}{a\sqrt{(\operatorname{cosec}^2\theta - 1)}} = \frac{1}{a\cot\theta} = \frac{1}{a}\tan\theta$$

3. *Given that* $s = 2\sin\theta$, *simplify (i)* $\sqrt{4-s^2}$ *(ii)* $\frac{s}{4-s^2}$.

(i) $\sqrt{4-s^2} = \sqrt{4-4\sin^2\theta} = 2\sqrt{(1-\sin^2\theta)} = 2\cos\theta$

(ii) $\frac{s}{4-s^2} = \frac{2\sin\theta}{4-4\sin^2\theta} = \frac{2\sin\theta}{4(1-\sin^2\theta)} = \frac{2\sin\theta}{4\cos^2\theta} = \frac{1}{2}\sec\theta\tan\theta$.

4. *Prove the following identities*

(a) $\sec^4 x - \operatorname{cosec}^4 x = \frac{\sin^2 x - \cos^2 x}{\cos^4 x\sin^4 x}$. *(b)* $\frac{\cos x}{\sqrt{1+\tan^2 x}} + \frac{\sin x}{\sqrt{1+\cot^2 x}} = 1$.

(c) $\tan^2 x + \sin^2 x = (\sec x + \cos x)(\sec x - \cos x)$ *(d)* $\frac{1-\cos^2 x}{\sec^2 x - 1} = 1 - \sin^2 x$

(e) $\tan x + \cot x = \sec x\operatorname{cosec} x$ *(f)* $\tan^2 A - \sin^2 A = \sin^4 A\sec^2 A$

(g) $\sqrt{\frac{1-\cos P}{1+\cos P}} = \operatorname{cosec} P - \cot P$

Solutions

(a) $L.H.S = \sec^4 x - \cos ec^4 x = \left(\sec^2 x - \cos ec^2 x\right)\left(\sec^2 x + \cos ec^2 x\right)$

$$= \left(\frac{1}{\cos^2 x} - \frac{1}{\sin^2 x}\right)\left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right)$$

$$= \left(\frac{\sin^2 x - \cos^2 x}{\cos^2 x \sin^2 x}\right)\left(\frac{\sin^2 x + \cos^2}{\cos^2 x \sin^2 x}\right)$$

$$= \frac{\sin^2 x - \cos^2 x}{\cos^4 x \sin^4 x} = R.H.S$$

(b) $\dfrac{\cos x}{\sqrt{1+\tan^2 x}} + \dfrac{\sin x}{\sqrt{1+\cot^2 x}} = 1$

$$L.H.S = \frac{\cos x}{\sqrt{1+\tan^2 x}} + \frac{\sin x}{\sqrt{1+\cot^2 x}} = \frac{\cos x}{\sqrt{\sec^2 x}} + \frac{\sin x}{\sqrt{\cos ec^2 x}} = \frac{\cos x}{\sec x} + \frac{\sin x}{\cos ec\, x}$$

$$= \cos^2 x + \sin^2 x = 1 \; R.H.S$$

(c) $\tan^2 x + \sin^2 x = \left(\sec x + \cos x\right)\left(\sec x - \cos x\right)$

$$L.H.S = \tan^2 x + \sin^2 x = \sec^2 x - 1 + 1 - \cos^2 x = \sec^2 x - \cos^2 x$$

$$= \left(\sec x + \cos x\right)\left(\sec x - \cos x\right) = R.H.S$$

(d) $\dfrac{1-\cos^2 x}{\sec^2 x - 1} = 1 - \sin^2 x$

$$L.H.S = \frac{1-\cos^2 x}{\sec^2 x - 1} = \frac{\sin^2 x}{\tan^2 x} = \sin^2 x \div \left(\frac{\sin^2 x}{\cos^2 x}\right) = \cos^2 x = 1 - \sin^2 x = R.H.S$$

(e) $\tan x + \cot x = \sec x \cos ec\, x$

$$L.H.S = \tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} = \frac{1}{\cos x \sin x}$$

$$= \sec x \cos ec\, x = R.H.S$$

(f) $\tan^2 A - \sin^2 A = \sin^4 A \sec^2 A$

$$L.H.S = \tan^2 A - \sin^2 A = \frac{\sin^2 A}{\cos^2 A} - \sin^2 A = \frac{\sin^2 A\left(1-\cos^2 A\right)}{\cos^2 A}$$

$$= \frac{\sin^2 A.\sin^2 A}{\cos^2 A} = \sin^4 A\sec^2 A = R.H.S$$

(g) $\sqrt{\frac{1-\cos P}{1+\cos P}} = \operatorname{cosec} P - \cot P$

$$L.H.S = \sqrt{\frac{1-\cos P}{1+\cos P}} = \sqrt{\frac{(1-\cos P)(1-\cos P)}{(1+\cos P)(1-\cos P)}} = \sqrt{\frac{(1-\cos P)^2}{1-\cos^2 P}} = \frac{1-\cos P}{\sin P}$$

$$= \operatorname{cosec} P - \cot P = R.H.S$$

Trial exercise

1. Eliminate θ given; i) $x = a\cos\theta, \quad y = b\tan\theta$ ii) $x = p\operatorname{cosec}\theta, \quad y = b + q\cot\theta$

2. Find the Cartesian equations defined by the following parametric equations.

   (i) $x = 5 + \frac{\sqrt{3}}{2}\cos\theta, y = -3 + \frac{\sqrt{3}}{2}\sin\theta$ (ii) $x = 1 - \sin\theta$, $y = 1 - \tan\theta$

3. Prove the following identity $\sqrt{(\sec^2 A - \tan^2 A)} + \sqrt{(\operatorname{cosec}^2 A - \cot^2 A)} = 2$.

COMPOUND ANGLES

You should take note of the following expansions.

(i) $\sin(A+B) = \sin A\cos B + \cos A\sin B$ (ii) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

(iii) $\cos(A+B) = \cos A\cos B - \sin A\sin B$ (iv) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

For $\tan(A+B)$, use $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$

Now, dividing through top and bottom by $\cos A\cos B$ gives

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B} \ldots \text{(v)}$$

For $\tan(A-B)$, use $\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)} = \frac{\sin A\cos B - \cos A\sin B}{\cos A\cos B + \sin A\sin B}$

Now, dividing through top and bottom by $\cos A\cos B$ gives

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}\ \ldots\ \text{(vi)}$$

The above six formulae are also summarized in the mathematical tables.

1. *Without using tables or calculator, find the values of the following;*

(i) $\sin 75^0$ *(ii)* $\cos 105^0$ *(iii)* $\tan 15^0$

Solutions

(i) $\sin 75^0 = \sin(30^0+45^0) = \sin 30^0\cos 45^0 + \cos 30^0 \sin 45^0$

$$\frac{1}{2}\times\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}=\frac{1}{4}\left(\sqrt{2}+\sqrt{6}\right)$$

(ii) $\cos 105^0 = \cos\left(60^0+45^0\right) = \cos 60^0\cos 45^0 - \sin 60^0\sin 45^0$

$$=\frac{1}{2}\times\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}=\frac{1}{4}\left(\sqrt{2}-\sqrt{6}\right)$$

(iii) $\tan 15^0 = \tan\left(45^0-30^0\right)$ *or* $\tan\left(60^0-45^0\right)$

$$=\frac{\tan 45^0-\tan 30^0}{1+\tan 45^0\tan 30^0}=\frac{1-\frac{\sqrt{3}}{3}}{1+\frac{\sqrt{3}}{3}}=\frac{3-\sqrt{3}}{3+\sqrt{3}}$$

$$=\frac{\left(3-\sqrt{3}\right)\left(3-\sqrt{3}\right)}{\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right)}=\frac{9-6\sqrt{3}+3}{9-3}=2-\sqrt{3}$$

2. *A and B are acute angles such that* $\sin A=\frac{12}{13}$ *and* $\tan B=\frac{3}{4}$. *Without using tables or calculator, find the value of (i)* $\cot(A-B)$ *(ii)* $\cos(A-B)$

Solution

From $\sin A=\frac{12}{13}$, $\cos A=\frac{5}{13}$, $\tan\frac{12}{5}$ and from $\tan B=\frac{3}{4}$, $\sin B=\frac{3}{5}$, $\cos B=\frac{4}{5}$.

(i) $\cot(A-B)=\dfrac{1}{\tan(A-B)}=\dfrac{1+\tan A\tan B}{\tan A-\tan B}=\dfrac{1+\frac{12}{5}\times\frac{3}{4}}{\frac{12}{5}-\frac{3}{4}}$

$$= \frac{\frac{14}{5}}{\frac{33}{20}} = \frac{56}{33}$$

(ii) $\cos(A-B) = \cos A\cos B + \sin A\sin B = \frac{5}{13}\times\frac{4}{5} + \frac{12}{13}\times\frac{3}{5} = \frac{56}{65}$

3. *Given that* $\sin P = \frac{3}{5}$ *and* $\cos Q = \frac{15}{17}$ *where P is acute and Q is obtuse, find the exact value of (i)* $\sin(P+Q)$ *(ii)* $\tan(P+Q)$.

Solution

For $\sin P = \frac{3}{5}$, $\cos P = \frac{4}{5}$, $\tan P = \frac{3}{4}$ and for Q is obtuse, $\cos Q = -\frac{15}{17}$, $\sin Q = \frac{8}{17}$, and $\tan Q = -\frac{8}{15}$.

(i) $\sin(P+Q) = \sin P\cos Q + \cos P\sin Q = \frac{3}{5}\times -\frac{15}{17} + \frac{4}{5}\times\frac{8}{17} = -\frac{13}{85}$

(ii) $\tan(P+Q) = \frac{\tan P + \tan Q}{1 - \tan P\tan Q} = \frac{\frac{3}{4} + -\frac{8}{15}}{1 - \frac{3}{4}\times -\frac{8}{15}} = \frac{\frac{13}{60}}{\frac{7}{5}} = \frac{13}{84}$

4. *Prove that* $\tan^2(A+45^0) = \frac{1+\sin 2A}{1-\sin 2A}$.

Solution

$$L.H.S = \tan^2(A+45^0) = \frac{\sin^2(A+45^0)}{\cos^2(A+45^0)} = \frac{(\sin A\cos 45^0 + \cos A\sin 45^0)^2}{(\cos A\cos 45^0 - \sin A\sin 45^0)^2}$$

$$= \left(\frac{\sin A + \cos A}{\cos A - \sin A}\right)^2 = \frac{\sin^2 A + 2\sin A\cos A + \cos^2 A}{\cos^2 A - 2\sin A\cos A + \sin^2 A}$$

$$= \frac{1 + 2\sin A\cos A}{1 - 2\sin A\cos A} = \frac{1+\sin 2A}{1-\sin 2A} = R.H.S$$

5. *Show that* $\tan(A+B+C)=\dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan A\tan C-\tan B\tan C}$. *Hence deduce that, if* $A+B+C=180^0$*, then* $\cot A\cot B+\cot A\cot C+\cot B\cot C=1$.

Solution

$$\tan(A+B+C)=\frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}=\frac{\dfrac{\tan A+\tan B}{1-\tan A\tan B}+\tan C}{1-\left(\dfrac{\tan A+\tan B}{1-\tan A\tan B}\right)\tan C}$$

$$=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan A\tan C-\tan B\tan C}.$$

$$\tan(A+B+C)=\tan 180^0=0$$

$$\Rightarrow\quad \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan A\tan C-\tan B\tan C}=0$$

$$\tan A+\tan B+\tan C-\tan A\tan B\tan C=0$$

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C$$

Dividing through out by $\tan A\tan B\tan C$ and rearranging gives

$\cot A\cot B+\cot A\cot C+\cot B\cot C=1$, as required.

6. *Deduce that if* $2\sin(x+\alpha)=\cos(x-\alpha)$*, then* $\tan x=\dfrac{1-2\tan\alpha}{2-\tan\alpha}$. *Hence solve the equation* $2\sin(x+20^0)=\cos(x-20^0)$ *for* $0^0\le x\le 360^0$.

Solution

$$2\sin(x+\alpha)=\cos(x-\alpha)\Rightarrow 2\sin x\cos\alpha+2\cos x\sin\alpha=\cos x\cos\alpha+\sin x\sin\alpha$$

$$\sin x(2\cos\alpha-\sin\alpha)=\cos x(\cos\alpha-2\sin\alpha)$$

$\tan x=\dfrac{\cos\alpha-2\sin\alpha}{2\cos\alpha-\sin\alpha}$; Divide top and bottom by $\cos\alpha$.

$$\tan x=\frac{1-2\tan\alpha}{2-\tan\alpha}.$$

For $2\sin\left(x+20^0\right)=\cos\left(x-20^0\right),\ \alpha=20^0$

$$\Rightarrow \tan x=\frac{1-2\tan 20^0}{2-\tan 20^0}=0.1663$$

$$x=9.44^0\ ,189.44^0$$

Trial Exercise

1. Without using tables or calculator, find the values of the following;

   (i) $\sin 15^0$ (ii) $\sin 105^0$ (iii) $\cot 15^0$ (iv) $\cos 165^0$

2. Show that $\tan 75^0=2+\sqrt{3}$.

3. Given that $\cot B=\frac{1}{2}$ and $\cot(A-B)=4$, show that $\cot A=\frac{2}{9}$.

4. If $\sin(x+\alpha)=2\cos(x-\alpha)$, prove that $\tan x=\frac{2-\tan\alpha}{1-2\tan\alpha}$. Hence solve

   $\sin(x+45)=2\cos(x-45)$ for $-180\le x\le 180$.

5. Expand $\sin(A+B+C)$ in terms of sines and cosines of angles A, B and C.

6. Given that $\sin A=\frac{3}{5}$ and $\cos B=\frac{5}{13}$ where A and B are acute angles, find the values of $\sin(A+B)$, $\cos(A-B)$ and $\tan(A-B)$.

7. Prove that $\dfrac{\tan(\theta-\alpha)+\tan(\theta+\alpha)}{1-\tan(\theta-\alpha)\tan(\theta+\alpha)}=\tan 2\theta$.

8. Show that if $2\sin x=\cos\left(x+60^0\right)$, then $\tan x=\frac{4-\sqrt{3}}{13}$.

9. Solve the equation $3\sin\left(A+10^0\right)=4\cos\left(A-10^0\right)$ for $0^0\le A\le 360^0$.

10. Given that $\tan\beta=\dfrac{3-4\tan\alpha}{4+3\tan\alpha}$, show $\sin(\alpha+\beta)=\frac{3}{5}$.

11. Solve the equation $\cos(45^0-x)=2\sin(30^0+x)$ for $-180^0\le x\le 180^0$.

12. Given that $\sin\theta+\cos\left(\theta-60^o\right)=\cos\theta$, show that $\tan\theta=2-\sqrt{3}$ and hence, solve for θ in the interval $0^o\le\theta\le 360^o$.

13. Show that $\tan(P-Q)=\dfrac{\tan P-\tan Q}{1+\tan P\tan Q}$. Hence solve the equation $\tan(x-45^0)=6\tan x$ for $-180^0<x<180^0$.

DOUBLE ANGLES

From the previously discussed section for compound angles, if $A=B$, then

$$\sin(A+A)=\sin 2A=\sin A\cos A+\cos A\sin A=2\sin A\cos A$$

$$\cos(A+A)=\cos 2A=\cos A\cos A-\sin A\sin A=\cos^2 A-\sin^2 A$$

Also $\cos 2A=2\cos^2 A-1=1-2\sin^2 A$

$$\tan(A+A)=\tan 2A=\frac{\tan A+\tan A}{1-\tan A\tan A}=\frac{2\tan A}{1-\tan^2 A}.$$

From above it can be deduced that;

- ✓ $\sin 4A=2\sin 2A\cos 2A$
- ✓ $\sin 10A=2\sin 5A\cos 2A$
- ✓ $\sin 5A=2\sin\frac{5}{2}A\cos\frac{5}{2}A$
- ✓ $\cos 6A=\cos^2 3A-\sin^2 3A=2\cos^2 3A-1=1-2\sin^2 3A$
- ✓ $\tan 6A=\dfrac{2\tan 3A}{1-\tan^2 3A}$

Examples

1. *Solve the equation* $2\cos 2\theta+3\sin\theta=1$ *for* $0^o\le\theta\le 360^o$.

$2(1-2\sin^2\theta)+3\sin\theta=1$, $4\sin^2\theta-3\sin\theta-1=0$

i.e express $\cos 2\theta$ in terms of $\sin^2\theta$ before proceeding.

$$4\sin^2\theta-4\sin\theta+\sin\theta-1=0,\quad (4\sin\theta+1)(\sin\theta-1)=0$$

$$\sin\theta=-\frac{1}{4},\ \theta=94.48^o,\ 345.52^o,\quad \sin\theta=1,\ \theta=90^o$$

2. *Prove that:* $\cos 3x=4\cos^3 x-3\cos x$, *hence, solve the equation* $1+\cos 3x=\cos x(1+\cos x)$ *for* $0^o\le x\le 360^o$.

Solution

L.H.S = $\cos 3x = \cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x$

$$= \cos x\left(2\cos^2 x - 1\right) - 2\cos x\left(1 - \cos^2 x\right)$$

$$= 4\cos^3 x - 3\cos x = \text{ R.H.S}$$

Now for $1 + \cos 3x = \cos x(1 + \cos x)$, $1 + 4\cos^3 x - 3\cos x = \cos x(1 + \cos x)$

$4\cos^3 x - \cos^2 x - 4\cos x + 1 = 0$, $\quad 4\cos x\left(\cos^2 x - 1\right) - \left(\cos^2 x - 1\right) = 0$

$\left(\cos^2 x - 1\right)\left(4\cos x - 1\right) = 0$, $\qquad \cos x = \pm 1, \quad \cos x = \dfrac{1}{4}$

$$x = 0^o, 180^o, 360^o, 75.5^o, 284.5^o$$

3. *Solve the equation* $3\sin 2x = 2\tan x$ *for* $0 \leq x \leq 360^0$.

Solution

$3\sin 2x = 2\tan x$

$3 \times 2\sin x \cos x = 2\dfrac{\sin x}{\cos x}$; $3\sin x \cos^2 x = \sin x$; Here don't divide as this can lead to loss of some solutions but instead factorise.

$\sin x\left(3\cos^2 x - 1\right) = 0$; $\sin x = 0$; $x = 0^0, 180^0, 360$

$\cos x = \dfrac{1}{\sqrt{3}}$ *or* $\cos x = -\dfrac{1}{\sqrt{3}}$

$$x = 54.74^0, 125.26^0, 234.74^0, 305.26^0$$

4. *Prove that* $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ *and hence solve the equation* $t^4 + 4t^3 - 6t^2 - 4t + 1 = 0$.

$$\text{L.H.S} = \tan 4\theta = \tan(2\theta + 2\theta) = \frac{2\tan 2\theta}{1-\tan^2 2\theta} = \frac{\dfrac{4\tan\theta}{1-\tan^2\theta}}{1-\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)^2}$$

$$= \frac{\dfrac{4\tan\theta}{1-\tan^2\theta}}{\dfrac{1-2\tan^2\theta+\tan^4\theta-4\tan^2\theta}{(1-\tan^2\theta)^2}} = \frac{4\tan\theta-4\tan^2\theta}{1-6\tan^2\theta+\tan^4\theta} = \text{R.H.S}$$

If $t = \tan\theta$ and we let $\tan 4\theta = 1$, then $t^4 + 4t^3 - 6t^2 - 4t + 1 = 0$

For $\tan 4\theta = 1$, $4\theta = 45^o, 225^o\ 405^o,\ 585^o \therefore\ \theta = \dfrac{45^o}{4}, \dfrac{225^o}{4}, \dfrac{405^o}{4}, \dfrac{585^o}{4}$

The roots of the polynomial are: $t = 0.1989,\ 1.4966,\ 5.0273,\ -0.6682$

5. *Without using tables, show that* $\dfrac{\cos 29^o + \sin 29^o}{\cos 29^o - \sin 29^o} = \tan 74$.

Solution

$$\frac{\cos 29^o + \sin 29^o}{\cos 29^o - \sin 29^o} = \frac{1+\tan 29^o}{1-\tan 29^o} = \frac{\tan 45^o + \tan 29^o}{1-\tan 45^o \tan 29^o}$$

$$= \tan(45^o + 29^o) = \tan 74^o = R.H.S$$

6. *Prove that (i)* $\operatorname{cosec} 2A + \cot 2A = \cot A$ *(ii)* $\dfrac{\sin 2\beta}{1+\cos 2\beta} = \tan\beta$.

Solution

(i) $\operatorname{cosec} 2A + \cot 2A = \cot A$

$$L.H.S = \operatorname{cosec} 2A + \cot 2A = \frac{1}{\sin 2A} + \frac{\cos 2A}{\sin 2A} = \frac{1+\cos 2A}{\sin 2A}$$

$$= \frac{1+2\cos^2 A - 1}{2\sin A\cos A} = \frac{2\cos^2 A}{2\sin A\cos A} = \cot A = R.H.S$$

(ii) $\dfrac{\sin 2\beta}{1+\cos 2\beta} = \tan\beta$

$$L.H.S = \frac{\sin 2\beta}{1+\cos 2\beta} = \frac{2\sin\beta\cos\beta}{1+2\cos^2\beta - 1} = \frac{2\sin\beta\cos\beta}{2\cos^2\beta} = \tan\beta = R.H.S$$

7. *Show that* $4\left(\cos^4\theta + \sin^4\theta\right) = 3 + \cos 4\theta$.

Solution

$$L.H.S = 4\left(\cos^4\theta + \sin^4\theta\right) = 4\left[\left(\cos^2\theta\right)^2 + \left(\sin\theta^2\right)^2\right]$$

$$=4\left[\left(\frac{1+\cos 2\theta}{2}\right)^2 + \left(\frac{1-\cos 2\theta}{2}\right)^2\right] = 4\left(\frac{1+2\cos 2\theta + \cos^2 2\theta}{4} + \frac{1-2\cos 2\theta + \cos^2 2\theta}{4}\right)$$

$$= 2 + 2\cos^2 2\theta = 2 + \cos 4\theta + 1 = 3 + \cos 4\theta = R.H.S$$

Trial Exercise

1. Solve the equation $5\sin 2x + 4 = 10\sin^2 x$ for $-180^0 < x < 180^0$.
2. Prove that $\dfrac{\sin 2x - 1 - \cos 2x}{2(1-\sin 2x)} = \dfrac{1}{\tan x - 1}$
3. Given that $\tan\phi = \dfrac{a}{b}$, prove that $\dfrac{a}{a+b} = \dfrac{\sin\phi}{\sqrt{2}\sin\left(\phi + 45^0\right)}$.
4. Solve the equation $\sin t\cos 3t + \sin 3t\cos t = 0.8$ for $0 \le t \le 360^0$.
5. Solve the equation $3\cos 4\theta + 7\cos 2\theta = 0$ for $0^0 \le \theta \le 180^0$.
6. Solve the equation $10\sin^2 t - 5\sin 2t = 4$ for $0^0 \le t \le 360^0$.
7. Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, hence, find the exact value of $\cos 3\theta$ if $\cos\theta = -\dfrac{2}{3}$.
8. Solve: $3\sin 2x + 4\cos^2 x = -1$ for $0^o \le x \le 180^o$.
9. Solve $\sin 6x = \sin 4x$ for $0^o \le x \le 90^o$.

10. Show that $2\sin 3x = 6\sin x - 8\sin^3 x$. Hence, given that $2\sin x = k + \frac{1}{k}$, deduce that

$2\sin 3x = -k^3 - \frac{1}{k^3}$.

11. Prove that $\frac{1 - \cos 2A + \sin 2A}{1 + \cos 2A + \sin 2A} = \tan A$.

12. Solve the equation: $\sin 2\theta + 1 = \cos 2\theta$ for $0° \leq \theta \leq 360°$.

13. Solve the equation: $\left(\cos\frac{1}{2}\theta + \sin\frac{1}{2}\theta\right)^2 = \frac{3}{2}$ for $0^0 \leq \theta \leq 360^0$.

14. If θ is acute and $\cot\theta = \frac{x^2 - y^2}{2xy}; x > 0, y > 0,$ find the value of $\sec\theta$ in the simplest form.

15. Prove that $\frac{1 + \cos 2x + \sin 2x}{1 - \cos 2x + \sin 2x} = \cot x$.

16. Show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$.

HALF ANGLES

From the previous discussion, when the angle is halved, then we have the following:

(i) $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

(ii) $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2}$

So $\cos^2\frac{A}{2} = \frac{1}{2}(\cos A + 1)$ and $\sin^2\frac{A}{2} = \frac{1}{2}(1 - \cos A)$

(iii) $\tan A = \frac{2\tan\frac{A}{2}}{1 - \tan^2\frac{A}{2}}$.

Proof

$$\tan A = \frac{\sin A}{\cos A} = \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos^2\frac{A}{2} - \sin^2\frac{A}{2}} = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$$ (On dividing top and bottom by $\cos^2\frac{A}{2}$)

Examples

1. *Prove that (i)* $\tan\frac{A}{2} = \frac{\sin A}{1+\cos A}$ *(ii)* $\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$.

Solution

(i) $L.H.S = \frac{\sin A}{1+\cos A} = \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{1+2\cos^2\frac{A}{2}-1} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \tan\frac{A}{2} = R.H.S$

(ii) $L.H.S = \sin\theta = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}} = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = R.H.S$ (By dividing top and bottom by $\cos^2\frac{\theta}{2}$)

2. *Without using tables or calculators, show that* $\tan^2 22.5^0 = 3 - 2\sqrt{2}$.

Solution

Using $\tan 2A = \frac{2\tan A}{1-\tan^2 A} \Rightarrow A = 22.5^0$

$$\tan 45^0 = \frac{2\tan A}{1-\tan^2 A} \Rightarrow \tan^2 A + 2\tan A - 1 = 0$$

$$\tan A = \frac{-2+\sqrt{4-4\times1\times-1}}{2\times1} = \tan A = -1+\sqrt{2} \; or \; -1-\sqrt{2} \text{ (discard)}$$

Now $\tan^2 22.5^0 = \left(-1+\sqrt{2}\right)^2 = 3 - 2\sqrt{2}$

Alternatively

$$\tan^2 22.5^0 = \frac{\sin^2 22.5^0}{\cos^2 22.5^0} = \frac{\frac{1}{2}\left(1-\sin 45^0\right)}{\frac{1}{2}\left(1+\cos 45^0\right)} = \frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}} = \frac{2-\sqrt{2}}{2+\sqrt{2}}$$

$$= \frac{\left(2-\sqrt{2}\right)\left(2-\sqrt{2}\right)}{\left(2+\sqrt{2}\right)\left(2-\sqrt{2}\right)} = \frac{6-4\sqrt{2}}{4-2} = 3-2\sqrt{2}\,.$$

3. *Solve the equation* $\cos\theta = \cos\frac{\theta}{2}$ *for* $0^0 \le \theta \le 360^0$.

Solution

$$\cos\theta = \cos\frac{\theta}{2} \;\Rightarrow \cos\theta - \cos\frac{\theta}{2} = 0$$

$$\cos\theta - \cos\frac{\theta}{2} = 0\;;\; 2\cos^2\frac{\theta}{2} - \cos\frac{\theta}{2} - 1 = 0$$

$$\cos\frac{\theta}{2} = 1 \qquad or\;\; \cos\frac{\theta}{2} = -0.5$$

$$\frac{\theta}{2} = 0^0\;\;;\theta = 0^0 \;\text{ or }\; \frac{\theta}{2} = -120^0\;, 120^0 \Rightarrow \theta = -240^0\;,\; 240^0$$

Trial Exercise

1. Solve the equations below for $0^0 \le \theta \le 360^0$;

(a) $2\sin^2\frac{\theta}{2} - \cos\theta + 1 = 0$ (b) $10\sin^2\frac{\theta}{2} - 5\sin\theta = 4$

(c) $2\cos\theta + 3\sin\theta - 2 = 0$ (d) $7\cos\theta - 4\sin\theta + 1 = 0$

2. Given that $x = \sec A - \tan A$, prove that $\tan\frac{A}{2} = \frac{1-x}{1+x}$.

3. Find $\sin 22.5^0$, $\cos 22.5^0$ and $\tan 22.5^0$.

4. Show that $\tan 75^0 = 2+\sqrt{3}$.

5. Given that $\tan A = \frac{4}{3}$, without using tables or calculator, find the possible values of $\tan\frac{A}{2}$ and $\sin\frac{A}{2}$.

The t- substitution

To solve equations of the form $a\cos\theta + b\sin\theta = c$, where *a*, *b* and *c* are constants, we use the substitution $t = \tan\frac{\theta}{2}$.

It is important to note that if $t = \tan\frac{\theta}{2}$, then $\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$ and $\tan\theta = \frac{2t}{1-t^2}$.

It has been left to the learner to prove each of the above.

From the above, it can be inferred that $\cos 4\theta = \frac{1-t^2}{1+t^2} \Rightarrow t = \tan 2\theta$; $\sin 5\theta = \frac{2t}{1+t^2} \Rightarrow$ $t = \tan\frac{5\theta}{2}$.

Example

Use the substitution $t = \tan\frac{\theta}{2}$ to solve the equations below for $0^0 \le \theta \le 360^0$.

(i) $2\cos\theta + 3\sin\theta - 2 = 0$ (ii) $7\cos\theta - 4\sin\theta + 1 = 0$

Solution

(i) $2\cos\theta + 3\sin\theta - 2 = 0 \Rightarrow 2\left(\frac{1-t^2}{1+t^2}\right) + 3\left(\frac{2t}{1+t^2}\right) - 2 = 0$

$-4t^2 + 6t = 0$; $t = 0$ *or* $t = \frac{3}{2}$

When $t = 0$; $\frac{\theta}{2} = 0^0 : \theta = 0^0$

When $t = \frac{3}{2}$; $\frac{\theta}{2} = 56.31^0, 236.31^0 \Rightarrow \theta = 112.62^0$

(ii) $7\cos\theta - 4\sin\theta + 1 = 0 \Rightarrow 7\left(\frac{1-t^2}{1+t^2}\right) - 4\left(\frac{2t}{1+t^2}\right) + 1 = 0$

$3t^2 + 4t - 4 = 0$; $t = \frac{2}{3}$ *or* $t = -2$

When $t = \frac{2}{3}$, $\frac{\theta}{2} = 33.69^0 \Rightarrow \theta = 67.38^0$

When $t = -2$, $\frac{\theta}{2} = 116.57^0 \Rightarrow \theta = 233.13^0$

Trial exercise

1. Use the substitution $t = \tan\frac{\theta}{2}$ to solve the equations below for $-360^0 \le \theta \le 360^0$.

   (a) $3\cos\theta + 4\sin\theta = 2$ (b) $7\cos\theta + \sin\theta = 5$

   (c) $52\cos\theta + 39\sin\theta = 60$ (d) $15\cos\theta + 2\sin\theta = 10$ (e) $5\sin\theta - 12\cos\theta = 6$

2. Use a substitution $t = \tan\frac{\theta}{2}$ to solve the equation $3\cos\theta - 5\sin\theta = -1$, for $0^o \le x \le 360^o$.

The harmonic form – use of the R- substitution

To solve equations of the form $a\cos\theta + b\sin\theta = c$, where *a*, *b* and *c* are constants, we can also use the form $a\cos\theta \pm b\sin\theta = R\cos(\theta \mp \alpha)$ *or* $R\sin(\theta \pm \alpha)$ where α is an acute angle whose value can be obtained.

Note:

- The above form can be used either when solving trigonometric equation in a given interval or
- When finding the minimum or maximum values of given expressions.

Example

1. *By using the form* $a\cos\theta \pm b\sin\theta = R\cos(\theta \mp \alpha)$, *solve the equations below for* $0^0 \le \theta \le 360^0$.

   (i) $2\cos\theta + 3\sin\theta - 2 = 0$ *(ii)* $7\cos\theta - 4\sin\theta + 1 = 0$

Solution

(i) $2\cos\theta + 3\sin\theta - 2 = 0$

Let $2\cos\theta + 3\sin\theta = R\cos(\theta - \alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$.

By comparing both sides, we have

$R\cos\alpha = 2$ …. (i) and $R\sin\alpha = 3$….. (ii)

Dividing equations (ii)by (i) we get $\tan\alpha = \frac{3}{2} \Rightarrow \alpha = 56.31^0$

Squaring equations (i) and (ii) and adding, $R^2 = 4 + 9 = 13 \Rightarrow R = \sqrt{13}$

So $2\cos\theta + 3\sin\theta - 2 = 0$ becomes $\sqrt{13}\cos(\theta - 56.31^0) = 2$

$\theta - 56.31^0 = -56.31^0$, 56.31^0, 303.69^0

$\theta = 0^0$, 112.62^0

(ii) $7\cos\theta - 4\sin\theta + 1 = 0$

Let $7\cos\theta - 4\sin\theta \quad = R\cos(\theta+\alpha) = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

By comparing both sides, we have

$R\cos\alpha = 7$ …. (i) and $R\sin\alpha = 4$ ….. (ii)

Dividing equations (ii)by (i) we get $\tan\alpha = \frac{4}{7} \Rightarrow \alpha = 29.74^0$

Squaring equations (i) and (ii) and adding, $R^2 = 49 + 16 = 65 \Rightarrow R = \sqrt{65}$

Now for $7\cos\theta - 4\sin\theta + 1 = 0 \Rightarrow \sqrt{65}\cos(\theta + 29.74^0) = -1$

$\theta + 29.74^0 = 97.13^0$, 262.87^0

$\theta = 67.39^0$, 233.13^0

2. *By expressing* $7\sin\theta + 5\cos\theta$ *in the form* $R\sin(\theta+\alpha)$,

(a) *solve the equation* $7\sin\theta + 5\cos\theta = 6$ *for* $0^0 \le \theta \le 180^0$.

(b) *determine the maximum and minimum values of* $7\sin\theta + 5\cos\theta$.

Solution

(a) Let $7\sin\theta + 5\cos\theta = R\sin(\theta+\alpha) = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

By comparison; $R\cos\alpha = 7$ ….. (i) and $R\sin\alpha = 5$ ….. (ii)

Dividing (ii) by (i) gives $\tan\alpha = \frac{5}{7} \Rightarrow \alpha = 35.54^0$

Squaring (i) and (ii) and adding $R^2 = 49 + 25 = 74$; $R = \sqrt{74}$

So $7\sin\theta + 5\cos\theta = \sqrt{74}\sin(\theta + 35.54^0)$

$7\sin\theta + 5\cos\theta = \sqrt{74}\sin(\theta + 35.54^0) = 6$

$\theta + 35.54^0 = 44.23^0, 135.77^0$

$\theta = 8.69^0$, 100.23^0

(b) For $7\sin\theta + 5\cos\theta = \sqrt{74}\sin(\theta + 35.54^0)$, the maximum value occurs when $\sin(\theta + 35.54^0) = +1$ (i.e maximum) . Thus the maximum value of the expression is $+\sqrt{74}$.

For $7\sin\theta + 5\cos\theta = \sqrt{74}\sin(\theta + 35.54^0)$, the minimum value occurs when $\sin(\theta + 35.54^0) = -1$ (i.e minimum) . Thus the minimum value of the expression is $-\sqrt{74}$.

3. *By expressing* $8\cos\theta - 15\sin\theta$ *in the form* $R\cos(\theta+\alpha)$ *where* α *is an acute angle, find the maximum and minimum values of* $\frac{3}{5 - (8\cos\theta - 15\sin\theta)}$, *giving the values of* θ *between* 0^0 *and* 360^0 *for which the maximum and minimum values occur.*

Solution

Let $8\cos\theta - 15\sin\theta = R\cos(\theta + \alpha) = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

By comparison, $R\cos\alpha = 8$ …. (i) and $R\sin\alpha = 15$ ….. (ii)

Dividing (i) and (ii); $\tan\alpha = \frac{15}{8}$; $\alpha = 61.93^0$

Squaring (i) and (ii) and adding, $R^2 = 64 + 225 = 289$; $R = 17$

$8\cos\theta - 15\sin\theta = 17\cos(\theta + 61.93^0)$

$$\frac{3}{5-(8\cos\theta - 15\sin\theta)} = \frac{3}{5-17\cos(\theta - 61.93^0)}$$

For maximum of $\frac{3}{5-17\cos(\theta-61.93^0)}$, $\cos(\theta - 61.93^0)$ must be minimum and this occurs when $\theta - 61.93^0 = 180^0 \Rightarrow \theta = 241.93^0$.

Therefore maximum value of the expression is $\frac{3}{5-17\times -1} = \frac{3}{22}$.

For minimum of $\frac{3}{5-17\cos(\theta-61.93^0)}$, $\cos(\theta - 61.93^0)$ must be maximum and this occurs when $\theta - 61.93^0 = 0^0 \Rightarrow \theta = 61.93^0$.

Therefore minimum value of the expression is $\frac{3}{5-17\times 1} = -\frac{3}{12} = -\frac{1}{4}$.

4. *Express* $6\cos^2\theta + 8\sin\theta\cos\theta$ *in the form* $R\cos(2\theta - \alpha)$ *hence, solve* $6\cos^2\theta + 8\sin\theta\cos\theta = 4$ *for* $0^o \le x \le 180^o$.

Solution

$3(2\cos^2\theta) + 4(2\sin\theta\cos\theta) \;=\; 3(1+\cos 2\theta) + 4\sin 2\theta$

$= 3\cos 2\theta + 4\sin 2\theta + 3$

Let $3\cos 2\theta + 4\sin 2\theta + 3 \equiv R\cos 2\theta\cos\alpha + R\sin 2\theta\sin\alpha$

So, $R\cos 2\theta \equiv 3$, $R\sin 2\theta \equiv 4$

$$\Rightarrow \quad \tan\alpha = \frac{R\sin\alpha}{R\cos\alpha} = \frac{3}{4} \qquad \therefore\ \alpha = 53.1^o$$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2 \quad \therefore\ R = 5$$

$$\Rightarrow \quad 3\cos 2\theta + 4\sin 2\theta + 3 = 5\cos(2\theta - 53.1^o) + 3$$

Thus: $6\cos^2\theta + 8\sin\theta\cos\theta = 4$

$$5\cos(2\theta - 53.1^o) + 3 = 4$$

$$\therefore \cos(2\theta - 53.1^o) = \frac{1}{5} \qquad 2\theta - 53.1^o = 78.5^o, 281.5^o$$

$$\Rightarrow \quad \theta = 65.8^o, 167.3^o$$

Trial exercise

1. By expressing in a suitable R- form solve the equations below for $-360^0 \le \theta \le 360^0$.
   (a) $3\cos\theta + 4\sin\theta = 2$ (b) $7\cos\theta + \sin\theta = 5$
   (c) $52\cos\theta + 39\sin\theta = 60$ (d) $15\cos\theta + 2\sin\theta = 10$
   (e) $5\sin\theta - 12\cos\theta = 6$ (f) $8\cos\theta - 15\sin\theta = 9$
   (g) $5\sin 2\theta + 7\cos 2\theta = 6$ (h) $4\sin\theta\cos\theta + 12\cos 2\theta = 10$
2. Find the maximum and minimum values of the expressions below and state the values of θ for which they occur.
   (a) $3\sin\theta - 5\cos\theta$ (b) $15\cos\theta - 2\sin\theta$
   (c) $\dfrac{4}{3 + 15\cos\theta - 8\sin\theta}$ (e) $\dfrac{3}{7 - (6\sin\theta - 8\cos\theta)}$.
3. Express $10\sin x\cos x + 12\cos 2x$ in the form $R\sin(2x + \alpha)$. Hence find the maximum and minimum values of $10\sin x\cos x + 12\cos 2x$.
4. Express $10\sin x\cos x + 12\cos 2x$ in the form $R\sin(2x + \alpha)$, hence or otherwise solve $10\sin x\cos x + 12\cos 2x + 7 = 0$ in the range $0^o \le x \le 360^o$.
5. Express $5 - 8\sin\theta + 15\cos\theta$ in the form $A + R\sin(\theta - \alpha)$ where α is an acute angle. Hence find the maximum value of $5 - 8\sin\theta + 15\cos\theta$.
6. Find the maximum and minimum values of $3\sqrt{2}\cos(\theta + 45^0) + 7\sin\theta$ by expressing in the form $R\cos(\theta - \alpha)$. State the smallest values of θ for which they occur.
7. Express $5 - 8\sin\theta + 15\cos\theta$ in the form $A + R\sin(\theta - \alpha)$ where α is an acute angle.

Hence find the maximum and minimum values of $\dfrac{3}{5-8\sin\theta+15\cos\theta}$, and state the values of θ for which they occur.

8. Solve the equation $\sqrt{3}\sin\theta-\cos\theta+2=0$ for $0^0<\theta<360^0$.

INVERSE TRIGONOMETRIC FUNCTIONS

It has previously been seen that $\sin^{-1}0.5$ gives all the angles whose sines are 0.5. Thus the inverse of the sine of *x*, written as $\arcsin x$ *or* $\sin^{-1}x$ means all the angles whose sines are *x*. We should note that (i) $\sin^{-1}(\sin 60^0)=60^0$ (ii) $\tan^{-1}(\tan 30^0)=30^0$ (iii) $\cos^{-1}(\cos x)=x$ (iv) $\sin^{-1}(\sin(P+Q))=P+Q$.

Examples

1. *Prove the following:*

(i) $\tan^{-1}\dfrac{1}{2}+\tan^{-1}\dfrac{1}{3}=\dfrac{\pi}{4}$ *(ii)* $\tan^{-1}\dfrac{1}{5}+\tan^{-1}\dfrac{1}{8}=\tan^{-1}\dfrac{1}{3}$

(iii) $2\tan^{-1}\dfrac{1}{3}+\tan^{-1}\dfrac{1}{7}=\dfrac{\pi}{4}$

Solution

(i) $\tan^{-1}\dfrac{1}{2}+\tan^{-1}\dfrac{1}{3}=\dfrac{\pi}{4}$

Let $A=\tan^{-1}\dfrac{1}{2}\Rightarrow\tan A=\dfrac{1}{2}$ and $B=\tan^{-1}\dfrac{1}{3}\Rightarrow\tan B=\dfrac{1}{3}$

Now L.H.S = $A+B=\tan^{-1}(\tan(A+B))=\tan^{-1}\left(\dfrac{\tan A+\tan B}{1-\tan A\tan B}\right)$

$$=\tan^{-1}\left(\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right)=\tan^{-1}1=\dfrac{\pi}{4}=R.H.S$$

It is very important to note in such situations that $\pi=180^0$.

(ii) $\tan^{-1}\dfrac{1}{5}+\tan^{-1}\dfrac{1}{8}=\tan^{-1}\dfrac{1}{3}$

Let $A=\tan^{-1}\dfrac{1}{5}\Rightarrow\tan A=\dfrac{1}{5}$ and $B=\tan^{-1}\dfrac{1}{8}\Rightarrow\tan B=\dfrac{1}{8}$

L.H.S = $A+B=\tan^{-1}(\tan(A+B))=\tan^{-1}\left(\dfrac{\tan A+\tan B}{1-\tan A\tan B}\right)$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\right) = \tan^{-1}\left(\frac{\frac{13}{40}}{\frac{39}{40}}\right) = \tan^{-1}\frac{1}{3} = R.H.S$$

(iii) $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

Let $A = \tan^{-1}\frac{1}{3} \Rightarrow \tan A = \frac{1}{3}$ and $B = \tan^{-1}\frac{1}{7} \Rightarrow \tan\ B = \frac{1}{7}$

L.H.S $= 2A + B = \tan^{-1}(\tan(2A+B)) = \tan^{-1}\left(\frac{\tan 2A + \tan B}{1-\tan 2A \tan B}\right)$

But $\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{2\times\frac{1}{3}}{1-\frac{1}{9}} = \frac{3}{4}$.

$$\text{L.H.S} = \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right) = \tan^{-1}\left(\frac{\frac{25}{28}}{\frac{25}{28}}\right) = \tan^{-1}1 = \frac{\pi}{4} = R.H.S$$

2. *Prove that:* $\tan^{-1}\frac{\sqrt{3}}{2} + \tan^{-1}\frac{\sqrt{3}}{5} = \frac{\pi}{3}$.

Let $\tan^{-1}\frac{\sqrt{3}}{2} = A$, $\tan^{-1}\frac{\sqrt{3}}{5} = B$ thus $\tan A = \frac{\sqrt{3}}{2}$ and $\tan B = \frac{\sqrt{3}}{5}$

$A + B = \tan^{-1}(\tan(A+B))$, $A + B = \tan^{-1}\left(\frac{\tan A + \tan B}{1-\tan A \tan B}\right)$

$$A + B = \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{5}}{1-\frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{5}}\right) \quad ; A + B = \tan^{-1}\left(\frac{\frac{7\sqrt{3}}{10}}{\frac{7}{10}}\right)$$

$A + B = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3} = R.H.S$

3. *Prove that* $\tan^{-1}\left(\frac{1+x}{1-x}\right) - \tan^{-1}x = \frac{\pi}{4}$.

Let $\tan^{-1}\left(\frac{1+x}{1-x}\right) = A, \Rightarrow \tan A = \frac{1+x}{1-x}$ and $\tan^{-1} x = B, \Rightarrow \tan B = x$

$$L.H.S = (A-B) = \tan^{-1}\frac{\tan A - \tan B}{1+\tan A \tan B} = \tan^{-1}\left(\frac{\frac{1+x}{1-x} - x}{1+(\frac{1+x}{1-x}).x}\right)$$

$$= \tan^{-1}\frac{1+x-x+x^2}{1-x+x+x^2} = \tan^{-1}1 = \frac{\pi}{4} = \text{R.H.S}$$

4. *Show that* $tan^{-1}\left(\frac{x-y}{1+xy}\right) + tan^{-1}\left(\frac{y-z}{1+yz}\right) = tan^{-1}\left(\frac{x-z}{1+xz}\right)$.

Solution

Let $\tan A = \frac{x-y}{1+xy}, \tan B = \frac{y-z}{1+yz}$; $\text{LHS} = \tan^{-1}(\tan(A+B)) = \tan^{-1}\left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right)$

$$= \tan^{-1}\left(\left(\frac{x-y}{1+xy} + \frac{y-z}{1+yz}\right) \div \left(1 - \frac{x-y}{1+xy} \times \frac{y-z}{1+yz}\right)\right)$$

$$= \tan^{-1}\left(\left(\frac{(x-y)(1+yz)+(y-z)(1+xy)}{(1+xy)(1+yz)}\right) \div \left(\frac{(1+xy)(1+yz)-(x-y)(y-z)}{(1+xy)(1+yz)}\right)\right)$$

$$= \tan^{-1}\left(\frac{(x-z)(1+y^2)}{(1+xz)(1+y^2)}\right) = \tan^{-1}\left(\frac{x-z}{1+xz}\right) = \text{R.H.S}$$

5. *If* $\tan\alpha = p$, $\tan\beta = q$, $\tan\gamma = r$, *prove that* $\tan(\alpha+\beta+\gamma) = \frac{p+q+r-pqr}{1-pr-rq-pq}$,

hence, show that $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{\pi}{4}$.

$$\text{L.H.S} = \tan(\alpha+\beta+\gamma) = \tan((\alpha+\beta)+\gamma) = \frac{\tan(\alpha+\beta)+\tan\gamma}{1-\tan(\alpha+\beta)\tan\gamma}$$

$$= \frac{\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} + \tan\gamma}{1-\left[\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\right]\tan\gamma} = \frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\alpha\tan\gamma-\tan\alpha\tan\beta-\tan\beta\tan\gamma}$$

$$\tan(\alpha+\beta+\gamma) = \frac{p+q+r-pqr}{1-pr-rq-pq}$$

$$For \quad \tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{\pi}{4}$$

$$\alpha+\beta+\gamma=\tan^{-1}\left(\frac{p+q+r-pqr}{1-pr-rq-pq}\right) \quad =\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{4}+\frac{2}{9}-\frac{1}{3}\times\frac{1}{4}\times\frac{2}{9}}{1-\frac{1}{3}\times\frac{1}{4}-\frac{1}{3}\times\frac{2}{9}-\frac{2}{9}\times\frac{1}{4}}\right)$$

$$=\tan^{-1}\frac{\frac{85}{108}}{\frac{85}{108}}=\tan^{-1}1 \;=\frac{\pi}{4}=RHS$$

Trial Exercise

1. Prove the following:

(a) $\tan^{-1}\frac{1}{3}+\sin^{-1}\frac{1}{\sqrt{5}}=\frac{\pi}{4}$ (b) $\tan^{-}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

(c) $\cot^{-1}\frac{1}{3}-\cot^{-1}3=\cos^{-1}\frac{3}{5}$ (d) $4\tan^{-1}\left(\frac{1}{5}\right)-\tan^{-1}\left(\frac{1}{239}\right)=\frac{\pi}{4}.$

(e) $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$ (g) $\cos^{-1}\frac{3}{\sqrt{10}}+\cos^{-1}\frac{2}{\sqrt{5}}=\frac{\pi}{4}.$

2. Solve for x if $\sin^{-1}x-\cos^{-1}x=\frac{\pi}{2}$.

3. Solve for x in the equation $\tan^{-1}(2x+1)+\tan^{-1}(2x-1)=\tan^{-1}2$.

4. Solve for x in (i) $\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x$ (ii) $\tan^{-1}x+\tan^{-1}(x-1)=\tan^{-1}\frac{9}{7}$

(iii) $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$ (iv) $\tan^{-1}(2x+1)+\tan^{-1}(2x-1)=\tan^{-1}2$.

FACTOR FORMULA

We have already seen that

$\sin(A+B)=\sin A\cos B+\cos A\sin B$ …. (i) $\sin(A-B)=\sin A\cos B-\cos A\sin B$…(ii)

$\cos(A+B)=\cos A\cos B-\sin A\sin B$...(iii) $\cos(A-B)=\cos A\cos B+\sin A\sin B$…(iv)

Now adding (i) and (ii) gives $\sin(A+B)+\sin(A-B)=2\sin A\cos B$ …. (1)

And subtracting (ii) from (i) gives $\sin(A+B)-\sin(A-B)=2\cos A\sin B$ …(2)

Also adding (iii) and (iv) gives $\cos(A+B)+\cos(A-B)=2\cos A\cos B$ …. (3)

And subtracting (iv) from (iii) gives $\cos(A+B)-\cos(A-B)=-2\sin A\sin B$… (4)

Letting $P=A+B$, $Q=A-B$ and solving simultaneously gives $A=\frac{P+Q}{2}$ and $B=\frac{P-Q}{2}$.

Therefore the equations (1), (2), (3) and (4) become

$$\sin P+\sin Q=2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}\ ;\ \ \sin P-\sin Q=2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

$$\cos P+\cos Q=2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2};\ \ \cos P-\cos Q=-\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

The above six equations are the statements of the factor formula. They have a wide range of applications in trigonometry such as in solving equations, simplification of some expressions and solution of a triangle.

Example

1. Simplify the following

(i) $\sin 4x+\sin 2x=2\sin\frac{4x+2x}{2}\cos\frac{4x-2x}{2}=2\sin 3x\cos x$

(ii) $\cos 4x+\cos 2x=2\cos\frac{4x+2x}{2}\cos\frac{4x-2x}{2}=2\cos 3x\cos x$

(iii) $\sin 7x-\sin 3x=2\cos\frac{7x+3x}{2}\sin\frac{7x-3x}{2}=2\cos 5x\sin 2x$

(iv) $\cos 4x-\cos 7x=-2\sin\frac{4x+7x}{2}\sin\frac{4x-7x}{2}=-2\sin\frac{11x}{2}\sin\left(\frac{-3x}{2}\right)$

$$=2\sin\frac{11x}{2}\sin\frac{3x}{2}$$

(v) $2\sin 3A\cos A=\sin(3A+A)+\sin(3A-A)=\sin 4A+\sin 2A$ (For such look out for the expression from which the terms came)

(vi) $\cos 5A\cos 3A=\frac{1}{2}(\cos 8A+\cos 2A)$

(vii) $\sin 3A \sin 4A = \sin 4A \sin 3A = -\frac{1}{2}(\cos 7A - \cos A)$; first rearrange.

(ix) $5\cos 3\theta \sin 7\theta = 5\sin 7\theta \cos 3\theta = \frac{5}{2}(\sin 10\theta + \sin 4\theta)$; also first rearrange.

(x) $\frac{1}{5}\sin 3\theta \sin\theta = \frac{1}{5} \times -\frac{1}{2}(\cos 4\theta - \cos 2\theta) = -\frac{1}{10}(\cos 4\theta - \cos 2\theta)$

2. Solve the equation $\sin 3x + \sin x = 0$ for $0^o \leq x \leq 360^o$.

Solution

$\sin 3x + \sin x = 0$; $2\sin 2x \cos x = 0$

$\sin 2x = 0$

$2x = 0^0, 180^0, 360^0, 540^0, 720^0$

$x = 0^0, 90^0, 180^0, 270^0, 360^0$

or $\cos x = 0$

$x = 90^0, 270^0$

3. Solve the equation $\cos 7\theta + \cos\theta + \cos 5\theta + \cos 3\theta = 0$ for $0^0 \leq \theta \leq 180^0$.

Solution

$\cos 7\theta + \cos\theta + \cos 5\theta + \cos 3\theta = 0$

$\Rightarrow 2\cos 4\theta \cos 3\theta + 2\cos 4\theta \cos\theta = 0$

$2\cos 4\theta(\cos 3\theta + \cos\theta) = 0$,

Either $\cos 4\theta = 0$ $4\theta = 90^o, 270^o, 450^o$ such that $\theta = 22.5^o, 67.5^o, 112.5^o$

or $\cos 3\theta + \cos\theta = 0$, $2\cos 2\theta \cos\theta = 0$

$\cos 2\theta = 0$, $2\theta = 90^o, 270^o$ so $\theta = 45^o, 135^o$

Or $\cos\theta = 0$, $\Rightarrow \theta = 90^o$

4. Prove that: $\frac{\sin 7\theta + \sin 5\theta + \sin 3\theta + \sin\theta}{\cos 7\theta + \cos 5\theta + \cos 3\theta + \cos\theta} = \tan 4\theta$.

$$L.H.S = \frac{\sin 7\theta + \sin\theta + \sin 5\theta + \sin 3\theta}{\cos 7\theta + \cos\theta + \cos 5\theta + \cos 3\theta} = \frac{2\sin 4\theta\cos 3\theta + 2\sin 4\theta\cos\theta}{2\cos 4\theta\cos 3\theta + 2\cos 4\theta\cos\theta}$$

$$= \frac{2\sin 4\theta(\cos 3\theta + \cos\theta)}{2\cos 4\theta(\cos 3\theta + \cos\theta)} = \tan 4\theta = R.H.S$$

5. Prove that: $\dfrac{\sin 8\theta\cos\theta - \sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta - \sin 3\theta\sin 4\theta} = \tan 2\theta$.

$$L.H.S = \frac{\sin 8\theta\cos\theta - \sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta - \sin 3\theta\sin 4\theta} = \frac{\frac{1}{2}(\sin 9\theta + \sin 7\theta) - \frac{1}{2}(\sin 9\theta + \sin 3\theta)}{\frac{1}{2}(\cos 3\theta + \cos\theta) + \frac{1}{2}(\cos 7\theta - \cos(-\theta))}$$

$$= \frac{(\sin 9\theta + \sin 7\theta) - (\sin 9\theta + \sin 3\theta)}{(\cos 3\theta + \cos\theta) + (\cos 7\theta - \cos\theta)}$$

$$= \frac{\sin 7\theta - \sin 3\theta}{\cos 7\theta + \cos 3\theta} = \frac{2\cos 5\theta\sin 2\theta}{2\cos 5\theta\cos 2\theta} = \tan 2\theta = \text{R.H.S}$$

6. Prove the identity $\cos^2 A - \cos^2 B = \sin(A+B)\sin(B-A)$.

L.H.S = $\cos^2 A - \cos^2 B = (\cos A + \cos B)(\cos A - \cos B)$

$$= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} \times -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$= 2\cos\frac{A+B}{2}\sin\frac{A+B}{2} \times -2\sin\frac{A-B}{2}\cos\frac{A-B}{2}$$

$$= \sin(A+B) \times -\sin(A-B) = \sin(A+B)\sin(B-A) = R.H.S$$

You can re-try the above by making use of the double angles i.e.

$\cos^2 A = \frac{1}{2}(\cos 2A + 1)$ and $\cos^2 B = \frac{1}{2}(\cos 2B + 1)$

Trial Exercise

1. Solve the equations below for $0^0 \le x \le 360^0$:

   (i) $\sin 2x + \sin 3x + \sin 5x = 0$ (ii) $\cos 6x + \cos 2x + \cos 4x = 0$

2. Show that (i) $\dfrac{\sin 3x + \sin x - \cos x}{\cos 3x - \cos x + \sin x} = -\cot x$ (ii) $\dfrac{\sin x - 2\sin 2x + \sin 3x}{\sin x + 2\sin 2x + \sin 3x} = -\tan^2\dfrac{x}{2}$

3. Prove that $\frac{\cos 5\theta - 2\cos 3\theta + \cos\theta}{\sin\theta - \sin 5\theta} \equiv \tan\theta.$

4. Solve the equations for $0^0 \leq \theta \leq 360^0$;

(i) $\tan(3\theta - 45) = 1$ (ii) $\sin(\theta + 15^0)\cos(\theta - 15^0) = 0.5$

(iii) $3\sin^2\theta - \sin\theta\cos\theta - 4\cos^2\theta = 0$

5. Prove that $\frac{\cos 2(\alpha+\beta) + \cos 2\alpha + \cos 2\beta + 1}{\cos 2(\alpha+\beta) - \cos 2\alpha - \cos 2\beta + 1} = -\cot\alpha\cot\beta$.

6. Solve for θ in the range $0^0 \leq \theta \leq 720^0$ $in\ cos 4\theta + cos\ 2\theta + cos\ 6\theta = 0.$

More examples

1. Given that $A + B + C = 180^o$, prove that

$\cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A \sin B \cos C$.

$$L.H.S = \cos 2A + \cos 2B - \cos 2C = 2\cos(A+B)\cos(A-B) - \left[2\cos^2 C - 1\right]$$

$$= 2\cos(180^o - C)\cos(A-B) - 2\cos^2 C + 1$$

$$= 2\cos C\left[-\cos(A-B) - \cos C\right] + 1$$

$$= 2\cos C\left[-\cos(A-B) + \cos(A+B)\right] + 1$$

$$= 2\cos C(2\sin A \sin -B) + 1 \quad \text{But } \sin(-B) = -\sin B$$

$$= 1 - 4\sin A \sin B \cos C = R.H.S$$

ALT: $\cos 2A + \cos 2B - \cos 2C \quad = 1 - 2\sin^2 A + -2\sin(B+C)\sin(B-C)$

$$= 1 - 2\sin^2 A - 2\sin(180^0 - A)\sin(B-C)$$

$$= 1 - 2\sin^2 A - 2\sin A \sin(B-C)$$

$$= 1 - 2\sin A(\sin A + \sin(B-C))$$

$$= 1 - 2\sin A(\sin(B+C) + \sin(B-C))$$

$$= 1 - 2\sin A(\sin B \cos C)$$

$$= 1 - 4\sin A \sin B \cos C \; = R.H.S$$

2. Prove that, if A, B and C are angles of a triangle, then

(a) $\cos A - \cos(B - C) = -2\cos B \cos C$

(b) $\cos A + \cos B + \cos C - 1 = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

Solution

(a) $\cos A - \cos(B - C) = -2\cos B \cos C$

$$\text{L.H.S} = \cos A - \cos(B - C) = \cos\left(180^0 - (B + C)\right) - \cos(B - C)$$

$$= -\cos(B + C) - \cos(B - C) = -\left(\cos(B + C) + \cos(B - C)\right)$$

$$= -2\cos B \cos C = R.H.S$$

(b) $\cos A + \cos B + \cos C - 1 = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

$$L.H.S \quad = \cos A + \cos B + \cos C - 1$$

$$= 2\cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right) + 1 - 2\sin^2\frac{C}{2} - 1$$

$$= 2\cos\left(90^0 - \frac{C}{2}\right)\cos\left(\frac{A - B}{2}\right) - 2\sin^2\frac{C}{2}$$

$$= 2\sin\frac{C}{2}\cos\left(\frac{A - B}{2}\right) - 2\sin^2\frac{C}{2} = 2\sin\frac{C}{2}\left(\cos\left(\frac{A - B}{2}\right) - \sin\frac{C}{2}\right)$$

$$= 2\sin\frac{C}{2}\left(\cos\frac{A - B}{2} - \sin\left(90^0 - \frac{A + B}{2}\right)\right)$$

$$= 2\sin\frac{C}{2}\left(\cos\frac{A - B}{2} - \cos\frac{A + B}{2}\right)$$

$$= 2\sin\frac{C}{2} \times -2\sin\frac{A}{2}\sin\left(-\frac{B}{2}\right) = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = R.H.S$$

3. Show that for any triangle ABC,

(i) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

(ii) $\cos 4A - \cos 4B - \cos 4C = -1 - 4\sin 2B \sin 2C \cos 2A$

Solution

(i) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$$L.H.S = \sin A + \sin B + \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} + 2\sin(\frac{B+C}{2})\cos(\frac{B-C}{2})$$

$$= 2\sin\frac{A}{2}\cos\frac{A}{2} + 2\sin(90^0 - \frac{A}{2})\cos(\frac{B-C}{2}) = 2\sin\frac{A}{2}\cos\frac{A}{2} + 2\cos\frac{A}{2}\cos(\frac{B-C}{2})$$

$$= 2\cos\frac{A}{2}\left[\sin\frac{A}{2} + \cos(\frac{B-C}{2})\right] = 2\cos\frac{A}{2}\left[\sin\frac{180^o - (B+C)}{2} + \cos(\frac{B-C}{2})\right]$$

$$= 2\cos\frac{A}{2}\left[\cos(\frac{B+C}{2}) + \cos(\frac{B-C}{2})\right] = 2\cos\frac{A}{2}\left[2\cos\frac{B}{2}\cos\frac{C}{2}\right]$$

$$= \quad 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \quad = R.H.S$$

(ii) $\cos 4A - \cos 4B - \cos 4C = -1 - 4\sin 2B \sin 2C \cos 2A$

$$L.H.S = \cos 4A - \cos 4B - \cos 4C = \cos 4A - (\cos 4B + \cos 4C)$$

$$= 2\cos^2 2A - 1 - 2\cos(2B + 2C)\cos(2B - 2C)$$

$$= 2\cos^2 2A - 2\cos(360^0 - 2A)\cos(2B - 2C) - 1$$

$$= 2\cos^2 2A - 2\cos 2A\cos(2B - 2C) - 1$$

$$= 2\cos 2A[\cos 2A - \cos(2B - 2C)] - 1$$

$$= 2\cos 2A \times -2\sin(A + B - C)\sin(A - B + C) - 1$$

$$= -4\cos 2A\sin(180^0 - 2C)\sin(180^0 - 2B) - 1$$

$$= -4\sin 2B\sin 2C\cos 2A - 1 = R.H.S$$

4. Prove that for any triangle ABC, Prove that:
$\cos^2 2A + \cos^2 2B + \cos^2 2C - 1 = 2\cos 2A \cos 2B \cos 2C$.

Solution

$L.H.S = \cos^2 2A + \cos^2 2B + \cos^2 2C - 1 \quad = \frac{1}{2}(1+\cos 4A) + \frac{1}{2}(1+\cos 4B) + \cos^2 2C - 1$

$= \frac{1}{2}(2 + 2\cos 2(A+B)\cos 2(A-B)) + \cos^2 2C - 1$

But $\cos 2(A+B) = \cos(360 - 2C) = \cos 2C$

$\therefore \ L.H.S = \cos 2C(\ \cos 2(A-B) + \cos 2C) \quad = \cos 2C(\ \cos 2(A-B) + \cos 2(A+B))$

$= \cos 2C(\ 2\cos 2A \cos 2B) \ = 2\cos 2A \cos 2B \cos 2C = R.H.S$

RELATIONSHIP BETWEEN THE SIDES AND ANGLES OF A TRIANGLE

(SOLUTION OF A TRIANGLE)

A triangle has six (6) parts as an element, that is to say three sides and three angles. We want to study the relationship between these.

The sine rule ; $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, where R is the radius of the circum-circle of the triangle with vertices A, B and C.

The cosine rule;

$a^2 = b^2 + c^2 - 2bc\cos A$ or $b^2 = a^2 + c^2 - 2ac\cos B$ or $c^2 = a^2 + b^2 - 2ab\cos C$

The above two rules can be proved. This has been left to the learners.

Example

1. Solve the triangle ABC in which AB= 5 cm, AC= 4 cm and angle ACB = 60^0.

Solution

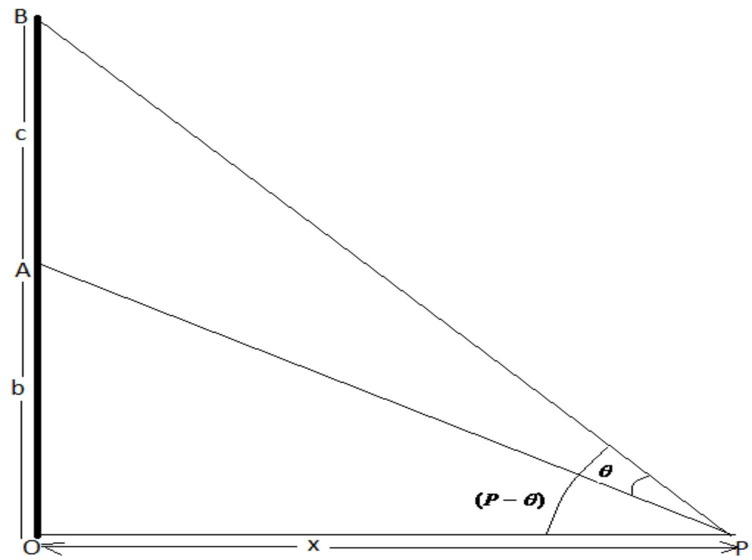
Here $c = 5\ cm$, $b = 4\ cm\ \ and\ \ C = 60^0$

Using $\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{4}{\sin B} = \frac{5}{\sin 60^0}$; $B = 43.85^0$

$\angle A = 180^0 - (60 + 43.85)^0 = 76.15^0$

Also $\dfrac{a}{\sin A} = \dfrac{c}{\sin C} \Rightarrow \dfrac{a}{\sin 76.15^0} = \dfrac{5}{\sin 60^0}$; $a = 5.6056\ cm$

2. A vertical pole BAO stands with its base O on a horizontal plane, where, $BA = c$ and $AO = b$. A point P is situated on the horizontal plane at a distance x from O, and angle $APB = \theta$. Show that $\tan\theta = \dfrac{cx}{x^2 + b^2 + bc}$.



$$\tan P = \frac{b+c}{x},\quad \tan(P-\theta) = \frac{b}{x};\quad \frac{\tan P - \tan\theta}{1 + \tan P \tan\theta} = \frac{b}{x},$$

$$b(1 + \tan P \tan\theta) = x(\tan P - \tan\theta)$$

$$\tan\theta = \frac{x\tan P - b}{x + b\tan P},\quad \tan\theta = \frac{x\left(\frac{b+c}{x}\right) - b}{x + b\left(\frac{b+c}{x}\right)} = \frac{c}{\frac{x^2 + b^2 + bc}{x}}$$

$$\therefore\ \tan\theta = \frac{cx}{x^2 + b^2 + bc} \text{ as required}$$

3. Show that in any triangle ABC,

(a) $\dfrac{a^2 - b^2}{c^2} = \dfrac{\sin(A-B)}{\sin(A+B)}$ (b) $\dfrac{a+b+c}{a+b-c} = \cot\dfrac{A}{2}\cot\dfrac{B}{2}$

Solution

(a) $$\frac{a^2-b^2}{c^2}=\frac{\sin(A-B)}{\sin(A+B)}$$

L.H.S = $$\frac{a^2-b^2}{c^2}=\frac{(2R\sin A)^2-(2R\sin B)^2}{(2R\sin C)^2}$$

$$=\frac{\sin^2 A-\sin^2 B}{\sin^2 C}=\frac{(\sin A+\sin B)(\sin A-\sin B)}{\sin^2 C}$$

$$=\frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}\times 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{\sin^2 C}$$

$$=\frac{\sin(180^0-C)\sin(A-B)}{\sin C\times\sin C}=\frac{\sin C\sin(A-B)}{\sin C\times\sin C}$$

$$=\frac{\sin(A-B)}{\sin(A+B)}=\text{R.H.S ; since }\sin C=\sin(A+B)$$

(b) $$\frac{a+b+c}{a+b-c}=\cot\frac{A}{2}\cot\frac{B}{2}$$

L.H.S = $$\frac{a+b+c}{a+b-c}=\frac{2R\sin A+2R\sin B+2R\sin C}{2R\sin A+2R\sin B-2R\sin C}$$

$$=\frac{\sin A+\sin B+\sin C}{\sin A+\sin B-\sin C}=\frac{\sin A+2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}{\sin A+2\cos\left(\frac{B+C}{2}\right)\sin\frac{B-C}{2}}$$

$$=\frac{2\sin\frac{A}{2}\cos\frac{A}{2}+2\sin\left(90^0-\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2\sin\frac{A}{2}\cos\frac{A}{2}+2\cos\left(90^0-\frac{A}{2}\right)\sin\left(\frac{B-C}{2}\right)}$$

$$=\frac{2\cos\frac{A}{2}\left(\sin\frac{A}{2}+\cos\left(\frac{B-C}{2}\right)\right)}{2\sin\frac{A}{2}\left(\cos\frac{A}{2}+\sin\left(\frac{B-C}{2}\right)\right)}$$

$$= \cot\frac{A}{2}\times\frac{\sin\left(90^0-\frac{B+C}{2}\right)+\cos\left(\frac{B-C}{2}\right)}{\cos\left(90^0-\frac{B+C}{2}\right)+\sin\frac{B-C}{2}}$$

$$= \cot\frac{A}{2}\times\frac{2\cos\frac{B}{2}\cos\frac{C}{2}}{2\sin\frac{B}{2}\cos\frac{C}{2}} = \cot\frac{A}{2}\cot\frac{B}{2} = R.H.S$$

4. Prove that $a\cos ec\,\tfrac{1}{2}A = (b+c)\sec\tfrac{1}{2}(B-C)$, hence find the angles B *and* C, given that $a=5.3$, $b+c=11.8$ and $A=46^o$.

Solution

$a\cos ec\,\tfrac{1}{2}A = (b+c)\sec\tfrac{1}{2}(B-C)$,which on rearranging gives

$$\frac{a}{(b+c)} = \frac{\sec\tfrac{1}{2}(B-C)}{\cos ec\,\tfrac{1}{2}A}$$

Now L.H.S = $\dfrac{a}{(b+c)} = \dfrac{2R\sin A}{2R\sin B + 2R\sin C}$

$$\frac{\sin A}{\sin B+\sin C} = \frac{2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A}{2\sin\tfrac{1}{2}(B+C)\cos\tfrac{1}{2}(B-C)} \quad \text{but } \sin\tfrac{1}{2}(B+C) = \cos\tfrac{1}{2}A$$

$$= \frac{\cos\tfrac{1}{2}A}{\cos\tfrac{1}{2}(B-C)} = \frac{\sec\tfrac{1}{2}(B-C)}{\cos ec\,\tfrac{1}{2}A} = \text{R.H.S}$$

Thus:

$$5.3\cos ec23 = 11.8\sec\tfrac{1}{2}(B-C),\quad \cos\tfrac{1}{2}(B-C) = \frac{11.8\times\sin 23}{5.3} = 0.8699$$

$$B-C = 59.1^o, \text{ but } B+C = 134^o$$

So, $B = 96.55^o$, $C = 37.45^o$

5. Find the expressions for $\sin A$ *and* $\sin\dfrac{A}{2}$ in terms of the sides of a triangle ABC and the semi-perimeter *s*.

From the cosine rule, $a^2 = b^2 + c^2 - 2bc\cos A \;\Rightarrow \cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

But $\sin^2 A = 1 - \cos^2 A = (1 + \cos A)(1 - \cos A)$

$$\sin^2 A = \left(1 + \frac{b^2 + c^2 - a^2}{2bc}\right)\left(1 - \frac{b^2 + c^2 - a^2}{2bc}\right)$$

$$= \left(\frac{b^2 + c^2 + 2bc - a^2}{2bc}\right)\left(\frac{2bc - b^2 - c^2 + a^2}{2bc}\right)$$

$$= \left(\frac{(b + c)^2 - a^2}{2bc}\right)\left(\frac{a^2 - (b - c)^2}{2bc}\right)$$

$$= \frac{(b + c + a)(b + c - a)(a + b - c)(a - b + c)}{4b^2c^2}$$

Now semi-perimeter, $s = \dfrac{a + b + c}{2} \Rightarrow 2s = a + b + c$

So $b + c - a = 2s - 2a$; $a + b - c = 2s - 2c$; $a - b + c = 2s - 2b$

$\therefore \quad \sin^2 A = \dfrac{2s(2s - 2a)(2s - 2b)(2s - 2c)}{4b^2c^2} = \dfrac{4s(s - a)(s - b)(s - c)}{b^2c^2}$

So $\sin A = \sqrt{\dfrac{4s(s - a)(s - b)(s - c)}{b^2c^2}} = \dfrac{2\sqrt{s(s - a)(s - b)(s - c)}}{bc}$.

Similarly $\sin B = \dfrac{2\sqrt{s(s - a)(s - b)(s - c)}}{ac}$ and $\sin C = \dfrac{2\sqrt{s(s - a)(s - b)(s - c)}}{ab}$.

Also $\sin^2 \dfrac{A}{2} = \dfrac{1}{2}(1 - \cos A) = \dfrac{1}{2}\left(1 - \dfrac{b^2 + c^2 - a^2}{2bc}\right) = \dfrac{2bc - b^2 - c^2 + a^2}{4bc}$

$$= \frac{a^2 - (b - c)^2}{4bc} = \frac{(a + b - c)(a - b + c)}{4bc}$$

$$= \frac{(2s - 2c)(2s - 2b)}{4bc} = \frac{(s - b)(s - c)}{bc}$$

$\therefore \quad \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$.

Similarly $\sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}$ and $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$.

Trial exercise

1. Solve the triangle ABC in which AB= 4.75 cm, BC= 3.84 cm and angle ACB = 40.2^0.

2. In the cyclic quadrilateral ABCD, AB = 7 cm, BC = 8 cm, CD = 8 cm and AD = 15 cm. Calculate the angle ABC and the length of AC.

3. In the triangle ABC, AB = 9 cm, AC = 12 cm, angle ABC = 2θ and

   angle ACB = θ. Find the (i) length of BC, (ii) area of the triangle ABC.

4. The sides BC, CA, AB of a triangle ABC are of lengths $x+y, x, x-y$ respectively. Prove that $\cos A = \frac{x-4y}{2(x-y)}$.

5. Prove that in any triangle ABC

   (i) $\cos(A-C) - \cos B = 2\cos A\cos C$ (ii) $\sin B + \sin C - \sin A = 4\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

6. Given triangle PQR, prove that $\tan\left(\frac{Q-R}{2}\right) = \frac{q-r}{q+r}\cot\frac{P}{2}$, hence, solve the triangle with two sides $5\,cm$ and $7\,cm$ and the included angle is 45^o.

7. Prove that in any triangle PQR, $\tan\left(\frac{P-Q}{2}\right) = \frac{p-q}{p+q}\cot\frac{R}{2}$. Hence find the values of angles P and Q.

8. Prove that in any triangle ABC $a = b\cos C + c\cos B$ and $a\sin\frac{B-C}{2} = (b-c)\cos\frac{A}{2}$.

9. (a) In a triangle ABC, prove that $a\cos 2B + b\cos C = c\cos B - 2b\cos A\cos B$.

   (b) Prove that (i) $8\cos 3\theta\cos 2\theta\cos\theta - 1 = \frac{\sin 7\theta}{\sin\theta}$ (ii) $4\cos 3\theta\cos\theta + 1 = \frac{\sin 5\theta}{\sin\theta}$.

(Hint: Here first write $1 = \frac{\sin\theta}{\sin\theta}$ and then apply the factor formula.)

10. Given that A, B and C are angles of a triangle, deduce that

(i) $\frac{b-c}{b+c} = \cot\left(\frac{B+C}{2}\right)\tan\left(\frac{B-C}{2}\right)$ (ii) $\sin\frac{B-C}{2} = \frac{b-c}{a}\cos\frac{A}{2}$

(iii) $\cos\frac{B-C}{2} = \frac{b+c}{a}\sin\frac{A}{2}$ (iv) $a+b = c\cos\frac{A-B}{2}\operatorname{cosec}\frac{C}{2}$

11. In any triangle ABC, show that $\frac{a+b-c}{a-b+c} = \tan\frac{B}{2}\cot\frac{C}{2}$.

12. Prove that in any triangle ABC, $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$.

13. In a triangle ABC, prove that $\tan(B-C) = \frac{2(b^2-c^2)\cot\frac{A}{2}}{(b+c)^2-(b-c)^2\cot^2\frac{A}{2}}$.

14. Two points X and Y 600m apart lie on a horizontal ground. A vertical pole 200m tall stands along the line XY. If the angle of elevation of the top, T, of the pole from X is 30^0 find the (i) angle of elevation of T from Y. ii) distance YT.

15. Show that in any triangle ABC, if $2s = a+b+c$, then $1 - \tan\frac{1}{2}A\ \tan\frac{1}{2}B = \frac{c}{s}$.

16. Prove that in any triangle PQR, $\frac{1}{p}\cos^2\tfrac{1}{2}P + \frac{1}{q}\cos^2\tfrac{1}{2}Q + \frac{1}{r}\cos^2\tfrac{1}{2}R = \frac{(p+q+r)^2}{4pqr}$.

17. The area of a triangle is 336 m^2. The sum of the three sides is 84 m and one side is 28 m. Calculate the lengths of the other two sides.

18. Find the angles of the triangle with side 10 cm, 12 cm and 14 cm respectively.

END