TOPIC 10: THE CIRCLE

The circle.

The circle is the Locus of a point which moves so that it is equidistant from a fixed point.

The equal distance from the fixed point is called the radius, and the fixed point is called the center.

The equation of circle.

Consider a circle of radius r whose centre is at (0,0) fig.1

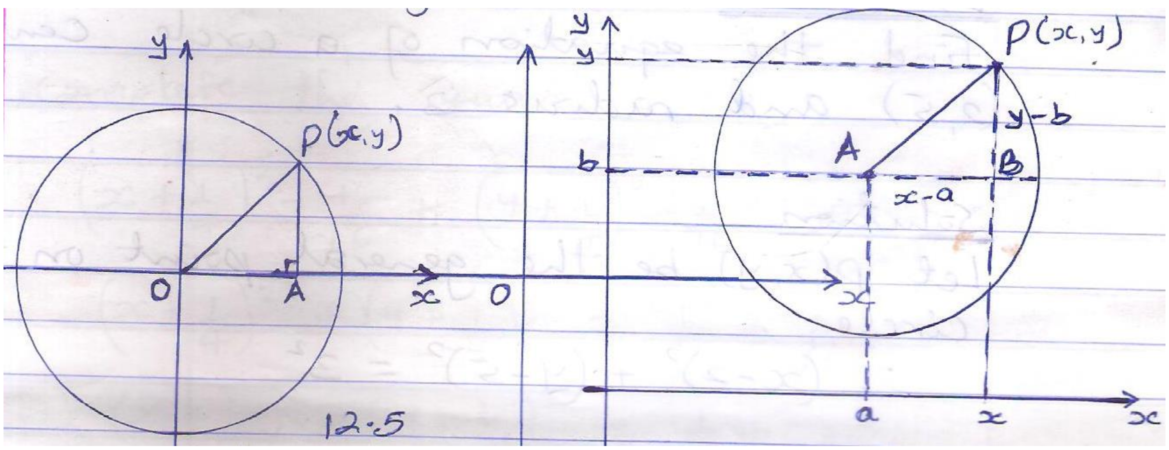


Fig.1 fig .2

Using Pythagoras' theorem

$OA^2 + AP^2 = OP^2$

Therefore, $x^2 + y^2 = r^2$

This is the equation of a circle centre (0,0), radius r.

If the centre of the circle is at (a,b) and the radius is r, the general equation can be obtained.(fig.2).

Let P (x,y) be the general point on the circle and A(a,b) be the centre.

From Fig 1, AB = x - a and BP = y - b.

Applying Pythagoras' theorem to triangle ABP,

$AB^2 + BP^2 = AP^2$

$\therefore (x - a)^2 + (y - b)^2 = r^2$

Therefore the equation of the circle, radius r, whose centre is at (a,b) is

$(x - a)^2 + (y - b)^2 = r^2$

Example 9

Find the equation of a circle centre at (2,5) and radius 3

Solution

Let P(x,y) be the general point on the circle,

$\therefore (x - 2)^2 + (y - 5)^2 = 3^2$

$x^2 - 4x + 4 + y^2 - 10y + 25 - 9 = 0$

$x^2 + y^2 - 4x - 10y + 20 = 0.$

Note:

(a) The coefficients of x^2 $and y^2 are\ equal,$
(b) The only other terms are linear, the form mx + ny + c,

Finding the centre and radius of a circle.

Given the equation of the circle we can find its centre and radius by completing the squares as in the example below.

Example 10

Find the radius and the coordinates of the centre of the circle $2x^2 + 2y^2 + x + y = 0$

Solution

$2x^2 + 2y^2 + x + y = 0 - - - - (1)$

Divide both sides of equation (1) by 2, in order to make the coefficients of x^2 and y^2 equal to 1.

$$x^2 + y^2 + \frac{1}{2}x + \frac{1}{2}y = 0$$

Re-arrange the terms, grouping those in x and y

$$x^2 + \frac{1}{2}x + \ y^2 + \frac{1}{2}y = 0$$

Complete the squares;

$$\left(x + \frac{1}{4}\right)^2 - \frac{1}{16} + \left(y + \frac{1}{4}\right)^2 - \frac{1}{16} = 0$$

$$\left(x + \frac{1}{4}\right)^2 + \left(y + \frac{1}{4}\right)^2 = \frac{1}{8}$$

Comparing this with the equation of the circle, radius r, centre (a,b)

(x-a)$^2 + (y - b)^2 = r^2, we$ obtain a $= -\frac{1}{4},\ b = -\frac{1}{4}, r = \frac{1}{\sqrt{8}} = \frac{\sqrt[2]{2}}{8}$

Therefore, the radius is $\frac{1}{4}\sqrt{2},$ and the centre is at $\left(-\frac{1}{4}, -\frac{1}{4}\right)$.

Qn: Find the radius and the coordinates of the centre of the circle.

$$x^2 + y^2 + 4x - 6y + 12 = 0$$

Exercise 4.

1. Find the equation of the circle with centre (2,-3) and radius 2
2. Find the radius and the coordinates of the centres of the following circles:
   (b) $8x^2 + 8y^2 - 16x + 48y + 79 = 0$
3. Find the equation of the circle whose centre is at (2,1) and which passes through the point (4,-3).
4. The point (-3,-6) and (3,2) are the ends of a diameter of a circle. Find the coordinates of the centre, and radius. Deduce the equation of the circle.
5. Find the radii of the two circles, with centres at the origin, which touch the circle
6. Show that the distance of the centre of the circle $x^2 + y^2 - 6x - 4y + 4 = 0\ from$ the y-axis is equal to the radius. What does this prove about the y-axis and the circle?

7. Find the equation of a circle passing the points A(-5,2), B (-3,-4) and C(1,8).
8. The centre of the circle lies on the line x=2y-2. If it touches the positive axes, find its equation.
9. Using the general equation of a circle, $x^2 + y^2 + 2gx + 2fy + c = 0, Find$ the equation of the circle passing through the points (-1,-2), (1,2) and (2,3).
10. A point moves so that its distance from the origin is twice its distance from the point (3,0). Show that the Locus is a circle. Find its centre and radius.

Tangents to a circle.

Example 11.

Show that the point (3,2) lies on the circle$\mathrm{x}^2 + \mathrm{y}^2 - 8\mathrm{x} + 2\mathrm{y} + 7 = 0.$ Find the equation of the tangent at this point.

Solution

Substitute for x = 3 and y =2, in the given equation if the L.H.S = R.H.S, then the point lies on the circle.

L.H.S= $3^2 + 2^2 - (8x3) + (2x2) + 7$

=9+4-24+4+7

= 24-24 = 0.

R.H.S = 0

L.H.S = R.H.S, therefore (3,2) lies on the circle.

By completing squares; we get centre and radius,

$x^2 - 8x + y^2 + 2y + 7 = 0$

$(x - 4)^2 - 16 + (y + 1)^2 - 1 + 7 = 0$

$(x - 4)^2 + (y + 1)^2 = 10$

Therefore centre is at (4,-1)

Gradient of the radius = $\frac{-1-2}{4-3}$

= -3

Gradient of the tangent = $\frac{1}{3}$

Equation of the tangent is given by

$$\frac{y-2}{x-3}=\frac{1}{3}$$

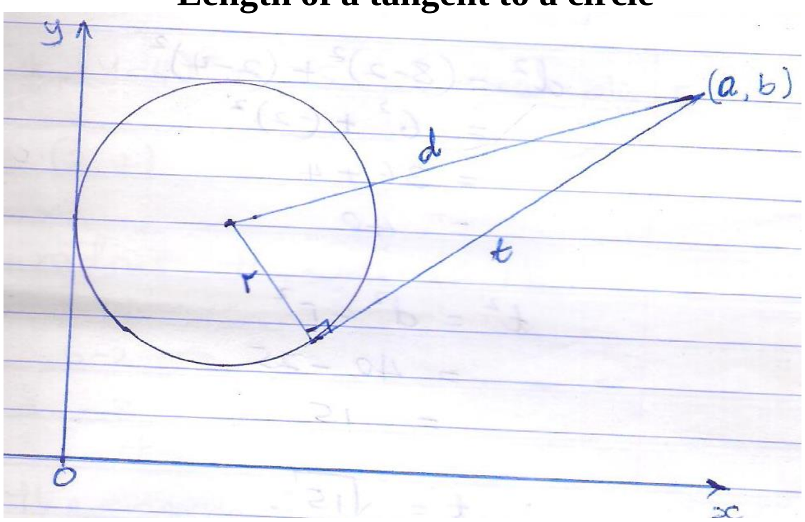
3y-6 x-3

Required equation is 3y – x - 3 =0

Note: The tangent is perpendicular to the radius at the point of tangency.

Length of a tangent to a circle

Fig.3



If the equation of the circle is given, then by completing the square we can find the centre and radius,

The length of a tangent t drawn from (a,b)can be obtained using Pythagoras' theorem, that is

$d^2 = r^2 + t^2$, Fig.6,where d is the distance between the centre of the circle and the point (a,b)

Therefore t = $\sqrt{d^2 - r^2}$

Example 12.

Find the length of the tangent from the point (8,2) to the circle $x^2 + y^2 - 4x - 8y - 5 = 0$

Solution

$x^2 + y^2 - 4x - 8y - 5 = 0$

$x^2 - 4x + y^2 - 8y - 5 = 0$

$(x-2)^2 - 4 + (y-4)^2 16 - 5 = 0$

$(x-2)^2 + (y-4)^2 = 25$

∴ Centre is at (2, 4) and r^2 =25

$d^2 = (8-2)^2 + (2-4)^2$

$=6^2 + (-2)^2$

= 36+4

= 40

$t^2 = d^2 - r^2$

= 40 - 25

=15

: . t = $\sqrt{15}$

Normals to a circle

Example 13

Show that the point (6,1) lies on the circle

$x^2 + y^2 - 4x - 8y - 5 = 0$. Find the equation of the normal to this circle at (6,1)

Solution

The gradient of the normal at (6,1) is the gradient of the radius through (6,1). We shall obtain the centre and radius by completing the squares.

$x^2 + y^2 - 4x - 8y - 5 = 0$

At (6.1)

L.H.S = $6^2 + 1^2 - (4x6) - (8x1) - 5$

= 36 + 1 -24 – 8 – 5 = 0.

R.H.S = 0

L.H.S = R.H.S, therefore (6,1) lies on the circle $x^2 + y^2 - 4x - 8y - 5 = 0$

By completing the squares, we get

$(x-2)^2 + (y-4)^2 = 25$ (Example 5)

Centre is at (2, 4)

Gradient of radius through (6,1)

$= \frac{1-4}{6-2} \quad = -\frac{3}{4}$

Equation of the normal is given by

$$\frac{y-1}{x-6} = -\frac{3}{4}$$

4y - 4 = -3x + 18

Therefore, 4y + 3x – 22 = 0 is the equation of the normal at (6,1).

Exercise 6

1. Show that the given points lie on the following circles and find the equations of the tangent to the circles at these points:
   (a) $x^2 + y^2 - 4x - 8y - 5 = 0$ (6,1)
   (b) $x^2 + y^2 + 2x + 4y - 12 = 0$ (3, -1)
   (c) $2x^2 + 2y^2 - 8x - 5y - 1 = 0$ $(1, -1)$
2. Find the length of the tangents from the origin to the circle $x^2 + y^2 - 10x + 2y + 13 = 0$. Show that these two tangents and the radii through the points of contact form a square.

3. Find the length of the tangents from the given points to the following circles;
   (a) $x^2 + y^2 + 6x + 10y - 2 = 0,\quad (-2,3)$
   (b) $x^2 + y^2 - 4x - 6y + 9 = 0\ ;\quad (2,2)$
4. The tangent to the circle $x^2 + y^2 - 2x - 6y + 5 = 0$ at the point (3,4) meets the x –axis at M find;
   (a) The distance of M from the centre of the circle.
   (b) The equation of the normal to the circle at (3,4).
5. Show that the length of the tangents to the circle $x^2 + y^2 - 4x - 6y + 12 = 0$ from the point P(X,Y) is $\sqrt{(X^2 + Y^2 - 4x - 6Y + 12)}$. Find the Locus of P when it moves so that the length of the tangents to the circle is equal to its distance from the origin.
6. Show the point (2,3) is outside, on or inside the circle $x^2 + y^2 + 6x + 10y - 2 = 0$
7. Find the equation of the tangents to the circle $x^2 + y^2 - 8x - 6y + 9 = 0$ which are parallel to the straight line 4x - 3y + 2 = 0.
8. Find the length of the tangent from the origin to the circle $x^2 + y^2 - 10x + 2y + 13 = 0$.

Intersection of circles

Example 14

Find the points of intersection of the circles $x^2 + y^2 - 4x - 2y + 1 = 0$ and

$$x^2 + y^2 + 4x - 6y - 10 = 0$$

Solution

First get equation of the common chord by getting the difference between the equations.

$$x^2 + y^2 - 4x - 2y + 1 = 0 \quad ----(1)$$

$$x^2 + y^2 + 4x - 6y - 10 = 0 \quad ----(2)$$

(1)– (2) ; - 8x + 4y + 11=0

4y = 8x - 11

y = $\frac{8x-11}{4}$ ---------- (3)

Substitute for y in (1)

$$x^2+\left(\frac{8x-11}{4}\right)2-4x-2\left(\frac{8x-11}{4}\right)+1=0$$

$$x^2+\frac{64x^2-176x+121}{16}-4x-4x+\frac{11}{2}+1=0$$

$$16x^2+16x^2-176x+121-64x-64x+88+16=0$$

$$80x^2-304x+225=0$$

X= $\frac{304\pm\sqrt{(304)^2-4\text{x }80\text{ x }225}}{2\text{x}80}$

= $\frac{304\pm\sqrt{20416}}{160}$

= $\frac{304\pm142.88}{160}$

= $2.793\ or\ 1.007$

= 2.8 0r 1

y = $\frac{(8x2.8)-11}{4}=2.85$

or y = $\frac{(8x2.8)-11}{4}$

= $\frac{8-11}{4}=\ -\frac{3}{4}$

Therefore, the points of intersection are (2.8, 2.9) and (1, $-\frac{3}{4}$)

Orthogonal circles

Two circles are orthogonal if their tangents, at the point of intersection of the circles, are at right angles. Since the radius through a point of contact is perpendicular to the tangent, it follows that the tangent to one circle is a radius of the other.

If the centres of two orthogonal circles of radii R and r are a distance d apart .

Fig 8, it follows that

$\boldsymbol{d^2} = \boldsymbol{R^2} + \boldsymbol{r^2}$ $(Pythagoras' theorem)$

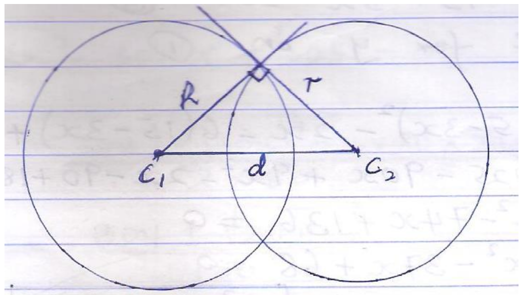


Fig.8

Example 15

Show that the circles $x^2 + y^2 + 10x - 4y - 3 = 0$ and $x^2 + y^2 - 2x - 6y + 5 = 0$ are orthogonal.

Solution

By completing squares, find the centres and radii of the circles

$x^2 + y^2 + 10x - 4y - 3 = 0$

$x^2 + 10x + y^2 - 4y - 3 = 0$

$(x + 5)^2 - 25 + (y - 2)^2 - 4 - 3 = 0$

$(x + 5)^2 + (y - 2)^2 = 32$

$\therefore$ C_1 (-5,2) and R = $\sqrt{32}$

$x^2 + y^2 - 2x - 6y + 5 = 0$

$x^2 - 2x + y^2 - 6y + 5 = 0$

$(x - 1)^2 - 1 + (y - 3)^2 - 9 + 5 = 0$

$(x - 1)^2 + (y - 3)^2 = 5$

$\therefore$ C_2(1,3) and r = $\sqrt{5}$.

Distance between centres, d is given by

d = $\sqrt{(-5-1)^2 + (2-3)^2}$

$d^2 = 36 + 1$

= 37

$R^2 + r^2 = 32 + 5$ = 37

$d^2 = R^2 + r^2$

Since $d^2 = R^2 + r^2$, therefore the two circles are orthogonal.

Exercise 6

1. Find the equations of the tangent and normal to the circle $3x^2 + 3y^2 + 6x - 4y - 15 = 0$ at the point (-2,3).
2. Calculate the length of the tangent from the point (8,7) to the circle $x^2 + y^2 - 6x - 2y + 1 = 0$.
3. Find the values of C for which the line y = x + c is a tangent to the circle $x^2 + y^2 - 4x + 2y - 3 = 0$.
4. Show that the circles $x^2 + y^2 - 6x - 8y + 9 = 0$ and $x^2 + y^2 = 9$ are orthogonal.

5. Prove that the line y = 2x is a tangent to the circle $x^2 + y^2 - 8x - y + 5 = 0$ and find the coordinates of the point of contact.

6. A triangle has vertices (0,6), (4,0), (6,0). Find the equation of the circle through the mid- points of the sides and show that it passes through the origin.

7. Find the circumcentre of the triangle PQR with vertices P(0,2), Q(8, -2), R(9,5). Hence find the equation of the circle through points P, Q and R. Verify that point (2,6) lies on the circle.

8. A circle A passes through the point (t+2, 3t) and has the centre at (t, 3t). Circle B has radius 2 and has its centre at (t+2,3t).

   (a) Determine the equations of circles A and B in terms of t.

   (b) If t =1, show that circles A and B intersect at (2, 3+$\sqrt{3}$) and (2, 3- $\sqrt{3}$).

(c) Show that the area of the region of intersection of the two circles A and B is

$8\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)$ sq. units.

9. Form the equation of a circle that passes through the points A(-1,4), B (2,5) and C(0,1).