TOPIC 9: DIFFERENTIAL EQUATIONS

The differential equation defines a family of curves

A general solution involves one or more arbitrary constant is the equation of any member of the family

A particular solution is the equation of one member of the family

The order of a differential equation is determined by the highest differential coefficient present .

Note that the syllabus only allows us to cover first order

1. First order separating the variables

$$x^2\frac{dy}{dx}=y(y-1)$$

Note that if x and y can be separated

$$\frac{dy}{y(y-1)}=\frac{1}{x^2}dx$$

$$\frac{1}{y(y-1)}=\frac{A}{y}+\frac{B}{y-1}$$

$$1\equiv A(y-1)+By$$

For y =1   y =0

1=B   1=-A

A=-1

$$-\int\frac{dy}{y}+\int\frac{dy}{y-1}=\int x^{-2}dx$$

$$-Iny+In(y-1)=\frac{-1}{x}+C$$

Example

$$x\frac{dy}{dx}-3=2\left(y+\frac{dy}{dx}\right)$$

$$x\frac{dy}{dx}-3=2y+2\frac{dy}{dx}$$

$$(x-2)\frac{dy}{dx}=(2y+3)$$

$$\frac{dy}{2y+3}=\frac{1}{x-2x}dx$$

$$\frac{1}{2}\int\frac{2dy}{2y+3}=\int\frac{dx}{x-2}$$

$$\frac{1}{2}In(2y+3)=In(x-2)+InA$$

$$In\sqrt{2y+3}=In(x-2)$$

$$\sqrt{2y+3}=A(x-2)$$

First Order Exact Equation

An exact equation is one with one side originates from some derivative

Example

$$x^2\frac{dy}{dx}+2xy=1$$

Taking the L.H.S

$$\frac{d}{dx}\left(x^2.y\right)=x^2\frac{dy}{dx}+y.2x$$

Since they are equal with above expression it is taken to be exact

So $\frac{d}{dx}\left(x^2.y\right)=1$

$$\int\frac{d}{dx}\left(x^2y\right)dx=\int 1dx$$

$$x^2y=x+C$$

Example

$$x^2Cosu\frac{du}{dx}+2xSinu=\frac{1}{x}$$

$$\frac{d}{dx}\left(x^2.Sinu\right)=x^2Cosu\frac{du}{dx}+\left(Sinu\right).2x$$

$$\therefore\frac{d}{dx}\left(x^2Sinu\right)=\frac{1}{x}$$

$$\int\frac{d}{dx}\left(x^2Sinu\right)dx=\int\frac{1}{x}dx$$

x^2Sinu=Inx+c

Integrating Factor

Differential equations which are not exact can be made exact if you use an integrating factor. This is only possible if the equation is not exact and can be written in this form

$\frac{dy}{dx}+Py=Q$ where P and Q are function in terms of what your differentiating with respect to

Example

$$\frac{dy}{dx}-y\tan x=x$$

$$I.F=e^{\int\tan x dx}$$

$$=e^{\int\frac{-\sin x}{\cos x}dx}=e^{InCosx}=Cosx$$

$$\therefore\cos x\frac{dy}{dx}-y\tan x\cos x=x\cos x$$

$$\frac{dy}{dx}(y\cos x)=x\cos x$$

$$\int\frac{d}{dx}(y\cos x)dx=\int x\cos xdx$$

ycosx=xsinx-$\int\sin xdx$

Let u=A $\qquad\frac{du}{dx}=1$

$$Cosx = \frac{dv}{dx}$$

V =Sinx

ycosx =xsinx+cosx+C

Example

$$x\frac{dy}{dx} - y = \frac{x}{x-1}$$

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{1}{x-1}$$

$$I.F = e^{\int \frac{-1}{x}dx} = e^{\int \frac{-1}{x}dx} = e^{-Inx} = e^{-Inx} = \frac{1}{x}$$

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \frac{1}{x(x-1)}$$

$$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1}{x(x-1)}$$

$$\int \frac{d}{dx}\left(\frac{y}{x}\right)dx = \int \frac{dx}{x(x-1)}$$

$$\frac{y}{x} = \int \frac{dx}{x(x-1)}$$

$$\frac{1}{x(x-1)} \equiv \frac{A}{x} + \frac{B}{x-1}$$

$$1 \equiv A(x-1) + Bx$$

For x=1 | for x=0

1=B | 1=-A

A=-1

$$\frac{1}{x(x-1)} \equiv \frac{1}{x-1} - \frac{1}{x}$$

$$\int \frac{dx}{x(x-1)} = \int \frac{dx}{x-1} - \int \frac{dx}{x}$$

=In(x-1)-Inx +C

$$= In\left(\frac{x-1}{x}\right) + C$$

$$\Rightarrow \frac{y}{x} In\left(\frac{x-1}{x}\right) + C$$

Example

$$(x-2)\frac{dy}{dx} + 3y = \frac{2}{x-2}$$

$$\frac{dy}{dx} + \left(\frac{3}{x-2}\right)y = \frac{2}{(x-2)^2}$$

$$= I.F = e^{\int \frac{3}{x-2}dx} = e^{3\int \frac{dx}{x-2}} = e^{3In(x-2)} = e^{In(x-2)^3} = (x-2)^3$$

$$(x-2)^3 \frac{dy}{dx} + 3(x-2)^2 y = 2(x-2)$$

$$\frac{d}{dx}\left(y(x-2)^3\right) = \int 2x - 4dx$$

$$y(x-2)^3 = x^2 - 4x + C$$

First Order Homogeneous Equations

In this part all the terms are taken to have same dimensions

Example

$$xy\frac{dy}{dx} = x^2 + y^2$$

Dividing through by x^2

$$\frac{xy}{x^2}\frac{dy}{dx} = 1 + \frac{y^2}{x^2}$$

$$\left(\frac{y}{x}\right)\frac{dy}{dx}=1+\left(\frac{y}{x}\right)^2$$

Let u = $\frac{y}{x}$

y = ux

$$\frac{dy}{dx}=u+x\frac{du}{dx}$$

$$u\left(u+x\frac{du}{dx}\right)=u+u^2$$

$$u^2+ux\frac{du}{dx}=1+u^2$$

$$ux\frac{du}{dx}=1$$

$$\int udu=\int\frac{1}{x}dx$$

$\tan^{-1}$u=Inx +InA

Tan^{-1}u= InAx

U= tan(InAx0

$$\frac{y}{x}=\tan(InAx)$$

$$y=x\tan(InAx)$$

EXAMPLE

$$\frac{du}{d\theta}+uCot\theta=2Cos\theta \quad Given \quad u=3 \quad when \quad \theta=\frac{\pi}{2}$$

$$I.F=e^{\int Cot\theta d\theta}=e^{\int\frac{Cot\theta d\theta}{Sin\theta}}=e^{\int In\sin\theta}=Sin\theta$$

$$Sin\theta\frac{du}{d\theta}+u\cot\theta\sin\theta=2\sin\theta\cos\theta$$

$$\frac{d}{d\theta}(uSin\theta)=\sin 2\theta$$

$$\int \frac{d}{d\theta}(uSin\theta)d\theta = \int \sin 2\theta d\theta$$

$$uSin\theta = -\frac{1}{2}Cos2\theta + C$$

$$When\ \ \theta = \frac{\pi}{2}\ \ ,\ u = 3$$

$$3\sin\frac{\pi}{2} = \frac{1}{2}Cos\ \pi + C$$

$$U\sin\theta = -\frac{1}{2}Cos\ \theta + \frac{5}{2}\text{ Particular solution}$$

Exercise

1. $x\frac{dy}{dx} = y + xy$
2. $2\sin\theta\frac{d\theta}{dr} = Cos\theta - Sin\theta$
3. $e^{x}\frac{du}{dx} = y^{2} + 4 = 0$
4. $r\sec^{2}\theta + 2\tan\theta\frac{d\theta}{dr} = \frac{2}{r}$
5. $x^{2}\frac{dy}{dx} = 3x^{2} + xy$
6. $x\frac{dy}{dx} = y\sqrt{x^{2} + y^{2}}$
7. $\frac{dy}{dx} = x\frac{(x - y + 2)}{x + y}$
8. $\frac{dy}{dx} = \frac{2x + y - 2}{2x + y + 1}$

Formation Of Differential Equatorial

In this category of question you are supposed to form a different equation using the information given

There are some key words to note for example rate , increasing or decreasing, growth decay. In some cases they use gradient, velocity , acceleration etc

A learner should note what is changing and with respect to what also not whether that leads to increase or decrease of the variable if it increases the rate will be positive and negative when it decrease.

According to Newton's law of cooling the rate at which the temperature of a body falls is proportional to the amount by which the temperature exceeds that of its surrounding temperature .

Suppose the temperature of the object falls from 200^0c to 100^0c in 40 minutes in a surrounding of 10^0c . Prove that after t minutes the body is given by $T=10+190e^{-kt}$ where $k=\frac{1}{40}In\left(\frac{19}{9}\right)$

Calculate the time it takes to reach 50^0c

$$\frac{dT}{dt}\alpha\left(T-10^0\right)$$

$$-\frac{dT}{dt}=K\left(T-10^0\right)$$

$$\frac{dT}{T-10}=-kdt$$

$$\int\frac{dT}{T-10}=\int-kdt$$

In(T-10) = -Kt+C

When T = 200, t=0

T =100 , t=40

In(200-10) = 0+C

In190=C

In(T-10)= -Kt + In190

$$In\left(\frac{T-10}{190}\right)=-kt$$

In(100-10) =-40k+In190

In(90)-In(190) =-40K

$$In\left(\frac{90}{190}\right) = -40K$$

$$In\left(\frac{9}{19}\right) = -40K$$

$$-\frac{1}{40}In\left(\frac{19}{9}\right) = K$$

$$\frac{1}{40}In\left(\frac{19}{9}\right) = K$$

$$In\left(\frac{T-10}{190}\right) = -kt$$

$$e^{In\left(\frac{T-10}{190}\right)} = e^{-kt}$$

$$\frac{T-10}{190} = e^{-kt}$$

$T\text{-}10=190e^{-kt}$

$T= 10+190e^{-Kt}$

$$In\left(\frac{T-10}{190}\right) = -kt$$

$$In\left(\frac{50-10}{190}\right) = -\frac{1}{40}In\left(\frac{19}{9}\right)t$$

$$\frac{In\left(\frac{4}{19}\right)}{\frac{1}{40}In\left(\frac{19}{9}\right)} = -t$$

T=83.4 Minutes

Note

These are the common questions set of formation of differentiation equation

1. Newton's law of cooling

   Relating change in Temperature and the excess temperature due temperature of the body and the surrounding T is temperature at any time and A Temperature of the surrounding.

   $$\frac{dT}{dt}\alpha(T-A)$$

   $$\frac{dT}{dt}=-K(T-A) \text{ Or } \frac{dT}{dt}=K(T-A)$$

   K is negative when cooling takes place and positive when temperature increases

2. **Falling bodies**

   If the body falls from rest in a medium which causes the velocity to decrease at a rate proportional to the velocity then $-\frac{dv}{dt}\alpha V$

   $$\frac{dv}{dt}=-KV$$

   But if it increases as it falls

   $$\frac{dv}{dt}\alpha V \qquad \text{and } \frac{dv}{dt}=KV$$

3. Growth of yeast cells in a culture it the number of cells are n at any time t. If the rate at which the number increases is proportion to the number of cells. $\frac{dn}{dt}\alpha\, n$

   $\frac{dn}{dt}=Kn$ Since it grows or increases rate will be positive.

4. A chemical mixture contains two substances A and B whose weight, are W_A and W_B and whose combined weight remains constant B is converted into A at a rate which is inversely proportional to the weight B and proportional to square of A in the mixture at any time ***t***. The weight B present at time t can be found using

   $\frac{dw_B}{dt}\alpha\frac{w_A^{\ 2}}{W_B}$ Since its B which is changing to A therefore it is the variable and increasing grad function will be positive

$$\frac{dw_B}{dt} = \frac{Kw_A{}^2}{W_B}$$

But $W=W_A+W_B$

Where W is a constant

$W_A=W-\mathbf{W}_B$

$$\frac{dw_B}{dt} \alpha \frac{K(w-w_B)^2}{W_B}$$

Example

A substance loses mass at a rate which is proportional to the amount M present at a time t

(a) For a differential equation connecting m, t and a constant of proportionality K

(b) If initially the mass of the substance is M_0 show that $M=M_oe^{-Kt}$

(c) Given that half of the substance is lost in 1600 years. Determine the number of years 15g of the substance would reduce to 13.6g

(d) Since it is loosing mass

$$-\frac{dm}{dt} \alpha\ m$$

$$\frac{dm}{dt} = -Km$$

$$\frac{dm}{dm} = -Kdt$$

$$\int \frac{dm}{dm} = \int -Kdt$$

Inm =-Kt +C

$t=0$, $m=m_o$

$\text{Inm}_o=0+C$

$C=\text{Inm}_o$

$\text{Inm}-\text{Inm}_o=-Kt$

$$In\left(\frac{m}{m_o}\right) = -Kt$$

$$\frac{m}{m_o} = e^{-Kt}$$

$$m = m_o e^{-Kt}$$

t =1600 , m = ½ m_o

$$In\left(\frac{\frac{1}{2}m_0}{m_0}\right) = -1600k$$

$$In\frac{1}{2} = -1600k$$

In1-In2 =-1600K

-In2 = -1600k

$$\frac{1}{1600}In2 = K$$

M_o=15g

M=13.6g

$$In\left(\frac{13.6}{15}\right) = \frac{In2}{-1600}t$$

$$\frac{-1600In\left(\frac{13.6}{15}\right)}{In2} = t$$

t =4years

Example

A bacteria in a culture increase at a rate proportional to the number of bacterial present .

If the number increases from 1000 to 2000 in one hour.

(a) How may bacterial will be present after 1 ½ hours

(b) How long will it take for the number of bacteria in the culture to become 4000

Let the number of bacterial present be P.

$$\frac{dp}{dt}\alpha P$$

$$\frac{dp}{dt}=KP$$

$$\frac{dp}{p}=Kdt$$

$$\int\frac{dp}{p}=\int Kdt$$

InP =Kt +C

t= 0, P=1000

t=1 P=2000

In(1000)=0+C

C=In1000

InP-In1000=Kt

$$In\left(\frac{p}{1000}\right)=Kt$$

$$In\left(\frac{2000}{1000}\right)=K$$

In2= K

$$In\left(\frac{p}{1000}\right)=Kt$$

$$e^{In\left(\frac{p}{1000}\right)}=e^{Kt}$$

$$\frac{p}{1000}=e^{Kt}$$

$$p=1000e^{(In2)t}$$

$$p=1000e^{(In2)\left(3/2\right)}$$

$$In\left(\frac{4000}{1000}\right) = (In2)t$$

$$\frac{In(4)}{(In(2)} = t$$

Example

The acceleration of a particle after t seconds is given by a =5+Cos ½t . If initially the particle is moving at 1ms^{-1}. Find the velocity after $2\pi\, Seconds$ and the distance it had covered by then.

$$a = \frac{dv}{dt}$$

$$\frac{dv}{dt} = (5 + Cos\frac{1}{2}t)$$

$$\int dv = \int 5 + Cos\frac{1}{2}tdt$$

$$v = 5t + 2Sin\frac{1}{2}t + C$$

t= 0 , V=1

1=0+C

C=1

$$v = 5t + 2\sin\frac{1}{2}t + 1$$

$$v = 5(2\pi) + 2\sin\pi + 1$$

$$v = (10\pi + 1)ms^{-1}$$

$$v = \frac{dx}{dt}$$

$$\frac{dx}{dt} = 5t + 2\sin\frac{1}{2}t + 1$$

$$dx = \left(5t + 2\sin\frac{1}{2}t + 1\right)dt$$

$$\int dx = \int\left(5t + 2\sin\frac{1}{2}t + 1\right)dt$$

$$x = \frac{5}{2}t^2 - 4Cos\frac{1}{2}t + t + C$$

From above

t =0 , x=0

0=0-4+0+C

C=4

$$x = \frac{5}{2}(2\pi)^2 - 4Cos\pi + 2\pi + 4$$

x= (112.98)m

Example

A rumour spreads through a town at a rate which is proportional to the product of the number of people who have heard the rumour and those who have not heard it. Given that x is a fraction of those who have heard the rumour at any time *t*

(i) Form a differential equation connecting *x, t* and constant *k*

(ii) If initially a fraction C of the population had heard the rumour deduce that

$$x = \frac{C}{C + (1-C)e^{-Kt}}$$

(iii) Given 15% had heard the rumour at 9.00 a.m and another 15% by noon . Find what further fraction of the population would have heard the rumour by 3.00P.m

The population who have not heard the romour =1-x

$$\frac{dx}{dt}\alpha x(1-x)$$

$$\frac{dx}{dt} = Kx(1-x)$$

$$\frac{dx}{x(1-x)} = Kdt$$

$$\int \frac{dx}{x(1-x)} = \int Kdt$$

$$\frac{1}{x(1-x)} \equiv \frac{A}{x} + \frac{B}{(1-x)}$$

$$1 \equiv A(1-x) + Bx$$

For x=0

1=A

When x=1

1=B

$$\Rightarrow \frac{1}{x} + \frac{1}{1-x}$$

$$\int \frac{dx}{x(1-x)} = \int \frac{dx}{x} + \int \frac{dx}{1-x}$$

$$\therefore Inx - In(1-x) = Kt + D$$

$$In\left(\frac{1}{1-x}\right) = Kt + D$$

t = 0 x=c

$$In\left(\frac{C}{1-C}\right) = 0 + D$$

$$D = In\left(\frac{C}{1-C}\right) =$$

$$In\left(\frac{x}{1-x}\right) = Kt + In\left(\frac{C}{1-C}\right)$$

$$In\left(\frac{x}{1-x}\right) - In\left(\frac{C}{1-C}\right) = Kt$$

$$let\ \frac{C}{1-C} = A$$

$$In\left(\frac{x}{1-x}\right) - InA = Kt$$

$$In\frac{x}{A(1-x)}=Kt$$

$$e^{In\frac{x}{A(1-x)}}=e^{Kt}$$

$$\frac{x}{A(1-x)}=e^{Kt}$$

$$x=InA(1-x)e^{Kt}$$

$$x=Ae^{kt}-Axe^{Kt}$$

$$x+Axe^{Kt}=Ae^{kt}$$

$$x(1+Ae^{Kt})=Ae^{kt}$$

$$x=\frac{Ae^{kt}}{1+Ae^{Kt}}=\frac{\left(\frac{C}{1-C}\right)e^{kt}}{1+\left(\frac{C}{1-C}\right)e^{Kt}}$$

$$x=\frac{Ce^{kt}}{(1-C)+Ce^{kt}}$$

$$x=\frac{Ce^{kt}.Ce^{-kt}}{(1-C)Ce^{-kt}+Ce^{kt}Ce^{-kt}}$$

$$x=\frac{C}{C+(1-C)e^{-kt}}$$

C=15% t =0 time 9:00am , x = 30%, t =3hrs

$$x=\frac{\frac{15}{100}}{\frac{15}{100}+\left(\frac{85}{100}\right)e^{-3k}}$$

$$\frac{30}{100}=\frac{\frac{15}{100}}{\frac{15}{100}+\left(\frac{85}{100}\right)e^{-3k}}$$

$$\frac{30}{100}=\frac{15}{15+85e^{-3k}}$$

$$\frac{3}{10}=\frac{3}{3+17e^{-3k}}$$

$$3+17e^{-3K}=10$$

$$17e^{-3K}=7$$

$$e^{-3K}=\frac{7}{17}$$

t = 6

$$x=\frac{\frac{15}{100}}{\frac{15}{100}+\left(\frac{85}{100}\right)e^{-6K}}$$

$$x=\frac{15}{15+85e^{2(-3K)}}=\frac{3}{3+17e^{(-3K)^2}}$$

$$x=\frac{3}{3+17\left(\frac{7}{17}\right)^2}$$

$$x=\frac{3}{3+\frac{17\times 49}{17}}$$

$$x=\frac{3\times 17}{3\times 17+49}$$

$$x=\frac{52}{52+49}=\frac{52}{101}=0.5148\approx 51.4\%$$

Further fraction $=51.4\%-15\%$

$=36\%$

Exercise

1. The rate of change of atmospheric pressure P with respect to altitude h in km is proportional to the pressure P if the pressure P_0 at sea level. Find the formula of the pressure at any height.

2. At3:00pm the temperature of a hot metal was 80^0 and that of the surroundings 20^0C. At 3:03 pm the temperature of the metal had dropped to 42^0C. The rate of cooling of the metal was directly proportional to the difference between its temperature T and that of the surroundings.
   (a) Write down the differential equation to represent the rate of cooling of the metal
   (b) Solve the differential equation
   (c) Find the temperature of the metal at 3:05pm
3. An Athlete runs at a speed proportional to the square root of the distance he still has to cover .If the athlete starts running at $10ms^{-1}$ and has a distance of 1600m to cover . Find how long he will take to cover this distance
4. The differential equation $\frac{dp}{dt} = kP(c-P)$ show a rate at which information flows in a students population C, P represents the number who have heard the information in t days and k is a constant.
   (a) Solve the differential equation
   (b) A school has a population of 1000 students initially 20 students had heard the information . A day later 50 students had heard the information. How many students heard the information by the tenth day.
5. A research investigates the effect of a certain chemicals on a virus infection crops, revealed that the rate at which the virus population is destroyed is directly proportional to the population at the time . Initially the population was P_o at t months later it was found to be P
   (a) Form a differential equation connecting P and t
   (b) Given that the virus population reduced to one third of the initial population in 4 months solve the equation above.