S5 MATHEMATICS- CONTINUED

Topic 4: PROBABILITY THEORY

LESSON 1: LAWS OF PROBABILITY

In this lesson, you should be able to:

(i) state the laws of probability.
(ii) apply the laws of probability to solve problems.

Definition: Probability is an attempt to use mathematics to estimate by means of a numerical answer the chance that some event will happen.

Event: This is an occurrence in a defined context

Trial: This is a single attempt to obtain a defined event

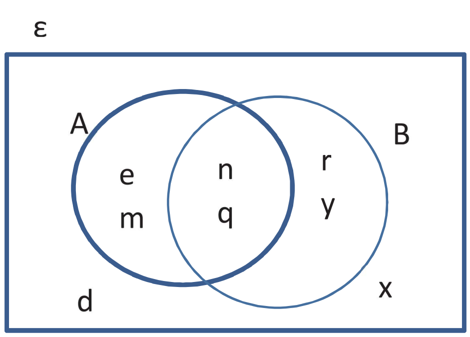
Sample space: This is the total or summation of all the events. Generation of a sample space include; **Table of outcomes** and **Tree diagrams**.

LAWS OF PROBABILITY

Consider an experiment whose sample space is (S). For each event (E) of the sample space to occur, we assume that the probability of that event satisfies the following laws;

1. $0 \leq P(E) \leq 1$. Probability of an event E to occur it can't be negative and can't be greater than 1
2. P(S) = 1. Probability of the sample space is 1
3. P(A ∩ B) = P(B ∩ A). also P(A U B) = P(B U A)

Consider $A = \{ m, n, q, e\}$ and $B = \{y, r, n, q\}$



A ∩ B = $\{n , q\}$
B ∩ A = $\{n , q\}$
BUA = A U B= $\{e, m, n, q, y, r\}$

$P(A \cap B) = \frac{2}{8} = \frac{1}{4}$

$P(A \cup B) = \frac{6}{8} = \frac{3}{4}$.

4. $P(A \cap B)^I = P(A^I \cup B^I)$:

   $A \cap B = \{n, q\}$ $\quad$ $\boldsymbol{(A \cap B)^I = \{e, m, y, r, d, x\}}$

   $A^I = \{r, y, d, x\}$ $\quad$ $B^I = \{e, m, d, x\}$

   $\boldsymbol{A^I \cup B^I = \{r, y, d, x, e, m\}}$

   Therefore; $\boldsymbol{(A \cap B)^I = A^I \cup B^I}$.

   $\boldsymbol{P(A \cap B)^I = P(A^I \cup B^I) = \frac{6}{8} = \frac{3}{4}}$.

5. $P(A \cup B)^I = P(A^I \cap B^I)$:

   $A \cup B = \{e, m, q, n, r, y\}$ $\quad$ $(A \cup B)^I = \{d, x\}$

   $A^I = \{r, y, d, x\}$ $\quad$ $B^I = \{e, m, d, x\}$

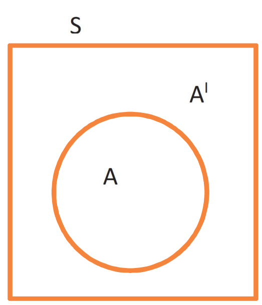
   $(A^I \cap B^I) = \{d, x\}$

   Therefore ;$P(A \cup B)^I = P(A^I \cap B^I) = \frac{2}{8} = \frac{1}{4}$.

   Note: Rules 4 and 5 are called the **Demorgan's laws.**

ADDITION RULES OF PROBABILITY

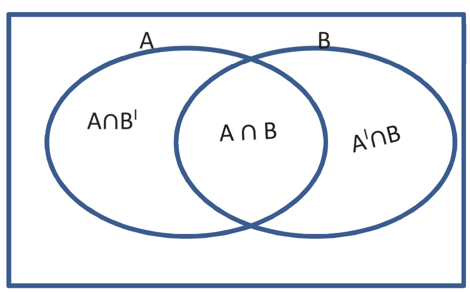
1. $P(A) + P(A^/) = 1$. Where $A^/$ means A complement (outside of set A)



   $P(A) + P(A^/) = P(S)$. but $P(S) = 1$

   $\therefore P(A) + P(A^/) = 1$ or $P(A) = 1 - P(A^/)$

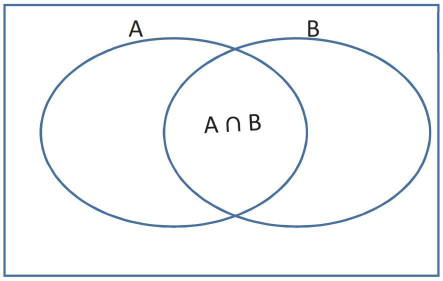
2. $P(A) = P(A \cap B) + P(A \cap B^/)$



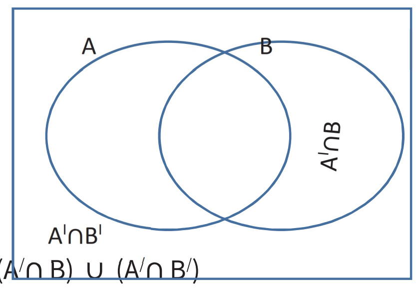
Similarly;

$P(B) = P(A \cap B) + P(A' \cap B)$

3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



4. $P(A') = P(A' \cap B) + P(A' \cap B')$



$A' = (A' \cap B) \cup (A' \cap B')$

$P(A') = P(A' \cap B) \cup P(A' \cap B')$

Similarly;

$P(B') = P(A \cap B') \cup P(A' \cap B')$

Contingency table

	A	**A'**	
B	**P(A ∩ B)**	**P(A'∩ B)**	**P(B)**
B'	**P(A ∩ B')**	**P(A'∩ B')**	**P(B')**
	P(A)	**P(A')**	**1**

ADD

ADD

This contingency table is a table which Summarises the addition laws of probability.

P(A) + P(A^I) = 1,

P(A ∩ B) + P(A ∩ B^I) = P(A),

P(A ∩ B) + P(A^I ∩ B) = P(B),

P(A^I ∩ B) + P(A^I ∩ B^I) = P(A^I)

and P(A ∩ B^I) + P(A^I ∩ B^I) = P(B^I).

Example:

1. Given that P(AUB) = $\frac{7}{8}$, P(A∩B) = $\frac{1}{4}$ and P(A$^/$) = $\frac{5}{8}$. Find;
   (i) P(A)
   (ii) P(B)
   (iii) P(A∩B$^/$)
   (iv) P(A$^/$UB$^/$)

Solution:

(i) P(A) = 1 – P(A$^/$)

$= 1 - \frac{5}{8}$

$= \frac{3}{8}$

(ii) P(A∪B) = P(A) + P(B) – P(A∩B)

$\frac{7}{8} = \frac{3}{8} + P(B) - \frac{1}{4}$

$\frac{7}{8} - \frac{3}{8} + \frac{1}{4} = P(B)$

$\frac{3}{4} = P(B).$

(iii) P(A) = P(A∩B) + P(A∩B$^/$)

$\frac{3}{4} = \frac{1}{4} + P(A\cap B^/)$

$\frac{3}{4} - \frac{1}{4} = P(A\cap B^/)$

$\frac{1}{8} = P(A\cap B^/)$

(iv) $P(A^{/} \cup B^{/}) = P(A \cap B)^{I}$

$= 1 - P(A \cap B)$

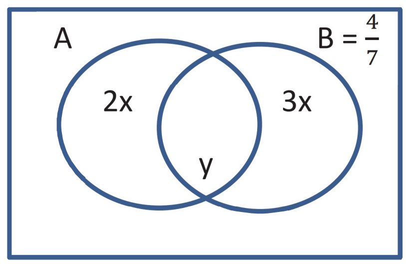
$= 1 - \frac{1}{4} = \frac{3}{4}.$

Example 2:

Two events A and B are such that $P(A^{/} \cap B) = 3x$, $P(A \cap B^{/}) = 2x$, $P(A^{/} \cap B^{/}) = x$ and $P(B) = \frac{4}{7}$. Using the Venn diagram, find the values of;

(i) x
(ii) $P(A \cap B)$

(i) Let y be the intersection



$P(B) = P(A \cap B) + P(A^{/} \cap B)$

$\frac{4}{7} = y + 3x$

$\frac{4}{7} - 3x = y$ …………(i)

$P(\varepsilon) = 1$

$2x + y + 3x + x = 1$

$6x + (\frac{4}{7} - 3x) = 1$

$3x = 1 - \frac{4}{7}$

$3x = \frac{3}{7}$

$x = \frac{1}{7}.$

(ii) $P(A \cap B) = y$

$y = \frac{4}{7} - \frac{3}{7}$

$y = \frac{1}{7}.$

EXERCISE:

1. Two events M and N are such that $P(M) = 0.7$, $P(M \cap N) = 0.45$ and $P(M^{/} \cap N) = 0.18$.
   Find; (i) $P(N^{/})$
   (ii) P (M U N).

2. The probability that that Peter reads the New vision is 0.75. And the probability that he reads the New vision but not the Monitor is 0.65. The probability that he reads neither of the two papers is 0.15. Find the probability that he reads the Monitor.

3. The probability that a student passes Mathematics is $\frac{2}{3}$, the probability that he passes Physics is $\frac{4}{9}$. If the probability that he passes at least one of them is $\frac{4}{5}$. Find the probability that he passes both subjects.

Lesson 2: Probability of the 'OR', 'AND' situations

In this lesson you should be able to:

(i) define the 'or' , 'and' situations
(ii) define mutually exclusive and independent events
(iii) calculate probabilities using mutually exclusive and independent events

THE OR SITUATION

If A and B are two events, the probability that either event A or B or both occur is denoted by P(AUB).

THE AND SITUATION

For two events A and B, the probability that both events A and B occur together is $P(A \cap B)$.

Example:

Two dice are thrown. What is the probability of scoring either a double or a sum greater than 8?

Solution;

Table of outcomes can be used to generate the sample space.

	First die					
Second die	1,1	1,2	1,3	1,4	1,5	1,6
	2,1	2,2	2,3	2,4	2,5	2,6
	3,1	3,2	3,3	3,4	3,5	3,6
	4,1	4,2	4,3	4,4	4,5	4,6
	5,1	5,2	5,3	5,4	5,5	5,6
	6,1	6,2	6,3	6,4	6,5	6,6

Table of sums

2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

Let A represent a double and B a sum greater than 8

A = $\{(1,1)(2,2)(3,3)(4,4)(5,5)(6,6)\}$

B = $\{(3,6)(4,5)(4,6)(5,4)(5,5)(5,6)(6,3)(6,4)(6,5)(6,6)\}$

n(A) = 6, n(B) = 10, n(A∩B) = 2

P(A∪B) = P(A) + P(B) – P(A∩B)

$= \frac{6}{36} + \frac{10}{36} - \frac{2}{36} = \frac{7}{18}$.

∴P(Double or sum greater than 8) = $\frac{7}{18}$.

Mutually Exclusive events

Two events A and B are said to be mutually exclusive if they cannot occur at the same time. i.e. the probability that they both occur is zero.

If A and B are mutually exclusive then P(A∩B) = 0

P(AUB) = P(A) + P(B).

Example:

1. Given that A and B are mutually exclusive events such that P(A) = 0.5 and P(AUB) = 0.9 find; (i) P(B)
   (ii) $P(A^/ \cap B^/)$

Solution:

For mutually exclusive events;

P(AUB) = P(A) + P(B)

P(B) = P(AUB) – P(A)

= 0.9 – 0.5

= 0.4

(ii) $P(A^/ \cap B^/) = P(AUB)^/$

= 1 – P(AUB)

= 1 – 0.9

= 0.1

2. In a race, the probability that Grace wins is 0.4, the probability that Mahad wins is 0.2 and the probability that Denis wins is 0.3. Find the probability that;
   (i) Grace or Denis wins
   (ii) Neither Denis nor Mahad wins

Solution:

(i) P(GUD) = P(G) + P(D) $[Since\ G, U\ \&D\ are\ mutually\ exclusive\ events\ ]$

$= 0.4 + 0.3$

$= 0.7$

(ii) $P(D^{/}\cap M^{/}) = P(DUM)^{I}$

$= 1 – P(DUM)$

$= 1 - [P(D) + P(M)]$

$= 1 – (0.3 + 0.2)$

$= 1 – 0.5$

$= 0.5$

INDEPENDENT EVENTS

Events are said to be independent if and only if the occurrence of one event does not influence the occurrence of the other event. Or the nonoccurrence of one does not influence the nonoccurrence of the other.

For independent events;

$P(A \text{ and } B) = P(A \cap B) = P(A) \times P(B)$.

Example1:

Given that A and B are independent events such that $P(A) = \frac{2}{5}$, $P(AUB) = \frac{4}{5}$.

Find; (i) P(B)

(ii) $P(A^{/}UB^{i})$.

Solution:

(i) $P(AUB) = P(A) + P(B) – P(A\cap B)$

Since A and B are independent events,

Then $P(A\cap B) = P(A) \times P(B)$

$\mapsto \frac{4}{5} = \frac{2}{5} + P(B) - \frac{2}{5}P(B)$

$\frac{4}{5} - \frac{2}{5} = P(B)\left[1 - \frac{2}{5}\right]$

$\frac{2}{5} = \frac{3}{5}P(B)$

$P(B) = \frac{2}{3}$.

(ii) $P(A^{/}UB^{/}) = P(A \cap B)^{I}$

$= 1 - P(A\cap B)$

$= 1 - \frac{2}{5} x \frac{2}{3}$

$= 1 - \frac{4}{15} = \frac{11}{15}$

Example 2:

Given that A and B are independent events, show that A^{I} and B^{I} are also independent.

Solution:

For independent events; $P(A\cap B) = P(A) \times P(B)$.

We need to show that $P(A^{/}\cap B^{/}) = P(A^{/}) \times p(B^{/})$

$P(A' \cap B') = P(AUB)'$
$= 1 - P(AUB)$
$= 1 - [P(A) + P(B) - P(A \cap B)]$
$= 1 - [P(A) + P(B) - P(A) \times P(B)]$
$= 1 - P(A) - P(B) + P(A) \times P(B)$
$= 1 - P(A) - P(B)[1 - P(A)]$
But; $1 - P(A) = P(A')$
$\Rightarrow P(A' \cap B') = P(A') - P(B)P(A')$
$= P(A')[1 - P(B)]$
Also; $1 - P(B) = P(B')$
$\therefore P(A' \cap B') = P(A')P(B')$. Hence they are independent.

Exercise:

1. Events A and B are independent and $P(A) = 0.3$, $P(AUB) = 0.6$. Find;
   (i) $P(A \cap B)$
   (ii) $P(B)$

2. If A and B are mutually exclusive events that $P(A) = 0.2$, $P(AUB)' = 0.3$. Find;
   (i) $P(B)$
   (ii) $P(A' \cap B)$
   (iii) $P(A \cap B')$

3. Events A and B are independent such that $P(A \cap B) = \frac{1}{4}$, $P(AUB) = \frac{3}{4}$. Find;
   (i) $P(A)$
   (ii) $P(B)$

4. The probability that two independent events occur together is $\frac{2}{15}$. The probability that either or both events occur is $\frac{2}{3}$. Find the individual probabilities of the two events.

LESSON 3: CONDITIONAL PROBABILITY

Learning Outcomes

Learners should be able to:

(i) define conditional probability.
(ii) draw tree diagrams.
(iii) apply conditional probability and tree diagrams to solve probabilities.

A conditional probability is a situation where an event will occur on the condition that another event has occurred. If A and B are events then the conditional probability of A given B is;

$$\mathbf{P}\left({}^{A}/_{B}\right)=\frac{P(A\cap B)}{P(B)}.$$

This is provided $P(B) \neq 0$.

Example 1.

Given that A and B are events such that $P(B) = \frac{1}{6}$, $P(A\cap B) = \frac{1}{12}$ and $P\left({}^{B}/_{A}\right) = \frac{1}{3}$.

Find: (i) P(A)

(ii) $P\left({}^{A}/_{B}\right)$

(iii) $P\left({}^{A}/_{B^{I}}\right)$

Solution: (i) $P\left({}^{B}/_{A}\right) = \frac{P(B\cap A)}{P(A)}$

$$\frac{1}{3} = \frac{\frac{1}{12}}{P(A)}$$

$$P(A) = \frac{1}{4}.$$

(ii) $P\left({}^{A}/_{B}\right) = \frac{P(A\cap B)}{P(B)}$

$$= \frac{1}{12} \div \frac{1}{6}$$

$$= \frac{1}{2}.$$

(iii) $P\left({}^{A}/_{B^{I}}\right) = \frac{P(A\cap B^{I})}{P(B^{I})}$

$$= \frac{P(A) - P(A\cap B)}{1-P(B)}$$

$$= \left(\frac{1}{4} - \frac{1}{12}\right) \div \left(1 - \frac{1}{6}\right)$$

$$= \frac{1}{6} \div \frac{5}{6}$$

$$= \frac{5}{6}.$$

Example 2:

Given that A and B are two events such that $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$, $P(AUB) = \frac{1}{2}$

Find:

(i) $P\left({}^{A}/_{B}\right)$

(ii) $P\left({}^{A}/_{B^{I}}\right)$

Solution:

(i) $P\left(A/_B\right) = \frac{P(A\cap B)}{P(B)}$

$P(AUB) = P(A) + P(B) - P(A\cap B)$

$\frac{1}{2} = \frac{1}{3} + \frac{1}{4} - P(A\cap B)$

$P(A\cap B) = \frac{7}{12} - \frac{1}{2}$

$= \frac{1}{12}$

$P\left(A/_B\right) = \frac{1}{12} \div \frac{1}{4}$

$= \frac{1}{3}.$

(ii) $P\left(A/_{B^I}\right) = \frac{P(A\cap B^I)}{P(B^I)}$

$= \frac{P(A) - P(A\cap B)}{1 - P(B)}$

$= \left(\frac{1}{3} - \frac{1}{12}\right) \div \left(1 - \frac{1}{4}\right)$

$= \frac{1}{4} \div \frac{3}{4}$

$= \frac{1}{3}.$

Tree Diagrams

To draw a tree diagram, we begin from a single point and produce a pair of branches for the first trial. We continue drawing branches for each of the subsequent outcomes.

To obtain the required probability look at the branches and trace out the path associated with events of interest then;

- Take product of the probabilities along that path
- Do the same for other similar paths of interest
- Sum the products

Note: At every junction where branches meet, the probabilities sum up to one.

Example 1

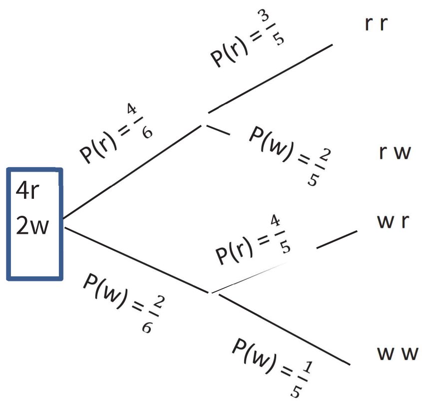
A box contains 4 red and 2 white balls. If two balls are picked from the box one at a time without replacement. Find the probability that;

(i) The second ball is red

(ii) Both balls are white

(iii) Both balls are of the same colour

Solution:



(i) The second ball is red

$= P(rr) \text{ or } P(wr)$

$= \left(\frac{4}{6}\ x\ \frac{3}{5}\right) + \left(\frac{2}{6}\ x\ \frac{4}{5}\right)$

$= \frac{2}{5} + \frac{4}{15}$

$= \frac{2}{3}$

(ii) Both balls are white

$= P(ww)$

$= \frac{2}{6}\ x\ \frac{1}{5} = \frac{1}{15}.$

(iii) Both balls are of same colour

$= P(rr) \text{ or } P(ww)$

$= \left(\frac{4}{6}\ x\ \frac{3}{5}\right) + \left(\frac{2}{6}\ x\ \frac{1}{5}\right)$

$= \frac{2}{5} + \frac{1}{15}$

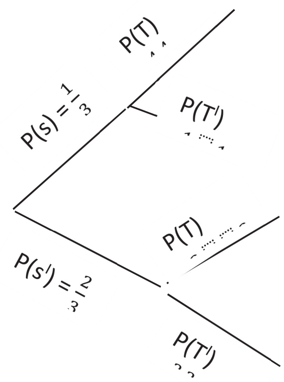
$= \frac{7}{15}$

Example 2:

The probability that it will be sunny tomorrow is $\frac{1}{3}$. If it is sunny, the probability that Simon plays tennis tomorrow is $\frac{4}{5}$. If it is not sunny, the probability that he plays tennis is $\frac{2}{5}$. Find the probability that Simon play tennis tomorrow.

Solution:

Let S represent sunny, and T represent playing tennis.



$$P(T) = P(S \cap T) + P(S' \cap T)$$
$$= \left(\frac{1}{3} x \frac{2}{5}\right) + \left(\frac{2}{3} x \frac{2}{5}\right)$$
$$= \frac{2}{15} + \frac{4}{15}$$
$$= \frac{6}{15} = \frac{2}{5}.$$

Exercise:

1. Show that $P\left(^{A}/_{B}\right) + P\left(^{A^{I}}/_{B}\right) = 1$
2. The events A and B are independent with P(A) = $\frac{1}{2}$ and P(A U B) = $\frac{2}{3}$, find;
   (i) P(B)
   (ii) P$\left(^{A}/_{B}\right)$
   (iii) P$\left(^{B^{I}}/_{A}\right)$

3. (a) A bag contains 30 white (W), 20 blue (B) and 20 red (R) balls. Three balls are drawn at random one after the other without replacement. Determine the probability that the first ball is white and the third ball is also white.

   (b) Events A and B are such that P(A) = $\frac{4}{7}$, P(A U B$'$) = $\frac{1}{3}$ and P$\left(^{A}/_{B}\right)$ = $\frac{5}{14}$.
   Find; (i) P(B). (ii) P(A$'$∩ B$'$).

4. A and B are two identical boxes, Box A contains one diamond ring and two gold rings. Box B has 3 diamond and 4 gold rings. A box is chosen at random and from it one ring is randomly taken and put into the other box. And a ring is then randomly drawn from the later box.

Determine the probability that;

(i) Both rings are diamond

(ii) The first ring is gold

(iii) The first ring is diamond given that the 2^{nd} is gold.

5. A box contains two types of balls, red and black. When a ball is picked from the box, the probability that it is red is $\frac{7}{12}$. Two balls are selected at random from the box without replacement.

   Find the probability that;

   (i) The second ball is black

   (ii) The first ball is red given that the second is black.

6. An interview involves written, oral and practical tests. The probability that an interviewee passes the written test is 0.8, the oral test is 0.6 and the practical test is 0.7.

   What is the probability that the interviewee will pass

   (i) The entire interview

   (ii) Exactly two of the interview tests.

Topic 5: Discrete Probability Distribution

Lesson 1. Random Variables

In this lesson you should be able to learn how to:

(i) describe a random experiment.
(ii) state the properties of a discrete random variable.
(iii) generate a probability distribution function of a discrete random variable from a given experiment.

When an experiment is performed, it is common that the main interest is some function of the outcome as opposed to the actual outcome itself.

For example, in tossing a pair of dice, we are often interested in the sum of the two dice and not really about the separate values of the dice.

We may be interested in knowing that the sum is 8 but not concerned about the actual outcomes i.e. (2,6), (3,5), (4,4), (5,3) or (6,2).

Random variable is a real valued function representing the outcome of a random experiment. X is used to denote the random variable and x denotes its outcomes.

Examples

Random Experiment	Random Variable
Tossing a coin 3 times	X = number of heads obtained
Rolling a pair of dice	X = sum, is 8
Measure the height of students	X = height is greater than 140 cm

Discrete Random Variable

A discrete random variable is one which takes on a countable number of values (outcomes). It assumes that each value has a certain probability.

Examples of a discrete random experiment and corresponding set of outcomes may be:

(a) a score when a die is tossed: X = 1, 2, 3, 4, 5, 6.
(b) the number of boys in a family of five children: X = 0, 1, 2, 3, 4, 5.
(c) the number of heads when two coins are tossed: X = 0, 1, 2.

If X is a discrete random variable, then the function f(x) = P(X = x). The function is called the probability distribution of x.

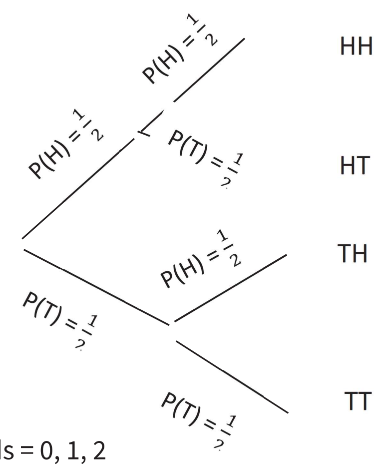
Properties of a discrete random variable

1. $P(X = x) = f(x) \geq 0$. For all values of x
2. Summation of the probability for all values of x is 1
   i.e. $\sum_{all\,x} P(X = x) = 1$ **or** $\sum_{all\,x} f(x) = 1$
   Total probability is 1

Example 1

If a random variable X is the number of heads obtained when an unbiased coin is tossed twice:

(i) form a sample space.
(ii) find the probability distribution.



Number of heads = 0, 1, 2

When x = 0: $P(X = 0) = P(TT)$
$= \frac{1}{2} \text{ x } \frac{1}{2} = \frac{1}{4}$

When x = 1: $P(X = 1) = P(HT)$ or $P(TH)$
$= \frac{1}{2} \text{ x } \frac{1}{2} + \frac{1}{2} \text{ x } \frac{1}{2}$
$= \frac{1}{4} + \frac{1}{4}$
$= \frac{1}{2}$

When x = 2: $P(X = 2) = P(HH)$
$= \frac{1}{2} \text{ x } \frac{1}{2} = \frac{1}{4}$

∴ The probability distribution is

X	0	1	2
P(X = x)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Exercise

1. Write the probability distribution of the score when an ordinary die is thrown.
2. Two discs are drawn at random without replacement from a bag containing 3 blue and 4 yellow discs. If X is a random variable for the number of blue discs drawn, construct a probability distribution for X
3. In a bag there are 3 green counters, 4 black counters and 2 red counters. Two counters are picked at random from the bag: one after the other without replacement. Find the probability distribution for the green counters.