DIFFERENTIAL EQUATIONS

A differential equation is a mathematical equation that relates some function with its derivative. In applications the function usually represent physical quantities, the derivatives represent their rate of change.

Examples of differential equation are

$\frac{dy}{dx} = y, \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8 = 0, \frac{dy}{dx} = \sin x + \cos x$, Etc.

Order of a differential equation

The order of a differential equation is the order of the highest derivative appearing in that equation.

For example:

(1). $\frac{dy}{dx} = x + y, \frac{dy}{dx} = \sin x + \cos x$,

$y - \frac{dy}{dx} = x$, etc

are all first order differential equations

(2). $\frac{d^2y}{dx^2} = 5 + x, 5\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^3 = 7,$

$$7\left(\frac{d^2y}{dx^2}\right)^2 + x\frac{dy}{dx} + 6 = 0$$

are all second order differential equation

(3). $\frac{d^3y}{dx^3} + \sin x = 0,\ \left(\frac{d^3y}{dx^3}\right)^2 + 5\left(\frac{dy}{dx}\right)^2 + \sin x = 0$

Are all third order differential equations.

The Degree of a Differential Equation

The degree of a differential equation is the highest integral power to which the highest order derivative is raised when a differential equation is written as a polynomial in unknown function and its derivative.

Example:

(i) $\frac{dy}{dx} - xy = 0,\ \frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 2 = 0$

$\frac{y}{x}.\frac{dy}{dx} = 1,\ \frac{d^3y}{dx^3} = \sin x - \cos x$

are all first degree differential equations

(ii) $\left(\frac{dy}{dx}\right)^2 + xy = 0,\ \ 2\left(\frac{d^2y}{dx^2}\right)^2 + 6x\left(\frac{dy}{dx}\right)^3 = 0$

Are all second degree differential equations

(iii) $\left(\frac{d^2y}{dx^2}\right)^3 = 5\left(\frac{dy}{dx}\right) + \sin x,$

$$\left(\frac{dy}{dx}\right)^3 + x\left(\frac{dy}{dx}\right)^4 + xy = 0$$

Are all third degree differential equations.

Example

Determine the order and the degree of the following differential equations

$(a)\ \frac{dy}{dx} + xy = 5x$

$(b)\ \frac{d^2y}{dx^2} - 5x\frac{dy}{dx} = e^x + x^2$

(c) $x^2\left(\frac{d^3y}{dx^3}\right)^2 + 3x\frac{d^2y}{dx^2} + 7x = 0$

$(d)\ y - 5x\left(\frac{dy}{dx}\right)^2 = e^x + x^2\cos x$

Solution

(a) First order, first degree differential equation

(b) 2nd order, first degree differential equations

(c) third order second degree differential equation.

(d) first order second degree differential equation.

A Solution of a Differential Equation

A solution to a differential equation is defined as any relationship connecting the two variables of the equation and not involving their derivatives. A solution to a first order differential equation is said to be general solution (complete solution). If an arbitrary constant remains in the solution. The solution obtained by giving a particular value to an arbitrary constant is called a particular solution.

First Order Differential Equations

(1) Method of separating the variables

Example I

Find the general solution of the following differential equations

a) $(a)\frac{dy}{dx} = y$

b) $(x+2)\frac{dy}{dx} = y$

c) $\frac{dv}{du} = v(v-1)$

d) $\frac{dy}{dx} = (1+x)(1+y^2)$

e) $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$

f) $(x-1)\frac{dy}{dx} = 2x^3y$

Solution

(**a**) $\dfrac{dy}{dx} = y$

$$\int \frac{dy}{y} = \int dx$$
$$\ln y = \ln x + C$$
$$\ln y - \ln x = C$$
$$\ln\left(\frac{y}{x}\right) = C$$
$$\log_e \frac{y}{x} = C$$
$$e^C = \frac{y}{x}$$
$$y = e^C x$$
$$y = kx$$

$(b)\dfrac{dv}{du} = v(v-1)$

$$\int \frac{dv}{v(v-1)} = \int du$$
$$consider \frac{1}{v(v-1)} = \frac{A}{v} + \frac{B}{v-1}$$
$$A(v-1) + Bv = 1$$

If $v = 1$

$B = 1$

If $v = 0$

$-A = 1$

$A = -1$

$$\Rightarrow \int -\frac{1}{v} + \frac{1}{v-1} dv = \int du$$
$$-\ln v + \ln(v-1) = u + c$$
$$\ln\left(\frac{v-1}{v}\right) = u + c$$
$$\log_e\left(\frac{v-1}{v}\right) = u + c$$
$$e^{u+c} = \frac{v-1}{v}$$
$$e^u \times e^c = \frac{v-1}{v}$$
$$ve^u = v - 1$$
$$1 = v - vAe^u$$
$$1 = v(1 - Ae^u)$$
$$v = \frac{1}{1 - Ae^u}$$

$$(d)\frac{dy}{dx} = (1+x)(1+y^2)$$
$$\frac{dy}{1+y^2} = (1+x)dx$$
$$tan^{-1}y = x + \frac{x^2}{2} + A$$
$$y = \tan\left(x + \frac{x^2}{2} + A\right)$$

(f) $(x-1)\frac{dy}{dx} = 2x^3y$

$$\frac{dy}{y} = \frac{2x^3}{x-1}dx$$
$$consider \frac{2x^3}{x-1}$$

$$\begin{array}{r|l} & 2x^2 + 2x + 2 \\ \hline x-1 & 2x^3 \\ & -\ 2x^3 - 2x^2 \\ \hline & 2x^2 \\ & 2x^2 - 2x \\ \hline & 2x \\ & 2x - 2 \\ \hline & 2 \end{array}$$

$$\int \frac{1}{y}dy = \int (2x^2 + 2x + 2) + \frac{2}{x-1}dx$$
$$\ln y = \frac{2x^3}{3} + x^2 + 2x + 2\ln(x-1) + c$$
$$\ln y - \ln(x-1)^2 = \frac{2x^2}{3} + x^2 + 2x$$
$$\ln \frac{y}{(x-1)^2} = \left(\frac{2x^3}{3} + x^2 + 2x\right)$$
$$(x-1)^2 e^{\left(\frac{2}{3}x^3 + x^2 + 2x\right)} = y$$

Example II

Solve the following differential equations.

$(a)\ x\dfrac{dy}{dx} = x - 1$

$(b)\dfrac{dy}{d\theta} = \tan y \tan\theta$

$$(c)\ \theta\frac{d\theta}{dr} = cos^2\theta$$

$$(d)\ e^t\frac{dx}{dt} = \sin t$$

$$(e)\ e^x\frac{dy}{dx} + y^2 + 4 = 0$$

Solution

$$x\frac{dy}{dx} = x - 1$$

$$dy = \frac{x-1}{x}dx$$

$$dy = 1 - \frac{1}{x}$$

$$y = x - \ln x + A$$

(b) $\theta\frac{dy}{d\theta} = \tan y \tan\theta$

$$\frac{dy}{\tan y} = \tan\theta\, d\theta$$

$$\int\frac{\cos y}{\sin y}dy = \int\frac{\sin\theta}{\cos\theta}d\theta$$

$$\ln(\sin y) = -\ln\cos\theta + c$$

$$\ln(\sin y) + \ln\cos\theta = C$$

$$\ln\sin y\cos\theta = C$$

$$e^c = \sin y\cos\theta$$

$$A = \sin y\cos\theta$$

(c) $\frac{\theta d\theta}{dr} = \cos^2\theta$

$$\int\frac{\theta d\theta}{\cos^2\theta} = \int dr$$

$$\int\theta\sec^2\theta\ \ d\theta = \int dr$$

Consider $\int\theta\sec^2\theta\, d\theta$

$$u = \theta$$

$$\frac{du}{d\theta} = 1$$

$$\frac{dv}{d\theta} = \sec^2\theta$$

$$v = \tan\theta$$

$$\int u\frac{dv}{d\theta}d\theta = uv - \int v\frac{du}{d\theta}d\theta$$

$$\int\theta\sec^2\theta\, d\theta = \theta\tan\theta - \int\tan\theta\, d\theta$$

$$= \theta\tan\theta - \int\frac{\sin\theta}{\cos\theta}d\theta$$

$$= \theta\tan\theta - -\ln\cos\theta + C$$

$$= \theta\tan\theta + \ln\cos\theta + C$$

(d) $e^t\frac{dx}{dt} = \sin t$

$$\int dx = \int\frac{\sin t}{e^t}dt$$

$$x = \int e^{-t}\sin t\, dt$$

Sign change	Differentiate	$\int$(integrate)
+	e^{-t}	$\sin t$
-	$-e^{-t}$	$-\cos t$
+	e^{-t}	$-\sin t$

$$\int e^{-t}\sin t = -e^{-t}\cos t - e^{-t}\sin t - \int e^{-t}\sin t\, dt$$

$$2\int e^{-t}\sin t\, dt = -e^{-t}(\cos t + \sin t)$$

$$\int e^{-t}\sin t\, dt = \frac{1}{2} - e^{-t}(\cos t + \sin t) + C$$

$$x = -\frac{1}{2}e^{-t}(\cos t + \sin t) + c$$

(*e*) $e^x\frac{dy}{dx} = -(y^2 + 4)$

$$\frac{dy}{y^2 + 4} = \frac{-dx}{e^x}$$

$$\int\frac{1}{y^2 + 4}dy = \int -e^{-x}dx$$

But $\int\frac{1}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

$$\Rightarrow\int\frac{1}{4 + y^2}dy = \frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) + C$$

$$\Rightarrow\frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) = e^{-x} + C$$

$$\tan^{-1}\left(\frac{y}{2}\right) = 2e^{-x} + A$$

$$\frac{y}{2} = \tan(A + 2e^{-x})$$

$$y = 2\tan(A + 2e^{-x})$$

Example III

Find the particular solution of the following differential equation which satisfy the given condition.

$$(a)\ (1 + \cos 2\theta)\frac{dy}{d\theta} = 2, \qquad y\left(\frac{\pi}{4}\right) = 1$$

$$(b)\frac{dy}{dx} = x(y - 2)\ when\ x = 0, y = 5$$

$(c)\ (1+x^2)\dfrac{dy}{dx} = 1+y^2 \quad y(2) = 3$

$(d)\ \dfrac{dy}{dx} = \sqrt{1-y^2} \quad , \qquad y\left(\dfrac{\pi}{6}\right) = 0$

Solution

$$(1+\cos 2\theta)\frac{dy}{d\theta} = 2$$

$$dy = \frac{2}{1+\cos 2\theta}d\theta$$

$$dy = \frac{2}{1+2\cos^2\theta - 1}d\theta$$

$$dy = \frac{2}{2\cos^2\theta}d\theta$$

$$dy = \sec^2\theta\, d\theta$$

$$y = \tan\theta + C$$

$$\theta = \frac{\pi}{4}, y = 1$$

$$\Rightarrow 1 = \tan\frac{\pi}{4} + C$$

$$C = 0$$

$$y = \tan\theta$$

(b) $\frac{dy}{dx} = x(y-2)$

$$\int \frac{dy}{y-2} = \int x\, dx$$

$$\ln(y-2) = \frac{x^2}{2} + C$$

$$when\ x = 0, y = 5$$

$$\ln 3 = 0 + C$$

$$\ln(y-2) = \frac{x^2}{2} + \ln 3$$

$$\ln\left(\frac{y-2}{3}\right) = \frac{x^2}{2}$$

$$e^{\frac{x^2}{2}} = \frac{y-2}{3}$$

$$y = 2 + 3e^{\frac{x^2}{2}}$$

(c) $(1+x^2)\dfrac{dy}{dx} = 1+y^2$

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$$

$$\tan^{-1} y = \tan^{-1} x + C$$

$$\tan^{-1}(3) = \tan^{-1}(2) + C$$

$y(2) = 3$

$$C = \tan^{-1} 3 - \tan^{-1} 2$$

Let $\tan^{-1}(3) = A$

$\tan^{-1}2 = B$

$C = A - B$

$\tan C = \tan(A - B)$

$$\tan C = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\tan C = \frac{3-2}{1+3\times 2}$$

$$\tan C = \frac{1}{7}$$

$$C = \tan^{-1}\left(\frac{1}{7}\right)$$

$$\tan^{-1} y = \tan^{-1} x + \tan^{-1}\frac{1}{7}$$

$$\tan^{-1} y - \tan^{-1} x = \tan^{-1}\frac{1}{7}$$

$$\frac{y-x}{1+xy} = \frac{1}{7}$$

$$7y - 7x = 1 + xy$$

$$7y - xy = 7x + 1$$

$$y(7-x) = 7x + 1$$

$$y = \frac{7x+1}{7-x}$$

$$\frac{dy}{dx} = \sqrt{1-y^2}$$

$$\frac{dy}{\sqrt{1-y^2}} = dx$$

$$\sin^{-1} y = x + C$$

$$y\left(\frac{\pi}{6}\right) = 0$$

$$\sin^{-1} 0 = \frac{\pi}{6} + C$$

$$-\frac{\pi}{6} = C$$

$$\sin^{-1} y = x - \frac{\pi}{6}$$

$$y = \sin\left(x - \frac{\pi}{6}\right)$$

Example IV (UNEB Question)

Solve the differential equation $\dfrac{dy}{dx} = y\tan 2x$; $y(0) = 2$

Solution

$$\frac{dy}{y} = \tan 2x\,dx$$
$$\int \frac{dy}{y} = \int \tan 2x\,dx$$
$$\int \frac{dy}{y} = \int \frac{\sin 2x}{\cos 2x}dx$$
$$\ln y = -\frac{1}{2}\ln\cos 2x + C$$
$$x = 0, \qquad y = 2$$
$$\ln 2 = -\frac{1}{2}\ln\cos 0 + C$$
$$(\ln 2) = -`\frac{1}{2}\ln 1 + C$$
$$C = (\ln 2)$$
$$\ln y = -\frac{1}{2}\ln\cos 2x + (\ln 2)$$
$$\ln y - \ln 2 = -\frac{1}{2}\ln\cos 2x$$
$$\ln\left(\frac{y}{2}\right) = -\frac{1}{2}\cos 2x$$
$$\ln\frac{y}{2} = \left(\frac{1}{\sqrt{\cos 2x}}\right)$$
$$\frac{y}{2} = \frac{1}{\sqrt{\cos 2x}}$$
$$y = \frac{2}{\sqrt{\cos 2x}}$$

Exact Differential Equations

A differential equation is said to be exact if and only if the left hand side can be expressed as an exact differential.

Example I

Solve the differential equation $x^2\frac{dy}{dx} + 2xy = 1$

Solution

$$x^2\frac{dy}{dx} + 2xy = 1$$
$$\frac{d}{dx}(x^2y) = 1$$
$$\int d(x^2y) = \int dx$$
$$x^2y = x + c$$

Example II

Solve the differential equation

$$(1-2x)e^y\frac{dy}{dx} - 2e^y = \sec^2 x$$

Solution

$$(1-2x)e^y\frac{dy}{dx} - 2e^y = \sec^2 x$$
$$\frac{d}{dx}(1-2x)e^y = \sec^2 x$$
$$d(1-2x)e^y = \sec^2 x\,dx$$
$$\int d(1-2x)e^y = \int \sec^2 x\,dx$$
$$(1-2x)e^y = \tan x + C$$

Example III

Solve the differential equation

$$x^2\cos y\frac{dy}{dx} + 2x\sin y = \frac{1}{x}$$

Solution

$$x^2\cos y\frac{dy}{dx} + 2x\sin y = \frac{1}{x}$$
$$\frac{d}{dx}(x^2\sin y) = \frac{1}{x}$$
$$\frac{d}{dx}(x^2\sin y) = \frac{1}{x}$$
$$\int d(x^2\sin y) = \int \frac{1}{x}dx$$
$$x^2\sin y = \ln x + C$$

Example 4

Solve the following exact differential equations

$$(a)\ \ln y + \frac{x}{y}\frac{dy}{dx} = \sec x\tan x$$
$$(b)\ \ xy^2 + x^2y\frac{dy}{dx} = \sec^2 x$$
$$(c)\ e^ur^2 + 2re^u\frac{dr}{du} = -cosec^2u$$

Solution

$$(a)\ \ln y + \frac{x}{y}\frac{dy}{dx} = \sec x\tan x$$
$$\frac{d}{dx}((\ln y)x) = \sec x\tan x$$
$$d(x(\ln y)) = \sec x\tan x\,dx$$
$$x(lny) = \sec x + C$$

$$(b)\ xy^2 + x^2y\frac{dy}{dx} = \sec^2 x$$
$$\frac{d}{dx}\left(\frac{x^2y^2}{2}\right) = \sec^2 x$$

$$d\left(\frac{x^2y^2}{2}\right) = \sec^2 x\,dx$$

$$\int d\left(\frac{x^2y^2}{2}\right) = \int \sec^2 x\,dx$$

$$\frac{x^2y^2}{2} = \tan x + C$$

(c) $e^u r^2 + 2re^u \frac{dr}{du} = -cosec^2 u$

$$\frac{d}{du}(e^u r^2) = -cosec^2 u$$

$$d(e^u r^2) = -cosec^2 u\,du$$

$$\int d(e^u r^2) = \int -\operatorname{cosec}^2 u\,du$$

$$e^u r^2 = \cot x + c$$

Example 5

Solve the differential equation

$$e^x y^2 + 2ye^x \frac{dy}{dx} = -cosec^2 x$$

Solution

$$e^x y^2 + 2ye^x \frac{dy}{dx} = -cosec^2 x$$

$$\frac{d}{dx}(e^x y^2) = -cosec^2 x$$

$$\int d(e^x y^2) = \int -cosec^2 x\,dx$$

$$e^x y^2 = \cot x + c$$

First order linear differential equation

A first order linear differential equation is a differential equation of the form

$$\frac{dy}{dx} + p(x)y = Q(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

$where\ p(x)\ and\ q(x) are\ functions\ of\ x$

Solution

Process of solving first order linear differential equation

(1) Put the differential equation in the standard linear form of a differential equation

$$\frac{dy}{dx} + p(x)y = Q(x)\ldots\ldots. \text{(i)}$$

(2) Find the integrating factor

$$I = e^{\int P(x)dx}$$

(3) Multiply both sides of equation (i) by the integrating factor

4. Integrate both sides and makes sure that you properly deal with the constant of integration

Example I

Solve the following linear differential equation

$$(a) x\frac{dy}{dx} - 2y = 0$$

$$(b) dy + (2x+1)y = e^{-x^2}$$

$$(c)\ x\frac{dy}{dx} + y = x^3$$

Solution

(a) $x\frac{dy}{dx} - 2y = 0$

$$\frac{dy}{dx} - \frac{2y}{x} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$I = e^{\int -\frac{2}{x}dx}$$

$$I = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2} = \frac{1}{x^2}$$

Multiplying both sides of equation (1) by $1/x^2$

We have:

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2y}{x^3} = 0$$

$$\frac{d}{dx}\left(\frac{1}{x^2}y\right) = 0$$

$$\int d\left(\frac{1}{x^2}y\right) = \int 0\,dx$$

$$\frac{1}{x^2}y = 0 + C$$

$$y = Cx^2$$

$(b) \frac{dy}{dx} + (2x+1)y = e^{-x^2}$

Comparing $\frac{dy}{dx} + (2x+1)y = e^{-x^2}$ with $\frac{dy}{dx} + p(x)y = Q(x)$ gives

$P(x) = 2x + 1$, $Q(x) = e^{-x^2}$

$$I = e^{\int 2x+1dx}$$

$$I = e^{x^2+x}$$

$$e^{x^2+x}\frac{dy}{dx} + (2x+1)e^{x^2+x}y = e^x$$
$$\frac{d}{dx}\left(e^{x^2+x}y\right) = e^x dx$$
$$e^{x^2+x}y = e^x + C$$

(c) $x\frac{dy}{dx} + y = x^3$

$$\frac{dy}{dx} + \frac{y}{x} = x^2 \dots\dots\dots\dots\dots\dots.(1)$$

Comparing $\frac{dy}{dx} + P(x)y = Q(x)$ with Eqn (1);

$$P(x) = \frac{1}{x}$$
$$I = e^{\int\frac{1}{x}dx}$$
$$I = e^{\ln x} = x$$

(d) $x\frac{dy}{dx} + y = x^3$

$$\frac{d}{dx}(xy) = x^3$$
$$\int d(xy) = \int x^3 dx$$
$$xy = \frac{x^4}{4} + A$$

Example II

Solve the following linear differential equation

$(a)\frac{dr}{d\theta} + 2r\cot\theta = cosec^2\theta$

$(b)3y + (x-2)\frac{dy}{dx} = \frac{2}{x-2}$

$(c)\ x\frac{dy}{dx} + 2y = \frac{\cos x}{x}$

$(d)\sin x\frac{dy}{dx} + y = \sin^2 x$

Solution

$(a)\frac{dr}{d\theta} + 2r\cot\theta = cosec^2\theta$ ……… (i)

$$\frac{dr}{d\theta} + P(\theta)r = Q(\theta)$$
$$P(\theta) = 2\cot\theta$$
$$I = e^{\int 2\cot\theta d\theta}$$
$$I = e^{\int\frac{2\cos\theta}{\sin\theta}d\theta}$$
$$I = e^{2\ln\sin\theta}$$
$$I = e^{\ln\sin^2\theta} = \sin^2\theta$$
$$\frac{dr}{d\theta} + 2r\cot\theta = cosec^2\theta$$
$$\Rightarrow \frac{dr}{d\theta} + \frac{2r\cos\theta}{\sin\theta} = \frac{1}{\sin^2\theta}$$

Multiplying both sides by the integrating factor we have,

$$\sin^2\theta\frac{dr}{d\theta} + 2r\sin\theta\cos\theta = 1$$
$$\frac{d}{d\theta}(\sin^2\theta\, r) = 1$$
$$d(\sin^2\theta\, r) = d\theta$$
$$r\sin^2\theta = \theta + C$$

$(b)3y + (x-2)\frac{dy}{dx} = \frac{2}{x-2}$

$$\frac{dy}{dx} + \frac{3y}{x-2} = \frac{2}{(x-2)^2} \dots\dots\dots\dots\dots..(1)$$
$$\frac{dy}{dx} + P(x)y = Q(x)$$
$$I = e^{\int P(x)dx}$$
$$I = e^{\int\frac{3}{x-2}dx}$$
$$I = e^{\ln(x-2)^3}$$
$$I = (x-2)^3$$

Multiply both sides of equation (1) by the integrating factor

$$(x-2)^3\frac{dy}{dx} + 3(x-2)^2y = 2(x-2)$$
$$\frac{d}{dx}(x-2)^3y = 2(x-2)$$
$$d(x-2)^3y = 2(x-2)dx$$
$$\int(x-2)^3y = \int 2(x-2)\,dx$$
$$(x-2)^3y = x^2 - 4x + C$$

(b) $x\frac{dy}{dx} + 2y = \frac{\cos x}{x}$

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{\cos x}{x^2} \dots\dots\dots\dots\dots\dots\dots\dots(1)$$
$$I = e^{\int\frac{2}{x}dx}$$
$$I = e^{2\ln x} = e^{\ln x^2} = x^2$$

Multiplying both sides of Eqn (1) by the integrating factor, we have

$$x^2\frac{dy}{dx}+2xy=\cos x$$

$$\int d\,(x^2y)=\int \cos x\,dx$$

$$x^2y=\sin x+C$$

(d) $\sin x\frac{dy}{dx}+y=\sin^2 x$

$$\frac{dy}{dx}+\frac{y}{\sin x}=\sin x$$

$$I=e^{\int\frac{1}{\sin x}dx}$$

Consider

$$\int\frac{1}{\sin x}dt$$

$$let\ t=\tan\frac{x}{2}$$

$$dt=\frac{1}{2}\sec^2\frac{x}{2}dx$$

$$dx=\frac{2dt}{\sec^2\frac{x}{2}}$$

$$dx=\frac{2dt}{1+t^2}$$

$$\int\frac{1}{\sin x}dx=\int\frac{(1+t^2)}{2t}.\frac{2dt}{1+t^2}$$

$$=\int\frac{1}{t}dt$$

$$\ln\tan\frac{x}{2}$$

$$I=e^{\ln\tan\frac{x}{2}}=\tan\frac{x}{2}$$

Multiplying both sides of Eqn (i) by the integrating factor, we have

$$\tan\frac{x}{2}\frac{dy}{dx}+\frac{\tan\frac{x}{2}}{\sin x}y=\sin x\tan\frac{x}{2}$$

$$\tan\tfrac{x}{2}\frac{dy}{dx}+\frac{\tan\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}y=2\sin\tfrac{x}{2}\cos\tfrac{x}{2}\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$$

$$\tan\frac{x}{2}\frac{dy}{dx}+\frac{1}{2}\sec^2\frac{x}{2}y=2\sin^2\frac{x}{2}$$

$$\frac{d}{dx}\left(\tan\frac{x}{2}y\right)=2\sin^2\frac{x}{2}$$

$$\int d\left(\tan\frac{x}{2}y\right)=\int 2\sin^2\frac{x}{2}dx$$

But $\cos x=1-2\sin^2\frac{x}{2}$

$$2\sin^2\tfrac{x}{2}=1-\cos x$$

$$\int d(\tan\frac{x}{2}y)=1-\cos x$$

$$\tan\tfrac{x}{2}y=x-\sin x+C$$

Example II

Find the particular solution of the following differential equations which satisfy the given condition

(a) $(x+1)\frac{dy}{dx}-3y=(x+1)^4$

$y=16\ \ when\ x=1$

(b) $\frac{du}{d\theta}+u\cot\theta=2\cos\theta$

$u\left(\frac{\pi}{2}\right)=3$

Solution

$$(x+1)\frac{dy}{dx}-3y=(x+1)^4$$

$$\frac{dy}{dx}-\frac{3y}{x+1}=(x+1)^3\ \text{………… (i)}$$

$$\frac{dy}{dx}+P(x)y=Q(x)$$

$$I=e^{\int P(x)dx}$$

$$I=e^{\int-\frac{3}{x+1}dx}$$

$$I=e^{-3\ln(x+1)}=e^{\ln(x+1)^{-3}}$$

$$I=\frac{1}{(x+1)^3}$$

Multiplying both sides of Eqn (i) by the integrating factor, we have

$$\frac{1}{(x+1)^3}\frac{dy}{dx}-\frac{3y}{(x+1)^4}=1$$

$$\frac{d}{dx}\left(\frac{1}{(x+1)^3}y\right)=1$$

$$\int d\left(\frac{1}{(x+1)^3}y\right)=\int dx$$

$$\frac{1}{(x+1)^3}y=x+C$$

$$y=(x+1)^3(x+C)$$

If $x=1$, $y=16$

$$16=(2)^3(1+C)$$

$$C=1$$

$$y=(x+1)^3(x+1)$$

(b) $\frac{du}{d\theta} + u\cot\theta = 2\cos\theta$

$u\left(\frac{\pi}{2}\right) = 3$

$I = e^{\int \frac{\cos\theta}{\sin\theta} d\theta}$

$I = e^{\ln\sin\theta} = \sin\theta$

$$\Rightarrow \sin\theta \frac{du}{d\theta} + \frac{u\cos\theta}{\sin\theta}\sin\theta = 2\sin\theta\cos\theta$$

$$\sin\theta \frac{du}{d\theta} + u\cos\theta = \sin 2\theta$$

$$\frac{d}{d\theta}(\sin\theta\ u) = \sin 2\theta$$

$$\int d(\sin\theta\, u) = \int \sin 2\theta\, d\theta$$

$$(\sin\theta)\, u = -\frac{1}{2}\cos 2\theta + C$$

$$u(\tfrac{\pi}{2}) = 3$$

$$\theta = \frac{\pi}{2},\ u = 3$$

$$(\sin\tfrac{\theta}{2})(3) = \frac{-1}{2}\cos\pi + c$$

$$3 = \frac{1}{2} + C$$

$$C = \frac{5}{2}$$

$$(\sin\theta)\, u = -\frac{1}{2}\cos 2\theta + \frac{5}{2}$$

$$(\sin\theta)\, u = \frac{1}{2}(5 - \cos 2\theta)$$

$$u = \frac{1}{2}\operatorname{cosec}\theta\ (2 - \cos 2\theta)$$

Example III (UNEB Question)

Solve $\frac{dy}{dx} + 2y\tan x = \cos^2 x$

$y(0) = 2$

Solution

$\frac{dy}{dx} + 2y\tan x = \cos^2 x$ ……………… (i)

$I = e^{\int \tan x\, dx}$

$I = e^{\int \frac{2\sin x}{\cos x} dx}$

$I = e^{-2\ln\cos x}$

$I = e^{\ln(\cos x)^{-2}} = \frac{1}{\cos^2 x}$

Multiplying both sides of Eqn (i) by the integrating factor, we have

$$sec^2 x\frac{dy}{dx} + 2(sec^2 x\tan x)y = 1$$

$$\frac{d}{dx}(sec^2 x\ y) = 1$$

$$d(\sec^2 x\ \ y) = dx$$

$$\sec^2 x\ \ y = x + C$$

$$If\ x = 0\ \ y = 2$$

$$(\sec^2 0)\ (2) = 0 + C$$

$$2 = C$$

$$C = 2$$

$$(\sec^2 x)y = x + 2$$

$$y = \frac{x+2}{sec^2 x}$$

$$y = (x+2)\cos^2 x$$

Example IV (UNEB Question)

Solve the differential equation

$\frac{dy}{dx} + (y+3)\cot x = e^{-2x} cosec\ x$

Solution

$$\frac{dy}{dx} + (y+3)\cot x = -e^{2x} cosec\ x$$

$$\frac{dy}{dx} + (\cot x)\, y + 3\cot x = -e^{2x} cosec\ x$$

$$\frac{dy}{dx} + (\cot x)y = -e^{2x} cosec\ x - 3\cot x$$

$$I = e^{\cot x dx}$$

$$I = e^{\int \frac{\cos x}{\sin x} dx}$$

$$I = e^{\ln\sin x} = \sin x$$

$$\Rightarrow (\sin x)\frac{dy}{dx} + (\cos x)y = -e^{2x} - 3\cos x$$

$$\frac{d}{dx}(\sin x\ \ y) = (-e^{2x} - 3\cos x)$$

$$\int d(\sin x\, y) = \int (-e^{2x} - 3\cos x)dx$$

$$(\sin x)y = -\frac{1}{2}e^{2x} - 2\sin x + C$$

Example

A curve passes through the point (1, 0). The gradient of the normal to the curve at any point (x, y) is

$\sec y - x\tan y$. Find the equation of the curve.

Solution

$\frac{dy}{dx} = gradient\ of\ the\ tangent$

Gradient of the normal $n = -\frac{dx}{dy}$

$$-\frac{dx}{dy} = \sec y - x\tan y$$

$$\frac{dx}{dy} = -\sec y + x\tan y$$

$$\frac{dx}{dy} - x\tan y = -\sec y$$

$$I = e^{-\int \tan y dy}$$

$$I = e^{-\int \frac{\sin y}{\cos y} dy}$$

$$I = e^{\ln(\cos y)}$$

$$I = e^{\ln(\cos y)} \quad = \cos y$$

$\frac{dx}{dy} - x\tan y = -\sec^2 y$ .... (i)

Multiplying both sides of Eqn (i) by the integrating factor

$$\Rightarrow \cos y\frac{dy}{dx} - x\sin y = -1$$

$$\frac{d}{dy}(\cos yx) = -1$$

$$\int d(\cos y)\, x = \int -1 dy$$

$$x\cos y = -y + C$$

The curve passes through (1, 0)

$$\Rightarrow x = 1, y = 0$$

$$(1)\cos 0 = -0 + C$$

$$1 = C$$

$(\cos y)x = -y + 1$ is the equation of the curve.

First order homogeneous equations

The function $f(x,y) = x^3 - 3x^2y - 5xy^2 + 2y^3$ is a function which is such that each of its terms is degree 3. Such a function is said to be homogeneous function of degree 3.

In general a homogeneous function $f(x, y)$ is said to be a homogenous of degree n if $f(tx, ty) = t^n f(x, y)$.

Once we have verified that the differential equation is homogenous. We use the substitutions

$$\boxed{y = vx}$$

$$\boxed{\frac{dy}{dx} = V + x\frac{dv}{dx}}$$

Example I

Solve the following homogeneous equations.

a) $x^2\frac{dy}{dx} = 3x^2 + xy$

b) $xy\frac{dy}{dx} = x^2 - y^2$

c) $x^2\frac{dy}{dx} = x^2 + xy + y^2$

d) $3x^2\frac{dy}{dx} = y^2$

Solution

$$x^2\frac{dy}{dx} = 3x^2 + xy$$

$$let\ y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow x^2\left(v + x\frac{dv}{dx}\right) = 3x^2 + x(vx)$$

$$x^2\left(v + x\frac{dv}{dx}\right) = x^2(3 + v).$$

$$v + x\frac{dv}{dx} = 3 + v$$

$$x\frac{dv}{dx} = 3.$$

$$xdv = 3dx$$

$$dv = \frac{3}{x}dx$$

$v = 3\ln x + c$

$v = \ln x^3 + c$

Let $c = \ln k$

$v = \ln x^3 + \ln k$

$$\frac{y}{x} = (\ln x^3 k)$$

$$y = x(\ln x^3 k).$$

(b) $xy\frac{dy}{dx} = x^2 - y^2.$

$$y = vx, \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$x(vx)\left(v + x\frac{dv}{dx}\right) = x^2 - v^2x^2.$$

$$x^2(v)\left(v+x\frac{dv}{dx}\right)=x^2(1-v^2).$$

$$v^2+vx\frac{dv}{dx}=1-v^2.$$

$$vx\frac{dv}{dx}=1-2v^2$$

$$\int\frac{v}{1-2v^2}dv=\int\frac{dx}{x}.$$

$$-\frac{1}{4}\ln(1-2v^2)=\ln x+C.$$

$$-\frac{1}{4}\ln(1-2v^2)=\ln(xk).$$

$$\ln(1-2v^2)^{-\frac{1}{4}}=\ln xk.$$

$$\frac{1}{(1-2v^2)^{\frac{1}{4}}}=xk.$$

$$\frac{1}{1-2v^2}=x^4A.$$

But $v=\frac{y}{x}$.

$$\frac{1}{1-\frac{2y^2}{x^2}}=x^4A.$$

$$\Rightarrow\frac{1}{\frac{x^2-2y^2}{x^2}}=x^4A.$$

$$\frac{x^2}{x^2-2y^2}=x^4A.$$

$$\frac{1}{A}=x^2(x^2-2y^2)$$

$$B=x^2(x^2-2y^2).$$

where $\frac{1}{A}=B$

(c) $x^2\frac{dy}{dx}=x^2+xy+y^2.$

$$\frac{dy}{dx}=v+x\frac{dv}{dx},y=vx$$

$$x^2\left(v+x\frac{dv}{dx}\right)=x^2+vx^2+v^2x^2$$

$$x^2\left(v+x\frac{dv}{dx}\right)=(1+v+v^2)x^2.$$

$$x\frac{dv}{dx}=1+v^2.$$

$$\int\frac{dv}{1+v^2}=\int\frac{dx}{x}$$

$$\tan^{-1}v=\ln x+C$$

$$\tan^{-1}\left(\frac{y}{x}\right)=\ln(xk)$$

$$\frac{y}{x}=\tan(\ln(xk))$$

$$y=x\tan(\ln xk).$$

(b) $3x^2\frac{dy}{dx}=y^2.$

$$y=Vx$$

$$\frac{dy}{dx}=v+x\frac{dv}{dx}.$$

$$3x^2\left(v+x\frac{dv}{dx}\right)=v^2x^2$$

$$3v+3x\frac{dv}{dx}=v^2$$

$$3x\frac{dv}{dx}=v^2-3v.$$

$$\frac{3dv}{v^2-3v}=\frac{dx}{x}$$

Consider.

$$\frac{3}{v^2-3v}=\frac{3}{v(v-3)}$$

$$\frac{3}{v(v-3)}=\frac{A}{v}+\frac{B}{v-3}$$

$$A(v-3)+Bv=3.$$

If $v=3$, $B=1$

If $v=0$, $-3A=3$

$$A=-1.$$

$$\frac{3}{v(v-3)}=\frac{-1}{v}+\frac{1}{v-3}$$

$$\int\frac{3dv}{v^2-3v}=\ln v+\ln(v-3)\quad=\ln\left(\frac{v-3}{v}\right)$$

$$\Rightarrow\ln\left(\frac{v-3}{v}\right)=\ln xk.$$

$$\frac{v-3}{v}=xk.$$

$$\frac{\frac{y}{x}-3}{\frac{y}{x}}=xk$$

$$\frac{\frac{y-3x}{x}}{\frac{y}{x}}=xk$$

$$\frac{y-3x}{y}=xk.$$

$$y=yxk+3x$$

$$y = \frac{3x}{1 - xk}.$$

Example II

Solve the following differential equations.

$a)\ 4xy\dfrac{dy}{dx} = 3x^2 + y^2$

$b)\ x\dfrac{dy}{dx} - y = \sqrt{x^2 - y^2}$

$c)\ \dfrac{dy}{dx} = \dfrac{y}{x} + \tan\dfrac{y}{x}.$

Solution

$(a)\ 4xy\dfrac{dy}{dx} = 3x^2 + y^2.$

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$4xy\frac{dy}{dx} = 3x^2 + y^2$$

$$\Rightarrow (4x)(vx)\left(v + x\tfrac{dv}{dx}\right) = 3x^2 + v^2x^2.$$

$$x^2(4v)\left(v + x\frac{dv}{dx}\right) = x^2(3 + v^2)$$

$$4v^2 + 4vx\frac{dv}{dx} = 3 + v^2$$

$$4vx\frac{dv}{dx} = 3 - 3v^2$$

$$\frac{4v}{3(1 - v^2)} = \frac{dx}{x}$$

$$\frac{4vdv}{3 - 3v^2} = \ln x + C$$

$$-\frac{2}{3}\ln(3 - 3v^2) = \ln xk$$

$$\ln\left(\frac{1}{(3 - 3v^2)^{\frac{2}{3}}}\right) = \ln xk$$

$$\frac{1}{\left(3 - \frac{3y^2}{x^2}\right)^{\frac{2}{3}}} = xk$$

$$\frac{1}{\left(\frac{3x^2 - 3y^2}{x^2}\right)^2} = x^3k^3$$

$$\frac{1}{\frac{9(x^2 - y^2)^2}{x^4}} = x^3A$$

$$x = A\big(9(x^2 - y^2)\big)^2$$

(b) $x\frac{dy}{dx} - y = \sqrt{x^2 - y^2}$

$$y = vx.$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$x\left(v + x\frac{dv}{dx}\right) - vx = \sqrt{x^2 - v^2x^2}.$$

$$x\left(v + x\frac{dv}{dx} - v\right) = x\sqrt{1 - v^2}$$

$$x\frac{dv}{dx} = \sqrt{1 - v^2}$$

$$\frac{dv}{\sqrt{1 - v^2}} = \frac{dx}{x}$$

$$\sin^{-1} v = Inx + A$$

$$\sin^{-1}\left(\frac{y}{x}\right) = (\ln(xk))$$

$$\frac{y}{x} = \sin(\ln(xk)).$$

$$y = x\sin(\ln xk).$$

(c) $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$

$$y = vx, \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\left(v + x\frac{dv}{dx}\right) = v + \tan v.$$

$$x\frac{dv}{dx} = \tan v$$

$$\frac{dv}{\tan v} = \frac{dx}{x}$$

$$\int\frac{\cos v}{\sin v}dv = \int\frac{1}{x}dx$$

$$\ln(\sin v) = \ln x + k.$$

$$\ln\sin v = \ln xk.$$

$$\sin v = xk$$

$$\sin\left(\frac{y}{x}\right) = xk$$

$$\frac{y}{x} = \sin^{-1}(xk)$$

$$y = x\sin^{-1}(xk)$$

Solve the following differential equations.

a) $a)\ xy\frac{dy}{dx} + 4x^2 + y^2 = 0\ ,\ y(7) = -7$

b) $b)\ (x^2 - y^2)\ \frac{dy}{dx} = xy\ ,\quad y(4) = 2$

c) $c)\ (x + y)\frac{dy}{dx} = x - y,\quad y(3) = -2.$

Solution

$$(a)\ xy\frac{dy}{dx} + 4x^2 + y^2 = 0.$$

$$y = vx,\qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$x(vx)\left(v + x\frac{dv}{dx}\right) + 4x^2 + v^2x^2 = 0$$

$$x^2(v)\left(v + x\frac{dv}{dx}\right) = x^2(-4 - v^2)$$

$$v^2 + vx\frac{dv}{dx} = -4 - v^2.$$

$$vx\frac{dv}{dx} = -4 - 2v^2.$$

$$\frac{vdv}{4 + 2v^2} = \frac{-dx}{x}$$

$$\frac{1}{4}\ln(4 + 2v^2) = -\ln x + C$$

$$\frac{1}{4}\ln\left(4 + \frac{2y^2}{x^2}\right) = -\ln x + C.$$

$$\ln\left(\frac{4x^2 + 2y^2}{x^2}\right)^{\frac{1}{4}} = \ln\frac{1}{x} + C$$

$$\ln\left(\frac{4x^2 + 2y^2}{x^2}\right)^{\frac{1}{4}} = \ln\frac{1}{x}A.$$

$$\frac{4x^2 + 2y^2}{x^2} = \frac{1}{x^4}k.$$

$$4x^2 + 2y^2 = \frac{k}{x^2}$$

When $x = 7$, $y = -7$

$$4(7^2) + 2(-7)^2 = \frac{k}{7^2}$$

$$294 = \frac{k}{7^2}$$

$k = 294 \times 49$

$k = 14406$

$$4x^2 + 2y^2 = \frac{14406}{x^2}.$$

(b) $(x^2 - y^2)\frac{dy}{dx} = xy$

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$(x^2 - v^2x^2)\left(v + x\frac{dv}{dx}\right) = x(vx)$$

$$(1 - v^2)\left(v + x\frac{dv}{dx}\right) = v$$

$$v + x\frac{dv}{dx} = \frac{v}{1 - v^2}$$

$$x\frac{dv}{dx} = \frac{v}{1 - v^2} - v.$$

$$x\frac{dv}{dx} = \frac{v - v + v^3}{1 - v^2}$$

$$\frac{(1 - v^2)}{v^3}dv = \frac{dx}{x}$$

$$\int \frac{1}{v^3} - \frac{1}{v}dv = \ln x + C$$

$$\frac{-1}{2}v^{-2} - \ln v = \ln x + c$$

$$-\frac{1}{2}\left(\frac{x^2}{y^2}\right) - \ln\frac{y}{x} = \ln x + C$$

$y(4) = 2.$

$x = 4, y = 2$

$$-\frac{1}{2}\left(\frac{16}{4}\right) - \ln\left(\frac{2}{4}\right) = \ln 4 + C$$

$$-2 - \ln\left(\frac{1}{2}\right) = \ln 4 + C$$

$$-2 - \ln\frac{1}{2} - \ln 4 = C.$$

$$-2 - \left(\ln\frac{1}{2} + \ln 4\right) = C.$$

$$-2 - \ln 2 = C.$$

$$C = -2 - \ln 2.$$

$$-\frac{1}{2}\left(\frac{x^2}{y^2}\right) - \ln\frac{y}{x} = \ln x - \ln 2 - 2$$

$$2 - \frac{1}{2}\left(\frac{x^2}{y^2}\right) = \ln\frac{y}{x} + \ln(x) - \ln 2$$

$$\frac{4y^2 - x^2}{2y^2} = \ln\left(\frac{y}{x}\right)x - \ln(2)$$

$$\frac{4y^2 - x^2}{2y^2} = \ln\left(\frac{y}{2}\right)$$

$$4y^2 - x^2 = 2y^2 \ln\left(\frac{y}{2}\right)$$

(c) $(x+y)\frac{dy}{dx} = x - y, \quad y(3) = -2.$

$$y = vx, \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

$$(x + vx)\left(v + x\frac{dv}{dx}\right) = x - vx.$$

$$x(1+v)\left(v + x\frac{dv}{dx}\right) = x\,(1 - v).$$

$$v + x\frac{dv}{dx} = \frac{1-v}{1+v}$$

$$x\frac{dv}{dx} = \frac{1-v}{1+v} - v$$

$$x\frac{dv}{dx} = \frac{1 - v - v - v^2}{1+v}$$

$$x\frac{dv}{dx} = \frac{1 - 2v - v^2}{1+v}$$

$$\int \frac{-(1+v)}{v^2 + 2v - 1} = \int \frac{dx}{x}$$

$$-\frac{1}{2}\ln(v^2 + 2v - 1) = \ln x + C$$

$$-\frac{1}{2}\left(\ln\left(\frac{y^2}{x^2} + \frac{2y}{x} - 1\right)\right) = \ln x + C.$$

$$-\frac{1}{2}\ln\left(\frac{y^2 + 2xy - x^2}{x^2}\right) = \ln x + c$$

$$\ln\frac{x}{\sqrt{y^2 + 2xy - x^2}} = (\ln x) + C$$

$\ln\frac{x}{\sqrt{y^2 - 2xy - x^2}} = \ln(A\,x)$ For $C = \ln A$,

$$\frac{x}{\sqrt{y^2 - 2xy - x^2}} = Ax$$

$$\frac{1}{\sqrt{y^2 - 2xy - x^2}} = A$$

$$\frac{1}{A} = \sqrt{y^2 - 2xy - x^2}.$$

$$k = \sqrt{y^2 - 2xy - x^2}.$$

When $x = 3$

$$y = -2.$$

$$K = \sqrt{4 - 2(-6) - 9}$$

$$K = \sqrt{16 - 9}$$

$$K = \sqrt{7}$$

$$\sqrt{7} = \sqrt{y^2 - 2xy - x^2}$$

$$7 = y^2 - 2xy - x^2$$

Example IV

Solve the following differential equations.

$a)\ \frac{dy}{dx} = \frac{2x + y - 2}{2x + y + 1}$

$b)\ (x+y)\frac{dy}{dx} = x + y - 2$

$c)\ \frac{dy}{dx} = \frac{x - y + 1}{x - y + 3.}$

Solution

(a) $\frac{dy}{dx} = \frac{2x + y - 2}{2x + y + 1}$

Let $2x + y = z$

$$2dx + dy = dz$$

$$2 + \frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dy}{dx} = \left(\frac{dz}{dx} - 2\right)$$

$$\Rightarrow \frac{dz}{dx} - 2 = \frac{z-2}{z+1}$$

$$\frac{dz}{dx} = \frac{z-2}{z+1} + 2$$

$$\frac{dz}{dx} = \frac{z - 2 + 2(z+1)}{z+1}$$

$$\frac{dz}{dx} = \frac{3z}{z+1}$$

$$\frac{(z+1)}{z}dz = 3dx$$

$$\int\left(1 + \frac{1}{z}\right)dz = \int 3dx$$

$$z + \ln z = 3x + C.$$

$$(2x + y) + \ln(2x + y) = 3x + C.$$

(b) $(x+y)\frac{dy}{dx} = x + y - 2$

$$\frac{dy}{dx} = \frac{x + y - 2}{x + y}$$

$$let\ z = x + y.$$

$$dz = dx + dy.$$

$$\frac{dz}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx} = \left(\frac{dz}{dx} - 1\right)$$

$$\left(\frac{dz}{dx}-1\right)=\frac{z-2}{z}$$
$$\frac{dz}{dx}=\left(\frac{z-2}{z}+1\right)$$
$$\frac{dz}{dx}=\frac{z-2+z}{z}$$
$$\frac{dz}{dx}=\frac{2z-2}{z}$$
$$\frac{z}{2z-2}dz=dx$$
$$\int\frac{z}{z-1}dz=\int 2dx$$
$$\int 1+\frac{1}{z-1}dz=2x+c$$
$$z+\ln(z-1)=2x+C$$
$$x+y+\ln(x+y-1)=2x+C$$
$$\ln(x+y-1)=x-y+C$$
$$e^{x-y+C}=x+y-1$$
$$e^{x-y}.e^{c}=x+y-1$$
$$A\,e^{x-y}=x+y-1$$

(c) $\frac{dy}{dx}=\frac{x-y+1}{x-y+3}$

$$z=x-y.$$
$$dz=dx-dy.$$
$$\frac{dz}{dx}=1-\frac{dy}{dx}$$
$$\frac{dy}{dx}=1-\frac{dz}{dx}$$
$$1-\frac{dz}{dx}=\frac{z+1}{z+3}$$
$$1-\left(\frac{z+1}{z+3}\right)=\frac{dz}{dx}$$
$$\frac{(z+3)-z-1}{z+3}=\frac{dz}{dx}$$
$$\frac{2}{z+3}=\frac{dz}{dx}$$
$$2dx=(z+3)dz$$
$$2x=\frac{z^2}{2}+3z+C.$$
$$2x=\frac{(x-y)^2}{2}+3(x-y)+C$$
$$4x=x^2-2xy+y^2+6x-6y+A$$
$$K=x^2+y^2-2xy+2x-6y$$

Application of differential equations (modeling) in differential equations

We can now move into the application of differential equations both in class and in general. Almost all the differential equations that you will use in your job (for the engineers and their audience) are there because somebody, at a certain time, someone modeled a situation to come up with a differential equation that you are using. First order differential equations can be applied to real world systems.

1. Newton's law of cooling
2. Population growth and decay
3. Radioactive decay and carbon dating.
4. Mixture of two salts solution.
5. Salt solutions
6. Series circuits.
7. Draining tank
8. Economics and finance
9. Mathematics policies
10. Men and women
11. Drug distribution in human body.
12. A pursuit problem harvesting of renewable natural resources.

Radioactive decay and carbon dating

Example 1

The rate at which a radioactive nuclei decay is proportional to number of such nuclei that are present in a given sample. Half of the original number of the radioactive nuclei has undergone disintegration in the period of 1500yrs.

a) What percentage of the original radioactive nuclei will remain after 3000yrs?
b) In how many years will one tenth of the original number remain?

Solution

$$\frac{dN}{dt}\propto \text{N}$$
$$\frac{d\text{N}}{dt}=-k\text{N}.$$
$$\int\frac{d\text{N}}{\text{N}}=\int -kdt.$$
$$\ln\text{N}=-kt+C.$$
$$at\ t=0,\text{N}=\text{N}_{0.}$$
$$\ln\text{N}_0=-k(0)+C$$

$C = \ln(N_0)$

$\ln N = -kt + \ln(N_0).$

$$\ln N - \ln N_0 = -kt.$$

$$\ln\left(\frac{N}{N_0}\right) = -kt.$$

$$e^{-kt} = \frac{N}{N_0}$$

$$N = N_0 e^{-kt}$$

When $t = 1500$, $N = \frac{N_0}{2}$.

$$\frac{N_0}{2} = N_0 e^{-k(1500)}$$

$$\frac{1}{2} = e^{-1500k.}$$

$$\ln\frac{1}{2} = -1500k.$$

$$K = \frac{-1}{1500}\ln\left(\frac{1}{2}\right).$$

$$\Rightarrow N = N_0 e^{\frac{t}{1500}\ln\left(\frac{1}{2}\right)}$$

When $t = 3000$.

$$N = N_0 e^{\frac{3000}{1500}\ln\left(\frac{1}{2}\right)}$$

$$N = N_0 e^{2\ln\frac{1}{2.}}$$

$$N = N_0\, e^{\ln\left(\frac{1}{4}\right)}$$

$$N = \frac{N_0}{4}$$

$$\text{Percentage} = \frac{\frac{N_0}{4}}{N_0} \times 100 = 25\%.$$

(b) $N = N_0 e^{\frac{t}{1500}\ln\left(\frac{1}{2}\right)}$

When $N = \frac{N_o}{10}$,

$$\frac{N_o}{10} = N_0 e^{\frac{t}{1500}\ln\left(\frac{1}{2}\right)}$$

$$\frac{1}{10} = e^{\frac{t}{1500}\ln\left(\frac{1}{2}\right)}$$

$$\ln\left(\frac{1}{10}\right) = \frac{t}{1500}\ln\left(\frac{1}{2}\right)$$

$t = 4982.892$ years

Example II

A radioactive substance disintegrates at a rate proportional to its mass. One half of a given mass of a substance disintegrates in 136 days. Calculate the time required for $\frac{7}{8}$ of a substance to disintegrate. If the original mass of a substance was 100gm, calculate the mass after 34 days.

Solution

$$\frac{dm}{dt} \propto m.$$

$$\frac{dm}{dt} = -km.$$

$$\int \frac{dm}{m} = \int -kdt$$

$$\ln m = -kt + C.$$

When $t = 0$, $m = m_0$.

$\ln m_0 = -k(0) + C.$

$C = \ln m_{0.}$

$\ln m = -kt + \ln m_0$

$\ln m - \ln m_0 = -kt.$

$\ln\left(\frac{m}{m_0}\right) = -kt$

$e^{-kt} = \frac{m}{m_0}$

$m = m_0 e^{-kt}.$

When $t = 136$ days, $m = \frac{m_0}{2}$

$$\frac{m_o}{2} = m_0 e^{-k(136)}$$

$$\frac{1}{2} = e^{-136k}$$

$$\ln\tfrac{1}{2} = -136k$$

$$k = -\frac{1}{136}\ln\left(\frac{1}{2}\right)$$

$$k = -\frac{1}{136}\ln\left(\frac{1}{2}\right).$$

$$m = m_0 e^{\frac{t}{136}\ln\left(\frac{1}{2}\right)}$$

For $\frac{7}{8}m_0$ to disintegrate,

$$m = \frac{1}{8}m_{0.}$$

$$\Rightarrow \frac{1}{8}m_0 = m_0 e^{\frac{t}{136}\ln\left(\frac{1}{2}\right)}$$

$$\frac{1}{8} = e^{\frac{t}{136}\ln\left(\frac{1}{2}\right)}$$

$$\ln\frac{1}{8} = \frac{t}{136}\ln\frac{1}{2}.$$

$$t = \frac{136\left(\ln\frac{1}{8}\right)}{\left(\ln\frac{1}{2}\right)}$$

t = 408 days.

Example III

Radioactive isotope has an initial mass 200mg, two years later, its mass is 50mg. Find the expression for the amount of isotope remaining at any time.

Solution

$$\frac{dm}{dt} \propto m.$$
$$\frac{dm}{dt} = -km.$$
$$\int \frac{dm}{m} = \int -kdt.$$
$$\ln m = -kt + C.$$

When $t = 0$, $m = m_o$

$$\ln m_0 = -k(0) + C.$$
$$C = \ln(m_0).$$
$$\ln m = -kt + \ln m_0$$
$$\ln m - \ln m_0 = -kt$$
$$\ln\left(\frac{m}{m_0}\right) = -kt.$$
$$e^{-kt} = \frac{m}{m_0}$$
$$m = m_0 e^{-kt}.$$
$$m_0 = 200.$$
$$m = 200e^{-kt}.$$
$$when\ t = 2, m = 50mg.$$
$$50 = 200e^{-2k}$$
$$\frac{1}{4} = e^{-2k}$$
$$\ln\left(\frac{1}{4}\right) = -2k.$$
$$k = -\frac{1}{2}\ln\left(\frac{1}{4}\right).$$
$$m = 200e^{\frac{t}{2}\ln\left(\frac{1}{4}\right)}$$

Newton's law of cooling

It states that the rate of heat loss from the body is directly proportional to the difference between the body temperature θ and the temperature θ_0 of the surrounding air.

$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$
$$\frac{d\theta}{dt} = -k(\theta - \theta_0).$$

Example I

A hot body at temperature of 100^0C is placed in a room of temperature 20^0C. Ten minutes later, its temperature is 60. Write down a differential equation to represent the rate of change of temperature θ of a body with time t. Determine the temperature of a body after further 10 minutes.

Solution

$$\frac{d\theta}{dt} \propto (\theta - 20)$$
$$\frac{d\theta}{dt} = -k(\theta - 20)$$
$$\int \frac{d\theta}{(\theta - 20)} = \int -kdt$$
$$\ln(\theta - 20) = -kt + C.$$

When $t = 0 \quad \theta = 100.$

$$\ln(100 - 20) = C.$$
$$C = \ln 80.$$
$$\ln(\theta - 20) = -kt + \ln 80.$$
$$\ln(\theta - 20) - \ln 80 = -kt$$
$$\ln\left(\frac{\theta - 20}{80}\right) = -kt$$
$$e^{-kt} = \frac{\theta - 20}{80}$$
$$\theta = 20 + 80e^{-kt}.$$

When $t = 10, \quad \theta = 60$

$$60 = 20 + 80e^{-10k}$$
$$\frac{1}{2} = e^{-10k}$$
$$\ln\left(\frac{1}{2}\right) = -10k$$
$$k = \frac{-1}{10}\ln\left(\frac{1}{2}\right)$$
$$\theta = 20 + 80e^{\frac{t}{10}\left(\ln\frac{1}{2}\right)}$$

After further 10 minutes, (t = 10 + 10)

$$t = 20\ min$$

$\theta = 20 + 80e^{\frac{20}{10}\ln\left(\frac{1}{2}\right)}$

$\theta = 20 + 80e^{2\ln\left(\frac{1}{2}\right)}$

$\theta = 20 + 80e^{\ln\left(\frac{1}{4}\right)}$

$\theta = 20 + 20$

$\theta = 40^0C.$

Example II

At 3:00 pm, the temperature of a hot metal was 80^0C and that of the surrounding 20^0C. At 3:03 pm the temperature of the metal had dropped to 42^0C.

Write a differential to represent the rate of cooling of a metal.

Solve the differential equation using the given condition.

Find the temperature of a metal at 3:05 pm.

Solution

$\frac{d\theta}{dt} \propto (\theta - 20)$

$\frac{d\theta}{dt} = -k(\theta - 20)$

$\int \frac{d\theta}{\theta - 20} = \int -kdt.$

$\ln(\theta - 20) = -kt + C.$

When $t = 0, \quad \theta = 80$

$\ln(80 - 20) = 0 + C$

$C = \ln 60.$

$\ln(\theta - 20) = -kt + (\ln 60)$

$\ln(\theta - 20) - \ln 60 = -kt.$

$\ln\left(\frac{\theta - 20}{60}\right) = -kt$

$e^{-kt} = \frac{\theta - 20}{60}$

$\theta = 20 + 60e^{-kt}.$

When t = 3 minutes, $\theta = 42$

$42 = 20 + 60e^{-k(3)}$

$42 = 20 + 60e^{-3k}$

$22 = 60e^{-3k}.$

$\frac{22}{60} = e^{-3k}$

$\frac{11}{30} = e^{-3k}$

$\ln\left(\frac{11}{30}\right) = -3k.$

$k = -\frac{1}{3}\ln\left(\frac{11}{30}\right)$

$\theta = 20 + 60e^{\frac{t}{3}\ln\left(\frac{11}{30}\right)}$

$when\ t = 5\ minutes.$

$\theta = 20 + 60e^{\frac{5}{3}\ln\left(\frac{11}{30}\right)}$

$\theta = 31.3°$

Example III

A police patrol on Kampala Jinja road found a dead body of a man lying in the middle of the road at exactly 7:00am and its body temperature was 30^0c, 10 minutes later the police surgeon measured the body temperature and found it to be 28.5^0C. If the normal body temperature is 37^0c estimate the time at which the man was killed given the temperature of the surrounding air is 25^0c.

Solution

$\frac{d\theta}{dt} \propto (\theta - 25).$

$\frac{d\theta}{dt} = -k(\theta - 25)$

$\int \frac{d\theta}{(\theta - 25)} = \int -kdt.$

$\ln(\theta - 25) = -kt + C.$

When $t = 0 \ \ \theta = 30.$

$\ln(30 - 25) = C.$

$C = \ln 5$

$\ln(\theta - 25) = -kt + (\ln 5)$

$\ln\left(\frac{\theta - 25}{5}\right) = -kt.$

$e^{-kt} = \frac{\theta - 25}{5}$

$\theta = 25 + 5e^{-kt}.$

When $t = 10 \ \ \theta = 28.5$

$28.5 = 25 + 5e^{-10k}$

$3.5 = 5e^{-10k}$

$e^{-10k} = \frac{3.5}{5}$

$e^{-10k} = \frac{35}{50}$

$-10k = \ln\left(\frac{7}{10}\right)$

$k = -\frac{1}{10}\ln\left(\frac{7}{10}\right).$

$\theta = 25 + 5e^{\frac{t}{10}\ln\frac{7}{10}}$

Before death, $\theta = 37$

$$37 = 25 + e^{\frac{t}{10}\ln\left(\frac{7}{10}\right)}$$

$12 = 5e^{\frac{t}{10}\ln\left(\frac{7}{10}\right)}$

$\ln\frac{12}{5} = \frac{t}{10}\ln\frac{7}{10}$

$$t = \frac{10\ln\frac{12}{5}}{\ln\frac{7}{10}} = -24.54\ minutes$$

$$\approx -25\ minutes$$

$$\begin{array}{r} 7:\ \ 00 \\ -\ \ \ 25 \\ \hline 6:35\ am \end{array}$$

So the man was killed at around 6:35 am.

Population growth rate and decay

Example I

a) Bacteria in a culture increase at a rate proportional to the number of bacterial present, if the number increases from 1000 to 2000 in one hour. How many bacteria will be present after one and half hours.

b) How long will it take for the number of bacteria in a culture to become 4000.

Solution

$$\frac{dx}{dt} \propto x.$$

$$\frac{dx}{dt} = -kx.$$

$$\int \frac{dx}{x} = \int -kdt.$$

$$\ln x = -kt + C$$

When $t = 0,\ \ x = 1000$

$$\ln 1000 = C.$$

$$\ln x = -t + \ln 1000.$$

$$(\ln x - \ln 1000) = -kt$$

$$\ln\left(\frac{x}{1000}\right) = -kt$$

$e^{-kt} = \frac{x}{1000}$

$x = 1000e^{-kt}$

When $t = 1,\ \ x = 2000$

$$2000 = 1000e^{-k}$$

$$2 = e^{-k(1)}$$

$$(\ln 2) = -k$$

$$k = -(\ln 2).$$

$$x = 1000e^{t\ln 2}$$

When x = 4000, $t = t_1$

$4000 = 1000e^{t_1(\ln 2)}$

$$4 = e^{t_1(\ln 2)}$$

$$(\ln 4) = t_1(\ln 2)$$

$$t_1 = \frac{\ln 4}{\ln 2}$$

2 hours

Example II

On 1st January 2015, Kidepo national park had 25 lions and 60 antelopes, the lions feed on antelopes. The rate at which antelopes are eaten besides dying natural death (natural circumstances) is proportional to a sum of 5 and the number of antelopes present. On 30th June, 40 antelopes are present.

Form a differential equation and solve it.

b) How many antelopes are in a park by 15 September 2015.

Solution

x = number of antelopes present

$$\frac{dx}{dt} \propto (x + 5)$$

$$\frac{dx}{dt} = -k(5 + x)$$

$$\int \frac{dx}{5 + x} = \int -kdt$$

$\ln(5 + x) = -kt + C$

When t = 0, x = 60

$$\ln(5 + 60) = -k(0) + C$$

$$\ln(65) = C.$$

$$\ln(5 + x) = -kt + \ln(65)$$

$$\ln(5 + x) - \ln(65) = -kt.$$

$$\ln\left(\frac{5 + x}{65}\right) = -kt.$$

$$e^{-kt} = \frac{5 + x}{65}$$

$$x = 65e^{-kt} - 5$$

When t = 6 months, x = 40

$$40 = 65e^{-6k} - 5.$$

$$45 = 65e^{-6k}.$$

$$\frac{9}{13} = e^{-6k}.$$

$$\ln\left(\frac{9}{13}\right) = -6k$$

$$k = -\frac{1}{6}\ln\left(\frac{9}{13}\right)$$

$$x = 65e^{\frac{t}{6}\ln\left(\frac{9}{13}\right)} - 5$$

On 15th September 2015, t = 8.5

$$x = 65\, e^{\frac{8.5}{6}\ln\left(\frac{9}{13}\right)} - (5)$$

x = 33.6 antelopes.

Example III

A research to investigate the effect of a certain chemical on virus infection crops revealed that the rate at which the virus population is destroyed is directly proportional to the population at that time initially the population was P_0 at t months later it was found to be P.

a) Form a differential equation connecting P and t. Given the virus population reduced to one third of the initial population in 4 months.

Solve the differential equation above:

b) Find how long it would it would take for only 5% of the original population to remain.

c) What percentage of the original virus population will be left after $2\frac{1}{2}\ months$.

Solution

$$\frac{dp}{dt} \propto P.$$

$$\frac{dp}{dt} = -kp$$

$$\int \frac{dp}{p} = \int -kdt.$$

$$\ln P = -kt + C.$$

When t = 0, P = P_O

$$\ln P_0 = C.$$

$$\ln P = -kt + \ln P_{0.}$$

$$\ln P - \ln P_0 = -kt$$

$$\ln\left(\frac{P}{P_0}\right) = -kt.$$

$$e^{-kt} = \frac{P}{P_0}$$

$$P = P_0e^{-kt}.$$

When t = 4, $P = \frac{P_o}{3}$

$$\Rightarrow \frac{P_0}{3} = P_0e^{-4k.}$$

$$\frac{1}{3} = e^{-4k}.$$

$$\ln\left(\frac{1}{3}\right) = -4k.$$

$$k = -\frac{1}{4}\ln\left(\frac{1}{3}\right).$$

$$P = P_0e^{\frac{t}{4}\ln\left(\frac{1}{3}\right)}.$$

When $P = \frac{5P_0}{100.}$

$$\frac{5}{100}P_o = P_0e^{\frac{t}{4}\ln\left(\frac{1}{3}\right)}$$

$$\frac{5}{100} = e^{\frac{t}{4}\ln\left(\frac{1}{3}\right).}$$

$$\ln\frac{5}{100} = \frac{t}{4}\ln\left(\frac{1}{3}\right).$$

$$t = \frac{4\ln\left(\frac{5}{100}\right)}{\ln\left(\frac{1}{3}\right).}$$

t = 10.907 minutes

$$P = P_oe^{\frac{t}{4}\ln\frac{1}{3}}$$

When $t = 2\frac{1}{2}$,

$$P = P_oe^{\frac{5}{8}\ln\frac{1}{3}}$$

$$= \frac{P}{P_o}\times 100$$

$$= \frac{P_oe^{\frac{5}{8}\ln\frac{1}{3}}}{P_o}\times 100$$

= 0.50328 × 100

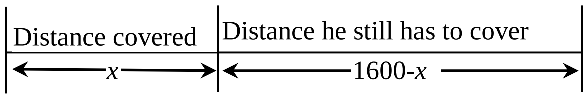
= 50.328%

Example IV

An athlete runs at a speed proportional to the square root of the distance he still has to cover. If the athlete starts running at $10ms^{-1}$ and has a distance of 1600m to cover. Find how long he will take to cover that distance.

Solution

Assume the athlete has covered a distance x.



$$\frac{dx}{dt} \propto \sqrt{1600 - x}.$$

$$\frac{dx}{dt} = -k\sqrt{1600 - x}.$$

$$\int \frac{dx}{\sqrt{1600 - x}} = \int -kdt.$$

Consider.

$$\int \frac{dx}{\sqrt{1600 - x}}$$

$$let\ u = \sqrt{1600 - x}.$$

$$u^2 = 1600 - x.$$

$$2udu = -dx.$$

$$dx = -2udu$$

$$\int \frac{dx}{\sqrt{1600 - x}} = \int \frac{-2udu}{u}$$

$$= -2u + C.$$

$$= -2\sqrt{1600 - x}.$$

From $\int \frac{dx}{\sqrt{1600-x}} = \int -kdt.$

$-2\sqrt{1600 - x} = -kt + C$ ……………………… (1)

From $\frac{dx}{dt} = -k\sqrt{1600 - x}$

When $x = 0$, $\frac{dx}{dt} = 10$

$$10 = -k\sqrt{1600 - 0}.$$

$$10 = -40k$$

$$k = -\frac{1}{4}$$

From Eqn (1), $-2\sqrt{1600 - x} = \frac{1}{4}t + C$

When $t = 0$, $x = 0$

$$-2\sqrt{1600 - 0} = \frac{1}{4} \times 0 + C.$$

$$-80 = C.$$

$$-2\sqrt{1600 - x} = \frac{1}{4}t - 80.$$

$$0 = \frac{1}{4}t - 80.$$

$t = 320$ s

Example V

A boy starts to sip a 900ml from a bottle at a rate of 10cm^3/min. Given that the rate of consumption is inversely proportional to the square root of the volume of soda remaining at time t. find the time he takes to empty the bottle.

Solution

$$\frac{dv}{dt} \propto \frac{1}{\sqrt{900 - V}}$$

$$\frac{dV}{dt} = \frac{-k}{\sqrt{900 - V}}$$

$$\int \left(\sqrt{900 - V}\right) dv = \int -kdt.$$

$$\int \left(\sqrt{900 - V}\right) dv = -kt + C.$$

Consider $\int \sqrt{900 - v}\, dv.$

Let $P = \sqrt{900 - v}$

$$P^2 = 900 - v$$

$$2pdp = -dv.$$

$$dv = -2pdp.$$

$$\int \sqrt{900 - v}\ \ dv = \int -2P^2 dp$$

$$= \frac{-2}{3}P^3 + C.$$

$$= \frac{-2}{3}\left(\sqrt{900 - v}\right)^3$$

From $\int \sqrt{900 - v}\, dv = -kt + C.$

$$\frac{-2}{3}\left(\sqrt{900 - v}\right)^3 = -kt + C$$

When $t = 0$, $v = 0$

$$-\frac{2}{3}\left(\sqrt{900}\right)^3 = C.$$

$c = -18000$

$$\Rightarrow -\frac{2}{3}\left(\sqrt{900 - v}\right)^3 = -kt - 18000.$$

From $\frac{dv}{dt}$ = 10cm^3/minute

When $t = 1$ min, $v = 10$

$$-\frac{2}{3}\left(\sqrt{900-10}\right)^3 = -k(1) - 18000$$

$$-\frac{2}{3}\left(\sqrt{890}\right)^3 = -k - 18000$$

$$k = \frac{2}{3}\left(\sqrt{890}\right)^3 - 18000$$

$$\Rightarrow \frac{-2}{3}\left(\sqrt{900-v}\right)^3 = \left(\frac{-2}{3}\left(\sqrt{890}\right)^3 + 18000\right)t - 18000$$

When $t = t_1$, $v = 900$

$$0 = \left(\frac{-2}{3}\left(\sqrt{890}\right)^3 + 18000\right)t_1 - 18000$$

$$18000 = \left(18000 - \frac{2}{3}\left(\sqrt{890}\right)^3\right)t.$$

$$t = \frac{18000}{18000 - \frac{2}{3}\left(\sqrt{890}\right)^3}$$

$t = 60.167$ s

Example VI

A rumour is circulating in Kampala town that Besigye has been arrested at a rate which is proportional to the product of people who have heard it and that of those who have a not heard it. Given that x is a fraction of the population of the town who have heard the rumour after time t. Form a differential equation connecting x, t and k.

If initially a fraction C had heard the rumour deduce that: $x = \dfrac{C}{C + (1-C)e^{-kt}}$

Given that 15% had heard the rumour at 9:00a.m another 15% by noon. What fraction of the population would have heard the rumour by 3pm?

Solution

$$\frac{dx}{dt} \propto x\,(1-x).$$

$$\frac{dx}{dt} = kx(1-x).$$

Note: If x is a fraction of the number f people who have heard the rumour then the fraction $(1 - x)$ have not heard the rumour.

$$\int \frac{dx}{x(1-x)} = \int kdt$$

$$\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x}.$$

$$A(1-x) + Bx = 1.$$

If $x = 1$, $B = 1$.

If $x = 0$, $A = 1$

$$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$$

$$\Rightarrow \int \frac{1}{x} + \frac{1}{1-x}\,dx = kt + A$$

$$\ln x - \ln(1-x) = kt + A.$$

$$\ln\frac{x}{1-x} = kt + A.$$

When $t = 0$, $x = c$

$$\ln\left(\frac{c}{1-c}\right) = A.$$

$$\ln\left(\frac{x}{1-x}\right) = kt + A$$

$$\ln\left(\frac{x}{1-x}\right) - \ln\left(\frac{c}{1-c}\right) = kt$$

$$\frac{\ln\left(\frac{x}{1-x}\right)}{\frac{c}{1-c}} = kt$$

$$\ln\frac{x(1-c)}{c(1-x)} = kt.$$

$$e^{kt} = \frac{x - xc}{c - cx.}$$

$$e^{kt} = \frac{x - xc}{c - cx.}$$

$$ce^{kt} - ce^{kt}x = x - xc.$$

$$ce^{kt} = x - xc + ce^{kt}x.$$

$$x\left(1 - c + ce^{kt}\right) = ce^{kt}.$$

$$x = \frac{ce^{kt}}{1 - c + ce^{kt}}$$

$$x = \frac{ce^{kt}(e^{-kt})}{\left[(1-c) + e^{kt}\right](e^{-kt})}$$

$$x = \frac{c}{(1-c)e^{-kt} + c}$$

$$\Rightarrow \frac{c}{c + (1-c)e^{-kt}}$$

$c = 0.15$

$$\Rightarrow x = \frac{0.15}{0.15 + 0.85e^{-kt}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

When $t = 3$hrs, $x = 0.3$

$$0.3 = \frac{0.15}{0.15 + 0.85e^{-3k}}$$

$$\frac{0.3}{0.15} = \frac{1}{0.15 + 0.85e^{-3k}}$$

$2 = \frac{1}{0.15 + 0.85e^{-3k}}$

$1 = 0.3 + 1.7e^{-3k}$

$0.7 = 1.7e^{-3k}$

$\frac{7}{17} = e^{-3k}$

$\ln\left(\frac{7}{17}\right) = -3k$

$k = \frac{-1}{3}\ln\left(\frac{7}{17}\right)$

From Eqn (1), $x = \dfrac{0.15}{0.15 + 0.85e^{\frac{t}{3}\ln\left(\frac{7}{17}\right)}}$

When t = 6hrs, x = 0.51

The fraction of $\frac{51}{100}$ will have heard the rumour by 3:00pm

Example VII

The rate at which a disease spreads through a certain community is found to be directly proportional to the fraction x of the community infected after t month but inversely proportional to the fraction not yet infected.

Set up a differential equation connecting x and t. Show that the general solution to the equation can be expressed as $e^{kt} = Axe^{-x}$, where k and A are constants. When first noticed one half of the community was infected and by this instant the disease is spreading at a fraction $\frac{1}{4}$ per month

Show that the particular solution to the differential equation is $e^{t} = 16x^4e^{2-4x}$.

Find how long (in days) from the instant it was first noticed, it takes the community to be completely infected given that a month has 30 days.

Solution

Let x be fraction of the community infected.

$1 - x$ = fraction of the community not yet infected.

$\frac{dx}{dt} \propto \frac{x}{1-x}$

$\frac{dx}{dt} = \frac{kx}{1-x.}$

$\int \frac{(1-x)}{x}dx = \int kdt$

$\int \frac{1}{x} - 1 = kt + C.$

$\ln x - x = kt + C.$

$\ln x = x + kt + C.$

$\log_e x = x + kt + C.$

$e^{kt}e^{x}e^{c} = x$

$e^{kt} = \frac{x}{e^{x}e^{c}}$

$e^{kt} = e^{-c}e^{-x}x$

$let\ e^{-c} = A$

$e^{kt} = Ae^{-x}x.$

$e^{kt} = Axe^{-x}.$

When t = 0, $x = \frac{1}{2}$

$e^{0} = A\left(\frac{1}{2}\right)e^{-\frac{1}{2}}$

$1 = A\left(\frac{1}{2}\right)e^{-\frac{1}{2}}$

$A = \frac{2}{e^{-\frac{1}{2}}}$

$A = 2e^{\frac{1}{2}}$

$\Rightarrow\ e^{kt} = 2xe^{\frac{1}{2}}e^{-x}$

From $\frac{dx}{dt} = \frac{kx}{k-x}$

When first noticed, x = 0 $\Rightarrow\ \frac{dx}{dt} = \frac{1}{4}$

$\frac{1}{4} = \frac{k\left(\frac{1}{2}\right)}{\frac{1}{2}}$

$k = \frac{1}{4.}$

$\Rightarrow e^{\frac{1}{4}t} = 2e^{\frac{1}{2}}xe^{-x}$

$\left(e^{\frac{1}{4}t}\right)^4 = \left(2e^{\frac{1}{2}}xe^{-x}\right)^4$

$e^{t} = 16e^{2}x^4e^{-4x}$

$e^{t} = 16x^4e^{2-4x}$

To be completely

$x = 1$

$$e^t = 16x^4 e^{2-4x}$$
$$e^t = 16(1)^4 e^{2-4}$$
$$e^t = 16e^{-2}$$
$t = \ln(16e^{-2})$

$t = \ln 16 + \ln e^{-2}$

$t = \ln 16 - 2$

$t = 0.775$ months

$t = 23.18$ days

Example VIII

The number of car accidents x in years on the high way was found to approximate to $\frac{dx}{dx} = kx$ where t is the time in years and k is constant. At the beginning of 2000, the number of accidents was 50. If the number of accidents increased to 60 at the beginning of 2002. Estimate the number that was expected at the beginning of 2005.

Solution

$$\frac{dx}{dt} = kx.$$
$$\int \frac{dx}{x} = \int kdt$$
$$\ln x = kt + C.$$
When $t = 0 \;\; x = 50.$
$$\ln 50 = 0 + C$$
$$C = \ln 50$$
$$\ln x = kt + \ln 50.$$
$$\ln x - \ln 50 = kt.$$
$$\ln\left(\frac{x}{50}\right) = kt.$$
$$\log_e \frac{x}{50} = kt$$
$$e^{kt} = \frac{x}{50}$$
$$x = 50e^{kt}.$$
$$when\ t = 2\ yrs,\;\; x = 60$$
$$60 = 50e^{2k}.$$
$$\frac{6}{5} = e^{2k}$$
$$\ln\left(\frac{6}{5}\right) = 2k.$$
$$k = \frac{1}{2}\ln\left(\frac{6}{5}\right).$$
$$x = 50e^{\frac{t}{2}\ln\left(\frac{6}{5}\right)}$$
When $t = 5$ yrs
$$x = 50e^{\frac{5}{2}\ln\left(\frac{6}{5}\right)} = 78.87 \text{ accidents}$$

Example IX

The population x of Kampala town follows a logistic model $\frac{dx}{dt} = \frac{1}{100}x - \frac{1}{10^8}x^2$, where t is the time measured in years. Given that the population of the town was 100,000 in 1990. Determine the population as a function of time (t).

In what year does the 1990 population double?

Assuming that the given differential equation applies for $t > 1990$, how large will the population ultimately be?

Solution

$$\frac{dx}{\frac{1}{100}x - \frac{1x^2}{10^8}} = dt$$
$$\frac{10^8}{10^6 x - x^2}dx = dt$$
$$\frac{10^8}{10^6 x - x^2} = \frac{10^8}{x(10^6 - x)} = \frac{A}{x} + \frac{B}{10^6 - x}$$
$$A(10^6 - x) + Bx = 10^8$$
If $x = 0$, $A = 100$

If $x = 10^6$, $B = 100$
$$\Rightarrow \int \frac{100}{x} + \frac{100}{10^6 - x}dx = \int dt.$$
$$100(\ln x) - 100\ln(10^6 - x) = t + C.$$
$$100\left(\ln\frac{x}{10^6 - x}\right) = t + C.$$
When $t = 0$, $x = 100{,}000$
$$100\ln\left(\frac{100}{10^6 - 100{,}000}\right) = C.$$
$$100\ln\left(\frac{100}{900{,}000}\right) = C.$$
$$100\ln\left(\frac{1}{9}\right) = C.$$
$$100\ln\left(\frac{x}{10^6 - x}\right) = t + 100\ln\frac{1}{9}$$
$$100\ln\frac{x}{10^6 - x} - 100\ln\left(\frac{1}{9}\right)$$

$$100\ln\left(\frac{x}{(10^6-x)\big/\frac{1}{9}}\right)=t.$$

$$\ln\frac{9x}{10^6-x}=\frac{t}{100.}$$

$$e^{\frac{t}{100}}=\frac{9x}{10^6-x}$$

For the population to double, $x = 200{,}000$, $t = t_1$

$$e^{\frac{t}{100}}=\frac{9\times 200{,}000}{800{,}000}$$

$$t=100\ln\left(\frac{1800000}{800000}\right)$$

$$t\simeq 81\,yrs$$

$$1990+81=2071\,yrs.$$

$$e^{\frac{t}{100}}=\frac{9x}{10^6-x}$$

$$10^6e^{\frac{t}{100}}-e^{\frac{t}{100}}x=9x.$$

$$10^6e^{\frac{t}{100}}=9x+xe^{\frac{t}{100}}$$

$$x=\frac{10^6e^{\frac{t}{100}}}{9+e^{\frac{t}{100}}}$$

$$x=\frac{10^6}{9e^{-\frac{t}{100}}+1}$$

$$x=\frac{10^6}{1+9e^{\frac{-t}{100}}}$$

As $t\longrightarrow\infty$, $x=\frac{10^6}{1+0}$

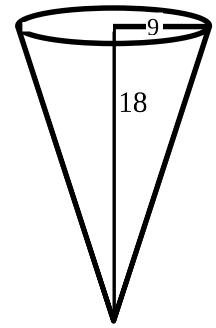
$$x=10^6$$

Example X

An inverted cone of radius 9cm and height 18cm is initially full of water. The water starts to leak through the vertex at a rate proportional to h^n where h is the depth of water remaining and n is a constant. Given that the depth decreases at a constant rate of $\frac{1}{3}$cm/s.

Find the:

i) The value of n.
ii) The time taken to empty the cone.
iii) The time taken for the volume to decrease from $\frac{1}{2}V_0\ to\ \frac{1}{8}V_0$ where V_o is the original volume



Solution

$$\frac{r}{h}=\frac{9}{18}$$

$$\frac{r}{h}=\frac{1}{2}$$

$$2r=h$$

$$r=\frac{h}{2}$$

$$V=\frac{1}{3}\pi r^2h.$$

$$V=\frac{1}{3}\pi\left(\frac{h^2}{4}\right)h$$

$$V=\frac{1}{12}\pi h^3$$

$$\frac{dv}{dt}\propto h^n.$$

$$\frac{dv}{dt}=-kh^n$$

$$\frac{dv}{dh}.\frac{dh}{dt}=-kh^n$$

The negative sign shows you that as time increases, the volume of water reduces

$$\frac{1}{4}\pi h^2.\frac{-1}{3}=-kh^n$$

$$k=\frac{\pi}{12}$$

$$\frac{1}{12}\pi h^2=kh^n$$

$n = 2$

$$\frac{dh}{dt}=\frac{-1}{3}.$$

$$\int_{18}^{0}dh=\int_{t_1}^{t_2}-\frac{1}{3}dt.$$

$$h\Big|_{18}^{0}=\frac{-1}{3}(t)\Big|_{t_1}^{t_2}$$

$$0 - 18 = -\frac{1}{3}(t_2 - t_1)$$
$$-54 = -(t_2 - t_1)$$
$$54 = t_2 - t_1$$

$\Rightarrow$ 54 minutes are taken to empty the bowel

$$V = \frac{1}{12}\pi h^3.$$
$$V_0 = \frac{1}{12}\pi(18^3)$$
$$V_0 = 486\pi$$
$$V_0 = 486\pi$$
$$\frac{1}{2}V_0 = 243\pi.$$
$$\tfrac{1}{8}V_0 = \tfrac{1}{8} \times 486\pi = 60.75\pi.$$
$$243\pi = \frac{1}{12}\pi h^3.$$
$$2916 = h^3$$
$$\Rightarrow h = \sqrt[3]{2916}.$$
$$60.75\pi = \frac{1}{12}\pi h^3$$
$$729 = h^3.$$
$$9 = h$$
$$\frac{dh}{dt} = \frac{-1}{3}$$
$$\int_{\sqrt[3]{2916}}^{9} dh = \int_{t_1}^{t_2} -\frac{1}{3}\,dt.$$
$$h\Big|_{\sqrt[3]{2916}}^{9} = \frac{-1}{3}(t)\Big|_{t_1}^{t_2}$$
$$\left(9 - \sqrt[3]{2916}\right) = -\frac{1}{3}(t_2 - t_1)$$
$$t_2 - t_1 = 3\left(9 - \sqrt[3]{2916}\right)$$
$$= 15.859 \text{ seconds}$$

Revision Exercise

1. Form differential equations by eliminating the arbitrary constants A, B and C.
   (a) $y = x + \dfrac{A}{x}$
   (b) $y = x^2\ln x + Ax^2$
   (c) $y^2 = A\cos x$
   (d) $y = x^A$
   (e) $y = A\cos x + B\sin x$
   (f) $y = Ae^{-x} + Be^{3x}$
   (g) $y^2 = Ax^2 + Bx + C$
   (h) $y = (Ax + B)e^x + C$
   (i) $y = 3x^2 + Ax$
   (j) $y = \dfrac{A}{x}$
   (k) $y = 4x^2 - A$
   (l) $y = Ae^{x^2}$
2. Solve the following differential equations giving in terms of x
   (a) $\dfrac{dy}{dx} + y^2 = 12x$, when $x = -2$, $y = 30$
   (b) $\dfrac{dy}{dx} = y^2$, when $x = 3$, $y = -1$
   (c) $3y^2\dfrac{dy}{dx} = 2x + 1$, when $x = 2$, $y = 2$
   (d) $(\cos y)\dfrac{dy}{dx} = x^2\operatorname{cosec}^2 y$; when $x = \dfrac{1}{2}$, $y = \dfrac{11}{2}$
   (e) $x\dfrac{dy}{dx} = 2$: $x > 0$ and when $x = 1$, $y = -3$
   (f) $x\dfrac{dy}{dx} = 2 + \dfrac{dy}{dx}$: $x > 1$ and when $x = 2$, $y = 1$
   (g) $\dfrac{dy}{dx} = 4xy$; when $x = 0$, $y = 4$
3. Solve the following differential equations giving y in terms of x
   (a) $\dfrac{dy}{dx} = 3x^2 + 1$
   (b) $\dfrac{dy}{dx} = \cos \tfrac{1}{2}x$
   (c) $x\dfrac{dy}{dx} = x^2 + 1$
   (d) $(x - 1)\dfrac{dy}{dx} = x + 1$
   (e) $\dfrac{dy}{dx} = \cos^2 y$

4. Use the substitution $y = uv$ where v is a function of x to solve $x\frac{dy}{dx} = 2x - y$, stating:
   (a) the general solution and
   (b) the particular solution for which $y = 5$ when $x = 1$
5. Use the substitution $4x + y = z$ to solve $\frac{dy}{dx} = 4x + y$ given that $y = 2$ when $x = 0$.
6. Use the substitution $y = uv$ where v is a function of x to find the general solution of $\frac{dy}{dx} = \frac{x^2 + y^2}{x(x+y)}$.
7. Use the substitution $z = 2x - 3y$ to solve $(2x - 3y + 3)\frac{dy}{dx} = 2x - 2y + 1$ given that $y = 1$ when $x = 1$.
8. Use the substitution $y = uv$ where v is a function of x to find:
   (a) the general solution of $x^2 - y^2 + 2xy\frac{dy}{dx} = 0$
   (b) Find the particular solution for which $y = 4$ when $x = 2$
9. Use the substitution $y = uv$ where v is a function of x to find the general solution of $(x - y)\frac{dy}{dx} = 2x + y$
10. By substituting $x = X - 1$ and $y = Y + 3$, reduce the differential equation $\frac{dy}{dx} = \frac{4x - y + 7}{2x + y - 1}$ to a homogeneous equation and hence find the general solution in terms of x and y.
11. Use the substitution $y = \frac{1}{z}$ to solve the differential equation $x^2\frac{dy}{dx} = y^2$.
12. Use the substitution $y = uv$ where v is a function of x to solve the differential equation $x\frac{dy}{dx} - y = x^2\cos x$
13. Use the substitution $y = vx^2$ where v is a function of x to solve the D.E $x\frac{dy}{dx} - 2y = x$
14. If $z = xe^y$, where y is a function of x. Find an expression for $\frac{dz}{dx}$. Hence find y in terms of x given that $xe^y\frac{dy}{dx} + e^y = 2x$ and that $y = 0$ when $x = 2$.
15. Solve the following differential equations, subject to the given conditions.
   (a) $\frac{dy}{dx} = (x + 1)^2$: $y = 0$ and $x = 2$
   (b) $\frac{dy}{dx} = \frac{1}{\sqrt{(2x+3)}}$; $y = 4$ when $x = 3$
   (c) $\sec x\frac{dy}{dx} = x$: $y = 0$ when $x = \pi$
   (d) $(x + y^2)\frac{dy}{dx} = 2x$: $y = 0$ when $x = -1$
16. Solve the differential equation $\frac{dy}{dx} = \frac{y^2 - 1}{2\tan x}$, given that $y = 3$ when $\frac{\pi}{2}$. Hence express y in terms of x.
17. Find the general solution of the differential equation $\frac{dy}{dx} = \frac{y^2}{(x^2 - x - 2)}$ in the region $x > 2$. Find also the particular solution which satisfies $y = 1$ when $x = 5$.
18. Find the solution of the differential equation $\frac{dy}{dx} = \frac{\sin^2 x}{y^2}$ which also satisfies $y = 1$ when $x = 0$.
19. Find the solution of the differential equation $x\frac{dy}{dx} = 3x - 2y$. For $x > 0$ given that $y = \frac{3}{4}$ when $x = 1$.
20. Find the general solution of the given differential equations expressing y in terms of x in each case.
   (a) $3y^2\frac{dy}{dx} - 1$
   (b) $\frac{dy}{dx} = 6xy^2$
   (c) $\frac{dy}{dx} = e^y\sin x$
   (d) $\frac{dy}{dx} = e^{x-y}$
   (e) $\frac{dy}{dx} = x\sec y$
21. Find the expressions for y in terms of x
   (a) $x^2\frac{dy}{dx} - y^2 = 0$; $y = -1$ when $x = 1$
   (b) $\frac{dy}{dx} + 2xy = 0$; $y = 5$ when $x = 0$
   (c) $\cot x\frac{dy}{dx} = y$; $y = 2$ when $x = 0$
   (d) $\frac{dy}{dx} = xe^{-2y}$; $y = 0$ when $x = 0$
22. Find the general solutions of the given differential equation expressing y in terms of x.

(a) $(1-x)\dfrac{dy}{dx} = xy$

(b) $(x^2-1)\dfrac{dy}{dx} = (x^2+1)y$

(c) $\dfrac{dy}{dx} = e^x(1+y^2)$

(d) $e^y\dfrac{dy}{dx} + 2x = 2xe^y$

23. Find y in terms of x given that $e^{2x}y\dfrac{dy}{dx}+2x=0$ and that $y = 1$ when $x = 0$

24. Find y in terms of x given that $xy\dfrac{dy}{dx}=\sqrt{y^2-9}$ and that $y = 5$ when $x = e^4$.

25. At time t, the rate of increase in the concentration C of a micro-organism in a controlled environment is equal to k times the concentration, where k is a positive constant when $t = 0$, $C = C_O$. write down a differential equation involving c, t and k. Hence find c in terms of C_O, t and k. Find also in terms of k the time at which the concentration has increased by 50% from its value at $t = 0$.

26. A body is kept in a room of constant temperature. The temperature of the body falls at a rate of $k\theta$°C per minute, where k is constant and θ is the difference between the temperature of the body and that of the room at time t. Express this information in form of a differential equation and hence show that $\theta=\theta_0e^{-kt}$, where θ_0 is the temperature difference when $t = 0$. The temperature of the body falls 5°C in the first minute and 4°C in the second minute. Show that the fall of temperature in the third minute is 3.2°C.

27. A rectangular tank with its base horizontal is filled with water to a depth h at time $t = 0$. Water leaks out of the tank from a small hole in the base at a rate proportional to the square of the depth of the water. If the depth of water is ½h at t, find the further time it will take before the tank is empty.

28. A radioactive substance decays so that the rate of decrease of mass at any time is proportional to the mass proportional at that time. Denoting by x the mass remaining at time t, write down a differential equation satisfied by x. Show that $x=x_0e^{-kt}$ where x_o is the initial mass and k is a constant. The mass is reduced to 4/5 of its initial value in 30 days. Calculate, correct to the nearest day, the time required for the mass to be reduced to half its initial value. A mass of 625 milligrams of the substance is prepared. Determine the mass which is present 90 days after the preparation.

29. According to Newton's law, the rate of cooling of a body in air is proportional to the difference to the difference between the temperature T of the body and the temperature T_o of the air. If the air temperature is kept constant at 20°C and the body cools from 100°C to 60°C in 20 minutes, in what further time will the body cool to 30°C?

30. Two liquids X and Y, are flowing into a trough at the same constant rate of 10 and 20 litres per minute respectively. The liquid in the trough is stirred continuously and pumped out at the rate of 30 litres per minute. initually, the trough contains 200 litres of X and 100 litres of Y. after t minutes, the tank contains x litres of X. By considering the change in x in a small interval of time δt, show that $\dfrac{dx}{dt}=10-\dfrac{x}{10}$. Hence find an expression for x in terms of t. Find correct to the nearest litre, the quantity of liquid X in the trough after 10 minutes. After how long to the nearest second will there be less of X in the mixture than liquid Y?

31. In a certain chemical reaction in which a compound X is formed from a compound Y, the masses of X, Y present at time t are x and y respectively. The sum of the masses of X and Y is a, where a is constant, and at any time the rate at which x is increasing is proportional to the product of the two masses at that time. Show that $\dfrac{dx}{dt}=kx(a-x)$; where k is constant. If $x=\dfrac{9}{5}$ at t = 0 and $x=\dfrac{9}{2}$ at t = ln 2, show that $k=\dfrac{2}{a}$.

32. A plant grows in a pot which contains a volume V of soil. At time, t, the mass of the plant is m and the volume of soil utilised by the roots is αm, where α is constant. The rate of increase of the mass of the plant is proportional to the mass of the plant times the volume of soil not yet utilised by the roots. Obtain a differential equation for m, and verify that it can be written in the form $V\beta\dfrac{dt}{dm}=\dfrac{1}{m}+\dfrac{\alpha}{V-\alpha m}$, where β is a constant. The mass of the plant is initially $\dfrac{V}{4\alpha}$. Find in terms of V and β, the time taken for the plant to double its mass. Find also the mass of the plane at time t.

33. At any instant, a spherical meteorite is gaining mass because of two effects (i) mass is condensing onto it at a rate which is proportional to the surface area of the meteorite at that instant, (ii) the gravitational field of the meteorite attracts mass onto itself, the rate being proportional to the

meteorite mass at that instant. Assuming that the two effects can be added together and the meteorite remains spherical and of constant density. Show that its radius r at time t satisfies the differential equation $\frac{dr}{dt} = A + Br$, where A and B are constants. If $r = r_o$ at $t = 0$, show that $r = r_o e^{Bt} + \frac{A}{B}(e^{Bt} - 1)$.

Answers

1. (a) $x\frac{dy}{dx} = 2x - y$ (b) $x\frac{dy}{dx} = 2y + x^2$
   (c) $2\frac{dy}{dx} + y\tan x = 0$ (d) $x\ln x\frac{dy}{dx} = y\ln y$
   (e) $\frac{d^2y}{dx^2} + y = 0$ (f) $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dy^2} + \frac{dy}{dx} = 0$
   (g) $3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right) + y\frac{d^3y}{dx^3} = 0$
   (h) $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$
   (i) $x\frac{dy}{dx} = 3x^2 + y$ (j) $x\frac{dy}{dx} + y = 0$
   (k) $\frac{dy}{dx} = 8x$ (l) $\frac{dy}{dx} = 2xy$
2. (a) $y = 6x^2 - 4x - 2$ (b) $y = \frac{1}{2-x}$
   (c) $y = (x^2 + x + 2)^{1/3}$ (d) $y = \sin^{-1}\left[\left(x^3 + \frac{7}{8}\right)^{1/3}\right]$
   (e) $y = 2\ln x - 3$ (f) $y = 2\ln(x-1) + 1$
   (g) $y = 4e^{2x^2}$
3. (a) $y = x^3 + x + c$ (b) $y = 2\sin\frac{1}{2}x + c$
   (c) $y = \frac{1}{2}x^2 + \ln|x| + c$ (d) $y = x + \ln(x-1)^2 + c$
   (e) $y = \tan^{-1}(x + k)$
4. (a) $y = x + \frac{A}{x}$ (b) $y = x + \frac{4}{x}$
5. $y = 6e^x - 4x - 4$ 6. $y = x\ln\left[\frac{|Ax|}{(x-y)^2}\right]$
7. $6x^2 + 9y^2 = 12xy + 18y - 6x - 9$
8. (a) $x^2 + y^2 = Ax$ (b) $x^2 + y^2 = 10x$
9. $\ln(y^2 + 2x^2) - \sqrt{2\tan^{-1}(\frac{y}{x\sqrt{2}})} = A$
10. $(y - x - 4)^3(y + 4x + 1)^2 = A$
11. $y = \frac{x}{(1+kx)}$
12. $y = kx + x\sin x$
13. $y = kx^2 - x$
14. $xe^y\frac{dy}{dx} + e^y$; $y = \ln\left(x - \frac{2}{x}\right)$
15. (a) $y = \frac{1}{3}(x+1)^3 - 9$ (b) $y = 1 + \sqrt{(2x+3)}$
    (c) $y = x\sin x + \cos x + 1$
    (d) $y = \ln\frac{1}{2}(1 + x^2)$
16. $y = \frac{2+\sin x}{2-\sin x}$
17. $e^{3/y} = A\left(\frac{x+1}{x-2}\right)$, $2e^{\frac{3(1-y)}{y}} = \frac{x+1}{x-2}$
18. $4y^3 = 6x - 3\sin 2x + 4$ 19. $y = x - \frac{1}{4x^2}$
20. (a) $y = (x^2 - x + c)^{1/3}$
    (b) $y = \frac{1}{(c - 3x^2)}$ (c) $y = -\ln(\cos x + c)$
    (d) $y = \ln(e^x + c)$ (e) $y = \sin^{-1}(\frac{1}{2}x^2 + c)$
21 (a) $y = \frac{x}{(1-2x)}$ (b) $y = 5e^{-x^2}$
   (c) $y = 2\sec x$ (d) $y = \frac{1}{2}\ln(x^2 + 1)$
22 (a) $y = \frac{Ae^{-x}}{(1-x)}$ (b) $y = \frac{Ae^x(x-1)}{(x+1)}$
   (c) $y = \tan(e^x + c)$ (d) $y = \ln(1 + Ae^{x^2})$
23. $y = e^{-x}\sqrt{(2x+1)}$
24. $y = \sqrt{[(\ln x)^2 + 9]}$
25. $\frac{dC}{dt} = kC$, $C = C_0e^{kt}$, $t = \frac{1}{k}\ln\frac{3}{2}$
26. $\frac{d\theta}{dt} = -k\theta$ 27. $(1 + \sqrt{2})t$
28. $\frac{dx}{dt} = -kx$, 93 days, 320 mg
29. 40 minutes
30. $x = 100(1 + e^{\frac{-t}{10}})$; 137 litres; 6 min 56 sec.
32. $t = \frac{\ln 3}{V\beta}$; $m = \frac{V}{\alpha(1 + 3e^{-V\beta t})}$