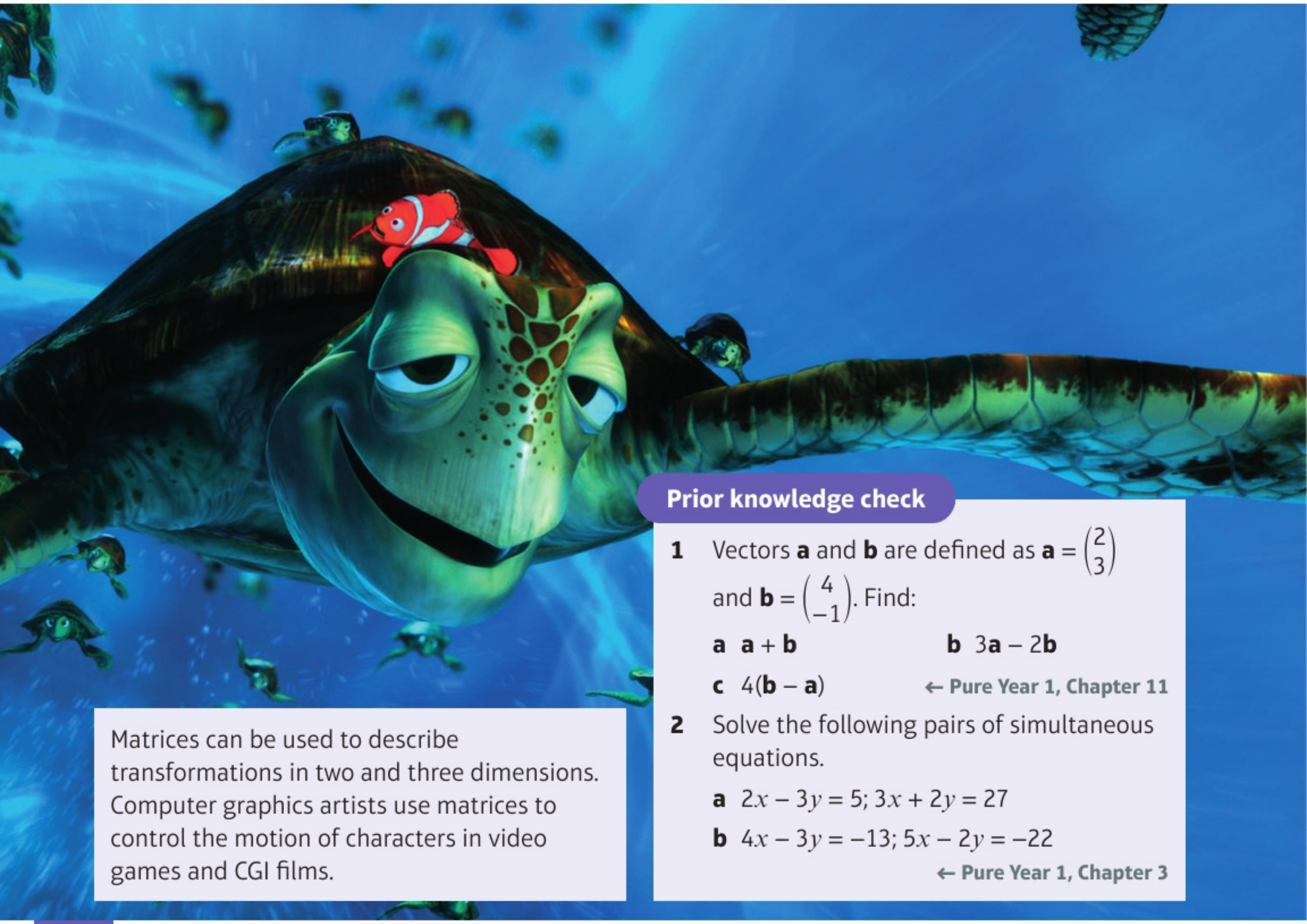
Matrices

6

Objectives

After completing this chapter you should be able to:

- Understand the concept of a matrix → pages 95–99
- Define the zero and identity matrices → pages 95–99
- Add and subtract matrices → pages 95–99
- Multiply a matrix by a scalar → pages 96–99
- Multiply matrices → pages 99–103
- Calculate the determinant of a matrix → pages 104–108
- Find the inverse of a matrix → pages 108–116
- Use matrices to solve systems of equations → pages 116–121
- Interpret simultaneous equations geometrically → pages 118–121



Matrices can be used to describe transformations in two and three dimensions. Computer graphics artists use matrices to control the motion of characters in video games and CGI films.

Prior knowledge check

1 Vectors **a** and **b** are defined as $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$. Find:

a $\mathbf{a} + \mathbf{b}$ **b** $3\mathbf{a} - 2\mathbf{b}$

c $4(\mathbf{b} - \mathbf{a})$ ← Pure Year 1, Chapter 11

2 Solve the following pairs of simultaneous equations.

a $2x - 3y = 5;\ 3x + 2y = 27$

b $4x - 3y = -13;\ 5x - 2y = -22$

← Pure Year 1, Chapter 3

6.1 Introduction to matrices

A matrix is an **array** of **elements** (which are usually numbers) set out in a pair of brackets.

You can describe the **size** of a matrix using the number of rows and columns it contains.

Links A **vector** is a simple example of a matrix with just one column.
← Pure Year 1, Chapter 11; Pure Year 2, Chapter 12

For example $\begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix}$ is a 2 × 2 matrix and $\begin{pmatrix} 1 & 4 & -1 & 1 \\ 2 & 3 & 0 & 2 \end{pmatrix}$ is a 2 × 4 matrix. Generally, you can refer to a matrix as $n \times m$ where n is the number of rows and m is the number of columns.

- **A square matrix is one where the numbers of rows and columns are the same.**
- **A zero matrix is one in which all of the elements are zero. The zero matrix is denoted by 0.**
- **An identity matrix is a square matrix in which the elements on the leading diagonal (starting top left) are all 1 and the remaining elements are 0. Identity matrices are denoted by $\mathbf{I}_k$ where k describes the size. The 3 × 3 identity matrix is**
  $$\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Notation Matrices are usually represented with bold capital letters such as **M** or **A**.

Example 1

Write down the size of each matrix in the form $n \times m$.

a $\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 & 2 \end{pmatrix}$

c $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ **d** $\begin{pmatrix} 3 & 2 \\ -1 & 1 \\ 0 & -3 \end{pmatrix}$

a $\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$ — There are two rows and two columns.

The size is 2 × 2.

b $\begin{pmatrix} 1 & 0 & 2 \end{pmatrix}$ — There is just one row and three columns.

The size is 1 × 3.

c $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ — There are two rows and one column.

The size is 2 × 1.

d $\begin{pmatrix} 3 & 2 \\ -1 & 1 \\ 0 & -3 \end{pmatrix}$ — There are three rows and two columns.

The size is 3 × 2.

- **To add or subtract matrices, you add or subtract the corresponding elements in each matrix. You can only add or subtract matrices that are the same size.**

Notation Matrices which are the same size are said to be **additively conformable**.

Example 2

Find: **a** $\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} -1 & 4 \\ 5 & 3 \end{pmatrix}$

b $\begin{pmatrix} 1 & -3 & 4 \\ 2 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 2 & 1 \\ 5 & 2 & 3 \end{pmatrix}$

a $\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} -1 & 4 \\ 5 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 3 \\ 5 & 6 \end{pmatrix}$

Top row:
$2 + -1 = 1$
$-1 + 4 = 3$

Bottom row:
$0 + 5 = 5$
$3 + 3 = 6$

b $\begin{pmatrix} 1 & -3 & 4 \\ 2 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 2 & 1 \\ 5 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & -5 & 3 \\ -3 & -1 & -2 \end{pmatrix}$

Top row:
$1 - 0 = 1$
$-3 - 2 = -5$
$4 - 1 = 3$

Bottom row:
$2 - 5 = -3$
$1 - 2 = -1$
$1 - 3 = -2$

- **To multiply a matrix by a scalar, you multiply every element in the matrix by that scalar.**

Notation A **scalar** is a number rather than a matrix. In questions on matrices, scalars will be represented by non-bold letters and numbers.

Example 3

$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$, $\mathbf{B} = (6 \quad 0 \quad -4)$

Find: **a** $2\mathbf{A}$ **b** $\frac{1}{2}\mathbf{B}$

c Explain why you cannot work out $\mathbf{A} + \mathbf{B}$.

a $2\mathbf{A} = \begin{pmatrix} 2 & 4 \\ -2 & 0 \end{pmatrix}$

Note that $2\mathbf{A}$ gives the same answer as $\mathbf{A} + \mathbf{A}$.

Top row:
$2 \times 1 = 2$
$2 \times 2 = 4$

Bottom row:
$2 \times -1 = -2$
$2 \times 0 = 0$

b $\frac{1}{2}\mathbf{B} = (3 \quad 0 \quad -2)$

$\frac{1}{2} \times 6 = 3$
$\frac{1}{2} \times 0 = 0$
$\frac{1}{2} \times (-4) = -2$

c $\mathbf{A}$ and $\mathbf{B}$ are not the same size, so you can't add them.

You could also say that **A** and **B** are not additively conformable.

Example 4

$\mathbf{A} = \begin{pmatrix} a & 0 \\ 1 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & b \\ 0 & 3 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 6 & 6 \\ 1 & c \end{pmatrix}$.

Given that $\mathbf{A} + 2\mathbf{B} = \mathbf{C}$, find the values of the constants a, b and c.

$$\begin{pmatrix} a & 0 \\ 1 & 2 \end{pmatrix} + 2\begin{pmatrix} 1 & b \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 6 \\ 1 & c \end{pmatrix}$$

$$\begin{pmatrix} a+2 & 2b \\ 1 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 6 \\ 1 & c \end{pmatrix}$$ If two matrices are equal, then all of their corresponding elements are equal.

$a + 2 = 6 \Rightarrow a = 4$ Compare top left elements.

$2b = 6 \Rightarrow b = 3$ Compare top right elements.

$c = 8$ Compare bottom right elements.

Exercise 6A

1 Write the size of each matrix in the form $n \times m$.

a $\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ **c** $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & -1 \end{pmatrix}$

d $(1 \quad 2 \quad 3)$ **e** $(3 \quad -1)$ **f** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2 Write down the 4×4 identity matrix, $\mathbf{I}_4$.

3 Two matrices **A** and **B** are given as:

$\mathbf{A} = \begin{pmatrix} 1 & 3 & a \\ 2 & -1 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 3 & 6 \\ b & -1 & 4 \end{pmatrix}$

If $\mathbf{A} = \mathbf{B}$, write down the values of a and b.

4 For the matrices

$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 & 1 \\ -1 & -2 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$

find:

a $\mathbf{A} + \mathbf{C}$ **b** $\mathbf{B} - \mathbf{A}$ **c** $\mathbf{A} + \mathbf{B} - \mathbf{C}$

5 For the matrices

$\mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{B} = (1 \quad -1)$, $\mathbf{C} = (-1 \quad 1 \quad 0)$,

$\mathbf{D} = (0 \quad 1 \quad -1)$, $\mathbf{E} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\mathbf{F} = (2 \quad 1 \quad 3)$

find where possible:

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{A} - \mathbf{E}$ **c** $\mathbf{F} - \mathbf{D} + \mathbf{C}$ **d** $\mathbf{B} + \mathbf{C}$

e $\mathbf{F} - (\mathbf{D} + \mathbf{C})$ **f** $\mathbf{A} - \mathbf{F}$ **g** $\mathbf{C} - (\mathbf{F} - \mathbf{D})$

6 Given that $\begin{pmatrix} a & 2 \\ -1 & b \end{pmatrix} - \begin{pmatrix} 1 & c \\ d & -2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$, find the values of the constants a, b, c and d.

P **7** Given that $\begin{pmatrix} 1 & 2 & 0 \\ a & b & c \end{pmatrix} + \begin{pmatrix} a & b & c \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} c & 5 & c \\ c & c & c \end{pmatrix}$, find the values of a, b and c.

8 Given that $\begin{pmatrix} 5 & 3 \\ 0 & -1 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ 2 & 0 \\ 1 & 4 \end{pmatrix}$, find the values of a, b, c, d, e and f.

9 For the matrices $\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 0 & 2 \\ 3 & 4 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 6 & 3 & -4 \\ 1 & 1 & 2 \\ -2 & 0 & -3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -2 & 0 & 3 \\ 2 & 8 & -6 \\ -1 & 1 & 1 \end{pmatrix}$, find:

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{B} - \mathbf{C}$ **c** $\mathbf{C} + \mathbf{A}$

d A matrix $\mathbf{M} = \begin{pmatrix} 5 & -6 & b \\ 4 & a & 6 \\ 2 & 0 & c \end{pmatrix}$. Find the values of a, b and c if:

i $\mathbf{A} + \mathbf{M} = \begin{pmatrix} 6 & -7 & -2 \\ 6 & 3 & 8 \\ 5 & 4 & 6 \end{pmatrix}$ **ii** $\mathbf{M} - \mathbf{B} = \begin{pmatrix} -1 & -9 & -5 \\ -2 & 7 & -1 \\ -3 & 2 & 2 \end{pmatrix}$

10 For the matrices $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 4 & -6 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, find:

a $3\mathbf{A}$ **b** $\frac{1}{2}\mathbf{A}$ **c** $2\mathbf{B}$

d Explain why it is not possible to find $\mathbf{A} - \mathbf{B}$.

11 The matrices **A** and **B** are defined as:

$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix}$

Find:

a $3\mathbf{A} + 2\mathbf{B}$ **b** $2\mathbf{A} - 4\mathbf{B}$ **c** $5\mathbf{A} - 2\mathbf{B}$ **d** $\frac{1}{2}\mathbf{A} + \frac{3}{2}\mathbf{B}$

12 The matrices **M** and **N** are defined as:

$\mathbf{M} = \begin{pmatrix} 2 & 4 & -1 \\ 1 & -3 & -1 \\ 0 & 2 & 2 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} 6 & -2 & 5 \\ 3 & -3 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

Find:

a $\mathbf{M} + 2\mathbf{N}$ **b** $3\mathbf{M} - \mathbf{N}$ **c** $4\mathbf{M} + 5\mathbf{N}$ **d** $\frac{2}{3}\mathbf{M} - \frac{1}{2}\mathbf{N}$

13 Find the value of k and the value of x such that $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} + k\begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 7 \\ x & 0 \end{pmatrix}$.

14 Find the values of a, b, c and d such that $2\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix} - 3\begin{pmatrix} 1 & c \\ d & -1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ -4 & -4 \end{pmatrix}$.

15 Find the values of a, b, c and d such that $\begin{pmatrix} 5 & a \\ b & 0 \end{pmatrix} - 2\begin{pmatrix} c & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 1 \\ 3 & d \end{pmatrix}$.

P **16** Find the value of k such that $\begin{pmatrix} -3 \\ k \end{pmatrix} + k\begin{pmatrix} 2k \\ 2k \end{pmatrix} = \begin{pmatrix} k \\ 6 \end{pmatrix}$.

P **17** The matrices **A** and **B** are defined as:

$$\mathbf{A} = \begin{pmatrix} p & 0 & 0 \\ 0 & q^2 & r \\ 0 & 0 & 5 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 2q & 0 & 0 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{pmatrix}$$

where p, q and r are positive constants.
Given that $\mathbf{A} - k\mathbf{B} = \mathbf{I}_3$, where $\mathbf{I}_3$ is the 3×3 identity matrix, find:

a the value of k **b** the values of p, q and r.

P **18** The matrices **P** and **Q** are defined as:

$$\mathbf{P} = \begin{pmatrix} 0 & 2 & c \\ a & 0 & 0 \\ 0 & \mathrm{b} & -1 \end{pmatrix} \text{ and } \mathbf{Q} = \begin{pmatrix} 0 & -1 & -1 \\ 3 & d & 0 \\ 0 & 2 & e \end{pmatrix}$$

where a, b, c, d, and e are constants.
Given that $\mathbf{P} - k\mathbf{Q} = \mathbf{0}$, where $\mathbf{0}$ is the zero matrix, find the values of a, b, c, d, e and k.

6.2 Matrix multiplication

Two matrices can be multiplied together. Unlike the operations we have seen so far, this is completely different from normal arithmetic multiplication.

- **Matrices can be multiplied together if the number of columns in the first matrix is equal to the number of rows in the second matrix.**

Notation If **AB** exists, then matrix **A** is said to be **multiplicatively conformable** with matrix **B**.

The **product matrix** will have the same number of rows as the first matrix, and the same number of columns as the second matrix.

AB = C ← If **A** has size $n \times m$ and **B** has size $m \times k$ then the product matrix, **C**, has size $n \times k$.

The **order** in which you multiply matrices is important. This has two consequences:

- In general **AB** ≠ **BA** (even if **A** and **B** are both square matrices).
- If **AB** exists, **BA** does not necessarily exist (for example if **A** is a 3×2 matrix and **B** is a 2×4 matrix).

- **To find the product of two multiplicatively conformable matrices, you multiply the elements in each row in the left-hand matrix by the corresponding elements in each column in the right-hand matrix, then add the results together.**

$$\begin{pmatrix} 5 & -1 & 2 \\ 8 & 3 & -4 \end{pmatrix} \times \begin{pmatrix} 2 & 2 \\ 9 & -3 \\ 7 & 4 \end{pmatrix} = \begin{pmatrix} 15 & 21 \\ 15 & -9 \end{pmatrix}$$

You are multiplying a 2×3 matrix by a 3×2 matrix, so the product matrix will have size 2×2.
To find the bottom left element, work out $8 \times 2 + 3 \times 9 + (-4) \times 7 = 16 + 27 - 28 = 15$

Example 5

Given that $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

a find **AB** **b** explain why it is not possible to find **BA**.

a First calculate the size of **AB**.

$(2 \times 2) \times (2 \times 1)$ gives 2×1

$\mathbf{AB} = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$

$p = 1 \times (-3) + (-2) \times 2 = -7$

$q = 3 \times (-3) + 4 \times 2 = -1$

So $\mathbf{AB} = \begin{pmatrix} -7 \\ -1 \end{pmatrix}$

b **BA** cannot be found, since the number of columns in **B** is not the same as the number of rows in **A**.

The number of rows is two from here.

The number of columns is one from here.

The top number is the total of the first row of **A** multiplied by the first column of **B**.

The bottom number is the total of the second row of **A** multiplied by the first column of **B**.

Watch out Remember that order is important. **B** is not multiplicatively conformable with **A**, but **A** is multiplicatively conformable with **B**.

Example 6

Given that $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 1 \\ 0 & -2 \end{pmatrix}$, find:

a **AB** **b** **BA**

a **A** is a 2×2 matrix and **B** is a 2×2 matrix so they can be multiplied and the product will be a 2×2 matrix.

$\mathbf{AB} = \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$a = (-1) \times 4 + 0 \times 0 = -4$

$b = (-1) \times 1 + 0 \times (-2) = -1$

$c = 2 \times 4 + 3 \times 0 = 8$

$d = 2 \times 1 + 3 \times (-2) = -4$

So $\mathbf{AB} = \begin{pmatrix} -4 & -1 \\ 8 & -4 \end{pmatrix}$

b **BA** will also be a 2×2 matrix.

$\begin{pmatrix} 4 & 1 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ -4 & -6 \end{pmatrix}$

This time there are four elements to be found.

a is the total of the first row multiplied by the first column.
b is the total of the first row multiplied by the second column.
c is the total of the second row multiplied by the first column.
d is the total of the second row multiplied by the second column.

First row times first column
$4 \times (-1) + 1 \times 2 = -2$

First row times second column
$4 \times 0 + 1 \times 3 = 3$

Second row times second column
$0 \times 0 + (-2) \times 3 = -6$

Second row times first column
$0 \times (-1) + (-2) \times 2 = -4$

You can enter matrices directly into your calculator to multiply them quickly

Example 7

$\mathbf{A} = \begin{pmatrix} -1 \\ a \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b & 2 \end{pmatrix}$

Given that $\mathbf{BA} = (0)$, find $\mathbf{AB}$ in terms of a.

(0) is a 1 × 1 zero matrix. You could also write it as **0**.

$\mathbf{BA} = \begin{pmatrix} b & 2 \end{pmatrix}\begin{pmatrix} -1 \\ a \end{pmatrix} = (-b + 2a)$

BA is a 1 × 1 matrix.

So $\mathbf{BA} = (0)$ implies that $b = 2a$.

$\mathbf{AB} = \begin{pmatrix} -1 \\ a \end{pmatrix}\begin{pmatrix} b & 2 \end{pmatrix} = \begin{pmatrix} -b & -2 \\ ab & 2a \end{pmatrix}$

AB is a 2 × 2 matrix.

Substituting $b = 2a$ gives $\mathbf{AB} = \begin{pmatrix} -2a & -2 \\ 2a^2 & 2a \end{pmatrix}$.

Watch out Although you can multiply matrices using a calculator, you need to know how the process works so that you can deal with matrices containing unknowns.

Example 8

$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$. Find **BCA**.

$\mathbf{BC} = \begin{pmatrix} 3 & -2 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = (2)$

$(\mathbf{BC})\mathbf{A} = (2)\begin{pmatrix} 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 4 \end{pmatrix}$

This product could have been calculated by first working out **CA** and then multiplying **B** by this product. In general, matrix multiplication is **associative** (meaning that the bracketing makes no difference provided the order stays the same), so $(\mathbf{BC})\mathbf{A} = \mathbf{B}(\mathbf{CA})$.

Exercise 6B

1 Given the sizes of the following matrices:

Matrix	A	B	C	D	E
Size	2 × 2	1 × 2	1 × 3	3 × 2	2 × 3

find the sizes of these matrix products.

a **BA** b **DE** c **CD**

d **ED** e **AE** f **DA**

2 Use your calculator to find these products:

a $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ b $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 5 \\ -1 & -2 \end{pmatrix}$

3 The matrix $\mathbf{A} = \begin{pmatrix} -1 & -2 \\ 0 & 3 \end{pmatrix}$ and the matrix $\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

Use your calculator to find:

a **AB** b $\mathbf{A}^2$

Hint $\mathbf{A}^2$ means $\mathbf{A} \times \mathbf{A}$.

4 The matrices **A**, **B** and **C** are given by:

$\mathbf{A} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$, $\mathbf{C} = (-3, -2)$

Without using your calculator, determine whether or not the following products exist and find the products of those that do.

a **AB** **b** **AC** **c** **BC**
d **BA** **e** **CA** **f** **CB**

5 Find $\begin{pmatrix} 2 & a \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 2 \end{pmatrix}$, giving your answer in terms of a.

6 Find $\begin{pmatrix} 3 & 2 \\ -1 & x \end{pmatrix}\begin{pmatrix} x & -2 \\ 1 & 3 \end{pmatrix}$, giving your answer in terms of x.

7 The matrices **A**, **B** and **C** are defined as:

$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -3 & 1 \\ 1 & 2 \end{pmatrix}$.

Use your calculator to find:

a **AB** – **C** **b** **BC** + 3**A** **c** 4**B** – 3**CA**

8 The matrices **M** and **N** are defined as:

$\mathbf{M} = \begin{pmatrix} 3 & k \\ k & 1 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} 1 & k \\ k & -1 \end{pmatrix}$. Find, in terms of k:

a **MN** **b** **NM** **c** 3**M** – 2**N** **d** 2**MN** + 3**N**

P **9** The matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Find:

a $\mathbf{A}^2$

b $\mathbf{A}^3$

c Suggest a form for $\mathbf{A}^k$.

Links You might be asked to prove this general form for $\mathbf{A}^k$. → Section 8.3

P **10** The matrix $\mathbf{A} = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$.

a Find, in terms of a and b, the matrix $\mathbf{A}^2$.

Given that $\mathbf{A}^2 = 3\mathbf{A}$,

b find the value of a.

11 $\mathbf{A} = (-1 \quad 3)$, $\mathbf{B} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 4 & -2 \\ 0 & -3 \end{pmatrix}$

Find: **a** **BAC** **b** $\mathbf{AC}^2$

12 $\mathbf{A} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & -2 & -3 \end{pmatrix}$

Find: **a** $\mathbf{ABA}$ **b** $\mathbf{BAB}$

(P) **13** **a** Write down $\mathbf{I}_2$.

b Given that matrix $\mathbf{A} = \begin{pmatrix} 2 & -2 \\ 1 & 3 \end{pmatrix}$, show that $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$.

(P) **14** $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 & 2 \\ -1 & 0 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$.

Show that $\mathbf{AB} + \mathbf{AC} = \mathbf{A}(\mathbf{B} + \mathbf{C})$.

(E/P) **15** $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ and $\mathbf{I}$ is the 2×2 identity matrix.

Prove that $\mathbf{A}^2 = 2\mathbf{A} + 5\mathbf{I}$. **(2 marks)**

(E) **16** A matrix $\mathbf{M}$ is given as $\mathbf{M} = \begin{pmatrix} 1 & 2 & c \\ a & -1 & 1 \\ 1 & b & 0 \end{pmatrix}$.

Find $\mathbf{M}^2$ in terms of a, b and c. **(3 marks)**

(E/P) **17** A matrix $\mathbf{A}$ is given as $\mathbf{A} = \begin{pmatrix} 1 & -1 & b \\ a & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$.

Given that $\mathbf{A}^2 = \begin{pmatrix} -4 & -3 & -8 \\ 9 & 1 & -6 \\ 4 & -1 & 7 \end{pmatrix}$, find the values of a and b. **(3 marks)**

(E/P) **18** $\mathbf{A} = \begin{pmatrix} p & 3 \\ 6 & p \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} q & 2 \\ 4 & q \end{pmatrix}$, where p and q are constants. Prove that $\mathbf{AB} = \mathbf{BA}$. **(3 marks)**

(E/P) **19** The matrix $\mathbf{A} = \begin{pmatrix} 3 & p \\ -4 & q \end{pmatrix}$ is such that $\mathbf{A}^2 = \mathbf{I}$. Find the values of p and q. **(3 marks)**

Challenge

A 2×2 matrix $\mathbf{A}$ has the property that $\mathbf{A}^2 = \mathbf{0}$. Find a possible matrix $\mathbf{A}$ such that:

a at least one of the elements in $\mathbf{A}$ is non-zero

b all of the elements in $\mathbf{A}$ are non-zero.

6.3 Determinants

You can calculate the **determinant** of a square matrix. The determinant is a scalar value associated with that matrix.

- **For a 2 × 2 matrix M =** $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$**, the determinant of M is** $ad - bc$**.**
- **If det M = 0 then M is a singular matrix.**
- **If det M ≠ 0 then M is a non-singular matrix.**

Notation You can write the determinant of **M** as det **M**, |**M**| or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. It is also sometimes written as Δ.

Links Singular matrices do not have an **inverse**. → Section 6.4

Example 9

Given that $\mathbf{A} = \begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix}$, find det **A**.

det **A** = $ad - bc = 6 \times 2 - 5 \times 1 = 12 - 5 = 7$

Example 10

$$\mathbf{A} = \begin{pmatrix} 4 & p+2 \\ -1 & 3-p \end{pmatrix}$$

Given that **A** is singular, find the value of p.

det **A** = $4(3 - p) - (p + 2)(-1)$
det **A** = $12 - 4p + p + 2 = 14 - 3p$
A is singular so det **A** = 0.
$14 - 3p = 0 \Rightarrow p = \frac{14}{3}$

Watch out Although you can find the determinant using a calculator, you need to know how the process works so that you can deal with matrices containing unknowns.

Finding the determinant of a 3 × 3 matrix is more difficult.

- **You find the determinant of a 3 × 3 matrix by reducing the 3 × 3 determinant to 2 × 2 determinants using the formula:**

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Watch out There is a minus sign in front of the second term.

In this expression for the determinant, each of the elements a, b and c is multiplied by its **minor**.

- **The minor of an element in a 3 × 3 matrix is the determinant of the 2 × 2 matrix that remains after the row and column containing that element have been crossed out.**

Example 11

Find the minors of the elements 5 and 7 in the matrix

$$\begin{pmatrix} 5 & 0 & 2 \\ -1 & 8 & 1 \\ 6 & 7 & 3 \end{pmatrix}$$

$$\begin{pmatrix} \cancel{5} & \cancel{0} & \cancel{2} \\ \cancel{-1} & 8 & 1 \\ \cancel{6} & 7 & 3 \end{pmatrix}$$

To find the minor of 5, you begin by crossing out the row and the column containing 5.

$$\begin{vmatrix} 8 & 1 \\ 7 & 3 \end{vmatrix} = 8 \times 3 - 7 \times 1 = 24 - 7 = 17$$

When you have crossed out the row and the column containing 5, you are left with the elements $\begin{pmatrix} 8 & 1 \\ 7 & 3 \end{pmatrix}$ and you evaluate the determinant of this 2 × 2 matrix.

The minor of 5 is 17.

$$\begin{pmatrix} 5 & \cancel{0} & 2 \\ -1 & \cancel{8} & 1 \\ \cancel{-6} & \cancel{7} & \cancel{3} \end{pmatrix}$$

To find the minor of 7, you begin by crossing out the row and the column containing 7.

$$\begin{vmatrix} 5 & 2 \\ -1 & 1 \end{vmatrix} = 5 \times 1 - 2 \times (-1) = 5 + 2 = 7$$

When you have crossed out the row and the column containing 7, you are left with the elements $\begin{pmatrix} 5 & 2 \\ -1 & 1 \end{pmatrix}$ and you evaluate the determinant of this matrix.

The minor of 7 is 7.

Example 12

Find the value of $\begin{vmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ -1 & 4 & 3 \end{vmatrix}$.

$$\begin{vmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ -1 & 4 & 3 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix} + 4\begin{vmatrix} 3 & 2 \\ -1 & 4 \end{vmatrix}$$

$$= 1(6 - 4) - 2(9 + 1) + 4(12 + 2)$$

$$= 1 \times 2 - 2 \times 10 + 4 \times 14$$

$$= 2 - 20 + 56 = 38$$

The determinant of this 2 × 2 matrix is the minor of the top left element.

The determinant of this 2 × 2 matrix is the minor of the top centre element.

The determinant of this 2 × 2 matrix is the minor of the top right element.

Example 13

The matrix $\mathbf{A} = \begin{pmatrix} 3 & k & 0 \\ -2 & 1 & 2 \\ 5 & 0 & k+3 \end{pmatrix}$, where k is a constant.

a Find det **A** in terms of k.

Given that **A** is singular,

b find the possible values of k.

a $\begin{vmatrix} 3 & k & 0 \\ -2 & 1 & 2 \\ 5 & 0 & k+3 \end{vmatrix} = 3\begin{vmatrix} 1 & 2 \\ 0 & k+3 \end{vmatrix} - k\begin{vmatrix} -2 & 2 \\ 5 & k+3 \end{vmatrix} + 0\begin{vmatrix} -2 & 1 \\ 5 & 0 \end{vmatrix}$

$= 3(k+3) - k(-2(k+3) - 10)$

$= 3k + 9 + 2k^2 + 16k$

$= 2k^2 + 19k + 9$

b As **A** is singular,

$2k^2 + 19k + 9 = 0$

$(2k + 1)(k + 9) = 0$

$k = -\frac{1}{2}, -9$

Part **b** will require you to solve det **A** = 0, so multiply this expression out, collect together terms and express the result as a quadratic.

Problem-solving

As **A** is singular, its determinant is 0. This gives a quadratic equation, which you solve, giving two possible values of k.

Exercise 6C

1 Find the determinants of the following matrices.

a $\begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 4 & 2 \\ 1 & 2 \end{pmatrix}$ **c** $\begin{pmatrix} -2 & 1 \\ 3 & 0 \end{pmatrix}$

d $\begin{pmatrix} -4 & -4 \\ 1 & 1 \end{pmatrix}$ **e** $\begin{pmatrix} 7 & -4 \\ 0 & 3 \end{pmatrix}$ **f** $\begin{pmatrix} -1 & -1 \\ -6 & -10 \end{pmatrix}$

P 2 Find the values of a for which these matrices are singular.

a $\begin{pmatrix} a & 1+a \\ 3 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 1+a & 3-a \\ a+2 & 1-a \end{pmatrix}$ **c** $\begin{pmatrix} 2+a & 1-a \\ 1-a & a \end{pmatrix}$

E/P 3 Given that k is a real number and that $\mathbf{M} = \begin{pmatrix} -2 & 1-k \\ k-1 & k \end{pmatrix}$,

find the exact values of k for which **M** is a singular matrix. **(3 marks)**

E/P 4 $\mathbf{P} = \begin{pmatrix} 3k & 4-k \\ k-2 & -k \end{pmatrix}$, where k is a real constant.

Given that **P** is a singular matrix, find the possible values of k. **(3 marks)**

5 The matrix $\mathbf{A} = \begin{pmatrix} a & 2a \\ b & 2b \end{pmatrix}$ and the matrix $\mathbf{B} = \begin{pmatrix} 2b & -2a \\ -b & a \end{pmatrix}$.

a Find det **A** and det **B**.

b Find **AB**.

6 Use your calculator to find the values of these determinants.

a $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$ **b** $\begin{vmatrix} 0 & 4 & 0 \\ 5 & -2 & 3 \\ 2 & 1 & 4 \end{vmatrix}$ **c** $\begin{vmatrix} 1 & 0 & 1 \\ 2 & 4 & 1 \\ 3 & 5 & 2 \end{vmatrix}$ **d** $\begin{vmatrix} 2 & -3 & 4 \\ 2 & 2 & 2 \\ 5 & 5 & 5 \end{vmatrix}$

7 Without using your calculator, find the values of these determinants.

a $\begin{vmatrix} 4 & 3 & -1 \\ 2 & -2 & 0 \\ 0 & 4 & -2 \end{vmatrix}$ **b** $\begin{vmatrix} 3 & -2 & 1 \\ 4 & 1 & -3 \\ 7 & 2 & -4 \end{vmatrix}$ **c** $\begin{vmatrix} 5 & -2 & -3 \\ 6 & 4 & 2 \\ -2 & -4 & -3 \end{vmatrix}$

P **8** The matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 & -4 \\ 2k+1 & 3 & k \\ 1 & 0 & 1 \end{pmatrix}$.

Given that $\mathbf{A}$ is singular, find the value of the constant k.

P **9** The matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ k & 2 & 4 \\ -2 & 1 & k+3 \end{pmatrix}$, where k is a constant.

Given that the determinant of $\mathbf{A}$ is 8, find the possible values of k.

10 The matrix $\mathbf{A} = \begin{pmatrix} 2 & 5 & 3 \\ -2 & 0 & 4 \\ 3 & 10 & 8 \end{pmatrix}$ and the matrix $\mathbf{B} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & -2 & -1 \end{pmatrix}$.

a Show that $\mathbf{A}$ is singular.

b Find $\mathbf{AB}$.

c Show that $\mathbf{AB}$ is also singular.

E/P **11** Show that, for all values of a, b and c, the matrix $\begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix}$ is singular. **(3 marks)**

E/P **12** Show that, for all real values of x, the matrix $\begin{pmatrix} 2 & -2 & 4 \\ 3 & x & -2 \\ -1 & 3 & x \end{pmatrix}$ is non-singular. **(3 marks)**

E/P **13** Find all the values of x for which the matrix $\begin{pmatrix} x-3 & -2 & 0 \\ 1 & x & -2 \\ -2 & -1 & x+1 \end{pmatrix}$ is singular. **(4 marks)**

E/P **14** The matrix $\mathbf{M} = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}$ and the matrix $\mathbf{N} = \begin{pmatrix} -1 & k \\ 4 & 3 \end{pmatrix}$, where k is a constant.

a Evaluate the determinant of $\mathbf{M}$. **(1 mark)**

b Given that the determinant of $\mathbf{N}$ is 7, find the value of k. **(2 marks)**

c Using the value of k found in part **b**, find $\mathbf{MN}$. **(1 mark)**

d Verify that $\det \mathbf{MN} = \det \mathbf{M} \det \mathbf{N}$. **(1 mark)**

E/P **15** The matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 4 \\ -4 & 2 & 1 \end{pmatrix}$ and the matrix $\mathbf{B} = \begin{pmatrix} 3 & 1 & 2 \\ k & 4 & 5 \\ 0 & 2 & 3 \end{pmatrix}$, where k is a constant.

a Evaluate the determinant of **A**. **(2 marks)**

Given that the determinant of **B** is 2,

b find the value of k. **(3 marks)**

Using the value of k found in part **b**,

c find **AB** **(2 marks)**

d verify that det **AB** = det **A** det **B**. **(2 marks)**

Challenge

a Find all the possible 2×2 singular matrices whose elements are the numbers 1 and –1.

b Find all the possible 2×2 singular matrices whose elements are the numbers 1 and 0.

Hint In part **a**, there are 8 possible matrices.

6.4 Inverting a 2 × 2 matrix

You can find the **inverse** of any non-singular matrix.

- **The inverse of a matrix M is the matrix $\mathbf{M}^{-1}$ such that $\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$.**

You can use the following formula to find the inverse of a 2×2 matrix.

- **If $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathbf{M}^{-1} = \dfrac{1}{\det \mathbf{M}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.**

Note If det **M** = 0, you will not be able to find the inverse matrix, since $\dfrac{1}{\det \mathbf{M}}$ is undefined.

Example 14

$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}$

For each of the matrices **A**, **B** and **C**, determine whether or not the matrix is singular. If the matrix is non-singular, find its inverse.

$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$ so $\det \mathbf{A} = 3 \times 1 - 2 \times (-1)$

Use the determinant formula with $a = 3$, $b = 2$, $c = -1$ and $d = 1$.

$\det \mathbf{A} = 5$

Since $5 \neq 0$, **A** is non-singular.

Swap a and d and change the signs of b and c.

So $\mathbf{A}^{-1} = \dfrac{1}{5}\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$ or $\begin{pmatrix} 0.2 & -0.4 \\ 0.2 & 0.6 \end{pmatrix}$

$\mathbf{A}^{-1}$ can be left in either form.

$\mathbf{B} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$ so $\det \mathbf{B} = 2 \times 1 - 1 \times 2 = 0$

Remember if det **B** = 0 then **B** is singular.

So **B** is singular and $\mathbf{B}^{-1}$ cannot be found.

$C = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}$ so $\det C = 1 \times 0 - 3 \times 2 = -6$

This is non-zero and so C is a non-singular matrix.

$C^{-1} = -\frac{1}{6}\begin{pmatrix} 0 & -3 \\ -2 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{6} \end{pmatrix}$

Note that a determinant can be a negative number.

Swap a and d and change the signs of b and c. Then multiply by $\frac{1}{\det \mathbf{C}}$

You can find the inverse of a matrix using your calculator.

- **If A and B are non-singular matrices, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.**

Example 15

P and **Q** are non-singular matrices. Prove that $(\mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1}$.

Let $C = (PQ)^{-1}$ then $(PQ)C = I$.

$P^{-1}PQC = P^{-1}I$

$(P^{-1}P)QC = P^{-1}I$

So $QC = P^{-1}$

$Q^{-1}QC = Q^{-1}P^{-1}$

$IC = Q^{-1}P^{-1}$

$C = Q^{-1}P^{-1}$

So $(PQ)^{-1} = Q^{-1}P^{-1}$ as required.

Use the definition of inverse $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$.

Multiply on the left by $\mathbf{P}^{-1}$.

Remember $\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}$, $\mathbf{IQ} = \mathbf{Q}$ and $\mathbf{P}^{-1}\mathbf{I} = \mathbf{P}^{-1}$.

Multiply on the left by $\mathbf{Q}^{-1}$.

Use $\mathbf{Q}^{-1}\mathbf{Q} = \mathbf{I}$.

Example 16

A and **B** are non-singular 2 × 2 matrices such that $\mathbf{BAB} = \mathbf{I}$.

a Prove that $\mathbf{A} = \mathbf{B}^{-1}\mathbf{B}^{-1}$.

Given that $\mathbf{B} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$:

b find the matrix **A** such that $\mathbf{BAB} = \mathbf{I}$.

a $BAB = I$

$B^{-1}BAB = B^{-1}I$

$(B^{-1}B)AB = B^{-1}I$

$AB = B^{-1}$

$AB\,B^{-1} = B^{-1}\,B^{-1}$

$AI = B^{-1}\,B^{-1}$

And hence $A = B^{-1}\,B^{-1}$ as required.

Multiply on the left by $\mathbf{B}^{-1}$.

Remember $\mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$ and $\mathbf{B}^{-1}\mathbf{I} = \mathbf{B}^{-1}$.

Multiply on the right by $\mathbf{B}^{-1}$ and remember $\mathbf{B}\,\mathbf{B}^{-1} = \mathbf{I}$.

b $\mathbf{B} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ so det $\mathbf{B} = 2 \times 3 - 5 \times 1 = 1$

So $\mathbf{B}^{-1} = \frac{1}{1}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

From part **a**,

$\mathbf{A} = \mathbf{B}^{-1}\mathbf{B}^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

$\mathbf{A} = \begin{pmatrix} 14 & -25 \\ -5 & 9 \end{pmatrix}$

First find $\mathbf{B}^{-1}$.

Use the result from part **a** and matrix multiplication to find **A**.

Exercise 6D

1 Determine which of these matrices are singular and which are non-singular. For those that are non-singular find the inverse matrix.

a $\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix}$ **c** $\begin{pmatrix} 2 & 5 \\ 0 & 0 \end{pmatrix}$

d $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$ **e** $\begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}$ **f** $\begin{pmatrix} 4 & 3 \\ 6 & 2 \end{pmatrix}$

2 Find inverses of these matrices, giving your answers in terms of a and b.

a $\begin{pmatrix} a & 1+a \\ 1+a & 2+a \end{pmatrix}$ **b** $\begin{pmatrix} 2a & 3b \\ -a & -b \end{pmatrix}$

P **3** **a** Given that $\mathbf{ABC} = \mathbf{I}$, prove that $\mathbf{B}^{-1} = \mathbf{CA}$.

b Given that $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & -6 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix}$, find **B**.

4 **a** Given that $\mathbf{AB} = \mathbf{C}$, find an expression for **B**, in terms of **A** and **C**.

b Given further that $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 3 & 6 \\ 1 & 22 \end{pmatrix}$, find **B**.

5 **a** Given that $\mathbf{BAC} = \mathbf{B}$, where **B** is a non-singular matrix, find an expression for **A** in terms of **C**.

b When $\mathbf{C} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$, find **A**.

6 The matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}$ and $\mathbf{AB} = \begin{pmatrix} 4 & 7 & -8 \\ -8 & -13 & 18 \end{pmatrix}$. Find the matrix **B**.

P **7** The matrix $\mathbf{B} = \begin{pmatrix} 5 & -4 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{AB} = \begin{pmatrix} 11 & -1 \\ -8 & 9 \\ -2 & -1 \end{pmatrix}$. Find the matrix **A**.

P 8 The matrix $\mathbf{A} = \begin{pmatrix} 3a & b \\ 4a & 2b \end{pmatrix}$, where a and b are non-zero constants.

a Find $\mathbf{A}^{-1}$, giving your answer in terms of a and b.

The matrix $\mathbf{B} = \begin{pmatrix} -a & b \\ 3a & 2b \end{pmatrix}$ and the matrix $\mathbf{X}$ is given by $\mathbf{B} = \mathbf{XA}$.

b Find $\mathbf{X}$, giving your answer in terms of a and b.

E/P 9 The non-singular matrices $\mathbf{A}$ and $\mathbf{B}$ are such that $\mathbf{AB} = \mathbf{BA}$, and $\mathbf{ABA} = \mathbf{B}$.

a Prove that $\mathbf{A}^2 = \mathbf{I}$. **(3 marks)**

Given that $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, by considering a matrix $\mathbf{B}$ of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

b show that $a = d$ and $b = c$. **(3 marks)**

E/P 10 $\mathbf{M} = \begin{pmatrix} 2 & 3 \\ k & -1 \end{pmatrix}$ where k is a constant.

a For which values of k does $\mathbf{M}$ have an inverse? **(2 marks)**

b Given that $\mathbf{M}$ is non-singular, find $\mathbf{M}^{-1}$ in terms of k. **(3 marks)**

E/P 11 Given that $\mathbf{A} = \begin{pmatrix} 4 & p \\ -2 & -2 \end{pmatrix}$ where p is a constant and $p \neq 4$,

a find $\mathbf{A}^{-1}$ in terms of p. **(2 marks)**

b Given that $\mathbf{A} + \mathbf{A}^{-1} = \begin{pmatrix} 5 & \frac{9}{2} \\ -3 & -4 \end{pmatrix}$, find the value of p. **(3 marks)**

E/P 12 $\mathbf{M} = \begin{pmatrix} k & -3 \\ 4 & k+3 \end{pmatrix}$ where k is a real constant.

a Find $\det \mathbf{M}$ in terms of k. **(2 marks)**

b Show that $\mathbf{M}$ is non-singular for all values of k. **(3 marks)**

c Given that $10\mathbf{M}^{-1} + \mathbf{M} = \mathbf{I}$ where $\mathbf{I}$ is the 2×2 identity matrix, find the value of k. **(3 marks)**

E/P 13 Given that $\mathbf{A} = \begin{pmatrix} a & 2 \\ 3 & 2a \end{pmatrix}$ where a is a real constant,

a find $\mathbf{A}^{-1}$ in terms of a **(3 marks)**

b write down two values of a for which $\mathbf{A}^{-1}$ does not exist. **(1 mark)**

6.5 Inverting a 3 × 3 matrix

Finding the inverse of a 3 × 3 matrix is more complicated. You need to know the following definition.

- **The transpose of a matrix is found by interchanging the rows and the columns.**

Notation The transpose of the matrix **M** is written as $\mathbf{M}^T$.

For example, if $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$, $\mathbf{A}^T = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$.

- **Finding the inverse of a 3 × 3 matrix A usually consists of the following 5 steps.**

Step 1 **Find the determinant of A, det A.**

Step 2 **Form the matrix of the minors of A. In this chapter, the symbol M is used for the matrix of the minors unless this causes confusion with another matrix in the question.**

In forming the matrix of minors, M, each of the nine elements of the matrix A is replaced by its minor.

Step 3 **From the matrix of minors, form the matrix of cofactors by changing the signs of some elements of the matrix of minors according to the rule of alternating signs illustrated by the pattern**

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

You leave the elements of the matrix of minors corresponding to the + signs in this pattern unchanged. You change the signs of the elements corresponding to the – signs.

Notation A cofactor is a minor with its appropriate sign.

In this chapter, the symbol **C** is used for the matrix of the cofactors unless this causes confusion with another matrix in the question.

Step 4 **Write down the transpose, $\mathbf{C}^T$, of the matrix of cofactors.**

Step 5 **The inverse of the matrix A is given by the formula**

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\mathbf{C}^T$$

Each element of the matrix $\mathbf{C}^T$ is divided by the determinant of **A**.

Example 17

The matrix $\mathbf{A} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & 1 \\ 2 & -1 & 0 \end{pmatrix}$. Find $\mathbf{A}^{-1}$.

Step 1

$\det \mathbf{A} = 1\begin{vmatrix} 4 & 1 \\ -1 & 0 \end{vmatrix} - 3\begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & 4 \\ 2 & -1 \end{vmatrix}$

$= 1(0 + 1) - 3(0 - 2) + 1(0 - 8)$

$= 1 + 6 - 8 = -1$

The first step of finding the inverse of a matrix is to evaluate its determinant.

Step 2

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 4 & 1 \\ -1 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 4 \\ 2 & -1 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & -8 \\ 1 & -2 & -7 \\ -1 & 1 & 4 \end{pmatrix}$$

The second step is to form the matrix of minors. The minor of an element is found by deleting the row and the column in which the element lies, then finding the determinant of the resulting 2×2 matrix.

For example, to find the minor of 4 in $\begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & 1 \\ 2 & -1 & 0 \end{pmatrix}$, delete the row and column containing 4, $\begin{pmatrix} 1 & \cancel{3} & 1 \\ \cancel{0} & \cancel{4} & \cancel{1} \\ 2 & \cancel{-1} & 0 \end{pmatrix}$. The minor is the determinant of the elements left, $\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2$.

Step 3

$$\mathbf{C} = \begin{pmatrix} 1 & 2 & -8 \\ -1 & -2 & 7 \\ -1 & -1 & 4 \end{pmatrix}$$

You find the matrix of cofactors by adjusting the signs of the minors using the pattern $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$. Here you leave the elements $\begin{pmatrix} 1 & & -8 \\ & -2 & \\ -1 & & 4 \end{pmatrix}$ unchanged but change the signs of $\begin{pmatrix} & -2 & \\ 1 & & -7 \\ & 1 & \end{pmatrix}$.

Step 4

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -2 & -1 \\ -8 & 7 & 4 \end{pmatrix}$$

Step 5

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\mathbf{C}^{\mathrm{T}} = \frac{1}{-1}\begin{pmatrix} 1 & -1 & -1 \\ 2 & -2 & -1 \\ -8 & 7 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & 1 \\ -2 & 2 & 1 \\ 8 & -7 & -4 \end{pmatrix}$$

You divide each term of the transpose of the matrix of cofactors, $\mathbf{C}^{\mathrm{T}}$, by the determinant of $\mathbf{A}$, -1.

Example 18

The matrix $\mathbf{A} = \begin{pmatrix} 3 & 2 & -2 \\ -2 & k & 0 \\ -1 & -3 & 3 \end{pmatrix}$, $k \neq 0$. Find $\mathbf{A}^{-1}$.

Step 1

$\det \mathbf{A} = 3\begin{vmatrix} k & 0 \\ -3 & 3 \end{vmatrix} - 2\begin{vmatrix} -2 & 0 \\ -1 & 3 \end{vmatrix} + (-2)\begin{vmatrix} -2 & k \\ -1 & -3 \end{vmatrix}$

$= 3(3k - 0) - 2(-6 - 0) - 2(6 + k)$

$= 9k + 12 - 12 - 2k = 7k$

Watch out Make sure you understand the steps needed to find the inverse of a 3×3 matrix. You won't be able to use your calculator if the matrix contains unknowns.

As you are given that $k \neq 0$, the matrix is non-singular and the inverse can be found.

Step 2

$$M = \begin{pmatrix} \begin{vmatrix} k & 0 \\ -3 & 3 \end{vmatrix} & \begin{vmatrix} -2 & 0 \\ -1 & 3 \end{vmatrix} & \begin{vmatrix} -2 & k \\ -1 & -3 \end{vmatrix} \\ \begin{vmatrix} 2 & -2 \\ -3 & 3 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ -1 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ -1 & -3 \end{vmatrix} \\ \begin{vmatrix} 2 & -2 \\ k & 0 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ -2 & 0 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ -2 & k \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 3k & -6 & k+6 \\ 0 & 7 & -7 \\ 2k & -4 & 3k+4 \end{pmatrix}$$

The second step is to find the matrix of minors in terms of k.

Step 3

$$C = \begin{pmatrix} 3k & 6 & k+6 \\ 0 & 7 & 7 \\ 2k & 4 & 3k+4 \end{pmatrix}$$

You obtain the matrix of the cofactors from the matrix of the minors by changing the signs of the elements corresponding to the – signs in the pattern $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$.

Step 4

$$C^T = \begin{pmatrix} 3k & 0 & 2k \\ 6 & 7 & 4 \\ k+6 & 7 & 3k+4 \end{pmatrix}$$

Step 5

$$A^{-1} = \frac{1}{\det A} C^T = \frac{1}{7k}\begin{pmatrix} 3k & 0 & 2k \\ 6 & 7 & 4 \\ k+6 & 7 & 3k+4 \end{pmatrix}$$

You can leave the answer in this form or write the inverse matrix as $\begin{pmatrix} \frac{3}{7} & 0 & \frac{2}{7} \\ \frac{6}{7k} & \frac{1}{k} & \frac{4}{7k} \\ \frac{k+6}{7k} & \frac{1}{k} & \frac{3k+4}{7k} \end{pmatrix}$.

Example 19

The matrix $\mathbf{A} = \begin{pmatrix} -2 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$ and the matrix $\mathbf{B}$ is such that $(\mathbf{AB})^{-1} = \begin{pmatrix} 8 & -17 & 9 \\ -5 & 10 & -6 \\ -3 & 5 & -4 \end{pmatrix}$.

a Show that $\mathbf{A}^{-1} = \mathbf{A}$.

b Find $\mathbf{B}^{-1}$.

a $$A^2 = \begin{pmatrix} -2 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} -2 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4+0-3 & -6+3+3 & 6+0-6 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ -2+0+2 & 3-1-2 & -3+0+4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$AA = I$

$A^{-1}AA = A^{-1}$

$A = A^{-1}$ as required

Problem-solving

Proving $\mathbf{A} = \mathbf{A}^{-1}$ is equivalent to proving $\mathbf{A}^2 = \mathbf{I}$. You still need to add working to show that $\mathbf{A}^2 = \mathbf{I}$ implies that $\mathbf{A} = \mathbf{A}^{-1}$.

Multiply both sides by $\mathbf{A}^{-1}$.

Since $\mathbf{A}^{-1}\mathbf{AA} = (\mathbf{A}^{-1}\mathbf{A})\mathbf{A} = \mathbf{IA} = \mathbf{A}$.

b $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

$(\mathbf{AB})^{-1}\mathbf{A} = \mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{A} = \mathbf{B}^{-1}\mathbf{I} = \mathbf{B}^{-1}$

$\mathbf{B}^{-1} = (\mathbf{AB})^{-1}\mathbf{A}$

$$= \begin{pmatrix} 8 & -17 & 9 \\ -5 & 10 & -6 \\ -3 & 5 & -4 \end{pmatrix}\begin{pmatrix} -2 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -16+0+9 & 24-17-9 & -24+0+18 \\ 10+0-6 & -15+10+6 & 15+0-12 \\ 6+0-4 & -9+5+4 & 9+0-8 \end{pmatrix}$$

$$= \begin{pmatrix} -7 & -2 & -6 \\ 4 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

Multiply both sides of this formula on the right by $\mathbf{A}$ and use $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ to obtain an expression for $\mathbf{B}^{-1}$ in terms of $(\mathbf{AB})^{-1}$ and $\mathbf{A}$, both of which you already know.

You could check your answer by multiplying these matrices quickly using your calculator.

Notation If $\mathbf{A}^{-1} = \mathbf{A}$, then the matrix $\mathbf{A}$ is said to be **self-inverse**.

Exercise 6E

1 Use your calculator to find the inverses of these matrices.

a $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ **c** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}$

2 Without using a calculator, find the inverses of these matrices.

a $\begin{pmatrix} 1 & -3 & 2 \\ 0 & -2 & 1 \\ 3 & 0 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 2 & 3 & 2 \\ 3 & -2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$ **c** $\begin{pmatrix} 3 & 2 & -7 \\ 1 & -3 & 1 \\ 0 & 2 & -2 \end{pmatrix}$

3 The matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ and the matrix $\mathbf{B} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$.

a Find $\mathbf{A}^{-1}$. **b** Find $\mathbf{B}^{-1}$.

Given that $(\mathbf{AB})^{-1} = \begin{pmatrix} -\frac{2}{3} & \frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

c verify that $\mathbf{B}^{-1}\mathbf{A}^{-1} = (\mathbf{AB})^{-1}$.

E/P **4** The matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 & 3 \\ k & 1 & 1 \\ 1 & 1 & 4 \end{pmatrix}$.

a Show that $\det \mathbf{A} = 3(k+1)$. **(3 marks)**

b Given that $k \neq -1$, find $\mathbf{A}^{-1}$. **(4 marks)**

E/P **5** The matrix $\mathbf{A} = \begin{pmatrix} 5 & a & 4 \\ b & -7 & 8 \\ 2 & -2 & c \end{pmatrix}$.

Given that $\mathbf{A} = \mathbf{A}^{-1}$, find the values of the constants a, b and c. **(6 marks)**

6 The matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 4 & -3 & 0 \\ -3 & 3 & 1 \end{pmatrix}$.

a Show that $\mathbf{A}^3 = \mathbf{I}$. **b** Hence find $\mathbf{A}^{-1}$.

7 The matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 3 & -3 & 1 \\ 0 & 3 & 2 \end{pmatrix}$.

a Show that $\mathbf{A}^3 = 13\mathbf{A} - 15\mathbf{I}$. **b** Deduce that $15\mathbf{A}^{-1} = 13\mathbf{I} - \mathbf{A}^2$. **c** Hence find $\mathbf{A}^{-1}$.

8 The matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 4 & 3 & -2 \\ 0 & 3 & -4 \end{pmatrix}$.

a Show that $\mathbf{A}$ is singular.

The matrix $\mathbf{C}$ is the matrix of the cofactors of $\mathbf{A}$.

b Find $\mathbf{C}$. **c** Show that $\mathbf{A}\mathbf{C}^{\mathrm{T}} = \mathbf{0}$.

E/P **9** $\mathbf{M} = \begin{pmatrix} 2 & k & 3 \\ -1 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix}$ where k is a real constant.

a For which values of k does $\mathbf{M}$ have an inverse? **(2 marks)**

b Given that $\mathbf{M}$ is non-singular, find $\mathbf{M}^{-1}$ in terms of k. **(4 marks)**

E/P **10** $\mathbf{A} = \begin{pmatrix} p & 2p & 3 \\ 4 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}$ where p is a real constant.

Given that $\mathbf{A}$ is non-singular, find $\mathbf{A}^{-1}$ in terms of p. **(4 marks)**

6.6 Solving systems of equations using matrices

You can use the inverse of a 3 × 3 matrix to solve a system of three simultaneous linear equations in three unknowns.

- **If $\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{v}$ then $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A}^{-1}\mathbf{v}$.**

If $\mathbf{A}$ is non-singular, a unique solution for $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ can be found for any vector $\mathbf{v}$.

Example 20

Use an inverse matrix to solve the simultaneous equations:

$$-x + 6y - 2z = 21$$
$$6x - 2y - z = -16$$
$$-2x + 3y + 5z = 24$$

Write the system of equations using matrices:

$$\begin{pmatrix} -1 & 6 & -2 \\ 6 & -2 & -1 \\ -2 & 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 21 \\ -16 \\ 24 \end{pmatrix}$$

Find the inverse of the left-hand matrix:

$$\frac{1}{189}\begin{pmatrix} 7 & 36 & 10 \\ 28 & 9 & 13 \\ -14 & 9 & 34 \end{pmatrix}$$

Left-multiply the right-hand matrix by this inverse:

$$\frac{1}{189}\begin{pmatrix} 7 & 36 & 10 \\ 28 & 9 & 13 \\ -14 & 9 & 34 \end{pmatrix}\begin{pmatrix} 21 \\ -16 \\ 24 \end{pmatrix} = \frac{1}{189}\begin{pmatrix} -189 \\ 756 \\ 378 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$$

Hence $x = -1$, $y = 4$ and $z = 2$.

You can confirm that this is equivalent to the original equations by multiplying out the left-hand side:

$$\begin{pmatrix} -1 & 6 & -2 \\ 6 & -2 & -1 \\ -2 & 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x + 6y - 2z \\ 6x - 2y - z \\ -2x + 3y + 5z \end{pmatrix}$$

Use your calculator to find the inverse matrix and then to multiply the matrices.

Problem-solving

Once you have found the inverse matrix, you could use it to solve a similar system of equations with a different answer vector.

Example 21

A colony of 1000 mole-rats is made up of adult males, adult females and youngsters. Originally there were 100 more adult females than adult males.

After one year:

- the number of adult males had increased by 2%
- the number of adult females had increased by 3%
- the number of youngsters had decreased by 4%
- the total number of mole-rats had decreased by 20

Form and solve a matrix equation to find out how many of each type of mole-rat were in the original colony.

Problem-solving

Assign a letter to each unknown value, then work your way through the question text using the information to formulate equations.

x = number of adult males
y = number of adult females
z = number of youngsters

$$x + y + z = 1000$$
$$x - y = -100$$
$$1.02x + 1.03y + 0.96z = 980$$

This represents the total number of mole-rats in the original colony.

There are 100 more adult females than adult males.

This equation represents the number of mole-rats of each type after 1 year. You could also consider the percentage changes in each population and the total overall change of –20:
$0.02x + 0.03y - 0.04z = -20$

So $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1.02 & 1.03 & 0.96 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1000 \\ -100 \\ 980 \end{pmatrix}$

Convert the three equations into a matrix equation.

$\mathbf{A}^{-1} = \frac{1}{13}\begin{pmatrix} -96 & 7 & 100 \\ -96 & -6 & 100 \\ 205 & -1 & -200 \end{pmatrix}$

Use your calculator to find $\mathbf{A}^{-1}$.

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 1000 \\ -100 \\ 980 \end{pmatrix}$

$= \frac{1}{13}\begin{pmatrix} -96 & 7 & 100 \\ -96 & -6 & 100 \\ 205 & -1 & -200 \end{pmatrix}\begin{pmatrix} 1000 \\ -100 \\ 980 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 100 \\ 200 \\ 700 \end{pmatrix}$

Use your calculator to multiply $\mathbf{A}^{-1}$ by the answer vector.

There were 100 adult males, 200 adult females and 700 youngsters in the original colony.

Write your answer in the context of the question, and check that it makes sense. You could also check that your answer matches the information given in the question. For example, the initial number of mole-rats is 100 + 200 + 700 = 1000, as given in the question.

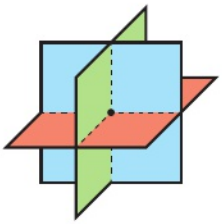
You need to be able to determine whether a system of three linear equations in three unknowns is **consistent** or **inconsistent**.

- **A system of linear equations is consistent if there is at least one set of values that satisfies all the equations simultaneously. Otherwise, it is inconsistent.**

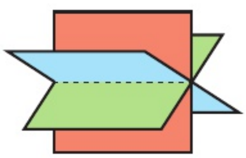
If the matrix corresponding to a set of linear equations is non-singular, then the system has one unique solution and is consistent. However, if the matrix is singular, there are two possibilities: either the system is consistent and has infinitely many solutions, or it is inconsistent and has no solutions.

You can visualise the different situations by considering the points of intersection of the planes corresponding to each equation. Here are some of the different possible configurations:

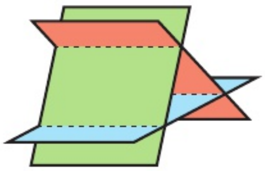
Links An equation in the form $ax + by + cz = d$ is the equation of a plane in three dimensions. → Section 9.2



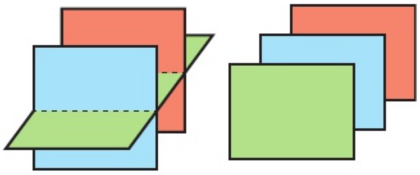
The planes meet at a **point**. The system of equations is **consistent** and has **one solution** represented by this point. This is the only case when the corresponding matrix is **non-singular**.



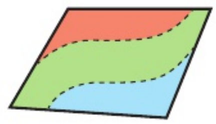
The planes form a **sheaf**. The system of equations is **consistent** and has **infinitely many solutions** represented by the line of intersection of the three planes.



The planes form a **prism**. The system of equations is **inconsistent** and has **no solutions**.



Two or more of the planes are parallel and non-identical. The system of equations is **inconsistent** and has **no solutions**.



All three equations represent the same plane. In this case the system of equations is **consistent** and has **infinitely many solutions**.

Hint If one row of the corresponding matrix is a **linear multiple** of another row, then these two rows will represent parallel planes.

Example 22

A system of equations is shown below:

$$\begin{aligned} 3x - ky - 6z &= k \\ kx + 3y + 3z &= 2 \\ -3x - y + 3z &= -2 \end{aligned}$$

For each of the following values of k, determine whether the system of equations is consistent or inconsistent. If the system is consistent, determine whether there is a unique solution or an infinity of solutions. In each case, identify the geometric configuration of the planes corresponding to each value of k.

a $k = 0$ **b** $k = 1$ **c** $k = -6$

a $k = 0$: $\begin{vmatrix} 3 & 0 & -6 \\ 0 & 3 & 3 \\ -3 & -1 & 3 \end{vmatrix} = -18$

Use your calculator to find the determinant of the corresponding matrix.

The corresponding matrix is non-singular so the system is consistent and has a unique solution. The planes meet at a single point.

Problem-solving

If the matrix is singular, you need to consider the equations to determine whether the system is consistent. Eliminate one of the variables from two different pairs of equations. If the resulting two equations are consistent then the system will be consistent.

b $k = 1$: $\begin{vmatrix} 3 & -1 & -6 \\ 1 & 3 & 3 \\ -3 & -1 & 3 \end{vmatrix} = 0$

$$\begin{aligned} 3x - y - 6z &= 1 && (1) \\ x + 3y + 3z &= 2 && (2) \\ -3x - y + 3z &= -2 && (3) \end{aligned}$$

$(1) + 2 \times (2)$: $5x + 5y = 5$ (4)

$(2) - (3)$: $4x + 4y = 4$ (5)

Eliminate z from equations (1) and (2) to form equation (4), and eliminate z from equations (2) and (3) to form equation (5).

Equations (4) and (5) are consistent so the system is consistent and has an infinity of solutions. The planes meet in a sheaf.

Equations (4) and (5) are consistent because one is a linear multiple of the other. Any values of x and y that satisfy one equation will also satisfy the other.

c $k = -6$: $\begin{vmatrix} 3 & 6 & -6 \\ -6 & 3 & 3 \\ -3 & -1 & 3 \end{vmatrix} = 0$

$3x + 6y - 6z = -6$ (1)
$-6x + 3y + 3z = 2$ (2)
$-3x - y + 3z = -2$ (3)
(1) + (3): $5y - 3z = -8$ (4)
$2 \times$ (1) + (2): $15y - 9z = -10$ (5)

Equations (4) and (5) are inconsistent so the system is inconsistent and has no solutions. The planes form a prism.

Problem-solving

This method is equivalent to showing that one equation is a linear combination of the other two. In this case $4 \times (1) + 3 \times (2) = -5 \times (3)$.

Eliminate x in two different ways to get two equations in y and z.

$3 \times (4)$ gives $15y - 9z = -24$. There are no values of y and z that can satisfy this equation and equation (5) simultaneously.

Exercise 6F

1 Solve the following systems of equations using inverse matrices.

a $\begin{aligned} 2x - 6y + 4z &= 32 \\ 3x + 2y - 9z &= -49 \\ -2x + 4y + z &= -3 \end{aligned}$ **b** $\begin{aligned} -4x + 6y - 2z &= -22 \\ 3x + 3y - 2z &= 1 \\ -6x - 7y + 3z &= 3 \end{aligned}$

c $\begin{aligned} 4x + 7y - 2z &= 21 \\ -10x - y - 7z &= 0 \\ -2x + y - 4z &= 9 \end{aligned}$ **d** $\begin{aligned} -3x - 6y + 4z &= -23 \\ -3x - 6y - 10z &= -1 \\ 3x + 7y - 3z &= 27 \end{aligned}$

E/P **2** Three planes A, B and C are defined by the following equations.

A: $x - 3y - 4z = 3$
B: $6x + 5y - 7z = 30$
C: $x + 4y + 6z = -3$

By constructing and solving a suitable matrix equation, show that these three planes intersect at a single point and find the coordinates of that point. **(5 marks)**

E/P **3** Phyllis invested £3000 across three savings accounts, A, B and C. She invested £190 more in account A than in account B.
After two years, account A had increased in value by 1%, account B had increased in value by 2.5% and account C had decreased in value by 1.5%. The total value of Phyllis's savings had increased by £41.
Form and solve a matrix equation to find out how much money was invested by Phyllis in each account. **(7 marks)**

E/P **4** A colony of bats is made up of brown bats, grey bats and black bats. Initially there are 2000 bats and there are 250 more brown bats than grey bats.
After one year:

- the number of brown bats had fallen by 1%
- the number of grey bats had fallen by 2%

- the number of black bats had increased by 4%
- overall there were 40 more bats

Form and solve a matrix equation to find out how many of each colour bat there were in the initial colony. **(7 marks)**

E/P **5** Three planes A, B and C are defined by the following equations:

$$A: \quad x + ay + 2z = a$$
$$B: \quad x - y - z = a$$
$$C: \quad z + 4y + 4z = 0$$

Given that the planes do not meet at a single point,

a find the value of a **(4 marks)**

b determine whether the three equations form a consistent system, and give a geometric interpretation of your answer. **(4 marks)**

Hint If the three planes do not meet at a single point, the corresponding 3 × 3 matrix must be singular.

E/P **6** The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 1 & 4 & q \\ 2 & 3 & -3 \\ q & q & -2 \end{pmatrix}$

a Given that det $\mathbf{M} = 0$, show that $q^2 + 9q - 10 = 0$. **(4 marks)**

A system of simultaneous equations is shown below:

$$x + 4y + qz = -16$$
$$2x + 3y - 3z = \tfrac{1}{2}q$$
$$qx + qy - 2z = -2$$

b For each of the following values of q, determine whether the system of equations is consistent or inconsistent. If the system is consistent, determine whether there is a unique solution or an infinity of solutions. In each case, identify the geometric configuration of the planes corresponding to each equation.

i $q = -10$ **ii** $q = 2$ **iii** $q = 1$ **(7 marks)**

Mixed exercise 6

P **1** The matrix $\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{AB} = \begin{pmatrix} 4 & 1 & 9 \\ 1 & 9 & 4 \end{pmatrix}$. Find the matrix $\mathbf{B}$.

E/P **2** The matrix $\mathbf{A} = \begin{pmatrix} a & b \\ 2a & 3b \end{pmatrix}$, where a and b are non-zero constants.

a Find $\mathbf{A}^{-1}$, giving your answer in terms of a and b. **(2 marks)**

The matrix $\mathbf{Y} = \begin{pmatrix} a & 2b \\ 2a & b \end{pmatrix}$ and the matrix $\mathbf{X}$ is given by $\mathbf{XA} = \mathbf{Y}$.

b Find $\mathbf{X}$, giving your answer in terms of a and b. **(3 marks)**

E **3** The 2×2, non-singular matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{X}$ satisfy $\mathbf{XB} = \mathbf{BA}$.

a Find an expression for $\mathbf{X}$ in terms of $\mathbf{A}$ and $\mathbf{B}$. **(1 mark)**

b Given that $\mathbf{A} = \begin{pmatrix} 5 & 3 \\ 0 & -2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$, find $\mathbf{X}$. **(2 marks)**

E/P **4** A matrix $\mathbf{A}$ is given as $\mathbf{A} = \begin{pmatrix} a & 2 & -1 \\ -1 & 1 & -1 \\ b & 2 & 1 \end{pmatrix}$.

Given that $\mathbf{A}^2 = \begin{pmatrix} -4 & 2 & -4 \\ -5 & -3 & -1 \\ 4 & 10 & -4 \end{pmatrix}$, find the values of a and b. **(3 marks)**

E/P **5** $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ t & 3 & 1 \\ -2 & -1 & 1 \end{pmatrix}$

Given that $\mathbf{A}$ is singular, find the value of t. **(3 marks)**

E/P **6** $\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ x & 2 & 0 \\ 3 & 1 & 1 \end{pmatrix}$

Find $\mathbf{M}^{-1}$ in terms of x. **(4 marks)**

E/P **7** $\mathbf{A} = \begin{pmatrix} k & -2 \\ -4 & k \end{pmatrix}$ where k is a real constant.

a For which values of k does $\mathbf{A}$ have an inverse? **(2 marks)**

b Given that $\mathbf{A}$ is non-singular, find $\mathbf{A}^{-1}$ in terms of k. **(3 marks)**

E/P **8** $\mathbf{B} = \begin{pmatrix} k & 6 \\ -1 & k-2 \end{pmatrix}$ where k is a real constant.

a Find det $\mathbf{B}$ in terms of k. **(2 marks)**

b Show that $\mathbf{B}$ is non-singular for all values of k. **(3 marks)**

c Given that $21\mathbf{B}^{-1} + \mathbf{B} = -8\mathbf{I}$ where $\mathbf{I}$ is the 2×2 identity matrix, find the value of k. **(3 marks)**

E/P **9** Given that $\mathbf{M} = \begin{pmatrix} 2 & -m \\ m & -1 \end{pmatrix}$ where m is a real constant,

a write down two values of m such that $\mathbf{M}$ is singular **(2 marks)**

b find $\mathbf{M}^{-1}$ in terms of m, given that $\mathbf{M}$ is non-singular. **(3 marks)**

E/P **10** $\mathbf{A} = \begin{pmatrix} 3 & 4 & 5 \\ 1 & a & -1 \\ -2 & 1 & 1 \end{pmatrix}$ where a is a real constant.

a For which values of a does $\mathbf{A}$ have an inverse? **(2 marks)**

b Given that $\mathbf{A}$ is non-singular, find $\mathbf{A}^{-1}$ in terms of a. **(4 marks)**

E/P **11** Three planes A, B and C are defined by the following equations.

$$\begin{aligned} A: &\quad x + y + z = 6 \\ B: &\quad x - 4y + 2z = -2 \\ C: &\quad 2x + y - 3z = 0 \end{aligned}$$

By constructing and solving a suitable matrix equation, show that these three planes intersect at a single point and find the coordinates of that point. **(5 marks)**

E/P **12** A sheep farmer has three types of sheep: Hampshire, Dorset horn and Wiltshire horn. Initially his flock had 2500 sheep in it. There were 300 more Hampshire sheep than Wiltshire horn.

After one year:

- the number of Hampshire sheep had increased by 6%
- the number of Dorset horn had increased by 4%
- the number of Wiltshire horn had increased by 3%
- overall the flock size had increased by 110

Form and solve a matrix equation to find out how many of each type of sheep there were in the initial flock. **(7 marks)**

E/P **13** **a** Determine the values of the real constants a and b for which there are infinitely many solutions to the simultaneous equations

$$\begin{aligned} 2x + 3y + z &= 6 \\ -x + y + 2z &= 7 \\ ax + y + 4z &= b \end{aligned}$$

(6 marks)

b Give a geometric interpretation of the three planes formed by these equations. **(1 mark)**

Challenge

Given that **A** and **B** are 2 × 2 matrices, prove that det(**AB**) = det **A** det **B**.

Summary of key points

1 A **square matrix** is one where the numbers of rows and columns are the same.

2 A zero matrix is one in which all of the numbers are zero. The zero matrix is denoted by 0.

3 An identity matrix is a square matrix in which the numbers in the leading diagonal (starting top left) are 1 and all the rest are 0. Identity matrices are denoted by $\mathbf{I}_k$ where k describes the size. The 3 × 3 identity matrix is $\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

4 To add or subtract matrices, you add or subtract the corresponding elements in each matrix. You can only add or subtract matrices that are the same size.

5 To multiply a matrix by a scalar, you multiply every element in the matrix by that scalar.

6
- Matrices can be multiplied together if the number of columns in the first matrix is equal to the number of rows in the second matrix. In this case the first is said to be multiplicatively conformable with the second.
- To find the product of two multiplicatively conformable matrices, you multiply the elements in each row in the left-hand matrix by the corresponding elements in each column in the right-hand matrix, then add the results together.

7 For a 2 × 2 matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant of **M** is $ad - bc$.

8
- If det **M** = 0 then **M** is a **singular** matrix.
- If det **M** ≠ 0 then **M** is a **non-singular** matrix.

9 You find the determinant of a 3 × 3 matrix by reducing the 3 × 3 determinant to 2 × 2 determinants using the formula:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

10 The **minor** of an element in a 3 × 3 matrix is the determinant of the 2 × 2 matrix that remains after the row and column containing that element have been crossed out.

11 The **inverse** of a matrix **M** is the matrix $\mathbf{M}^{-1}$ such that $\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$.

12 If $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathbf{M}^{-1} = \frac{1}{\det\mathbf{M}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

13 If **A** and **B** are non-singular matrices, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

14 The **transpose** of a matrix is found by interchanging the rows and the columns.

15 Finding the inverse of a 3 × 3 matrix **A** usually consists of the following 5 steps.

Step 1 Find the determinant of **A**, det **A**.

Step 2 Form the matrix of the minors of **A**, **M**.

In forming the matrix **M**, each of the nine elements of the matrix **A** is replaced by its minor.

Step 3 From the matrix of minors, form the matrix of **cofactors**, **C**, by changing the signs of some elements of the matrix of minors according to the **rule of alternating signs** illustrated by the pattern

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

You leave the elements of the matrix of minors corresponding to the + signs in this pattern unchanged. You change the signs of the elements corresponding to the – signs.

Step 4 Write down the transpose, $\mathbf{C}^{\mathrm{T}}$, of the matrix of cofactors.

Step 5 The inverse of the matrix **A** is given by the formula

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\mathbf{C}^{\mathrm{T}}$$

16 If $\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{v}$ then $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A}^{-1}\mathbf{v}$.

17 A system of linear equations is **consistent** if there is at least one set of values that satisfies all the equations simultaneously. Otherwise, it is **inconsistent**.