Pure Mathematics

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This edition revised by MF Macrae

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International Edition (East Africa)

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Preface

This edition

This international edition of *Pure Mathematics* is a compilation of two books[1] written by the authors listed on the title page. The purpose of the compilation is to create a single textbook that satisfies the pure mathematics components of the Advanced level mathematics syllabi of Examination Boards in East Africa, and meets the standard needed for examinations at that level. In addition, much of the content, especially in the latter chapters, will meet many of the requirements of foundation courses in pure mathematics of many East African universities.

Those who are familiar with earlier editions of *Pure Mathematics* will note that some topics have been omitted. This is because they are no longer examined. On the other hand it was necessary to add a chapter on linear programming, which is nowadays included in most syllabi.

The book provides some coverage of applications of mathematics, for example in topics such as velocity, acceleration, simple harmonic motion, vectors, probability, and numerical methods. However, for a more comprehensive treatment of applied mathematics, it is advisable to refer to appropriate texts in mechanics and statistics.

The work of editing and revision has been greatly facilitated by the extremely high quality and accuracy of the original materials. Ever since the first edition of *Pure Mathematics*, the work has been regularly and assiduously revised and updated. As a result, books by 'Backhouse and Houldsworth' have become an international byword for reliability, thoroughness and clarity in advanced mathematical writing. This edition has greatly benefited from the excellent work that has preceded it.

M. F. Macrae, 2010

Using this book

Preliminary chapters, P1 to P3

Proficiency in advanced mathematics depends on learners being able to manipulate algebra with confidence. Chapters P1 and P2 provide practice in algebraic skills that ideally should have been mastered at *School Certificate* level. Chapter P3 provides an overview of the use of calculators and computers. It is expected that readers will own and use scientific calculators, and this is assumed in various places throughout the book. Although the preliminary chapters are optional, a quick study of their content will pay useful dividends.

The main text, Chapters 1 to 38

We have used certain conventions as follows.

Questions marked Q: These questions, which appear within shaded boxes, are a very important part of the exposition and explanation.

Questions

Q1 Write down the coordinates of the points D, E, O in Fig. 1.1.

Q2 Sketch your own axes and plot the points P(2, 4), Q(–5, 7), R(4, –2), S(0, 3), T(2, 0).

Readers should work through them all. **Q** questions may be used in a number of ways:

- by individual learners on a self-study basis;
- by small study-groups, working together to improve mutual understanding;
- by the teacher, who will find that many of these questions are suitable for oral class work.

Answers to the **Q** questions are grouped at the beginning of the answers to each chapter. Use these answers to assist your understanding.

Exercises: are generally graded, with more straightforward questions appearing near the beginning of each exercise.

Questions marked with an asterisk *: These occasionally appear in the exercises. The * indicates that the question

[1] *Pure Mathematics, Books 1 and 2*, Longman, Harlow

contains a useful result which may not have been explicitly covered in the text.

Symbols such as §2.6: From time to time there are cross references such as *See §2.6*. This means, for example, that you should turn to section 2.6 in Chapter 2 to find some related material.

Currency: Some questions involve money. We have used dollars, \$, as a 'neutral' currency denomination. These are not linked to any actual dollar currency.

Answers

Intelligent use of the answers section is an important part of learning at this level.

Diagrams: To save space, diagrams are generally not supplied with the answers.

Accuracy: Where possible, exact answers are given. Therefore surds and π are retained in answers where they occur. Where answers are *not* exact, they are rounded to three significant figures, or, for angles measured in degrees, to the nearest tenth of a degree.

Mathematical notation

The following notation is used in this book. It follows the conventions employed by most courses and examining bodies.

1 Set notation

$\in$	is an element of
$\notin$	is not an element of
$\{a, b, c, \ldots\}$	the set with elements $a, b, c \ldots$
$\{x: \ldots\}$	the set of elements x, such that …
$n(A)$	the number of elements in set A
$\varnothing$	the empty set
$\mathscr{E}$	the universal set
A'	the complement of set A
$\mathbb{N}$	the set of natural numbers (including zero) $0, 1, 2, 3 \ldots$
$\mathbb{Z}$	the set of integers $0, \pm1, \pm2, \pm3 \ldots$
$\mathbb{Z}^+$	the set of positive integers $+1, +2, +3 \ldots$
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\subseteq$	is a subset of
$\subset$	is a proper subset of
$\cup$	union
$\cap$	intersection
$[a, b]$	the closed interval $\{x \in \mathbb{R}: a \leqslant x \leqslant b\}$
(a, b)	the open interval $\{x \in \mathbb{R}: a < x < b\}$

2 Miscellaneous symbols

$=$	is equal to
$\neq$	is not equal to
$>, <$	is greater than, is less than
$\geqslant, \leqslant$	is greater than or equal to, is less than or equal to
$\approx$	is approximately equal to

3 Operations

$a + b$	a plus b
$a - b$	a minus b
$a \times b, ab, a.b$	a multiplied by b
$a \div b, \frac{a}{b}, a/b$	a divided by b
$\sum_{i=1}^{i=n} a_i$	$a_1 + a_2 + a_3 + \ldots + a_n$

4 Functions

$\mathrm{f}(x)$	the value of the function f at x
$\mathrm{f}: A \to B$	f is a function which maps each element of set A onto a member of set B
$\mathrm{f}: x \mapsto y$	f maps the element x onto an element y
f^{-1}	the inverse of the function f
$\mathrm{g} \circ \mathrm{f}$ or gf	the composite function $\mathrm{g}(\mathrm{f}(x))$
$\lim_{x \to a} \mathrm{f}(x)$	the limit of $\mathrm{f}(x)$ as x tends to a
δx	an increment of x
$\frac{\mathrm{d}y}{\mathrm{d}x}$	the derivative of y with respect to x
$\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$	the nth derivative of y with respect to x
$\mathrm{f}'(x), \mathrm{f}''(x), \ldots \mathrm{f}^{(n)}(x)$	the first, second, … nth derivatives of $\mathrm{f}(x)$
$\int y \, \mathrm{d}x$	the indefinite integral of y with respect to x
$\int_a^b y \, \mathrm{d}x$	the definite integral, with limits a and b
$[\mathrm{F}(x)]_a^b$	$\mathrm{F}(b) - \mathrm{F}(a)$

5 Exponential and logarithmic functions

e^x or $\exp x$	the exponential function
$\log_a x$	logarithm of x in base a logarithms
$\ln x$	$\log_e x$
$\lg x$	$\log_{10} x$

6 Circular and hyperbolic functions

$\sin x, \cos x, \tan x$	the circular functions sine, cosine, tangent
$\operatorname{cosec} x, \sec x, \cot x$	the reciprocals of the above functions
$\sin^{-1} x$ or $\arcsin x$	the inverse of the function $\sin x$ (with similar abbreviations for the inverses of the other circular functions)
$\sinh x$ etc.	the hyperbolic functions

7 Other functions

$\sqrt{a}$	the positive square root of a
$\lvert a \rvert$	the modulus of a
$n!$	n factorial; $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$ $(0! = 1)$
$\binom{n}{r}$	$\frac{n!}{r!(n-r)!}$ when $n, r \in \mathbb{N}$ and $0 \leqslant r \leqslant n$
$\binom{n}{r}$	$\frac{n(n-1)\ldots(n-r+1)}{r!}$ when $n \in \mathbb{Q}$ and $r \in \mathbb{N}$

8 Complex numbers

i	the square root of -1
z or w	a typical complex number, e.g. $x + \mathrm{i}y$, where $x, y \in \mathbb{R}$
$\mathrm{Re}(z)$	the real part of z; $\mathrm{Re}(x + \mathrm{i}y) = x$
$\mathrm{Im}(z)$	the imaginary part of z; $\mathrm{Im}(x + \mathrm{i}y) = y$
$\lvert z \rvert$	the modulus of z; $\lvert x + \mathrm{i}y \rvert = \sqrt{(x^2 + y^2)}$
$\arg z$	the argument of z
z^*	the complex conjugate of z

9 Matrices

$\mathbf{M}$	a typical matrix $\mathbf{M}$
$\mathbf{M}^{-1}$	the inverse of a matrix $\mathbf{M}$ (provided it exists)
$\mathbf{M}^{\mathrm{T}}$	the transpose of matrix $\mathbf{M}$
$\det \mathbf{M}$	the determinant of a square matrix $\mathbf{M}$
$\mathrm{adj}(\mathbf{M})$	the adjoint of a square matrix $\mathbf{M}$
$\mathbf{I}$	the identity matrix

10 Vectors

$\mathbf{a}$	the vector $\mathbf{a}$
$\lvert \mathbf{a} \rvert$ or a	the magnitude of vector $\mathbf{a}$
$\hat{\mathbf{a}}$	the unit vector with the same direction as $\mathbf{a}$
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors parallel to the cartesian coordinate axes
$\overline{\mathrm{AB}}$ or $\mathbf{AB}$	the vector represented by the line segment AB
$\lvert \overline{\mathrm{AB}} \rvert$ or AB	the length of the vector $\overline{\mathrm{AB}}$
$\mathbf{a}.\mathbf{b}$	the scalar product or dot product of $\mathbf{a}$ and $\mathbf{b}$
$\mathbf{a} \wedge \mathbf{b}$ or $\mathbf{a} \times \mathbf{b}$	the vector product of $\mathbf{a}$ and $\mathbf{b}$

11 General

$\therefore$	therefore
L.H.S.	left hand side
R.H.S.	right hand side
$\ldots$	and so on
$\hat{\mathrm{ABC}}$ or $\angle\mathrm{ABC}$	angle ABC

Chapter P1

Elementary algebra

Introduction

To benefit from the *Pure Mathematics* course, you must have a command of elementary algebra. This chapter provides practice in algebra that you have already met.

If you are confident in your command of algebra you may omit this chapter. However, you may find it useful to 'brush up' on particular topics.

P1.1 Simplification

Example 1 *Simplify* $(x+h)^2 + (x-h)^2$.

$$\begin{aligned}(x+h)^2 + (x-h)^2 &= x^2 + 2xh + h^2 + x^2 - 2xh + h^2 \\ &= 2x^2 + 2h^2 \\ &= 2(x^2 + h^2)\end{aligned}$$

Exercise P1a

Simplify:

1 $(x+h)^2 - (x-h)^2$

2 $(x+h)^3 + (x-h)^3$

3 $(x+h)^3 - (x-h)^3$

4 $x(1-2x^2) + 2x(1-x^2)$

5 $(2\sqrt{t} - 3)(1 + \sqrt{t})$

6 $\dfrac{3t^2 + 2t^3}{(1+t)^2} + \dfrac{t^2}{(1+t)^3}$

7 $\dfrac{\sqrt{x}}{(\sqrt{x}-1)} \times \dfrac{\sqrt{x}}{(\sqrt{x}+1)}$

8 $\dfrac{\dfrac{1}{x} + \dfrac{1}{y}}{1 - \dfrac{1}{xy}}$

P1.2 Factorisation

Example 2 *Factorise* $2x(2x+1)^2 + (2x+1)(4x^2-3)$.

When factorising an expression like this, it is important to identify any common factors. In this example $(2x+1)$ is a common factor.

$$\begin{aligned}&2x(2x+1)^2 + (2x+1)(4x^2-3) \\ &\quad = (2x+1)[2x(2x+1) + 4x^2 - 3] \\ &\quad = (2x+1)(4x^2 + 2x + 4x^2 - 3) \\ &\quad = (2x+1)(8x^2 + 2x - 3) \\ &\quad = (2x+1)(2x-1)(4x+3)\end{aligned}$$

(Note that if an expression can be factorised, this should be done.)

Exercise P1b

Factorise:

1 $35x^2 + x - 6$

2 $2x^2 - 98$

3 $2x^2 - xy - y^2$

4 $xy + ay + xb + ab$

5 $xy + 3y - 2x - 6$

6 $x(x+1)^2 + (x+1)(x^2-3)$

7 $(x+3)(x^2+3) + x(x+3)^2$

8 $5(x+1)^2 + 7x(x+1)$

9 $(x+3)^2 - (x-7)^2$

10 $(x-2)^3 + 5x(x-2)^2$

P1.3 Fractions

Example 3 *Express as a single fraction:*

a $\dfrac{1}{2+x} + \dfrac{2}{1-3x}$,

b $\dfrac{1}{a^3b} + \dfrac{1}{ab^3}$.

a $$\begin{aligned}\frac{1}{2+x} + \frac{2}{1-3x} &= \frac{1}{2+x} \times \frac{1-3x}{1-3x} + \frac{2}{1-3x} \times \frac{2+x}{2+x} \\ &= \frac{(1-3x) + 2(2+x)}{(2+x)(1-3x)} \\ &= \frac{1 - 3x + 4 + 2x}{(2+x)(1-3x)} \\ &= \frac{5-x}{(2+x)(1-3x)}\end{aligned}$$

b $$\begin{aligned}\frac{1}{a^3b} + \frac{1}{ab^3} &= \frac{b^2}{a^3b^3} + \frac{a^2}{a^3b^3} \\ &= \frac{a^2+b^2}{a^3b^3}\end{aligned}$$

In this part, notice that a^3b^3 is the lowest common multiple of the original denominators a^3b and ab^3.

Exercise P1c

Express as a single fraction:

1 $\dfrac{1}{x}-\dfrac{1}{y}$

2 $\dfrac{x}{y}+\dfrac{y}{x}$

3 $\dfrac{1}{a^2}+\dfrac{1}{a}$

4 $\dfrac{1}{ab^2}+\dfrac{1}{a^2b}$

5 $\dfrac{1}{x-h}+\dfrac{1}{x+h}$

6 $\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}$

7 $\dfrac{1}{1-x}-\dfrac{2}{2+x}$

8 $\dfrac{x}{x^2+2}-\dfrac{2}{2+x}$

9 $\dfrac{n}{n+1}+\dfrac{1}{(n+1)(n+2)}$

10 $\dfrac{1}{(x+1)^2}+\dfrac{1}{(x+1)}+1$

Further simplification

Exercise P1d

Simplify:

1 $\dfrac{2T-2t}{T^2-t^2}$

2 $y-2t=\dfrac{1}{t}(x-t^2)$

3 $\dfrac{1-1/t}{1-t}$

4 $\dfrac{T-t}{1/T-1/t}$

5 $N(4N^2-1)+3(2N+1)^2$

6 $\dfrac{a}{\surd(a+b)}+\dfrac{b}{\surd(a+b)}$

7 $\dfrac{a/b+c/d}{1+ac/(bd)}$

8 $\dfrac{(x+h)^3-x^3}{h}$

9 $\dfrac{1}{\surd(1+x^2)}-\dfrac{x^2}{\surd(1+x^2)(1+x^2)}$

10 $\sqrt{\left\{\dfrac{1-2t/(1+t^2)}{1+2t/(1+t^2)}\right\}}$

P1.4 Completing the square

Completing the square is a very useful technique which appears in several different contexts. It depends on the identity

$$(A+B)^2=A^2+2AB+B^2$$

as the following example illustrates.

In each of the questions below, there is a number missing wherever a box has been printed.

Example 4 *Complete:* $(\square x+\square)^2=25x^2+70x+\square$.

Comparing this incomplete statement with the identity above and, in particular, comparing the term $25x^2$ with A^2 in the identity, we see that $A=5x$.

Comparing the middle terms, namely $70x$ and $2AB$, and bearing in mind that $A=5x$, we can see that $B=7$.

Lastly, the final term on the right-hand side should be B^2 and so the missing number in the last box is 49.

The complete statement is

$$(5x+7)^2=25x^2+70x+49$$

Exercise P1e

Complete the following:

1 $(x+3)^2=x^2+\square x+9$

2 $(x-5)^2=x^2-\square x+\square$

3 $(3x+2)^2=9x^2+\square x+4$

4 $(x+\square)^2=x^2+10x+\square$

5 $(x-\square)^2=x^2-14x+\square$

6 $(2x+\square)^2=\square x^2+12x+\square$

7 $(x+\frac{1}{2})^2=x^2+\square x+\square$

8 $(\frac{1}{2}x-\square)^2=\square x^2-x+\square$

9 $(\square x^2+3)^2=100x^4+60x^2+\square$

10 $(\frac{1}{3}x+\square y)^2=\square x^2+\frac{1}{3}xy+\square y^2$

P1.5 Changing the subject of a formula

Example 5 a *Make y the subject of* $\dfrac{y-k}{a}=\dfrac{x-h}{b}$.

b *Make x the subject of* $m=\dfrac{x+a}{b-x}$.

a $\dfrac{y-k}{a}=\dfrac{x-h}{b}$

Multiply both sides by a:

$$y-k=a\times\frac{(x-h)}{b}$$
$$=\frac{a}{b}(x-h)$$

Add k to both sides:

$$y=\frac{a}{b}(x-h)+k$$

b This is slightly harder because x appears more than once. The objective of the first few steps is to rearrange the equation so that x appears once only.

$$m=\frac{x+a}{b-x}$$

Multiply both sides by $b-x$:

$$m(b-x)=x+a$$
$$\therefore\ mb-mx=x+a$$

Add mx to both sides and subtract a from both sides:

$$mb-a=x+mx$$
$$=x(1+m)$$

(This has achieved the first objective; x now appears once only.)

Now, divide both sides by $(1+m)$:

$$\frac{mb-a}{1+m}=x \qquad \text{i.e.} \qquad x=\frac{mb-a}{1+m}$$

Exercise P1f

In each question, the letter which is to be made the subject is printed in brackets at the end of the question.

1 $y=mx+c$ (m)

2 $b=a(1-e)$ (e)

3 $y^2=(x+a)^2-(x-a)^2$ (x)

4 $\dfrac{y-k}{K-k}=\dfrac{x-h}{H-h}$ (y)

5 $3mc=(4+3m)(c-4)$ (c)

6 $ax-x+1-b=0$ (x)

7 $T=2\pi\sqrt{\dfrac{l}{g}}$ (l)

8 $T=2\pi\sqrt{\dfrac{l}{g}}$ (g)

9 $2x+2y+2mx-4my+1=0$ (m)

10 $2x-3y-3mx+2my-2m+4=0$ (m)

P1.6 Linear and quadratic equations

Example 6 *Solve the equation* $\frac{1}{2}(2x-3)-\frac{1}{3}(x-2)=\frac{7}{6}$.

$$\tfrac{1}{2}(2x-3)-\tfrac{1}{3}(x-2)=\tfrac{7}{6}$$

Multiply both sides by 6:

$$3(2x-3)-2(x-2)=7$$
$$6x-9-2x+4=7$$

(Be very careful over the + sign in front of the 4: this is a very common source of error!)

Simplifying the left-hand side gives

$$4x-5=7$$
$$\therefore\ 4x=12$$

and hence

$$x=3$$

It is useful to check the answer by substituting $x=3$ in the original equation. The L.H.S. gives

$$\tfrac{1}{2}(2x-3)-\tfrac{1}{3}(x-2)=\tfrac{1}{2}(6-3)-\tfrac{1}{3}(3-2)$$
$$=\tfrac{1}{2}\times3-\tfrac{1}{3}\times1$$
$$=1\tfrac{1}{2}-\tfrac{1}{3}$$
$$=\tfrac{7}{6}$$

Example 7 *Solve the equation $tx - t^2 = Tx - T^2$, expressing x in terms of t and T.*

$$tx - t^2 = Tx - T^2$$
$$\therefore tx - Tx = t^2 - T^2$$

Factorising this gives

$$x(t - T) = (t - T)(t + T)$$

Dividing both sides by $(t - T)$ we have

$$x = t + T$$

(However, note that the final step, namely dividing by $(t - T)$, is only permissible if $t \neq T$, because you must never divide by zero. If t *does* equal T, the original equation is true for all values of x.)

Exercise P1g

Solve the following equations:

1 $2x + 1 = 16 - 3x$

2 $\dfrac{2x-1}{3} - \dfrac{x-7}{5} = 2$

3 $\dfrac{x}{x+1} - \dfrac{1}{x-2} = 1$

4 $\dfrac{x-5}{x+1} = \dfrac{x-7}{x-2}$

5 $2x^2 - 17x + 21 = 0$

6 $x^2 = 5x + 14$

7 $\dfrac{1}{x} - \dfrac{1}{x+1} = \dfrac{1}{x+4}$

In questions 8–10, express x in terms of the other letters.

8 $\dfrac{x - ct}{2} = \dfrac{cT - x}{3}$

9 $5x^2 - 16tx + 3t^2 = 0$

10 $tx^2 + (tT - 1)x - T = 0$

P1.7 Simultaneous equations

You will have solved simultaneous equations before, but the method of substitution may be new. In Example 8, substitution is simply an alternative to other possible methods, but in Example 9, it is the only way to solve the equation.

Example 8 *Solve the simultaneous equations*

$$7x + 2y = 11 \qquad (1)$$
$$4x + y = 7 \qquad (2)$$

Equation (2) can be rearranged to give

$$y = 7 - 4x$$

Substituting $(7 - 4x)$ for y in equation (1):

$$7x + 2(7 - 4x) = 11$$

Removing the brackets,

$$7x + 14 - 8x = 11$$
$$\therefore 14 - x = 11$$
$$\therefore x = 3$$

Putting $x = 3$ in equation (2) gives

$$12 + y = 7$$
$$\therefore y = -5$$

Hence the solution is $x = 3, y = -5$.

(Check this by substituting these values into equation (1).)

Example 9 *Solve the simultaneous equations*

$$x^2 + y^2 = 25r^2$$
$$2y + x = 10r$$

giving the answers in terms of r.

From the second equation we have

$$x = 10r - 2y$$

Substituting this into the first equation gives

$$(10r - 2y)^2 + y^2 = 25r^2$$

Removing the brackets,

$$100r^2 - 40ry + 4y^2 + y^2 = 25r^2$$
i.e. $$100r^2 - 40ry + 5y^2 = 25r^2$$
$$\therefore 5y^2 - 40ry + 75r^2 = 0$$

After dividing both sides by 5, this becomes

$$y^2 - 8ry + 15r^2 = 0$$
$$\therefore (y - 3r)(y - 5r) = 0$$

Therefore either $y - 3r = 0$, or $y - 5r = 0$.

$$\therefore y = 3r \quad \text{or} \quad 5r$$

Substituting these values into the equation $2y + x = 10r$, gives

either $6r + x = 10r$, i.e., $x = 4r$
or $10r + x = 10r$, i.e., $x = 0$

Hence the solution is

$x = 0$ and $y = 5r$

or

$x = 4r$ and $y = 3r$

(Note that each solution consists of a value of x and a value of y.)

Exercise P1h

Solve the following equations:

1 $7x + 4y = 10$, $5x + 3y = 7$

2 $6x + y = 9$, $4x - y = 11$

3 $5x + 2y + 1 = 0$, $y = 7x + 3$

4 $y^2 = 4x$, $y = x$

5 $xy = 64$, $4x - y = 60$

6 $y^2 = 4x + 1$, $y = x + 1$

In questions 7–10, express x and y in terms of the other letters.

7 $2y = x + 4c$, $5y = x + 25c$

8 $ty = x + t^2$, $Ty = x + T^2$, (where $t \neq T$)

9 $xy = 1$, $t^2x - y = t^3 - 1/t$

10 $x^2 - y^2 = 16a^2$, $y = 3x - 12a$

P1.8 Equations of higher degree

Equations of degree more than two can be difficult to solve unless they can be factorised. However, when factors can be found, the same idea which is used in the solution of quadratic equations by factorisation can be used, namely that a product of real numbers can only be zero if one of the factors is zero (see §P2.5 on page 13).

Example 10 *Solve the equation $x^3 - 5x^2 + 6x = 0$.*

Since x is a factor of each term in the equation, we can rewrite it as

$$x(x^2 - 5x + 6) = 0$$

and on factorising the quadratic, we have

$$x(x - 2)(x - 3) = 0$$

Hence

$$x = 0 \quad \text{or} \quad x - 2 = 0 \quad \text{or} \quad x - 3 = 0$$

Therefore $x = 0$, or 2, or 3.

(Notice that although it is tempting to 'divide through' by x, if we do so, we lose the solution $x = 0$.)

Example 11 *Solve the equation $x^4 - 5x^2 - 36 = 0$.*

Although this is an equation of degree 4, we can treat it as a quadratic in x^2.

$$X^2 - 5X - 36 = 0, \quad \text{where } X = x^2$$

Factorising:

$$(X - 9)(X + 4) = 0$$
i.e. $$(x^2 - 9)(x^2 + 4) = 0$$

The factor $(x^2 + 4)$ cannot be zero (not unless we use complex numbers, see §10.3 on page 131), so if x is a real number we deduce that

$$x^2 - 9 = 0$$
$$\therefore x^2 = 9$$

Therefore $x = +3$, or -3.

Exercise P1i

Solve the following equations. In questions 6–10 express x in terms of the other letters.

1 $x^3 - 4x = 0$

2 $x^3 = 7x^2$

3 $x^3 - x^2 - 20x = 0$

4 $x^4 - 17x^2 + 16 = 0$

5 $9x^4 + 5x^2 - 4 = 0$

6 $x^3 + kx^2 = 0$

7 $(x - a)^3 - b^2(x - a) = 0$

8 $x^3 + a^2x = 0$

9 $x^4 - a^4 = 0$

10 $(x - p)^3 = q^3$

Chapter P2

Useful topics in algebra

Introduction

This chapter builds on Chapter P1 by revising and extending further algebraic topics, most of which you have probably already met. Make sure that you are familiar with these topics.

P2.1 Surds

It is not immediately obvious that

$$\frac{3\sqrt{5}}{2\sqrt{7}},\quad \frac{\sqrt{45}}{\sqrt{28}},\quad \frac{3}{14}\sqrt{35},\quad \frac{15}{2\sqrt{35}},$$
$$\frac{3}{2}\sqrt{\frac{5}{7}},\quad \frac{\sqrt{45}}{2\sqrt{7}},\quad \frac{3\sqrt{5}}{\sqrt{28}},\quad \sqrt{\frac{45}{28}}$$

all represent the same number. Again, it may not be clear on first sight that $1/(\sqrt{2}-1)$ and $\sqrt{2}+1$ are equal.

Since square roots frequently occur in trigonometry and coordinate geometry, it is useful to be able to recognise a number when it is written in different ways. The purpose of this section is to give you practice in this.

You may have found an approximate value of $\sqrt{2} \approx 1.414\,213\,562$ on a calculator and may know that this decimal does not terminate or recur. The ancient Greeks did not use decimals, but they discovered that $\sqrt{2}$ could not be expressed as a fraction of two integers (see §2.4 on page 44). $\sqrt{3}$, $\sqrt{5}$, $\sqrt[3]{6}$ are other examples. Such a root is called a **surd**. In general, a number which cannot be expressed as a fraction of two integers is called an **irrational** number.

Question

Q1 Square:

a $\sqrt{2}$	**b** $\sqrt{6}$	**c** $\sqrt{a}$
d $\sqrt{(ab)}$	**e** $3\sqrt{2}$	**f** $4\sqrt{5}$
g $2\sqrt{a}$	**h** $\sqrt{2}\times\sqrt{3}$	**i** $\sqrt{5}\times\sqrt{7}$
j $\sqrt{2}\times\sqrt{8}$	**k** $\sqrt{12}\times\sqrt{3}$	**l** $\sqrt{a}\times\sqrt{b}$

Note that the answers to parts **d** and **l** are the same, i.e.

$$\sqrt{(ab)} = \sqrt{a}\times\sqrt{b}$$

This result is used in Example 1.

Example 1 *Write $\sqrt{63}$ as the simplest possible surd.*

The factors of 63 are $3^2\times 7$.

$\therefore \sqrt{63} = \sqrt{(3^2\times 7)} = \sqrt{3^2}\times\sqrt{7} = 3\sqrt{7}$

Example 2 *Express $6\sqrt{5}$ as a single square root.*

$6\sqrt{5} = \sqrt{36}\times\sqrt{5} = \sqrt{(36\times 5)} = \sqrt{180}$

Example 3 *Simplify $\sqrt{50}+\sqrt{2}-2\sqrt{18}+\sqrt{8}$.*

$$\begin{aligned}\sqrt{50}+\sqrt{2}-2\sqrt{18}+\sqrt{8} &= 5\sqrt{2}+\sqrt{2}-2\times 3\sqrt{2}+2\sqrt{2}\\ &= 8\sqrt{2}-6\sqrt{2}\\ &= 2\sqrt{2}\end{aligned}$$

It is usual not to write surds in the denominator of a fraction when possible. The process of clearing *irrational* numbers is called **rationalisation**.

Example 4 *Rationalise the denominators of*

a $\dfrac{1}{\sqrt{2}}$, **b** $\dfrac{1}{3-\sqrt{2}}$.

a Multiply numerator and denominator by $\sqrt{2}$. Thus

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1}{2}\sqrt{2}$$

b Multiply numerator and denominator by the denominator with the sign of $\sqrt{2}$ changed:

$$\begin{aligned}\frac{1}{3-\sqrt{2}} &= \frac{1}{3-\sqrt{2}}\times\frac{3+\sqrt{2}}{3+\sqrt{2}}\\ &= \frac{3+\sqrt{2}}{9-2}\\ &= \frac{1}{7}(3+\sqrt{2})\end{aligned}$$

Exercise P2a (Oral)

1 Square:

- **a** $\sqrt{5}$ — **b** $\sqrt{\frac{1}{2}}$
- **c** $4\sqrt{3}$ — **d** $\frac{1}{2}\sqrt{2}$
- **e** $\sqrt{\dfrac{a}{b}}$ — **f** $\sqrt{3}\times\sqrt{5}$
- **g** $\sqrt{3}\times\sqrt{7}$ — **h** $\dfrac{\sqrt{p}}{\sqrt{q}}$
- **i** $\dfrac{1}{2\sqrt{p}}$ — **j** $\dfrac{3\sqrt{a}}{\sqrt{(2b)}}$

2 Express in terms of the simplest possible surds:

- **a** $\sqrt{8}$ — **b** $\sqrt{12}$
- **c** $\sqrt{27}$ — **d** $\sqrt{50}$
- **e** $\sqrt{45}$ — **f** $\sqrt{1210}$
- **g** $\sqrt{75}$ — **h** $\sqrt{32}$
- **i** $\sqrt{72}$ — **j** $\sqrt{98}$
- **k** $\sqrt{60}$ — **l** $\sqrt{512}$

3 Express as single square roots:

- **a** $3\sqrt{2}$ — **b** $2\sqrt{3}$
- **c** $4\sqrt{5}$ — **d** $2\sqrt{6}$
- **e** $3\sqrt{8}$ — **f** $6\sqrt{6}$
- **g** $8\sqrt{2}$ — **h** $10\sqrt{10}$
- **i** $\dfrac{\sqrt{2}}{2}$ — **j** $\dfrac{\sqrt{3}}{3}$
- **k** $\dfrac{\sqrt{2}}{2\sqrt{3}}$ — **l** $\dfrac{2}{\sqrt{6}}$

4 Rationalise the denominators of the following fractions:

- **a** $\dfrac{1}{\sqrt{5}}$ — **b** $\dfrac{1}{\sqrt{7}}$
- **c** $-\dfrac{1}{\sqrt{2}}$ — **d** $\dfrac{2}{\sqrt{3}}$
- **e** $\dfrac{3}{\sqrt{6}}$ — **f** $\dfrac{1}{2\sqrt{2}}$
- **g** $-\dfrac{3}{2\sqrt{3}}$ — **h** $\dfrac{9}{4\sqrt{6}}$
- **i** $\dfrac{1}{\sqrt{2}+1}$ — **j** $\dfrac{1}{2-\sqrt{3}}$
- **k** $\dfrac{1}{4-\sqrt{10}}$ — **l** $\dfrac{2}{\sqrt{6}+2}$
- **m** $\dfrac{1}{\sqrt{5}-\sqrt{3}}$ — **n** $\dfrac{3}{\sqrt{6}-\sqrt{5}}$
- **o** $\dfrac{1}{3-2\sqrt{2}}$ — **p** $\dfrac{1}{3\sqrt{2}-2\sqrt{3}}$

Exercise P2b

Do not use a calculator in this exercise.

1 Simplify:

- **a** $\sqrt{8}+\sqrt{18}-2\sqrt{2}$
- **b** $\sqrt{75}+2\sqrt{12}-\sqrt{27}$
- **c** $\sqrt{28}+\sqrt{175}-\sqrt{63}$
- **d** $\sqrt{1000}-\sqrt{40}-\sqrt{90}$
- **e** $\sqrt{512}+\sqrt{128}+\sqrt{32}$
- **f** $\sqrt{24}-3\sqrt{6}-\sqrt{216}+\sqrt{294}$

2 Given that $\sqrt{2}=1.414\ldots$ and $\sqrt{3}=1.732\ldots$, evaluate correct to 3 significant figures:

- **a** $\sqrt{648}$ — **b** $\sqrt{5.12}$
- **c** $\dfrac{1}{\sqrt{3}-\sqrt{2}}$ — **d** $(3+\sqrt{2})^2$
- **e** $\sqrt{\frac{1}{2}}-\sqrt{\frac{1}{8}}$ — **f** $\sqrt{0.0675}$

3 Express in the form $A+B\sqrt{C}$:

- **a** $\dfrac{2}{3-\sqrt{2}}$ — **b** $(\sqrt{5}+2)^2$
- **c** $(1+\sqrt{2})(3-2\sqrt{2})$ — **d** $(\sqrt{3}-1)^2$
- **e** $(1-\sqrt{2})(3+2\sqrt{2})$ — **f** $\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{4}}+\sqrt{\frac{1}{8}}$
- **g** $\sqrt{\frac{1}{3}}-\sqrt{\frac{1}{27}}$ — **h** $\dfrac{1}{\sqrt{5}}+\sqrt{\dfrac{1}{125}}$
- **i** $\dfrac{\sqrt{3}+2}{2\sqrt{3}-1}$ — **j** $\dfrac{\sqrt{5}+1}{\sqrt{5}-1}$
- **k** $\dfrac{\sqrt{8}+3}{\sqrt{18}+2}$ — **l** $\sqrt{3}+2+\dfrac{1}{\sqrt{3}-2}$

4 Rationalise the denominators of

- **a** $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$ — **b** $\dfrac{\sqrt{5}+1}{\sqrt{5}-\sqrt{3}}$
- **c** $\dfrac{2\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}}$ — **d** $\dfrac{\sqrt{2}+2\sqrt{5}}{\sqrt{5}-\sqrt{2}}$
- **e** $\dfrac{\sqrt{6}+\sqrt{3}}{\sqrt{6}-\sqrt{3}}$ — **f** $\dfrac{\sqrt{10}+2\sqrt{5}}{\sqrt{10}+\sqrt{5}}$

5 Express in surd form and rationalise the denominators (see §16.3 on page 196):

- **a** $\dfrac{1}{1+\cos 45^\circ}$ — **b** $\dfrac{2}{1-\cos 30^\circ}$
- **c** $\dfrac{1+\tan 60^\circ}{1-\tan 60^\circ}$ — **d** $\dfrac{1+\tan 30^\circ}{1-\tan 30^\circ}$
- **e** $\dfrac{1+\sin 45^\circ}{1-\sin 45^\circ}$ — **f** $\dfrac{1}{(1-\sin 45^\circ)^2}$

P2.2 Indices

Laws of indices

You should already know the three laws of indices for positive integers:

(1) $a^m \times a^n = a^{m+n}$
(2) $a^m \div a^n = a^{m-n}$, $(m > n)$
(3) $(a^m)^n = a^{mn}$

We shall now assume that these laws are true for *any* indices, and see what meaning must be assigned to fractional and negative indices as a result of this assumption.

Rational indices

We know that $4^3 = 4 \times 4 \times 4$, but so far $4^{1/2}$ has not been given any meaning. If rational indices are to be used, they must obey the laws of indices. This being so, what meaning should be given to $4^{1/2}$? By the first law of indices,

$$4^{1/2} \times 4^{1/2} = 4^1 = 4$$

Therefore $4^{1/2}$ is defined as the square root of 4 (to avoid ambiguity we take it to be the positive square root) and so $4^{1/2} = 2$. Similarly, $a^{1/2} = \sqrt{a}$.

To see what value should be given to $8^{1/3}$, consider

$$8^{1/3} \times 8^{1/3} \times 8^{1/3} = 8^1 = 8$$

Therefore $8^{1/3}$ is defined as $\sqrt[3]{8}$, which is 2. Similarly, $a^{1/3} = \sqrt[3]{a}$.

In general, taking n factors of $a^{1/n}$,

$$a^{1/n} \times a^{1/n} \times \ldots \times a^{1/n} = a$$

so that

$$a^{1/n} = \sqrt[n]{a}$$

Next consider $8^{2/3}$. We know that $8^{1/3} = 2$, so

$$8^{2/3} = 8^{1/3} \times 8^{1/3} = 2 \times 2 = 4$$

Therefore we take $8^{2/3}$ to be the square of the cube root of 8, and in general $a^{m/n}$ must be taken to be the mth power of $\sqrt[n]{a}$ (or the nth root of a^m), and we may write

$$a^{m/n} = \sqrt[n]{a^m}$$

Question

Q2 Find the values of

a $9^{1/2}$ **b** $27^{1/3}$ **c** $27^{2/3}$ **d** $4^{1/2}$
e $4^{3/2}$ **f** $9^{5/2}$ **g** $8^{4/3}$ **h** $16^{3/4}$

Zero and negative indices

So far 2^0 has been given no meaning. Again it is desirable for it to be given a meaning consistent with the laws of indices. Divide 2^1 by 2^1 using the second law:

$$2^1 \div 2^1 = 2^0$$

But $2^1 \div 2^1 = 1$, so 2^0 has the value 1. In the same way, $a^n \div a^n = a^0$, so

$$a^0 = 1 \quad (a \neq 0)$$

Question

Q3 Why does the above not hold for $a = 0$?

To find what meaning is given to 2^{-1}, divide 2^0 by 2^1, using the second law of indices:

$$2^0 \div 2^1 = 2^{-1}$$

But $2^0 \div 2^1 = 1 \div 2 = \frac{1}{2}$, therefore we take 2^{-1} to be $\frac{1}{2}$.

Similarly,

$$2^{-3} = 2^0 \div 2^3 = \frac{1}{2^3}$$

Thus 2^{-3} is the reciprocal of 2^3.

In the same way

$$a^{-n} = \frac{1}{a^n}$$

that is, a^{-n} is the reciprocal of a^n.

Example 5 *Find the value of* $(27/8)^{-2/3}$.

Using the last result, $(27/8)^{-2/3} = (8/27)^{2/3}$.

Taking the cube root,

$$\left(\frac{8}{27}\right)^{2/3} = \left(\frac{2}{3}\right)^2$$

$$\therefore \left(\frac{27}{8}\right)^{-2/3} = \frac{4}{9}$$

Example 6 *Simplify* $\dfrac{(1+x)^{1/2}-\frac{1}{2}x(1+x)^{-1/2}}{1+x}$.

Multiply numerator and denominator by $2(1+x)^{1/2}$.

$$\frac{(1+x)^{1/2}-\frac{1}{2}x(1+x)^{-1/2}}{1+x}=\frac{2(1+x)-x}{2(1+x)^{3/2}}$$
$$=\frac{2+x}{2(1+x)^{3/2}}$$

Exercise P2c (Oral)

1 Find the values of

a $25^{1/2}$ b $27^{1/3}$ c $64^{1/6}$

d $49^{1/2}$ e $\left(\frac{1}{4}\right)^{1/2}$ f $1^{1/4}$

g $(-8)^{1/3}$ h $(-1)^{1/5}$ i $8^{4/3}$

j $27^{2/3}$ k $25^{3/2}$ l $49^{3/2}$

m $\left(\frac{1}{4}\right)^{3/2}$ n $\left(\frac{4}{9}\right)^{1/2}$ o $\left(\frac{27}{8}\right)^{1/3}$

p $\left(\frac{16}{81}\right)^{1/4}$

2 Find the values of

a 7^0 b 3^{-1} c 5^0

d 4^{-1} e 2^{-3} f $\left(\frac{1}{2}\right)^{-1}$

g $\left(\frac{1}{3}\right)^{-2}$ h $\left(\frac{4}{9}\right)^{0}$ i 3^{-3}

j $(-6)^{-1}$ k $\left(-\frac{1}{6}\right)^{0}$ l $\left(\frac{2}{3}\right)^{-2}$

m $\left(-\frac{1}{2}\right)^{-2}$ n $\dfrac{1}{3^{-1}}$ o $\dfrac{2^{-1}}{3^{-2}}$

p $\dfrac{2^0\times 3^{-2}}{5^{-1}}$

3 Find the values of

a $8^{-1/3}$ b $8^{-2/3}$ c $4^{-1/2}$

d $4^{-3/2}$ e $27^{-2/3}$ f $\left(\frac{1}{4}\right)^{-1/2}$

g $\left(\frac{1}{8}\right)^{-1/3}$ h $\left(\frac{1}{27}\right)^{-2/3}$ i $\left(\frac{4}{9}\right)^{-1/2}$

j $\left(\frac{8}{27}\right)^{-1/3}$ k $\left(\frac{16}{81}\right)^{-1/4}$ l $\left(\frac{27}{8}\right)^{-4/3}$

Exercise P2d

1 Find the values of

a $256^{1/2}$ b $1296^{1/2}$ c $64^{1/3}$

d $216^{1/3}$ e $(2\frac{1}{4})^{1/2}$ f $(1\frac{7}{9})^{1/2}$

g $8^{-1/3}$ h $4^{-3/2}$ i $64^{-2/3}$

j $81^{-3/4}$ k $\left(\frac{121}{16}\right)^{1/2}$ l $\left(\frac{1}{16}\right)^{-3/2}$

m $\left(\frac{8}{27}\right)^{2/3}$ n $1.331^{1/3}$ o $0.04^{-3/2}$

p $\dfrac{4^{-3/2}}{8^{-2/3}}$

2 Find the values of

a $\dfrac{16^{1/3}\times 4^{1/3}}{8}$ b $\dfrac{27^{1/2}\times 243^{1/2}}{243^{4/5}}$

c $\dfrac{32^{3/4}\times 16^{0}\times 8^{5/4}}{128^{3/2}}$ d $\dfrac{6^{1/2}\times 96^{1/4}}{216^{1/4}}$

e $\dfrac{12^{1/3}\times 6^{1/3}}{81^{1/6}}$ f $\dfrac{8^{1/6}\times 4^{1/3}}{32^{1/6}\times 16^{1/12}}$

3 Simplify:

a $8^n\times 2^{2n}\div 4^{3n}$

b $3^{n+1}\times 9^n\div 27^{(2/3)n}$

c $16^{(3/4)n}\div 8^{(5/3)n}\times 4^{n+1}$

d $9^{-(1/2)n}\times 3^{n+2}\times 81^{-1/4}$

e $6^{(1/2)n}\times 12^{n+1}\times 27^{-(1/2)n}\div 32^{(1/2)n}$

f $10^{(1/3)n}\times 15^{(1/2)n}\times 6^{(1/6)n}\div 45^{(1/3)n}$

4 Simplify:

a $\dfrac{x^{-2/3}\times x^{1/4}}{x^{1/6}}$ b $\dfrac{\sqrt{(xy)}\times x^{1/3}\times 2y^{1/4}}{(x^{10}y^9)^{1/12}}$

c $\dfrac{x^{2n+1}\times x^{1/2}}{\sqrt{x^{3n}}}$ d $\dfrac{x^{3n+1}}{x^{2n+2\frac{1}{2}}\times\sqrt{x^{2n-3}}}$

e $\dfrac{x^{p+(1/2)q}\times y^{2p-q}}{(xy^2)^p\times\sqrt{x^q}}$ f $\dfrac{x^{-2/3}\times y^{-1/3}}{(x^4y^2)^{-1/6}}$

5 Simplify:

a $\dfrac{x^2(x^2+1)^{-1/2}-(x^2+1)^{1/2}}{x^2}$

b $\dfrac{-\frac{1}{2}x(1-x)^{-1/2}+(1-x)^{1/2}}{x^2}$

c $\dfrac{\frac{1}{2}x^{1/2}(1+x)^{-1/2}-\frac{1}{2}x^{-1/2}(1+x)^{1/2}}{x}$

d $\dfrac{(1+x)^{1/3}-\frac{1}{3}x(1+x)^{-2/3}}{(1+x)^{2/3}}$

e $\dfrac{\sqrt{(1-x)}\,\frac{1}{2}(1+x)^{-1/2}+\frac{1}{2}(1-x)^{-1/2}\sqrt{(1+x)}}{1-x}$

P2.3 Logarithms

You are probably familiar with the use of logarithms for multiplication and division, but there are certain properties of logarithms that are useful in more advanced work. Having just considered indices, this is the appropriate place for logarithms because *a logarithm is an index*.

From a calculator we can see that the logarithm of 3, to base 10, is 0.47712 (correct to five decimal places). This means that $10^{0.47712} = 3$, working to five significant figures. The statement 'the logarithm of 3, to the base 10, is 0.47712' is abbreviated to $\log_{10} 3 = 0.47712$.

Similarly, $10^{0.90309} \approx 8$, which may be expressed as $\log_{10} 8 \approx 0.90309$.

Now $2^3 = 8$, and this statement may also be written in logarithmic notation. Here the base is 2 and the index (i.e. logarithm) is 3, thus $\log_2 8 = 3$.

Question

Q4 What are the bases and logarithms in the following statements?

a	$10^2 = 100$	**b**	$10^{1.6021} \approx 40$
c	$9 = 3^2$	**d**	$4^3 = 64$
e	$1 = 2^0$	**f**	$8 = (1/2)^{-3}$
g	$a^b = c$		

Exercise P2e (Oral)

1 Express the following statements in logarithmic notation:

a	$2^4 = 16$	**b**	$27 = 3^3$
c	$125 = 5^3$	**d**	$10^6 = 1\,000\,000$
e	$1728 = 12^3$	**f**	$64 = 16^{3/2}$
g	$10^4 = 10\,000$	**h**	$4^0 = 1$
i	$0.01 = 10^{-2}$	**j**	$\frac{1}{2} = 2^{-1}$
k	$9^{3/2} = 27$	**l**	$8^{-2/3} = \frac{1}{4}$
m	$81 = (1/3)^{-4}$	**n**	$e^0 = 1$
o	$16^{-1/4} = \frac{1}{2}$	**p**	$(1/8)^0 = 1$
q	$27 = 81^{3/4}$	**r**	$4 = (1/16)^{-1/2}$
s	$(-2/3)^2 = 4/9$	**t**	$(-3)^{-1} = -\frac{1}{3}$
u	$c = a^5$	**v**	$a^3 = b$
w	$p^q = r$	**x**	$a = b^c$

2 Express in index notation:

a	$\log_2 32 = 5$	**b**	$\log_3 9 = 2$
c	$2 = \log_5 25$	**d**	$\log_{10} 100\,000 = 5$
e	$7 = \log_2 128$	**f**	$\log_9 1 = 0$
g	$-2 = \log_3 \frac{1}{9}$	**h**	$\log_4 2 = \frac{1}{2}$
i	$\log_e 1 = 0$	**j**	$\log_{27} 3 = \frac{1}{3}$
k	$2 = \log_a x$	**l**	$\log_3 a = b$
m	$\log_a 8 = c$	**n**	$y = \log_x z$
o	$p = \log_q r$		

3 Evaluate:

a	$\log_2 64$	**b**	$\log_{10} 100$
c	$\log_{10} 10^7$	**d**	$\log_a a^2$
e	$\log_8 2$	**f**	$\log_4 1$
g	$\log_{27} 3$	**h**	$\log_{2/3} \frac{4}{9}$
i	$\log_5 125$	**j**	$\log_{0.1} 10$
k	$\log_e e^3$	**l**	$\log_e \dfrac{1}{e}$

Two numbers can be multiplied by adding their logarithms and divided by subtracting them. The rules are familiar, but it is worth proving them as an example of logarithmic notation.

Questions

Q5 Write in logarithmic notation: $a = c^x$, $b = c^y$, $ab = c^{x+y}$, $a/b = c^{x-y}$.

Deduce that

$$\log_c a + \log_c b = \log_c ab, \quad \text{and that}$$
$$\log_c a - \log_c b = \log_c (a/b)$$

The logarithm of the nth power of a number is obtained by multiplying its logarithm by n. A method of proving this rule is suggested in the next question.

Q6 Write in logarithmic notation: $a = c^x$, $a^n = c^{nx}$.

Deduce that $\log_c a^n = n \log_c a$.

In Q5 and Q6, the suffix c has been used to denote the base of the logarithms. However, when the same base is used throughout a piece of work (for example the answer to a single question or exercise) the suffix may be omitted. Using this convention, the results we have found above can be summarised as follows:

$$\log a + \log b = \log (a \times b)$$
$$\log a - \log b = \log (a/b)$$
$$n \times \log a = \log (a^n)$$

These three results are used in the next example.

Example 7 *Express* $\log_{10} \dfrac{a^2b^3}{100\sqrt{c}}$ *in terms of* $\log_{10} a$, $\log_{10} b$, $\log_{10} c$.

First note that $\sqrt{c} = c^{1/2}$.

Using the two rules of Q5,

$$\log_{10} \frac{a^2b^3}{100c^{1/2}} = \log_{10} a^2 + \log_{10} b^3 - \log_{10} 100 - \log_{10} c^{1/2}$$

Then by the rule of Q6, and writing $\log_{10} 100 = 2$,

$$\log_{10} \frac{a^2b^3}{100c^{1/2}} = 2 \log_{10} a + 3 \log_{10} b - 2 - \frac{1}{2} \log_{10} c$$

The logarithm, to base *ten*, of x is frequently written $\lg x$. This abbreviation is used in the next example and in the exercise which follows.

Example 8 *Simplify* $\dfrac{\lg 125}{\lg 25}$.

[Note that 125 and 25 are both powers of 5, so their logarithms can be expressed in terms of lg 5.]

$$\frac{\lg 125}{\lg 25} = \frac{\lg 5^3}{\lg 5^2} = \frac{3 \lg 5}{2 \lg 5} = \frac{3}{2}$$

Example 9 *Use tables or a calculator to find an approximate value of* $\log_2 7$.

Write $x = \log_2 7$, then $2^x = 7$. Since $2^x = 7$, their logarithms to the base of ten are equal, therefore

$$\lg 2^x = \lg 7$$
$$\therefore x \lg 2 = \lg 7$$
$$\therefore x = \frac{\lg 7}{\lg 2}$$
$$= 2.8074 \text{ (correct to five significant figures)}$$

Therefore $\log_2 7 \approx 2.8074$.

Exercise P2f

Note $\lg x = \log_{10} x$.

1 Express in terms of $\log a$, $\log b$, $\log c$:

- **a** $\log ab$
- **b** $\log \dfrac{a}{c}$
- **c** $\log \dfrac{1}{b}$
- **d** $\log a^2b^{3/2}$
- **e** $\log \dfrac{1}{b^4}$
- **f** $\log \dfrac{a^{1/3}b^4}{c^3}$
- **g** $\log \sqrt{a}$
- **h** $\log \sqrt[3]{b}$
- **i** $\log \sqrt{(ab)}$
- **j** $\lg (10a)$
- **k** $\lg \dfrac{1}{100b^2}$
- **l** $\log \sqrt{\left(\dfrac{a}{b}\right)}$
- **m** $\log \sqrt{\left(\dfrac{ab^3}{c}\right)}$
- **n** $\log \dfrac{b\sqrt{a}}{\sqrt[3]{c}}$
- **o** $\lg \sqrt{\left(\dfrac{10a}{b^5c}\right)}$

2 Express as single logarithms:

- **a** $\log 2 + \log 3$
- **b** $\log 18 - \log 9$
- **c** $\log 4 + 2 \log 3 - \log 6$
- **d** $3 \log 2 + 2 \log 3 - 2 \log 6$
- **e** $\log c + \log a$
- **f** $\log x + \log y - \log z$
- **g** $2 \log a - \log b$
- **h** $2 \log a + 3 \log b - \log c$
- **i** $\frac{1}{2} \log x - \frac{1}{2} \log y$
- **j** $\log p - \frac{1}{3} \log q$
- **k** $2 + 3 \lg a$
- **l** $1 + \lg a - \frac{1}{2} \lg b$
- **m** $2 \lg a - 3 - \lg 2c$
- **n** $3 \lg x - \frac{1}{2} \lg y + 1$

3 Simplify:

- **a** $\lg 1000$
- **b** $\frac{1}{2} \log_3 81$
- **c** $\frac{1}{3} \log_2 64$
- **d** $-\log_2 \frac{1}{2}$
- **e** $\frac{1}{3} \log 8$
- **f** $\frac{1}{2} \log 49$
- **g** $-\frac{1}{2} \log 4$
- **h** $3 \log 3 - \log 27$
- **i** $5 \log 2 - \log 32$
- **j** $\dfrac{\log 8}{\log 2}$
- **k** $\dfrac{\log 81}{\log 9}$
- **l** $\dfrac{\log 49}{\log 343}$

4 Solve the equations:

- **a** $2^x = 5$
- **b** $3^x = 2$
- **c** $3^{4x} = 4$
- **d** $2^x \times 2^{x+1} = 10$
- **e** $(1/2)^x = 6$
- **f** $(2/3)^x = 1/16$

5 Evaluate, taking $\log \pi = 0.4971$ and $e = 2.718$:

a $\log_2 9$ b $\log_{12} 6$
c $\log_3 \pi$ d $\log_e 10$
e $\log_e \pi$ f $\log_3 \frac{1}{2}$

6 Show that $\log_a b = 1/\log_b a$,

a using the result $\log_a b \times \log_b c = \log_a c$,
b from first principles.

7 Evaluate:

a $2.56^{1.21}$ b $1.57^{0.576}$
c $2.718^{3.142}$ d $0.561^{2/5}$
e $0.513^{3/2}$ f $0.0057^{1.39}$

P2.4 The functions $x \mapsto a^x$ and $x \mapsto \log_a x$

Chapter 2 explains more fully the terms used in this section.

We use the word **function** to describe $x \mapsto 10^x$ and $x \mapsto \log_{10} x$, because, in each case, for a given value of x, the rule produces a unique result. In the case of $x \mapsto 10^x$, the domain is $\mathbb{R}$ and the range is $\mathbb{R}^+$, and for $x \mapsto \log_{10} x$, the domain is $\mathbb{R}^+$ and the range is $\mathbb{R}$. In most instances, the actual calculation of 10^x or $\log_{10} x$ will be very complicated, but this does not matter; a calculator can be used where it is appropriate (the same remarks apply to the function $x \mapsto \sqrt{x}$). More generally, if a is a fixed, positive, real number, $x \mapsto a^x$ and $x \mapsto \log_a x$, are perfectly satisfactory functions. (Note that the domains are $\mathbb{R}$ and $\mathbb{R}^+$, respectively.)

Questions

Q7 If $f(x) = 10^x$ and $g(x) = \log_{10} x$, find the values of

a $f(1)$ b $f(2)$ c $f(-1)$
d $g(10)$ e $g(1)$ f $g(\sqrt{10})$

Q8 If $F(x) = a^x$ and $G(x) = \log_a x$, find

a $F(1)$ b $F(2)$ c $F(-1)$
d $G(a)$ e $G(1)$ f $G(\sqrt{a})$

The following special cases are very common and you should commit them to memory:

$\log_a 1 = 0$
$\log_a a = 1$
$\log_a (1/a) = -1$

Remember that a logarithm is an index. The logarithm of q to base a is the power to which a must be raised to equal q, e.g. $\log_{10} 1000 = 3$, and $\log_2 (1/8) = -3$. Thus if $a^p = q$, then $\log_a q = p$, and these are equivalent statements, since they are simply alternative ways of stating the relationship between a, p and q. We can combine these statements in two ways:

$$\log_a (a^p) = \log_a (q) = p$$

and

$$a^{\log_a q} = a^p = q$$

So, if $f(x) = \log_a x$ and $g(x) = a^x$, then the **composite functions** fg and gf are given by

$$fg(x) = f(a^x) = \log_a (a^x) = x$$

and

$$gf(x) = g(\log_a x) = a^{\log_a x} = x$$

In other words, the composite function merely gives the original value of x. The function f 'undoes' the effect of function g, and function g 'undoes' the effect of function f. That is the functions f and g are **inverses** of one another.

This effect can easily be seen on a calculator. Enter any positive number, say 5, press the 'log' function key (the display should show 0.69897), and then press the '10^x' function key. The display should return to the value originally entered, i.e. 5. Repeat this with other numbers; try it also with the functions in the reverse order. If you have a scientific calculator with function keys for e^x and $\log_e x$ (these appear as *exp* and *ln* on some calculators) try the same routine with this pair of inverse functions.

Sketches of the graphs of $y = a^x$ and $\log_a x$ are shown in Fig. P2.1. As with all inverse functions, the graphs are reflections of one another in the line $y = x$.

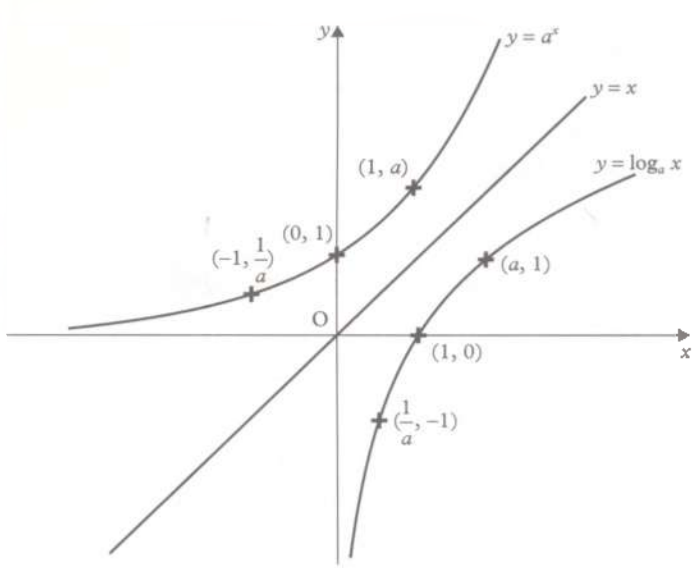


Figure P2.1

P2.5 Roots of quadratic equations

If an algebraic equation, in which the unknown quantity is x, is satisfied by putting $x = c$, we say that c is a **root** of the equation. For example $x^2 - 5x + 6 = 0$ is satisfied by putting $x = 2$, so one root of this equation is 2 (the other is 3).

It is often useful to be able to obtain information about the roots of an equation without actually solving it. For instance, if α and β are the roots of the equation $3x^2 + x - 1 = 0$, the value of $\alpha^2 + \beta^2$ can be found without first finding the values of α and β. This is done by finding the values of $\alpha + \beta$ and $\alpha\beta$, and expressing $\alpha^2 + \beta^2$ in terms of $\alpha + \beta$ and $\alpha\beta$.

The equation whose roots are α and β may be written

$$(x-\alpha)(x-\beta) = 0$$
$$\therefore x^2 - \alpha x - \beta x + \alpha\beta = 0$$
$$\therefore x^2 - (\alpha+\beta)x + \alpha\beta = 0 \qquad (1)$$

But suppose that α and β are also the roots of the equation

$$\boldsymbol{ax^2 + bx + c = 0}$$

which may be written

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad (2)$$

Since equations (1) and (2) have the same roots, they are the same equation, written in two different ways (since the coefficients of x^2 are both 1). Therefore

a the coefficients of x must be equal,

$$\therefore \boldsymbol{\alpha + \beta = -\frac{b}{a}}$$

b the constant terms must be equal,

$$\therefore \boldsymbol{\alpha\beta = \frac{c}{a}}$$

Note: If you are asked to write down an equation whose roots are known, equation (1) gives it in a convenient form. It may be written:

$$x^2 - (\text{sum of the roots})x + (\text{product of the roots}) = 0$$

Questions

Q9 Write down the sums and products of the roots of the following equations:

a $3x^2 - 2x - 7 = 0$
b $5x^2 + 11x + 3 = 0$
c $2x^2 + 5x = 1$
d $2x(x+1) = x + 7$

Q10 Write down equations, the sums and products of whose roots are respectively:

a 7, 12
b 3, −2
c $-\frac{1}{2}, -\frac{3}{8}$
d $\frac{2}{3}, 0$

Q11 Write down the sum and product of the roots of the equation

$$3x^2 + 9x + 7 = 0.$$

Example 10 *The roots of the equation $3x^2 + 4x - 5 = 0$ are α, β. Find the values of* **a** $1/\alpha + 1/\beta$, **b** $\alpha^2 + \beta^2$.

Both $1/\alpha + 1/\beta$ and $\alpha^2 + \beta^2$ can be expressed in terms of $\alpha + \beta$ and $\alpha\beta$.

$$\alpha + \beta = -\tfrac{4}{3}, \qquad \alpha\beta = -\tfrac{5}{3}$$

a $$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{-\frac{4}{3}}{-\frac{5}{3}} = \frac{4}{5}.$$

b $$\alpha^2 + \beta^2 = \alpha^2 + 2\alpha\beta + \beta^2 - 2\alpha\beta = (\alpha+\beta)^2 - 2\alpha\beta = (-\tfrac{4}{3})^2 - 2(-\tfrac{5}{3})$$

$$\therefore \alpha^2 + \beta^2 = \frac{16}{9} + \frac{10}{3} = \frac{46}{9}$$

Alternatively, since α and β are roots of the equation $3x^2 + 4x - 5 = 0$,

$$3\alpha^2 + 4\alpha - 5 = 0$$
$$3\beta^2 + 4\beta - 5 = 0$$

Adding,

$$3(\alpha^2 + \beta^2) + 4(\alpha + \beta) - 10 = 0$$
$$\therefore 3(\alpha^2 + \beta^2) - \tfrac{16}{3} - 10 = 0$$
$$\therefore \alpha^2 + \beta^2 = \tfrac{16}{9} + \tfrac{10}{3} = \tfrac{46}{9}.$$

Example 11 *The roots of the equation $2x^2 - 7x + 4 = 0$ are α, β. Find an equation with integral coefficients whose roots are $\alpha/\beta, \beta/\alpha$.*

Since α, β are the roots of the equation $2x^2 - 7x + 4 = 0$, we have

$$\alpha + \beta = \tfrac{7}{2}, \qquad \alpha\beta = 2$$

To find the required equation, express the sum and product of α/β and β/α in terms of $\alpha + \beta$ and $\alpha\beta$. Then substitute 7/2 for $\alpha + \beta$ and 2 for $\alpha\beta$. See below

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{\frac{49}{4} - 4}{2} = \frac{33}{8}$$

Therefore the sum of the roots is $\frac{33}{8}$.

$$\frac{\alpha}{\beta} \times \frac{\beta}{\alpha} = 1$$

Therefore the product of the roots is 1.

Hence the equation with roots $\alpha/\beta, \beta/\alpha$ is

$$x^2 - \tfrac{33}{8}x + 1 = 0$$

Multiplying through by 8, in order to obtain integral coefficients, the required equation is

$$8x^2 - 33x + 8 = 0$$

P2.6 Symmetrical functions

The functions of α and β that have been used in this chapter all show a certain symmetry. Consider, for example,

$$\alpha + \beta, \qquad \alpha\beta, \qquad \frac{1}{\alpha} + \frac{1}{\beta}, \qquad \alpha^2 + \beta^2, \qquad \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$$

Notice that if α and β are interchanged:

$$\beta + \alpha, \qquad \beta\alpha, \qquad \frac{1}{\beta} + \frac{1}{\alpha}, \qquad \beta^2 + \alpha^2, \qquad \frac{\beta}{\alpha} + \frac{\alpha}{\beta}$$

the resulting functions are the same. When a function of α and β is unchanged when α and β are interchanged, it is called a **symmetrical** function of α and β. Such functions occurring in this chapter may be expressed in terms of $\alpha + \beta$ and $\alpha\beta$, as in the next example.

Example 12 *Express in terms of $\alpha + \beta$ and $\alpha\beta$:*
a $\alpha^3 + \beta^3$, **b** $(\alpha - \beta)^2$.

a α^3 and β^3 occur in the expansion of $(\alpha + \beta)^3$.

$$(\alpha+\beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$$
$$\therefore \alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha^2\beta - 3\alpha\beta^2$$
$$\therefore \alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$$

b $(\alpha - \beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2$.
α^2 and β^2 occur in the expansion of $(\alpha + \beta)^2$.

$$(\alpha+\beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$$
$$\therefore (\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$$

Exercise P2g

1 Find the sums and products of the roots of the following equations:

a	$2x^2 - 11x + 3 = 0$	**b**	$2x^2 + x - 1 = 0$
c	$3x^2 = 7x + 6$	**d**	$x^2 + x = 1$
e	$t(t-1) = 3$	**f**	$y(y+1) = 2y + 5$
g	$x + \frac{1}{x} = 4$	**h**	$\frac{1}{t} + \frac{1}{t+1} = \frac{1}{2}$

2 Find equations, with integral coefficients, the sums and products of whose roots are respectively:

a	3, 4	**b**	−5, 6
c	$\frac{3}{2}, -\frac{5}{2}$	**d**	$-\frac{7}{3}, 0$
e	0, −7	**f**	1.2, 0.8
g	$-\frac{1}{3}, \frac{1}{36}$	**h**	−2.5, −1.6

3 The roots of the equation $2x^2 + 3x - 4 = 0$ are α, β. Find the values of

a	$\alpha^2 + \beta^2$	**b**	$(1/\alpha) + (1/\beta)$
c	$(\alpha + 1)(\beta + 1)$	**d**	$(\beta/\alpha) + (\alpha/\beta)$

4 If the roots of the equation $3x^2 - 5x + 1 = 0$ are α, β, find the values of

a	$\alpha\beta^2 + \alpha^2\beta$	**b**	$\alpha^2 - \alpha\beta + \beta^2$
c	$\alpha^3 + \beta^3$	**d**	$\alpha^2/\beta + \beta^2/\alpha$

5 The equation $4x^2 + 8x - 1 = 0$ has roots α, β. Find the values of

a	$(1/\alpha^2) + (1/\beta^2)$	**b**	$(\alpha - \beta)^2$
c	$\alpha^3\beta + \alpha\beta^3$	**d**	$\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2}$

6 If the roots of the equation $x^2 - 5x - 7 = 0$ are α, β, find equations whose roots are

a α^2, β^2 **b** $\alpha + 1, \beta + 1$ **c** $\alpha^2\beta, \alpha\beta^2$

7 The roots of the equation $2x^2 - 4x + 1 = 0$ are α, β. Find equations with integral coefficients whose roots are

a $\alpha - 2, \beta - 2$ **b** $1/\alpha, 1/\beta$ **c** $\alpha/\beta, \beta/\alpha$

8 Find an equation, with integral coefficients, whose roots are the squares of the roots of the equation $2x^2 + 5x - 6 = 0$.

9 The roots of the equation $x^2 + 6x + q = 0$ are α and $\alpha - 1$. Find the value of q.

10 The roots of the equation $x^2 - px + 8 = 0$ are α and $\alpha + 2$. Find two possible values of p.

11 The roots of the equation $x^2 + 2px + q = 0$ differ by 2. Show that $p^2 = 1 + q$.

12 If the roots of the equation $ax^2 + bx + c = 0$ are α, β, find expressions in terms of a, b, c for

a $\alpha^2\beta + \alpha\beta^2$ **b** $\alpha^2 + \beta^2$
c $\alpha^3 + \beta^3$ **d** $(1/\alpha) + (1/\beta)$
e $(\alpha/\beta) + (\beta/\alpha)$ **f** $\alpha^4 + \beta^4$

13 The equation $ax^2 + bx + c = 0$ has roots α, β. Find equations whose roots are

a $-\alpha, -\beta$ **b** $\alpha + 1, \beta + 1$
c α^2, β^2 **d** $-1/\alpha, -1/\beta$
e $\alpha - \beta, \beta - \alpha$ **f** $2\alpha + \beta, \alpha + 2\beta$

14 Prove that, if the difference between the roots of the equation

$$ax^2 + bx + c = 0$$

is 1, then $a^2 = b^2 - 4ac$.

15 Prove that, if one root of the equation $ax^2 + bx + c = 0$ is twice the other, then $2b^2 = 9ac$.

16 Prove that, if the sum of the squares of the roots of the equation

$$ax^2 + bx + c = 0$$

is 1, then $b^2 = 2ac + a^2$.

17 Prove that, if the sum of the reciprocals of the roots of the equation

$$ax^2 + bx + c = 0$$

is 1, then $b + c = 0$.

If, in addition, one root of the equation is twice the other, use the result of question 15 to find one set of values of a, b, c.
Solve the equation.

18 In the equation $ax^2 + bx + c = 0$, make the substitutions

a $x = y - 1$ **b** $x = y^2$ **c** $x = \sqrt{y}$

and simplify the equations.

If the roots of the equation $ax^2 + bx + c = 0$ are α, β, what are the roots of the three equations in y? [Express y in terms of x, and give your answers in terms of α, β.]

19 If the roots of the equation $ax^2 + bx + c = 0$ are α, β, make substitutions, as in question 18, to find equations whose roots are

a $\alpha + 2, \beta + 2$
b $1/\alpha, 1/\beta$
c $1 \pm \sqrt{\alpha}, 1 \pm \sqrt{\beta}$

P2.7 The remainder theorem

An expression of the form

$$ax^n + bx^{n-1} + \ldots + k$$

where $a, b, \ldots k$ are real numbers and n is a positive integer is called a **polynomial** of degree n. (The expression $5x^7 - 3x^2 + 1.5x - 0.3$, for example, is a polynomial of degree 7.)

If we divide the polynomial $x^3 - 3x^2 + 6x + 5$ by $x - 2$:

$$\begin{array}{r|l} & x^2 - x + 4 \\ \cline{2-2} x-2 & x^3 - 3x^2 + 6x + 5 \\ & x^3 - 2x^2 \\ \cline{2-2} & \quad -x^2 + 6x \\ & \quad -x^2 + 2x \\ \cline{2-2} & \qquad\quad 4x + 5 \\ & \qquad\quad 4x - 8 \\ \cline{2-2} & \qquad\qquad 13 \end{array}$$

the result may be expressed in the identity

$$x^3 - 3x^2 + 6x + 5 = (x - 2)(x^2 - x + 4) + 13$$

Here $x^2 - x + 4$ is called the **quotient** and 13 the **remainder**.

The **remainder theorem** gives a method of finding the remainder without going through the process of division.

Suppose it is required to find the remainder when $x^4 - 5x + 6$ is divided by $x - 2$. If the division were performed, we could write

$$x^4 - 5x + 6 = (x - 2) \times \text{quotient} + \text{remainder}$$

If we put $x = 2$ in this identity we obtain

$$16 - 10 + 6 = 0 \times \text{quotient} + \text{remainder}$$

$\therefore$ the remainder = 12

Applying this process to any such expression divided by $x - a$, we may write

$$\text{expression} = (x - a) \times \text{quotient} + \text{remainder}$$

Putting $x = a$ in this identity, it follows that

the remainder = the value of the expression when $x = a$.

Function notation may be used to state the remainder theorem:

If a polynomial f(x) is divided by $x - a$, the remainder is f(a).

Question

Q12 For what type of expression is the above method valid?

Example 13

Find the remainder when

$$x^5 - 4x^3 + 2x + 3$$

is divided by **a** $x - 1$, **b** $x + 2$.

Let $f(x) = x^5 - 4x^3 + 2x + 3$, then

a the remainder when $f(x)$ is divided by $x - 1$ is

$$f(1) = 1 - 4 + 2 + 3 = 2$$

b the remainder when $f(x)$ is divided by $x + 2$ is

$$f(-2) = -32 + 32 - 4 + 3 = -1$$

Example 14

Find the remainder when $4x^3 - 6x + 5$ *is divided by* $2x - 1$.

As $2x - 1$ is not in the form $x - a$, imagine the division to have been performed, then

$$4x^3 - 6x + 5 = (2x - 1) \times \text{quotient} + \text{remainder}$$

Putting $x = \frac{1}{2}$ in this identity,

$$\tfrac{1}{2} - 3 + 5 = 0 \times \text{quotient} + \text{remainder}$$

Therefore the remainder is $2\frac{1}{2}$.

Example 15

Factorise the expression $2x^3 + 3x^2 - 32x + 15$.

Let $f(x) = 2x^3 + 3x^2 - 32x + 15$.

[$x - a$ will be a factor of $f(x)$ only if there is no remainder on division, i.e. if $f(a) = 0$.]

$f(1) = 2 + 3 - 32 + 15 \neq 0$
$\therefore x - 1$ is *not* a factor.

$f(-1) = -2 + 3 + 32 + 15 \neq 0$
$\therefore x + 1$ is *not* a factor.

$x - 2$ and $x + 2$ cannot be factors, as 2 is not a factor of the constant term 15.

$f(3) = 54 + 27 - 96 + 15 = 0$
$\therefore x - 3$ is a factor.

On division (or by inspection),

$$2x^3 + 3x^2 - 32x + 15 = (x - 3)(2x^2 + 9x - 5)$$

Therefore the factors of $2x^3 + 3x^2 - 32x + 15$ are $(x - 3)(x + 5)(2x - 1)$.

Example 16

When the expression $x^5 + 4x^2 + ax + b$ *is divided by* $x^2 - 1$, *the remainder is* $2x + 3$. *Find the values of* a *and* b.

Suppose the division to have been performed, then

$$x^5 + 4x^2 + ax + b = (x^2 - 1) \times \text{quotient} + 2x + 3$$

Putting $x = 1$, $\quad 1 + 4 + a + b = 2 + 3$.
Putting $x = -1$, $\quad -1 + 4 - a + b = -2 + 3$.

These equations may be rewritten $a + b = 0$ and $-a + b = -2$.

Adding,

$2b = -2$
$\therefore b = -1 \quad$ and $\quad a = 1$

Exercise P2h

1 Find the values of $f(0)$, $f(1)$, $f(-1)$, $f(2)$, $f(-2)$ when

a $f(x) = x^3 + 3x^2 - 4x - 12$

b $f(x) = 3x^3 - 2x - 1$

c $f(x) = x^5 + 2x^4 + 3x^3$

d $f(x) = x^4 - 4x^2 + 3$

State one factor of each expression.

2 Find the remainders when

a $x^3 + 3x^2 - 4x + 2$ is divided by $x - 1$

b $x^3 - 2x^2 + 5x + 8$ is divided by $x - 2$

c $x^5 + x - 9$ is divided by $x + 1$

d $x^3 + 3x^2 + 3x + 1$ is divided by $x + 2$

e $4x^3 - 5x + 4$ is divided by $2x - 1$

f $4x^3 + 6x^2 + 3x + 2$ is divided by $2x + 3$

3 Find the values of a in the expressions below when the following conditions are satisfied:

a $x^3 + ax^2 + 3x - 5$ has remainder -3 when divided by $x - 2$

b $x^3 + x^2 + ax + 8$ is divisible by $x - 1$

c $x^3 + x^2 - 2ax + a^2$ has remainder 8 when divided by $x - 2$

d $x^4 - 3x^2 + 2x + a$ is divisible by $x + 1$

e $x^3 - 3x^2 + ax + 5$ has remainder 17 when divided by $x - 3$

f $x^5 + 4x^4 - 6x^2 + ax + 2$ has remainder 6 when divided by $x + 2$

4 Show that $2x^3 + x^2 - 13x + 6$ is divisible by $x - 2$. Hence find the other factors of the expression.

5 Show that $12x^3 + 16x^2 - 5x - 3$ is divisible by $2x - 1$ and find the factors of the expression.

6 Factorise:

a $x^3 - 2x^2 - 5x + 6$

b $x^3 - 4x^2 + x + 6$

c $2x^3 + x^2 - 8x - 4$

d $2x^3 + 5x^2 + x - 2$

e $2x^3 + 11x^2 + 17x + 6$

f $2x^3 - x^2 + 2x - 1$

7 Find the values of a and b if $ax^4 + bx^3 - 8x^2 + 6$ has remainder $2x + 1$ when divided by $x^2 - 1$.

8 The expression $px^4 + qx^3 + 3x^2 - 2x + 3$ has remainder $x + 1$ when divided by $x^2 - 3x + 2$. Find the values of p and q.

9 The expression $ax^2 + bx + c$ is divisible by $x - 1$, has remainder 2 when divided by $x + 1$, and has remainder 8 when divided by $x - 2$. Find the values of a, b, c.

10 $x - 1$ and $x + 1$ are factors of the expression $x^3 + ax^2 + bx + c$, and it leaves a remainder of 12 when divided by $x - 2$. Find the values of a, b, c.

Calculators and computers

Introduction

As this book will show, for most aspects of advanced mathematics, it is not necessary to use electronic equipment. However, we have included this chapter because the use of calculators and, increasingly, computers can speed up mathematical processes, for example by assisting us with calculations and drawing graphs. It is expected that, at the very least, you will have access to a scientific calculator.

Calculators

Most of us have an electronic calculator. However, we do not always use it very efficiently. Sometimes we use it when it is not necessary. Often we don't know how to get the best out of it. There are many kinds of calculators. Some have more buttons than others; sometimes they work in slightly different ways. So you need to learn about *your* calculator. Find out what it will do and how it will save you from performing laborious calculations. This chapter will help you use your calculator efficiently, whether it is a simple four-function calculator or a more specialised scientific calculator. Throughout the book, you will be directed when you may use calculators, *and when not*.

Computers

The use of computers has grown considerably in recent years and will continue to develop and grow in future. This chapter briefly discusses the importance of computers in information and communications technology (ICT). In particular, it explains how to use a spreadsheet program to record and operate on numerical data.

P3.1 The four-function calculator

Fig. P3.1 shows the main parts of a simple four-function calculator which has percentage and square root keys as well as a memory. Other calculators, such as scientific calculators and programmable calculators, have many more keys and functions. All of them, however, have the basic functions shown in Fig. P3.1; these are the functions you will use most often.

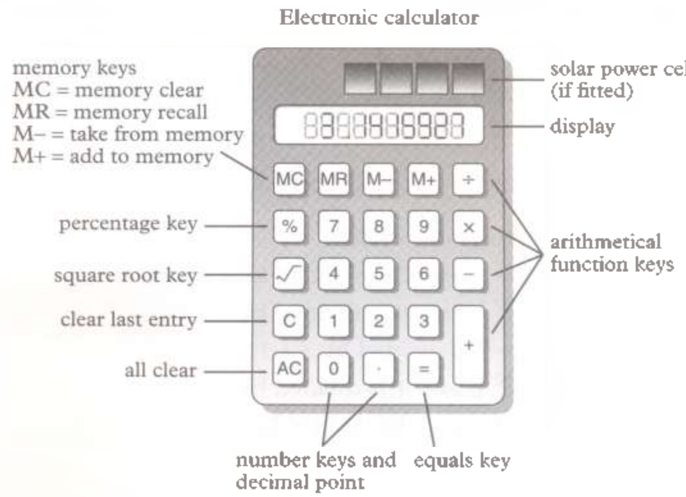


Figure P3.1

Power

Previously calculators got their power from batteries. Nowadays they have solar cells that provide free power from any source of light (e.g. daylight or even candlelight).

Display

The display shows the answers. The digits in the display are usually made of small line segments as shown in Fig. P3.1.

Keyboard

The keyboard has four main sets of keys or buttons:

1. *Number keys*
   Press these keys: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 and the decimal point key (usually shown as a dot ·) to enter numbers into the calculator.
2. *Basic calculation keys*
   Press these keys: +, –, ×, ÷, %, √ and = to operate on the numbers you have entered, and to display answers.
3. *Clearing keys*
   The C key clears the last number you entered. Press C if you enter a wrong number by mistake. On some calculators CE is written instead of C. The AC key clears the whole calculation that you are working on. Use this if you want to start from the beginning again. Often the AC key is linked to the calculator's ON key and is written as ON/AC or just as AC, as in Fig. P3.1. Press AC before starting any calculation and 0 is shown on the display.

4 *Memory keys*

a Memory plus key
Press [M+] to store the displayed number in the memory of the calculator. If there is any previous number in the memory, it adds the displayed number to it.

b Memory minus key
Press [M−] to subtract the displayed number from the number in the memory. The answer obtained from the addition or subtraction will be the new number in the memory.

c Memory recall key
If there is a number in the memory, the calculator usually shows a small M in one corner of the display. Press [MR] to display the number in the memory.

d Memory clear key
Press [MC] to clear the number stored in the memory.

Exercise P3a (activity)

1 Copy and complete Table P3.1 below. For each key sequence first guess what you think the outcome will be. Then use your calculator to get a result. If anything unexpected happens, make a note in the right-hand column.

key sequence	guess	result on calculator	notes	key sequence	guess	result on calculator	notes
a [2][3][+][4][5][=] [1][2][−][5][=] [6][×][4][=] [3][6][÷][9][=]				b [5][+][2][=][=][=] [1][9][−][3][=][=][=] [5][×][2][=][=][=] [3][2][÷][2][=][=][=]			
c [1][÷][4][×][4][=] [1][÷][3][×][3][=] [2][÷][9][×][9][=]				d [5][×][×][=] [5][×][×][=][=] [7][+][+][=] [7][+][+][=][=][=]			
e [49][√] [81][√][√] [7][+][36][√][=] [49][√][−][5][=]				f [2][÷][5][%] [12][×][25][%] [12][+][25][%] [12][−][25][%]			
g [2][+][5][×][3][=] [5][+][3][×][2][=] [3][+][5][÷][2][=] [2][+][3][÷][5][=] [2][×][5][+][3][=] [5][×][3][+][2][=] [5][÷][2][+][3][=] [3][÷][5][+][2][=]				h [6][+][3][C][4][=] [2][×][8][C][7][=] [8][÷][5][C][4][=] [9][+][5][−][3][C][2][=] [7][−][2][×][8][C][9][=]			
i [2][×][8][AC][3][=] [8][÷][5][AC][8][−][5][=]				j [7][−][−][3][=] [7][−][+][3][=] [7][−][×][3][=] [7][+][−][3][=]			
k [MC][2][M+][8][M+][3][M+][MR] [MC][29][M+][8][M−][MR] [MC][25][M+][4][M+][MC][MR] [MC][2][×][7][M+][5][×][3][M+][MR] [MC][8][×][×][M+][6][×][×][M+][MR][√]							

Table P3.1

2 Display on your calculator:
   a the highest possible number,
   b the lowest positive number.

3 To find what a snail lives in:
   a calculate $5 \times 31 \times 499$;
   b turn your calculator upside down and read the display.

4 To find out what plants grow in:
   a calculate $\sqrt{50\,481\,025}$;
   b turn your calculator upside down and read the display.

5 a Use your calculator to complete Table P3.2.

powers of 7	value
7^1	7
7^2	49
7^3	343
7^4	
7^5	
7^6	
7^7	
7^8	

Table P3.2

   b Look at the final digits of the values displayed in Table P3.2. Is there a pattern? If so, what is it?
   c Is there a recognisable pattern in the final two digits?
   d Try the above with a different starting number, e.g. 3, 6, 11, or 13. Are there any patterns?

6 '100 up' is a game for one person.

To start: Enter any 2-digit prime number into your calculator.

Aim: To get the calculator to display a number in the form 100. ***** where * may be any digit.

Rule: You must *multiply* the number shown in the calculator display by any number of your choice.

Scoring: Record each multiplication as a trial. Try to achieve your aim in as few trials as possible, i.e. you should keep your score as *low as possible*.

Here is a sample game:

	display	press keys	trial no. (score)
START	29	$\times 3$	1
	87	$\times 1.2$	2
	104.4	$\times 0.9$	3
	93.96	$\times 1.05$	4
	98.658	$\times 1.02$	5
FINISH	100.63116		

The score for this game is 5. Starting with 29, can you do better? Try to beat 5, then play some games starting with other prime numbers.

P3.2 The operations [+], [−], [×], [÷]

Addition and subtraction

Use the [+], [−] keys to add and subtract and [=] to display the result.

Example 1 *Calculate* 356 + 717.

Keystrokes:
[AC] [3] [5] [6] [+] [7] [1] [7] [=]

Display:
0 3 35 356 356 7 71 717 *1073* (answer)

Rough check: 400 + 700 = 1100

It is a good idea to start any new calculation by pressing the [AC] key. This clears any previous calculation or data which the calculator may contain. When using a calculator it is possible to make keying-in mistakes. So make a habit of doing a rough check. Do the check mentally.

Example 2 *Calculate* 89 – 54 – 17.

Keystrokes:

AC 8 9 − 5 4 − 1 7 =

Display:

0 8 89 89 5 54 *35* 1 17 *18* (answer)

Rough check: 90 – 50 – 20 = 20

Notice the value 35 in the display. This is an intermediate result (89 – 54 = 35). It appears when the second operation is entered.

Example 3 *Calculate* 9 – 16 + 18.

Keystrokes:

AC 9 − 1 6 + 1 8 =

Display:

0 9 9 1 16 *–7* 1 8 *11* (answer)

Rough check: 10 – 20 + 20 = 10

Notice that calculators give a *negative* outcome when subtracting a larger number from a smaller number. Thus 9 – 16 gives –7 as an intermediate result during the above calculation.

Exercise P3b

1 Do the following on your calculator. Write down what appears in the display when you press each key and underline the final answer.

- a 7 + 2
- b 9 – 5
- c 57 – 29
- d 38 + 48
- e 94 – 38 – 26
- f 18 + 37 + 42
- g 123 + 456 – 543
- h 38 – 82 + 71
- i 32.7 – 8.4
- j 3.4 + 7.8 + 4.3

2 Look at the following. Six of them are incorrect.

- i 6 + 7 = 14
- ii 48 + 19 = 912
- iii 22 – 12 = 10
- iv 950 – 42 = 53
- v 235 + 680 = 3015
- vi 8.9 + 4.5 = 13.4
- vii 87 – 59 = 82
- viii 36 + 48 = –12

a Decide which ones appear to be incorrect.

b Use your calculator to correct them.

3 Look at the following before doing them. What kind of answer do you expect? Do the calculations.

- a 2 – 7
- b 5 – 15
- c 16 – 49
- d 36 – 73
- e 8 – 75
- f 44 – 260
- g 256 – 911
- h 56 – 46 – 66

4 An athlete buys some clothes. The bill is shown in Fig. P3.2. Check that the shop assistant has added up everything correctly.

Sports World

Tracksuit	$32.99
Sweatshirt	$13.99
Shorts	$22.49
Running shoes	$38.65
Total	$108.12

Figure P3.2

5 What will the shopping in Fig. P3.3 cost?

Bottle orange juice	$4.29
Jar coffee	$5.82
Packet of tea	$1.95
Sugar	$1.80
Margarine	$2.99
Jar peanut butter	$2.65
Oranges	$3.89
Meat	$8.50
Shampoo	$4.05
Bottle apple juice	$3.76
Chicken	$14.50

Figure P3.3

Multiplication and division

Use the [×] and [÷] buttons to multiply and divide numbers. Given a multiplication (or a division) in the form $a \times b$ (or $a \div b$), press the keys [a] [×] [b] (or [a] [÷] [b]) then either [+], [−], [×], [÷] or [=] will display the answer.

Example 4 *Calculate* 68×29.

Keystrokes:
[AC] [6] [8] [×] [2] [9] [=]

Display:
0 6 68 68 2 29 *1972* (answer)

Rough check: $70 \times 30 = 2100$

Example 5 *Calculate* $725 \div 25 \times 14$.

Keystrokes:
[AC] [7] [2] [5] [÷] [2] [5] [×] [1] [4] [=]

Display:
0 7 72 725 725 2 25 *29* 1 14 *406* (answer)

Rough check: $700 \div 20 \times 10 = 350$

Notice that $725 \div 25 = 29$.
29 appears in the display at an intermediate stage of the calculation.

Calculators have a limited number of spaces in the display (usually eight spaces). Because of this there is a limit to the size of answer they can display.
Try the calculation $68\,000 \times 29\,000$ on a calculator.
Fig. P3.4 shows what could result on some eight-digit calculators.

a
b


Figure P3.4

c
d


Figure P3.4 (continued)

Note that $68\,000 \times 29\,000 = 1\,972\,000\,000$ (requiring 10 digits).

The calculators in **a** and **b** show the digits 1972 but are unable to display the full number properly. So they print a small E (error) to warn the user.

The calculators in **c** and **d** group the display into two parts: 1.972 and 09. This is short for $1.972 \times 1\,000\,000\,000$. Note the nine zeros in the second number. They correspond to the 09. This type of calculator is called a *scientific calculator*. Scientific calculators can cope with very large numbers, because they give the outcome in *standard form*:
$68\,000 \times 29\,000 = 1.972 \times 10^9$.

Example 6 *How many seconds are there in a 31-day month?*

Number of seconds
$= 60 \times 60 \times 24 \times 31$
$= 2\,678\,400$ [calculator]

Exercise P3c

1 Do the following on your calculator. Write down what appears in the display as you press each button. Underline your final answer.

- **a** 67×88
- **b** $4234 \div 58$
- **c** $513 \div 19$
- **d** 46^2
- **e** $49 \times 67 \times 13$
- **f** $7938 \div 81 \div 14$
- **g** $102 \times 104 \div 78$
- **h** $495 \div 33 \times 41$

2 Do the following calculations.

Give each answer

i as displayed on the calculator,

ii rounded off to 2 decimal places.

a 6.74×9.08

b 51.73×24.79

c $28\,341 \div 85$

d $74.184 \div 40.08$

e $4 \div 3 \times 6$

f $7 \div 11 \times 44$

g $773 \div 4.17 \times 5.308$

h $3.142 \times 4.5 \times 4.5$

3 Look at the following. Six of them are incorrect.

i $5 \times 9 = 30$

ii $100\,000 \div 100 = 1000$

iii $67 \times 84 = 3216$

iv $690 \div 15 = 45$

v $1000 \div 30 = 33$

vi $1.7 \times 1.5 \times 1.3 = 33.15$

vii $360 \div 18 \div 5 = 4$

viii $4123 \div 814 = 5$

a Decide which ones appear to be incorrect.

b Use your calculator to correct them.

4 Multiply 30 000 by 50 000 on your calculator. What is displayed?

5 a Calculate $10\,000\,000 \div 0.9$,

b Calculate $10\,000\,000 \div 0.9 \div 0.9$.

6 How many seconds are there in a 365-day year?

7 a Write your age to the nearest year.

b Calculate how many days you have lived (assume 365 days in a year).

c Calculate how many hours you have lived.

d Calculate how many minutes you have lived.

e Is it possible for your calculator to calculate the number of seconds you have lived?

8 A health inspector gets a salary of \$65 952 per annum. How much does this represent **a** per month, **b** per day? Give answers to the nearest dollar.

9 Twelve members of a club hired a bus to go to a match. If the bus company charged \$315, how much did each member have to pay?

10 An aeroplane travels 550 km in 1 hour.

a How many km does it travel in one minute?

b How many metres does it travel in one minute?

c How many metres does it travel in one second?

P3.3 Further calculator techniques

Mixed operations

Look back to Exercise P3a, Question 1, part g. In some cases calculators appear to give two answers to the same problem. According to the rules of precedence in arithmetic:

$$2 + 5 \times 3 = 2 + 15 = 17$$

and $$5 \times 3 + 2 = 15 + 2 = 17.$$

However, the calculator gives on the one hand:

Keystrokes:	$\boxed{2}$	$\boxed{+}$	$\boxed{5}$	$\boxed{\times}$	$\boxed{3}$	$\boxed{=}$	
Display:	2	2	5	7	3	*21*	(answer)

and on the other:

Keystrokes:	$\boxed{3}$	$\boxed{\times}$	$\boxed{5}$	$\boxed{+}$	$\boxed{2}$	$\boxed{=}$	
Display:	3	3	5	15	2	*17*	(answer)

This is because the calculator follows the operations in the order it receives them.

Example 7 *Calculate* $34 + 8 \times 52$.

There are no brackets, but multiplication is done before addition. Rearrange the numbers as follows:

$$\begin{aligned} 34 + 8 \times 52 &= 8 \times 52 + 34 \\ &= (8 \times 52) + 34 \\ &= 450 \text{ [calculator]} \end{aligned}$$

Brackets

Example 8 *Calculate* $2.3 \times (8.9 - 2.1)$.

The brackets show that the subtraction is to be done first. Rearrange the numbers so that the subtraction comes before the multiplication:

$$\begin{aligned} &2.3 \times (8.9 - 2.1) \\ &= (8.9 - 2.1) \times 2.3 \\ &= 15.64 \text{ [calculator]} \end{aligned}$$

In the above examples it is possible to 'turn the calculation round' because in general $a \times b = b \times a$. However, with division this is not possible. Read the following example carefully.

Example 9 *Calculate* 84 ÷ (37 − 23).

The subtraction in the brackets must be done first. The outcome is held in memory, to be recalled when it is needed. The sequence of working is as follows:

37 $\boxed{-}$ 23 $\boxed{\text{M+}}$ 84 $\boxed{\div}$ $\boxed{\text{MR}}$ $\boxed{=}$ 6

Check the above sequence on your own calculator and note the changes in display.

It is essential to enter numbers and operations in an order which will enable the calculator to give correct results. This means doing calculations in brackets first and storing them if necessary. Thereafter, do multiplications and divisions before additions and subtractions.

Exercise P3d

1 In six of the following cases, calculators will give incorrect results if the numbers and operations are entered in the given order. In those cases rearrange the numbers so that calculators will compute correct results.
- a $89 \times 6 - 231$
- b $45 + 68 \div 17$
- c $22 + 42 \div 3$
- d $63 + 18 \times 5$
- e $18 \times (17 - 15)$
- f $(19 + 9) \div 7$
- g $487 \times (6 + 3)$
- h $100 \times (31 - 14)$

2 Calculate the following, rearranging the order *where necessary.*
- a $95 \times 7 - 436$
- b $101 + 51 \times 9$
- c $55 + 75 \div 5$
- d $666 \div 36 + 2.5$
- e $49 \times (19 - 3)$
- f $(434 - 343) \div 13$
- g $8.438 + 36.2 \div 2.6$
- h $8.8 \times (6.12 - 3.47)$

3 All of the following require part of the calculation to be stored (either on paper or in memory).
- a $68 - 14 \times 3$
- b $216 \div (25 - 7)$
- c $708 \div (28 + 31)$
- d $444 - 261 \div 29$
- e $46.7 \div (15.28 - 3.59)$
- f $381.04 - 12.6 \times 7.8$

Squares and square roots

Squares

Many calculators have a 'squares' key, usually shown as $\boxed{x^2}$. However, if your calculator does not have one of these, then it is usually possible to get squares by pressing the $\boxed{\times}$ key twice. For example, try $\boxed{9}$ $\boxed{\times}$ $\boxed{\times}$ $\boxed{=}$; this should give 81.

Square roots

Simply enter a number and use the $\boxed{\surd}$ key.
For example $\boxed{2}$ $\boxed{\surd}$ gives 1.414 213 6.

Exercise P3e

1 Use your calculator to find the value of the following. Round your numbers to 3 s.f.
- a 67^2
- b 76^2
- c 9.8^2
- d 783^2
- e 6.77^2
- f 48.72^2

2 Use your calculator to find the value of the following. Round your answers to 3 s.f.
- a $\sqrt{3}$
- b $\sqrt{5}$
- c $\sqrt{760}$
- d $\sqrt{6.82}$
- e $\sqrt{24.9}$
- f $\sqrt{3884}$

3 *Calculator fun*
- a Find $\sqrt{53\,787\,556}$. Turn the calculator upside down.
  Is the display on your hand or on your foot?
- b Find $19^2 - 4^2$. Turn the calculator upside down. Is the display male or female?
- c Make up more like these.

P3.4 The scientific calculator

You may prefer to leave this section until you have read Chapter 18.

In addition to the operations on a four-function calculator, a **scientific calculator** has a wide range of trigonometrical, logarithmic and statistical functions. Fig. P3.5 shows a typical layout of a scientific calculator.

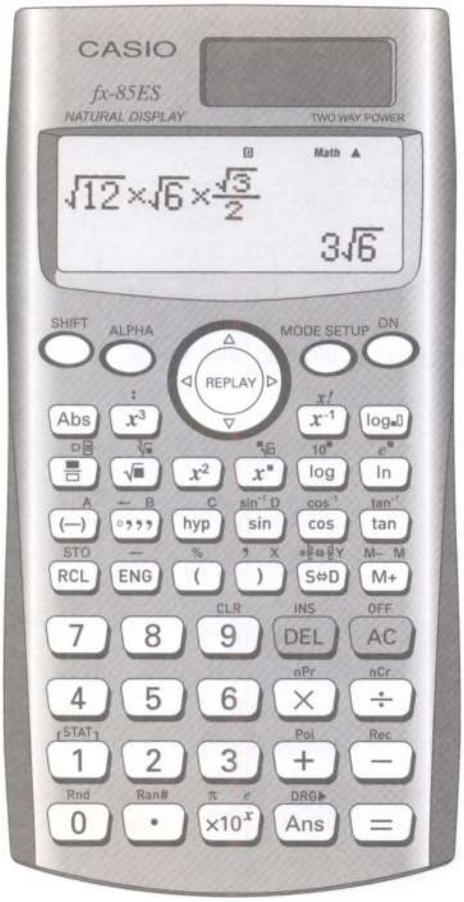


Figure P3.5

A full description of what a scientific calculator can do is not possible within the scope of this book. You should refer to your calculator manual for complete instructions. What follows is a brief account of how to use a scientific calculator to assist with mathematical calculations that involve trigonometry.

In this section we will concentrate on the **trigonometric** keys, the **shift** key and the **reciprocal** key. Fig. P3.6 shows how they are arranged on a typical scientific calculator. The [SHIFT] key is usually the top left-hand button (also see Fig. P3.5) and the reciprocal key [1/x] or [x^{-1}] is often just below it.

[SHIFT]

[1/x]

$\sin^{-1}$ $\cos^{-1}$ $\tan^{-1}$

[sin] [cos] [tan]

Figure P3.6

Note that $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ are **the inverse functions** of sin, cos and tan (sine, cosine and tangent).

Chapter 18 fully explains inverse trigonometrical functions. On a scientific calculator they are usually written just above the [sin], [cos] and [tan] buttons.[1]
Sin^{-1} (0.8) means 'the angle whose sine is 0.8'. To find this on the calculator, enter 0.8, then press [SHIFT] followed by [sin]. The display will give 53.130 102, i.e. an angle of 53.1° to the nearest one-tenth of a degree. Therefore sin 53.1° = 0.8.

Example 10 (discussion) *Given Fig. P3.7, calculate BC.*

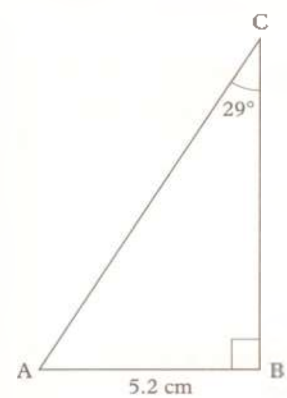


Figure P3.7

In Fig. P3.7, $\tan 29° = \dfrac{AB}{BC} = \dfrac{5.2}{BC}$

$$\Rightarrow \quad BC = \frac{5.2}{\tan 29°}\text{ cm}$$

There are various ways to do this on a calculator. Whichever way, it is usually best to start with the trigonometric term.

i Here is a key sequence (and the intermediate displays) that gives the answer:

calculator key	*display*
[29]	29
[tan]	0.554 309
[M+]	0.554 309
[5.2]	5.2
[÷]	5.2
[MR]	0.554 309
[=]	9.381 048 3 (answer)

$\Rightarrow$ BC = 9.4 cm (to 2 s.f.)

[1]Remember that there are different makes of calculator. Your one may be slightly different to the type described here.

ii Here is another way. In this case we invert the calculation then use the reciprocal button, $\boxed{1/x}$, to turn it up the correct way. Follow the steps carefully.*

calculator key	*display*
$\boxed{29}$	29
$\boxed{\tan}$	0.554 309
$\boxed{\div}$	0.554 309
$\boxed{5.2}$	5.2
$\boxed{=}$	0.106 597 8
$\boxed{1/x}$	9.381 048 3 (answer)

Again, BC = 9.4 cm (to 2 s.f.)

Do you see how this method works? This method is more efficient because it involves one less key stroke.

iii However, there is an even more efficient way. In $\triangle ABC$, notice that $\hat{A} = 61°$ (angles of a triangle). Therefore:

$$\tan 61° = \frac{BC}{5.2}$$

$$\Rightarrow \qquad BC = 5.2 \times \tan 61°$$

Again, on the calculator, begin with the trigonometrical function:

calculator key	*display*
$\boxed{61}$	61
$\boxed{\tan}$	1.804 047 8
$\boxed{\times}$	1.804 047 8
$\boxed{5.2}$	5.2
$\boxed{=}$	9.381 048 3 (answer)

Again, BC = 9.4 cm (to 2 s.f.).

This is the most efficient method (fewest key strokes).

Notice in every case we round the final answer to the degree of accuracy implied in the data of the question.

Example 11 (discussion) *Given the data of* **Fig. P3.8**, *calculate $\widehat{QRS}$.*

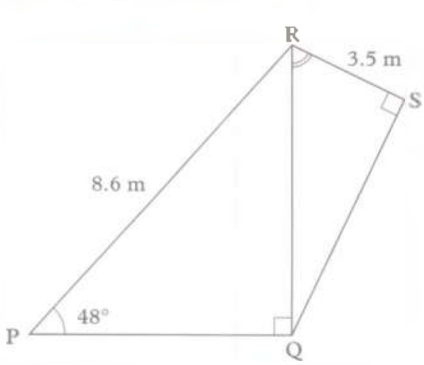


Figure P3.8

In $\triangle PQR$, $\quad \sin 48° = \dfrac{QR}{8.6}$

$$\Rightarrow \qquad QR = 8.6 \times \sin 48° \text{ m}$$

In $\triangle QRS$, $\quad \cos \widehat{QRS} = \dfrac{RS}{QR} = \dfrac{3.5}{8.6 \times \sin 48°}$

$$\widehat{QRS} = \cos^{-1}\left(\frac{3.5}{8.6 \times \sin 48°}\right)$$

On the calculator, start at sin 48° and 'work backwards':

calculator key	*display*
$\boxed{48}$	48
$\boxed{\sin}$	0.743 144 8
$\boxed{\times}$	0.743 144 8
$\boxed{8.6}$	8.6
$\boxed{\div}$	6.391 045 5
$\boxed{3.5}$	3.5
$\boxed{=}$	1.826 013
$\boxed{1/x}$	0.547 641 2
$\boxed{\text{SHIFT}}$	0.547 641 2
$\boxed{\cos}$	56.794 659 (answer)

$\widehat{QRS} = 57°$ (to 2 s.f.)

Notice the following:

i the order of working — it appears to be backwards;

ii the last two steps, $\boxed{\text{SHIFT}}$ then $\boxed{\cos}$, is equivalent to $\cos^{-1}$.

Exercise P3f

1 On a scientific calculator find the value of the following. Give each answer **i** as displayed by the calculator, **ii** rounded to 3 s.f.

a	sin 35°	**b**	cos 77.3°
c	tan 77.3°	**d**	$\tan^{-1} 3.6$
e	$\sin^{-1} 0.85$	**f**	$\cos^{-1} 2.8$
g	1/67	**h**	1/0.67
i	1/0.000 067		

2 Explain why 'E' showed in part **f** of question 1.

3 Enter any number into your scientific calculator. If you press $\boxed{1/x}$ twice, guess what the final display will be. Do it. Check your guess.

If you press $\boxed{1/x}$ an odd number of times, what happens?

If you press it an even number of times, what happens?

4 *(Discussion – using the $\boxed{x^y}$ button)*

a Choose any number from the set {2, 4, 5, 6} and enter it into your scientific calculator. Press the $\boxed{x^y}$ button once. Now enter the number 3 followed by $\boxed{=}$.

How does the answer relate to your initial number? In relation to the x and y on the $\boxed{x^y}$ button, which number is x and which is y?

b Now choose a different number from the set and enter it. Guess what will happen after you press 3 followed by $\boxed{=}$? Check on the calculator.

c Explore the use of the $\boxed{x^y}$ button for simple values of y other than 3. Use it to find the value of **i** 7^3, **ii** 2.8^4, **iii** 17^5.

5 Copy **Fig. P3.7** but change the data so that $A\hat{B}C = 38°$ and AC = 7.4 cm. Calculate BC in three different ways.

6 Copy **Fig. P3.8** but change the dimensions so that PQ = 6.3 m, RS = 3.7 m and $\hat{P} = 63°$. Calculate $Q\hat{R}S$.

7 **(optional)**

In **Fig. P3.9**, △ABC is any general triangle, with sides a, b, c opposite angles A, B, C.

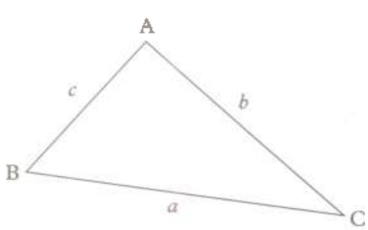


Figure P3.9

Here are three formulae* that relate to any such triangle:

(1) Area of $\triangle ABC = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B = \frac{1}{2}bc \sin A$

(2) $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ (called the *sine rule*)

(3) $a^2 = b^2 + c^2 - 2bc \cos A$
or $b^2 = a^2 + c^2 - 2ac \cos B$
or $c^2 = a^2 + b^2 - 2ab \cos C$
(called the *cosine rule*)

*These formulae will be proved and used in Chapter 18.

a If $a = 7.2$ cm, $b = 9.7$ cm and C = 52°, calculate the area of △ABC.

b A triangle has an area of 68.7 cm² and two of its sides measure 14.3 cm and 10.8 cm. What is the acute angle between these sides?

c Use the sine rule to calculate a if A = 64°, B = 53° and b = 18 m.

d Use the cosine rule to calculate a if b = 4.4 cm, c = 6.9 cm and $\hat{A} = 73°$.

[You will need to use the memory buttons in this part.]

P3.5 Overview of ICT

In the past few decades the impact of **information and communications technology (ICT)** on many people's personal and professional lives has been immense. Mobile phones, computers and computer programs have played a central role in ICT development and will play an even greater role in the future.

Although computers were originally developed to speed up calculation, a huge leap forward in ICT development took place when it became possible to connect them to the **internet** on a **world wide web (www)** via telephone, satellite and broad band radio systems. Think of the internet as a global communication network that provides national and international access to information and to other people. Millions of people use the internet on a daily basis to access information, to purchase and provide goods and services, and to communicate with each other by electronic mail (**email**).

Nowadays, to meet ICT demands, computers have become smaller, faster, more versatile, more reliable and more affordable. Computer chips are used in cars, mobile phones, television sets, music players, cameras and are carried by many people in the form of bank cards. The tendency for greater portability, greater connectivity and 'more power for less cost' will continue. For example, most mobile phones now have considerable computing, photographic, entertainment and internet capability.

ICT is mainly used for 'connecting people', usually by the spoken or written word. **Fig. P3.10** shows some of the kinds of computers available, and **Fig. P3.11** shows some of the functions and programs that demonstrate the computer's central place in ICT.

a desktop computer

b laptop computer

c mobile phone with in-built computer

Figure P3.10

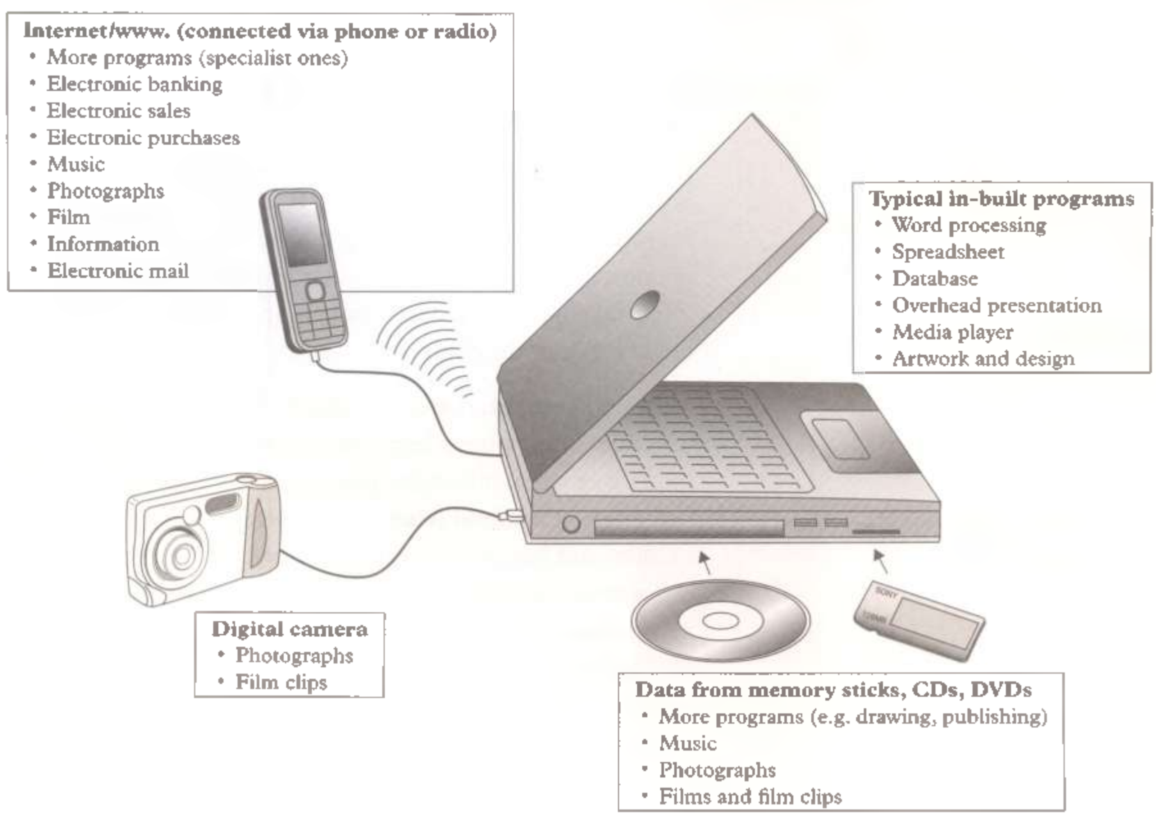


Figure P3.11

P3.6 Computers and mathematics

This section will only be meaningful if you have access to a computer that has a spreadsheet program, preferably Microsoft Excel as used here.

Follow through the Class Activities on your computer.

As already mentioned, computers had a historically important role in speeding up calculation and handling numerical information. Of the software listed in Fig. P3.11, **spreadsheet** programs currently have the greatest everyday application to mathematics, statistics and economics.

A spreadsheet has the appearance of an extensive matrix of cells (Fig. P3.12). Data, either written or numerical, are entered into each cell.

There are many spreadsheet programs. The one used in this chapter is the most common: Microsoft Excel. In this program we identify each cell by an ordered pair: column letter, row number. The cell highlighted in Fig. P3.12 is C7. This is similar to an ordered pair, (x, y), on the cartesian plane. Use the arrow keys on the keyboard, or the computer mouse, to locate a cell.

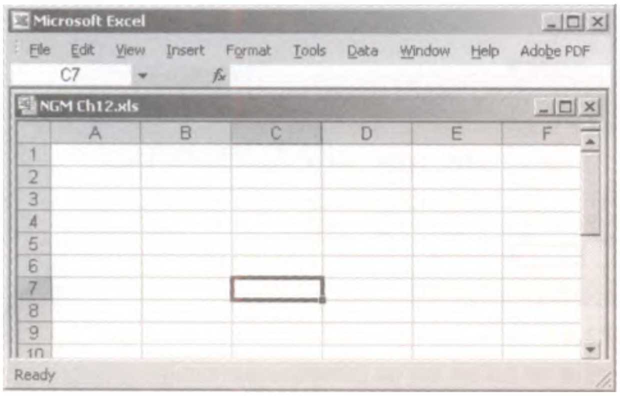


Figure P3.12

Entering data into a spreadsheet

Computer activity and discussion

Table P3.3 shows the numbers of students by gender from District XXX that obtained tertiary education awards for the period 2003–04 to 2007–08.

	2003–04	2004–05	2005–06	2006–07	2007–08
Males	1324	1421	1843	1981	2102
Females	1171	1406	1316	1494	1651

Table P3.3 Tertiary students by gender in District XXX from 2003–04 to 2007–08

Enter the data in Table P3.3 onto a spreadsheet file. Work through this section, copying the various methodologies given below. At the end, save your file as District XXX.

Method:
Open a blank spreadsheet.

- Go to Cell A1. Type in the table title. Press Enter.
- Go to Cell B3. Enter **2003–04**. Then, move across to cells C3, D3, E3, F3 one at a time, entering the years from **2004–05** to **2007–08**.
- Go to Cell A4. Enter the title **Males**. Move across from B4 to F4; enter the numerical data for males from Table P3.3. [Do not put spaces between the digits.]
- Go to Cell A5. Enter the title **Females**, then move across from B5 to F5, entering the numerical data for females from Table P3.3.

Your spreadsheet should now look like Fig. P3.13.

In Fig. P3.13 we used the 'bold' command to emphasise the title, the years and the left-hand column. To change or edit the contents of a cell, go to that cell, click on it, change the contents as desired, and then press Enter.

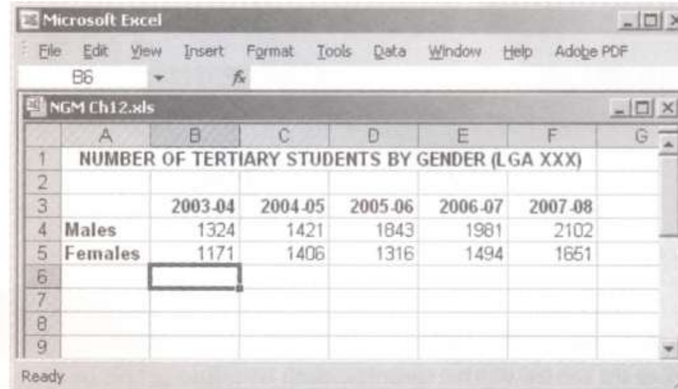


Figure P3.13

Drawing graphs

To draw a graph of the data in Fig. P3.13:

- Select or highlight all of the relevant cells by clicking and dragging from A3 to F5.
- Click on the *Chart Wizard* icon.
- Choose a graph type, in this case a **bar graph**, then follow the instructions on the screen. Fig. P3.14 is an Excel **bar graph** of the male and female tertiary students for the given years.

Alternatively, you may wish to draw a **pie chart** that shows the proportion of male to female students in a given year, e.g. 2003/04:

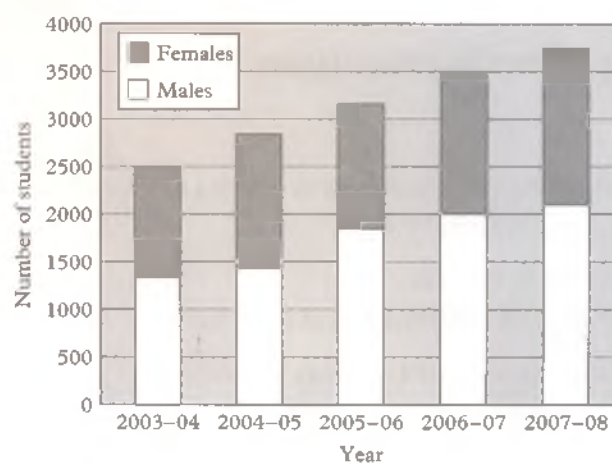


Figure P3.14

- Select the data for 2003/04 (cells B4 to B5).
- Follow the steps under the *Chart Wizard*, this time selecting the instructions for a pie chart. Fig. P3.15 shows one of many kinds of pie chart that Excel can draw.

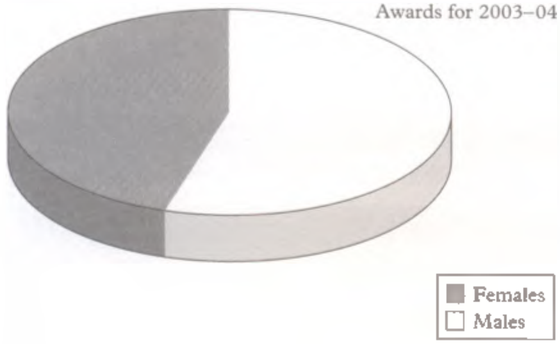
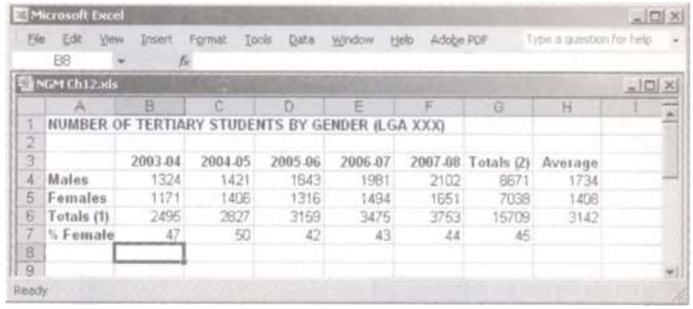


Figure P3.15



	A	B	C	D	E	F	G	H
1	NUMBER OF TERTIARY STUDENTS BY GENDER (LGA XXX)							
2								
3		2003-04	2004-05	2005-06	2006-07	2007-08	Totals (2)	Average
4	Males	1324	1421	1843	1981	2102	8671	1734
5	Females	1171	1406	1316	1494	1651	7038	1408
6	Totals (1)	2495	2827	3159	3475	3753	15709	3142
7	% Female	47	50	42	43	44	45	
8								
9								

Figure P3.17

Depending on the data, we can use the spreadsheet program to draw other graphs, such as a line graph or a scattergram. For example, Fig. P3.16 is a **line graph** of the data, drawn by selecting cells B3 to F5, then following the instructions in the *Chart Wizard*.

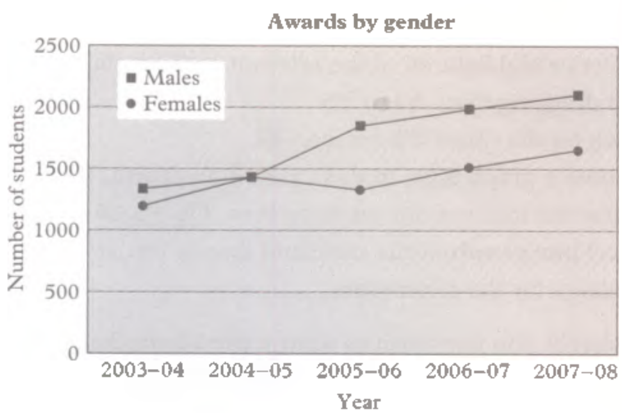


Figure P3.16

Using formulae functions

Sum

To find the **sum** of the numbers in cells B4 to F4 and place the total in Cell G4:

- First click in Cell G4
- Then type =SUM(B4:F4) and press Enter.

Notice that =SUM(B4:F4) is short for 'the sum of the values in cells B4 to F4'. Another way to find this sum is to select cells B4 to F4 and press the Σ icon (Σ is the Greek letter S, which is short for 'sum'). Try this for cells B5 to F5.

Use both of these methods to check the data in the **Totals (1)** row and **Totals (2)** column in Fig. P3.17.

Average

To find **the average** of the numbers from B4 to F4 on the spreadsheet:

- Click the cell where you want the average to go (H4)
- Type =AVERAGE(B4:F4), then press Enter.

The result is shown in Cell H4 in Fig. P3.17. Similarly, cells H5 and H6 give the averages of cells B5 to F5 and B6 to F6 respectively.

Percentage

The spreadsheet program allows you to make up your own formula. For example, to find the **percentage** of females in 2003–04:

- Click in Cell B7
- Type =B5*100/B6, then press Enter.

The outcome, 47, is the percentage of females to the nearest whole number. Note that the formula =B5*100/B6 is short for (B5 × 100) ÷ B6, i.e. B5 as a percentage of B6. On a computer * and / are the symbols for multiplication and division.

Check the percentage formulae in the other boxes.

Notes:

1. The spreadsheet program that produced Fig. P3.13 to Fig. P3.17 was adjusted to give data to the nearest whole number.
2. When writing a formula, always begin with an = sign.
3. The above is a simplified account of some basic operations with a spreadsheet. There are many other operations, clever shortcuts and other ways of working with data on a spreadsheet. Practise on your computer and don't be too proud to ask for tips from other users.

Exercise P3g

1 *Puzzle: magic square.* Work on a 3 cell × 3 cell grid on a spreadsheet.
Enter the digits 1 to 9, one to each cell, so that the three numbers in every row, every column *and* in the two main diagonals all have the same total.

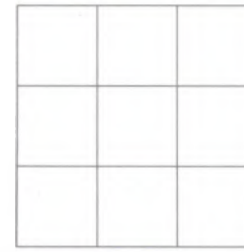


Figure P3.18

2 Table P3.4 gives similar data to Table P3.3 for District YYY.

	2003–04	2004–05	2005–06	2006–07	2007–08
Males	2504	2701	2855	3019	3186
Females	2399	2650	2777	2982	3177

Table P3.4 Tertiary students by gender in District YYY from 2003-04 to 2007-08

a Make a copy of the file you worked on for District XXX.
Rename the copy District YYY.

b Replace the written and numerical data in cells A1 to F5 with the data in Table P3.4. (Do not touch the other cells.)

c Notice what happens automatically to the data in the **Totals**, **Average** and **% Female** rows and columns.

d What does this tell you about the power of the spreadsheet program?

3 Table P3.5 shows the attendances (days absent) of 10 students during a school term. The table also shows the end of term test results of the same students.

Student	A	B	C	D	E	F	G	H	I	J
Days absent	0	4	0	0	2	15	8	0	10	5
Test result	71	66	80	56	61	43	50	74	44	60

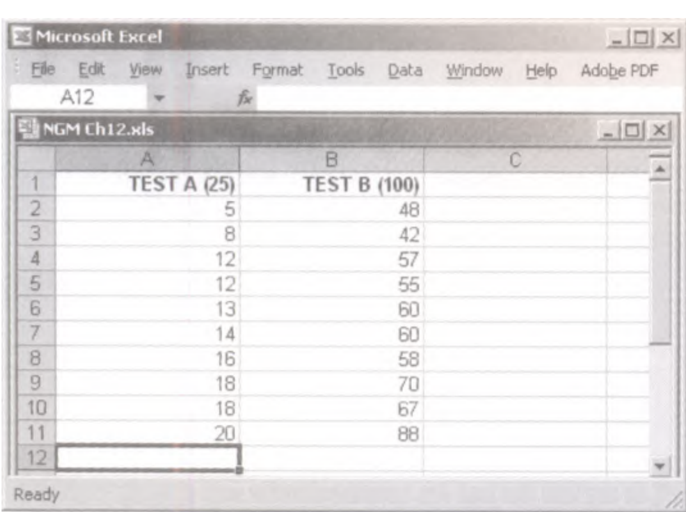
Table P3.5 Student absences and test results

a Enter the data on a spreadsheet.

b Use the program to find the average score for the test.

c Produce a scattergram of the data. Try to decide whether there appears to be a relationship between days absent and test results.

4 Fig. P3.19 shows the results of two tests: Test A out of 25 and Test B out of 100, as presented on a spreadsheet.



	A	B	C
1	TEST A (25)	TEST B (100)	
2	5	48	
3	8	42	
4	12	57	
5	12	55	
6	13	60	
7	14	60	
8	16	58	
9	18	70	
10	18	67	
11	20	88	
12			

Figure P3.19

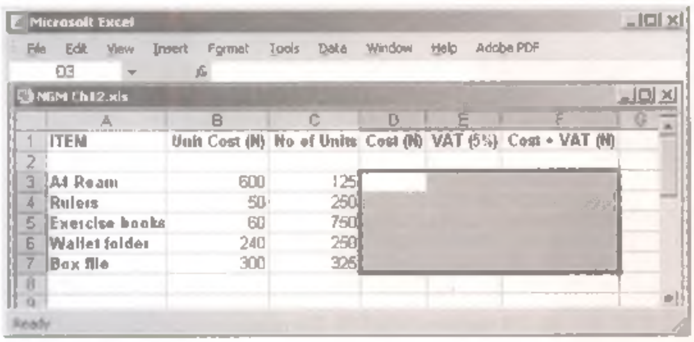
Test A was on the geography of East Africa and was given *before* any teaching had taken place. The teacher then taught about the geography of East Africa and afterwards gave the class Test B. The data show corresponding marks of ten of the students: i.e. the student in Row 4 got 12/25 in Test A and 57/100 in Test B.

a In Column C, scale up the scores in Test A to marks out of 100.
E.g. try =A3*4 for Cell C3 and so on for the rest of the column.

b Produce a scattergram for the data.

c Use the scattergram to decide whether the teacher had been effective.

5 A school uses its office computer to keep its accounts. Fig. P3.20 shows part of the stationery account.



	A	B	C	D	E	F
1	ITEM	Unit Cost (N)	No of Units	Cost (N)	VAT (5%)	Cost + VAT (N)
2						
3	A4 Ream	600	125			
4	Rulers	50	250			
5	Exercise books	60	750			
6	Wallet folder	240	250			
7	Box file	300	325			

Figure P3.20

a Open a spreadsheet and enter the data as in Fig. P3.20.

b In Row 3, the formulae for the ***Cost***, ***VAT*** and ***Cost*** + ***VAT*** columns are, respectively: =B3*C3 (entered into D3), =D3*0.05 (entered into E3) and =D3+E3 (entered into F3). Apply these formulae to all the rows.

c Use the program to complete the shaded cells, showing the total costs of the stationery.

d How would you change the VAT formula if the Government increases VAT to 12%?

6 *Challenge.* In this challenge you can use a spreadsheet to record your work.

a

1			
		4	
	2		
			3

Figure P3.21

The challenge is to complete Fig. P3.21 so that the cells in every row, every column *and* every 2×2 box each contain the digits 1 to 4 once and once only. The digits can be in any order. [This puzzle can be done by logic. As in question 1, you can try this with pencil and paper.]

b Make up a similar puzzle of your own. (Make up the answer first!)

Chapter 1

Coordinate geometry (1)

The straight line

1.1 Coordinates

The first thing that a student of advanced mathematics will discover is that number, and the methods of algebra, are used in geometry to a much greater extent than before, and with great clarity and economy. To do this we must have a way of describing exactly and briefly the position of a point in a plane (i.e. a flat surface).

We may think of a pirate, who buried some treasure on a large flat island, and was able to locate it years later. Starting at the most westerly point, going 400 paces due East, and then from there 100 paces due North gave the exact spot at which to dig.

This is roughly the method we use to fix the position of a point on a plane. Two straight lines cutting at right angles fix our directions, and we start our measurement from their point of intersection O (**Fig. 1.1**).

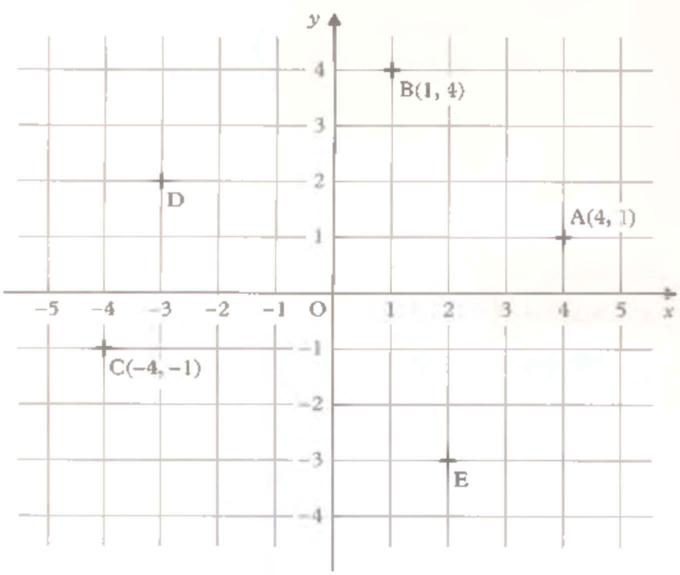


Figure 1.1

The point O is called the **origin**. The ***x*-axis** is drawn across the page, and the ***y*-axis** is drawn up the page; units of distance are marked off on them, positive in one direction, negative in the other. The plane containing these axes is called the **cartesian** plane, after René Descartes (1588–1648) who did much to lay the foundations of the subject we now call Coordinate Geometry. When the axes are drawn in a vertical plane (for instance, when a teacher draws them on a board), the x-axis is always drawn as a horizontal line and the y-axis as a vertical line. For this reason, they are often called the **horizontal axis** and the **vertical axis**, respectively.

Consider the point A in **Fig. 1.1**. To reach A from O we travel 4 units in the direction of Ox, and then 1 unit in the direction of Oy.

The **x-coordinate** (or *abscissa*) of A is +4.
The **y-coordinate** (or *ordinate*) of A is +1.

We say that the **coordinates** of A are (4, 1), or that A is the point (4, 1). The x-coordinate always comes first, thus we can distinguish between the points A(4, 1) and B(1, 4). By use of the sign of the coordinates we distinguish between the points A(4, 1) and C(−4, −1).

Questions

Q1 Write down the coordinates of the points D, E, O in **Fig. 1.1**.

Q2 Sketch your own axes and plot the points P(2, 4), Q(−5, 7), R(4, −2), S(0, 3), T(2, 0).

1.2 The length of a straight line

Example 1 *Find the length of the straight line joining* A(2, 1) *and* B(5, 5).

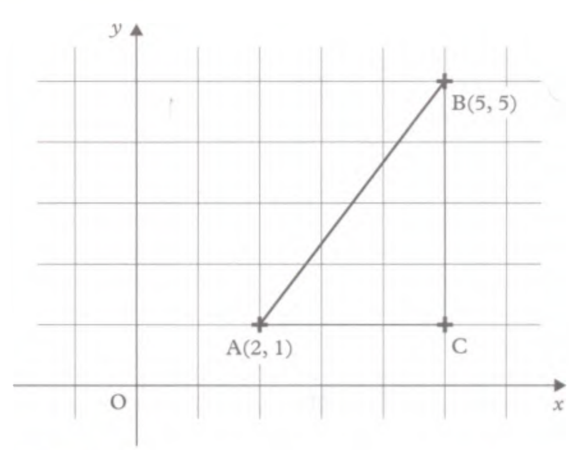


Figure 1.2

AC and CB are drawn parallel to the x-axis and y-axis respectively (**Fig. 1.2**). Applying Pythagoras' theorem to the right-angled triangle ABC,

$$
\begin{aligned}
AB^2 &= AC^2 + CB^2 \\
&= (5-2)^2 + (5-1)^2 \\
&= 9 + 16
\end{aligned}
$$

$\therefore AB = \sqrt{25} = 5$

Notice that, if A had been the point (−2, 1) in the above example, the length of AC would still be the *difference* between the x-coordinates of A and B, since it would be $5 - (-2) = 5 + 2 = 7$.

Question

Q3 Find the lengths of the straight lines joining the following pairs of points:

- **a** A(3, 2) and B(8, 14)
- **b** C(−1, 3) and D(4, 7)
- **c** E(p, q) and F(r, s)

1.3 The mid-point of a straight line

Example 2 *Find the mid-point of the straight line joining* A(2, 1) *and* D(6, 5).

Let M, the mid-point of AD, have coordinates (p, q). FM and ED are drawn parallel to Oy.

AFE is drawn parallel to Ox (**Fig. 1.3**).

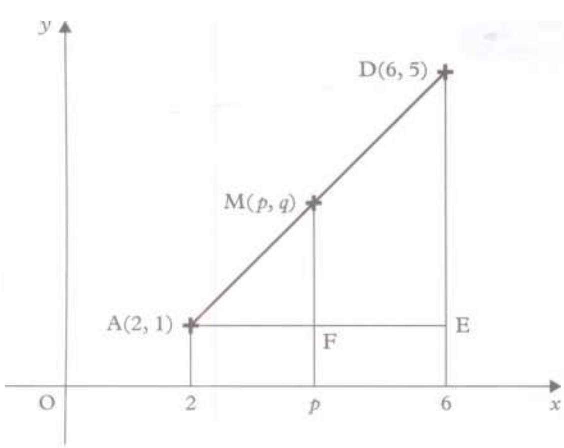


Figure 1.3

In the triangle ADE, applying the mid-point theorem, since M is the mid-point of AD, and MF is parallel to DE, F is the mid-point of AE. Thus

AF = FE

$\therefore p - 2 = 6 - p$

$\therefore p = \dfrac{6+2}{2}$

$\therefore p = 4$

The x-coordinate of M is the average of those of A and D. The y-coordinate of M is found similarly.

$q = \dfrac{5+1}{2}$

$\therefore q = 3$

$\therefore$ the mid-point of AD is (4, 3).

In practice the working is presented in shortened form thus:

the mid-point of AD is $\left(\dfrac{6+2}{2}, \dfrac{5+1}{2}\right)$, i.e. (4, 3)

Question

Q4 Find the coordinates of the mid-points of the straight lines joining the following pairs of points:

- **a** A(4, 2) and B(6, 10)
- **b** C(−5, 6) and D(3, 2)
- **c** E(−6, −1) and F(3, −4)
- **d** G(p, q) and H(r, s)

Exercise 1a

1 Find the lengths of the straight lines joining the following pairs of points:
- **a** A(1, 2) and B(5, 2)
- **b** C(3, 4) and D(7, 1)
- **c** E(−2, 3) and F(4, 3)
- **d** G(−6, 1) and H(6, 6)
- **e** J(−4, −2) and K(3, −7)
- **f** L(−2, −4) and M(−10, −10)

2 Find the coordinates of the mid-points of the lines AB, CD, etc., in question 1.

3 Find the distance of the point (−15, 8) from the origin.

4 P, Q, R are the points (5, −3), (−6, 1), (1, 8) respectively. Show that triangle PQR is isosceles, and find the coordinates of the mid-point of the base.

5 Repeat question 4 for the points L(4, 4), M(−4, 1), N(1, −4).

6 A and B are the points (−1, −6) and (5, −8) respectively. Which of the following points lie on the perpendicular bisector of AB?
   a P(3, −4)
   b Q(4, 0)
   c R(5, 2)
   d S(6, 5)

7 Three of the following four points lie on a circle whose centre is at the origin. Which are they, and what is the radius of the circle?
   A(−1, 7), B(5, −5), C(−7, 5), D(7, −1).

8 A and B are the points (12, 0) and (0, −5) respectively. Find the length of AB, and the length of the median, through the origin O, of the triangle OAB.

1.4 The gradient of a straight line

Consider the straight line passing through A(1, 1) and B(7, 2) (**Fig. 1.4**). If we think of the x-axis as horizontal, and the line through A and B as a road, then someone walking from A to B would rise a vertical distance CB while moving a horizontal distance AC.

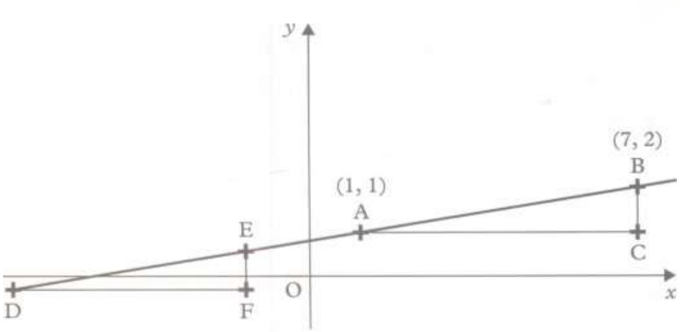


Figure 1.4

The gradient of the road is CB/AC = (2 − 1)/(7 − 1) = 1/6. Instead of the two points A and B we could have taken any other two points on the line, D and E; the gradient would then be FE/DF, which is the same as CB/AC, since the triangles ABC and DEF are similar.

Definition

The gradient of a straight line is $\dfrac{\textit{the increase in } y}{\textit{the increase in } x}$ *in moving from one point on the line to another.*

In moving from A to B, since both x and y increase by positive amounts, the gradient is positive.

But now consider the gradient of PQ (**Fig. 1.5**). In moving from P to Q, the increase in x is +2, but since y decreases, we may say the increase in y is −3. Thus the gradient of PQ is $-\frac{3}{2}$.

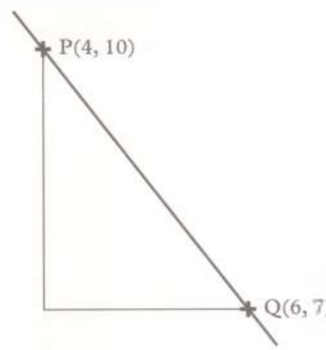


Figure 1.5

It may be helpful to think of the idea of positive and negative gradient as follows. In travelling along a line with x increasing (i.e. moving from left to right across the page) if going *up*hill the gradient is *positive*: whereas if going *down*hill the gradient is *negative*. When calculating gradients a figure is not always necessary, but one similar to **Fig. 1.5** will help in the first few examples.

Example 3 *Find the gradient of the line joining* R(4, 8) *and* S(5, −2).

$$\text{The gradient of RS} = \frac{y\text{-coord. of R} - y\text{-coord. of S}}{x\text{-coord. of R} - x\text{-coord. of S}}$$
$$= \frac{8 - (-2)}{4 - 5}$$
$$= \frac{10}{-1} = -10$$

[Remember that the coordinates of R must appear first in the denominator *and* numerator (or second in both). In this case {8 − (−2)}/(4 − 5) and (−2 − 8)/(5 − 4) both give the correct gradient.]

The gradient of the line joining A(2, 1) and B(2, 9) presents us with a problem. Proceeding as in Example 3, above, we might say that

$$\text{the gradient of AB} = \frac{y\text{-coord. of A} - y\text{-coord. of B}}{x\text{-coord. of A} - x\text{-coord. of B}} = \frac{1-9}{0} = \frac{-8}{0}$$

On the other hand we might also say that

$$\text{the gradient of AB} = \frac{y\text{-coord. of B} - y\text{-coord. of A}}{x\text{-coord. of B} - x\text{-coord. of A}} = \frac{9-1}{0} = \frac{+8}{0}$$

What meaning should we attach to expressions like $-8/0$ and $+8/0$ and how can the line AB have two apparently different gradients? This is just one of the difficulties which can arise when trying to divide by zero. Division by zero is never allowed in mathematics. We say that an expression like $8/0$ 'is undefined'. So what are we to do about the gradient of the line AB? For a 'vertical line' such as AB, no numerical value can be given to its gradient. However, we simply say that 'AB is parallel to the y-axis'.

Questions

Q5 Find the gradients of the lines joining the following pairs of points:

- **a** (4, 3) and (8, 12)
- **b** (−2, −3) and (4, 6)
- **c** (5, 6) and (10, 2)
- **d** (−3, 4) and (8, −6)
- **e** (−5, 3) and (2, 3)
- **f** (p, q) and (r, s)
- **g** $(0, a)$ and $(a, 0)$
- **h** $(0, 0)$ and (a, b)

Q6 A and B are the points (3, 4) and (7, 1) respectively. Use Pythagoras' theorem to prove that OA is perpendicular to AB. Calculate the gradients of OA and AB, and find their product.

Q7 Repeat Q6 for the points A(5, 12) and B(17, 7).

1.5 Parallel and perpendicular lines

The gradient of a straight line was defined in §1.4 on page 35. It can also be defined as the tangent of the angle between the line and the positive direction of the x-axis.

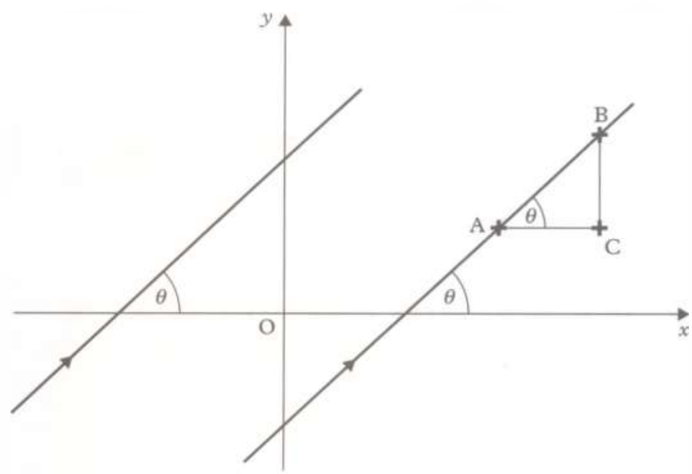


Figure 1.6

In **Fig. 1.6** the gradient of AB is CB/AC, which is tan θ. If you are familiar with the tangent of an obtuse angle you will appreciate that this covers negative gradient as well.

Since parallel lines make equal corresponding angles with the x-axis, *parallel lines have equal gradients.*

Q6 and 7 of §1.4 will have led you to discover a useful property of the gradients of perpendicular lines. We will now prove this.

Consider the two straight lines AB and CD which cut at right angles at E. EF is drawn perpendicular to the x-axis (**Fig. 1.7**).

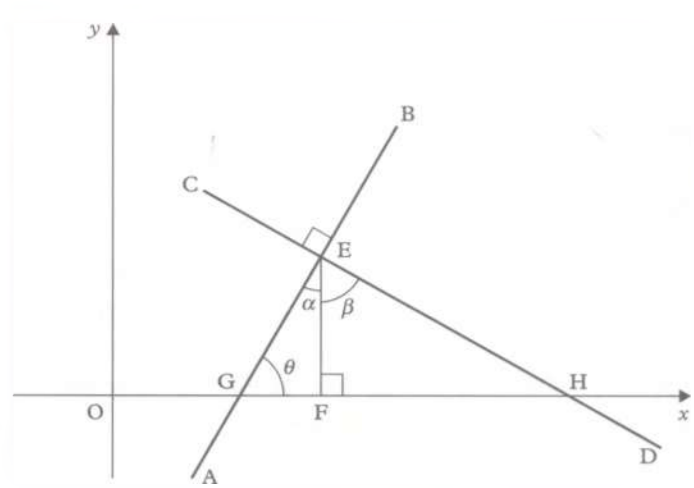


Figure 1.7

$\alpha + \theta = 90°$
$\alpha + \beta = 90°$
$\therefore \theta = \beta$

Let the gradient of AB be m, then

$$m = \frac{FE}{GF} = \tan\theta$$

$$\text{The gradient of CD} = -\frac{FE}{FH} = -\frac{1}{\tan\beta} = -\frac{1}{\tan\theta} = -\frac{1}{m}$$

$\therefore$ the gradient of AB $\times$ the gradient of CD $= m \times \left(-\frac{1}{m}\right) = -1$

In general, *if two lines are perpendicular, the product of their gradients is* -1. Or in other words, if the gradient of one is m, the gradient of the other is $-1/m$.

Questions

Q8 Write down the gradients of lines perpendicular to lines of gradient

a 3 **b** $\frac{1}{4}$ **c** -6 **d** $-\frac{2}{3}$
e $2m$ **f** $-b/a$ **g** $-m/2$

Q9 Find if AB is parallel or perpendicular to PQ in the following cases:

a A(1, 4), B(6, 6), P(2, −1), Q(12, 3)
b A(−1, −1), B(0, 4), P(−4, 3), Q(6, 1)
c A(0, 3), B(7, 2), P(6, −1), Q(−1, −2)

1.6 The meaning of equations

The statement 'P is the point (x, y)' means that P can be anywhere in the plane. Previously, if we have been asked to find P, we have been given some data which enabled us to find one pair of numerical values for x and y, and so to fix the position of P.

Suppose however that the data is in the form of the equation $y = x^2 - 2x$. This does not give one pair of values for x and y, it gives as many as we like to find. But P is not now free to be anywhere in the plane, since for any chosen value of x there is only one corresponding value of y. P is now restricted to positions whose coordinates (x, y) satisfy the relationship $y = x^2 - 2x$.

You will be familiar with tables of values as shown below, in which certain suitable values of x are chosen, and the corresponding values of y calculated.

Table of values for $y = x^2 - 2x$

x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
x^2	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	$2\frac{1}{4}$	4	$6\frac{1}{4}$	9
$-2x$	2	1	0	-1	-2	-3	-4	-5	-6
y	3	$1\frac{1}{4}$	0	$-\frac{3}{4}$	-1	$-\frac{3}{4}$	0	$1\frac{1}{4}$	3

From this we find that the points $(-1, 3)$, $(-\frac{1}{2}, 1\frac{1}{4})$, $(0, 0)$, etc., have coordinates which satisfy the relationship $y = x^2 - 2x$. By plotting these points and drawing a smooth curve through them (**Fig. 1.8**), we obtain all the possible positions of P corresponding to the values of x from $x = -1$ to $x = 3$.

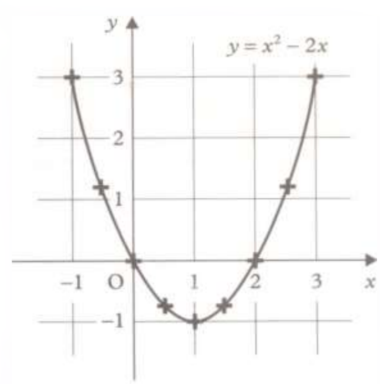


Figure 1.8

Just as we use coordinates to name a point, so we use an equation as a name of a curve. I.e. we often refer to 'the curve $y = x^2 - 2x$'.

It must be stressed that the equation is the condition that the point (x, y) should lie on the curve. Thus, only if $b = a^2 - 2a$ does the point (a, b) lie on the curve $y = x^2 - 2x$, and in that case we say that the coordinates of the point *satisfy* the equation.

If $q \neq p^2 - 2p$, the point (p, q) does *not* lie on the curve $y = x^2 - 2x$.

Example 4 *Do the points* $(-3, 9)$ *and* $(14, 186)$ *lie on the curve* $y = x^2$?

a The point $(-3, 9)$:
When $x = -3$, $y = x^2 = (-3)^2 = +9$,

$\therefore$ $(-3, 9)$ does lie on the curve $y = x^2$.

b The point $(14, 186)$:
When $x = 14$, $y = x^2 = 14^2 = 196$,

$\therefore$ $(14, 186)$ does *not* lie on the curve $y = x^2$.

The next example shows another way of presenting this idea.

Example 5 *Does the point* $(-7, 6)$ *lie on the curve* $x^2 - y^2 = 14$?

[We use L.H.S. as an abbreviation for 'the left-hand side' of the equation and R.H.S. for 'the right-hand side'.]

$$x^2 - y^2 = 14$$

When $x = -7$ and $y = +6$,

$$\text{L.H.S.} = (-7)^2 - 6^2 = 49 - 36 = 13$$

$$\text{R.H.S.} = 14$$

The coordinates of the point do not satisfy the equation. Therefore $(-7, 6)$ does *not* lie on the curve $x^2 - y^2 = 14$.

Questions

Q10 Find the y-coordinates of the points on the curve $y = 2x^2 - x - 1$ for which $x = 2, -3, 0$.

Q11 Find the x-coordinates of the points on the curve $y = 2x + 3$ for which the y-coordinates are $7, 3, -2$.

Q12 Find the points at which the curve in Q10 cuts **a** the x-axis, and **b** the y-axis.

Q13 Determine whether the following points lie on the given curves:

a $y = 6x + 7$, $(1, 13)$
b $y = 2x + 2$, $(13, 30)$
c $3x + 4y = 1$, $(-1, \frac{1}{2})$
d $y = x^3 - 6$, $(2, -2)$
e $xy = 36$, $(-9, -4)$
f $x^2 + y^2 = 25$, $(3, -4)$

The relationship between a curve and its equation gives rise to two main groups of problems.

Firstly there are those problems in which we are given the equation, and from it we are required to find the curve. You will already be familiar with this type when finding the graphical solution of quadratic and other equations.

Secondly there are those problems in which we are given some purely geometrical facts about the curve, and from these we are required to discover the equation. It is this second type of problem with which we are now mainly concerned, but first we shall discuss a few more simple equations, to see what they represent.

$y = x$.
This equation is satisfied by the coordinates of the points $(0, 0)$, $(1, 1)$, $(2, 2)$, $(3, 3)$, etc. It represents a straight line through the origin. Its gradient is 1.

$x = 2$.
Whatever the value of its y-coordinate, a point will lie on this curve, provided that its x-coordinate is 2. The points $(2, 0)$, $(2, 1)$, $(2, 2)$, $(2, 3)$, etc., lie on the curve, which is a straight line parallel to the y-axis, 2 units from it, on the side on which x is positive.

Question

Q14 Make a rough sketch of the lines represented by the following equations. Write down the gradient of each:

a $y = 3$ **b** $y = 2x$ **c** $y = 3x$
d $y = \frac{1}{2}x$ **e** $y = -x$

1.7 The equation $y = mx + c$

We come now to the second type of problem mentioned above, in which we discover the equation of a straight line from some of its geometrical properties. The examples we do will, in turn, help us to interpret straight line equations more skilfully.

Example 6 *Find the equation of the straight line of gradient 4 which passes through the origin.*

If P(x, y) is any point on the line, other than O, the gradient of the line may be written y/x (**Fig. 1.9**).

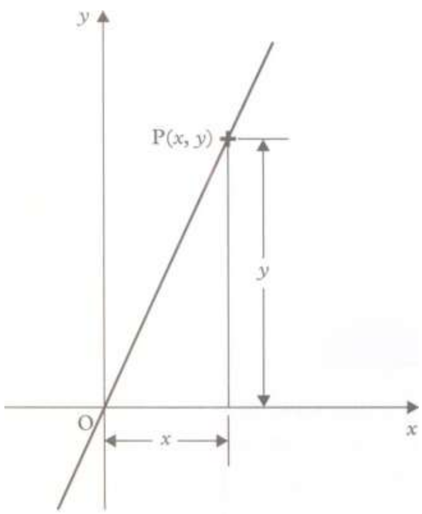


Figure 1.9

$$\therefore \frac{y}{x} = 4$$

Hence $y = 4x$ is the required equation.

Example 7 *Find the equation of the straight line of gradient 3 which cuts the y-axis at* (0, 1).

Let P(x, y) be any point on the line other than (0, 1). The gradient of the line may be written $(y-1)/x$ (**Fig. 1.10**).

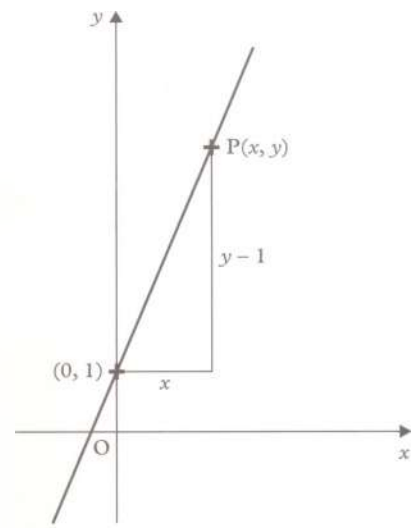


Figure 1.10

$$\therefore \frac{y-1}{x} = 3$$

Hence $y = 3x + 1$ is the required equation.

Questions

Q15 Write down the equations of the straight lines through the origin having gradients
a $\frac{1}{3}$ **b** -2 **c** m

Q16 Rearrange the following equations in the form $y = mx$, and hence write down the gradients of the lines they represent:
a $4y = x$ **b** $5x + 4y = 0$
c $3x = 2y$ **d** $\frac{x}{4} = \frac{y}{7}$
e $\frac{x}{p} - \frac{y}{q} = 0$

Question

Q17 By the method of Example 7, find the equations of the straight lines of given gradients cutting the y-axis at the named points:
a gradient 3, (0, 2)
b gradient 3, (0, 4)
c gradient 3, (0, −1)
d gradient $\frac{1}{3}$, (0, 2)
e gradient $\frac{1}{3}$, (0, 4)

If a straight line cuts the y-axis at the point (0, c), the distance of this point from the origin is called the **intercept** on the y-axis.

Then the general equation of a straight line of gradient m, making an intercept c on the y-axis (**Fig. 1.11**) is

$$\frac{y-c}{x} = m$$

i.e. $\quad \boldsymbol{y = mx + c}$

This line is parallel to $y = mx$, which passes through the origin, and it is m, the coefficient of x, which in each case determines the gradient. The effect of altering the value of the number c (c being the intercept on the y-axis) is to raise or lower the line, without altering its gradient; the sign of c determines whether the line cuts the y-axis above or below the origin.

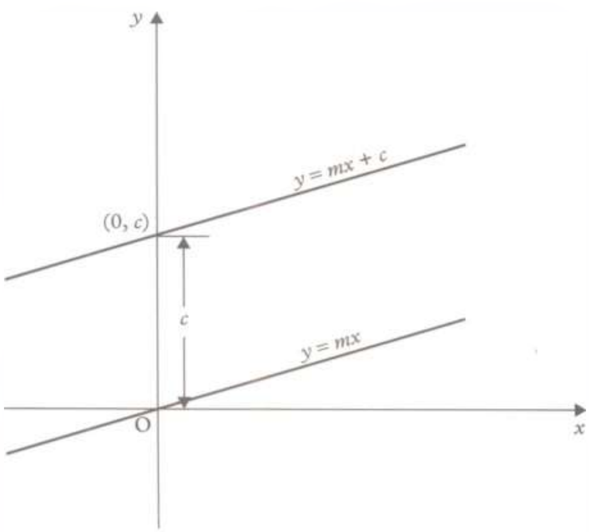


Figure 1.11

It is tempting to think that $y = mx + c$ is the form in which all straight line equations are written. But remember that on page 36 we ran into trouble trying to find the gradient of a line parallel to the y-axis. For such a line it is impossible to find a numerical value for m, and the equation is $x = k$, where k is a constant.

The various straight line equations we have met are summarised below. Notice that only terms of the first degree in x and y and a constant term occur. This, in fact, is how we recognise a straight line, or *linear*, equation.

$\boldsymbol{y = mx + c}$	is a line of gradient m, passing through $(0, c)$.
$\boldsymbol{y = mx}$	is a line of gradient m, passing through the origin.
$\boldsymbol{y = c}$	is a line of zero gradient (i.e. parallel to the x-axis).
$\boldsymbol{x = k}$	is a line parallel to the y-axis.

Example 8 *Find the gradient of the straight line* $7x + 4y + 2 = 0$, *and its intercepts on the axes.*

The equation may be written

$$4y = -7x - 2$$

or $\quad y = -\frac{7}{4}x - \frac{1}{2}$

This is now in the form $y = mx + c$, where $m = -\frac{7}{4}$, and $c = -\frac{1}{2}$, and we see that the gradient is $-\frac{7}{4}$, and that the intercept on the y-axis is $-\frac{1}{2}$. In fact, to find the intercepts on each axis it is better to go back to the original equation

$$7x + 4y + 2 = 0$$

To find the intercept on the y-axis:
putting $x = 0$, $\quad 4y + 2 = 0$, $\quad \therefore y = -\frac{1}{2}$.

To find the intercept on the x-axis:
putting $y = 0$, $\quad 7x + 2 = 0$, $\quad \therefore x = -\frac{2}{7}$.

The intercepts on the x-axis and y-axis are $-\frac{2}{7}$ and $-\frac{1}{2}$ respectively.

Questions

Q18 Arrange the following equations in the form $y = mx + c$, hence write down the gradient of each line; also find the intercepts on the y-axis:

a $3y = 2x + 6$ **b** $x - 4y + 2 = 0$
c $3x + y + 6 = 0$ **d** $7x = 3y + 5$
e $y + 4 = 0$ **f** $lx + my + n = 0$

Q19 Write down the equations of **a** the x-axis, **b** the y-axis, **c** a straight line parallel to the y-axis through $(4, 0)$, **d** a straight line parallel to the x-axis making an intercept of -7 on the y-axis.

Exercise 1b

1 Find the gradients of the straight lines joining the following pairs of points:
a $(4, 6)$ and $(9, 15)$
b $(5, -11)$ and $(-1, 3)$
c $(-2\frac{1}{2}, -\frac{1}{2})$ and $(4\frac{1}{2}, -1)$
d $(7, 0)$ and $(-3, -2)$

2 Show that the three given points are in each case collinear, i.e. they lie on the same straight line:
a $(0, 0), (3, 5), (21, 35)$
b $(-3, 1), (1, 2), (9, 4)$
c $(-3, 4), (1, 2), (7, -1)$
d $(1, 2), (0, -1), (-2, -7)$

3 Find the gradients of the straight lines which make the following angles with the x-axis, the angle in each case being measured anti-clockwise from the positive direction of the x-axis:
a $45°$ **b** $135°$ **c** $60°$ **d** $150°$

4 Find if AB is parallel or perpendicular to PQ in the following cases:

a	A(4, 3),	B(8, 4),	P(7, 1),	Q(6, 5)
b	A(−2, 0),	B(1, 9),	P(2, 5),	Q(6, 17)
c	A(8, −5),	B(11, −3),	P(1, 1),	Q(−3, 7)
d	A(−6, −1),	B(−6, 3),	P(2, 0),	Q(2, −5)
e	A(4, 3),	B(−7, 3),	P(5, 2),	Q(5, −1)
f	A(3, 1),	B(7, 3),	P(−3, 2),	Q(1, 0)

5 Show that A(−3, 1), B(1, 2), C(0, −1), D(−4, −2) are the vertices of a parallelogram.

6 Show that P(1, 7), Q(7, 5), R(6, 2), S(0, 4) are the vertices of a rectangle. Calculate the lengths of the diagonals, and find their point of intersection.

7 Show that D(−2, 0), E($\frac{1}{2}$, $1\frac{1}{2}$), F($3\frac{1}{2}$, $-3\frac{1}{2}$) are the vertices of a right-angled triangle, and find the length of the shortest side, and the mid-point of the hypotenuse.

8 Find the y-coordinates of the points on the curve $y = x^2 + 1$ for which the x-coordinates are −3, 0, 1, 5. Find the coordinates of points on the curve whose y-coordinates are 5, and 17. Sketch the curve.

9 Find the coordinates of the points on the curve $y = x^3$ for which $x = -3, -1, 1, 3$; and also of the points for which $y = -8, 0, +8$. Sketch the curve.

10 Determine whether the following points lie on the given curve:

a $y = 3x - 5$, $(-1, -8)$

b $5x - 2y + 7 = 0$, $(1, -1)$

c $y = x^3$, $(-4, 64)$

d $x^2y = 1$, $(-2, \frac{1}{4})$

11 Find the intercepts on the axes made by the straight line $3x - 2y + 10 = 0$. Hence find the area of the triangle enclosed by the axes and this line.

12 Find the coordinates of the points at which the following curves cut the axes:

a $y = x^2 - x - 12$ b $y = 6x^2 - 7x + 2$

c $y = x^2 - 6x + 9$ d $y = x^3 - 9x^2$

e $y = (x + 1)(x - 5)^2$ f $y = (x^2 - 1)(x^2 - 9)$

13 Plot the following points on squared paper, and write down the equations of the straight lines passing through them, in the form $y = mx + c$:

a (−1, −1), (0, 0), (4, 4)

b (−1, 1), (0, 0), (1, −1)

c (−4, −2), (0, 0), (8, 4)

d (0, −4), (4, −2), (6, −1)

e (−5, 2), (−5, 0), (−5, −2)

f (−3, 7), (3, 3), (6, 1)

14 Write down the equation of the straight line

a through (5, 11) parallel to the x-axis

b which is the perpendicular bisector of the line joining (2, 0) and (6, 0)

c through (0, −10) parallel to $y = 6x + 3$

d through (0, 2) parallel to $y + 8x = 0$

e through (0, −1) perpendicular to $3x - 2y + 5 = 0$

15 Find the equation of the straight line joining the origin to the mid-point of the line joining A(3, 2) and B(5, −1).

16 P(−2, −4), Q(−5, −2), R(2, 1), S are the vertices of a parallelogram. Find the coordinates of M, the point of intersection of the diagonals, and of S.

17 a Write down the gradient of the straight line joining (a, b) and (p, q). Write down the two conditions that these points should lie on the line $y = 7x - 3$. From these deduce the gradient of the line.

b Repeat for the line $3x + 2y - 1 = 0$, and check your result by writing the equation in the form $y = mx + c$.

1.8 The use of suffixes

When we wish to refer to points whose coordinates are not given, it is convenient to write them as

(x_1, y_1) read as 'x one, y one'

(x_2, y_2) read as 'x two, y two', etc.

It is important to write the number (the suffix) at the bottom of the letter, so as to avoid confusion between x_2 and x^2, x_3 and x^3, and so on.

This is a suitable point at which to summarise some of the early results of this chapter, using this notation.

If A and B are the points (x_1, y_1) and (x_2, y_2) respectively,

the length of AB is $\sqrt{\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}}$

the mid-point of AB is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

the gradient of AB is $\dfrac{y_1 - y_2}{x_1 - x_2}$ or $\dfrac{y_2 - y_1}{x_2 - x_1}$

the condition for A to lie on $ax + by + c = 0$ is
$$ax_1 + by_1 + c = 0$$

1.9 Finding the equation of a straight line

The method of Example 7 in §1.7 on page 38 can be used to find the equation of any straight line provided **a** that we know one point through which the line passes, and **b** that we know, or can calculate, the gradient. Examples 9 and 10 show this.

Example 9 *Find the equation of the straight line of gradient $-\frac{2}{3}$, which passes through $(-4, 1)$.*

Let $P(x, y)$ be any point on the line other than $(-4, 1)$ (**Fig. 1.12**).

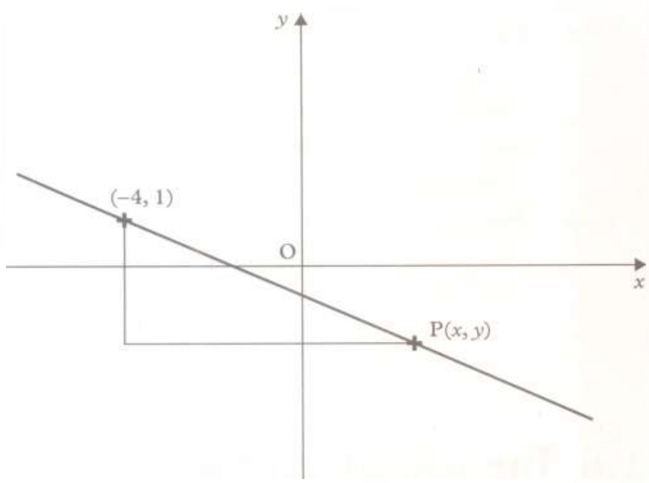


Figure 1.12

The gradient of the line may be written

$$\frac{y-1}{x-(-4)} = \frac{y-1}{x+4}$$

But the gradient is given as $-\frac{2}{3}$,

$$\therefore \frac{y-1}{x+4} = -\frac{2}{3}$$
$$\therefore 3(y-1) = -2(x+4)$$
$$\therefore 3y-3 = -2x-8$$

Hence the required equation is $2x + 3y + 5 = 0$.

Example 10 *Find the equation of the straight line joining the points $(-5, 2)$ and $(3, -4)$.*

The gradient of the line $= \dfrac{2-(-4)}{-5-3} = \dfrac{6}{-8} = -\dfrac{3}{4}$.

If $P(x, y)$ is any point on the line other than $(3, -4)$, the gradient may be written

$$\frac{y-(-4)}{x-3} = \frac{y+4}{x-3}$$
$$\therefore \frac{y+4}{x-3} = -\frac{3}{4}$$
$$\therefore 4(y+4) = -3(x-3)$$
$$\therefore 4y+16 = -3x+9$$

Hence the required equation is $3x + 4y + 7 = 0$.

Examples 9 and 10 illustrate the most direct approach. The equation as first written is the direct statement of the condition that the point (x, y) should lie on the given line.

Another method is given below as an alternative solution to Example 9. We know that the equation $y = mx + c$ represents a straight line of gradient m. So the equation $y = -\frac{2}{3}x + c$ represents any line of gradient $-\frac{2}{3}$, according to the value of the constant c. Our problem is to find the appropriate value of c for the given line. To do this we use the fact that if the point (x_1, y_1) lies on the straight line $y = mx + c$, its coordinates satisfy the equation of the line, i.e. $y_1 = mx_1 + c$.

Example 9 *(Alternative solution)*

The equation is of the form $y = -\frac{2}{3}x + c$.
Since $(-4, 1)$ lies on this line,

$$1 = -\tfrac{2}{3}(-4) + c$$
$$\therefore c = 1 - \tfrac{8}{3} = -\tfrac{5}{3}$$

Hence the required equation is $y = -\frac{2}{3}x - \frac{5}{3}$, or $2x + 3y + 5 = 0$.

Questions

Q20 Use the methods of Examples 9 (first solution) and 10 to find the equations of the straight lines

- **a** through $(4, -3)$, of gradient $\frac{5}{2}$
- **b** joining $(-3, 8)$ and $(1, -2)$

Q21 Use the method of Example 9 (alternative solution) to find the equations of the straight lines

- **a** through $(5, -2)$, of gradient $\frac{3}{2}$
- **b** joining $(-2, 5)$ and $(3, -7)$

Q22 Write down the equation of the straight line through (x_1, y_1) of gradient m.

1.10 Points of intersection

If the two straight lines $x + y - 1 = 0$ and $2x - y - 8 = 0$ cut at the point P(a, b) then the coordinates of P satisfy the equation of each line. Therefore we may write

$$a + b - 1 = 0$$
$$2a - b - 8 = 0$$

The solution of these equations is $a = 3$, $b = -2$, which tells us that the given lines cut at $(3, -2)$. In practice we obtain the result by solving the equations simultaneously for x and y.

Questions

Q23 Find the points of intersection of the following pairs of straight lines:

- **a** $2x - 3y = 6$ and $4x + y = 19$
- **b** $y = 3x + 2$ and $2x + 3y = 17$
- **c** $y = c$ and $y = mx + c$
- **d** $x = -a$ and $y = mx + c$

Q24 Can you find the point of intersection of $3x - 2y - 10 = 0$ and $4y = 6x - 7$?

Q25 Find the points of intersection of the curve $y = 12x^2 + x - 6$ and the x-axis.

Exercise 1c

1 Find the equations of the straight lines of given gradients, passing through the points named:

- **a** $4, (1, 3)$ **b** $3, (-2, 5)$
- **c** $\frac{1}{3}, (2, -5)$ **d** $-\frac{3}{4}, (7, 5)$
- **e** $\frac{1}{2}, (\frac{1}{3}, -\frac{1}{2})$ **f** $-\frac{5}{3}, (\frac{1}{4}, -3)$

2 Find the equations of the straight lines joining the following pairs of points:

- **a** $(1, 6)$ and $(5, 9)$
- **b** $(3, 2)$ and $(7, -3)$
- **c** $(-3, 4)$ and $(8, 1)$
- **d** $(-1, -4)$ and $(4, -3)$
- **e** $(\frac{1}{2}, 2)$ and $(3, \frac{1}{3})$
- **f** $(-\frac{1}{2}, 0)$ and $(5, 11)$

3 Find the points of intersection of the following pairs of straight lines:

- **a** $x + y = 0, y = -7$
- **b** $y = 5x + 2, y = 3x - 1$
- **c** $3x + 2y - 1 = 0, 4x + 5y + 3 = 0$
- **d** $5x + 7y + 29 = 0, 11x - 3y - 65 = 0$

4 Find the equation of the straight line

- **a** through $(5, 4)$, parallel to $3x - 4y + 7 = 0$
- **b** through $(-2, 3)$, parallel to $5x - 2y - 1 = 0$
- **c** through $(4, 0)$, perpendicular to $x + 7y + 4 = 0$
- **d** through $(-2, -3)$, perpendicular to $4x + 3y - 5 = 0$

5 Find the equation of the perpendicular bisector of AB, where A and B are the points $(-4, 8)$ and $(0, -2)$ respectively.

6 Repeat question 5 for the points A$(7, 3)$ and B$(-6, 1)$.

7 Find the equation of the straight line joining A$(10, 0)$ and B$(0, -7)$. Also find the equation of the median through the origin, O, of the triangle OAB.

8 P, Q, R are the points $(3, 4)$, $(7, -2)$, $(-2, -1)$ respectively. Find the equation of the median through R of the triangle PQR.

9 Calculate the area of the triangle formed by the line $3x - 7y + 4 = 0$ and the axes.

10 Find the circumcentre of the triangle with vertices $(-3, 0)$, $(7, 0)$, $(9, -6)$. Show that the point $(1, 2)$ lies on the circumcircle.

11 Find the equation of the straight line through P$(7, 5)$ perpendicular to the straight line AB whose equation is $3x + 4y - 16 = 0$. Calculate the length of the perpendicular from P to AB.

12 ABCD is a rhombus. A is the point $(2, -1)$, and C is the point $(4, 7)$. Find the equation of the diagonal BD.

13 L$(-1, 0)$, M$(3, 7)$, N$(5, -2)$ are the mid-points of the sides BC, CA, AB respectively of the triangle ABC. Find the equation of AB.

14 Find the points of intersection of $x^2 = 4y$ and $y = 4x$.

15 The straight line $x - y - 6 = 0$ cuts the curve $y^2 = 8x$ at P and Q. Calculate the length of PQ.

Chapter 2

Functions

2.1 Real numbers

Any student of mathematics who has progressed this far will be familiar with the **real numbers**. All the weighing, measuring and calculating that are used in commerce and science require the use of the real numbers. To the mathematician, they are the numbers, both positive and negative, which can be represented by points on the 'real number line'. Fig. 2.1 shows some of them.

Figure 2.1

2.2 Integers

One of the most basic mathematical skills is that of counting: ... 'one, two, three, ...'. In mathematics these numbers are called the **counting** or **natural** numbers. However, in order to develop mathematical ideas beyond very elementary arithmetic, it is necessary to extend the concept of natural numbers in two important directions. One of these is the extension to negative, as well as positive, numbers. Mathematicians refer to the positive and negative whole numbers, together with zero, as **integers**. An integer is therefore any number of the form $\ldots, -4, -3, -2, -1, 0, +1, +2, +3, \ldots$.

2.3 Rational and non-rational numbers

The other important extension of the natural numbers is the idea of fractions, e.g. $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \ldots$. In mathematics we extend this idea to include fractions which are greater than one, e.g. 7/5, 22/7 (often called improper fractions), and they can be positive or negative. The collective name for all such numbers is **rational numbers** (rational is the adjective derived from the word 'ratio').

However that is not the complete story. The rational numbers do not 'fill' the number line. There are points on the number line, e.g. the one representing $\sqrt{2}$, which do not represent rational numbers. In other words, some real numbers are **non-rational** numbers. In the next section we shall prove that $\sqrt{2}$ is non-rational or **irrational**.

Before we can do this, we must state clearly and unambiguously what we mean by a rational number. *A rational number is a number of the form a/b, in which a and b are integers with no common factor.* (If there is a common factor, it should be divided out, e.g. 12/15 should be simplified to 4/5.) *The number b must not be zero.* Notice however that b can be 1; this enables us to regard any integer, including zero itself, as a rational number. An integer is simply a rational number whose denominator b is equal to 1. Notice that a can be larger than b; 5/3 is a perfectly acceptable rational number.

2.4 The irrationality of $\sqrt{2}$

The Greek mathematicians of the 4th century BC knew about Pythagoras' theorem so they knew that the hypotenuse of a right-angled triangle, whose other two sides have a length of 1 unit, would have a length of $\sqrt{2}$ units. They discovered the proof that $\sqrt{2}$ is irrational, which is expressed in modern terms as follows:

Firstly we assume that $\sqrt{2}$ *can* be expressed as a rational number. That is, we assume that two integers, a and b, with no common factor, can be found such that

$$\sqrt{2} = \frac{a}{b}$$

Multiplying both sides by b gives

$$\sqrt{2}b = a$$

and squaring both sides we have

$$2b^2 = a^2$$

This equation tells us that a^2 is a multiple of 2, that is, it is an even number. Now, the squares of even numbers are even and the squares of odd numbers are odd, so we can deduce that a itself is an even number. Consequently it can be written as $2c$, where c is a natural number. Substituting $2c$ for a in the last equation, we have

$$2b^2 = (2c)^2 = 4c^2$$

and dividing through by 2 gives

$$b^2 = 2c^2$$

As before we can now deduce that b^2, and hence b itself, is an even number.

Thus the initial assumption that $\sqrt{2}$ is a rational number has led us to the conclusion that both a and b are even numbers, that is, they have a common factor of 2. But a and b have *no* common factor, so we have contradicted ourselves. It follows that *either* the argument is faulty (you should go through it again to see that this is not the case), *or* the original assumption is false. Hence $\sqrt{2}$ is not a rational number.

This proof is an example of a very important type of argument called *reductio ad absurdum*.

With only minor amendments it can be adapted to prove that the square root of any prime number is irrational. If such a square root is multiplied by a rational number, the result is also irrational. Numbers such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{6}$ are often called **surds** (see Chapter P2 on page 6).

There are other irrational numbers, π for example, but we shall not go into the details here.

Question

Q1 Are the following statements true or false? If you think they are false explain clearly why you have come to this conclusion.

- **a** All prime numbers are odd numbers.
- **b** Any natural number can be expressed as a rational number.
- **c** The square root of a natural number is an irrational number.
- **d** $\pi = 22/7$, so π is a rational number.

2.5 Infinity

Use a calculator to work out the value of $1/n$ for $n = 0.1$, 0.001, 0.0001, 0.000 0001. (Even if you do not have a calculator it is easy to find the answers!) You should find that $1/n$ gets bigger and bigger as n gets smaller and smaller; we say that $1/n$ 'tends to infinity as n tends to zero'. The symbol ∞ is normally used for infinity. However the idea of 'infinity' can be very risky. Consider, for example the two lists

$$\begin{array}{ccccccc} 1, & 2, & 3, & 4, & 5, & 6, & \ldots \\ 1, & 4, & 9, & 16, & 25, & 36, & \ldots \end{array}$$

How many terms are there in each of these lists? You might think 'infinity', but look carefully. Since each number in the second list is the square of the corresponding number in the first, each list contains the same number of terms. Yet the second list clearly omits many of the terms which are in the first, so it appears that the second list contains fewer terms than the first. 'Infinity' is a problematic concept and should be handled with great care. Mathematicians usually try to avoid it. In particular they never divide by zero; instead they usually say that an expression like 1/0 is **not defined**. Infinity itself is not a number.

Example 1 *Find the values of x for which the expression $\dfrac{2x+5}{x^2-x-6}$ is not defined.*

The expression is undefined if

$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0$$

i.e. either $x - 3 = 0$ or $x + 2 = 0$

The expression is undefined when $x = 3$ or -2.

Question

Q2 Find the values of x for which the following expressions are undefined:

a $\dfrac{x}{2x+5}$ **b** $\dfrac{1}{x^2+8x+15}$

c $\dfrac{x}{x^2-25}$ **d** $\dfrac{10}{x^2-3}$

2.6 Sets

In the previous sections we have already needed to refer to particular collections, or **sets**, of numbers. A set is any clearly defined collection of objects (in this chapter the objects will always be numbers, but in later chapters you will meet sets of other mathematical objects or **elements**). The members of a set may be defined by listing them, or by describing them carefully in words. It is usual to enclose the list of members of a set in curly brackets, e.g.

$\{2, 4, 6, 8\}$	is the set of even numbers less than ten
$\{2, 3, 5, 7\}$	is the set of prime numbers less than ten
$\{3, 6, 9, \ldots, 99\}$	is the set of multiples of three, less than a hundred

Notice that when the pattern has been clearly established, as in the last case, the three dots indicate that the pattern continues until the last term is reached. In some cases there may be no last term, for example the set of square numbers,

$$\{1, 4, 9, 16, 25, 36, \ldots\}$$

When listing the members of a set, an individual member is never repeated. Thus the set of prime factors of 1200 is $\{2, 3, 5\}$.

When we wish to show that a particular element belongs to a certain set, the symbol $\in$ is used. Thus if P is the set of prime numbers we may write

$$37 \in P$$

and this means '37 is a member of the set of prime numbers'. In contrast,

$$36 \notin P$$

means '36 is not a member of the set of prime numbers'.

The symbol : is often used in this context to mean 'such that'. Thus if we use $\mathbb{N}$ to indicate the set of natural numbers, the mathematical statement

$$A = \{x^3 : x \in \mathbb{N}\}$$

means 'A is the set whose members have the form x^3, where x is such that it belongs to the set of natural numbers'. Thus $A = \{1, 8, 27, 64, \ldots\}$. Or again,

$$B = \{3n^2 : n \in \mathbb{N}\}$$

means 'B is the set whose members have the form $3n^2$, where n is a member of the set of natural numbers', i.e. $B = \{3, 12, 27, 48, 75, \ldots\}$.

$$C = \{x : -3 \leqslant x \leqslant +3\}$$

means that C is the set which contains any real number x between -3 and $+3$, inclusive.

Some very important sets have standard symbols:

$\mathbb{N}$ is **the set of natural numbers**, $\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$,
$\mathbb{Z}$ is **the set of integers**, positive or negative, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, +1, +2, +3, \ldots\}$,
$\mathbb{Z}^+$ is **the set of positive integers**, $\mathbb{Z}^+ = \{+1, +2, +3, +4, +5, +6, \ldots\}$,
$\mathbb{Q}$ is **the set of rational numbers**, (see §2.3),
$\mathbb{R}$ is **the set of real numbers**.

In Chapter 10 (§10.3, page 131) you will meet $\mathbb{C}$, the set of complex numbers.

2.7 The algebra of sets

Given two sets A and B, the set consisting of all those elements which belong both to A and B is called the **intersection of A and B**. The symbol for it is $A \cap B$. Thus if

$$A = \{2, 4, 6, 8, 10, 12\} \quad \text{and} \quad B = \{3, 6, 9, 12\}$$

the intersection of A and B is the set $\{6, 12\}$ and we write

$$A \cap B = \{6, 12\}$$

The set consisting of those elements which belong to A or B, *or both*, is called the **union of A and B** and the symbol for it is $A \cup B$. (The symbol $\cup$ can be remembered as the initial letter of union.) It is important to remember that when we list the members of a set we never repeat any individual element. Thus in the case of the sets A and B in the previous paragraph,

$$A \cup B = \{2, 3, 4, 6, 8, 9, 10, 12\}$$

Example 2 *Given that A is the set of odd numbers less than 20, and B is the set of prime numbers less than 20, list the members of A, B, $A \cap B$, $A \cup B$.*

$$\begin{aligned} A &= \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\} \\ B &= \{2, 3, 5, 7, 11, 13, 17, 19\} \\ A \cap B &= \{3, 5, 7, 11, 13, 17, 19\} \\ A \cup B &= \{1, 2, 3, 5, 7, 9, 11, 13, 15, 17, 19\} \end{aligned}$$

Notice that if P is the set of odd numbers and Q is the set of multiples of 2 then there would be no number which belongs to $P \cap Q$. A set with no members is called an **empty set**; the symbol for it is $\varnothing$. Thus in the example above we write $P \cap Q = \varnothing$. ($\varnothing$ is pronounced 'ur', as in hurt.)

Sometimes it is convenient to have a special symbol for *all* the elements which are involved in a particular topic, or in a particular question. The symbol for this is $\mathscr{E}$; it is called **the universal set**. In this context, it is also frequently useful to have a symbol for all the elements of the universal set $\mathscr{E}$ which are *not* in a given set A. The symbol used for this A' and this set is called the **complement** of set A. For example, given that

$$\mathscr{E} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \quad \text{and that} \quad X = \{4, 8\}$$

the complement of X is the set

$$X' = \{1, 2, 3, 5, 6, 7, 9, 10\}$$

Notice that for any set P,

$$P \cap P' = \varnothing \quad \text{and} \quad P \cup P' = \mathscr{E}$$

If every member of a certain set H is also a member of a set K, then H is called a **subset** of K. For example, $\{2, 4, 6, 8\}$ is a subset of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and the symbol used for this is $\subset$. Thus $H \subset K$ reads 'H is a subset of K'.

The notation $n(A)$ means 'the number of elements in set A'. Thus in Example 2 above, $n(A) = 10$, $n(B) = 8$, $n(A \cap B) = 7$ and $n(A \cup B) = 11$. Notice that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

You should think carefully about this equation and should be able to see that it is true for *any* sets A and B.

Exercise 2a

1 Given that $A = \{1, 2, 3, 4, 5\}$, list the members of the following sets:

- **a** $\{x^2: x \in A\}$
- **b** $\{1/x: x \in A\}$
- **c** $\{2x: x \in A\}$
- **d** $\{4x + 1: x \in A\}$

2 Given that $A = \{-3, -2, -1, 0, +1, +2, +3\}$ list the members of the following sets:

- **a** $\{x^2: x \in A\}$
- **b** $\{x^3 - x: x \in A\}$
- **c** $\{x^4: x \in A\}$
- **d** $\{1/(x+5): x \in A\}$

3 In this question, $x \in \mathbb{Z}^+$. List the members of the following sets:

- **a** $\{x^2: x < 10\}$
- **b** $\{10x - x^2: x < 10\}$
- **c** $\{10 - x: x < 10\}$
- **d** $\{x/2: x < 10\}$

4 Are the following statements true or false? If you think a statement is false, give a clear reason for your conclusion.

- **a** All factors of an even integer are even.
- **b** All the factors of an odd integer are odd.
- **c** $\mathbb{Z} \subset \mathbb{Q}$.
- **d** Any odd square number can be expressed in the form $4m + 1$, where $m \in \mathbb{Z}^+$.

5 List the members of the set of real numbers for which the expression $\dfrac{1}{(x-1)(x-2)(x-3)}$ is undefined.

6 In this question, $\mathscr{E}$ is the set of positive integers less than 100 and the sets A and B are subsets of $\mathscr{E}$. A is the set of multiples of 5, and B is the set of multiples of 7.

- **a** List the members of A, B, $A \cap B$, $A \cup B$.
- **b** Describe in words the members of set $A \cap B$.
- **c** Write down the values of $n(A)$, $n(B)$, $n(A \cap B)$ and $n(A \cup B)$. Verify that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

7 Given that $\mathscr{E}$ is the set of natural numbers less than or equal to 20, list the members of the following subsets of $\mathscr{E}$:

- **a** A, the multiples of 3
- **b** B, the multiples of 4
- **c** A', the complement of A
- **d** B'
- **e** $(A \cup B)'$
- **f** $A' \cap B'$

Comment on your answers.

8 Express as recurring decimals the rational numbers

a $1/3$ **b** $2/7$ **c** $3/11$

9 Express the recurring decimal $0.\dot{7}$ as a rational number. (Hint: let $x = 0.7$ and consider $10x$.)

10 Express the following recurring decimals as rational numbers:

- **a** $0.\dot{1}\dot{2}$
- **b** $0.\dot{6}5\dot{7}$
- **c** $0.\dot{4}2857\dot{1}$

2.8 Functions

Consider exercises (1) and (2) below.

(1) *A stone is projected vertically upwards. Its height, h metres, after t seconds, is given approximately by the formula* $h = 20t - 5t^2$. *Use the formula to calculate its height after* 0, 1, 2, 3, 4 *seconds.*

The answers to this exercise are shown in the table below:

t	0	1	2	3	4
h	0	15	20	15	0

(2) *Given that* $x \in \{1, 2, 3, 4, 5\}$ *find the corresponding set of values of y, where y is given by the rule:*

a $y = x^2$, **b** $y = 1/x$, **c** $y = \sqrt{(5-x)}$.

The three answers to this exercise are

a $\{1, 4, 9, 16, 25\}$, **b** $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$, **c** $\{2, \sqrt{3}, \sqrt{2}, 1, 0\}$.

All exercises like these have certain features in common. In each case, a set of values is given for one of the variables. Then a rule is given and this is applied to the given set of numbers, to produce a set of values of the other variable. In mathematics there are standard terms which are used to describe these features. The variable for which the values are given (t in exercise (1), x in (2)) is called the **independent variable** and the set of values of the independent variable is called the **domain**. The rule which is applied to the independent variable is called the **function** and the variable which is produced by the rule is called the **dependent variable**. (In (1) h is the dependent variable and in (2) y is the dependent variable.) The set of values of the dependent variable is called the **range** of the function. In exercise (2) part **a**, the range is the set $\{1, 4, 9, 16, 25\}$.

The member of the range which corresponds to a certain member of the domain is usually called the **image** of that member, e.g. in (2)**a** above, 25 is the image of 5. Notice that there is no objection to having two distinct members of the domain with the same image, e.g. see (1) above, in which both $t = 1$ and $t = 3$ have the image 15. The converse however is not allowed; a member of the domain must not have more than one image. When each member of the range has exactly one corresponding member of the domain the function is called a **one-to-one** function. Thus if the domain is $\mathbb{R}$, the set of all real numbers, $y = x^3$ represents a one-to-one function, but $y = x^2$ does not. A function which is not one-to-one is said to be **many-to-one**.

Questions

Q3 For each of the rules below, state carefully the largest possible subset of $\mathbb{R}$ which would be a suitable domain. In each case describe the corresponding range.

a $y = 1/(x-3)$ **b** $y = \sqrt{(10-x)}$
c $y = \sqrt{(25-x^2)}$ **d** $y = 1/(25-x^2)$
e $y = 1/(25+x^2)$

Q4 Which of the rules below represent functions (distinguish between one-to-one functions and many-to-one functions)? In each part, the domain is $\mathbb{R}$.

a $y = x^4$ **b** $y = x^5$
c $y^2 + x^2 = 25$ **d** $y = x^3 - x$

2.9 Function notation

Sometimes we need to discuss several functions simultaneously. Suppose we have two functions, both having $\mathbb{R}$ as the domain, and suppose one of them squares each member of the domain and the other doubles each member of the domain. We write $f(x)$ to represent the image of x under the function f; our first function would be represented by $f(x) = x^2$ and the second by $g(x) = 2x$. $f(x)$ and $g(x)$ are examples of **function notation**. The usual letters to use for function notation are f, g, h and their corresponding capital letters. In the illustration above, $f(5) = 25$ and $g(5) = 10$. We can also write $f(a) = a^2$, $f(a+h) = (a+h)^2$, $g(k) = 2k$, and so on.

Example 3 *Given that* $h(x) = x^2 - x$, *find the values of* $h(10)$, $h(-3)$, $h(\frac{1}{2})$, $h(t+1)$, $h(5k)$.

$$h(10) = 10^2 - 10 = 100 - 10 = 90$$
$$h(-3) = (-3)^2 - (-3) = 9 + 3 = 12$$
$$h(\tfrac{1}{2}) = (\tfrac{1}{2})^2 - (\tfrac{1}{2}) = \tfrac{1}{4} - \tfrac{1}{2} = -\tfrac{1}{4}$$
$$h(t+1) = (t+1)^2 - (t+1) = t^2 + 2t + 1 - t - 1 = t^2 + t$$
$$h(5k) = (5k)^2 - 5k = 25k^2 - 5k$$

The function $f(x) = x^3$ can also be written as

$$f: x \mapsto x^3$$

This statement should be read 'f is a function which **maps** x onto x^3'. The function $g(x) = 2x$ now becomes $g: x \mapsto 2x$. When $x = 5$, we write $f: 5 \mapsto 125$ and $g: 5 \mapsto 10$.

2.10 Composite functions

In this section f and g will be used to represent the functions $f(x) = x^2$ and $g(x) = x + 5$. The domain of both functions will be $\mathbb{R}$.

Notice that $f(3) = 9$ and $g(9) = 14$. Thus if we start with $x = 3$ and apply to it first function f and then function g, we shall obtain the number 14. This could be written $g(f(3)) = g(9) = 14$, but it is usually abbreviated to $gf(3) = 14$ (alternatively, the notation $g \circ f(3) = 14$ may be used). Similarly $gf(10) = 100 + 5 = 105$. In general

$$gf(x) = x^2 + 5$$

The function $gf(x)$ is called a **composite function**. Notice that the order of the functions which make up a composite function is very important. With f and g defined as above,

$fg(x) = (x+5)^2 = x^2 + 10x + 25$

Remember that when a composite function is evaluated, the individual functions are taken from right to left.

Example 4 *Given that* F: $x \mapsto (10 + x)$, G: $x \mapsto x^3$ *and* H: $x \mapsto x/2$, *write down the functions* **a** FG, **b** GF, **c** FGH.

a FG: $x \mapsto (10 + x^3)$

b GF: $x \mapsto (10 + x)^3$

c H: $x \mapsto (x/2)$

GH: $x \mapsto (x/2)^3$, hence GH: $x \mapsto x^3/8$

FGH: $x \mapsto 10 + x^3/8$

Example 5 *Given that* $f(x) = 25 - x^2$ *and that* $g(x) = \sqrt{x}$, *find, where possible, the values of* **a** gf(0), **b** gf(4), **c** gf(13).

a $f(0) = 25$, $gf(0) = g(25) = 5$,

b $f(4) = 9$, $gf(4) = g(9) = 3$,

c $f(13) = 25 - 169 = -144$, but we cannot evaluate $g(-144)$ because a negative number does not have a real square root.

Example 5c illustrates a difficulty which can arise when forming a composite function. If the domain of the function $f(x)$, above, is $\mathbb{R}$ then its range is $\{y: y \in \mathbb{R}, y \leqslant 25\}$ and this includes negative numbers, which are not in the domain of the function $g(x) = \sqrt{x}$. This can only be avoided if we restrict the domain of $f(x)$ to $\{x: x \in \mathbb{R}, -5 \leqslant x \leqslant +5\}$. In general, when a composite function $gf(x)$ is formed, the range of the function $f(x)$ must be a subset of the domain of the function $g(x)$.

Some mathematicians insist that whenever a function is defined its domain should be explicitly stated and, strictly speaking, they are correct. However this soon becomes tedious and most people adopt the convention that it may be assumed that the domain is intended to be $\mathbb{R}$. You should use common sense to exclude any members of the domain which give rise to obvious difficulties (e.g. square roots of negative numbers, fractions with a zero denominator). This is the convention which is generally used in this book, although in *this* chapter, the domain will be described in full.

The term **co-domain** is sometimes used for any set which contains the range. For example, the function $f(x) = x^2$ maps real numbers onto real numbers and so one can say the domain is $\mathbb{R}$ and the co-domain is $\mathbb{R}$, but since all the images are positive (or, to be precise, non-negative) the range is the set of non-negative real numbers.

Exercise 2b

1 Given that $g(x) = x^3 + 1$, find the values of

a $g(0)$ **b** $g(5)$ **c** $g(\frac{3}{4})$ **d** $g(-2)$

2 The domain of the function $g(x) = 5x + 1$ is $\{0, 1, 2, 3, 4, 5\}$. Find its range.

3 The domain of the function $f(x) = x^2 + 1$ is $\mathbb{R}$. Find its range.

4 The domain of the function $f(x) = 1/(1 + x^2)$ is $\mathbb{R}$. Find its range.

5 The domain of the function $f(x) = 1/\sqrt{(25 - x)}$ is a subset of $\mathbb{R}$. Write down the largest possible set which is a suitable domain.

In questions 6–10 the domain is $\mathbb{R}$.

6 Given that f: $x \mapsto 5x + 1$ and that g: $x \mapsto x^2$, express the composite functions fg and gf in their simplest possible forms.

7 Given that $f(x) = x^2$, express as simply as possible

a $f(5 + h)$

b $\dfrac{f(5+h) - f(5)}{h}$, $(h \neq 0)$

8 If $f(x) = x^2$ express as simply as possible

$\dfrac{f(a+h) - f(a)}{h}$, $(h \neq 0)$.

9 Given that $f(x) = x^3$ find

a $f(2)$

b $f(-10)$

c $f(\frac{1}{2})$

d $f(5a)$

e $f(a/3)$

f $f(a + h)$

g $f(a + h) - f(a - h)$

h $\dfrac{f(a+h) - f(a-h)}{2h}$, $(h \neq 0)$

10 If $f(x) = 7x$ and $g(x) = x + 3$ and fg: $x \mapsto y$, express as simply as possible the rule which maps x onto y. Find the values of p, q, r such that

a fg: $5 \mapsto p$

b fg: $10 \mapsto q$

c fg: $r \mapsto 35$

Find also the function, F, which reverses the function fg, that is, it maps y onto x.

2.11 Graphs of functions

When the domain is the set of real numbers ℝ, it is always represented by the horizontal axis, and the corresponding values of the dependent variable are represented by points on the vertical axis. When x and y are used to represent typical members of the domain and the co-domain, these axes are called the x-axis and the y-axis respectively.

Fig. 2.2 shows the graph of a function $y = f(x)$. A typical member a of the domain and its image $f(a)$ are shown.

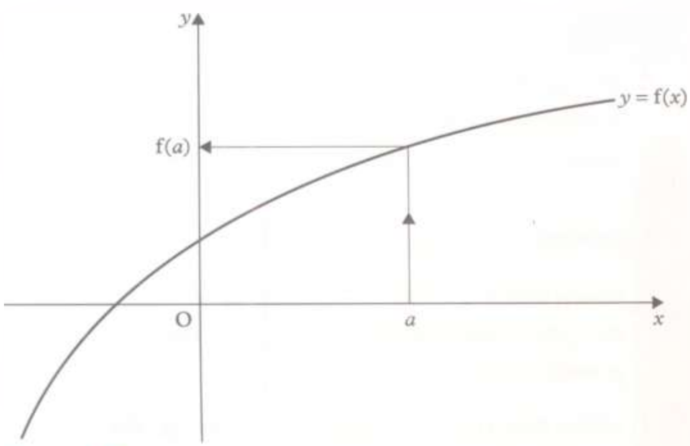


Figure 2.2

Bearing in mind that each member of the domain has exactly one image in the co-domain, a graph like the one shown in Fig. 2.3 does *not* represent the graph of a function. This is because in this figure, a has three possible images in the co-domain.

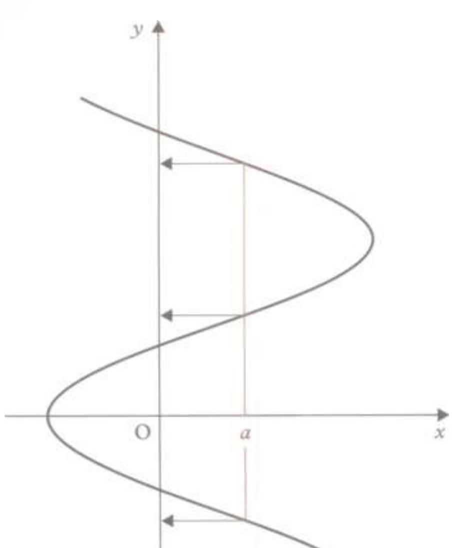


Figure 2.3

Also consider the circle, centre (0, 0), radius 10. The coordinates of any point P on the circle satisfy the equation $x^2 + y^2 = 100$, so a relation exists between the values of x and y at each point, and a graph can be drawn, but this is not the graph of a function because there are values of x for which there are two possible values of y, e.g. when $x = 6$, $y = +8$ or $y = -8$ (see Fig. 2.4).

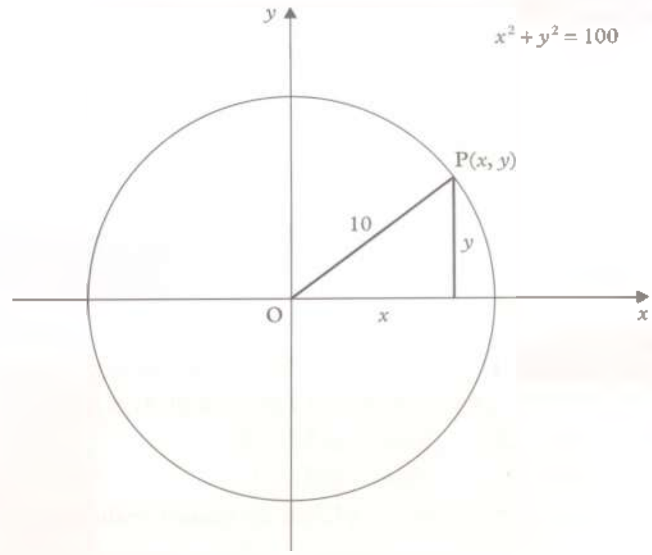


Figure 2.4

Graphs of some common functions

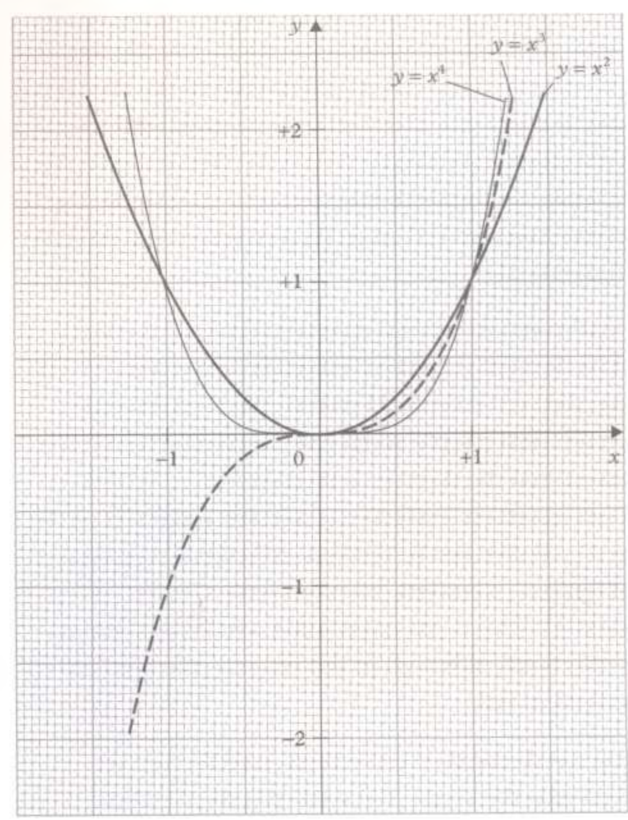


Figure 2.5

Fig. 2.5 illustrates the graphs of $y = x^2$, $y = x^3$ and $y = x^4$. If you are not familiar with these already you should draw and save graphs of these functions. Note that all of these graphs pass through the point (1, 1); $y=x^2$ and $y=x^4$ also pass through (−1, 1), while $y=x^3$ passes through (−1, −1). Notice also that the graphs 'flatten out' between $x=-1$ and $x=+1$, as the power increases. (Try plotting $y=x^{10}$: a calculator may be needed for some of the calculations.)

Fig. 2.6 shows the graphs of **i** $y=1/x$, $(x \neq 0)$ and **ii** $y=\sqrt{x}$, $(x \geqslant 0)$. (Remember that the square-root sign means the positive square root.)

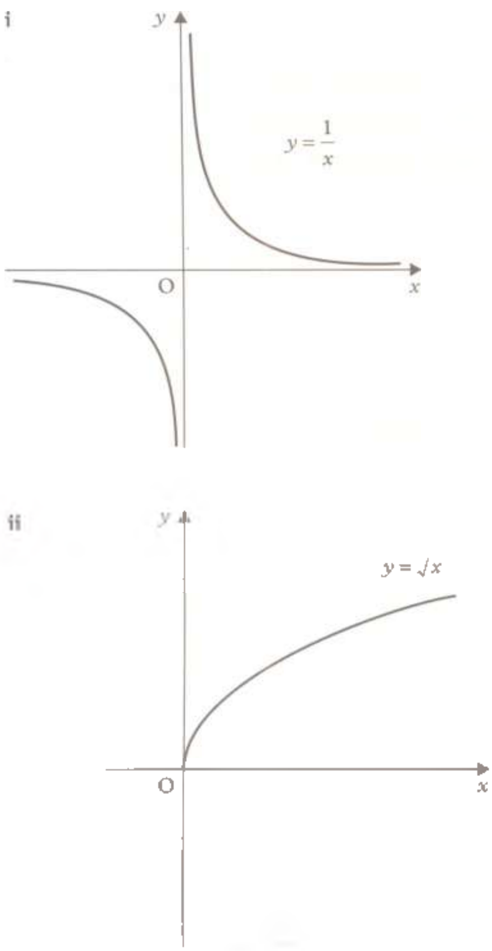


Figure 2.6

If you are not familiar with these graphs you should make careful copies, using a calculator where necessary, and save the graphs for future reference. Notice also that if the functions are changed to $y=1/(x-2)$, $(x \neq 2)$, and $y=\sqrt{(x-2)}$, $(x \geqslant 2)$, then the graphs have the same shape but they are translated 2 units to the right. In general, the graph of $y=\mathrm{f}(x-a)$ will have the same shape as $y=\mathrm{f}(x)$ but it will be translated a units to the right.

The **modulus** of x, written $|x|$, is *the magnitude or size of* x, thus $|+5|=+5$ and $|-7|=+7$. A table of values of $|x|$ for $x=-4$ to $x=+4$ is shown below:

x	−4	−3	−2	−1	0	+1	+2	+3	+4
$\lvert x\rvert$	+4	+3	+2	+1	0	+1	+2	+3	+4

Fig. 2.7 shows the graph of $y=|x|$.

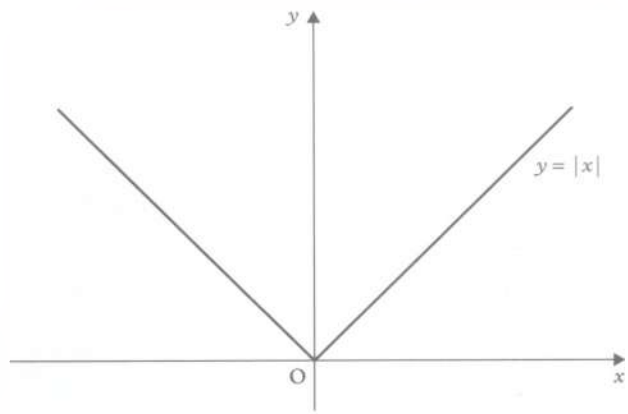


Figure 2.7

The instructions 'plot the graph of …' and 'sketch the graph of …' have very definite, but distinct, meanings. The instruction 'plot the graph of $y=x^2$, for $x=0$ to 5' means that the necessary values of y should be calculated, the points should be accurately plotted on graph paper and the points should be joined with a neat smooth curve. In contrast, a *sketch* of a curve should not be done on graph paper. Use plain, or ordinary lined paper instead. Only those points which have special importance should be marked. The sketch should not be limited to a small part of the domain. Instead, try to show the overall appearance of the graph throughout its domain.

Example 6 *Sketch the graph of* $y=\dfrac{1}{x-3}+2$, $(x \neq 3)$.

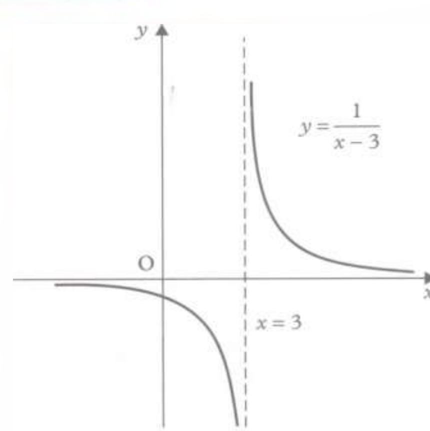


Figure 2.8i

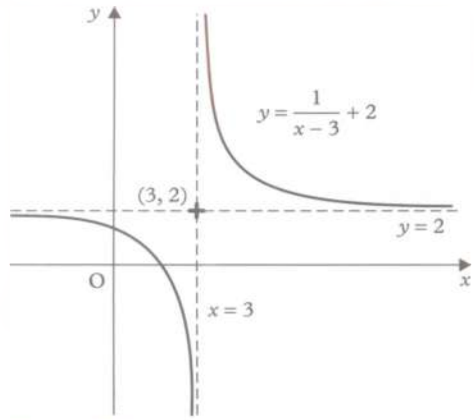


Figure 2.8ii (continued)

The graph of $y = 1/x$ in Fig. 2.6i is one of the standard graphs. Replacing x by $x - 3$ translates the graph 3 units to the right, and so a sketch of $y = 1/(x - 3)$ would be like Fig. 2.8i. When the final 2 is added to $1/(x - 3)$ the graph is translated 2 units vertically upwards. Hence the sketch graph of $y = 1/(x - 3) + 2$ will look like Fig. 2.8ii.

Notice that the curve in Fig. 2.8 is in two parts. The lines $x = 3$ and $y = 0$ (the x-axis) in Fig. 2.8i and and $x = 3$ and $y = 2$ in Fig. 2.8ii are called **asymptotes**. The curve approaches these lines without ever meeting them. See §28.2 on page 294 for further discussion of asymptotes.

Exercise 2c

Sketch the graphs of the following functions (detailed plotting is not required). Where possible, the sketch should be obtained by modifying one of the standard graphs in the preceding section.

1 $y = 2x + 1$

2 $y = (x + 2)^3$

3 $y = x^2 + 5$

4 $y = 1/(x + 4), \quad (x \neq -4)$

5 $y = -x^2$

6 $y = 5x^2$

7 $y = \surd(10 - x), \quad (x \leqslant 10)$

8 $y = 1/x^2, \quad (x \neq 0)$

So far x has always been used for the independent variable and y for the dependent variable, but x and y are not the only letters which may be used. In questions 9–15, t is used for the independent variable, hence the t-axis is horizontal, and z is used for the dependent variable.

9 $z = (t - 4)^3$

10 $z = 100 - t^2, \quad (-10 \leqslant t \leqslant +10)$

11 $z = |t - 3|$

12 $z = |(t + 4)(t - 4)|$

13 $z = |1/(1 + t)|$

14 $z = 1/(1 + |t|)$

15 $z = |t| - |t + 1|$

16 Fig. 2.9 shows the graph of an unspecified function $y = f(x)$. Trace the diagram and use the tracing to show, on a single diagram, sketch graphs of

a $y = f(x - 6)$

b $y = f(x + 3)$

c $y = f(x) + 2$

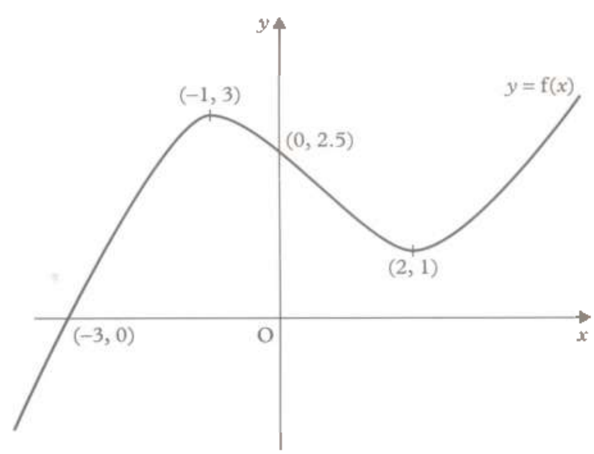


Figure 2.9

17 Use the tracing from question 16 to draw the graph of $y = f(x)$ and superimpose on it sketches of the following graphs, showing clearly their relationship to the graph of $y = f(x)$:

a $y = 5f(x)$

b $y = f(5x)$

c $y = -f(x)$

d $y = f(-x)$

***18** Describe, in words, the appearance of the following graphs, relative to the graph of $y = f(x)$:

a $y = f(x - a)$

b $y = f(x) + a$

c $y = k \times f(x)$

d $y = -f(x)$

e $y = f(-x)$

2.12 Further functions

Example 7 *In a certain county the cost of posting a parcel, weighing not more than 10 kg, is given by the table below. Explain why this table expresses the cost of the parcel as a function of its weight and draw a graph of the function.*

Not over	Cost ($)	Not over	Cost ($)
1 kg	5.50	6 kg	11.00
2 kg	7.15	7 kg	11.75
3 kg	8.65	8 kg	12.25
4 kg	9.50	9 kg	12.75
5 kg	10.25	10 kg	13.25

The table expresses the cost as a function of the weight because, if the weight is known, the table shows the cost of postage. A function is any rule which enables the dependent variable to be found, when the independent variable is known. It is not necessary to express the rule as a formula. The graph is shown in Fig. 2.10.

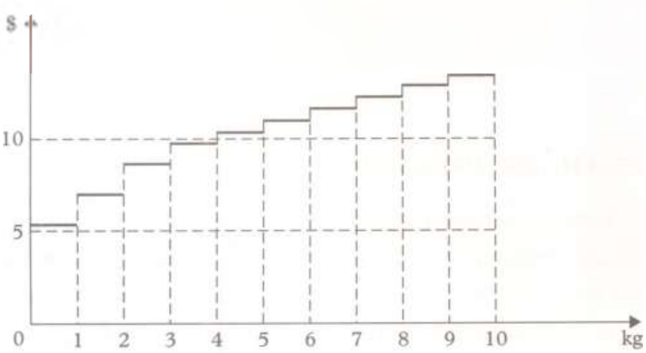


Figure 2.10

The function in Example 7 differs from the functions discussed earlier in the chapter, in that different rules apply to different parts of the domain. Many functions which arise from real life problems are like this. Examples 8 and 9 show other functions which display this characteristic.

Example 8 *The domain of function* f *is* $\mathbb{R}$.

$$f: x \mapsto 1 \quad \textit{when } x < 0, \quad \textit{and}$$
$$f: x \mapsto x^2 + 1 \quad \textit{when } x \geqslant 0.$$

Sketch the graph of this function.

See Fig. 2.11.

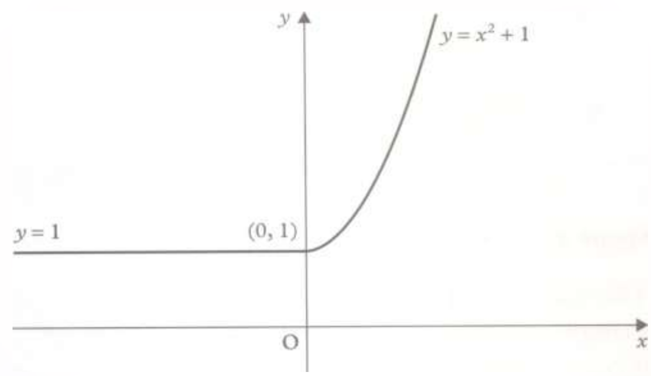


Figure 2.11

Example 9 *The domain of the function* f *is* $\mathbb{R}$.

$$f: x \mapsto 1, \quad \textit{if } x \in \mathbb{Z}, \qquad \textit{and} \qquad f: x \mapsto 2, \quad \textit{if } x \notin \mathbb{Z}.$$

Write down $f(+5)$, $f(-1)$, $f(0)$, $f(3.4)$, $f(\sqrt{2})$ *and* $f(\pi)$. *Sketch the graph of* $y = f(x)$.

$f(+5) = 1$, because $+5$ is an integer, i.e. $+5 \in \mathbb{Z}$. Similarly $f(-1) = 1$ and $f(0) = 1$.

But 3.4, $\sqrt{2}$ and π are not members of $\mathbb{Z}$, so $f(3.4) = f(\sqrt{2}) = f(\pi) = 2$.

The graph of this function consists of the points … $(-2, 1)$, $(-1, 1)$, $(0, 1)$, $(1, 1)$, $(2, 1)$ … and the straight line $y = 2$, with small gaps whenever x is an integer (Fig. 2.12). In the diagram, we have enlarged the gaps for the sake of clarity.

Figure 2.12

Odd and even functions

Functions whose graphs are symmetrical about the vertical axis (i.e. the y-axis in **Fig. 2.13**) are called **even functions**.

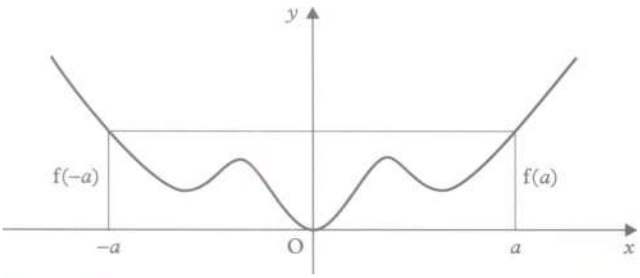


Figure 2.13

This means that for any value of a, $f(a) = f(-a)$. Obvious examples of even functions are functions of the form $f(x) = x^n$, where n is an even integer, hence the name, even function. Another important even function is $f(x) = \cos x$ (see §16.2 on page 195).

A function with the property $f(-a) = -f(a)$, for every member a of the domain, is called an **odd function**. The graph of an odd function will have rotational symmetry of order 2 about the origin, that is, after a 180° rotation about the origin, the graph will be superimposed upon itself (**Fig. 2.14**).

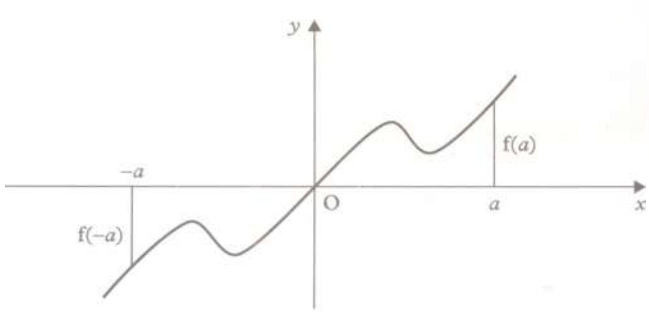


Figure 2.14

Functions of the form $f(x) = x^n$, where n is an odd number, will be odd functions. Another important odd function is $f(x) = \sin x$ (see §16.2 on page 195).

Example 10 *Prove that the sum of two even functions is an even function and that the sum of two odd functions is an odd function.*

Let $f(x)$ and $g(x)$ be two even functions. Then $f(x)$ and $g(x)$ have the property $f(-a) = f(a)$ and $g(-a) = g(a)$, for any member a of the domain.

Let $F(x)$ be the sum of $f(x)$ and $g(x)$, that is $F(x) = f(x) + g(x)$. Then if a is any member of the domain

$$\begin{aligned} F(-a) &= f(-a) + g(-a) \\ &= f(a) + g(a) \\ &= F(a) \end{aligned}$$

hence $F(x)$ is an even function.

Similarly if $f(x)$ and $g(x)$ are odd functions, then

$$\begin{aligned} F(-a) &= f(-a) + g(-a) \\ &= -f(a) - g(a) \\ &= -F(a) \end{aligned}$$

hence $F(x)$ is an odd function.

Questions

Q5 Prove that the product of two even functions is an even function.

Q6 Prove that the product of two odd functions is an even function.

Q7 Is the product of an even function and an odd function odd or even?

Periodic functions

A function whose graph repeats itself at regular intervals is called a **periodic function** (see **Fig. 2.15**). Such functions are especially important in science. The sound wave of a note of constant pitch, for example, is periodic.

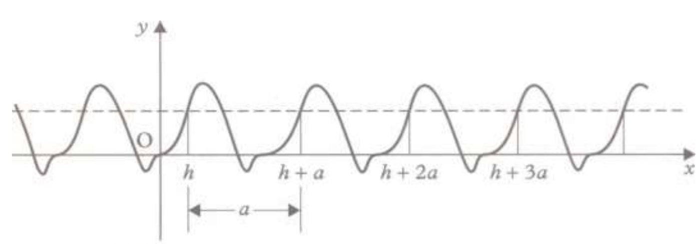


Figure 2.15

The length of the interval between repeats is called the **period** of the function. If the period is a, then for any value of h in the domain of the function,

$$f(h + a) = f(h)$$

The most common periodic functions are the trigonometric functions $\sin x$ and $\cos x$ (see §16.5 on page 198). They have a period of 360°, because

$$\sin (x + 360)° = \sin x° \quad \text{and} \quad \cos (x + 360)° = \cos x°.$$

Example 11 *Sketch the graph of the periodic function such that* $f(x) = x$, *for* $-1 < x \leqslant +1$, *where the period of* $f(x)$ *is* 2.

Between $x = -1$ and $x = +1$, the graph is the ordinary straight line $y = x$. Outside this interval, the graph repeats itself every 2 units (Fig. 2.16).

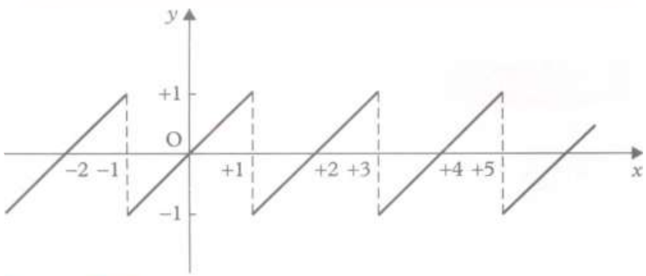


Figure 2.16

Example 12 *Sketch the graph of* $y = f(x)$ *where* $f(x) = \sqrt{(1 - x^2)}$, *when* $0 \leqslant x \leqslant 1$, *and* $f(x)$ *is an even function with a period of* 2.

The equation $y = \sqrt{(1 - x^2)}$ produces an arc of a circle between $x = 0$ and $x = +1$. Because the function is even, the graph is symmetrical about the vertical axis. Thus between $x = -1$ and $x = +1$, the graph is a semi-circle, and this is repeated at regular intervals of 2 units (Fig. 2.17).

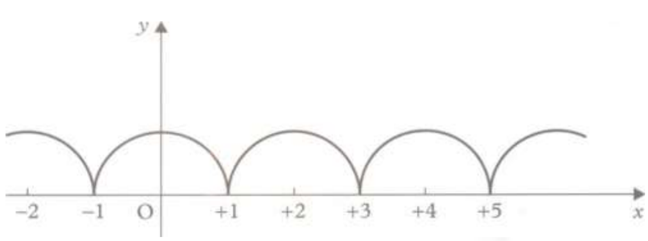


Figure 2.17

2.13 Inverse of a function

Consider the function $y = f(x)$, where $f(x) = \frac{1}{8}x^3 + 1$. Fig. 2.18 is a sketch of its graph.

If we are given a member of the range, say $y = 9$, is it possible to find the corresponding member of the domain? On the graph this would mean starting from $y = 9$ on the vertical axis, drawing a line horizontally to the point P on the curve and then drawing a vertical line down to the x-axis. The point where the line meets the axis gives the value of x which is required. In this particular example it is fairly easy to solve the problem algebraically.

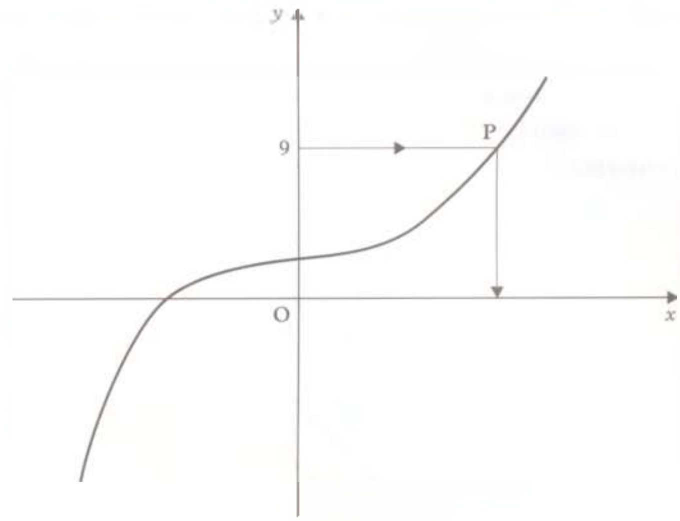


Figure 2.18

The value of x required is found by solving the equation

$$\tfrac{1}{8}x^3 + 1 = 9$$

$$\tfrac{1}{8}x^3 = 8$$

$$x^3 = 64$$

$$\therefore x = 4$$

It is quite simple to generalise this. Starting with the given value from the range of function f, we first subtract 1, then we multiply by 8 and finally we find the cube root. The whole operation is called the inverse of function f and it is written f^{-1}. Following the usual convention of writing x for a typical member of the domain of function f^{-1}, we can write the inverse function as follows:

$$f^{-1}(x) = \sqrt[3]{\{8(x - 1)\}}$$

Thus

$$f^{-1}(9) = \sqrt[3]{\{8(9 - 1)\}} = \sqrt[3]{(8 \times 8)} = \sqrt[3]{64} = 4$$

There is however one problem: when we draw the horizontal line from the given number to the graph of $y = f(x)$, this line must meet the curve *once only*. Otherwise there will be more than one possible answer and we are not allowed to use the word function to describe such a situation. For example, $f(x) = x^2$ is a perfectly acceptable function, but it maps both +5 and −5 onto the same image, namely 25. There is no objection to this; we simply agree to call it a many-to-one function. However, if we attempt to find $f^{-1}(25)$, there are two possible answers, namely +5 and −5. So $f(x)$ does not have an inverse function. This difficulty can be by-passed if we agree in advance to limit the domain of $f(x) = x^2$ to the non-negative real numbers. In that case we shall not apply it to −5 and the difficulty of having two possible answers will not arise.

To sum up: we can only have an inverse function if the original function is a one-to-one function. (However, the fact that an inverse function exists does not necessarily mean we shall be able to write down the rule which gives the inverse.)

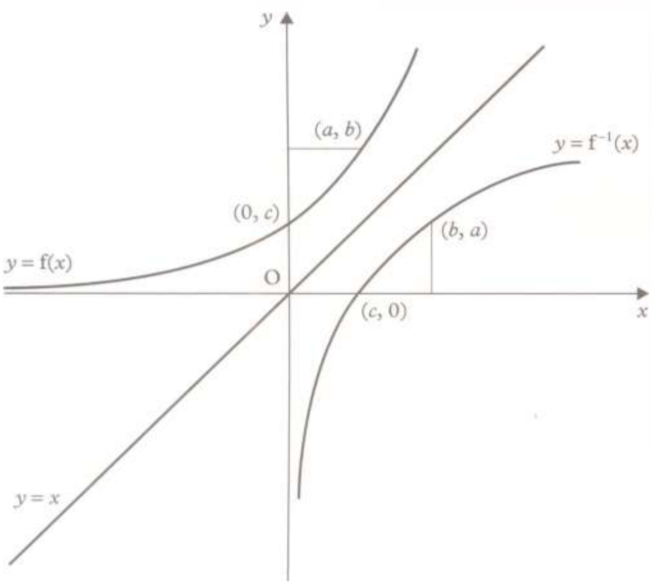


Figure 2.19

In general, if (a, b) is a point on the graph of $y = f(x)$, then (b, a) will be a point on the graph of $y = f^{-1}(x)$, and consequently the graph of $y = f^{-1}(x)$ will be the reflection of the graph of $y = f(x)$ in the line $y = x$ (see Fig. 2.19).

Here are some examples of some common functions and their inverses:

a	$f(x) = x + a$	$f^{-1}(x) = x - a,$
b	$f(x) = kx$	$f^{-1}(x) = x/k,$
c	$f(x) = x^2, \quad (x \geqslant 0)$	$f^{-1}(x) = \sqrt{x},$
d	$f(x) = a - x$	$f^{-1}(x) = a - x,$
e	$f(x) = 1/x$	$f^{-1}(x) = 1/x.$

Functions, like **d** and **e**, which are the same as their inverses are called **self-inverse functions**.

If a function f is applied to a number a, and then f^{-1} is applied, the final result will be the original number a. For example, using function **c** above, $f(3) = 9$ and $f^{-1}(9) = 3$. (This can be shown on a calculator. First key in any number a, then press a function button, say x^2, and then the button of the inverse function $\sqrt{x}$, and the original number a should be displayed. Although we have yet to study the following functions, you can observe the same phenomenon by pressing the buttons representing the following pairs of functions and their inverses: $\log x$, 10^x; $\ln x$, e^x.)

We have already seen that $fg(x)$ is the composite function, in which the function g is applied first and then function f is applied to the result. The inverse of this composite function is $g^{-1}f^{-1}(x)$. (This is rather like packing and unpacking a parcel. Suppose you wrap the parcel in paper and then tie it up with string. When the parcel is unpacked, first the string must be untied and, after that, the paper removed.)

Example 13 *Given that* $f(x) = 10x$ *and* $g(x) = x + 3$, *find* $fg(x)$ *and* $(fg)^{-1}(x)$. *Verify that if* $b = fg(a)$, *then* $(fg)^{-1}(b) = a$.

$$g(x) = x + 3$$
$$fg(x) = 10x(x + 3)$$

The inverses of g and f are $g^{-1}(x) = x - 3$ and $f^{-1}(x) = x/10$. Hence

$$\begin{aligned}(fg)^{-1}(x) &= g^{-1}f^{-1}(x)\\ &= g^{-1}\left(\frac{x}{10}\right)\\ &= \frac{x}{10} - 3\end{aligned}$$

In the general case, we are given $b = fg(a)$,

$\therefore\ b = 10(a + 3)$

and hence,

$$\begin{aligned}(fg)^{-1}(b) &= \frac{10(a+3)}{10} - 3\\ &= a + 3 - 3\\ &= a\end{aligned}$$

In some cases the inverse function can be found by thinking of $y = f(x)$ as an equation in which y is *known*, and solving the equation for the *unknown* x.
For instance, if

$$y = \frac{5x + 7}{3x + 2}$$

then

$$\begin{aligned}y(3x + 2) &= 5x + 7\\ 3xy + 2y &= 5x + 7\\ 3xy - 5x &= 7 - 2y\\ x(3y - 5) &= 7 - 2y\\ x &= \frac{7 - 2y}{3y - 5}\end{aligned}$$

So the inverse of $f(x) = (5x + 7)/(3x + 2)$ is $g(y) = (7 - 2y)/(3y - 5)$. However, since the letter x is normally used to represent the independent variable, we express this result as

$$f^{-1}(x) = \frac{7-2x}{3x-5}$$

A result such as this can be checked by verifying that $f^{-1}(f(x)) = x$. In this case,

$$\begin{aligned} f^{-1}(f(x)) &= \frac{7-2(5x+7)/(3x+2)}{3(5x+7)/(3x+2)-5} \\ &= \frac{7(3x+2)-2(5x+7)}{3(5x+7)-5(3x+2)} \\ &= \frac{21x+14-10x-14}{15x+21-15x-10} \\ &= \frac{11x}{11} \\ &= x \end{aligned}$$

Exercise 2d

1 Given that $f(x) = 5x+1$, find the values of

a $f^{-1}(36)$ **b** $f^{-1}(-14)$
c $f^{-1}(0)$ **d** $f^{-1}(a)$

2 Given that $g(t) = 1/(t-5)$, $(t \neq 5)$, find the values of

a $g^{-1}(\frac{1}{2})$ **b** $g^{-1}(2)$
c $g^{-1}(-1)$ **d** $g^{-1}(a)$

3 Find the inverses of the following functions:

a $f(x) = 12 - \frac{1}{2}x$ **b** $f(x) = \frac{1}{2}(x-3)$
c $f(x) = (2x+1)/5$ **d** $f(x) = (7-3x)/10$

4 Find the inverses of the following functions:

a $f: x \mapsto \frac{5}{9}(x-32)$ **b** $f: x \mapsto 180(x-2)$
c $f: x \mapsto 2\pi x$ **d** $f: x \mapsto [5(x+7)/3] - 9$

5 Find the inverses of the following functions:

a $F: t \mapsto t^2 + 5, \quad (t \geqslant 0)$
b $F: t \mapsto 5\sqrt{t}, \quad (t \geqslant 0)$
c $F: t \mapsto (t-5)^3$
d $F: t \mapsto \sqrt[3]{(t+1)}$

6 Find the inverses of the following functions:

a $g: x \mapsto \dfrac{1}{x-3}, \quad (x \neq 3)$

b $g: x \mapsto \dfrac{1}{2x+1}, \quad (x \neq -\frac{1}{2})$

c $g: x \mapsto \dfrac{3}{4-x}, \quad (x \neq 4)$

d $g: x \mapsto \dfrac{2x}{1+x}, \quad (x \neq -1)$

7 Show that the function $f(x) = 1/(1-x)$, $(x \neq 1)$, is the inverse of the function $g(x) = (x-1)/x$, $(x \neq 0)$.

8 Show that the function $H(x) = x/(x-1)$ is a self-inverse function.

9 Sketch the graph of the function $y = f(x)$, where $f(x) = x^3 + 1$. On the same diagram, sketch the graph of the inverse function $y = f^{-1}(x)$.

10 Fig. 2.20 shows the graph of a function $y = f(x)$. Copy the diagram carefully, using tracing paper if necessary, and on the same diagram, sketch the graph of the inverse function.

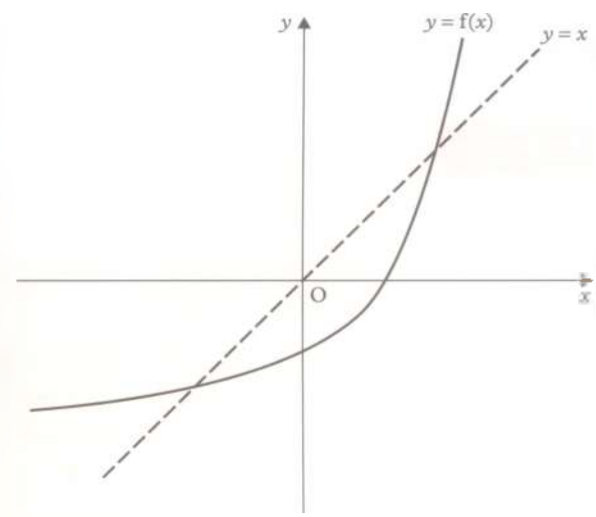


Figure 2.20

2.14 Investigating limits, using a calculator

In this section we shall investigate the limits of some functions using a calculator. It is important to understand that our investigations will only tell us the value of the function at the points we examine. To prove that the limits are what we think they are, we must turn to algebra, which we shall do in the next section. Nevertheless, the calculator can give us some very strong clues to the behaviour of certain functions.

The phrase 'x tends to zero', which is written '$x \to 0$', means that x can be made as small as we please. If any prearranged small number is chosen, then it must be possible to make x smaller than that number.

Example 14 *Investigate the function* $f(x) = x/\sin x$, *as* $x \to 0$, *using your calculator in degree mode.*

(Notice that this function is undefined at $x = 0$, since when $x = 0$, the function would give 0/0.)

x	1.0	0.5	0.1	0.01
$f(x)$	57.299	57.297	57.296	57.296

This function seems to approach 57.296 (to five significant figures) as x tends to zero.

When we say 'x tends to a', where a is a fixed real number, we mean that x can be made as close to a as we please; or, to put it another way, $|x - a| \to 0$. In the following example, x tends to 2.

Example 15 *Investigate the function* $f(x) = \dfrac{x^3 - 8}{x - 2}$, *as* $x \to 2$.

(Notice that $f(x)$ is undefined when $x = 2$ since with this value of x the function gives 0/0.)

Set out below are two tables. The first shows the values of $f(x)$ when x approaches 2 from below, and the second shows the values of $f(x)$ when x approaches 2 from above.

x	1.9	1.99	1.999	1.9999
$f(x)$	11.41	11.940	11.994	11.999

x	2.1	2.01	2.001	2.0001
$f(x)$	12.61	12.060	12.006	12.000

This suggests that $f(x)$ approaches 12, as x tends to 2.

A function $f(x)$ is said to tend to a **limit** L, if $|f(x) - L| \to 0$, as $x \to a$. The *same* number L must be reached whether x approaches the fixed number a from above or below. The function itself may, in some cases, be undefined at $x = a$. (In Example 15, above, we say that the limit of $f(x)$ is 12, as $x \to 2$.)

The phrase 'x tends to infinity', means that x gets bigger and bigger, without any limit on its size. If we choose a large number N, then it must be possible for x to exceed N. (Infinity itself is not a real number; see §2.5 on page 45.) Thus we can say that $1/n$ tends to zero as n tends to infinity. In other words $1/n$ gets smaller and smaller as n gets bigger and bigger. If we choose a very small number, say 10^{-6}, and ask whether we can make $1/n$ smaller than this number, the answer is 'yes'; all we have to do is to make n bigger than 10^{+6}. In writing, this statement is abbreviated to '$1/n \to 0$, as $n \to \infty$'.

Example 16 *Investigate the function* $f(x) = \dfrac{2x}{1 + x}$, *as* $x \to \infty$.

The table below shows some values of $f(x)$ for some increasingly large values of x. (The values of $f(x)$ are given to five significant figures.)

x	10	100	1000	10 000
$f(x)$	1.8182	1.9802	1.9980	1.9998

From this table it seems reasonable to suppose that $f(x) \to 2$, as $x \to \infty$.

Example 17 *Investigate* $f(n) = (1 + 1/n)^n$, *as* $n \to \infty$.

The table below shows the values of $f(n)$, for some increasingly large values of n. (The values of $f(n)$ have been corrected to four significant figures.)

n	1	5	10	100	1000	1 000 000
$f(n)$	2	2.488	2.594	2.705	2.717	2.718

The table suggests that the limit of this function is 2.718. (It is difficult to investigate the limit of this function rigorously, but it can be shown that it is a number called **e**. We will meet **e** properly in Chapter 31 on page 324. Like π, it plays a very important role in higher mathematics.)

In Q8–12, use your calculator to investigate each function, as x tends to the number stated.

Questions

Q8 $\dfrac{2x-7}{x-4}$, $x \to \infty$.

Q9 $\dfrac{x^2+5x-14}{x-2}$, $x \to 2$.

Q10 $\left(1+\dfrac{2}{x}\right)^x$, $x \to \infty$.

Q11 $\dfrac{x}{\sin x}$, $x \to 0$, using your calculator in radian mode.

Q12 $\dfrac{1-\cos x}{x^2}$, $x \to 0$, using your calculator in radian mode.

2.15 Finding limits algebraically

Some of the functions in the preceding sections can be examined more rigorously using algebra.

In Example 16, above, if we divide the numerator and the denominator by x, the function can be written

$$f(x) = \frac{2}{1/x + 1}$$

If now we let $x \to \infty$, the term $1/x$ will tend to zero and we can see that $f(x)$ will tend to 2. Notice that, since x is positive, the denominator will always be slightly bigger than 1, so $f(x)$ will always be slightly less than 2. We say that $f(x)$ tends to 2 *from below*. On the other hand, when $x \to -\infty$, the denominator will be slightly less than 1 and so $f(x)$ will approach 2 *from above*.

In Example 15, put $x = 2 + h$, where h is small (in due course, we shall let h tend to zero).

$$x^3 = (2+h)^3 = 8 + 12h + 6h^2 + h^3$$

hence

$$\begin{aligned}\frac{x^3-8}{x-2} &= \frac{12h+6h^2+h^3}{h}\\ &= 12 + 6h + h^2 \qquad (h \neq 0)\end{aligned}$$

Although we must not put h *equal* to zero, we can let h *tend* to zero, that is, we can let it get smaller and smaller. As it does so, the terms $6h$ and h^2 tend to zero and we see that the function tends to 12. This confirms the result of our investigation by calculator.

If $f(x)$ tends to L as x tends to a, we frequently say that the limit of $f(x)$, as x tends to a, is L. This is usually abbreviated to

$$\lim_{x \to a} f(x) = L$$

Thus the outcome of Example 15 could be written

$$\lim_{x \to 2}\left(\frac{x^3-8}{x-2}\right) = 12$$

2.16 Continuity

Looking back at Examples 7 and 8 (§2.12 on page 53), you will notice that there is an important difference between them. The graph of Example 8 could be drawn with a single sweep of the pencil, whereas in Example 7 the pencil must be lifted off the page at each integer point of the domain. We say that the function in Example 8 is **continuous**, but the function in Example 7 is **discontinuous** at 1, 2, 3, ...

Fig. 2.21 shows sketches of the graphs of $y = x^2$, $y = 1/x$ and $y = 1/x^2$.

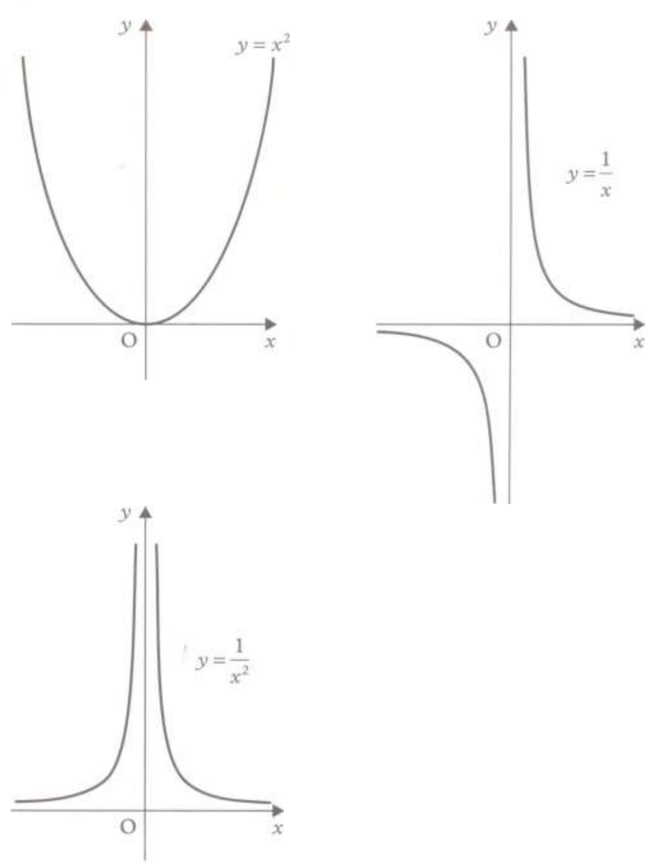


Figure 2.21

$f(x) = x^2$ is a continuous function, but the other two are discontinuous at $x = 0$ (they are, of course, both undefined at this point).

The function given by

$$f(x) = +1, \quad \text{when } x \geqslant 0$$
$$f(x) = -1, \quad \text{when } x < 0$$

is defined at every point of $\mathbb{R}$, but it is discontinuous at $x = 0$. A sketch of its graph is shown in **Fig. 2.22**.

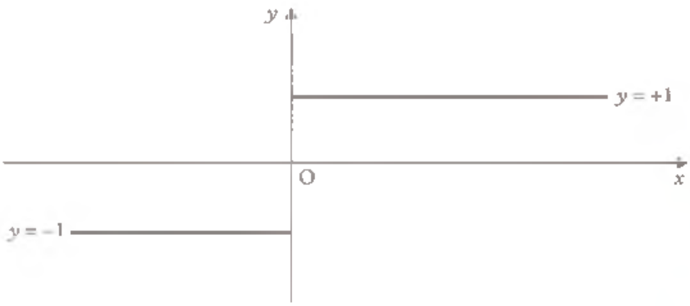


Figure 2.22

In all these cases the break in the graph has been quite obvious, but a discontinuity can be more subtle than this. Consider for example the function

$$F(x) = \frac{x^2 - 4}{x - 2} \qquad (x \neq 2)$$

For all values of x, except $x = 2$, this function is equal to $(x + 2)$. Consequently its graph is the straight line $y = x + 2$, with a gap in it at $x = 2$ (**Fig. 2.23**).

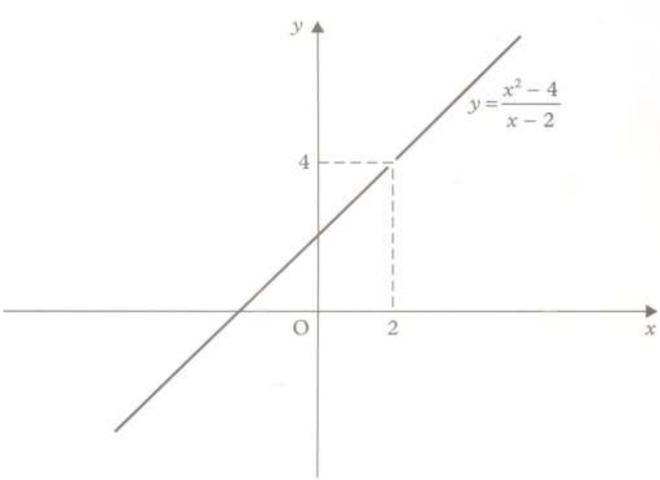


Figure 2.23

Although it is correct to say that $\lim_{x \to 2} F(x) = 4$, we must not actually put x *equal* to 2. At the moment the graph is undefined at this point. Now, if we wish, we can 'plug the gap' by *defining* F(2) as 4. In doing so we shall have made $F(x)$ continuous at $x = 2$. However, if we wish to be difficult, we could choose to define F(2) as something else, say $F(2) = 0$; in this case $F(x)$ is discontinuous at $x = 2$.

Notice that in the case of $f(x) = 1/x$ and $f(x) = 1/x^2$, we can decide to define the function at $x = 0$, if we wish, but there is no number which we could assign to it which would make these functions continuous at $x = 0$.

We can express this more formally by saying that, if

$$\lim_{x \to a} f(x) \neq f(a)$$

then $f(x)$ is discontinuous at $x = a$. But if

$$\lim_{x \to a} f(x) = f(a)$$

then the function is continuous at $x = a$. A function which is continuous at every point in its domain is called a **continuous function**.

Exercise 2e

1 Find the limits of the following expressions as $x \to \infty$:

a $\dfrac{5x+1}{10+2x}$ b $\dfrac{x+1}{x^2}$ c $\dfrac{x^2+1}{x}$ d $\dfrac{5}{1+x}$

2 Find the limits of the following expressions as $x \to 5$:

a $\dfrac{x^2-4x-5}{x-5}$ b $\dfrac{x^2-25}{x-5}$

c $\dfrac{x^3-125}{x-5}$ d $\dfrac{x^2-25}{(x-5)^2}$

3 The following functions are not defined at $x = 0$. Define them, if possible, so that each function is continuous at $x = 0$.

a $f(x) = \dfrac{x^2+x}{x}$ b $f(x) = x^2 + \dfrac{5}{x}$

c $f(x) = \dfrac{|x|}{x}$ d $f(x) = \dfrac{10+(6/x)}{5+(2/x)}$

4 Which of the following functions are continuous at $x = 0$? Sketch the graph in each case.

a $f(x) = x$, when $x \geqslant 0$
$\quad = 0$, when $x < 0$

b $f(x) = x$, when $x \geqslant 0$
$\quad = 1$, when $x < 0$

c $f(x) = x + 1$, when $x \geqslant 0$
$\quad = 0$, when $x < 0$

d $f(x) = 2^x$, when $x \geqslant 0$
$\quad = 1$, when $x < 0$

5 The function $f(x) = \dfrac{x^3+x^2-9x-9}{x^2-9}$ is undefined for two members of $\mathbb{R}$.

Find these two members of $\mathbb{R}$ and define $f(x)$ at each of these points, so that it becomes a continuous function.

Chapter 3

Differentiation (1)

Gradient function

3.1 Gradient of a curve

So far we have discussed the gradient of a straight line only. A student walking up the ramp AB (Fig. 3.1) is climbing a gradient of $\frac{2}{7}$.

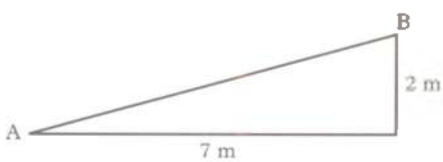


Figure 3.1

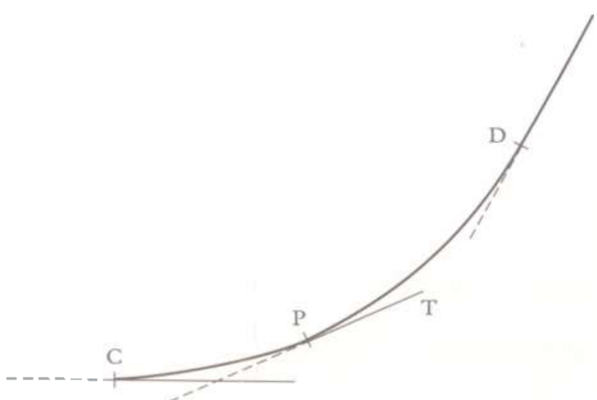


Figure 3.2

Let us now consider a student walking up the slope represented by the curve CPD (Fig. 3.2). Between C and D the gradient is steadily increasing. If, when she had reached the point P, the gradient had stopped increasing, and had remained constant from then on, she would have climbed up the slope represented by the straight line PT, the tangent to the curve at P. Thus in walking up the slope CD, on reaching the point P (and only at that point) the student will be climbing a gradient represented by the gradient of PT.

Definition

The gradient of a curve at any point is the gradient of the tangent to the curve at that point.

3.2 Gradient at a point

If we wish to find approximately the gradient of a curve at a certain point, we could draw the curve, draw the tangent at that point by eye, and measure its gradient. However, to develop our study of curves and their equations, it is important that we should discover a method of calculating exactly the gradient of a curve at any point. To do this we shall think of a tangent to a curve in the following way.

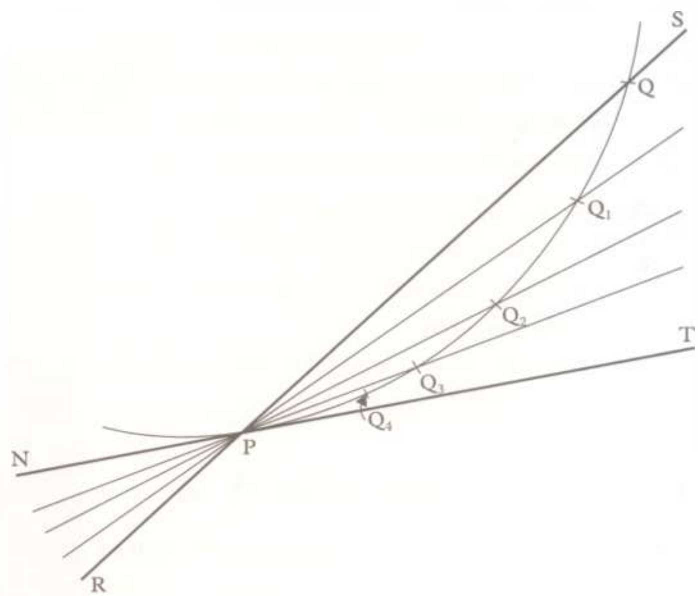


Figure 3.3

First we start with two distinct points on a curve, P and Q (Fig. 3.3), and the chord PQ is drawn and produced in both directions. Now consider RPQS as a straight rod hinged at P, which is rotated clockwise about P to take up successive positions shown by PQ_1, PQ_2, PQ_3, etc. Notice that the points at which it cuts the curve, Q_1, Q_2, Q_3, are successively nearer the fixed point P. The nearer this second point of intersection approaches P, the nearer the gradient of the chord approaches the gradient of the tangent NPT. *By taking* Q *sufficiently close to* P, *we can make the gradient of the chord* PQ *as near as we please to the gradient of the tangent at* P.

To see how this happens, place the edge of a ruler along RPQS and then rotate it clockwise about P. You will see the second point of intersection approach P along the curve, until it actually coincides with P when the ruler lies along the tangent NPT. Using an arrow to denote 'tends to' or 'approaches' we may write:

as Q $\to$ P along the curve,
the gradient of the chord PQ $\to$ the gradient of the tangent at P,

the tangent at P is called the **limit** of the chord PQ (or more exactly of the secant RPQS), and
the gradient of the curve at P is the limit of the gradient of the chord PQ.

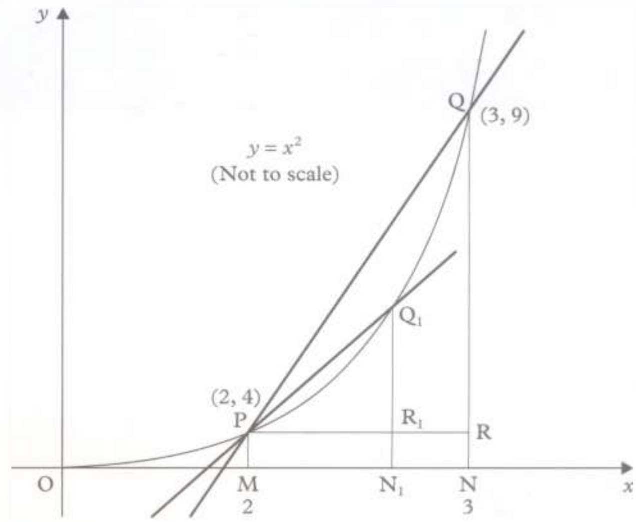


Figure 3.4

Questions

Q1 A regular polygon of n sides is inscribed in a circle.
What is the limit of the polygon as $n \to \infty$?

Q2 OP is a radius of a circle centre O. A straight line PQR cuts the circumference at Q.
What is the limit of the angle QPR as Q approaches P along the circumference?

Q3 P is a point on the straight line $y = \frac{1}{3}x$. Q is the foot of the perpendicular from P to the x-axis.
As P approaches O, the origin, what happens to PQ and QO?
What can you say about the value of PQ/QO?

The gradient of $y = x^2$ at (2, 4)

We shall use the idea of a tangent being the limit of a chord, to find the gradient of the curve $y = x^2$ at a particular point, namely (2, 4).

P is the point (2, 4) on the curve $y = x^2$ (**Fig. 3.4**). Q is another point on the curve, which we take first as (3, 9). Then, as the chord PQ rotates clockwise about P, Q moves along the curve to Q_1, and then gets nearer and nearer to P. By considering the gradient of PQ as this happens we use the following method to deduce the gradient of the tangent at P.

$$\begin{aligned}\text{The gradient of PQ} &= \frac{RQ}{PR} = \frac{RQ}{MN}\\ &= \frac{NQ - NR}{ON - OM}\\ &= \frac{9-4}{3-2} = 5\end{aligned}$$

If Q now moves to Q_1, whose coordinates are $(2\frac{1}{2}, 6\frac{1}{4})$,

$$\begin{aligned}\text{the gradient of PQ}_1 &= \frac{N_1Q_1 - N_1R_1}{ON_1 - OM}\\ &= \frac{6\frac{1}{4} - 4}{2\frac{1}{2} - 2}\\ &= \frac{2\frac{1}{4}}{\frac{1}{2}} = 4\frac{1}{2}\end{aligned}$$

Now let Q approach yet closer to P along the curve. The table below gives the gradient of the chord PQ as it approaches the gradient of the tangent at P.

ON (x-coord. of Q)	NQ (y-coord. of Q)	PR (ON – 2)	RQ (NQ – 4)	$\frac{RQ}{PR}$ Gradient of PQ
3	9	1	5	5
$2\frac{1}{2}$	$6\frac{1}{4}$	$\frac{1}{2}$	$2\frac{1}{4}$	$\frac{2\frac{1}{4}}{\frac{1}{2}} = 4\frac{1}{2}$
2.1	4.41	0.1	0.41	$\frac{0.41}{0.1} = 4.1$
2.01	4.0401	0.01	0.0401	$\frac{0.0401}{0.01} = 4.01$
2.001	4.004 001	0.001	0.004 001	$\frac{0.004\,001}{0.001} = 4.001$

Comparing the first and last columns of this table, we see that for each position of Q, the gradient of PQ exceeds 4 by the same amount as the x-coordinate of Q exceeds 2. The values we have taken so far suggest that by taking Q sufficiently near P (i.e. by taking the x-coordinate of Q sufficiently near 2) we can make the gradient of PQ as near 4 as we please (see §2.14 on page 57). This suggests that the limit of the gradient of PQ is 4, and that the gradient of the tangent at P is 4.

Questions

Q4 Draw a figure similar to **Fig. 3.4**, taking P as the point (1, 1). Taking the x-coordinate of Q successively as 2, $1\frac{1}{2}$, 1.1, 1.01, make out a table similar to the one on page 62. What appears to be the limit of the gradient of PQ in this case?

Q5 Add a last line to your table for Q4 by taking the x-coordinate of Q to be $1 + h$.
What happens to Q as $h \to 0$?
What happens to the gradient of PQ as $h \to 0$?
Deduce the gradient of $y = x^2$ at (1, 1).

Q6 Add a last line to the table on the previous page, taking the x-coordinate of Q as $(2 + h)$. Deduce the gradient of $y = x^2$ at (2, 4).

The gradient function of $y = x^2$

We now use the method suggested in Q5 to find the gradient of $y = x^2$ at *any* point.

P is the point (a, a^2), and Q is another point on the curve whose x-coordinate is $a + h$ (**Fig. 3.5**).

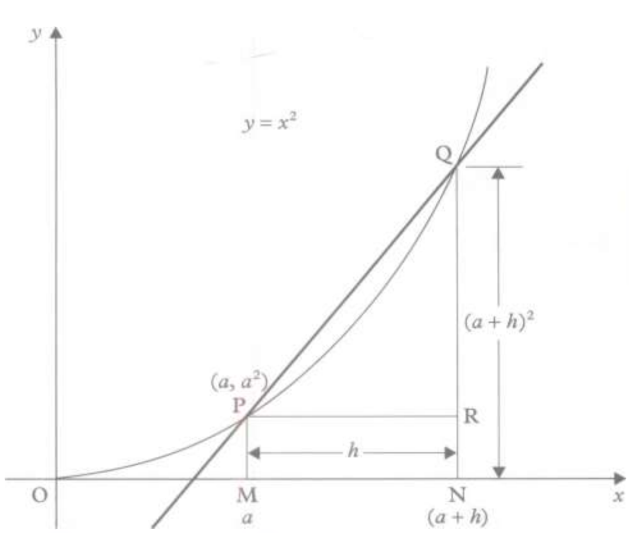


Figure 3.5

$$\begin{aligned} RQ &= NQ - NR \\ &= (a+h)^2 - a^2 \\ &= 2ah + h^2 \end{aligned}$$

and $PR = h$

The gradient of the chord PQ is

$$\frac{RQ}{PR} = \frac{2ah + h^2}{h} = 2a + h$$

If the chord rotates clockwise about P, Q approaches P along the curve, and the gradient of the chord PQ $\to$ the gradient of the tangent at P, and $h \to 0$.

But as $h \to 0$, the gradient of the chord PQ, $(2a + h) \to 2a$.

It follows that the gradient of the tangent at P is $2a$.

Thus the gradient of $y = x^2$ at (a, a^2) is $2a$, and since a is the x-coordinate of the point (a, a^2), the gradient of $y = x^2$ at (x, x^2) is $2x$.

Just as x^2 is the expression in which we substitute a value of x to find the corresponding y-coordinate and plot a point on the curve $y = x^2$, so we now have another expression, $2x$, in which we can substitute the value of x to find the gradient at that point.

$2x$ is called the **gradient function** of the curve $y = x^2$.

Example 1 *Find the coordinates of the points on the curve $y = x^2$, given by $x = 4$ and -10, and find the gradient of the curve at these points.*

$y = x^2$

When $x = 4$, $y = 4^2 = 16$.
The gradient function $= 2x$
$\therefore$ the gradient $= 8$, when $x = 4$
$\therefore$ the point is (4, 16), and the gradient is 8.

When $x = -10$, $y = x^2 = +100$.
The gradient function $= 2x = -20$
$\therefore$ the point is $(-10, 100)$, and the gradient is -20.

Questions

Q7 Calculate the gradients of the tangents to $y = x^2$ at the points given by $x = -1\frac{1}{2}, -1, +\frac{1}{2}, +2$.

Q8 Use the method of §3.4 on page 65 to find the gradient functions of the following curves, making a sketch in each case, and compare each result with the gradient function of $y = x^2$: **a** $y = 3x^2$, **b** $y = 5x^2$, **c** $y = \frac{1}{2}x^2$, **d** $y = cx^2$, where c is a constant, **e** $y = x^2 + 3$, **f** $y = x^2 + k$, where k is a constant.

A convenient way of writing the statement 'the gradient function of $y = x^2$ is $2x$' follows

'if $y = x^2$
grad $y = 2x$'

3.3 Differentiation

The process of finding the gradient function of a curve is known as **differentiation**. It is useful if we take 'grad' also to be an instruction to differentiate. Thus,

grad $(x^2) = 2x$

The differentiation of x^3

P is any point (a, a^3) on the curve $y = x^3$. Q is another point on the curve with x-coordinate $(a + h)$ (**Fig. 3.6**).

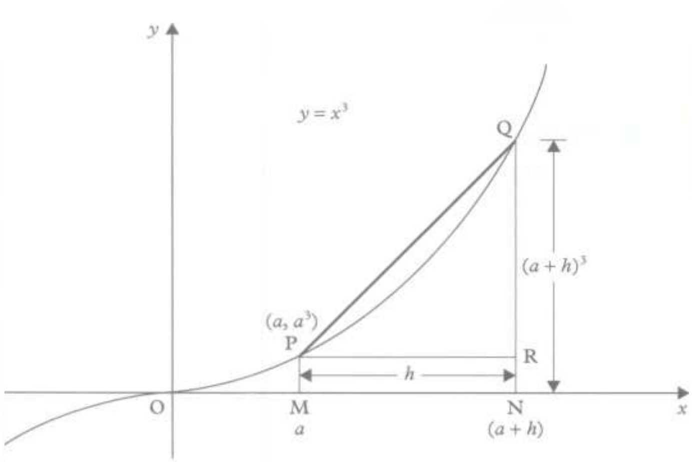


Figure 3.6

$$\begin{aligned} RQ &= NQ - NR \\ &= (a + h)^3 - a^3 \\ &= a^3 + 3a^2h + 3ah^2 + h^3 - a^3 \\ &= 3a^2h + 3ah^2 + h^3 \\ PR &= h \end{aligned}$$

$$\text{The gradient of PQ} = \frac{RQ}{PR} = \frac{3a^2h + 3ah^2 + h^3}{h} = 3a^2 + 3ah + h^2$$

As Q approaches P along the curve, $h \to 0$, and the terms $3ah$ and h^2 each tend to zero. Therefore the gradient of PQ $\to 3a^2$.

It follows that the gradient of $y = x^3$ at (a, a^3) is $3a^2$, or

grad $x^3 = 3x^2$

Questions

Q9 Use the above method to find grad x^4. [Hint: $(a + h)^4 = a^4 + 4a^3h + 6a^2h^2 + 4ah^3 + h^4$.]

Q10 Differentiate $2x^3$ by the same method.

Summary of results

We have now confirmed the following:

grad $x^2 = 2x$
grad $x^3 = 3x^2$
grad $x^4 = 4x^3$

These results suggest that the rule for differentiating a power of x is *multiply by the index, and reduce the index by 1*. This means that grad x^5, would be $5x^4$, grad x^6 would be $6x^5$, and so on.

At this stage we shall assume that

grad $x^n = nx^{n-1}$

when $n \in \mathbb{Z}^+$.

It is now time to link up these ideas with our earlier work on a straight line, and to extend them further.

$y = c$

Straight lines of this form, such as $y = 4$ and $y = -2$, are parallel to the x-axis, and have zero gradient. It follows that grad 4 = 0 and grad −2 = 0. Thus, *if we differentiate a constant we get* 0.

[Note that this agrees with the general result, grad $x^n = nx^{n-1}$. Since $x^0 = 1$ (see §P2.2 on page 8), we may write grad 4 = grad $4x^0 = 0 \times 4x^{-1} = 0$.]

$y = kx$, $y = kx^n$

We know that the straight line $y = mx + c$ has gradient m, e.g. $y = x$ has gradient 1, and $y = 3x$ has gradient 3. Thus

$$\text{grad } x = 1$$

[Again, this agrees with the general result, since grad $x^1 = 1 \times x^0 = 1$.] Also,

$$\text{grad } 3x = 3 \times \text{grad } x = 3 \times 1 = 3$$

and as Q8 showed,

$$\text{grad } 3x^2 = 3 \times \text{grad } x^2 = 3 \times 2x = 6x$$

This illustrates the general property that *if a function has a constant factor, that constant remains unchanged as a factor of the gradient function* (**Fig. 3.7**).

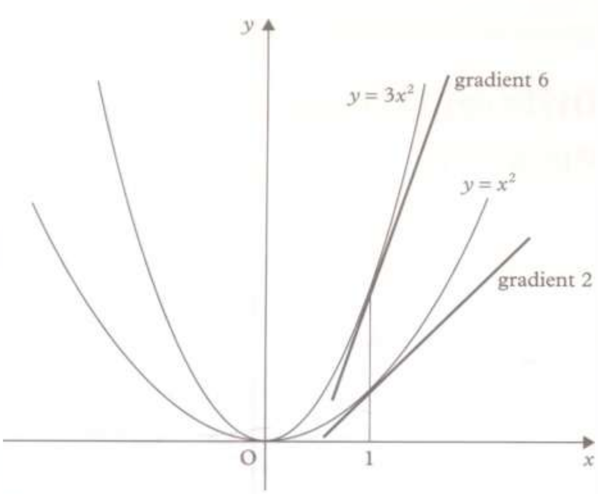


Figure 3.7

Question

Q11 Differentiate:

a $4x^3$

b $5x^4$

c ax^2

d $4x^n$

e Kx^{n+1}

3.4 Differentiation of a polynomial

So far we have differentiated functions of one term only. What happens if there are two or more terms?

$y = mx + c$

The straight lines $y = 3x$, $y = 3x + 4$, and $y = 3x - 2$ all have gradient 3. Thus

$$\begin{aligned} \text{grad } 3x &= 3 \\ \text{grad } (3x + 4) &= 3 \\ \text{grad } (3x - 2) &= 3 \end{aligned}$$

The above lines are parallel, and as we discovered in §1.7 on page 38, the effect of giving the different values $c = 0, +4$ and -2, is to raise or lower the line, but not to change its gradient.

Clearly the same applies to the curves $y = x^2$, $y = x^2 + 4$ and $y = x^2 - 2$ (**Fig. 3.8**). At the point on each curve for which $x = a$, the tangents are parallel, each having gradient $2a$.

$$\begin{aligned} \text{grad } x^2 &= 2x \\ \text{grad } (x^2 + 4) &= 2x \\ \text{grad } (x^2 - 2) &= 2x \end{aligned}$$

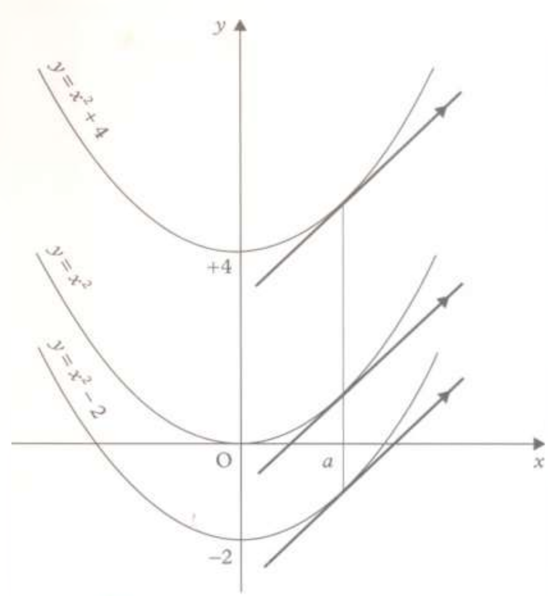


Figure 3.8

In the above cases where the function consists of two terms, we should get the same result by differentiating each term separately. Thus,

$$\begin{aligned} \text{grad } (x^2 + 4) &= \text{grad } x^2 + \text{grad } 4 \\ &= 2x + 0 \\ &= 2x \end{aligned}$$

This leads us to investigate whether this method is valid in general. Consider:

$y = x^2 + 3x - 2$

To find the gradient function of this curve, let P be any point $(a, a^2 + 3a - 2)$ on it. Q is another point on the curve with x-coordinate $(a + h)$ (Fig. 3.9).

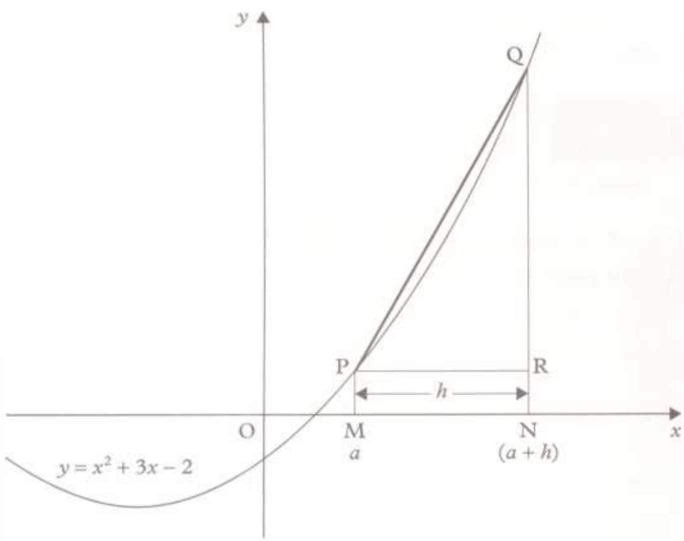


Figure 3.9

$$\begin{aligned} RQ &= NQ - NR \\ &= \{(a+h)^2 + 3(a+h) - 2\} - \{a^2 + 3a - 2\} \\ &= a^2 + 2ah + h^2 + 3a + 3h - 2 - a^2 - 3a + 2 \\ &= 2ah + h^2 + 3h \\ PR &= h \end{aligned}$$

$$\begin{aligned} \text{The gradient of PQ} &= \frac{RQ}{PR} \\ &= \frac{2ah + h^2 + 3h}{h} \\ &= 2a + h + 3 \end{aligned}$$

As Q approaches P along the curve, $h \to 0$ and the gradient of PQ $\to 2a + 3$.

It follows that the gradient of $y = x^2 + 3x - 2$ at $(a, a^2 + 3a - 2)$ is $2a + 3$, or

$$\text{grad } (x^2 + 3x - 2) = 2x + 3$$

This is the same as differentiating each term separately:

$$\begin{aligned} \text{grad } (x^2 + 3x - 2) &= \text{grad } x^2 + \text{grad } 3x + \text{grad } -2 \\ &= 2x + 3 + 0 \\ &= 2x + 3 \end{aligned}$$

This illustrates the general property that *the gradient function of the sum of a number of terms is obtained by differentiating each term separately.*

Question

Q12 Differentiate:

a $x^3 + 2x^2 + 3x$

b $4x^4 - 3x^2 + 5$

c $ax^2 + bx + c$

A special method of dealing with products and quotients will be met later. Meanwhile we must reduce a function in this form to the sum of a number of terms before differentiating. (Check that to differentiate each factor separately in the following examples does *not* lead to the correct result.)

$$\text{grad } \{x^2(2x + 3)\} = \text{grad } (2x^3 + 3x^2) = 6x^2 + 6x$$

$$\text{grad } \left\{\frac{x^3 + 4x^2}{x}\right\} = \text{grad } (x^2 + 4x) = 2x + 4$$

Question

Q13 Differentiate:

a $x^2(4x - 2)$

b $(x + 3)(x - 4)$

c $\dfrac{5x^3 + 3x^2}{x^2}$

3.5 Differentiation and function notation

In the preceding sections we have considered a variety of functions and we have found their corresponding gradient functions. The gradient function is often called the **derived function**, or **derivative**.

If we have a given function $f(x)$ it is very convenient to have a standard notation for its corresponding gradient function; the normal way of doing this is to write $f'(x)$. Thus if $f(x) = x^3 + 5x^2 + 3x - 7$ then we write its derivative $f'(x) = 3x^2 + 10x + 3$. Alternatively

$$\begin{aligned} f&: x \mapsto x^3 + 5x^2 + 3x - 7 \\ f'&: x \mapsto 3x^2 + 10x + 3 \end{aligned}$$

The process of finding the derived functions in the case of $f(x) = x^2$ and $f(x) = x^3$, has been written out in full in §3.2 on page 61. The general case follows below:

Fig. 3.10 shows the graph of a general function $y = f(x)$. M and N are the points $(a, 0)$ and $(a + h, 0)$ respectively. P and Q are the points on the curve given by $x = a$ and $x = a + h$. So $MP = f(a)$ and $NQ = f(a + h)$.

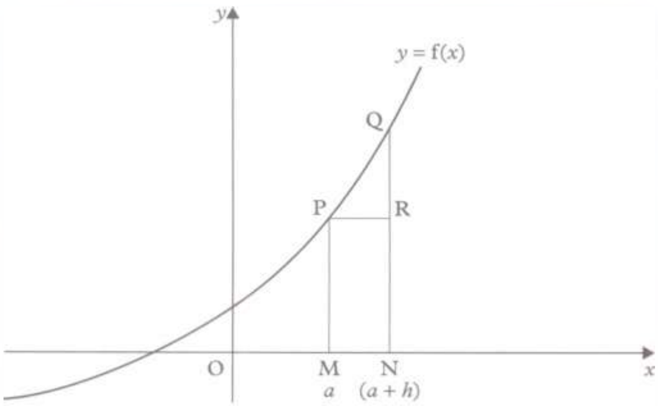


Figure 3.10

$$\begin{aligned}RQ &= NQ - NR\\ &= NQ - MP\\ &= f(a+h) - f(a)\end{aligned}$$

The gradient of PQ

$$= \frac{RQ}{PR}$$

$$= \frac{f(a+h) - f(a)}{h}$$

Hence

the gradient of the tangent at P $= \lim\limits_{h\to 0} [f(a+h) - f(a)]/h$,

and the derived function $f'(x)$ is given by

$$\mathbf{f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}} \qquad (1)$$

In saying this, we assume that this limit exists and that it is the same whether h tends to zero from above or from below (see §2.14 on page 57).

If you ever need to differentiate a given function **from first principles**, you should start the proof by using the formula marked (1).

Example 2 *Find, from first principles, the derivative of the function $f(t) = kt^4$, where k is a constant.*

$$f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}$$

$$f(t+h) = k(t+h)^4 = k(t^4 + 4t^3h + 6t^2h^2 + 4th^3 + h^4)$$

$$\begin{aligned}\therefore\ f(t+h) - f(t) &= kt^4 + 4kt^3h + 6kt^2h^2 + 4kth^3 + kh^4 - kt^4\\ &= 4kt^3h + 6kt^2h^2 + 4kth^3 + kh^4\end{aligned}$$

$$\frac{f(t+h) - f(t)}{h} = 4kt^3 + 6kt^2h + 4kth^2 + kh^3$$

and hence

$$f'(t) = \lim_{h\to 0} (4kt^3 + 6kt^2h + 4kth^2 + kh^3) = 4kt^3$$

Example 3 *Find from first principles, $f'(x)$ when $f(x) = |x|$.*

$$\begin{aligned}f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\\ &= \lim_{h\to 0} \frac{|x+h| - |x|}{h}\end{aligned}$$

Now if x and $x + h$ are both positive, then $|x+h| = x + h$ and $|x| = x$. Consequently

$$\begin{aligned}f'(x) &= \lim_{h\to 0} \frac{x+h-x}{h}\\ &= \lim_{h\to 0} \left(\frac{h}{h}\right)\\ &= +1\end{aligned}$$

But, if x and $x + h$ are both negative, $|x+h| = -(x+h)$ and $|x| = -x$. In this case

$$\begin{aligned}f'(x) &= \lim_{h\to 0} \left(\frac{-x-h-(-x)}{h}\right)\\ &= \lim_{h\to 0} \left(\frac{-h}{h}\right)\\ &= -1\end{aligned}$$

The remaining case, namely $f'(0)$, is rather tricky!

$$\begin{aligned}f'(0) &= \lim_{h\to 0} \frac{|0+h| - 0}{h}\\ &= \lim_{h\to 0} \left(\frac{|h|}{h}\right)\end{aligned}$$

But $|h|/h = +1$ if $h > 0$, or -1 if $h < 0$. Consequently, the limit as $h \to 0$ from above is $+1$, but it is -1 when h tends to 0 from below. Hence $f'(0)$ cannot be found. This may seem rather strange, but it makes sense when we consider the graph of $y = |x|$ (**Fig. 3.11**).

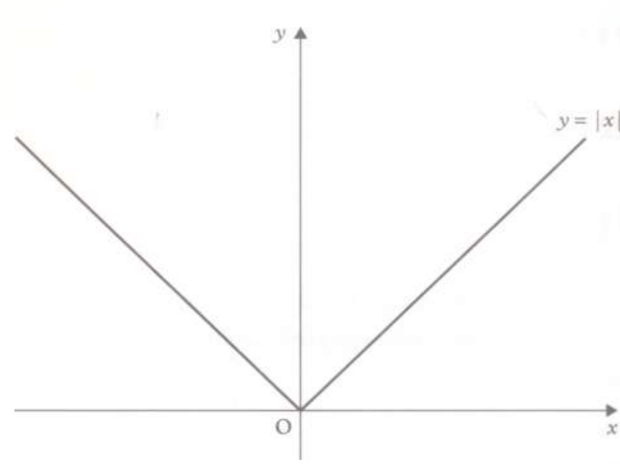


Figure 3.11

It is clear from the graph that when $x > 0$, the gradient is $+1$, while if $x < 0$ the gradient is -1. At $x = 0$, however, the graph comes to a point and its gradient here is undefined.

Exercise 3a

Write down the gradient functions of the following curves:

1 $y = x^{12}$ **2** $y = 3x^7$ **3** $y = 5x$

4 $y = 5x + 3$ **5** $y = 3$ **6** $y = 5x^2 - 3x$

Write down the derived function $f'(x)$, for each of the following functions:

7 $f(x) = 3x^4 - 2x^3 + x^2 - x + 10$

8 $f(x) = 2x^4 + \frac{1}{3}x^3 - \frac{1}{4}x^2 + 2$

9 $f(x) = ax^3 + bx^2 + cx$

10 $f(x) = 2x(3x^2 - 4)$

11 $f(x) = \dfrac{10x^5 + 3x^4}{2x^2}$

Differentiate the following functions:

12 $-x$ **13** $+10$

14 $4x^3 - 3x + 2$ **15** $\frac{1}{2}ax^2 - 2bx + c$

16 $2(x^2 + x)$ **17** $3x(x - 1)$

18 $\frac{1}{3}(x^3 - 3x + 6)$ **19** $(x + 1)(x - 2)$

Find the derivatives of the following functions:

20 $f: x \mapsto 3(x + 1)(x - 1)$

21 $f: x \mapsto \dfrac{(x + 3)(2x + 1)}{4}$

22 $f: x \mapsto \dfrac{2x^3 - x^2}{3x}$

23 $f: x \mapsto \dfrac{x^4 + 3x^2}{2x^2}$

Find the y-coordinate, and the gradient, at the points on the following curves corresponding to the given values of x:

24 $y = x^2 - 2x + 1$, $x = 2$

25 $y = x^2 + x + 1$, $x = 0$

26 $y = x^2 - 2x$, $x = -1$

27 $y = (x + 2)(x - 4)$, $x = 3$

28 $y = 3x^2 - 2x^3$, $x = -2$

29 $y = (4x - 5)^2$, $x = \frac{1}{2}$

Find the coordinates of the points on the following curves at which the gradient has the given values:

30 $y = x^2$; 8

31 $y = x^3$; 12

32 $y = x(2 - x)$; 2

33 $y = x^2 - 3x + 1$; 0

34 $y = x^3 - 2x + 7$; 1

35 $y = x^3 - 6x^2 + 4$; -12

36 $y = x^4 - 2x^3 + 1$; 0

37 $y = x^2 - x^3$; -1

38 $y = x(x - 3)^2$; 0

3.6 Tangents and normals

Definition

A normal to a curve at a point is the straight line through the point at right angles to the tangent at the point (**Fig. 3.12**).

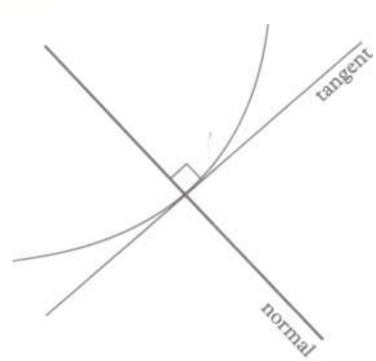


Figure 3.12

We are now able to find the equations of tangents and normals.

Example 4 *Find the equation of the tangent to the curve $y = x^3$ at the point (2, 8).*

$$y = x^3$$
$$\therefore \text{grad } y = 3x^2$$

When $x = 2$,

$$\text{grad } y = 12$$

Thus the gradient of the tangent at (2, 8) is +12. Its equation is

$$\frac{y-8}{x-2} = 12$$
$$\therefore y - 8 = 12x - 24$$

$\therefore$ the equation of the tangent is $12x - y - 16 = 0$

We can generalise Example 4 as shown in Example 5.

Example 5 *Find the equation of the tangent to the curve $y = \mathrm{f}(x)$ at the point (a, b).*

Putting $x = a$ in the equation gives

$$b = \mathrm{f}(a)$$

The gradient at the given point is obtained by differentiating and putting $x = a$.

Hence the gradient required is $\mathrm{f}'(a)$.

The equation of the tangent has the form

$$\frac{y-b}{x-a} = m \quad \text{where } m \text{ is the gradient.}$$

Hence the equation of the tangent is

$$y - \mathrm{f}(a) = \mathrm{f}'(a)(x - a)$$

Example 6 *Find the equation of the normal to the curve $y = (x^2 + x + 1)(x - 3)$ at the point where it cuts the x-axis.*

$$y = (x^2 + x + 1)(x - 3)$$

When $y = 0$,

$$(x^2 + x + 1)(x - 3) = 0$$

But $x^2 + x + 1 = 0$ has no real roots,

$\therefore x = +3$

$\therefore$ the curve cuts the x-axis at (3, 0)

$$y = x^3 - 2x^2 - 2x - 3$$
$$\therefore \text{grad } y = 3x^2 - 4x - 2$$

When $x = 3$,

$$\text{grad } y = 27 - 12 - 2 = 13$$

The gradient of the tangent at (3, 0) is +13. Therefore the gradient of the normal at (3, 0) is $-\frac{1}{13}$ (see §1.5 on page 36) and its equation is

$$\frac{y-0}{x-3} = -\frac{1}{13}$$
$$\therefore 13y = -x + 3$$

$\therefore$ the equation of the normal is $x + 13y - 3 = 0$

Exercise 3b

1 Find the equations of the tangents to the following curves at the points corresponding to the given values of x:

a $y = x^2, x = 2$
b $y = 3x^2 + 2, x = 4$
c $y = 3x^2 - x + 1, x = 0$
d $y = 3 - 4x - 2x^2, x = 1$
e $y = 9x - x^3, x = -3$

2 Find the equations of the normals to the curves in question 1 at the given points.

3 Find the equation of the tangent and the normal to the curve $y = x^2(x - 3)$ at the point where it cuts the x-axis. Sketch the curve.

4 Repeat question 3 for the curve $y = x(x - 4)^2$.

5 Find the equation of the tangent to the curve $y = 3x^3 - 4x^2 + 2x - 10$ at the point of intersection with the y-axis.

6 Repeat question 5 for the curve $y = x^2 - 4x + 3$.

7 Find the values of x for which the gradient function of the curve

$$y = 2x^3 + 3x^2 - 12x + 3$$

is zero. Hence find the equations of the tangents to the curve which are parallel to the x-axis.

8 Repeat question 7 for the curve

$$y = 2x^3 - 9x^2 + 10.$$

Chapter 4

Applications of differentiation (1)

Velocity and acceleration

4.1 Gradient and velocity

You will already have met 'travel graphs'. One such graph is shown in Fig. 4.1, representing a man walking to see his mother who lives 5 km away, staying 2 hours, and then returning home.

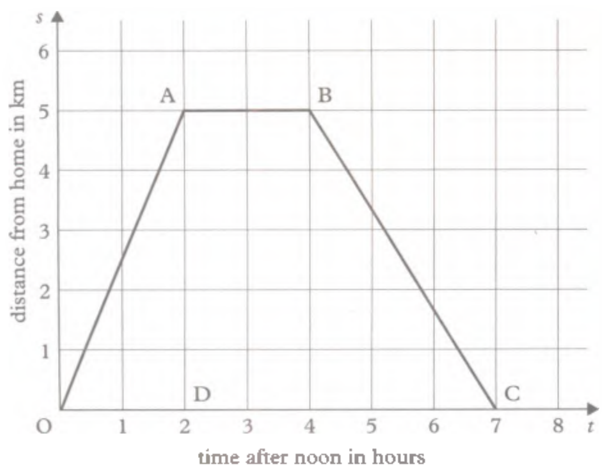


Figure 4.1

On his outward journey, represented by OA, he travels 5 km in 2 hours. His velocity, $\frac{5}{2}$ km/h or $\frac{5}{2}$ kmh^{-1}, is represented by DA/OD, the gradient of OA.

Whilst with his mother his velocity is zero; this is represented by the gradient of AB.

On his return journey, the gradient of BC gives his velocity as $-\frac{5}{3}$ km/h. The negative sign denotes that he is now travelling in the opposite direction; he is *decreasing* the distance from home.

This type of graph in which the distance, s, is plotted against the time, t, is called a **distance-time graph**.

Variable velocity

When velocity is variable, as in a car journey, we may be concerned with the average velocity, which we define as follows.

Definition

Average velocity is $\dfrac{\textit{total distance travelled}}{\textit{total time taken}}$ *or* $\dfrac{\textit{increase in } s}{\textit{increase in } t}$.

When the speed of a car changes, the speedometer shows the speed *at any instant*. We must therefore deal with the idea of *velocity at an instant*.

Suppose that a car, starting from rest, increases its velocity steadily up to 80 km/h. Then the distance-time graph is similar to the curve OPQ in Fig. 4.2. The point P corresponds to the instant at which the speedometer needle reaches the 60 km/h mark. If from that instant onward the velocity had been kept constant at 60 km/h, then the distance-time graph would have consisted of the curve OP and the straight line PT of gradient 60.

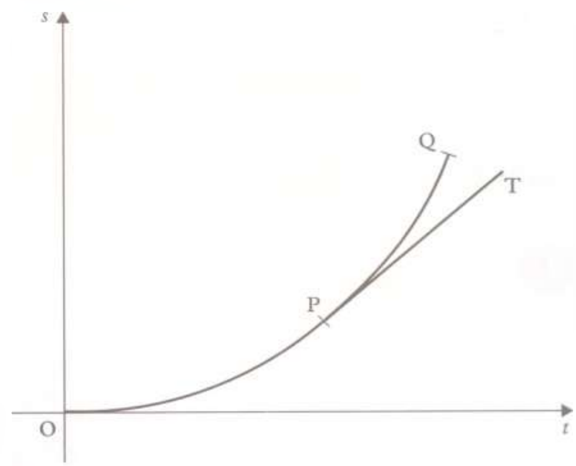


Figure 4.2

It would appear that PT is the tangent at P to the original distance-time curve OPQ (like cotton under tension leading off a reel). In that case its gradient would be the same as the gradient of the curve OPQ at P. This suggests that, when the velocity is variable, the velocity at an instant represented by the gradient of the distance-time curve at the corresponding point. However, we must find a precise definition.

Velocity at an instant

Consider a stone falling from rest, its velocity steadily increasing. Under certain conditions it will be s m below its starting point t seconds after the start, where s is given by the formula $s = 4.9t^2$. From this we may make a table of values giving the position of the stone at different times.*

Value of t	0	0.5	1.0	1.5	2.0	2.5	3.0
Value of s	0	1.2	4.9	11.0	19.6	30.6	44.1

*Throughout §4.1, including Q1 to 5, we work to one decimal place.

Fig. 4.3 shows part of the distance-time graph.

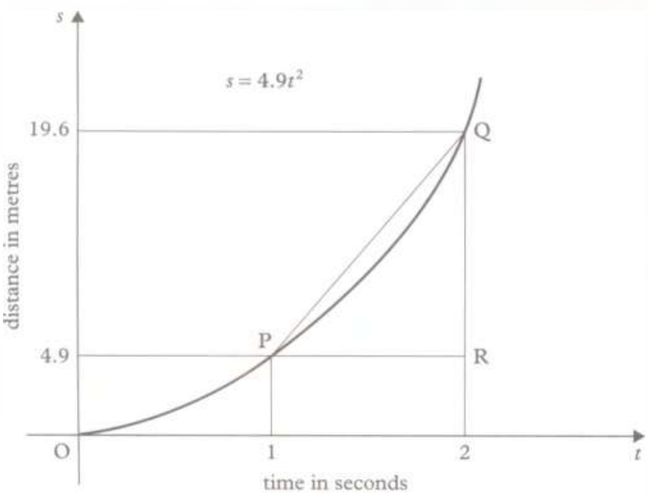


Figure 4.3

From $t = 1$ to $t = 2$, the *average velocity* is represented by the gradient of the chord PQ.

$$\frac{RQ}{PR} = \frac{19.6 - 4.9}{2 - 1} = 14.7$$

$\therefore$ the average velocity is 14.7 m/s (or 14.7 ms^{-1}).

Questions

Q1 How far does the stone move in the interval $t = 1$ to $t = 1.5$? What is the average velocity during this interval?

Q2 Repeat Q1 for the intervals **a** $t = 1$ to $t = 1.1$, and **b** $t = 1$ to $t = 1 + h$.

The smaller we make the time interval (letting Q $\to$ P along the curve), the nearer the average velocity (the gradient of PQ) approaches the velocity given by the gradient of the curve at P.

We have seen that the gradient of the curve at P is the limit of the gradient of PQ as Q $\to$ P (§3.2 on page 61). This leads to the following definition.

Definition

The velocity at an instant is the limit of the average velocity for an interval following that instant, as the interval tends to zero.

Questions

Q3 From your answer to Q2**b** determine the actual velocity at the instant when $t = 1$.

Q4 Calculate the distance moved, and the average velocity during the following intervals:

a $t = 2$ to $t = 3$

b $t = 2$ to $t = 2.5$

c $t = 2$ to $t = 2.1$

d $t = 2$ to $t = 2 + h$

Find the velocity when $t = 2$.

The definition given above identifies the velocity at an instant with the gradient of the space-time graph for the corresponding value of t. If we are given s in terms of t we can therefore find an expression for the velocity of the stone at any instant by differentiation, that is, if $s = f(t)$, then the velocity v is given by

$$v = f'(t)$$

In the above case, $f(t) = 4.9t^2$ and so the velocity, v m/s, is given by

$$v = f'(t) = 9.8t$$

Thus when $t = 0$, $v = 0$,
when $t = 1$, $v = 9.8$,
when $t = 2$, $v = 19.6$, etc.

Question

Q5 A stone is thrown vertically downwards from the top of a cliff, and the depth below the top, s m, after t s, is given by the formula $s = 2t + 4.9t^2$.

a Where is the stone after 1, 2, 3, 4 s?

b What is its velocity at these times?

c What is its average velocity during the 3rd second (from $t = 2$ to $t = 3$)?

4.2 The symbols δs and δt

The idea of gradient helped us to arrive at the definition of velocity at an instant. Now, without reference to graphical ideas, we shall again demonstrate that velocity is found by differentiating the expression for s in terms of t. To do this we will introduce some important new symbols.

Again consider the stone which falls s metres from rest in t seconds. Suppose that it falls a further small distance δs metres in the additional small interval of time δt seconds.

[The symbol δt, read as 'delta t', is used to denote a small increase, or *increment*, in time. Note that δt is a single symbol; it does not mean δ multiplied by t. Similarly δs is the corresponding *increment* in distance.]

The average velocity for the time interval δt (i.e. from t to $t + \delta t$) is $\delta s/\delta t$ m/s. We now obtain an expression for this in terms of t.

Since the stone falls $(s + \delta s)$ metres in $(t + \delta t)$ seconds

$$s + \delta s = 4.9(t + \delta t)^2$$

i.e. $\quad s + \delta s = 4.9t^2 + 9.8t \times \delta t + 4.9 \times (\delta t)^2$

But $\quad s = 4.9t^2$

and subtracting,

$$\delta s = 9.8t \times \delta t + 4.9 \times (\delta t)^2$$

To find the average velocity between time t and time $(t + \delta t)$ we divide each side by δt, giving

$$\frac{\delta s}{\delta t} = 9.8t + 4.9 \times \delta t$$

As $\delta t \to 0$ the R.H.S. $\to 9.8t$.

By the definition of velocity at an instant, the velocity, v m/s, at time t, is the limit of $\delta s/\delta t$ as $\delta t \to 0$, hence

$$v = 9.8t$$

The fact that this process is identical with that of finding the gradient function of $s = 4.9t^2$ can be seen from **Fig. 4.4**.

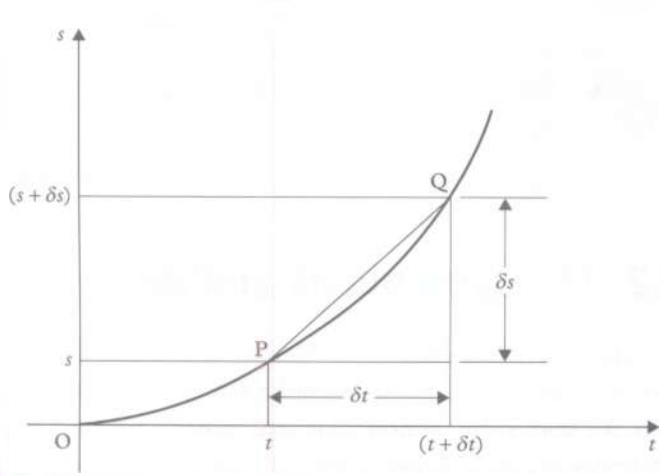


Figure 4.4

Exercise 4a

1 A stone is thrown vertically upwards at 35 m/s. It is s m above the point of projection t s later, where $s = 35t - 4.9t^2$.

 a What is the distance moved, and the average velocity during the 3rd second (from $t = 2$ to $t = 3$)?

 b Find the average velocities for the intervals $t = 2$ to $t = 2.5$, $t = 2$ to $t = 2.1$, $t = 2$ to $t = 2 + h$.

 c Find the actual velocity at the end of the 2nd second.

2 A stone is thrown vertically upwards at 24.5 ms^{-1} from a point on the level with but just beyond a cliff ledge. Its height above the ledge t s later is $4.9t(5 - t)$ m. If its velocity is v m/s, differentiate to find v in terms of t.

 a When is the stone at the ledge level?

 b Find its height and velocity after 1, 2, 3, and 6 s.

 c What meaning is attached to a negative value of s? A negative value of v?

 d When is the stone momentarily at rest? What is its greatest height?

 e Find the total distance moved during the 3rd second.

3 A particle moves along a straight line so that it is s m from a fixed point O on the line t s after a given instant, where $s = 3t + t^2$. After $(t + \delta t)$ s it is $(s + \delta s)$ m from O. Find the average velocity during the time interval t to $(t + \delta t)$ as was done opposite, and deduce an expression for the velocity v m/s, at time t. Check by differentiation.

 a Where is the particle and what is its velocity at the instant from which time is measured (i.e. when $t = 0$)?

 b When is the particle at O?

 c When is the particle momentarily at rest? Where is it then?

 d What is the velocity the first time the particle is at O?

4 A particle moves along a straight line OA in such a way that it is s m from O t s after the instant from which time is measured, where $s = 6t - t^2$. A is to be taken as being on the positive side of O.

 a Where is the particle when $t = 0, 2, 3, 4, 6, 7$? What is the meaning of a negative value of s?

 b Differentiate the given expression to find the velocity, v m/s, in terms of t. Find the value of

v when $t = 0, 2, 4, 6$. What is the meaning of a negative value of v?

c When and where does the particle change its direction of motion?

5 A slow train which stops at every station passes a signal box at noon. Its motion between the two adjacent stations is such that it is s km past the signal box t min past noon, where $s = \frac{1}{3}t + \frac{1}{9}t^2 - \frac{1}{27}t^3$. Find

a the time of departure from the first station, and the time of arrival at the second,

b the distance of each station from the signal box,

c the average velocity between the stations,

d the velocity with which the train passes the signal box.

6 Repeat question 5 in the case where $s = \frac{1}{72}t(36 - 3t - 2t^2)$.

7 A stone is thrown vertically downwards at $19.6\ \text{ms}^{-1}$ from the top of a tower 24.5 m high. It is s m below the top after t s, where $s = 19.6t + 4.9t^2$. Calculate the velocity with which it strikes the ground below.

4.3 Acceleration

Constant acceleration

Earlier in this chapter we used the formula $s = 4.9t^2$ for a stone falling from rest. On differentiation $v = \text{grad } s = 9.8t$. The stone's velocity is 9.8, 19.6, 29.4, 39.2 … m/s at the end of successive seconds, and it is steadily increasing by 9.8 m/s in each second. This *rate* at which the stone's velocity increases is called its **acceleration**. This formula is based on the assumption that gravity is producing a *constant* acceleration of 9.8 m per second per second, written usually as $9.8\ \text{m/s}^2$ or $9.8\ \text{ms}^{-2}$.

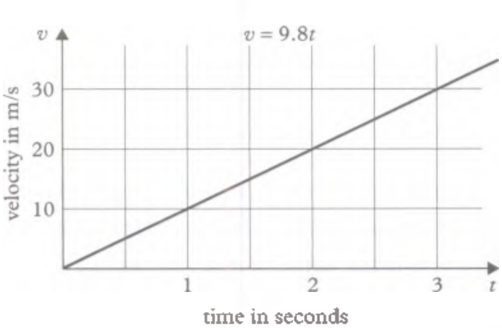


Figure 4.5

Fig. 4.5 shows the corresponding **velocity-time graph**. The equation $v = 9.8t$ (in the form $y = mx$) represents graphically a straight line through the origin of gradient 9.8. In this case the acceleration is represented by the gradient of the velocity-time graph.

Question

Q6 A stone is thrown vertically downwards with a velocity of 10 m/s, and gravity produces on it an acceleration of $9.8\ \text{m/s}^2$.

a What is the velocity after 1, 2, 3, t s?

b Sketch the velocity-time graph.

If a particle has an initial velocity u m/s and a constant acceleration $a\ \text{m/s}^2$, then its velocity after t s is $(u + at)$ m/s and the equation $v = u + at$ (being of the form $y = mx + c$) represents a straight line of gradient a.

Thus when acceleration is constant, it is represented by the *gradient* of the straight-line *velocity-time graph*.

Exercise 4b

In this exercise, acceleration is constant.

1 At the start and end of a two-second interval, a particle's velocity is observed to be 5, $10\ \text{ms}^{-1}$. What is its acceleration?

2 A body starts with velocity 15 m/s, and at the end of the 11th second its velocity is 48 m/s. What is its acceleration?

3 Express an acceleration of $5\ \text{m/s}^2$ in **a** km/h per s, **b** km/h^2.

4 A car accelerates from 5 km/h to 41 km/h in 10 s. Express this acceleration in **a** km/h per s, **b** m/s^2, **c** km/h^2.

5 A car can accelerate at $4\ \text{ms}^{-2}$. How long will it take to reach 90 km/h from rest?

6 Sketch the velocity-time curve for a cyclist who, starting from rest, reaches 3 m/s in 5 s, travels at that speed for 20 s, and then comes to rest in a further 2 s. What is his acceleration when braking? What is the gradient of the corresponding part of the graph?

7 An express train reducing its velocity to $40\ \text{kmh}^{-1}$, has to apply the brakes for 50 s. If the retardation produced is $0.5\ \text{ms}^{-2}$, find its initial velocity in kmh^{-1}.

*Note: **retardation** in the rate of **decrease** of velocity. Think of it as negative acceleration.*

Variable acceleration

A car starts from rest and moves a distance s m in t seconds, where $s = \frac{1}{6}t^3 + \frac{1}{4}t^2$. If its velocity after t s is v m/s, then $v = \text{grad } s = \frac{1}{2}t^2 + \frac{1}{2}t$. The following table gives some corresponding values of v and t:

t	0	1	2	3	4
v	0	1	3	6	10

The increases in velocity during the first four seconds are 1 m/s, 2 m/s, 3 m/s, 4 m/s respectively. Since the rate of increase of the velocity is not constant in this case, we shall first investigate the average rate of increase over a given time interval.

Definition

Average acceleration is $\dfrac{\textit{increase in } v}{\textit{increase in } t}$.

Thus from $t = 0$ to $t = 2$,

the average acceleration $= \dfrac{3-0}{2} = 1\frac{1}{2}$ m/s^2

and from $t = 2$ to $t = 4$,

the average acceleration $= \dfrac{10-3}{2} = 3\frac{1}{2}$ m/s^2.

Clearly the acceleration itself is increasing with the time, and the next step is to define what is meant by the acceleration at an instant.

Definition

The acceleration at an instant is the limit of the average acceleration for an interval following that instant, as the interval tends to zero.

Using the notation of §4.2 on page 71, if δv is the small increase in velocity which occurs in time δt, then the average acceleration for that interval is $\delta v/\delta t$, and the acceleration at time t is the limit of this as $\delta t \to 0$.

Reference to the velocity-time graph given in Fig. 4.6 shows that the average acceleration $\delta v/\delta t$ is the gradient of the chord PQ, and the limit is the gradient of the graph at P.

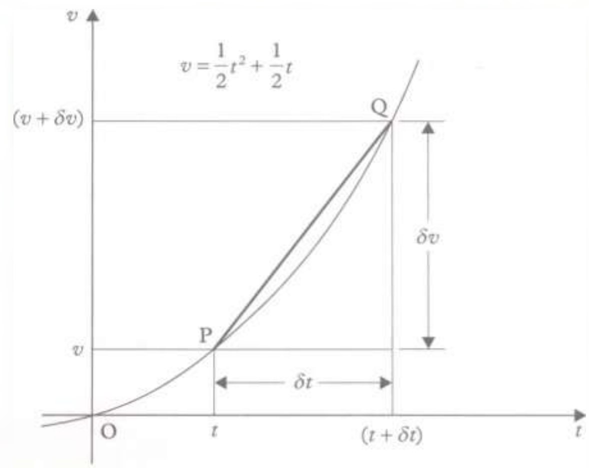


Figure 4.6

Thus an expression for the acceleration at time t may be found by differentiating the expression for v, that is, if $v = g(t)$, then a the acceleration is given by $a = g'(t)$.

Notice that if we start with the distance given by $s = f(t)$, then we differentiate once to obtain the velocity v and we differentiate again to find the acceleration a. We are already familiar with the symbol $f'(t)$ for the derivative of $f(t)$; when this in turn is differentiated we write $f''(t)$. Thus we can sum up the preceding statement as follows:

$s = f(t)$
$v = f'(t)$ (said 'f dash t')
$a = f''(t)$ (said 'f double dash t')

Example 1 *A car starts from rest and moves a distance s m in t s, where $s = \frac{1}{6}t^3 + \frac{1}{4}t^2$. What is the initial acceleration, and the acceleration at the end of the 2nd second?*

$$s = f(t) = \tfrac{1}{6}t^3 + \tfrac{1}{4}t^2$$
$$v = f'(t) = \tfrac{1}{2}t^2 + \tfrac{1}{2}t$$
$$a = f''(t) = t + \tfrac{1}{2}$$

When $t = 0$, $a = \frac{1}{2}$ and when $t = 2$, $a = 2\frac{1}{2}$.

Hence the required accelerations are $\frac{1}{2}$ m/s^2, and $2\frac{1}{2}$ m/s^2.

Before reading Example 2 refer again to the definitions of *average velocity* and *average acceleration*. In particular note that

a average velocity is not the same as the average of the initial and final velocities (unless the acceleration is constant); and

b average acceleration is not necessarily the same as the average of the initial and final accelerations.

Example 2 *A particle moves along a straight line in such a way that its distance from a fixed point O on the line after t s is s m, where $s = \frac{1}{6}t^4$. Find* **a** *its velocity after 3 s, and after 4 s,* **b** *its average velocity during the 4th second,* **c** *its acceleration after 2 s, and after 4 s, and* **d** *its average acceleration from $t = 2$ to $t = 4$.*

$$s = f(t) = \tfrac{1}{6}t^4$$
$$v = f'(t) = \tfrac{2}{3}t^3$$
$$a = f''(t) = 2t^2$$

a When $t = 3$, $v = \frac{2}{3} \times 3^3 = 18\ \text{ms}^{-1}$ and when $t = 4$, $v = \frac{2}{3} \times 4^3 = 42\frac{2}{3}\ \text{ms}^{-1}$.

Hence after 3 s and 4 s, the velocity is $18\ \text{ms}^{-1}$ and $42\frac{2}{3}\ \text{ms}^{-1}$ respectively.

b When $t = 3$, $s = \frac{81}{6} = 13\frac{1}{2}$ m and when $t = 4$, $s = \frac{256}{6} = 42\frac{2}{3}$ m.

$\therefore$ the average velocity during the 4th second is

$$\frac{42\frac{2}{3} - 13\frac{1}{2}}{1} = 29\tfrac{1}{6}\ \text{ms}^{-1}$$

c When $t = 2$, $a = 2 \times 2^2 = 8\ \text{ms}^{-2}$ and when $t = 4$, $a = 2 \times 4^2 = 32\ \text{ms}^{-2}$.

d When $t = 2$, $v = \frac{2}{3} \times 2^3 = 5\frac{1}{3}\ \text{ms}^{-1}$ and when $t = 4$, $v = \frac{2}{3} \times 4^3 = 42\frac{2}{3}\ \text{ms}^{-1}$.

The change in velocity $= 37\frac{1}{3}\ \text{ms}^{-1}$.

$\therefore$ the average acceleration from $t = 2$ to $t = 4$ is

$$\frac{37\frac{1}{3}}{2}\ \text{ms}^{-2} = 18\tfrac{2}{3}\ \text{ms}^{-2}$$

Exercise 4c

1 A stone is thrown vertically upwards, and after t s its height is h m, where $h = 10.5t - 4.9t^2$. Determine, with particular attention to the signs, the height, velocity and acceleration of the stone **a** when $t = 1$, **b** when $t = 2$, and **c** when $t = 3$. Also state clearly in each case whether the stone is going up or down, and whether its speed is increasing or decreasing.

2 A stone is thrown downwards from the top of a cliff, and after t s it is s m below the top, where $s = 20t + 4.9t^2$. Find how far it has fallen, its velocity, and its acceleration at the end of the first second.

3 A ball is thrown vertically upwards and its height after t s is s m where $s = 25.2t - 4.9t^2$. Find
- **a** its height and velocity after 3 s,
- **b** when it is momentarily at rest,
- **c** the greatest height reached,
- **d** the distance moved in the 3rd second,
- **e** the acceleration when $t = 2\frac{4}{7}$.

4 A particle moves in a straight line so that after t s it is s m from a fixed point O on the line, where $s = t^4 + 3t^2$. Find
- **a** the acceleration when $t = 1$, $t = 2$, and $t = 3$,
- **b** the average acceleration between $t = 1$ and $t = 3$.

5 At the instant from which time is measured a particle is passing through O and travelling towards A, along the straight line OA. It is s m from O after t s where $s = t(t - 2)^2$.
- **a** When is it again at O?
- **b** When and where is it momentarily at rest?
- **c** What is the particle's greatest displacement from O, and how far does it move during the first 2 s?
- **d** What is the average velocity during the 3rd second?
- **e** At the end of the 1st second where is the particle, which way is it going, and is its speed increasing or decreasing?

6 Repeat question 5e for the instant when $t = -1$.

7 A particle moves along a straight line so that after t s, its distance from O a fixed point on the line is s m where $s = t^3 - 3t^2 + 2t$.
- **a** When is the particle at O?
- **b** What is its velocity and acceleration at these times?
- **c** What is its average velocity during the 1st second?
- **d** What is its average acceleration between $t = 0$ and $t = 2$?

Chapter 5

Applications of differentiation (2)

Maxima and minima

5.1 The symbols δx, δy and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

In Chapter 4 we met the symbols δs and δt. To extend the scope of differentiation we denote small increases in x and y as δx and δy in the same way. If P is the point (x, y) on a curve, and Q is another point, and if the increase in x in moving from P to Q is δx, then the corresponding increase in y is δy; thus Q is the point $(x + \delta x, y + \delta y)$ (**Fig. 5.1**).

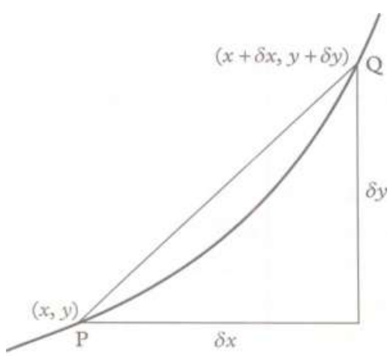


Figure 5.1

The gradient of the chord PQ is $\dfrac{\delta y}{\delta x}$, and the gradient of the curve at P is the limit of $\dfrac{\delta y}{\delta x}$, as $\delta x \to 0$. Up to now we have denoted this limit as 'grad y' to keep in mind the fundamental idea of gradient in relation to differentiation. We will in future adopt the usual practice of writing this limit as $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, the symbol $\dfrac{\mathrm{d}}{\mathrm{d}x}$ being an instruction to differentiate.*

Thus, if $y = x^2$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x$; or we may write $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x$.

The gradient function will be referred to as the *derived function*, or *derivative* (see §3.5 on page 66).

Question

Q1 Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when

a $y = x^2 - 4x$ **b** $y = 3x^2 - 3$

c $y = 2x^3 - 5x^2 + 1$ **d** $y = x(x-2)$

e $y = x(x+1)(x-3)$

**Note.* This notation $\dfrac{\mathrm{d}}{\mathrm{d}x}$ indicates that we are differentiating with respect to x. Thus $\dfrac{\mathrm{d}}{\mathrm{d}y}(y^3) = 3y^2$, and $\dfrac{\mathrm{d}}{\mathrm{d}t}(2t^2) = 4t$.

The notation $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is often called 'Leibnitz notation' after Gottfried Leibnitz (1646–1716), who invented it.

5.2 Greatest and least values

Fig. 5.2 represents the path of a stone thrown from O, reaching its greatest height AB, and striking the ground at C. Between O and A, when the stone is climbing, the gradient is positive but steadily decreases to zero at A. Past A the stone is descending, and the path has a negative gradient.

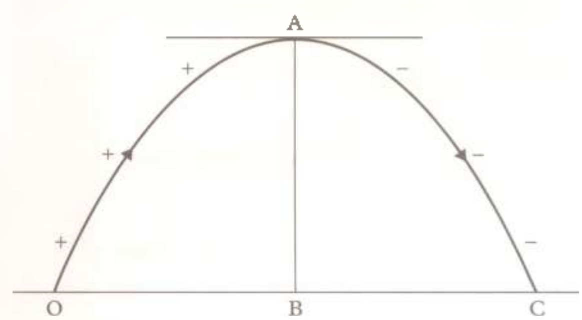


Figure 5.2

The curve $y = x^2$, which we have used earlier, is called a parabola. A more general equation of this type of curve is of the form $y = ax^2 + bx + c$. When a is positive, we get a curve like a valley, such as DEF in **Fig. 5.3**, on which y has a least value (GE). When a is negative, we get a curve like a hill, such as OAC in **Fig. 5.3**, on which y has a greatest value (BA).

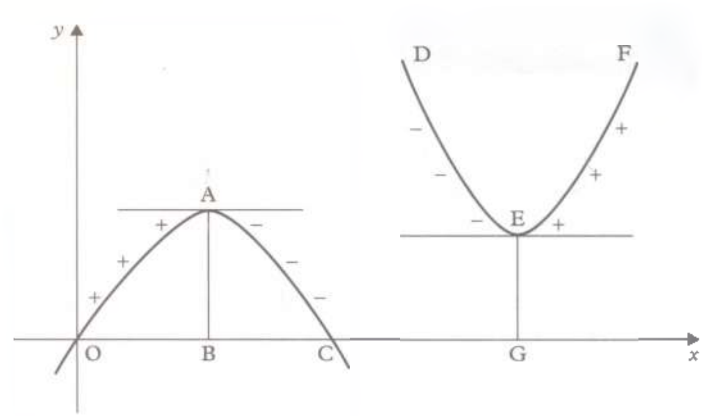


Figure 5.3

If we imagine travelling along each curve in **Fig. 5.3** from left to right (the direction in which x increases), we notice that in passing through A, where y has a greatest value, the gradient is zero and is changing sign *from positive to negative.* On the other hand, in passing through E, where y has a least value, the gradient is zero and is changing sign *from negative to positive*. This enables us to investigate the highest or lowest point on a parabola without plotting the curve in detail.

Example 1 *Find the greatest or least value of y on the curve $y = 4x - x^2$. Sketch the curve.*

$$y = 4x - x^2$$
$$\frac{dy}{dx} = 4 - 2x$$
$$= 2(2 - x)$$

The gradient is zero when

$$2(2 - x) = 0$$
$$x = 2$$

and $\quad y = 4 \times 2 - 2^2 = 4$

We must now investigate the sign of the gradient on either side of the point (2, 4) to discover whether it is a highest (Fig. 5.4) or lowest (Fig. 5.5) point on the curve. We look back to the gradient in the form $2(2 - x)$.

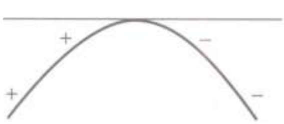


Figure 5.4

Figure 5.5

Just to the left of (2, 4), x is just less than 2, and $\frac{dy}{dx}$ is positive.

Just to the right of (2, 4), x is just greater than 2, and $\frac{dy}{dx}$ is negative.

Thus Fig. 5.4 gives the shape of the curve at (2, 4), and the greatest value of y is +4.

To make a rough sketch of the curve, we find where it cuts the axes.

$$y = 4x - x^2$$

When $x = 0$,

$$y = 0$$

$\therefore$ the curve passes through (0, 0).

When $y = 0$,

$$4x - x^2 = 0$$
$$x(4 - x) = 0$$
$$x = 0 \quad \text{or} \quad 4$$

$\therefore$ the curve passes through (0, 0) and (4, 0).

From this information we can make the sketch (Fig. 5.6).

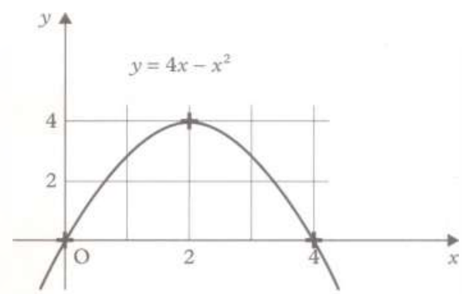


Figure 5.6

Question

Q2 Find the coordinates of the points on the following curves where the gradient is zero:

a $y = 4x - 2x^2$

b $y = 3x^2 + 2x - 5$

c $y = 4x^2 - 6x + 2$

At this stage you must be clear about the meaning of 'greater than' and 'less than' in respect of negative numbers. For example, −3.1 is *less* than −3, and −2.9 is *greater* than −3.

In Q3 and Example 2, we use the notation $f'(x)$ for the derived function; it is a useful alternative to the $\frac{dy}{dx}$ notation and you should be prepared to use it.

Question

Q3 Find the values of x for which the following derived functions are zero. Determine whether the corresponding graphs have a highest or a lowest point for these values of x:

a $f'(x) = 5 - 3x$ **b** $f'(x) = 6x - 7$

c $f'(x) = 2x + 3$ **d** $f'(x) = -4 - 5x$

The investigation of the sign of the gradient may be conveniently set out as shown in the following example.

Example 2 *Find the greatest or least value of the function* $f(x) = x^2 + 4x + 3$ *and the value of x for which it occurs.*

$$\begin{aligned} f(x) &= x^2 + 4x + 3 \\ f'(x) &= 2x + 4 \\ &= 2(x+2) \end{aligned}$$

The gradient is zero when $f'(x) = 0$, i.e. when $x = -2$ and

$$f(-2) = (-2)^2 + 4(-2) + 3 = -1$$

Value of x	L	-2	R
Sign of $f'(x)$	$-$	0	$+$

[L for 'left', R for 'right']

When $x = -2$, $x^2 + 4x + 3$ has the least value -1.

This method can be used to solve some practical problems, as in the following example.

Example 3 1000 m *of fencing is to be used to make a rectangular shamba. Find the greatest possible area, and the corresponding dimensions.*

If the length is x m, the width will be $(500 - x)$ m, and the area, A m^2, is given by

$$A = x(500 - x)$$
or $$A = 500x - x^2$$

[This problem could now be solved by drawing accurately the graph of area plotted against length (**Fig. 5.7**), and reading off the greatest area (NM) and the corresponding length (ON).

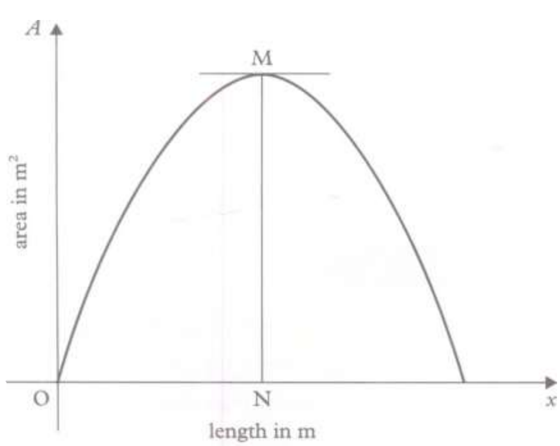


Figure 5.7

In practice it is much quicker to continue, as in Example 2, by finding the greatest value of $500x - x^2$, without plotting a graph.]

$$\begin{aligned} \frac{dA}{dx} &= 500 - 2x \\ &= 2(250 - x) \end{aligned}$$

which is zero when

$$x = 250$$

and $A = 250(500 - 250) = 62\,500$

Value of x	L	250	R
Sign of $\frac{dA}{dx}$	$+$	0	$-$

The greatest area is 62 500 m^2, when the length is 250 m and the width is 250 m.

Exercise 5a

1 Find $\frac{dy}{dx}$ when

- **a** $y = 3x^2 - 2x + 5$
- **b** $y = 5x^2 + 4x - 6$
- **c** $y = 2x(1 - x)$
- **d** $y = (x + 1)(3x - 2)$
- **e** $y = 3(2x - 1)(4x + 3)$

2 Find the coordinates of the points on the following curves where the gradient is zero:

- **a** $y = x^2 + 5x - 2$
- **b** $y = 5 + 9x - 7x^2$
- **c** $y = x(3x - 2)$
- **d** $y = (2 + x)(3 - 4x)$

3 Find the values of x for which the following derived functions are zero, and determine whether the corresponding graphs have a highest or a lowest point for these values of x:

- **a** $f'(x) = 2x - 5$
- **b** $f'(x) = \frac{1}{2}x + 3$
- **c** $f'(x) = \frac{1}{3} - \frac{1}{4}x$
- **d** $f'(x) = -5 - \frac{1}{5}x$

4 Find the greatest or least values of the following functions:

- **a** $x^2 - x - 2$
- **b** $x(4 - x)$
- **c** $15 + 2x - x^2$
- **d** $(2x + 3)(x - 2)$

5 Sketch the graphs of the functions in question 4.

6 A ball is thrown vertically upwards from ground level and its height after t s is $(15.4t - 4.9t^2)$ m. Find the greatest height it reaches, and the time it takes to get there.

7 A farmer has 100 m of metal railing with which to form two adjacent sides of a rectangular plot, the other two sides being two existing walls, meeting at right angles. What dimensions will give him the maximum possible area?

8 A stone is thrown into a mud bank and penetrates $(1200t - 36\,000t^2)$ cm in t s after impact. Calculate the maximum depth of penetration.

9 A rectangular shamba is to be made out of 1000 m of fencing, using an existing straight hedge for one of the sides. Find the maximum area possible, and the dimensions necessary to achieve this.

10 An aeroplane flying level at 250 m above the ground suddenly swoops down to drop supplies, and then regains its former altitude. It is h m above the ground t s after beginning its dive, where $h = 8t^2 - 80t + 250$. Find its least altitude during this operation, and the interval of time during which it was losing height.

11 **Fig. 5.8** represents the end view of the outer cover of a match box, AB and EF being gummed together, and assumed to be the same length. If the total length of edge (ABCDEF) is 12 cm, calculate the lengths of AB and BC which will give the maximum possible cross-section area.

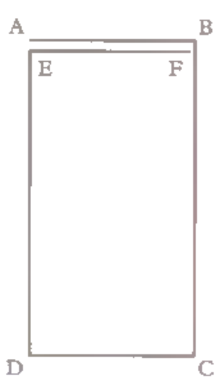


Figure 5.8

5.3 To differentiate the function $f(x) = x^{-1}$

In §3.3 on page 64 we found that if $f(x) = x^n$, where $n \in \mathbb{Z}^+$, then $f'(x) = nx^{n-1}$ (although we only *proved* this for $n = 1, 2, 3$ and 4). In this section we shall prove that it is also true when $n = -1$; that is, we shall prove that if $f(x) = 1/x = x^{-1}$, then $f'(x) = -x^{-2} = 1/x^2$. We start by using the expression for $f'(x)$ in §3.5 on page 66,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Now, in this case,

$$\begin{aligned} f(x+h) - f(x) &= \frac{1}{x+h} - \frac{1}{x} \\ &= \frac{x - (x+h)}{(x+h)x} \\ &= \frac{-h}{x(x+h)} \end{aligned}$$

Hence

$$\frac{f(x+h) - f(x)}{h} = \frac{-1}{x(x+h)}$$

and thus

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{-1}{x(x+h)} \\ &= -\frac{1}{x^2} \\ &= -x^{-2} \end{aligned}$$

We have proved that if $f(x) = x^{-1}$, then $f'(x) = -x^{-2}$. This verifies that the general result that if $f(x) = x^n$, then $f'(x) = nx^{n-1}$ is true when $n = -1$. We shall now assume that it is true for $n \in \mathbb{Z}$; that is, when n is a positive or negative integer, or zero.*

Question

Q4 Write down the derivative of

a x^{-4}

b $\frac{3}{x^2}$

c $\frac{2}{x^3}$

d $\frac{1}{2x^3}$

e $\frac{1}{x^m}$

f $2x^2 - 3x + 4 + \frac{5}{x}$

g $\frac{x^3 + 3x - 4}{x^2}$

**Note.* $n = 0$ is a special case. The rule suggests that the gradient of $y = x^0$ is zero. $x^0 = 1$, (see §P2.2 on page 8) so the graph of $y = x^0$ is a straight line parallel to the x-axis, i.e. its gradient is zero. Therefore the result predicted by the rule is correct.

5.4 Maxima and minima

In §5.2 on page 76 we were dealing with a type of curve whose gradient was zero only at one point. With a more complicated curve (**Fig. 5.9**) the gradient may be zero at a number of points, and the possible shapes fall into three categories. In this case, moving along the curve from left to right, that is with x increasing,

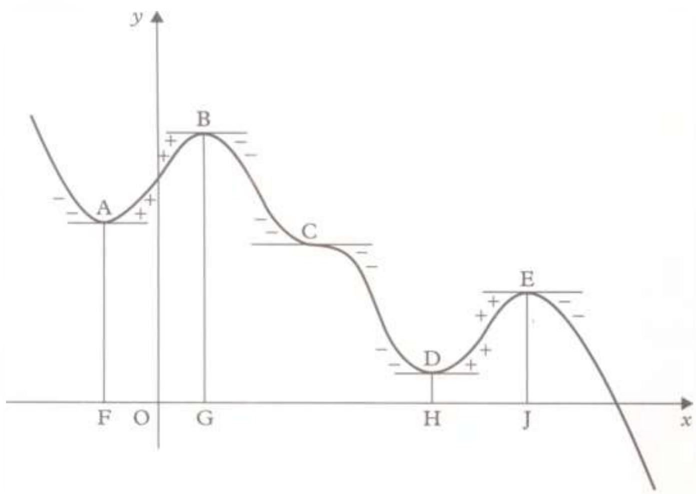


Figure 5.9

a at A and D, the gradient is changing from negative to positive, and these are called **minimum points**; FA and HD are **minimum values** of y (or **minima**),

b at B and E, the gradient is changing from positive to negative, and these are called **maximum points**; GB and JE are **maximum values** of y (or **maxima**).

Notice that the words maximum and minimum are used in the sense of greatest and least only in the immediate vicinity of the point. This local meaning is brought out clearly in this curve, since a maximum value, JE, is in fact less than a minimum value, FA. For this reason the expressions *local* maximum and *local* minimum are often used.

c At C the gradient is zero, but is *not* changing sign. This is a **point of inflexion**, which may be likened to the point on an S-bend at which a road stops turning left and begins to turn right, or vice versa. The gradient of a curve at a point of inflexion need not be zero (you should be able to find four more in **Fig. 5.9**). However at this stage we are concerned only with searching for maxima and minima, but we need to bear in mind points of inflexion as a third possibility at points where the gradient is zero.

At any point where the gradient of a curve is zero, y is said to have a **stationary value**. Any maximum or minimum point is called a **turning point**, and y is said to have a **turning value** there.

Question

Q5 Copy **Figs. 5.10–5.12**, and on each draw the tangents at all points where the gradient is zero, and mark in the sign of the gradient for each segment of the curve. State whether the points marked are maxima, minima, or points of inflexion.

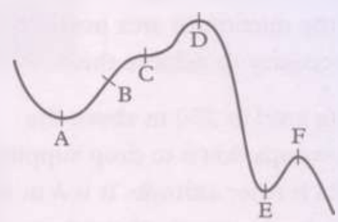


Figure 5.10

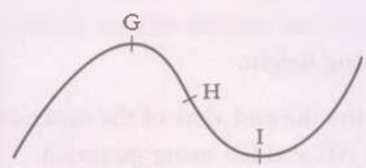


Figure 5.11

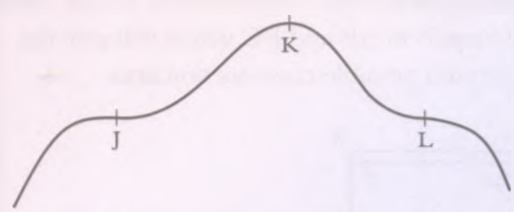


Figure 5.12

Consider the functions $f(x) = x^3$ and $g(x) = x^4$; sketches of their graphs are shown in **Fig. 5.13**.

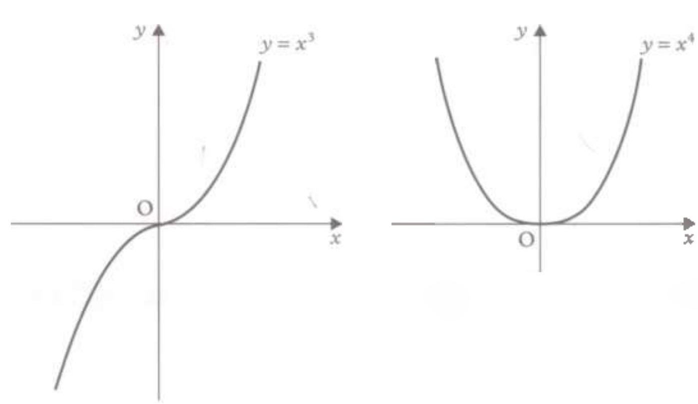


Figure 5.13

The derived functions are $f'(x) = 3x^2$ and $g'(x) = 4x^3$ and, in both cases, the derivative is zero when $x = 0$. This is confirmed by the graphs which both have zero gradient at the origin. Notice, however, that $f'(x) = 3x^2$ is never negative, which is in accordance with the observation that the graph of $y = x^3$ (see **Fig. 5.13**) always slopes upwards to the right, and has a point of inflexion at (0, 0). On the other hand, $g'(x)$ is negative for $x < 0$ and positive for $x > 0$. This also is in accordance with the graph of $y = x^4$ (see **Fig. 5.13**) which slopes downwards on the left and upwards on the right, and has a local minimum at (0, 0).

Example 4 *Investigate the stationary values of the function $x^4 - 4x^3$.**

Let $y = x^4 - 4x^3$

$$\frac{dy}{dx} = 4x^3 - 12x^2$$
$$= 4x^2(x-3)$$

which is zero when $x = 0$ or $+3$.

When $x = 0$, $y = 0$, and when $x = 3$, $y = -27$. Thus the stationary values of the function occur at (0, 0) and at (+3, −27).

[We now find the shape of the curve at these points by investigating the sign of the gradient just to the left and just to the right of each. Looking back to the factorised form of $\frac{dy}{dx}$, we see that $4x^2$ is positive for all values of x other than zero, so we are concerned with the sign of the factor $x - 3$ only.

When x is just less than 0, $x - 3$ is negative,
and when x is just greater than 0, $x - 3$ is negative.
When x is just less than +3, $x - 3$ is negative,
and when x is just greater than +3, $x - 3$ is positive.

These signs are entered in the table.]

Value of x	L	0	R	L	+3	R
Sign of $\frac{dy}{dx}$	−	0	−	−	0	+
		infl.			min.	

The stationary values of $x^4 - 4x^3$ are 0 and −27; (0, 0) is a point of inflexion; (3, −27) is a minimum point.

The next example shows that it is advisable to arrange the gradient function in a convenient factorised form. It also brings out an important point in the investigation of the sign of the gradient for negative values of x.

Example 5 *Find the turning values of y on the graph $y = f(x)$, where*

$$f(x) = 5 + 24x - 9x^2 - 2x^3$$

and distinguish between them.

$$f(x) = 5 + 24x - 9x^2 - 2x^3$$
$$f'(x) = 24 - 18x - 6x^2 = -6(x^2 + 3x - 4)$$
$$= -6(x+4)(x-1)$$

which is zero when $x = -4$ or 1.

When $x = -4$,

$$y = 5 + 24 \times (-4) - 9 \times (-4)^2 - 2 \times (-4)^3 = -107$$

and when $x = 1$,

$$y = 5 + 24 - 9 - 2 = 18$$

Thus the turning values of y occur at (−4, −107) and (1, 18).

[When completing the gradient table remember the negative factor −6, and find the sign of each factor $(x + 4)$ and $(x - 1)$. Then see if there are one, two or three negative factors, and so determine the sign of $f'(x)$.

Pay particular attention to the point (−4, −107), and the sign of the factor $(x + 4)$. To the *left*, when x is just *less* than −4 (e.g. −4.1), $(x + 4)$ is negative, $(x - 1)$ is also negative, thus $f'(x)$ has three negative factors and is negative. To the *right*, when x is just greater than −4 (e.g. −3.9), $(x + 4)$ is now positive, $(x - 1)$ is still negative, thus $f'(x)$ has two negative factors, and is positive.]

Value of x	L	−4	R	L	1	R
Sign of $f'(x)$	−	0	+	+	0	−
		min.			max.	

The turning values of y are −107 and 18.
−107 is a minimum value; 18 is a maximum value.

**Note.* Questions often do not specify a symbol for the dependent variable. The solution to such a question should normally start with a phrase like 'Let $y = x^4 - 4x^3$', as in this example, or, alternatively, 'Let $f(x) = \ldots$'.

Exercise 5b

1 Write down the values of x for which the following derived functions are zero, and prepare in each case a gradient table as in the foregoing examples, showing whether the corresponding points on the graphs are maxima, minima or points of inflexion:

- **a** $f'(x) = 3x^2$
- **b** $f'(x) = -4x^3$
- **c** $f'(x) = (x-2)(x-3)$
- **d** $f'(x) = (x+3)(x-5)$
- **e** $f'(x) = (x+1)(x+6)$
- **f** $f'(x) = -(x-1)(x-3)$
- **g** $f'(x) = -x^2 + x + 12$
- **h** $f'(x) = -x^2 - 5x + 6$
- **i** $f'(x) = 15 - 2x - x^2$
- **j** $f'(x) = 5x^4 - 27x^2$
- **k** $f'(x) = 1 - 4/x^2$

2 Find any maximum or minimum values of the following functions:

- **a** $f(x) = 4x - 3x^3$
- **b** $f(x) = 2x^3 - 3x^2 - 12x - 7$
- **c** $f(x) = x^2(x-4)$
- **d** $f(x) = x + 1/x$
- **e** $f(x) = x(2x-3)(x-4)$

3 Find the turning points on the following curves, and state whether y has a maximum or minimum value at each:

- **a** $y = x(x^2 - 12)$
- **b** $y = x^3 - 5x^2 + 3x + 2$
- **c** $y = x^2(3-x)$
- **d** $y = 4x^2 + 1/x$
- **e** $y = x(x-8)(x-15)$

4 Investigate the stationary values of y on the following curves:

- **a** $y = x^4$
- **b** $y = 3 - x^3$
- **c** $y = x^3(2-x)$
- **d** $y = 3x^4 + 16x^3 + 24x^2 + 3$

5 Fig. 5.14 represents a rectangular sheet of metal 8 cm by 5 cm. Equal squares of side x cm are removed from each corner, and the edges are then turned up to make an open box of volume V cm^3. Show that $V = 40x - 26x^2 + 4x^3$. Hence find the maximum possible volume, and the corresponding value of x.

Figure 5.14

6 Repeat question 5 when the dimensions of the sheet of metal are 8 cm by 3 cm, showing that in this case $V = 24x - 22x^2 + 4x^3$.

7 The size of a parcel is limited by the fact that the sum of its length and girth (perimeter of cross-section) must not exceed 2 metres. What is the volume of the largest parcel of square cross-section acceptable for posting? (Let the cross-section be a square of side x metres.)

8 Repeat question 7 for a parcel of circular cross-section, leaving π in your answer.

9 A factory wishes to make a cylindrical container, of thin metal, to hold 10 cm^3, using the least possible area of metal. If the outside surface is S cm^2, and the radius is r cm, show that $S = 2\pi r^2 + 20/r$ and hence find the required radius and height for the container. (Leave π in your answer.)

10 Repeat question 9 showing that whatever may be the given volume, the area of metal will always be least when the height is twice the radius.

11 64 cm^3 of butter is to be made into a cuboid of square cross-section. Calculate the required length if the total surface area is to be as small as possible.

12 An open cardboard box with a square base is required to hold 108 cm^3. What should be the dimensions if the area of cardboard used is as small as possible?

5.5 Curve sketching

We have seen in §5.4 above how maxima and minima problems may be solved without direct use of the relevant graph. Frequently however the determination of maximum and minimum points is a valuable aid in sketching a curve. (See §2.11 on page 50 for a note on the difference between *sketching* and *plotting* a curve.)

Example 6 *Sketch the curve* $y = 4x^3 - 3x^4$.

a To find where the curve meets the x-axis, put $y = 0$, then

$$4x^3 - 3x^4 = 0$$
$$\therefore x^3(4 - 3x) = 0$$

Therefore the curve meets the x-axis at the points $(0, 0)$ and $(\frac{4}{3}, 0)$.

b To find where the curve meets the y-axis, put $x = 0$. The curve meets the y-axis at the origin.

c To find stationary points:

$$y = 4x^3 - 3x^4$$
$$\therefore \frac{dy}{dx} = 12x^2 - 12x^3$$
$$= 12x^2(1 - x)$$

which is zero when $x = 0$ or 1.

Therefore $(0, 0)$ and $(1, 1)$ are stationary points.

Value of x	L	0	R	L	1	R
Sign of $\frac{dy}{dx}$	+	0	+	+	0	–
		infl.			max.	

Hence $(0, 0)$ is a point of inflexion and $(1, 1)$ is a maximum.

The results in **a**, **b** and **c** are used to sketch the curve, as in **Fig. 5.15**.

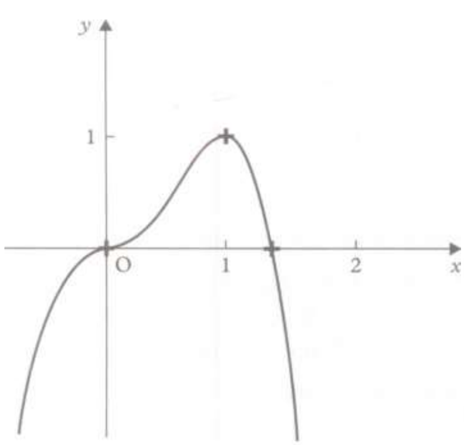


Figure 5.15

Exercise 5c

Find where the following curves meet the axes. Find, also, the coordinates of their stationary points and use these results to sketch the curves.

1 $y = 3x^2 - x^3$

2 $y = x^3 - 6x^2$

3 $y = x^3 - 2x^2 + x$

4 $y = (x + 1)^2(2 - x)$

5 $y = x^2(x - 2)^2$

6 $y = x^4 - 8x^3$

7 $y = x^4 - 10x^2 + 9$

8 $y = x^4 + 32x$

9 $y = 4x^5 - 5x^4$

10 $y = 3x^5 - 5x^3$

11 $y = 2x^5 + 5x^2$

Example 7 shows another useful approach to curve sketching.

Example 7 *Sketch the curve* $y = (x + 1)(x - 1)(2 - x)$.

a To find where the curve meets the x-axis, put $y = 0$, then

$$(x + 1)(x - 1)(2 - x) = 0$$

Therefore the curve meets the x-axis at $(-1, 0)$, $(1, 0)$, $(2, 0)$.

b To find where the curve meets the y-axis, put $x = 0$. Thus the curve meets the y-axis at $(0, -2)$.

c To examine the behaviour of the curve 'at infinity', expand the R.H.S. of the equation:

$$y = (x^2 - 1)(2 - x) = -x^3 + 2x^2 + x - 2$$

Now, if x is large, the sign of y will be determined by the term of highest degree, $-x^3$. (If $x = 100$, say, $y = -1\,000\,000 + 20\,000 + 100 - 2$; or if $x = -100$, $y = 1\,000\,000 + 20\,000 - 100 - 2$. In either case the term in x^3 predominates.)

If x is large and positive, y is large and negative, and if x is large and negative, y is large and positive. The behaviour of the curve as $x \to +\infty$ and $x \to -\infty$ is shown in **Fig. 5.16**.

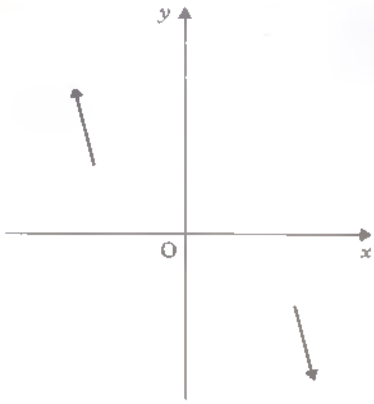


Figure 5.16

The curve is then sketched, as in Fig. 5.17.

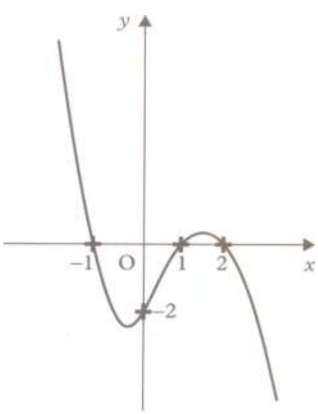


Figure 5.17

5.6 Distance, velocity and acceleration graphs

Useful physical interpretations of the graphical ideas discussed in §5.4 on page 80 are obtained from the distance-time, velocity-time, and acceleration-time graphs for the motion of a particle, if we plot one above the other as in Example 8.

Example 8 *O is a point on a straight line. A particle moves along the line so that it is s m from O, t s after a certain instant, where $s = t(t-2)^2$. Describe the motion before and after $t = 0$.*

The *distance-time graph* has the equation $s = t(t-2)^2$. By the methods of §5.5 above we find that the graph has a max. point $(\frac{2}{3}, \frac{32}{27})$, a min. point $(2, 0)$, and passes through $(0, 0)$. See the upper sketch in Fig. 5.18.

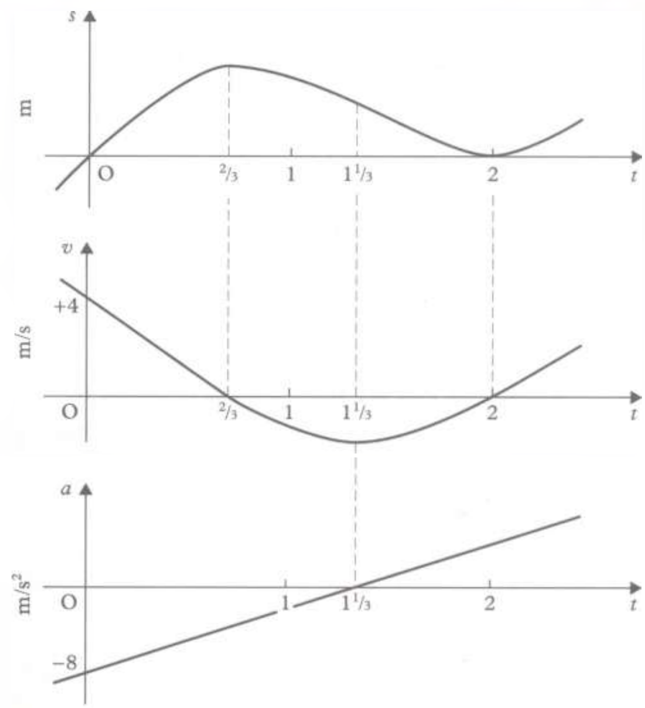


Figure 5.18

The equation may be written $s = t^3 - 4t^2 + 4t$.

$$\therefore \frac{ds}{dt} = 3t^2 - 8t + 4 = (3t-2)(t-2)$$

Hence the *velocity-time graph* has the equation $v = (3t-2)(t-2)$. This graph has a min. point $(1\frac{1}{3}, -1\frac{1}{3})$, and passes through $(\frac{2}{3}, 0)$, $(2, 0)$, and $(0, 4)$ as in the middle sketch in Fig. 5.18.

Differentiating once again, $\frac{dv}{dt} = 6t - 8$, and so the *acceleration-time graph* has the equation $a = 6t - 8$, and is the bottom sketch in Fig. 5.18.

Notice that the local max. and min. values of s occur when $v \left(\text{i.e. } \frac{ds}{dt} \right)$ is zero, and that the local min. value of v occurs when $a \left(\text{i.e. } \frac{dv}{dt} \right)$ is zero.

It is easy to visualise the motion of the particle as being along the Os axis of the distance-time graph, its distance from O at any instant being given by the height of the graph for the corresponding value of t. Before $t = 0$, the particle is approaching O from the negative side; at $t = 0$, it is passing through O with velocity 4 m/s, and acceleration −8 m/s², hence its speed is decreasing. It comes momentarily to rest $\frac{32}{27}$ m from O (on the positive side) when $t = \frac{2}{3}$; it returns to O, where it is momentarily at rest when $t = 2$, and thereafter it moves away from O in the positive direction.

Some further points regarding the sign and direction of the velocity and acceleration in Example 8 deserve emphasis. Consider the three graphs between $t = 0$ and $t = 1\frac{1}{3}$. Throughout this interval the acceleration is negative, and the velocity decreases from +4 m/s to $-1\frac{1}{3}$ m/s. The effect of the negative acceleration is to *decrease* the speed when the velocity is positive ($t = 0$ to $t = \frac{2}{3}$), and to *increase* the speed when the velocity is negative ($t = \frac{2}{3}$ to $t = 1\frac{1}{3}$). Also note the distinction between the *speed* and the *velocity*, speed being the numerical value of the velocity, irrespective of direction.

Question

Q6 In Example 8, give the signs of the velocity, and acceleration.
State if the speed is increasing or decreasing, when **a** $t = 1\frac{1}{2}$, **b** $t = 3$, **c** $t = 1\frac{1}{3}$.

Exercise 5d

1 Make a rough sketch of each of the following curves by finding the points of intersection with the axes, and by investigating the behaviour of y as $x \to +\infty$ and as $x \to -\infty$. (Do *not* find maximum and minimum points).

- **a** $y = (x + 2)(x - 3)$
- **b** $y = (5 + x)(1 - x)$
- **c** $y = x(x + 1)(x + 2)$
- **d** $y = (2 + x)(1 + x)(3 - x)$
- **e** $y = (x - 1)(x - 3)^2$
- **f** $y = (x + 4)^2(x - 3)$
- **g** $y = -x(x - 7)^2$
- **h** $y = x^2(5 - x)$
- **i** $y = (x - 2)^3$
- **j** $y = (x - 3)^4$
- **k** $y = -x(x - 4)^3$

2 A particle moves along a straight line OB so that t s after passing O it is s m from O, where $s = t(2t - 3)(t - 4)$. Deduce expressions for the velocity and acceleration in terms of t, and sketch the distance-, velocity-, and acceleration-time graphs as in **Fig. 5.18**. Briefly describe the motion, and when $t = 2$ find

- **a** where the particle is,
- **b** if it is going towards or away from B,
- **c** its speed,
- **d** if its speed is increasing or decreasing,
- **e** the rate of change of the speed.

3 Answer the questions in question 2 for the instant when $t = 1$.

4 With the data of question 2, when is the particle moving at its greatest speed away from B, and where is it then?

5 A particle is moving along a straight line OA in such a way that t s after passing through O for the first time it is s m from O where

$$s = -t(t - 8)(t - 15)$$

A is taken to be on the positive side of O. Deduce expressions for the velocity and acceleration in terms of t, and sketch the three graphs as in **Fig. 5.18**. Briefly describe the motion.

- **a** Describe in detail the motion and position of the particle when $t = 10$.
- **b** When is it moving towards A?
- **c** When is it travelling at its greatest speed towards A?

6 A car in a traffic jam starts from rest with constant acceleration 2 ms^{-2}, and when its velocity reaches 6 ms^{-1} it remains constant at that figure for 4 s, and it is then reduced to zero in 6 s at a constant retardation. Sketch the distance-, velocity-, and acceleration-time graphs for this motion.

Chapter 6

Integration (1)

Reverse of differentiation

6.1 Integration and integrals

Suppose that instead of an equation of a curve, we take as our starting point a gradient function. For example, what is represented geometrically by the equation $\frac{dy}{dx} = \frac{1}{3}$?

The constant gradient $\frac{1}{3}$ indicates a straight line. $y = \frac{1}{3}x$ is the equation of the straight line of gradient $\frac{1}{3}$ through the origin, and, on differentiation, gives $\frac{dy}{dx} = \frac{1}{3}$. But $y = \frac{1}{3}x$ is not the only possibility. Any straight line of gradient $\frac{1}{3}$ may be written as $y = \frac{1}{3}x + c$, where c is a constant, and this is the most general equation which gives $\frac{dy}{dx} = \frac{1}{3}$.

Thus the equation $\frac{dy}{dx} = \frac{1}{3}$ relates to the equation $y = \frac{1}{3}x + c$, namely *all straight lines of gradient* $\frac{1}{3}$ (Fig. 6.1).

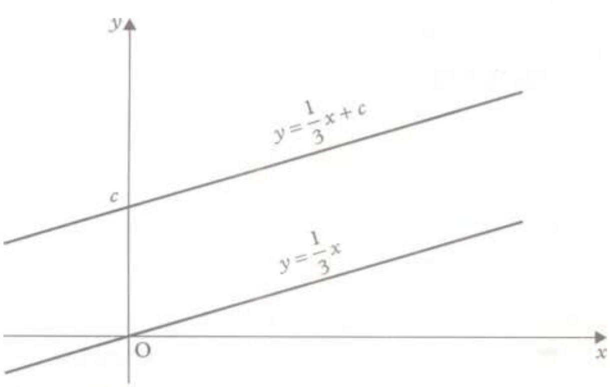


Figure 6.1

Let us take another example, $\frac{dy}{dx} = 2x$. We know that $y = x^2$ is a curve with this gradient function; but the most general equation leading to $\frac{dy}{dx} = 2x$ on differentiation is $y = x^2 + c$, where c is a constant.

Thus the equation $\frac{dy}{dx} = 2x$ represents the equation $y = x^2 + c$, namely the family of curves 'parallel' to $y = x^2$ (see Fig. 3.8 on page 65).

We have found that if $\frac{dy}{dx} = \frac{1}{3}$, then $y = \frac{1}{3}x + c$

Also

$$\text{if } \frac{dy}{dx} = 2x, \quad \text{then} \quad y = x^2 + c$$

The process of finding the expression for y in terms of x when given the gradient function – in other words, the **reverse of differentiation** – is called **integration**.

$x^2 + c$ is called the **integral** of $2x$ with respect to x.

The constant c, which can only be found if further data is given, is called the **arbitrary constant** of integration.

We know that when we differentiate a power of x, the index is reduced by 1, since $\frac{d}{dx}(x^n) = nx^{n-1}$. In this reverse process of integration we must therefore increase the index by 1, thus

$$\text{if } \frac{dy}{dx} = x, \qquad y = \frac{x^2}{2} + c$$

and

$$\text{if } \frac{dy}{dx} = 5x^2, \quad y = 5 \times \frac{x^3}{3} + c$$

You should check these by differentiating. It will then be clear why the denominators 2 and 3 arise. The rule for integrating a power of x is 'increase the index by 1, and divide by the new index'.

Question

Q1 Integrate with respect to x:

a 2	**b** m	**c** $3x^2$
d $3x$	**e** $3x^4$	**f** $3 + 2x$
g $x - x^2$	**h** $ax + b$	

Just as we have assumed that the rule for differentiating x^n is valid for $n \in \mathbb{Z}$, i.e. when n is any integer, positive or negative, so we shall make a similar assumption about the rule for integrating x^n, with the notable exception of x^{-1}. In other words, for all positive and negative integral values of n, other than -1,

$$\textbf{if } \frac{dy}{dx} = x^n, \quad \textbf{then} \quad y = \frac{x^{n+1}}{n+1} + c$$

Thus if $\frac{dy}{dx} = 1/x^2 = x^{-2}$, then

$$y = \frac{x^{-2+1}}{-2+1} + c = \frac{x^{-1}}{-1} + c = -\frac{1}{x} + c$$

Check this last result by differentiating. You should make a habit of doing this. It is important to remember that the arbitrary constant is an essential part of each integral.

Questions

Q2 Integrate with respect to x:

a $\frac{1}{x^3}$ **b** x^{-4}

c $\frac{2}{x^2}$ **d** $\frac{1}{x^n}$

Q3 Why is the rule for integrating not valid when $n = -1$?

Reverting to our earlier examples, $\frac{dy}{dx} = \frac{1}{3}$ and $\frac{dy}{dx} = 2x$ are called **differential equations**, and $y = \frac{1}{3}x + c$ and $y = x^2 + c$ respectively are the **general solutions**.

We saw that the differential equation $\frac{dy}{dx} = \frac{1}{3}$ represents all straight lines of gradient $\frac{1}{3}$. To be able to find the equation of a particular straight line of gradient $\frac{1}{3}$, we must find the appropriate value of c in the general solution $y = \frac{1}{3}x + c$, and to do this we need to know one point through which the line passes. You should now read again the alternative solution of Example 9 in §1.9 on page 42. It will be seen that the process of finding the equation of a straight line of given gradient passing through a given point may be thought of as finding a particular solution of a differential equation.

Question

Q4 $\frac{dy}{dx} = 4$. Find y in terms of x, given that $y = 10$ when $x = -2$. What does the solution represent graphically?

Exercise 6a

1 Integrate:

a with respect to x:

$\frac{1}{2}$, $\frac{1}{2}x^2$, $x^2 + 3x$, $(2x+3)^2$, x^{-5}, $\frac{-2}{x^4}$;

b with respect to t:

at, $\frac{1}{3}t^3$, $(t+1)(t-2)$, $\frac{1}{t^{n+1}}$, $\frac{1}{t^2} + 3 + 2t$;

c with respect to y: $-ay^{-2}$, $\frac{k}{y^2}$, $\frac{(y^2+2)(y^2-3)}{y^2}$.

2 Solve the following differential equations:

a $\frac{dy}{dx} = 3ax^2$ **b** $\frac{ds}{dt} = 3t^3$

c $\frac{ds}{dt} = u + at$ **d** $\frac{dx}{dt} = \left(1 + \frac{1}{t}\right)\left(1 - \frac{1}{t}\right)$

e $\frac{dy}{dt} = \frac{t^3 - 3t + 4}{t^3}$ **f** $\frac{dA}{dx} = \frac{(1+x^2)(1-2x^2)}{x^2}$

3 What is the gradient function of a straight line passing through $(-4, 5)$ and $(2, 6)$? Find its equation.

4 A curve passes through the point $(3, -1)$ and its gradient function is $2x + 5$. Find its equation.

5 A curve passes through the point $(2, 0)$ and its gradient function is $3x^2 - 1/x^2$. Find its equation.

6 The gradient of a curve at the point (x, y) is $3x^2 - 8x + 3$. If it passes through the origin, find the other points of intersection with the x-axis.

7 The gradient of a curve at the point (x, y) is $8x - 3x^2$, and it passes through the origin. Find where it cuts the x-axis, and find the equation of the tangent parallel to the x-axis.

8 Find s in terms of t if $\frac{ds}{dt} = 3t - 8/t^2$, given that $s = 1\frac{1}{2}$ when $t = 1$.

9 Find A in terms of x if $\frac{dA}{dx} = (3x+1)(x^2-1)/x^5$. What is the value of A when $x = 2$, if $A = 0$ when $x = 1$?

6.2 Velocity and acceleration

In Chapter 4 we used the formula $s = 4.9t^2$ for a stone falling from rest. It was explained that this is based on the assumption that the acceleration of the stone is 9.8 metres per second per second, or 9.8 m/s². We can now use the process of integration to see how the formula is found from this assumption.

If the acceleration is given by

$$\frac{dv}{dt} = 9.8$$

then

$$v = 9.8t + c$$

Now if the stone falls from rest at the instant from which we measure the time, $v = 0$ when $t = 0$. Substituting these values in the last equation, we get $c = 0$.

$$\therefore\ v = 9.8t$$

This may be written

$$\frac{ds}{dt} = 9.8t$$

from which

$$s = 4.9t^2 + k$$

If s measures the distance below the initial position of the stone, $s = 0$ when $t = 0$. Substituting these values in the last equation, we get $k = 0$.

$$\therefore\ s = 4.9t^2$$

Question

Q5 A stone is thrown vertically downwards from the top of a cliff at 15 ms^{-1}. Assuming that its acceleration due to gravity is 9.81 ms^{-2}, find expressions for its velocity and position t s later, by solving the differential equation $\frac{dv}{dt} = 9.81$.

It again needs emphasising that displacement (s), velocity and acceleration in a straight line are positive in one direction, negative in the other, and it is important to decide at the outset which is to be taken as the positive direction. E.g. take upwards as positive in Q6.

Question

Q6 A stone is thrown vertically upwards from the edge of a cliff at 19.6 m/s. Assuming that gravity produces a downwards acceleration of 9.8 m/s², deduce the velocity and position of the stone after 1, 3 and 5 s. Explain the sign of each answer, taking upwards as positive.

Example 1 *Fig. 6.2 represents part of a conveyor belt, the dots being small articles on it at 1 m spacing. Initially the belt is at rest with the article R 7 m short of O, a fixed mark on a wall. The belt accelerates from rest so that its velocity is $0.1t$ m/s, t s after starting. Find* **a** *the position of R when $t = 10$, and* **b** *the distance moved by R between $t = 3$ and $t = 5$.*

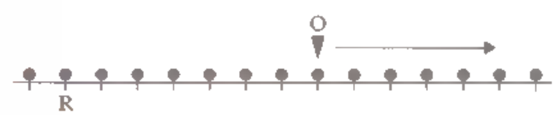


Figure 6.2

a If the distance from O at time t s is s m (positive to the right of O, negative to the left), then it is true of each article that its velocity, $\frac{ds}{dt} = 0.1t$, and also, by integration, that

$$s = 0.05t^2 + c$$

However, this last equation does not give us the distance of any particular article from O, until we have discovered the appropriate value of c. Since when $t = 0$, $s = c$, the arbitrary constant of integration in this case represents the initial position of an article.

In the case of R, when $t = 0$,

$$s = -7$$

Substituting in the last equation, $-7 = 0 + c$,

$$\therefore\ c = -7$$

Therefore the distance of R from O at time t s is s m where

$$s = 0.05t^2 - 7$$

When $t = 10$,

$$s = 0.05 \times 100 - 7 = -2$$

$\therefore$ R is 2 m short of O at this instant.

b The distance moved by each article in any given interval is the same, therefore we are not concerned with any particular numerical value for the constant of integration, and we shall leave c in our working.

As before, since $\frac{ds}{dt} = 0.1t$,

$$s = 0.05t^2 + c$$

When $t = 3$,

$$s = 0.05 \times 3^2 + c$$

When $t = 5$,

$$s = 0.05 \times 5^2 + c$$

The distance moved between $t = 3$ and $t = 5$ is

$$\begin{aligned}&(0.05 \times 5^2 + c) - (0.05 \times 3^2 + c) \text{ m}\\ &= 0.05 \times 25 + c - 0.05 \times 9 - c \text{ m}\\ &= 0.8 \text{ m}\end{aligned}$$

b (*Alternative layout*) The following square bracket notation is an instruction to substitute and subtract. It shortens the working.

$$\frac{ds}{dt} = 0.1t$$

$$\therefore s = 0.05t^2 + c$$

The distance moved between $t = 3$ and $t = 5$ is

$$\begin{aligned}\Big[0.05t^2 + c\Big]_3^5 \text{ m} &= (0.05 \times 25 + c) - (0.05 \times 9 + c) \text{ m}\\ &= 1.25 + c - 0.45 - c \text{ m}\\ &= 0.8 \text{ m}\end{aligned}$$

Question

Q7 Evaluate:

a $\Big[3t + 8\Big]_2^5$

b $\Big[3t^2 - t + k\Big]_1^4$

c $\Big[t^2 - t\Big]_{-2}^{+1}$

d $\Big[t^3 - 3t^2 + t\Big]_{-3}^{-2}$

Questions

Q8 A particle moves in a straight line with velocity $2t^2$ m/s, t s after the start. Find the distance moved in the 3rd second.

Q9 With the data of Example 1, answer the following questions.

a Find the position of R when $t = 20$.

b Find the position when $t = 10$ of the article initially at O.

c An article N is 2.2 m past O when $t = 2$; find its position when $t = 10$.

d An article T is 99.95 m short of O when $t = 1$; find its initial position.

Exercise 6b

1 A stone is thrown vertically downwards at 20 m/s from the top of a cliff. Assuming that gravity produces on it an acceleration of 9.81 m/s^2, find, from the differential equation $\frac{dv}{dt} = 9.81$, expressions for its velocity and position t s later.

2 A stone is thrown vertically upwards from ground level at 12 ms^{-1}, at a point immediately above a well. Taking the downwards direction as positive, deduce, from the differential equation $\frac{dv}{dt} = 9.8$, expressions for the stone's velocity and position t s later. Find the velocity and position after 1, 2, 3 s, explaining the sign of each answer.

3 Find the displacement (s) in terms of time (t) from the following data:

a $\frac{ds}{dt} = 3$, $s = 3$ when $t = 0$

b $v = 4t - 1$, $s = 0$ when $t = 2$

c $v = (3t - 1)(t + 2)$, $s = 1$ when $t = 2$

d $v = t^2 + 5 - \frac{2}{t^2}$, $s = \frac{1}{3}$ when $t = 1$

4 Evaluate:

a $\Big[8t + c\Big]_1^5$ **b** $\Big[3t^2 + 5t\Big]_2^{10}$

c $\Big[t^2 - 4t\Big]_{-3}^{0}$ **d** $\Big[2t^3 - t^2 - t\Big]_{-2}^{-1}$

5 Find s in terms of t, and the distance moved in the stated interval (the units being metres and seconds), given that

a $\dfrac{ds}{dt} = 4t + 3$, $t = 0$ to $t = 2$

b $v = t^2 - 3$, $t = 2$ to $t = 3$

c $v = (t-1)(t-2)$, $t = -1$ to $t = 0$

d $v = t + 3 - \dfrac{1}{t^2}$, $t = 10$ to $t = 20$

6 If a particle moves in a straight line so that its acceleration in terms of the time is At (A being a constant), find expressions for the velocity and displacement at time t.

7 Deduce expressions for v and s from the following data, determining the constants of integration whenever possible:

a $a = 3t$, $s = 0$ and $v = 3$ when $t = 0$

b $a = 2 + t$, $s = -3$ and $v = 0$ when $t = 0$

c $a = 10 - t$, $v = 2$ when $t = 1$, $s = 0$ when $t = 0$

d $a = \frac{1}{2}t$, $v = 5$ when $t = 0$

e $a = t^2$, $s = 10$ when $t = 1$

8 A system of particles moves along a straight line OA so that t s after a certain instant their velocity is v m/s where $v = 3t$.

a One of the particles is at O when $t = 0$. Find its position when $t = 3$.

b A second particle is 4 m past O when $t = 1$. Find its position when $t = 0$.

c A third particle is 10 m short of O when $t = 2$. Find its position when $t = 4$.

d Find the distance moved by the particles during the 3rd second.

9 A particle moves along a straight line OA with velocity $(6 - 2t)$ m/s. When $t = 1$ the particle is at O.

a Find an expression for its distance from O in terms of t, and deduce the net change in position which takes place between $t = 0$ and $t = 5$.

b By finding the time at which it is momentarily at rest, calculate the actual distance through which it moves during the same interval.

c Sketch the distance-time and velocity-time graphs from $t = 0$ to $t = 6$.

10 A stone is thrown vertically upwards from ground level with a velocity of 12.6 ms^{-1}. If the acceleration due to gravity is 9.8 ms^{-2}, deduce, from the differential equation $\dfrac{dv}{dt} = -9.8$, expressions for its velocity and its height t s later. Find

a the time to the highest point,

b the greatest height reached,

c the distance moved through by the stone during each of the first two seconds of motion.

11 A train runs non-stop between two stations P and Q, and its velocity t hours after leaving P is $60t - 30t^2$ km/h. Find

a the distance between P and Q,

b the average velocity for the journey,

c the maximum velocity attained.

12 A stopping train travels between two adjacent stations so that its velocity is v km/min, t min after leaving the first, where $v = \frac{4}{3}t(1 - t)$. Find

a the average velocity for the journey in km/h,

b the maximum velocity in km/h.

13 The formula connecting the velocity and time for the motion of a particle is $v = 1 + 4t + 6t^2$. Find the average velocity and the average acceleration for the interval $t = 1$ to $t = 3$, the units being metres and seconds.

14 A racing car starts from rest and its acceleration after t s is $(k - \frac{1}{6}t)$ ms^{-2} until it reaches a velocity of 60 ms^{-1} at the end of 1 minute. Find the value of k, and the distance travelled in the first minute.

15 A particle starting from rest at O moves along a straight line OA so that its acceleration after t s is $(24t - 12t^2)$ m/s^2.

a Find when it again returns to O and its velocity then.

b Find its maximum displacement from O during this interval.

c What is its maximum velocity and its greatest speed during this interval?

16 P and R are two adjacent railway stations, and Q is a signal box on the line between them. A train which stops at P and R has a velocity of $(\frac{3}{8} + \frac{1}{2}t - \frac{1}{2}t^2)$ km/min at t min past noon, and it passes Q at noon. Find

a the times of departure from P and arrival at R,

b an expression for the distance of the train from P in terms of t,

c the average velocity between P and R, in km/h,

d the maximum velocity attained, in km/h.

6.3 Area under a curve

Another important aspect of integration is that it enables us to calculate exactly the areas enclosed by curves.

Consider the area enclosed by the axes, the line $x = 3$, and part of the curve $y = 3x^2 + 2$. This is the area TUVO in Fig. 6.3.

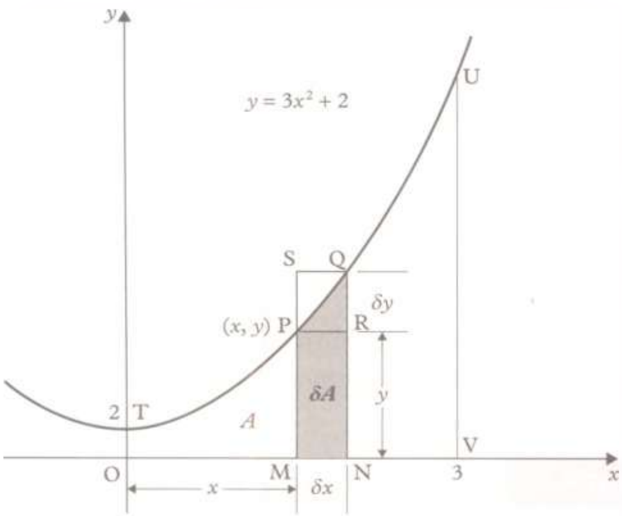


Figure 6.3

P is the point (x, y) on the curve, PM is its y-coordinate, and the area TPMO we shall call A. Now if we move P along the curve, A increases or decreases as x increases or decreases. The size of A depends upon the value of x, i.e. A is a function of x, and our aim is to find an expression for A in terms of x.

With the usual notation Q is the point $(x + \delta x, y + \delta y)$ adjacent to P, and QN is its y-coordinate. If we move the right-hand boundary of A from PM to QN, we increase x by δx, and the resulting increase in A, the shaded area PQNM, we call δA. In other words δA is the increment in A corresponding to the increment δx in x. It can be seen from Fig. 6.3 that δA lies between the areas of the two rectangles PRNM, $y\,\delta x$, and SQNM, $(y + \delta y)\delta x$. This may be written*

$$y\,\delta x < \delta A < (y + \delta y)\,\delta x$$

and dividing by δx, which is positive,

$$y < \frac{\delta A}{\delta x} < (y + \delta y)$$

*This statement is called an inequality. $<$ means 'is less than'; $>$ means 'is greater than'. See Chapter 9 on page 116. Note that an inequality is reversed by changing the sign of each term. Thus $1 < 2 < 3$, but $-1 > -2 > -3$; this explains the reference to δx being positive.

Since $\delta x \to 0$, then $\delta y \to 0$, and so $(y + \delta y) \to y$. Thus we find that $\dfrac{\delta A}{\delta x}$ lies between y and something which we can make as near to y as we please, by making δx sufficiently small. Therefore the limit of $\dfrac{\delta A}{\delta x}$ is y, and writing the limit of $\dfrac{\delta A}{\delta x}$ as $\dfrac{dA}{dx}$, we get

$$\frac{\mathbf{d}A}{\mathbf{d}x} = y$$

$$\therefore \frac{dA}{dx} = 3x^2 + 2$$

and by integration,

$$A = x^3 + 2x + c$$

If we were to bring in the right-hand boundary of the area A from PM to TO, we should reduce A to zero; that is to say, when $x = 0$, $A = 0$. Substituting these values in the last equation we find that $c = 0$.

$$\therefore A = x^3 + 2x$$

and we have achieved our aim of expressing A in terms of x; now to find the area TUVO. In this case, the right-hand boundary of A has been pushed out from PM to UV, and x is increased to 3.

When $x = 3$,

$$A = 3^3 + 2 \times 3 = 33 \text{ square units}$$

$\therefore$ the area TUVO = 33 square units.

Example 2 *Find the area enclosed by the x-axis, the curve $y = 3x^2 + 2$ and the straight lines $x = 3$ and $x = 5$.*

The required area is UWZV in Fig. 6.4, i.e. the difference between the areas TWZO and TUVO. Using A as above,

$$\frac{dA}{dx} = y = 3x^2 + 2$$

$$\therefore A = x^3 + 2x$$

(We have shown above that the constant of integration is zero.)

When $x = 5$,

$$A = 5^3 + 2 \times 5 = 135 \text{ (Area TWZO)}$$

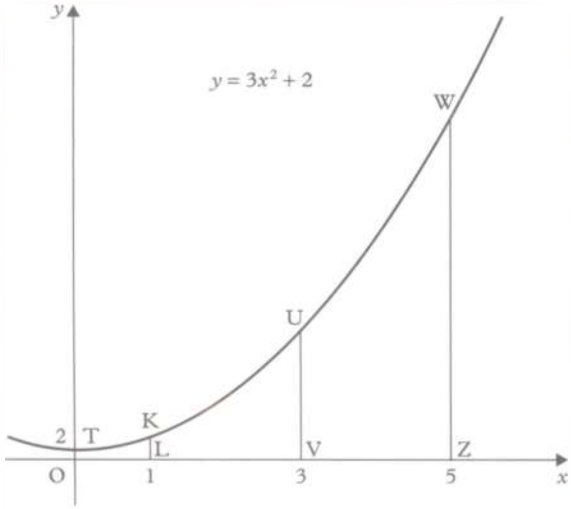


Figure 6.4

and when $x = 3$,

$$A = 3^3 + 2 \times 3 = 33 \text{ (Area TUVO)}$$

$\therefore$ the area UWZV $= 135 - 33 = 102$ square units.

Question

Q10 Find the area enclosed by the x-axis, the curve $y = 3x^2 + 2$, and the following straight lines:

- **a** the y-axis and $x = 4$
- **b** $x = 1$ and $x = 2$
- **c** $x = -1$ and $x = 3$
- **d** $x = -3$ and $x = -2$

In all the working so far in this chapter we have used the symbol A to denote an area having the y-axis as its left-hand boundary. Suppose that instead we had, in Fig. 6.3, defined a similar area A' having the line $x = 1$ as its left-hand boundary. By the same process of reasoning we should arrive at the result

$$\frac{dA'}{dx} = y = 3x^2 + 2$$

$$\therefore A' = x^3 + 2x + k$$

But $A' = 0$ when $x = 1$, and substituting these values we get $k = -3$.

$$\therefore A' = x^3 + 2x - 3$$

Now A' is measured to the right from the line KL ($x = 1$) in Fig. 6.4, and Example 2 might just as well be done using A' instead of A, finding the area UWZV as the difference between the areas KWZL and KUVL. Thus, when $x = 5$,

$$A' = 5^3 + 2 \times 5 - 3 = 135 - 3$$

and when $x = 3$,

$$A' = 3^3 + 2 \times 3 - 3 = 33 - 3$$

$\therefore$ the area UWZV $= (135 - 3) - (33 - 3) = 102$ unit^2.

In each solution we have determined the constant of integration. Using A, it is zero, and using A', it is -3. But as is clear from the second solution, the constant drops out on subtraction. We could in fact have measured A from any convenient left-hand boundary, and found the area UWZV by subtraction, without evaluating the constant of integration.

We shall from now onwards assume the relationship $\frac{dA}{dx} = y$ to calculate areas of this nature, and the square bracket notation introduced in §6.2 on page 88 may now be put to further use, as shown in the next example.

Example 3 *Find the area enclosed by the x-axis, $x = 1$, $x = 3$ and the graph $y = x^3$.* (Fig. 6.5).

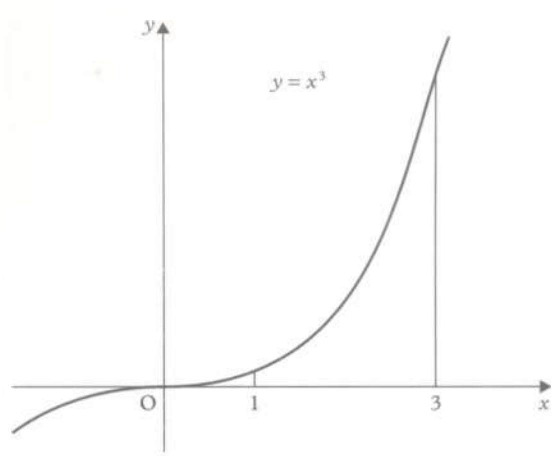


Figure 6.5

$$\frac{dA}{dx} = y = x^3$$

$$\therefore A = \tfrac{1}{4}x^4 + c$$

The required area $= \left[\tfrac{1}{4}x^4 + c\right]_1^3$,

$= (\tfrac{81}{4} + c) - (\tfrac{1}{4} + c)$

$= \tfrac{81}{4} + c - \tfrac{1}{4} - c$

$= 20$ unit^2

Note in Example 3:

1 The area evaluated in Example 3 is called *the area under the curve* $y = x^3$ *from* $x = 1$ *to* $x = 3$.

2 1 and 3 are called, respectively, the *lower* and *upper limits of integration.*

3 The integral $\frac{1}{4}x^4 + c$, involving the arbitrary constant of integration, is called an **indefinite integral**.

4 However when limits are given, and the integral can be evaluated, e.g. $\left[\frac{1}{4}x^4 + c\right]_1^3$, it is called a **definite integral**.

5 Since the constant of integration drops out in a definite integral, it is not necessary to write it in the bracket.

Questions

Q11 Evaluate the following definite integrals:

a $\left[3x^2 + 2x\right]_{1/2}^{1}$

b $\left[x^4 - 2x^2\right]_{-1}^{2}$

c $\left[x^3 - 3x\right]_{-2}^{0}$

d $\left[2x^2 + 4x\right]_{-3}^{-1}$

e $\left[x^4 - 2x^3 + x^2 - x\right]_{-2}^{0}$

f $\left[x^2 + 3x - \frac{1}{x^3}\right]_{+1/2}^{+1}$

Q12 Find the area under $y = \frac{1}{2}x$ from $x = 0$ to $x = 10$ by integration. Check by another method.

Q13 Find the area under

a $y = x^2$ from $x = 0$ to $x = 3$

b $y = 2x^2 + 1$ from $x = 2$ to $x = 5$

Examples 4 and 5 illustrate the advisability of making a rough sketch when not sure of the shape and position of any curve. They also bring out two important points.

In these examples and in all subsequent questions and exercises we shall assume that the area is expressed in square units (unit^2).

Example 4 *Find the area under the curve* $y = x^2(x - 2)$ **a** *from* $x = 0$ *to* $x = 2$, *and* **b** *from* $x = 2$ *to* $x = \frac{8}{3}$.

Consideration of the sign of the highest degree term, and the points of intersection with the x-axis, enables an adequate sketch to be made (**Fig. 6.6**).

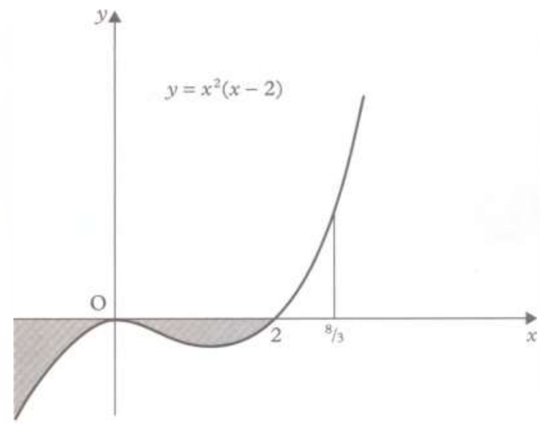


Figure 6.6

$$\frac{dA}{dx} = y = x^3 - 2x^2$$

$$\therefore\ A = \frac{1}{4}x^4 - \frac{2}{3}x^3 + c$$

a The required area

$$= \left[\frac{1}{4}x^4 - \frac{2}{3}x^3\right]_0^2$$

$$= \left(\frac{1}{4}\times 2^4 - \frac{2}{3}\times 2^3\right) - (0)$$

$$= -1\frac{1}{3}$$

b The required area

$$= \left[\frac{1}{4}x^4 - \frac{2}{3}x^3\right]_2^{8/3}$$

$$= \left(\frac{1}{4}\times\frac{8^4}{3^4} - \frac{2}{3}\times\frac{8^3}{3^3}\right) - \left(\frac{1}{4}\times 2^4 - \frac{2}{3}\times 2^3\right)$$

$$= (0) - \left(-1\frac{1}{3}\right)$$

$$= +1\frac{1}{3}$$

Part **a** of Example 4 shows that *the area under a curve is negative below the x-axis*. Check that $\left[\frac{1}{4}x^4 - \frac{2}{3}x^3\right]_0^{8/3}$ is zero. This is because it represents the sum of the two areas that are, numerically equal but of opposite sign ($-1\frac{1}{3}$ and $+1\frac{1}{3}$).

This shows that a sketch of the relevant curve helps to avoid misleading results arising from perfectly correct calculation.

Questions

Q14 Confirm that the total area enclosed by $y = x^2(x-2)$, the x-axis, $x = 1$ and $x = 3$ is $4\frac{1}{2}$.

What is the value of $\left[\frac{1}{4}x^4 - \frac{2}{3}x^3\right]_1^3$?

Q15 Sketch the curve $y = x(x-1)(x-2)$. Find the total area enclosed between this curve and the x-axis.

Example 5 **a** *Find the area under $y = 1/x^2$ from $x = 1$ to $x = 2$.* **b** *Can any meaning be attached to the phrase 'the area under $y = 1/x^2$ from $x = -1$ to $x = +2$'?*

a $\dfrac{dA}{dx} = y = \dfrac{1}{x^2} = x^{-2}$ $\quad\therefore A = -x^{-1} + c$

The required area $= \left[-\dfrac{1}{x}\right]_1^2$

$= (-\frac{1}{2}) - (-1)$

$= -\frac{1}{2} + 1$

$= \frac{1}{2}$

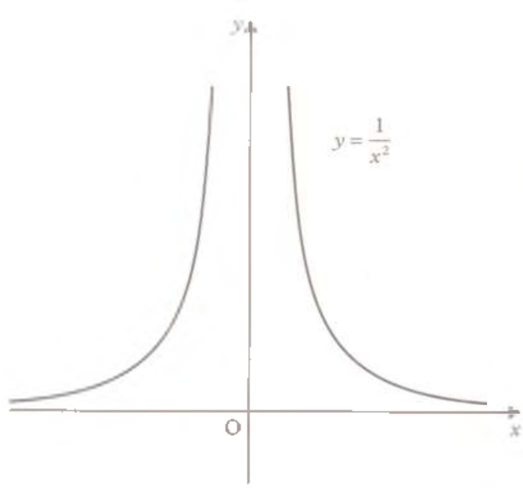


Figure 6.7

b **Fig. 6.7** is a sketch of $y = \dfrac{1}{x^2}$, and we see that if we try to find the area under the graph from $x = -1$ to $x = 2$, between these limits is the value $x = 0$ for which y has no value, and the curve consists of two separate branches. It is possible to evaluate $\left[-\dfrac{1}{x}\right]_{-1}^{+2}$ but the result, $-1\frac{1}{2}$, is meaningless. If we break up the area into two parts and integrate from -1 to 0 and from 0 to 2, in each case we get the meaningless term $\dfrac{1}{0}$. (See §2.5 on page 45.)

The second part of Example 5 shows that to calculate the area under a curve, the curve must have no breaks between the limits of x involved. I.e. the function must be continuous (see §2.16 on page 59) for all values of x between these limits.

Exercise 6c

1 Evaluate:

a $\left[\dfrac{x^4}{4}\right]_{1/2}^2$ **b** $\left[3x^3 - 4x\right]_{-1}^{+1}$

c $\left[\frac{1}{6}x^3 - 3x^2 + \frac{1}{2}x\right]_{-2}^{-1}$ **d** $\left[x^3 - \dfrac{1}{x^2}\right]_{-4}^{-3}$

2 Find the area enclosed by $x + 4y - 20 = 0$ and the axes, by integration. Check by another method.

3 Find the areas enclosed by the x-axis, and the following curves and straight lines:

a $y = 3x^2,\ x = 1,\ x = 3$

b $y = x^2 + 2,\ x = -2,\ x = 5$

c $y = x^2(x-1)(x-2),\ x = -2,\ x = -1$

d $y = 3/x^2,\ x = 1,\ x = 6$

4 Find the area under $y = 4x^3 + 8x^2$ from $x = -2$ to $x = 0$.

5 Sketch the curve $y = x^2 - 5x + 6$ and find the area cut off below the x-axis.

6 Sketch the curve $y = x(x+1)(2-x)$, and find the area of each of the two segments cut off by the x-axis.

7 Sketch the following curves and find the areas enclosed by them, and by the x-axis, and the given straight lines:

a $y = x(4-x),\ x = 5$ **b** $y = -x^3,\ x = -2$

c $y = x^3(x-1),\ x = 2$ **d** $y = 1/x^2 - 1,\ x = 2$

8 Find the area of the segment cut off from $y = x^2 - 4x + 6$ by the line $y = 3$.

9 Repeat question 8 for the curve $y = 7 - x - x^2$, and $y = 5$.

10 Find the points of intersection of the following curves and straight lines, and find the area of the segment cut off from each curve by the corresponding straight line:

a $y = \frac{1}{2}x^2,\ y = 2x$ **b** $y = 3x^2,\ 3x + y - 6 = 0$

c $y = (x+1)(x-2),\ x - y + 1 = 0$

11 Find the areas enclosed by the following curves and straight lines:

a $y = \frac{1}{2}x^3$, the y-axis, and $y = 32$

b $y = x^3 - 1$, the axes and $y = 26$

c $y = 1/x^2 - 1,\ y = -1,\ x = \frac{1}{2}$ and $x = 2$

12 Find the area enclosed by the curves $y = 2x^2$ and $y = 12x^2 - x^3$.

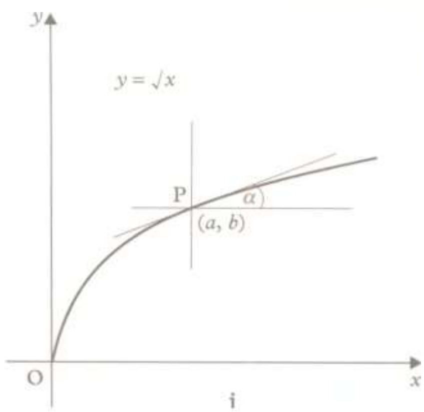
Chapter 7

Differentiation (2)

Chain rule

7.1 To differentiate the function $f(x) = x^n$ $(n \in \mathbb{Q})$

In this chapter we shall use fractional and negative indices. If you are not prepared for this first read §P2.2 on page 8. We are already familiar with the rule that the derivative of x^n is nx^{n-1}, but so far we have used it only when n has been a positive or negative integer or zero, i.e. for $n \in \mathbb{Z}$. We now need to extend this rule. First we shall prove its validity for the special case $n = \frac{1}{2}$.



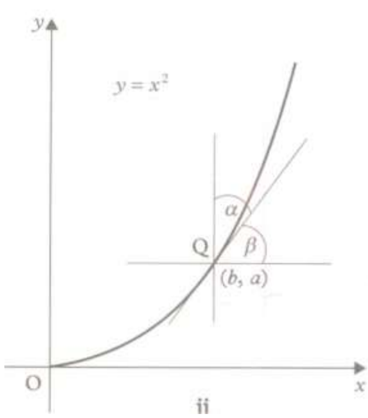


Figure 7.1

Fig. 7.1 shows the graphs of the function $f(x) = x^{1/2} = \sqrt{x}$ and its inverse function $f^{-1}(x) = x^2, x \geqslant 0$. We saw in Chapter 2 that the graph of the inverse function is the reflection of the graph of $y = f(x)$ in the line $y = x$. The point $Q(b, a)$ on the graph $y = x^2$ is the reflection of the point $P(a, b)$ on the graph $y = x^{1/2}$. Notice, in particular, that the tangent at P to $y = x^{1/2}$ is inclined at an angle α to the x-axis, whereas the tangent at Q to $y = x^2$ is inclined at an angle α to the y-axis. Thus, in Fig. 7.1ii, α is equal to $(90° - \beta)$. Also notice that, since $P(a, b)$ is on $y = \sqrt{x}$, and $Q(b, a)$ is on $y = x^2$, $a = b^2$, or $\sqrt{a} = b$.

The gradient of $y = \sqrt{x}$ at P, $f'(a)$, is equal to $\tan\alpha$, but at the moment we do not know how to find $f'(x)$. However, if we consider the graph of $y = x^2$, we know that $\tan\beta$ is given by the derivative of $y = x^2$ when $x = b$, which we can find. The derivative is $2x$ and hence

$$\tan\beta = 2b$$

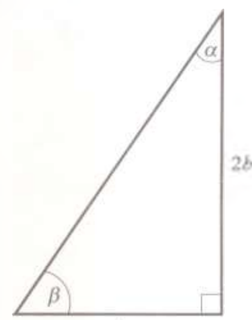


Figure 7.2

But $\tan\alpha = \tan(90° - \beta) = 1/(2b)$, (see Fig. 7.2), therefore

$$f'(a) = \frac{1}{2b} = \frac{1}{2\sqrt{a}} = \tfrac{1}{2}a^{-1/2}$$

So we have proved that, for the function $f(x) = x^{1/2}$, the derivative

$$f'(x) = \tfrac{1}{2}x^{-1/2}$$

and this is in accordance with the general rule we have previously been using for differentiating x^n. From now on we shall assume that

$$\textbf{if } f(x) = x^n,\ f'(x) = nx^{n-1} \quad \textbf{when } n \in \mathbb{Q},$$

i.e. when n is any rational number. It is important to remember that, although this assumption is indeed valid, we have so far justified the use of a general rule for differentiation simply by demonstrating its truth for particular values of n. At a more advanced level of study a proof can be provided.

Example 1 *Differentiate* **a** $\dfrac{2}{x^3}$, **b** $\dfrac{1}{\sqrt{x}}$.

a Let $\quad y = \dfrac{2}{x^3} = 2x^{-3}$

$$\therefore \frac{dy}{dx} = 2(-3)x^{-4}$$

$$\therefore \frac{d}{dx}\left(\frac{2}{x^3}\right) = \frac{-6}{x^4}$$

b Let $\quad y = \dfrac{1}{\sqrt{x}} = x^{-1/2}$

$$\therefore \frac{dy}{dx} = -\frac{1}{2}x^{-3/2}$$

$$\therefore \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = -\frac{1}{2\sqrt{x^3}}$$

Example 2 *Integrate* $\dfrac{3}{\sqrt[3]{x}}$.

If $\dfrac{dy}{dx} = \dfrac{3}{\sqrt[3]{x}} = 3x^{-1/3}$

$$y = 3\frac{x^{-1/3+1}}{-1/3+1} + c$$

$$= \tfrac{9}{2}\sqrt[3]{x^2} + c$$

Questions

Q1 Differentiate:

a x^{-4} **b** $2x^{-3}$ **c** $\dfrac{1}{x^2}$ **d** $\dfrac{4}{x}$

e $-\dfrac{2}{x^2}$ **f** $\dfrac{1}{3x^3}$ **g** $-\dfrac{1}{x^4}$ **h** $\dfrac{3}{5x^5}$

Integrate:

i x^{-3} **j** $2x^{-2}$ **k** $\dfrac{1}{x^2}$

l $\dfrac{2}{x^3}$ **m** $\dfrac{1}{3x^3}$ **n** $\dfrac{2}{5x^4}$

Q2 Differentiate:

a $x^{1/2}$ **b** $2x^{-1/3}$ **c** $\sqrt{x}$ **d** $\sqrt[3]{x}$

e $\dfrac{1}{\sqrt[3]{x}}$ **f** $\dfrac{-2}{\sqrt[3]{x}}$ **g** $2\sqrt{x^3}$ **h** $\dfrac{2}{3\sqrt{x}}$

Integrate:

i $x^{-1/4}$ **j** $2x^{3/2}$ **k** $\sqrt{x}$

l $3\sqrt{x}$ **m** $\dfrac{1}{\sqrt{x}}$ **n** $\dfrac{1}{\sqrt{x^3}}$

7.2 The chain rule

The process of differentiating a function was explained in Chapter 3. When faced with a simple expression you should differentiate it term by term after expansion. E.g. if

$$y = (x+3)^2 = x^2 + 6x + 9$$

then

$$\frac{dy}{dx} = 2x + 6 = 2(x+3)$$

The process of expansion can lead to laborious multiplication when something like $(x+3)^7$ is given. You might guess that $7(x+3)^6$ would be its derivative — and this would be correct but only because $x+3$ has the same derivative as x. Guessing is usually untrustworthy!

The derivative of $(3x+2)^4$ is *not* $4(3x+2)^3$.
The derivative of $(x^2+3x)^7$ is *not* $7(x^2+3x)^6$.

Question

Q3 In each part of this question, find $\dfrac{dy}{dx}$ by removing the brackets and them differentiating. Factorise each answer and try to guess its relationship to the original expression.

a $y = (x+4)^2$ **b** $y = (x+2)^3$
c $y = (3x+1)^2$ **d** $y = (5-2x)^2$
e $y = (x+4)^3$ **f** $y = (x^3+1)^2$
g $y = (5+x^2)^3$ **h** $y = (2+1/x)^2$
i $y = (1-x^3)^2$ **j** $y = (\tfrac{1}{2}x-7)^3$

Suppose y is a function of t, and t is itself a function of x. If δy, δt, and δx are corresponding small increments in the variables y, t, and x, then

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta t} \times \frac{\delta t}{\delta x} \qquad (1)$$

When δy, δt, and δx tend to zero,

$$\frac{\delta y}{\delta x} \to \frac{dy}{dx}, \qquad \frac{\delta y}{\delta t} \to \frac{dy}{dt}, \qquad \frac{\delta t}{\delta x} \to \frac{dt}{dx}$$

and equation (1) becomes

$$\mathbf{\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}}$$

This important result is known as the **chain rule**. It will affect almost every exercise in differentiation which you will meet from here onwards, so it is most important to master it. The following examples will give you some practice in its use.

Example 3 *Differentiate* $(3x+2)^4$.

Let $y=(3x+2)^4$ and $t=3x+2$, then $y=t^4$.

$$\frac{dt}{dx}=3, \qquad \frac{dy}{dt}=4t^3$$

By the chain rule,

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$$

$$\therefore \frac{dy}{dx}=4t^3\times 3$$

$$\therefore \frac{d}{dx}\{(3x+2)^4\}=12(3x+2)^3$$

Example 4 *Differentiate* $(x^2+3x)^7$.

Let $y=(x^2+3x)^7$ and $t=x^2+3x$, then $y=t^7$.

$$\frac{dt}{dx}=2x+3 \qquad \frac{dy}{dt}=7t^6$$

$$\therefore \frac{dy}{dx}=7t^6(2x+3)$$

$$\therefore \frac{d}{dx}\{(x^2+3x)^7\}=7(2x+3)(x^2+3x)^6$$

In the very simple instance of Example 3 a similar method will apply for integration, i.e. $\int(3x+2)^4\,dx$ does equal $\frac{1}{5}(3x+2)^5\times\frac{1}{3}$, but this is a special case. A corresponding division rule in integration does *not* apply. The integration of these awkward composite functions is explained later in Chapter 24 (§24.4, page 261).

It is not necessary to show the actual substitution, as has been done in the examples above, but it is advisable. The bracket is really treated as a single term — the t of our formula — and then you must remember to 'multiply by the derivative of the bracket'. Differentiation of reciprocals and roots of functions is pure chain rule technique.

In the function notation, the chain rule becomes $(fg)'(x)=f'(g(x))\times g'(x)$, but this lacks the elegant simplicity of the statement

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$$

(Remember this as 'differentiate y with respect to t and then multiply by $\frac{dt}{dx}$'.)

Example 5 *Differentiate* $\frac{1}{1+\sqrt{x}}$.

Let $y=(1+\sqrt{x})^{-1}$

$$\therefore \frac{dy}{dx}=-1\times(1+\sqrt{x})^{-2}\times\left\{\frac{d}{dx}(1+\sqrt{x})\right\}$$

$$=-1\times(1+\sqrt{x})^{-2}\times(\tfrac{1}{2}x^{-1/2})$$

$$\therefore \frac{d}{dx}\left(\frac{1}{1+\sqrt{x}}\right)=\frac{-1}{2\sqrt{x}(1+\sqrt{x})^2}$$

Example 6 *Differentiate* $\sqrt{(1+x^2)}$.

Let $y=(1+x^2)^{1/2}$

$$\therefore \frac{dy}{dx}=\tfrac{1}{2}(1+x^2)^{-1/2}\times 2x$$

$$\therefore \frac{d}{dx}\{\sqrt{(1+x^2)}\}=\frac{x}{\sqrt{(1+x^2)}}$$

Exercise 7a

1 Differentiate:

a $(2x+3)^2$ b $2(3x+4)^4$
c $(2x+5)^{-1}$ d $(3x-1)^{2/3}$
e $(3-2x)^{-1/2}$ f $(3-4x)^{-3}$

2 Integrate:

a $(3x+2)^3$ b $(2x+3)^2$
c $(3x-4)^{-2}$ d $(2x+3)^{1/2}$

3 Differentiate:

a $\frac{1}{(3x+2)}$ b $\frac{1}{(2x+3)^2}$
c $\frac{1}{\sqrt{(3x+1)}}$ d $\frac{1}{(2x-1)^{2/3}}$

4 Integrate:

a $\frac{1}{(2x-3)^2}$ b $\frac{1}{\sqrt{(3x+2)}}$
c $\frac{1}{(2x-1)^{3/4}}$

5 Differentiate:

a $(3x^2+5)^3$ b $(3x^3+5x)^2$
c $(7x^2-4)^{1/3}$ d $(6x^3-4x)^{-2}$
e $(3x^2-5x)^{-2/3}$

6 Differentiate:

a $\frac{1}{(3x^2+2)}$ b $\frac{3}{\sqrt{(2+x^2)}}$

c $\dfrac{-1}{(1+\sqrt{x})^2}$ d $\left(1-\dfrac{1}{x}\right)^3$

e $\dfrac{1}{(x^2-1)^{1/3}}$

7 Differentiate:

a $(3\sqrt{x}-2x)^3$ b $\left(\dfrac{2}{\sqrt{x}}-1\right)^{-1}$

c $\left(2x^2-\dfrac{3}{x^2}\right)^{1/3}$ d $\left(x-\dfrac{1}{x}\right)^{1/2}$

8 Differentiate:

a $\dfrac{1}{x^{3/2}-1}$ b $\sqrt{\left(1-\dfrac{1}{x}\right)}$

c $\sqrt[3]{(1-\sqrt{x})}$ d $\left(\sqrt{x}-\dfrac{1}{\sqrt{x}}\right)^2$

9 Differentiate:

a $\dfrac{1}{(x^2-7x)^3}$ b $\dfrac{1}{(x^2-\sqrt{x})^2}$

c $\sqrt{\left(\dfrac{1}{1-x^2}\right)}$ d $\left(\dfrac{1}{1-\sqrt{x}}\right)^2$

10 Differentiate:

a $\sqrt{\left(x^2-\dfrac{1}{x^2}\right)}$ b $\dfrac{2}{x+2\sqrt{x}}$

c $\left(1-\dfrac{2}{\sqrt{x}}\right)^{1/3}$ d $\sqrt{\left(1-\dfrac{1}{\sqrt{x}}\right)}$

7.3 Rates of change

The chain rule can be used to investigate related rates of change. Suppose a spherical balloon is inflated at the rate of 2 cm^3 every second. What is the rate of growth of the radius?

The solution of this type of problem has obvious calculus* possibilities because $\dfrac{dy}{dx}$ is the rate of change of y with respect to x, and with the formula of the preceding section we have a ready means of connecting rates of change of dependent variables.

If the radius of the balloon is r, then the volume, $V=\frac{4}{3}\pi r^3$.

We are given that $\dfrac{dV}{dt}$, the rate of change of the volume with respect to time, t, is 2 cm^3/s. By the chain rule,

$$\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt} \quad\text{and}\quad \frac{dV}{dr}=4\pi r^2$$

which leads to

$$\frac{dr}{dt}=\frac{2}{4\pi r^2}$$

i.e. the rate of change of the radius is $1/(2\pi r^2)$ cm/s. At some time you might have blown up a balloon or a tyre and noticed that it grows much more quickly at the beginning than near the end. The rate of change of the radius at any particular time could be calculated when the value of r is known. In the problem chosen, the radius after t s could be calculated from $\frac{4}{3}\pi r^3=2t$. The arithmetic is harder than the calculus.

Example 7 *A container in the shape of a right circular cone of height* 10 cm *and base radius* 1 cm *is catching the drips from a tap leaking at the rate of* 0.1 cm^3/s. *Find the rate at which the surface area of water is increasing when the water is half-way up the cone.*

Suppose the height of the water at any time is h cm, and that the radius of the surface of water at that time is r cm (Fig. 7.3).

Figure 7.3

By similar triangles,

$$\frac{r}{1}=\frac{h}{10}$$
$$\therefore r=\tfrac{1}{10}h$$

The surface area of water, $A=\pi r^2=\pi h^2/100$ and we wish to find $\dfrac{dA}{dt}$ when $h=5$. By the chain rule,

*The system of analysis using differentiation and/or integration is known as **infinitesimal calculus**, or calculus for short.

$$\frac{dA}{dt} = \frac{dA}{dh} \times \frac{dh}{dt} = \frac{2\pi h}{100} \times \frac{dh}{dt} \qquad (1)$$

The volume of water, $V = \frac{1}{3}\pi r^2 h = \pi h^3/300$, and using the chain rule again,

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt} = \frac{3\pi h^2}{300} \times \frac{dh}{dt}$$

We are given that $\frac{dV}{dt} = 0.1$,

$$\therefore \frac{dh}{dt} = \frac{dV}{dt} \times \frac{300}{3\pi h^2} = 0.1 \times \frac{100}{\pi h^2} = \frac{10}{\pi h^2} \qquad (2)$$

From (1) and (2)

$$\frac{dA}{dt} = \frac{2\pi h}{100} \times \frac{10}{\pi h^2} = \frac{1}{5h}$$

and, when $h = 5$,

$$\frac{dA}{dt} = \frac{1}{25} = 0.04$$

$\therefore$ when the water is half-way up, the rate of change of the surface area is equal to $0.04 \text{ cm}^2/\text{s}$.

Exercise 7b

1 The side of a cube is increasing at the rate of 6 cm/s. Find the rate of increase of the volume when the length of a side is 9 cm.

2 The area of surface of a sphere is $4\pi r^2$, r being the radius. Find the rate of change of the area in square cm per second when $r = 2$ cm, given that the radius increases at the rate of 1 cm/s.

3 The volume of a cube is increasing at the rate of $2 \text{ cm}^3/\text{s}$. Find the rate of change of the side of the base when its length is 3 cm.

4 The area of a circle is increasing at the rate of $3 \text{ cm}^2\text{s}^{-1}$. Find the rate of change of the circumference when the radius is 2 cm.

5 At a given instant the radii of two concentric circles are 8 cm and 12 cm. The radius of the outer circle increases at the rate of 1 cm/s and that of the inner at 2 cm/s. Find the rate of change of the area enclosed between the two circles.

6 If $y = (x^2 - 3x)^3$, find $\frac{dy}{dt}$ when $x = 2$, given $\frac{dx}{dt} = 2$.

7 A hollow right circular cone is held vertex downwards beneath a tap leaking at the rate of $2 \text{ cm}^3/\text{s}$. Find the rate of rise of water level when the depth is 6 cm given that the height of the cone is 18 cm and its radius 12 cm.

8 An ink blot on a piece of paper spreads at the rate of $\frac{1}{2} \text{ cm}^2\text{s}^{-1}$. Find the rate of increase of the radius of the circular blot when the radius is $\frac{1}{2}$ cm.

9 A hemispherical bowl is being filled with water at a uniform rate. When the height of the water is h cm the volume is $\pi(rh^2 - \frac{1}{3}h^3) \text{ cm}^3$, r cm being the radius of the hemisphere. Find the rate at which the water level is rising when it is half way to the top, given that $r = 6$ and that the bowl fills in 1 minute.

10 An inverted right circular cone of vertical angle 120° is collecting water from a tap at a steady rate of $18\pi \text{ cm}^3\text{min}^{-1}$. Find

- a the depth of the water after 12 minutes,
- b the rate of increase of the depth at this instant.

11 From the formula $v = \sqrt{(60s + 25)}$ the velocity, v, of a body can be calculated when its distance, s, from the origin is known. Find the acceleration when $v = 10$.

12 If $y = (x - 1/x)^2$, find $\frac{dx}{dt}$ when $x = 2$, given $\frac{dy}{dt} = 1$.

13 A rectangle is twice as long as it is broad. Find the rate of change of the perimeter when the breadth of the rectangle is 1 m and its area is changing at the rate of $18 \text{ cm}^2\text{s}^{-1}$, assuming the expansion is uniform.

14 A horse-trough has triangular cross-section of height 25 cm and base 30 cm, and is 2 m long. A horse is drinking steadily, and when the water level is 5 cm below the top it is being lowered at the rate of 1 cm/min. Find the rate of consumption in litres per minute.

7.4 Products and quotients

You are now able to differentiate quite elaborate functions, but no method has yet been suggested for a product such as $f(x) = (x + 1)^7(x - 3)^4$. We could multiply out the brackets and differentiate each term separately, but this would be extremely laborious. Although it is possible to differentiate each of the factors, we need a method for tackling the product as it stands. (We cannot simply write down the product of the two derivatives. If you are tempted to do so, you should consider the product $f(x) = x^3 \times x^4$, which is equal to x^7 and hence its derivative is $f'(x) = 7x^6$; but this is not the same as the product of the two derivatives $3x^2$ and $4x^3$.)

A brief return to fundamental ideas will produce a formula to help us with functions of this kind.

Let y be the product of two functions u and v of a variable x. Then $y = uv$ and

$$y + \delta y = (u + \delta u)(v + \delta v)$$

where a small increment δx in x produces increments δu in u, δv in v and δy in y.

$$y + \delta y = uv + v\delta u + u\delta v + \delta u\delta v$$

and since $y = uv$,

$$\delta y = v\delta u + u\delta v + \delta u\delta v$$

Dividing by δx,

$$\frac{\delta y}{\delta x} = v\frac{\delta u}{\delta x} + u\frac{\delta v}{\delta x} + \frac{\delta u}{\delta x} \times \delta v$$

As $\delta x \to 0$, δu, δv and δy also approach 0,

$$\frac{\delta y}{\delta x} \to \frac{dy}{dx} \qquad \frac{\delta u}{\delta x} \to \frac{du}{dx} \qquad \frac{\delta v}{\delta x} \to \frac{dv}{dx}$$

$$\therefore \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} + \frac{du}{dx} \times 0$$

$$\therefore \boldsymbol{\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}}$$

This formula must be remembered.

It is also necessary to remember that, should one of the factors in the product be a composite function, its derivative must be found as carefully as those in §7.2 on page 96 before insertion in this product formula.

Question

Q4 Use the above method to differentiate the following functions:

a $(x + 1)(x + 2)$

b $(x^2 + 1)x^2$

c $(x - 2)^2(x^2 - 2)$

d $(x + 1)^2(x + 2)^2$

Check your results by multiplying out and then differentiating.

The most common mistakes made in this type of question are due to careless algebra. Therefore particular attention should be paid to details of simplification.

Example 8 *Differentiate the expression $y = (x^2 - 3)(x + 1)^2$ and simplify the result.*

Let $u = (x^2 - 3)$ and let $v = (x + 1)^2$, then

$$\frac{du}{dx} = 2x \qquad \text{and} \qquad \frac{dv}{dx} = 2(x + 1)$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= (x + 1)^2 \times 2x + (x^2 - 3) \times 2(x + 1)\\ &= 2(x + 1)\{x(x + 1) + (x^2 - 3)\}\\ &= 2(x + 1)\{2x^2 + x - 3\}\\ &= 2(x + 1)(2x + 3)(x - 1)\end{aligned}$$

Example 9 *Differentiate $(x^2 + 1)^3(x^3 + 1)^2$.*

If $u = (x^2 + 1)^3$ and $v = (x^3 + 1)^2$, then let $y = uv$.

$$\frac{du}{dx} = 3(x^2 + 1)^2 \times 2x \qquad \text{and} \qquad \frac{dv}{dx} = 2(x^3 + 1) \times 3x^2$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= (x^3 + 1)^2 \times 6x(x^2 + 1)^2 + (x^2 + 1)^3 \times 6x^2(x^3 + 1)\\ &= 6x(x^3 + 1)(x^2 + 1)^2\{(x^3 + 1) + x(x^2 + 1)\}\end{aligned}$$

$$\therefore \frac{d}{dx}\{(x^2 + 1)^3(x^3 + 1)^2\} = 6x(x^3 + 1)(x^2 + 1)^2(2x^3 + x + 1)$$

Example 10 *Find the x-coordinates of the stationary points of the curve $y = (x^2 - 1)\sqrt{(1 + x)}$.*

$$y = (x^2 - 1)(x + 1)^{1/2}$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= (x + 1)^{1/2} \times 2x + (x^2 - 1) \times \tfrac{1}{2}(x + 1)^{-1/2}\\ &= \frac{2(x + 1) \times 2x + (x^2 - 1)}{2(x + 1)^{1/2}}\\ &= \frac{(x + 1)(4x + x - 1)}{2(x + 1)^{1/2}}\\ &= \frac{(5x - 1)(x + 1)}{2(x + 1)^{1/2}}\\ &= \tfrac{1}{2}(5x - 1)(x + 1)^{1/2}\end{aligned}$$

$\therefore$ for stationary points $x = \frac{1}{5}$ or -1.

There is a formula for quotients corresponding to that for products. It is proved in a similar way.

If $y = u/v$ then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

You may prefer to ignore this formula and deal with the quotient u/v as the product uv^{-1}.

Example 11 *Differentiate* $\frac{(x-3)^2}{(x+2)^2}$.

Let $y = (x-3)^2/(x+2)^2$ and let $u = (x-3)^2$ and $v = (x+2)^2$, then $y = u/v$.

$$\frac{du}{dx} = 2(x-3) \quad \text{and} \quad \frac{dv}{dx} = 2(x+2)$$

$$\therefore \frac{dy}{dx} = \frac{(x+2)^2 \times 2(x-3) - (x-3)^2 \times 2(x+2)}{(x+2)^4}$$

$$= \frac{2(x+2)(x-3)\{(x+2)-(x-3)\}}{(x+2)^4}$$

$$= \frac{2(x-3) \times 5}{(x+2)^3}$$

$$\therefore \frac{d}{dx}\left\{\frac{(x-3)^2}{(x+2)^2}\right\} = \frac{10(x-3)}{(x+2)^3}$$

Example 12 *Differentiate* $\frac{x}{\sqrt{(1+x^2)}}$.

Let $y = x/\sqrt{(1+x^2)}$ and let $u = x$ and $v = \sqrt{(1+x^2)}$, then $y = u/v$.

$$\frac{du}{dx} = 1 \quad \text{and} \quad \frac{dv}{dx} = \frac{2x}{2\sqrt{(1+x^2)}}$$

$$\therefore \frac{dy}{dx} = \frac{\sqrt{(1+x^2)} \times 1 - x \times \frac{x}{\sqrt{(1+x^2)}}}{(1+x^2)}$$

$$= \frac{1+x^2-x^2}{(1+x^2)^{3/2}}$$

$$\therefore \frac{d}{dx}\left\{\frac{x}{\sqrt{(1+x^2)}}\right\} = \frac{1}{(1+x^2)^{3/2}}$$

Questions

Q5 Prove the formula for quotients by the δu, δv method.

Q6 Differentiate:

a $(x^2+1)(x+3)^{-2}$ as a product,

b $\frac{x^2+1}{(x+3)^2}$ as a quotient.

Simplify the results and compare them.

Simplification was an essential part of answering Example 11. Since the gradient of a function is often needed for a specific purpose, you should get into the habit of factorising and simplifying as far as possible. It will be necessary to do this when checking with the answers at the back of the book! Practice in the algebra involved in differentiating a quotient is given in Chapters P1 and P2.

Exercise 7c

Differentiate with respect to x the following functions:

1 $x^2(x+1)^3$

2 $x(x^2+1)^4$

3 $(x+1)^2(x^2-1)$

4 $\frac{x}{x+1}$

5 $\frac{1-x^2}{1+x^2}$

6 $\frac{x^2+1}{(x+1)^2}$

7 $(1+x^2)^2(1-x^2)$

8 $x^2\left(1-\frac{1}{\sqrt{x}}\right)$

9 $(1-x^2)^2(1-x^3)$

10 $(x-1)\sqrt{(x^2+1)}$

11 $x^2\sqrt{(1+x^2)}$

12 $\frac{x^2}{\sqrt{(1+x^2)}}$

13 $\frac{(x-1)^2}{\sqrt{x}}$

14 $\frac{2x^2-x^3}{\sqrt{(x^2-1)}}$

15 $\sqrt{(x+2)}\sqrt{(x+3)}$

16 $\frac{\sqrt{x}}{\sqrt{(x+1)}}$

17 $\frac{1-\sqrt{x}}{1+\sqrt{x}}$

18 $\sqrt{\left(\frac{1+x}{2+x}\right)}$

19 $\sqrt{(x+1)}\sqrt{(x+2)^3}$

20 $\sqrt{\left\{\frac{(x+1)^3}{x+2}\right\}}$

7.5 Implicit functions

So far we have dealt only with *explicit* functions of x. E.g. $y = x^2 - 5x + 4/x$ where y is given as an expression in x. If, however, y is given *implicitly* by an equation such as $x = y^4 - y - 1$, we cannot express y in terms of x.

Consider an easier case. If $x = y^2$, $y = x^{1/2}$.

$$\therefore \frac{dy}{dx} = \tfrac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2y}$$

But $\dfrac{dx}{dy} = 2y$, so *in this case,*

$$\frac{dy}{dx} = \frac{1}{dx/dy}$$

(Strictly speaking, the equation $x = y^2$, does not define y as a *function* of x, since, for each positive value of x, there are *two* values of y, namely the positive and negative square roots of x.)

Now consider the general case.

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x/\delta y}$$

where δx and δy are the increments in x and y respectively.

Now as $\delta x, \delta y \to 0$, so $\dfrac{\delta y}{\delta x} \to \dfrac{dy}{dx}$, and $\dfrac{\delta x}{\delta y} \to \dfrac{dx}{dy}$

$$\therefore \mathbf{\frac{dy}{dx} = \frac{1}{dx/dy}}.$$

When it is impracticable to express either variable explicitly in terms of the other, we can still differentiate both sides with respect to x, as in Example 13 below. A term like y^n can be differentiated by first differentiating with respect to y then, as the chain rule demands, multiplying by $\dfrac{dy}{dx}$. Thus

$$\frac{d}{dx}(y^n) = \frac{d}{dy}(y^n)\frac{dy}{dx} = ny^{(n-1)}\frac{dy}{dx}$$

Similarly, if we have a term of the form $x^m y^n$, then we use the product rule and obtain

$$\frac{d}{dx}(x^m y^n) = x^m\frac{d}{dx}(y^n) + y^n\frac{d}{dx}(x^m)$$
$$= nx^m y^{(n-1)}\frac{dy}{dx} + mx^{(m-1)}y^n$$

Example 13 *Find the gradient of the curve*

$$x^2 + 2xy - 2y^2 + x = 2$$

at the point $(-4, 1)$.

To find the gradient, differentiate with respect to x.

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(2xy) - \frac{d}{dx}(2y^2) + \frac{d}{dx}(x) = \frac{d}{dx}(2)$$

$$\therefore 2x + \left(2y + 2x\frac{dy}{dx}\right) - 4y\frac{dy}{dx} + 1 = 0$$

$$\therefore \frac{dy}{dx}(2x - 4y) = -1 - 2x - 2y$$

When $x = -4$, $y = 1$,

$$\frac{dy}{dx}(-8 - 4) = -1 + 8 - 2$$

$$\therefore \frac{dy}{dx} = \frac{+5}{-12}$$

$\therefore$ the gradient at $(-4, 1)$ is $-\frac{5}{12}$.

Questions

Q7 Differentiate with respect to x:

a x **b** y **c** x^2 **d** y^2
e xy **f** x^2y **g** xy^2

Q8 Find $\dfrac{dy}{dx}$ if $x^2 + y^2 - 6xy + 3x - 2y + 5 = 0$.

7.6 Parameters

Sometimes both x and y are given as functions of a third variable, a **parameter**. In such cases the gradient is given in terms of the variable parameter. In Example 14, t is the parameter.

Example 14 *If* $x = t^3 + t^2$, $y = t^2 + t$ *find* $\dfrac{dy}{dx}$ *in terms of the parameter* t.

$$\frac{dx}{dt} = 3t^2 + 2t \qquad \frac{dy}{dt} = 2t + 1$$

Now $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx}$, but $\dfrac{dt}{dx} = \dfrac{1}{dx/dt}$.

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$\therefore \frac{dy}{dx} = \frac{2t + 1}{t(3t + 2)}$$

Question

Q9 Show that the above parametric representation is of the curve $y^3 = x^2 + xy$.

Find $\frac{dy}{dx}$ for this curve and show that it agrees with the above result.

Example 15 *Find the gradient of the curve* $x = \frac{t}{1+t}$, $y = \frac{t^3}{1+t}$ *at the point* $(\frac{1}{2}, \frac{1}{2})$.

$$\frac{dx}{dt} = \frac{(1+t)\times 1 - t\times 1}{(1+t)^2} = \frac{1}{(1+t)^2}$$

$$\frac{dy}{dt} = \frac{(1+t)\times 3t^2 - t^3\times 1}{(1+t)^2} = \frac{3t^2+2t^3}{(1+t)^2}$$

$$\therefore \frac{dy}{dx} = 3t^2 + 2t^3$$

At $(\frac{1}{2}, \frac{1}{2})$, $t = 1$,

$$\therefore \frac{dy}{dx} = 3 + 2$$

$\therefore$ the gradient at $(\frac{1}{2}, \frac{1}{2})$ is 5.

Exercise 7d

1 Find the gradient of the curve $2x^2 + 3y^2 = 14$ at the points where $x = 1$.

2 Find the x-coordinates of the stationary points of the curve represented by the equation $x^3 - y^3 - 4x^2 + 3y = 11x + 4$.

3 Find the gradient of the curve $x^2 - 3yx + 2y^2 - 2x = 4$ at the point $(1, -1)$.

4 Find the gradient of the tangent at the point $(2, 3)$ to the curve $xy = 6$.

5 a If $x = t^2$, $y = t^3$ find $\frac{dy}{dx}$ in terms of t.

b If $y = x^{3/2}$ find $\frac{dy}{dx}$.

Is there any connection between these two results?

6 At what points are the tangents to the circle $x^2 + y^2 - 6y - 8x = 0$ parallel to the y-axis?

7 Find $\frac{dy}{dx}$ when

a $x^2y^3 = 8$ b $xy(x - y) = 4$

8 Find $\frac{dy}{dx}$, in terms of t, when

a $x = at^2$, $y = 2at$

b $x = (t+1)^2$, $y = (t^2 - 1)$

9 If $x = t/(1-t)$ and $y = t^2/(1-t)$ find $\frac{dy}{dx}$ in terms of t.

10 Find $\frac{dy}{dx}$ in terms of x, y when $x^2 + y^2 - 2xy + 3y - 2x = 7$.

11 If $x = 2t/(t+2)$, $y = 3t/(t+3)$, find $\frac{dy}{dx}$ in terms of t.

12 Find $\frac{dy}{dx}$ in terms of x, y when $3(x-y)^2 = 2xy + 1$.

7.7 Small changes

We have seen that, as $\delta x \to 0$, $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$. Therefore, if δx is small,

$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$$

$$\therefore \delta y \approx \frac{dy}{dx}\delta x$$

Three applications of this formula follow in Examples 16–18.

Example 16 *The side of a square is* 5 cm. *Find the increase in the area of the square when the side expands* 0.01 cm.

Let the area of the square be A cm^2 when the side is x cm. Then $A = x^2$. Now

$$\delta A \approx \frac{dA}{dx}\delta x \quad \text{and} \quad \frac{dA}{dx} = 2x$$

$$\therefore \delta A \approx 2x\delta x$$

When $x = 5$ and $\delta x = 0.01$,

$$\delta A \approx 2 \times 5 \times 0.01 = 0.1$$

$\therefore$ the increase in area ≈ 0.1 cm^2.

In this case the increase in area can be found accurately very easily:

$$\delta A = 5.01^2 - 5^2 = 0.1001$$

You are strongly advised to use the calculus method for the moment, even if there seems to be quicker way. This method is an important introduction to certain topics which follow later.

Note that the error by the calculus method is, *in this case*, $(0.01)^2 = (\delta x)^2$.

Example 17 *A 2% error is made in measuring the radius of a sphere. Find the percentage error in surface area.*

Let the surface area be S and the radius be r, then

$$S = 4\pi r^2 \qquad \therefore \frac{\mathrm{d}S}{\mathrm{d}r} = 8\pi r$$

$$\therefore \delta S \approx 8\pi r \delta r$$

But the error in r is 2%, therefore $\delta r = \frac{2}{100} \times r$.

$$\therefore \delta S \approx 8\pi r \times \frac{2r}{100} = \frac{16\pi r^2}{100}$$

$$\therefore \frac{\delta S}{S} \approx \frac{16\pi r^2}{100} \div 4\pi r^2 = \frac{4}{100}$$

$\therefore$ the error in the surface area $\approx 4\%$.

Example 18 *Find an approximation for $\sqrt{9.01}$.*

Let $y = \sqrt{x}$, so $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{2\sqrt{x}}$.

$$\therefore \delta y \approx \frac{1}{2\sqrt{x}} \times \delta x$$

When $x = 9$, and $\delta x = 0.01$,

$$\delta y \approx \tfrac{1}{6} \times 0.01 \approx 0.001\,67$$

$$\therefore \sqrt{9.01} \approx 3.001\,67.$$

Exercise 7e

1 The surface area of a sphere is $4\pi r^2$. If the radius of the sphere is increased from 10 cm to 10.1 cm, what is the approximate increase in surface area?

2 An error to 3% is made in measuring the radius of the sphere. Find the percentage error in volume.

3 Find **a** $\sqrt[3]{8.01}$, **b** $\sqrt{25.1}$ by the method of Example 18.

4 If l cm is the length of a pendulum and t s the time of one complete swing, it is known that $l = kt^2$. If the length of the pendulum is increased by x%, x being small, find the corresponding percentage increase in time of swing.

5 If the pressure and volume of a gas are p and v then Boyle's law states pv = constant (k). If δp and δv denote corresponding small changes in p and v express $\dfrac{\delta p}{p}$ in terms of $\dfrac{\delta v}{v}$.

6 An error of $2\frac{1}{2}$% is made in the measurement of the area of a circle. What percentage error results in **a** the radius, **b** the circumference?

7 The height of a cylinder is 10 cm and its radius is 4 cm. Find the approximate increase in volume when the radius increases to 4.02 cm.

8 One side of a rectangle is three times the other. If the perimeter increases by 2% what is the percentage increase in area?

9 The radius of a closed cylinder is equal to its height. Find the percentage increase in total surface area corresponding to unit percentage increase in height.

10 Find **a** $\sqrt{627}$, **b** $\sqrt[3]{1005}$, by the method of Example 18.

11 The volume of a sphere increases by 2%. Find the corresponding percentage increase in surface area.

12 As x increases, prove that the area of a circle of radius x and the area of a square of side x increase by the same percentage, provided that the increase in x is small.

7.8 Second derivative

We know that velocity, v, is the rate of change of displacement, s, with respect to time, t, and may be denoted by $\dfrac{\mathrm{d}s}{\mathrm{d}t}$. Acceleration is the rate of change of velocity with respect to time, and we have up to now denoted this by $\dfrac{\mathrm{d}v}{\mathrm{d}t}$; but $\dfrac{\mathrm{d}}{\mathrm{d}t}(v)$ may also be written as $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}s}{\mathrm{d}t}\right)$, and thus acceleration is seen to be the second derivative of s with respect to t.

The second derivative arises in a wide variety of contexts and we therefore need a more convenient notation.

$$\frac{\mathbf{d}}{\mathbf{d}x}\left(\frac{\mathbf{d}y}{\mathbf{d}x}\right) \textbf{ is written as } \frac{\mathbf{d}^2y}{\mathbf{d}x^2}$$

which is said 'd two y by d x squared'.

Acceleration, $\frac{d^2s}{dt^2}$, may be written in yet another way by using the fact that

$$\frac{dv}{dt} = \frac{dv}{ds} \times \frac{ds}{dt} = \frac{dv}{ds} \times v$$

Thus we have arrived at the following alternative notations,

$$a = \frac{\mathbf{d}v}{\mathbf{d}t} = \frac{\mathbf{d}^2s}{\mathbf{d}t^2} = v\frac{\mathbf{d}v}{\mathbf{d}s}$$

the last form, $v\frac{dv}{ds}$, being applicable when velocity or acceleration is a function of s rather than of t.

Remember that if $y = f(x)$, $\frac{dy}{dx}$ is written as **f′(x)**, and $\frac{d^2y}{dx^2}$ as **f″(x)**.

Question

Q10 a If $y = x^2 - 1/x^2$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

b Given that $v = 3(4 - s^2)^{1/2}$, show that $a = -9s$.

c If $f(x) = x/(x - 1)$, find $f'(x)$ and $f''(x)$.

If $\frac{dy}{dx}$ is found in terms of a parameter t, $\frac{d^2y}{dx^2}$ requires differentiation with respect to x, so

$$\frac{\mathbf{d}^2y}{\mathbf{d}x^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} = \frac{\mathbf{d}}{\mathbf{d}t}\left(\frac{\mathbf{d}y}{\mathbf{d}x}\right) \div \frac{\mathbf{d}x}{\mathbf{d}t}$$

Question

Q11 If $x = a(t^2 - 1)$, $y = 2a(t + 1)$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t.

Chapter 8

Integration (2)

Area and volume

8.1 Some standard curves

In §5.5 on page 82 we met some simple aids to curve sketching. By this stage, you should be thoroughly familiar with some standard curves which will be frequently occurring in the work which follows.

Fig. 8.1 shows some variations on the curve $y = x^2$, which is a *parabola*. The line of symmetry of the parabola is called the *axis*, and it cuts the curve at the *vertex*. Thus for the curve $y = x^2 + c$, the axis is the y-axis, and the vertex is $(0, c)$. Any equation of the form $y = ax^2 + bx + c$, where a, b, and c are constants $(a \neq 0)$, represents a parabola with the axis parallel to the y-axis (see Chapter 10, page 130).

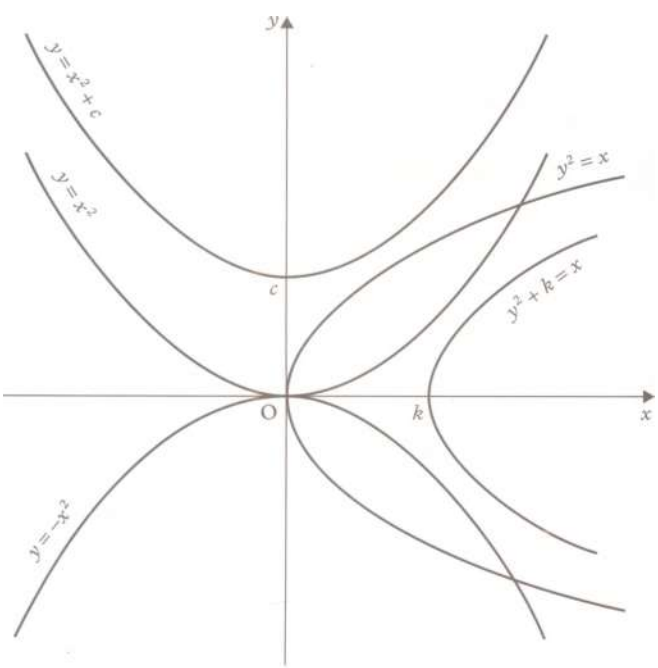


Figure 8.1

Typical shapes of curves for which y is given as a cubic function of x are shown in Fig. 8.2.

i represents $y = (x + 3)(x + 1)(x - 2)$, the x^3 term in the expansion being positive;

ii represents $y = (3 + x)(1 + x)(2 - x)$, the x^3 term in the expansion being negative (see §5.5, Example 7, page 83).

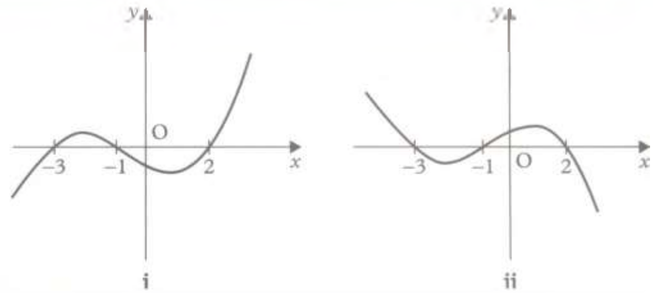


Figure 8.2

Fig. 8.3 shows **i** $y = x(x - 2)^2$, and **ii** $y = -(x + 1)^3$, illustrating that when the function of x has a squared factor, the curve touches the x-axis; and with a cubed factor, the curve touches *and crosses* the x-axis.

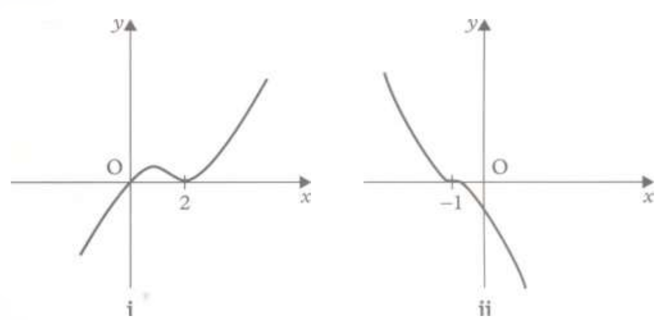


Figure 8.3

Fig. 8.4 shows how a sketch of the curve $y = x^2 + 1/x$ may be built up by adding the y-coordinates of the two known curves $y = x^2$ and $y = 1/x$.

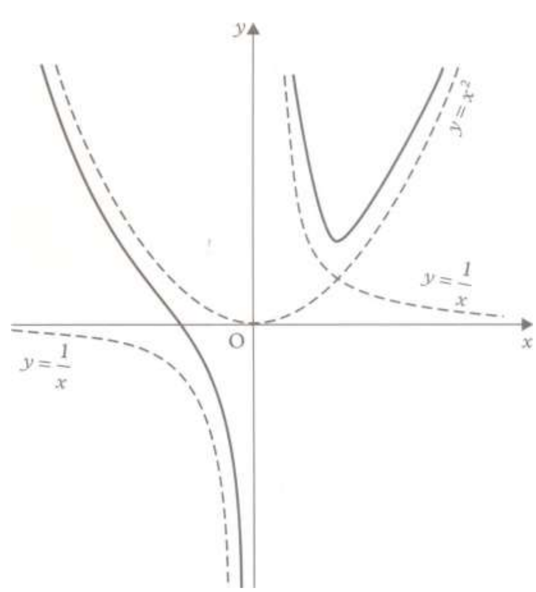


Figure 8.4

8.2 Integration of x^n ($n \in \mathbb{Q}$)

In Chapter 7 on page 95 the differentiation of x^n was assumed to include cases where n is a fraction, and so we can now integrate powers of x with fractional indices. Thus, if

$$\frac{dy}{dx} = \sqrt{x} = x^{1/2}$$

then

$$y = \frac{x^{3/2}}{3/2} + c = \tfrac{2}{3}x^{3/2} + c$$

In general,

if $$\frac{dy}{dx} = x^n \ (n \in \mathbb{Q})$$

then $$y = \frac{x^{n+1}}{n+1} + c$$

Exercise 8a

1 Sketch the following curves:

a $y = 4x^2$ b $y = -x^2 + 9$
c $y - 1 = x^2$ d $x = -y^2$
e $x - y^2 + 4 = 0$ f $2x + y^2 + 16 = 0$

2 Sketch the following curves showing where each meets the x-axis:

a $y = (x-1)(x-2)(x-3)$
b $y = (1-x)(x-2)(x-3)$
c $y = (x+1)(x-2)^2$
d $y = x^2(3-x)$
e $y = (x+2)(1-x)^2$
f $y^2 = x^6$
g $x = y^3$
h $x + y^3 = 0$
i $x = y(y-3)^2$

3 Sketch the following curves:

a $y = -x^4$ b $y = \dfrac{1}{x^2}$
c $y = x^2 + \dfrac{1}{x^2}$ d $y = x^3 + \dfrac{1}{x}$
e $y = x^3 + \dfrac{1}{x^2}$ f $y = x^2 - \dfrac{1}{x}$
g $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

4 Integrate with respect to x:

a $x^{1/3}$ b $4\sqrt{x}$
c $2x^{1/5}$ d $k\sqrt[3]{x}$
e $x^{-1/2}$ f $\dfrac{1}{\sqrt[3]{x}}$
g $x^{-1/6}$ h $\dfrac{2}{\sqrt[5]{x}}$
i $\sqrt[3]{x^2}$ j $x^{7/3}$
k $(\sqrt{x})^3$ l $x^{-4/3}$
m $x^{1/a}$ n $\dfrac{1}{\sqrt[n]{x}}$
o $\dfrac{x^3 + 2x^2 - 3x}{\sqrt{x}}$ p $\sqrt{x} + \dfrac{2}{\sqrt{x}}$
q $(\sqrt{x}+2)(\sqrt{x}-3)$ r $\sqrt{(x+2)}$
s $x\sqrt{(x^2-3)}$

5 Evaluate the following:

a $\left[x^{-1/2}\right]_1^4$ b $\left[x^{3/2} + 2x^{1/2}\right]_4^9$
c $\left[\tfrac{2}{3}(x+4)^{3/2}\right]_0^5$

8.3 Area as the limit of a sum

We have already discussed the use of integration in finding the area under a curve (§6.3 on page 91). The word *integration* implies the putting together of parts to make up a whole. This fundamental aspect of the process is brought out in the following alternative approach to the area under a curve.

Suppose that we wish to find the area under the curve in Fig. 8.5 from $x = 0$ to $x = 3$. We divide this area into three equal strips by the lines $x = 1$ and $x = 2$.

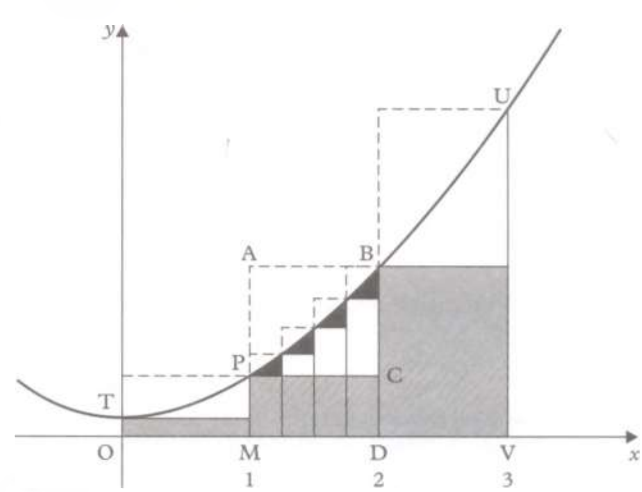


Figure 8.5

The required area TUVO lies between the sum of the areas of the three shaded 'inside' rectangles, and the sum of the three 'outside' rectangles bounded at the top by the broken lines. For example, the middle strip PBDM lies between the areas PCDM and ABDM.

We shall for the time being confine our attention to the 'inside' rectangles. The sum of these falls short of the required area by the sum of PBC and the two corresponding areas. We now divide TUVO into 12 strips (for clarity only 4 of these are shown in **Fig. 8.5**).
The sum of the 12 'inside' rectangles is clearly a better approximation to the area under the curve, since an error such as PBC has been reduced to a much smaller error represented by the black roughly triangular areas. Thus by taking a sufficient number of strips (in other words, by making the width of each strip sufficiently small) we can make the sum of the areas of the 'inside' rectangles as near as we please to the area under the curve.

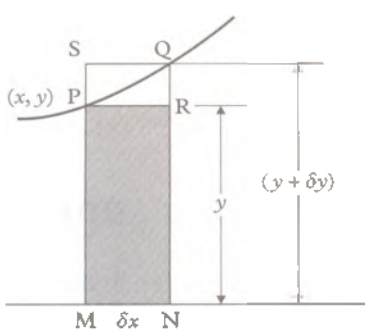


Figure 8.6

If we were to divide the area TUVO into a very large number of strips, then a typical one would be PQNM (**Fig. 8.6**), where $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ are two points on the curve. A typical 'inside' rectangle is PRNM, of area $y\delta x$, and the process of increasing the number of strips is the same as letting $\delta x \to 0$. The required area TUVO is found by adding all the 'inside' rectangular areas $y\delta x$ between $x = 0$ and $x = 3$, and then finding the *limit* of this sum as $\delta x \to 0$. Using the symbol $\sum$ to denote 'the sum of' (see §13.8 on page 168),

$$\text{as } \delta x \to 0, \quad \sum_{x=0}^{x=3} y\delta x \to \text{the area TUVO}$$

*Hence area TUVO = the *limit*, as $\delta x \to 0$, of $\sum_{x=0}^{x=3} y\delta x$.

*For simplicity we have confined our attention to the 'inside' rectangles. **Fig. 8.6** also shows a typical 'outside' rectangle SQNM of area $(y + \delta y)\delta x$; as $\delta x \to 0$, $\sum_{x=0}^{x=3} (y + \delta y)\delta x$ tends to the same limit.

Example 1 *Calculate the area under $y = x + 1$ from $x = 0$ to $x = 10$.*

Divide the area into n strips of equal width parallel to Oy (**Fig. 8.7**); the width of each strip will be $10/n$. To find the sum of the areas of the inner shaded rectangles we must first calculate their heights.

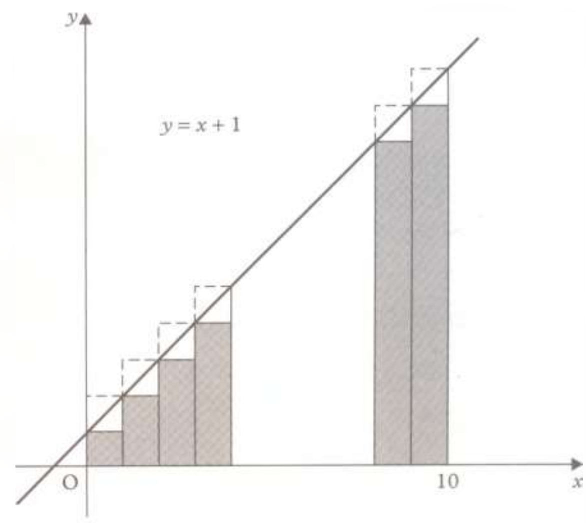


Figure 8.7

For the three smallest,

when $x = 0$, $\quad y = x + 1 = 1$

when $x = \dfrac{10}{n}$, $\quad y = \dfrac{10}{n} + 1$

when $x = 2 \times \dfrac{10}{n}$, $\quad y = \dfrac{20}{n} + 1$

and for the largest,

when $x = 10 - \dfrac{10}{n}$, $\quad y = 11 - \dfrac{10}{n}$

The sum of the areas of the inner rectangles is

$$\left\{\frac{10}{n}(1) + \frac{10}{n}\left(\frac{10}{n} + 1\right) + \frac{10}{n}\left(\frac{20}{n} + 1\right) + \ldots + \frac{10}{n}\left(11 - \frac{10}{n}\right)\right\}$$
$$= \frac{10}{n}\left\{1 + \left(\frac{10}{n} + 1\right) + \left(\frac{20}{n} + 1\right) + \ldots + \left(11 - \frac{10}{n}\right)\right\}$$

We have not written all of the terms in the above statement. We have used three dots (...) to represent them. There are as many terms in the curly brackets as there are strips, namely n, and they form an arithmetic progression (see §13.2 on page 159) with common difference $10/n$. We can now add the terms in the brackets using the formula

$$S_n = \frac{n}{2}(a+l) \qquad \text{(See §13.4 on page 162)}$$
$$= \frac{n}{2}\left(1+11-\frac{10}{n}\right) = \frac{n}{2}\left(12-\frac{10}{n}\right)$$

∴ the sum of the 'inside' rectangles

$$= \frac{10}{n}\times\frac{n}{2}\left(12-\frac{10}{n}\right)$$
$$= 60-\frac{50}{n}$$

As $n \to \infty$, the limit of the sum is 60,

∴ the area under $y = x + 1$ from $x = 0$ to $x = 10$ is 60 unit^2*.

Question

Q1 Calculate the sum of the areas of the n 'outside' rectangles in Example 1, and find the limit of this sum as $n \to \infty$.

Example 2 *Calculate the area under the curve $y = x^2$, from $x = 0$ to $x = a$.*

Here again divide the interval $0 \leqslant x \leqslant a$, into n equal sub-intervals, each of length a/n (**Fig. 8.8**). To find the area inside the shaded region, first calculate the heights of the $(n-1)$ rectangles. Since the equation of the curve is $y = x^2$, these heights are

$$\left(\frac{a}{n}\right)^2, \quad \left(\frac{2a}{n}\right)^2, \quad \left(\frac{3a}{n}\right)^2, \quad \ldots \quad \left(\frac{(n-1)a}{n}\right)^2$$

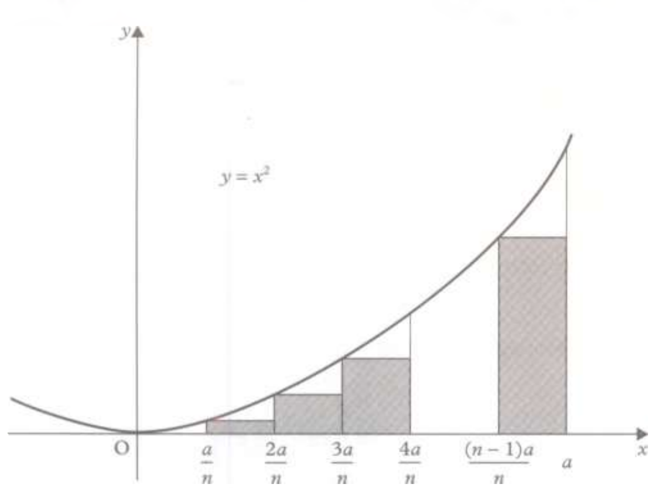


Figure 8.8

*In the rest of this chapter it will be assumed that all areas are measured in basic unit^2.

and, since the width of each rectangle is a/n, the sum of the areas of the rectangles is

$$\frac{a}{n}\times\frac{a^2}{n^2}+\frac{a}{n}\times\frac{4a^2}{n^2}+\frac{a}{n}\times\frac{9a^2}{n^2}+\ldots+\frac{a}{n}\times\frac{(n-1)^2a^2}{n^2}$$
$$=\frac{a^3}{n^3}+\frac{4a^3}{n^3}+\frac{9a^3}{n^3}+\ldots+\frac{(n-1)^2a^3}{n^3}$$
$$=\frac{a^3}{n^3}(1+4+9+\ldots(n-1)^2)$$

Now, it can be shown (see §13.7 on page 166) that

$$1+4+9+\ldots(n-1)^2 = \frac{1}{6}(n-1)\times n\times(2n-1)$$
$$= \frac{1}{6}(2n^3-3n^2+n)$$

Hence the sum of the areas of these rectangles is

$$S = \frac{a^3}{n^3}\times\frac{1}{6}(2n^3-3n^2+n)$$
$$= \frac{a^3}{6}\left(2-\frac{3}{n}+\frac{1}{n^2}\right)$$

and hence, when $n \to \infty$,

$$S \to \frac{a^3}{3}$$

Hence the area under the curve $y = x^2$, from $x = 0$ to $x = a$, is given by $a^3/3$.

It is interesting to note that this result, proved by a similar method, was known to the ancient Greeks, long before the development of calculus.

8.4 Integral notation

Example 1 could be done by integration. Before doing this, we introduce the symbol $\int(\ldots)\,dx$ to denote integration with respect to x. The symbol $\int$, which is an elongated S, for 'sum', is a reminder that integration is essentially summation.

The area under $y = x + 1$ from $x = 0$ to $x = 10$ is

$$\int_0^{10} y\,dx = \int_0^{10}(x+1)\,dx$$
$$= \left[\tfrac{1}{2}x^2+x\right]_0^{10}$$
$$= (\tfrac{1}{2}\times 10^2+10)-(0)$$
$$= 60$$

Similarly the result of Example 2 can be obtained by integration, as follows:

$$\int_0^a x^2\,dx = \left[\tfrac{1}{3}x^3\right]_0^a$$
$$= \tfrac{1}{3}a^3$$

For indefinite integrals, where there are no limits, a similar notation is used. Thus

$$\int (3x^2 + 4)\,dx = x^3 + 4x + c$$

Questions

Q2 Find the following indefinite integrals:

a $\int (3x - 4)\,dx$

b $\int \frac{8x^5 - 3x}{x^3}\,dx$

c $\int \sqrt[3]{x}\,dx$

d $\int (2\sqrt{t} - 3)(1 - \sqrt{t})\,dt$

Q3 Evaluate the following definite integrals:

a $\int_{1/2}^{1} (60t - 16t^2)\,dt$

b $\int_1^2 \frac{1}{2x^4}\,dx$

c $\int_1^4 \frac{(y+3)(y-3)}{\sqrt{y}}\,dy$

We have shown above that when $y = x + 1$, the limit of $\sum_{x=0}^{x=10} y\delta x$, as $\delta x \to 0$, is identical with, and is more readily evaluated as $\int_0^{10} y\,dx$.

We shall now assume that for any curve which is continuous between $x = a$ and $x = b$, the area under the curve from $x = a$ to $x = b$ is

$$\text{the limit, as } \delta x \to 0, \text{ of } \sum_{x=a}^{x=b} y\delta x = \int_a^b y\,dx^*$$

Notice that, in general, if $f(x)$ is a continuous function and $F(x)$ is the function whose derivative is $f(x)$, i.e. $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b$$
$$= F(b) - F(a)$$

If, in addition, $f(x) \geqslant 0$, when $a \leqslant x \leqslant b$, then this integral gives the area under the curve $y = f(x)$, from $x = a$ to $x = b$. If, however, $f(x)$ is not always positive in this interval, then the graph of $y = f(x)$ must be consulted, in order to distinguish between the positive and negative areas.

Think of every area bounded by a curve as a summation, first writing down the area of one of the typical strips, or *elements of area*, into which it is most conveniently divided, and then evaluating the limit of the sum of those strips by integration. A convenient way of laying out the working is shown in the following examples. These extend the work of Chapter 6 in the following ways:

a by using elements of area parallel to the x-axis, we may integrate with respect to y;

b by finding the element of area cut off between two curves we may evaluate in only one step the area enclosed between them.

Example 3 *Find the area enclosed by $y = 4x - x^2$, $x = 1$, $x = 2$ and the x-axis* (**Fig. 8.9**).

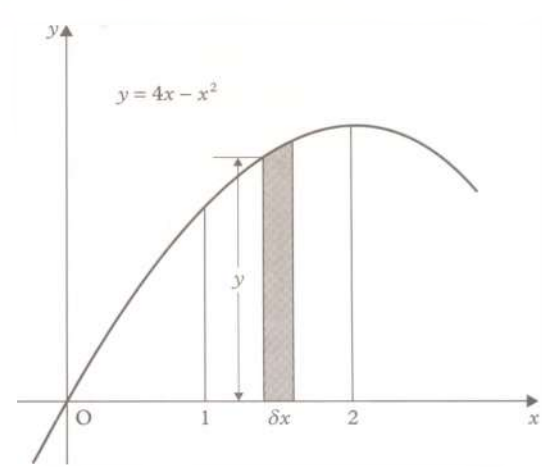


Figure 8.9

The element of area is $y\delta x = (4x - x^2)\delta x$

$$\therefore \text{ the required area} = \int_1^2 (4x - x^2)\,dx$$
$$= \left[2x^2 - \tfrac{1}{3}x^3\right]_1^2$$
$$= (8 - \tfrac{8}{3}) - (2 - \tfrac{1}{3})$$
$$= 3\tfrac{2}{3}$$

*Note the similarity between this statement and that relating to gradient, namely the limit, as $\delta x \to 0$, of $\frac{\delta y}{\delta x} = \frac{dy}{dx}$.

Example 4 *Find the area enclosed by that part of $y = x^2$ for which x is positive, the y-axis, and the lines $y = 1$ and $y = 4$.*

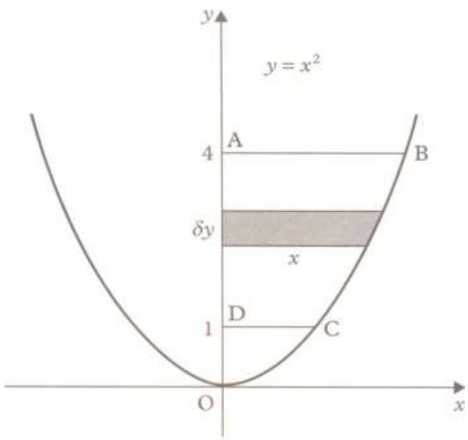


Figure 8.10

The required area is ABCD in Fig. 8.10. The equation may be written $x = \pm\sqrt{y}$, and for the part of the curve with which we are concerned $x = +\sqrt{y} = +y^{1/2}$.

The element of area is $x\delta y$.

$$\therefore \text{ the required area } = \int_1^4 x\,dy$$
$$= \int_1^4 y^{1/2}\,dy$$
$$= \left[\tfrac{2}{3}y^{3/2}\right]_1^4$$
$$= (\tfrac{2}{3}\times 8) - (\tfrac{2}{3})$$
$$= 4\tfrac{2}{3}$$

Example 5 *Find the area enclosed between the two curves $y = 4 - x^2$ and $y = x^2 - 2x$.*

To find the limits of integration we must find the x-coordinates of the points of intersection of the two curves.

When $x^2 - 2x = 4 - x^2$,

$$2x^2 - 2x - 4 = 0$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = -1 \quad \text{or} \quad +2$$

Fig. 8.11 is a sketch of the two curves, showing a typical element of area (shaded).

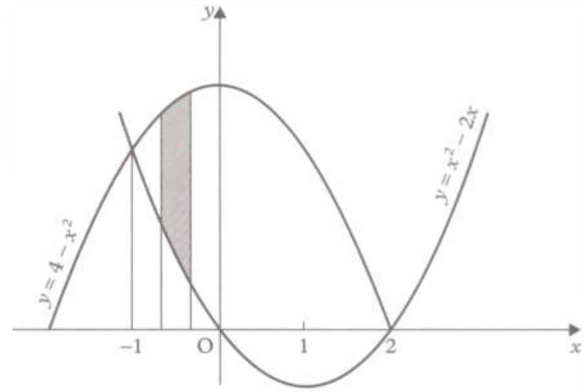


Figure 8.11

If we write $Y = 4 - x^2$ and $y = x^2 - 2x$, the element of area is $(Y - y)\delta x$.

$$\therefore \text{ the required area } = \int_{-1}^{+2} (Y - y)\,dx$$
$$= \int_{-1}^{+2} \{(4 - x^2) - (x^2 - 2x)\}\,dx$$
$$= \int_{-1}^{+2} (4 + 2x - 2x^2)\,dx$$
$$= \left[4x + x^2 - \tfrac{2}{3}x^3\right]_{-1}^{+2}$$
$$= (4\times 2 + 2^2 - \tfrac{2}{3}\times 2^3) - (-4 + 1 + \tfrac{2}{3})$$
$$= 8 + 4 - 5\tfrac{1}{3} + 4 - 1\tfrac{2}{3}$$
$$= 9$$

Exercise 8b

1 Find the following integrals:

a $\int x(x-3)\,dx$

b $\int \frac{2(x-1)}{x^3}\,dx$

c $\int \left(at^2 + b + \frac{c}{t^2}\right)dt$

d $\int \left(x^4 - \sqrt[3]{x^2} + 2 - \frac{1}{x^2}\right)dx$

e $\int \left(y + \frac{1}{\sqrt{y}}\right)\left(y + \frac{1}{y}\right)dy$

f $\int \frac{(s+1)^2}{\sqrt[3]{s}}\,ds$

2 Evaluate:

a $\int_{-2}^{+3}(v^2+3)\,dv$ **b** $\int_{1}^{4}(y^2+\sqrt{y})\,dy$

c $\int_{0}^{1}\sqrt{x}(x+2)\,dx$ **d** $\int_{1}^{2}\left(3+\frac{1}{t^2}+\frac{1}{t^4}\right)dt$

e $\int_{1}^{9}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$ **f** $\int_{4}^{11}\sqrt{(x+5)}\,dx$

3 Find the area under each of the following curves between the given limits:

a $y=x^2+3,\quad x=-1$ to $x=2$

b $y=x^2(3-x),\quad x=4$ to $x=5$

c $y=x^2+1/x^2,\quad x=\frac{1}{2}$ to $x=1$

4 Find the area enclosed by the y-axis and the following curves and straight lines:

a $x=y^2, y=3$

b $y=x^3, y=1, y=8$

c $x-y^2-3=0, y=-1, y=2$

d $x=1/\sqrt{y}, y=2, y=3$

5 Find the area enclosed by each of the following curves and the y-axis:

a $x=(y-1)(y-4)$ (Why is this negative?)

b $x=3y-y^2$ **c** $x=y(y-2)^2$

6 Find the area enclosed by $y^2=4x$ and the straight lines $x=1$ and $x=4$.

7 Find the area enclosed by $y^2=x+9$ and the y-axis, by taking an element of area **a** parallel to the y-axis, and **b** parallel to the x-axis.

8 Find the area enclosed by $9x^2+y-16=0$ and the x-axis, by integrating **a** with respect to x, and **b** with respect to y.

9 Calculate the areas enclosed by

a $y=1/x^2, y=1$ and $y=4$

b $x=1/y^2, y=1, y=4$, and $x=0$

10 Find the area of the segment cut off from each of the following curves by the given straight line:

a $y=x^2-2x+2, y=5$

b $y=x^2-6x+9, y=1$

c $y=-x^2+3x-4, y=-4$

d $y=x(x-2), y=x$

e $y=4-3x-x^2, 2x+y+2=0$

f $y=x^2-6x+2, x+y-2=0$

11 Find the area enclosed by each of the following pairs of curves:

a $y=x(x-1)$ and $y=x(2-x)$

b $y=x(x+3)$ and $y=x(5-x)$

c $y=x^2-5x$ and $y=3x^2-6x$

d $y^2=4x$ and $x^2=4y$

e $y=x^2-3x-7$ and $y=5-x-x^2$

f $y=2x^2+7x+3$ and $y=9+4x-x^2$

12 Find the area of the segment cut off from $y=1/x^2$ by $10x+4y-21=0$, given that one of the points of intersection of the straight line and the curve is $(-\frac{2}{5},\frac{25}{4})$.

13 By reference to a clear diagram, show that if $f(x)$ is an odd function, then

$$\int_{-a}^{+a} f(x)\,dx = 0$$

Show also that if $g(x)$ is an even function, then

$$\int_{-a}^{+a} g(x)\,dx = 2\int_{0}^{+a} g(x)\,dx$$

14 Prove, using the method of Example 2 on page 109, that

$$\int_{0}^{a} x^3\,dx = \frac{a^4}{4}$$

[You will need to quote that
$1+8+27+\ldots+(N-1)^3=N^2(N-1)^2/4$.]

8.5 Solids of revolution

If we take a triangular piece of cardboard ABC with a right angle at B, and rotate it through 360 degrees about AB, it will sweep out the volume of a right circular cone (**Fig. 8.12**). The cone can be thought of as the **solid of revolution** generated by rotating the area ABC about the line AB.

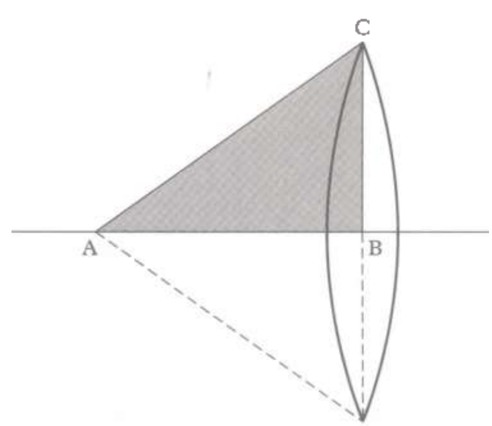


Figure 8.12

Question

Q4 State the solid generated by rotating through 360 degrees:

a the above triangle ABC **i** about BC, **ii** about AC,

b the area of a semi-circle about the bounding diameter,

c a quadrant of a circle about a boundary radius,

d the area of a circle centre (3, 3) radius 1, about the y-axis,

e a rectangle about one of its sides.

The method of calculating the volume of a solid of revolution is best illustrated by discussing an example. The ideas involved are the same as those of §8.3 on page 107. See Example 6.

Example 6 *Find the volume of the solid generated by rotating about the x-axis the area under $y = \frac{3}{4}x$ from $x = 0$ to $x = 4$.*

A typical element of area under $y = \frac{3}{4}x$ is $y\delta x$, shown shaded in **Fig. 8.13**. Rotating this area about the x-axis generates a typical *element of volume*, a cylinder of volume $\pi y^2\delta x$. The corresponding 'slice' of the solid (**Fig. 8.14**) has one circular face of radius y, and the other of radius $y + \delta y$, and its volume lies between that of an 'inside' cylinder $\pi y^2\delta x$, and an 'outside' cylinder $\pi(y + \delta y)^2\delta x$.

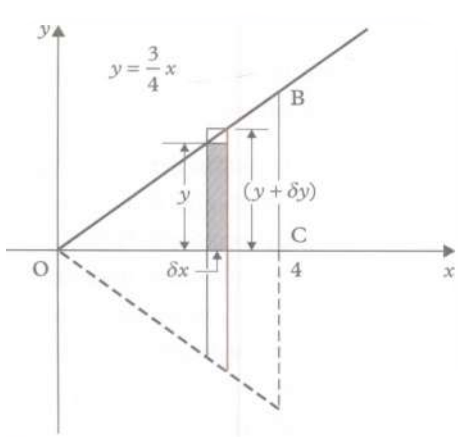


Figure 8.13

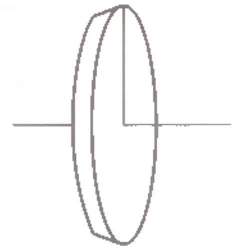


Figure 8.14

The sum of the volumes of all the 'inside' (or 'outside') cylinders is an approximation to the volume required. By making δx sufficiently small, we can make this sum approach as close as we please to the volume of the solid of revolution. This may therefore be written as

$$\text{the limit, as } \delta x \to 0, \text{ of } \sum_{x=0}^{x=4} \pi y^2 \delta x$$

This may be evaluated as $\int_0^4 \pi y^2 \, dx$.

Since $y = \frac{3}{4}x$, $y^2 = \frac{9}{16}x^2$ and

$$\text{the element of volume} = \pi y^2 \delta x = \pi \frac{9x^2}{16} \delta x$$

$$\therefore \text{ the required volume} = \int_0^4 \pi \frac{9x^2}{16} \, dx$$

$$= \left[\pi \frac{3x^3}{16}\right]_0^4$$

$$= \pi \frac{3 \times 4^3}{16}$$

$$= 12\pi^*$$

Question

Q5 Find the volume of the solid generated by rotating about the x-axis

a the area under $y = x^2$ from $x = 1$ to $x = 2$,

b the area under $y = x^2 + 1$ from $x = -1$ to $x = +1$.

The volumes of solids generated by rotating areas about the y-axis may be evaluated by integration with respect to y. This, and other aspects of this work, are illustrated by the following examples.

*As before, the dimensions are in $unit^3$; this is assumed and not stated.

Example 7 *Find the volume of the solid generated by rotating about the y-axis the area in the first quadrant enclosed by $y = x^2$, $y = 1$, $y = 4$ and the y-axis* (**Fig. 8.15**).

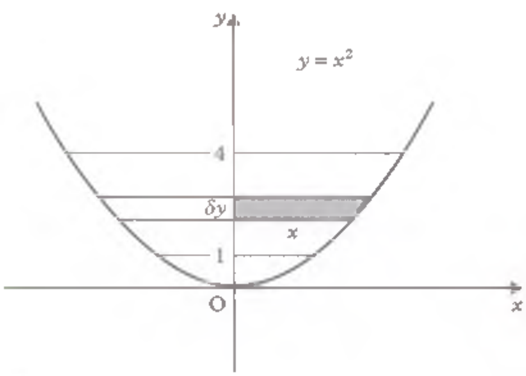


Figure 8.15

The element of volume $= \pi x^2 \delta y = \pi y \delta y$

$$\therefore \text{ the required volume } = \int_1^4 \pi y \, dy$$

$$= \left[\tfrac{1}{2}\pi y^2\right]_1^4$$

$$= \tfrac{1}{2}\pi \times 16 - \tfrac{1}{2}\pi$$

$$= \frac{15\pi}{2}$$

Example 8 *The area of the segment cut off by $y = 5$ from the curve $y = x^2 + 1$ is rotated about $y = 5$; find the volume generated* (**Fig. 8.16**).

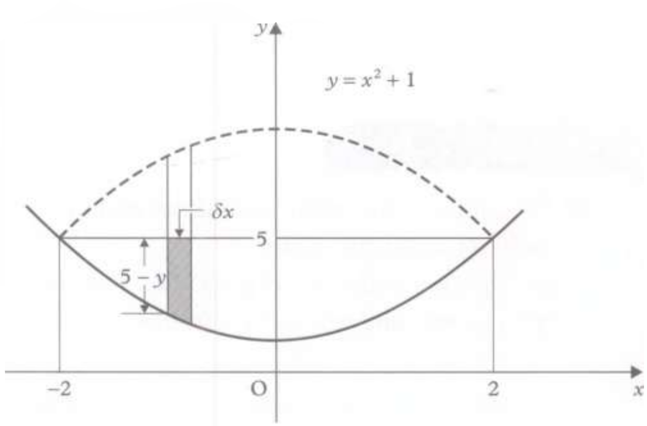


Figure 8.16

The points of intersection occur when

$$x^2 + 1 = 5$$
$$x^2 = 4$$
$$x = -2 \quad \text{or} \quad +2$$

The element of volume $= \pi(5 - y)^2 \delta x$
$= \pi(5 - x^2 - 1)^2 \delta x$
$= \pi(16 - 8x^2 + x^4)\delta x$

$\therefore$ the required volume

$$= \int_{-2}^{+2} \pi(16 - 8x^2 + x^4)\, dx$$

$$= \left[\pi(16x - \tfrac{8}{3}x^3 + \tfrac{1}{5}x^5)\right]_{-2}^{+2}$$

$$= \pi(32 - 21\tfrac{1}{3} + 6\tfrac{2}{5}) - \pi(-32 + 21\tfrac{1}{3} - 6\tfrac{2}{5})$$

$$= 34\tfrac{2}{15}\pi$$

Example 9 *The area of the segment cut off by $y = 5$ from the curve $y = x^2 + 1$ is rotated about the x-axis; find the volume generated* (**Fig. 8.17**).

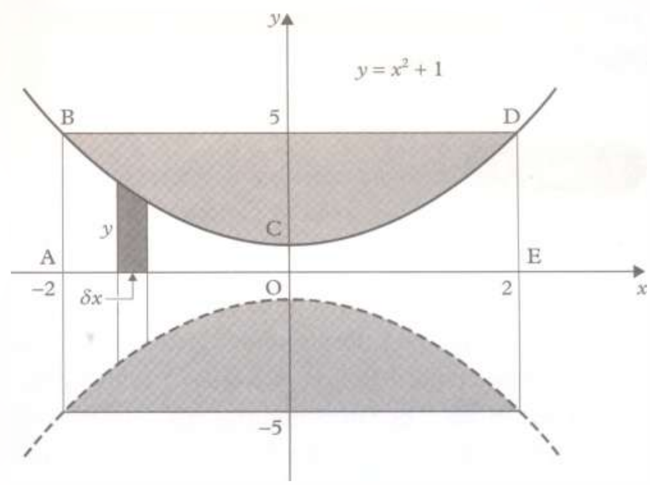


Figure 8.17

The solid generated is a cylinder fully open at each end, but with the internal diameter decreasing towards the middle; its volume is found by subtracting the volume of the cavity from the volume of the solid cylinder of the same external dimensions.

The required volume = the volume generated by rotation, about the x-axis, of the rectangle ABDE (1)
− the volume generated by rotation, about the x-axis, of the area under $y = x^2 + 1$ from $x = -2$ to $x = +2$, i.e. ABCDE (2)

Volume (1) $= \pi r^2 h = \pi \times 5^2 \times 4 = 100\pi$

Element of Volume (2) $= \pi y^2 \delta x$
$= \pi(x^4 + 2x^2 + 1)\delta x$

$$\therefore \text{Volume (2)} = \int_{-2}^{+2} \pi(x^4 + 2x^2 + 1)\,dx$$

$$\therefore \text{Volume (2)} = \left[\pi\left(\frac{x^5}{5} + \frac{2}{3}x^3 + x\right)\right]_{-2}^{+2}$$

$$= \pi(6\tfrac{2}{5} + 5\tfrac{1}{3} + 2) - \pi(-6\tfrac{2}{5} - 5\tfrac{1}{3} - 2)$$

$$= 27\tfrac{7}{15}\pi$$

$$\therefore \text{the required volume} = 100\pi - 27\tfrac{7}{15}\pi$$
$$= 72\tfrac{8}{15}\pi$$

Exercise 8c

Leave π in the answers.

1 Find the volumes of the solids generated by rotating about the x-axis each of the areas bounded by the following curves and lines:

- a $x + 2y - 12 = 0$, $x = 0$, $y = 0$
- b $y = x^2 + 1$, $y = 0$, $x = 0$, $x = 1$
- c $y = \sqrt{x}$, $y = 0$, $x = 2$
- d $y = x(x - 2)$, $y = 0$
- e $y = x^2(1 - x)$, $y = 0$
- f $y = 1/x$, $y = 0$, $x = 1$, $x = 4$

2 Find the volumes of the solids generated by rotating about the y-axis each of the areas bounded by the following curves and lines:

- a $y = 2x - 4$, $y = 2$, $x = 0$
- b $x = \sqrt{(y - 1)}$, $x = 0$, $y = 4$
- c $x - y^2 - 2 = 0$, $x = 0$, $y = 0$, $y = 3$
- d $y^2 = x + 4$, $x = 0$
- e $y = 1 - x^3$, $x = 0$, $y = 0$
- f $xy = 1$, $x = 0$, $y = 2$, $y = 5$

3 Find the volumes of the solids generated when each of the areas enclosed by the following curves and lines is rotated about the given line:

- a $y = x$, $x = 0$, $y = 2$, about $y = 2$
- b $y = \sqrt{x}$, $y = 0$, $x = 4$, about $x = 4$
- c $y^2 = x$, $x = 0$, $y = 2$, about $y = 2$
- d $y = 2 - x^2$, $y = 1$, about $y = 1$
- e $y = x^3 - 2x^2 + 3$, $y = 3$, about $y = 3$
- f $y = 1/x^2$, $y = 4$, $x = 1$, about $y = 4$

4 Repeat Q3 for the following areas:

- a $x - 3y + 3 = 0$, $x = 0$, $y = 2$, about the x-axis
- b $x - y^2 - 1 = 0$, $x = 2$, about the y-axis
- c $y^2 = 4x$, $y = x$, about $y = 0$
- d $y = 1/x$, $y = 1$, $x = 2$, about $y = 0$

5 Obtain, by integration, the formula for the volume of a right circular cone of base radius r, height h. (Consider the area enclosed by $y = (h/r)x$, $x = 0$ and $y = h$.)

6 The equation of a circle centre the origin and radius r is $x^2 + y^2 = r^2$. By considering the area of this circle cut off in the first quadrant being rotated about either the x- or y-axis, deduce the formula for the volume of a sphere radius r.

7 A hemispherical pot of internal radius 13 cm contains water to a maximum depth of 8 cm. Find the volume of the water.

8 A spherical pot has an inside diameter of 20 cm. Calculate the volume of water it contains when the maximum depth is 18 cm.

9 A container has one plane face, and its volume is equivalent to that generated when the area enclosed by $x = \frac{1}{64}y^3 + 1$, the y-axis and $y = 8$ is rotated through 2 right angles about the y-axis, the units being cm. Calculate its volume.

10 The area under $y = \frac{1}{9}x^2 + 1$ from $x = 0$ to $x = 3$, and the area enclosed by $y = 0$, $y = 2$, $x = 3$, and $x = 4$, are rotated about the y-axis, and the solid generated represents a metal tray, the units being cm. Calculate the volume of metal.

11 The area enclosed by $y = x^2 - 6x + 18$ and $y = 10$ is rotated about $y = 10$. Find the volume generated.

12 The area enclosed by $y = x^2 + 1/x$, the x-axis and $x = -2$, is rotated about the x-axis. Find the volume generated.

13 The area enclosed by $y = 4/x^2$, $y = 1$ and $y = 4$ is rotated about the x-axis. Find the volume generated.

14 The area enclosed by $y = x^2 - 6x + 18$ and $y = 10$ is rotated about the y-axis. Find the volume generated. [Take an element of area parallel to the x-axis of length $(x_2 - x_1)$. Express the typical element of volume in terms of y by using the fact that x_1 and x_2 are the roots of $x^2 - 6x + (18 - y) = 0$; see §P2.7 on page 15.]

15 Repeat question 14 for the area enclosed by $4y = 4x^2 - 20x + 25$ and $4y = 9$.

Chapter 9

Inequalities and linear programming

Introduction

A basic understanding of **inequalities** and **graphs of inequalities** is important for some of the more advanced work that follows in Chapter 28. This chapter revises elementary ideas of inequalities and extends them to finding 'best solutions' to real-life problems by transforming them into mathematical models that take into account restrictions that can be represented by graphs of linear inequalities.

9.1 Inequalities

This section revises the elementary work on inequalities that were covered when studying at School Certification level.

In mathematics, we probably use the equals sign, =, more than any other. For example, $5 + 3 = 8$. However, many quantities are *not* equal:

$$5 + 5 \neq 8$$

where $\neq$ means *is not equal to*. We can also write:

$$5 + 5 > 8$$

where $>$ means *is greater than*. Similarly we can write the following:

$$3 + 3 \neq 8$$
$$3 + 3 < 8$$

where $<$ means *is less than*.

In many countries, voters in elections must be not less than 18 years of age. If a person of age y years can vote, then $y > 18$ or $y = 18$. We can write this as a single statement:

$$y \geqslant 18$$

where $\geqslant$ means *is greater than or equal to*. Note $\geqslant$ also means *is not less than*.

In towns and cities there is usually a speed limit of 50 km/h. If a car is travelling legally at s km/h, then either $s < 50$ or $s = 50$ (the maximum legal speed). We can write this as one inequality:

$$s \leqslant 50$$

where $\leqslant$ means *is less than or equal to*. Note that $\leqslant$ also means *is not greater than*.

Statements like $y \geqslant 18$ and $s \leqslant 50$ are called **inequalities**.

Inequality symbols

The common inequality symbols are:

- $>$ is greater than
- $<$ is less than
- $\geqslant$ is greater than or equal to (not less than)
- $\leqslant$ is less than or equal to (not greater than)
- $\neq$ is not equal to

Questions

Q1 Rewrite the following using $>$, $<$, $\geqslant$ or $\leqslant$ instead of the words.

- **a** 8 is less than 14
- **b** 0 is greater than $-\frac{1}{4}$
- **c** x is greater than 5
- **d** y is not greater than 5
- **e** 17 is not greater than m
- **f** 17 is not less than n

Q2 State whether each of the following is true or false.

- **a** $12 > 11$ **b** $7 < 18$
- **c** $-12 > -11$ **d** $-7 < -18$
- **e** $7 \leqslant 14 \div 2$ **f** $7 \geqslant 2 \times 3\frac{1}{2}$
- **g** $9 \geqslant -4 \times -2$ **h** $-4 \times -5 \leqslant -21$

9.2 Linear inequalities in one variable

Solution of linear inequalities

In most cases, solve inequalities in the same way as you would solve equations.

Example 1 *Solve the inequality* $8 \geqslant 3x - 2$.

$$8 \geqslant 3x - 2$$

Add 2 to both sides:

$$10 \geqslant 3x$$

Divide both sides by 3:

$$3\tfrac{1}{3} \geqslant x$$

i.e. $x \leqslant 3\frac{1}{3}$

Example 2 *Solve the inequality* $\frac{5-3x}{2} \geqslant \frac{2x+3}{5}$.

$$\frac{5-3x}{2} \geqslant \frac{2x+3}{5}$$

Multiply through by 10.

$$\frac{10(5-3x)}{2} \geqslant \frac{10(2x+3)}{5}$$
$$5(5-3x) \geqslant 2(2x+3)$$
$$25 - 15x \geqslant 4x + 6$$

Add $15x$ to both sides:

$$25 \geqslant 19x + 6$$

Subtract 6 from both sides:

$$19 \geqslant 19x$$

Divide both sides by 19:

$$1 \geqslant x$$

i.e. $x \leqslant 1$

Multiplication and division by negative numbers

When multiplying or dividing both sides of an inequality by a negative number, we must take special care. Read Examples 3 and 4 carefully.

Example 3 *Find the range of values of x for which* $1 - 6x > 5$.

$$1 - 6x > 5$$

Subtract 1 from both sides:

$$-6x > 4$$

Divide both sides by -6 *and reverse the inequality*:

$$x < -\tfrac{2}{3}$$

Example 4 *Solve* $\frac{3}{4}x - \frac{4}{5}x \leqslant \frac{1}{2}$.

$$\tfrac{3}{4}x - \tfrac{4}{5}x \leqslant \tfrac{1}{2}$$
$$\Rightarrow \quad \tfrac{15}{20}x - \tfrac{16}{20}x \leqslant \tfrac{1}{2}$$
$$\Rightarrow \quad -\tfrac{1}{20}x \leqslant \tfrac{1}{2}$$

Multiply both sides by -20 *and reverse the inequality*:

$$x \geqslant -10$$

Examples 1, 2, 3, 4 show that

a we may *add* or *subtract* any number to both sides of an inequality,

b we may *multiply* or *divide* both sides of an inequality by any *positive* number,

c if we *multiply* or *divide* both sides of an inequality by a *negative* number, we must *reverse* the inequality.

Exercise 9a

1 Solve the following inequalities.

a $x + 2 \geqslant 3$ b $x - 3 < 1$
c $3 - x \geqslant 1$ d $3x \leqslant 12$
e $-5x < 30$ f $4x + 20 > 0$

2 Solve the following inequalities.

a $5 - 2x \leqslant x - 1$
b $8x - 5 \leqslant 6x + 7$
c $p + 2 < 96 - p$
d $7x - (5x - 3) \geqslant 9$
e $3x + 4(x - 3) > x - 6$
f $2(x + 4) > 3(x - 1)$
g $\frac{1}{2}(3x - 2) \geqslant x - 6$
h $\frac{y+8}{3} - \frac{2y-4}{7} < 1$
i $\frac{x}{2} + \frac{3}{4} \leqslant \frac{5x}{6} - \frac{7}{12}$
j $\frac{7}{z+2} > \frac{-2}{5-4z}$

3 Solve the inequality $3(2x + 1) \geqslant \frac{1}{3}(2x - 9)$.

4 Solve the inequality $\frac{x-2}{3} - \frac{x+1}{2} < 2$.

5 Solve the inequality $\frac{1}{3}x + \frac{5}{8} \geqslant \frac{1}{2}x - \frac{5}{24}$.

Line graphs of inequalities in one variable

We can use a simple **number line** to show inequalities in one variable.

Fig. 9.1 shows two ways of doing this for the inequality $x < 3$.

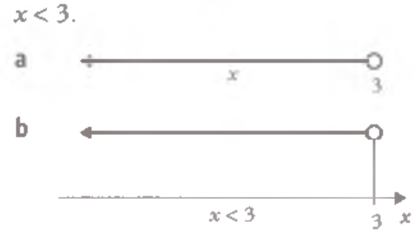


Figure 9.1

In each part of **Fig. 9.1** the heavy arrowed line represents the **range of values** of x.

The *empty circle* at 3 shows that the value 3 *is not included*.

Fig. 9.2 shows the inequality $x \geqslant -2$.

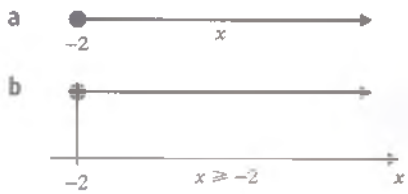


Figure 9.2

The *solid circle* at -2 shows that the value -2 *is included*.

Example 5 shows how to use diagrams like **Fig. 9.1** and **Fig. 9.2** to represent the **solution set** of a given inequality.

Example 5 *Solve $4 \leqslant 7 - 2x$ and show the solution set on the number line.*

$4 \leqslant 7 - 2x$

Subtract 7 from both sides:

$-3 \leqslant -2x$

Divide both sides by -2 *and reverse the inequality*:

$1\frac{1}{2} \geqslant x$

i.e. $\quad x \leqslant 1\frac{1}{2}$

Fig. 9.3 shows the solution set on the number line.

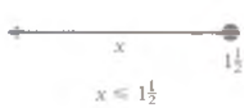


Figure 9.3

Combining inequalities

Example 6 *Represent the inequality $-6 \leqslant x < 2$ on the number line.*

An expression like $-6 \leqslant x < 2$ combines two inequalities:

$x \geqslant -6 \quad$ and $\quad x < 2$

Fig. 9.4 shows the combined inequality on the number line.

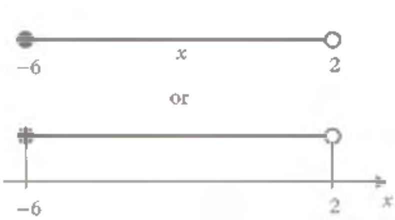


Figure 9.4

Example 7 *What is the range of values of x for which $2x - 1 > 3$ and $x - 3 < 5$ are both satisfied? Show the range on a number line.*

We are given that

$$
\begin{aligned}
& 2x - 1 > 3 \quad \textit{and} \quad x - 3 < 5 \\
\Rightarrow \quad & 2x > 4 \quad \textit{and} \quad x < 8 \\
\Rightarrow \quad & x > 2 \quad \textit{and} \quad x < 8 \\
\Rightarrow \quad & 2 < x < 8
\end{aligned}
$$

$2 < x < 8$ is the required range of values.

Fig. 9.5 shows the range on the number line.

Figure 9.5

Example 8 *Illustrate the set $\{-7 < x < 0\} \cap \{-2 < x < 2\}$ on a number line.*

Fig. 9.6a and **Fig. 9.6b** show the solutions sets $\{-7 < x < 0\}$ and $\{-2 < x < 2\}$ separately.

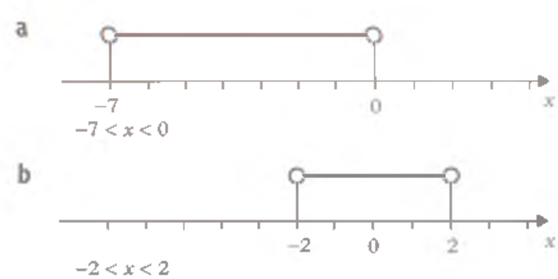


Figure 9.6

The intersection of the two sets is the region of the number line that satisfies both sets. This is shown in **Fig. 9.7**.

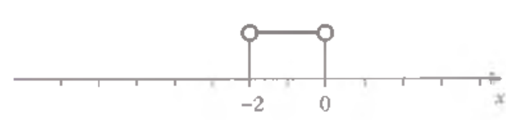


Figure 9.7

Exercise 9b

1 Sketch line graphs of the following inequalities.

a $x > 3$ b $x > -5$
c $x \geqslant -1$ d $x \leqslant 0$
e $x < 7$ f $x \geqslant 4$

2 State the range of values shown in each part of **Fig. 9.8**.

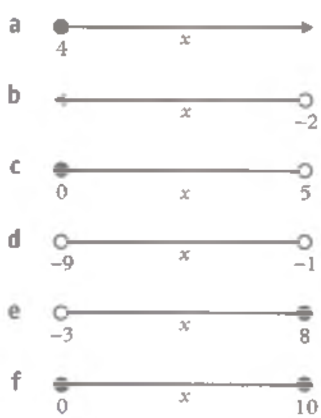


Figure 9.8

3 Make sketch graphs of the solution sets of the inequalities in Questions 1 and 2 of Exercise 9a.

4 Sketch graphs of the following inequalities.

a $-2 \leqslant x < 5$
b $3 < x \leqslant 11$
c $0 \leqslant x \leqslant 8$
d $-7 < x < -4$
e $-4 < x \leqslant 0$
f $-9 \leqslant x < 6$

5 a Solve the inequality $x + 1\frac{1}{2}x \geqslant 1\frac{2}{3}x - 1\frac{1}{6}$.
b Show the result on a line graph.

6 x is such that $3x + 4 < 7$ and $5x \geqslant 2x - 9$.
a What range of values of x satisfies both inequalities?
b Draw a graph which represents this inequality.

7 **Fig. 9.9** shows two ranges of values for a variable x.

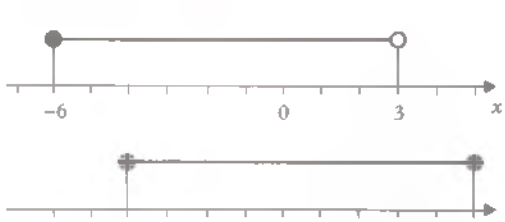


Figure 9.9

a Combine the two parts of **Fig. 9.9** into a single range of values of x.
b State the range in the form $c \square x \square d$, where c and d are numbers and the boxes contain inequality symbols. In your statement, identify the inequalities that should appear in the boxes.

8 Illustrate the following sets on a number line.
a $\{-9 \leqslant x \leqslant -2\} \cap \{-5 \leqslant x < 4\}$
b $\{0 < y \leqslant 10\} \cap \{3 \leqslant y < 8\}$

In each case state the resulting range of values.

9.3 Linear inequalities in two variables

Expressions such as $3x + 2y < 5$ or $x - 4y + 7 \geqslant 0$ are examples of **linear inequalities in two variables**, x and y.

Cartesian graphs of inequalities

(x, y) represents any point on a cartesian plane which has coordinates x and y. If (x, y) is such that $x \geqslant 2$ then (x, y) may lie anywhere in the unshaded region in **Fig. 9.10**. Small crosses show some possible positions of (x, y). The region on the left is shaded to show that it is *not* wanted.

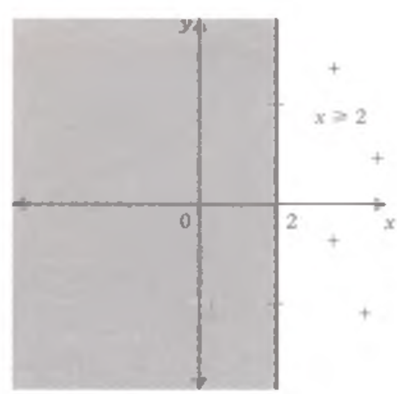


Figure 9.10

(An alternative convention is to shade the required region. This book uses the convention of shading the unrequired regions. Always label the required region.)

In **Fig. 9.10** the boundary of the shaded region is a line through the x-axis where $x = 2$. $x = 2$ at every point on this line. The **equation of the line** is $x = 2$.

The points shown by crosses are members of the set of points (x, y) for which $x \geqslant 2$. The unshaded region in Fig. 9.10 is the cartesian graph of the inequality $x \geqslant 2$.

Example 9 *On a cartesian plane, sketch the region which contains the set of points such that $y > -3$.*

The unshaded part of Fig. 9.11 is the required region.

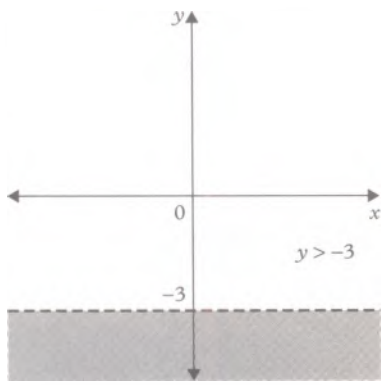


Figure 9.11

Note The boundary line has the equation $y = -3$. In Fig. 9.11 this line is broken to show that the points on the line are *not* included in the required region.

Example 10 *Combine Fig. 9.10 with Fig. 9.11 to show the region which represents the set of points for which $x \geqslant 2$ and $y > -3$.*

In Fig. 9.12 the unshaded part is the required region. The other parts are shaded to show that they are not wanted.

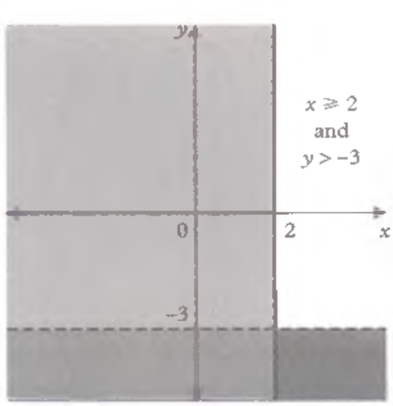


Figure 9.12

Note In Fig. 9.12 the unshaded region contains those points which satisfy both conditions.

Exercise 9c

1 On a cartesian plane, sketch and label the lines represented by the following equations.

a $x = 3$
b $y = 5$
c $y = -4$
d $x = -2$
e $x = 0$
f $y = 0$
g $y = -1$
h $x = -6$

2 On a cartesian plane sketch the regions which represent the following sets of points. Use the rule that shaded regions contain unwanted points and that solid boundary lines mean that the equality is included.

a $x \geqslant -1$
b $x < 2$
c $y > -2$
d $y \leqslant 3$
e $x \leqslant 0$
f $y \geqslant 0$
g $y < 6$
h $x > 5$

3 In Fig. 9.13 the points belonging to the shaded region are not wanted. State the set of points represented by the unshaded region.

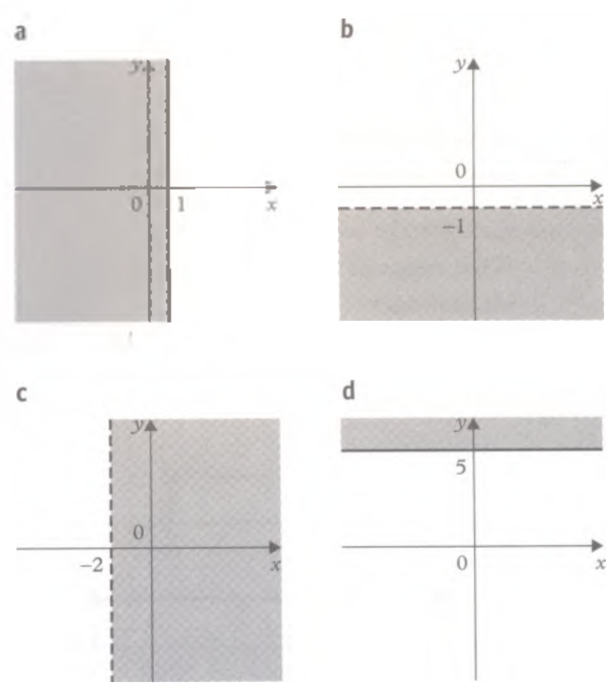


Figure 9.13

4 In each part of Fig. 9.14 the unshaded region represents a set of points defined by two inequalities. Write down the inequalities.

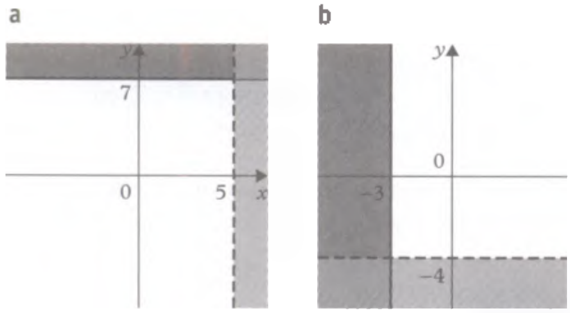


Figure 9.14

5 On a cartesian plane sketch the region which represents the set of points for which

a $x < 2$ and $y \geqslant 5$

b $x > 1$ and $y < -1$

Simultaneous linear inequalities

(x, y) represents any point on the cartesian plane which has coordinates x and y. In Fig. 9.15 the unshaded region represents the set of points for which $x \geqslant 1$ and $y < 2$.

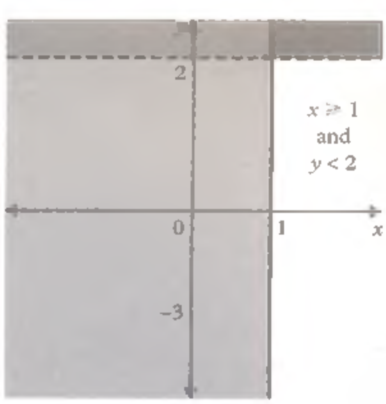


Figure 9.15

In Fig. 9.15,

1 $x \geqslant 1$ is the set of all points to the right of the boundary line $x = 1$. The line $x = 1$ is drawn solid to show that the points on the line are included. The region to the left of the line is shaded to show that it is *not* required.

2 $y < 2$ is the set of all points below the boundary line $y = 2$. The line $y = 2$ is drawn broken to show that the points on the line are *not* included. The region above the line is shaded to show that it is *not* required.

Example 11 *Show on a graph the region that contains the set of points for which $2x + y < 3$.*

First make y the subject of the given inequality.

$$y < 3 - 2x$$

The line $y = 3 - 2x$ is the boundary between the required region and the set of points which are not required.

If $y = 3 - 2x$,
then when $x = 0, y = 3$
and when $y = 0, x = 1\frac{1}{2}$

Since the points on the line $y = 3 - 2x$ are not included, a broken line is drawn through the points $(0, 3)$ and $(1\frac{1}{2}, 0)$.

Since $y < 3 - 2x$, the points *below* the line $y = 3 - 2x$ are required. In Fig. 9.16 the region above the line is shaded to show that it is *not* required.

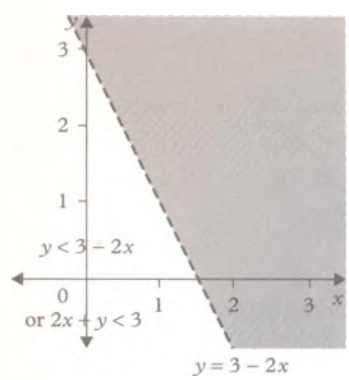


Figure 9.16

Note the following.

1 y is made the subject of the inequality in order to determine the required region.

2 Boundary lines cross the axes at the points where $x = 0$ and $y = 0$. It is usually most convenient to draw boundary lines through such points since their coordinates are easily calculated.

3 A boundary line may be solid or broken, depending on whether the equality is included or not.

4 The region which is *not* required is shaded.

Example 12 *Write down the three inequalities which define the unshaded area labelled A in* **Fig. 9.17**.

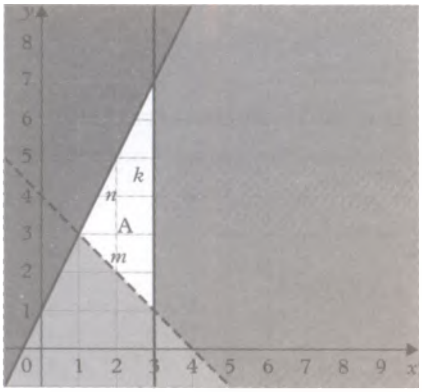


Figure 9.17

The lines are labelled k, m, n for convenience.

Line k

k is the line $x = 3$. k is solid. Points to the right of k are not required. Hence the corresponding inequality is $x \leqslant 3$.

Line m

m has a gradient of -1 and cuts the y-axis at $(0, 4)$. Its equation is $y = -x + 4$. m is a broken line. Points below m are not required. Hence $y > 4 - x$ is the corresponding inequality.

Line n

n has a gradient of 2 and cuts the y-axis at $(0, 1)$. Its equation is $y = 2x + 1$. n is solid. Points above n are not required. Hence $y \leqslant 2x + 1$ is the corresponding inequality.

The inequalities which define the region A are:

$x \leqslant 3$
$y > 4 - x$
$y \leqslant 2x + 1$

Graphical solution of simultaneous inequalities

Example 13 *Show on a graph the region that contains the solutions of the simultaneous inequalities* $2x + 3y < 6$, $y - 2x \leqslant 2$, $y \geqslant 0$.

Consider the first inequality, $2x + 3y < 6$. It may be rewritten as

$$3y < 6 - 2x$$
$$\Rightarrow \quad y < \tfrac{1}{3}(6 - 2x)$$

$y = \frac{1}{3}(6 - 2x)$ is a boundary line

when $x = 0$, $y = 2$,
when $y = 0$, $x = 3$.

Points below the broken line through $(0, 2)$ and $(3, 0)$ satisfy the inequality $2x + 3y < 6$. Similarly, points on and below the solid line through $(0, 2)$ and $(-1, 0)$ satisfy the inequality $y - 2x \leqslant 2$.

Likewise, points on and above the x-axis (i.e. the line $y = 0$) satisfy the inequality $y \geqslant 0$. The solutions are contained in the unshaded triangular region shown in **Fig. 9.18**.

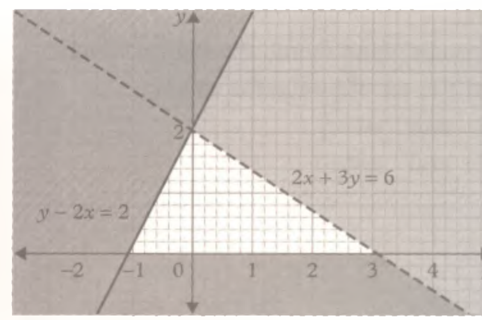


Figure 9.18

Example 14 *Solve graphically the simultaneous inequalities* $4x + 3y < 12$, $y \geqslant 0$, $x > 0$ *for integral values of* x *and* y.

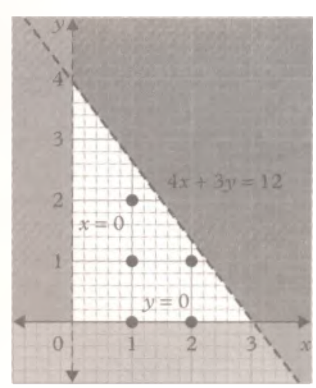


Figure 9.19

In **Fig. 9.19**,

$4x + 3y = 12$ (broken)
$y = 0$ (solid)
$x = 0$ (broken)

are the boundary lines. The solutions lie within the unshaded region.

In **Fig. 9.19** the solutions are shown by the five points marked by spots: $(1, 0)$, $(1, 1)$, $(1, 2)$, $(2, 0)$, $(2, 1)$.

Example 15 *$y - x \geqslant 1$; $2x < 5$; $5y > -4x$ are simultaneous inequalities.*

a *Show on a graph the region which contains the solution set of the inequalities.*

b *If the solution set contains integral values of x and y only, list its members.*

a The boundary lines of the region are

p: $y - x = 1$
q: $2x = 5$
r: $5y = -4x$

The unshaded region in Fig. 9.20 gives the solution set of all points (x, y) which satisfy the three inequalities.

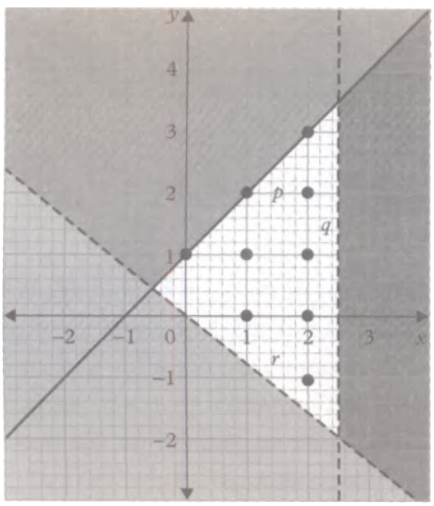


Figure 9.20

b In Fig. 9.20 the solution set is shown by spots, indicating that the values of x and y are integral. The solution set is as follows: {(0, 1), (1, 0), (1, 1), (1, 2), (2, −1), (2, 0), (2, 1), (2, 2), (2, 3)}.

Exercise 9d

1 In Fig. 9.21 the lines m, $x + y = 2$ and $x + 2y = 5$ are the boundaries of the unshaded region that contains the solutions of three simultaneous inequalities.

a What is the equation of the line m?

b Write down the three inequalities which define the unshaded region, A.

c Write down the coordinates of the solutions given that they are integral values of x and y only.

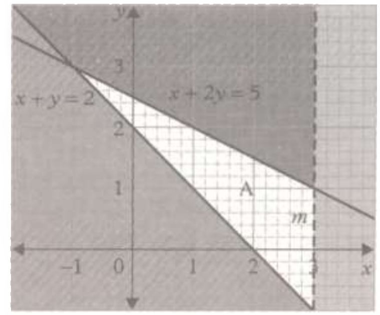


Figure 9.21

2 Write down the three inequalities which define the unshaded area labelled A in Fig. 9.22.

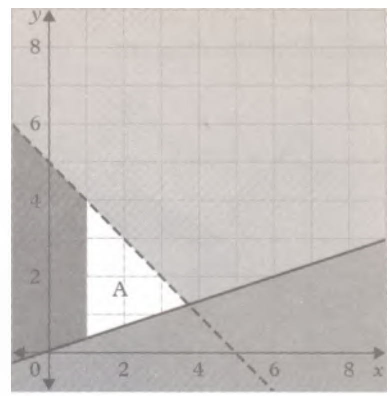


Figure 9.22

3 What are the three inequalities which define the unshaded region R in Fig. 9.23?

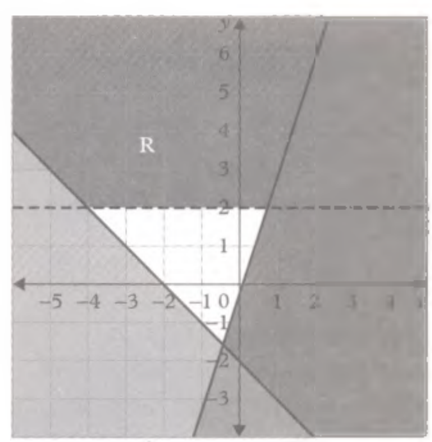


Figure 9.23

4 Using graph paper, draw the regions defined by each of the following. (Use solid and broken lines as explained earlier; leave each required region unshaded.)

a $y \geqslant 0, y < 3x, x + y \leqslant 4$

b $x \geqslant -3, y \leqslant 2, x - y < 2$

c $y \leqslant 5, x - y \leqslant 1, 4x + 3y \geqslant 12$

d $x \geqslant 0, y \geqslant 0, x + y < 6, y - x < 2$

e $y < 3, x < 4, 2x + y + 2 \geqslant 0, x - y - 2 \geqslant 0$

5 Solve each of the following graphically for integral values of x and y.
a $y \geqslant 0, x - y \geqslant 1, 3x + 4y < 12$
b $y \geqslant 1, y - x < 5, 2x + y \leqslant 0$
c $y > -2, x > 0, 2x + y < 4$
d $x + y \leqslant 2, x - y \leqslant 2, 2x + y \geqslant 2$
e $y > 0, y < 4, 4x + 3y > 0, 5x + 2y < 10$

6 In Fig. 9.24 find
a the coordinates of the point P where the line $2y = x + 8$ crosses the y-axis.
b the equation of the line which passes through the origin O and the point $(-3, 6)$,
c the three inequalities which define the triangular region R in the diagram.

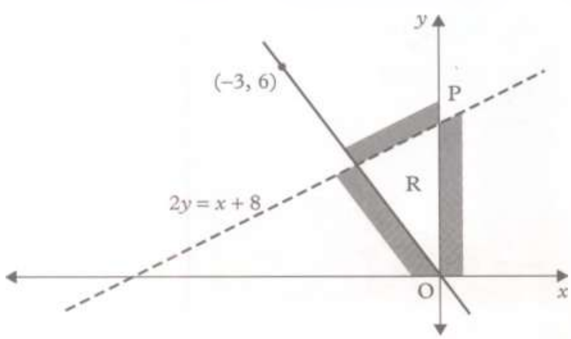


Figure 9.24

9.4 Linear programming

Example 16 *A student has \$25. She buys pens at \$2.50 each and pencils at \$1 each. She gets at least five of each and the money spent on pens is over \$5 more than that spent on pencils.*

Find
a *how many ways the money can be spent,*
b *the greatest number of pens that can be bought,*
c *the greatest number of pencils that can be bought.*

Let the student buy x pens at \$2.50 each and y pencils at \$1 each.

From the first two sentences,

$$2.5x + y \leqslant 25$$
$$\Rightarrow \quad 5x + 2y \leqslant 50 \qquad (1)$$

Since she gets at least 5 of each,

$$x \geqslant 5 \text{ and} \qquad (2)$$
$$y \geqslant 5 \qquad (3)$$

Also, from the same sentence,

$$2.5x - y > 5$$
$$\Rightarrow \quad 5x - 2y > 10 \qquad (4)$$

Inequalities (1), (2), (3) and (4) are shown in Fig. 9.25.

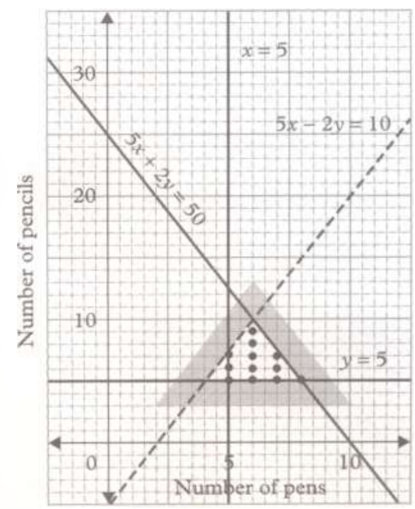


Figure 9.25

a The solution set of the four inequalities is given by the twelve points marked inside the unshaded region. For example, the point (7, 6) shows that the student can buy seven pens and six pencils and still satisfy the restrictions on the two variables. Hence there are twelve ways of spending the money.
b The greatest number of pens that can be bought is eight, corresponding to the point (8, 5).
c The greatest number of pencils is nine, corresponding to the point (6, 9).

Example 17 *The student in Example 16 wants to buy as many items as possible. How many can she get and how much change will there be from the \$25?*

The number of items bought is $x + y$.
If the total is n, then $x + y = n$.

The general equation $x + y = n$ can be represented graphically by a family of parallel lines. Fig. 9.26 shows some members of the family when n has the values 5, 10, 12.

Fig. 9.26 shows that as n increases, the lines appear to move to the right. Considering Fig. 9.26, the greatest possible value of n corresponds to the line which is parallel to $x + y = n$ and as far as possible to the right, but which also passes through the unshaded region.

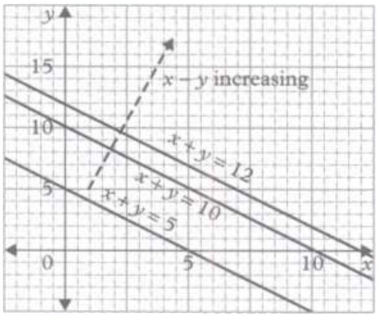


Figure 9.26

Fig. 9.27 is a repeat of Fig. 9.26 with some of the family $x + y = n$ added.

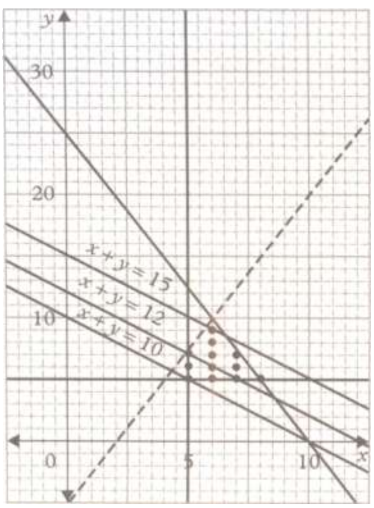


Figure 9.27

From Fig. 9.27, the greatest value of n is 15, where the line $x + y = 15$ passes through the point (6, 9). The 15 items are made up as follows:

6 pens at \$2.50	\$15
9 pencils at \$1	\$ 9
total cost:	\$24

There will be \$1 change from \$25.

The kind of problem in Examples 16 and 17 involves making decisions in a situation in which there are **restrictions**. Each restriction, such as the limit on the amount of money available, can be represented by a linear inequality. Hence the solution to the problem can be found graphically. This method is called **linear programming**. Linear programming can be used to solve a variety of realistic problems.

Example 18 *To start a transport company, a businessman needs at least 5 buses and 10 minibuses. He does not want to have more than 30 vehicles altogether. A bus takes up 3 units of garage space, a minibus takes up 1 unit of garage space and there are only 54 units of garage space available.*

If x and y are the numbers of buses and minibuses respectively: **a** *write down four inequalities which represent the restrictions on the businessman;* **b** *draw a graph which shows a region representing possible values of x and y.*

Running costs are \$135 a day for a bus and \$72 a day for a minibus. **c** *Write down an expression for the total cost per day, \$ C.* **d** *Find the maximum daily cost and the corresponding numbers of buses and minibuses.*

a From the first sentence,

$$x \geqslant 5$$
$$y \geqslant 10$$

From the second sentence,

$$x + y \leqslant 30$$

From the third sentence,

$$3x + y \leqslant 54.$$

b In Fig. 9.28, R is the region which contains the possible values of x and y.

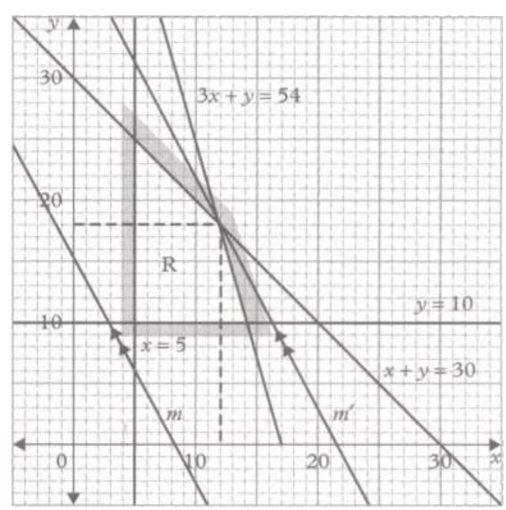


Figure 9.28

c $C = 135x + 72y$

d In Fig. 9.28, m is a line of gradient $-\frac{15}{8}$ that passes through (0, 15) and (8, 0). (See Note 1 below.). As m moves to the right, the cost increases. When m reaches m' the line passes through R at the point (12, 18), the point of maximum cost.

Maximum cost
$= 12 \times \$135 + 18 \times \$72$
$= \$1620 + \$1296$
$= \$2916$

It costs \$2916 to run 12 buses and 18 minibuses.

Notes

1 The *'cost equation'* is $C = 135x + 72y$. Make y the subject of this equation:

$$72y = -135x + C$$
$$\Rightarrow \quad y = -\tfrac{135}{72}x + \tfrac{C}{72}$$
$$\Rightarrow \quad y = -\tfrac{15}{8}x + k$$

Various costs are represented by a family of lines with gradient $-\frac{15}{8}$. To draw the *'basic family member'*, m, join (0, 15) to (8, 0). Then draw m' parallel to m using a set square and a ruler.

2 The working would be clearer and more accurate if a larger scale had been used in **Fig. 9.28**.

Exercise 9e

1 Redraw **Fig. 9.28** using a scale of 2 cm to 5 units on both axes. Use your graph to answer the following.
When the transport company is running at full efficiency, the daily profit on a bus is four times that on a minibus. Find the numbers of buses and minibuses the businessman should buy to maximise his profit.

2 Notebooks cost \$3 and pencils \$1.80. A girl has \$18 to spend and needs at least 3 notebooks and 3 pencils. She decides to spend as much as possible of her \$18.
- a How many ways can she spend her money?
- b Do any of the ways give her change? If so, how much?

3 A car repair workshop uses large numbers of two types of spare part, one costing \$30 and the other \$40. The workshop owner allows \$3000 to buy spare parts and he needs twice as many cheap ones as dear ones. There must be at least 50 cheap and 20 expensive parts.
- a What is the largest number of spare parts he can buy, and in what way?
- b If he decides to get as many of the expensive parts as conditions allow, how many of each type can he get?

4 A storeman fills a new warehouse with two types of goods, A and B. They both come in tall boxes which cannot be stacked. A box of A takes up $\frac{1}{2}$ m^2 of floor space and costs \$500. A box of B takes up $1\frac{1}{2}$ m^2 of floor space and costs \$3000. The storeman has up to 100 m^2 of floor space available and can spend up to \$150 000 altogether.
He wants to buy at least 50 boxes of A and 20 boxes of B.
- a How many boxes of each should he buy in order to
  - i spend all the money available and also to use as much space as possible,
  - ii use all the space for the least cost?
- b What is the cost in the second case?

5 Following an illness, a patient is required to take pills containing minerals and vitamins. The contents and costs of two types of pill, Feelgood and Getbetta, together with the patient's daily requirement, are shown in **Table 9.1**.

	Mineral	Vitamin	Cost
Feelgood	160 mg	4 mg	\$2
Getbetta	40 mg	3 mg	\$1
Daily requirement	800 mg	30 mg	

Table 9.1

A daily prescription contains x Feelgood pills and y Getbetta pills.
- a State the inequalities to be satisfied by x and y.
- b Use a graphical method to show the solution set of x and y.
- c Find the cheapest way of prescribing the pills and the cost.

6 While exploring for oil, it was necessary to carry at least 18 tonnes of supplies and 80 people into a desert region. There were two types of lorry available, Landmasters and Sandrovers. Each Landmaster could carry 900 kg of supplies and 6 people; each Sandrover could carry 1350 kg of supplies and 5 people.
If there were only 12 of each type in good running order, find the smallest number of lorries necessary for the journey.

7 A shopkeeper orders packets of soap powder. The cost price of a large packet is \$2.70 and that of a small packet is \$1.20. She is prepared to spend up to \$60 altogether and needs twice as many small

packets as large packets with a minimum of 10 large and 20 small packets.

a What is the greatest number of packets she can buy?

The profit is $0.30 on a large packet and $0.15 on a small packet.

b Which arrangement gives the greatest profit?

c What is that profit?

8 A dressmaker plans to buy new machines for her factory. **Table 9.2** shows the cost, the necessary floor space and the output of each machine.

Machine	Cost	Floor space	Output in components/hour
Machine A	$300 000	3 m^2	10 per hour
Machine B	$400 000	$2\frac{1}{2}$ m^2	15 per hour

Table 9.2

She can spend $36 million altogether and she has 27 m^2 of floor space. Trade restrictions are such that she has to buy at least 3 of Machine A and 4 of Machine B.

a What is the maximum number of machines she can buy?

b What arrangement gives the biggest output?

9 A builder has $960 000 and 8 ha of land available for building houses. Large houses cost $24 000 each to build and need 0.25 ha; small houses cost $15 000 each and occupy 0.1 ha. Permission to build is given so long as there are at least 16 large houses and 30 small houses.

a Find the greatest number of large houses that can be built.

b Find the distribution that

i gives the greatest number of houses altogether,

ii uses up all the land available.

10 A shopkeeper stocks two brands of drinks called Kula and Sundown, both of which are produced in cans of the same size. He wishes to order fresh supplies and finds that he has room for up to 1000 cans. He knows that Sundown is more popular and so proposes to order at least twice as many cans of Sundown as Kula. He wishes, however, to have at least 100 cans of Kula and not more than 800 cans of Sundown.

Taking x to be the number of cans of Kula and y to be the number of cans of Sundown which he orders, write down the four inequalities involving x and/or y which satisfy these conditions.

The point (x, y) represents x cans of Kula and y cans of Sundown. Using a scale of 1 cm to represent 100 cans on each axis, construct and indicate clearly, by shading the unwanted regions, the region in which (x, y) must lie.

The profit on a can of Kula is $3 and a can of Sundown is $2. Use your graph to estimate the number of cans of each that the shopkeeper should order to give the maximum profit.

Chapter 10

Quadratic equations and complex numbers

10.1 Quadratic equations

The quadratic equation $ax^2 + bx + c = 0$

It is assumed that you are familiar with solving quadratic equations by factorisation, as in Example 1 below, and that you may be familiar with 'completing the square', shown in Example 2.

Example 1 *Solve* $2x^2 + 7x - 15 = 0$.

$$2x^2 + 7x - 15 = 0$$
$$(2x - 3)(x + 5) = 0$$

hence,

either $2x - 3 = 0, \quad x = 1\frac{1}{2}$
or $x + 5 = 0, \quad x = -5$

When factorisation is difficult, use the technique of completing the square as in Example 2. This method depends on the identity

$$(x + k)^2 = x^2 + 2kx + k^2$$

Example 2 *Solve* $5x^2 - 6x - 2 = 0$.

$$5x^2 - 6x - 2 = 0$$

Add 2 to both sides,

$$5x^2 - 6x = 2$$

divide through by 5,

$$x^2 - \frac{6}{5}x = \frac{2}{5}$$

complete the square, by adding $\left(\frac{3}{5}\right)^2$ to both sides,

$$x^2 - \left(\frac{6}{5}\right)x + \left(\frac{3}{5}\right)^2 = \left(\frac{3}{5}\right)^2 + \frac{2}{5} = \frac{9+10}{25} = \frac{19}{25}$$

factorise the left-hand side,

$$\left(x - \frac{3}{5}\right)^2 = \frac{19}{25}$$

take the square root of both sides,

$$\left(x - \frac{3}{5}\right) = \pm\sqrt{\frac{19}{25}} = \pm\frac{\sqrt{19}}{5}$$

and finally, add $\frac{3}{5}$ to both sides,

$$x = \frac{3}{5} \pm \frac{\sqrt{19}}{5} = \frac{3 \pm \sqrt{19}}{5}$$

[Unless told otherwise, leave surds in the answers to such questions.]

Notice that the roots can be used to find the factors of the original expression. Thus in Example 2,

$$5x^2 - 6x - 2 = 5(x^2 - \tfrac{6}{5}x - 2)$$
$$= 5\left(x - \frac{3 + \sqrt{19}}{5}\right)\left(x - \frac{3 - \sqrt{19}}{5}\right)$$

The quadratic formula

The method of Example 2, above, can be generalised, as follows. To solve

$$ax^2 + bx + c = 0$$

subtract c from both sides,

$$ax^2 + bx = -c$$

divide through by a,

$$x^2 + \left(\frac{b}{a}\right)x = -\left(\frac{c}{a}\right)$$

complete the square (by adding $b^2/(4a^2)$ to both sides),

$$x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

and factorise the left-hand side, which gives

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Take the square root (but note this is only possible if the right-hand side is non-negative i.e. if $b^2 - 4ac \geqslant 0$),

$$\left(x + \frac{b}{2a}\right) = \pm\sqrt{\left(\frac{b^2 - 4ac}{4a^2}\right)} = \frac{\pm\sqrt{(b^2 - 4ac)}}{2a} \qquad (1)$$

Now subtract $b/(2a)$ from both sides

$$x = -\frac{b}{2a} \pm \frac{\sqrt{(b^2 - 4ac)}}{2a}$$

So, if $\boldsymbol{ax^2 + bx + c = 0}$ then $\boldsymbol{x = \dfrac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}}$

This formula is usually the most convenient way of solving quadratic equations which cannot easily be solved by factorisation.

Example 3 *Solve* $2x^2 - 6x - 3 = 0$.

In this example $a = 2$, $b = -6$ and $c = -3$, hence $b^2 - 4ac = 36 - 4 \times 2 \times (-3)$, that is, $b^2 - 4ac = 60$. Substituting these values into the formula

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

gives

$$x = \frac{+6 \pm \sqrt{60}}{4}$$

$$= \frac{6 \pm 2\sqrt{15}}{4}$$

$$\therefore x = \frac{3 \pm \sqrt{15}}{2}$$

Notice the importance of the step marked (1) in the proof of the quadratic formula. Three possibilities can arise:

i $b^2 - 4ac > 0$; a *real* value of $\sqrt{(b^2 - 4ac)}$ can be found and so the equation has two real distinct roots,

ii $b^2 - 4ac = 0$; the solution is $x = -b/(2a)$,

iii $b^2 - 4ac < 0$; there is no real value of $\sqrt{(b^2 - 4ac)}$ and so there are no real roots.

In case **ii**, the expression $x^2 + (b/a)x + (c/a)$ is the square of $x + b/(2a)$ and it is convenient to say that the quadratic equation has 'two identical roots'. We shall return to case **iii** in §10.3 on page 131.

Because of its important role in determining the nature of the roots, the term $(b^2 - 4ac)$ is called the **discriminant** of the equation.

Question

Q1 Calculate the discriminant of each of the quadratics below and state whether the equation has **i** two distinct real roots, **ii** two identical roots, or **iii** no real roots.

a $3x^2 + 5x - 1 = 0$

b $49x^2 + 42x + 9 = 0$

c $2x^2 + 8x + 9 = 0$

d $2x^2 + 7x + 4 = 0$

The quadratic function $f(x) = ax^2 + bx + c$

Using the method of completing the square, the form $ax^2 + bx + c$ can always be reduced to the form $a(x - p)^2 + q$. See Example 4 below.

Example 4 *Express the function* $f(x) = 2x^2 - 12x + 23$ *in the form* $a(x - p)^2 + q$.

$$\begin{aligned} 2x^2 - 12x + 23 &= 2(x^2 - 6x + 11.5) \\ &= 2[(x-3)^2 - 9 + 11.5] \\ &= 2[(x-3)^2 + 2.5] \\ &= 2(x-3)^2 + 5 \end{aligned}$$

In this example, $a = 2$, $p = 3$ and $q = 5$. One advantage of this form is that, since $(x - 3)^2 \geqslant 0$, we can read off that $f(x) \geqslant 5$ and that the least value of the function occurs when $x = 3$.

Example 5 *Find, by completing the square, the greatest value of the function* $f(x) = 1 - 6x - x^2$.

$$\begin{aligned} f(x) &= 1 - 6x - x^2 \\ &= 10 - (9 + 6x + x^2) \\ &= 10 - (3 + x)^2 \end{aligned}$$

Since $(3 + x)^2$ is the square of a real number it cannot be negative; it is zero when $x = -3$, otherwise it is positive. Consequently $10 - (3 + x)^2$ is always less than or equal to 10.

$\therefore$ the greatest value of the function is 10 and this occurs when $x = -3$.

In general

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right] \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] \\ &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a} \end{aligned}$$

and thus $f(x) = ax^2 + bx + c$ may be written $a(x - p)^2 + q$, where

$$p = -\frac{b}{2a} \quad \text{and} \quad q = -\frac{b^2 - 4ac}{4a}$$

The least (or greatest) value of $f(x)$ is $f(p) = q$. If $a > 0$, $f(p)$ is the least value; if $a < 0$, it is the greatest value.

Question

Q2 Find, by completing the square, the range of the function
$f(t) = 10 + 20t - 5t^2$

The graph of $y = ax^2 + bx + c$

We have seen above that this equation can be expressed in the form

$$y = a(x - p)^2 + q$$

Now, we know that the graph of $y = x^2$ is a parabola and that the graph of $y = (x - p)^2$ is the same shape, but it is displaced p units to the right (**Fig. 10.1**).

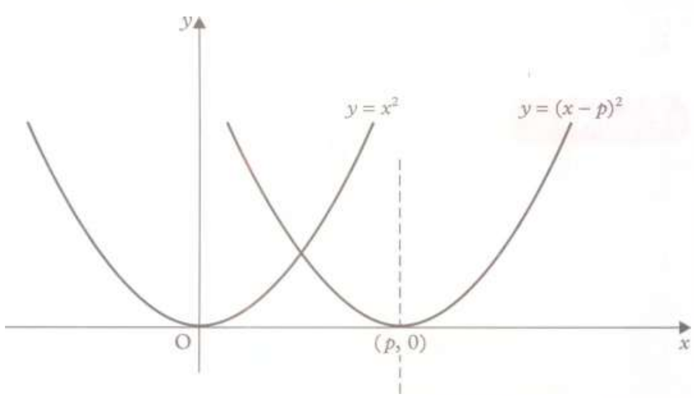


Figure 10.1

Multiplying $(x - p)^2$ by a merely 'stretches' the graph parallel to the y-axis, although if a is negative it will also turn it upside down. Adding q to $a(x - p)^2$ translates the graph q units vertically upwards. Thus the graph of

$$y = a(x - p)^2 + q$$

looks like **Fig. 10.2**. In this diagram $a > 0$, $p > 0$ and $q > 0$ (i.e. $b^2 - 4ac < 0$).

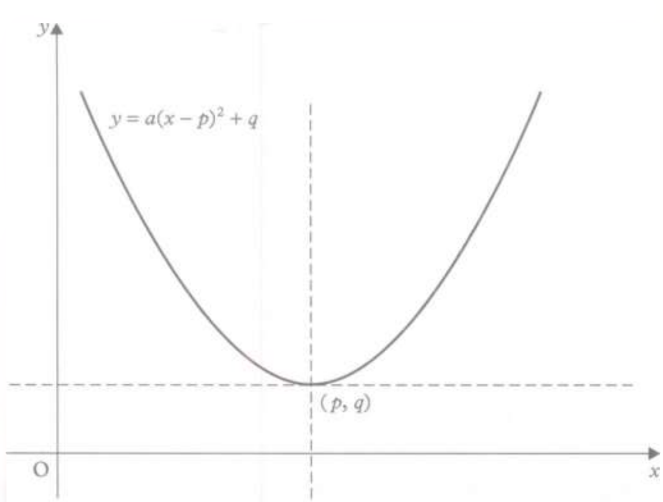


Figure 10.2

Notice that if $q < 0$ (i.e. $b^2 - 4ac > 0$) but a and p are positive, then the graph would look like **Fig. 10.3**.

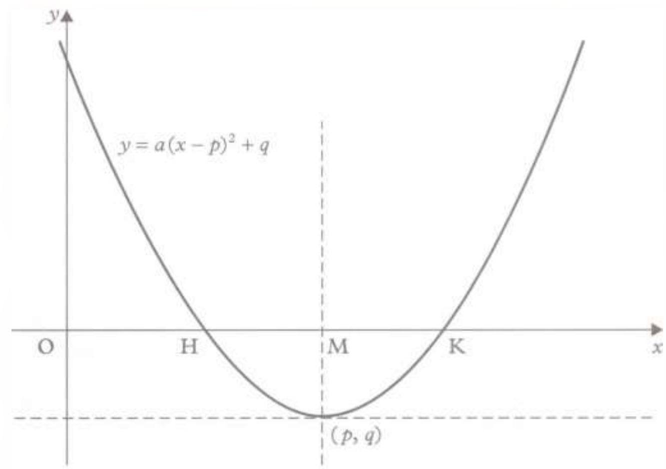


Figure 10.3

In this case, M is the point $(-b/(2a), 0)$ and H and K are the points where $ax^2 + bx + c = 0$ and, as we have already seen, at these points

$$x = -\frac{b}{2a} \pm \frac{\sqrt{(b^2 - 4ac)}}{2a}$$

Notice that these values of x can only be real if $b^2 - 4ac \geqslant 0$.

Question

Q3 Sketch the graph of $y = ax^2 + bx + c$, when

a $b^2 > 4ac$ and $a < 0$

b $b^2 = 4ac$ and $a > 0$

In each diagram mark clearly the coordinates of the vertex.

Exercise 10a

Leave surds in the answers.

1 Solve, by factorisation:

a $2x^2 - 5x + 3 = 0$

b $x^2 + 4x - 21 = 0$

c $4x^2 - 25 = 0$

d $7x^2 + 5x = 0$

2 Solve, by completing the square:

a $2x^2 - 6x - 1 = 0$

b $5x^2 + 12x + 6 = 0$

c $x^2 + 7x - 3 = 0$

d $10 + 3x - 2x^2 = 0$

3 Solve, by using the formula:

a $3t^2 - 7t - 1 = 0$

b $5z^2 + 3z - 7 = 0$

c $4 + 13y + y^2 = 0$

d $3p^2 = 7p + 2$

4 Solve, where possible, by any suitable method:

a $15 - 30x + 4x^2 = 0$

b $11x^2 = 48x$

c $9x^2 = 8x - 2$

d $7x^2 - 38x + 15 = 0$

5 Using the results of question 2, factorise:

a $2x^2 - 6x - 1$

b $5x^2 + 12x + 6$

c $x^2 + 7x - 3$

d $10 + 3x - 2x^2$

6 Sketch the graphs of

a $y = 2x^2 - 5x + 3$

b $y = 2x^2 - 6x - 1$

c $y = 3x^2 - 7x - 1$

d $y = 3x^2 - 7x + 5$

[Hint: use the answers to 1**a**, 2**a** and 3**a**.]

7 Sketch the graphs of

a $y = 9x^2 - 30x + 25$

b $y = x^2 - 6x + 13$

c $y = 5 - x^2$

d $y = 36 + 48x - 9x^2$

8 Given that $3x^2 - kx + 12$ is positive for all values of x, find the range of possible values for k.

9 Given that α and β are the roots of the quadratic equation, $x^2 - 7x + 3 = 0$, find α and β from the formula, and verify that $\alpha + \beta = 7$ and $\alpha\beta = 3$.

10 By completing the square, find the greatest values of

a $2 - 2x - x^2$

b $-7 + 12x - 3x^2$

and the least values of

c $13 + 6x + 3x^2$

d $15 + 8x + \frac{1}{2}x^2$

10.2 Imaginary numbers

We have seen that the equation $x^2 + 1 = 0$, or $x^2 = -1$, has no real roots. For the moment, do not worry about this. Instead, we will write i for $\surd(-1)$. We could then say that $x^2 + 1 = 0$ has two roots, namely $x = \pm\mathrm{i}$. We call $\surd(-1)$ an **imaginary number**.

Having introduced i there is no need to invent further symbols for the square roots of other negative numbers. Consider, for example, $\surd(-25)$.

$$\begin{aligned}\surd(-25) &= \surd(25 \times -1)\\ &= \surd 25 \times \surd(-1)\\ &= 5\mathrm{i}\end{aligned}$$

So an equation in the form

$$x^2 + n^2 = 0 \quad \text{or} \quad x^2 = -n^2, \quad \text{where } n \in \mathbb{R},$$

has two roots, $x = \pm n\mathrm{i}$.

[In some contexts, especially electricity where i is used to represent the current in an electrical circuit, the symbol j is used instead of i.]

Question

Q4 Solve the equations:

a $x^2 + 64 = 0$

b $x^2 + 7 = 0$

c $4x^2 + 9 = 0$

d $(x + 3)^2 = -25$

10.3 Complex numbers

We can now return to the problem of solving

$$ax^2 + bx + c = 0 \qquad \text{when } b^2 < 4ac$$

Previously we decided that no real roots exist in this case.

First, consider a particular example. Solve

$$x^2 - 4x + 5 = 0$$

Completing the square gives

$$\begin{aligned}x^2 - 4x &= -5\\ (x - 2)^2 - 4 &= -5\\ (x - 2)^2 &= -1\end{aligned}$$

Previously, at this stage we were unable to proceed further because we could not find the square root of -1. Now, we can use imaginary numbers. Hence

$$\begin{aligned}(x - 2) &= \pm\mathrm{i}\\ \therefore\ x &= 2 \pm \mathrm{i}\end{aligned}$$

Question

Q5 Solve $x^2 - 6x + 34 = 0$.

The general solution of the equation $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \surd(b^2 - 4ac)}{2a}$$

When $b^2 < 4ac$ this can be written

$$x = \frac{-b \pm \surd\{-1(4ac - b^2)\}}{2a}$$
$$= \frac{-b}{2a} \pm \frac{\surd(4ac - b^2)}{2a}\mathrm{i}$$

Notice that both $-b/(2a)$ and $\surd(4ac - b^2)/(2a)$ are real numbers.

Numbers of the form $p + \mathrm{i}q$, where p and q are real numbers, are called **complex numbers**. The standard symbol for the set of complex numbers is $\mathbb{C}$.

Example 6 *Solve* $x^2 - 6x + 13 = 0$, *where* $x \in \mathbb{C}$.

$$x^2 - 6x + 13 = 0$$

Using the formula

$$x = \frac{+6 \pm \surd(36 - 4 \times 1 \times 13)}{2}$$
$$= \frac{6 \pm \surd(-16)}{2}$$
$$= \frac{6 \pm 4\mathrm{i}}{2}$$

$\therefore x = 3 \pm 2\mathrm{i}$.

In the complex number $p + \mathrm{i}q$, the number p is called the **real part** of the complex number and q is called its **imaginary part**. (Thus the real part of $5 + 4\mathrm{i}$ is 5, and the imaginary part is 4.) It is usual to represent a complex number by the letter z, although w is also sometimes used. The real part of a complex number z can then be abbreviated to $\mathrm{Re}(z)$ and the imaginary part is written $\mathrm{Im}(z)$. Thus if $z = 2 + 7\mathrm{i}$, then $\mathrm{Re}(z) = 2$ and $\mathrm{Im}(z) = 7$, or, if $w = 4 - 3\mathrm{i}$, then we can write $\mathrm{Re}(w) = 4$ and $\mathrm{Im}(w) = -3$.

It is important to notice that two complex numbers are equal if, and only if, their real parts are equal and their imaginary parts are equal, for if

$$a + \mathrm{i}b = c + \mathrm{i}d$$

then

$$a - c = \mathrm{i}(d - b)$$

and, squaring both sides,

$$(a - c)^2 = -(d - b)^2$$

Now, since a, b, c and d are real numbers, $(a - c)^2$ and $(d - b)^2$ are either positive or they are zero. It is impossible for them to be positive, because we would then have a positive number on the left-hand side and a negative number on the right. Therefore $(a - c)^2$ and $(d - b)^2$ are both zero, i.e.

$$a = c \quad \text{and} \quad b = d$$

[You may feel that this is a rather trivial point, but, as we will see later, this is a very valuable feature of complex numbers. It may seem less trivial if it is compared with a similar situation in rational numbers. Here it *is* possible to have $a/b = c/d$, even though $a \neq c$ and $b \neq d$, for example, $2/3 = 10/15$.]

Since it was necessary to introduce complex numbers in order to include the roots of all quadratic equations, you might think that further types of number are necessary to find the roots of equations of higher degree. However, this is not so. It can be proved that a polynomial equation of degree n has exactly n roots (possibly repeated) in $\mathbb{C}$ (the proof is beyond the scope of this book).

Question

Q6 Solve the following equations with the quadratic formula or by completing the square:

a $z^2 - 4z + 13 = 0$
b $9z^2 + 25 = 0$
c $2z^2 = 2z - 13$
d $34z^2 - 6z + 1 = 0$

The algebra of complex numbers

The operations addition, subtraction, multiplication, and division, which we have used so far are concerned with real numbers. It is now necessary to define what we mean by these operations with regard to complex numbers. It is easiest for us to define these operations by using the usual laws of algebra together with the relation $\mathrm{i}^2 = -1$. Thus

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$
$$(a + ib) - (c + id) = (a - c) + i(b - d)$$
$$(a + ib) \times (c + id) = ac + aid + ibc + i^2bd$$
$$= (ac - bd) + i(ad + bc)$$

At this stage it is worth comparing the corresponding operations with real numbers in the form $a + \sqrt{2}b$ (a, b rational):

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + \sqrt{2}(b + d)$$
$$(a + b\sqrt{2}) - (c + d\sqrt{2}) = (a - c) + \sqrt{2}(b - d)$$
$$(a + b\sqrt{2}) \times (c + d\sqrt{2}) = ac + ad\sqrt{2} + bc\sqrt{2} + 2bd$$
$$= (ac + 2bd) + \sqrt{2}(ad + bc)$$

This helps us to find a way of expressing $(a + ib)/(c + id)$ in the form $p + iq$. You will probably recall the corresponding process with $(a + b\sqrt{2})/(c + d\sqrt{2})$. The method is to multiply numerator and denominator in such a way that the new denominator involves a difference of two squares:

$$\frac{a + b\sqrt{2}}{c + d\sqrt{2}} \times \frac{c - d\sqrt{2}}{c - d\sqrt{2}} = \frac{(ac - 2bd) + \sqrt{2}(bc - ad)}{c^2 - 2d^2}$$
$$= \frac{ac - 2bd}{c^2 - 2d^2} + \sqrt{2}\frac{bc - ad}{c^2 - 2d^2}$$

Similarly, the expression $(a + ib)/(c + id)$ may be expressed in the form $p + iq$ by multiplying numerator and denominator by $c - id$ because

$$(c + id) \times (c - id) = c^2 - i^2d^2 = c^2 + d^2$$

In other words,

$$\frac{a + ib}{c + id} = \frac{a + ib}{c + id} \times \frac{c - id}{c - id}$$
$$= \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}$$

Definition

Two complex numbers in the form $x + iy$, $x - iy$ are called ***conjugate complex numbers****.*

The symbol z^* is used to represent the complex conjugate of z, so if $z = x + iy$, then we write

$$z^* = x - iy$$

Question

Q7 Express $(2 + 3i)/(1 + i)$ in the form $p + iq$ ($p, q \in \mathbb{R}$). [Multiply numerator and denominator by $1 - i$.]

Do not attempt to memorise expressions for the sum, difference, product, and quotient of two complex numbers: simply use the usual laws of algebra, together with the relation $i^2 = -1$.

Exercise 10b

Simplify:

1 **a** i^3 **b** i^4 **c** i^5 **d** i^6 **e** $\dfrac{1}{i^2}$ **f** $\dfrac{1}{i}$ **g** $\dfrac{1}{i^3}$

2 **a** $(3 + i) + (1 + 2i)$ **b** $(5 - 3i) + (4 + 3i)$ **c** $(2 - 3i) - (1 + 2i)$ **d** $(1 + i) - (1 - i)$

3 **a** $(2 + 3i)(4 + 5i)$ **b** $(2 - i)(3 + 2i)$ **c** $(1 + i)(1 - i)$ **d** $(3 + 4i)(3 - 4i)$ **e** $(u + iv)(u - iv)$ **f** $(x + 2iy)(2x + iy)$ **g** $i(2p + 3iq)$ **h** $(p + 2iq)(p - 2iq)$

4 Express with real denominators:

a $\dfrac{1 - i}{1 + i}$ **b** $\dfrac{1}{2 - 3i}$ **c** $\dfrac{3i - 2}{1 + 2i}$ **d** $\dfrac{5 + 4i}{5 - 4i}$ **e** $\dfrac{1}{x + iy}$ **f** $\dfrac{1}{x - iy}$ **g** $\dfrac{1}{2 + 3i} + \dfrac{1}{2 - 3i}$

Simplify the expressions in questions 5 and 6:

5 **a** $(2 + 3i)^2$ **b** $(4 - 5i)^2$ **c** $(x + iy)^2$

6 **a** $(1 + i)^3$ **b** $(1 - i)^3$ **c** $1/(1 + i)^3$

7 Solve the quadratic equations:
a $z^2 - 4z + 29 = 0$ **b** $4z^2 + 7 = 0$ **c** $2z^2 + 3z + 5 = 0$ **d** $4z^2 + 4z + 5 = 0$

8 If α and β are the roots of $z^2 - 10z + 29 = 0$, find α and β by using the formula. Verify that $\alpha + \beta = 10$ and $\alpha\beta = 29$.

9 If α and β are the roots of $az^2 + bz + c = 0$, find, by using the formula, expressions for α and β, in terms of a, b and c. Verify that $\alpha + \beta = -b/a$ and that $\alpha\beta = c/a$.

10 Solve the cubic equation $2z^3 + 3z^2 + 8z - 5 = 0$.

10.4 Complex numbers as ordered pairs

We have seen that a complex number involves a *pair* of real numbers and that the *order* of the pair is important because in general $a + ib \neq b + ia$. We therefore define a complex number as an ordered pair of real numbers which we shall write as $[a, b]$. The fundamental operations of addition and multiplication are defined by the rules:

$$[a, b] + [c, d] = [a + c, b + d]$$
$$[a, b] \times [c, d] = [ac - bd, ad + bc]$$

Subtraction and division are defined in terms of addition and multiplication thus, for any type of number,

$p - q$ is the number x such that $q + x = p$ and
$p \div q$ is the number y such that $q \times y = p$

Now

$$[c, d] + [a - c, b - d] = [a, b]$$
$$\therefore\ [a, b] - [c, d] = [a - c, b - d].$$

Questions

Q8 Show that for the complex numbers $[a, b]$, $[c, d]$,

$$[a, b] \div [c, d] = \left(\frac{ac + bd}{c^2 + d^2}, \frac{bc - ad}{c^2 + d^2}\right)$$

Q9 Note that to every real number a there corresponds a unique complex number $[a, 0]$. Find, from the definitions of the four operations on complex numbers

a $[a, 0] + [c, 0]$
b $[a, 0] \times [c, 0]$
c $[a, 0] - [c, 0]$
d $[a, 0] \div [c, 0]$

10.5 The Argand diagram

The last section showed that complex numbers can be put on a satisfactory logical basis. However, manipulation of complex numbers is most easily carried out as before: the ordered pair notation is simply a device for defining these numbers without reference to $\surd(-1)$. We could write $\surd(-1)$ as the ordered pair $[0, 1]$ but this would be rather clumsy and it is easier to write $\surd(-1) = \mathrm{i}$.

Question

Q10 Prove from the definition of multiplication of complex numbers that
$[0, 1] \times [0, 1] = [-1, 0]$

Although the idea of an ordered pair may appear to have been a digression, it leads us to the next step in our treatment of the subject. **The Argand diagram** is named after J. R. Argand, who published his work on the graphical representation of complex numbers in 1806.

Corresponding to every complex number $[x, y]$ or $x + \mathrm{i}y$, there is a point (x, y) in the cartesian plane; and corresponding to any point (x, y) in the plane, there is a complex number $x + \mathrm{i}y$. In an Argand diagram, think of the x-axis as the real axis and the y-axis as the imaginary axis. At first this correspondence between complex numbers and points on the plane may seem obvious and not very useful, but in fact it proves to be of considerable importance in the theory of complex numbers.

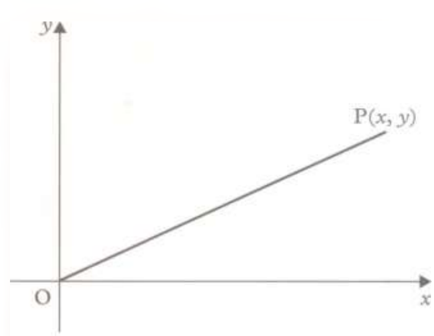


Figure 10.4

The value of this correspondence is increased by the fact that with every point $P(x, y)$ in the plane there is associated a *radius vector* OP (see Fig. 10.4). This means that corresponding to every complex number $x + \mathrm{i}y$ there is a radius vector OP where P is (x, y). Further, corresponding to every radius vector OP in the plane there is complex number $x + \mathrm{i}y$.

Look at Fig. 10.5. The points A, B, A′, B′ are respectively $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$. Corresponding to

OA there is the complex number	$1 + 0\mathrm{i}$	or	1
OB there is the complex number	$0 + 1\mathrm{i}$	or	i
OA′ there is the complex number	$-1 + 0\mathrm{i}$	or	-1
OB′ there is the complex number	$0 + (-1)\mathrm{i}$	or	$-\mathrm{i}$

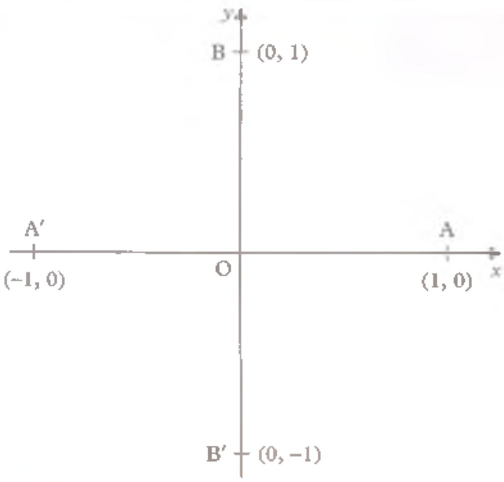


Figure 10.5

Looking down the right-hand side of the last four lines, each number is equal to the previous one multiplied by i. Meanwhile, the corresponding radius vector has rotated in the positive (anti-clockwise) sense through one right angle. Would the same thing happen if any complex number were multiplied by i?

Question

Q11 Find the complex numbers obtained by multiplying $x + iy$ once, twice and three times by i. Does the corresponding radius vector rotate through one right angle each time?

Two quantities are required to specify a vector through the origin: magnitude and direction. The magnitude r of OP (Fig. 10.6) presents no difficulty on an Argand diagram.

$$r = \sqrt{(x^2 + y^2)}$$

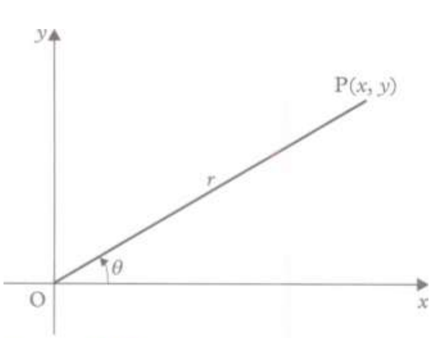


Figure 10.6

This quantity is called the **modulus** of the complex number $x + iy$. 'The modulus of $x + iy$' is abbreviated to $|x + iy|$ hence

$$|x + iy| = \sqrt{(x^2 + y^2)}$$

Question

Q12 Write down the moduli of

- **a** $3 + 4i$
- **b** $-i$
- **c** $\cos\theta + i\sin\theta$
- **d** $\frac{1}{2} - \frac{1}{2}\sqrt{3}i$
- **e** -3
- **f** $1 + i$

The direction of the radius vector OP is not quite so easy to identify because there are infinitely many positive and negative angles which would do.

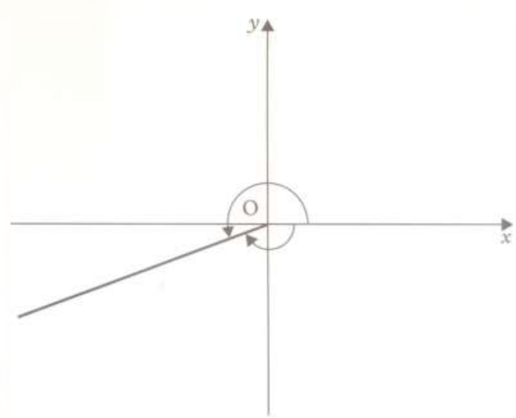


Figure 10.7

The problem of which angle to choose can be illustrated by a radius vector in the third quadrant (Fig. 10.7). It is simply a matter of convention whether we take the positive reflex angle or the negative obtuse angle. In fact the numerically smaller angle is used. The angle between the radius vector OP and the positive x-axis is called the **argument** of the complex number $x + iy$. This is abbreviated to **arg (x + iy)** and has, as we have said before, infinitely many values. The value uniquely specified by the above convention is called the **principal value** of the argument and is written arg $(x + iy)$, so that

$$-180° < \arg(x + iy) \leqslant 180°$$

[In some textbooks, the argument is called the amplitude but this term is less acceptable because of possible confusion with the amplitude of a current, motion, or wave.]

Question

Q13 Find the principal values of the arguments of

a $\cos 45° + i \sin 45°$

b $+1$

c $-i$

d $1 - i$

e $\frac{1}{2} + \frac{1}{2}\sqrt{3}i$

f $\cos 120° + i \sin 120°$

g $\cos 20° - i \sin 20°$

h $\sin 20° + i \cos 20°$

A complex number can be completely specified by its modulus and argument, because, as we can see from **Fig. 10.8**, $x = r\cos\theta$ and $y = r\sin\theta$. Thus if $|z| = r$ and $\arg z = \theta$, then

$$\begin{aligned} z &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$

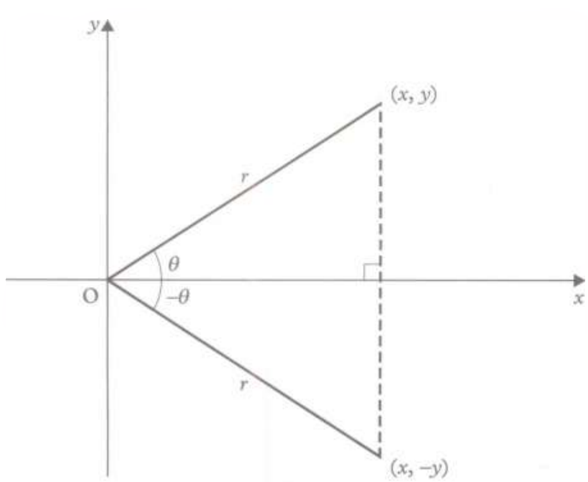


Figure 10.8

Notice, also, that if we are given a complex number $z = x + iy$, then its complex conjugate, $z^* = x - iy$. In other words z^* is the reflection of z in the real axis. Hence $|z^*| = |z|$ and $\arg(z^*) = -\arg z$. ($\bar{z}$ may also be used to denote the complex conjugate of z.)

Example 7 *Given $|z| = 10$ and $\arg z = 120°$, write down z.*

$$\begin{aligned} z &= 10(\cos 120° + i \sin 120°) \\ &= 10\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ &= -5 + 5\sqrt{3}i \end{aligned}$$

Exercise 10c

1 Mark on an Argand diagram, the radius vectors corresponding to

a $1 + i$

b $-3 + 2i$

c $-3 - 2i$

d $3 - 4i$

e $-4 + 3i$

f $\cos 60° + i \sin 60°$

g $\cos 120° + \sin 120°$

h $\cos 180° + i \sin 180°$

Write down the moduli of these complex numbers and give the principal values of their arguments.

2 Write down, in the form $x + iy$, the complex numbers whose moduli are equal to one and whose arguments are

a 0°	**b** 90°
c 180°	**d** 270°
e 360°	**f** 30°
g −30°	**h** 120°
i −120°	**j** 150°

3 Given that $z = 3 + 4i$ and $w = 12 + 5i$, write down the moduli and arguments of

a z	**b** w
c $1/z$	**d** $1/w$
e zw	**f** z^*
g w^*	**h** $(zw)^*$
i z^2	**j** w^2

4 Simplify: $(1 + i)^2$, $(1 + i)^3$, $(1 + i)^4$.

Draw in an Argand diagram the radius vectors corresponding to $(1 + i)$, $(1 + i)^2$, $(1 + i)^3$, $(1 + i)^4$. Find the principal values of the arguments of these complex numbers.

5 Repeat question 4

a for the complex number $\frac{1}{2}\sqrt{3} + \frac{1}{2}i$,

b for the complex number $\sqrt{3} + i$.

6 Given the complex number $z = a + ib$, where a and $b \in \mathbb{R}$, find z^2 and $1/z$ in terms of a and b. Verify that $|z^2| = |z|^2$ and $|1/z| = 1/|z|$.

7 Prove that if $|z| = r$, then $zz^* = r^2$.

8 Given that $z = a + ib$ and $w = c + id$, where a, b, c and $d \in \mathbb{R}$, find zw in terms of a, b, c, and d, and verify that $|zw| = |z| \times |w|$.

Chapter 11

Matrices and determinants (1)

11.1 Introduction to matrices

If you are already familiar with this topic check through §11.2 to §11.5 to ensure that your knowledge of the basic work is absolutely secure. If it is totally new you may find it helpful to supplement the exercises in this book with further practice from a more elementary textbook such as *New General Mathematics*.

A matrix is simply a rectangular array of numbers. A matrix containing m rows and n columns is called an $\boldsymbol{m \times n}$ **matrix**. Matrices are often used to store information. E.g. the matrix **P**, below, records the sales of three books, A, B and C, each of which is published in hardback and paperback form, on one particular day. (This is a 2×3 matrix.)

	A	B	C
Hardback	5	2	1
Paperback	10	7	4

This matrix tells us that, on the day in question, 7 copies of the paperback edition of book B were sold, and so on.

There are conventions in mathematics about the way matrices are written. Firstly, if the layout of the matrix has been standardised, the labels of the rows and columns may be discarded. Secondly, the array of numbers should be enclosed in large round brackets. The letter used as the name of the matrix (**P** in the example above) is printed in bold type in text books. So the matrix in the preceding paragraph is written

$$\mathbf{P} = \begin{pmatrix} 5 & 2 & 1 \\ 10 & 7 & 4 \end{pmatrix}$$

The matrix **Q**, below, represents the sales of the same books on the next day:

$$\mathbf{Q} = \begin{pmatrix} 3 & 0 & 1 \\ 8 & 7 & 4 \end{pmatrix}$$

On this day, for example, 3 copies of the hardback version of book A were sold.

One very common use for matrices* in mathematics is to store the coordinates of points in coordinate geometry. In the example below, the first row of the 2×4 matrix **M** gives the x-coordinate and the second row gives the y-coordinate of four points, A, B, C and D, in order.

$$\mathbf{M} = \begin{pmatrix} 0 & -3 & \frac{1}{2} & 5 \\ 1 & 2 & -1 & 4 \end{pmatrix}$$

Matrix **M** tells us that A is the point (0, 1), B is (−3, 2), C is $(\frac{1}{2}, -1)$ and D is (5, 4). Unlike the previous example, the entries in this matrix do not have to be whole numbers. In general, the elements in a matrix can be any real numbers (in more advanced work, even complex numbers may be used).

*Plural of matrix.

11.2 Matrix addition

In the last section, we used **P** and **Q** to represent the sales of books on two consecutive days. If the book shop owner wishes to know the number of books sold on the two days taken together, all he has to do is to add the corresponding elements, i.e. the numbers which appear in the corresponding positions in the two matrices. If he is good at arithmetic, he should obtain

$$\begin{pmatrix} 8 & 2 & 2 \\ 18 & 14 & 8 \end{pmatrix}$$

It is natural to call the matrix obtained in this way the sum of **P** and **Q**, and so we write

$$\mathbf{P} + \mathbf{Q} = \begin{pmatrix} 8 & 2 & 2 \\ 18 & 14 & 8 \end{pmatrix}$$

The difference of **P** and **Q** is obtained in a similar fashion:

$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

What meaning could the bookseller attach to this matrix?

In the preceding paragraphs we have described **P** and **Q** as '2×3 matrices' and **M** as 'a 2×4 matrix'. This was because **P** and **Q** each had two rows and three columns, and **M** had two rows and four columns. A matrix which has m rows and n columns is called an $m \times n$ matrix and we say that the **order** of the matrix is $m \times n$. It is only possible to add (or subtract) matrices which have the same order, i.e. they must each have the same number of rows and the same number of columns. If $m = n$, that is, the number of rows equals the number of columns, the matrix is called a **square** matrix.

Example 1 *Find* **A** + **B** *and* **A** − **B** *when*

a $\mathbf{A} = \begin{pmatrix} 3 & 5 & \frac{1}{2} & 4 \\ 4 & -1 & 2 & 0 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} -1 & 4 & 0 & 3 \\ 0 & 2 & 1 & 5 \end{pmatrix}$,

b $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 3 & -6 \\ 5 & 1 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 0 & -6 \\ 6 & 7 \\ 3 & 0 \end{pmatrix}$.

a
$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} 3-1 & 5+4 & \frac{1}{2}+0 & 4+3 \\ 4+0 & -1+2 & 2+1 & 0+5 \end{pmatrix} = \begin{pmatrix} 2 & 9 & \frac{1}{2} & 7 \\ 4 & 1 & 3 & 5 \end{pmatrix}$$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} 3+1 & 5-4 & \frac{1}{2}-0 & 4-3 \\ 4-0 & -1-2 & 2-1 & 0-5 \end{pmatrix} = \begin{pmatrix} 4 & 1 & \frac{1}{2} & 1 \\ 4 & -3 & 1 & -5 \end{pmatrix}$$

b
$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} 2+0 & 0-6 \\ 3+6 & -6+7 \\ 5+3 & 1+0 \end{pmatrix} = \begin{pmatrix} 2 & -6 \\ 9 & 1 \\ 8 & 1 \end{pmatrix}$$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} 2-0 & 0+6 \\ 3-6 & -6-7 \\ 5-3 & 1-0 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ -3 & -13 \\ 2 & 1 \end{pmatrix}$$

A matrix in which every element is zero is called a **zero matrix**. When a zero matrix is added to another matrix with the same number of rows and columns, that matrix will be unchanged:

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

The zero matrix, then, has the property **A** + **0** = **A**, which is very similar to the way the number zero behaves in ordinary algebra. (When you write **0** for the zero matrix, write a wavy line under it, $\underset{\sim}{0}$, to distinguish it from the number zero.)

11.3 Multiplication by a scalar

If **M** is the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ then, proceeding as in the last section,

$$\mathbf{M}+\mathbf{M}+\mathbf{M}+\mathbf{M}+\mathbf{M} = \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix}$$

In ordinary algebra we reduce $x + x + x + x + x$ to $5x$ and we do the same in matrix algebra. So we write

$$5\mathbf{M} = \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix}$$

In general, to multiply a matrix **A** by a real number k (often called a **scalar** in this context), we multiply each number, or element, in the matrix **A** by k. Two examples are given below to illustrate this:

$$k\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} ka & kb & kc \\ kd & ke & kf \end{pmatrix}$$

and

$$x\begin{pmatrix} x+y & x-y \\ 2x & 3y \end{pmatrix} = \begin{pmatrix} x^2+xy & x^2-xy \\ 2x^2 & 3xy \end{pmatrix}$$

Question

Q1 Given that $\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, find $5\mathbf{A} + 4\mathbf{B}$.

11.4 Matrix multiplication

Returning to the illustration of the book sales on page 137, suppose the matrix **S**, below, represents the total sales of the hardback books in one week,

$$\mathbf{S} = (20 \quad 25 \quad 10)$$

and, let us suppose the prices of the three books are \$5, \$6 and \$7, respectively, then the total value of the books sold is

$$\$(20 \times 5 + 25 \times 6 + 10 \times 7) = \$320$$

In matrix algebra, to learn and apply the rule for multiplying matrices requires care and patience. In its simplest form, the rule for multiplying a single row by a single column, each containing the same number of elements is expressed as follows:

$$(a \quad b \quad c \quad d)\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = (ap + bq + cr + ds)$$

If there are more than four elements, just continue to multiply each element of the row by the corresponding element in the column and add the product to the total. Notice that the result of the operation is a 1×1 matrix, that is, it is a single number (but it is still a matrix, so do not leave out the brackets).

The illustration of the book sales, above, can be expressed in matrix algebra as follows.

The *sales* are represented by the 1×3 matrix **S**, above, the *prices* are shown in a 3×1 column matrix **P**, where $\mathbf{P} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$ and the total value of the books sold is found by evaluating the matrix product **SP**.

$$\mathbf{SP} = (20 \quad 25 \quad 10)\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$$

$$= (100 + 150 + 70)$$

$$= (320)$$

Now suppose the sales of the same books in the following week are represented by the matrix **R**, where $\mathbf{R} = (30 \quad 15 \quad 5)$, then the value of the total sales in the second week is given by the matrix product **RP**.

$$\mathbf{RP} = (30 \quad 15 \quad 5)\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$$

$$= (150 + 90 + 35)$$

$$= (275)$$

We can combine these two sets of figures into a single matrix product, namely,

$$\begin{pmatrix} 20 & 25 & 10 \\ 30 & 15 & 5 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 320 \\ 275 \end{pmatrix}$$

When we read this, it must be clearly understood that the first row of the first matrix and the first row of the product represent the first week's figures and the second row in each case represents the second week's figures.

Let us now suppose that our bookseller discovered that the price list he had been using was out of date and the prices he should have been charging were \$5.50, \$6.50 and \$7.50. He would, of course, want to know how much he should have got for his two weeks' sales. Proceeding as before, he would calculate the matrix product:

$$\begin{pmatrix} 20 & 25 & 10 \\ 30 & 15 & 5 \end{pmatrix}\begin{pmatrix} 5.50 \\ 6.50 \\ 7.50 \end{pmatrix} = \begin{pmatrix} 347.50 \\ 300.00 \end{pmatrix}$$

He could go a stage further and display both sets of figures side by side. Here, it must be understood, the second column of the price matrix corresponds to the second column of the product.

$$\begin{pmatrix} 20 & 25 & 10 \\ 30 & 15 & 5 \end{pmatrix}\begin{pmatrix} 5 & 5.50 \\ 6 & 6.50 \\ 7 & 7.50 \end{pmatrix} = \begin{pmatrix} 320 & 347.50 \\ 275 & 300.00 \end{pmatrix}$$

It is unlikely that booksellers would learn matrix algebra in order to do their accounts! Nevertheless this example serves to introduce the multiplication of matrices, which is absolutely fundamental in the study of matrix algebra. The study of matrices was an important factor in the development of mathematics in the twentieth century. Although it originated as a branch of pure mathematics it has turned out to be an extremely useful subject and today it is extensively used in applied mathematics, commerce and physics.

Let us now take another look at matrix multiplication. Here we multiply a 3×2 matrix **A** by a 2×1 matrix **B**. (Notice that, for multiplication to be possible, it is essential that the number of *columns* in the *first* matrix should be the same as the number of *rows* in the *second* matrix.) Remember to work across each row and down each column.

If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times 7 & + & 2 \times 8 \\ 3 \times 7 & + & 4 \times 8 \\ 5 \times 7 & + & 6 \times 8 \end{pmatrix}$$

$$= \begin{pmatrix} 23 \\ 53 \\ 83 \end{pmatrix}$$

Now we examine the product of a 3×2 matrix **P** and a 2×2 matrix **Q**, bearing in mind that in picking out the pairs of corresponding elements for multiplying together, we work across each row of **P** and down each column of **Q**.

If $\mathbf{P}=\begin{pmatrix}1&2\\3&4\\5&6\end{pmatrix}$ and $\mathbf{Q}=\begin{pmatrix}7&8\\9&0\end{pmatrix}$.

$$\mathbf{PQ}=\begin{pmatrix}1\times7+2\times9 & 1\times8+2\times0\\3\times7+4\times9 & 3\times8+4\times0\\5\times7+6\times9 & 5\times8+6\times0\end{pmatrix}$$

$$=\begin{pmatrix}25&8\\57&24\\89&40\end{pmatrix}$$

Note that **a** for each row of matrix **P** there is a row in the product **PQ**, and that for each column of matrix **Q** there is a column in the product **PQ**, and **b**, for example, the element 89 in the third row and first column of **PQ** is the sum of the products of the corresponding elements of the third row of **P** and the first column of **Q**.

We can now set out the following general features of matrix multiplication:

1 In any matrix product **CD**, if the first matrix **C** has m rows and n columns and the second matrix **D** has n rows and p columns, then the product **CD** has m rows and p columns.

2 The element which lies in the ith row and jth column of **CD** is the sum of the products of the corresponding elements of the ith row of **C** and the jth column of **D**.

Example 2 *Find, where possible, the products* **PQ** *and* **MN**, *given that*

a $\mathbf{P}=\begin{pmatrix}2&3&4\\1&5&2\end{pmatrix}$, $\mathbf{Q}=\begin{pmatrix}1&-1\\0&2\\1&3\end{pmatrix}$

b $\mathbf{M}=\begin{pmatrix}2&1&0\\3&4&7\end{pmatrix}$, $\mathbf{N}=\begin{pmatrix}3&1\\4&5\end{pmatrix}$

a $$\mathbf{PQ}=\begin{pmatrix}2&3&4\\1&5&2\end{pmatrix}\begin{pmatrix}1&-1\\0&2\\1&3\end{pmatrix}$$

$$=\begin{pmatrix}2\times1+3\times0+4\times1 & 2\times(-1)+3\times2+4\times3\\1\times1+5\times0+2\times1 & 1\times(-1)+5\times2+2\times3\end{pmatrix}$$

$$=\begin{pmatrix}6&16\\3&15\end{pmatrix}$$

b It is impossible to form the product **MN**, because **M** has three columns, while **N** has only two rows.

Question

Q2 Find the following matrix products:

a $\begin{pmatrix}3&1\\2&0\\4&1\end{pmatrix}\begin{pmatrix}1&0\\2&1\end{pmatrix}$ **b** $\begin{pmatrix}1&5&6\\2&3&4\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix}$

c $\begin{pmatrix}1&2&3\end{pmatrix}\begin{pmatrix}2&-1\\3&1\\4&2\end{pmatrix}$ **d** $\begin{pmatrix}1&0&3\\2&1&1\end{pmatrix}\begin{pmatrix}3&2\\1&0\\1&1\end{pmatrix}$

In the algebra of real numbers, the order of the terms in a product does not matter, for instance 3×5 and 5×3 both equal 15. We say that in the algebra of real numbers multiplication is *commutative*, that is, $ab=ba$ for any pair of real numbers. This is not the case in matrix algebra.

For example, if $\mathbf{A}=\begin{pmatrix}1&2\\3&4\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}5&6\\7&8\end{pmatrix}$, then

$$\mathbf{AB}=\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}5&6\\7&8\end{pmatrix}=\begin{pmatrix}19&22\\43&50\end{pmatrix}$$

but $$\mathbf{BA}=\begin{pmatrix}5&6\\7&8\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}=\begin{pmatrix}23&34\\31&46\end{pmatrix}$$

i.e. $\mathbf{AB}\neq\mathbf{BA}$

So in matrix algebra, the order of the matrices in a product *does* matter. We say that in matrix algebra, multiplication is *not commutative*.

Exercise 11a

1 Given that $\mathbf{A}=\begin{pmatrix}3&1&2\\5&1&7\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}4&-1&2\\3&1&3\end{pmatrix}$

evaluate:

a 3**A** **b** 2**B**
c 3**A** + 2**B** **d** 3**A** − 2**B**

2 A newspaper agent records the number of papers sold on each day of one week, as follows:

	Mon	Tue	Wed	Thu	Fri	Sat
The Post	120	250	350	300	420	200
The News	120	300	420	200	300	500

Write this as a 2×6 matrix **S**.

The Post costs 12¢ and *The News* costs 15¢. Write this information as a 1×2 row matrix **P**. It is only possible to form *one* of the products **PS** and **SP**. Evaluate the product which it is possible to form and explain the meaning of the first element in the product matrix.

3 Find, where possible, the following products. When it is not possible to form the product, state this clearly and give the reason for your conclusion.

a $\begin{pmatrix} 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 4 & 7 \\ 1 & 6 \end{pmatrix}$

b $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

c $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \end{pmatrix}$

d $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 7 \\ 8 \end{pmatrix}$

4 Given that $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 4 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & 2 \end{pmatrix}$, find **AB** and **BA**. State the property of matrix multiplication which is illustrated by the answer.

5 Given that $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, find **IA** and **AI**. In the algebra of real numbers there is a number which has a property which is very similar to the property shown by **I** in this question. State the number and describe this property.

6 Given that $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix}$, find **AB** and **BA**.

7 Given that $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$, use the result of question 6 to solve the matrix equation $\mathbf{AX} = \mathbf{C}$.

[Hint: multiply both sides of the equation by **B**.]

8 Repeat question 6, given that $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 8 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & -2 \\ -8 & 5 \end{pmatrix}$.

9 Repeat question 7, given that $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 8 & 4 \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

10 Evaluate the matrix products:

a $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \\ 1 & 0 & 7 \end{pmatrix}$

b $\begin{pmatrix} 2 & \frac{1}{2} & \frac{3}{4} \\ 1 & 0 & \frac{1}{4} \\ 3 & 1 & 0 \end{pmatrix}\begin{pmatrix} 8 & -4 \\ 0 & 12 \\ 4 & 0 \end{pmatrix}$

11 Matrices **M** and **N** are members of a set **S** which is defined as follows:

$$S = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

Prove that the product **MN** is also a member of set S.

$\left[\text{Hint: let } \mathbf{M} = \begin{pmatrix} p & q \\ -q & p \end{pmatrix} \text{ and } \mathbf{N} = \begin{pmatrix} r & s \\ -s & r \end{pmatrix}. \right]$

12 Matrices **P** and **Q** are members of a set R which is defined as follows:

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R},\ ad - bc = 1 \right\}$$

Prove that the product **PQ** is also a member of set R.

13 S is the set of matrices of the form $\begin{pmatrix} a & -kb \\ b/k & a \end{pmatrix}$ where a and b can be *any* real numbers, but k is the *same* real number for *all* members of S. If **A** and **B** are two distinct members of set S, show that the product **AB** also belongs to set S.

14 Given that $\mathbf{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and that $\mathbf{Q} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, evaluate the products **PQ** and **QP**. Comment on your answers.

11.5 Matrix algebra

The rules for adding and subtracting a pair of $m \times n$ matrices, which were introduced in §11.3 on page 138, are very simple and unremarkable. It follows that, if **A** and **B** are a pair of such matrices,

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

so *matrix addition is commutative*. Also if **0** is the $m \times n$ zero matrix, then

$$\mathbf{A} + \mathbf{0} = \mathbf{A}$$

If **C** is another $m \times n$ matrix, then it follows from the associative property of real numbers under addition that

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$$

The technical term for this is that *matrix addition is associative*. (This means is that the position of the brackets does not matter. If this seems trivial, contrast it with $(24 \div 12) \div 2$ which does *not* equal $24 \div (12 \div 2)$. Division is *not* an associative operation in real numbers.)

Multiplication of matrices, which was introduced in §11.4 on page 138, is a more complicated operation and, as a result, the rules of matrix multiplication are more interesting. We have already seen that it is possible to have a pair of matrices **A** and **B**, for which $\mathbf{AB} \neq \mathbf{BA}$, so *matrix multiplication is not commutative*.

We have also seen (Exercise 11a, question 5) that if **A** is any 2×2 matrix and if $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$. This is very similar to the way the real number 1 behaves in ordinary algebra, that is, $1 \times x = x \times 1 = x$, where x is any real number. The matrix **I** is usually called the **unit matrix** (in recognition of its similarity to the number 1) or the **identity matrix**. More generally, if **A** is any $n \times n$ matrix, then the corresponding unit matrix is an $n \times n$ matrix, with 1's along the **leading diagonal** (the one that goes from the top left-hand corner to the bottom right-hand corner), and 0's elsewhere.

So the 3×3 unit matrix is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Question

Q3 If $\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$ and **I** is the 3×3 unit matrix, verify that $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$.

In ordinary algebra, if we have a pair of numbers p and q such that $pq = 1$ (for example $4 \times \frac{1}{4} = 1$) we say that q is the **inverse** of p, and conversely p is the inverse of q. (Similarly $\frac{1}{2}$ is the inverse of 2 and 3/5 is the inverse of 5/3.) The same term is used in matrix algebra to describe a pair of matrices **A** and **B** such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$. We say that **A** is the inverse of **B** and **B** is the inverse of **A**. For such a statement to be possible, both **A** and **B** must be square matrices which have the same number of rows and columns as each other. (If this is not obvious, write down a pair of matrices for which it is not true and try to evaluate both **AB** and **BA**.)

If we are given any square matrix **A**, the task of finding its inverse can be very difficult. In this section we shall investigate the simplest case, where **A** is a 2×2 matrix.

Suppose we are given a 2×2 matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The problem is to find a 2×2 matrix **B**, such that $\mathbf{AB} = \mathbf{I}$. Let us write **B** as $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$. (In the work that follows, remember that a, b, c and d are known, but p, q, r and s are unknown; the task is to find p, q, r and s.)

$$\mathbf{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}$$

This product is to be equal to the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so we can write down four equations

$$\begin{aligned} ap + br &= 1 \qquad (1) \\ cp + dr &= 0 \qquad (2) \\ aq + bs &= 0 \qquad (3) \\ cq + ds &= 1 \qquad (4) \end{aligned}$$

from which to find p, q, r and s.

Multiplying (1) by d and (2) by b, we have

$$\begin{aligned} adp + bdr &= d \\ bcp + bdr &= 0 \end{aligned}$$

Subtracting,

$$(ad - bc)p = d$$

Provided $ad - bc$ is not zero we may divide by it, hence

$$p = \frac{d}{\Delta} \qquad (\text{where } \Delta = ad - bc)$$

Substituting this in equation (2) gives

$$\frac{cd}{\Delta} + dr = 0$$

$$\therefore\ r = -\frac{c}{\Delta}$$

You should now solve equations (3) and (4) to find q and s. The solutions are $q = -b/\Delta$ and $s = a/\Delta$.

Hence the inverse matrix **B** is given by

$$\mathbf{B} = \begin{pmatrix} d/\Delta & -b/\Delta \\ -c/\Delta & a/\Delta \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

This is the required inverse of the matrix **A** and the standard abbreviation of this is $\mathbf{A}^{-1}$. Consequently we write:

$$\text{if } \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ then } \mathbf{A}^{-1} = \frac{1}{\boldsymbol{ad-bc}}\begin{pmatrix} \boldsymbol{d} & \boldsymbol{-b} \\ \boldsymbol{-c} & \boldsymbol{a} \end{pmatrix}.$$

This is an important result and you should memorise it.

The method for finding the inverse of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be summarised as follows:

> the elements on the leading diagonal, a and d, are interchanged, the elements on the other diagonal, b and c, have their signs changed, and the matrix is divided by $ad - bc$.

Question

Q4 Using the matrices **A** and $\mathbf{A}^{-1}$ above, verify that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Notice that in finding the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the term $\Delta = ad - bc$ has a very important role to play. We refer to this term as the **determinant** of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

It is convenient to write 'the determinant of matrix **M**' as det **M**, so if $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then we write det $\mathbf{M} = ad - bc$.

Matrices for which det $\mathbf{M} = 0$ are called **singular** matrices. A singular matrix has no inverse because we cannot divide by zero.

Example 3 *Given that* $\mathbf{M} = \begin{pmatrix} 7 & 9 \\ 5 & 7 \end{pmatrix}$, *write the simultaneous equations*

$$7x + 9y = 3$$
$$5x + 7y = 1$$

in the form $\mathbf{MX} = \mathbf{C}$, *where* **X** *is the column matrix* $\begin{pmatrix} x \\ y \end{pmatrix}$ *and* **C** *is the column matrix* $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$. *Hence solve the equations.*

In matrix notation the equations can be expressed

$$\begin{pmatrix} 7 & 9 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

This is in the form

$$\mathbf{MX} = \mathbf{C}$$

as required. Multiply both sides of this matrix equation by $\mathbf{M}^{-1}$ and we have

$$\mathbf{M}^{-1}(\mathbf{MX}) = \mathbf{M}^{-1}\mathbf{C} \qquad (1)$$

Now the left-hand side of this equation can be simplified, as follows:

$$\mathbf{M}^{-1}(\mathbf{MX}) = (\mathbf{M}^{-1}\mathbf{M})\mathbf{X}$$

using the associative property of matrix multiplication, and

$$(\mathbf{M}^{-1}\mathbf{M})\mathbf{X} = \mathbf{IX} = \mathbf{X}$$

using the properties of the inverse and identity matrices.

Equation (1) can now be reduced to

$$\mathbf{X} = \mathbf{M}^{-1}\mathbf{C}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 7 & -9 \\ -5 & 7 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 12 \\ -8 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Hence $x = 3$ and $y = -2$.

This method can be developed for tackling the more general problem of solving n simultaneous equations in n unknowns. It is an example of the way the basic properties of matrix algebra can be combined into a logical argument.

Exercise 11b (Oral)

Find the determinants of the following matrices:

1 **a** $\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 2 \\ 5 & 8 \end{pmatrix}$

c $\begin{pmatrix} 4 & -2 \\ 1 & 7 \end{pmatrix}$ **d** $\begin{pmatrix} 6 & \frac{1}{4} \\ 8 & \frac{1}{2} \end{pmatrix}$

2 **a** $\begin{pmatrix} 3 & 12 \\ 2 & 8 \end{pmatrix}$ **b** $\begin{pmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{6} & \frac{1}{3} \end{pmatrix}$

c $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ **d** $\begin{pmatrix} a & b/k \\ ck & d \end{pmatrix}$, $k \neq 0$

3 State which of these matrices are singular:

a $\begin{pmatrix} 3 & -2 \\ 9 & 6 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 2 \\ 9 & 6 \end{pmatrix}$

c $\begin{pmatrix} 34 & 119 \\ 26 & 91 \end{pmatrix}$ **d** $\begin{pmatrix} x & x^2 \\ x^2 & x^3 \end{pmatrix}$

4 Find the values of x for which the following matrices have no inverse:

a $\begin{pmatrix} x & 7 \\ 8 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} x & 8 \\ 2 & x \end{pmatrix}$

c $\begin{pmatrix} x-2 & 1 \\ 2 & x-3 \end{pmatrix}$ **d** $\begin{pmatrix} x & 2 \\ -2 & x \end{pmatrix}$

5 State the inverse of each of these matrices (read each column in turn):

a $\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}$ **b** $\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$

c $\begin{pmatrix} 6 & 11 \\ 2 & 7 \end{pmatrix}$ **d** $\begin{pmatrix} x & -1 \\ 1 & x \end{pmatrix}$

Exercise 11c

1 Find, where possible, the inverses of the following matrices:

a $\begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 8 & 2 \\ 11 & 3 \end{pmatrix}$

c $\begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 1 \end{pmatrix}$ **d** $\begin{pmatrix} 6 & 3 \\ 8 & 4 \end{pmatrix}$

2 Find the inverses of the following matrices:

a $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ **b** $\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$

c $\begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}$ **d** $\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$

3 Find the inverse of the matrix **M**, where $\mathbf{M} = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$ and hence solve the matrix equation $\mathbf{MX} = \mathbf{C}$, in which $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

4 Repeat question 3 for $\mathbf{M} = \begin{pmatrix} 9 & 2 \\ 8 & 4 \end{pmatrix}$.

5 Write the simultaneous equations

$$7x + 9y = 1$$
$$10x + 13y = 2$$

in matrix form, and, using the method employed in questions 3 and 4, solve the equations.

6 Solve the matrix equation $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A} = \begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ to find the (unknown) matrix **X**.

7 Given that $\mathbf{P} = \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, find the matrix **M**, where $\mathbf{M} = \mathbf{P}^{-1}\mathbf{AP}$.
Hence, or otherwise, find $\mathbf{M}^5$.

8 By writing $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, prove that, for any two 2×2 matrices **M** and **N**, det **MN** = det **M** det **N**.

9 Verify that if $\mathbf{M} = \begin{pmatrix} -5 & 10 & 8 \\ 4 & -7 & -6 \\ -3 & 6 & 5 \end{pmatrix}$ and

$\mathbf{N} = \begin{pmatrix} -1 & 2 & 4 \\ 2 & 1 & -2 \\ -3 & 0 & 5 \end{pmatrix}$, then $\mathbf{MN} = \mathbf{NM} = \mathbf{I}$,

where **I** is the 3×3 unit matrix.
Use this to solve the matrix equation

$$\begin{pmatrix} -5 & 10 & 8 \\ 4 & -7 & -6 \\ -3 & 6 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}$$

10 Express the simultaneous equations

$$\begin{aligned} -x + 2y + 4z &= 7 \\ 2x + y - 2z &= -2 \\ -3x \qquad + 5z &= 7 \end{aligned}$$

in the form of a matrix equation $\mathbf{NX} = \mathbf{C}$, where $\mathbf{N}$ is the 3×3 matrix in No. 9 and $\mathbf{X}$ and $\mathbf{C}$ are suitable column matrices. Hence, using the information from question 9, solve these equations by the matrix method.

11 Given that $\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 4 & -3 \end{pmatrix}$, verify that

$\mathbf{A}^3 = 11\mathbf{A} - 14\mathbf{I}$, where $\mathbf{I}$ is the 3×3 unit matrix. Hence find $\mathbf{A}^{-1}$.

12 Solve, by elimination, the simultaneous equations

$$\begin{aligned} 2x + y \qquad &= a \\ y + \; z &= b \\ 4y - 3z &= c \end{aligned}$$

in terms of a, b and c.
Express the three simultaneous equations in the form $\mathbf{AX} = \mathbf{C}$, where $\mathbf{A}$ and $\mathbf{C}$ are suitably chosen matrices and $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, and give your answer in the form $\mathbf{X} = \mathbf{BC}$.
Hence write down the inverse of matrix $\mathbf{A}$.

11.6 Transformations and matrices

In two dimensions, a linear transformation is a transformation which moves any point P, with coordinates (x, y), to a new position P′, whose coordinates (x', y') are given by a pair of linear equations, that is equations of the form

$$x' = ax + by$$
$$y' = cx + dy$$

In matrix notation this can be written

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Example 4 *A transformation is defined by the matrix equation*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Draw a diagram showing the unit square OIRJ, whose vertices are at (0, 0), (1, 0), (1, 1) and (0, 1) respectively, and its image O′I′R′J′ under the transformation. Describe in words the effect of the transformation on the unit square.

$$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

Note that these four operations can be combined into a single one, in which the matrix $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ is applied to the 2×4 matrix $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, that is,

$$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & -2 \end{pmatrix}$$

From Fig. 11.1 we can see that OIRJ has been enlarged by a scale-factor of 2 and it has been reflected in the x-axis.

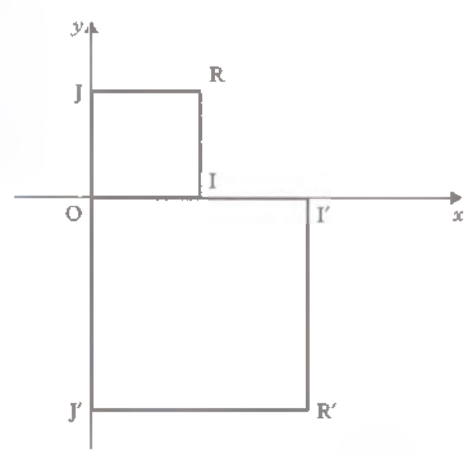


Figure 11.1

If we are given a description, in words, of a certain transformation, it can be quite difficult to find the corresponding matrix. In some simple cases the matrix can be found by considering the effect of the transformation on a triangle OPM, whose vertices are the points (0, 0), (x, y) and $(x, 0)$ respectively. See (a), (b) and (c) below. It should be noted, at this stage, that the image of (0, 0) under this type of transformation is always (0, 0).

(a) Rotation, about O, through 90° anti-clockwise

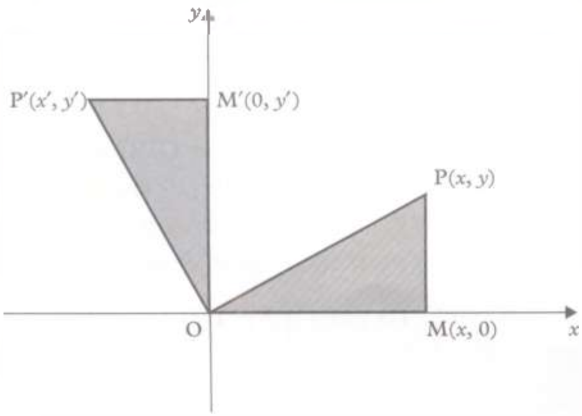


Figure 11.2

From Fig. 11.2, we can see that the new y-coordinate is OM′ and that this is equal in length to OM (since OM′ is OM rotated through 90°) and OM is the original x-coordinate, so $y' = x$. The new x-coordinate is equal to P′M′ in magnitude, but it is negative. However, P′M′ is equal in length to the original y-coordinate and so, $x' = -y$. Hence the new coordinates (x', y') are given by the pair of equations

$$\begin{aligned} x' &= -y \\ y' &= x \end{aligned}$$

and these can be written in matrix form as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

In the next two cases the detailed explanation is omitted. You should make sure that you understand how the matrix equations are obtained from the diagram.

(b) Reflection in the x-axis (see Fig. 11.3)

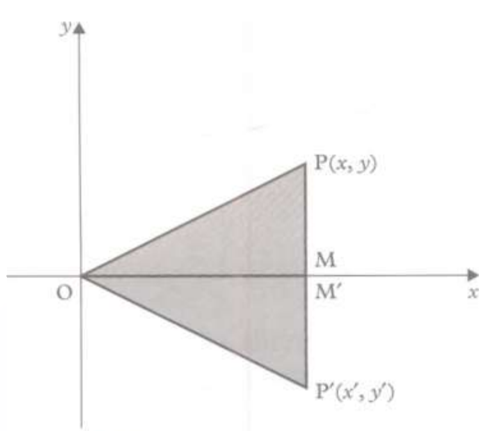


Figure 11.3

$$\begin{aligned} x' &= x \\ y' &= -y \end{aligned}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(c) Reflection in the line $y = x$ (see Fig. 11.4)

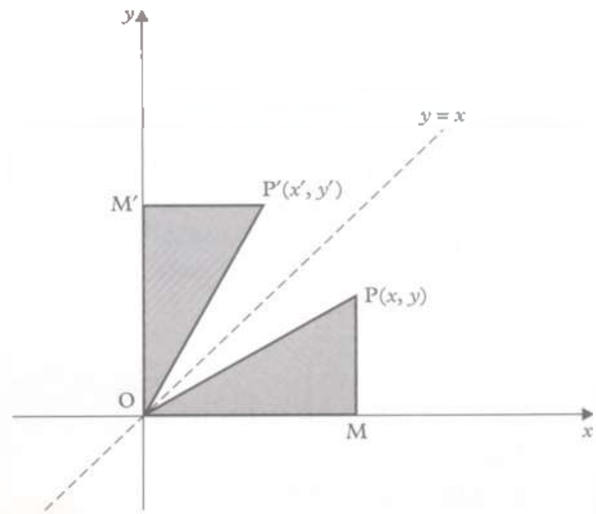


Figure 11.4

$$\begin{aligned} x' &= y \\ y' &= x \end{aligned}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Question

Q5 Find the matrices which correspond to the following transformations:

- **a** a rotation about the origin, through 90°, clockwise,
- **b** a reflection in the line $x + y = 0$,
- **c** an enlargement by a factor of 5, with the origin as the centre of the enlargement.

11.7 General properties of linear transformations

In the last section we looked at some simple transformations and found the corresponding matrices. Before we consider more complicated transformations, we must look more closely at the general properties of transformations which are defined by matrix equations of the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Where appropriate, the notation $(x, y) \mapsto (x', y')$ will be used to indicate that, under the transformation, the point (x, y) moves to the point (x', y'). It is the normal practice to say '(x', y') is the image of (x, y) under the transformation' and that '(x, y) is mapped onto the point (x', y')'.

The following four properties of such transformations are very important. Make sure you understand them before proceeding further.

(1) The image of (0, 0) is (0, 0)

We can see from the matrix product $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ that the image of (0, 0) is (0, 0), for all values of a, b, c and d. We say that the origin is *invariant* under any linear transformation; $(0, 0) \mapsto (0, 0)$.

(2) The images of (1, 0) and (0, 1) are (a, c) and (b, d) respectively

[Throughout this chapter the points (1, 0) and (0, 1) will be labelled I and J respectively; a similar convention is used in Chapter 15.]

As before we need only look at the matrix products

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

to see that $(1, 0) \mapsto (a, c)$ and $(0, 1) \mapsto (b, d)$.

This property is especially valuable because it means that, if we are given the description of a transformation, we only have to look at its effect on the unit square OIRJ, and in particular, the images of I and J, to find the values of a, b, c and d. (You should confirm this by looking back at the transformations in §11.6 on page 145.) Fig. 11.5 shows the unit square OIRJ and its image, for a general transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

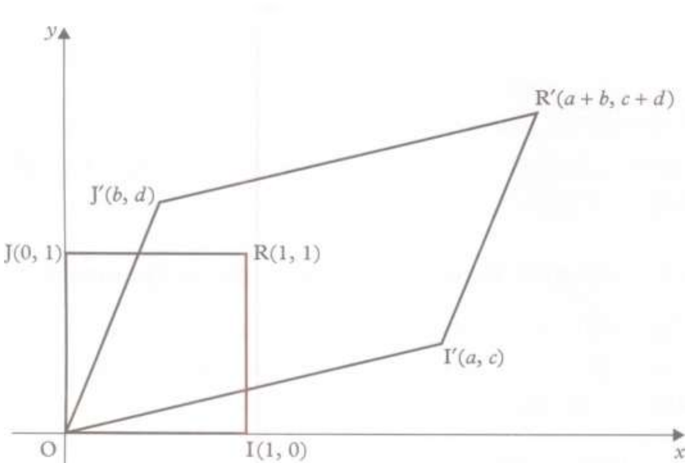


Figure 11.5

(3) The area of the parallelogram OI′R′J′ is ($ad - bc$)

This is left as an exercise. It can be proved fairly easily if the parallelogram is 'framed' in a rectangle which has O and R′ as a pair of diagonally opposite vertices. The region surrounding the parallelogram should then be dissected into suitable rectangles and right-angled triangles.

Notice that $(ad - bc)$ is the determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Notice also that it is possible for $(ad - bc)$ to be negative. This will happen when the unit square is 'turned inside-out', as in a reflection.

(4) Any set of parallel lines is transformed into a set of lines which are also parallel to one another

Let the original set of lines have equations of the form $y = mx + k$, where m is constant, thereby ensuring that the lines in the original set all have the same gradient, i.e., they are parallel to one another. We shall show that these are transformed into a set of lines whose gradient does not depend upon the value of k, i.e. the gradient is the same for any line from the original set of lines.

The new coordinates (x', y') are given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Solving this equation, as in Example 3 on page 143, we obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} \quad \text{where } \Delta = ad - bc.$$

Hence,

$$x = \frac{dx' - by'}{\Delta} \qquad y = \frac{-cx' + ay'}{\Delta}$$

Now (x, y) is a point on the line $y = mx + k$, and consequently its coordinates satisfy this equation. Substituting for x and y we find

$$\frac{-cx' + ay'}{\Delta} = \frac{m(dx' - by')}{\Delta} + k$$

$$-cx' + ay' = mdx' - mby' + k\Delta$$

$$(a + bm)y' = (c + dm)x' + k\Delta$$

so the coordinates (x', y') of P′ satisfy the equation

$$(a + bm)y = (c + dm)x + k\Delta$$

This is the equation of a straight line and its gradient, $(c + dm)/(a + bm)$, does not depend on k. Consequently all members of the original set of lines are transformed into another set of lines, all of which have the same gradient as each other, namely $(c + dm)/(a + bm)$.

In Fig. 11.6 the first diagram shows the original plane with a set of equally spaced lines parallel to the x-axis and another set parallel to the y-axis. The second diagram shows these two sets of lines after the transformation. The unit square is labelled OI_1RJ_1 in the first diagram and its image $OI_1'R'J_1'$ appears in the second.

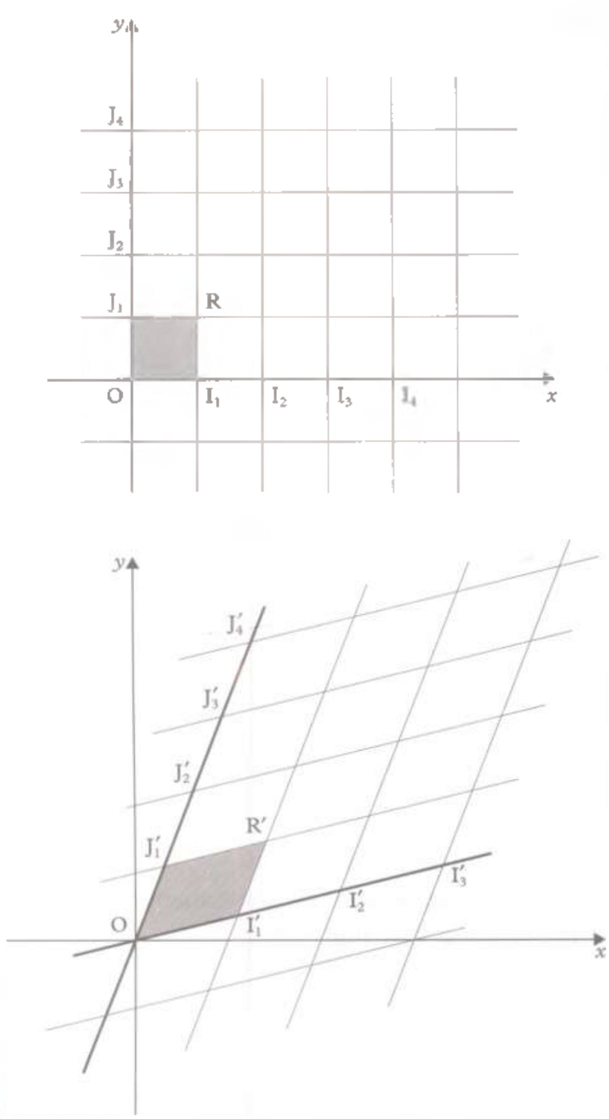


Figure 11.6

Notice that each little square in the original diagram has an area of one square unit and that each of these is transformed into a parallelogram whose area is $(ad - bc)$. Consequently any region in the original diagram will be transformed into a region whose area is $(ad - bc)$ times greater than the area of the original region.

Example 5 *A linear transformation is defined by*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Find the images of (1, 0) *and* (0, 1) *and find the factor by which areas are increased by the transformation. Find also the point whose image is* (4, 6).

The image of (1, 0) is given by the first column of the matrix. Hence $(1, 0) \mapsto (3, 5)$. The image of (0, 1) is given by the second column. Hence $(0, 1) \mapsto (2, 4)$. The area is increased by a factor equal to the determinant, i.e. $(3 \times 4 - 2 \times 5) = 2$.

Let the point (4, 6) be the image of (x, y), then

$$\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

Multiplying both sides of this equation by the inverse matrix, we obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 6 \end{pmatrix} \quad \text{where } \Delta = 12 - 10 = 2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Hence (4, 6) is the image of (2, −1).

Property (2), on the previous page, is especially useful because it enables us to write down, with very little working, the matrices which represent some common transformations, which we can add to the list **(a)**, **(b)**, **(c)** in §11.6 on page 145.

(d) Rotation through an angle α about the origin

Since $OI' = 1$, we can see that $a = \cos\alpha$ and $c = \sin\alpha$. Also, since $OJ' = 1$, $b = -\sin\alpha$ and $d = \cos\alpha$. (See Fig. 11.7.) Hence the required matrix is

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

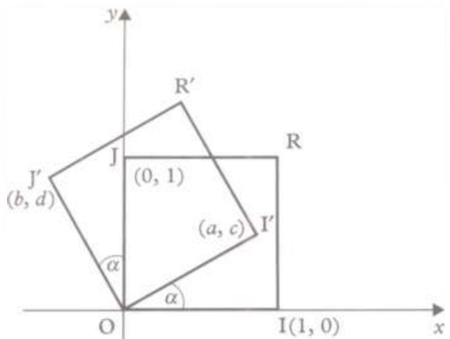


Figure 11.7

(e) Reflection in the line $y = mx$, where $m = \tan \alpha$

The required matrix is

$$\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$$

Proof of this is left to you. It is not difficult, provided you draw a careful diagram.

(f) The transformation under which the unit square is mapped onto the parallelogram with vertices O, I′ (1, 0), R′ (3, 1) and J′ (2, 1)

See Fig. 11.8; a transformation such as this is called a **shear**, parallel to the x-axis.

Using the same method as before, the required matrix is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

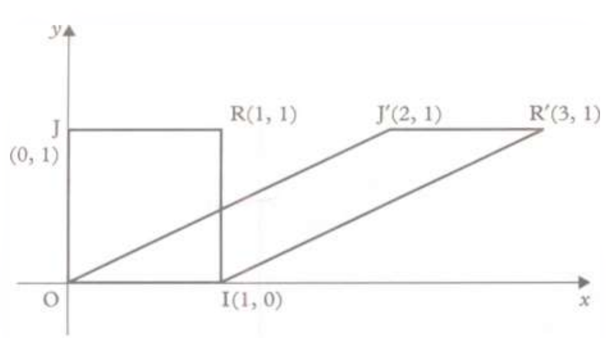


Figure 11.8

Question

Q6 Write down the matrix which represents the shear parallel to the y-axis, under which the unit square is mapped onto the parallelogram with vertices O, I′ (1, 5), R′ (1, 6) and J′ (0, 1).

Notice that the same letter may be used to represent both the transformation and its corresponding matrix — indeed this causes less confusion than using two different letters. Thus we can say 'the transformation **E** is an enlargement with a scale factor k' and we can also say that the matrix representing this transformation is **E**, where $\mathbf{E} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.

11.8 Composite transformations

Suppose that we have two transformations **P** and **Q**, which are given by the matrix equations

$$\mathbf{P}: \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\mathbf{Q}: \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and suppose that **P** is applied first, mapping (x, y) onto (x', y') and that **Q** is then applied, mapping (x', y') onto (x'', y'') i.e.

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

Then, substituting for $\begin{pmatrix} x' \\ y' \end{pmatrix}$ we obtain

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

So the matrix which represents the composite transformation 'do **P**, then do **Q**' is the matrix product

$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

Notice that the matrix which represents the *first* transformation is the matrix on the *right* in this product. This composite matrix is always written **QP**. Remember that **P** is applied first and **Q** second. This may seem strange, but it is logical if we look at the way the matrix product, above, was formed. Notice also that it is the same convention as that used in forming composite functions (see §2.10 on page 48).

Example 6 *Write down the matrices* **R** *and* **S**, *which represent a reflection in the line $y = x$, and a rotation through 90°, anti-clockwise about the origin, respectively. Find the matrix which represents the composite transformation* **SR** *and draw a diagram showing the unit square and its image under the transformation* **SR**. *Describe* **SR** *in words.*

$$\mathbf{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(see §11.6 on page 145)

$$\mathbf{SR} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

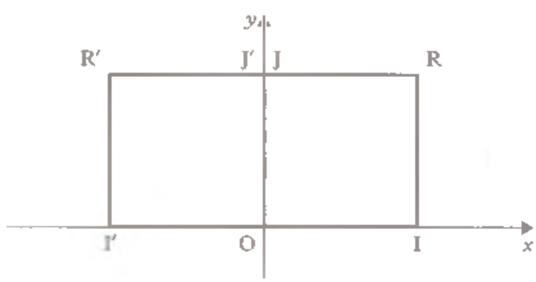


Figure 11.9

The transformation **SR** is a reflection in the line $x = 0$ (see **Fig. 11.9**).

Example 7 *Write down the matrices* **A** *and* **B** *which represent rotations about the origin, through angles α and β, respectively. Find the matrix which represents the transformation* **AB** *and describe this transformation in words. Write down another matrix which represents this transformation and hence find expressions, in terms of* $\sin\alpha$, $\cos\alpha$, $\sin\beta$ *and* $\cos\beta$, *for* $\sin(\alpha+\beta)$ *and* $\cos(\alpha+\beta)$.

$$\mathbf{A} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \quad \text{and}$$

$$\mathbf{B} = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \quad \text{(see §11.6 on page 145).}$$

The composite transformation is given by the product

$$\mathbf{AB} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{pmatrix}$$

The composite transformation is a rotation through an angle β followed by a rotation through an angle α. This can be simplified by replacing it by a single rotation through an angle $(\alpha+\beta)$. (In this case the order of the transformations is immaterial; in other words the transformations are commutative.) The single rotation through an angle $(\alpha+\beta)$ can be represented by the matrix

$$\begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$$

Comparing this with the matrix **AB**, above, we see that

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Example 8 *Write down the matrix* **R** *which represents a reflection in the line $y = mx$, where $m = \tan\alpha$. Prove that $\mathbf{R}^2 = \mathbf{I}$, and hence write down the inverse of the matrix* **R**. *Verify that this agrees with the result obtained by using the normal method for finding $\mathbf{R}^{-1}$ (see §11.5 on page 141).*

$$\mathbf{R} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$$

$$\therefore \mathbf{R}^2 = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2 2\alpha + \sin^2 2\alpha & \cos 2\alpha\sin 2\alpha - \sin 2\alpha\cos 2\alpha \\ \cos 2\alpha\sin 2\alpha - \sin 2\alpha\cos 2\alpha & \cos^2 2\alpha + \sin^2 2\alpha \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

Since $\mathbf{R}^2 = \mathbf{I}$, the inverse of **R** is **R** itself, so

$$\mathbf{R}^{-1} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$$

(This is not very surprising because we have reflected an object in a given line, and then reflected it again in the same line. This returns the object to its original position. In other words $\mathbf{R}^2$ leaves the object unchanged. Any matrix **M** with the property $\mathbf{M}^{-1} = \mathbf{M}$ is called a **self-inverse matrix**.)

The determinant of **R** is given by

$$\begin{aligned} \det \mathbf{R} &= -\cos^2 2\alpha - \sin^2 2\alpha \\ &= -(\cos^2 2\alpha + \sin^2 2\alpha) \\ &= -1 \end{aligned}$$

Hence, applying the method in §11.5 on page 141 for inverting a matrix, we obtain

$$\mathbf{R}^{-1} = \frac{1}{-1}\begin{pmatrix} -\cos 2\alpha & -\sin 2\alpha \\ -\sin 2\alpha & +\cos 2\alpha \end{pmatrix}$$

$$\mathbf{R}^{-1} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} = \mathbf{R}$$

Exercise 11d

1 Describe the transformations represented by

a $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **b** $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

c $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ **d** $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

e $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

2 A certain transformation is represented by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Draw a diagram showing the unit square and its image under this transformation. The triangle whose vertices are A(3, 2), B(7, 2) and C(6, 5) is mapped onto A′B′C′, by this transformation. Find the coordinates of A′, B′ and C′. Find also the areas of the triangles ABC and A′B′C′.

3 Matrices **P** and **Q** are given below:

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \qquad \mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Find the product **QPQ** and describe the transformation it represents.

4 A circle, centre O, radius a, is subject to a transformation whose matrix is $\begin{pmatrix} 1 & 0 \\ 0 & b/a \end{pmatrix}$.

Draw a diagram showing the circle and its image and write down the area inside each of the curves.

5 Write down the matrices which represent

a an anti-clockwise rotation, about the origin, through an acute angle whose sine is 3/5,

b an enlargement by a factor of 5, followed by a reflection in the line $y = x$.

6 Describe the transformation given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

where a and b are real numbers. State the condition required if this matrix represents a pure rotation.

7 By considering the effect on the unit square, describe the transformation represented by the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Hence, or otherwise, find λ and m such that

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ m \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ m \end{pmatrix}$$

expressing m in the form $\tan\alpha$. Hence prove that $\tan 22\tfrac{1}{2}° = \sqrt{2} - 1$.

8 Show that $\mathbf{A} = \begin{pmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{pmatrix}$ is 'self-inverse', that is, $\mathbf{A}^2 = \mathbf{I}$, the unit matrix. Hence describe the transformation which **A** represents.

9 Write down the matrix which represents a reflection in the line $y = (\tan\alpha)x$. Hence show that a reflection in a line which is inclined at an angle α to the x-axis, followed by a reflection in a line which is inclined at an angle β to the x-axis, is equivalent to a reflection. State the angle which the mirror line of this reflection makes with the x-axis.
[You will need the formulae

$$\sin(P - Q) = \sin P\cos Q - \cos P\sin Q$$
$$\cos(P - Q) = \cos P\cos Q + \sin P\sin Q$$

See Chapter 17.]

10 State the transformation which is represented by the matrix **A**, where

$$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and find the matrix $\mathbf{A}^2$. Describe the transformation represented by $\mathbf{A}^2$ and hence write down expressions for $\cos 2\theta$ and $\sin 2\theta$, in terms of $\cos\theta$ and $\sin\theta$.

Chapter 12

Permutations and combinations

12.1 Arrangements

This chapter aims to help you to solve problems that involve arrangements and selections. In the course of the work, a notation is introduced, and a formula is obtained for use in the proof of the binomial theorem (Chapter 14).

Example 1 *Five cards are labelled A, B, C, D and E. In how many different ways can three of these five cards be placed in a row from left to right?*

The first card can be any one of the five:

A, B, C, D, E.

When the first card has been placed, there are four cards left to choose from. The possible ways of placing the first two cards are:

AB,	AC,	AD,	AE,
BA,	BC,	BD,	BE,
CA,	CB,	CD,	CE,
DA,	DB,	DC,	DE,
EA,	EB,	EC,	ED.

Thus, for *each* of the 5 ways of choosing the first, there are 4 ways in which the second card may be chosen. Therefore there are 5×4 (i.e. 20) ways of choosing the first two cards.

Now for each of the 20 ways of placing the first two cards, there are 3 cards left to choose from (e.g. if the first two cards were AB, the third could be C, D, or E). Therefore there are 20×3 ways of placing the third card.

Thus, three cards chosen from A, B, C, D and E may be placed in a row from left to right in 60 different ways.

Example 2 *Three schools have teams of six or more runners in a long distance race. In how many ways can the first six places be taken by the three schools, if there are no dead heats?*

First it should be made clear that there is no question of the individuality of the runners, but only which school each of the first six runners belongs to.

The first place can be taken by any of the 3 schools.

When the first runner has come in, the second place can be taken by any of the 3 schools, so the first two places can be taken in 3×3, or 3^2, ways.

Similarly, the third place can be taken by any of the 3 schools, so the first three places can be taken in $3^2 \times 3$, or 3^3, ways.

Continuing the argument for the fourth, fifth and sixth places, it follows that the first six places may be taken in 3^6, or 729, ways by the three schools.

Example 3 *How many even numbers, greater than 2000, can be formed with the digits 1, 2, 4, 8, if each digit may be used only once in each number?*

If the number is greater than 2000, the first digit can be chosen in 3 ways (i.e.: 2, 4, or 8).

Then, whichever has been chosen to be the first digit, there are 2 ways in which the last digit may be chosen, in order to make the number even. Thus there are 3×2 ways of choosing the first and last digits.

When the first and last digits have been chosen, there are 2 digits, either of which may be the second digit of the number. Thus there are $3 \times 2 \times 2$ ways of choosing the first, last, and second digit.

Now, when three digits have been chosen, there is only 1 left to fill the remaining place, and so there are $3 \times 2 \times 2 \times 1$, i.e. 12, even numbers greater than 2000 which may be formed from the digits 1, 2, 4, 8, without repetitions.

The following table is useful for showing the argument briefly:

Position of digit	First	Last	Second	Third
Number of possibilities	3	2	2	1

It is to be understood, in this and later tables, that the choice is made in the order of the first line.

Exercise 12a

1 Ten students are running a race. In how many ways can the first three places be filled, if there are no dead heats?

2 In how many ways can four letters of the word BRIDGE be arranged in a row, if no letter is repeated?

3 Five letters from the word DRILLING are arranged in a row. Find the number of ways in which this can be done, when the first letter is I and the last is L,
 a if no letter may be repeated,
 b if each letter may occur as many times as it does in DRILLING.

4 A teacher, who works a five-day week, can travel to work on foot, by cycle or by bus. In how many ways can she arrange a week's travelling to work?

5 How many five-figure odd numbers can be made from the digits 1, 2, 3, 4, 5, if no digit is repeated?

6 A girl has two skirts, four blouses and three pairs of shoes. How many different outfits, consisting of skirt, blouse, and a pair of shoes, can she make out of these?

7 In a class of thirty pupils, one prize is awarded for English, another for physics, and a third for mathematics. In how many ways can the prize-winners be chosen?

8 A man has five different flags. In how many ways can he fly them one above the other?

9 The computer department in a large company assigns a personal code number to each employee in the form of a three-digit number, using the digits 0 to 9 inclusive. Code numbers starting with 0 are reserved for members of the management. How many code numbers are available for non-management employees?

10 There are sixteen books on a shelf. In how many ways can these be arranged if twelve of them are volumes of an encyclopaedia, and must be kept together, in order?

11 A typist has six envelopes and six letters. In how many ways can one letter be placed in each envelope without getting every letter in the right envelope?

12 How many codes of the form AB1 2CD (i.e. two letters, followed by a single digit, a space, another digit and two more letters) can be formed from the symbols A, B, C, D, 1 and 2, if each symbol is used once only?

13 In how many ways can the letters of the word NOTATION be arranged?

14 How many odd numbers, greater than 500 000, can be made from the digits 2, 3, 4, 5, 6, 7, without repetitions?

15 Three letters from the word RELATION are arranged in a row. In how many ways can this be done? How many of these contain exactly one vowel?

16 Seven men and six women are to be seated in a row on a platform. In how many ways can they be arranged if no two men sit next to each other? In how many ways can the arrangement be made if there are six men and six women, subject to the same restriction?

17 A man stays three days at a hotel and the menu is the same for breakfast each day. He may have any one of three types of egg dish, or two types of fish, or meat. In how many ways can he order his three breakfasts if he does not have egg two days running nor repeat any dish?

18 A girl has five blue balls, four green balls and three red balls. In how many ways can she arrange four of them in a row, if the balls of any one colour are indistinguishable?

19 I have fifteen books of three different sizes, five of each. In how many ways can I arrange them on my shelf if I keep books of the same size together?

20 Four husbands and wives sit on a bench. In how many ways can they be arranged if
 a there is no restriction,
 b each husband and wife sits next to each other?

12.2 Factorial notation

There are times when a problem on arrangements leads to an answer involving a product of more factors than it is convenient to write down. The next example shows how this can happen.

Example 4 *In how many ways can the cards of one suit, from a pack of playing cards, be placed in a row?*

Position of card in row	First	Second	...	Twelfth	Thirteenth
Number of possibilities	13	12	...	2	1

The table abbreviates the type of argument used in the last three examples, and it leads to the conclusion that the cards of one suit can be placed in a row in

$13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ ways

To shorten the answer in Example 4, the product could be evaluated, giving 6 227 020 800. However it is easier to write this in **factorial notation**:

$13!$

(which is read, 'factorial thirteen', or by some, 'thirteen shriek'!). Thus,

$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

and similarly for any other positive integer.

Factorial notation is used frequently in this chapter and Chapter 14. You should become thoroughly used to it before going on to the next section.

Example 5 a *Evaluate* $\dfrac{9!}{2!7!}$, b *Write* $40 \times 39 \times 38 \times 37$ *in factorial notation.*

a Written in full,

$$\frac{9!}{2!7!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{9 \times 8}{2 \times 1} = 36$$

b $40 \times 39 \times 38 \times 37$

$$= 40 \times 39 \times 38 \times 37 \times \frac{36 \times 35 \times \ldots \times 2 \times 1}{36 \times 35 \times \ldots \times 2 \times 1} = \frac{40!}{36!}$$

Exercise 12b

1 Evaluate:

a $3!$ b $4!$ c $5!$ d $\dfrac{10!}{8!}$

e $\dfrac{7!}{4!}$ f $\dfrac{12!}{9!}$ g $\dfrac{11!}{7!4!}$ h $\dfrac{6!2!}{8!}$

i $(2!)^2$ j $\dfrac{6!}{(3!)^2}$ k $\dfrac{10!}{3!7!}$ l $\dfrac{10!}{2!3!5!}$

2 Express in factorial notation:

a $6 \times 5 \times 4$

b 10×9

c $12 \times 11 \times 10 \times 9$

d $n(n-1)(n-2)$

e $(n+2)(n+1)n$

f $\dfrac{10 \times 9}{2 \times 1}$

g $\dfrac{7 \times 6 \times 5}{3 \times 2 \times 1}$

h $\dfrac{52 \times 51 \times 50}{3 \times 2 \times 1}$

i $\dfrac{n(n-1)}{2 \times 1}$

j $\dfrac{(n+1)n(n-1)}{3 \times 2 \times 1}$

k $\dfrac{2n(2n-1)}{2 \times 1}$

l $n(n-1) \ldots (n-r+1)$

3 Express in factors:

a $20! + 21!$

b $26! - 25!$

c $14! - 2(13!)$

d $15! + 4(14!)$

e $(n+1)! + n!$

f $(n-1)! - (n-2)!$

g $n! + 2(n-1)!$

h $(n+2)! + (n+1)! + n!$

4 Simplify:

a $\dfrac{15!}{11!4!} + \dfrac{15!}{12!3!}$

b $\dfrac{21!}{7!14!} + \dfrac{21!}{8!13!}$

c $\dfrac{16!}{9!7!} + \dfrac{2 \times 16!}{10!6!} + \dfrac{16!}{11!5!}$

d $\dfrac{35!}{16!19!} + \dfrac{3 \times 35!}{17!18!}$

e $\dfrac{n!}{(n-r)!r!} + \dfrac{n!}{(n-r+1)!(r-1)!}$

f $\dfrac{n!}{(n-r)!r!} + \dfrac{2 \times n!}{(n-r+1)!(r-1)!} + \dfrac{n!}{(n-r+2)!(r-2)!}$

12.3 Permutations

In Example 4, we found that 13 playing cards could be placed in a row in 13! ways. If we consider n unlike objects placed in a row, using the same method,

Position of object in row	1st	2nd	...	$(n-1)$th	nth
Number of possibilities	n	$n-1$	...	2	1

we find that they may be arranged in $n!$ ways.

The arrangements of the n objects are called **permutations**. Thus

ABC, ACB, BCA, BAC, CAB, CBA,

are the 3! permutations of the three letters A, B, C.

Again, in Example 1, we found that 3 cards chosen from 5 unlike cards could be arranged in 60 ways. In other words there are 60 permutations of 3 cards chosen from 5 unlike cards.

A permutation is an arrangement of a number of objects in a particular order. In practice, the order may be in space, such as from left to right in a row. Or it may be in time, such as reaching the winning post in a race, or entering a number on a cell phone.

How many permutations are there of r objects chosen from n unlike objects?

The method is indicated in the table below.

Order of choice of object	1st	2nd	3rd	...	$(r-1)$th	rth
Number of possibilities	n	$(n-1)$	$(n-2)$	...	$(n-r+2)$	$(n-r+1)$

Thus there are

$$n(n-1)(n-2)\ldots(n-r+2)(n-r+1)$$

permutations of the objects. But

$$\begin{aligned} & n(n-1)(n-2)\ldots(n-r+2)(n-r+1) \\ &= \frac{n(n-1)(n-2)\ldots(n-r+2)(n-r+1)\times(n-r)\ldots 2\times 1}{(n-r)\ldots 2\times 1} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

Therefore there are $\boldsymbol{n!/(n-r)!}$ permutations of r objects chosen from n unlike objects, if r is less than n.

(We have already found that there are $n!$ permutations of n unlike objects.)

Example 6 *There are 20 books on a shelf, but two of them have red covers, and they must not be put together. In how many ways can the books be arranged?*

This is best approached by finding out the number of ways in which the two books *are* together, and subtracting this from the number of ways in which the 20 books can be arranged if there is no restriction.

Suppose the two red books are tied together, then there are 19 objects, which can be arranged in 19! ways. Now if the order of the two red books is reversed, there will again be 19! arrangements; so that there are $2\times 19!$ ways of arranging the books with the red ones next to each other.

With *no* restriction, 20 books can be arranged in 20! ways; therefore the number of arrangements in which the red books are not together is

$$\begin{aligned} 20! - 2\times 19! &= 19!(20-2) \\ &= 19!\times 18 \end{aligned}$$

Example 7 *In how many ways can 8 people sit at a round table?*

Since the table is round, the position of people relative to the *table* is not important. Thus, supposing they sit down, and then all move one place to the left, the arrangement is still the same.

Therefore one person may be considered to be fixed, and the other 7 can then be arranged about him or her in 7! ways.

Thus there are 5040 ways in which 8 people can sit at a round table.

Example 8 *In how many ways can the letters of the word* BESIGE *be arranged?*

First, give the three E's suffixes: $BE_1SIE_2GE_3$. Then, treating the E's as different, the 7 letters may be arranged in 7! ways.

Now, in every distinct arrangement, the 3 E's may be rearranged amongst themselves in 3! ways, without altering the positions of the B, S, I, or G. For instance, SEIBEEG would have been counted 3! times in the 7! arrangements as

$SE_1IBE_2E_3G$, $SE_2IBE_3E_1G$, $SE_3IBE_1E_2G$,
$SE_1IBE_3E_2G$, $SE_2IBE_1E_3G$, $SE_3IBE_2E_1G$.

Therefore the number of distinct arrangements of the letters in BESIEGE is $7!/3! = 840$.

In the Exercise 12c there are some examples which are best tackled from first principles, as in Example 9.

Example 9 *How many even numbers, greater than 50 000, can be formed with the digits 3, 4, 5, 6, 7, 0, without repetitions?*

Compared with Example 3 page 152, there are two extra difficulties: the number can have either 5 or 6 digits, and the number cannot begin with 0. Therefore the problem is split up into four parts:

1 Numbers with 5 digits, the first digit being even.

Position of digit in number	1st	5th	2nd	3rd	4th
Number of possibilities	1	2	4	3	2

This gives $1 \times 2 \times 4 \times 3 \times 2 = 48$ possibilities.

2 Numbers with 5 digits, the first digit being odd.

Position of digit in number	1st	5th	2nd	3rd	4th
Number of possibilities	2	3	4	3	2

This gives $2 \times 3 \times 4 \times 3 \times 2 = 144$ possibilities.

3 Numbers with 6 digits, the first digit being even.

Position of digit in number	1st	6th	2nd	3rd	4th	5th
Number of possibilities	2	2	4	3	2	1

This gives $2 \times 2 \times 4 \times 3 \times 2 \times 1 = 96$ possibilities.

4 Numbers with 6 digits, the first digit being odd.

Position of digit in number	1st	6th	2nd	3rd	4th	5th
Number of possibilities	3	3	4	3	2	1

This gives $3 \times 3 \times 4 \times 3 \times 2 \times 1 = 216$ possibilities.

Therefore the total number of possibilities is $48 + 144 + 96 + 216 = 504$.

Exercise 12c

1 Seven boys and two girls are to sit together on a long bench. In how many ways can they arrange themselves so that the girls do not sit next to each other?

2 Eight women and two men are to sit at a round table. In how many ways can they be arranged? If, however, the two men sit directly opposite each other, in how many ways can the ten people be arranged?

3 How many arrangements can be made of the letters in the word TROTTING? In how many of these are the N and the G next to each other?

4 On a bookshelf, four books are bound in leather and sixteen in cloth. If the books are to be arranged so that the leather-bound ones are together, in how many ways can this be done? If, in addition, the cloth-bound books are to be kept together, in how many ways can the shelf be arranged?

5 There is room for ten books on a shelf, but there are fifteen to choose from. Of these, however, a dictionary and a telephone directory must go at the ends. In how many ways can the books be arranged?

6 Ten beads of different colours are arranged on a ring. If a salesman claims that no two of his rings are the same, what is the greatest number of rings he could have? (A ring can be turned over.)

7 In his cowhouse, a farmer has seven stalls for cows, and four for calves. If he has ten cows and five calves, in how many ways can he arrange the animals in his cowhouse?

8 At a conference of five nations, each delegation consists of three members. If each delegation sits together, with their leader in the middle, in how many ways can the members be arranged at a round table?

9 How many numbers, divisible by 5, can be made with the digits 2, 3, 4, 5, no digit being used more than once in each number?

10 In a cricket team of 11 players, the captain has settled the first four places in the batting order, and has decided that the four bowlers will occupy the last four places. In how many ways can the batting order be made out?

11 How many arrangements can be made of the letters in the word TERRITORY?

12 A woman has ten ornaments to choose from. One of these is a clock. There is only room for seven ornaments on a shelf and the clock must go in the centre. How many arrangements can be made on the shelf?

13 How many odd numbers, greater than 60 000, can be made from the digits 5, 6, 7, 8, 9, 0, if no number contains any digit more than once?

14 A code word consists of three letters, followed by two digits. How many code words can be made, if no letter nor digit is repeated in any code word?

15 How many numbers of five digits can be made from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, when each number contains exactly one even digit and no digit more than once?

16 A card player holds five spades, four hearts, two diamonds and two clubs. If he keeps the cards of each suit together, in how many ways can he arrange the cards he holds

a if the suits are in the above order,

b if the suits may be arranged in any order?

17 Find the number of ways in which the letters of ISOSCELES can be arranged if the two E's are separated.

18 Find how many numbers greater than 400 000 can be made, using all the digits of 416 566.

19 In how many ways can four red beads, three green beads, and five beads of different colours be strung on a circular wire?

20 Six Tanzanians and two Ugandans are seated in a compartment of a railway carriage with four seats either side. In how many ways can the passengers sit if

a the Ugandans do not sit opposite each other,

b the Ugandans do not sit next to each other?

12.4 Combinations

In the last section, attention was given to permutations, where the order of a set of objects was of importance. However in other circumstances, the order of selection is not important. If, for instance, eight tourists find there is only room for five of them at a hotel, they will be chiefly interested in which five of them stay there, rather than in any order of arrangement.

When a selection of objects is made when their order is not important, it is referred to as a **combination**. Thus, ABC, ACB, CBA, are different permutations, but they are the same combination of letters.

Example 10 *In how many ways can 13 cards be selected from a pack of 52 playing cards?*

First of all, suppose that thirteen cards from the pack are laid on a table in an order from left to right. From the last section, it follows that this can be done in 52!/39! ways.

Now each combination of cards can be arranged in 13! ways, therefore

the number of permutations
$$= 13! \times (\text{the number of combinations})$$

$$\therefore \frac{52!}{39!} = 13! \times (\text{the number of combinations})$$

Therefore the number of combinations of 13 cards chosen from a pack of playing cards is 52!/(39!13!).

In how many ways can r objects be chosen from n unlike objects?

In §12.3 on page 154 it was shown that there are $n!/(n-r)!$ permutations of r objects chosen from n unlike objects.

Now each combination of r objects can be arranged in $r!$ ways, therefore

the number of permutations
$$= r! \times (\text{the number of combinations})$$

$$\therefore \frac{n!}{(n-r)!} = r! \times (\text{the number of combinations})$$

Hence the number of combinations of r objects chosen from n unlike objects is

$$\frac{n!}{(n-r)!r!}$$

For brevity, the number of combinations of r objects chosen from n unlike objects is written ^nC_r, thus

$$^nC_r = \frac{n!}{(n-r)!r!}$$

^nC_r is also sometimes written as $_nC_r$ or $\binom{n}{r}$ (see §14.4 on page 174).

Questions

Q1 What are the values of **a** $^8C_3$, $^8C_5$; **b** $^{10}C_6$, $^{10}C_4$?

Q2 In how many ways can $n-r$ objects be chosen from n unlike objects?

Q3 Show that $^nC_r = {^nC_{n-r}}$.

Example 11 *A mixed hockey team containing 5 men and 6 women is to be chosen from 7 men and 9 women. In how many ways can this be done?*

Five men can be selected from 7 men in ${}^7C_5$ ways, and 6 women can be selected from 9 women in ${}^9C_6$ ways.

Now for each of the ${}^7C_5$ ways of selecting the men, there are ${}^9C_6$ ways of selecting the women, therefore there are ${}^7C_5 \times {}^9C_6$ ways of selecting the team.

$$\begin{aligned}{}^7C_5 \times {}^9C_6 &= \frac{7!}{2!5!} \times \frac{9!}{3!6!}\\ &= 21 \times 84\end{aligned}$$

Therefore the team can be chosen in 1764 ways.

Exercise 12d

1 Evaluate: **a** ${}^{10}C_2$, **b** ${}^6C_4$, **c** ${}^7C_3$, **d** ${}^9C_5$, **e** ${}^8C_4$.
Express in factors:
f ${}^nC_2$, **g** ${}^nC_3$, **h** ${}^nC_{n-2}$, **i** ${}^{n+1}C_2$, **j** ${}^{n+1}C_{n-1}$.

2 In how many ways can a football team be selected from thirteen players?

3 There are ten possible players to represent a tennis club. Of these the captain and the secretary must be in the team. In how many ways can the team be selected?

4 Ten boxes each hold one white ball and one coloured ball, every colour being different. Find the number of ways in which one ball may be taken from each box if half those taken are white.

5 Nine people are going to travel in two taxis. The larger has five seats, and the smaller has four. In how many ways can the party be split up?

6 A girl wants to ask eight friends to her house, but there is only room for four of them. In how many ways can she choose whom to invite if two of them are sisters and must not be separated? (Consider two cases, **a** when both sisters are invited, **b** when neither sister is invited.)

7 In a game of mixed hockey there are ten married couples and two unmarried women playing. In how many ways can the two teams be made up, if no husband may play against his wife?

8 A ferry which holds ten people has to carry thirteen men and seven women across a river. Find the number of ways in which they may be taken across if all the women go on the first trip.

9 Twelve people each spin a coin. Find the number of ways in which exactly five heads may be obtained.

10 Two canoes each hold six people. In how many ways can a party of six boys and six girls divide themselves so that there are equal numbers of boys and girls in each canoe?

11 In how many ways can four black and eight white blocks be arranged in a pile?

12 A committee of six is to be formed from nine women and three men. In how many ways can the members be chosen so as to include at least one man?

13 Ten men are present at a tennis club. In how many ways can four be chosen to play doubles if two men refuse to play on the same court?

14 A student is going to take six books to read. She chooses these from eleven books, one of which contains two volumes, which she will take or leave together. Find the number of ways in which she can make her choice.

15 Four people are to play tennis and four others are to play golf. Find the number of ways in which they may be chosen if eleven people are available.

16 Twelve students are to dine at three tables. In how many ways can they be split up if each table holds four?

17 Twelve people are to travel by three cars, each of which holds four. Find the number of ways in which they may be divided if two people refuse to travel in the same car.

18 A committee of ten is to be chosen from nine men and six women. In how many ways can it be formed if at least four women are to be on the committee?

19 In how many ways can eleven people be chosen for an international hockey team if they are selected from seven Ugandans, six Tanzanians and five Rwandese, and if at least one of each nationality must be in the team?

Series

13.1 Sequences

Examine the lists of numbers in **Q1**. Each list is written down in a definite order, and there is a simple rule by which the terms are obtained. Such a list of terms is called a **sequence**.

Question

Q1 Write down the next two terms in each of the following sequences:

a $1, 3, 5, 7, \dots$

b $2, 5, 8, 11, \dots$

c $1, 2, 4, 8, \dots$

d $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \dots$

e $1^3, 2^3, 3^3, 4^3, \dots$

f $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$

g $1, 4, 9, 16, \dots$

h $1, 2, 6, 24, 120, \dots$

i $1, \frac{2}{3}, \frac{3}{9}, \frac{4}{27}, \dots$

j $4, 2, 0, -2, \dots$

k $1, -1, 1, -1, \dots$

l $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \dots$

Suppose you are asked to add up the integers from 1 to 100. This can be done by elementary arithmetic, but it would be very tedious. Fortunately there is a short-cut.

First write the numbers down in their natural order:

$$1 + 2 + 3 + \dots + 98 + 99 + 100$$

Now write the numbers down again in the opposite order, so that we have:

$$\begin{array}{rrrcrrr} 1\,+ & 2\,+ & 3\,+ & \dots\,+ & 98\,+ & 99\,+ & 100 \\ 100\,+ & 99\,+ & 98\,+ & \dots\,+ & 3\,+ & 2\,+ & 1 \\ \hline 101\,+ & 101\,+ & 101\,+ & \dots\,+ & 101\,+ & 101\,+ & 101 \end{array}$$

The numbers in each column have been added together, and, since there are 100 terms in the top line, the total is $100 \times 101 = 10\,100$. But this is twice the sum required, therefore the sum of the integers from 1 to 100 is 5050.

If the terms of a sequence are considered as a sum, for instance

$$1 + 2 + 3 + \dots + 98 + 99 + 100$$

or

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \dots$$

the expression is called a **series**. A series may end after a finite number of terms, in which case it is called a **finite series**. Sometimes it may not end; it is then called an **infinite series**.

13.2 Arithmetical progressions

The method of §13.1, for finding the sum of a series, may only be applied to a certain type, which is usually called an **arithmetical progression** (often abbreviated to A.P.). For example,

$$1 + 3 + 5 + \dots + 99$$
$$7 + 11 + 15 + \dots + 79$$
$$3 - 2 - 7 - \dots - 42$$
$$1\tfrac{1}{8} + 1\tfrac{1}{4} + 1\tfrac{3}{8} + \dots + 3\tfrac{1}{2}$$
$$-2 - 4 - 6 - \dots - 16$$

are finite arithmetical progressions. In such a series, any term may be obtained from the previous term by adding a certain number, called the **common difference**. Thus the common differences in the above progressions are 2, 4, −5, $\frac{1}{8}$, −2.

Example 1 *Find the third, tenth, twenty-first and nth terms of the A.P. with first term 6 and common difference 5.*

Position of term	1st	2nd	3rd	4th	10th	21st	nth
Value	6	6+5	6+2×5	6+3×5	6+9×5	6+20×5	$6+(n-1)\times5$

Note that to find the nth term $n - 1$, common differences are added to the first term. (Throughout this chapter assume that n represents a positive integer.)

The third, tenth, twenty-first, and nth terms are 16, 51, 106, and $5n + 1$.

Example 2 *Find the sum of the first twenty terms of the A.P.* $-4 - 1 + 2 + \dots$.

To find the twentieth term, add 19 times the common difference to the first term: $-4 + 19 \times 3 = 53$.

Write S_{20} for the sum of the first twenty terms, then using the method of §13.1 opposite,

$$S_{20} = -4 - 1 + 2 + \dots + 53$$

Again,

$$S_{20} = 53 + 50 + 47 + \ldots - 4$$

Adding,

$$2S_{20} = 49 + 49 + 49 + \ldots + 49 = 20 \times 49$$
$$\therefore S_{20} = 490$$

The sum of the first twenty terms of the A.P. is 490.

Exercise 13a

1 Which of the following series are arithmetical progressions? Write down the common differences of those that are.

a $7 + 8\frac{1}{2} + 10 + 11\frac{1}{2}$
b $-2 - 5 - 8 - 11$
c $1 + 1.1 + 1.2 + 1.3$
d $1 + 1.1 + 1.11 + 1.111$
e $\frac{1}{2} + \frac{2}{3} + \frac{7}{6} + \frac{3}{2}$
f $1^2 + 2^2 + 3^2 + 4^2$
g $n + 2n + 3n + 4n$
h $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$
i $1\frac{1}{8} + 2\frac{1}{4} + 3\frac{3}{8} + 4\frac{1}{2}$
j $19 + 12 + 5 - 2 - 9$
k $1 - 2 + 3 - 4 + 5$
l $1 + 0.8 + 0.6 + 0.4$

2 Write down the terms indicated in each of the following A.P.s:

a $3 + 11 + \ldots$, 10th, 19th
b $8 + 5 + \ldots$, 15th, 31st
c $\frac{1}{4} + \frac{7}{8} + \ldots$, 12th, nth
d $50 + 48 + \ldots$, 100th, nth
e $7 + 6\frac{1}{2} + \ldots$, 42nd, nth
f $3 + 7 + \ldots$, 200th, $(n + 1)$th

3 Find the number of terms in the following A.P.s:

a $2 + 4 + 6 + \ldots + 46$
b $50 + 47 + 44 + \ldots + 14$
c $2.7 + 3.2 + \ldots + 17.7$
d $6\frac{1}{4} + 7\frac{1}{2} + \ldots + 31\frac{1}{4}$
e $407 + 401 + \ldots - 133$
f $2 - 9 - \ldots - 130$
g $2 + 4 + \ldots + 4n$
h $x + 2x + \ldots + nx$
i $a + (a + d) + \ldots + \{a + (n - 1)d\}$
j $a + (a + d) + \ldots + l$

4 Find the sums of the following A.P.s:

a $1 + 3 + 5 + \ldots + 101$
b $2 + 7 + 12 + \ldots + 77$
c $-10 - 7 - 4 - \ldots + 50$
d $71 + 67 + 63 + \ldots - 53$
e $2.01 + 2.02 + 2.03 + \ldots + 3.00$
f $1 + 1\frac{1}{6} + 1\frac{1}{3} + \ldots + 4\frac{1}{2}$
g $x + 3x + 5x + \ldots + 21x$
h $a + (a + 1) + \ldots + (a + n - 1)$
i $a + (a + d) + \ldots + \{a + (n - 1)d\}$

5 Find the sums of the following arithmetical progressions as far as the terms indicated:

a $4 + 10 + \ldots$ 12th term
b $15 + 13 + \ldots$ 20th term
c $1 + 2 + \ldots$ 200th term
d $20 + 13 + \ldots$ 16th term
e $6 + 10 + \ldots$ nth term
f $1\frac{1}{4} + 1 + \ldots$ nth term

6 The second term of an A.P. is 15, and the fifth is 21. Find the common difference, the first term and the sum of the first ten terms.

7 The fourth term of an A.P. is 18, and the common difference is -5. Find the first term and the sum of the first sixteen terms.

8 Find the difference between the sums of the first ten terms of the A.P.s whose first terms are 12 and 8, and whose common differences are respectively 2 and 3.

9 The first term of an A.P. is -12, and the last term is 40. If the sum of the progression is 196, find the number of terms and the common difference.

10 Find the sum of the odd numbers between 100 and 200.

11 Find the sum of the even numbers, divisible by three, lying between 400 and 500.

12 The twenty-first term of an A.P. is $5\frac{1}{2}$, and the sum of the first twenty-one terms is $94\frac{1}{2}$.
Find the first term, the common difference and the sum of the first thirty terms.

13 Show that the sum of the integers from 1 to n is $\frac{1}{2}n(n + 1)$.

14 The twenty-first term of an A.P. is 37 and the sum of the first twenty terms is 320.
What is the sum of the first ten terms?

15 Show that the sum of the first n terms of the A.P. with first term a and common difference d is $\frac{1}{2}n\{2a + (n - 1)d\}$.

13.3 Geometrical progressions

Another common series is the **geometrical progression**, or G.P. The following are examples of G.P.s:

$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{512}$
$3+6+12+\ldots+192$
$\frac{16}{27}-\frac{8}{9}+\frac{4}{3}-\ldots+\frac{27}{4}$

In such a progression, the ratio of a term to the previous one is a constant, called the **common ratio**. Thus, the common ratios of the above progressions are respectively $\frac{1}{2}$, 2 and $-\frac{3}{2}$.

Question

Q2 Write down the third and fourth terms of the progressions which begin **i** $2+4+\ldots$, **ii** $12+6+\ldots$,
a if they are A.P.s, **b** if they are G.P.s.

Example 3 *Find the third, tenth, twenty-first and nth terms of the G.P. which begins* $3+6+\ldots$.

Position of term	1st	2nd	3rd	4th	10th	21st	nth
Value	3	3×2	3×2^2	3×2^3	3×2^9	3×2^{20}	$3\times2^{n-1}$

Note that to find the nth term, the first term is multiplied by the $(n-1)$th power of the common ratio.

The third, tenth, twenty-first, and nth terms are 12, 1536, 3 145 728, and $3\times2^{n-1}$.

Example 4 *Find the sum of the first eight terms of the geometrical progression* $2+6+18+\ldots$.

To find the eighth term, multiply the first term by the seventh power of the common ratio: 2×3^7.

Let S_8 be the sum of the first eight terms of the expression.

$\therefore S_8 = 2+2\times3+2\times3^2+\ldots+2\times3^7$

Now multiply both sides by the common ratio and write the terms obtained one place to the right, so that we have

$$S_8 = 2+2\times3+2\times3^2+\ldots+2\times3^7$$
$$3S_8 = \quad 2\times3+2\times3^2+\ldots+2\times3^7+2\times3^8$$

Subtracting the top line from the lower,

$$2S_8 = -2+2\times3^8$$
$$\therefore S_8 = 3^8-1$$

Therefore the sum of the first eight terms is 6560.

Exercise 13b

1 Which of the following series are geometrical progressions? Write down the common ratios of those that are.

a $3+9+27+81$
b $1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}$
c $-1+2-4+8$
d $1-1+1-1$
e $1+1\frac{1}{2}+1\frac{1}{4}+1\frac{1}{8}$
f $a+a^2+a^3+a^4$
g $1+1.1+1.21+1.331$
h $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{36}$
i $2+4-8-16$
j $\frac{3}{4}+\frac{9}{2}+27+162$

2 Write down the terms indicated in each of the following geometrical progressions. Do not simplify your answers.

a $5+10+\ldots$, 11th, 20th
b $10+25+\ldots$, 7th, 19th
c $\frac{2}{3}+\frac{3}{4}+\ldots$, 12th, nth
d $3-2+\ldots$, 8th, nth
e $\frac{2}{7}-\frac{3}{7}+\ldots$, 9th, nth
f $3+1\frac{1}{2}+\ldots$, 19th, $2n$th

3 Find the number of terms in the following geometrical progressions:

a $2+4+8+\ldots+512$
b $81+27+9+\ldots+\frac{1}{27}$
c $0.03+0.06+0.12+\ldots+1.92$
d $\frac{8}{81}-\frac{4}{27}+\frac{2}{9}-\ldots-1\frac{11}{16}$
e $5+10+20+\ldots+5\times2^n$
f $a+ar+ar^2+\ldots+ar^{n-1}$

4 Find the sums of the geometrical progressions in question 3. Simplify, but do not evaluate, your answers.

5 Find the sums of the following geometrical progressions as far as the terms indicated. Simplify, but do not evaluate, your answers.

a $4+12+36+\ldots$, 12th term
b $15+5+1\frac{2}{3}+\ldots$, 20th term

c $1 - 2 + 4 - \ldots$, 50th term
d $24 - 12 + 6 - \ldots$, 17th term
e $1.1 + 1.21 + 1.331 + \ldots$, 23rd term
f $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$, 13th term
g $3 + 6 + 12 + \ldots$, nth term
h $1 - \frac{1}{3} + \frac{1}{9} - \ldots$, nth term

6 The third term of a geometrical progression is 10, and the sixth is 80. Find the common ratio, the first term and the sum of the first six terms.

7 The third term of a geometrical progression is 2, and the fifth is 18.
Find two possible values of the common ratio, and the second term in each case.

8 The three numbers, $n - 2$, n, $n + 3$, are consecutive terms of a geometrical progression.
Find n, and the term after $n + 3$.

9 A man starts saving on 1st April. He saves 1¢ the first day, 2¢ the second, 4¢ the third, and so on, doubling the amount every day. If he managed to keep on saving under this system until the end of the month (30 days), how much would he have saved? Give your answer in dollars, correct to three significant figures.

10 The first term of a G.P. is 16 and the fifth term is 9. What is the value of the seventh term?

11 Show that the sum of the series
$4 + 12 + 36 + 108 + \ldots$ to 20 terms is greater than 3×10^9.

12 The numbers $n - 4$, $n + 2$, $3n + 1$ are in geometrical progression.
Find the two possible values of the common ratio.

13 What is the common ratio of the G.P.
$(\sqrt{2} - 1) + (3 - 2\sqrt{2}) + \ldots$?
Find the third term of the progression.

14 Find the ratio of the sum of the first 10 terms of the series

$$\log x + \log x^2 + \log x^4 + \log x^8 + \ldots$$

to the first term.

13.4 Formulae for the sums of A.P.s and G.P.s

The methods of Examples 2 and 4 will now be applied to general A.P.s and G.P.s to obtain formulae for their sums.

a If the first term of an A.P. is a, and the nth term is l, we may find the sum S_n of the first n terms.

We have

$$S_n = a + (a + d) + \ldots + (l - d) + l \quad \text{(where there are } n \text{ terms)}$$

And in reverse order,

$$S_n = l + (l - d) + \ldots + (a + d) + a$$

Adding,

$$2S_n = (a + l) + (a + l) + \ldots + (a + l) + (a + l)$$

There are n terms on the right-hand side,

$$\therefore 2S_n = n(a + l)$$

$$\therefore \boldsymbol{S_n = \frac{n(a + l)}{2}}$$

b If the first term of an A.P. is a, and the common difference is d, the nth term is $a + (n - 1)d$. Substituting $l = a + (n - 1)d$ in the formula above,

$$S_n = \frac{n}{2}\{a + a + (n - 1)d\}$$

$$\therefore \boldsymbol{S_n = \frac{n}{2}\{2a + (n - 1)d\}}$$

c If the first term of a G.P. is a and the common ratio is r, we may find the sum S_n of the first n terms.

The nth term is ar^{n-1}, therefore

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$$
$$\therefore rS_n = \quad ar + ar^2 + \ldots + ar^{n-1} + ar^n$$

Subtracting,

$$S_n - rS_n = a - ar^n$$
$$\therefore S_n(1 - r) = a(1 - r^n)$$
$$\therefore \boldsymbol{S_n = a\left(\frac{1 - r^n}{1 - r}\right)}$$

An alternative formula for the sum of a G.P. is obtained by multiplying numerator and denominator by -1:

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right)$$

This is more convenient if r is greater than 1.

Example 5 *In an arithmetical progression, the thirteenth term is 27, and the seventh term is three times the second term. Find the first term, the common difference and the sum of the first ten terms.*

[There are two unknowns (the first term and the common difference). We have two pieces of information:

a the thirteenth term is 27.
b the seventh term is three times the second term.

Thus we can form two equations which will enable us to find the two unknowns.]

Let the first term be a, and let the common difference be d.

Then the thirteenth term is $a + 12d$, therefore

$$a + 12d = 27$$

The seventh term is $a + 6d$, and the second term is $a + d$, therefore

$$a + 6d = 3(a + d)$$
$$\therefore 3d = 2a$$

Substituting in the first equation,

$$a + 8a = 27$$
$$\therefore a = 3$$

and so

$$d = 2$$

Therefore the first term is 3, and the common difference is 2.

To find the sum of the first ten terms, we know that

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$
$$\therefore S_{10} = \tfrac{10}{2}(6 + 9 \times 2)$$
$$= 5 \times 24$$

Therefore the sum of the first ten terms is 120.

Example 6 *In a geometrical progression, the sum of the second and third terms is 6, and the sum of the third and fourth terms is –12. Find the first term and the common ratio.*

[As in the last example, there are two unknowns (the first term and the common ratio). We have two pieces of information:

a the sum of the second and third terms is 6,
b the sum of the third and fourth terms is –12.

We may therefore write down two equations and these will enable us to find the two unknowns.]

Let the first term be a, and let the common ratio be r. Then the second term is ar, and the third term is ar^2, therefore

$$ar + ar^2 = 6$$

The third term is ar^2, and the fourth term is ar^3, therefore

$$ar^2 + ar^3 = -12$$

Factorising the left-hand sides of the equations,

$$ar(1 + r) = 6$$
$$ar^2(1 + r) = -12$$

We may eliminate a by dividing:

$$\frac{ar(1+r)}{ar^2(1+r)} = -\frac{6}{12}$$
$$\therefore \frac{1}{r} = -\frac{1}{2}$$
$$\therefore r = -2$$

Substituting $r = -2$ in $ar(1 + r) = 6$,

$$a(-2)(-1) = 6$$
$$\therefore a = 3$$

Therefore the first term is 3, and the common ratio is –2.

Example 7 *The sum of a number of consecutive terms of an arithmetical progression is $-19\frac{1}{2}$, the first term is $16\frac{1}{2}$, and the common difference is –3. Find the number of terms.*

With the notation of §13.4 on the previous page,

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

Substituting $S_n = -19\frac{1}{2}$, $a = 16\frac{1}{2}$, $d = -3$:

$$-\frac{39}{2} = \frac{n}{2}\{33 - 3(n-1)\}$$
$$\therefore -39 = n(36 - 3n)$$
$$\therefore 3n^2 - 36n - 39 = 0$$

Dividing through by 3,

$$n^2 - 12n - 13 = 0$$
$$\therefore (n - 13)(n + 1) = 0$$
$$\therefore n = 13 \quad \text{or} \quad -1$$

Therefore the number of terms is 13.

Example 8 *What is the smallest number of terms of the geometrical progression, 8 + 24 + 72 + …, that will give a total greater than 6 000 000?*

With the notation of §13.4 above,

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right)$$

Substituting $a = 8$ and $r = 3$,

$$S_n = 8\left(\frac{3^n - 1}{3 - 1}\right) = 4(3^n - 1)$$

If we solve the equation

$$4(3^n - 1) = 6\,000\,000$$

the first integer greater than the value of n found from this will be the number of terms required.

To solve the equation:

$$3^n - 1 = 1\,500\,000$$
$$\therefore\ 3^n = 1\,500\,001$$

Taking logarithms (base 10) of both sides,

$$n \lg 3 = \lg 1\,500\,001$$
$$\therefore\ n = \frac{\lg 1\,500\,001}{\lg 3}$$
$$\approx \frac{6.1761^*}{0.4771}$$
$$= 12.94, \quad \text{correct to four significant figures}$$

Therefore the number of terms required to make a total exceeding 6 000 000 is 13.

13.5 Arithmetic and geometric means

If three numbers a, b, c are in arithmetical progression, b is called the **arithmetic mean** of a and c. The common difference of the progression is given by $b - a$ or $c - b$. Therefore

$$b - a = c - b$$
$$\therefore\ 2b = a + c$$

Therefore the arithmetic mean of a and c is $(a + c)/2$. This is the ordinary 'average' of a and c.

If three numbers a, b, c are in geometrical progression, b is called the **geometric mean** of a and c. The common ratio is given by b/a or c/b. Therefore

$$\frac{b}{a} = \frac{c}{b}$$

$$\therefore\ b^2 = ac$$

Therefore the geometric mean of a and c is $\sqrt{(ac)}$.
If a rectangle is drawn with sides a and c, then b is the side of a square whose area is equal to that of the rectangle.

Questions

Q3 Find **a** the arithmetic mean, **b** the geometric mean of 4 and 64.

Q4 The reciprocal of the harmonic mean of two numbers is the arithmetic mean of their reciprocals. Find the harmonic mean of 5 and 20. Also find the arithmetic and geometric means of 5 and 20.

Q5 Find an expression for the harmonic mean of a and c.

Exercise 13c

1 Find the sum of the even numbers up to and including 100.

2 How many terms of the series $2 - 6 + 18 - 54 + \dots$ are needed to make a total of $\frac{1}{2}(1 - 3^8)$?

3 The fifth term of an A.P. is 17 and the third term is 11. Find the sum of the first seven terms.

4 The fourth term of a G.P. is -6 and the seventh term is 48. Write down the first three terms of the progression.

5 Find the sum of the first eight terms of the G.P. $5 + 15 + \dots$.

6 What is the difference between the sums to ten terms of the A.P. and G.P. whose first terms are $-2 + 4 \dots$?

7 The sum of the second and fourth terms of an arithmetical progression is 15, and the sum of the fifth and sixth terms is 25.
Find the first term and the common difference.

8 The second term of an arithmetical progression is three times the seventh, and the ninth term is 1. Find the first term, the common difference, and which is the first term less than 0.

*If a calculator is used it is not necessary, or desirable, to write these numbers down.

9 In a geometrical progression, the sum of the second and third terms is 9, and the seventh term is eight times the fourth. Find the first term, the common ratio, and the fifth term.

10 The fourth term of an arithmetical progression is 15, and the sum of the first five terms is 55. Find the first term and the common difference, and write down the first five terms.

11 The sum of the first three terms of an arithmetical progression is 3, and the sum of the first five terms is 20.
Find the first five terms of the progression.

12 The sum of the first two terms of a geometrical progression is 3, and the sum of the second and third terms is -6.
Find the first term and the common ratio.

13 How many terms of the A.P. $15 + 13 + 11 + \ldots$ are required to make a total of -36?

14 Which is the first term of the geometrical progression $5 + 10 + 20 + \ldots$ to exceed 400 000?

15 Find how many terms of the G.P. $1 + 3 + 9 + \ldots$ are required to make a total of more than a million.

16 The sum of the first six terms of an arithmetical progression is 21, and the seventh term is three times the sum of the third and fourth.
Find the first term and the common difference.

17 In an arithmetical progression, the sum of the first five terms is 30, and the third term is equal to the sum of the first two.
Write down the first five terms of the progression.

18 Find the difference between the sums of the first ten terms of the geometrical and arithmetical progressions which begin, $6 + 12 + \ldots$.

19 The sum of the first n terms of a certain series is $n^2 + 5n$, for all integral values of n.
Find the first three terms and prove that the series is an arithmetical progression.

20 The second, fourth, and eighth terms of an A.P. are in geometrical progression, and the sum of the third and fifth terms is 20.
Find the first four terms of the progression.

21 A man pays a premium of \$100 at the beginning of every year to an Insurance Company on the understanding that at the end of fifteen years he can receive back the premiums which he has paid with 5% compound interest. What should he receive? Give your answer correct to three significant figures.

22 A woman earned \$2000 in a certain year from a certain source and her annual earnings from this time continued to increase at the rate of 5%. Find to the nearest \$ the whole amount she received from this source in this year and the next seven years. Give your answer correct to three significant figures.

13.6 Proof by induction

It sometimes happens that a result is found by some means which does not provide a proof. For example, consider the following table:

n	1	2	3	4	5
Sum of the integers up to n	1	3	6	10	15
n^3	1	8	27	64	125
Sum of the cubes of the integers up to n	1	9	36	100	225

Here the terms in the fourth row are the squares of the corresponding terms in the second row. Thus it might be natural to suppose that

$$1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$$

Now $1 + 2 + \ldots + n$ is an arithmetical progression whose sum is $\frac{1}{2}n(n+1)$. Therefore we might suppose that

$$1^3 + 2^3 + \ldots + n^3 = \tfrac{1}{4}n^2(n+1)^2$$

In proof by induction, it is shown that if the result holds for some particular value of n, say k, then it also holds for $n = k + 1$. It is then verified that the result does hold for some value of n, usually 1 or 2.

Example 9 *Prove by induction that* $1^3 + 2^3 + \ldots + n^3 = \frac{1}{4}n^2(n+1)^2$.

Suppose the result holds for a particular value of n, say k; that is,

$$1^3 + 2^3 + \ldots + k^3 = \tfrac{1}{4}k^2(k+1)^2$$

Then, adding the next term of the series, $(k+1)^3$, to both sides, we obtain

$$1^3+2^3+\ldots+k^3+(k+1)^3=\tfrac{1}{4}k^2(k+1)^2+(k+1)^3$$
$$=(k+1)^2\left(\frac{k^2}{4}+k+1\right)$$
$$=(k+1)^2\left(\frac{k^2+4k+4}{4}\right)$$
$$\therefore\ 1^3+2^3+\ldots+(k+1)^3=\tfrac{1}{4}(k+1)^2(k+2)^2$$

Now this is the formula with $n=k+1$. Therefore if the result holds for $n=k$, then it also holds for $n=k+1$; but if $n=1$,

$$\text{L.H.S.}=1^3=1 \quad\text{and}\quad \text{R.H.S.}=\tfrac{1}{4}\times1^2\times2^2=1$$

Therefore, since the result is true for $n=1$, it follows, by what has been shown above, that it must also be true for $n=2$. From this it follows that the result is true for $n=3$, and so on, for all positive integral values of n.

Exercise 13d

Prove the following results by induction:

1 $1+2+\ldots+n=\frac{1}{2}n(n+1)$

2 $1^2+2^2+\ldots+n^2=\frac{1}{6}n(n+1)(2n+1)$

3 $1\times2+2\times3+\ldots+n(n+1)=\frac{1}{3}n(n+1)(n+2)$

4 $1\times3+2\times4+\ldots+n(n+2)=\frac{1}{6}n(n+1)(2n+7)$

5 $3+8+\ldots+(n^2-1)=\frac{1}{6}n(n-1)(2n+5)$

6 $a+ar+\ldots+ar^{n-1}=a\left(\dfrac{1-r^n}{1-r}\right)$

7 $\dfrac{1}{1\times2}+\dfrac{1}{2\times3}+\ldots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$

8 $\dfrac{1}{1\times3}+\dfrac{1}{2\times4}+\ldots+\dfrac{1}{n(n+2)}=\dfrac{3}{4}-\dfrac{2n+3}{2(n+1)(n+2)}$

9 $\dfrac{3}{4}+\dfrac{5}{36}+\ldots+\dfrac{2n-1}{n^2(n-1)^2}=1-\dfrac{1}{n^2}$

10 $\dfrac{1}{1\times2\times3}+\dfrac{1}{2\times3\times4}+\ldots+\dfrac{1}{n(n+1)(n+2)}$
$=\dfrac{1}{4}-\dfrac{1}{2(n+1)(n+2)}$

11 $\dfrac{d}{dx}(x^n)=nx^{n-1}$
[Use the formula for differentiating a product.]

12 $1^2+3^2+5^2+\ldots+(2n-1)^2=\frac{1}{3}n(4n^2-1)$

13 $1^3+3^3+5^3+\ldots+(2n-1)^3=n^2(2n^2-1)$

14 $4^2+7^2+10^2+\ldots+(3n+1)^2=\frac{1}{2}n(6n^2+15n+11)$

15 Show that $\binom{n}{r}+\binom{n}{r-1}=\binom{n+1}{r}$, and prove by induction that

$$(1+x)^n=1+nx+\ldots+\binom{n}{r}x^r+\ldots+x^n$$

where $\binom{n}{r}=\dfrac{n!}{(n-r)!\,r!}$ and r is a positive integer, less than or equal to n.

13.7 Further series

Certain series can be summed by means of the results:

$$1+2+\ldots+n=\tfrac{1}{2}n(n+1)$$
$$1^2+2^2+\ldots+n^2=\tfrac{1}{6}n(n+1)(2n+1)$$
$$1^3+2^3+\ldots+n^3=\tfrac{1}{4}n^2(n+1)^2$$

which appear in the last section and exercise.

It should be noted that they may be used to sum the series to more or less than n terms. For instance,

$$1^3+2^3+\ldots+(2n+1)^3=\tfrac{1}{4}(2n+1)^2\{(2n+1)+1\}^2$$
$$=\tfrac{1}{4}(2n+1)^2(2n+2)^2$$
$$=\tfrac{1}{4}(2n+1)^24(n+1)^2$$
$$=(2n+1)^2(n+1)^2$$

Question

Q6 Find the sums of the following series:

a $1+2+\ldots+2n$
b $1^2+2^2+\ldots+(n+1)^2$
c $1^3+2^3+\ldots+(n-1)^3$
d $1+2+\ldots+(2n-1)$
e $1^2+2^2+\ldots+(2n)^2$
f $1^3+2^3+\ldots+(2n-1)^3$

Example 10 *Find the sum of the series*
$1^3+3^3+5^3+\ldots+(2n+1)^3$

This series can be thought of as
$1^3+2^3+3^3+4^3+5^3+\ldots+(2n+1)^3$ with the even terms missing.

We found above that

$$1^3+2^3+3^3+4^3+5^3+\ldots+(2n+1)^3$$
$$=(2n+1)^2(n+1)^2$$

and so it remains to find the sum of the series

$$2^3+4^3+6^3+\ldots+(2n)^3$$
$$=2^3\times1^3+2^3\times2^3+2^3\times3^3+\ldots+2^3\times n^3$$
$$=8(1^3+2^3+3^3+\ldots+n^3)$$
$$=8\times\tfrac{1}{4}n^2(n+1)^2=2n^2(n+1)^2$$

Therefore $1^3+3^3+5^3+\ldots+(2n+1)^3$
$$=(2n+1)^2(n+1)^2-2n^2(n+1)^2$$
$$=(n+1)^2\{(2n+1)^2-2n^2\}$$
$$=(n+1)^2(4n^2+4n+1-2n^2)$$

Therefore the sum is $(n+1)^2(2n^2+4n+1)$.

Example 11 *Find the sum of n terms of the series* $2\times3+3\times4+4\times5+\ldots$.

The mth term of this series is $(m+1)(m+2)$, or m^2+3m+2. Therefore we require the sum of

$$\begin{aligned}&1^2+3\times1+2\\+&2^2+3\times2+2\\+&3^2+3\times3+2\\+&\ldots\ldots\ldots\ldots\ldots\\+&n^2+3\times n+2\end{aligned}$$

The sums of the three columns are respectively

$$1^2+2^2+3^2+\ldots+n^2=\tfrac{1}{6}n(n+1)(2n+1)$$
$$3(1+2+3+\ldots+n)=\tfrac{3}{2}n(n+1)$$
$$(2+2+2+\ldots+2)=2n$$

Therefore the sum of the series is

$$\frac{1}{6}n(n+1)(2n+1)+\frac{3}{2}n(n+1)+2n$$
$$=\frac{n}{6}\{(n+1)(2n+1)+9(n+1)+12\}$$
$$=\frac{n}{6}(2n^2+3n+1+9n+9+12)$$
$$=\frac{n}{6}(2n^2+12n+22)$$

$$\frac{1}{6}n(n+1)(2n+1)+\frac{3}{2}n(n+1)+2n=\frac{n}{3}(n^2+6n+11)$$

Therefore the sum of the first n terms of the series $2\times3+3\times4+4\times5+\ldots$ is $\tfrac{1}{3}n(n^2+6n+11)$.

The $\sum$ notation

It is useful to have a short way of writing expressions like

$$1^2+2^2+\ldots+n^2$$

This is done by writing

$$\sum m^2$$

which means, 'the sum of all the terms like m^2'. For extra precision numbers are placed below and above the $\sum$ to show where the series begins and ends. Thus

$$\sum_1^n m^2=1^2+2^2+\ldots+n^2$$

and

$$\sum_2^5 m(m+2)=2\times4+3\times5+4\times6+5\times7$$

Exercise 13e

1 Write in full:

a $\sum_1^4 m^3$ **b** $\sum_2^n m^2$ **c** $\sum_1^n (m^2+m)$

d $\sum_1^3 \frac{1}{m(m+1)}$ **e** $\sum_2^5 2^m$ **f** $\sum_1^4 (-1)^m m^2$

g $\sum_1^n m^m$ **h** $\sum_3^6 \frac{(-1)^m}{m}$

i $\sum_n^{n+2} m(m-1)$ **j** $\sum_{n-2}^n \frac{m}{m+1}$

2 Write in the $\sum$ notation:

a $1+2+3+\ldots+n$

b $1^4+2^4+\ldots+n^4+(n+1)^4$

c $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$

d $3^2+3^3+3^4+3^5$

e $2\times7+3\times8+4\times9+5\times10+6\times11$

f $1+\frac{2}{3}+\frac{3}{9}+\frac{4}{27}+\frac{5}{81}$

g $\frac{1\times3}{4}+\frac{2\times5}{6}+\frac{3\times7}{8}+\frac{4\times9}{10}+\frac{5\times11}{12}$

h $-1+2-3+4-5+6$

i $1-2+4-8+16-32$

j $1\times3-2\times5+3\times7-4\times9+5\times11$

3 Use the results quoted at the beginning of §13.7 above to find the sums of the following series:

a $1+2+3+\ldots+(2n+1)$

b $1^2+2^2+3^2+\ldots+(n-1)^2$

c $1^3+2^3+3^3+\ldots+(2n)^3$

d $3+5+7+\ldots+(2n+1)$

e $2+5+8+11+\ldots$, to n terms

f $5+9+13+17+\ldots$, to n terms

g $2+5+10+\ldots+(n^2+1)$

h $1\times2+2\times3+3\times4+4\times5+\ldots$, to n terms

i $1\times3+2\times4+3\times5+4\times6+\ldots$, to n terms

j $2^2+4^2+6^2+\ldots+(2n)^2$

k $1^2+3^2+5^2+\ldots+(2n-1)^2$

l $2+10+30+\ldots+(n^3+n)$

m $2+12+36+\ldots+(n^3+n^2)$

13.8 Infinite geometrical progressions

Consider the geometrical progression

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^{n-1}}$$

The sum of these n terms, obtained by the formula of §13.4 on page 162, is given by

$$S_n=\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}=2\left(1-\left(\frac{1}{2}\right)^n\right)$$

Now as n increases, $(\frac{1}{2})^n$ approaches zero; and $(\frac{1}{2})^n$ can be made as close to zero as we like, if n is large enough. Therefore the sum of n terms approaches 2 as n increases.

This is what is meant by writing that the infinite series

$$1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^{m-1}}+\ldots=2$$

The limit 2 is called its **sum to infinity**.

In general, the sum of the geometrical progression

$$a+ar+ar^2+\ldots+ar^{n-1}=a\left(\frac{1-r^n}{1-r}\right)$$

If $-1<r<+1$, i.e. $|r|<1$, we assume that r^n approaches zero as n increases. The sum to infinity of the series where $-1<r<+1$ is given by:

$$a+ar+ar^2+\ldots+ar^{n-1}+\ldots=\frac{a}{1-r}$$

Example 12 *Express as fractions in their lowest terms:* **a** $0.0\dot{7}$, **b** $0.\dot{4}\dot{5}$.

a $0.0\dot{7}$ means 0.0777 …, which may be written

$$\frac{7}{100}+\frac{7}{1000}+\frac{7}{10\ 000}+\ldots$$

This is a geometrical progression, and in the notation of §13.4 on page 162, $a=\frac{7}{100}$ and $r=\frac{1}{10}$. Therefore

$$S_n=\frac{7}{100}\left(\frac{1-(\frac{1}{10})^n}{1-\frac{1}{10}}\right)$$

Therefore the sum to infinity, S_∞, is given by

$$S_\infty=\frac{7}{100}\left(\frac{1}{\frac{9}{10}}\right)=\frac{7}{100}\times\frac{10}{9}=\frac{7}{90}$$

$$\therefore 0.0\dot{7}=\frac{7}{90}$$

b $0.\dot{4}\dot{5}$ means 0.454 545 …, which may be written

$$\frac{45}{100}+\frac{45}{10\ 000}+\frac{45}{1\ 000\ 000}+\ldots$$

In this geometrical progression, $a=\frac{45}{100}$, and $r=\frac{1}{100}$.

$$\therefore S_n=\frac{45}{100}\left(\frac{1-(\frac{1}{100})^n}{1-\frac{1}{100}}\right)$$

$$\therefore S_\infty=\frac{45}{100}\left(\frac{1}{\frac{99}{100}}\right)=\frac{45}{100}\times\frac{100}{99}=\frac{5}{11}$$

$$\therefore 0.\dot{4}\dot{5}=\frac{5}{11}$$

Using this method, any recurring decimal can be expressed as a rational number. (See §2.3 on page 44.)

Exercise 13f

1 Write down the sums of the first n terms of the following series, and deduce their sums to infinity:

a $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$ b $12+6+3+1\frac{1}{2}+\ldots$

c $\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10\ 000}+\ldots$

d $\frac{13}{100}+\frac{13}{10\ 000}+\frac{13}{1\ 000\ 000}+\ldots$

e $0.5+0.05+0.005+\ldots$

f $0.54+0.0054+0.000\ 054+\ldots$

g $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$

h $54-18+6-2+\ldots$

2 Express the following recurring decimals as rational numbers:

a $0.\dot{8}$ b $0.\dot{1}\dot{2}$ c $3.\dot{2}$

d $2.\dot{6}\dot{9}$ e $1.00\dot{4}$ f $2.\dot{9}\dot{6}\dot{0}$

3 If the sum to infinity of a G.P. is three times the first term, what is the common ratio?

4 The sum to infinity of a G.P. is 4 and the second term is 1. Find the first, third, and fourth terms.

5 The second term of a G.P. is 24 and its sum to infinity is 100. Find the two possible values of the common ratio and the corresponding first terms.

Chapter 14

The binomial theorem (1)

14.1 Pascal's triangle

It is well known that

$$(a+b)^2 = a^2 + 2ab + b^2.$$

The aim of this chapter is to show how to expand higher powers of $a+b$.

You will probably not be able to write down similar expressions for $(a+b)^3$ and $(a+b)^4$ without doing some work on paper, and so the long multiplication is given below. The reason for printing the coefficients in heavy type will appear later.

$$\begin{array}{l}
\mathbf{1}a^2 + \mathbf{2}ab \;\; + \mathbf{1}b^2 \\
\qquad\quad a + b \\
\hline
\mathbf{1}a^3 + \mathbf{2}a^2b + \mathbf{1}ab^2 \\
\qquad\quad \mathbf{1}a^2b + \mathbf{2}ab^2 + \mathbf{1}b^3 \\
\hline
\mathbf{1}a^3 + \mathbf{3}a^2b + \mathbf{3}ab^2 + \mathbf{1}b^3
\end{array}$$

$$\begin{array}{l}
\mathbf{1}a^3 + \mathbf{3}a^2b + \mathbf{3}ab^2 \;\; + \mathbf{1}b^3 \\
\qquad\qquad\qquad a + b \\
\hline
\mathbf{1}a^4 + \mathbf{3}a^3b + \mathbf{3}a^2b^2 + \mathbf{1}ab^3 \\
\qquad\quad \mathbf{1}a^3b + \mathbf{3}a^2b^2 + \mathbf{3}ab^3 + \mathbf{1}b^4 \\
\hline
\mathbf{1}a^4 + \mathbf{4}a^3b + \mathbf{6}a^2b^2 + \mathbf{4}ab^3 + \mathbf{1}b^4
\end{array}$$

The results so far obtained are summarised below.

$$\begin{aligned}
(a+b)^2 &= \qquad\qquad \mathbf{1}a^2 + \mathbf{2}ab + \mathbf{1}b^2 \\
(a+b)^3 &= \qquad \mathbf{1}a^3 + \mathbf{3}a^2b + \mathbf{3}ab^2 + \mathbf{1}b^3 \\
(a+b)^4 &= \mathbf{1}a^4 + \mathbf{4}a^3b + \mathbf{6}a^2b^2 + \mathbf{4}ab^3 + \mathbf{1}b^4
\end{aligned}$$

It is clearer, however, if the coefficients are written alone.

$$\begin{array}{ccccccccc}
 & & \mathbf{1} & & \mathbf{2} & & \mathbf{1} & & \\
 & \mathbf{1} & & \mathbf{3} & & \mathbf{3} & & \mathbf{1} & \\
\mathbf{1} & & \mathbf{4} & & \mathbf{6} & & \mathbf{4} & & \mathbf{1}
\end{array}$$

You may now be able to guess the next line and, more important, you may be able to see how the table can be continued, obtaining each line from the previous one.

To show the construction of the table of coefficients, the last three lines of the long multiplications are written below, leaving out the letters.

$$\begin{array}{cccc}
1 & 2 & 1 & \\
 & 1 & 2 & 1 \\
\hline
1 & 3 & 3 & 1
\end{array}
\qquad\qquad
\begin{array}{ccccc}
1 & 3 & 3 & 1 & \\
 & 1 & 3 & 3 & 1 \\
\hline
1 & 4 & 6 & 4 & 1
\end{array}$$

Every coefficient in the table is obtained by adding the two on either side of it in the row above. In this way the next line can be obtained:

$$\begin{array}{ccccccccccc}
 & \mathbf{1} & & \mathbf{4} & & \mathbf{6} & & \mathbf{4} & & \mathbf{1} & \\
\mathbf{1} & & \mathbf{5} & & \mathbf{10} & & \mathbf{10} & & \mathbf{5} & & \mathbf{1}
\end{array}$$

For completeness, notice that

$$(a+b)^0 = \mathbf{1} \quad \text{and} \quad (a+b)^1 = \mathbf{1}a + \mathbf{1}b$$

Therefore the table of coefficients may be written in a triangle (known as **Pascal's triangle**, after the French mathematician and philosopher Blaise Pascal 1623–1662) as follows:

$$\begin{array}{ccccccccccccc}
 & & & & & & \mathbf{1} & & & & & & \\
 & & & & & \mathbf{1} & & \mathbf{1} & & & & & \\
 & & & & \mathbf{1} & & \mathbf{2} & & \mathbf{1} & & & & \\
 & & & \mathbf{1} & & \mathbf{3} & & \mathbf{3} & & \mathbf{1} & & & \\
 & & \mathbf{1} & & \mathbf{4} & & \mathbf{6} & & \mathbf{4} & & \mathbf{1} & & \\
 & \mathbf{1} & & \mathbf{5} & & \mathbf{10} & & \mathbf{10} & & \mathbf{5} & & \mathbf{1} & \\
\mathbf{1} & & \mathbf{6} & & \mathbf{15} & & \mathbf{20} & & \mathbf{15} & & \mathbf{6} & & \mathbf{1} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

When an expression is written as a series of terms, it is said to be **expanded**, and the series is called its **expansion**. Thus the expansion of $(a+b)^3$ is

$$a^3 + 3a^2b + 3ab^2 + b^3$$

Certain points should be noted about the expansion of $(a+b)^n$. They should be verified for the cases $n = 2, 3, 4$, in the expansions obtained so far.

- **a** Reading from either end of each row, the *coefficients* are the same.
- **b** There are $(n+1)$ terms.
- **c** Each term is of degree n.
- **d** The coefficients are obtained from the row in Pascal's triangle beginning $1, n$.

Example 1 *Expand $(a+b)^6$ in descending powers of a.*

There will be 7 terms, involving

$$a^6, \quad a^5b, \quad a^4b^2, \quad a^3b^3, \quad a^2b^4, \quad ab^5, \quad b^6,$$

each of which is of degree 6. Their coefficients, obtained from Pascal's triangle, are respectively

$$1, \quad 6, \quad 15, \quad 20, \quad 15, \quad 6, \quad 1.$$

Therefore the expansion of $(a+b)^6$ in descending powers of a is

$$a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

Example 2 *Expand $(2x + 3y)^3$ in descending powers of x.*

Here $a = 2x$, $b = 3y$, and so there will be four terms involving

$$(2x)^3, \quad (2x)^2(3y), \quad (2x)(3y)^2, \quad (3y)^3.$$

Their coefficients, obtained from Pascal's triangle are respectively

$$1, \quad 3, \quad 3, \quad 1.$$

Therefore the expansion of $(2x + 3y)^3$, in descending powers of x is

$$(2x)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2 + (3y)^3$$

which simplifies to

$$8x^3 + 36x^2y + 54xy^2 + 27y^3$$

Example 3 *Obtain the expansion of $(2x - \frac{1}{2})^4$, in descending powers of x.*

Here $a = 2x$ and $b = -\frac{1}{2}$, therefore the five terms of the expansion will involve

$$(2x)^4, \quad (2x)^3(-\tfrac{1}{2}), \quad (2x)^2(-\tfrac{1}{2})^2, \quad (2x)(-\tfrac{1}{2})^3, \quad (-\tfrac{1}{2})^4$$

and their coefficients will be respectively

$$1, \quad 4, \quad 6, \quad 4, \quad 1.$$

$$\begin{aligned}\therefore (2x - \tfrac{1}{2})^4 &= (2x)^4 + 4(2x)^3(-\tfrac{1}{2}) + 6(2x)^2(-\tfrac{1}{2})^2 \\ &\quad + 4(2x)(-\tfrac{1}{2})^3 + (-\tfrac{1}{2})^4 \\ &= 16x^4 + 4(8x^3)(-\tfrac{1}{2}) + 6(4x^2)(\tfrac{1}{4}) \\ &\quad + 4(2x)(-\tfrac{1}{8}) + \tfrac{1}{16}\end{aligned}$$

Therefore the expansion of $(2x - \frac{1}{2})^4$, in descending powers of x, is

$$16x^4 - 16x^3 + 6x^2 - x + \tfrac{1}{16}$$

Note that terms are alternately +ve and –ve, according to the even or odd degree of $(-\frac{1}{2})$.

Example 4 *Use Pascal's triangle to obtain the value of $(1.002)^5$, correct to six places of decimals.*

1.002 may be written $(1 + 0.002)$, so that the expansion of $(a + b)^5$ may be used, with $a = 1$ and $b = 0.002$.

The terms in the expansion will involve

$$1, \quad (0.002), \quad (0.002)^2, \quad (0.002)^3, \quad (0.002)^4, \quad (0.002)^5$$

and their coefficients will be

$$1, \quad 5, \quad 10, \quad 10, \quad 5, \quad 1,$$

respectively. The last three terms will make no difference to the answer, correct to six places of decimals. Therefore

$$\begin{aligned}(1.002)^5 &\approx 1 + 5(0.002) + 10(0.002)^2 \\ &= 1 + 0.010 + 0.000\,040\end{aligned}$$

and so $(1.002)^5 = 1.010\,040$, correct to six places of decimals.

Exercise 14a

This exercise is intended to give practice in using Pascal's triangle. Calculators should not be used.

1 Expand:

a $(a+b)^5$	**b** $(x+y)^3$	**c** $(x+2y)^4$
d $(1-z)^4$	**e** $(2x+3y)^4$	**f** $(4z+1)^3$
g $(a-b)^6$	**h** $(a-2b)^3$	**i** $(3x-y)^4$
j $(2x+\frac{1}{3})^3$	**k** $\left(x-\dfrac{1}{x}\right)^5$	**l** $\left(\dfrac{x}{2}+\dfrac{2}{x}\right)^4$
m $(a+b)^7$	**n** $(a^2-b^2)^5$	**o** $(a-b)^3(a+b)^3$

2 Simplify, leaving surds in the answers, where appropriate:

a $(1+\sqrt{2})^3 + (1-\sqrt{2})^3$
b $(2+\sqrt{3})^4 + (2-\sqrt{3})^4$
c $(1+\sqrt{2})^3 - (1-\sqrt{2})^3$
d $(2+\sqrt{6})^4 - (2-\sqrt{6})^4$
e $(\sqrt{2}+\sqrt{3})^4 + (\sqrt{2}-\sqrt{3})^4$
f $(\sqrt{6}+\sqrt{2})^3 - (\sqrt{6}-\sqrt{2})^3$

3 Write down the expansion of $(2+x)^5$ in ascending powers of x. Taking the first three terms of the expansion, put $x = 0.001$, and find the value of $(2.001)^5$ correct to five places of decimals.

4 Write down the expansion of $(1 + \frac{1}{4}x)^4$. Taking the first three terms of the expansion, put $x = 0.1$, and find the value of $(1.025)^4$, correct to three places of decimals.

5 Expand $(2-x)^6$ in ascending powers of x. Taking $x = 0.002$, and using the first three terms of the expansion, find the value of $(1.998)^6$ as accurately as you can. Examine the fourth term of the expansion to find to how many places of decimals your answer is correct.

14.2 The binomial theorem

Introduction

The last section showed how to use Pascal's triangle to expand $(a+b)^n$ for a known value of n. If n is large, this may involve a considerable amount of addition. When only the first few terms are required, it is much quicker to use a formula that will be obtained in the next section.

The last section began with the expansions of $(a+b)^2$ and $(a+b)^3$. Now, consider the expansions of $(a+b)(c+d)$ and $(a+b)(c+d)(e+f)$.

It is easily seen that

$$(a+b)(c+d) = ac + ad + bc + bd$$

To obtain the expansion of $(a+b)(c+d)(e+f)$, each term of $ac + ad + bc + bd$ is multiplied by e and f, giving

$$ace + ade + bce + bde + acf + adf + bcf + bdf$$

Note that each term contains one factor from each bracket, and that the expansion consists of the sum of all such combinations.

Now the expansion of $(a+b)(c+d)(e+f)(g+h)$ would be obtained by multiplying each term of the expansion by g and by h. So, continuing this method of expansion, it follows that, if the product of n factors is expanded, each term contains one factor from each bracket, and that the expansion consists of the sum of all such combinations.

The expansion of $(a+b)^5$ will be obtained by an argument making use of this fact.

$$(a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b)$$

a Choosing an a from each bracket we obtain a^5.

b The term in a^4 is obtained by choosing a b from one bracket, and a's from the other four. This can be done in ${}^5C_1$ ways, giving ${}^5C_1a^4b$ (see §12.4 on page 157).

c The term in a^3 is obtained by choosing b's from two brackets, and a's from the other three. This can be done in ${}^5C_2$ ways, giving ${}^5C_2a^3b^2$.

d Similarly, the terms in a^2 and a are ${}^5C_3a^2b^3$ and ${}^5C_4ab^4$.

e Choosing a b from each bracket we obtain b^5.

$$\therefore\ (a+b)^5 = a^5 + {}^5C_1a^4b + {}^5C_2a^3b^2 + {}^5C_3a^2b^3 + {}^5C_4ab^4 + b^5$$

Expansion of $(a+b)^n$

It follows from the above argument that the expansion of $(a+b)^n$ is:

$$(\boldsymbol{a}+\boldsymbol{b})^n = \boldsymbol{a}^n + {}^n\boldsymbol{C}_1\boldsymbol{a}^{n-1}\boldsymbol{b} + \ldots + {}^n\boldsymbol{C}_r\boldsymbol{a}^{n-r}\boldsymbol{b}^r + \ldots + \boldsymbol{b}^n$$

where ${}^nC_r = \dfrac{n!}{(n-r)!r!}$ and n is a positive integer.

This is known as the **binomial theorem**.

The expansion of $(a+b)^n$ is obtained as follows.

$$(a+b)^n = (a+b)(a+b)\ldots(a+b), \quad \text{to } n \text{ factors.}$$

a Choosing an a from each bracket we obtain a^n.

b The term in a^{n-1} is obtained by choosing a b from one bracket, and a's from the other $n-1$. This can be done in ${}^nC_1$ ways, giving ${}^nC_1a^{n-1}b$.

c The term in a^{n-2} is obtained by choosing a b from two brackets, and a's from the other $n-2$. This can be done in ${}^nC_2$ ways, giving ${}^nC_2a^{n-2}b^2$.

d The term in a^{n-r} is obtained by choosing a b from r brackets, and a's from the other $n-r$. This can be done in ${}^nC_r$ ways, giving ${}^nC_ra^{n-r}b^r$.

e Choosing a b from each bracket we obtain b^n.

This proves the theorem.

When only the first few terms of an expansion are required, the theorem is used in the form

$$\boldsymbol{(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots + b^n}$$

This follows immediately, since

$${}^nC_1 = n, \qquad {}^nC_2 = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2!} \qquad \text{and}$$

$${}^nC_3 = \frac{n!}{(n-3)!3!} = \frac{n(n-1)(n-2)}{3!}$$

To understand the name of the theorem, it is necessary to know that an expression with one term is called a *mononomial*, one which has two terms is a *binomial*, and one with three terms is a *trinomial*. Thus the theorem about the expansion of a power of two terms is called the binomial theorem.

Example 5 *Find the coefficient of x^{10} in the expansion of $(2x-3)^{14}$.*

The term in $(2x)^{10}(-3)^4$ is the only one needed, and by the binomial theorem it is

$${}^{14}C_4(2x)^{10}(-3)^4$$

Therefore the coefficient of x^{10} is $\dfrac{14!}{10!4!}2^{10}\times 3^4$.

It is important to note that we could equally well have written the term as

$^{14}C_{10}(2x)^{10}(-3)^4$

because $^{14}C_{10} = {}^{14}C_4$. This is clear if they are written in factorial notation:

$$^{14}C_{10} = \frac{14!}{4!10!} \qquad \text{and} \qquad {}^{14}C_4 = \frac{14!}{10!4!}$$

Alternatively, as shown in §12.4 on page 157, $^{14}C_{10}$ is the number of ways of choosing ten objects from fourteen unlike objects; but if ten are chosen, four are left, and so it must also be the number of ways of choosing four objects from fourteen unlike objects, which is $^{14}C_4$.

Question

Q1 Show that $^nC_{n-r} = {}^nC_r$.

Notice in Example 5 that the numbers whose factorials appear in the coefficient

$$\frac{14!}{10!4!}$$

are all indices. 14 is the index of $2x - 3$, 10 is the index of $2x$ and 4 is the index of -3. That this is always the case should be clear if the term in $a^{n-r}b^r$ in the expansion of $(a+b)^n$ is written with factorial notation:

$$\frac{n!}{(n-r)!r!}a^{n-r}b^r$$

Example 6 *Obtain the first four terms of the expansion of* $(1+\frac{1}{2}x)^{10}$ *in ascending powers of x. Hence find the value of* $(1.005)^{10}$, *correct to four decimal places.*

Using the second form of the binomial theorem,

$$\left(1+\frac{1}{2}x\right)^{10} = 1 + 10\left(\frac{x}{2}\right) + \frac{10\times 9}{2\times 1}\left(\frac{x}{2}\right)^2 + \frac{10\times 9\times 8}{3\times 2\times 1}\left(\frac{x}{2}\right)^3 + \ldots$$

$$= 1 + 5x + \frac{45}{4}x^2 + 15x^3 + \ldots$$

Now $\frac{1}{2}x = 0.005$, if $x = 0.01$; so substituting this value of x,

$$\begin{aligned}(1.005)^{10} &\approx 1 + 5(0.01) + 11.25(0.01)^2 + 15(0.01)^3\\ &= 1 + 0.05 + 0.001\,125 + 0.000\,015\\ &= 1.051\,140\end{aligned}$$

Therefore $(1.005)^{10} \approx 1.0511$, correct to four places of decimals.

Example 7 *Obtain the expansion of* $(1+x-2x^2)^8$, *as far as the term in* x^3.

$(1+x-2x^2)^8$ may be written $\{1+(x-2x^2)\}^8$, which may then be expanded by the binomial theorem.

$$\begin{aligned}\{1+(x-2x^2)\}^8 &= 1 + 8(x-2x^2) + \frac{8\times 7}{2!}(x-2x^2)^2\\ &\quad + \frac{8\times 7\times 6}{3!}(x-2x^2)^3 + \ldots\\ &= 1 + 8(x-2x^2) + 28(x^2-4x^3+4x^4)\\ &\quad + 56(x^3 + \text{other terms}) + \ldots\\ &= 1 + 8x - 16x^2 + 28x^2 - 112x^3 + 56x^3\\ &\quad + \text{terms in } x^4 \text{ and higher powers}\end{aligned}$$

$\therefore (1+x-2x^2)^8 = 1 + 8x + 12x^2 - 56x^3$ as far as the term in x^3.

Exercise 14b

Calculators should not be used in this exercise.

1 Write down the terms indicated, in the expansions of the following, and simplify your answers:

- **a** $(x+2)^8$, term in x^5
- **b** $(3u-2)^5$, term in u^3
- **c** $(2t-\frac{1}{2})^{12}$, term in t^7
- **d** $(2x+y)^{11}$, term in x^3

2 Write down, and simplify, the terms indicated, in the expansions of the following in ascending powers of x:

- **a** $(1+x)^9$, 4th term
- **b** $(2-x/2)^{12}$, 4th term
- **c** $(3+x)^7$, 5th term
- **d** $(x+1)^{20}$, 3rd term

3 Write down, and simplify, the coefficients of the terms indicated, in the expansions of the following:

- **a** $(\frac{1}{2}t+\frac{1}{2})^{10}$, term in t^4
- **b** $(4+\frac{1}{4}x)^6$, term in x^3
- **c** $(2x-3)^7$, term in x^5
- **d** $(3+\frac{1}{3}y)^{11}$, term in y^5

4 Write down the coefficients of the terms indicated, in the expansions of the following in ascending powers of x:

- **a** $(1+x)^{16}$, 3rd term
- **b** $(2-x)^{20}$, 18th term
- **c** $(3+2x)^6$, 4th term
- **d** $(2+\frac{3}{2}x)^8$, 5th term

5 Write down the terms involving

a $x^4\left(\frac{1}{x}\right)^2$

b $x^3\left(\frac{1}{x}\right)^3$, in the expansion of $\left(x+\frac{1}{x}\right)^6$

6 Write down the constant terms in the expansions of

a $\left(x-\frac{1}{x}\right)^8$ b $\left(2x^2-\frac{1}{2x}\right)^6$

7 Find the coefficients of the terms indicated in the expansions of the following:

a $\left(x+\frac{1}{x}\right)^6$, term in x^4

b $\left(2x+\frac{1}{x}\right)^7$, term in $\frac{1}{x^5}$

c $\left(x-\frac{2}{x}\right)^8$, term in x^6

8 Find the ratio of the term in x^5 to the term in x^6, in the expansion of $(2x+3)^{20}$.

9 Find the ratio of the term in x^7 to the term in x^8 in the expansion of $(3x+\frac{2}{3})^{17}$.

10 Find the ratio of the term in a^r to the term in a^{r+1} in the expansion of $(a+b)^n$.

11 Write down the first four terms of the expansions of the following, in ascending powers of x:

a $(1+x)^{10}$ b $(1+\frac{1}{2}x)^9$

c $(1-x)^{11}$ d $(x+1)^{12}$

e $(2+\frac{1}{2}x)^8$ f $(2-\frac{1}{2}x)^7$

12 Use the binomial theorem to find the values of

a $(1.01)^{10}$, correct to three places of decimals

b $(2.001)^{10}$, correct to six significant figures

c $(0.997)^{12}$, correct to three places of decimals

d $(1.998)^8$, correct to two places of decimals

13 Expand the following as far as the terms in x^3:

a $(1+x+x^2)^3$ b $(1+2x-x^2)^6$

c $(1-x-x^2)^4$ d $(2+x+x^2)^5$

e $(1-x+x^2)^8$ f $(2+x-2x^2)^7$

g $(3-2x+x^2)^4$ h $(3+x+x^3)^4$

14.3 Convergent series

The series

$$1+x+x^2+\ldots+x^{n-1}$$

is a geometrical progression, with common ratio x, and may be summed by the method of §13.4 on page 162. In this way

$$1+x+x^2+\ldots+x^{n-1}=\frac{1-x^n}{1-x}$$

If x lies between -1 and $+1$, we will assume that x^n approaches zero as n increases, which makes the right-hand side of the identity approach $1/(1-x)$.

Thus when we write

$$1+x+x^2+\ldots+x^r+\ldots=\frac{1}{1-x}$$

we mean that the left-hand side can be made to differ as little as we please from the right-hand side, providing enough terms are taken. Remember, however, that x lies between -1 and $+1$.

A series of terms, whose sum approaches a finite value as the number of terms is increased indefinitely is called a **convergent** series, and the finite value is called its **sum to infinity**.

Thus $1+x+x^2+\ldots+x^r+\ldots$ is a convergent series, provided x lies between -1 and $+1$, and its sum is $1/(1-x)$.

To emphasise the necessity for the condition

$$-1<x<+1 \qquad (\text{i.e. } |x|<1)$$

the behaviour of the series for other values of x is examined below.

a If $x=1$, $1+x+x^2+\ldots+x^{n-1}=n$. Therefore as n increases, the value of the series increases indefinitely.

b If $x=-1$,

$$1+x+x^2+\ldots+x^{n-1}=1-1+1-\ldots+(-1)^{n-1}$$

which is equal to 1 or 0, according to whether n is odd or even.

c If x is greater than 1, x^n is greater than 1, and can be made as large as we like, if n is sufficiently large. Therefore the sum of the series, $(1-x^n)/(1-x)$, can be made as large as we like.

d When x is less than -1, $1-x$ is positive and x^n is numerically greater than 1. If n is even, x^n is positive, therefore $1-x^n$ is negative and so the sum $(1-x^n)/(1-x)$ is negative. If n is odd, x^n is negative, therefore $1-x^n$ is positive and so the sum is positive. Hence the sum is alternately positive and negative.

It is beyond the scope of this book to give tests to discover whether any particular series is convergent. However, this section has been included to remind you that series are *not always* convergent.

14.4 The binomial theorem for any index

We have shown that

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \ldots + b^n$$

where n is a positive integer.

Now we will assume that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

for *any* rational value of n *provided* $-1 < x < +1$, i.e. $|x| < 1$. The proof is beyond the scope of this book.

Remember that, if n is a positive integer, there will only be a finite number of terms (see §14.2 on page 171).

Note that the index, n, is often called the **exponent**.

The coefficient of x^r in the expansion of $(1+x)^n$ is usually written $\binom{n}{r}$, that is,

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1\times 2 \times 3\times \ldots \times r}$$

(Notice also that for each factor in the top line, there is a corresponding factor in the bottom line.)

Unlike ${}^nC_r$, the symbol $\binom{n}{r}$ may be used when n is *not* a positive integer.

Example 8 *Use the binomial theorem to expand $1/(1-x)$ in ascending powers of x, as far as the term in x^3.*

(This example has been chosen because the result has already been established in §14.3 on page 173.)

Since $1/(1-x) = (1-x)^{-1}$, the binomial theorem may be used. Thus

$$(1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \frac{(-1)(-2)(-3)}{3!}(-x)^3 + \ldots$$

$$\therefore \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \text{ provided } |x| < 1.$$

Example 9 *Obtain the first five terms of the expansion of $\sqrt{(1+2x)}$ in ascending powers of x. State the values of x for which the expansion is valid.*

Since $\sqrt{(1+2x)} = (1+2x)^{1/2}$, the binomial theorem may be used.

$$(1+2x)^{1/2} = 1 + \frac{1}{2}(2x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}(2x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}(2x)^3 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}(2x)^4 + \ldots$$

$$\therefore \sqrt{(1+2x)} = 1 + x - \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 - \tfrac{5}{8}x^4 + \ldots$$

For the expansion to be valid, $-1 < 2x < +1$, i.e. $|x| < \frac{1}{2}$.

Example 10 *Expand $1/(2+x)^2$ in ascending powers of x, as far as the term in x^3, and state for what values of x the expansion is valid.*

First note that $1/(2+x)^2 = (2+x)^{-2}$. However, the binomial theorem has been stated for $(1+x)^n$. Therefore a factor must be taken out, in order to leave the bracket in this form.

$$(2+x)^{-2} = \{2(1+\tfrac{1}{2}x)\}^{-2} = 2^{-2}(1+\tfrac{1}{2}x)^{-2} = \tfrac{1}{4}(1+\tfrac{1}{2}x)^{-2}$$

and this may now be expanded.

Alternatively: $\dfrac{1}{(2+x)^2} = \dfrac{1}{2^2(1+\frac{1}{2}x)^2} = \frac{1}{4}(1+\frac{1}{2}x)^{-2}$.

$$\frac{1}{4}\left(1+\frac{1}{2}x\right)^{-2} = \frac{1}{4}\left\{1 + (-2)\left(\frac{x}{2}\right) + \frac{(-2)(-3)}{2!}\left(\frac{x}{2}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{x}{2}\right)^3 + \ldots\right\}$$

$$\therefore \frac{1}{(2+x)^2} = \frac{1}{4}\left(1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3 + \ldots\right)$$

For the expansion to be valid, $-1 < \frac{1}{2}x < +1$, i.e. $|x| < 2$.

Exercise 14c

Calculators should not be used in this exercise.

1 Evaluate the following binomial coefficients:

a $\binom{5}{3}$ **b** $\binom{-2}{4}$ **c** $\binom{\frac{1}{2}}{2}$ **d** $\binom{-\frac{1}{4}}{3}$

2 Expand the following in ascending powers of x, as far as the terms in x^3, and state the values of x for which the expansions are valid.

a $(1+x)^{-2}$ **b** $(1+x)^{1/3}$ **c** $(1+x)^{3/2}$

d $(1-2x)^{1/2}$ **e** $\left(1+\frac{x}{2}\right)^{-3}$ **f** $(1-3x)^{-1/2}$

g $\frac{1}{1+3x}$ **h** $\sqrt{(1-x^2)}$ **i** $\sqrt[3]{(1-x)}$

j $\frac{1}{\sqrt{(1+2x)}}$ **k** $\frac{1}{(1+x/2)^2}$ **l** $\sqrt{(1-2x)^3}$

m $\frac{1}{2+x}$ **n** $\sqrt{(2-x)}$ **o** $\sqrt[3]{(3+x)}$

p $\frac{1}{\sqrt{(2+x^2)}}$ **q** $\frac{1}{(3-x)^2}$ **r** $\frac{3}{\sqrt[3]{(3-x^3)}}$

3 Use the binomial theorem to find the values of the following:

a $\sqrt{(1.001)}$, correct to six places of decimals

b $\frac{1}{(1.02)^2}$, correct to four places of decimals

c $\sqrt{(0.998)}$, correct to six places of decimals

d $\sqrt[3]{(1.03)}$, correct to four places of decimals

e $\frac{1}{\sqrt{(0.98)}}$, correct to four places of decimals

4 Find the first four terms of the expansions of the following in ascending powers of x:

a $\frac{1+x}{1-x}$ **b** $\frac{x+2}{(1+x)^2}$ **c** $\frac{1-x}{\sqrt{(1+x)}}$

d $\sqrt{\frac{1+x}{1-x}}$, [Multiply numerator and denominator by $\sqrt{(1+x)}$.]

e $\frac{2x-3}{x+2}$ **f** $\sqrt{\frac{(1-x)^3}{1+x}}$ **g** $\frac{x+3}{\sqrt[3]{(1-3x)}}$

5 Find the first four terms of the expansion of $(1-8x)^{1/2}$ in ascending powers of x. Substitute $x=\frac{1}{100}$ and obtain the value of $\sqrt{23}$ correct to five significant figures.

6 Expand $(1-x)^{1/3}$ in ascending powers of x as far as the fourth term. By taking the first two terms of the expansion and substituting $x=\frac{1}{1000}$, find the value of $\sqrt[3]{37}$, correct to six significant figures. [Hint: $27\times 37=999$.]

7 Obtain the first four terms of the expansion of $(1-16x)^{1/4}$. Substitute $x=1/10\,000$ and use the first two terms to find $\sqrt[4]{39}$. To how many significant figures is your answer accurate?

Chapter 15

Vectors (1)

15.1 Introduction to vectors

Consider the following sentences:

a The temperature is 15°C.
b The journey lasted 2 hours.
c The plane is flying due East at 800 km/h.
d A horizontal force of 2 newtons was applied to the ruler at right-angles to its length.
e Move the table 10 m to the right.

The first two sentences differ from the others in one very important respect: the first two are complete when the magnitude of the quantity is given, but in the others it is necessary to define both the magnitude and the direction. A quantity which is completely specified by its magnitude alone is called a **scalar** quantity. One which requires both the magnitude and the direction to be given is called a **vector** quantity. (Strictly speaking, a vector quantity must also obey the triangle law of addition; see §15.6 on page 179.)

Consider sentences **d** and **e** in more detail. The effect of the force applied to the ruler will be determined by the point at which the force is applied. If it is applied to the end of the ruler, the ruler will start to rotate, but if it is applied to the mid-point of the ruler, the ruler will start to slide without rotating. So when we describe a force we have to give not only its magnitude and direction, but also its line of action. (This is usually done by describing a point through which the force passes.) Vectors which have a definite line of action are called **localised vectors**.

On the other hand, when the table in sentence **e** is moved, then, as we can see in Fig. 15.1, every point on the table moves 10 m to the right. All the line segments AA′, BB′, CC′, DD′ and PP′ are equal in length and they are parallel to one another. Any one of them can be used to describe the movement which has been applied to the table.

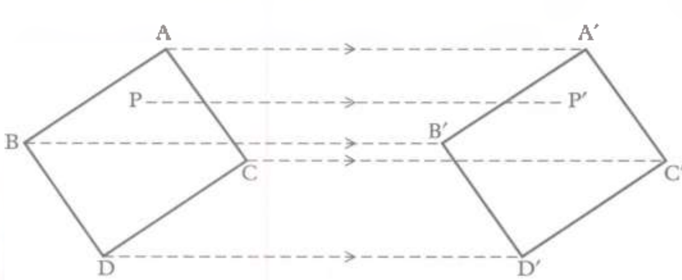


Figure 15.1

Vectors which do not have a particular line of action are called **free vectors**; all free vectors which have the same magnitude and direction are equivalent to one another. In the example above, we write $\overline{AA'} = \overline{BB'} = \overline{CC'} = \overline{DD'} = \overline{PP'}$. The use of bold letters for vectors, e.g.: $\mathbf{AA'} = \mathbf{BB'} = \mathbf{CC'} = \mathbf{DD'} = \mathbf{PP'}$ is also very common.

In this chapter all the vectors, with one important exception, will usually be free vectors. The main exception is the position vector (see §15.7 on page 181), which must always start from the origin.

15.2 Displacement vectors

Looking at a map of East Africa, we see that Dar es Salaam is about 1000 km from Kampala, and it is approximately South East of Kampala. This is an example of a common type of vector quantity, namely a **displacement**. The displacement of one point from another can be defined, as in the example above, by giving the distance and the direction. Alternatively, when using cartesian coordinates, the displacement can be defined by giving the increase in the x-coordinate and the increase in the y-coordinate.

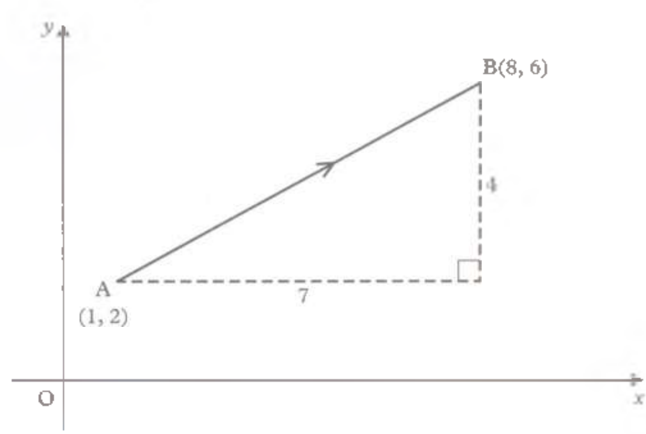


Figure 15.2

In Fig. 15.2, A is the point (1, 2) and B is the point (8, 6), so the displacement from A to B is '7 across and 4 up'.

The normal notation for this is $\begin{pmatrix} 7 \\ 4 \end{pmatrix}$. The upper number is the increase in the x-coordinate, and the lower one is the increase in the y-coordinate. It is also necessary to make it clear that the displacement goes *from* A *to* B, and so we write

$$\overline{AB} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

The displacement *from* B *to* A is written $\overline{BA}$, and, in this case is equal to $\begin{pmatrix} -7 \\ -4 \end{pmatrix}$.

Question

Q1 Write down the displacement vector $\overline{AB}$ for each of the following pairs of points:

- **a** A(3, 5), B(5, 9)
- **b** A(9, 7), B(12, 4)
- **c** A(12, 5), B(5, 4)
- **d** A(2, 3), B(2, 5)
- **e** A(5, 1), B(8, 1)

Fig. 15.3 shows that the displacement from $A(x_1, y_1)$ to $B(x_2, y_2)$ is

$$\overline{AB} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}.$$

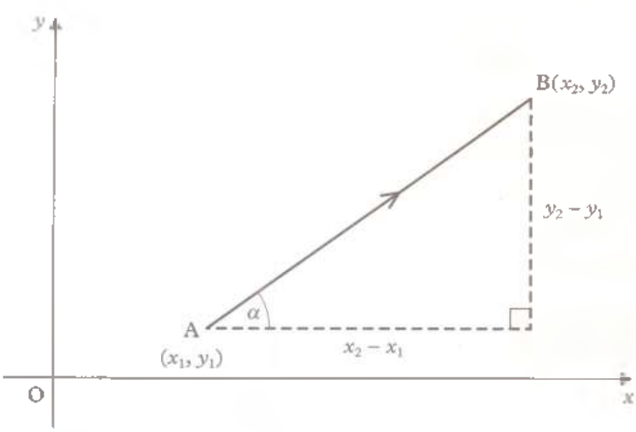


Figure 15.3

Notice that the magnitude of the vector (i.e. its length) is given by

$$AB = \sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2\}}$$

and that its direction is defined by the angle α which it makes with the x-axis, where

$$\tan \alpha = \frac{y_2 - y_1}{x_2 - x_1}$$

(note that this is the gradient of the line AB). The magnitude is never negative and the angle is usually given in the range $-180° \leqslant \alpha \leqslant +180°$. In the special case when $x_2 = x_1$, $\tan \alpha$ is not defined, because the denominator is zero. However, its graph would show that the vector is parallel to the y-axis.

Example 1 *Find the magnitude and direction of the displacement vector* $\overline{AB}$*, where A and B are the points (2, 1) and (8, 9) respectively. Find also the magnitude and direction of* $\overline{BA}$*, giving the angle correct to the nearest tenth of a degree.*

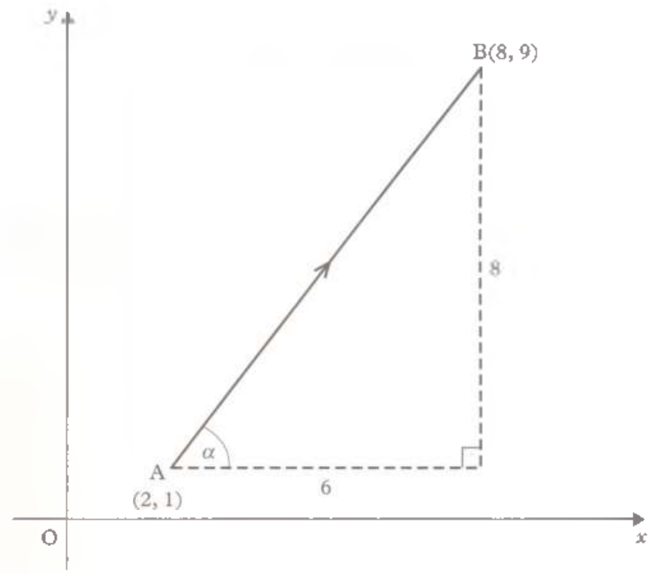


Figure 15.4

From **Fig. 15.4**, we can see that

$$\overline{AB} = \begin{pmatrix} 8-2 \\ 9-1 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

$$\therefore AB^2 = 6^2 + 8^2 = 36 + 64 = 100$$
$$AB = 10$$

We can also see that the direction is given by

$$\tan \alpha = \frac{8}{6}$$

$\therefore \alpha = 53.1°$, correct to the nearest 0.1°

Similarly,

$$\overline{BA} = \begin{pmatrix} 2-8 \\ 1-9 \end{pmatrix} = \begin{pmatrix} -6 \\ -8 \end{pmatrix}$$

$\overline{BA}$ is inclined at $-126.9°$ to the x-axis

Question

Q2 Find the magnitude and direction of each of the vectors in Q1.

15.3 Unit vectors

Any vector whose magnitude is 1, for example $\begin{pmatrix}0.6\\0.8\end{pmatrix}$, is called a **unit vector**. The unit vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ are especially important because they are parallel to the x-axis and y-axis respectively. They are called **base vectors**, and we use the letters **i** and **j** for them (**i** and **j** are always printed in bold type; when writing them they should be underlined).

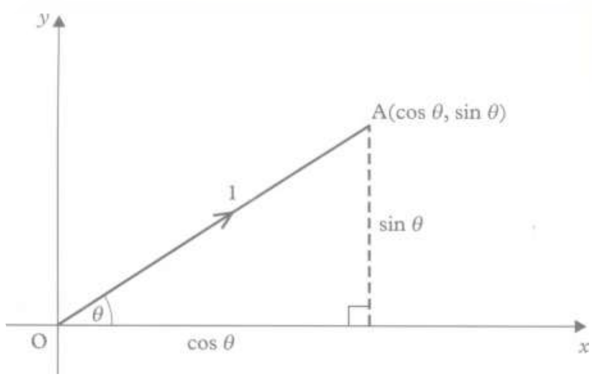


Figure 15.5

The unit vector $\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}$ is very useful as it is inclined at an angle θ to the x-axis (see Fig. 15.5).

15.4 Multiplication by a scalar

In Fig. 15.6, the displacement $\overline{AB}$ has been enlarged by a factor k, that is $\overline{AB}' = k\overline{AB}$. If $\overline{AB} = \begin{pmatrix}a\\b\end{pmatrix}$, then AP = a and PB = b. Also, since the triangles APB and AP′B′ are similar, AB′ = kAB, AP′ = ka and P′B′ = kb, and so

$$\overline{AB}' = \begin{pmatrix}ka\\kb\end{pmatrix}$$

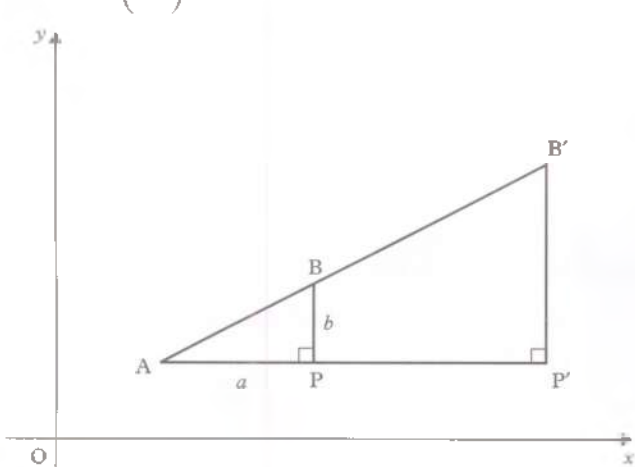


Figure 15.6

Thus we can write $\begin{pmatrix}20\\30\end{pmatrix} = 10\begin{pmatrix}2\\3\end{pmatrix}$, $\begin{pmatrix}5\\0\end{pmatrix} = 5\begin{pmatrix}1\\0\end{pmatrix} = 5\mathbf{i}$, etc.

In general $k\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}ka\\kb\end{pmatrix}$

15.5 Equal vectors

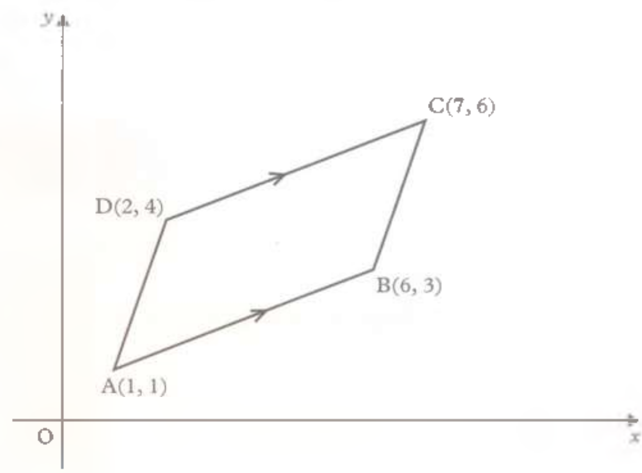


Figure 15.7

In Fig. 15.7, $\overline{AB} = \begin{pmatrix}5\\2\end{pmatrix}$ and $\overline{DC} = \begin{pmatrix}5\\2\end{pmatrix}$, that is the displacement from A to B is the same as that from D to C. In this sense we can say that the vectors $\overline{AB}$ and $\overline{DC}$ are equal. Vectors are equal when they have the same magnitude and direction. Notice that $\overline{AD}$ and $\overline{BC}$ are also equal and ABCD is a parallelogram.

Example 2 *Given that A is the point (1, 3) and that* $\overline{AB}$ *and* $\overline{AD}$ *are* $\begin{pmatrix}4\\-1\end{pmatrix}$ *and* $\begin{pmatrix}2\\3\end{pmatrix}$ *respectively, find the coordinates of the vertices B, C and D of the parallelogram ABCD (Fig. 15.8).*

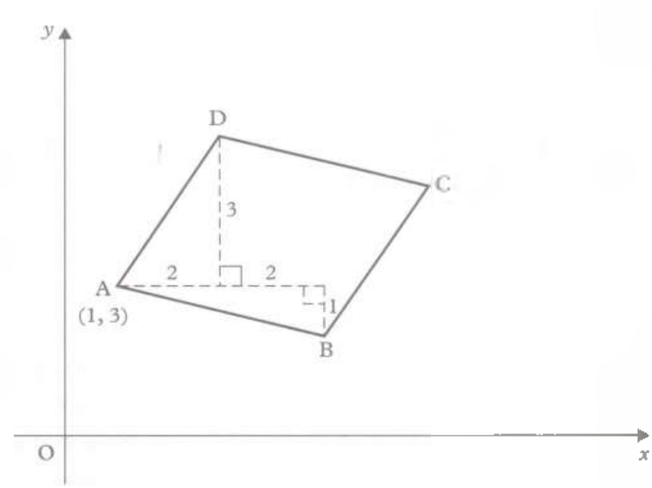


Figure 15.8

B is the point $(1 + 4, 3 - 1) = (5, 2)$

D is the point $(1 + 2, 3 + 3) = (3, 6)$

$$\mathbf{DC} = \mathbf{AB} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

hence

C is the point $(3 + 4, 6 - 1) = (7, 5)$

Example 3 *Given the points* A(1, 1), B(5, 4), C(8, 9) *and* D(0, 3), *show that* ABCD *is a trapezium* (Fig. 15.9).

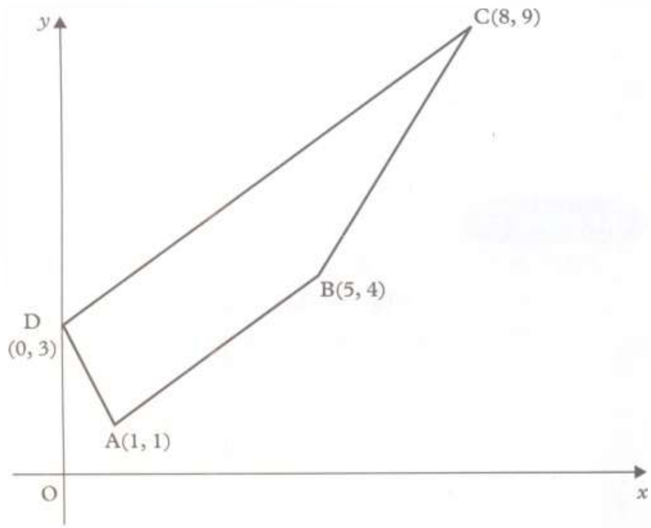


Figure 15.9

$$\overline{AB} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad \text{and} \quad \overline{DC} = \begin{pmatrix} 8 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$\therefore 2\overline{AB} = \overline{DC}$

Hence $\overline{DC}$ is parallel to $\overline{AB}$ (and twice as long). So ABCD is a trapezium.

15.6 Addition and subtraction of vectors

If we make the displacement $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and follow this with the displacement $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$, then overall we have moved 7 units to the right and 4 units up. We could also achieve the same result by making the displacement $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ *first* and the displacement $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ *second*. We write

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

and we say that we have 'added' the vectors.

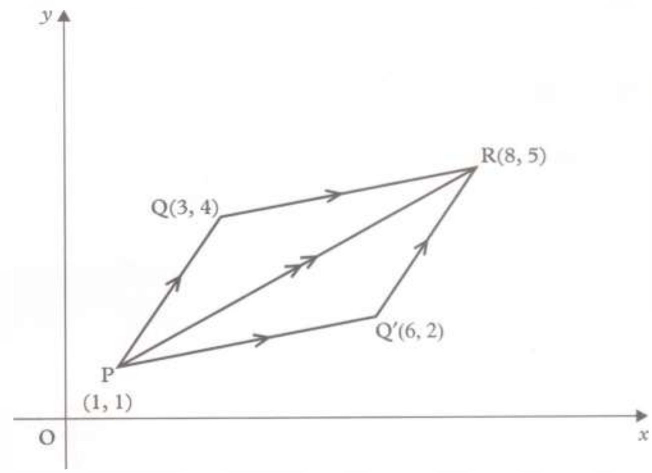


Figure 15.10

In Fig. 15.10, $\overline{PQ} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\overline{QR} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ and $\overline{PR} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$. Notice that $\overline{PQ} + \overline{QR} = \overline{PR}$ (this is the 'triangle law of addition', which you may have met in physics). We could also say that $\overline{PQ'} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$, $\overline{Q'R} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and that $\overline{PQ'} + \overline{Q'R} = \overline{PR}$.

In general

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix}$$

If $\overline{AB} = \begin{pmatrix} h \\ k \end{pmatrix}$, then $\overline{BA} = \begin{pmatrix} -h \\ -k \end{pmatrix}$. Notice that $\overline{BA} = -\overline{AB}$, and also that $\overline{AB} + \overline{BA} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; the vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is called the **zero vector** and is denoted by $\mathbf{0}$.

Any vector $\begin{pmatrix} x \\ y \end{pmatrix}$ can be expressed as $\begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix}$, and this in turn can be written $x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix} = x\mathbf{i} + y\mathbf{j}$.

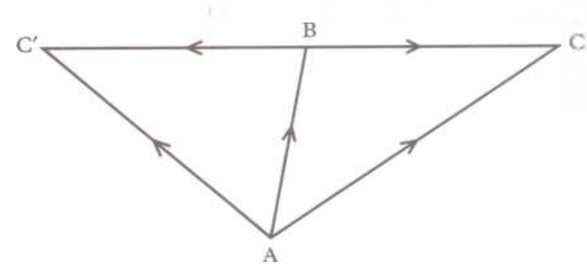


Figure 15.11

To subtract vectors, see Fig. 15.11, where C′ is the point on CB produced, such that BC′ = CB.

$$\overline{AB} - \overline{BC} = \overline{AB} + (-\overline{BC})$$
$$= \overline{AB} + \overline{BC'}$$
$$= \overline{AC'}$$

Thus

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} - \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 - a_2 \\ b_1 - b_2 \end{pmatrix}$$

It is frequently convenient to use a single letter to represent a vector. When this is done, a lower case letter (i.e. not a capital letter) is used and it is always printed in bold type in textbooks. (In handwriting, underline the letter with a wavy line, $\underset{\sim}{x}$.). For example we may write

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\mathbf{y} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \quad \text{and}$$

$$\mathbf{x} - \mathbf{y} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

In **i**, **j** notation, the statement above could be written

$\mathbf{x} = 2\mathbf{i} + \mathbf{j}$
$\mathbf{y} = \mathbf{i} + 5\mathbf{j}$
$\mathbf{x} + \mathbf{y} = 3\mathbf{i} + 6\mathbf{j}$ and
$\mathbf{x} - \mathbf{y} = \mathbf{i} - 4\mathbf{j}$

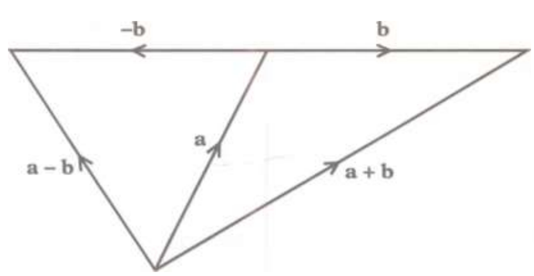


Figure 15.12

This is especially useful for labelling diagrams. For example, Fig. 15.12 illustrates the sum, **a** + **b**, and the difference, **a** – **b**, of the vectors **a** and **b**.

When using the single letter notation, an *italic* letter is always used to denote the *magnitude* of the vector which is represented by the same letter in **bold** type, e.g. if $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$, then $a = 5$.

Example 4 *In Fig. 15.13 each set of parallel lines is equally spaced and it is given that* $\overline{OP} = \mathbf{p}$ *and* $\overline{OU} = \mathbf{u}$. *Express the following vectors in terms of* **p** *and* **u**: **a** $\overline{OQ}$, **b** $\overline{QW}$, **c** $\overline{OW}$, **d** $\overline{OM}$, **e** $\overline{OS}$, **f** $\overline{OA}$.

a $\overline{OQ} = 2\mathbf{p}$, **b** $\overline{QW} = \mathbf{u}$,
c $\overline{OW} = 2\mathbf{p} + \mathbf{u}$, **d** $\overline{OM} = -2\mathbf{p}$,
e $\overline{OS} = \overline{OM} + \overline{MS} = -2\mathbf{p} + \mathbf{u}$, **f** $\overline{OA} = -2\mathbf{p} - 2\mathbf{u}$.

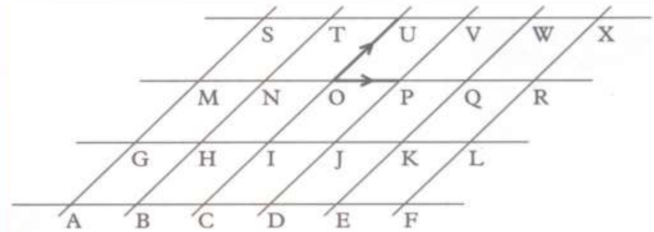


Figure 15.13

Example 5 *In triangle* OAB (*Fig. 15.14*), **OA** = **a** *and* **OB** = **b**. *Given that* P *and* Q *are the mid-points of* OA *and* OB, *express* **PQ** *and* **AB** *in terms of* **a** *and* **b**. *State the geometrical relationship between* **PQ** *and* **AB**.

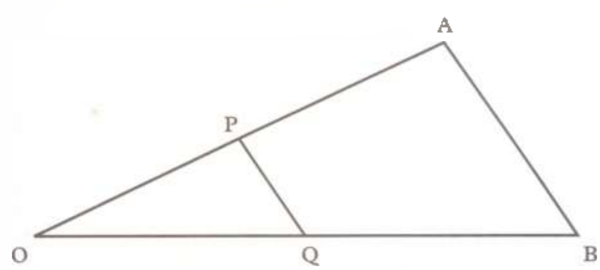


Figure 15.14

Since P and Q are the mid-points of **OA** and **OB**, we can write

$$\mathbf{OP} = \tfrac{1}{2}\mathbf{a} \quad \text{and} \quad \mathbf{OQ} = \tfrac{1}{2}\mathbf{b}$$

Now

$$\mathbf{PQ} = \mathbf{PO} + \mathbf{OQ}$$
$$= -\tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{b}$$
$$= \tfrac{1}{2}(\mathbf{b} - \mathbf{a})$$
$$\mathbf{AB} = \mathbf{AO} + \mathbf{OB}$$
$$= -\mathbf{a} + \mathbf{b}$$
$$\therefore \mathbf{AB} = 2\mathbf{PQ}$$

In other words, **AB** is parallel to **PQ** and twice its length.

From a mathematical point of view, the beauty of the argument in Example 5 is that it does not depend upon the actual dimensions of the triangle.

Exercise 15a

1 Given that $\mathbf{x} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$ write down as column vectors:

a $2\mathbf{x}$ b $3\mathbf{y}$

c $-\mathbf{y}$ d $\frac{1}{2}\mathbf{y}$

e $\mathbf{x}+\mathbf{y}$ f $2\mathbf{x}+3\mathbf{y}$

g $\mathbf{x}-\mathbf{y}$ h $3\mathbf{x}-2\mathbf{y}$

2 Find the magnitude and direction of the vectors:

a $3\mathbf{i}+4\mathbf{j}$ b $-5\mathbf{i}+12\mathbf{j}$

c $-10\mathbf{j}$ d $\mathbf{i}-\mathbf{j}$

3 The vector $\overline{XY}$ has magnitude 10 units and it is inclined at 30° to the x-axis. Express $\overline{XY}$ as a column vector.

4 The vector **PQ** has magnitude 5 units and is inclined at 150° to the x-axis. Express **PQ** in the form $a\mathbf{i} + b\mathbf{j}$, where $a, b \in \mathbb{R}$.

5 A and B are the points (3, 7) and (15, 13) respectively. P is a point on AB such that $\overline{AP} = s\overline{AB}$. Write down the vector $\overline{OP}$ in terms of s. Find the coordinates of P, when

a $s = \frac{3}{4}$ b $s = \frac{3}{2}$ c $s = -2$

6 In Fig. 15.15, OABC is a quadrilateral and P, Q, R and S are the mid-points of the sides OA, AB, BC and CO, respectively. Given that **OA** = **a**, **OB** = **b** and **OC** = **c**, express the following vectors in terms of **a**, **b** and **c**:

a **PS** b **AC** c **QR**

What can you deduce about the lines PS, AC and QR?

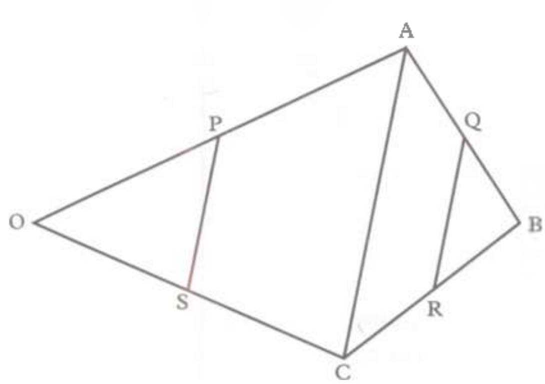


Figure 15.15

7 In question 6 above, X is the mid-point of PR, and Y is the mid-point of QS. Express **OX** and **OY** in terms of **a**, **b** and **c**. State clearly in words the deduction which can be made from these expressions.

15.7 Position vectors

In the preceding exercise, you will have noticed that the vector from the origin O to a point P is frequently required. This vector **OP** is called the **position vector** of the point P; it is always denoted by the single letter **p** (similarly, the position vectors of points A, B, C, … would be written **a**, **b**, **c**, …). It is important to notice that position vectors are localised; they *must* start from the origin.

If the coordinates of P are (x, y) then **p** is the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$. Notice that the displacement vector **PQ** is related to the position vectors of P and Q, as follows:

$$\mathbf{PQ} = \mathbf{PO} + \mathbf{OQ} = -\mathbf{p} + \mathbf{q} = \mathbf{q} - \mathbf{p}$$

Similarly, $\mathbf{AB} = \mathbf{b} - \mathbf{a}$, $\mathbf{XY} = \mathbf{y} - \mathbf{x}$ and so on. Expressions like these are very common in vector geometry and you should memorise such statements.

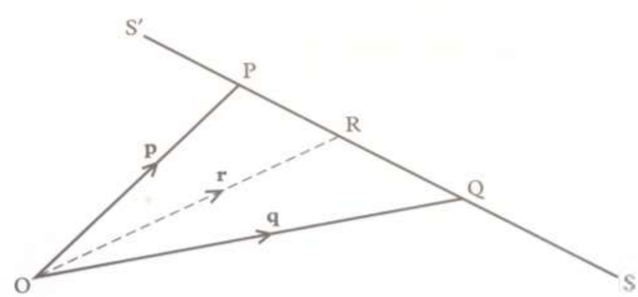


Figure 15.16

In Fig. 15.16, R is a point on the line PQ, such that $\mathbf{PR} = t\mathbf{PQ}$.

Applying the results above to **PR** and **PQ**, we have

$$\mathbf{PR} = t\mathbf{PQ}$$

hence

$$\begin{aligned} \mathbf{r} - \mathbf{p} &= t(\mathbf{q} - \mathbf{p}) \\ \therefore \mathbf{r} &= \mathbf{p} + t(\mathbf{q} - \mathbf{p}) \\ &= (1-t)\mathbf{p} + t\mathbf{q} \end{aligned}$$

Since R lies *between* P and Q, $0 < t < 1$. But if $t = 1$, then R will coincide with Q, and if $t = 0$, then R will coincide with P. The position vector of a point such as S, is given by a similar expression, e.g.

$$\mathbf{s} = (1-t)\mathbf{p} + t\mathbf{q}$$

but here the number t is greater than 1. A point such as S′ can be obtained by using a negative value for t.

Example 6 *In Fig. 15.17,* $\mathbf{OS} = 2\mathbf{r}$ *and* $\mathbf{OQ} = \frac{3}{2}\mathbf{p}$. *Given that* $\mathbf{QK} = m\mathbf{QR}$ *and* $\mathbf{PK} = n\mathbf{PS}$*, find two distinct expressions, in terms of* $\mathbf{p}$, $\mathbf{r}$, m *and* n*, for* $\mathbf{OK}$*. By equating these expressions, find the values of* m *and* n *and hence calculate the ratios* QK:KR *and* PK:KS.

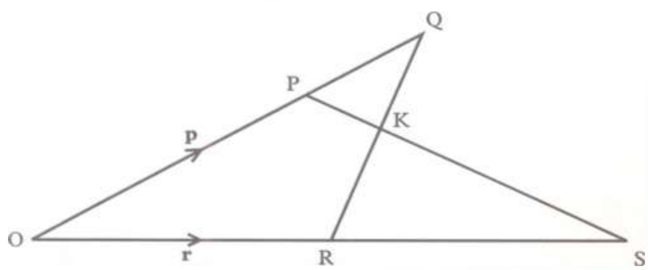


Figure 15.17

$$\mathbf{QR} = \mathbf{QO} + \mathbf{OR}$$
$$= -\tfrac{3}{2}\mathbf{p} + \mathbf{r}$$

One expression for $\mathbf{OK}$ is given by

$$\mathbf{OK} = \mathbf{OQ} + \mathbf{QK} = \mathbf{OQ} + m\mathbf{QR}$$
$$= \tfrac{3}{2}\mathbf{p} + m(\mathbf{r} - \tfrac{3}{2}\mathbf{p})$$

Similarly, $\mathbf{OK} = \mathbf{OP} + \mathbf{PK}$, hence

$$\mathbf{OK} = \mathbf{p} + n(-\mathbf{p} + 2\mathbf{r})$$

Equating the two expressions for $\mathbf{OK}$, we have

$$\tfrac{3}{2}\mathbf{p} + m(\mathbf{r} - \tfrac{3}{2}\mathbf{p}) = \mathbf{p} + n(-\mathbf{p} + 2\mathbf{r})$$

and rearranging this gives

$$(\tfrac{1}{2} - \tfrac{3}{2}m + n)\mathbf{p} = (2n - m)\mathbf{r}$$

but since $\mathbf{p}$ and $\mathbf{r}$ are not parallel, the two sides of this equation cannot be equal unless they are both zero. (You should think carefully about this statement, and make sure you fully understand it. This argument is very common when vectors are used in geometrical problems.) Hence

$$\tfrac{1}{2} - \tfrac{3}{2}m + n = 0 \qquad (1)$$

and $$2n - m = 0 \qquad (2)$$

Substituting $m = 2n$ in equation (1), we have

$$\tfrac{1}{2} - \tfrac{3}{2} \times 2n + n = 0$$
$$\tfrac{1}{2} - \quad 3n + n = 0$$
$$2n = \tfrac{1}{2}$$
$$n = \tfrac{1}{4}$$

and hence

$$m = \tfrac{1}{2}$$

But $\mathbf{QK} = m\mathbf{QR}$ (given) so $\mathbf{QK} = \frac{1}{2}\mathbf{QR}$ and hence $\mathbf{QK} = \mathbf{KR}$. Therefore

$$\text{QK:KR} = 1:1$$

Also $\mathbf{PK} = n\mathbf{PS}$, so $\mathbf{PK} = \frac{1}{4}\mathbf{PS}$ and hence

$$\text{PK:KS} = 1:3$$

Example 7 *At noon, two boats P and Q are at points whose position vectors are* $4\mathbf{i} + 8\mathbf{j}$ *and* $4\mathbf{i} + 3\mathbf{j}$ *respectively. Both boats are moving with constant velocity; the velocity of P is* $4\mathbf{i} + \mathbf{j}$ *and the velocity of Q is* $2\mathbf{i} + 5\mathbf{j}$*, (all distances are in kilometres and the time is measured in hours). Find the position vectors of P and Q, and* $\mathbf{PQ}$ *after* t *hours, and hence express the distance PQ between the boats in terms of* t*. Show that the least distance between the boats is* $\sqrt{5}$ *km.*

After t hours the displacement of P from its starting point is $t(4\mathbf{i} + \mathbf{j})$, hence

$$\mathbf{p} = (4\mathbf{i} + 8\mathbf{j}) + t(4\mathbf{i} + \mathbf{j})$$
$$= (4 + 4t)\mathbf{i} + (8 + t)\mathbf{j}$$

Similarly

$$\mathbf{q} = (4 + 2t)\mathbf{i} + (3 + 5t)\mathbf{j}$$

Hence

$$\mathbf{PQ} = \mathbf{q} - \mathbf{p} = -2t\mathbf{i} + (-5 + 4t)\mathbf{j}$$
$$\therefore \text{PQ}^2 = (-2t)^2 + (-5 + 4t)^2$$
$$= 20t^2 - 40t + 25$$

Hence the distance between the boats is given by

$$\text{PQ} = \sqrt{(20t^2 - 40t + 25)} \text{ km}$$

To find the least distance, consider

$$\text{PQ}^2 = 20(t^2 - 2t + 1) + 5$$
$$= 20(t - 1)^2 + 5$$

Since $(t - 1)^2$ cannot be negative, its least value is zero and this is obtained by putting $t = 1$. Hence the least value of PQ^2 is 5. (See §10.1 on page 128.)

$\therefore$ The shortest possible distance between the boats is $\sqrt{5}$ km.

Exercise 15b

1 Given that A is the point (2, 5) and that B is the point (10, −1), find the position vector of a point P on AB, such that

a $\overrightarrow{AP} = \overrightarrow{PB}$ **b** $2\overrightarrow{AP} = \overrightarrow{PB}$

c $\overrightarrow{AP} = 4\overrightarrow{AB}$ **d** AP:PB = 2:3

e AP:PB = 4:1 **f** AP:PB = m:n

2 Repeat question 1 for A(−7, 3) and B(−1, −15).

3 A, B, C are three collinear points whose position vectors are **a**, **b** and **c** respectively and **AC** = 3**AB**. Express **c** in the form **c** = m**a** + n**b**.
Find the scalars m and n and verify that $m + n = 1$.
Show also that if **a** = p**b** + q**c** then $p + q = 1$.

4 Repeat question 3 given that **AC** = −2**AB**.

5 A stationary observer O observes a ship S at noon, at a point whose coordinates relative to O are (20, 15); the units are kilometres. The ship is moving at a steady 10 km/h on a bearing 150° (a bearing is measured clockwise from North). Express its velocity as a column vector. Write down, in terms of t, its position after t hours. Hence find the value of t when it is due East of O. How far is it from O at this instant?

6 Find numbers m and n such that

$$m\begin{pmatrix}3\\5\end{pmatrix} + n\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}4\\9\end{pmatrix}.$$

7 In Fig. 15.18, **OP** = **p** and **OR** = **r**.
P is the mid-point of OQ and PX:XR = 1:3. Express **x** in terms of **p** and **r**. Taking **OY** to be h**OX**, find **QY** in terms of **p**, **r** and h and hence find the ratio QY:YR.

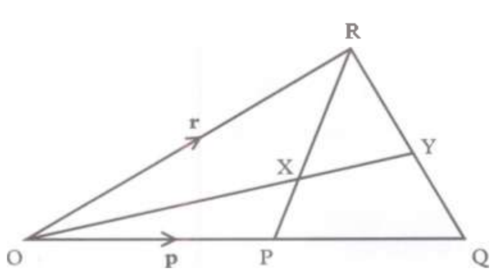


Figure 15.18

8 In Fig. 15.19, OBC is a triangle and the line NL produced meets the line OC produced at M.

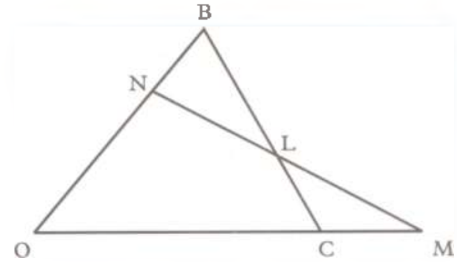


Figure 15.19

Given that $\overrightarrow{ON} = \frac{3}{4}\overrightarrow{OB}$ and $\overrightarrow{BL} = \frac{2}{3}\overrightarrow{BC}$, express the vector $\overrightarrow{NL}$ in terms of **b** and **c**, the position vectors of the points B and C with respect to the origin O. Find an expression for the position vector of any point R on the line NL. Hence express $\overrightarrow{OM}$ as a multiple of $\overrightarrow{OC}$. Find the ratio CM/MO and verify that

$$\frac{ON}{NB} \times \frac{BL}{LC} \times \frac{CM}{MO} = -1$$

9 In a triangle OAB, X is a point on OB such that **OX** = 2**XB** and Y is a point on AB such that 2**BY** = 3**YA**. Express **x** and **y** in terms of **a** and **b**. Find the position vector of any point on XY and hence find the position vector of the point Z, where XY produced meets OA produced. Calculate the value of AZ/OZ.

10 Prove that if **a** and **b** are the position vectors of points A and B, then the position vector of a point P on AB, where AP:PB = m:n is given by $(m + n)\mathbf{p} = n\mathbf{a} + m\mathbf{b}$.

11 Prove that if **p** = h**a** + k**b** represents the point P on the line AB, then $h + k = 1$.

12 Given that A, B and C are three collinear points whose position vectors satisfy the equation $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = \mathbf{0}$, prove that $\alpha + \beta + \gamma = 0$.

Sections 15.8 to 15.12 may be left until later in the course.

15.8 The ratio theorem

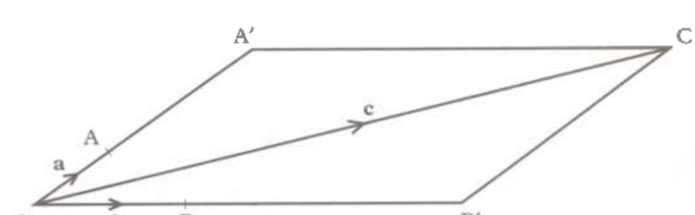


Figure 15.20

In **Fig. 15.20**, $\overline{OA'} = h\mathbf{a}$, $\overline{OB'} = k\mathbf{b}$ and $\overline{OC} = h\mathbf{a} + k\mathbf{b}$. We say that $\overline{OC}$ is a **linear combination** of **a** and **b**. Any point C, whose position vector is a linear combination of **a** and **b**, will be a point in the plane of O, A and B. (So far we have only considered vectors in two dimensions; this last statement becomes very important when we start to consider three dimensions.)

However, if $\overline{OC} = h\mathbf{a} + k\mathbf{b}$ and $h + k = 1$, it can be shown that C lies on the line AB, as follows:

$$\begin{aligned} \mathbf{c} &= h\mathbf{a} + k\mathbf{b} \\ &= (1-k)\mathbf{a} + k\mathbf{b} \\ &= \mathbf{a} + k(\mathbf{b} - \mathbf{a}) \end{aligned}$$

Using capital letter notation this last equation becomes

$$\overline{OC} = \overline{OA} + k\overline{AB}$$

hence

$$\overline{AC} = k\overline{AB}$$

so C is the point on AB such that $\overline{AC} = k\overline{AB}$ (see **Fig. 15.21**).

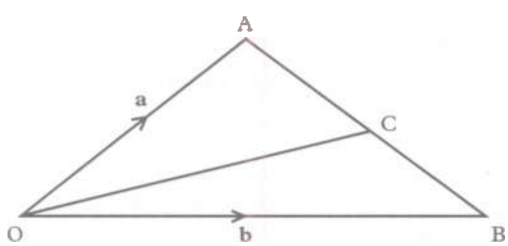


Figure 15.21

Similarly $\overline{BC} = h\overline{BA}$. Hence AC:CB = k:h.

Conversely if we are given that C is a point on the line AB such that AC:CB = m:n, then we can write

$$\frac{AC}{CB} = \frac{m}{n}$$

i.e. $\quad n\overline{AC} = m\overline{CB}$

$$\therefore n(\mathbf{c} - \mathbf{a}) = m(\mathbf{b} - \mathbf{c})$$

$$n\mathbf{c} - n\mathbf{a} = m\mathbf{b} - m\mathbf{c}$$

$$m\mathbf{c} + n\mathbf{c} = n\mathbf{a} + m\mathbf{b}$$

$$(m+n)\mathbf{c} = n\mathbf{a} + m\mathbf{b}$$

$$\therefore \mathbf{c} = \left(\frac{n}{m+n}\right)\mathbf{a} + \left(\frac{m}{m+n}\right)\mathbf{b}$$

This is usually called the **ratio theorem**. Notice that the sum of the coefficients $n/(m+n)$ and $m/(m+n)$ is 1.

Example 8 *If* $\mathbf{c} = \frac{2}{5}\mathbf{a} + \frac{3}{5}\mathbf{b}$, *show that* C *is a point on* AB *and that* AC:CB = 3:2.

Since $\frac{2}{5} + \frac{3}{5}$ is equal to 1, C lies on the line AB. Also,

$$AC{:}CB = \tfrac{3}{5}{:}\tfrac{2}{5} = 3{:}2$$

The centroid of a triangle

In **Fig. 15.22**, ABC is any triangle and P is the mid-point of BC. G is the point on AP such that AG:GP = 2:1. The origin is not shown in the diagram.

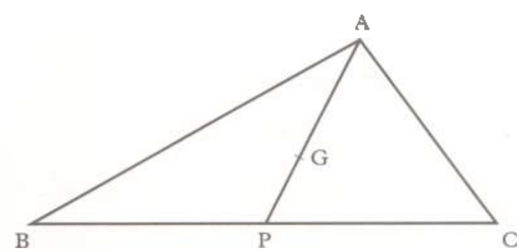


Figure 15.22

Since BP:PC = 1:1,

$$\mathbf{p} = \tfrac{1}{2}\mathbf{b} + \tfrac{1}{2}\mathbf{c}$$

and since AG:GP = 2:1,

$$\begin{aligned} \mathbf{g} &= \tfrac{1}{3}\mathbf{a} + \tfrac{2}{3}\mathbf{p} \\ &= \tfrac{1}{3}\mathbf{a} + \tfrac{2}{3}\left(\tfrac{1}{2}\mathbf{b} + \tfrac{1}{2}\mathbf{c}\right) \\ &= \tfrac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c}) \end{aligned}$$

This last expression is symmetrical in **a**, **b** and **c** (that is, the letters **a**, **b** and **c** can be interchanged without altering **g**), so the same result could be obtained by dividing the median from B to AC (or that from C to AB) in the ratio 2:1. Hence the point G, whose position vector is given by

$$\mathbf{g} = \tfrac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$$

is the point of intersection of the three medians. G is called the **centroid** of the triangle.

Question

Q3 Find the centroid of the triangle whose vertices are A(1, 2), B(3, 7) and C(2, 3).

Menelaus' theorem

In Fig. 15.23, OAB is any triangle and PQR is a straight line intersecting the sides of the triangle as shown.

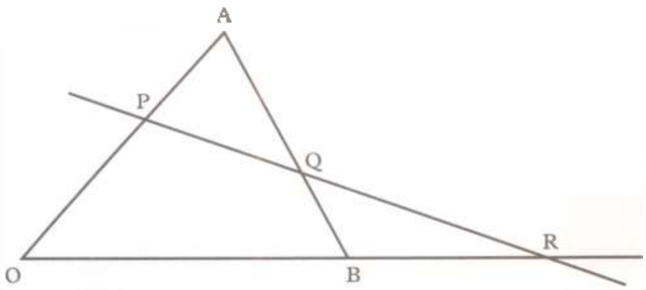


Figure 15.23

Menelaus' theorem states:
if $\mathbf{OP} = \alpha\mathbf{PA}$, $\mathbf{AQ} = \beta\mathbf{QB}$ and $\mathbf{BR} = \gamma\mathbf{RO}$, then $\alpha\beta\gamma = -1$.
(Notice that since R is on OB produced, γ is a negative number.)

This famous theorem appeared in a treatise published by Menelaus in 100 AD, although it was probably known to Euclid almost 400 years earlier. These great mathematicians would not, of course, have expressed the proof in vector notation.

Menelaus' theorem can be proved by vector methods, as follows:

$\mathbf{OP} = \alpha\mathbf{PA}$, hence $\mathbf{p} = \alpha(\mathbf{a} - \mathbf{p})$.

$$\therefore (1+\alpha)\mathbf{p} = \alpha\mathbf{a} \qquad (1)$$

$\mathbf{AQ} = \beta\mathbf{QB}$, hence $\mathbf{q} - \mathbf{a} = \beta(\mathbf{b} - \mathbf{q})$.

$$\therefore (1+\beta)\mathbf{q} = \mathbf{a} + \beta\mathbf{b} \qquad (2)$$

$\mathbf{BR} = \gamma\mathbf{RO}$, hence $\mathbf{r} - \mathbf{b} = -\gamma\mathbf{r}$.

$$\therefore (1+\gamma)\mathbf{r} = \mathbf{b} \qquad (3)$$

From equation (1) we have

$$\mathbf{a} = \left(\frac{1+\alpha}{\alpha}\right)\mathbf{p}$$

and from equation (3),

$$\mathbf{b} = (1+\gamma)\mathbf{r}$$

Substituting these expressions for **a** and **b** in equation (2) gives

$$(1+\beta)\mathbf{q} = \left(\frac{1+\alpha}{\alpha}\right)\mathbf{p} + \beta(1+\gamma)\mathbf{r}$$

$$\therefore \mathbf{q} = \frac{(1+\alpha)}{\alpha(1+\beta)}\mathbf{p} + \frac{\beta(1+\gamma)}{(1+\beta)}\mathbf{r}$$

However, Q is a point on PR, so, using the ratio theorem (see §15.8 on page 183),

$$\frac{(1+\alpha)}{\alpha(1+\beta)} + \frac{\beta(1+\gamma)}{(1+\beta)} = 1$$

$$(1+\alpha) + \alpha\beta(1+\gamma) = \alpha(1+\beta)$$

$$1 + \alpha + \alpha\beta + \alpha\beta\gamma = \alpha + \alpha\beta$$

$$\therefore \alpha\beta\gamma = -1$$

The result looks slightly more elegant, and it is perhaps easier to remember, if the diagram is re-lettered as in Fig. 15.24.

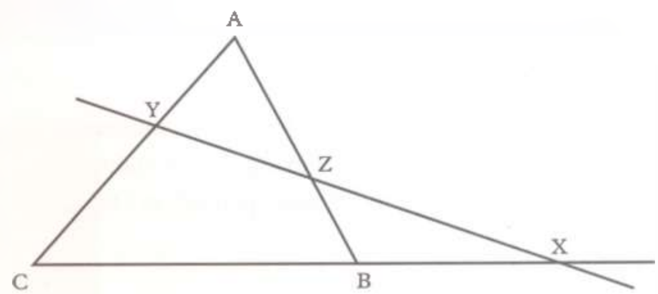


Figure 15.24

Menelaus' theorem can then be expressed

$$\frac{AZ}{ZB} \times \frac{BX}{XC} \times \frac{CY}{YA} = -1$$

15.9 Vectors in three dimensions

So far in this chapter, we have considered vectors in two dimensions only. However, the real world is three dimensional, so we now consider vectors in three dimensions.

First we will look at cartesian coordinates in three dimensions with three mutually perpendicular axes Ox, Oy and Oz. This cannot be drawn on a flat surface, so Fig. 15.25 at the top of the next page should be viewed with the page held so that the z-axis is vertical, the y-axis is horizontal, and the x-axis is *imagined* to be horizontal, but coming out of the page at right angles to the plane of the page. By convention, the three axes must form a 'right-handed set'. If the thumb, index finger and middle finger of the right hand are stretched out so that they are mutually perpendicular, it should be possible to make the thumb correspond to the x-axis, the index finger to the y-axis and the middle finger to the z-axis. (In a 'left-handed set' the x-axis would go into, instead of come out of, the page.)

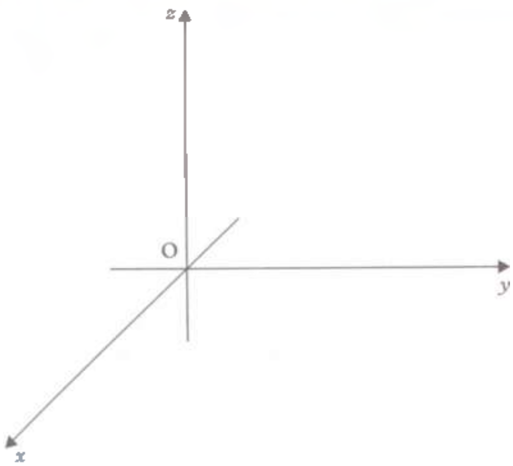


Figure 15.25

A point A(2, 3, 5) is located in the usual way, namely by starting from the origin and moving 2 units along Ox, 3 units parallel to Oy and 5 units parallel to Oz (see Fig. 15.26).

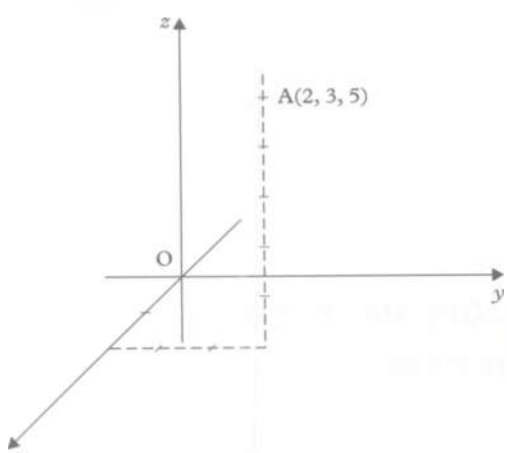


Figure 15.26

The position vector of the point A is written $\mathbf{OA} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$.

Similarly the displacement vector from A(2, 3, 5) to B(3, 6, 4) is written $\mathbf{AB} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$.

In general, if A is the point (x_1, y_1, z_1) and B is the point (x_2, y_2, z_2) then we write

$$\mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \quad \text{and} \quad \mathbf{AB} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$$

Fig. 15.27 represents a cuboid, in which AP is parallel to the x-axis, PQ is parallel to the y-axis and QB is parallel to the z-axis. Hence

$$AP = x_2 - x_1, \quad PQ = y_2 - y_1 \quad \text{and} \quad QB = z_2 - z_1$$

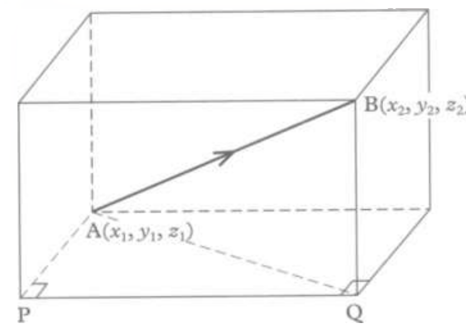


Figure 15.27

The length of vector **AB** can be found, using Pythagoras' theorem, as follows. In the right-angled triangle ABQ,

$$AB^2 = AQ^2 + QB^2$$

and, in the right-angled triangle APQ,

$$AQ^2 = AP^2 + PQ^2$$

hence

$$\begin{aligned} AB^2 &= AP^2 + PQ^2 + QB^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \end{aligned}$$

Multiplication of a vector by a scalar in three dimensions is a simple extension of the method used in two dimensions (see §15.4 on page 178), that is,

$$k\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} kx \\ ky \\ kz \end{pmatrix}$$

Addition and subtraction are also similar to the method used before, i.e.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

and

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}$$

All the results described so far in this chapter are equally valid in three dimensions [e.g. the centroid of a triangle ABC is given by $\mathbf{g} = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$].

The letter **k** is always used to represent the unit vector parallel to the z-axis. Consequently in **i**, **j**, **k** notation the vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ becomes $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Example 9 *If A and B are the points (1, 1, 1) and (13, 4, 5) respectively, find, in terms of* **i**, **j** *and* **k**, *the displacement vector* $\overline{AB}$. *Find also the unit vector parallel to* $\overline{AB}$.

$$\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{b} = 13\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$$
$$\overline{AB} = \mathbf{b} - \mathbf{a} = 12\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$$
$$\therefore AB^2 = 12^2 + 3^2 + 4^2 = 169$$
$$\therefore AB = 13$$

The magnitude of $\overline{AB}$ is 13 and so the vector $\frac{1}{13}\overline{AB}$ is a parallel vector of magnitude 1. Hence the required unit vector is $\frac{12}{13}\mathbf{i} + \frac{3}{13}\mathbf{j} + \frac{4}{13}\mathbf{k}$.

Example 10 *Using the points A and B in* Example 9, *find the point P on* $\overline{AB}$ *such that AP:PB = 1:3.*

We are given that AP:PB = 1:3, so $\overline{AP} = \frac{1}{4}\overline{AB}$, hence

$$4(\mathbf{p} - \mathbf{a}) = (\mathbf{b} - \mathbf{a})$$
$$\therefore 4\mathbf{p} = 4\mathbf{a} + \mathbf{b} - \mathbf{a}$$
$$= 3\mathbf{a} + \mathbf{b}$$
$$= 3(\mathbf{i} + \mathbf{j} + \mathbf{k}) + (13\mathbf{i} + 4\mathbf{j} + 5\mathbf{k})$$
$$= 16\mathbf{i} + 7\mathbf{j} + 8\mathbf{k}$$
$$\therefore \mathbf{p} = 4\mathbf{i} + \tfrac{7}{4}\mathbf{j} + 2\mathbf{k}$$

Hence P is the point $(4, \frac{7}{4}, 2)$.

Example 11 *Show that the points A(1, 2, 3), B(3, 8, 1), C(7, 20, −3) are collinear.*

$$\mathbf{AB} = (3\mathbf{i} + 8\mathbf{j} + \mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$$
$$= 2\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$$

Similarly

$$\mathbf{BC} = 4\mathbf{i} + 12\mathbf{j} - 4\mathbf{k}$$

hence

$$\mathbf{BC} = 2\mathbf{AB}$$

Consequently **AB** and **BC** are in the same direction and so ABC is a straight line (i.e. A, B and C are collinear).

Questions

Q4 Find the centroid of the triangle whose vertices are A(1, 2, 3), B(3, 7, 4), C(2, 0, 5).

Q5 Prove that A(1, 2, 1), B(4, 7, 8), C(6, 4, 12) and D(3, −1, 5) are the vertices of a parallelogram.

15.10 The vector equation of a line

Given any two points A and B, with position vectors **a** and **b**, the position vector of any point R on $\overline{AB}$ can be expressed as follows:

$$\overline{OR} = \overline{OA} + \overline{AR}$$

Let $\overline{AR} = t\overline{AB}$, where $t \in \mathbb{R}$, hence

$$\overline{OR} = \overline{OA} + t\overline{AB}$$
$$\therefore \mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$$
$$= (1 - t)\mathbf{a} + t\mathbf{b}$$

The letter t in this equation represents any real number and, for all values of t, **r** is the position vector of a point on $\overline{AB}$. The equation $\mathbf{r} = (1 - t)\mathbf{a} + t\mathbf{b}$ is called the **vector equation** of the line AB. The number t is called the **parameter**. For any value of the parameter, R is a point on AB.

Example 12 *Find the equation of the line through the points A(1, 2, 3) and B(4, 4, 4) and find the coordinates of the point where this line meets the plane* $z = 0$.

$$\overline{AB} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

Let R be any point on AB, so that

$$\overline{OR} = \overline{OA} + t\overline{AB}, \quad \text{where } t \in \mathbb{R}$$
$$\therefore \mathbf{r} = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + t(3\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$
$$= (1 + 3t)\mathbf{i} + (2 + 2t)\mathbf{j} + (3 + t)\mathbf{k}$$

This is the equation of the line.

The line meets the plane $z = 0$, where $(3 + t) = 0$. Thus the parameter at this point is $t = -3$. Substituting this in the equation of the line, we have

$$\mathbf{r} = -8\mathbf{i} - 4\mathbf{j} + 0\mathbf{k}$$

so the line meets the plane at the point (−8, −4, 0).

Any vector equation of the form $\mathbf{r} = \mathbf{a} + t\mathbf{u}$, where **a** and **u** are given vectors, represents the equation of a line passing through the point whose position vector is **a**. The direction of the line is parallel to the vector **u**.

If $\mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$ then

$$\mathbf{r} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t\begin{pmatrix} l \\ m \\ n \end{pmatrix}$$

If the point R has coordinates (x, y, z), then $\mathbf{r}$ can be written $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and hence the last equation becomes

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 + tl \\ y_1 + tm \\ z_1 + tn \end{pmatrix}$$

Thus the coordinates of R are $x = x_1 + tl, y = y_1 + tm, z = z_1 + tn$. These three equations are frequently arranged in the form

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = t$$

Example 13 *Given the equation of the line in the form*

$$\frac{x-2}{3} = \frac{y-4}{5} = \frac{z-7}{2}$$

express the equation in the form $\mathbf{r} = \mathbf{a} + t\mathbf{u}$ *and show that the line passes through the point* (8, 14, 11).

Let $\dfrac{x-2}{3} = \dfrac{y-4}{5} = \dfrac{z-7}{2} = t$, then

$x - 2 = 3t \qquad y - 4 = 5t \qquad z - 7 = 2t$

hence

$$x = 2 + 3t$$
$$y = 4 + 5t$$
$$z = 7 + 2t$$

that is, in vector form,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 + 3t \\ 4 + 5t \\ 7 + 2t \end{pmatrix}$$

which can be written in the form $\mathbf{r} = \mathbf{a} + t\mathbf{u}$ as follows:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix} + t\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$$

Compare this with the coordinates (8, 14, 11). When $2 + 3t = 8$, $t = 2$. [Now try this value of the parameter on the y- and z-coordinates.] When $t = 2$, $4 + 5t = 14$ and $7 + 2t = 11$. Hence the line passes through the point (8, 14, 11).

Questions

Q6 Find the unit vector which is parallel to the line $\dfrac{x-1}{3} = \dfrac{y-2}{4} = \dfrac{z-7}{12}$.

Q7 Show that the equations

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + m\begin{pmatrix} 4 \\ 6 \\ -2 \end{pmatrix} \quad \text{and}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 15 \\ -3 \end{pmatrix} + n\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$$

represent the same line.

15.11 Planes

If A, B and C are three given points it is always possible to find a plane which contains all three of them. (Imagine the tips of the thumb and first two fingers of the right hand as the three given points. A flat surface, say a book, can then be placed on these three points to represent the plane passing through them.) A fourth point, P, may or may not lie in the same plane. If it does, then, as was shown in §15.8 on page 183, the vector $\overrightarrow{AP}$ can be expressed as a linear combination of $\overrightarrow{AB}$ and $\overrightarrow{AC}$, that is scalars m and n can be found so that

$$\overrightarrow{AP} = m\overrightarrow{AB} + n\overrightarrow{AC}$$

hence

$$\mathbf{p} - \mathbf{a} = m(\mathbf{b} - \mathbf{a}) + n(\mathbf{c} - \mathbf{a})$$
$$\therefore\ \mathbf{p} = (1 - m - n)\mathbf{a} + m\mathbf{b} + n\mathbf{c}$$

In other words, $\mathbf{p}$ can be expressed as a linear combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$:

$$\mathbf{p} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$$

where $\alpha + \beta + \gamma = 1$ [since $(1 - m - n) + m + n = 1$].

(It is interesting to compare this with the statement 'if R is a point on the line AB then $\mathbf{r}$ can be expressed as a linear combination of $\mathbf{a}$ and $\mathbf{b}$, in other words $\mathbf{r} = \lambda\mathbf{a} + \mu\mathbf{b}$ where $\lambda + \mu = 1$'.)

Example 14 *Given that* A, B, C *are the points* (1, 1, 1), (5, 0, 0) *and* (3, 2, 1) *respectively, find the equation which must be satisfied by the coordinates* (x, y, z) *of any point,* P, *in the plane* ABC.

As P lies in the plane ABC, we may write $\overrightarrow{AP} = m\overrightarrow{AB} + n\overrightarrow{AC}$. Then, since

$$\overrightarrow{AB} = \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and}$$

$$\overrightarrow{AP} = \begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix}, \quad \begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix} = m\begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix} + n\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Thus

$$\left.\begin{aligned} x &= 1 + 4m + 2n \\ y &= 1 - \ \ m + \ \ n \\ z &= 1 - \ \ m \end{aligned}\right\} \qquad (1)$$

Eliminating n,

$$x - 2y = -1 + 6m$$

and eliminating m,

$$x - 2y + 6z = 5$$

This is the equation of the plane ABC.

In the equations (1) in Example 14, the scalars m and n are usually called 'the parameters of the plane'. For any values of m and n the coordinates (x, y, z) resulting from these equations are the coordinates of a point in the plane ABC. In the two-dimensional world of the plane ABC, we have two degrees of freedom; we can choose a value for m and we can choose a value for n. (Compare this with the one-dimensional world of the line, in §15.10 on page 187, in which there is only one degree of freedom; that is we can choose a value for the parameter t.)

Questions

Q8 Find the equation of the plane containing the points (1, 1, 0), (0, 1, 2), (2, 3, –8).

Q9 Find the equation of the plane which passes through the point (1, 2, 3) and which is parallel to the vectors $\begin{pmatrix} 2 \\ 4 \\ -10 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ -4 \\ 2 \end{pmatrix}$.

The intersection of two planes

Two non-parallel planes will always meet in a straight line. Given the equations of two such planes, say, $3x - 5y + z = 8$ and $2x - 3y + z = 3$, then the equation of the line of intersection can be found as follows.

For any point (x, y, z) which lies in *both* planes, the values of x, y and z fit both equations simultaneously. Hence eliminating z from both equations (by subtracting the second equation from the first) we obtain

$$x - 2y = 5$$

There are infinitely many pairs of values of x and y which satisfy this equation, but if we choose a value for x then the value of y is fixed and *vice versa*. (For example, if $x = 7$ then $y = 1$.)

Let $y = t$, then x must be $5 + 2t$ and substituting these expressions for x and y into the first of the original equations, we obtain

$$\begin{aligned} 3(5 + 2t) - 5t + z &= 8 \\ 15 + 6t - 5t + z &= 8 \\ \therefore\ z &= -7 - t \end{aligned}$$

Thus

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 + 2t \\ t \\ -7 - t \end{pmatrix}, \quad \text{i.e.} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -7 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

The latter is the equation of the line. It is parallel to the vector $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and it passes through the point (5, 0, –7).

A typical point on the line can be written $(5 + 2t, t, -7 - t)$ and it can easily be verified that, for all values of t, this point lies in both of the planes. If we substitute its coordinates into the first equation, we obtain

$$\begin{aligned} 3x - 5y + z &= 3(5 + 2t) - 5t + (-7 - t) \\ &= 15 + 6t - 5t - 7 - t \\ &= 8 \end{aligned}$$

and substituting in the second equation gives

$$\begin{aligned} 2x - 3y + z &= 2(5 + 2t) - 3t + (-7 - t) \\ &= 10 + 4t - 3t - 7 - t \\ &= 3 \end{aligned}$$

Exercise 15c

1 Given the points A and B below, write down the displacement vector, $\overline{AB}$, in each case:

- **a** A(1, 0, 2), B(3, 6, 4)
- **b** A(5, 0, 4), B(3, 0, 4)
- **c** A(2, 1, 3), B(6, 4, 3)
- **d** A(5, 4, 7), B(2, 8, 1)
- **e** A(k, $2k$, $3k$), B($3k$, $2k$, k)

2 For each part of question 1, write down the position vector of the mid-point of AB.

3 For each part of question 1, write down the position vector of the point P, such that $\overline{AP} = 5\overline{AB}$.

4 Find the equation of the plane through the points (1, 2, 0), (1, 1, 1) and (0, 3, 0).

5 Find the equation of the plane through the point (1, 1, 1) parallel to the vectors $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{i} + \mathbf{j}$.

6 Find the coordinates of the point where the line

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

meets the plane $x - 2y + 3z = 26$.

7 Show that the line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ lies in the plane

$$2x + 3y - 5z = -7$$

8 Find the point of intersection of the lines

$$\mathbf{r} = (1 + m)\mathbf{i} + (2 + m)\mathbf{j} + (4 + 2m)\mathbf{k}$$

and

$$\mathbf{r} = (1 + 3n)\mathbf{i} + 5n\mathbf{j} + (3 + 7n)\mathbf{k}$$

9 Show that the lines $\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} + m\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and

$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + n\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ do not meet.

(Non-parallel lines which do not meet are called *skew* lines.)

10 Given four points A, B, C and D, the point G, whose position vector $\mathbf{g}$ is defined by $\mathbf{g} = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})$, is called the centroid of A, B, C and D. Prove that G lies on the line joining D to M, the centroid of triangle ABC. Find the ratio DG:GM.

11 Find the equation of the line of intersection of the planes

$$4x + 3y + z = 10$$
$$x + y + z = 6$$

12 Show that the three planes whose equations are

$$2x + 3y + z = 8$$
$$x + y + z = 10$$
$$3x + 5y + z = 6$$

contain a common line.

15.12 Scalar product (dot product) of two vectors

So far we have added and subtracted vectors, and vectors have been multiplied by scalars. but we have not 'multiplied' one vector by another vector. In vector work there are two kinds of 'multiplication'; in one of them, the result is a scalar quantity, so this is called **scalar multiplication**, while in the other the result is a vector quantity. The latter kind, 'vector multiplication', is covered in Chapter 38.

Definition

Given two vectors **a** *and* **b** (*see* Fig. 15.28), *whose magnitudes are* a *and* b *respectively, the* **scalar product a.b** *is* $ab\cos\theta$, *where* θ *is the angle between the vectors.*

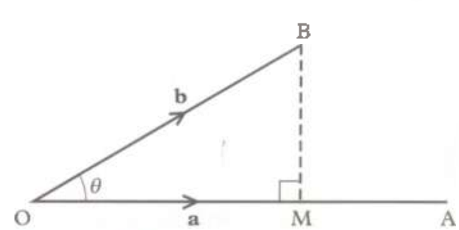


Figure 15.28

(The scalar product is always written with a very distinct dot between the **a** and the **b**. It is quite common to call this the **dot product** of **a** and **b**.)

At first sight $\mathbf{a}.\mathbf{b} = ab\cos\theta$ might seem a rather odd definition to choose, and you might reasonably ask why it should be this and not, say $ab\tan\theta$, or $ab\sin\theta$.

The definition, $ab \cos \theta$, is useful because it has many interesting mathematical properties, some of which will appear in the next few sections. Also, applied mathematicians and physicists find it a useful concept: e.g. the 'work done' when the point of application of a force $\mathbf{F}$ (a vector) undergoes a displacement $\mathbf{x}$ (a vector) is given by $\mathbf{F.x}$ (a scalar).

Notice that $\mathbf{b.a} = ba \cos \theta$, which of course is the same as $ab \cos \theta$, so the order of $\mathbf{a}$ and $\mathbf{b}$ in the scalar product does not matter, in other words scalar multiplication is *commutative*. (This may seem to be a rather trivial remark, nevertheless it is very important; in contrast, vector multiplication is not commutative.) We shall frequently require the scalar products of the base vectors $\mathbf{i}$ and $\mathbf{j}$, so the following results should be memorised (bearing in mind that $\cos 0° = 1$ and $\cos 90° = 0$):

$$\mathbf{i.i} = \mathbf{j.j} = \mathbf{k.k} = 1$$
$$\mathbf{i.j} = \mathbf{j.k} = \mathbf{k.i} = 0$$

For any vector $\mathbf{a}$, the scalar product $\mathbf{a.a}$ is equal to a^2, and for any pair of perpendicular vectors $\mathbf{a}$ and $\mathbf{b}$ the scalar product $\mathbf{a.b}$ is zero (because $\cos 90°$ is zero). Conversely if we know that the scalar product of a pair of vectors is zero, then we can deduce that the vectors are perpendicular (or one of the vectors is zero).

Example 15 *Given that* OA = 6, OB = 4 *and* ∠AOB = 60°, *calculate the value of* $\overline{\text{OA}}.\overline{\text{OB}}$.

$$\begin{aligned}\overline{\text{OA}}.\overline{\text{OB}} &= 6 \times 4 \times \cos 60° \\ &= 6 \times 4 \times 0.5 \\ &= 12\end{aligned}$$

There is an alternative form of the definition of scalar product. Note that in Fig. 15.28 $b \cos \theta = \text{OM}$; the length OM is often called the projection of $\overline{\text{OB}}$ onto $\overline{\text{OA}}$. Consequently we can say that *the scalar product*, $\mathbf{a.b}$, *is the product of* OA *and the projection of* $\overline{\text{OB}}$ *onto* $\overline{\text{OA}}$. The A and B in this statement can, of course, be interchanged.

Although $\mathbf{a.b}$ has been called a 'product' and the process has been called scalar 'multiplication', it is necessary to establish that this 'multiplication' obeys the same rules that we are familiar with, from working with real numbers.

We have already seen that the order of $\mathbf{a}$ and $\mathbf{b}$ in the scalar product does not matter, so the *commutative** law, $\mathbf{a.b} = \mathbf{b.a}$, is obeyed.

Since $\mathbf{a.b}$ is scalar, it is impossible to attach any meaning to a triple product $(\mathbf{a.b}).\mathbf{c}$; consequently there is no question of scalar products obeying the *associative** law. However $(\mathbf{a.b})\mathbf{c}$ could be taken to mean the scalar $\mathbf{a.b}$ multiplied by the vector $\mathbf{c}$, as in §15.4 on page 178, so great care is needed.

It is, however, very important that we should be able to remove brackets from $\mathbf{a.(b + c)}$ and obtain $\mathbf{a.b + a.c}$. The law

$$\mathbf{a.(b + c)} = \mathbf{a.b + a.c}$$

is called the *distributive** law and this is proved below.

Proof of the distributive law

In Fig. 15.29, $\overline{\text{OA}} = \mathbf{a}$, $\overline{\text{OB}} = \mathbf{b}$, $\overline{\text{OC}} = \mathbf{c}$ and $\overline{\text{OR}} = \mathbf{b + c}$.

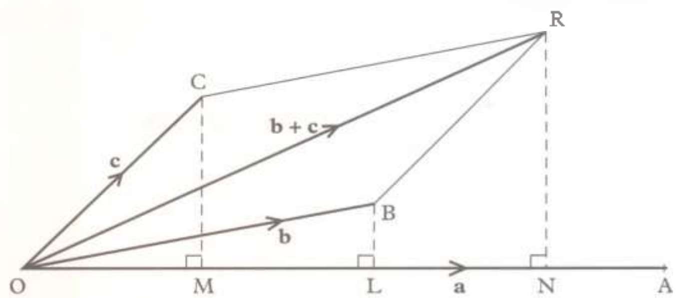


Figure 15.29

$\mathbf{a.b} = \text{OA} \times \text{OL}$ (the product of OA and the projection of $\overline{\text{OB}}$ onto $\overline{\text{OA}}$)

and similarly

$$\mathbf{a.c} = \text{OA} \times \text{OM}$$

Adding,

$$\mathbf{a.b + a.c} = \text{OA} \times (\text{OL} + \text{OM})$$

but since OC and BR are opposite sides of a parallelogram, the projection of $\overline{\text{OC}}$ onto $\overline{\text{OA}}$ is equal to the projection of $\overline{\text{BR}}$ onto $\overline{\text{OA}}$. Hence OM = LN. Thus

$$\begin{aligned}\mathbf{a.b + a.c} &= \text{OA} \times (\text{OL} + \text{LN}) = \text{OA} \times \text{ON} \\ &= \overline{\text{OA}}.\overline{\text{OR}} \\ &= \mathbf{a.(b + c)}\end{aligned}$$

With this law proved, we may now proceed to remove brackets according to the normal rules of algebra, e.g.

$$\begin{aligned}\mathbf{(a + b).(c + d)} &= \mathbf{a.(c + d) + b.(c + d)} \\ &= \mathbf{a.c + a.d + b.c + b.d}\end{aligned}$$

In particular, if we wish to form the scalar product of $\mathbf{a}$ and $\mathbf{b}$, where

*The terms commutative, associative, distributive may be new. They are explained in more detail in Chapter 25.

$$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} \quad \text{and} \quad \mathbf{b} = 5\mathbf{i} + 6\mathbf{j} + 7\mathbf{k}$$

then, bearing in mind that $\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$ and $\mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{i} = 0$, we have

$$\begin{aligned}\mathbf{a}.\mathbf{b} &= (2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}).(5\mathbf{i} + 6\mathbf{j} + 7\mathbf{k}) \\ &= 2 \times 5 + 3 \times 6 + 4 \times 7 = 56\end{aligned}$$

In general,

$$\begin{aligned}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} &= (x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}).(x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) \\ &= x_1x_2 + y_1y_2 + z_1z_2\end{aligned}$$

Example 16 *Given that* $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}$ *and* $\mathbf{b} = 8\mathbf{i} - 6\mathbf{j}$, *find* a^2, b^2 *and* $\mathbf{a}.\mathbf{b}$. *Hence find the angle between the vectors* $\mathbf{a}$ *and* $\mathbf{b}$.

$$\begin{aligned}a^2 = \mathbf{a}.\mathbf{a} &= (4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}).(4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}) \\ &= 16 + 9 + 144 \\ &= 169\end{aligned}$$

Similarly, $b^2 = 100$.

Hence $a = 13$ and $b = 10$.

$$\begin{aligned}\mathbf{a}.\mathbf{b} &= (4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}).(8\mathbf{i} - 6\mathbf{j}) \\ &= 32 - 18 \\ &= 14\end{aligned}$$

However, by definition, $\mathbf{a}.\mathbf{b} = ab \cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, consequently

$$14 = 13 \times 10 \cos\theta$$

$$\therefore \cos\theta = \frac{14}{130}$$

$$\therefore \theta = 83.8°$$

The angle between $\mathbf{a}$ and $\mathbf{b}$ is 83.8°, correct to the nearest tenth of a degree.

Example 17 *Prove that* $\mathbf{p} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ *is perpendicular to* $\mathbf{q} = 5\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$.

$$\begin{aligned}\mathbf{p}.\mathbf{q} &= (2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}).(5\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) \\ &= 10 + 6 - 16 \\ &= 0\end{aligned}$$

Since neither $\mathbf{p}$ nor $\mathbf{q}$ is zero, we can deduce that

$$\cos\theta = 0$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{q}$, so $\theta = 90°$. Hence $\mathbf{p}$ is perpendicular to $\mathbf{q}$.

Questions

Q10 Given that $a = 10$ and $b = 15$ and that the angle between $\mathbf{a}$ and $\mathbf{b}$ is 120°, calculate the value of $\mathbf{a}.\mathbf{b}$.

Q11 Write down the condition for the vectors

$$\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k} \quad \text{and}$$
$$\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$$

to be perpendicular.

Q12 Find the angle between the vectors

$$\mathbf{p} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \quad \text{and}$$
$$\mathbf{q} = 2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$$

***Q13** The unit vector $\mathbf{u}$ makes angles α, β and γ with the x-, y- and z-axes respectively. By considering $\mathbf{u}.\mathbf{i}$, or otherwise, show that

$$\mathbf{u} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$$

and prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

($\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the **direction-cosines** of $\mathbf{u}$.)

Q14 Find the direction-cosines of the unit vector parallel to $3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$ and calculate the angles this vector makes with the axes.

It is frequently convenient to have a symbol for the *unit* vector in the direction of a given vector $\mathbf{r}$. The normal symbol for this is $\hat{\mathbf{r}}$. So if we use r to represent the magnitude of $\mathbf{r}$, the unit vector $\hat{\mathbf{r}}$ is given by

$$\hat{\mathbf{r}} = \frac{1}{r}\mathbf{r}$$

e.g. if we are given that $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j}$, then $r = 5$ and

$$\hat{\mathbf{r}} = \frac{1}{5}\mathbf{r} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

Exercise 15d

1 a Find the scalar product of

$$\mathbf{a} = \begin{pmatrix} 7.2 \\ 9.6 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 12 \\ -9 \end{pmatrix},$$

b find the magnitudes of **a** and **b**,

c calculate the angle between **a** and **b**.

2 Given that P, Q and R are the points (8, 10), (6, 20) and (16, 16) respectively, calculate the value of the scalar product **PQ.PR**. Hence calculate the size of the angle QPR.

3 The points A, B and C have coordinates (4, −1, 5), (8, 0, 6) and (5, −3, 3) respectively. Prove that the angle BAC is a right angle.

4 Given the vectors **a** and **b**, where

$$\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k} \quad \text{and} \quad \mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$$

prove that the vector

$$\mathbf{c} = (y_1z_2 - y_2z_1)\mathbf{i} + (z_1x_2 - z_2x_1)\mathbf{j} + (x_1y_2 - x_2y_1)\mathbf{k}$$

is perpendicular to both **a** and **b**.

5 Points P, Q and R have position vectors **p**, **q** and **r**. If $\mathbf{p} = (1 - \alpha)\mathbf{q} + \alpha\mathbf{r}$, for some number α, describe the position of P relative to Q and R.

OABC are four non-coplanar points in space. A, B, C have position vectors **a**, **b**, **c** relative to O. The position vector of V is 2**a** − **c**, and of W is −2**a** + 3**b**. If VW meets the plane OBC in U, find the position vector of U and show that U is on BC.

Use scalar products to show that if V is in the plane through O perpendicular to OB, and W is in the plane through O perpendicular to OC, then U is in the plane through O perpendicular to OA.

Chapter 16

The general angle and Pythagoras' theorem

16.1 The general angle

Consider a bicycle wheel which is free to rotate about a fixed axis, and suppose that one spoke has been painted. If the wheel starts from rest and makes one revolution, the painted spoke turns through 360°, and if the wheel makes another revolution the spoke turns through 360° again. Thus we may say that the wheel has turned through a total of 720°, and by using angles greater than 360° the number of revolutions may be specified, as well as the position of the marked spoke.

On the x-axis of a graph the positive direction is usually taken to the right and the negative direction to the left. Similarly, if a wheel rotates anti-clockwise, we take that to be positive, and a clockwise rotation to be negative. Angles measured from the x-axis in an anti-clockwise sense are positive, and those measured in a clockwise sense are negative (see **Fig. 16.1**).

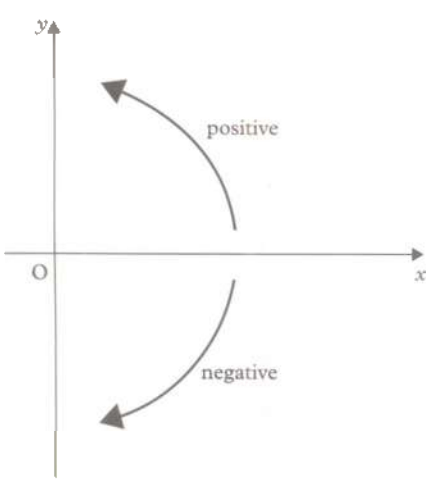


Figure 16.1

Trigonometrical ratios of angles of any magnitude are required in connection with oscillating bodies and rotation about an axis, and in physics they arise in connection with such topics as alternating currents. As you may have had the ratios defined for only a limited range of angles, we will now develop a general definition.

The axes divide the plane into four quadrants, and, as angles are measured in an anti-clockwise direction from the x-axis, the quadrants are numbered as in **Fig. 16.2**. For the present, a point $P(x, y)$ and its coordinates will be given a suffix corresponding to the quadrant it lies in.

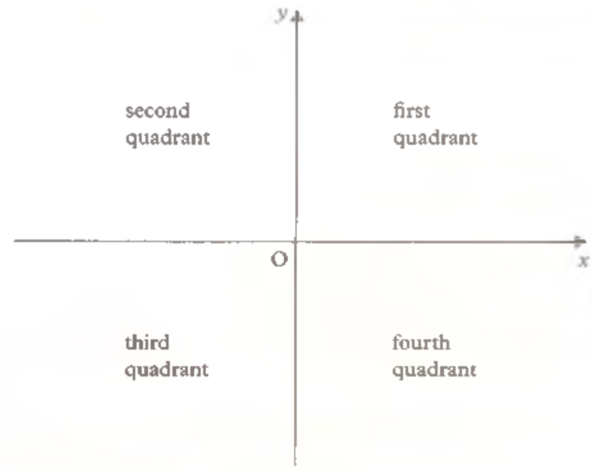


Figure 16.2

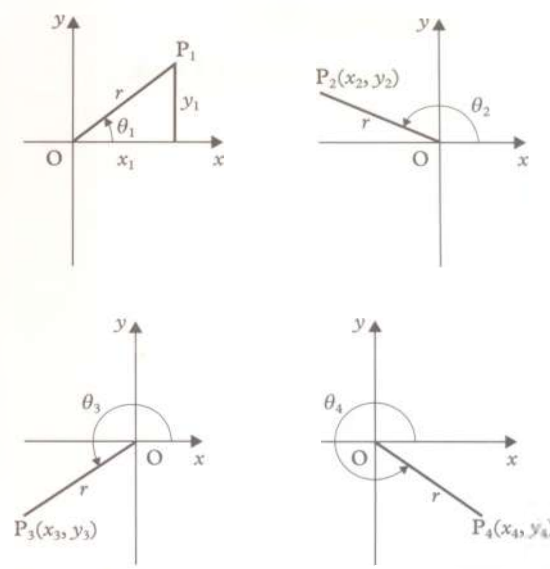


Figure 16.3

For an acute angle θ_1 (see **Fig. 16.3**),

$$\sin\theta_1 = \frac{y_1}{r}, \qquad \cos\theta_1 = \frac{x_1}{r}, \qquad \tan\theta_1 = \frac{y_1}{x_1}$$

In each case, r is the length of the vector **OP**, and, as in Chapter 15, it is always taken to be positive. Now

$$\frac{\sin\theta_1}{\cos\theta_1} = \frac{y_1/r}{x_1/r} = \frac{y_1}{x_1} = \tan\theta_1$$

so for an angle θ of *any* magnitude we define the six trigonometrical ratios as follows:

$$\sin\theta = \frac{y}{r}, \qquad \cos\theta = \frac{x}{r}, \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}$$

For an angle θ_2 in the second quadrant (see **Fig. 16.3**), y_2 is positive (abbreviated +ve) but x_2 is negative (abbreviated −ve), therefore

$\sin\theta_2$ is +ve, $\cos\theta_2$ is −ve, $\tan\theta_2$ is −ve

For θ_3 in the third quadrant, x_3 and y_3 are both negative, hence

$\sin\theta_3$ is −ve, $\cos\theta_3$ is −ve, $\tan\theta_3$ is +ve

For θ_4 in the fourth quadrant, x_4 is positive, and y_4 is negative, hence

$\sin\theta_4$ is −ve, $\cos\theta_4$ is +ve, $\tan\theta_4$ is −ve

These results can be summarised by writing which ratios are positive in each quadrant:

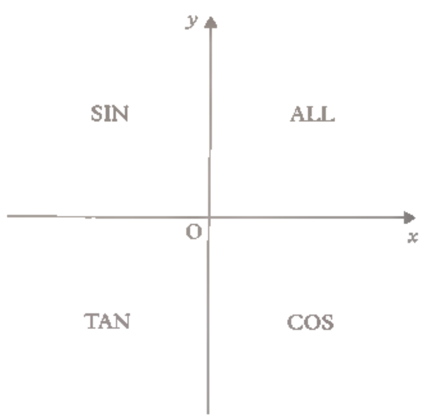


Figure 16.4

The signs of the ratios can be worked out as above quite easily. However, there are also simple ways of remembering the order of the first letters in the four quadrants: e.g., starting at ALL(**A**) and going clockwise gives **ACTS**. The signs of cosec θ, sec θ, cot θ are the same as their reciprocals.

A useful point to note is that angles for which OP is equally inclined to the positive or negative x-axis have trigonometrical ratios of the same magnitude, their signs being determined as above. Thus the ratios of 150°, 210°, 330° are *numerically* the same as the ratios of 30°, since in each case the acute angle between OP and the x-axis is 30°, as shown below.

$\sin 150° = +\sin 30°$ $\quad$ $\sin 210° = -\sin 30°$
$\cos 150° = -\cos 30°$ $\quad$ $\cos 210° = -\cos 30°$
$\tan 150° = -\tan 30°$ $\quad$ $\tan 210° = +\tan 30°$

$\sin 330° = -\sin 30°$
$\cos 330° = +\cos 30°$
$\tan 330° = -\tan 30°$

Question

Q1 Express in terms of the trigonometrical ratios of acute angles:

a	sin 170°	**b**	tan 300°
c	cos 200°	**d**	sin (−50°)
e	cos (−20°)	**f**	sin 325°
g	tan (−140°)	**h**	cos 164°
i	cosec 230°	**j**	tan 143°
k	cos (−130°)	**l**	sin 250°
m	tan (−50°)	**n**	cot 200°
o	cos 293°	**p**	sin (−230°)
q	sec 142°	**r**	cot 156°
s	cosec (−53°)	**t**	sec (−172°)

16.2 Graphs of sin θ, cos θ, tan θ

It is instructive to draw the graphs of sin θ, cos θ, and tan θ. **Fig. 16.5** shows how the graph of sin θ may be drawn from the definition. Construct a circle of unit radius, then $\sin\theta = y$. Dotted lines show this for $\theta = 30°$, 60°, 90°, and the rest of the figure is drawn similarly.

You can see that the graph of sin θ repeats itself at intervals of 360°. (This should be clear from the way it was drawn, because points on the graph separated by 360° correspond to the same point on the circle.) If a function repeats itself at regular intervals, like sin θ, it is called a **periodic** function, and the interval is called its **period** (see *Periodic Functions* in §2.12 on page 54).

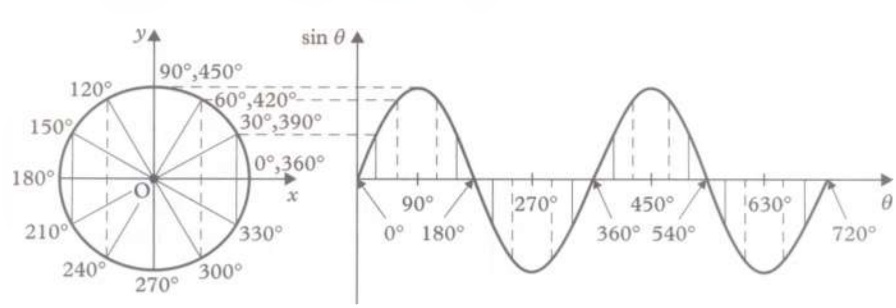


Figure 16.5

The graph of $\cos\theta$ is drawn in a similar way to that of $\sin\theta$. In this case, since $\cos\theta = x/r$, the values of x are used instead of y.

The graph of $\tan\theta$ may also be drawn from a unit circle, but in this case a tangent is drawn at the point (1, 0) (see Fig. 16.6). If P is any point on the circle, and OP meets the tangent at Q, then the y-coordinate of Q is equal to $\tan\theta$.

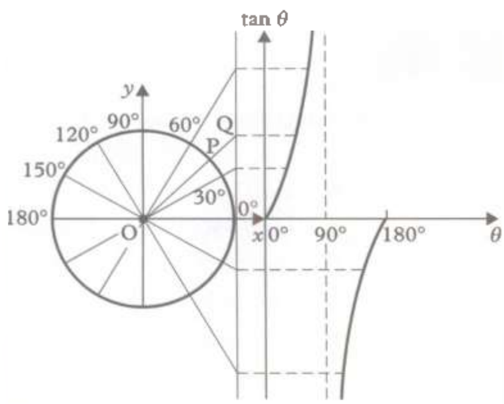


Figure 16.6

Questions

Q2 Complete the graph of $\tan\theta$ up to $\theta = 720°$.

Q3 What are the periods of $\cos\theta$ and $\tan\theta$?

16.3 Trigonometrical ratios of 30°, 45°, 60°

The trigonometrical ratios of 30°, 45°, and 60° are frequently used and are obtained from two figures. Fig. 16.7 represents an equilateral triangle with an altitude constructed. The sides of the triangle are 2 units, and so, by Pythagoras' theorem, the altitude is $\sqrt{3}$ units. The ratios of 30° and 60° may now be read off. Fig. 16.8 represents a right-angled isosceles triangle with two sides of unit length. By Pythagoras' theorem the hypotenuse is $\sqrt{2}$ units, and so the ratios of 45° may be read off.

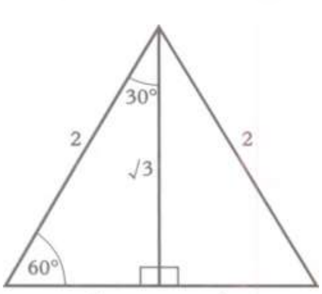


Figure 16.7

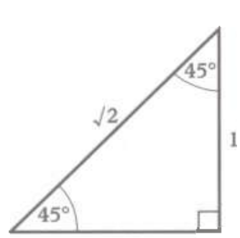


Figure 16.8

Question

Q4 Write down the values of **a** $\sin 30°$, **b** $\cos 30°$, **c** $\cos 45°$, **d** $\tan 30°$, **e** $\sec 60°$, **f** $\operatorname{cosec} 60°$, **g** $\tan 45°$, **h** $\operatorname{cosec} 45°$.

16.4 Trigonometrical equations

Most equations in algebra have a finite number of roots, but in many cases trigonometrical equations have an unlimited number. For instance, the equation $\sin\theta = 0$ is satisfied by $\theta = 0°, \pm 180°, \pm 360°, \pm 540°$ and so on, indefinitely. In this book it will be specified for what range of values the roots are required.

Example 1 *Solve the equation $\sin\theta = -\frac{1}{2}$ for values of θ from 0° to 360° inclusive.*

The acute angle whose sine is $\frac{1}{2}$ is 30° and Fig. 16.9 indicates the angles between 0° and 360° whose sines are $\pm\frac{1}{2}$. But $\sin\theta$ is negative only in the third and fourth quadrants. Therefore the roots of the equation in the required range are 210° and 330°.

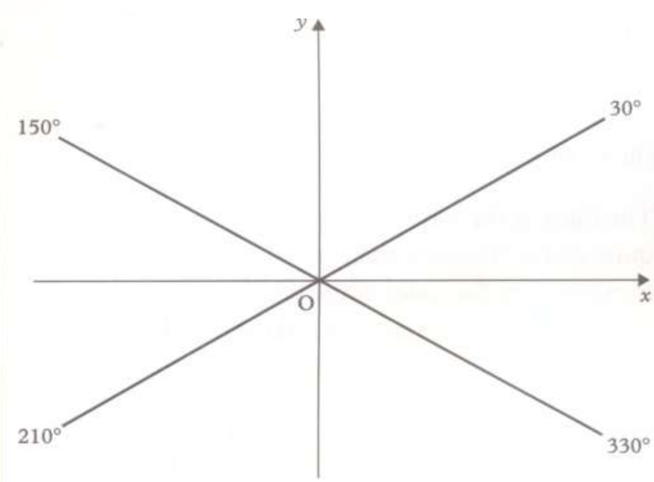


Figure 16.9

Example 2 *Solve the equation $\cos 2\theta = 0.6428$, for values of θ between −180° and +180°.*

[Note that since θ must lie between −180° and +180°, 2θ may lie between −360° and +360°.]

From a calculator or tables it can be seen that the acute angle whose cosine is 0.6428 is 50° (see note on accuracy on page xi), and since $\cos 2\theta$ is positive only in the first and fourth quadrants

$$2\theta = -310°, \quad -50°, \quad 50°, \quad 310°$$
$$\therefore\ \theta = -155°, \quad -25°, \quad 25°, \quad 155°$$

Example 3 *Solve the equation** $2\sin^2\theta = \sin\theta$, *for values of* θ *from* $0°$ *to* $360°$ *inclusive.*

[This equation is a quadratic equation for $\sin\theta$, and may be solved by factorisation.]

$$2\sin^2\theta - \sin\theta = 0$$
$$\therefore \sin\theta\,(2\sin\theta - 1) = 0$$
$$\therefore \sin\theta = 0 \quad \text{or} \quad \sin\theta = \tfrac{1}{2}$$

If $\sin\theta = 0$, $\theta = 0°, 180°, 360°$. If $\sin\theta = \frac{1}{2}$, $\theta = 30°, 150°$.

Therefore the roots of the equation, from $0°$ to $360°$ inclusive are $0°, 30°, 150°, 180°$, and $360°$.

(Note that if we had divided both sides of the equation by $\sin\theta$, giving $2\sin\theta = 1$, we should have lost some of the roots, namely those for which $\sin\theta = 0$.)

Example 4 *Solve the equation* $\tan\theta = 2\sin\theta$, *for values of* θ *from* $0°$ *to* $360°$ *inclusive.*

[Equations are often solved by factorisation, so look for a common factor.]

Remembering that $\tan\theta = \sin\theta/\cos\theta$ we may write

$$\frac{\sin\theta}{\cos\theta} = 2\sin\theta$$
$$\therefore 2\sin\theta\cos\theta = \sin\theta$$
$$\therefore 2\sin\theta\cos\theta - \sin\theta = 0$$
$$\therefore \sin\theta\,(2\cos\theta - 1) = 0$$
$$\therefore \sin\theta = 0 \quad \text{or} \quad \cos\theta = \tfrac{1}{2}$$

If $\sin\theta = 0$, $\theta = 0°, 180°, 360°$. If $\cos\theta = \frac{1}{2}$, $\theta = 60°, 300°$.

Therefore the required values of θ are $0°, 60°, 180°, 300°$, and $360°$.

Exercise 16a

1 Write down the values of the following, leaving surds in your answers (calculators should not be used in this question):

a	$\cos 270°$	b	$\sin 540°$	c	$\cos(-180°)$
d	$\tan 135°$	e	$\sin 150°$	f	$\cos 210°$
g	$\tan 120°$	h	$\cos(-30°)$	i	$\sin(-120°)$
j	$\sin 405°$	k	$\cos(-135°)$	l	$\sin 225°$
m	$\tan(-60°)$	n	$\sin(-270°)$	o	$\tan 210°$

2 Sketch the graph of $\sin\theta$, for values of θ from $-360°$ to $360°$.

*To avoid brackets $(\sin\theta)^2$ is written $\sin^2\theta$.

3 Sketch the graph of $\cos\theta$, for values of θ from $0°$ to $720°$, and state its period.

4 Draw the graph of $\tan\theta$, for values of θ from $0°$ to $720°$. (This has been started in Fig. 16.6.) What is the period of $\tan\theta$?

5 Sketch the graphs of **a** $\cos 2\theta$, **b** $\sin\frac{1}{2}\theta$, **c** $\sin\frac{3}{2}\theta$, **d** $\cos(\theta + 60°)$, **e** $\sin(\theta - 45°)$, for values of θ from $0°$ to $360°$, stating the period of each.

6 Find the values of θ from $180°$ to $360°$, inclusive, which satisfy the following equations:

a	$\cos\theta = -\frac{1}{2}$	b	$\tan\theta = 1$
c	$\operatorname{cosec}\theta = 2$	d	$\sin\theta = -0.7660$
e	$\cos\theta = 0.6$	f	$\tan\theta = -\sqrt{3}$
g	$\cos(\theta + 60°) = 0.5$	h	$\sin(\theta - 30°) = -\sqrt{3}/2$

7 Solve the following equations for values of θ from $0°$ to $360°$, inclusive:

a	$\sin^2\theta = \frac{1}{4}$	b	$\tan^2\theta = \frac{1}{3}$
c	$\sin 2\theta = \frac{1}{2}$	d	$\tan 2\theta = -1$
e	$\cos 3\theta = \sqrt{3}/2$	f	$\sin 3\theta = -1$
g	$\sin^2 2\theta = 1$	h	$\sec 2\theta = 3$
i	$\tan^2 3\theta = 1$	j	$4\cos 2\theta = 1$
k	$\sin(2\theta + 30°) = 0.8$	l	$\tan(3\theta - 45°) = \frac{1}{2}$

8 Solve the following equations for values of θ from $-180°$ to $+180°$, inclusive:

a $\tan^2\theta + \tan\theta = 0$
b $2\cos^2\theta = \cos\theta$
c $3\sin^2\theta + \sin\theta = 0$
d $2\sin^2\theta - \sin\theta - 1 = 0$
e $2\cos^2\theta + 3\cos\theta + 1 = 0$
f $4\cos^3\theta = \cos\theta$
g $\tan\theta = \sin\theta$
h $\sec\theta = 2\cos\theta$
i $\cot\theta = 5\cos\theta$
j $4\sin^2\theta = 3\cos^2\theta$
k $3\cos\theta = 2\cot\theta$
l $\tan\theta = 4\cot\theta + 3$
m $5\sin\theta + 6\operatorname{cosec}\theta = 17$
n $3\cos\theta + 2\sec\theta + 7 = 0$

9 Write down the maximum and minimum values of the following expressions, giving the smallest positive or zero value of θ for which they occur:

a	$\sin\theta$	b	$3\cos\theta$
c	$2\cos\frac{1}{2}\theta$	d	$-\frac{1}{2}\sin 2\theta$
e	$1 - 2\sin\theta$	f	$3 + 2\cos 3\theta$
g	$\dfrac{1}{2+\sin\theta}$	h	$\dfrac{1}{4-3\cos\theta}$

i $\sec \frac{3}{2}\theta$ j $\tan^2 \theta$

k $\dfrac{1}{1 + \text{cosec } \theta}$ l $\dfrac{2}{3 - 2 \cot \theta}$

m $\dfrac{\cos \theta}{\cos \theta + \sin \theta}$

10 State, with reasons, which of the following equations have no roots:

a $2 \sin \theta = 3$ b $\sin \theta + \cos \theta = 0$

c $\sin \theta + \cos \theta = 2$ d $3 \sin \theta + \text{cosec } \theta = 0$

e $4 \text{ cosec}^2 \theta - 1 = 0$ f $\text{cosec } \theta = \sin \theta$

g $\sec \theta = \sin \theta$

11 Sketch on the same axes, for values of θ from $-360°$ to $360°$, the graphs of **a** $\sin \theta$, cosec θ; **b** $\cos \theta$, $\sec \theta$; **c** $\tan \theta$, $\cot \theta$.

12 Sketch the graphs of the following functions and state the period in each case:

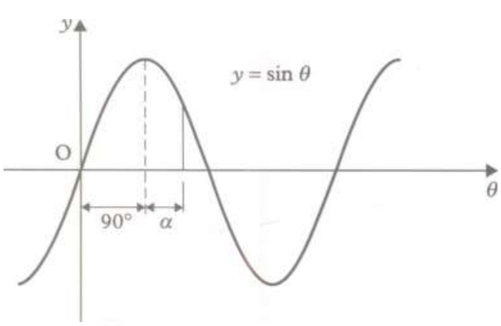
a $y = \sin 2x$ b $y = \cos (x/3)$

c $y = \tan 3x$ d $y = \tan (x/2)$

e $y = \sin (2x/3)$

16.5 Trigonometrical ratios of $-\theta$, $180° \pm \theta$, $90° \pm \theta$

If you have drawn the graphs of $y = \sin \theta$ and $y = \cos \theta$ you may have noticed that they are the same, except for the positions of the y-axes relative to the curves.



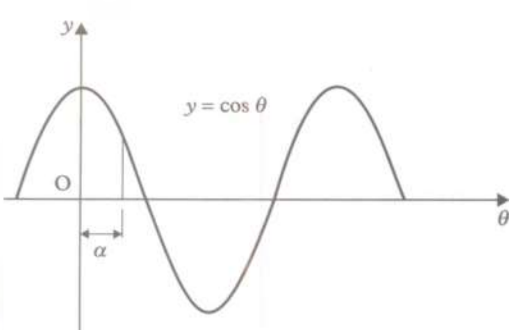


Figure 16.10

Fig. 16.10 suggests that, for any angle α,

$$\cos \alpha = \sin (90° + \alpha)$$

and other relationships like this may be found from the graphs. Some people find the graphs help them to remember such relationships, but now it will be shown how they may be obtained from first principles.

For any value of θ, in the notation of §16.1 on page 194 we have by definition

$$\sin \theta = \frac{y}{r}, \qquad \cos \theta = \frac{x}{r}$$

Consider:

a ratios of $-\theta$. In Fig. 16.3, page 194, the angle $-\theta$ is obtained by replacing (x, y) by $(x, -y)$,

$$\therefore \sin (-\theta) = -\frac{y}{r} = -\sin \theta$$

$$\cos (-\theta) = \frac{x}{r} = \cos \theta$$

i.e. $\sin \theta$ is an odd function and $\cos \theta$ is an even function (see §2.12 on page 53).

b ratios of $(180° - \theta)$. Replace (x, y) by $(-x, y)$, hence

$$\sin (180° - \theta) = \frac{y}{r} = \sin \theta$$

$$\cos (180° - \theta) = -\frac{x}{r} = -\cos \theta$$

c ratios of $(180° + \theta)$. Replace (x, y) by $(-x, -y)$, hence

$$\sin (180° + \theta) = -\frac{y}{r} = -\sin \theta$$

$$\cos (180° + \theta) = -\frac{x}{r} = -\cos \theta$$

[Note that in all the above cases, OP is inclined at an angle θ to the positive or negative x-axis, the ratios of these angles have the same magnitude as those of θ, and their signs are determined as on page 000 if θ is acute.]

d ratios of $(90° - \theta)$. Replace (x, y) by (y, x), hence

$$\sin (90° - \theta) = \frac{x}{r} = \cos \theta$$

$$\cos (90° - \theta) = \frac{y}{r} = \sin \theta$$

e ratios of $(90° + \theta)$. Replace (x, y) by $(-y, x)$, hence

$$\sin (90° + \theta) = \frac{x}{r} = \cos \theta$$

$$\cos (90° + \theta) = -\frac{y}{r} = -\sin \theta$$

Question

Q5 Express the following in terms of the trigonometrical ratios of θ:

a	$\tan(90° - \theta)$	**b**	$\operatorname{cosec}(180° - \theta)$
c	$\sec(90° + \theta)$	**d**	$\cot(90° + \theta)$
e	$\sec(-\theta)$	**f**	$\operatorname{cosec}(180° + \theta)$
g	$\cos(270° - \theta)$	**h**	$\sin(360° + \theta)$
i	$\tan(-\theta)$	**j**	$\sin(\theta - 90°)$
k	$\cos(\theta - 180°)$	**l**	$\sec(270° + \theta)$

16.6 Pythagoras' theorem

You will be familiar with Pythagoras' theorem. In trigonometry it retains its importance and leads to relationships between trigonometrical ratios.

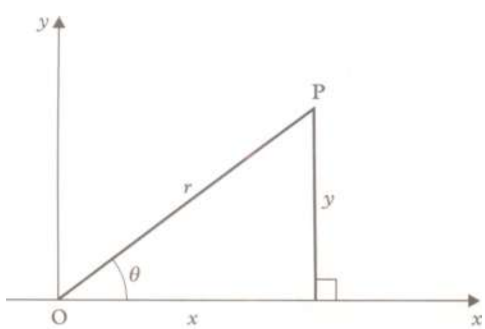


Figure 16.11

In Fig. 16.11, the triangle is right-angled and so, by Pythagoras' theorem,

$$x^2 + y^2 = r^2$$

But $\cos\theta = x/r$ and $\sin\theta = y/r$, so we divide by r^2 obtaining

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$\therefore \boldsymbol{\cos^2\theta + \sin^2\theta \equiv 1}$$

(If P is not in the first quadrant, OP^2 is still $x^2 + y^2$ by the distance formula of §1.8 on page 41 and the proof continues as before.)

The $\equiv$ symbol is used to stress that the relationship is an identity, i.e. it holds for *all* values of θ.

Two similar identities can be deduced from this. Dividing through by $\cos^2\theta$,

$$1 + \frac{\sin^2\theta}{\cos^2\theta} \equiv \frac{1}{\cos^2\theta}$$

but $\tan\theta = \sin\theta/\cos\theta$ and $\sec\theta = 1/\cos\theta$, therefore

$$\boldsymbol{1 + \tan^2\theta \equiv \sec^2\theta}$$

Dividing the original identity by $\sin^2\theta$,

$$\frac{\cos^2\theta}{\sin^2\theta} + 1 \equiv \frac{1}{\sin^2\theta}$$

but $\cos\theta/\sin\theta = \cot\theta$ and $1/\sin\theta = \operatorname{cosec}\theta$, therefore

$$\boldsymbol{\cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta}$$

Example 5 *Solve the equation* $1 + \cos\theta = 2\sin^2\theta$, *for values of* θ *between* $0°$ *and* $360°$.

[The square on the right-hand side indicates that the equation is a quadratic, and to solve it, we must write it in terms of either $\cos\theta$ or $\sin\theta$.]

We know that

$$\cos^2\theta + \sin^2\theta = 1$$

hence $\sin^2\theta = 1 - \cos^2\theta$

so substituting $2 - 2\cos^2\theta$ for $2\sin^2\theta$, we obtain

$$1 + \cos\theta = 2 - 2\cos^2\theta$$

This quadratic for $\cos\theta$ is solved by factorisation:

$$2\cos^2\theta + \cos\theta - 1 = 0$$

$$\therefore (2\cos\theta - 1)(\cos\theta + 1) = 0$$

$$\therefore \cos\theta = \tfrac{1}{2} \quad \text{or} \quad -1$$

If $\cos\theta = \frac{1}{2}$, $\theta = 60°, 300°$. If $\cos\theta = -1$, $\theta = 180°$.

Therefore the roots of the equation between $0°$ and $360°$ are $60°$, $180°$, and $300°$.

Example 6 *Simplify* $1/\sqrt{(x^2 - a^2)}$ *when* $x = a\operatorname{cosec}\theta$.

Substituting $x = a\operatorname{cosec}\theta$, we obtain

$$\frac{1}{\sqrt{(a^2\operatorname{cosec}^2\theta - a^2)}}$$

The $\operatorname{cosec}^2\theta$ in the denominator suggests the use of the identity

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

With this the expression $(a^2\operatorname{cosec}^2\theta - a^2)$ may be simplified, giving

$$a^2\operatorname{cosec}^2\theta - a^2 = a^2(\cot^2\theta + 1) - a^2 = a^2\cot^2\theta$$

Thus the original expression becomes

$$\frac{1}{\sqrt{(a^2\cot^2\theta)}} = \frac{1}{a\cot\theta} = \frac{1}{a}\tan\theta$$

Example 7 *Eliminate θ from the equations $x = a \sin\theta$, $y = b \tan\theta$.*

[Since $\sin\theta$ and $\tan\theta$ are the reciprocals of $\operatorname{cosec}\theta$ and $\cot\theta$ we use the identity $\operatorname{cosec}^2\theta = \cot^2\theta + 1$.]

$$\operatorname{cosec}\theta = \frac{a}{x} \quad \text{and} \quad \cot\theta = \frac{b}{y}$$

Substituting into the identity $\operatorname{cosec}^2\theta = \cot^2\theta + 1$,

$$\frac{a^2}{x^2} = \frac{b^2}{y^2} + 1$$

Exercise 16b

1 If $s = \sin\theta$, simplify:

a $\sqrt{(1-s^2)}$ **b** $\dfrac{s}{\sqrt{(1-s^2)}}$ **c** $\dfrac{1-s^2}{s}$

2 If $c = \cos\theta$, simplify:

a $\sqrt{(1-c^2)}$ **b** $\dfrac{\sqrt{(1-c^2)}}{c}$ **c** $\dfrac{c}{1-c^2}$

3 If $t = \tan\theta$, simplify:

a $\sqrt{(1+t^2)}$ **b** $t(1+t^2)$ **c** $\dfrac{t}{\sqrt{(1+t^2)}}$

4 If $c = \operatorname{cosec}\theta$, simplify:

a $\sqrt{(c^2-1)}$ **b** $\dfrac{\sqrt{(c^2-1)}}{c}$ **c** $\dfrac{c}{c^2-1}$

5 If $x = a\sin\theta$, simplify:

a $a^2 - x^2$ **b** $\dfrac{1}{\sqrt{(a^2-x^2)}}$ **c** $\dfrac{a^2-x^2}{x}$

6 If $y = b\cot\theta$, simplify:

a $b^2 + y^2$ **b** $y\sqrt{(b^2+y^2)}$ **c** $\dfrac{y}{b^2+y^2}$

7 If $z = a\sec\theta$, simplify:

a $z^2 - a^2$ **b** $\dfrac{1}{\sqrt{(z^2-a^2)}}$ **c** $\dfrac{\sqrt{(z^2-a^2)}}{z}$

In questions 8–13, solve the equations, giving values of θ from 0° to 360° inclusive.

8 $3 - 3\cos\theta = 2\sin^2\theta$ **9** $\cos^2\theta + \sin\theta + 1 = 0$

10 $\sec^2\theta = 3\tan\theta - 1$ **11** $\operatorname{cosec}^2\theta = 3 + \cot\theta$

12 $3\tan^2\theta + 5 = 7\sec\theta$

13 $2\cot^2\theta + 8 = 7\operatorname{cosec}\theta$

14 If $\sin\theta = \frac{3}{5}$, find without using tables or calculators, the values of **a** $\cos\theta$, **b** $\tan\theta$.

15 If $\cos\theta = -\frac{8}{17}$, and θ is obtuse, find without using tables or calculators, the values of **a** $\sin\theta$, **b** $\cot\theta$.

16 If $\tan\theta = \frac{7}{24}$ and θ is reflex, find without using tables or calculators, the values of **a** $\sec\theta$, **b** $\sin\theta$.

Prove the following identities:

17 $\tan\theta + \cot\theta = 1/(\sin\theta\cos\theta)$

18 $\operatorname{cosec}\theta + \tan\theta\sec\theta = \operatorname{cosec}\theta\sec^2\theta$

19 $\sec^2\theta - \operatorname{cosec}^2\theta = \tan^2\theta - \cot^2\theta$

20 $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$

21 $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$

22 $2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$

23 $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta\operatorname{cosec}^2\theta$

24 $\sec^4\theta - \operatorname{cosec}^4\theta = \dfrac{\sin^2\theta - \cos^2\theta}{\cos^4\theta\sin^4\theta}$

25 $\dfrac{1}{\tan^2\theta + 1} + \dfrac{1}{\cot^2\theta + 1} = 1$

26 $(\sec^2\theta - 1)(\operatorname{cosec}^2\theta - 1) = 1$

27 $\sqrt{(\sec^2\theta - 1)} + \sqrt{(\operatorname{cosec}^2\theta - 1)} = \sec\theta\operatorname{cosec}\theta$

28 $\sqrt{(\sec^2\theta - \tan^2\theta)} + \sqrt{(\operatorname{cosec}^2\theta - \cot^2\theta)} = 2$

29 $\dfrac{1-\cos^2\theta}{\sec^2\theta - 1} = 1 - \sin^2\theta$

30 $\dfrac{\sec\theta - \operatorname{cosec}\theta}{\tan\theta - \cot\theta} = \dfrac{\tan\theta + \cot\theta}{\sec\theta + \operatorname{cosec}\theta}$

31 $\dfrac{\cos\theta}{\sqrt{(1+\tan^2\theta)}} + \dfrac{\sin\theta}{\sqrt{(1+\cot^2\theta)}} = 1$

Eliminate θ from the following equations:

32 $x = a\cos\theta, y = b\sin\theta$

33 $x = a\cot\theta, y = b\operatorname{cosec}\theta$

34 $x = a\tan\theta, y = b\cos\theta$

35 $x = 1 - \sin\theta, y = 1 + \cos\theta$

36 $x = a\sec\theta, y = b + c\cos\theta$

37 $x = a\operatorname{cosec}\theta, y = b\sec\theta$

38 $x = 1 + \tan\theta, y = \cos\theta$

39 $x = \sin\theta + \cos\theta, y = \sin\theta - \cos\theta$

40 $x = \sec\theta + \tan\theta, y = \sec\theta - \tan\theta$

Chapter 17

Trigonometrical identities

17.1 Compound angle formulae

Formulae for sin $(A \pm B)$, cos $(A \pm B)$

Place a rectangular piece of cardboard PQRS in a vertical plane with two edges horizontal, and then turn it through an angle B (see Fig. 17.1). Take the diagonal PR as the unit of length and let angle RPQ be A.

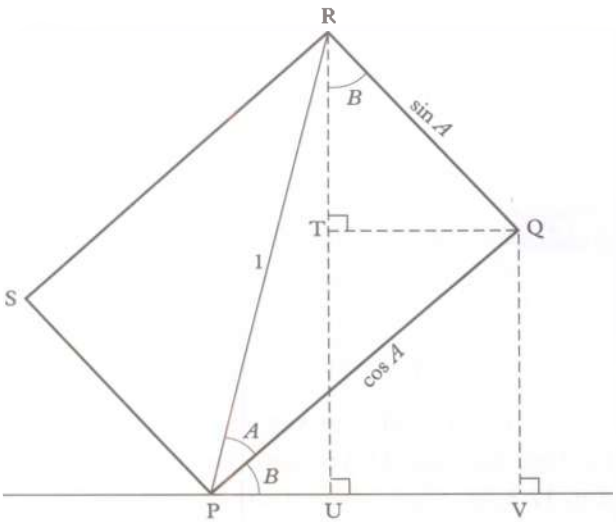


Figure 17.1

What is the height of R *above* P?

One way to find this is to construct a perpendicular RU from R to the horizontal through P, then from the triangle RPU, RU = $\sin(A+B)$.

Alternatively, since RQ = $\sin A$, PQ = $\cos A$ and angle QRU = B, the height of R above P can be found in two parts. First, the height of R above Q, RT = $\sin A \cos B$ (from triangle RTQ). Secondly, the height of Q above P, QV = $\cos A \sin B$ (from triangle PQV). Thus, equating the height of R above P obtained in the two ways,

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

How far to the right of P *is* R?

In triangle RPU, PU = $\cos(A+B)$.

Alternatively, the distance of Q to the right of P, PV = $\cos A \cos B$ (from triangle PQV), and the distance of R to the left of Q, QT = $\sin A \sin B$ (from triangle RTQ). So, equating the distance of R to the right of P obtained in these two ways,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

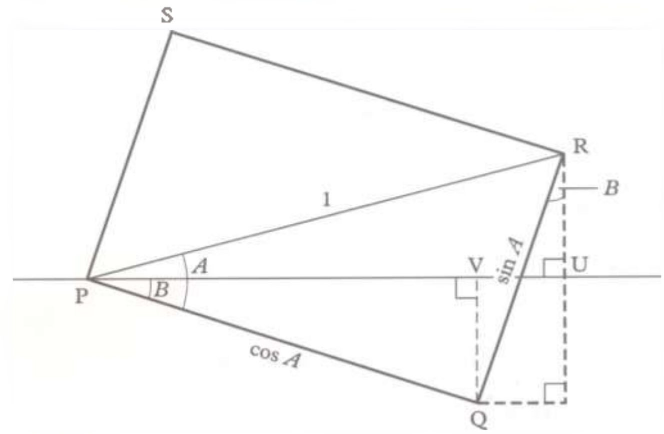


Figure 17.2

Consider now what happens if PQ is tilted through an angle B below the horizontal, as in Fig. 17.2. The height of R above P is now $\sin(A-B)$. R is a distance $\sin A \cos B$ above Q, but Q is a distance $\cos A \sin B$ below P, therefore

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

Further, R is a distance $\cos(A-B)$ to the right of P. Q is a distance $\cos A \cos B$ to the right of P, but R is now a distance $\sin A \sin B$ to the right of Q, therefore

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

Formulae for tan $(A \pm B)$

Two more identities will be deduced from the four just obtained. They give $\tan(A+B)$ and $\tan(A-B)$ in terms of $\tan A$ and $\tan B$.

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

Therefore, using the formulae for $\sin(A+B)$ and $\cos(A+B)$

$$\tan(A+B) = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

Dividing numerator and denominator of the right-hand side by $\cos A \cos B$,

$$\tan(A+B) = \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}}$$

$$= \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A}{\cos A} \times \dfrac{\sin B}{\cos B}}$$

$$\therefore \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Similarly,

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

The six addition formulae

For convenience, the six identities shown above are printed together:

$$\mathbf{\sin(A+B) = \sin A \cos B + \cos A \sin B}$$

$$\mathbf{\sin(A-B) = \sin A \cos B - \cos A \sin B}$$

$$\mathbf{\cos(A+B) = \cos A \cos B - \sin A \sin B}$$

$$\mathbf{\cos(A-B) = \cos A \cos B + \sin A \sin B}$$

$$\mathbf{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}}$$

$$\mathbf{\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}}$$

These are usually called the **addition formulae** or **compound angle formulae**. When memorising these, note the following:

a the formulae for the ratios of $(A-B)$ are the same as those for $(A+B)$, except for the changes in signs,

b the signs on the two sides of each of the sine formulae are the same, but in the cosine formulae they are different,

c in the tangent formulae, the signs in the numerators are the same as in the corresponding sine formulae, and those in the denominators are the same as in the cosine formulae.

The above identities have many applications apart from their use in trigonometry. They, or identities which will be derived from them, are needed in calculus, coordinate geometry and mechanics. Some applications will be found in Chapters 19 and 22.

Example 1 *Find, without using tables or calculators, the value of*

$$\sin(120° + 45°)$$

leaving surds in the answer.

Using the formula for $\sin(A+B)$,

$$\sin(120° + 45°) = \sin 120° \cos 45° + \cos 120° \sin 45°$$

Reference to Fig. 16.7 and Fig. 16.8 on page 196 should remind the reader how to obtain the ratios of 30°, 45°, and 60°. Thus we have

$$\sin 120° = \sin 60° = \frac{\surd 3}{2}$$

$$\cos 120° = -\cos 60° = -\tfrac{1}{2}$$

$$\cos 45° = \sin 45° = \frac{1}{\surd 2} = \frac{\surd 2}{2}$$

$$\therefore \sin(120° + 45°) = \frac{\surd 3}{2} \times \frac{\surd 2}{2} + \left(-\frac{1}{2}\right) \times \frac{\surd 2}{2}$$

$$\therefore \sin(120° + 45°) = \frac{\surd 2}{4}(\surd 3 - 1)$$

Example 2 *If $\sin A = \frac{3}{5}$ and $\cos B = \frac{15}{17}$, where A is obtuse and B is acute, find the exact value of* $\sin(A+B)$.

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

So it is necessary to find the values of $\cos A$ and $\sin B$. Figs. 17.3 and 17.4 indicate the method. In Fig. 17.3, the third side of the right-angled triangle is 4 (by Pythagoras' theorem), hence the x-coordinate of P is -4, therefore $\cos A = -\frac{4}{5}$.

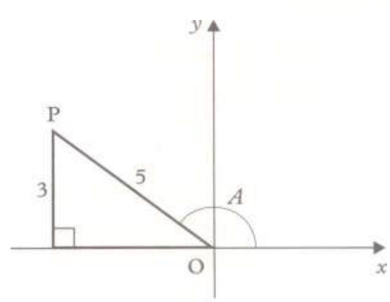


Figure 17.3

Similarly, in Fig. 17.4, the y-coordinate of P is 8, and therefore $\sin B = \frac{8}{17}$.

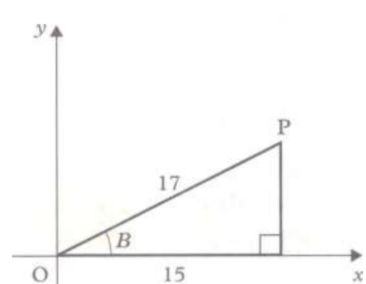


Figure 17.4

$$\therefore \sin(A+B) = \frac{3}{5}\times\frac{15}{17}+\left(-\frac{4}{5}\right)\times\frac{8}{17}$$

$$= \frac{45}{85}-\frac{32}{85}$$

$$\therefore \sin(A+B) = \frac{13}{85}$$

Example 3 *If* $\sin(x+\alpha) = \cos(x-\beta)$, *find* $\tan x$ *in terms of* α *and* β.

Since $\sin(x+\alpha) = \cos(x-\beta)$, we have

$$\sin x\cos\alpha + \cos x\sin\alpha = \cos x\cos\beta + \sin x\sin\beta$$

[Since $\tan x = \sin x/\cos x$, collect terms in $\sin x$ on one side of the equation, and terms in $\cos x$ on the other.]

Thus

$$\sin x\cos\alpha - \sin x\sin\beta = \cos x\cos\beta - \cos x\sin\alpha$$

$$\therefore \sin x(\cos\alpha - \sin\beta) = \cos x(\cos\beta - \sin\alpha)$$

$$\therefore \frac{\sin x}{\cos x} = \frac{\cos\beta-\sin\alpha}{\cos\alpha-\sin\beta}$$

$$\therefore \tan x = \frac{\cos\beta-\sin\alpha}{\cos\alpha-\sin\beta}$$

Exercise 17a

The questions in this exercise are intended to give practice in using the trigonometrical identities introduced in the preceding section. Do not use a calculator or tables in this exercise. Leave surds in the answers where appropriate.

1 Find the values of the following:

- **a** $\cos(45°-30°)$
- **b** $\sin(30°+45°)$
- **c** $\sin(60°+45°)$
- **d** $\cos 105°$
- **e** $\cos(120°+45°)$
- **f** $\sin 165°$
- **g** $\sin 15°$
- **h** $\cos 75°$

2 If $\sin A = \frac{3}{5}$ and $\sin B = \frac{5}{13}$, where A and B are acute angles, find the values of

- **a** $\sin(A+B)$
- **b** $\cos(A+B)$
- **c** $\cot(A+B)$

3 If $\sin A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$, where A is *obtuse* and B is acute, find the values of

- **a** $\sin(A-B)$
- **b** $\tan(A-B)$
- **c** $\tan(A+B)$

4 If $\cos A = \frac{3}{5}$ and $\tan B = \frac{12}{5}$, where A and B are both reflex angles, find the values of

- **a** $\sin(A-B)$
- **b** $\tan(A-B)$
- **c** $\cos(A+B)$

5 If $\tan(x+45°) = 2$, find the value of $\tan x$.

6 If $\tan(A+B) = \frac{1}{7}$ and $\tan A = 3$, find the value of $\tan B$.

7 If A and B are acute, $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$, find the value of $A+B$.

8 If $\tan A = -\frac{1}{7}$ and $\tan B = \frac{3}{4}$, where A is obtuse and B is acute, find the value of $A-B$.

9 Express as single trigonometrical ratios:

- **a** $\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$
- **b** $\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x$
- **c** $\frac{\sqrt{3}+\tan x}{1-\sqrt{3}\tan x}$
- **d** $\cos 16°\sin 42° - \sin 16°\cos 42°$
- **e** $\frac{1}{\cos 24°\cos 15° - \sin 24°\sin 15°}$
- **f** $\frac{1}{2}\cos 75° + \frac{\sqrt{3}}{2}\sin 75°$

10 Find the values of

- **a** $\cos 75°\cos 15° + \sin 75°\sin 15°$
- **b** $\sin 50°\cos 20° - \cos 50°\sin 20°$
- **c** $\frac{\tan 10° + \tan 20°}{1-\tan 10°\tan 20°}$
- **d** $\cos 70°\cos 20° - \sin 70°\sin 20°$
- **e** $\frac{1}{\sqrt{2}}\cos 15° - \frac{1}{\sqrt{2}}\sin 15°$
- **f** $\frac{\sqrt{3}}{2}\cos 15° - \frac{1}{2}\sin 15°$
- **g** $\frac{1-\tan 15°}{1+\tan 15°}$
- **h** $\cos 15° + \sin 15°$

11 Find the value of $\tan A$, when $\tan(A-45°) = \frac{1}{3}$.

12 Find the value of $\cot B$, when $\cot A = \frac{1}{4}$ and $\cot(A-B) = 8$.

13 From the following equations, find the values of $\tan x$:

a $\sin(x+45°) = 2\cos(x+45°)$

b $2\sin(x-45°) = \cos(x+45°)$

c $\tan(x-A) = \frac{3}{2}$, where $\tan A = 2$

d $\sin(x+30°) = \cos(x+30°)$

14 If $\sin(x+\alpha) = 2\cos(x-\alpha)$, prove that

$$\tan x = \frac{2-\tan\alpha}{1-2\tan\alpha}$$

15 If $\sin(x-\alpha) = \cos(x+\alpha)$, prove that $\tan x = 1$.

16 Solve, for values of x between 0° and 360°, the equations:

a $2\sin x = \cos(x+60°)$

b $\cos(x+45°) = \cos x$

c $\sin(x-30°) = \frac{1}{2}\cos x$

d $3\sin(x+10°) = 4\cos(x-10°)$

Prove the following identities:

17 $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

18 $\cos(A+B) - \cos(A-B) = -2\sin A\sin B$

19 $\tan A + \tan B = \dfrac{\sin(A+B)}{\cos A\cos B}$

20 $\tan(A+B+C)$

$$= \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1-\tan B\tan C - \tan C\tan A - \tan A\tan B}.$$

Hence prove that if A, B, C are angles of a triangle, then

$$\tan A + \tan B + \tan C = \tan A\tan B\tan C$$

17.2 The double angle formulae

The special cases of the identities on page 202, when $A = B$, are even more useful than the identities themselves. For convenience of reference, they are given together, below.

$$\begin{aligned}\sin 2A &= 2\sin A\cos A\\ \cos 2A &= \cos^2 A - \sin^2 A\\ &= 2\cos^2 A - 1\\ &= 1 - 2\sin^2 A\\ \tan 2A &= \frac{2\tan A}{1-\tan^2 A}\end{aligned}$$

Further, it is useful to remember that

$$\begin{aligned}\sin^2 A &= \tfrac{1}{2}(1-\cos 2A)\\ \cos^2 A &= \tfrac{1}{2}(1+\cos 2A)\end{aligned}$$

To prove the identities concerning $\cos 2A$, we put $B = A$ in the identity

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

which gives

$$\cos 2A = \cos^2 A - \sin^2 A$$

Now $\cos^2 A + \sin^2 A = 1$, so substituting $\sin^2 A = 1 - \cos^2 A$, we obtain

$$\begin{aligned}\cos 2A &= \cos^2 A - 1 + \cos^2 A\\ \therefore \cos 2A &= 2\cos^2 A - 1\end{aligned}$$

If we had substituted $\cos^2 A = 1 - \sin^2 A$ in the identity

$$\cos 2A = \cos^2 A - \sin^2 A$$

we would have obtained

$$\begin{aligned}\cos 2A &= 1 - \sin^2 A - \sin^2 A\\ \therefore \cos 2A &= 1 - 2\sin^2 A\end{aligned}$$

The expressions for $\cos^2 A$ and $\sin^2 A$ are obtained by changing the subjects in the formulae

$$\cos 2A = 2\cos^2 A - 1 \quad \text{and} \quad \cos 2A = 1 - 2\sin^2 A$$

The identities for $\sin 2A$ and $\tan 2A$ are obtained immediately, when the substitution $B = A$ is made in the formulae for $\sin(A+B)$ and $\tan(A+B)$.

Example 4 *Solve the equation* $3\cos 2\theta + \sin\theta = 1$, *for values of* θ *from* 0° *to* 360° *inclusive.*

[Quadratic equations often appear in various disguises. Here, $\sin\theta$ suggests that the equation may be a quadratic in $\sin\theta$, so we express $\cos 2\theta$ in terms of $\sin\theta$.]

We have $\cos 2\theta = 1 - 2\sin^2\theta$, so, substituting in the equation

$$3\cos 2\theta + \sin\theta = 1$$

it follows that

$$3(1 - 2\sin^2\theta) + \sin\theta = 1$$

This is a quadratic equation in $\sin\theta$, and is solved by factorisation.

$$\begin{aligned}&3 - 6\sin^2\theta + \sin\theta = 1\\ \therefore\ &6\sin^2\theta - \sin\theta - 2 = 0\\ \therefore\ &(3\sin\theta - 2)(2\sin\theta + 1) = 0\\ \therefore\ &\sin\theta = \tfrac{2}{3} \quad \text{or} \quad \sin\theta = -\tfrac{1}{2}\end{aligned}$$

If $\sin\theta = \frac{2}{3}$,

$\theta = 41.8°$ or $180° - 41.8°$, correct to one decimal place.

If $\sin\theta = -\frac{1}{2}$,

$\theta = 180° + 30°$ or $360° - 30°$

Therefore the values of θ between 0° and 360° which satisfy the equation are 41.8°, 138.2°, 210°, and 330°.

Example 5 *Prove that* $\sin 3A = 3\sin A - 4\sin^3 A$.

The left-hand side of the identity may be written as $\sin(A + 2A)$, so by using the formula for $\sin(A+B)$ we have

$$\sin(A+2A) = \sin A\cos 2A + \cos A\sin 2A$$

The right-hand side of the given identity is in terms of $\sin A$. This suggests that $\cos 2A$ should be expressed in terms of $\sin A$. (There is only one formula for $\sin 2A$, so it must be used.) Therefore

$$\begin{aligned}\sin 3A &= \sin A(1 - 2\sin^2 A) + \cos A(2\sin A\cos A)\\ &= \sin A - 2\sin^3 A + 2\sin A\cos^2 A\end{aligned}$$

$\cos^2 A$ can be expressed in terms of $\sin A$ using the identity $\cos^2 A = 1 - \sin^2 A$. Therefore

$$\begin{aligned}\sin 3A &= \sin A - 2\sin^3 A + 2\sin A(1 - \sin^2 A)\\ &= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A\end{aligned}$$

$\therefore \mathbf{\sin 3A = 3\sin A - 4\sin^3 A}$

A similar formula for $\cos 3A$ in terms of $\cos A$ may be obtained from the expansion of $\cos(2A + A)$. The proof is left as an exercise.

$$\mathbf{\cos 3A = 4\cos^3 A - 3\cos A}$$

Exercise 17b (Oral)

Express more simply:

1 $2\sin 17°\cos 17°$

2 $\dfrac{2\tan 30°}{1 - \tan^2 30°}$

3 $2\cos^2 42° - 1$

4 $2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$

5 $1 - 2\sin^2 22\frac{1}{2}°$

6 $\dfrac{2\tan\frac{1}{2}\theta}{1 - \tan^2\frac{1}{2}\theta}$

7 $\cos^2 15° - \sin^2 15°$

8 $2\sin 2A\cos 2A$

9 $2\cos^2\frac{1}{2}\theta - 1$

10 $1 - 2\sin^2 3\theta$

11 $\dfrac{\tan 2\theta}{1 - \tan^2 2\theta}$

12 $\sin x\cos x$

13 $\dfrac{1 - \tan^2 20°}{\tan 20°}$

14 $\sec\theta\,\mathrm{cosec}\,\theta$

15 $1 - 2\sin^2\frac{1}{2}\theta$

Exercise 17c

Questions 1–6 in this exercise are intended to give practice in using the trigonometrical identities introduced above. Do not use a calculator or tables. Leave surds in the answers where appropriate.

1 Evaluate:

- **a** $2\sin 15°\cos 15°$
- **b** $\dfrac{2\tan 22\frac{1}{2}°}{1 - \tan^2 22\frac{1}{2}°}$
- **c** $2\cos^2 75° - 1$
- **d** $1 - 2\sin^2 67\frac{1}{2}°$
- **e** $\cos^2 22\frac{1}{2}° - \sin^2 22\frac{1}{2}°$
- **f** $\dfrac{1 - \tan^2 15°}{\tan 15°}$
- **g** $\dfrac{1 - 2\cos^2 25°}{1 - 2\sin^2 65°}$
- **h** $\sec 22\frac{1}{2}°\,\mathrm{cosec}\,22\frac{1}{2}°$

2 Find the values of $\sin 2\theta$ and $\cos 2\theta$ when

- **a** $\sin\theta = \frac{3}{5}$
- **b** $\cos\theta = \frac{12}{13}$
- **c** $\sin\theta = -\sqrt{3}/2$

3 Find the value of $\tan 2\theta$ when

- **a** $\tan\theta = \frac{4}{3}$
- **b** $\tan\theta = \frac{8}{15}$
- **c** $\cos\theta = -\frac{5}{13}$

4 Find the values of $\cos x$ and $\sin x$ when $\cos 2x$ is

- **a** $\frac{1}{8}$
- **b** $\frac{7}{25}$
- **c** $-\frac{119}{169}$

5 Find the values of $\tan\frac{1}{2}\theta$ when $\tan\theta$ is

- **a** $\frac{3}{4}$
- **b** $\frac{4}{3}$
- **c** $-\frac{12}{5}$

6 If $t = \tan 22\frac{1}{2}°$, use the formula for $\tan 2\theta$ to show that $t^2 + 2t - 1 = 0$. Deduce the value of $\tan 22\frac{1}{2}°$.

Solve the following equations for values of θ from 0° to 360° inclusive:

7 $\cos 2\theta + \cos\theta + 1 = 0$

8 $\sin 2\theta = \sin\theta$

9 $\cos 2\theta = \sin\theta$

10 $3\cos 2\theta - \sin\theta + 2 = 0$

11 $\sin 2\theta\cos\theta + \sin^2\theta = 1$

12 $\sin\theta = 6\sin 2\theta$

13 $2\sin\theta(5\cos 2\theta + 1) = 3\sin 2\theta$

14 $3\tan\theta = \tan 2\theta$

15 $3\cot 2\theta + \cot\theta = 1$

16 $4\tan\theta\tan 2\theta = 1$

17 Eliminate θ from the equations:

- **a** $x = \cos\theta, y = \cos 2\theta$
- **b** $x = 2\sin\theta, y = 3\cos 2\theta$
- **c** $x = \tan\theta, y = \tan 2\theta$
- **d** $x = 2\sec\theta, y = \cos 2\theta$

Prove the following identities:

18 $\dfrac{\cos 2A}{\cos A + \sin A} = \cos A - \sin A$

19 $\dfrac{\sin A}{\sin B} + \dfrac{\cos A}{\cos B} = \dfrac{2\sin(A+B)}{\sin 2B}$

20 $\dfrac{\cos A}{\sin B} - \dfrac{\sin A}{\cos B} = \dfrac{2\cos(A+B)}{\sin 2B}$

21 $\tan A + \cot A = 2\operatorname{cosec} 2A$

22 $\cot A - \tan A = 2\cot 2A$

23 $\dfrac{1}{\cos A + \sin A} + \dfrac{1}{\cos A - \sin A} = \tan 2A\operatorname{cosec} A$

24 $\dfrac{\sin 2A}{1+\cos 2A} = \tan A = \dfrac{1-\cos 2A}{\sin 2A}$

25 $\cos 3A = 4\cos^3 A - 3\cos A$

26 $\operatorname{cosec} 2x - \cot 2x = \tan x$

27 $\operatorname{cosec} 2x + \cot 2x = \cot x$

28 $\tan x = \sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$

29 $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$

30 $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

17.3 The *t*-formulae

We have already met (page 204) the following formulae for $\sin 2x$ and $\cos 2x$:

$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$

It is possible to express both $\sin 2x$ and $\cos 2x$ in terms of $\tan x$ and there are many occasions when this is very useful.

In the case of $\sin 2x$ we introduce a factor $\sin x/\cos x$, which is equal to $\tan x$.

$$\begin{aligned}\sin 2x &= 2\sin x\cos x\\ &= 2\frac{\sin x}{\cos x}\cos^2 x\\ &= 2\tan x\cos^2 x\\ &= 2\tan x \times \frac{1}{\sec^2 x}\end{aligned}$$

The reason for the last step is to enable us to replace $\sec^2 x$ by $1+\tan^2 x$ (see §16.6 on page 199). Hence

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$$

This identity is most frequently used in the form obtained by substituting θ for $2x$, i.e.

$$\sin\theta = \frac{2\tan\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}$$

$$\therefore \boldsymbol{\sin\theta = \frac{2t}{1+t^2}} \quad (\text{where } t = \tan\tfrac{1}{2}\theta)$$

This is usually called the ***t*-formula** for $\sin\theta$. The corresponding t-formulae for $\cos\theta$ and $\tan\theta$ are left as exercises (see Q1 and Q2 below).

Questions

Q1 Prove that, in the usual notation, $\boldsymbol{\cos\theta = \frac{1-t^2}{1+t^2}}$.

Q2 Prove that $\boldsymbol{\tan\theta = \frac{2t}{1-t^2}}$.

Q3 Use the t-formulae to solve the following equations, giving values of θ from 0° to 360° inclusive:

- **a** $2\cos\theta + 3\sin\theta - 2 = 0$
- **b** $7\cos\theta + \sin\theta - 5 = 0$
- **c** $3\cos\theta - 4\sin\theta + 1 = 0$
- **d** $3\cos\theta + 4\sin\theta = 2$

The t-formulae are often useful when solving certain integrals (see Chapter 34).

17.4 The form $a \cos\theta + b \sin\theta$

Examples 6 and 7 contain two applications of the identities of §17.1 on page 201.

Example 6 *Solve the equation* $3\cos\theta + 4\sin\theta = 2$, *for values of* θ *from* $0°$ *to* $360°$, *inclusive.*

The solution is obtained by dividing both sides of the equation by some number, so as to leave it in the form

$$\mathbf{\cos\alpha}\cos\theta + \mathbf{\sin\alpha}\sin\theta = \text{constant}$$

Comparing this with

$$3\cos\theta + \quad 4\sin\theta = 2$$

it follows that

$$\frac{\cos\alpha}{3} = \frac{\sin\alpha}{4}, \quad \text{i.e. } \tan\alpha = \frac{4}{3}$$

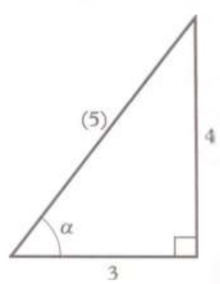


Figure 17.5

From a calculator or tables $\alpha = 53.13°$, and from Fig. 17.5 it follows that $\sin\alpha = \frac{4}{5}$ and $\cos\alpha = \frac{3}{5}$. Therefore we divide the original equation by 5, giving

$$\tfrac{3}{5}\cos\theta + \tfrac{4}{5}\sin\theta = \tfrac{2}{5}$$

$$\therefore \cos\theta\cos\alpha + \sin\theta\sin\alpha = 0.4$$

$$\therefore \cos(\theta - \alpha) = 0.4$$

$$\therefore \theta - 53.13° = 66.42° \quad \text{or} \quad 293.58°$$

Therefore the roots of the equation in the range from 0° to 360° are 119.6° and 346.7°, correct to the nearest tenth of a degree.*

*The numeral in the second decimal place should be included in the intermediate working, in order to avoid errors due to premature approximation.

Question

Q4 What advantage is there in using the formula for $\cos(A - B)$, rather than that for $\sin(A + B)$ in Example 6?

Example 7 *Find the maximum and minimum values of* $2\sin\theta - 5\cos\theta$, *and the corresponding values of* θ *between* $0°$ *and* $360°$.

This will be solved by writing

$$2\sin\theta - 5\cos\theta = k(\cos\alpha\sin\theta - \sin\alpha\cos\theta)$$

where k and α are to be found. Comparing the two forms of the expression,

$$\frac{\sin\alpha}{\cos\alpha} = \frac{5}{2}, \quad \text{i.e. } \tan\alpha = 2.5$$

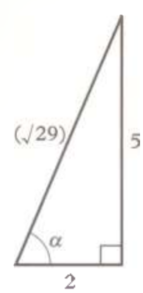


Figure 17.6

From a calculator or tables $\alpha = 68.20°$; and from Fig. 17.6, it follows that $\cos\alpha = 2/\surd 29$, and $\sin\alpha = 5/\surd 29$. So we may write

$$\begin{aligned} 2\sin\theta - 5\cos\theta &= \surd 29\left(\frac{2}{\surd 29}\sin\theta - \frac{5}{\surd 29}\cos\theta\right) \\ &= \surd 29(\sin\theta\cos\alpha - \cos\theta\sin\alpha) \\ &= \surd 29\sin(\theta - \alpha) \end{aligned}$$

The greatest value of $\sin x$ is 1, and this occurs when $x = 90°$. The least value of $\sin x$ is -1, when $x = 270°$. (Values of x less than 0° or greater than 360° have been ignored.)

Therefore $\surd 29\sin(\theta - \alpha)$ has a maximum value of $\surd 29$ when $\theta - \alpha = 90°$ and a minimum value of $-\surd 29$ when $\theta - \alpha = 270°$.

Therefore the maximum and minimum values of

$$2\sin\theta - 5\cos\theta$$

are $\surd 29$ and $-\surd 29$, when

$$\theta = 90° + \alpha = 158.2° \quad \text{and}$$
$$\theta = 270° + \alpha = 338.2° \text{ respectively.}$$

Exercise 17d

Solve the following equations for values of θ from 0° to 360° inclusive.

1 $\sqrt{3} \cos \theta + \sin \theta = 1$

2 $5 \sin \theta - 12 \cos \theta = 6$

3 $\sin \theta + \cos \theta = \frac{1}{2}$

4 $\cos \theta - 7 \sin \theta = 2$

5 $2 \sin \theta + 7 \cos \theta = 4$

6 $3 \tan \theta - 2 \sec \theta = 4$

7 $4 \cos \theta \sin \theta + 15 \cos 2\theta = 10$

8 $\cos \theta + \sin \theta = \sec \theta$

9 Prove that
$\cos \theta - \sin \theta = \sqrt{2} \cos (\theta + 45°) = -\sqrt{2} \sin (\theta - 45°)$.

10 Show that $\sqrt{3} \cos \theta - \sin \theta$ may be written as

$$2 \cos (\theta + 30°) \quad \text{or} \quad 2 \sin (60° - \theta)$$

Find the maximum and minimum values of the expression, and state the values of θ between 0° and 360° for which they occur.

11 Show that $3 \cos \theta + 2 \sin \theta$ may be written in the form $\sqrt{13} \cos (\theta - \alpha)$, where $\tan \alpha = \frac{2}{3}$. Hence find the maximum and minimum values of the function, giving the corresponding values of θ from −180° to +180°.

12 Show that $3 \cos \theta + 4 \sin \theta$ may be expressed in the form $R \cos (\theta - \alpha)$, where α is acute. Find the values of R and α.

13 By expressing $\cos \theta + 2 \sin \theta$ in the form $R \sin (\theta + \alpha)$, where α is acute, find the maximum and minimum values of the expression, giving the values of θ between −180° and 180° for which they occur.

Find the maximum and minimum values of the following expressions, stating the values of θ, from 0° to 360° inclusive, for which they occur.

14 $\cos \theta + \sin \theta$

15 $4 \sin \theta - 3 \cos \theta$

16 $\sqrt{3} \sin \theta + \cos \theta$

17 $8 \cos \theta - 15 \sin \theta$

18 $\sin \theta - 6 \cos \theta$

19 $\cos (\theta + 60°) - \cos \theta$

20 $3\sqrt{2} \cos (\theta + 45°) + 7 \sin \theta$

17.5 Factor formulae

Introduction

In algebra factors are very useful for solving equations and simplifying expressions. When dealing with trigonometrical ratios, it is often convenient to be able to factorise a sum of two terms. On the other hand, it is sometimes useful to express a product as a sum or difference of two terms, and we consider this first.

In §17.2 on page 204 it was shown that

$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$
$$\cos (A - B) = \cos A \cos B + \sin A \sin B$$

Adding,

$$\cos (A + B) + \cos (A - B) = 2 \cos A \cos B$$

and subtracting,

$$\cos (A + B) - \cos (A - B) = -2 \sin A \sin B$$

Keeping the formulae for $\cos (A + B)$ and $\cos (A - B)$ in mind, work through the next exercise.

Exercise 17e (Oral)

Express as a sum or difference of two cosines:

1 $-2 \sin x \sin y$

2 $2 \cos x \cos y$

3 $2 \cos 3\theta \cos \theta$

4 $-2 \sin (S + T) \sin (S - T)$

5 $2 \sin 5x \sin 3x$

6 $2 \cos (x + y) \cos (x - y)$

7 $2 \cos \dfrac{A + B}{2} \cos \dfrac{A - B}{2}$

8 $-2 \sin \dfrac{B + C}{2} \sin \dfrac{B - C}{2}$

9 $-2 \sin (x + 45°) \sin (x - 45°)$

10 $2 \cos(2x + 30°) \cos (2x - 30°)$

Following the same method as before, we have

$$\sin (A+B) = \sin A \cos B + \cos A \sin B$$
$$\sin (A-B) = \sin A \cos B - \cos A \sin B$$

Adding,

$$\mathbf{\sin (A+B) + \sin (A-B) = 2 \sin A \cos B}$$

and subtracting,

$$\mathbf{\sin (A+B) - \sin (A-B) = 2 \cos A \sin B}$$

Keeping the formulae for $\sin (A+B)$ and $\sin (A-B)$ in mind, work through the next exercise.

Exercise 17f (Oral)

Express as a sum or difference of two sines:

1 $2 \sin x \cos y$

2 $2 \cos x \sin y$

3 $2 \sin 3\theta \cos \theta$

4 $2 \sin (S+T) \cos (S-T)$

5 $2 \cos 5x \sin 3x$

6 $2 \cos (x+y) \sin (x-y)$

7 $-2 \cos 4x \sin 2x$

8 $2 \sin \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$

9 $2 \cos \dfrac{A+B}{2} \sin \dfrac{A-B}{2}$

10 $2 \sin \dfrac{R-S}{2} \cos \dfrac{R+S}{2}$

The four factor formulae

We now proceed to factorising a sum or difference of two cosines or sines. The last section showed that

$$\cos (A+B) + \cos (A-B) = 2 \cos A \cos B$$
$$\cos (A+B) - \cos (A-B) = -2 \sin A \sin B$$
$$\sin (A+B) + \sin (A-B) = 2 \sin A \cos B$$
$$\sin (A+B) - \sin (A-B) = 2 \cos A \sin B$$

Here, the right-hand sides of the identities are in factors, but it would be more convenient if the left-hand sides wer in the form $\cos P + \cos Q$, etc. Therefore let

$$P = A+B \quad \text{and} \quad Q = A-B$$

Adding,

$$P+Q = 2A \quad \therefore A = \frac{P+Q}{2}$$

Subtracting,

$$P-Q = 2B \quad \therefore B = \frac{P-Q}{2}$$

Substituting into the four identities above,

$$\mathbf{\cos P + \cos Q = 2 \cos \frac{P+Q}{2} \cos \frac{P-Q}{2}}$$

$$\mathbf{\cos P - \cos Q = -2 \sin \frac{P+Q}{2} \sin \frac{P-Q}{2}}$$

$$\mathbf{\sin P + \sin Q = 2 \sin \frac{P+Q}{2} \cos \frac{P-Q}{2}}$$

$$\mathbf{\sin P - \sin Q = 2 \cos \frac{P+Q}{2} \sin \frac{P-Q}{2}}$$

Remember how these identities were obtained: this will make it easier to remember them. Many people find it helpful to remember them in the form,

'cos plus cos, equals two cos semi-sum, cos semi-diff.'

Example 8 *Solve the equation* $\sin 3x + \sin x = 0$ *for values of* x *from* $-180°$ *to* $+180°$ *inclusive.*

$$\sin 3x + \sin x = 0$$

Therefore, using the formula for $\sin P + \sin Q$,

$$2 \sin 2x \cos x = 0$$
$$\therefore \sin 2x = 0 \quad \text{or} \quad \cos x = 0$$

Since x lies in the range $-180°$ to $180°$, $2x$ lies in the range $-360°$ to $360°$.

If $\sin 2x = 0$,

$$2x = -360°, -180°, 0°, 180°, 360°$$
$$\therefore x = -180°, \quad -90°, 0°, \quad 90°, 180°$$

If $\cos x = 0$, $x = -90°, 90°$.

Therefore the roots of the equation between $-180°$ and $+180°$, inclusive, are $-180°, -90°, 0°, 90°$ and $180°$.

Example 9 *Solve the equation*

$\cos(x+20°) - \cos(x+80°) = 0.5$, *for* $0° \leqslant x \leqslant 360°$.

(The difference of the two cosines suggests using one of the above identities.)

$$\cos(x+20°) - \cos(x+80°) = 0.5$$
$$\therefore -2\sin(x+50°)\sin(-30°) = 0.5$$

But $\sin(-30°) = -\sin 30° = -\frac{1}{2}$.

$$\therefore \sin(x+50°) = 0.5$$
$$x+50° = 30°, 150°, 390°, 510°, \ldots$$
$$x = -20°, 100°, 340°, \ldots$$

Therefore the roots of the equation between 0° and 360° are 100° and 340°.

Example 10 *Solve the equation*

$\sin(x+15°)\cos(x-15°) = 0.5$, *for values of x from* 0° *to* 360° *inclusive.*

(The product of a sine and a cosine suggests that the left-hand side may be expressed as the sum of two sines.)

$$\sin(x+15°)\cos(x-15°) = 0.5$$
$$\therefore 2\sin(x+15°)\cos(x-15°) = 1$$
$$\therefore \sin 2x + \sin 30° = 1$$
$$\therefore \sin 2x = 1 - \sin 30°$$
$$= 0.5$$
$$\therefore 2x = 30°, 150°, 390°, 510°, \ldots$$

Hence the values of x required are 15°, 75°, 195°, 255°.

Example 11 *Prove the identity*

$$\cos^2 A - \cos^2 B = \sin(A+B)\sin(B-A)$$

[A neat method is to use $\cos^2 A = \frac{1}{2}(1+\cos 2A)$, $\cos^2 B = \frac{1}{2}(1+\cos 2B)$.]

$$\cos^2 A - \cos^2 B = \tfrac{1}{2}(\cos 2A - \cos 2B)$$
$$= \tfrac{1}{2}\{-2\sin(A+B)\sin(A-B)\}$$
$$\therefore \cos^2 A - \cos^2 B = \sin(A+B)\sin(B-A)$$

Exercise 17g (Oral)

Express the following in factors:

1 $\cos x + \cos y$

2 $\sin 3x + \sin 5x$

3 $\sin 2y - \sin 2z$

4 $\cos 5x + \cos 7x$

5 $\cos 2A - \cos A$

6 $\sin 4x - \sin 2x$

7 $\cos 3A - \cos 5A$

8 $\sin 5\theta + \sin 7\theta$

9 $\sin(x+30°) + \sin(x-30°)$

10 $\cos(y+10°) + \cos(y-80°)$

11 $\sin 3\theta - \sin 5\theta$

12 $\cos(x+30°) - \cos(x-30°)$

13 $\cos\frac{3x}{2} - \cos\frac{x}{2}$

14 $\sin 2(x+40°) + \sin 2(x-40°)$

15 $\cos(90°-x) + \cos y$

16 $\sin A + \cos B$

17 $\sin 3x + \sin 90°$

18 $1 + \sin 2x$

19 $\cos A - \sin B$

20 $\frac{1}{2} + \cos 2\theta$

17.6 Further identities and equations

Example 12 *Solve the equation*

$\cos 6x + \cos 4x + \cos 2x = 0$, *for values of x from* 0° *to* 180° *inclusive.*

[Remember that equations are very often solved by factorisation, so check whether any of the three terms is a factor of the sum of the other pair. Note that $\cos 4x$ is a factor of $\cos 6x + \cos 2x$, so group $\cos 6x$ and $\cos 2x$ together.]

$$\cos 4x + \cos 6x + \cos 2x = 0$$
$$\therefore \cos 4x + 2\cos 4x\cos 2x = 0$$
$$\therefore \cos 4x(1 + 2\cos 2x) = 0$$
$$\therefore \cos 4x = 0 \quad \text{or} \quad \cos 2x = -\tfrac{1}{2}$$

If $\cos 4x = 0$, $\quad 4x = 90°, 270°, 450°, 630°$

$$\therefore x = 22\tfrac{1}{2}°, 67\tfrac{1}{2}°, 112\tfrac{1}{2}°, 157\tfrac{1}{2}°$$

If $\cos 2x = -\frac{1}{2}$, $\quad 2x = 120°, 240°$

$$\therefore x = 60°, 120°$$

Therefore the roots of the equation in the range 0° to 180° are $22\frac{1}{2}°$, 60°, $67\frac{1}{2}°$, $112\frac{1}{2}°$, 120°, $157\frac{1}{2}°$.

Example 13 *If A, B, C are the angles of a triangle, prove that*

$$\cos A + \cos B + \cos C - 1 = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

Split the left-hand side into two pairs of terms.
Now,

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

But since $A + B = 180° - C$,

$$\frac{A+B}{2} = 90° - \frac{C}{2} \qquad \therefore \cos \frac{A+B}{2} = \sin \frac{C}{2}$$

Seeing this factor $\sin (C/2)$ on the right-hand side, write $\cos C - 1 = -2 \sin^2 \dfrac{C}{2}$

Therefore

$$\cos A + \cos B + \cos C - 1 = 2 \sin \frac{C}{2} \cos \frac{A-B}{2} - 2 \sin^2 \frac{C}{2}$$
$$= 2 \sin \frac{C}{2}\left(\cos \frac{A-B}{2} - \sin \frac{C}{2}\right)$$

On the right-hand side of the identity to be proved, $\sin (C/2)$ is multiplied by a function of A and B, so in the last bracket we must express $\sin (C/2)$ in terms of A and B. This has been done above.

$\therefore \cos A + \cos B + \cos C - 1$

$$= 2 \sin \frac{C}{2}\left(\cos \frac{A-B}{2} - \cos \frac{A+B}{2}\right)$$
$$= -2\left(\cos \frac{A+B}{2} - \cos \frac{A-B}{2}\right) \sin \frac{C}{2}$$
$$= -2\left(-2 \sin \frac{A}{2} \sin \frac{B}{2}\right) \sin \frac{C}{2}$$

$$\therefore \cos A + \cos B + \cos C - 1 = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

Exercise 17h

Prove the following identities:

1 $\dfrac{\cos B + \cos C}{\sin B - \sin C} = \cot \dfrac{B-C}{2}$

2 $\dfrac{\cos B - \cos C}{\sin B + \sin C} = -\tan \dfrac{B-C}{2}$

3 $\dfrac{\sin B + \sin C}{\cos B + \cos C} = \tan \dfrac{B+C}{2}$

4 $\dfrac{\sin B - \sin C}{\sin B + \sin C} = \cot \dfrac{B+C}{2} \tan \dfrac{B-C}{2}$

5 $\sin x + \sin 2x + \sin 3x = \sin 2x\,(2 \cos x + 1)$

6 $\cos x + \sin 2x - \cos 3x = \sin 2x\,(2 \sin x + 1)$

7 $\cos 3\theta + \cos 5\theta + \cos 7\theta = \cos 5\theta\,(2 \cos 2\theta + 1)$

8 $\cos \theta + 2 \cos 3\theta + \cos 5\theta = 4 \cos^2 \theta \cos 2\theta$

9 $1 + 2 \cos 2\theta + \cos 4\theta = 4 \cos^2 \theta \cos 2\theta$

10 $\sin \theta - 2 \sin 3\theta + \sin 5\theta = 2 \sin \theta\,(\cos 4\theta - \cos 2\theta)$

11 $\cos \theta - 2 \cos 3\theta + \cos 5\theta = 2 \sin \theta\,(\sin 2\theta - \sin 4\theta)$

12 $\sin x - \sin (x + 60°) + \sin (x + 120°) = 0$

13 $\cos x + \cos (x + 120°) + \cos (x + 240°) = 0$

Solve the following equations, for values of x from 0° to 360° inclusive:

14 $\cos x + \cos 5x = 0$

15 $\cos 4x - \cos x = 0$

16 $\sin 3x - \sin x = 0$

17 $\sin 2x + \sin 3x = 0$

18 $\sin (x + 10°) + \sin x = 0$

19 $\cos (2x + 10°) + \cos (2x - 10°) = 0$

20 $\cos (x + 20°) - \cos (x - 70°) = 0$

Chapter 18

Further topics in trigonometry

Historical note

One of Euler's many contributions to mathematics was the invention of a standard notation for labelling triangles. In this notation the vertices are always labelled with capital letters, say A, B and C, which are also used to represent the sizes of the angles at these vertices. The corresponding lower case letters, a, b, c, are then used to represent the lengths of the sides opposite the vertices, i.e. the letter a is used to represent the length of the side BC opposite angle A (see Fig. 18.1).

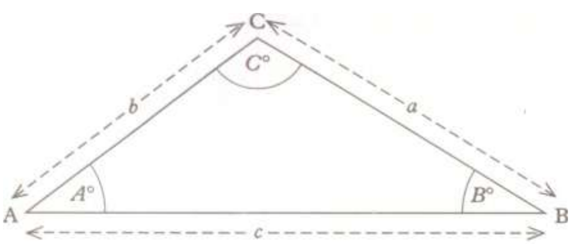


Figure 18.1

The traditional unit of measurement for angles is the degree which has been used for over 2000 years. The traditional sub-unit is the minute, which is 1/60th of a degree, and the standard symbol for it is a small dash. So 35° 12′ is equal to $35\frac{12}{60}°$. In decimals this becomes 35.2°. Use a calculator to convert minutes into a decimal fraction of a degree.

In the next two sections Euler's notation will be used to introduce two important rules, the **sine rule** and the **cosine rule**. These rules are used to 'solve' triangles; that is, given sufficient data to define a unique triangle, the sine and cosine rules can be used to calculate the sizes of the remaining sides and angles.

18.1 The sine rule

In the triangle in Fig. 18.2, CP is perpendicular to AB.

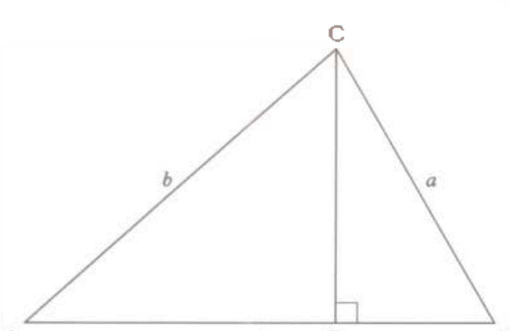


Figure 18.2

By elementary trigonometry the length of the altitude CP is equal to $b \sin A$ (from triangle APC) and it is also equal to $a \sin B$ (from triangle BPC). Equating these expressions, we have

$$a \sin B = b \sin A$$

and hence

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Applying the same argument to the line from A, perpendicular to BC, we obtain

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Putting these expressions together, we have,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

This is called the **sine rule**.

Note: in Fig. 18.2, we assumed that all the angles are acute. If one of them is obtuse, the proof must be modified. Suppose that B is an obtuse angle as shown in Fig. 18.3.

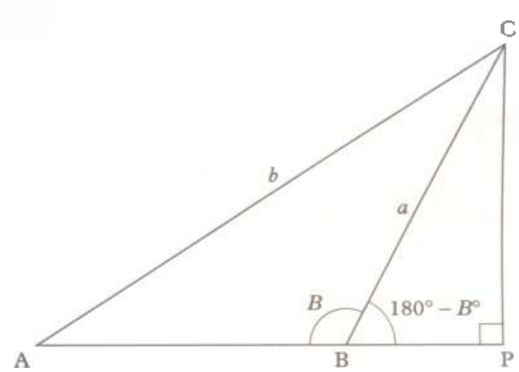


Figure 18.3

In *this* diagram, CP is the perpendicular line from C to AB *produced*. By elementary trigonometry $CP = a \sin \angle CBP = a \sin (180° - B)$. However $\sin (180° - B)$ is equal to $\sin B$ and so we can write

$$CP = a \sin B = b \sin A$$

and proceed with the proof as before.

Note: In this chapter assume that all dimensions are in cm, unless otherwise stated.

Example 1 *In triangle* PQR, $r = 5.75$ *and the sizes of angles* P *and* Q *are* 42° *and* 65° *respectively. Calculate the length of* PR.

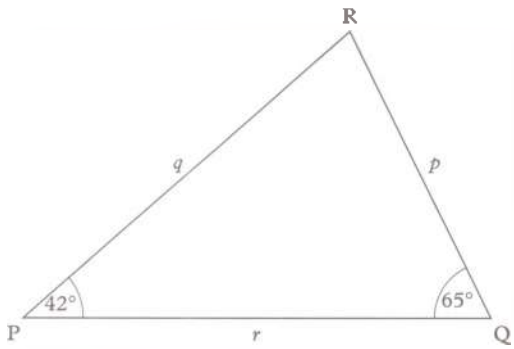


Figure 18.4

With these letters (see Fig. 18.4), the sine rule becomes

$$\frac{p}{\sin P} = \frac{q}{\sin Q} = \frac{r}{\sin R}$$

Notice that when two angles are given, the remaining angle can be calculated from the fact that the sum of the three angles of a triangle is 180°, so $R = 73°$. Substituting the data, and this value of R, we obtain

$$\frac{p}{\sin 42°} = \frac{q}{\sin 65°} = \frac{5.75}{\sin 73°}$$

In this example, the length of PR, i.e. q, is required. Making q the subject of the formula above, we obtain

$$q = \frac{5.75}{\sin 73°} \times \sin 65°$$

$= 5.45$, correct to three significant figures

Example 2 *In triangle* ABC, $a = 4.73$, $c = 3.58$ *and* $C = 42°\ 12'$. *Calculate the size of angle* A.

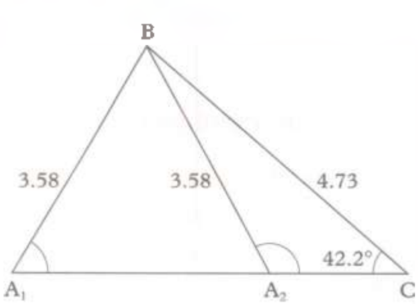


Figure 18.5

Firstly, we note that $42°\ 12' = 42\frac{12}{60}° = 42.2°$, and secondly, from Fig. 18.5, we can see that *two* triangles can be drawn with these data. (It is important to make a sketch so that this can be anticipated.)

By the sine rule,

$$\frac{4.73}{\sin A} = \frac{b}{\sin B} = \frac{3.58}{\sin 42.2}$$

In this case, the middle term is superfluous. The other two terms give

$$\frac{\sin A}{4.73} = \frac{\sin 42.2°}{3.58}$$

$$\therefore \sin A = \frac{\sin 42.2°}{3.58} \times 4.73 \quad (= 0.8875)^*$$

$$\therefore A = 62.560° \quad or \quad 117.440°$$

$$= 62.6° \quad or \quad 117.4°,$$

correct to the nearest tenth of a degree

Note:

1. The step marked with the asterisk indicates the numbers which appear on a calculator at this stage; it is not necessary to write them down. (Indeed, to write them down, correct to four significant figures, and then to use the *corrected* numbers to find A is poor calculator technique.)
2. The alternative value of A, namely, $A = 117.4°$, follows from the fact that $\sin\theta = \sin(180° - \theta)$, i.e. in this case, $\sin 62.6° = \sin 117.4°$. If we inspect the diagram, we can see that both answers are perfectly reasonable, because triangle A_1BA_2 is isosceles.

A case like this, where there are two possible answers, is called *the ambiguous case*.

The sine rule can be used when two angles are given (as in Example 1) or when one of the given sides is opposite the given angle (as in Example 2), but, as you should be able to see, it is useless when the lengths of the three sides are given, or when two sides and the *included* angle (i.e. the angle between them) are given. In these circumstances we must turn to the cosine rule. [You may prefer to work Exercise 18a, questions 1–3, first.]

18.2 The cosine rule

There are several possible proofs of **the cosine rule**. This one uses the idea of the scalar product (see §15.12 on page 190).

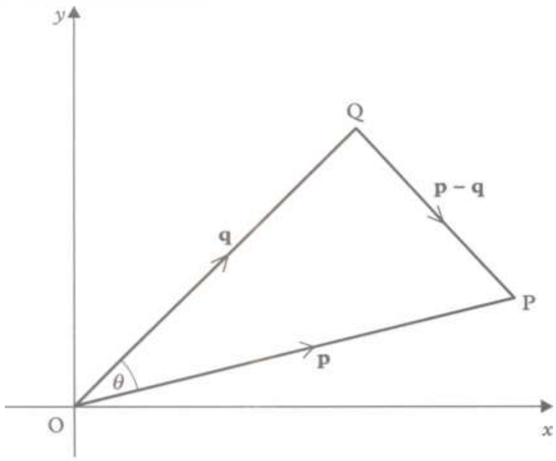


Figure 18.6

In the triangle OPQ (Fig. 18.6), the angle POQ is equal to θ and $\overline{\text{QP}} = \mathbf{p} - \mathbf{q}$. Consider the scalar product $\overline{\text{QP}}.\overline{\text{QP}}$:

$$\begin{aligned}\overline{\text{QP}}.\overline{\text{QP}} &= (\mathbf{p} - \mathbf{q}).(\mathbf{p} - \mathbf{q}) \\ &= \mathbf{p}.\mathbf{p} + \mathbf{q}.\mathbf{q} - 2\mathbf{p}.\mathbf{q} \\ &= p^2 + q^2 - 2pq \cos\theta \quad \text{(See §15.12 on page 190)}\end{aligned}$$

But $\overline{\text{QP}}.\overline{\text{QP}}$ is equal to QP^2,

$$\therefore \text{QP}^2 = p^2 + q^2 - 2pq \cos\theta$$

So, if we are given the values of p and q, and the size of the included angle θ, we can calculate the length of QP.

If the triangle is re-lettered ABC, as in Fig. 18.7, the cosine rule becomes

$$\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A}$$

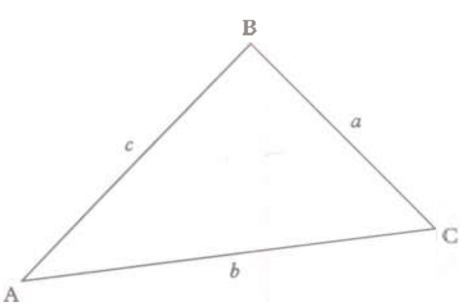


Figure 18.7

The letters a, b and c can be permuted to give the following alternative forms:

$$\boldsymbol{b^2 = c^2 + a^2 - 2ca \cos B}$$
$$\boldsymbol{c^2 = a^2 + b^2 - 2ab \cos C}$$

Example 3 *In triangle* PQR, $p = 14.3$, $r = 17.5$ *and* $Q = 25°\ 36'$. *Calculate the length of side* PR.

In this question we are given the lengths of two sides and the size of the included angle, so the cosine rule is appropriate. With these letters it takes the form

$$q^2 = r^2 + p^2 - 2rp \cos Q$$

Substituting the data gives

$$q^2 = 17.5^2 + 14.3^2 - 2 \times 17.5 \times 14.3 \cos 25.6°$$

Hence

$$q = 7.71, \quad \text{correct to three significant figures}$$

(On most calculators it should be possible, using the memory button, to do the whole calculation without having to write down any of the intermediate working. If in doubt consult the calculator's instruction booklet.)

Example 4 *In triangle* XYZ, XY = 3.5, YZ = 4.5 *and* ZX = 6.5. *Calculate the size of angle Y.*

In this case the lengths of the three sides are given. The cosine rule can be used, but first it must be rearranged to make $\cos Y$ the subject.

$$y^2 = z^2 + x^2 - 2zx \cos Y$$

$$\therefore 2zx \cos Y = z^2 + x^2 - y^2$$

and hence

$$\cos Y = \frac{z^2 + x^2 - y^2}{2zx}$$

Substituting the data,

$$\cos Y = \frac{3.5^2 + 4.5^2 - 6.5^2}{2 \times 3.5 \times 4.5}$$

$$\therefore Y = 108.0°, \text{ correct to the nearest tenth of a degree}$$

Once again, if you are using a calculator, the entire calculation should be done without writing down the intermediate steps. Be careful to press the 'equals' key when you have completed the top line (the calculator should display −9.75 at this stage).

18.3 Area of a triangle

We assume that you are familiar with the elementary formula for Δ, the area of a triangle, namely,

$$\Delta = \tfrac{1}{2}bh$$

where b is the length of the base and h is the height of the triangle.

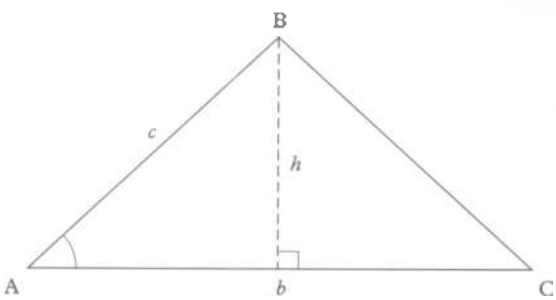


Figure 18.8

If we are given the lengths b and c and the size of the included angle A (see Fig. 18.8), then the height, h, can be expressed as

$$h = c \sin A$$

and the formula for the area can be written

$$\boldsymbol{\Delta = \tfrac{1}{2}bc \sin A}$$

(Note that this formula can be used for both acute and obtuse angles.)

Example 5 *In triangle* PQR, QR = 3.5, RP = 4 *and* PQ = 5. *Calculate the size of angle P and hence find the area of the triangle.*

Rearranging the cosine rule (see Example 4),

$$\cos P = \frac{q^2 + r^2 - p^2}{2qr}$$

and substituting the data, $p = 3.5$, $q = 4$ and $r = 5$, we have

$$\cos P = \frac{16 + 25 - 12.25}{2 \times 4 \times 5}$$

$$= \frac{28.75}{40}$$

$\therefore P = 44.0°$, correct to the nearest tenth of a degree

The area of the triangle is given by

$$\Delta = \tfrac{1}{2}qr \sin P$$

$$= \tfrac{1}{2} \times 4 \times 5 \times \sin P$$

$= 6.95$, correct to three significant figures

Note. When no units have been explicitly stated, as in the above example, it is assumed that the same units have been used consistently throughout the question, e.g. if the lengths QR, RP and PQ are in cm, then the area of PQR will be in cm^2.

The problem of calculating the area of a triangle when the lengths of the three sides are given is a very ancient one. The area can be calculated from the formula

$$\boldsymbol{\Delta = \sqrt{\{s(s-a)(s-b)(s-c)\}}}$$

where $s = \tfrac{1}{2}(a + b + c)$.

This formula is usually known as **Heron's formula**.

Questions

Q1 Calculate the area of the triangle in Example 3.

Q2 Use Heron's formula to calculate the area of the triangle in Example 5.

Q3 Calculate the areas of the triangles in which

a	$A = 60°$,	$b = 3$,	$c = 5$
b	$C = 110°$,	$a = 14$,	$b = 11$
c	$B = 90°$,	$c = 8.6$,	$b = 11.4$
d	$a = 8$,	$b = 11$,	$c = 13$
e	$a = 12.3$,	$b = 14.1$,	$c = 13.6$
f	$a = 17.6$,	$b = 16.9$,	$c = 16.1$
g	$a = 209$,	$b = 313$,	$c = 390$

Exercise 18a

Solve the following triangles:

1 (Sine formula, acute angled)

a	$a = 12$,	$B = 59°$,	$C = 73°$
b	$A = 75.6°$,	$b = 5.6$,	$C = 48.3°$
c	$A = 73.2°$,	$B = 61.7°$,	$c = 171$

2 (Sine formula, obtuse angled)

a	$A = 36°$,	$b = 2.37$,	$C = 49°$
b	$A = 123.2°$,	$a = 11.5$,	$C = 37.1°$
c	$a = 136$,	$B = 104.2°$,	$C = 43.1°$

3 (Sine formula, ambiguous case)

a	$b = 17.6$,	$C = 48° 15'$,	$c = 15.3$
b	$B = 129°$,	$b = 7.89$,	$c = 4.56$
c	$A = 28° 15'$,	$a = 8.5$,	$b = 14.8$

4 (Cosine formula, acute angled)

a	$a = 5$,	$b = 8$,	$c = 7$
b	$a = 10$,	$b = 12$,	$c = 9$
c	$a = 17$,	$b = 13$,	$c = 18$

5 (Cosine formula, acute angled)

a	$A = 60°$,	$b = 8$,	$c = 15$
b	$a = 14$,	$B = 53°$,	$c = 12$
c	$a = 11$,	$b = 9$,	$C = 43.2°$

6 (Cosine formula, obtuse angled)

a	$a = 8$,	$b = 10$,	$c = 15$
b	$a = 11$,	$b = 31$,	$c = 24$
c	$a = 27$,	$b = 35$,	$c = 46$

7 (Cosine formula, obtuse angled)

a	$a = 17$,	$B = 120°$,	$c = 63$
b	$A = 104° 15'$,	$b = 10$,	$c = 12$
c	$a = 31$,	$b = 42$,	$C = 104° 10'$

8 Two points A and B on a straight coastline are 1 km apart, B being due East of A. If a ship is observed on bearings 167° and 205° from A and B respectively, what is its distance from the coastline?

9 A student walks directly towards a building. The angle of elevation of a point on the top of the building increases from 10° to 15° as the student walks a distance of 50 m. What is the height of the building?

10 A triangle is taken with sides 10, 11, 15 cm. By how much does its largest angle differ from a right angle?

11 A ship first sails 4 nautical miles on a course of 069° then 5 nautical miles on a course of 295°. Calculate the distance and bearing of its new position from its original position.

12 A student travelling along a straight level road in the direction 053° observes a wireless mast on a bearing of 037°. A further 800 m along the road the bearing of the pylon is 296°. Calculate the distance of the mast from the road.

18.4 Radians

The fact that there are 90 degrees in a right angle is very familiar. This is an arbitrary number which has come down to us from the Babylonian civilisation. The following example illustrates the arbitrary nature of the number of degrees in a right angle.

Example 6 *An arc AB of a circle, centre O, subtends an angle of $x°$ at O. Find expressions in terms of x and the radius, r, for* **a** *the length of the arc AB,* **b** *the area of the sector OAB (see* **Fig. 18.9**).

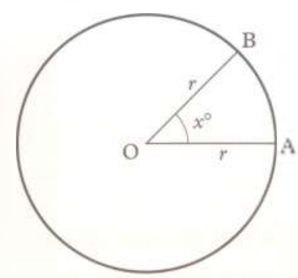


Figure 18.9

a The length of an arc of a given circle is proportional to the angle it subtends at the centre. But an angle of 360° is subtended by an arc of length $2\pi r$, therefore an angle of $x°$ is subtended by an arc of length

$$\frac{x}{360} \times 2\pi r$$

Therefore the length of arc AB is $(\pi/180)xr$.

b The area of a sector of a given circle is proportional to the angle at the centre. But a sector containing an angle of 360° is the whole circle, which has an area of πr^2, therefore a sector containing an angle of $x°$ has an area of

$$\frac{x}{360} \times \pi r^2$$

Therefore the area of the sector OAB is $\frac{1}{2}(\pi/180)xr^2$.

Example 6 shows that in both the length of an arc and the area of a sector, there appears a factor of $\pi/180$, which is due to the unit of measurement of the angle OAB. This suggests a new unit for measuring angles, which is called **a radian**, such that an

$$\text{angle in radians} = \frac{\pi}{180}(\text{angle in degrees}) \qquad (1)$$

If we let θ radians equal x degrees, then, referring to Fig. 18.9,

the **length of arc AB** $= r\theta$

and

the **area of sector OAB** $= \frac{1}{2}r^2\theta$

If we construct an angle of 1 radian, the arc AB will be of length r, and so *an arc of a circle equal to the radius subtends at the centre an angle of 1 radian.*

Radians are sometimes termed **circular measure**, and are denoted by rad. It follows from (1) above that

π **rad = 180°**

Hence 1 radian = 57.296 degrees and
1 degree = 0.017 453 radians, both correct to five significant figures.

The use of radians extends far beyond finding lengths of arcs and areas of sectors. Later sections show how they have applications in mechanics and calculus.

Exercise 18b (Oral)

1 Convert to degrees:

a $\frac{\pi}{2}$ rad b $\frac{\pi}{4}$ rad c $\frac{\pi}{3}$ rad
d $\frac{2\pi}{3}$ rad e $\frac{\pi}{6}$ rad f $\frac{3\pi}{2}$ rad
g $\frac{5\pi}{2}$ rad h 4π rad i 5π rad
j $\frac{4\pi}{3}$ rad k $\frac{7\pi}{2}$ rad l $\frac{3\pi}{4}$ rad

2 Convert to radians, leaving π in your answer:

a 360° b 90° c 45° d 15°
e 60° f 120° g 300° h 270°
i 540° j 30° k 150° l 450°

3 What is the length of an arc which subtends an angle of 0.8 rad at the centre of a circle of radius 10 cm?

4 An arc of a circle subtends an angle of 1.2 rad at any point on the remaining part of the circumference. Find the length of the arc, if the radius of the circle is 4 cm.

5 An arc of a circle subtends an angle of 0.5 rad at the centre. Find the radius of the circle, if the length of the arc is 3 cm.

6 Find, in radians, the angle subtended at the centre of a circle of radius 2.5 cm by an arc 2 cm long.

7 What is the area of a sector containing an angle of 1.5 rad, in a circle of radius 2 cm?

8 The radius of a circle is 3 cm. What is the angle contained by a sector of area 18 cm^2?

9 An arc subtends an angle of 1 rad at the centre of a circle, and a sector of area 72 cm^2 is bounded by this arc and two radii. What is the radius of the circle?

10 The arc of a sector in a circle, radius 2 cm, is 4 cm long. What is the area of the sector?

Exercise 18c

1 Express in radians, leaving π in your answers:

a $22\frac{1}{2}°$ b 1080° c 12′ d 37° 30′

2 Express in degrees:

a $\frac{2\pi}{5}$ rad b $\frac{\pi}{36}$ rad
c $\frac{7\pi}{12}$ rad d $\frac{7\pi}{2}$ rad

3 Find the length of an arc of a circle, which subtends an angle of 31° at the centre, if the radius of the circle is 5 cm.

4 The chord AB of a circle subtends an angle of 60° at the centre. What is the ratio of chord AB to arc AB?

5 An arc of a circle, radius 2.5 cm, is 3 cm long. What is the angle subtended by the arc at the centre

a in radians, b in degrees?

6 A segment is cut off a circle of radius 5 cm by a chord AB, 6 cm long. What is the length of the minor arc AB?

7 What is the area of a sector containing an angle of 1.4 rad in a circle whose radius is 2.4 cm?

8 A chord AB subtends an angle of 120° at O, the centre of a circle with radius 12 cm. Find the area of

a sector AOB
b triangle AOB
c the minor segment AB

9 An arc AB of a circle with radius 6 cm subtends an angle of 40° at the centre. Find the area bounded by the diameter BC, CA and the arc AB.

10 Two equal circles of radius 5 cm are situated with their centres 6 cm apart. Calculate the area that lies within both circles.

11 A chord PQ of a circle with radius r, subtends an angle θ at the centre. Show that the area of the minor segment PQ is $\frac{1}{2}r^2(\theta - \sin\theta)$, and write down the area of the major segment PQ in terms of r and θ.

12 A circle of radius r is drawn with its centre on the circumference of another circle of radius r. Show that the area common to both circles is $2r^2(\pi/3 - \sqrt{3}/4)$.

18.5 Angular velocity

The label on an electric motor may say that it does 12 000 revolutions per minute (rev/min or rpm). On the other hand the drum of a barograph turns at the rate of 49 degrees per day*. In either case the rate of turning, which is called average **angular velocity**, is given by

$$\text{average angular velocity} = \frac{\text{angle turned}}{\text{time taken}}$$

Questions

Q4 Find the average angular velocity of the second hand of a watch

a in degrees per second (deg/s),

b in rev/min.

Q5 Convert

a 500 rpm into deg/s,

b 1 rev/week into deg/h.

In many cases of turning, however, the angular velocity is not constant, so consider the average angular velocity in a small interval of time δt. If the angle turned through in this time is $\delta\theta$ radians,

$$\text{average angular velocity} = \frac{\delta\theta}{\delta t}\ \text{rad/s}$$

But as $\delta t \to 0$,

$$\frac{\delta\theta}{\delta t} \to \frac{d\theta}{dt}$$

$$\therefore \text{average angular velocity} \to \frac{d\theta}{dt}$$

$\frac{d\theta}{dt}$ is a measure of angular velocity and is denoted by ω (the Greek letter omega).

Therefore

$$\omega = \frac{d\theta}{dt}$$

[In motion in a straight line average velocity $= \frac{\text{distance}}{\text{time}}$.

If a distance δs is travelled in a time δt, average velocity $= \frac{\delta s}{\delta t}$.

But $\frac{\delta s}{\delta t} \to \frac{ds}{dt}$ as $\delta t \to 0$ and so the velocity at an instant is given by $v = \frac{ds}{dt}$. In this way there is a parallel between linear motion and angular motion.]

If a particle moves in a circle of radius r with speed v and angular velocity ω about the centre, the relation between r, v, ω can be obtained from §18.4 on page 216. If s is the distance of the particle measured along the circumference of the circle from a fixed point,

$$s = r\theta$$

Differentiating with respect to time (remember r is constant),

$$\frac{ds}{dt} = r\frac{d\theta}{dt}$$

$$\therefore v = r\omega$$

Remember that ω must be measured in radians/unit time. Three sets of possible units for v, r, ω are shown in the table below:

v	r	ω
m/s	m	rad/s
km/h	km	rad/h
cm/min	cm	rad/min

*A barograph is an instrument that records atmospheric pressure. It rotates about once per week.

Example 7 *A belt runs round a pulley attached to the shaft of a motor. If the belt runs at 0.75 m/s and the radius of the pulley is 6 cm, find the angular velocity of the pulley* **a** *in rad/s,* **b** *in rev/min.*

a Using the result $v = r\omega$,

$$\omega = \frac{75}{6} = 12.5 \text{ rad/s}$$

b $12.5 \text{ rad/s} = \frac{12.5}{2\pi} \text{ rev/s}$

$$= \frac{12.5}{2\pi} \times 60 \text{ rev/min}$$

$$\approx 120 \text{ rev/min}$$

(The sign ≈ means 'is approximately equal to'.) Therefore the angular velocity is 12.5 rad/s or approximately 120 rev/min.

Exercise 18d

Use the result $v = r\omega$ where you can.

1 Express the angular velocity of the minute hand of a clock in
a rev/min **b** deg/s **c** rad/s

2 A wheel is turning at 200 rev/min. Express this angular velocity in
a deg/s **b** rad/s

3 A machine rotates the handle of an egg whisk 32 times in 5 seconds. Each time the handle rotates, the paddles rotate four times. At what speed are the paddles rotating in
a rev/min, **b** rad/s?

4 The Earth rotates on its axis approximately $365\frac{1}{4}$ times in a year. Calculate its angular velocity in rad/h, correct to three significant figures.

5 The cutters of an electric shaver rotate about 3000 times a minute, and the distance from the axis to the tip of the cutter is 0.65 cm. Find
a the angular velocity of the cutter in rad/s,
b the speed of the tip of a cutter in cm/s.

6 A disc rotates through 334° in $1\frac{1}{2}$ s approximately. What is the average angular speed of the disc in rad/s, and what is the speed of a point on the circumference of the disc if its diameter is 8 cm?

7 A motor runs at 1200 rev/min. What is its angular velocity in rad/s? If the shaft of the motor is 2.5 cm in diameter, at what speed is a point on the circumference of the shaft moving?

8 A point on the rim of a wheel of diameter 2.5 m is moving at a speed of 44 m/s relative to the axis. At what rate in **a** rad/s, **b** rev/min, is the wheel turning?

9 If a cotton reel drops 1.76 m in 0.7 s, the end of the cotton being held still, at what average angular velocity, in rev/min, is the reel turning, if its diameter is 3 cm?

10 A belt runs round two pulleys of diameters 26.25 cm and 15 cm. If the larger rotates 700 times in a minute, find the angular velocity of the smaller in rad/s.

11 In 1 year the Earth moves round the sun approximately in a circle of radius 150 000 000 km. Find its angular speed in rad/s, and obtain its speed along its orbit in km/s.

12 Taking the Earth to be a sphere of radius 6300 km which rotates about its axis once in 23.93 hours, what error will be made in calculating the velocity of a point on the equator, if it is assumed that the Earth rotates once in 24 hours? Express your answer in km/h, correct to two significant figures.

18.6 Inverse trigonometrical functions

Can you find an angle $x°$, such that $\sin x° = 0.5$? Problems like this arise frequently in mathematics. An answer can be easily obtained from tables or from a calculator. In this particular case, the angle $x°$ is an angle in one of the 'standard' triangles described in §16.3 on page 196, i.e. 30°. But this is not the complete solution; we can see from the graph of $y = \sin x$ (Fig. 18.10), that 150° is also a possibility.

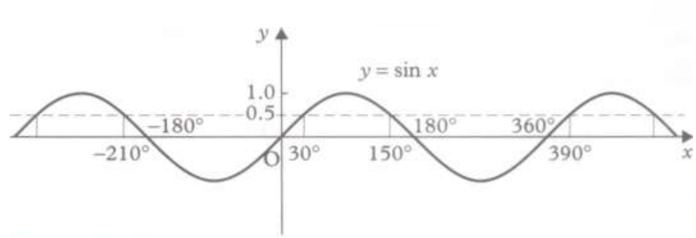


Figure 18.10

Moreover, since sin x has a period of 360° (it repeats itself every 360°), we can add any multiple of 360° to these two angles. Hence there are infinitely many values of x which satisfy the equation $\sin x = 0.5$; they can be expressed in the form

$$x = 30° + n360° \quad \text{or} \quad x = 150° + n360°$$

where n is any integer, positive or negative. If we were working in radians, this general solution would take the form

$$x = \frac{\pi}{6} + 2n\pi \quad \text{or} \quad x = \frac{5\pi}{6} + 2n\pi$$

Questions

Q6 Write down the general solution, in degrees, of the equation $\cos x° = -0.5$.

Q7 Write down, in radians, the general solution of the equation $\tan x = 1$.

In advanced trigonometry, it is useful to have an abbreviation for the phrase 'the angle whose sine is x', etc. The usual abbreviation for this is **arcsin x**. **arccos x**, **arctan x** are used for the **inverses** of the cos and tan functions. This is the standard notation on all computers and it is also found on many calculators. The notation **$\sin^{-1} x$, $\cos^{-1} x$** and **$\tan^{-1} x$**, is also used.*

The fact that there are infinitely many angles whose sine is x, causes some problems. For instance, if you were designing a calculator, which of the infinitely many possible answers would you choose to show on the display? (Try finding arcsin, arccos and arctan of ±0.2, ±0.4, ±0.8, etc., on a scientific calculator. Can you discover the principle which the manufacturer of your calculator used to select the angle shown on the display?)

Another serious problem is that if we are intending to describe arcsin x, arccos x and arctan x, as *functions*, then we must ensure that the function has *exactly one value*, for any given value of x (see §2.8 on page 47). Consequently we must define these functions more carefully than we have done so far.

Definitions

a **arcsin x** *is the angle (in radians) between* $-\frac{1}{2}\pi$ *and* $+\frac{1}{2}\pi$, *inclusive, whose sine is* x.

b **arccos x** *is the angle (in radians) between* 0 *and* π, *inclusive, whose cosine is* x.

c **arctan x** *is the angle (in radians) between* $-\frac{1}{2}\pi$ *and* $+\frac{1}{2}\pi$, *whose tangent is* x.

(The angles within these ranges are often called the *principal values*.)

If desired, these definitions may be expressed in degrees, but for advanced work in trigonometry, radians are more common than degrees.

Question

Q8 Why is the range $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$ unsuitable for arccos x?

Notice that, since there is no angle whose sine is greater than 1, an expression such as arcsin 2 is meaningless. The function arcsin x only makes sense if x is numerically smaller than (or equal to) 1. In other words, *the domain* of the function arcsin x is $\{x: -1 \leqslant x \leqslant +1\}$. The function arccos x has the same domain, but in the function arctan x, the variable x can take any (real) value, i.e. the domain of arctan x is $\mathbb{R}$ (see Fig. 18.11).

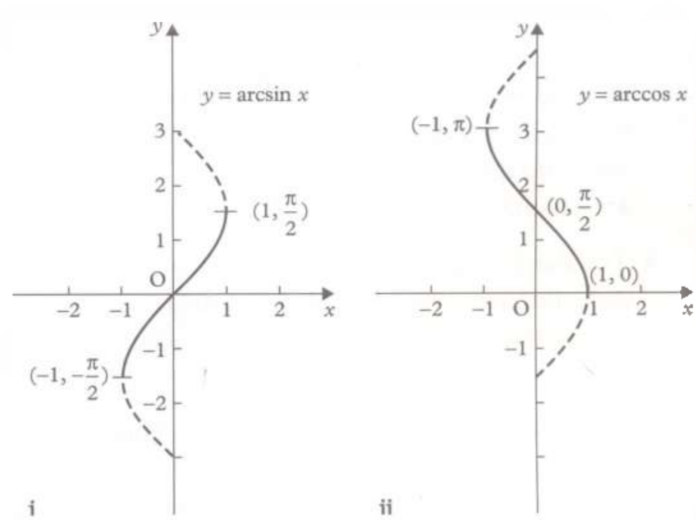


Figure 18.11

*Note that $\sin^{-1} x$ does not mean $\frac{1}{\sin x}$ or $\sin \frac{1}{x}$. These would be written $(\sin x)^{-1}$ and $\sin (x)^{-1}$ respectively.

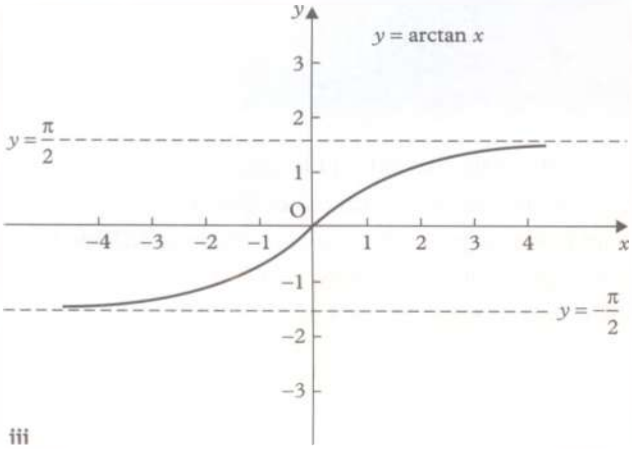


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Figure 18.11 (continued)

Like all inverse functions, the graphs of arcsin x, arccos x and arctan x are the reflections of the graphs of the corresponding functions in the line $y = x$.

In diagrams **i** and **ii**, the solid parts of the graphs represent the principal values of arcsin x and arccos x respectively; the broken parts of the graphs represent the other values.

Exercise 18e

All the questions in this exercise use the angles in the 'standard' triangles (see §16.3 on page 196).
Do not use a calculator.

Write down the general solutions of the following equations (in degrees):

1 $\sin x° = 1/\sqrt{2}$ **2** $\cos x° = 1$

3 $\tan x° = \sqrt{3}$ **4** $\sin x° = -1$

5 $\cos x° = -1/2$ **6** $\tan x° = -1/\sqrt{3}$

Write down, *in radians*, the general solutions of the following equations:

7 $\cos x = \frac{1}{2}$ **8** $\tan x = -1$

9 $\sin 2x = \frac{1}{2}$ **10** $\cos^2 x = \frac{3}{4}$

Write down, *in radians*, the values of

11 arcsin $(\sqrt{3}/2)$ **12** arccos $(1/\sqrt{2})$

13 arctan 1 **14** arcsin $(-\frac{1}{2})$

15 arccos $(-\sqrt{3}/2)$ **16** arctan (-1)

17 arcsin (-1) **18** arccos (-1)

19 arctan 0 **20** arccos 0

Differentiation (3)

Trigonometrical functions

19.1 Small angles

Fig. 19.1 shows that, for small acute angles, $\tan\theta$, θ and $\sin\theta$ are practically equal.

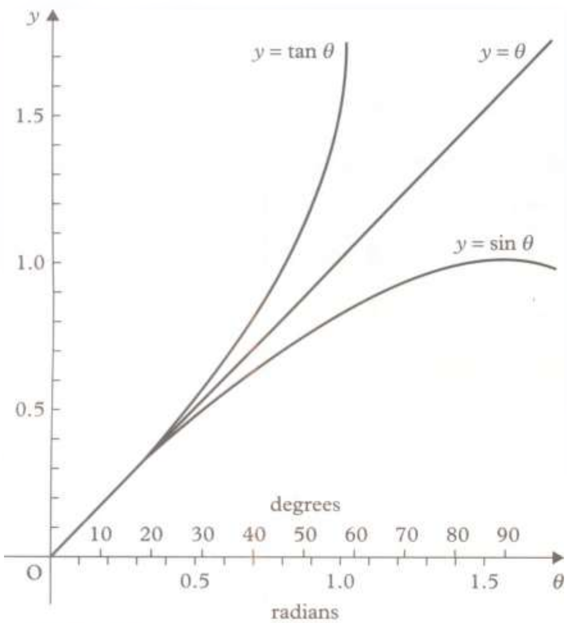


Figure 19.1

This can be confirmed on a scientific calculator:

Angle in degrees	10°	5°	1°
θ (radians)	0.174 532 9	0.087 266 4	0.017 453 2
$\tan\theta$	0.176 326 9	0.087 488 6	0.017 455 1
$\sin\theta$	0.173 648 1	0.087 155 7	0.017 452 4

We shall now consider this geometrically.

In Fig. 19.2, the chord AB subtends an angle θ at the centre of a circle of radius r, and the tangent at B meets OA at D. Consider the three areas: triangle AOB, sector AOB, triangle DOB.

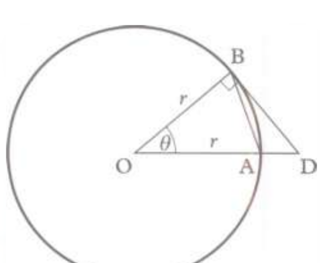


Figure 19.2

a In triangle AOB, two sides of length r include an angle θ, therefore its area is $\frac{1}{2}r^2\sin\theta$ (see §18.3 on page 215).

b From §18.4 on page 216, the area of sector AOB is $\frac{1}{2}r^2\theta$.

c In triangle DOB, B is a right angle, therefore BD $= r\tan\theta$ and so its area is $\frac{1}{2}r^2\tan\theta$.

From the figure it can be seen that

triangle AOB < sector AOB < triangle DOB

$$\therefore \tfrac{1}{2}r^2\sin\theta < \tfrac{1}{2}r^2\theta < \tfrac{1}{2}r^2\tan\theta$$

If we divide each term by $\frac{1}{2}r^2$,

$$\sin\theta < \theta < \tan\theta$$

providing θ is acute, as in Fig. 19.2. Again, if we divide each term by $\sin\theta$,

$$\frac{\sin\theta}{\sin\theta} < \frac{\theta}{\sin\theta} < \frac{\tan\theta}{\sin\theta}$$

But $\tan\theta = \sin\theta/\cos\theta$, therefore

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

Now as $\theta \to 0$, $\cos\theta \to 1$ and $\dfrac{1}{\cos\theta} \to 1$

Thus $\theta/\sin\theta$ lies between 1 and a function which approaches 1 as $\theta \to 0$.

$$\therefore \frac{\boldsymbol{\theta}}{\mathbf{sin}\,\boldsymbol{\theta}} \to \mathbf{1} \quad \text{as } \theta \to 0$$

(See Chapter 2, Example 17 on page 58 and Q11.)

This limit (or, more strictly, $(\sin\theta)/\theta \to 1$ as $\theta \to 0$) is required in the next section for the differentiation of $\sin x$.

Another way of expressing the statement that $\theta/\sin\theta \to 1$ as $\theta \to 0$, is to say that, for small values of θ,

$$\mathbf{sin}\,\boldsymbol{\theta} \approx \boldsymbol{\theta}$$

An approximation for $\cos\theta$ is obtained from the identity

$$\cos\theta = 1 - 2\sin^2\tfrac{1}{2}\theta$$

If θ is small, $\sin\frac{1}{2}\theta \approx \frac{1}{2}\theta$, therefore

$$\cos\theta \approx 1 - 2(\tfrac{1}{2}\theta)^2$$

Therefore, for small values of θ,

$$\mathbf{cos}\,\boldsymbol{\theta} \approx \mathbf{1} - \tfrac{1}{2}\boldsymbol{\theta}^2$$

Example 1 *Find the approximate value of* $\dfrac{1-\cos 2\theta}{\theta \tan \theta}$ *when θ is small.*

We cannot put $\theta = 0$, as the numerator and denominator would both be zero. Since

$$\cos \theta \approx 1 - \tfrac{1}{2}\theta^2,$$

$$\cos 2\theta \approx 1 - \tfrac{1}{2}(2\theta)^2 = 1 - 2\theta^2$$

Therefore the numerator $\approx 2\theta^2$. But the denominator $\approx \theta^2$, since $\tan \theta \approx \theta$. Therefore, when θ is small,

$$\frac{1-\cos 2\theta}{\theta \tan \theta} \approx \frac{2\theta^2}{\theta^2}$$

$$\therefore \frac{1-\cos 2\theta}{\theta \tan \theta} \approx 2 \quad \text{when } \theta \text{ is small}$$

Question

Q1 Find approximations for the following functions when θ is small:

a $\dfrac{\sin 3\theta}{2\theta}$ **b** $\dfrac{\sin 4\theta}{\sin 2\theta}$

c $\dfrac{1-\cos \theta}{\theta^2}$ **d** $\dfrac{\theta \sin \theta}{1-\cos 2\theta}$

e $\dfrac{\sin(\alpha+\theta)\sin\theta}{\theta}$ **f** $\dfrac{\sin(\alpha+\theta)-\sin\alpha}{\theta}$

g $\dfrac{\sin\theta\tan\theta}{1-\cos 3\theta}$ **h** $\sin\theta \operatorname{cosec} \tfrac{1}{2}\theta$

i $\dfrac{\tan(\alpha+\theta)-\tan\alpha}{\theta}$

19.2 Differentiation of sin *x* and cos *x*

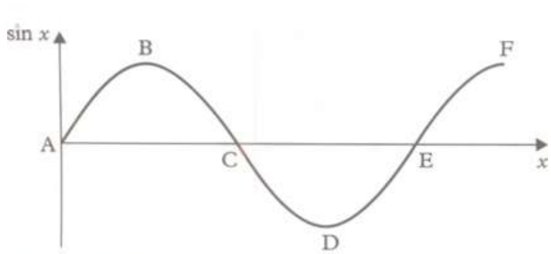


Figure 19.3

Fig. 19.4 is a sketch graph of sin *x*. We can use this to sketch a graph of its gradient. The gradient is zero at B, D, F, positive from A to B and from D to F, and negative from B to D, giving a graph like Fig. 19.4.

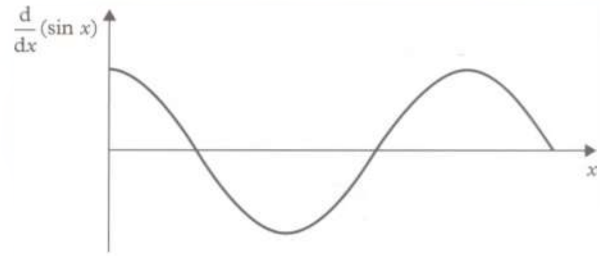


Figure 19.4

Questions

Q2 Does Fig. 19.4 look like any graph you have met so far?

Q3 Express $\sin A - \sin B$ in factors. (See §17.5 on page 208.)

We shall now find the derivative of sin *x* from first principles, using the definition in §3.5 on page 66, that is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

(You should review §3.5 on page 66 before proceeding further.)

In this case, $f(x) = \sin x$, and so

$$f(x+h) - f(x) = \sin(x+h) - \sin x$$

Using the factor formula (see Q3 above), this can be written

$$f(x+h) - f(x) = 2\cos\frac{x+h+x}{2}\sin\frac{h}{2}$$

$$\therefore \frac{f(x+h)-f(x)}{h} = \frac{2\cos\dfrac{2x+h}{2}\sin\dfrac{h}{2}}{h} \qquad (1)$$

$$= \frac{\cos(x+\frac{1}{2}h)\sin\frac{1}{2}h}{\frac{1}{2}h}$$

But we know that when $h \to 0$,

$$\cos(x+\tfrac{1}{2}h) \to \cos x \quad \text{and} \quad \frac{\sin\frac{1}{2}h}{\frac{1}{2}h} \to 1$$

Therefore, when $h \to 0$, the right-hand side of equation (1) tends to cos *x*. So, for this function,

$$\mathbf{f'(x) = \cos x}$$

In Leibnitz notation, this is written

$$y = \sin x \qquad \frac{dy}{dx} = \cos x$$

Or, more concisely,

$$\boldsymbol{\frac{d}{dx}(\sin x) = \cos x}$$

Questions

Q4 At what stage in the above is it necessary to have x in radians?

Q5 Prove from first principles that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$$

Remember that these results hold only if x is in radians.

Example 2 *Differentiate* **a** $\sin(2x+3)$, **b** $\cos^2 x$, **c** $\sin x^\circ$.

a Let $y = \sin(2x+3)$, $t = 2x+3$, then $y = \sin t$.

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}t} = \cos t \quad \text{and} \quad \frac{\mathrm{d}t}{\mathrm{d}x} = 2$$

But $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}t} \times \dfrac{\mathrm{d}t}{\mathrm{d}x} = (\cos t)2$,

$$\therefore \frac{\mathrm{d}}{\mathrm{d}x}\{\sin(2x+3)\} = 2\cos(2x+3)$$

b Let $y = \cos^2 x$, $t = \cos x$, then $y = t^2$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} = 2t(-\sin x) = -2\cos x \sin x$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}x}(\cos^2 x) = -\sin 2x$$

c Let $y = \sin x^\circ$.

$x^\circ = (\pi/180)x$ radians,

$$\therefore y = \sin \frac{\pi}{180}x$$

Put $t = (\pi/180)x$, then $y = \sin t$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} = (\cos t)\frac{\pi}{180}$$

$$= \frac{\pi}{180}\cos\frac{\pi}{180}x = \frac{\pi}{180}\cos x^\circ$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}x}(\sin x^\circ) = \frac{\pi}{180}\cos x^\circ$$

Question

Q6 Differentiate

a $\cos 3x$ **b** $\sin^2 x$
c $2\sin 2x$ **d** $\cos^3 x$

Example 3 *Integrate* **a** $\cos 2x$, **b** $3\sin\frac{1}{2}x$.

The method used here is to change cos to sin, or sin to cos, and to determine the coefficient by differentiation:

a $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sin 2x) = 2\cos 2x$.

$$\therefore \frac{\mathrm{d}}{\mathrm{d}x}(\tfrac{1}{2}\sin 2x) = \cos 2x$$

$$\therefore \int \cos 2x \,\mathrm{d}x = \tfrac{1}{2}\sin 2x + c$$

b $\dfrac{\mathrm{d}}{\mathrm{d}x}(3\cos\frac{1}{2}x) = -\frac{3}{2}\sin\frac{1}{2}x$.

$$\therefore \frac{\mathrm{d}}{\mathrm{d}x}(-2\times 3\cos\tfrac{1}{2}x) = 3\sin\tfrac{1}{2}x$$

$$\therefore \int 3\sin\tfrac{1}{2}x\,\mathrm{d}x = -6\cos\tfrac{1}{2}x + c$$

Exercise 19a

1 Differentiate:

a	$\cos 2x$	**b**	$\sin 6x$
c	$\cos(3x-1)$	**d**	$\sin(2x-3)$
e	$-3\cos 5x$	**f**	$2\sin 4x$
g	$-4\sin\frac{3}{2}x$	**h**	$2\sin\frac{1}{2}(x+1)$
i	$\sin x^2$		

2 Integrate:

a	$\sin 3x$	**b**	$\cos 3x$
c	$2\sin 4x$	**d**	$2\cos 2x$
e	$-\frac{1}{2}\sin 6x$	**f**	$6\cos 4x$
g	$\sin(2x+1)$	**h**	$3\cos(2x-1)$
i	$\frac{2}{3}\sin\frac{1}{2}x$		

3 Differentiate:

a	$\sin^2 x$	**b**	$4\cos^2 x$
c	$\cos^3 x$	**d**	$2\sin^3 x$
e	$3\cos^4 x$	**f**	$\sqrt{(\sin x)}$
g	$\sqrt{(\cos x)}$	**h**	$\cos^2 3x$
i	$\sin^2 2x$	**j**	$-2\sin^3 3x$
k	$3\sin^4 2x$	**l**	$\sqrt{(\sin 2x)}$

4 Differentiate:

a	$x\cos x$	**b**	$x\sin 2x$
c	$x^2\sin x$	**d**	$\sin x\cos x$
e	$\dfrac{\sin x}{x}$	**f**	$\dfrac{\cos 2x}{x}$
g	$\dfrac{x}{\sin x}$	**h**	$\dfrac{x^2}{\cos x}$

i $\dfrac{\sin x}{\cos x}$ j $\cot x$

k $\dfrac{1}{\cos x}$ l $\operatorname{cosec} x$

5 A particle moves in a straight line such that its velocity in m/s, t s after passing through a fixed point O, is $3\cos t - 2\sin t$. Find
- **a** its distance from O after $\frac{1}{2}\pi$ s,
- **b** its acceleration after π s,
- **c** the time when its velocity is first zero.

6 A particle is moving in a straight line in such a way that its distance from a fixed point O, t s after the motion begins, is $\cos t + \cos 2t$ cm. Find
- **a** the time when the particle first passes through O,
- **b** the velocity of the particle at this instant,
- **c** the acceleration when the velocity is zero.

7 The distance of a particle from a fixed point O is given by $s = 3\cos 2t + 4\sin 2t$.
Show that the velocity v and the acceleration a are given by $v^2 + 4s^2 = 100$, $a + 4s = 0$. Hence find
- **a** the greatest distance of the particle from O,
- **b** the acceleration at this instant.

8 The velocity at time t of a particle moving in a straight line is $6\cos 2t + \cos t$, and when $t = 0$, the particle is at O. Find
- **a** the time when v is first zero,
- **b** the distance from O at this instant,
- **c** the acceleration at the same instant.

9 Find the area between the curve $y = \sin 3x$ and the x-axis between $x = 0$ and $x = \frac{1}{3}\pi$.

10 Sketch the curve $y = 1 + \cos x$ from $x = -\pi$ to $x = \pi$, and find the area enclosed by the curve and the x-axis between these limits.

11 Find the maximum value of $y = x + \sin 2x$ which is given by a value of x between 0 and $\frac{1}{2}\pi$. Sketch the graph of y for $0 \leqslant x \leqslant \frac{1}{2}\pi$ and find the area bounded by the curve, the x-axis and the line $x = \frac{1}{2}\pi$.

12 Find the maximum value of $y = 2\sin x - x$ which is given by a value of x between 0 and $\frac{1}{2}\pi$. Sketch the graph of y for values of x from 0 to π, and find the area between the curve, the x-axis and the line $x = \frac{1}{2}\pi$.

13 Show that $\dfrac{d}{dx}\left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right) = \sin^2 x$ and deduce that

$$\int_0^{\pi} \sin^2 x \, dx = \tfrac{1}{2}\pi$$

14 Express $\cos^2 x$ in terms of $\cos 2x$, and hence show that

$$\int \cos^2 x \, dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + c$$

15 Show that $\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$, and deduce that

$$\begin{aligned}\int \cos^3 x \, dx &= \tfrac{1}{12}\sin 3x + \tfrac{3}{4}\sin x + c\\ &= \sin x - \tfrac{1}{3}\sin^3 x + c\end{aligned}$$

16 By expressing $\sin^3 x$ in terms of $\sin x$ and $\sin 3x$, show that

$$\begin{aligned}\int \sin^3 x \, dx &= \tfrac{1}{12}\cos 3x - \tfrac{3}{4}\cos x + c\\ &= \tfrac{1}{3}\cos^3 x - \cos x + c\end{aligned}$$

17 Express $2\cos 5x\cos 3x$ as a sum of two cosines and hence evaluate

$$\int_0^{\pi/4} 2\cos 5x \cos 3x \, dx$$

19.3 Derivatives of tan *x*, cot *x*, sec *x*, cosec *x*

Using the derivatives of $\sin x$ and $\cos x$, those of the four other trigonometrical ratios can be obtained by writing

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x} \qquad \operatorname{cosec} x = \frac{1}{\sin x}$$

This is left as an exercise if you have not already done question 4 (i)–(l) of Exercise 19a. The results are

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

Note:
- **a** the similarity of the pair of formulae on each line.
- **b** the associations between $\tan x$ and $\sec x$, and between $\cot x$ and $\operatorname{cosec} x$. The same associations occur in the identities $1 + \tan^2 x = \sec^2 x$, $\cot^2 x + 1 = \operatorname{cosec}^2 x$.
- **c** that the derivatives of ratios beginning with 'co', i.e. $\cos x$, $\cot x$, $\operatorname{cosec} x$, all have a negative sign.

Exercise 19b

1 Differentiate:

a $\tan 2x$ b $\cot 3x$
c $3 \sec 2x$ d $2 \operatorname{cosec} \frac{1}{2}x$
e $-\tan(2x+1)$ f $\frac{1}{3} \sec(3x-2)$
g $-2 \cot(3x+2)$ h $\cot x^2$
i $\tan \sqrt{x}$

2 Differentiate:

a $\tan^2 x$ b $\sec^2 x$
c $2 \cot^3 x$ d $3 \operatorname{cosec}^2 x$
e $-\tan^2 2x$* f $\frac{1}{2} \cot^2 3x$
g $\frac{1}{6} \sec^3 2x$ h $-2 \operatorname{cosec}^4 x$
i $\sqrt{(\tan x)}$

3 Differentiate:

a $x \tan x$ b $\sec x \tan x$
c $x^2 \cot x$ d $3x \operatorname{cosec} x$
e $\operatorname{cosec} x \cot x$ f $\dfrac{\tan x}{x}$
g $\dfrac{\sec x}{x^2}$ h $\sin x - x \cos x$
i $x \sec^2 x - \tan x$

4 Integrate:

a $\sec^2 2x$ b $3 \sec x \tan x$
c $-\operatorname{cosec}^2 \frac{1}{2}x$ d $\frac{1}{3} \operatorname{cosec} 3x \cot 3x$
e $2 \sec^2 x \tan x$ f $\dfrac{1}{\cos^2 x}$
g $\dfrac{\sin x}{\cos^2 x}$ h $\dfrac{1}{\sin^2 2x}$
i $\dfrac{\cos 2x}{\sin^2 2x}$

5 Sketch the graph of the curve $y = \sec^2 x - 1$ between $x = -\frac{1}{2}\pi$ and $x = \frac{1}{2}\pi$. Calculate the area enclosed by the curve, the x-axis and the line $x = \frac{1}{4}\pi$.

6 Find the volume generated by revolving the area bounded by the x-axis, the lines $x = \pm\frac{1}{4}\pi$ and the curve $y = \sec x$ about the x-axis.

7 Find the minimum values of the following functions which are given by values of x between 0 and $\frac{1}{2}\pi$:

a $\tan x + 3 \cot x$
b $\sec x + 8 \operatorname{cosec} x$
c $6 \sec x + \cot x$

8 By expressing $\tan^2 x$ in terms of $\sec^2 x$, show that

$$\int \tan^2 x \, dx = \tan x - x + c$$

9 Express $\cot^2 x$ in terms of $\operatorname{cosec}^2 x$ and hence integrate $\cot^2 x$.

*The following method of working is recommended when using the chain rule:

$$\frac{d}{dx}(3 \sin^4 5x) = \frac{d}{dx}\{3(\sin 5x)^4\}$$
$$= 3 \times 4(\sin 5x)^3 \times \cos 5x \times 5$$
$$= 60 \sin^3 5x \cos 5x.$$

Chapter 20

Loci

20.1 Introduction to loci

'A goat is tied to a fixed point O by a rope 6 m long. If the goat moves so that the rope is always tight, describe its path.' Clearly the goat moves in a circle, centre O, radius 6 m.

Now consider this problem: 'A goat is tied to a ring which can slide freely on a rope 6 m long. The ends of the rope are attached to two fixed points A and B 4 m apart. Describe the goat's path.' A scale drawing could be made, using a piece of string 6 cm long with its ends attached to two drawing pins fixed, 4 cm apart. Use a pencil to trace the goat's path, being careful to keep the string tight. The diagram should look something like Fig. 20.1. Note that at all points on the path, AP + PB = 6 m.

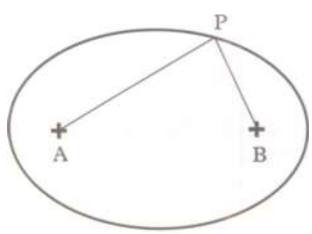


Figure 20.1

In this chapter, we shall use some of the techniques introduced earlier in the book to investigate problems like this. In particular, the moving point P will be represented by a point in the cartesian plane with coordinates (x, y). We shall try to find an equation which expresses, algebraically, the conditions governing the motion of P as it moves in a plane. (It is customary, in this context, to use P to represent a *moving* point. Any *fixed* points are usually represented by A, B or C, although in many cases the origin O is also a fixed point.) The path of P, as it moves according to the given conditions, is called the **locus** of P. The equation satisfied by the coordinates of P is called the **equation of the locus**. The plural of locus is **loci**.

20.2 Equation of a locus

In the first of the introductory problems above, the given condition is OP = 6, so, if O is the origin, the equation of the locus can be obtained by applying Pythagoras' theorem in Fig. 20.2.

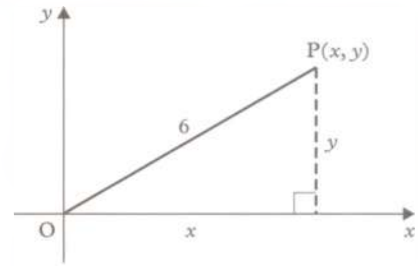


Figure 20.2

i.e. the equation of the locus is

$$x^2 + y^2 = 36$$

In the second problem, take the two fixed points to be $(-2, 0)$ and $(2, 0)$, respectively (Fig. 20.3).

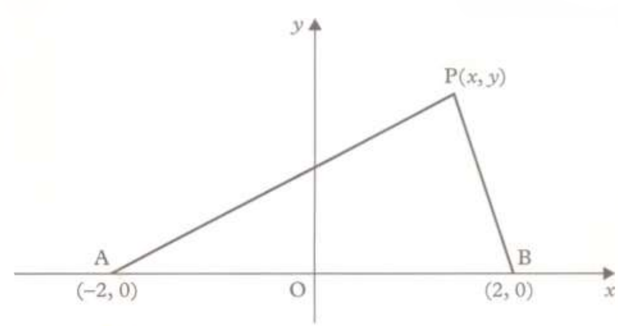


Figure 20.3

Applying the usual formula for the distance between two points we obtain

$$AP = \sqrt{\{(x+2)^2 + y^2\}} \quad \text{and} \quad BP = \sqrt{\{(x-2)^2 + y^2\}}$$

The condition which governs the movement of the point P is AP + PB = 6, so the equation of the locus is

$$\sqrt{\{(x+2)^2 + y^2\}} + \sqrt{\{(x-2)^2 + y^2\}} = 6$$

Question

Q1 Show that the above equation simplifies to

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

(We shall see in Chapter 30 that this is the equation of an ellipse.)

Example 1 *Find the equation of the locus of a point P which moves so that it is equidistant from two fixed points A and B whose coordinates are (3, 2) and (5, −1) respectively.*

Let P be the point (x, y).

The condition satisfied by P is

$$PA = PB$$

However, since we shall use Pythagoras' theorem to express the lengths of PA and PB in terms of x and y, it is neater to square this equation, obtaining

$$PA^2 = PB^2$$

Now

$$PA^2 = (x-3)^2 + (y-2)^2$$
$$PB^2 = (x-5)^2 + (y+1)^2$$

therefore the equation which must be satisfied by the coordinates of P is

$$(x-3)^2 + (y-2)^2 = (x-5)^2 + (y+1)^2$$

i.e. $x^2 - 6x + 9 + y^2 - 4y + 4 = x^2 - 10x + 25 + y^2 + 2y + 1$

or $4x - 6y - 13 = 0$

Therefore the equation of the locus of points equidistant from (3, 2) and (5, −1) is $4x - 6y - 13 = 0$.

This locus is actually the perpendicular bisector (or mediator) of AB. Because of the close connection between the locus and the equation connecting the points lying on the locus, the equation itself is often referred to as the locus.

Question

Q2 Find the equation of the locus in Example 1 by using the fact that it is the perpendicular bisector of AB.

Note: When drawing graphs it is often useful to take different scales on the two axes, but in coordinate geometry the scales should, if possible, be the same or the figures will be distorted.

Example 2 *Find the locus of a point P, whose distance from the point A(−1, 2) is twice its distance from the origin.*

Let $P(x, y)$ be a point on the locus (**Fig. 20.4**), then

$$PA = 2PO$$

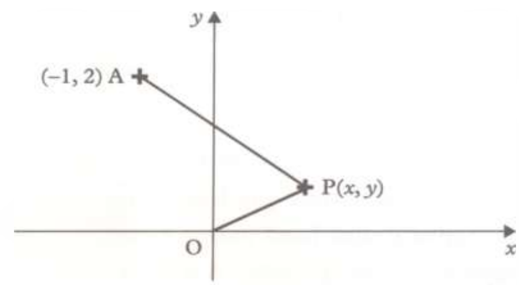


Figure 20.4

The lengths of PA and PO may be found using the method of §1.2 on page 33. Since both expressions involve a square root, it is neater to square first, giving

$$PA^2 = 4PO^2$$
$$\therefore (x+1)^2 + (y-2)^2 = 4(x^2 + y^2)$$
$$\therefore x^2 + 2x + 1 + y^2 - 4y + 4 = 4x^2 + 4y^2$$

Therefore the locus of P is $3x^2 + 3y^2 - 2x + 4y - 5 = 0$.

Exercise 20a

1 Find the equation of a circle with centre at the origin and radius 5 units.

2 What is the locus of a point which moves so that its distance from the point (3, 1) is 2 units?

3 What is the locus of a point which is equidistant from the origin and the point (−2, 5)?

4 What is the locus of a point which moves so that its distance from the point (−2, 1) is equal to its distance from the point (3, −2)?

5 What is the distance of the point (x, y) from the line $x = -1$? Find the locus of a point which is equidistant from the origin and the line $x = -1$.

6 Find the locus of a point which is equidistant from the point (0, 1) and the line $y = -1$.

7 Find the locus of a point which moves so that its distance from the point A(−2, 0) is three times its distance from the origin.

8 A point P moves so that its distance from A(2, 1) is twice its distance from B(−4, 5). What is the locus of P?

9 Find the locus of a point which moves so that its distance from the point (8, 0) is twice its distance from the line $x = 2$.

10 Find the locus of a point which moves so that its distance from the point (2, 0) is half its distance from the line $x = 8$.

11 Find the locus of a point which moves so that the sum of the squares of its distances from the points (−2, 0) and (2, 0) is 26 units.

12 Find the locus of a point which moves so that it is equidistant from the point $(a, 0)$ and the line $x = -a$.

13 A is the point (1, 0), and B is the point (−1, 0). Find the locus of a point P which moves so that PA + PB = 4.

14 A is the point (1, 0), and B is the point (−1, 0). Find the locus of a point P which moves so that PA − PB = 2.

15 A rectangle is formed by the axes and the lines $x = 4$ and $y = 6$. Find the locus of a point which moves so that the sum of the squares of its distances from the axes is equal to the sum of the squares of its distances from the other two sides.

Further examples

Example 3 *Show that the equation of the circle on the line segment joining* A(3, −5) *and* B(2, 6) *as diameter is* $(x-3)(x-2)+(y+5)(y-6)=0$.

Let P(x, y) be any point on the circle. The vector $\overline{AP}$ is perpendicular to the vector $\overline{BP}$, and hence the scalar product $\overline{AP}.\overline{BP}$ is zero (see §15.12 on page 190).

Now, $\overline{AP} = (x-3)\mathbf{i} + (y+5)\mathbf{j}$ and
$\overline{BP} = (x-2)\mathbf{i} + (y-6)\mathbf{j}$, so

$$\begin{aligned}\overline{AP}.\overline{BP} &= \{(x-3)\mathbf{i} + (y+5)\mathbf{j}\}.\{(x-2)\mathbf{i} + (y-6)\mathbf{j}\}\\ &= (x-3)(x-2) + (y+5)(y-6)\end{aligned}$$

But this scalar product is zero, so the equation of the circle is

$$(x-3)(x-2)+(y+5)(y-6)=0$$

Question

Q3 Show that the equation in Example 3 may also be found by using the result that the product of the gradients of two perpendicular lines is −1.

Example 4 *Point* P *moves on the curve* $y^2 = 4x$ *and* A *is the point* (1, 0). *Find the locus of the mid-point of* AP.

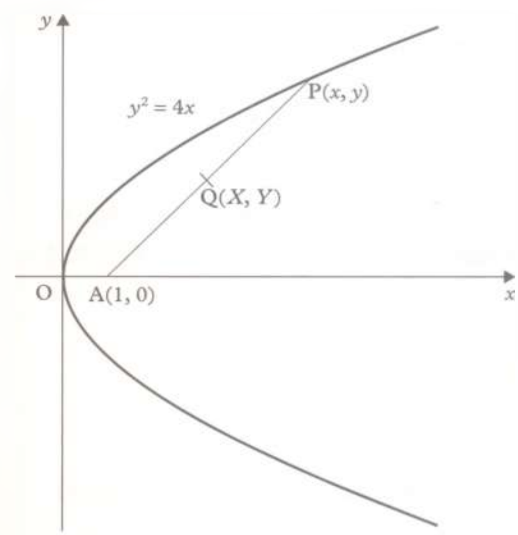


Figure 20.5

Let P be the point (x, y), and let Q(X, Y) be the mid-point of AP (Fig. 20.5). Then the coordinates of Q are given by

$$X = \frac{x+1}{2} \quad \text{and} \quad Y = \frac{y}{2}$$

Since P lies on the given curve, we have

$$y^2 = 4x$$

but $x = 2X - 1$ and $y = 2Y$, therefore

$$\begin{aligned}4Y^2 &= 4(2X-1)\\ Y^2 &= 2X-1\end{aligned}$$

Therefore the locus of the mid-point of AP is $y^2 = 2x - 1$.

Example 5 *A straight line AB of length 10 units is free to move with its ends on the axes. Find the locus of a point P on the line at a distance of 3 units from the end on the x-axis.*

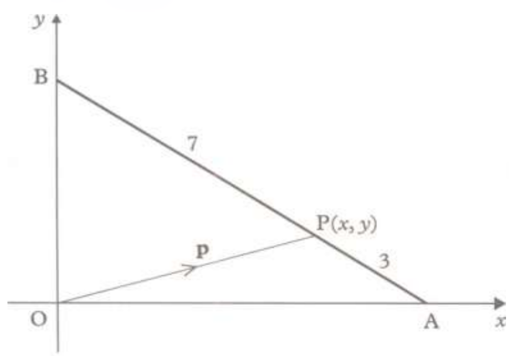


Figure 20.6

Let A be the point $(X, 0)$ and B the point $(0, Y)$ and note that, by Pythagoras' theorem (Fig. 20.6),

$$OA^2 + OB^2 = AB^2$$

and therefore

$$X^2 + Y^2 = 100 \qquad (1)$$

Also, let the coordinates of the point P be (x, y). We are given that BP:PA = 7:3, and so **p**, the position vector of the point P, is given by

$$\mathbf{p} = \tfrac{7}{10}\mathbf{a} + \tfrac{3}{10}\mathbf{b}$$

where **a** and **b** are the position vectors of the points A and B (see the ratio theorem, §15.8 on page 183), hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{7}{10}\begin{pmatrix} X \\ 0 \end{pmatrix} + \frac{3}{10}\begin{pmatrix} 0 \\ Y \end{pmatrix}$$

so, $x = \tfrac{7}{10}X$ and $y = \tfrac{3}{10}Y$. From these equations we see that

$$X = \frac{10}{7}x \quad \text{and} \quad Y = \frac{10}{3}y$$

Substituting these into equation (1) we obtain

$$\frac{100}{49}x^2 + \frac{100}{9}y^2 = 100$$

and hence the required equation is

$$\frac{x^2}{49} + \frac{y^2}{9} = 1$$

Exercise 20b

1 Find the equations of the circles on the diameters whose ends are

- **a** $(-3, 2)$ and $(4, -5)$
- **b** $(\frac{1}{2}, 1)$ and $(-\frac{3}{2}, 4)$
- **c** $(0, a)$ and $(a, 0)$
- **d** (x_1, y_1) and (x_2, y_2)

2 P is a point on a line of length 12 units, which moves so that its ends lie on the axes. Find the locus of P when it is

- **a** the mid-point of the line,
- **b** the point of trisection of the line nearer to the y-axis.

3 L and M are the feet of perpendiculars from a point P on to the axes. Find the locus of P when it moves so that LM is of length 4 units.

4 A variable line through the point (3, 4) cuts the axes at Q and R, and the perpendiculars to the axes at Q and R intersect at P.
What is the locus of the point P?

5 A variable point P lies on the curve $xy = 12$. Q is the mid-point of the line joining P to the origin. Find the locus of Q.

6 P is a variable point on the curve $y = 2x^2 + 3$, and O is the origin. Q is the point of trisection of OP nearer the origin. Find the locus of Q.

7 A line parallel to the x-axis cuts the curve $y^2 = 4x$ at P and the line $x = -1$ at Q.
Find the locus of the mid-point of PQ.

8 Variable lines through the points O(0, 0) and A(2, 0) intersect at right angles at the point P. Show that the locus of the mid-point of OP is $y^2 + x(x-1) = 0$.

9 Find the locus of a point which moves so that the sum of the squares of its distances from the lines $x + y = 0$ and $x - y = 0$ is 4.

10 A is the point (1, 0), B is the point (2, 0) and O is the origin. A point P moves so that angle BPO is a right angle, and Q is the mid-point of AP.
What is the locus of Q?

11 A line parallel to the y-axis meets the curve $y = x^2$ at P and the line $y = x + 2$ at Q.
Find the locus of the mid-point of PQ.

12 M is a variable point on the x-axis, and A is the point (2, 3). A line through A, perpendicular to AM, meets the y-axis at N. Perpendiculars to the axes at M and N meet at P. Find the locus of the point P.

13 M and N are points on the axes, and the line MN passes through the point (3, 2). P is a variable point which moves so that the mid-point of the line joining P to the origin is the mid-point of MN. Find the locus of the point P.

14 A straight line LM, of length 4 units, moves with L on the line $y = x$ and M on the x-axis. Find the locus of the mid-point of LM.

15 A straight line LM meets the x-axis in M and the line $y = x$ in L, and passes through the point (6, 4). What is the locus of the mid-point of LM?

20.3 Tangents and normals

If a tangent touches a curve at the point P, the line through P perpendicular to the tangent is called a **normal**. (See §3.6 on page 68.)

Example 6 *Find the equations of the tangent and normal to the curve $y = 3x^2 - 8x + 5$, at the point where $x = 2$.*

[The equation of a line can be found from its gradient and the coordinates of a point through which it passes. Therefore begin by finding these.]

$$y = 3x^2 - 8x + 5$$

The gradient of the tangent, $\frac{dy}{dx}$, is given by

$$\frac{dy}{dx} = 6x - 8$$

At the point of contact $x = 2$, and so

$$\frac{dy}{dx} = 6 \times 2 - 8 = 4$$

The y-coordinate of the point of contact may be found by substituting $x = 2$ in the equation of the curve:

$$y = 3 \times 2^2 - 8 \times 2 + 5 = 1$$

Therefore the coordinates of the point of contact are (2, 1).

Using the equation of a line in the form

$$y - y_1 = m(x - x_1)$$

the equation of the tangent is

$$y - 1 = 4(x - 2)$$

i.e. $4x - y - 7 = 0$

The normal is perpendicular to the tangent, and so its gradient is $-\frac{1}{4}$. Therefore its equation may be written

$$y - 1 = -\tfrac{1}{4}(x - 2)$$

i.e. $x + 4y - 6 = 0$

Thus the equations of the tangent and normal to the curve $y = 3x^2 - 8x + 5$ at the point (2, 1) are respectively $4x - y - 7 = 0$ and $x + 4y - 6 = 0$.

Note: When finding the equation of the tangent, the actual gradient of the curve *at* (2, 1) was used. If we had taken the gradient to be $6x - 8$, the equation $y - 1 = (6x - 8)(x - 2)$ would not have represented a straight line.

Example 7 *Find the equations of the tangents to the curve $xy = 6$ which are parallel to the line $2y + 3x = 0$.*

The gradient of the line $2y + 3x = 0$ is $-\frac{3}{2}$. Therefore we must find the points on the curve $xy = 6$ where the gradient is $-\frac{3}{2}$.

$$y = \frac{6}{x}$$

$$\therefore \frac{dy}{dx} = -\frac{6}{x^2}$$

If $\frac{dy}{dx} = -3/2$,

$$-\frac{6}{x^2} = -\frac{3}{2}$$

$$\therefore 3x^2 = 12, \quad \text{and so} \quad x^2 = 4$$

$$\therefore x = \pm 2$$

When $x = 2$, $y = \frac{6}{2} = 3$; and when $x = -2$, $y = -\frac{6}{2} = -3$. Thus the gradient of the curve is $-\frac{3}{2}$ at the points (2, 3) and (−2, −3).

The equations of the tangents may be found using the form $y - y_1 = m(x - x_1)$:

$$y - 3 = -\tfrac{3}{2}(x - 2) \quad \text{and} \quad y + 3 = -\tfrac{3}{2}(x + 2)$$

Therefore the equations of the tangents to the curve $xy = 6$ which are parallel to the line $2y + 3x = 0$ are $3x + 2y - 12 = 0$ and $3x + 2y + 12 = 0$.

Sometimes questions about tangents may be solved without using calculus. **Fig. 20.7** shows a curve with chords PQ passing through a fixed point P with variable points Q. When P and Q are distinct, we obtain distinct roots from solving the equations of the curve and PQ simultaneously. When P and Q coincide, producing a tangent, there will be a repeated root.

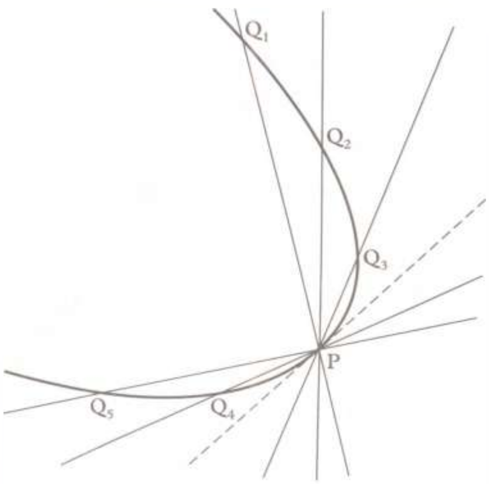


Figure 20.7

Example 8 *Show that if the line $y = mx + c$ is a tangent to the curve $4x^2 + 3y^2 = 12$, then $c^2 = 3m^2 + 4$.*

[If the line $y = mx + c$ is a tangent, then the point of contact must be given by an equation with a repeated root.]

Substituting $y = mx + c$ in the equation $4x^2 + 3y^2 = 12$, we obtain

$$4x^2 + 3(mx + c)^2 = 12$$
$$\therefore\ 4x^2 + 3m^2x^2 + 6mxc + 3c^2 = 12$$
$$\therefore\ (4 + 3m^2)x^2 + 6mcx + 3c^2 - 12 = 0$$

Now if the equation $ax^2 + bx + c = 0$ has equal roots then $b^2 = 4ac$ (see §10.1 on page 128). Therefore if $y = mx + c$ is a tangent,

$$36m^2c^2 = 4(4 + 3m^2)(3c^2 - 12)$$
$$\therefore\ 9m^2c^2 = 12c^2 - 48 + 9m^2c^2 - 36m^2$$
$$\therefore\ 12c^2 = 36m^2 + 48$$

Therefore if $y = mx + c$ is a tangent to the curve $4x^2 + 3y^2 = 12$, then $c^2 = 3m^2 + 4$.

[This means that the line $y = mx \pm \sqrt{(3m^2 + 4)}$ will touch the curve for all values of m. Hence we may find the tangents parallel to $y = 2x$ by substituting $m = 2$, which gives $y = 2x \pm 4$.]

Questions

Q4 Find the equations of the tangents to the curve $4x^2 + 3y^2 = 12$ which are
- a parallel to $y = x$,
- b inclined at 60° to the x-axis.

Q5 Solve the following pairs of simultaneous equations:
- a $y = x$, $y^2 = x^3 + x^2$
- b $y = 2x$, $y^2 = x^3 + x^2$

What is the significance of the repeated root in each case?

Exercise 20c

1 Find the equations of the tangents and normals to the following curves at the points indicated:
- a $y = x^2$, (2, 4)
- b $y = 3x^2 - 2x + 1$, where $x = 1$
- c $y = x + 1/x$, (−1, −2)
- d $y^2 = 4x$, (1, −2)
- e $y = x^2 - 2x$, where $x = -2$
- f $xy = 4$, where $y = 2$
- g $y^3 = x^2$, (1, 1)

2 Show that the following lines touch the given curves and find the coordinates of the points of contact:
- a $y^2 = 8x$, $y - 2x - 1 = 0$
- b $x^2 + y^2 = 8$, $x - y - 4 = 0$
- c $xy = 4$, $x + 9y - 12 = 0$
- d $9x^2 - 4y^2 = 36$, $5x - 2y + 8 = 0$

3 At what points does the curve $y = x^2 - 4x + 3$ cut the x-axis? Find the equations of the tangents and normals at these points.

4 Find the equations of the tangents at the points of intersection of the line $y = x + 1$ and the curve $y = x^2 - x - 2$.

5 Find the equations of the normals to the curve $y = x^2 - 1$ at the points where it cuts the x-axis. What are the coordinates of the point of intersection of these normals?

6 Find the coordinates of the points of intersection of the curves $y^2 = x$ and $x^2 = y$. What are the equations of the tangents to the curves at these points?

7 What is the equation of the normal to the curve $y = x^2 - 4x - 12$ at the point where it cuts the y-axis? Where does this normal meet the x-axis?

8 Find the equations of the tangents to the curve $y = x^3 - 3x^2$ which are parallel to the line $y = 9x$.

9 Find the equations of the tangents to the curve $xy = 4$, which are inclined at 135° to the x-axis.

10 Show that the equation of the tangent to the curve $y = x^2$ at the point (h, k) may be written $y - 2hx + h^2 = 0$. Find the values of h for which the tangent passes through the point $(1, 0)$, and obtain the equations of these tangents.

11 Show that the equation of the tangent to the curve $xy = c^2$ at the point (h, k) may be written $xk + yh - 2c^2 = 0$. Find the equation of the tangent which passes through the point $(0, c)$.

12 Show that, if the line $y = x + c$ is a tangent to the circle $x^2 + y^2 = 4$, then $c^2 = 8$.

13 Prove that the condition that the line $y = mx + c$ should touch the ellipse $x^2 + 4y^2 = 4$ is $c^2 = 4m^2 + 1$. Hence find the equations of the tangents to the ellipse which are parallel to the line $3x - 8y = 0$.

14 Show that the line $y = mx + c$ touches the curve $b^2x^2 - a^2y^2 = a^2b^2$ if $c^2 = a^2m^2 - b^2$. Hence find the equations of the tangents to the curve $9x^2 - 25y^2 = 225$ which are parallel to the line $x - y = 0$.

15 Find the condition that the line $lx + my + n = 0$ should touch the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

Chapter 21

Coordinate geometry (2)

The circle

21.1 Equation of a circle

The work of previous chapters will now be applied to the circle. We begin by obtaining the equation of a circle, radius r, with its centre at the origin.

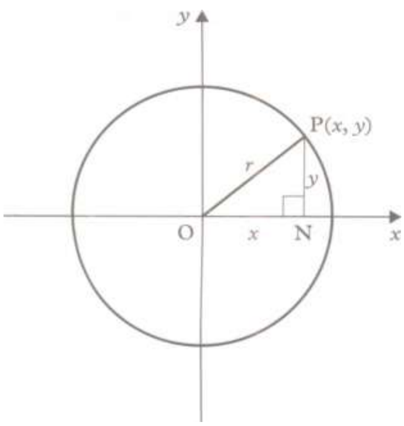


Figure 21.1

We require an equation connecting the coordinates (x, y) of any point P on the circle (see Fig. 21.1). Let N be the foot of the perpendicular from P to the x-axis, so that $ON = x$ and $NP = y$.

Then by Pythagoras' theorem,

$$ON^2 + NP^2 = r^2$$
$$\therefore x^2 + y^2 = r^2$$

Therefore the equation of the circle, radius r, with its centre at the origin is

$$\boldsymbol{x^2 + y^2 = r^2}$$

This is the simplest form in which the equation of a circle can be written. To be more general, consider the circle, radius r, whose centre is at the point $C(a, b)$.

Let $P(x, y)$ be any point on the circle, and draw CN and NP parallel to the x- and y-axes, as shown in Fig. 21.2.

$CN = x - a$ and $NP = y - b$.
By Pythagoras' theorem in triangle CNP,

$$CN^2 + NP^2 = CP^2$$
$$\therefore (x-a)^2 + (y-b)^2 = r^2$$

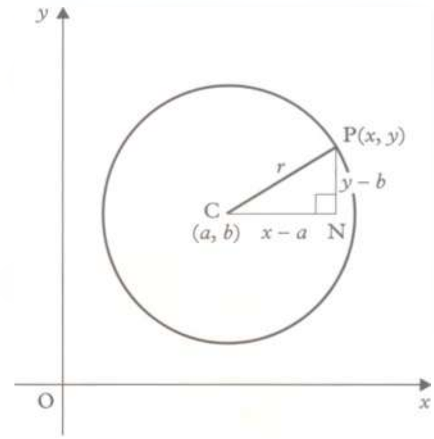


Figure 21.2

Therefore the **general equation of a circle**, radius r, whose centre is at (a, b) is

$$\boldsymbol{(x-a)^2 + (y-b)^2 = r^2}$$

Using this result, the equation of the circle with centre at $(4, -1)$ and radius 2 may be written

$$(x-4)^2 + (y+1)^2 = 2^2$$

Expanding the squares:

$$x^2 - 8x + 16 + y^2 + 2y + 1 = 4$$

Collecting the terms:

$$x^2 + y^2 - 8x + 2y + 13 = 0$$

The equation of a circle is usually given in this form. Note that

a the coefficients of x^2 and y^2 are equal,

b all other terms are linear (such as may occur in the equation of a straight line).

Question

Q1 Express the equation $(x-a)^2 + (y-b)^2 = r^2$ in the form

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Write down g, f, c, in terms of a, b, r.

5 The points (8, 4) and (2, 2) are the ends of a diameter of a circle. Find the coordinates of the centre, and the radius. Deduce the equation of the circle.

6 What is the equation of the circle, centre (2, –3), which touches the x-axis?

7 Find the radii of the two circles, with centres at the origin, which touch the circle $x^2 + y^2 - 8x - 6y + 24 = 0$.

8 Show that the distance of the centre of the circle $x^2 + y^2 - 6x - 4y + 4 = 0$ from the y-axis is equal to the radius. What does this prove about the y-axis and the circle?

9 Find the equations of the circles which touch the x-axis, have radius 5, and pass through the point (0, 8).

10 What is the equation of the circle whose centre lies on the line $x - 2y + 2 = 0$, and which touches the positive axes?

11 A circle passes through the points A(–5, 2), B(–3, –4), C(1, 8). Find the point of intersection of the perpendicular bisectors of AB and BC. What is the equation of the circle?

12 The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ passes through the points A(–1, –2), B(1, 2), C(2, 3). Write down three equations which must be satisfied by g, f, c. Solve these equations and write down the equation of the circle ABC.

13 Find the equations of the circles which pass through

- a the origin, (–1, 3), (–4, 2)
- b (3, 1), (8, 2), (2, 6)
- c (6, –5), (2, –7), (–6, –1)

14 A point moves so that its distance from the origin is twice its distance from the point (3, 0). Show that the locus is a circle, and find its centre and radius.

15 A is the point (3, –1), and B is the point (5, 3). Show that the locus of a point P, which moves so that $PA^2 + PB^2 = 28$, is a circle. Find its centre and radius.

21.2 Tangents to a circle

Elementary geometry will frequently help to simplify working in coordinate geometry. For example, it gives a simple way of obtaining the equation of a tangent at a point on a circle, using the fact that a tangent is perpendicular to the radius through the point of contact. See the next example.

Example 3 *Verify that the point (3, 2) lies on the circle $x^2 + y^2 - 8x + 2y + 7 = 0$, and find the equation of the tangent at this point.*

Substituting the coordinates (3, 2) into the equation $x^2 + y^2 - 8x + 2y + 7 = 0$,

$$\text{L.H.S.} = 9 + 4 - 24 + 4 + 7 = 0 = \text{R.H.S.}$$

Therefore (3, 2) lies on the circle.

[The gradient of the tangent can be found from the gradient of the radius through (3, 2). To find this, obtain the coordinates of the centre of the circle.]

The equation of the circle may be written

$$x^2 - 8x \quad + y^2 + 2y \quad = -7$$

$$\therefore x^2 - 8x + 16 + y^2 + 2y + 1 = -7 + 16 + 1$$

$$\therefore (x - 4)^2 + (y + 1)^2 = 10$$

Therefore the centre of the circle is (4, –1). Hence the gradient of the radius through (3, 2) is $(-1 - 2)/(4 - 3) = -3$.

Therefore the gradient of the tangent is $\frac{1}{3}$.
Using the formula $y - y_1 = m(x - x_1)$, the equation of the tangent at (3, 2) is

$$y - 2 = \tfrac{1}{3}(x - 3)$$

$$\therefore 3y - 6 = x - 3$$

Therefore the equation of the tangent to the circle at (3, 2) is $x - 3y + 3 = 0$.

Example 1 *Find the radius and the coordinates of the centre of the circle* $2x^2 + 2y^2 - 8x + 5y + 10 = 0$.

[We can find the centre and radius if the equation is expressed in the form $(x-a)^2 + (y-b)^2 = r^2$.]

Divide both sides of the equation of the circle

$$2x^2 + 2y^2 - 8x + 5y + 10 = 0$$

by 2, in order to make the coefficients of x^2 and y^2 equal to 1:

$$x^2 + y^2 - 4x + \tfrac{5}{2}y + 5 = 0$$

Rearrange the terms, grouping those in x and y:

$$x^2 - 4x \quad + \quad y^2 + \tfrac{5}{2}y \qquad = -5$$

Complete the squares (see §P1.4 on page 2):

$$x^2 - 4x + 4 + y^2 + \tfrac{5}{2}y + (\tfrac{5}{4})^2 = -5 + 4 + \tfrac{25}{16}$$
$$\therefore\ (x-2)^2 + (y + \tfrac{5}{4})^2 = \tfrac{9}{16}$$
$$\therefore\ (x-2)^2 + (y + \tfrac{5}{4})^2 = (\tfrac{3}{4})^2$$

Comparing this with the equation of the circle, radius r, centre (a, b):

$$(x-a)^2 + (y-b)^2 = r^2$$

we obtain

$$a = 2, \quad b = -\tfrac{5}{4}, \quad r = \tfrac{3}{4}$$

Therefore the radius of the circle is $\frac{3}{4}$ and its centre is at the point $(2, -\frac{5}{4})$.

Example 2 *Find the equations of the circles which pass through the points* A(0, 2) *and* B(0, 8), *and which touch the x-axis.*

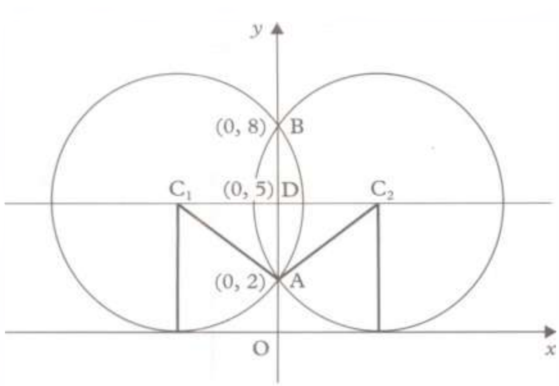


Figure 21.3

Fig. 21.3 suggests a method. The centre of the circle must lie on the perpendicular bisector of the chord AB, i.e. on the line $y = 5$.

Since the circle touches the x-axis, its radius is 5.

If D is the point (0, 5) and C is the centre of either circle, then triangle ADC is right-angled and DC = 4 by Pythagoras' theorem. Therefore the centres of the circles are (−4, 5) and (4, 5) and so their equations are

$$(x \pm 4)^2 + (y-5)^2 = 5^2$$

Therefore the equations of the circles are $x^2 + y^2 \pm 8x - 10y + 16 = 0$.

Exercise 21a

1 Find the equations of the circles with the following centres and radii:

- **a** centre (2, 3), radius 1
- **b** centre (−3, 4), radius 5
- **c** centre $(\frac{2}{3}, -\frac{1}{3})$, radius $\frac{2}{3}$
- **d** centre (0, −5), radius 5
- **e** centre (3, 0), radius $\sqrt{2}$
- **f** centre $(-\frac{1}{4}, \frac{1}{3})$, radius $\frac{1}{2}\sqrt{2}$

2 Find the radii and the coordinates of the centres of the following circles:

- **a** $x^2 + y^2 + 4x - 6y + 12 = 0$
- **b** $x^2 + y^2 - 2x - 4y + 1 = 0$
- **c** $x^2 + y^2 - 3x = 0$
- **d** $x^2 + y^2 + 3x - 4y - 6 = 0$
- **e** $2x^2 + 2y^2 + x + y = 0$
- **f** $36x^2 + 36y^2 - 24x - 36y - 23 = 0$
- **g** $x^2 + y^2 - 2ax - 2by = 0$
- **h** $x^2 + y^2 + 2gx + 2fy + c = 0$

3 Which of the following equations represent circles?

- **a** $x^2 + y^2 - 5 = 0$
- **b** $x^2 + y^2 + 10 = 0$
- **c** $3x^2 + 2y^2 + 6x - 8y + 100 = 0$
- **d** $ax^2 + ay^2 = 1$
- **e** $x^2 + y^2 + 8x + xy + 4 = 0$
- **f** $x^2 + y^2 + bxy = 1$
- **g** $x^2 + y^2 + c = 0$
- **h** $x^2 + dy^2 - 8x + 10y + 50 = 0$

Which of them can represent circles if suitable values are given to the constants a, b, c, d?

4 Find the equation of the circle whose centre is at the point (2, 1) and which passes through the point (4, −3).

Example 4 *Find the length of the tangents from the point (5, 7) to the circle $x^2 + y^2 - 4x - 6y + 9 = 0$.*

[**Fig. 21.4** suggests a method. The tangent is perpendicular to the radius through the point of contact, so t can be found by Pythagoras' theorem if d and r are known.]

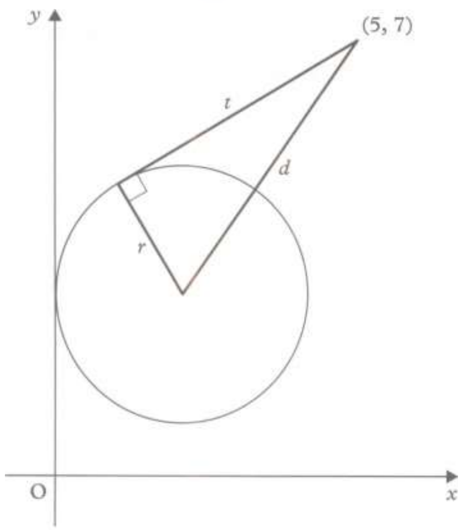


Figure 21.4

In **Fig. 21.4**, the radius, length r, is perpendicular to the tangent, length t, from the point (5, 7). If the distance of (5, 7) from the centre of the circle is d, then by Pythagoras' theorem $d^2 = t^2 + r^2$, or

$$t^2 = d^2 - r^2$$

To find the coordinates of the centre of the circle $x^2 + y^2 - 4x - 6y + 9 = 0$:

$$x^2 - 4x + 4 + y^2 - 6y + 9 = 4$$
$$\therefore (x-2)^2 + (y-3)^2 = 2^2$$

Therefore the centre is (2, 3) and the radius is 2.

Now, by Pythagoras' theorem,

$$d^2 = (5-2)^2 + (7-3)^2 = 9 + 16 = 25$$

But $r^2 = 4$, $\quad \therefore t^2 = 25 - 4 = 21$

Therefore the length of the tangents from (5, 7) to the circle is $\sqrt{21}$.

Question

Q2 Calculate the lengths of the tangents to the circle in Example 4 from **a** (4, 3), **b** (2, 2). What do you conclude from these results? If in doubt, mark these points in a figure containing the circle.

Exercise 21b

1 Verify that the given points lie on the following circles and find the equations of the tangents to the circles at these points:

a $x^2 + y^2 + 6x - 2y = 0$, (0, 0)
b $x^2 + y^2 - 8x - 2y = 0$, (3, 5)
c $x^2 + y^2 + 2x + 4y - 12 = 0$, (3, −1)
d $x^2 + y^2 + 2x - 2y - 8 = 0$, (2, 2)
e $2x^2 + 2y^2 - 8x - 5y - 1 = 0$, (1, −1)

2 Find the lengths of the tangents from the given points to the following circles:

a $x^2 + y^2 + 4x - 6y + 10 = 0$, (0, 0)
b $x^2 + y^2 - 4x - 8y - 5 = 0$, (8, 2)
c $x^2 + y^2 + 6x + 10y - 2 = 0$, (−2, 3)
d $x^2 + y^2 - 10x + 8y + 5 = 0$, (5, 4)
e $x^2 + y^2 = a^2$, (x_1, y_1)
f $x^2 + y^2 + 2gx + 2fy + c = 0$, (0, 0)

3 The tangent to the circle $x^2 + y^2 - 2x - 6y + 5 = 0$ at the point (3, 4) meets the x-axis at M. Find the distance of M from the centre of the circle.

4 Find the equations of the tangents to the circle $x^2 + y^2 - 6x + 4y + 5 = 0$ at the points where it meets the x-axis.

5 The tangent to the circle $x^2 + y^2 - 4x + 6y - 77 = 0$ at the point (5, 6) meets the axes at A and B. Find the coordinates of A and B. Deduce the area of triangle AOB.

6 Find the length of the tangents from the origin to the circle

$$x^2 + y^2 - 10x + 2y + 13 = 0$$

Use this answer to show that these two tangents and the radii through the points of contact form a square.

7 Find the length of the tangents to the circle $x^2 + y^2 - 4 = 0$ from the point P(X, Y). Find the equation of the locus of P, when it moves so that the length of the tangents to the circle is equal to the distance of P from the point (1, 0).

8 Show that the length of the tangents to the circle $x^2 + y^2 - 4x - 6y + 12 = 0$ from the point P(X, Y) is $\sqrt{(X^2 + Y^2 - 4X - 6Y + 12)}$. Find the locus of P when it moves so that the length of the tangents to the circle is equal to its distance from the origin.

9 Show that the point (x_1, y_1) is outside, on or inside the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

according as to whether $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ is positive, zero or negative.

10 Prove that the line $x - y - 3 = 0$ is a common tangent to the circles

$$x^2 + y^2 - 2x - 4y - 3 = 0 \quad \text{and}$$
$$x^2 + y^2 + 4x - 2y - 13 = 0$$

What are the coordinates of the point in which it meets the other common tangent?

21.3 Intersection of two circles

Example 5 *Find the equation of the common chord of the circles $x^2 + y^2 - 4x - 2y + 1 = 0$ and $x^2 + y^2 + 4x - 6y - 10 = 0$.*

The coordinates of the points of intersection A and B of the circles satisfy the two equations

$$x^2 + y^2 - 4x - 2y + 1 = 0$$
$$x^2 + y^2 + 4x - 6y - 10 = 0$$

Therefore, by subtraction, the coordinates of A and B satisfy the equation

$$-8x + 4y + 11 = 0$$

However, this equation represents a straight line, and it is satisfied by the coordinates of A and B, therefore it is the equation of the common chord.

Two circles may not intersect but, by subtracting one equation from the other, the equation of a line may still be obtained. What then does the line represent? Q3 suggests an answer.

Questions

Q3 What are the squares of the lengths of the tangents from the point $P(X, Y)$ to the circles $x^2 + y^2 - 1 = 0$, $x^2 + y^2 - 6x - 8y + 21 = 0$? What is the locus of P such that the lengths of the tangents from P to the circles are equal?

Q4 Write down the equation of the line joining the origin to the point of intersection of the lines $17x - 15y + 7 = 0$ and $19x - 13y + 7 = 0$.

21.4 Orthogonal circles

If the tangents to two circles at their points of intersection are perpendicular, the circles are said to be **orthogonal**. Since the radius through a point of contact is perpendicular to the tangent, it follows that the tangent to one circle is a radius of the other. Thus if the centres of two orthogonal circles of radii R and r are a distance d apart (Fig. 21.5), then by Pythagoras' theorem:

$$d^2 = R^2 + r^2$$

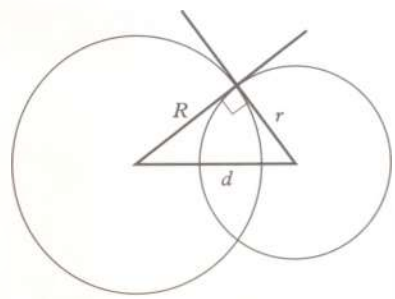


Figure 21.5

Example 6 *Show that the circles $x^2 + y^2 - 6x + 4y + 2 = 0$ and $x^2 + y^2 + 8x + 2y - 22 = 0$ are orthogonal.*

The centres of the circles are $(3, -2)$ and $(-4, -1)$, and their radii are $\sqrt{11}$ and $\sqrt{39}$.

The sum of the squares of the radii is 50, and the square of the distance between the centres is $7^2 + 1^2 = 50$, therefore the circles are orthogonal.

Exercise 21c

1 Show that the common chord of the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 4x - 2y - 4 = 0$ passes through the origin.

2 Find the coordinates of the point where the common chord of the circles $x^2 + y^2 - 4x - 8y - 5 = 0$ and $x^2 + y^2 - 2x - 4y - 5 = 0$ meets the line joining their centres.

3 Show that the following pairs of circles are orthogonal:

a $x^2 + y^2 - 6x - 8y + 9 = 0$, $x^2 + y^2 = 9$

b $x^2 + y^2 - 4x + 2 = 0$, $x^2 + y^2 + 6y - 2 = 0$

c $x^2 + y^2 - 6y + 8 = 0$, $x^2 + y^2 - 4x + 2y - 14 = 0$

d $x^2 + y^2 + 10x - 4y - 3 = 0$, $x^2 + y^2 - 2x - 6y + 5 = 0$

4 A circle passing through the point $(4, 0)$ is orthogonal to the circle $x^2 + y^2 = 4$. Find the locus of the centre of the variable circle.

Chapter 22

Coordinate geometry (3)

Polar coordinates

22.1 Equation of a straight line

Straight lines are very important in coordinate geometry. It is therefore useful to be able to find their equations quickly. Example 9 in Chapter 1 on page 42 was done by two methods. What follows is an extension of the second method.

Example 1 *Find the equation of the line with gradient* $-\frac{2}{3}$*, which passes through the point* $(1, -4)$.

[Think: the line has equation $y = -\frac{2}{3}x + c$, therefore it can be written

$$3y + 2x = \text{constant}$$

Since the line passes through $(1, -4)$, the constant is found by substituting these coordinates in the LHS.]

The equation of the line is

$$3y + 2x = -12 + 2$$

i.e. $2x + 3y + 10 = 0$

Note. Check that the line **a** has gradient $-\frac{2}{3}$, **b** passes through $(1, -4)$.

Given the equation of a line, it is easy to find the equation of a perpendicular line through a given point. For example, to find the equation of the line perpendicular to $4x + 5y + 7 = 0$ which passes through $(6, -5)$, interchange the coefficients of x and y, changing one of the signs, and balance the equation as before. Thus the perpendicular is $5x - 4y = 50$.

Example 2 *Find the equation of the line joining the points* $(a, 0)$, $(0, b)$.

The gradient of the line is $-b/a$. Therefore, using the method of Example 1, its equation is $ay + bx = ab$.

Dividing through by ab, the equation becomes

$$\frac{x}{a} + \frac{y}{b} = 1$$

which is known as the **intercept** form of the equation of a line.

Exercise 22a

1 Find the equations of the lines with the given gradients which pass through the given points.

a gradient 1, through $(3, 2)$
b gradient -2, through $(1, -3)$
c gradient $\frac{1}{2}$, through $(0, -6)$
d gradient $-\frac{1}{3}$, through $(-2, 5)$
e gradient $-\frac{7}{5}$, through $(3, -6)$
f gradient $\frac{3}{4}$, through $(-1, 1)$
g gradient $-\frac{5}{6}$, through $(-3, -4)$
h gradient $\frac{4}{3}$, through $(-2, 5)$
i gradient $1/t$, through $(at^2, 2at)$
j gradient $-t$, through $(at^2, 2at)$
k gradient $-\cot\theta$, through $(a\cos\theta, a\sin\theta)$
l gradient $-1/t^2$, through $(ct, c/t)$

2 Find the equations of the perpendiculars to

a $3x + 2y - 1 = 0$, through $(2, 2)$
b $4x - 3y + 7 = 0$, through the origin
c $5x + 6y + 11 = 0$, through $(-3, 5)$
d $3x - 2y - 7 = 0$, through $(-1, 3)$
e $ty - x = at^2$, through (h, k)
f $ax + by + c = 0$, through (x_1, y_1)
g $t^2y + x = 2ct$, through $(ct, c/t)$

3 Find the equations of the lines which make the following intercepts on the x- and y-axes respectively.

a 3, 2
b -1, 2
c $\frac{1}{2}$, $\frac{1}{5}$
d $-\frac{1}{3}$, $\frac{1}{4}$

4 Find the equations of the lines joining the following pairs of points.

a $(0, 2)$, $(3, 0)$
b $(-1, 0)$, $(0, 5)$
c $(-\frac{1}{2}, 0)$, $(0, \frac{2}{3})$

5 The perpendicular from the origin to a straight line is of length p and makes an angle α with the x-axis (see Fig. 22.1).
What intercepts does the line make on the axes?
Write down the equation of the line.

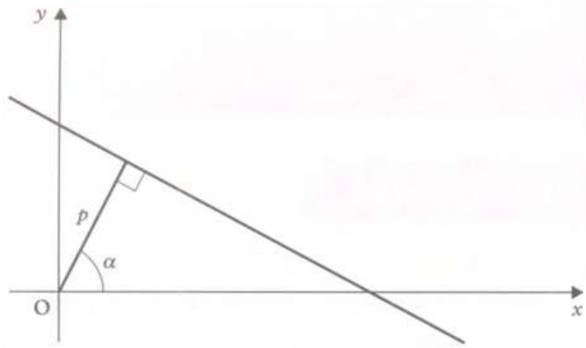


Figure 22.1

6 Use the above methods to find the equations of the following straight lines.
   a with gradient 3, through (4, 3)
   b with gradient $-\frac{1}{2}$, through (2, −1)
   c with gradient $\frac{2}{5}$, through (1, 1)
   d with gradient $-\frac{3}{4}$, through (0, −3)
   e joining (3, 2) and (2, −4)
   f joining (1, 3) and (−3, −6)
   g joining (−1, 2) to the mid-point of (3, 5) and (5, −1)
   h through (2, 1), perpendicular to $2x - y = 0$
   i through (−1, 3) perpendicular to $3x + 4y - 2 = 0$
   j the altitude through A of the triangle A(1, 3), B(2, −1), C(3, 5)
   k the altitude through B of the triangle in j
   l through (h, k), perpendicular to $t^2y + x = 2ct$

7 Show that the rectangular hyperbolas $xy = 1$ and $x^2 - y^2 = 1$ are orthogonal. (See §21.4 on page 238.)

8 Show that the ellipse $16x^2 + 25y^2 = 400$ and the hyperbola $4x^2 - 5y^2 = 20$ are orthogonal.

22.2 Polar coordinates

Let P be any point and let OP = r, where O is the origin (see Fig. 22.2) and let OP make an angle θ with the x-axis. r and θ are called the **polar coordinates** of the point P, written (r, θ). The x-axis is sometimes called the initial line.

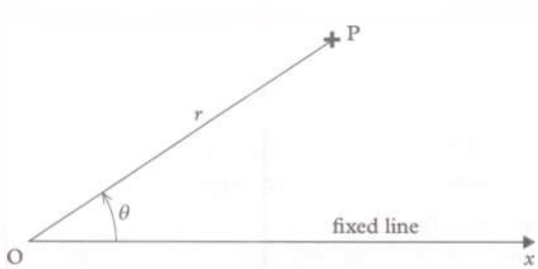


Figure 22.2

Notice that, while a bearing is usually measured clockwise from North, the polar coordinate θ is normally represented in an anti-clockwise direction starting from the x-axis.

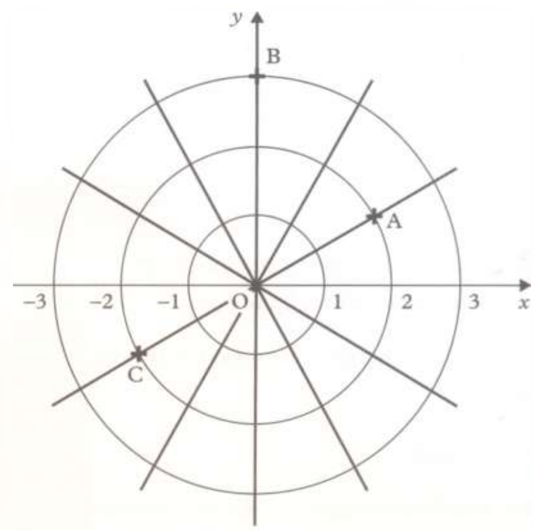


Figure 22.3

Thus in Fig. 22.3, the polar coordinates of A are (2, 30°) and those of B are (3, 90°). A point may be described in different ways, for instance C may be written as (2, 210°), (2, −150°), (−2, 30°) and so on. To avoid this confusion it is usual to take, r to be positive and θ to lie in the range $-180° < \theta \leqslant 180°$.

Example 3 *Sketch the curve whose polar equation is* $r = a(1 + 2\cos\theta)$.

Take values of θ, and calculate $1 + 2\cos\theta$, as below.

θ	0°	30°	45°	60°	90°
$2\cos\theta$	2	1.732	1.414	1	0
$1 + 2\cos\theta$	3	2.732	2.414	2	1

θ	120°	135°	150°	180°
$2\cos\theta$	−1	−1.414	−1.732	−2
$1 + 2\cos\theta$	0	−0.414	−0.732	−1

If α is any angle then $\cos(-\alpha) = \cos\alpha$; therefore the same values of r are obtained for negative values of θ. Thus the curve is symmetrical about the x-axis. Fig. 22.4 is the required sketch.

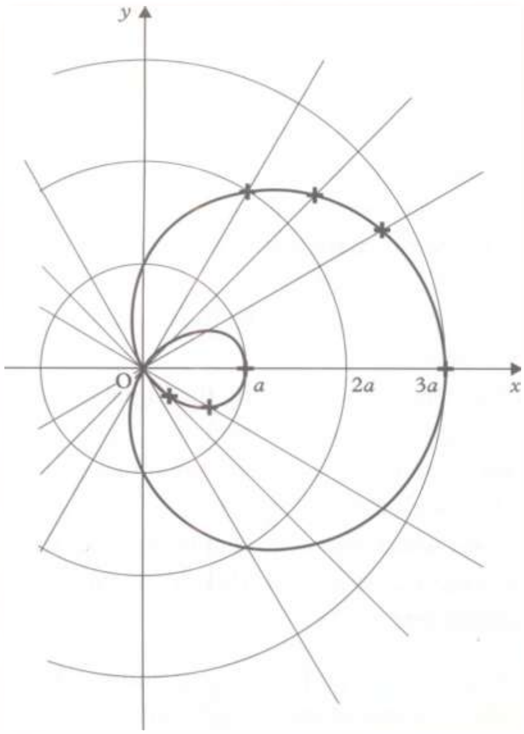


Figure 22.4

Example 4 *Find the polar equation of a line such that the perpendicular to it from the origin is of length p and makes an angle α with the x-axis.*

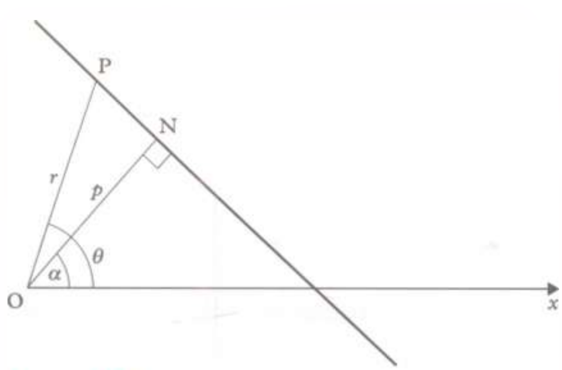


Figure 22.5

In Fig. 22.5, N is the foot of the perpendicular from the origin to the line. Let P be any point (r, θ) on the line.

In triangle ONP, N is a right angle and angle PON $= \theta - \alpha$ (or $\alpha - \theta$ if $\alpha > \theta$).

$\therefore\ r\cos(\theta - \alpha) = p \qquad (\text{or } r\cos(\alpha - \theta) = p)$

Therefore, in either case, the polar equation of the line is $r\cos(\theta - \alpha) = p$.

22.3 Relations between polar and cartesian coordinates

In Fig. 22.6, P is the point (x, y) in cartesian coordinates and (r, θ) in polar coordinates, and PM is an ordinate.

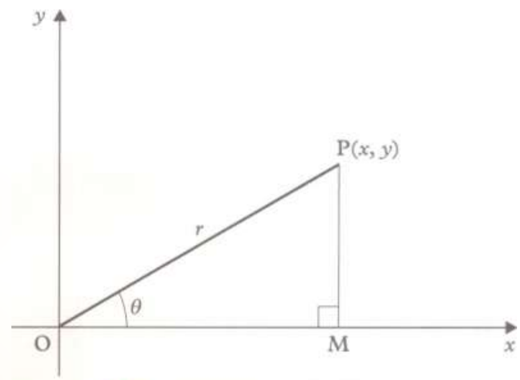


Figure 22.6

Now, by the definitions of cosine and sine given in §16.1 on page 194,

$$\cos\theta = \frac{x}{r} \quad \text{and} \quad \sin\theta = \frac{y}{r}$$

Therefore x and y are given in terms of r and θ by the equations

$$\boldsymbol{x = r\cos\theta \quad \text{and} \quad y = r\sin\theta}$$

If, on the other hand, we are given the values of x and y, we can, by inspecting the diagram in Fig. 22.6, write down the values of r and θ.

By Pythagoras' theorem,

$$r^2 = x^2 + y^2$$
$$\therefore\ \boldsymbol{r = \pm\sqrt{(x^2 + y^2)}}$$

Usually we take the positive square root. However on some occasions it is necessary to use the negative value. (For instance, in Example 3 on page 240, at the point $x = a, y = 0$, r is equal to $-a$.)

θ is found by elementary trigonometry. In Fig. 22.6, θ is given by

$$\boldsymbol{\tan\theta = \frac{y}{x}}$$

[Again, be careful. For example, the point $x = -1, y = -1$, gives $\tan\theta = +1$, but in this case θ is equal to $-135°$, not $45°$. If in doubt, check from the diagram. Compare this with the modulus and argument of a complex number (see §10.5 on page 134).]

Example 5 *Find the cartesian equations of*

a $r = a(1 + 2\cos\theta)$, **b** $r\cos(\theta - \alpha) = p$.

a $r = a(1 + 2\cos\theta)$

[The $\cos\theta$ suggests the relation $x = r\cos\theta$, so multiply through by r.]

$$\therefore r^2 = a(r + 2r\cos\theta)$$

$$\therefore x^2 + y^2 = a\{\surd(x^2 + y^2) + 2x\}$$

$$\therefore x^2 + y^2 - 2ax = a\surd(x^2 + y^2)$$

Therefore the cartesian equation of $r = a(1 + 2\cos\theta)$ is

$$(x^2 + y^2 - 2ax)^2 = a^2(x^2 + y^2)$$

b $r\cos(\theta - \alpha) = p$

$\cos(\theta - \alpha)$ may be expanded (see §17.2 on page 204),

$$\therefore r\cos\theta\cos\alpha + r\sin\theta\sin\alpha = p$$

Therefore the cartesian equation of $r\cos(\theta - \alpha) = p$ is

$$x\cos\alpha + y\sin\alpha = p$$

Note. In part **b** of Example 5 the perpendicular from the origin to this line is of length p and makes an angle α with the x-axis. This form of the equation of a straight line is known as the **normal** or **perpendicular** form.

Example 6 *Find the polar equation of the circle whose cartesian equation is $x^2 + y^2 = 4x$.*

$$x^2 + y^2 = 4x$$

Put $x = r\cos\theta$, $y = r\sin\theta$, then

$$r^2\cos^2\theta + r^2\sin^2\theta = 4r\cos\theta$$

$$\therefore r^2 = 4r\cos\theta$$

Therefore the polar equation of the circle is $r = 4\cos\theta$.

Exercise 22b

1 Sketch the curves given by the following polar equations.

a $r = a(1 + \cos\theta)$ **b** $r = a\cos 2\theta$
c $r = a(1 - \sin\theta)$ **d** $r = a\sin 3\theta$
e $r = a\sec\theta$ **f** $r = a\tan\theta$
g $r = a\cos\frac{1}{2}\theta$ **h** $r = a(1 + \sin 2\theta)$

2 Find the polar equations of the following loci.

a a circle, centre at the origin, radius a
b a straight line through the origin, inclined at an angle α to the initial line
c a straight line perpendicular to the initial line, at a distance a from the origin
d a straight line parallel to the initial line at a distance a
e a circle on the line joining the origin to $(a, 0)$ as a diameter
f a circle, radius a, touching the initial line at the origin and lying above it
g a circle, radius a, centre on the initial line at a distance c from the origin
h a point which moves so that its distance from the origin is equal to its distance from the straight line $x = 2a$

3 P_1 is the point (r_1, θ_1), P_2 is (r_2, θ_2) and $\theta_2 > \theta_1$. Show that the area of the triangle OP_1P_2 is $\frac{1}{2}r_1r_2\sin(\theta_2 - \theta_1)$. Show that if the cartesian coordinates of P_1 and P_2 are (x_1, y_1) and (x_2, y_2), then the area of OP_1P_2 is $\frac{1}{2}(x_1y_2 - x_2y_1)$.

4 Deduce from the result of question 3, that the area of the triangle $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ is

$$\tfrac{1}{2}\{(x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1)\}$$

[If new axes are drawn at (x_3, y_3), the coordinates of P_1 and P_2 referred to them are $(x_1 - x_3, y_1 - y_3)$ and $(x_2 - x_3, y_2 - y_3)$.]

5 Find the polar equations of the following loci.

a $x^2 + y^2 = a^2$ **b** $x^2 - y^2 = a^2$
c $y = 0$ **d** $y^2 = 4a(a - x)$
e $x^2 + y^2 - 2y = 0$ **f** $xy = c^2$

6 Find the cartesian equations of the following loci.

a $r = 2$ **b** $r = a(1 + \cos\theta)$
c $r = a\cos\theta$ **d** $r = a\tan\theta$
e $r = 2a(1 + \sin 2\theta)$ **f** $2r^2\sin 2\theta = c^2$
g $l/r = 1 + e\cos\theta$ **h** $r = 4a\cot\theta\operatorname{cosec}\theta$

7 Express the following straight lines in the form $x\cos\alpha + y\sin\alpha = p$. State the distance of each line from the origin and give the angle which the perpendicular from the origin makes with the x-axis. (See Example 5**b**.)

a $x + \surd 3y = 2$ **b** $x - y = 4$
c $3x + 4y - 10 = 0$ **d** $5x - 12y + 26 = 0$
e $x + 3y - 2 = 0$ **f** $ax + by + c = 0$

22.4 Distance of a point from a line

Given a point $P_1(x_1, y_1)$ and the line

$$ax + by + c = 0,$$

to find the distance of P from the line we first find the distance, r, of P_1 from a point P_2 on the line, such that $\overline{P_1P_2}$ makes an angle α with the x-axis (see **Fig. 22.7**).

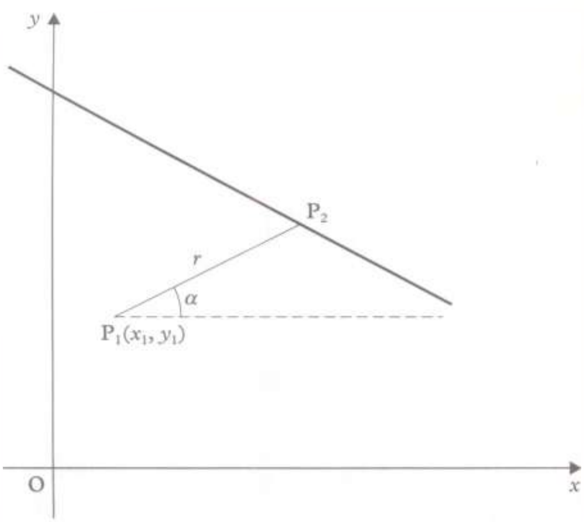


Figure 22.7

P_2 has coordinates $(x_1 + r\cos\alpha, y_1 + r\sin\alpha)$, but, since P_2 lies on the line, its coordinates satisfy the equation $ax + by + c = 0$. Therefore

$$a(x_1 + r\cos\alpha) + b(y_1 + r\sin\alpha) + c = 0$$

$$\therefore r(a\cos\alpha + b\sin\alpha) = -(ax_1 + by_1 + c)$$

$$\therefore r = -\frac{ax_1 + by_1 + c}{a\cos\alpha + b\sin\alpha} \qquad (1)$$

Now consider the case when P_1P_2 is perpendicular to the line $ax + by + c = 0$. The gradient of $ax + by + c = 0$ is $-a/b$, therefore the gradient of P_1P_2 is b/a.

$$\therefore \tan\alpha = \frac{b}{a}$$

$$\therefore \sec^2\alpha = 1 + \frac{b^2}{a^2} = \frac{a^2 + b^2}{a^2}$$

$$\therefore \cos\alpha = \pm\frac{a}{\sqrt{(a^2 + b^2)}}$$

and, since $\tan\alpha = b/a$,

$$\sin\alpha = \pm\frac{b}{\sqrt{(a^2 + b^2)}}$$

so in the denominator of equation (1)

$$a\cos\alpha + b\sin\alpha = \pm\left(\frac{a^2}{\sqrt{(a^2 + b^2)}} + \frac{b^2}{\sqrt{(a^2 + b^2)}}\right)$$
$$= \pm\sqrt{(a^2 + b^2)}$$

Therefore the perpendicular distance of (x_1, y_1) from the line $ax + by + c = 0$ is

$$\pm\frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}$$

This should always be a positive quantity. In other words, the perpendicular distance is

$$\boldsymbol{\left|\frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}\right|}$$

Example 7 *Find the distances of the points* **a** $(1, 3)$, **b** $(-3, 4)$, **c** $(4, -2)$ *from the line* $2x + 3y - 6 = 0$.

The distance of (x_1, y_1) from the line $ax + by + c = 0$ is

$$\left|\frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}\right|$$

Therefore the distances of $(1, 3)$, $(-3, 4)$, $(4, -2)$ from $2x + 3y - 6 = 0$ are respectively

a $\left|\dfrac{2\times1 + 3\times3 - 6}{\sqrt{(2^2 + 3^2)}}\right| = \dfrac{5}{\sqrt{13}}$,

b $\left|\dfrac{2\times(-3) + 3\times4 - 6}{\sqrt{(2^2 + 3^2)}}\right| = 0$,

c $\left|\dfrac{2\times4 + 3\times(-2) - 6}{\sqrt{(2^2 + 3^2)}}\right| = \dfrac{4}{\sqrt{13}}$.

The formula is more easily remembered if you notice the following:

1. the numerator is obtained by substituting the coordinates of the point into the equation of the line (remember that the perpendicular distance is zero if the point lies on the line),
2. the denominator is the square root of the sum of the squares of the coefficients.

Question

Q1 Find the distances of the given points from the following lines.

a $(3, 2)$, $3x - 4y + 4 = 0$
b $(2, -1)$, $5x + 12y = 0$
c $(0, -3)$, $x + 5y + 2 = 0$
d $(2, 5)$, $x + y - 1 = 0$
e $(-4, 2)$, $3y = 5x - 6$
f $(2, 1)$, $y = \frac{2}{3}x + \frac{1}{3}$
g $(0, a)$, $3y = 4x$
h (p, q), $3x + 4y - 3p = 0$
i (X, Y), $12x - 5y + 7 = 0$
j (x_1, y_1), $8x = 15y$

Example 8 *Find the equations of the bisectors of the angles between the lines $4x + 3y - 12 = 0$ and $y = 3x$.*

[The angle bisectors are the locus of a point which is equidistant from the two lines. This gives a method of finding their equations.]

Let $P(X, Y)$ be a point on the locus, then the distances of P from the lines $4x + 3y - 12 = 0$ and $y - 3x = 0$ are

$$\pm\frac{4X + 3Y - 12}{\sqrt{(4^2 + 3^2)}} \quad \text{and} \quad \pm\frac{Y - 3X}{\sqrt{(3^2 + 1^2)}}$$

But P is equidistant from the two lines, therefore

$$\frac{4X + 3Y - 12}{5} = \pm\frac{Y - 3X}{\sqrt{10}}$$

[One ± sign has been dropped, since there are only two distinct equations: one given by the same sign on each side, the other by different signs.]

Simplifying these equations we obtain

$$4\sqrt{10}X + 3\sqrt{10}Y - 12\sqrt{10} = 5Y - 15X$$

and

$$4\sqrt{10}X + 3\sqrt{10}Y - 12\sqrt{10} = -5Y + 15X$$

Therefore the equations of the angle bisectors of the lines are

$$(4\sqrt{10} + 15)x + (3\sqrt{10} - 5)y - 12\sqrt{10} = 0$$

and

$$(4\sqrt{10} - 15)x + (3\sqrt{10} + 5)y - 12\sqrt{10} = 0$$

Example 9 *Find the equations of the tangents to the circle*

$$x^2 + y^2 - 4x - 2y - 8 = 0$$

which are parallel to the line $3x + 2y = 0$.

[Use the property that the perpendicular distance of a tangent from the centre of a circle is equal to the radius.]

The required tangents are parallel to the line $3x + 2y = 0$. Therefore their equations are in the form

$$3x + 2y + c = 0$$

where c is a constant to be determined for each tangent.

To find the centre and radius of the circle

$$x^2 + y^2 - 4x - 2y - 8 = 0$$

$$\therefore x^2 - 4x + 4 + y^2 - 2y + 1 = 8 + 4 + 1$$

$$\therefore (x - 2)^2 + (y - 1)^2 = 13$$

Therefore the centre is $(2, 1)$ and the radius is $\sqrt{13}$.

The distance of the point (x_1, y_1) from the line $ax + by + c = 0$ is $|(ax_1 + by_1 + c)/\sqrt{(a^2 + b^2)}|$. Therefore the distance of the centre of the circle $(2, 1)$ from the line $3x + 2y + c = 0$ is

$$\left|\frac{3\times 2 + 2\times 1 + c}{\sqrt{(3^2 + 2^2)}}\right|$$

However, if the line is a tangent, this distance is equal to the radius, therefore

$$\left|\frac{8 + c}{\sqrt{13}}\right| = \sqrt{13}$$

$$\therefore \pm(8 + c) = 13$$

Taking the positive sign, $8 + c = 13$, and so $c = 5$. With the negative sign, $-8 - c = 13$, and so $c = -21$.

Therefore the equations of the tangents parallel to $3x + 2y = 0$ are $3x + 2y + 5 = 0$ and $3x + 2y - 21 = 0$.

Exercise 22c

1 Find the distances of the given points from the following lines.

a $(2, 5)$, $4x + 3y - 2 = 0$
b $(-1, 3)$, $12x - 5y = 0$
c $(-2, 0)$, $4x + y - 2 = 0$
d $(3, 5)$, $x - y + 2 = 0$
e $(-1, 7)$, $2x = 5y + 1$
f $(0, 0)$, $3x = 4y + 6$
g $(2, 3)$, $y = \frac{4}{5}x + \frac{1}{5}$
h $(1, 4)$, $\frac{1}{2}x + \frac{1}{3}y = 1$
i $(0, 0)$, $x \cos\alpha + y \sin\alpha = p$
j (X, Y), $5x - 12y + 1 = 0$
k $(c, 2c)$, $8x = 15y$
l (x_1, y_1), $y = \frac{3}{4}x - \frac{1}{2}$

2 Find the equations of the bisectors of the angles between

a $3x + 4y - 7 = 0$ and $y - 1 = 0$
b $4x - 3y + 1 = 0$ and $3x - 4y + 3 = 0$
c $5x + 12y = 0$ and $12x + 5y - 4 = 0$
d $x + y - 1 = 0$ and the x-axis

3 Find the equations of the bisectors of the acute angles between

a $3x - 4y + 2 = 0$ and $x + 3 = 0$
b $5x + 12y + 9 = 0$ and $5x - 12y + 6 = 0$
c $x + y + 1 = 0$ and $x = 7y$

[Draw figures to determine which equations give the required lines.]

4 What is the locus of a point which moves so that it is equidistant from the point $(2, -3)$ and the line $x + 2y = 0$?

5 Find the locus of a point which is equidistant from the line $3x - 4y + 7 = 0$ and the point $(3, 4)$ on the line.

6 What is the locus of a point which moves so that its distance from $(2, 2)$ is half its distance from $x + y + 4 = 0$?

7 Find the equations of the tangents to the circle $x^2 + y^2 + 4x + 8y - 5 = 0$ which are parallel to the line $4y - 3x = 0$.

8 Show that the line $3x + 2y = 0$ touches the circle $x^2 + y^2 + 6x + 4y = 0$, and find the equations of the perpendicular tangents.

9 Find the equation of the circle in the first quadrant with radius 2 which touches the y-axis and the line $3y - 4x - 3 = 0$.

10 Prove that the line $y = mx + c$ touches the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$. Also find the condition that the line $lx + my + n = 0$ should touch the circle.

22.5 Parameters

Consider a circle, radius a, centre at the origin (see **Fig. 22.8**). Let $P(x, y)$ be any point on the circle, and let the angle between PO and the x-axis be θ, then

$$x = a\cos\theta \quad \text{and} \quad y = a\sin\theta$$

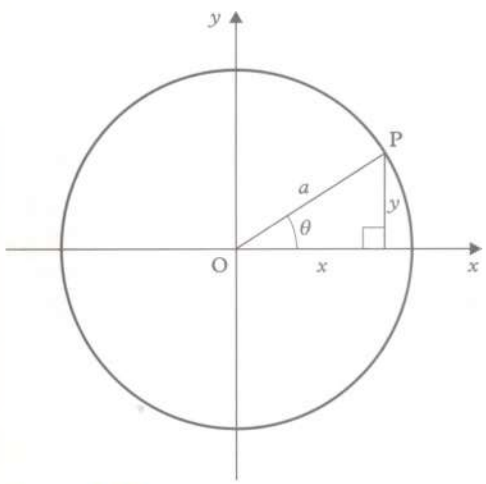


Figure 22.8

These equations, which give the coordinates of any point on the curve in terms of θ, are called **parametric equations**, and θ is called a **parameter**.

If we wish to refer to a particular point on the curve, a single value of θ, will determine it. Thus $\theta = 60°$ gives the point $(a/2, \sqrt{3}a/2)$. On the other hand, for any value of x, say $\frac{1}{2}a$, there are two corresponding points: $(a/2, \sqrt{3}a/2)$ and $(a/2, -\sqrt{3}a/2)$. Another advantage of parameters is that we may write down the coordinates of a general point on the curve $(a\cos\theta, a\sin\theta)$. If we wrote (x_1, y_1), we should also have to bear in mind the equation $x_1{}^2 + y_1{}^2 = a^2$.

Another example of parameters was used in §22.4 on page 243. The point

$$(x_1 + r\cos\alpha, y_1 + r\sin\alpha)$$

lies on the straight line through (x_1, y_1) with gradient $\tan\alpha$. In this case the parameter, r, is a distance. However, it is not always possible to give a simple interpretation of a parameter in terms of angles or distances.

Example 10 *Plot the graph of the curve given parametrically by the equations $x = t^2 - 4$, $y = t^3 - 4t$, for values of t from −3 to +3.*

A table of values is shown below.

t	−3	−2	−1	0	1	2	3
$x = t^2 - 4$	5	0	−3	−4	−3	0	5
$y = t^3 - 4t$	−15	0	3	0	−3	0	15

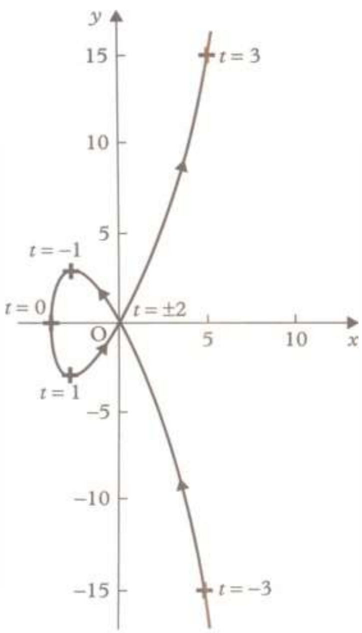


Figure 22.9

The graph is shown in Fig. 22.9. The values of the parameter, t, have been written against the corresponding points. The arrows indicate the direction of motion of a point on the curve as t increases from −3 to +3.

Example 11 *Sketch the curve given parametrically by $x = \sin\theta$, $y = \sin 2\theta$.*

A few values of θ will give all the points we need.

θ	0	45°	90°	135°	180°
$x = \sin\theta$	0	0.7071	1	0.7071	0
$y = \sin 2\theta$	0	1	0	−1	0

Plotting these points and joining them by a curve we obtain the part of the curve in Fig. 22.10 which lies to the right of the y-axis.

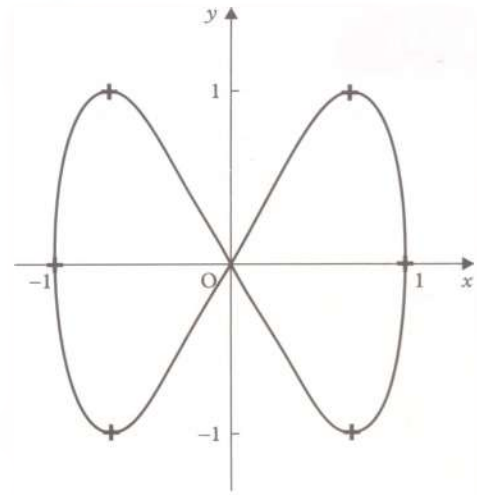


Figure 22.10

Since $\sin(-\alpha) = -\sin\alpha$, the negative values of θ change the signs of x and y. Therefore the rest of the curve is drawn using symmetry.

Question

Q2 Sketch the locus given by $x = t^2$, $y = 1 - t^2$, for real values of t. Is it the line $x + y = 1$?

The graph of the curve given by the parametric equations $x = t^2 - 4$, $y = t^3 - 4t$ was plotted for values of t from −3 to +3 in Example 10 (Fig. 22.9). The cartesian equation connecting x and y can be found by eliminating t from the two parametric equations

$$x = t^2 - 4 \qquad y = t^3 - 4t$$

Notice that $y = tx$. Therefore we can substitute $t = y/x$ in either of the equations above. Choosing the simpler:

$$x = \frac{y^2}{x^2} - 4$$

$$\therefore x^3 = y^2 - 4x^2$$

Therefore the cartesian equation of the locus is $y^2 = x^2(x + 4)$.

Example 12 *Find the cartesian equation of the locus given parametrically by the equations $x = \sin\theta$, $y = \sin 2\theta$ (see* Example 11).

$y = \sin 2\theta$, but $\sin 2\theta = 2\sin\theta\cos\theta$, therefore

$$y = 2\sin\theta\cos\theta$$
$$\therefore y^2 = 4\sin^2\theta\cos^2\theta$$

Since $x = \sin\theta$, $1 - x^2 = \cos^2\theta$. Therefore the cartesian equation of the locus is $y^2 = 4x^2(1 - x^2)$.

The process of obtaining parametric equations from a given cartesian equation is generally not so easy. However, Example 13 shows one method.

Example 13 *Obtain parametric equations for the locus* $y^2 = x^3 - x^2$.

Put $y = tx$ in the equation $y^2 = x^3 - x^2$, then

$$t^2x^2 = x^3 - x^2$$
$$\therefore\ t^2 = x - 1$$
$$\therefore\ x = t^2 + 1$$

Therefore the locus may be represented by the parametric equations $x = t^2 + 1, y = t^3 + t$.

Note. This method is not suitable for all equations. It works well when the terms are of degree n and $n - 1$.

Exercise 22d

1 Plot the curves given parametrically by the equations.

- **a** $x = t^2 + 1, y = t + 2$; from $t = -3$ to $t = +3$
- **b** $x = t^2, y = t^3$; from $t = -3$ to $t = +3$
- **c** $x = t, y = 1/t$; taking $t = \pm 4, \pm 3, \pm 2, \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}$
- **d** $x = 1 + t, y = 3 - 2t$
- **e** $x = at^2, y = 2at$ **f** $x = t^2, y = 1/t$
- **g** $x = \dfrac{2 + 3t}{1 + t}, y = \dfrac{3 - 2t}{1 + t}$
- **h** $x = 3(t + 1/t), y = 2(t - 1/t)$
- **i** $x = 3 \cos\theta, y = 2 \sin\theta$
- **j** $x = 4 \sec\theta, y = 3 \tan\theta$

2 Find the values of the parameters and the other coordinates of the given points on the following curves.

- **a** $x = t, y = 2/t$; where $y = 1\frac{1}{2}$
- **b** $x = at^2, y = 2at$; where $x = \frac{9}{4}a$
- **c** $x = \dfrac{1 + t}{1 - t},\ y = \dfrac{2 + 3t}{1 - t}$; where $y = -\frac{4}{3}$
- **d** $x = a \cos\theta, y = b \sin\theta$; where $x = \frac{1}{2}a$

3 Find the cartesian equations of the loci in question 1.

4 By substituting $y = tx$, find parametric equations for the loci whose cartesian equations are

- **a** $y^4 = x^5$ **b** $y = x^2 + 2x$
- **c** $y^2 = x^2 + 2x$ **d** $x^2 = x^3 - y^3$
- **e** $x^3 + y^3 = 3xy$

5 Show that the parametric equations

- **a** $x = 1 + 2t, y = 2 + 3t$
- **b** $x = 1/(2t - 3), y = t/(2t - 3)$

both represent the same straight line, and find its cartesian equation.

6 Show that the line given parametrically by the equations $x = \dfrac{2 - t}{1 + 2t}$, $y = \dfrac{3 + t}{1 + 2t}$ passes through the points $(6, 7)$ and $(-2, -1)$. Find the values of t corresponding to these points.

7 P is the variable point $(t^2, 3t)$ and O is the origin. Find the coordinates of Q, the mid-point of OP, and hence obtain the locus of Q as P varies.

8 P is the variable point $(at^2, 2at)$ on the parabola $y^2 = 4ax$, and Q is the foot of the perpendicular from P to the y-axis.
Find the locus of the mid-point of PQ.

9 The line joining the origin to the variable point $P(t, 1/t)$ meets the line $x = 1$ at Q.
Find the locus of the mid-point of PQ.

10 Find the coordinates of the points nearest to the origin on the curve $x = t, y = 1/t$.
What is their distance from the origin?

11 Find the coordinates of the points on the curve $x = at^2, y = 2at$ where the distance from the point $(5a, -2a)$ is stationary. Distinguish between maxima, minima and points of inflexion.

12 Find the equations of the chords joining the points with parameters p and q on the following curves.

- **a** $x = t^2, y = 2t$ **b** $x = t, y = -1/t$
- **c** $x = t^3, y = t$ **d** $x = t + 1/t, y = 2t$

13 Determine the point on the parabola $x = at^2$, $y = 2at$ where the distance to the line $x - y + 4a = 0$ is least and find the least distance.

14 Find the values of t at the points of intersection of the line $2x - y - 4 = 0$ with the parabola $x = t^2, y = 2t$ and give the coordinates of these points.

15 Find the points of intersection of the parabola $x = t^2, y = 2t$ with the circle $x^2 + y^2 - 9x + 4 = 0$.

Example 14 *Find the equation of the tangent to the curve $xy = c^2$ at the point $P(ct, c/t)$. Show that, if this tangent meets the axes at Q and R, then P is the mid-point of QR.*

The gradient of the curve is given by

$$\frac{dy}{dx} = \frac{dy}{dt} \bigg/ \frac{dx}{dt}$$

But $y = c/t$, $\quad \therefore \dfrac{dy}{dt} = -\dfrac{c}{t^2}$

and $x = ct$, $\quad \therefore \dfrac{dx}{dt} = c$

$$\therefore \frac{dy}{dx} = \frac{-c/t^2}{c} = -\frac{1}{t^2}$$

Therefore the equation of the tangent at P is

$$yt^2 + x = 2ct$$

This tangent meets the axes at $Q(2ct, 0)$ and $R(0, 2c/t)$ therefore $P(ct, c/t)$ is the mid-point of QR.

Example 15 *Find the coordinates of the points where the line $4x - 5y + 6a = 0$ cuts the curve given parametrically by $(at^2, 2at)$.*

If the line $4x - 5y + 6a = 0$ meets the curve at the point $(at^2, 2at)$, then its coordinates must satisfy the equation of the line. Therefore

$$4at^2 - 10at + 6a = 0$$
$$\therefore 2t^2 - 5t + 3 = 0$$
$$\therefore (2t - 3)(t - 1) = 0$$
$$\therefore t = \tfrac{3}{2} \quad \text{or} \quad 1$$

Therefore the coordinates of the points of intersection are $(\frac{9}{4}a, 3a)$ and $(a, 2a)$.

Exercise 22e

1 Find the equations of the tangents and normals to the following curves at the given points.
   a $x = t^2, y = t^3, (1, -1)$
   b $x = t^2, y = 1/t, (\frac{1}{4}, 2)$
   c $x = at^2, y = 2at, (a, -2a)$
   d $x = ct, y = c/t, (-c, -c)$
   e $x = t^2 - 4, y = t^3 - 4t, (-3, -3)$
   f $x = 3\cos\theta, y = 2\sin\theta, (\frac{3}{2}, \sqrt{3})$

2 Find the equations of the tangents and normals to the following curves at the point whose parameter is t.
   a $x = t^3, y = 3t^2$
   b $x = at^2, y = 2at$
   c $x = 4t^3, y = 3t^4$
   d $x = ct, y = c/t$
   e $x = a\cos t, y = b\sin t$
   f $x = a\sec t, y = b\tan t$

3 Find the equations of the chords joining the points whose parameters are p and q on the following curves. Find the equations of the tangents at the points p by finding the limiting equations of the chords as q approaches p.
   a $x = t^2, y = 2t$
   b $x = 1/t, y = t^2$
   c $x = ct, y = c/t$
   d $x = a\cos t, y = b\sin t$
   [Hint: cancel a factor of $p - q$ in the gradients.]

4 Find the equation of the normal to the parabola $x = at^2, y = 2at$ at the point $(4a, 4a)$. Find also the coordinates of the point where the normal meets the curve again.

5 Find the coordinates of the point where the normal to the rectangular hyperbola $x = ct, y = c/t$ at $(2c, \frac{1}{2}c)$ meets the curve again.

6 Find the coordinates of the point where the tangent to the curve $x = 1/t, y = t^2$ at $(1, 1)$ meets the curve again.

7 Find the equation of the tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$. For what values of t does the tangent pass through the point $(8a, 6a)$? Write down the equations of the tangents to the parabola from $(8a, 6a)$.

8 Find the equations of the tangents to the hyperbola $x = ct, y = c/t$ from the point $(\frac{3}{2}c, \frac{1}{2}c)$.

9 Find the equations of the normals to the parabola $x = at^2, y = 2at$ from the point $(14a, -16a)$.

10 The normal to the hyperbola $x = ct, y = c/t$ at the point P with parameter p meets the curve again at Q. Find the coordinates of Q.

11 Show that, if a tangent to the curve $x = 1/t, y = t^2$ meets the axes in A and B, then PB = 2AP.

12 Show that the tangent at the point t on the astroid $x = a\cos^3 t, y = a\sin^3 t$ is the line $y\cos t + x\sin t = a\sin t\cos t$. Show that the tangent meets the axes in points whose distance apart is a.

22.6 The parabola

Definition

The locus of a point equidistant from a given point and a given line is called a **parabola**. *The given point is the* **focus** *of the parabola and the given line is its* **directrix**.

As no new method is required, work on the parabola is given in the form of exercises. Note that any result proved may be used in later questions. Later, Chapter 30 treats the parabola in greater detail.

Exercise 22f

1 Use compasses and graph paper to plot a parabola from the definition.

2 Given a parabola, take axes with the x-axis through the focus, perpendicular to the directrix, and the origin where the x-axis meets the curve. Let the focus be $(a, 0)$ and show that the equation of the parabola is $y^2 = 4ax$.
[It follows from the definition that the equation of the directrix is $x = -a$.]

3 Verify that the point $(at^2, 2at)$ lies on the parabola $y^2 = 4ax$ for all values of t, and that every point on the parabola is given thus.

4 Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

Refer to Fig. 22.11 when doing questions 5–18.

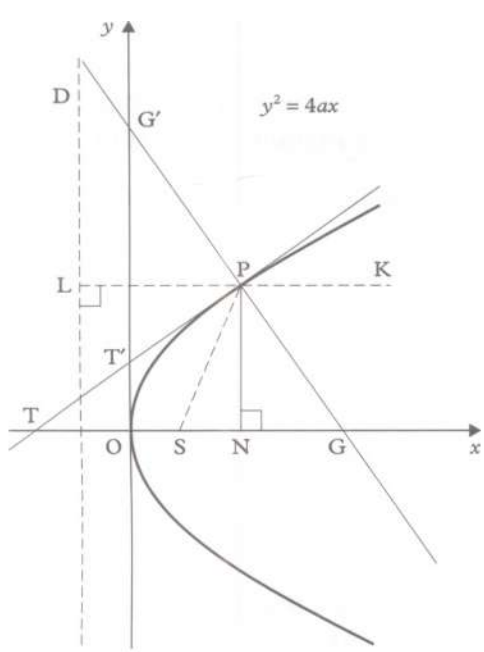


Figure 22.11

In Fig. 22.11, the tangent and normal at the point P on the parabola $y^2 = 4ax$ meet the x-axis at T and G, and the y-axis at T′ and G′. PN is parallel to the y-axis. S is the focus. LD is the directrix and L is the foot of the perpendicular from P to the directrix.

5 Show that ST′ = T′L and deduce that
   a $L\hat{P}T' = S\hat{P}T'$ [use the definition of the curve],
   b $S\hat{P}G = K\hat{P}G$.
   [This proves the optical property of a parabolic mirror, i.e. that light from a point source at the focus is reflected in rays parallel to the axis.]

6 Show that L, T′, S are collinear (i.e. lie on a straight line), and that LS is perpendicular to PT.

7 Show that TS = SP = SG.

8 Show that LPST is a rhombus and that LPGS is a parallelogram.

9 Show that NG = $2a$.

10 If the parameters of the points P and Q are p and q, show that the tangents to the parabola meet at the point $(apq, ap + aq)$.

11 If PQ passes through the focus prove that, with the notation of question 10, $pq = -1$.

12 Show that the tangents at the ends of a focal chord meet on the directrix.

13 Show that if the tangents at the ends of a focal chord meet the tangent at the vertex at U and V, then $U\hat{S}V$ is a right angle.

14 Show that the locus of the mid-point of a focal chord is $y^2 = 2a(x - a)$.

15 Show that if the tangents to the parabola at P and Q meet on the line $x = ah$, then the locus of the mid-point of the chord PQ is $y^2 = 2a(x + ah)$.

16 If the tangents to the parabola at P and Q intersect on the line $y = k$, find the locus of the mid-point of PQ.

17 Find the values of t for which the normal at $(at^2, 2at)$ passes through the point $(5a, 2a)$. Hence find the equations of the normals to the parabola from $(5a, 2a)$.

18 Find the equations of the tangents to the parabola from the point $(4a, 5a)$.

Chapter 23
Iterative methods for solving equations

Introduction

One of the most common tasks in mathematics is to solve equations. In this book we have already solved a variety of different equations. We have solved quadratic equations by factorisation or by the formula, we have solved other polynomial equations by factorising them and we have solved some carefully selected trigonometrical equations.

Consider, however, the following problem. **Fig. 23.1** represents a circle, whose centre is at O, and whose radius is one unit. Can we find the value of θ, in radians, so that the area of the shaded segment is exactly 0.5 square units?

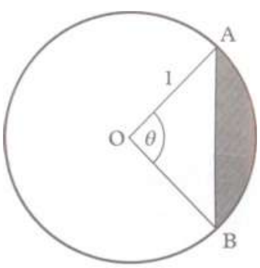


Figure 23.1

Since the angle θ is measured in radians, the area of the *sector* OAB is $\frac{1}{2}\theta r^2$, and since $r = 1$, this is just $\frac{1}{2}\theta$. The area of the *triangle* OAB can be obtained from the standard formula, $\frac{1}{2}ab \sin C$; in this case $a = 1$, $b = 1$, and $C = \theta$, so the area of the triangle OAB is $\frac{1}{2}\sin\theta$. The area of the shaded segment is the difference of these two areas, i.e.

$$\tfrac{1}{2}\theta - \tfrac{1}{2}\sin\theta$$

The problem is to find the value of θ so that this area is 0.5 square units. In other words we need to solve the equation

$$\tfrac{1}{2}\theta - \tfrac{1}{2}\sin\theta = \tfrac{1}{2}$$

or $\qquad \theta - \sin\theta = 1$

None of our methods for solving equations (apart from drawing a graph) would help us to solve this equation; indeed it is impossible to find an *exact* solution. However, it is obvious that such an angle exists. With a little experimentation using tables or a calculator, it is possible to see that an **approximate solution** is $\theta = 2$.

In this chapter we shall develop methods by which approximate solutions to equations can be obtained. An approximate solution can be very useful, and, as in the above example, it may be the only solution available. The value of such an answer is improved by giving an estimate of its degree of accuracy.

In Example 5 on page 257 we shall return to the equation $\theta - \sin\theta = 1$, but first we shall solve a simpler problem: to find the square root of a given number without using square root tables, or a calculator.

23.1 An iterative method for finding square roots

What is the square root of 18? Or, in other words, solve the equation

$$x^2 = 18$$

Since we are not going to use tables or the square root function on a calculator, the first step is to check through the square numbers:

$$\begin{array}{llllllll} x: & 1, & 2, & 3, & 4, & \boxed{?}, & 5, & 6, & \ldots \\ x^2: & 1, & 4, & 9, & 16, & \boxed{18}, & 25, & 36, & \ldots \end{array}$$

Note that $\sqrt{18}$ lies between 4 and 5, and it is nearer 4 than 5. So we might say

> 'the square root of 18 is 4, correct to the nearest whole number'

This is an approximate answer and it indicates its degree of accuracy.

We shall now use this 'first approximation' to obtain a better 'second approximation'. This in turn will be used to obtain an even better 'third approximation'. Such a procedure is called **successive approximation**, or **iteration**.

We use the following method to find successive approximations:

$$\begin{array}{llll} \text{If} & x = \sqrt{18}, & \text{then} & \dfrac{18}{x} = \sqrt{18} \\ \text{If} & x \neq \sqrt{18}, & \text{then} & \\ \textit{either} & x < \sqrt{18}, & \text{i.e.} & \dfrac{18}{x} > \sqrt{18} \\ \textit{or} & x > \sqrt{18}, & \text{i.e.} & \dfrac{18}{x} < \sqrt{18} \end{array}$$

In both cases $\quad x < \sqrt{18} < \dfrac{18}{x}$

Therefore, taking $\sqrt{18} = 4$ as a first approximation

$$4 < \sqrt{18} < \frac{18}{4}$$

In other words, $\sqrt{18}$ lies between 4 and 4.5. For the second approximation we take the mean of these numbers, i.e.

$$\frac{1}{2}\left(4+\frac{18}{4}\right)=4.25$$

Now we repeat the process, using $\sqrt{18} \approx 4.25$. We now know that $\sqrt{18}$ must lie between 4.25 and 18/4.25, and so we take as our 'third approximation' the average of *these* two numbers. In other words the third approximation is

$$\frac{1}{2}\left(4.25+\frac{18}{4.25}\right)=4.24265, \quad \text{correct to 6 s.f.*}$$

(The arithmetic at this stage is becoming quite heavy, and a calculator or tables may be helpful. However, square root tables and the square root function on the calculator are *not* allowed!)

We now have a very good approximate value of $\sqrt{18}$. The exact value lies between 4.24265 and 18/4.24265 (= 4.24263). So we are now able to say that

$$\sqrt{18} = 4.243, \quad \text{correct to 4 s.f.}$$

This procedure can be summed up as follows: writing x_r for the rth approximation, the $(r+1)$th approximation is given by

$$x_{r+1}=\frac{1}{2}\left(x_r+\frac{18}{x_r}\right)$$

This is called an **iterative formula** for finding $\sqrt{18}$. More generally, the iterative formula for finding the square root of any positive number, N, is

$$x_{r+1}=\frac{1}{2}\left(x_r+\frac{N}{x_r}\right)$$

Question

Q1 Use the iterative formula above, to find the square roots of

a 17 **b** 40 **c** 85 **d** 96

correct to four significant figures.

If a programmable calculator or a computer is available, you should try to write programs to solve some of the equations in this chapter by iteration. Iterative methods are ideally suited to such an approach, because the same basic sequence of steps is repeated over and over again. This can be done very rapidly and accurately on a programmable calculator or a computer.

*Six significant figures.

23.2 Further iterative formulae

If we were given the iterative formula

$$x_{r+1}=\frac{1}{2}\left(x_r+\frac{18}{x_r}\right)$$

but we did not know how it had been constructed, would it be possible to discover the equation which it is designed to solve? The answer is 'Yes', provided the sequence

$$x_1, x_2, x_3, x_4, \ldots$$

tends to a limit. Suppose that $x_n \to X$, as $n \to \infty$, then for a large value of n, the iterative formula could be written

$$X=\frac{1}{2}\left(X+\frac{18}{X}\right)$$

This equation could then be simplified, as follows:

$$2X = X+\frac{18}{X}$$

$$\therefore X=\frac{18}{X}$$

$$\therefore X^2 = 18$$

So, as expected, we see that the equation which is solved by the iterative formula above is

$$x^2 = 18$$

Example 1 *Starting with $x_1 = 4$, use the iterative formula*

$$x_{r+1}=5-\frac{2}{x_r}$$

to find x_2, x_3, and x_4, giving these values correct to three significant figures. Find the equation which is solved by this iterative formula.

$$x_2 = 5-\frac{2}{4}$$
$$= 4.5, \quad \text{exactly}$$

$$x_3 = 5-\frac{2}{4.5}$$
$$\approx 4.55556$$
$$= 4.56, \quad \text{correct to 3 s.f.}$$

$$x_4 = 5-\frac{2}{4.55556}$$
$$\approx 4.56098$$
$$= 4.56, \quad \text{correct to 3 s.f.}$$

The successive values of x_r appear to be tending to a limit, namely 4.56.

(*Note.* When using a calculator, you should store the successive values x_2, x_3, x_4 etc. in the memory. It is poor technique to use the *corrected* value of x_r to calculate x_{r+1}.)

To find the equation which this iterative formula solves, we write this limit as X, then, for large values of r, the iterative formula becomes

$$X = 5 - \frac{2}{X}$$

When this is simplified we obtain

$$X^2 - 5X + 2 = 0$$

So $x = 4.56$ is a root, correct to 3 s.f., of the equation

$$x^2 - 5x + 2 = 0$$

(The equation in Example 1 is a quadratic equation, and using an iterative method to solve it is not really appropriate. If this equation is solved by the formula, the solution would be

$$x = \frac{5 \pm \sqrt{17}}{2} = 4.56 \text{ or } 0.44, \quad \text{correct to 2 d.p.*}$$

The iterative formula produced the first of these, but not the second. However, we could use the fact that the sum of the roots is 5 to calculate the second root, i.e. $5 - 4.56 = 0.44$.)

As we have seen above, if the sequence $x_1, x_2, x_3, x_4, \ldots$ converges, then we can deduce the equation from the iterative formula. This suggests that if we have a given equation and we wish to construct a suitable iterative formula, all we need to do is to rearrange the equation in the form

$$x = f(x)$$

and the corresponding iterative formula will be

$$x_{r+1} = f(x_r)$$

Example 2 *Form an iterative formula to solve the equation*

$$x^3 - 5x + 1 = 0$$

and use it to find the root which lies between 0 *and* 1, *correct to three significant figures.*

The given equation can be arranged in the form

$$5x = x^3 + 1$$

$$x = \frac{x^3 + 1}{5}$$

consequently we shall take as the iterative formula

$$x_{r+1} = \frac{x_r^3 + 1}{5}$$

and, starting with $x_1 = 0$, we obtain

$$x_2 = \frac{1}{5} = 0.2$$

$$x_3 = \frac{1.008}{5} = 0.2016$$

$$x_4 = \frac{0.2016^3 + 1}{5}$$

$$= 0.201639, \quad \text{correct to 6 s.f.}$$

In view of the very small change from x_3 to x_4, it is reasonable to conclude that we are *very* near to the exact answer. Consequently we can be confident that the root of the equation is 0.202, correct to 3 s.f.

Do not assume that *any* rearrangement of a given equation produces a suitable iterative formula. Consider, for example, the following equation:

$$x^2 - 5x + 3 = 0$$

It is easy to verify that this has a root between 4 and 5.

The rearrangement

$$x = \frac{x^2 + 3}{5}$$

produces the iterative formula

$$x_{r+1} = \frac{x_r^2 + 3}{5}$$

If we start at $x_1 = 5$, the succeeding values of x_r, correct to 4 s.f., are

$$x_2 = \frac{25 + 3}{5} = 5.6$$

$$x_3 = \frac{5.6^2 + 3}{5} = 6.872$$

$$x_4 = \frac{6.872^2 + 3}{5} = 10.04$$

$$x_5 = \frac{10.04^2 + 3}{5} = 20.78$$

These values of x_r are getting further and further away from the root we were expecting. We say the sequence $x_1, x_2, x_3, \ldots$ is *diverging*. However, the rearrangement of the original equation was by no means the only possible one. Consider, for example,

$$x = 5 - \frac{3}{x}$$

*Two decimal places.

This gives the iterative formula

$$x_{r+1} = 5 - \frac{3}{x_r}$$

and if we start, as before, with $x_1 = 5$, we obtain

$$x_2 = 5 - \frac{3}{5} = 4.4$$

$$x_3 = 5 - \frac{3}{4.4} = 4.318$$

$$x_4 = 5 - \frac{3}{4.318} = 4.305$$

$$x_5 = 5 - \frac{3}{4.305} = 4.303$$

$$x_6 = 5 - \frac{3}{4.303} = 4.303$$

(The root given by the quadratic formula is 4.303.)

So this second rearrangement is suitable.

We can see from the above that not all rearrangements of a given equation lead to a suitable iterative formula. It would be more satisfactory if we had a method for discriminating between a formula which produces a divergent sequence and one which produces a convergent sequence. We will look at this in §23.3.

Exercise 23a

1 Use the iterative formula in §23.1 on page 250 to find the square roots of
a 12 **b** 30 **c** 50 **d** 75
giving your answers correct to 3 s.f.

2 Use the iterative formula

$$x_{r+1} = \frac{2x_r}{3} + \frac{4}{x_r^2}$$

starting at $x_1 = 2$, to find x_2, x_3 and x_4, giving your answers correct to 3 s.f. Find, in its simplest form, the equation which is solved by this iterative formula.

3 Adapt question 2 so that it can be used to find $20^{1/3}$.

4 Show that the equation $x^2 - 5x + 1 = 0$ can be arranged as $x = (x^2 + 1)/5$, or as $x = 5 - 1/x$. Hence write down two possible iterative formulae which might be used for solving this quadratic. Starting from $x_1 = 0.2$, find the values of x_2, x_3 and x_4 which are produced by each of these iterative formulae.
Only one of these sequences appears to converge. Use this sequence to write down the (two) roots of the quadratic equation.

5 The cubic equation $x^3 - 10x + 1 = 0$ can be rearranged in the form $x = (x^3 + 1)/10$.
Use this rearrangement to form an iterative formula and use it to find, correct to 4 s.f., the root which lies between 0 and 1.
(Start with $x_1 = 0$.)

6 Solve the equation $\theta = \sin\theta + 1$, by an iterative method, starting from $\theta = 2$. (θ is measured in *radians*.)

7 Show that the equation $x^2 - 8x + 10 = 0$, has a root between 1 and 2.
Show that the iterative formula $x_{r+1} = 8 - 10/x_r$, can be formed from this equation, and, starting from $x_1 = 1$, calculate the values of x_2, x_3 and x_4. Comment on your results.

8 The iterative formulae

a $x_{r+1} = \dfrac{2x_r^3 + 10}{3x_r^2}$ and **b** $x_{r+1} = \dfrac{10}{x_r^2}$

can both be obtained by rearranging the equation $x^3 - 10 = 0$.
Starting from $x_1 = 2$, find the values of x_2, x_3 and x_4, which are produced by these iterative formulae. One of these sequences converges. Use this to find $\sqrt[3]{10}$, correct to 4 s.f.

9 The fifth root of a real number N can be calculated from the iterative formula

$$x_{r+1} = \left(4x_r + \frac{N}{x_r^4}\right)\Big/5$$

Use this formula to find the fifth root of 50, correct to 3 s.f. [Hint: start with $x_1 = 2$.]

10 The product of the roots of the quadratic equation

$$x^2 - px + q = 0$$

is q. Therefore if x_r is an approximate value of one of the roots, the other could be written q/x_r. Use the fact that the sum of the roots of this quadratic equation is p to find a new approximation to the first root. Hence deduce the iterative formula

$$x_{r+1} = p - \frac{q}{x_r}$$

Use this iterative formula to solve the quadratic equation

$$x^2 - 7x + 3 = 0$$

giving your answers correct to 3 s.f.

23.3 Iteration — the test for convergence

In the preceding sections we saw that an iterative formula

$$x_{r+1} = f(x_r)$$

can be used to produce a sequence of values of x_r

$$x_1, x_2, x_3, x_4, \ldots$$

the value of x_1 being selected by trial and error. We have also seen (but not formally proved) that, provided the sequence tends to a limit X, then $x = X$ is a root of the equation

$$x = f(x)$$

In this section we examine the conditions under which we can expect the sequence $x_1, x_2, x_3, x_4, \ldots$ to converge. (Example 1 on page 251 will be used as an illustration, so it is advisable to read through this example again before proceeding.)

Fig. 23.2 shows the graphs of $y = x$ and $y = f(x)$, where $f(x) = 5 - 2/x$. The graphs intersect at $P(X, Y)$.

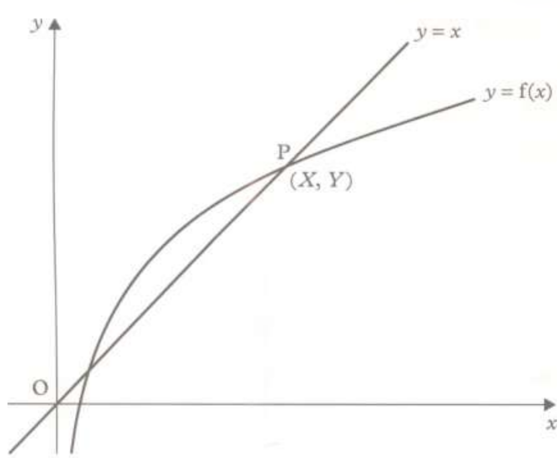


Figure 23.2

The x-coordinate of the point P, that is X, is a solution of the equation

$$x = f(x)$$

This is the root of the equation which we expect to obtain from the iterative formula

$$x_{r+1} = f(x_r)$$

Fig. 23.3 is an enlargement of the region around the point P in the previous diagram. It also shows the points $P_1, P_2, P_3, \ldots$, whose coordinates are $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots$ respectively, where $x_1, x_2, x_3, \ldots$ are the successive approximations given by the iterative formula.

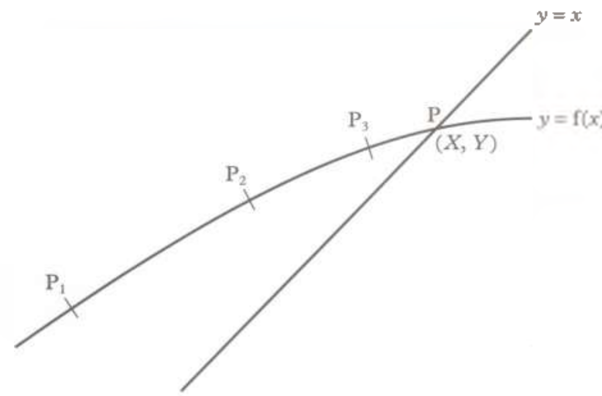


Figure 23.3

Since the point P_r (x_r, y_r) lies on the curve $y = f(x)$, the y-coordinate is given by

$$y_r = f(x_r)$$

This in turn is equal to x_{r+1}, so the coordinates of P_r can be written (x_r, x_{r+1}). This lets us produce the following geometrical method for constructing the points $P_1, P_2, P_3, \ldots$ (see Fig. 23.4).

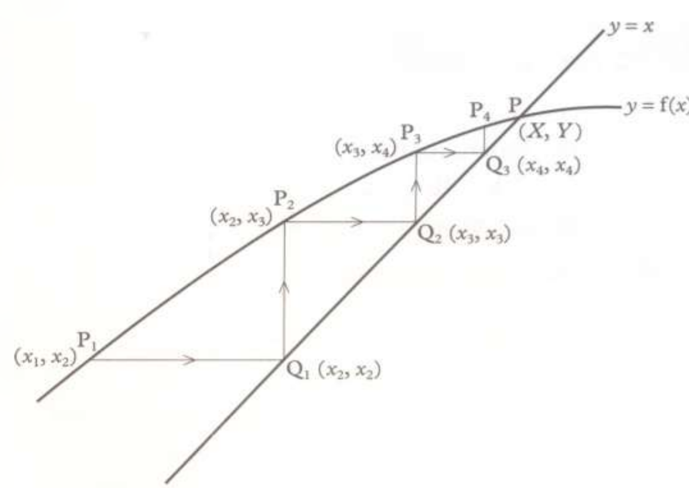


Figure 23.4

First mark the point (x_1, x_2), remembering that x_1 is selected on a trial-and-error basis. From P_1 draw a line parallel to the x-axis, and call the point where this meets the line, Q_1. The points P_1 and Q_1 have the same y-coordinate and Q_1 lies on the line $x = y$, so the coordinates of Q_1 are (y_1, y_1). But $y_1 = x_2$, so these coordinates could be written (x_2, x_2). From Q_1 we now draw a line parallel to the y-axis. The point where this meets the curve has the same x-coordinate as Q_1 and so its coordinates are (x_2, x_3). This is the point P_2. We now repeat the operation to construct the subsequent points $P_3, P_4, P_5, \ldots$. (Because space is limited, only the first few points are shown.)

In Fig. 23.4 we can see the points $P_1, P_2, P_3, \ldots$ getting closer and closer to the point P. Therefore the x-coordinates $x_1, x_2, x_3, \ldots$ will be getting closer and closer to X, or, to put it more formally, $x_r \to X$, as $r \to \infty$.

Although the function $f(x) = 5 - 2/x$ has been used in this illustration, a diagram like that in Fig. 23.4 could be drawn for other functions *provided* $f'(x)$ *lies between* 0 *and* 1. If the gradient is greater than 1 the picture is quite different. Fig. 23.5 shows the same construction applied to the graph of a function whose gradient is greater than 1. In this case, each step moves P_r further and further away from P.

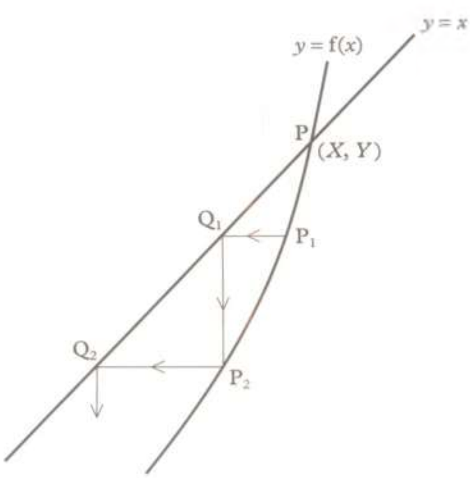


Figure 23.5

The diagrams in Fig. 23.6 show the corresponding constructions for graphs whose gradients are negative.

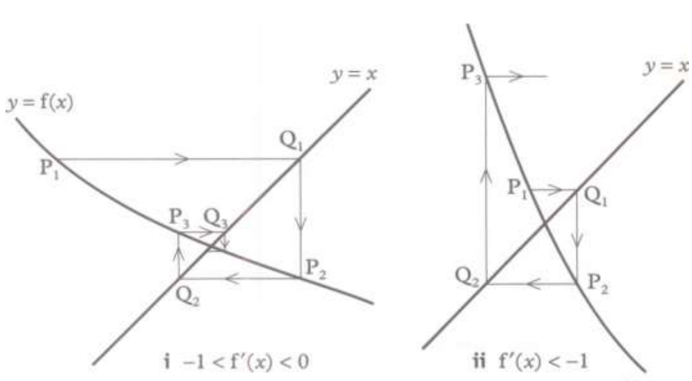


Figure 23.6

Fig. 23.6i (in which $-1 < f'(x) < 0$) shows the points $P_1, P_2, P_3, \ldots$ getting closer and closer to P. In other words, the sequence $x_1, x_2, x_3, \ldots$ converges when $|f'(x)| < 1$. In contrast Fig. 23.6ii (in which $f'(x) < -1$) shows these points moving further and further away from P, and so the sequence $x_1, x_2, x_3, \ldots$ diverges when $|f'(x)| > 1$.

From these diagrams we can conclude that the sequence $x_1, x_2, x_3, \ldots$ will converge if $|f'(x)| < 1$. To ensure that this sequence converges rapidly, the initial approximation should be as close as possible to the exact root and the function $f(x)$ should be selected so that $|f'(x)|$ is as small as possible.

(A more rigorous proof is beyond the scope of this book; if you wish to know more, consult a more specialised textbook. This topic usually comes under the heading 'Numerical methods'.)

Example 3 *Show that one of the iterative formulae*

a $x_{r+1} = (x_r^2 + 3)/5$, **b** $x_{r+1} = 5 - 3/x_r$,

produces a convergent sequence for $x \approx 5$, *and the other does not.*

In iterative formula **a**,

$$f(x) = \frac{x^2 + 3}{5}$$

$$f'(x) = \frac{2x}{5}$$

hence,

$$f'(5) = 2$$

Since $|f'(5)| > 1$, formula **a** will *not* produce a convergent sequence when $x \approx 5$.

In formula **b**,

$$f(x) = 5 - \frac{3}{x}$$

$$f'(x) = \frac{3}{x^2}$$

hence,

$$f'(5) = \frac{3}{25} = 0.12$$

In this case $|f'(5)| < 1$, so formula **b** *will* produce a convergent sequence when $x \approx 5$.

(*Note*. These formulae were used earlier in this chapter; see the end of §23.2 on page 251.)

Exercise 23b

Which of the following iterative formulae should, according to the test in the preceding section, produce a convergent sequence, $x_1, x_2, x_3, \ldots$, in the region of the value of x indicated? (These iterative formulae were used in Exercise 23a.)

1 $x_{r+1} = \dfrac{1}{2}\left(x_r + \dfrac{12}{x_r}\right)$; $x \approx 3$

2 $x_{r+1} = \dfrac{2x_r}{3} + \dfrac{4}{x_r^{\,2}}$; $x \approx 2$

3 $x_{r+1} = \dfrac{x_r^{\,2}+1}{5}$; $x \approx 0.2$

4 $x_{r+1} = 5 - \dfrac{1}{x_r}$; $x \approx 0.2$

5 $x_{r+1} = \dfrac{x_r^{\,3}+1}{10}$; $x \approx 1$

6 $\theta_{r+1} = \sin\theta_r + 1$; $\theta \approx 2$

7 $x_{r+1} = 8 - \dfrac{10}{x_r}$; $x \approx 1$

8 $x_{r+1} = \dfrac{2x_r^{\,3}+10}{3x_r^{\,2}}$; $x \approx 2$

9 $x_{r+1} = \dfrac{10}{x_r^{\,2}}$; $x \approx 2$

10 $x_{r+1} = \left(4x_r + \dfrac{50}{x_r^{\,4}}\right)\Big/5$; $x \approx 2$

23.4 The Newton–Raphson method

We now come to a particular method of iteration known as the Newton–Raphson method (frequently called Newton's method). Throughout this section we shall be considering the task of solving an equation of the form $F(x) = 0$ and the exact root we are seeking will be denoted by X.

As with all iterative methods, the first step is to find an approximate root. This can be done quite conveniently by drawing the graph of $y = F(x)$. The *exact* root is the x-coordinate of the point where the graph crosses the x-axis. Fig. 23.7 shows the graph of $y = F(x)$ and the point $P(X, 0)$.

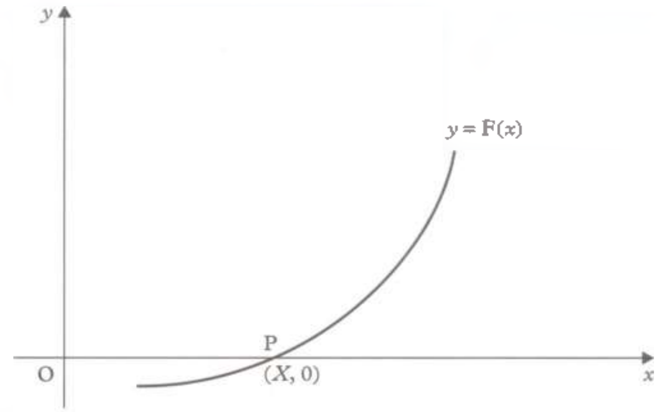


Figure 23.7

Now consider an enlargement of the region surrounding P as shown in Fig. 23.8.

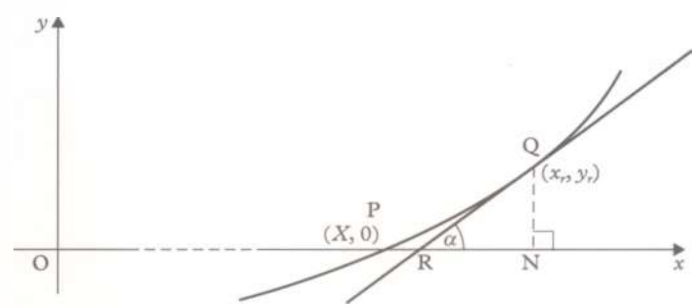


Figure 23.8

In this diagram, the point Q is near the point P and its x-coordinate x_r is an approximation to the exact root X, i.e. $x_r \approx X$. The coordinates of Q then are $(x_r, F(x_r))$. Newton's method consists of drawing a tangent to the curve at Q, and, if this line meets the x-axis at R, using the x-coordinate of R as the next approximation to X. In other words R is the point $(x_{r+1}, 0)$. It is clear from the diagram that x_{r+1} will be a better approximation than x_r.

(You are advised to draw the corresponding diagram for a graph whose gradient is negative, and also to consider the effect of $F(x_r)$ being negative. From these diagrams you should be able to see that Newton's method will yield the desired approximation, provided $F'(x)$ is not zero near the exact root.)

From the diagram in Fig. 23.8, we can produce a formula for x_{r+1}, in terms of the function $F(x)$ and x_r.

We know that

$$NQ = F(x_r)$$

and by elementary trigonometry

$$\frac{NQ}{RN} = \tan\alpha$$

i.e. $\mathrm{RN} = \dfrac{\mathrm{NQ}}{\tan\alpha}$

However, since the line RQ is the tangent to the curve at Q, $\tan\alpha$ is equal to the gradient at Q. In other words

$$\tan\alpha = \mathrm{F}'(x_r)$$

So we can write

$$\mathrm{RN} = \frac{\mathrm{F}(x_r)}{\mathrm{F}'(x_r)}$$

Now, from the diagram we can see that

$$\mathrm{OR} = \mathrm{ON} - \mathrm{RN}$$

$$\therefore \mathrm{OR} = x_r - \frac{\mathrm{F}(x_r)}{\mathrm{F}'(x_r)}$$

and since Newton's method is to use the x-coordinate of R as the new approximation, we have

$$\boldsymbol{x_{r+1} = x_r - \frac{\mathrm{F}(x_r)}{\mathrm{F}'(x_r)}}$$

Example 4 *Verify that the equation $x^3 - 5x - 40 = 0$ has a root between $x = 3$ and $x = 4$. Use the Newton–Raphson method to find this root correct to 3 s.f.*

In this example,

$$\mathrm{F}(x) = x^3 - 5x - 40$$

Putting $x = 3$ gives

$$\mathrm{F}(3) = 27 - 15 - 40 = -28$$

and, putting $x = 4$,

$$\mathrm{F}(4) = 64 - 20 - 40 = +4$$

Since $\mathrm{F}(x)$ has changed sign between $x = 3$ and $x = 4$, the graph of the function must cross the x-axis in this interval, so there is a root between 3 and 4. (This assumes that $\mathrm{F}(x)$ is continuous between these points. Special care must be taken if $\mathrm{F}(x)$ is known to have a discontinuity near the root being investigated.)

The Newton–Raphson iterative formula is

$$x_{r+1} = x_r - \frac{\mathrm{F}(x_r)}{\mathrm{F}'(x_r)}$$

and, in this case

$$\mathrm{F}(x) = x^3 - 5x - 40$$

Differentiating,

$$\mathrm{F}'(x) = 3x^2 - 5$$

So, the iterative formula to solve this equation is

$$x_{r+1} = x_r - \frac{x_r^{\,3} - 5x_r - 40}{3x_r^{\,2} - 5}$$

As $|\mathrm{F}(4)|$ is much smaller than $|\mathrm{F}(3)|$, the root appears to be nearer 4 than 3, so we start with $x_1 = 4$, then

$$x_2 = 4 - \frac{64 - 20 - 40}{48 - 5}$$
$$= 4 - \frac{4}{43}$$
$$= 3.907$$

(*Note.* The value of x_2 is given, correct to 4 s.f. If you are using a calculator, store each intermediate value in the memory for use in the next iteration. At this stage it would be unwise to claim that more than the first one or two s.f. have been determined.)

This value of x_2 should now be substituted into the Newton–Raphson formula. This gives

$$x_3 = 3.90445$$

In view of the very small change between x_2 and x_3, we can now say that the root is 3.90 correct to 3 s.f.

Example 4 shows some of the benefits of the Newton–Raphson formula. Firstly, provided $\mathrm{F}'(x)$ is not zero near the root, it is unnecessary to check whether the sequence converges. Secondly, the sequence converges very rapidly. In other words it is necessary to calculate only a few values of x_r to get a very accurate answer.

Example 5 *Use the Newton–Raphson formula to solve the equation*

$$\theta - \sin\theta = 1$$

giving your answer correct to 3 s.f.

(This is the equation in the Chapter introduction.) First, arrange the equation in the form

$$\theta - \sin\theta - 1 = 0$$

and note that the function needed is

$$\mathrm{F}(\theta) = \theta - \sin\theta - 1$$

Differentiating,

$$F'(\theta) = 1 - \cos\theta$$

The iterative formula we require is

$$\theta_{r+1} = \theta_r - \frac{\theta_r - \sin\theta_r - 1}{1 - \cos\theta_r}$$

Starting from $\theta_1 = 2$ (see §24.1 on page 259, and remembering that θ must be measured in radians),

$$\theta_2 = 2 - \frac{2 - \sin 2 - 1}{1 - \cos 2} = 1.93595$$

and

$$\theta_3 = 1.93456$$

and hence

$$\theta_4 = 1.93456$$

(These values have, for convenience, been rounded off to 6 s.f.) As the changes in $\theta_2, \theta_3, \theta_4$, have been so small, we can conclude that the root is 1.93 correct to 3 s.f.

Exercise 23c

Use the Newton–Raphson method to find the root of each equation that is near the given value. Give your answers correct to 3 s.f.

1 $x^3 - 4x^2 - x - 12 = 0, \quad x_1 = 5$

2 $x^4 - 3x^3 - 10 = 0, \quad x_1 = 3$

3 $2\sin\theta = \theta, \quad \theta_1 = 2$

4 $x^3 - 5x^2 = 4, \quad x_1 = 5$

5 $x^3 = 10x + 10, \quad x_1 = 3.5$

6 $3\tan\theta + 4\theta = 6, \quad \theta_1 = 1$

7 $x^4 - 4x^3 - x^2 + 4x - 10 = 0, \quad x_1 = 4$

8 $x^3 = 5x + 32, \quad x_1 = 4$

9 Verify that the equation $x^3 - 2x - 5 = 0$ has a root between $x = 2$ and $x = 3$, and find this root correct to 3 s.f.

10 Find, correct to 3 s.f., the smallest positive root of $5x^5 = 5x + 1$.

Chapter 24

Integration (3)

Introduction

In Chapters 3–7 we dealt with the differentiation of powers of x, polynomials, products and quotients, composite functions, trigonometrical functions, and we also discussed implicit functions and parameters.

Now that we come to extend integration we will find that it is not simply a matter of putting into reverse the techniques for differentiation. For example, we have learned a technique for differentiating $(3x^2+2)^4$ as it stands, but can we integrate this function without first expanding it? Consider the simple function x^n: we can differentiate this whenever $n \in \mathbb{Q}$, but, as we will show in the next chapter, dealing with $\int x^{-1}\,dx$ is not such a simple matter.

Integration is much less susceptible than differentiation to concise systematic treatment. It presents a broad front, and your experience of it will gradually expand, so that, by quick recognition of an increasing number of forms of *integrand* (i.e. the function to be integrated), you will learn how to select and use the most appropriate approach.

24.1 A function and its derivative

The first thing to search for in any but the simplest integrands is the presence of a function and its derivative. With this, we may often guess the integral to be a certain composite function, check by differentiation, and adjust the numerical factor. The following two examples show this approach.

Example 1 *Find* $\int x(3x^2+2)^4\,dx$.

[We note that the x outside the bracket is a constant $\times$ the derivative of the expression inside the bracket. We deduce that the integral is a function of $(3x^2+2)$.]

$$\frac{d}{dx}\{(3x^2+2)^5\} = 5(3x^2+2)^4 \times 6x = 30x(3x^2+2)^4$$

$$\therefore \frac{d}{dx}\left\{\frac{1}{30}(3x^2+2)^5\right\} = x(3x^2+2)^4$$

Hence

$$\int x(3x^2+2)^4\,dx = \frac{1}{30}(3x^2+2)^5 + c$$

Example 2 *Find* $\int \sin^2 4x \cos 4x\,dx$.

[We note that $\cos 4x$ is a constant $\times$ the derivative of $\sin 4x$, and we deduce that the integral is a function of $\sin 4x$.]

$$\frac{d}{dx}\{\sin^3 4x\} = 3(\sin 4x)^2 \times \cos 4x \times 4$$
$$= 12\sin^2 4x \cos 4x$$

Hence

$$\int \sin^2 4x \cos 4x\,dx = \frac{1}{12}\sin^3 4x + c$$

Questions

Q1 Differentiate.

a $(2x^2+3)^4$ b $\sqrt{(x^2-2x+1)}$

c $\dfrac{1}{(2x-1)^2}$ d $\sin(4x-7)$

e $\tan^3 x$ f $\cos^2 3x$

Q2 Find the following integrals, and check by differentiation.

a $\int x(x^2+1)^2\,dx$ b $\int (2x+1)^4\,dx$

c $\int (x^2+1)^3\,dx$ d $\int \frac{1}{2}\sin 3x\,dx$

e $\int x^2\sqrt{(x^3+1)}\,dx$ f $\int \sec^2 x \tan x\,dx$

24.2 Integration of odd powers of sin *x*, cos *x*, etc.

Pythagoras' theorem in the forms

$$\cos^2 x + \sin^2 x = 1, \cot^2 x + 1 = \operatorname{cosec}^2 x, \quad \text{and}$$
$$1 + \tan^2 x = \sec^2 x$$

(see §16.6 on page 199), may be used to change some integrands to a form suitable for the method of §24.1. In particular, it enables us to integrate odd powers of $\sin x$ and $\cos x$.

Example 3 *Find* $\int \sin^5 x\,dx$.

$$\int \sin^5 x\,dx = \int \sin^4 x \sin x\,dx$$
$$= \int (1-\cos^2 x)^2 \sin x\,dx$$
$$= \int (\sin x - 2\cos^2 x \sin x + \cos^4 x \sin x)\,dx$$

$$\therefore \int \sin^5 x\,dx = -\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + c$$

Questions

Q3 Find: **a** $\int \sin^3 x \, dx$ **b** $\int \cos^5 x \, dx$

Q4 Find: **a** $\int \cos^3 x \sin^2 x \, dx$
[Write $\cos^3 x$ as $\cos x(1 - \sin^2 x)$.]
b $\int \sin^3 x \cos^2 x \, dx$

Q5 Find $\int \sec x \tan^3 x \, dx$.
[Remember $\frac{d}{dx}(\sec x) = \sec x \tan x$.]

24.3 Integration of even powers of sin x, cos x*

Two very important formulae derived from the double-angle formulae are $\mathbf{\cos^2 x = \frac{1}{2}(1 + \cos 2x)}$ and $\mathbf{\sin^2 x = \frac{1}{2}(1 - \cos 2x)}$. (See §17.2 on page 204).

Their use in integrating even powers of sin x and cos x is shown in the latter part of Exercise 24a, which also gives practice in the use of other formulae, including the factor formulae. (See §17.5 on page 208.)

Exercise 24a

1 Differentiate.

a $(5x^2 - 1)^3$ **b** $\dfrac{1}{(2x^2 - x + 3)^2}$
c $\sqrt[3]{(x^2 + 4)}$ **d** $\cot 5x$
e $\cos(5x - 1)$ **f** $\sin^2 \dfrac{x}{3}$
g $\tan \sqrt{x}$ **h** $\sec^2 2x$ **i** $\sqrt{\operatorname{cosec} x}$

Find the integrals in questions 2–4.

2 a $\int x(x^2 - 3)^5 \, dx$ **b** $\int (3x - 1)^5 \, dx$
c $\int x(x + 2)^2 \, dx$ **d** $\int \dfrac{x}{(x^2 + 1)^2} \, dx$
e $\int \dfrac{x + 1}{(x^2 + 2x - 5)^3} \, dx$
f $\int (2x - 3)(x^2 - 3x + 7)^2 \, dx$
g $\int \dfrac{2x}{(4x^2 - 7)^2} \, dx$ **h** $\int 2x\sqrt{(3x^2 - 5)} \, dx$
i $\int (x^3 + 1)^2 \, dx$ **j** $\int \dfrac{x^2 - 1}{\sqrt{(x^3 - 3x)}} \, dx$
k $\int \dfrac{x - 1}{(2x^2 - 4x + 1)^{3/2}} \, dx$ **l** $\int (2x^2 - 1)^3 \, dx$

3 a $\int 3 \cos 3x \, dx$ **b** $\int \sin(2x + 3) \, dx$
c $\int \cos x \sin x \, dx$ **d** $\int \frac{1}{3} \cos 2x \, dx$
e $\int \sin 3x \cos^2 3x \, dx$ **f** $\int \sec^2 x \tan^2 x \, dx$
g $\int \sec^5 x \tan x \, dx$ **h** $\int \cos x \sqrt{\sin x} \, dx$
i $\int x \operatorname{cosec}^2 x^2 \, dx$ **j** $\int \dfrac{\cos \sqrt{x}}{\sqrt{x}} \, dx$
k $\int \operatorname{cosec}^3 x \cot x \, dx$

4 a $\int \cos^3 x \, dx$ **b** $\int \cos^5 \dfrac{x}{2} \, dx$
c $\int \sin^3 2x \, dx$ **d** $\int \cos^3 (2x + 1) \, dx$
e $\int \sin^5 x \cos^2 x \, dx$ **f** $\int \cos^3 x \sin^3 x \, dx$
g $\int \sec^4 x \, dx$ **h** $\int \operatorname{cosec} x \cot^3 x \, dx$
i $\int \tan^5 x \sec x \, dx$

5 Find $\int \tan x \sec^4 x \, dx$, **a** as a function of sec x, **b** as a function of tan x, and show that they are the same.

6 Show that the integral given in question **3c** may be obtained in three different forms.

Questions 7 onwards may be delayed (see footnote opposite).

7 Express **a** $\sin^2 \dfrac{x}{2}$ in terms of cos x
b $\cos^2 3x$ in terms of cos $6x$

8 Find **a** $\int \cos^2 x \, dx$ **b** $\int \sin^2 \dfrac{x}{2} \, dx$
c $\int \cos^2 3x \, dx$

9 Express $\sin^4 x$ in terms of cos $2x$, and $\cos^2 2x$ in terms of cos $4x$. Show that
$\int \sin^4 x \, dx = \frac{3}{8}x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + c$.

10 Find $\int \cos^4 x \, dx$.

11 Find the following integrals.

a $\int \sin^2 x \, dx$ **b** $\int \cos^2 \dfrac{x}{3} \, dx$
c $\int \sin^4 2x \, dx$ **d** $\int \cos^4 \dfrac{x}{2} \, dx$

12 Write down a formula for cos x in terms of cos $\dfrac{x}{2}$, and show that

$$\int \frac{1}{1 + \cos x} \, dx = \tan \frac{x}{2} + c$$

13 Find the following integrals.

a $\int \sqrt{(1 + \cos x)} \, dx$ **b** $\int \dfrac{\cot x}{\sqrt{(1 - \cos 2x)}} \, dx$
c $\int \sin 2x \sin^2 x \, dx$ **d** $\int 2 \sin x \cos \dfrac{x}{2} \, dx$

*This section and the latter part of Exercise 24a may be delayed and done along with later parts of the chapter.

14 a Factorise $\sin 3x + \sin x$. (See §17.5 on page 208.)

b Express $2 \sin 3x \cos 2x$ as the sum of two terms.

c Find $\int \sin 3x \cos 2x \, dx$.

15 Find the following integrals.

a $\int \sin x \cos 3x \, dx$

b $\displaystyle\int 2 \cos \frac{3x}{2} \cos \frac{x}{2} \, dx$

c $\int \sin 4x \sin x \, dx$

24.4 Change of variable

In Example 1 we found that

$$\int x(3x^2+2)^4 \, dx = \frac{1}{30}(3x^2+2)^5 + c$$

The integral is a function of $(3x^2+2)$. If we write $3x^2+2$ as u, then the integral is a function of u. This suggests that we might make the substitution $u = 3x^2+2$ in the integrand, and *integrate with respect to u*. Let us see how to do this.

Let $\quad y = \int x(3x^2+2)^4 \, dx$

then $\quad \dfrac{dy}{dx} = x(3x^2+2)^4$

If $u = 3x^2+2$, x may be expressed as a function of u. Then, by the chain rule,

$$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$$

$$\therefore \frac{dy}{du} = x(3x^2+2)^4 \frac{dx}{du}$$

Integrating with respect to u,

$$y = \int x(3x^2+2)^4 \frac{dx}{du} \, du$$

But $u = 3x^2+2$, $\therefore \dfrac{du}{dx} = 6x$ and $\dfrac{dx}{du} = \dfrac{1}{6x}$.

$$\begin{aligned} \therefore \int x(3x^2+2)^4 dx &= \int x(3x^2+2)^4 \frac{1}{6x} \, du \\ &= \int \tfrac{1}{6} u^4 \, du \\ &= \tfrac{1}{30} u^5 + c \\ &= \tfrac{1}{30}(3x^2+2)^5 + c \end{aligned}$$

Questions

Q6 Find $\int \sin^2 4x \cos 4x \, dx$; put $u = \sin 4x$.

Q7 Find $\int \sin^5 x \, dx$; put $u = \cos x$.

Comparing the above with the solutions of Examples 1, 2 and 3, it might appear that we have introduced a more difficult technique. However, the power of changing the variable lies in its application to a wide class of integrals not suitable for the methods of §24.1 and §24.2.

In general, let $f(x)$ be a function of x, and let

$$y = \int f(x) \, dx$$

Then $\quad \dfrac{dy}{dx} = f(x)$

If u is a function of x, then by the chain rule

$$\begin{aligned} \frac{dy}{du} &= \frac{dy}{dx} \times \frac{dx}{du} \\ \therefore \frac{dy}{du} &= f(x)\frac{dx}{du} \\ \therefore y &= \int f(x)\frac{dx}{du} \, du \end{aligned}$$

$$\therefore \boldsymbol{\int f(x) \, dx = \int f(x) \frac{dx}{du} \, du}$$

Thus an integral with respect to x may be transformed into an integral with respect to a related variable u, by using the above result, and substituting for $f(x)$ and $\dfrac{dx}{du}$ in terms of u.

Example 4 *Find* $\int x\sqrt{(3x-1)} \, dx$.

$$\begin{aligned} &\int x\sqrt{(3x-1)} \frac{dx}{du} \, du \\ &= \int \frac{1}{3}(u^2+1)u \frac{2u}{3} \, du \\ &= \int (\tfrac{2}{9}u^4 + \tfrac{2}{9}u^2) \, du \\ &= \tfrac{2}{45}u^5 + \tfrac{2}{27}u^3 + c \\ &= \tfrac{2}{135}u^3(3u^2+5) + c \end{aligned}$$

Let $\sqrt{(3x-1)} = u$

$x = \frac{1}{3}(u^2+1)$

$\dfrac{dx}{du} = \dfrac{2u}{3}$

$$\therefore \int x\sqrt{(3x-1)} \, dx = \tfrac{2}{135}(3x-1)^{3/2}(9x+2) + c$$

Question

Q8 Find the following integrals, using the given change of variable:

a $\int x\sqrt{(2x+1)}\,dx$, $\quad\sqrt{(2x+1)} = u$

b $\int x\sqrt{(2x+1)}\,dx$, $\quad 2x+1 = u$

c $\int x(3x-2)^6\,dx$, $\quad 3x-2 = u$

Exercise 24b

1 Find the following integrals, using the given change of variable.

a $\int 3x\sqrt{(4x-1)}\,dx$, $\quad\sqrt{(4x-1)} = u$

b $\int x\sqrt{(5x+2)}\,dx$, $\quad 5x+2 = u$

c $\int x(2x-1)^6\,dx$, $\quad 2x-1 = u$

d $\displaystyle\int \frac{x}{\sqrt{(x-2)}}\,dx$, $\quad\sqrt{(x-2)} = u$

e $\int (x+2)(x-1)^4\,dx$, $\quad x-1 = u$

f $\int (x-2)^5(x+3)^2\,dx$, $\quad x-2 = u$

g $\displaystyle\int \frac{x(x-4)}{(x-2)^2}\,dx$, $\quad x-2 = u$

h $\displaystyle\int \frac{x-1}{\sqrt{(2x+3)}}\,dx$, $\quad\sqrt{(2x+3)} = u$

2 Repeat questions **1a** and **1d** using a different change of variable in each case.

3 For each of the following pairs of integrals, **(i)** find one of them by a suitable change of variable, **(ii)** write down the other as a composite function of x, as in Example 1 on page 259:

a $\int x\sqrt{(3x-4)}\,dx$ and $\int x\sqrt{(3x^2-4)}\,dx$

b $\int x(x^2+5)^6\,dx$ and $\int x(x+5)^6\,dx$

c $\displaystyle\int \frac{x}{\sqrt{(x-1)}}\,dx$ and $\displaystyle\int \frac{x}{\sqrt{(x^2-1)}}\,dx$

4 Find the following integrals, using a suitable change of variable only where necessary.

a $\int x\sqrt{(2x^2+1)}\,dx$ $\qquad$ b $\displaystyle\int \frac{3x^2-1}{(x^3-x+4)^3}\,dx$

c $\int 2x\sqrt{(2x-1)}\,dx$ $\qquad$ d $\int \cos^3 2x\,dx$

e $\int \sin x\sqrt{\cos x}\,dx$ $\qquad$ f $\int \cot^2 x\,\mathrm{cosec}^2 x\,dx$

g $\int 2x(4x^2-1)^3\,dx$ $\qquad$ h $\displaystyle\int \frac{x}{\sqrt{(2x^2-5)}}\,dx$

i $\displaystyle\int \frac{3x}{\sqrt{(4-x)}}\,dx$ $\qquad$ j $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

24.5 Definite integrals and change of limits

The method of changing the variable can also be used with definite integrals. It is usually more convenient to change the limits to those of the new variable at the same time.

As a reminder that you must always check for the presence of a function and its derivative in an integrand, two examples of this type are provided.

Example 5 *Evaluate* $\displaystyle\int_{1/2}^{3} x\sqrt{(2x+3)}\,dx$.

$$*\int_{x=1/2}^{x=3} x\sqrt{(2x+3)}\,\frac{dx}{du}\,du$$

$$=\int_4^9 \tfrac{1}{2}(u-3)u^{1/2}\,\tfrac{1}{2}\,du$$

$$=\int_4^9 \left(\tfrac{1}{4}u^{3/2}-\tfrac{3}{4}u^{1/2}\right)du$$

$$=\left[\tfrac{1}{10}u^{5/2}-\tfrac{1}{2}u^{3/2}\right]_4^9$$

$$=(24.3-13.5)-(3.2-4)$$

$$=11.6$$

Let $2x+3=u$

$x=\tfrac{1}{2}(u-3)$

$\dfrac{dx}{du}=\tfrac{1}{2}$

x	u
3	9
$\tfrac{1}{2}$	4

Example 6 *Evaluate* **a** $\displaystyle\int_2^3 \frac{x}{\sqrt{(x^2-3)}}\,dx$,

b $\displaystyle\int_0^{\pi/4} \cos^3 x\sin x\,dx$.

a $$\int_2^3 \frac{x}{\sqrt{(x^2-3)}}\,dx=\left[(x^2-3)^{1/2}\right]_2^3$$

$$=(9-3)^{1/2}-(4-3)^{1/2}$$

$$=\sqrt{6}-1$$

b $$\int_0^{\pi/4} \cos^3 x\sin x\,dx=\left[-\tfrac{1}{4}\cos^4 x\right]_0^{\pi/4}$$

$$=\left(-\tfrac{1}{4}\times\tfrac{1}{4}\right)-\left(-\tfrac{1}{4}\right)=\tfrac{3}{16}$$

*Note that, in practice, this integral will of course first be written down as given (i.e. as an integral with respect to x). When you change the variable, you change dx to $\dfrac{dx}{du}\,du$; you then need to specify that the limits are still those of x.

Exercise 24c

1 Evaluate the following definite integrals by changing the variable and the limits.

a $\int_2^3 x\sqrt{(x-2)}\,dx$ b $\int_0^1 x(x-1)^4\,dx$

c $\int_1^2 \frac{x}{\sqrt{(2x-1)}}\,dx$ d $\int_1^2 (2x-1)(x-2)^3\,dx$

e $\int_{-3/8}^0 \frac{x+3}{\sqrt{(2x+1)}}\,dx$

2 Evaluate the following definite integrals either by writing down the integral as a function of x, or by using the given change of variable.

a $\int_0^{\pi/6} \sec^4 x \tan x\,dx$ $(\sec x = u)$

b $\int_0^{\pi/2} \sin^5 x\,dx$ $(\cos x = u)$

c $\int_{\pi/6}^{\pi/2} \frac{\cot x}{\sqrt{\operatorname{cosec}^3 x}}\,dx$ $(\operatorname{cosec} x = u)$

3 Evaluate:

a $\int_0^{1/2} \frac{x}{\sqrt{(1-x^2)}}\,dx$ b $\int_0^4 2x\sqrt{(4-x)}\,dx$

c $\int_{-1}^0 x(x^2-1)^4\,dx$ d $\int_0^{\pi/4} \sec^4 x\,dx$

e $\int_{1/2}^1 \frac{x-2}{(x+2)^3(x-6)^3}\,dx$

f $\int_{-1}^2 (x+1)(2-x)^4\,dx$

g $\int_{-\pi/2}^{\pi/2} \cos^3 x\,dx$ h $\int_{5/3}^{8/3} \frac{x+2}{\sqrt{(3x-4)}}\,dx$

i $\int_0^{\pi/2} \sin x\sqrt{\cos x}\,dx$

4 Calculate the area enclosed by the curve $y = x/\sqrt{(x^2-1)}$, the x-axis, $x = 2$ and $x = 3$.

5 Calculate the area under $y = \sin^3 x$ from $x = 0$ to $x = 2\pi/3$.

6 Calculate the volume of the solid generated when the area under $y = \cos x$, from $x = 0$ to $x = \pi/2$ is rotated through four right angles about the x-axis. (See §24.3 on page 260.)

7 The area of a uniform lamina is that enclosed by the curve $y = \sin x$, the x-axis, and the line $x = \pi/2$. Find the distance from the x-axis of the centre of gravity of the lamina. (See §24.3 on page 260.)

24.6 Integration using inverse trigonometrical functions

Inverse trigonometrical functions were introduced in §18.6 on page 219. You should revise this topic before proceeding further. Q9–12 are included to provide revision.

Questions

Q9 The following angles lie between 0 and 90° inclusive. Express them in degrees, and in radians in terms of π.

a $\tan^{-1} 1$ b $\sin^{-1} \frac{1}{2}$

c $\frac{1}{2}\sin^{-1} 1$ d $\cos^{-1} \frac{1}{2}$

e $\frac{1}{2}\cos^{-1} \frac{1}{2}$ f $\cos^{-1} 1$

g $2\cos^{-1} \frac{\sqrt{3}}{2}$ h $\frac{1}{3}\cos^{-1} 0$

i $\frac{2}{3}\cos^{-1} 1$ j $\sec^{-1} 2$

k $2\operatorname{cosec}^{-1}\sqrt{2}$

Q10 Express the following angles in radians, leaving π in the answers.

a 20° b 70°

c 150° d 300°

e 405°

Q11 Express the following angles in degrees.

a 1 radian b 0.03 radian

c 1.25 radians d 0.715 radian

e $\pi/5$ radian

Q12 Express the following (acute) angles in radians.

a $2\sin^{-1} 0.6$ b $\tan^{-1} 1.333$

c $\frac{2}{3}\cos^{-1} 0.3846$

The inverse sine function may be written as arcsin x, or as $\sin^{-1} x$. Both forms will be used in this book to familiarise you with them.

The expression $\sqrt{(1-x^2)}$ may be reduced to a rational form by changing the variable to u, where $x = \sin u$; thus

$$\sqrt{(1-x^2)} = \sqrt{(1-\sin^2 u)} = \sqrt{\cos^2 u} = \cos u$$

This is used in the following example.

Example 7 *Find* $\int \frac{1}{\sqrt{(1-x^2)}}\,dx$.

$$\int \frac{1}{\sqrt{(1-x^2)}}\frac{dx}{du}\,du$$

Let $x = \sin u$

$\frac{dx}{du} = \cos u$

$$= \int \frac{1}{\sqrt{(1-\sin^2 u)}}\cos u\,du$$

$$= \int \frac{1}{\cos u}\cos u\,du$$

$$= u + c$$

$$= \arcsin x + c$$

24.7 $\int \frac{1}{\sqrt{(a^2-b^2x^2)}}\,dx$

Note that the integral in Example 7 is not solvable by the change of variable $\sqrt{(1-x^2)} = u$. $\int \frac{1}{\sqrt{(1-x^2)}}\,dx$ merely becomes $\int \frac{-1}{\sqrt{(1-u^2)}}\,du$. However, changes of variable involving a trigonometrical substitution, as successfully used in Example 7, open the way to finding a very important group of integrals. Here are two examples of the type of substitution we will use.

If $x = 5\sin u$,

$\sqrt{(25-x^2)} = \sqrt{(25-25\sin^2 u)} = \sqrt{\{25(1-\sin^2 u)\}} = 5\cos u$

If $x = \frac{\sqrt{3}}{2}\sin u$,

$\sqrt{(3-4x^2)} = \sqrt{(3-4\times\frac{3}{4}\sin^2 u)} = \sqrt{\{3(1-\sin^2 u)\}} = \sqrt{3}\cos u$

Question

Q13 Reduce each of the following to the form $k\cos u$, and give u in terms of x in each case.

a $\sqrt{(9-x^2)}$ **b** $\sqrt{(1-25x^2)}$
c $\sqrt{(4-9x^2)}$ **d** $\sqrt{(7-x^2)}$
e $\sqrt{(1-3x^2)}$ **f** $\sqrt{(3-2x^2)}$

We see that when given $\sqrt{(a^2-b^2x^2)}$ we write

$$a^2 - b^2x^2 \quad \text{as} \quad a^2 - a^2\sin^2 u$$

thus $b^2x^2 = a^2\sin^2 u$, and $x = (a/b)\sin u$. Note that $u = \arcsin(bx/a)$ and, for the substitution to be valid and of use, u must be real and not $\pi/2$, so $|bx| < |a|$. This condition is implicit in $\sqrt{(a^2-b^2x^2)}$ being real and not zero.

Example 8 *Find* $\int \frac{1}{\sqrt{(9-4x^2)}}\,dx$.

$$\int \frac{1}{\sqrt{(9-4x^2)}}\frac{dx}{du}\,du$$

$9 - 4x^2$

$9 - 9\sin^2 u$

Let $x = \frac{3}{2}\sin u$

$\frac{dx}{du} = \frac{3}{2}\cos u$

$$= \int \frac{1}{\sqrt{(9-9\sin^2 u)}}\times\frac{3}{2}\cos u\,du$$

$$= \int \frac{1}{3\cos u}\times\frac{3}{2}\cos u\,du$$

$$= \int \tfrac{1}{2}\,du$$

$$= \tfrac{1}{2}u + c$$

$$= \frac{1}{2}\arcsin\left(\frac{2x}{3}\right) + c$$

(This answer could also be written $\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + c$.)

Questions

Q14 Find the following integrals.

a $\int \frac{1}{\sqrt{(4-x^2)}}\,dx$ **b** $\int \frac{1}{\sqrt{(1-3x^2)}}\,dx$

c $\int \frac{1}{\sqrt{(16-9x^2)}}\,dx$

Q15 Prove that

$$\int \frac{1}{\sqrt{(a^2-b^2x^2)}}\,dx = \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right) + c.$$

24.8 $\int \frac{1}{a^2+b^2x^2}\,dx$

In §24.7 above we made use of Pythagoras' theorem in the form $\cos^2 u + \sin^2 u = 1$. We shall now find that another form, $1 + \tan^2 u = \sec^2 u$, leads to other useful changes of variable.

Questions

Q16 Find $\int \frac{1}{1+x^2}\,dx$ by taking x as $\tan u$.

Q17 Reduce each of the following to the form $k\sec^2 u$, and give u in terms of x in each case.

a $9 + x^2$ **b** $1 + 4x^2$ **c** $25 + 9x^2$
d $3 + x^2$ **e** $1 + 5x^2$ **f** $7 + 3x^2$

Question

Q18 Find the following integrals.

a $\displaystyle\int \frac{1}{4+x^2}\,dx$ b $\displaystyle\int \frac{1}{1+16x^2}\,dx$

c $\displaystyle\int \frac{1}{3+4x^2}\,dx$

Example 9 Evaluate $\displaystyle\int_{\surd 3/2}^{3/2} \frac{1}{3+4x^2}\,dx.$

$$\int_{x=\surd 3/2}^{x=3/2} \frac{1}{3+4x^2}\frac{dx}{du}\,du$$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{3(1+\tan^2 u)}\frac{\surd 3}{2}\sec^2 u\,du$$

$$= \int_{\pi/4}^{\pi/3} \frac{\surd 3}{6}\,du$$

$$= \left[\frac{\surd 3u}{6}\right]_{\pi/4}^{\pi/3}$$

$$= \frac{\surd 3}{6}\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$

$$= \frac{\surd 3\pi}{72}$$

$3+4x^2$

$3+3\tan^2 u$

Let $x = \dfrac{\surd 3}{2}\tan u$

$\dfrac{dx}{du} = \dfrac{\surd 3}{2}\sec^2 u$

x	$\tan u$	u
$\frac{3}{2}$	$\surd 3$	$\frac{\pi}{3}$
$\frac{\surd 3}{2}$	1	$\frac{\pi}{4}$

Exercise 24d

(Questions 1–4 revise §18.6 on page 219.)

1 The following angles lie between 0 and 90° inclusive. Express them in degrees, and in radians in terms of π.

a $\arccos \dfrac{1}{\surd 2}$ b $\operatorname{arccot} 1$

c $\dfrac{1}{\surd 3}\operatorname{arccot}\surd 3$ d $\arcsin \dfrac{\surd 3}{2}$

e $\surd 3 \arcsin \frac{1}{2}$ f $\frac{1}{3}\operatorname{arcsec}\surd 2$

g $\frac{3}{2}\arctan 1$ h $\frac{1}{2}\operatorname{arccosec} 2$

2 Express the following angles in radians.

a 32° b 60° 21′

c 5° 41′ d 235° 16′

3 Express the following angles in degrees.

a 2 radians b 0.08 radian

c 1.362 radians d $\pi/6$ radian

4 Express the following (acute) angles in radians. (In this question, the notation, $\sin^{-1} x$, etc., is used to give you some practice in using it.)

a $\sin^{-1} 0.8$ b $\frac{1}{2}\cos^{-1}\left(\dfrac{5}{13}\right)$

c $2\tan^{-1} 0.625$

5 Express the following in the form $k\cos u$, and give u in terms of x in each case.

a $\surd(16-x^2)$ b $\surd(1-9x^2)$

c $\surd(9-4x^2)$ d $\surd(10-x^2)$

e $\surd(1-6x^2)$ f $\surd(5-3x^2)$

6 Find the following integrals.

a $\displaystyle\int \frac{1}{\surd(25-x^2)}\,dx$ b $\displaystyle\int \frac{1}{\surd(1-4x^2)}\,dx$

c $\displaystyle\int \frac{1}{\surd(4-9x^2)}\,dx$ d $\displaystyle\int \frac{1}{\surd(3-x^2)}\,dx$

e $\displaystyle\int \frac{1}{\surd(1-7x^2)}\,dx$ f $\displaystyle\int \frac{1}{\surd(2-3x^2)}\,dx$

7 Express the following in the form $k\sec^2 u$, and give u in terms of x in each case.

a $16+x^2$ b $1+9x^2$

c $4+3x^2$ d $2+x^2$

e $1+3x^2$ f $5+2x^2$

8 Find the following integrals.

a $\displaystyle\int \frac{1}{25+x^2}\,dx$ b $\displaystyle\int \frac{1}{1+36x^2}\,dx$

c $\displaystyle\int \frac{1}{16+3x^2}\,dx$ d $\displaystyle\int \frac{1}{5+x^2}\,dx$

e $\displaystyle\int \frac{1}{1+6x^2}\,dx$ f $\displaystyle\int \frac{1}{3+10x^2}\,dx$

9 Find the following integrals.

a $\displaystyle\int \frac{1}{9+2x^2}\,dx$ b $\displaystyle\int \frac{3}{\surd(4-5x^2)}\,dx$

c $\displaystyle\int \frac{1}{\surd(3-2x^2)}\,dx$ d $\displaystyle\int \frac{2}{3+5x^2}\,dx$

10 Evaluate the following integrals, leaving π in your answers.

a $\displaystyle\int_1^{\surd 3} \frac{2}{1+x^2}\,dx$ b $\displaystyle\int_0^{\surd 2} \frac{1}{\surd(4-x^2)}\,dx$

c $\displaystyle\int_{1/2}^{1} \frac{3}{\surd(1-x^2)}\,dx$ d $\displaystyle\int_0^{3} \frac{1}{9+x^2}\,dx$

e $\displaystyle\int_0^{1/6} \frac{1}{\surd(1-9x^2)}\,dx$ f $\displaystyle\int_{-2}^{\surd 3} \frac{1}{5\surd(4-x^2)}\,dx$

11 **a** Find $\int \frac{1}{\sqrt{(9-x^2)}}\,dx$ using

i $x = 3\sin u$, **ii** $x = 3\cos u$.

b Evaluate $\int_{3/2}^{3} \frac{1}{\sqrt{(9-x^2)}}\,dx$ using

i $x = 3\sin u$, **ii** $x = 3\cos u$.

12 Find the following integrals, using the given change of variable.

a $\int \frac{1}{\sqrt{\{4-(x+1)^2\}}}\,dx$, $x+1 = 2\sin u$

b $\int \frac{1}{9+(x-3)^2}\,dx$, $x-3 = 3\tan u$

13 Find the following integrals.

a $\int \frac{1}{(x+3)^2+25}\,dx$

b $\int \frac{1}{\sqrt{\{4-(x-1)^2\}}}\,dx$

c $\int \frac{1}{3(x-2)^2+5}\,dx$

d $\int \frac{1}{\sqrt{\{9-3(x+1)^2\}}}\,dx$

***14** **a** $2x^2 - 12x + 21$ may be written $2(x^2-6x+9)+21-18 = 2(x-3)^2+3$.
Write the following expressions in the form $a(x+b)^2+c$ (see §10.3 on page 131).

i $x^2 - 6x + 16$

ii $3x^2 - 12x + 14$

iii $2x^2 - 4x + 5$

b Find the following integrals.

i $\int \frac{1}{x^2-2x+5}\,dx$

ii $\int \frac{1}{2x^2+4x+11}\,dx$

iii $\int \frac{1}{x^2-4x+13}\,dx$

iv $\int \frac{1}{4x^2-8x+7}\,dx$

15 **a** $1 + 6x - 3x^2$ may be written $4 - 3(x^2-2x+1) = 4 - 3(x-1)^2$.
Write the following expressions in the form $a - b(x+c)^2$.

i $3 - 2x - x^2$ **ii** $5 + 4x - x^2$

iii $7 + 2x - 2x^2$

b Find the following integrals.

i $\int \frac{1}{\sqrt{(3-2x-x^2)}}\,dx$

ii $\int \frac{1}{\sqrt{(1+8x-4x^2)}}\,dx$

iii $\int \frac{1}{\sqrt{(12+4x-x^2)}}\,dx$

iv $\int \frac{1}{\sqrt{(-2x^2+12x-9)}}\,dx$

16 Evaluate:

a $\int_2^3 \frac{1}{x^2-4x+5}\,dx$

b $\int_{-1}^{1} \frac{1}{\sqrt{(3-2x-x^2)}}\,dx$

***17** Find the following integrals by writing each integrand as two fractions.

a $\int \frac{3-x}{\sqrt{(1-x^2)}}\,dx$ **b** $\int \frac{2x+3}{\sqrt{(4-x^2)}}\,dx$

18 Show that
$\int \sqrt{(1-x^2)}\,dx = \frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{(1-x^2)} + c$.
Find the following integrals.

a $\int \frac{x^2}{\sqrt{(1-x^2)}}\,dx$ **b** $\int \frac{1}{(x^2+9)^2}\,dx$

c $\int \frac{x}{\sqrt{(4-x^4)}}\,dx$

19 Show that $\int \frac{1}{(1-x^2)^{3/2}}\,dx = x(1-x^2)^{-1/2} + c$.

Find the following integrals.

a $\int \frac{1}{(1-9x^2)\sqrt{(1-9x^2)}}\,dx$

b $\int \frac{1}{x^2\sqrt{(1-x^2)}}\,dx$ **c** $\int \frac{1}{x\sqrt{(x^2-1)}}\,dx$

Chapter 25
Exponential and logarithmic functions

25.1 Exponential functions

We often use the word *exponent* instead of *index*. Functions in which the variable is in the index (such as 2^x, $10^{\sin x}$) are called **exponential functions**.

The graph of $y = a^x$

First consider the function 2^x. A table of values follows.

Table of values, $y = 2^x$

x	-3	-2	-1	0	1	2
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

As $x \to -\infty$, $2^x \to 0$, and so the curve approaches the x-axis but does not meet it. Fig. 25.1 is a sketch of $y = 2^x$.

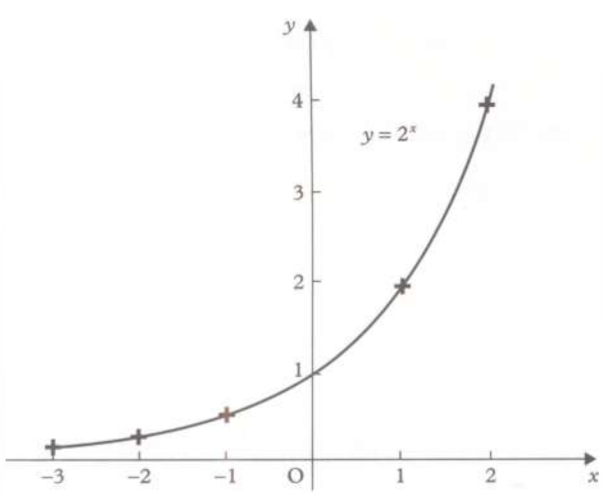


Figure 25.1

Questions

Q1 Copy and extend the above table to include values of 1.5^x (from $x = -3$ to $x = +3$), and of 2.5^x, 3^x (both from $x = -2$ to $x = +2$). Sketch, with the same axes, the graphs of $y = 1^x$, $y = 1.5^x$, $y = 2^x$, $y = 2.5^x$, $y = 3^x$. What do you notice about the gradient of $y = a^x$ at (0, 1) as a takes different values greater than 1?

Q2 How would you deduce the shape of the graph of $y = (\frac{1}{2})^x$ from Fig. 25.1?

The gradient of $y = a^x$ at (0, 1); a limit

For the time being we will consider only exponential functions of the form a^x, where a is a constant real number greater than 1. Since $a^0 = 1$, the graph of $y = a^x$ (Fig. 25.2) passes through the point A(0, 1), and we let the gradient of the curve at this point be m.

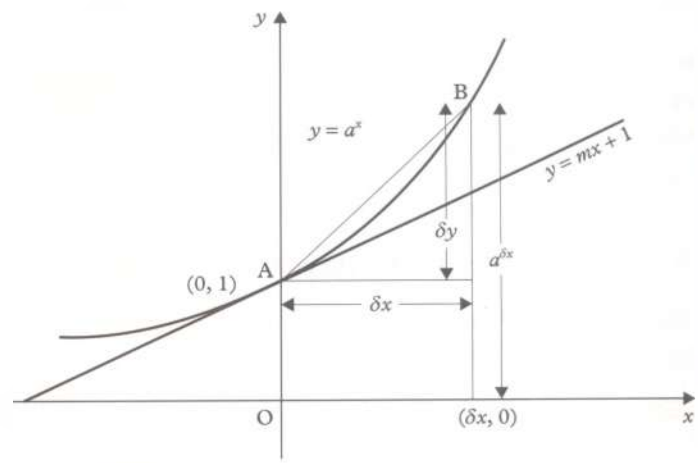


Figure 25.2

With the usual notation, if B is the point $(\delta x, a^{\delta x})$, then the gradient of AB is

$$\frac{\delta y}{\delta x} = \frac{a^{\delta x} - 1}{\delta x}$$

Now as $\delta x \to 0$, the gradient of AB $\to m$.

It follows that the **limit**, as $\delta x \to 0$, of $\dfrac{a^{\delta x} - 1}{\delta x}$ is m, the gradient of $y = a^x$ at (0, 1).

The form of $\dfrac{d}{dx}(a^x)$

The limit just established lets us investigate the gradient of $y = a^x$ at any point P(x, y) on the curve. With the usual notation, if Q is the point $(x + \delta x, y + \delta y)$,

$$y + \delta y = a^{x+\delta x}$$
$$\therefore \delta y = a^{x+\delta x} - a^x = a^x(a^{\delta x} - 1)$$

$$\therefore \text{the gradient of PQ, } \frac{\delta y}{\delta x} = a^x\left(\frac{a^{\delta x} - 1}{\delta x}\right) \qquad (1)$$

As $\delta x \to 0$, the gradient of PQ $\to$ the gradient of the tangent at P. Also, since we have shown that

$$\left(\frac{a^{\delta x} - 1}{\delta x}\right) \to m$$

then the R.H.S. of (1) $\rightarrow ma^x$.

$$\therefore \frac{dy}{dx} = ma^x$$

Thus $\dfrac{d}{dx}(a^x) = ma^x$, where m is the gradient of $y = a^x$ at the point (0, 1).

We have already seen (Q1) that as a increases, the gradients of the curves $y = a^x$ at (0, 1) increase. For every value of a there is an appropriate value of m. It is reasonable to suppose that we should be able to express m in terms of a. However for the time being we must be satisfied with some numerical approximations for m which we will now find.

Approximate derivatives of 2^x and 3^x

The following table was used to draw the graphs of $y = 2^x$ and $y = 3^x$ in Fig. 25.3.

Table of values for $y = 2^x$ and $y = 3^x$

x	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	$\frac{3}{2}$	2
2^x	0.25	0.35	0.5	0.71	0.84	1	1.19	1.41	2	2.83	4
3^x	0.11	0.19	0.33	0.58	0.76	1	1.32	1.73	3	5.20	9

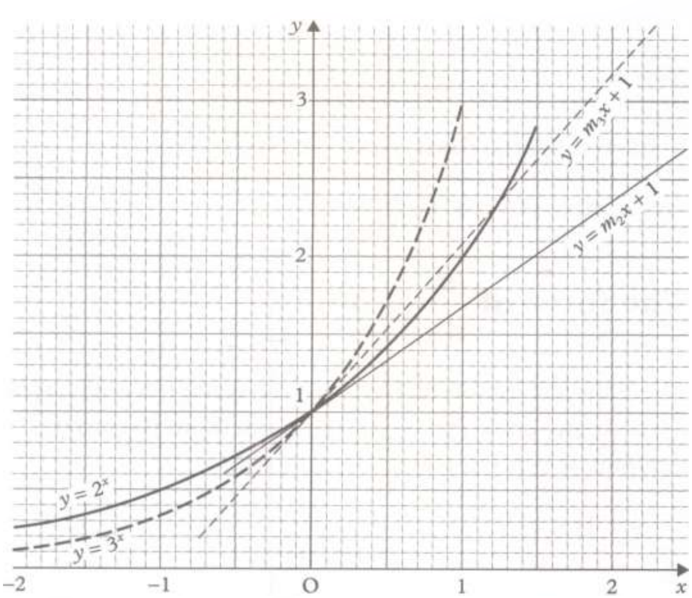


Figure 25.3

$y = m_2x + 1$ and $y = m_3x + 1$ are the respective tangents at (0, 1).

Questions

Q3 a Measure the gradients of the two tangents in Fig. 25.3, and deduce approximate expressions for $\dfrac{d}{dx}(2^x)$ and $\dfrac{d}{dx}(3^x)$.

b Now calculate the gradient of $y = 2^x$ where $x = 1$, and the gradient of $y = 3^x$ where $x = \frac{1}{2}$. Check from the graph.

Q4 Tangents were drawn to a graph of $y = 2^x$, and their gradients were measured and entered in the following table:

x	-3	-2	-1	0
$y = 2^x$	0.125	0.25	0.5	1
$\dfrac{dy}{dx}$	0.08	0.18	0.37	0.62

x	1	$\frac{3}{2}$	2	$\frac{5}{2}$
$y = 2^x$	2	2.83	4	5.66
$\dfrac{dy}{dx}$	1.33	1.90	2.82	3.68

Show graphically that these results indicate that $\dfrac{dy}{dx} \propto y$, and find an approximate expression for $\dfrac{d}{dx}(2^x)$.

25.2 The exponential function e^x

The previous section showed that

$$\frac{d}{dx}(2^x) \approx 0.7 \times 2^x$$

and

$$\frac{d}{dx}(3^x) \approx 1.1 \times 3^x$$

Since, in general, $\dfrac{d}{dx}(a^x) = ka^x$, these results suggest that for simplicity we should find a value of a between 2 and 3 for which $k = 1$. This number is called **e**, its value is approximately 2.71828, and it will play an important part in the development of mathematics from this point.

Let us now summarise what we know about e.

Definition

e is the number such that the gradient of $y = e^x$ at (0, 1) is 1.
e^x is called the **exponential function**.

Thus $\dfrac{d}{dx}(e^x) = e^x$, or if $y = e^x$, $\dfrac{dy}{dx} = y$.

Also $\int e^x \, dx = e^x + c$.

Since x may be any real number, the *domain* of the exponential function is $\mathbb{R}$. However e^x is always positive, so the *range* of the exponential function is $\mathbb{R}^+$.
(See §2.8 on page 47.)

Question

Q5 Letting 1 cm represent 0.1 on each axis, plot the graph of $y = e^x$. Take values of x at intervals of 0.05 from -0.5 to $+0.5$, and use tables or a calculator to find e^x and e^{-x}.

Obtain the gradient of the tangent to the curve at the point given by $x = 0.08$
a by drawing and measurement, **b** by measuring the ordinate, **c** by differentiation.

Example 1 *Find* $\dfrac{dy}{dx}$ *when* $y = e^{3x^2}$.

[Here we have a composite function of x. This example is written out in full as a reminder of the technique involved, but you should be able to differentiate in one step.]

$$y = e^{3x^2}$$

Let $u = 3x^2$, then $y = e^u$.

$$\therefore \frac{du}{dx} = 6x \quad \text{and} \quad \frac{dy}{du} = e^u$$

Now, by the chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx} = e^u \times 6x$.

$$\therefore \frac{dy}{dx} = 6x\, e^{3x^2}$$

Example 2 *Find* **a** $\int e^{x/2}\, dx$, **b** $\int x^2 e^{x^3}\, dx$, **c** $\dfrac{d}{dx}(e^{3y})$.

a Since $\dfrac{d}{dx}(e^{x/2}) = \frac{1}{2}e^{x/2}$, then $\int e^{x/2}\, dx = 2e^{x/2} + c$.

b Since $\dfrac{d}{dx}(e^{x^3}) = 3x^2 e^{x^3}$, then $\int x^2 e^{x^3}\, dx = \frac{1}{3}e^{x^3} + c$.

c $\dfrac{d}{dx}(e^{3y}) = \dfrac{d}{dy}(e^{3y}) \times \dfrac{dy}{dx} = 3e^{3y} \times \dfrac{dy}{dx}$.

Questions

Q6 Differentiate with respect to x.
a $(2x^3 + 1)^5$ **b** $\sin(2x^3)$
c e^{2x^3} **d** e^{y^2}
e e^{-x^2} **f** $e^{\tan x}$
g $e^{\sqrt{x}}$ **h** $e^{\sin y}$

Q7 Find the following integrals, and check by differentiation.
a $\displaystyle\int \frac{x}{(x^2+1)^2}\, dx$ **b** $\int x \sin(x^2)\, dx$
c $\int x\, e^{x^2}\, dx$ **d** $\int \sin x\, e^{\cos x}\, dx$
e $\int 2e^{x/3}\, dx$ **f** $\int 3e^{2x}\, dx$
g $\int \frac{1}{2}x\, e^{3x^2}\, dx$ **h** $\int \operatorname{cosec}^2 2x\, e^{\cot 2x}\, dx$

Exercise 25a

1 Make rough sketches of the graphs of the following functions.
a e^{2x} **b** 2^{-x} **c** $2^{1/x}$
d 2^{x^2} **e** $3e^x$ **f** e^{x+1}

2 With the same axes, sketch the graphs of $y = e^{\sin x}$ and $y = e^{\cos x}$. Are these functions periodic?
a Is the function $e^{\sin x}$ odd, even or neither?
b Is the function $e^{\cos x}$ odd, even or neither?

3 Is the function e^{-x^2} odd, even or neither? If its domain is $\mathbb{R}$, what is its range? Sketch the graph of $y = e^{-x^2}$.

4 Use the following table of values to draw the graph of $y = 3^x$, taking 2 cm to represent 1 unit on each axis:

x	-2	$-\frac{7}{4}$	$-\frac{3}{2}$	$-\frac{5}{4}$	-1	$-\frac{3}{4}$
$y = 3^x$	0.11	0.15	0.19	0.25	0.33	0.44

x	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
$y = 3^x$	0.58	0.76	1	1.32	1.73	2.28

x	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	2
$y = 3^x$	3	3.95	5.20	6.84	9

Draw the tangents to the curve at the points given by $x = -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}$, measure their gradients, and confirm graphically that $\dfrac{dy}{dx} = ky$.

Deduce an approximate expression for $\dfrac{d}{dx}(3^x)$.

Differentiate with respect to x in questions 5–8.

5 a $4e^x$ b e^{3x} c e^{2x+1}
d e^{2x^2} e e^{-2x} f e^{3y}
g e^{x^2+3} h $e^{x^{-2}}$ i $e^{5/x}$
j $e^{\sqrt[3]{x}}$ k e^{ax^2+b} l $e^{\sqrt{t}}$

6 a $e^{\cos x}$ b $e^{\sec x}$ c $e^{3\tan y}$
d $e^{\sin 2x}$ e $e^{-\cot x}$ f $e^{\operatorname{cosec}^2 x}$
g $e^{\sqrt{\cos x}}$ h $e^{a\sin bx}$ i $e^{\sin 3t}$
j $e^{\tan x^2}$

7 a $e^{\sqrt{(x^2+1)}}$ b $e^{(1-x^2)^{-1}}$ c $e^{\sin^2 4x}$
d $e^{\tan(x^2+1)}$ e $e^{\sec^2 3x}$ f $\dfrac{1}{e^{\operatorname{cosec} x}}$
g $\dfrac{1}{e^{x^{-2}}}$ h $e^{x\sin x}$ i e^{xy}
j e^{e^x}

8 a $x^2 e^x$ b $\dfrac{e^x}{x}$ c $\dfrac{x}{2}e^{\sin x}$
d $e^{x^2}\operatorname{cosec} x$ e $\dfrac{e^x}{\sin x}$ f $\dfrac{\cos x}{x e^x}$
g e^{xe^x} h $e^{ax}\sec bx$ i $\dfrac{e^{ax}}{\sin bx}$
j $\tan^n e^x$ k $e^x(\cos x + \sin x)$

9 Find the following integrals.
a $\int 3e^{x/2}\,dx$ b $\int e^{-x}\,dx$
c $\int e^{x/3}\,dx$ d $\int 2e^{3x-1}\,dx$
e $\displaystyle\int \frac{x}{2}e^{x^2}\,dx$ f $\int x^2 e^{-x^3}\,dx$
g $\int \sin x\, e^{\cos x}\,dx$ h $\int (1+\tan^2 x)\,e^{\tan x}\,dx$
i $\displaystyle\int \frac{e^{\cot x}}{\sin^2 x}\,dx$ j $\int x^{-2}\,e^{1/x}\,dx$

10 Find the equation of the tangent to the curve $y = e^x$ at the point given by $x = a$. Deduce the equation of the tangent to the curve which passes through the point $(1, 0)$.

11 Find the volume of the solid generated by rotation about the x-axis of the area enclosed by $y = e^x$, the axes, $x = 1$.

12 Find $\dfrac{d}{dx}(x\,e^x)$, and deduce $\int x\,e^x\,dx$.

13 Investigate any maximum or minimum values of the function $x\,e^x$, and then sketch the graph of $y = x\,e^x$. Find the equation of the tangent to this curve at the point where $x = -2$.

In questions 14–17, A and B are constants; in each case show that the differential equation (see §6.1 on page 86) is satisfied by the given solution

14 $\dfrac{d^2y}{dx^2} = 4y;\quad y = A\,e^{2x} + B\,e^{-2x}$

15 $\dfrac{d^2s}{dt^2} + 4\dfrac{ds}{dt} = 0;\quad s = A + B\,e^{-4t}$

16 $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 3y = 0;\quad y = A\,e^{-x} + B\,e^{-3x}$

17 $\dfrac{d^2y}{dt^2} - 6\dfrac{dy}{dt} + 9y = 0;\quad y = e^{3t}(A + Bt)$

18 If $f(x) = e^{4x}\cos 3x$, show that $f'(x) = 5e^{4x}\cos(3x+\alpha)$, where $\tan\alpha = \frac{3}{4}$. Deduce expressions of a similar form for $f''(x)$ and $f'''(x)$.

19 If $f(x) = e^{5x}\sin 12x$, show that $f'(x) = 13e^{5x}\sin(12x+\beta)$, where $\tan\beta = 12/5$. Write down an expression for $f''(x)$.

20 Show that the 9th derivative of $e^x \sin x$ is $16\sqrt{2}\,e^x\sin(x+\pi/4)$.

25.3 Natural logarithms

Since a logarithm is an index (or exponent), the discussion of exponential functions leads naturally to consideration of logarithms. It is worth restating some of the ideas covered in §P2.3 on page 10.

Definition

The logarithm of b to the base a, written $\log_a b$, is the power to which the base must be raised to equal b.

Thus, since $10^2 = 100$, $2 = \log_{10} 100$,
and if $a^x = b$, $x = \log_a b$.

You should already be familiar with the following basic rules:

$$\log_c(ab) = \log_c a + \log_c b$$
$$\log_c(a/b) = \log_c a - \log_c b$$
$$\log_c(a^n) = n\log_c a$$

Remember also that if $y = \log_a x$ then $x = a^y$, and that if we eliminate y from these two equations, we obtain

$$x = a^{\log_a x}$$

On the other hand, eliminating x gives

$$y = \log_a(a^y)$$

We shall now show that $\log_a b = \dfrac{\log_c b}{\log_c a}$.

Let $x = \log_a b$,

$\therefore a^x = b$

Taking logarithms to the base c of each side,

$$\log_c (a^x) = \log_c b$$

$$\therefore x \log_c a = \log_c b$$

$$\therefore x = \frac{\log_c b}{\log_c a}$$

i.e. $\boldsymbol{\log_a b = \dfrac{\log_c b}{\log_c a}}$

Questions

Q8 Express as a single logarithm:

a $2 \log_{10} a - \frac{1}{3} \log_{10} b + 2$,

b $\log_c (1+x) - \log_c (1-x) + A$, where $A = \log_c B$.

Q9 Express in terms of $\log_c a$.

a $\log_c (2a)$ b $\log_c a^2$ c $\log_c \dfrac{1}{a}$

d $\log_c \dfrac{2}{a}$ e $\log_c \sqrt{a}$ f $\log_c \dfrac{a}{2}$

g $\log_c \dfrac{1}{a^2}$ h $\log_c (2a)^{-1}$

Q10 Solve the equations:

a $3^{2x} = 27$ b $1.2^x = 3$

Q11 a Prove that $\log_2 10 = \dfrac{1}{\log_{10} 2}$.

b Evaluate $\log_e 100$ correct to 3 significant figures, taking e as 2.718.

Logarithms to the base e are called **natural logarithms**. The choice of e as the base provides a new function of fundamental importance: $\boldsymbol{\log_e x}$.

Question

Q12 Evaluate $\log_e 2$ correct to three significant figures, taking e as 2.718 and using logarithms to the base 10. Check your answer with a table of natural logarithms, or a calculator.

Exercise 25b

1 Express as a single logarithm:

a $2 \log_{10} a - 2 + \log_{10} 2a$

b $3 \log_e x + 3 - \log_e 3x$

c $4 \log_e (x-3) - 3 \log_e (x-2)$

d $\frac{1}{2} \log_e (1+y) + \frac{1}{2} \log_e (1-y) + \log_e k$

2 Express in terms of $\log_e a$:

a $\log_e 3a$ b $\log_e a^3$

c $\log_e (a/3)$ d $\log_e (1/a^3)$

e $\log_e (3/a)$ f $\log_e (\frac{1}{3}a^{-1})$

g $\log_e (\sqrt[3]{a})$

3 Express as the sum or difference of logarithms:

a $\log_e \cot x$ b $\log_e \tan^2 x$

c $\log_e (x^2 - 4)$ d $\log_e \sqrt{\left(\dfrac{x+1}{x-1}\right)}$

e $\log_e (3 \sin^2 x)$

4 Solve the equations:

a $\frac{3}{2} \log_{10} a^3 - \log_{10} \sqrt{a} - 2 \log_{10} a = 4$

b $\log_{10} y - 4 \log_y 10 = 0$

5 Solve the equations:

a $2^{2/x} = 32$ b $3^{x+1} = 12$

6 Evaluate $\log_e 3$ correct to three significant figures, taking e as 2.718 and using logarithms to the base 10.

Solve the equations in questions 7–10.

7 $3^{2(1+x)} - 28 \times 3^x + 3 = 0$

8 $\log_{10} a + \log_a 100 = 3$

9 $\log_{10} (19x^2 + 4) - 2 \log_{10} x - 2 = 0$

10 $\log_{10} x + \log_{10} y = 1, x + y = 11$

Notation – $\ln x$ for $\log_e x$

The notation $\log_e x$ emphasises that the base of the logarithms is e. However **ln x** *has now been universally adopted as the standard abbreviation (the* n *signifying that these are natural logarithms). From here onwards we shall use this notation.*

Question

Q13 Express as a single term:

a $\ln x + \ln y - 1$ b $\ln\left(\dfrac{e^3}{x}\right) + \ln\left(\dfrac{x}{e}\right)$

In the function $\ln x$, the independent variable x *must* be a positive real number (it must not be zero, nor must it be negative). In other words the *domain* of the logarithmic function is $\mathbb{R}^+$ (or some subset of $\mathbb{R}^+$). When $x > 1$, $\ln x$ is positive, it is zero when $x = 1$ and it is negative when $0 < x < 1$; in other words, its *range* is $\mathbb{R}$. In this context, if the domain is not explicitly stated, always assume that it has been chosen so that only logarithms of positive numbers are required. For example, in the function $\ln(1 + x)$, it should be assumed that $x > -1$, or again, in the function $\ln(2 - x)$, it should be assumed that $x < 2$.

You should memorise the following important identities:

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln x} = x$$

Question

Q14 Simplify:

a	$e^{2 \ln x}$	b	$e^{-\ln x}$
c	$e^{(1/2) \ln x}$	d	$\ln(e^{\sin x})$
e	$\frac{1}{2} \ln(e^{x^2})$	f	$\ln(\sqrt{e^x})$

25.4 The derivative of ln *x*

Figure 25.4 shows the graph of $y = \ln x$ (or $x = e^y$); this curve is the reflection in the line $y = x$ of the graph of $y = e^x$ (shown as a dotted curve). This is to be expected because $\ln x$ and e^x are inverse functions (see §2.13 on page 55).

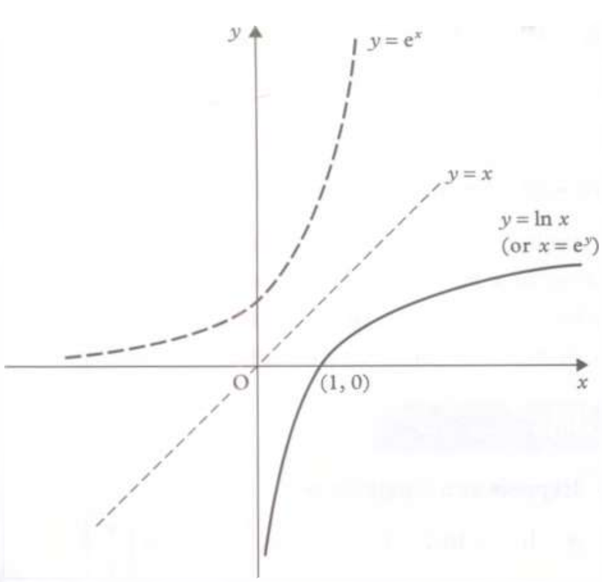


Figure 25.4

To find $\frac{d}{dx}(\ln x)$, we write $y = \ln x$ as

$$x = e^y$$

Differentiating each side with respect to x,*

$$\frac{d}{dx}(x) = \frac{d}{dy}(e^y) \times \frac{dy}{dx}$$

$$\therefore 1 = e^y \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$$\therefore \mathbf{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

Example 3 *Find* $\frac{dy}{dx}$ *if* **a** $y = \ln 2x$, **b** $y = \ln x^2$, **c** $y = \ln(x^2 + 2)$, **d** $\ln \frac{x}{\sqrt{(x^2+1)}}$.

a $y = \ln 2x = \ln 2 + \ln x$

$$\therefore \frac{dy}{dx} = \frac{1}{x}$$

b $y = \ln x^2 = 2 \ln x$

$$\therefore \frac{dy}{dx} = \frac{2}{x}$$

c $y = \ln(x^2 + 2)$

Let $u = x^2 + 2$, then $y = \ln u$.

$$\frac{du}{dx} = 2x \quad \text{and} \quad \frac{dy}{du} = \frac{1}{u}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times 2x$$

$$\therefore \frac{dy}{dx} = \frac{2x}{x^2 + 2}$$

d $y = \ln \frac{x}{\sqrt{(x^2+1)}} = \ln x - \frac{1}{2} \ln(x^2 + 1)$

$$\therefore \frac{dy}{dx} = \frac{1}{x} - \frac{1}{2} \times \frac{2x}{x^2+1} = \frac{x^2 + 1 - x^2}{x(x^2+1)}$$

$$\therefore \frac{dy}{dx} = \frac{1}{x(x^2+1)}$$

*Or we may differentiate each side with respect to y.

$\frac{dx}{dy} = e^y = x, \quad \therefore \frac{dy}{dx} = 1 \Big/ \frac{dx}{dy} = \frac{1}{x}.$

Questions

Q15 Differentiate with respect to x:

a $(x^2-2)^5$ **b** $\operatorname{cosec} x^2$
c e^{x^2} **d** $\ln(x^2-2)$
e $\ln \sin^2 x$ **f** $\ln \sin x^2$

Q16 If $y = \ln\{x\sqrt{(x+1)}\}$ find $\frac{dy}{dx}$

a by differentiating the logarithm of a product as it stands, **b** by first writing y as the sum of two logarithms.

Q17 Differentiate with respect to x:

a $\ln(3x)$ **b** $\ln(4x)$
c $\ln(3x+1)$ **d** $\ln y$
e $\ln(2x^3)$ **f** $\ln(x^3-2)$
g $\ln(x-1)^3$ **h** $\ln(4t)$
i $\ln(3\sin x)$ **j** $\ln\cos 3x$
k $\ln(2\cos^3 x)$ **l** $\ln(4\sin^2 3x)$
m $\ln\sqrt{(x^2-1)}$ **n** $\ln\dfrac{x}{(x-1)^2}$

25.5 $\frac{d}{dx}(a^x)$ and $\int a^x\,dx$

In §25.1 on page 267 we found that $\frac{d}{dx}(a^x) = ma^x$, m being the gradient of $y = a^x$ at $(0, 1)$.

When $a = e$, $m = 1$; we can now find m for other values of a.

Let $y = a^x$, then

$$\ln y = \ln a^x = x \ln a$$

Differentiating with respect to x:

$$\frac{d}{dy}(\ln y)\times\frac{dy}{dx} = \frac{d}{dx}(x\ln a)$$

$$\therefore \frac{1}{y}\times\frac{dy}{dx} = \ln a$$

$$\therefore \frac{dy}{dx} = y\ln a$$

$$\therefore \frac{d}{dx}(a^x) = a^x \ln a$$

Questions

Q18 Find $\frac{d}{dx}(4^x)$ copying the above method in full.

Find the gradient of the curve $y = 4^x$ at $(2, 16)$.

Q19 Find the gradient at $(0, 1)$ of the following curves, to 4 d.p.: **a** $y = 2^x$, **b** $y = 3^x$.

Q20 Differentiate with respect to x: **a** 10^x, **b** 2^{3x+1}.

It follows from the previous page that

$$\int a^x \ln a\,dx = a^x + k$$

$$\therefore \int a^x\,dx = \frac{a^x}{\ln a} + c$$

Questions

Q21 Find $\frac{d}{dx}(5^x)$ and deduce $\int 5^x\,dx$.

Q22 Find $\frac{d}{dx}(2^{x^2})$ and deduce $\int x\,2^{x^2}\,dx$.

Q23 Find the following integrals:

a $\int 3^{2x}\,dx$ **b** $\int x^2 e^{x^3}\,dx$
c $\int 2^{\tan x}\sec^2 x\,dx$

Exercise 25c

1 Differentiate with respect to x:

a $\ln(4x)$ **b** $4\ln x$ **c** $\ln(2x-3)$
d $\ln(\frac{1}{3}y)$ **e** $\ln\dfrac{x-1}{2}$ **f** $\ln x^4$
g $\ln(x^2-1)$ **h** $\ln 3x^2$ **i** $3\ln x^2$
j $\ln(x+1)^2$ **k** $\ln(2t^3)$ **l** $\ln\dfrac{1}{x}$
m $\ln(\frac{1}{2}x)$ **n** $\ln\sqrt{x}$ **o** $\ln\dfrac{1}{2x}$
p $\ln\dfrac{2}{x}$ **q** $\ln x^{-2}$ **r** $\log_{10} x$
s $\ln\dfrac{1}{t^3}$ **t** $\ln\sqrt[3]{x}$

2 Differentiate with respect to x:

a $\ln\cos x$ **b** $\ln\sin^2 x$
c $\ln\tan 3x$ **d** $\ln\cos^3 2x$
e $\ln(2\cot^2 x)$ **f** $\ln(3\cos^2 2x)$
g $\ln\tan\dfrac{x}{2}$ **h** $\ln\sec x$

i $\ln(\sec x + \tan x)$ j $\ln \operatorname{cosec} x^2$

k $\ln \dfrac{\sin x + \cos x}{\sin x - \cos x}$

3 Find:

a $\dfrac{d}{dx} \ln \sqrt{\dfrac{1-x}{1+x}}$ b $\dfrac{d}{dx} \ln \{x\sqrt{(x^2-1)}\}$

c $\dfrac{d}{dx} \ln \dfrac{(x+1)^2}{\sqrt{(x-1)}}$ d $\dfrac{d}{dx} \ln \{x + \sqrt{(x^2-1)}\}$

4 Differentiate with respect to x:

a $\ln t$ b $x \ln x$ c $x^2 \ln x$ d $\dfrac{\ln x}{x}$

e $x \ln y$ f $y \ln x$ g $\dfrac{\ln x}{x^2}$ h $\dfrac{x}{\ln x}$

i $(\ln x)^2$ j $\ln(\ln x^k)$ k $\ln e^{\sin x}$

5 Differentiate with respect to x:

a 5^x b 2^{x^2} c 3^{2x-1} d $e^{\ln x}$

6 a Find $\dfrac{d}{dx}(3^x)$ and deduce $\int 3^x \, dx$.

b Find $\dfrac{d}{dx}(2^{x^2})$ and deduce $\int x\, 2^{x^2} \, dx$.

7 Find the following integrals:

a $\int 10^x \, dx$ b $\int 2^{3x} \, dx$

c $\int x\, 3^{x^2} \, dx$ d $\int 2^{\cos x} \sin x \, dx$

8 Find $\dfrac{d}{dx}(x \ln x)$ and deduce $\int \ln x \, dx$.

9 Find $\dfrac{d}{dx}(x\, 2^x)$ and deduce $\int x\, 2^x \, dx$.

10 Find a $\dfrac{d}{dx} \ln(x-2)$ b $\dfrac{d}{dx} \ln(2-x)$

Sketch on the same axes the graphs of $\ln(x-2)$, $\ln(2-x)$ and $y = \dfrac{1}{x-2}$.

11 Sketch on the same axes the following curves:

a $y = \ln x$, $y = \ln(-x)$, $y = \dfrac{1}{x}$

b $y = \ln \dfrac{1}{x}$, $y = \ln\left(-\dfrac{1}{x}\right)$, $y = -\dfrac{1}{x}$

c $y = \ln(x-3)$, $y = \ln(3-x)$, $y = \dfrac{1}{x-3}$

d $y = \ln\left(\dfrac{1}{x-3}\right)$, $y = \ln\left(\dfrac{1}{3-x}\right)$, $y = \dfrac{1}{3-x}$

12 Given that $x \in \mathbb{R}^+$, write down the range of the following functions:

a $y = \ln(1+x^2)$ b $y = \ln(1/x)$

Sketch the graph of each of these functions.

25.6 $\int \dfrac{1}{x}\, dx$

The result $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + c$ holds for all rational values of n, except $n = -1$. We are now able to consider the case when $n = -1$. Since we have seen that $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$, it follows that

$$\int \frac{1}{x}\, dx = \ln x + c$$

or $$\int \frac{1}{x}\, dx = \ln(kx) \qquad \text{where } c = \ln k$$

Example 4 *Find the following integrals:*

a $\int \dfrac{1}{2x}\, dx$, b $\int \dfrac{1}{2x-1}\, dx$.

a $\int \dfrac{1}{2x}\, dx = \dfrac{1}{2} \int \dfrac{1}{x}\, dx$

$= \frac{1}{2} \ln x + c$

$= \ln(k\sqrt{x}) \qquad$ where $c = \ln k$

b $\int \dfrac{1}{2x-1}\, dx$

This is best approached in reverse by guessing the form of the integral.

$$\frac{d}{dx}\{\ln(2x-1)\} = \frac{2}{2x-1}$$

$$\therefore \int \frac{1}{2x-1}\, dx = \tfrac{1}{2} \ln(2x-1) + c$$

Question

Q24 Find the following integrals.

a $\int \dfrac{2}{x}\, dx$ b $\int \dfrac{1}{3x}\, dx$

c $\int \dfrac{1}{3x-2}\, dx$ d $\int \dfrac{1}{3x-6}\, dx$

Questions

Q25 Find the following integrals.

a $\displaystyle\int \frac{1}{2x+3}\,dx$, using the substitution $u = 2x + 3$

b $\displaystyle\int \frac{1}{1-x}\,dx$, using the substitution $u = 1 - x$

Q26 Evaluate $\displaystyle\int_1^2 \frac{3}{x}\,dx$.

Q27 a Show that the answer to Example 4**b** may be written $\ln\{A(2x-1)^{1/2}\}$ and express c in terms of A.

b If $\frac{1}{2}\ln(x - \frac{1}{2}) + c$ may be written as $\ln\{k\sqrt{(2x-1)}\}$, express c in terms of k.

An integral of the form $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$ may be reduced to the form $\displaystyle\int \frac{1}{u}\,du$, by the substitution $u = f(x)$.

$$\int \frac{\mathbf{f'(x)}}{\mathbf{f(x)}}\,\mathbf{dx}$$

$$\begin{aligned}\int \frac{f'(x)}{f(x)}\,dx &= \int \frac{f'(x)}{f(x)} \times \frac{dx}{du}\,du \\ &= \int \frac{f'(x)}{u} \times \frac{1}{f'(x)}\,du \\ &= \int \frac{1}{u}\,du \\ &= \ln u + c \\ &= \ln f(x) + c\end{aligned}$$

Hence

$$\int \frac{\mathbf{f'(x)}}{\mathbf{f(x)}}\,\mathbf{dx} = \mathbf{\ln\{k\,f(x)\}}$$

From now onwards we must be prepared to recognise, in yet another form, the integrand involving a function of x and its derivative. As before, such an integral may be found by substitution, or often it may be written down directly.

Example 5 *Find* $\displaystyle\int \frac{x}{x^2+1}\,dx$.

Since $\dfrac{d}{dx}\ln(x^2+1) = \dfrac{2x}{x^2+1}$,

$$\begin{aligned}\int \frac{x}{x^2+1}\,dx &= \tfrac{1}{2}\ln(x^2+1) + c \\ &= \ln\{k\sqrt{(x^2+1)}\} \quad \text{where } k = \ln c\end{aligned}$$

Questions

Q28 Find the following integrals.

a $\displaystyle\int \frac{x^2}{(x^3-2)^2}\,dx$ **b** $\int x^2 \cos x^3\,dx$

c $\int x^2 e^{x^3}\,dx$ **d** $\displaystyle\int \frac{x^2}{x^3-2}\,dx$

e $\displaystyle\int \frac{x-1}{x^2-2x}\,dx$ **f** $\displaystyle\int \frac{2x}{3-x^2}\,dx$

g $\int \cot x\,dx$

Q29 Find $\displaystyle\int \frac{x}{x-1}\,dx$

a using the substitution $u = x - 1$,

b by first dividing the numerator by the denominator.

$\displaystyle\int_a^b \frac{1}{x}\,dx$ when a, b are negative*

Fig. 25.4 on page 272 reminds us that as the value of x goes from 0 to $+\infty$, the value of $\ln x$ goes from $-\infty$ to $+\infty$. $\ln x$ *is not defined for negative values of* x. This presents us with a problem demonstrated graphically below.

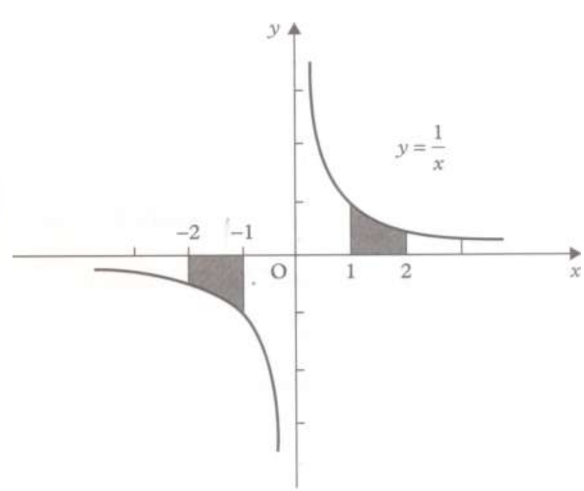


Figure 25.5

*This section should be delayed until after you have answered question 1 of Exercise 25d.

Referring to the graph of $y = 1/x$ in Fig. 25.5 it is apparent that the two shaded areas are equal in magnitude and of opposite sign. However, difficulties arise when we try to evaluate the integral with the negative limits. For example, is it true to say that

$$\int_{-2}^{-1} \frac{1}{x}\,dx = \Big[\ln x\Big]_{-2}^{-1}$$
$$= \text{'}\ln(-1) - \ln(-2)\text{'}?$$

We could now write this as $\ln\left(\frac{-1}{-2}\right) = \ln\frac{1}{2} = \ln 2^{-1} = -\ln 2$,

and thus obtain a correct value for the area. However the working is not valid, since the expression 'ln (−1) − ln (−2)' is meaningless.

This problem disappears when we realise that *for negative values of x*, although $\ln x$ is not defined, $\ln(-x)$ *does exist*, and

$$\frac{d}{dx}\ln(-x) = \frac{-1}{-x} = \frac{1}{x} \qquad \text{(See Fig. 25.6)}$$

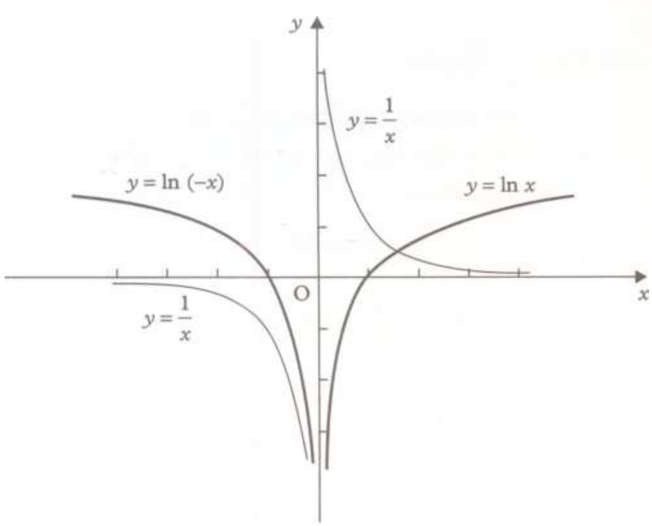


Figure 25.6

Thus, if a and b are negative, $\int_a^b \frac{1}{x}\,dx = \Big[\ln(-x)\Big]_a^b$.

(Using the modulus sign, we could write this as $\int_a^b \frac{1}{x}\,dx = \Big[\ln|x|\Big]_a^b$. This form of the result could be used for a and b both positive *and* for a and b both negative; notice however that a and b *must not* have opposite signs.)

Hence the left-hand shaded area in Fig. 25.5 $= \int_{-2}^{-1} \frac{1}{x}\,dx$

$$= \Big[\ln(-x)\Big]_{-2}^{-1}$$
$$= \ln 1 - \ln 2$$
$$= -\ln 2$$

Questions

Q30 Evaluate **a** $\int_{-4}^{-3} \frac{1}{x}\,dx$, **b** $\int_{-1}^{-1/2} \frac{1}{x}\,dx$.

Q31 Evaluate $\int_{-4}^{-2} \frac{1}{x}\,dx$, using the change of variable $x = -u$.

Q32 Can any meaning be assigned to $\int_{-2}^{+2} \frac{1}{x}\,dx$?

Example 6 *Find the area enclosed by the curve* $y = \frac{1}{x-2}$ *and*

a *the lines $x = 4$, $x = 5$, and the x-axis,*

b *the line $x = 1$, and the axes.* (Fig. 25.7.)

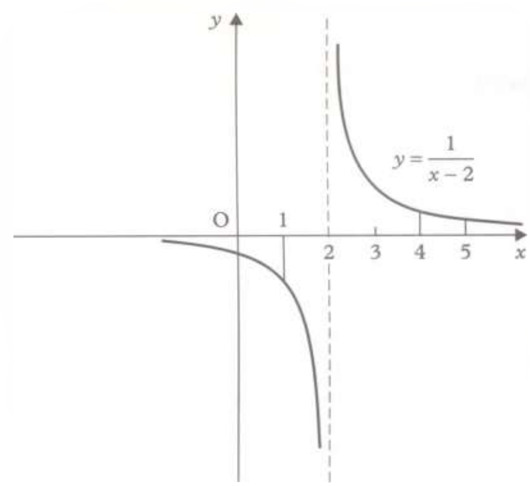


Figure 25.7

a The required area $= \int_4^5 \frac{1}{x-2}\,dx$

$$= \Big[\ln(x-2)\Big]_4^5$$
$$= \ln 3 - \ln 2$$
$$= \ln\tfrac{3}{2}$$

b If we proceed as in **a** but with the new limits, we obtain the meaningless

'$\Big[\ln(x-2)\Big]_0^1 = \ln(-1) - \ln(-2)$.'

We must note that when $x > 2$,

$$\int \frac{1}{x-2}\,dx = \ln(x-2) + c$$

but when $x < 2$,

$$\int \frac{1}{x-2}\,dx = \ln(2-x) + c$$

i.e. when $x > 2$ or $x < 2$, $\left(\int \frac{1}{x-2}\,dx = \ln|x-2| + c\right)$

$$\begin{aligned}\text{The required area } &= \int_0^1 \frac{1}{x-2}\,dx \\ &= \Big[\ln|x-2|\Big]_0^1 \\ &= \ln 1 - \ln 2 \\ &= -\ln 2\end{aligned}$$

Questions

Q33 Find $\frac{d}{dx}\{\ln(x-3)\}$ and $\frac{d}{dx}\{\ln(3-x)\}$.

Q34 Sketch the curve $y = \frac{1}{x-3}$, and evaluate:

a $\int_5^6 \frac{1}{x-3}\,dx$ **b** $\int_{-2}^2 \frac{1}{x-3}\,dx$

Q35 Sketch the curve $y = \frac{1}{2-x}$ and evaluate $\int_3^5 \frac{1}{2-x}\,dx$.

Exercise 25d

1 Find the following integrals.

a $\int \frac{1}{4x}\,dx$ **b** $\int \frac{5}{x}\,dx$

c $\int \frac{1}{2x-3}\,dx$ **d** $\int \frac{1}{2x+8}\,dx$

e $\int \frac{1}{3-2x}\,dx$ **f** $\int \frac{x}{1-x^2}\,dx$

g $\int \frac{3x}{x^2-1}\,dx$ **h** $\int \frac{2x+1}{x^2+x-2}\,dx$

i $\int \frac{2x-3}{3x^2-9x+4}\,dx$ **j** $\int \frac{x}{x+2}\,dx$

k $\int \frac{3x}{2x+3}\,dx$ **l** $\int \frac{2x}{3-x}\,dx$

m $\int \frac{x-1}{2-x}\,dx$ **n** $\int \frac{3-2x}{x-4}\,dx$

o $\int \tan x\,dx$ **p** $\int \cot\frac{x}{2}\,dx$

q $\int \cot(2x+1)\,dx$ **r** $\int -\tan\frac{x}{3}\,dx$

s $\int \frac{1-\sin 2x}{x-\sin^2 x}\,dx$ **t** $\int \frac{1-\tan x}{1+\tan x}\,dx$

u $\int \frac{2+\tan^2 x}{x+\tan x}\,dx$

2 **a** Sketch the curves $y = \ln(2x-1)$, and $y = \ln(1-2x)$.

b Find $\frac{d}{dx}\ln(2x-1)$ and $\frac{d}{dx}\ln(1-2x)$.

c Evaluate $\int_1^2 \frac{1}{2x-1}\,dx$ and $\int_{-2}^0 \frac{1}{2x-1}\,dx$.

3 Sketch the curve $y = \frac{1}{x-4}$ and evaluate:

a $\int_1^2 \frac{1}{x-4}\,dx$ **b** $\int_5^6 \frac{1}{x-4}\,dx$

4 **a** Find $\frac{d}{dx}\ln\left(\frac{1}{3-x}\right)$ and $\frac{d}{dx}\ln\left(\frac{1}{x-3}\right)$.

b Sketch on the same axes the graphs of $y = -\ln(3-x)$, $y = -\ln(x-3)$, $y = 1/(3-x)$, and find the area enclosed by the latter, the lines $x = 5$, $x = 6$, and the x-axis.

c Find the area under $y = 1/(3-x)$ from $x = 0$ to $x = 1$.

5 Evaluate the following.

a $\int_2^8 \frac{1}{2x}\,dx$ **b** $\int_1^{4/3} \frac{1}{3x-2}\,dx$

c $\int_1^3 \frac{1}{x-5}\,dx$ **d** $\int_3^5 \frac{1}{1-2x}\,dx$

e $\int_{-0.25}^{0.25} \frac{1}{2x+1}\,dx$ **f** $\int_{-\sqrt{2}}^0 \frac{x}{x^2+2}\,dx$

g $\int_0^3 \frac{2x-1}{x^2-x+1}\,dx$ **h** $\int_4^6 \frac{x}{x-2}\,dx$

i $\int_{-7}^{-5} \frac{x+1}{x+3}\,dx$ **j** $\int_{-0.5}^0 \frac{2-x}{x-1}\,dx$

k $\int_{\pi/3}^{\pi/2} \cot\theta\,d\theta$ **l** $\int_0^{\pi/6} \tan 2x\,dx$

m $\int_{\pi/6}^{\pi/4} \frac{\sec^2\theta}{\tan\theta}\,d\theta$

Chapter 26

Partial fractions

Introduction

Early work in algebra teaches us how to 'simplify' an expression such as $\dfrac{1}{x-1}-\dfrac{1}{x+1}$ by reducing it to $\dfrac{2}{x^2-1}$ (see Chapter P1 on page 1).

We have now reached a stage when the reverse process is useful. Given a fraction such as $\dfrac{5}{x^2+x-6}$ whose denominator factorises, we may split it up into its component fractions, writing it as $\dfrac{1}{x-2}-\dfrac{1}{x+3}$. It is now said to be in **partial fractions**. Here is one example of why this is useful. No change of variable yet discussed would enable us to find $\displaystyle\int \frac{5}{(x-2)(x+3)}\,dx$ as it stands.

But using partial fractions,

$$\begin{aligned}\int \frac{5}{(x-2)(x+3)}\,dx &= \int \left\{\frac{1}{x-2}-\frac{1}{x+3}\right\}dx \\ &= \ln(x-2)-\ln(x+3)+c \\ &= \ln\left\{\frac{k(x-2)}{x+3}\right\}\end{aligned}$$

Questions

Q1 Express each of the following as a single fraction.

a $\dfrac{1}{1-x}+\dfrac{2}{1+x}$ **b** $\dfrac{2x-1}{x^2+1}-\dfrac{1}{x+1}$

c $\dfrac{3}{(x-1)^2}+\dfrac{1}{x-1}+\dfrac{2}{x+1}$

Q2 Express in partial fractions:

a $\dfrac{4}{(x-2)(x+2)}$

b $\dfrac{1}{1-x^2}$ **c** $\dfrac{1}{2\times 3}$ **d** $\dfrac{1}{n(n+1)}$

Unfortunately most partial fractions cannot be obtained by trial and error quite as easily as those in Q2. Therefore we need a method to find partial fractions. This will involve us using algebraic identities, so we will discuss these in §26.1.

26.1 Identities

Let us first distinguish clearly between an *equation* and an *identity*.

$x^2 = 4$ is an **equation**, which is satisfied only by the two values $x = \pm 2$.

But

$$x^2-4 \equiv (x+2)(x-2)$$

and

$$x^2+2x-2 \equiv (x+1)(x-1)+2(x+1)-3$$

are both **identities**, and for them the L.H.S. = R.H.S. *for any value of x*. Moreover, if the R.H.S. is multiplied out, the coefficients of x^2, x and the constant term will be identical on each side.

Example 1 *Find the values of the constants A, B, C such that*

$$5x+3 \equiv Ax(x+3)+Bx(x-1)+C(x-1)(x+3)$$

First method

Collecting like terms on the R.H.S.,

$$5x+3 \equiv (A+B+C)x^2+(3A-B+2C)x-3C$$

Equating coefficients of x^2,

$$0 = A+B+C \qquad (1)$$

Equating coefficients of x,

$$5 = 3A-B+2C \qquad (2)$$

Equating constant terms,

$$3 = -3C \qquad (3)$$

From (3), $C = -1$, and substituting this value into (1) and (2), and solving these equations simultaneously, we obtain $A = 2$, and $B = -1$.

Second method

$$5x+3 \equiv Ax(x+3)+Bx(x-1)\ +C(x-1)(x+3)$$

Putting $x = 0$,

$$3 = \quad 0 \quad + \quad 0 \quad - \quad 3C^*$$
$$\therefore C = -1$$

*Compare this with equation (3) above.

Putting $x = -3$,

$$\begin{aligned} -15+3 &= 0 + B(-3)(-4) + 0 \\ \therefore -12 &= 12B \\ \therefore B &= -1 \end{aligned}$$

Putting $x = 1$,

$$\begin{aligned} 5+3 &= A \times 1 \times 4 + 0 + 0 \\ \therefore A &= 2 \end{aligned}$$

Although the given identity holds for *any* value of x, we have chosen those particular values which make all but one term on the R.H.S. vanish each time.

Questions

Q3 $2x^2+9x-10 \equiv A(x-3)(x+4)+B(x+2)(x+4)$
$+ C(x+2)(x-3)$.

- **a** Obtain three equations in A, B, C by substituting $x=-1, 0, 1$ in this identity.
- **b** Find the values of A, B, C by substituting convenient values of x.

Q4 Find the values of the constants A, B, C in the following identities:

- **a** $22-4x-2x^2 \equiv A(x-1)^2+B(x-1)(x+3)$
$+C(x+3)$,
using the first method in Example 1,
- **b** $5x+31 \equiv A(x+2)(x-1)+B(x-1)(x-5)$
$+C(x-5)(x+2)$,
using the second method in Example 1,
- **c** $13x-11 \equiv A(3x-2)+B(2x+1)$.

Q5 Put $x=1$ to find the value of A in the identity

$$x^2+x+7 = A(x^2+2)+(Bx+C)(x-1)$$

Now substitute any other values of x to find B and C.

The substitution method is fast, but for greater speed it can often be combined with the method of equating coefficients (e.g. having found A in Q5, equate coefficients of x^2 to find B). The latter method also gives a better idea of the nature of identities. Consider the statement

$$x^2-5x+8 \equiv A(x+3)+B(x-1)^2$$

Applying the method of substitution we obtain $A=1$, $B=2$. However, when A and B are given these values we do *not* have an identity! This is because the method of equating coefficients shows that for only *two* unknowns there are *three* equations, $B=1$, $A-2B=-5$, and $3A+B=8$, which are not consistent. Thus we cannot find values for A and B to form the above 'identity'.

We shall soon be concerned with forming identities. Therefore the method of equating coefficients is a valuable check that the number of unknown constants corresponds to the number of equations to be satisfied.

Question

Q6 Can values of A, B, C be found which make the following pairs of expressions identical?

- **a** $2x+3$ and
$A(x+1)(x-2)+B(x+1)^2+C$
- **b** $x^2-8x+30$ and $A(x-3)^2+B(x+2)$

Exercise 26a

1 Express each of the following as a single fraction.

- **a** $\dfrac{3}{x+3}-\dfrac{2}{x-2}$
- **b** $\dfrac{1}{(x+2)^2}-\dfrac{2}{x+2}+\dfrac{1}{3x-1}$
- **c** $\dfrac{4}{2+3x^2}-\dfrac{1}{1-x}$
- **d** $\dfrac{3}{x^2+1}-\dfrac{1}{x-1}+\dfrac{2}{(x-1)^2}$

2 Express in partial fractions.

- **a** $\dfrac{2x}{(3+x)(3-x)}$
- **b** $\dfrac{a}{a^2-b^2}$
- **c** $\dfrac{1}{5\times 6}$
- **d** $\dfrac{1}{p(1-p)}$

3 Use the first method of Example 1 to find the values of the constants A, B, C in the following identities.

- **a** $31x-8 \equiv A(x-5)+B(4x+1)$
- **b** $8-x \equiv A(x-2)^2+B(x-2)(x+1)+C(x+1)$
- **c** $71+9x-2x^2 \equiv A(x+5)(x+2)+$
$B(x+2)(x-3)+C(x-3)(x+5)$
- **d** $2x^3-15x^2-10 \equiv A(x-2)(x+1)+$
$B(x+1)(2x^2+1)+C(2x^2+1)(x-2)$

4 Use the second method of Example 1 to find the values of the constants A, B, C in the following identities.

a $2x - 4 \equiv A(3 + x) + B(7 - x)$

b $8x + 1 \equiv A(3x - 1) + B(2x + 3)$

c $4x^2 + 4x - 26 \equiv A(x + 2)(x - 4) + B(x - 4)(x - 1) + C(x - 1)(x + 2)$

d $17x^2 - 13x - 16 \equiv A(3x + 1)(x - 1) + B(x - 1)(2x - 3) + C(2x - 3)(3x + 1)$

5 Can values of A, B, C, D be found which make the following pairs of expressions identical?

a $2x^2 - 22x + 53$ and $A(x - 5)(x - 3) + B(x - 3)(x + 2) + C(x + 2)(x - 5)$

b $x + 7$ and $A(x - 2) + B(x + 1)^2$

c $3x^2 + 7x + 11$ and $(Ax + B)(x + 2) + C(x^2 + 5)$

d $x + 1$ and $A(x - 2) + B(x^2 + 1)$

e $x^3 + 2x^2 - 4x - 2$ and $(Ax + B)(x - 2)(x + 1) + C(x + 1) + D(x - 2)$

6 Find the values of A, B, C if $x^3 - 1$ is expressed in the form

$$(x - 1)(Ax^2 + Bx + C)$$

Factorise:

a $x^3 + 1$ b $x^3 - 8$ c $x^3 + 27$

d $8x^3 - 27$ e $27x^3 + 125$

7 Express $x^3 + 1$ in the form $x(x - 1)(x - 2) + Ax(x - 1) + Bx + C$.

8 Find the values of a and b if $x^4 + 12x^3 + 46x^2 + ax + b$ is the square of a quadratic expression.

9 Write down the quadratic equation whose roots are α, β. If the same equation may also be written $ax^2 + bx + c = 0$, express $\alpha + \beta$ and $\alpha\beta$ in terms of a, b, c.

10 If α, β, γ are the roots of the equation $px^3 + qx^2 + rx + s = 0$, deduce expressions for $\alpha + \beta + \gamma$, $\beta\gamma + \gamma\alpha + \alpha\beta$, $\alpha\beta\gamma$ in terms of p, q, r, s.

26.2 Partial fractions

Partial fractions generally fall into three main types. Each is illustrated below by a worked example. You should work through each question, before going on to consider the next type.

In practice, the solution of a problem may depend upon the correct determination of partial fractions in the early stages. It is important to develop the habit of *checking* partial fractions from the start. They should be substituted into one fraction mentally, the checking numerator obtained with the original.

Type I — denominator with only linear factors

First we consider a fraction whose denominator consists of only linear factors.

Example 2 *Express* $\dfrac{11x + 12}{(2x + 3)(x + 2)(x - 3)}$ *in partial fractions.*

Let $\dfrac{11x + 12}{(2x + 3)(x + 2)(x - 3)} \equiv \dfrac{A}{2x + 3} + \dfrac{B}{x + 2} + \dfrac{C}{x - 3}$,

where A, B, C are constants to be found.

It follows that

$$\frac{11x + 12}{(2x + 3)(x + 2)(x - 3)} \equiv \frac{A(x + 2)(x - 3) + B(x - 3)(2x + 3) + C(2x + 3)(x + 2)}{(2x + 3)(x + 2)(x - 3)}$$

$\therefore 11x + 12$
$\equiv A(x + 2)(x - 3) + B(x - 3)(2x + 3) + C(2x + 3)(x + 2)$

Putting $x = 3$,

$33 + 12 = 0 + 0 + C \times 9 \times 5$
$\therefore C = 1$

Putting $x = -2$,

$-22 + 12 = 0 + B \times (-5) \times (-1) + 0$
$\therefore -10 = 5B$
$\therefore B = -2$

Putting $x = -\frac{3}{2}$,

$-\frac{33}{2} + 12 = A \times \frac{1}{2} \times \frac{-9}{2} + 0 + 0$
$\therefore -\frac{9}{2} = -\frac{9}{4}A$
$\therefore A = 2$

$$\therefore \frac{11x + 12}{(2x + 3)(x + 2)(x - 3)} \equiv \frac{2}{2x + 3} - \frac{2}{x + 2} + \frac{1}{x - 3}$$

[Since the R.H.S.

$$\equiv \frac{2(x + 2)(x - 3) - 2(x - 3)(2x + 3) + (2x + 3)(x + 2)}{(2x + 3)(x + 2)(x - 3)},$$

we check the coefficient in the numerator.]

Check: Coefficient of $x^2 = 2 - 4 + 2 = 0$
Coefficient of $x = -2 + 6 + 7 = 11$
Constant term $= -12 + 18 + 6 = 12$

Question

Q7 Express in partial fractions:

a $\dfrac{6}{(x+3)(x-3)}$

b $\dfrac{x}{(2+x)(2-x)}$

c $\dfrac{x-1}{3x^2-11x+10}$

d $\dfrac{3x+1}{(x+2)(x+1)(x-3)}$

e $\dfrac{3-4x}{2+3x-2x^2}$

Type II — denominator with a quadratic factor

Fractions which can be split solely into partial fractions are *proper*. I.e. *the degree of the numerator is less than the degree of the denominator.** Moreover, the partial fractions themselves are always proper.

Bearing this in mind we now consider a fraction whose denominator contains a quadratic factor which does not factorise.

Let $\dfrac{3x+1}{(x-1)(x^2+1)} \equiv \dfrac{A}{x-1} + \dfrac{\text{'numerator'}}{x^2+1}$

Then $3x+1 \equiv A(x^2+1) + \text{'numerator'} \times (x-1)$

From our previous work we know from equating coefficients that there are *three* equations to be satisfied. It follows that there are *three* constants to find.** Therefore the '*numerator*' must contain two of them. Thus the only way to write the second partial fraction, so that it is proper, is in the form $\dfrac{Bx+C}{x^2+1}$. See Example 3.

Example 3 *Express* $\dfrac{3x+1}{(x-1)(x^2+1)}$ *in partial fractions.*

Let $\dfrac{3x+1}{(x-1)(x^2+1)} \equiv \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1}$

$\therefore 3x+1 \equiv A(x^2+1) + (Bx+C)(x-1)$

Putting $x=1$, $4 = 2A + 0$, $\therefore A = 2$.

Putting $x=0$, $1 = A - C$, $\therefore 1 = 2 - C$, $\therefore C = 1$.

Equating coefficients of x^2, $0 = A + B$, $\therefore B = -2$.

$$\therefore \frac{3x+1}{(x-1)(x^2+1)} \equiv \frac{2}{x-1} + \frac{1-2x}{x^2+1} \equiv \frac{2}{x-1} - \frac{2x-1}{x^2+1}$$

Check: Coefficient of $x^2 = 2 - 2 = 0$
Coefficient of $x = -(-2-1) = +3$
Constant term $= 2 - 1 = +1$

*With an improper fraction, we divide first and obtain a quotient and partial fraction, thus

$$\frac{x^2+x+1}{(x-1)(x+2)} \equiv 1 + \frac{3}{(x-1)(x+2)} \equiv 1 + \frac{1}{x-1} - \frac{1}{x+2},$$

and

$$\frac{x^3+2x^2-7x-18}{x^2-9} \equiv x+2+\frac{2x}{(x+3)(x-3)} \equiv x+2+\frac{1}{x+3}+\frac{1}{x-3}.$$

**In general, the number of constants to be found is the same as the degree of the denominator of the original fraction.

Question

Q8 Express in partial fractions:

a $\dfrac{6-x}{(1-x)(4+x^2)}$

b $\dfrac{4}{(x+1)(2x^2+x+3)}$

c $\dfrac{5x+2}{(x+1)(x^2-4)}$

d $\dfrac{3+2x}{(2-x)(3+x^2)}$

Type III — denominator with a repeated factor

Here we take as an example $\dfrac{1}{(x+2)(x-1)^2}$. Written as $\dfrac{1}{(x+2)(x^2-2x+1)}$ this suggests Type II with partial fractions $\dfrac{A}{x+2} + \dfrac{Bx+K}{x^2-2x+1}$. Certainly we have the correct number of constants to identify this expression with the original fraction. However, the denominator of the second partial fraction factorises, and so we have not gone far enough.

$$\frac{Bx+K}{(x-1)^2} \equiv \frac{B(x-1)+B+K}{(x-1)^2}$$

$$\equiv \frac{B}{x-1} + \frac{B+K}{(x-1)^2}$$

Writing C for $B+K$, we obtain

$$\frac{Bx+K}{(x-1)^2} \equiv \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

This indicates the appropriate form when we have a repeated factor. (See also Q10 on page 282.)

Example 4 *Express* $\dfrac{1}{(x+2)(x-1)^2}$ *in partial fractions.*

Let $\dfrac{1}{(x+2)(x-1)^2} \equiv \dfrac{A}{x+2} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$

$$\therefore \frac{1}{(x+2)(x-1)^2} \equiv \frac{A(x-1)^2 + B(x-1)(x+2) + C(x+2)}{(x+2)(x-1)^2}$$

$\therefore 1 \equiv A(x-1)^2 + B(x-1)(x+2) + C(x+2)$

Putting $x = -2$, $\quad 1 = 9A, \quad \therefore A = \frac{1}{9}$.

Putting $x = 1$, $\quad 1 = 3C, \quad \therefore C = \frac{1}{3}$.

Equating coefficients of x^2, $\quad 0 = A + B, \quad \therefore B = -\frac{1}{9}$.

$$\therefore \frac{1}{(x+2)(x-1)^2} \equiv \frac{1}{9(x+2)} - \frac{1}{9(x-1)} + \frac{1}{3(x-1)^2}$$

Check: Expressing the R.H.S. as a single fraction with denominator $(x+2)(x-1)^2$, the numerator is $\frac{1}{9}\{(x-1)^2 - (x+2)(x-1) + 3(x+2)\}$.

Coefficient of $x^2 = \frac{1}{9}(1-1) = 0$
Coefficient of $x = \frac{1}{9}(-2-1+3) = 0$
Constant term $= \frac{1}{9}(1+2+6) = 1$

Questions

Q9 Express in partial fractions:

a $\dfrac{x+1}{(x+3)^2}$

b $\dfrac{2x^2-5x+7}{(x-2)(x-1)^2}$

Q10 Find the values of A, B, C, D, if

$$\frac{x^3-10x^2+26x+3}{(x+3)(x-1)^3} \equiv \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

Improper fractions

As already implied, an *improper* fraction is one whose *numerator is of degree equal to, or greater than, that of the denominator.* To deal with this we first divide the numerator to obtain a quotient and a proper fraction, and then split the latter into partial fractions. Thus

$$\frac{x^4-2x^3-x^2-4x+4}{(x-3)(x^2+1)} \equiv x+1+\frac{x^2-2x+7}{(x-3)(x^2+1)}, \quad \text{etc.}$$

Often, instead of doing long division, it is quicker to proceed as follows:

$$\frac{2x^2+1}{(x-1)(x+2)} \equiv \frac{2(x^2+x-2)-2x+5}{x^2+x-2}$$
$$\equiv 2 + \frac{5-2x}{(x-1)(x+2)}, \quad \text{etc.}$$

Questions

Q11 Express the following in the form of a quotient and a proper fraction.

a $\dfrac{x^3+2x^2-2x+2}{(x-1)(x+3)}$ (by long division)

b $\dfrac{3x^2-2x-7}{(x-2)(x+1)}$ (by the short method suggested above)

Q12 Express in partial fractions.

a $\dfrac{x^2-7}{(x-2)(x+1)}$

b $\dfrac{x^3-x^2-4x+1}{x^2-4}$

Exercise 26b

Express in partial fractions.

1 a $\dfrac{x-11}{(x+3)(x-4)}$ b $\dfrac{x}{25-x^2}$

c $\dfrac{3x^2-21x+24}{(x+1)(x-2)(x-3)}$ d $\dfrac{4x^2+x+1}{x(x^2-1)}$

e $\dfrac{8x^2+13x+6}{(x+2)(2x+1)(3x+2)}$ f $\dfrac{2x^3+x^2-15x-5}{(x+3)(x-2)}$

2 a $\dfrac{5x^2-10x+11}{(x-3)(x^2+4)}$ b $\dfrac{2x^2-x+3}{(x+1)(x^2+2)}$

c $\dfrac{3x^2-2x+5}{(x-1)(x^2+5)}$ d $\dfrac{11x}{(2x-3)(2x^2+1)}$

e $\dfrac{20x+84}{(x+5)(x^2-9)}$ f $\dfrac{2x^3-x-1}{(x-3)(x^2+1)}$

3 a $\dfrac{x-5}{(x-2)^2}$ b $\dfrac{5x+4}{(x-1)(x+2)^2}$

c $\dfrac{5x^2+2}{(3x+1)(x+1)^2}$ d $\dfrac{x^4+3x-1}{(x+2)(x-1)^2}$

4 a $\dfrac{3x^3+x+1}{(x-2)(x+1)^3}$ (see Q10 opposite)

b $\dfrac{3x^2+2x-9}{(x^2-1)^2}$

5 a $\dfrac{x^3+2x^2-10x-9}{x^2-9}$ b $\dfrac{3(x^2-3)}{(x-1)(x+2)}$

c $\dfrac{2x^4-4x^3-42}{(x-2)(x^2+3)}$ d $\dfrac{x^4-6x^2+3}{x(x+1)^2}$

6 $\dfrac{3x+7}{x(x+2)(x-1)}$ 7 $\dfrac{3}{x^2(x+2)}$

8 $\dfrac{2x^4-17x-1}{(x-2)(x^2+5)}$ 9 $\dfrac{68+11x}{(3+x)(16-x^2)}$

10 $\dfrac{2x+1}{x^3-1}$ 11 $\dfrac{2x^2+39x+12}{(2x+1)^2(x-3)}$

12 $\dfrac{x+4}{6x^2-x-35}$ 13 $\dfrac{x-2}{x^2(x-1)^2}$

14 $\dfrac{7x+2}{125x^3-8}$ 15 $\dfrac{x^2+2x+18}{x(x^2+3)^2}$

16 $\dfrac{1}{x^4+5x^2+6}$ 17 $\dfrac{1}{x^4-9}$

26.3 Summation of series

§13.4 on page 162 contains an introduction to the summation of series. Some series may be summed using partial fractions. See Example 5.

Example 5 a *Express* $\dfrac{2}{n(n+1)(n+2)}$ *in partial fractions, and* b *deduce that*

$$\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$$

a Let $\dfrac{2}{n(n+1)(n+2)}\equiv\dfrac{A}{n}+\dfrac{B}{n+1}+\dfrac{C}{n+2}$

$\therefore\ 2\equiv A(n+1)(n+2)+B(n+2)n+Cn(n+1)$

Putting $n=0$, $2=2A$, $\therefore A=1$.

Putting $n=-1$, $2=-B$, $\therefore B=-2$.

Putting $n=-2$, $2=2C$, $\therefore C=1$.

$$\therefore\ \frac{2}{n(n+1)(n+2)}\equiv\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}$$

Check: Coefficient of $n^2=1-2+1=0$.
Coefficient of $n=3-4+1=0$.
Constant term $=2$.

b If $S\equiv\dfrac{1}{1\times2\times3}+\dfrac{1}{2\times3\times4}+\ldots+\dfrac{1}{n(n+1)(n+2)}$

$$2S\equiv\frac{2}{1\times2\times3}+\frac{2}{2\times3\times4}+\ldots+\frac{2}{n(n+1)(n+2)}$$

[From Part **a** it follows that

$$2S\equiv\left(\frac{1}{1}-\frac{2}{2}+\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{2}{4}+\frac{1}{5}\right)+\ldots+\left(\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}\right)$$

We see that the majority of terms when grouped three together in a different way, such as $\frac{1}{3}-\frac{2}{3}+\frac{1}{3}$, have zero sum. We then pick out those terms which remain at the beginning and at the end. This is most easily done if we set out the working in columns.]

From Part **a**

$$\begin{aligned}
\frac{2}{1\times2\times3}&=\frac{1}{1}-\frac{2}{2}+\frac{1}{3}\\
\frac{2}{2\times3\times4}&=\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\\
\frac{2}{3\times4\times5}&=\frac{1}{3}-\frac{2}{4}+\frac{1}{5}\\
&\cdots\cdots\cdots\\
\frac{2}{(n-2)(n-1)n}&=\frac{1}{n-2}-\frac{2}{n-1}+\frac{1}{n}\\
\frac{2}{(n-1)n(n+1)}&=\frac{1}{n-1}-\frac{2}{n}+\frac{1}{n+1}\\
\frac{2}{n(n+1)(n+2)}&=\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}
\end{aligned}$$

Adding, $2S=\dfrac{1}{2}-\dfrac{1}{n+1}+\dfrac{1}{n+2}$

$$=\frac{1}{2}-\frac{1}{(n+1)(n+2)}$$

$$\therefore S=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$$

Note that as $n\to\infty$, $\dfrac{1}{2(n+1)(n+2)}\to0$ the infinite series

$$\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+\frac{1}{3\times4\times5}+\ldots$$

is *convergent*, and its *sum to infinity* is $\frac{1}{4}$ (see §14.3 on page 173).

Question

Q13 Show that

$$\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\ldots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$$

26.4 Integration

We have already shown in §3.1 on page 61 how partial fractions can be applied to integration. Two more examples follow.

Example 6 *Find* $\int \frac{2x-1}{(x+1)^2}\,dx$.

Let $\frac{2x-1}{(x+1)^2} \equiv \frac{A}{x+1}+\frac{B}{(x+1)^2}$;

we find that $A = 2$, $B = -3$.

$$\therefore \int \frac{2x-1}{(x+1)^2}\,dx = \int \left\{\frac{2}{x+1}-\frac{3}{(x+1)^2}\right\}dx$$

$$= 2\ln(x+1) + 3(x+1)^{-1} + c$$

Questions

Q14 Find **a** $\int \frac{1}{x^2-9}\,dx$ **b** $\int \frac{2x+2}{(2x-3)^2}\,dx$

Q15 a Find $\int \frac{x}{4-x^2}\,dx$ without using partial fractions.

b Find this integral using partial fractions.

Example 7 *Evaluate* $\int_2^3 \frac{5+x}{(1-x)(5+x^2)}\,dx$ *correct to three significant figures.*

Let $\frac{5+x}{(1-x)(5+x^2)} \equiv \frac{A}{1-x}+\frac{Bx+C}{5+x^2}$;

we find that $A = 1$, $B = 1$, $C = 0$.

$$\therefore \int_2^3 \frac{5+x}{(1-x)(5+x^2)}\,dx$$

$$= \int_2^3 \left\{\frac{1}{1-x}+\frac{x}{5+x^2}\right\}dx$$

$$= \left[-\ln(x-1)+\tfrac{1}{2}\ln(5+x^2)\right]_2^3$$

$$= (-\ln 2+\tfrac{1}{2}\ln 14) - (-\ln 1+\tfrac{1}{2}\ln 9)$$

$$= \tfrac{1}{2}\ln 14 - \ln 2 - \ln 3$$

$$= \tfrac{1}{2}(\ln 10 + \ln 1.4) - \ln 6$$

$$= -0.472 \quad \text{(correct to 3 s.f.)}$$

Example 7 also revises an important point.

If $x < 1$, $\int \frac{1}{1-x}\,dx = -\ln(1-x) + c$

However, if $x > 1$, as shown by the limits in Example 7,

$\int \frac{1}{1-x}\,dx = -\ln(x-1) + c$ (See §25.6 on page 274.)

Questions

Q16 Can $\int_0^2 \frac{5+x}{(1-x)(5+x^2)}\,dx$ be evaluated?

Q17 Evaluate **a** $\int_2^3 \frac{x-4}{(x+2)(x-1)}\,dx$

b $\int_1^2 \frac{3x^2+2x+2}{(x+1)(x^2+2)}\,dx$

Exercise 26c

1 Express $\frac{2}{n(n+2)}$ in partial fractions, and deduce that

$$\frac{1}{1\times3}+\frac{1}{2\times4}+\frac{1}{3\times5}+\ldots+\frac{1}{n(n+2)} = \frac{3}{4}-\frac{2n+3}{2(n+1)(n+2)}$$

2 Express $\frac{n+3}{(n-1)n(n+1)}$ in partial fractions, and deduce that

$$\frac{5}{1\times2\times3}+\frac{6}{2\times3\times4}+\frac{7}{3\times4\times5}+\ldots+\frac{n+3}{(n-1)n(n+1)} = 1\tfrac{1}{2}-\frac{n+2}{n(n+1)}$$

3 For the series given in question 2 write down **a** the nth term, **b** the sum of the first n terms, **c** the limit of this sum as $n \to \infty$.

4 Prove that the series $\frac{2}{1\times 2}+\frac{2}{2\times 3}+\frac{2}{3\times 4}+\ldots$ is convergent, and find its sum to infinity.

5 Find the sum of the first n terms of the following series.

a $\frac{1}{1\times 4}+\frac{1}{2\times 5}+\frac{1}{3\times 6}+\ldots$

b $\frac{1}{2\times 4}+\frac{1}{4\times 6}+\frac{1}{6\times 8}+\ldots$

c $\frac{1}{3\times 6}+\frac{1}{6\times 9}+\frac{1}{9\times 12}+\ldots$

d $\frac{1}{2\times 6}+\frac{1}{4\times 8}+\frac{1}{6\times 10}+\ldots$

e $\frac{1}{1\times 3\times 5}+\frac{1}{2\times 4\times 6}+\frac{1}{3\times 5\times 7}+\ldots$

f $\frac{1}{3\times 4\times 5}+\frac{2}{4\times 5\times 6}+\frac{3}{5\times 6\times 7}+\ldots$

6 Find the sum of the first n terms of the following series, remembering that $2n-1$, $2n+1$, etc. are odd for all integral values of n.

a $\frac{2}{1\times 3}+\frac{2}{3\times 5}+\frac{2}{5\times 7}+\ldots$

b $\frac{1}{1\times 3\times 5}+\frac{1}{3\times 5\times 7}+\frac{1}{5\times 7\times 9}+\ldots$

c $\frac{2}{1\times 3\times 5}+\frac{3}{3\times 5\times 7}+\frac{4}{5\times 7\times 9}+\ldots$

7 Find the following integrals.

a $\int \frac{1}{x(x-2)}\,dx$ **b** $\int \frac{1}{(x+3)(5x-2)}\,dx$

c $\int \frac{7x+2}{3x^3+x^2}\,dx$ **d** $\int \frac{x}{16-x^2}\,dx$

e $\int \frac{1}{x^2-4x-5}\,dx$ **f** $\int \frac{x-2}{x^2-4x-5}\,dx$

g $\int \frac{2x^2+2x+3}{(x+2)(x^2+3)}\,dx$

h $\int \frac{22-16x}{(3+x)(2-x)(4-x)}\,dx$

i $\int \frac{4x-33}{(2x+1)(x^2-9)}\,dx$

j $\int \frac{5x+2}{(x-2)^2(x+1)}\,dx$

k $\int \frac{x^2-8x+5}{(2x+1)(x^2+9)}\,dx$

l $\int \frac{6-9x}{27x^3+8}\,dx$

m $\int \frac{x^3-18x-21}{(x+2)(x-5)}\,dx$

n $\int \frac{37}{4(x-3)(1+4x^2)}\,dx$

8 (Attempt all the parts of this question at one sitting. This will show the comparison between the forms.) Find the following integrals.

a $\int \frac{1}{1+x^2}\,dx$ **b** $\int \frac{x}{1+x^2}\,dx$

c $\int \frac{1+x}{1+x^2}\,dx$ **d** $\int \frac{1}{1-x^2}\,dx$

e $\int \frac{x}{1-x^2}\,dx$ **f** $\int \frac{x}{\surd(1-x^2)}\,dx$

g $\int \frac{1}{\surd(1-x^2)}\,dx$ **h** $\int \frac{1+x}{\surd(1-x^2)}\,dx$

i $\int \frac{1}{1-x}\,dx,\ (x<1)$ **j** $\int \frac{1}{1-x}\,dx,\ (x>1)$

k $\int \frac{x}{1+x}\,dx$ **l** $\int \frac{1}{(1-x)^2}\,dx$

m $\int \frac{x}{(1-x)^2}\,dx$

9 Evaluate the following, correct to 3 s.f.

a $\int_3^5 \frac{2}{x^2-1}\,dx$ **b** $\int_{-1}^0 \frac{2}{(1-x)(1+x^2)}\,dx$

c $\int_2^3 \frac{x-9}{x(x-1)(x+3)}\,dx$

d $\int_0^3 \frac{13x+7}{(x-4)(3x^2+2x+3)}\,dx$

10 Find the volume of the solid generated when the area under $y=\frac{1}{x-2}$ from $x=3$ to $x=4$ is rotated through four right angles about the x-axis. If the solid is made of material of uniform density, where is its centre of gravity?

The binomial theorem (2)

27.1 Expansion of $(1+x)^n$ when n is not a positive integer

The binomial theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

for $-1 < x < +1$, was used in Chapter 14. However the general term in the expansion was not discussed for values of n other than positive integers. The term in x^r is found to be

$$\frac{n(n-1)\ldots(n-r+1)}{r!}x^r$$

The proof of this, for values of n other than positive integers, is outside the scope of this book. When n is a positive integer, the term in x^r was shown in §14.2 on page 171 to be

$${}^nC_r x^r = \frac{n!}{(n-r)!\,r!}x^r$$

Dividing numerator and denominator by $(n-r)!$, we obtain

$$\frac{n(n-1)\ldots(n-r+1)}{r!}x^r$$

Note that $n(n-1)\ldots(n-r+1)$ contains r factors. If you can remember that this expression is $n!/(n-r)!$ (when n is a positive integer), it may help you to remember that the last factor, $n-r+1$, is 1 greater than $n-r$.

Using the notation

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

the binomial expansion can be written

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots$$

Example 1 *Find the general terms in the expansions in ascending powers of x of* **a** $(1+x)^{-1}$, **b** $(1-2x)^{-3}$.

a $(1+x)^{-1}$. The general term is

$$\begin{aligned}&(-1)(-2)\ldots(-r)x^r/r!\\ &= (-1)^r 1\times 2\ldots r x^r/r!\\ &= (-1)^r x^r\end{aligned}$$

b $(1-2x)^{-3}$. The general term is

$$\begin{aligned}&(-3)(-4)\ldots(-2-r)\times(-2x)^r/r!\\ &= (-1)^r 3\times 4\ldots(r+2)\times(-1)^r 2^r x^r/r!\\ &= \frac{r!(r+1)(r+2)}{2}\times\frac{2^r x^r}{r!}\\ &= (r+1)(r+2)2^{r-1}x^r\end{aligned}$$

The expansion obtained in Example 1 part **a**,

$$(1+x)^{-1} = 1 - x + x^2 - \ldots + (-1)^r x^r + \ldots$$

should be memorised.

Note that the right-hand side is an infinite geometrical progression with first term 1 and common ratio $-x$. Therefore its sum to infinity is $1/(1+x)$, for $-1 < x < +1$ (see §13.8 on page 168).

An approximation for $(1-x)^{-1}$ can be obtained by taking the first three terms of the binomial expansion:

$$(1-x)^{-1} \approx 1 + x + x^2$$

This is quite accurate when x is small. For instance, when $x = 0.2$,

$$\text{L.H.S.} = 0.8^{-1} = 1.25$$

and $\text{R.H.S.} = 1 + 0.2 + 0.04 = 1.24$

Fig. 27.1 shows the graph of $y = (1-x)^{-1}$ (continuous curve) and the graph of $y = 1 + x + x^2$ (broken curve).

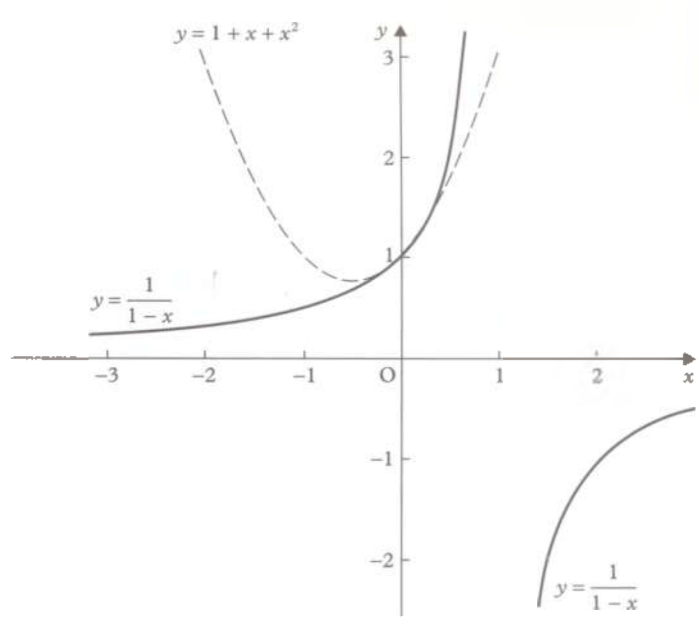


Figure 27.1

We can see from this diagram that the graphs are close together (showing that the approximation is quite accurate) for $|x| < 0.5$. They begin to diverge when $0.5 < |x| < 1$, and when $|x| > 1$, the curves are totally unrelated. The approximation between -1 and $+1$ could be improved by taking an extra term of the binomial expansion, i.e.

$$(1-x)^{-1} \approx 1 + x + x^2 + x^3$$

With $x = 0.2$, the R.H.S. is equal to 1.248, which is much closer to the exact value than the value we obtained from the previous approximation. Plot the graphs of $y = (1-x)^{-1}$ and $y = 1 + x + x^2 + x^3$.

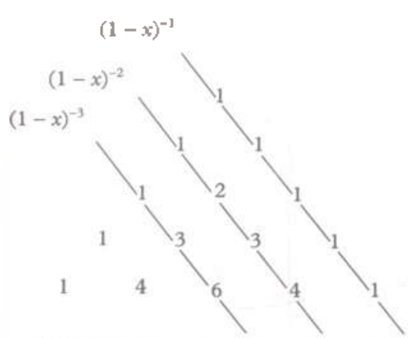
These graphs should be quite close together for $|x| < 1$. (If you have access to a computer you could try plotting some of the graphs obtained by including further terms of the binomial expansion).

Question

Q1 Write down and simplify the general terms in the expansions of

a $(1+x)^{-2}$ **b** $(1-3x)^{-1}$
c $(1-\frac{1}{2}x)^{-3}$ **d** $(1+x)^{-4}$

Remember that the coefficients in the expansions of $(1-x)^{-1}, (1-x)^{-2}, (1-x)^{-3}, \ldots$ are contained in Pascal's triangle (see §14.1 on page 169).



We use this in Examples 2 and 3.

Example 2 *Find the first three terms and the general term in the expansion in ascending powers of x of*

$$\frac{x+5}{(1+3x)(2-x)}.$$

Expressed in partial fractions,

$$\frac{x+5}{(1+3x)(2-x)} = \frac{2}{1+3x} + \frac{1}{2-x}$$

(check that this is correct.)

$$2(1+3x)^{-1} = 2\{1 - 3x + 9x^2 - \ldots + (-1)^r(3x)^r + \ldots\}$$
$$= 2 - 6x + 18x^2 - \ldots + (-1)^r \times 2(3x)^r + \ldots$$
$$(2-x)^{-1} = 2^{-1}(1-\tfrac{1}{2}x)^{-1}$$
$$= \tfrac{1}{2}\{1 + \tfrac{1}{2}x + \tfrac{1}{4}x^2 + \ldots + (\tfrac{1}{2}x)^r + \ldots\}$$
$$= \tfrac{1}{2} + \tfrac{1}{4}x + \tfrac{1}{8}x^2 + \ldots + (\tfrac{1}{2})^{r+1}x^r + \ldots$$

Therefore the sum of the two expansions is

$$2\tfrac{1}{2} - 5\tfrac{3}{4}x + 18\tfrac{1}{8}x^2 + \ldots + \{(-1)^r \times 2 \times 3^r + (\tfrac{1}{2})^{r+1}\}x^r + \ldots$$

For the expansion to be valid,

$$-1 < 3x < +1 \quad \text{and} \quad -1 < -\tfrac{1}{2}x < 1$$

Multiplying the pairs of inequalities by $\frac{1}{3}$ and -2 respectively,

$$-\tfrac{1}{3} < x < +\tfrac{1}{3} \quad \text{and} \quad 2 > x > -2^*$$

Therefore the expansion is valid when $-\frac{1}{3} < x < +\frac{1}{3}$.

Example 3 *Find the first three terms and the term in x^r in the expansion in ascending powers of x of $(x+2)(1+x)^{12}$.*

$$(1+x)^{12} = 1 + 12x + \frac{12 \times 11}{2!}x^2 + \ldots$$
$$+ {}^{12}C_{r-1}x^{r-1} + {}^{12}C_r x^r + \ldots + x^{12}$$

$$\therefore 2(1+x)^{12} = 2 + 24x + 132x^2 + \ldots + 2x^{12}$$

and

$$x(1+x)^{12} = x + 12x^2 + 66x^3 + \ldots + x^{13}$$

Adding

$$(x+2)(1+x)^{12} = 2 + 25x + 144x^2 + \ldots + x^{13}$$

To find the term in x^r, we must multiply

the term in x^r in the expansion of $(1+x)^{12}$ by 2
and
the term in x^{r-1} in the expansion of $(1+x)^{12}$ by x.

Thus the term in x^r is

$$2 \times {}^{12}C_r x^r + x \times {}^{12}C_{r-1}x^{r-1}$$
$$= \left\{\frac{2 \times 12!}{(12-r)!\,r!} + \frac{12!}{(13-r)!\,(r-1)!}\right\}x^r$$
$$= \frac{12!}{(12-r)!\,(r-1)!}\left(\frac{2}{r} + \frac{1}{13-r}\right)x^r$$
$$= \frac{12!\,(26-r)}{(13-r)!\,r!}x^r$$

* When an inequality is multiplied by a negative number, the direction of the inequality sign is reversed. (See §6.1 on page 86 and Chapter 9.)

Questions

Q2 Find the terms in x^r in the expansions in ascending powers of x of

a $(1-x)(1+x)^{20}$ **b** $(2x+3)(1-x)^{10}$

Q3 Find the general term in the expansion of $(2x-1)/(1+x)^2$ **a** by the method of Example 3, **b** by expressing the function in partial fractions.

Example 4 *Sum to infinity the series*

a $1-\dfrac{3}{2}x+\dfrac{3\times9}{2\times4}x^2-\dfrac{3\times9\times15}{2\times4\times6}x^3+\ldots,$

b $\dfrac{1}{4}-\dfrac{1\times2}{4\times8}+\dfrac{1\times2\times5}{4\times8\times12}-\dfrac{1\times2\times5\times8}{4\times8\times12\times16}+\ldots.$

a Let $S_1=1-\dfrac{3}{2}x+\dfrac{3\times9}{2\times4}x^2-\dfrac{3\times9\times15}{2\times4\times6}x^3+\ldots$

Note that there is a factor of 3^r in the numerator and a factor of 2^r in the denominator of the term in x^r.

$$\therefore\ S_1=1-\frac{1}{2}(3x)+\frac{1\times3}{2^2}\frac{(3x)^2}{2!}-\frac{1\times3\times5}{2^3}\frac{(3x)^3}{3!}+\ldots$$

$$=1+\left(-\frac{1}{2}\right)(3x)+\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{(3x)^2}{2!}$$

$$+\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{(3x)^3}{3!}+\ldots$$

$\therefore\ S_1=(1+3x)^{-1/2}$, provided $|x|<\frac{1}{3}$

b Let $S_2=\dfrac{1}{4}-\dfrac{1\times2}{4\times8}+\dfrac{1\times2\times5}{4\times8\times12}-\dfrac{1\times2\times5\times8}{4\times8\times12\times16}+\ldots$

Here the denominators are $4^r\times r!$

$$\therefore\ S_2=\frac{1}{4}-\frac{1\times2}{4^2}\times\frac{1}{2!}+\frac{1\times2\times5}{4^3}\times\frac{1}{3!}$$

$$-\frac{1\times2\times5\times8}{4^4}\times\frac{1}{4!}+\ldots$$

By altering some signs, we can arrange that the factors in the numerators become terms of an arithmetical progression.

$$\therefore\ S_2=\frac{1}{4}+\frac{1(-2)}{4^2}\times\frac{1}{2!}+\frac{1(-2)(-5)}{4^3}\times\frac{1}{3!}$$

$$+\frac{1(-2)(-5)(-8)}{4^4}\times\frac{1}{4!}+\ldots$$

$$\therefore\ S_2=\frac{1}{3}\times\frac{3}{4}+\frac{\frac{1}{3}(-\frac{2}{3})}{2!}\times\frac{3^2}{4^2}+\frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{3!}\times\frac{3^3}{4^3}$$

$$+\frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})}{4!}\times\frac{3^4}{4^4}+\ldots$$

The series has a sum to infinity since $-1<\frac{3}{4}<+1$.

$\therefore\ 1+S_2=(1+\frac{3}{4})^{1/3}$

$\therefore\ S_2=\sqrt[3]{\frac{7}{4}}-1$

Expansion of $(1+x)^n$ when $|x|>1$

Remember that the expansion of $(1+x)^n$ in ascending powers of x is only valid for $|x|<1$. If, however, $|x|>1$, the function can be expanded in ascending powers of $1/x$.

$$(1+x)^n=\{x(1+x^{-1})\}^n=x^n(1+x^{-1})^n$$

When $|x|>1$, it follows that $|x^{-1}|<1$, so we may write

$$(1+x)^n=x^n\left\{1+n\left(\frac{1}{x}\right)+\frac{n(n-1)}{2!}\left(\frac{1}{x}\right)^2+\ldots\right.$$

$$\left.+\frac{n(n-1)\ldots(n-r+1)}{r!}\left(\frac{1}{x}\right)^r+\ldots\right\}$$

Question

Q4 Expand the following in ascending powers of $1/x$, giving the ranges of values of x for which the expansions are valid:

a $(1+x)^{-1}$ **b** $(2+x)^{-2}$

c $(1+3x)^{-2}$ **d** $\dfrac{3}{(x-1)(x-2)}$

e $\dfrac{x}{x^2+1}$

Exercise 27a

1 Find the first three terms and the general terms in the expansions of the following functions in ascending powers of x. State the ranges of values of x for which the expansions are valid.

a $(1+3x)^{-1}$ **b** $(1-2x)^{-1}$

c $(1+x)^{-2}$ **d** $(1-\frac{1}{2}x)^{-2}$

e $(1+x)^{-3}$ **f** $(2+x)^{-1}$

g $\dfrac{1}{(3-x)^2}$ **h** $\dfrac{1}{(2-3x)^3}$

i $\sqrt{(1+x)}$

2 Express the following functions in partial fractions and find the first three terms and the general terms in their expansions in ascending powers of x. For what values of x are the expansions valid?

a $\dfrac{3}{(1-x)(1+2x)}$ b $\dfrac{1}{(1+x)(x+2)}$

c $\dfrac{x-1}{x^2+2x+1}$ d $\dfrac{5}{1-x-6x^2}$

e $\dfrac{x+3}{(x-2)^2}$ f $\dfrac{x+2}{x^2-1}$

3 Expand the following functions in ascending powers of x, giving the first three terms and the general term. State the necessary restrictions on the values of x.

a $\dfrac{1}{1+x^2}$ b $\dfrac{x}{1-x^2}$

c $\dfrac{1-x}{1+x}$ d $(1+x)(1-x)^{10}$

e $\dfrac{4}{(x+3)(1+x)}$ f $\dfrac{x+5}{(3-2x)(x-1)}$

g $\dfrac{x+7}{(x+1)^2(x-2)}$

4 Expand the following functions in ascending powers of $1/x$, giving the first three terms and the general terms. State the necessary restrictions on the values of x.

a $(2+x)^{-1}$ b $(3-x)^{-3}$

c $(1-2x)^{-2}$ d $\dfrac{x+2}{x+1}$

e $\dfrac{x-1}{(x+2)^2}$ f $\dfrac{1}{x^2-5x+6}$

g $\dfrac{2x+4}{(x-1)(x+3)}$ h $\dfrac{2x}{1-x^2}$

i $\dfrac{1}{1-x+x^2-x^3}$

5 Expand $(x-2)^{1/2}$ as a series of descending powers of x as far as the third term. By substituting $x=100$, evaluate $\sqrt{2}$ to 5 s.f. [Hint: $\sqrt{98}=7\sqrt{2}$.]

6 Obtain $\sqrt[3]{2}$ to 5 d.p. by substituting $x=1000$ in the expansion of $(x+24)^{1/3}$ in descending powers of x.

In questions 7–10, use the binomial expansion to find the values of

7 $(16.32)^{1/4}$ to 5 d.p. **8** $\sqrt{9.09}$ to 6 d.p.

9 $\dfrac{1}{(10.04)^2}$ to 4 s.f. **10** $\dfrac{1}{\sqrt{17}}$ to 4 d.p.

11 Expand the function $(1+2x)^{1/2}(1-3x)^{-1/3}$ in a series of ascending powers of x as far as the term in x^2. [Hint: multiply the first three terms of the expansion of $(1+2x)^{1/2}$ by those of $(1-3x)^{-1/3}$, ignoring terms in x^3 and higher powers of x.]

In questions 12–17 expand the functions in series of ascending powers of x as far as the terms indicated.

12 $\dfrac{1}{1+x+2x^2}$, (x^3) [Write $y=x+2x^2$.]

13 $\dfrac{1}{(1+2x+3x^2)^2}$, (x^3)

14 $\sqrt[3]{(1+3x)}\sqrt{(1+2x)}$, (x^2)

15 $\dfrac{\sqrt[3]{(1-x)^2}}{1+x}$, (x^3)

16 $\dfrac{x}{1-\sqrt{(1+2x)}}$, (x^2)

[Expand the denominator; then divide numerator and denominator by x.]

17 $\dfrac{x^2}{1-\sqrt[3]{(1-3x)}}$, (x^3)

18 The field H on the axis of a bar magnet of moment M at a distance d from its centre is approximately $2M/d^3$. Suppose that in calculating the value of H, values of M and d differ by ±2% and ±1% respectively. What is the greatest possible percentage error in calculating the value of H?

19 If a clock with a seconds pendulum registers x s too few per day, what is the time of one beat of the pendulum?

One beat of a seconds pendulum takes $\pi(l/g)^{1/2}$ s, where l is the length of the pendulum and g is a constant. If the length of the pendulum increases by 0.04% owing to expansion, calculate the number of seconds it will fail to register in a day.

20 If a pendulum beats seconds (see question 19) at a place where $g=981$ cm/s^2 and is then removed to a place where g is 0.05% less, how many seconds will it fail to register in a day?

21 The heat H produced by an electric current flowing through a resistance R with potential difference V for a time t is given by $H=JV^2t/R$, where J is a constant. If V, t, R are given percentage increases x, y, z which are so small that the squares and products of x, y, z may be neglected, find the percentage increase in the value of H.

22 The period of oscillation T of a vibration magnetometer is given by the formula

$$T = 2\pi\sqrt{\left(\frac{I}{MH}\right)}$$

If the quantities I, M, H are estimated with errors of p, q, r per cent, respectively, find the corresponding percentage error in T if the squares and products of p, q, r may be neglected.

Sum to infinity the series in questions 23–30 stating the necessary restrictions on the value of x.

23 $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$

24 $1 - x + \frac{1\times 3}{1\times 2}x^2 - \frac{1\times 3\times 5}{1\times 2\times 3}x^3 + \frac{1\times 3\times 5\times 7}{1\times 2\times 3\times 4}x^4 - \ldots$

25 $1 + 4x + 12x^2 + \ldots + (n+1)2^n x^n + \ldots$

26 $1 - \frac{2}{3} + \frac{2\times 3}{3\times 6} - \frac{2\times 3\times 4}{3\times 6\times 9} + \ldots$

27 $1 + \frac{1}{6} - \frac{1\times 2}{6\times 12} + \frac{1\times 2\times 5}{6\times 12\times 18} - \frac{1\times 2\times 5\times 8}{6\times 12\times 18\times 24} + \ldots$

28 $1 - 6x + 24x^2 - \ldots + (-1)^n(n+1)(n+2)2^{n-1}x^n + \ldots$

29 $1 - x - \frac{x^2}{2!} - 1\times 3\frac{x^3}{3!} - 1\times 3\times 5\frac{x^4}{4!} - \ldots$

30 $1 + \frac{1}{4} + \frac{1\times 4}{4\times 8} + \frac{1\times 4\times 7}{4\times 8\times 12} + \ldots$

27.2 Relations between binomial coefficients

We first show how to find the greatest coefficient in a binomial expansion. A similar method may be used to find the greatest term. In this section n and r represent positive integers.

Example 5 *Find the greatest coefficient in the expansion of* $(2x+3)^{12}$.

The coefficient of x^r is given by

$$u_r = \frac{12!}{(12-r)!\,r!}2^r 3^{12-r}$$

and the coefficient of x^{r+1} is given by

$$u_{r+1} = \frac{12!}{(11-r)!(r+1)!}2^{r+1}3^{11-r}$$

The ratio of these coefficients is

$$\frac{u_r}{u_{r+1}} = \frac{r+1}{12-r}\times\frac{3}{2}$$

Therefore $u_r < u_{r+1}$ if

$$\frac{r+1}{12-r}\times\frac{3}{2} < 1$$

$$\therefore\ 3r + 3 < 24 - 2r^*$$

$$\therefore\ 5r < 21$$

$$\therefore\ r < \tfrac{21}{5} = 4\tfrac{1}{5}$$

That is, $u_1 < u_2$, $u_2 < u_3$, $u_3 < u_4$, $u_4 < u_5$, but $u_5 \nless u_6$. Therefore the coefficient of x^5 is the largest. Its value is $792\times 2^5\times 3^7$.

Question

Q5 Find the greatest *term* in the above expansion when $x = 2$.

Some series involving the binomial coefficients are considered next. For brevity we shall write

$$(1+x)^n = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n$$

So far we have not assigned a meaning to ${}^nC_0$. Since, in general, ${}^nC_r = c_r$, it is most convenient to *define* ${}^nC_0 = 1$; and, if we *define* $0! = 1$, we can write

$$c_r = {}^nC_r = \frac{n!}{(n-r)!\,r!} \quad \text{for all values of } r \text{ from 0 to } n.$$

Note that c_r is only used for ${}^nC_r$. Other coefficients such as ${}^{n-1}C_r$ and ${}^{2n}C_r$ will not be abbreviated in this way.

Example 6 *Find the values of*

a $c_0 + c_1 + \ldots + c_n$,

b $c_0 - 2c_1 + 3c_2 - \ldots + (-1)^n(n+1)c_n$,

c $\frac{1}{2}c_0 + \frac{1}{3}c_1 + \frac{1}{4}c_2 + \ldots + c_n/(n+2)$.

a $(1+x)^n = c_0 + c_1x + \ldots + c_nx^n$.

Substituting $x = 1$,

$$c_0 + c_1 + \ldots + c_n = 2^n$$

b Remember that $\frac{\mathrm{d}(x^n)}{\mathrm{d}x} = nx^{n-1}$.

$$x(1+x)^n = c_0x + c_1x^2 + c_2x^3 + \ldots + c_nx^{n+1}$$

*$12 - r$ is positive so the inequality sign is unchanged.

Differentiating with respect to x,

$$(1+x)^n \times 1 + x \times n(1+x)^{n-1} = c_0 + 2c_1x + 3c_2x^2 + \ldots + (n+1)c_nx^n$$

Substituting $x = -1$,

$$c_0 - 2c_1 + 3c_2 - \ldots + (-1)^n(n+1)c_n = 0$$

c Remember that $\int x^n \,dx = x^{n+1}/(n+1) + k$.

$$(1+x)^n = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n$$
$$\therefore x(1+x)^n = c_0x + c_1x^2 + c_2x^3 + \ldots + c_nx^{n+1}$$

Integrating the R.H.S. with respect to x between 0 and 1, we obtain

$$\tfrac{1}{2}c_0 + \tfrac{1}{3}c_1 + \tfrac{1}{4}c_2 + \ldots + c_n/(n+2)$$

For the L.H.S., we write

$$x(1+x)^n = \{(1+x) - 1\}(1+x)^n$$

$$\int_0^1 x(1+x)^n \,dx = \int_0^1 \{(1+x)^{n+1} - (1+x)^n\} \,dx$$
$$= \left[\frac{(1+x)^{n+2}}{n+2} - \frac{(1+x)^{n+1}}{n+1}\right]_0^1$$
$$= \frac{2^{n+2}-1}{n+2} - \frac{2^{n+1}-1}{n+1}$$

which is the sum of the series.

Alternatively, the integral could have been evaluated by the substitution $u = 1 + x$.

Certain relations between the binomial coefficients may be obtained by equating coefficients (see §26.1 on page 278). For example, the identity

$$(1+x)^{n+2} \equiv (1+x)^2(1+x)^n$$

may be expanded in two different ways:

a $1 + {}^{n+2}C_1x + \ldots + {}^{n+2}C_rx^r + \ldots + {}^{n+2}C_{n+2}x^{n+2}$,

b $(1 + 2x + x^2)(c_0 + c_1x + \ldots + c_rx^r + \ldots + c_nx^n)$.

The term in x^r in **b** is obtained by multiplying c_rx^r by 1, $c_{r-1}x^{r-1}$ by $2x$, $c_{r-2}x^{r-2}$ by x^2. Equating coefficients of x^r in the expansions **a** and **b**,

$$^{n+2}C_r = c_r + 2c_{r-1} + c_{r-2} \qquad (2 \leqslant r \leqslant n)$$

Question

Q6 What relations are obtained by equating coefficients of

a x^{r+1} **b** x^{r+2}

Example 7 *Prove that* $c_0^2 + c_1^2 + \ldots + c_n^2 = {}^{2n}C_n$.

[The expression ${}^{2n}C_n$ suggests the use of $(1+x)^{2n}$, and the terms c_r^2 suggest the square of $(1+x)^n$.]

$$(1+x)^n(1+x)^n \equiv (1+x)^{2n}$$
$$\therefore (c_0 + c_1x + \ldots + c_nx^n)(c_0 + c_1x + \ldots + c_nx^n) \equiv (1+x)^{2n}$$

[${}^{2n}C_n$ is the coefficient of x^n on the R.H.S.]

Equating coefficients of x^n,

$$c_0c_n + c_1c_{n-1} + \ldots + c_{n-1}c_1 + c_nc_0 = {}^{2n}C_n$$

But $c_r = c_{n-r}$ (see §12.4 on page 157),

$$\therefore c_0^2 + c_1^2 + \ldots + c_{n-1}^2 + c_n^2 = {}^{2n}C_n$$

Exercise 27b

1 Find the greatest coefficients in the binomial expansions of the following.

a	$(x+2)^{10}$	**b**	$(3x+1)^8$
c	$(4x+3)^{12}$	**d**	$(2x+5)^{20}$
e	$(x+\frac{2}{3})^{11}$	**f**	$(3x-2)^9$
g	$(12-11x)^{-2}$	**h**	$(7-5x)^{-3}$

2 Find the greatest terms in the binomial expansions of

a $(2x+3y)^{12}$, when $x = 1, y = 3$

b $(x+2y)^{10}$, when $x = \frac{1}{2}, y = \frac{1}{3}$

c $(4x+5y)^8$, when $x = \frac{1}{3}, y = \frac{1}{2}$

d $(3x-5)^{-2}$, when $x = 1\frac{1}{2}$

Prove that

3 $c_0 - c_1 + c_2 - \ldots + (-1)^nc_n = 0$

4 $c_1 + 2c_2 + 3c_3 + \ldots + nc_n = n \times 2^{n-1}$

5 $c_0 + 2c_1 + 3c_2 + \ldots + (n+1)c_n = 2^{n-1}(n+2)$

6 $c_1 - 2c_2 + 3c_3 - \ldots + (-1)^{n+1}nc_n = 0$

7 $2 \times 1c_2 + 3 \times 2c_3 + \ldots + n(n-1)c_n = n(n-1)2^{n-2}$

8 $2c_1 - 6c_2 + \ldots + (-1)^{n+1}n(n+1)c_n = 0$

9 $1^2c_1 + 2^2c_2 + \ldots + n^2c_n = n(n+1)2^{n-2}$

10 $1^2c_0 + 2^2c_1 + \ldots + (n+1)^2c_n = (n+1)(n+4)2^{n-2}$

11 $\frac{1}{2}c_0 - \frac{1}{3}c_1 + \ldots + (-1)^nc_n/(n+2) = 1/\{(n+1)(n+2)\}$

12 $c_0 + \frac{1}{2}c_1 + \frac{1}{3}c_2 + \ldots + c_n/(n+1) = (2^{n+1}-1)/(n+1)$

13 $\frac{1}{2}c_0 - \frac{1}{6}c_1 + \ldots + (-1)^nc_n/(n+1)(n+2) = 1/(n+2)$

Chapter 28

Quadratic inequalities, graphs, curve sketching

28.1 Quadratic inequalities

In Chapter 9 we found that the following rules apply when operating with inequalities:

a we may *add* or *subtract* any number to both sides of an inequality
b we may *multiply or divide* both sides of an inequality by any *positive* number
c if we *multiply* or *divide* both sides of an inequality by a *negative* number, we must *reverse* the inequality

These rules relate to operations with directed numbers, i.e. positive and negative numbers, and they apply to any inequality. A **quadratic inequality** is an expression that contains one or more second order variables such as x^2, y^2, xy.

Questions

Q1 If $a = x$, then $a^2 = x^2$. Consider the inequality $a < x$ if **a** $a = 3, x = 5$, **b** $a = -7, x = 4$. Is $a^2 < x^2$?

Q2 If $b = y(\neq 0)$, then $1/b = 1/y$. Consider the inequality $b < y$ if **a** $b = 2, y = 4$, **b** $b = -3, y = 1$. Is $1/b < 1/y$?

Q3 If $a = x, b = y$, then $a - b = x - y$. Consider the inequalities $a < x, b < y$ if $a = 5, x = 6, b = 4, y = 7$. Is $a - b < x - y$?

Q4 If $a < x, b < y$, is $ab < xy$? Try $a = -3, x = 2, b = -4, y = 5$.

Example 1 *Find the values of x for which* $\frac{x+3}{x-1} > 2$.

Inequalities such as this are usually easier to solve if one of the sides is zero. So first subtract 2 from both sides.

$$\frac{x+3}{x-1} - 2 > 0$$

Putting the L.H.S. over a common denominator of $(x - 1)$, we obtain

$$\frac{x+3-2(x-1)}{x-1} > 0$$

$$\therefore \frac{5-x}{x-1} > 0$$

The numerator (or top line) of this fraction is zero when $x = 5$, and the denominator is zero when $x = 1$. To obtain the inequality we require, these terms must have the same signs, so consider the table below:

	$x < 1$	$1 < x < 5$	$x > 5$
$5 - x$	+	+	–
$x - 1$	–	+	+
$\frac{5-x}{x-1}$	–	+	–

From this table we can see that the required values of x are in the interval

$1 < x < 5$

[Remember that x represents a real number. Therefore the statement $1 < x < 5$ means that x can be *any* number (not just *whole* numbers) between 1 and 5.]

Question

Q5 Sketch the graphs of $y = \frac{x+3}{x-1}$ and $y = 2$, and use your diagram to illustrate the result of Example 1.

Example 2 *For what values of x is the function* $2x^2 + 5x - 3$

a *negative,* **b** *positive?*

Let $f(x) = 2x^2 + 5x - 3 = (2x - 1)(x + 3)$. $f(x)$ is zero when $x = \frac{1}{2}$ or $x = -3$, so consider the signs of the factors in the following intervals:

	$x < -3$	$-3 < x < \frac{1}{2}$	$\frac{1}{2} < x$
$x + 3$	–	+	+
$2x - 1$	–	–	+
$f(x)$	+	–	+

Alternatively, sketch the graph of the function $f(x)$. As $x \to \pm\infty$, $f(x) \to +\infty$; $f(\frac{1}{2}) = f(-3) = 0$. See **Fig. 28.1**.

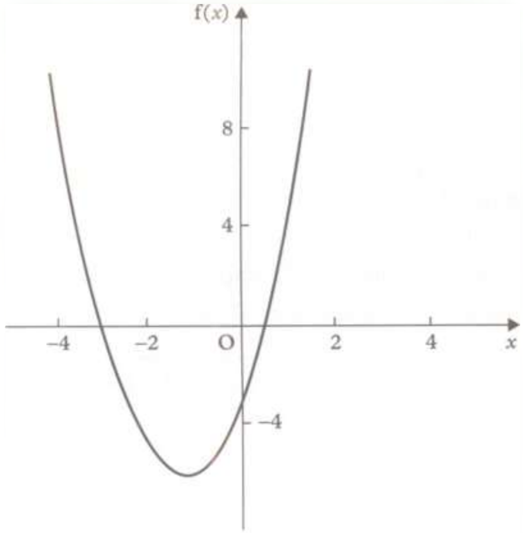


Figure 28.1

By either method, the function is negative if $-3 < x < \frac{1}{2}$ and positive if $x < -3$ or $x > \frac{1}{2}$.

Example 3 *Show that $3x^2 + 10x + 9$ cannot be negative and find its least value.*

Completing the square (see §10.1 on page 128),

$$3x^2 + 10x + 9 = 3(x^2 + \tfrac{10}{3}x + \tfrac{25}{9}) + 9 - \tfrac{25}{3}$$
$$= 3(x + \tfrac{5}{3})^2 + \tfrac{2}{3}$$

Since $(x + \frac{5}{3})^2$ is a square, the least value it can take is zero, so that $3x^2 + 10x + 9$ cannot be zero and its least value is $\frac{2}{3}$.

A similar method may be applied to functions with more than one variable.

Example 4 *Show that $a^2 + b^2 + c^2 - bc - ca - ab$ cannot be negative. Under what circumstances is it zero?*

$$a^2 + b^2 + c^2 - bc - ca - ab$$
$$= \tfrac{1}{2}(b^2 + c^2 - 2bc + c^2 + a^2 - 2ca + a^2 + b^2 - 2ab)$$
$$= \tfrac{1}{2}\{(b-c)^2 + (c-a)^2 + (a-b)^2\} \geqslant 0$$

The equality occurs only when each square is zero, that is when $a = b = c$.

Example 5 *If $(x^2 - x + 1)y = 2x$, within what interval does y lie?*

Writing $(x^2 - x + 1)y = 2x$ as a quadratic equation in x,

$$x^2y - x(y+2) + y = 0$$

The roots of the equation $ax^2 + bx + c = 0$ are real when $b^2 - 4ac \geqslant 0$ (see §10.1 on page 128). Therefore, for real values of x,

$$\{-(y+2)\}^2 - 4y^2 \geqslant 0$$
$$\therefore -3y^2 + 4y + 4 \geqslant 0$$
$$\therefore (2 + 3y)(2 - y) \geqslant 0$$

Hence, if x is real, y lies in the interval $-\frac{2}{3} \leqslant y \leqslant 2$ (see Fig. 28.2).

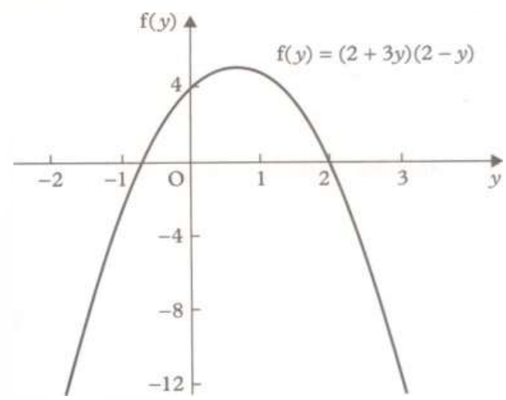


Figure 28.2

Exercise 28a

For what intervals do the following inequalities hold?

1 **a** $\dfrac{x+1}{2-x} < 1$ **b** $\dfrac{x+1}{2-x} > 1$

2 **a** $\dfrac{4-x}{x+2} < 3$ **b** $\dfrac{4-x}{x+2} > 3$

3 $(3-x)(x+2) > 0$

4 $(2x-5)(3x+7) > 0$

5 $2x^2 + x - 15 < 0$

6 $10 + x - 2x^2 < 0$

7 $(4x-3)(x+1) > 2$

8 $(5x-7)(x-3) < 16x$

9 $\dfrac{2x^2 - 7x - 4}{3x^2 - 14x + 11} > 2$

10 $\dfrac{(x-1)(x-3)}{(x+1)(x-2)} > 0$

Solve the following inequalities and find the extreme values of the functions concerned.

11 $x^2 - 5x + 7 > 0$

12 $4x - x^2 - 5 < 0$

13 $2x^2 + 3x + 2 > 0$

14 $5x - 3x^2 - 3 < 0$

Find the intervals of x and y for which there are no real points on the following loci.

15 $y^2 = x(1-x)$ **16** $3x^2 + 4y^2 = 12$

17 $y^2 = x(x^2 - 1)$ **18** $(x-2)(x-3)y = 2x - 5$

19 $(x^2+1)y = 3x + 4$

For what intervals are the equations in questions 20–22 satisfied by real values of θ?

20 $\sin\theta = \dfrac{x-1}{x+1}$ **21** $\cos\theta = \dfrac{x+3}{3-x}$

22 $\sin\theta + \cos\theta = x$

23 Show that $(a+b)^2 \geqslant 4ab$.

24 Verify the identity

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-bc-ca-ab)$$

and deduce that the arithmetic mean of three unequal positive numbers x, y, z $[\frac{1}{3}(x+y+z)]$ is greater than their geometric mean $[(xyz)^{1/3}]$.

25 Express $5x^2 - 12xy + 9y^2 - 4x + 4$ as the sum of two squares and show that the expression is positive except for one pair of values of x and y.

28.2 Rational functions of two quadratics

In this section we investigate functions of the form

$$\frac{ax^2+bx+c}{Ax^2+Bx+C}$$

where a, b, c, A, B, C are constants. The method of the last section will be used.

Example 6 *Sketch the curve* $y = \dfrac{(x-1)(x+2)}{(x+1)(x-3)}$.

First note the following:

a when $y = 0$, $x = 1$ or $x = -2$;

b when $x = 0$, $y = \frac{2}{3}$;

c when $x = -1$ or $x = 3$, the denominator of the fraction is zero so that there is no corresponding value of y; the function is *discontinuous* at these points. If x differs from 1 or 3 by a small amount, the denominator is small and so y is large. Therefore $y \to \infty$ as $x \to -1$ and $x \to 3$;

d given any value of x, other than 1 or 3, there exists one and only one value of y. Note that since the equation is a quadratic in x, there are in general two values of x corresponding to each value of y;

e the sign of y is found by inspecting the signs of the factors $x+2, x+1, x-1, x-3$;

	$x<-2$	$-2<x<-1$	$-1<x<1$	$1<x<3$	$3<x$
$x+2$	$-$	$+$	$+$	$+$	$+$
$x+1$	$-$	$-$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$+$	$+$
$x-3$	$-$	$-$	$-$	$-$	$+$
y	$+$	$-$	$+$	$-$	$+$

f $y = \dfrac{x^2+x-2}{x^2-2x-3}$.

If x is large, the terms in x and the constants are small compared with x^2 so that $y \approx x^2/x^2 = 1$. If we substitute $y = 1$ in the equation,

$$x^2 - 2x - 3 = x^2 + x - 2$$
$$\therefore x = -\tfrac{1}{3}$$

Therefore the graph crosses $y = 1$ at $(-\frac{1}{3}, 1)$.

The above findings are shown in Fig. 28.3. The shading denotes areas where the curve cannot lie (see note **e** above).

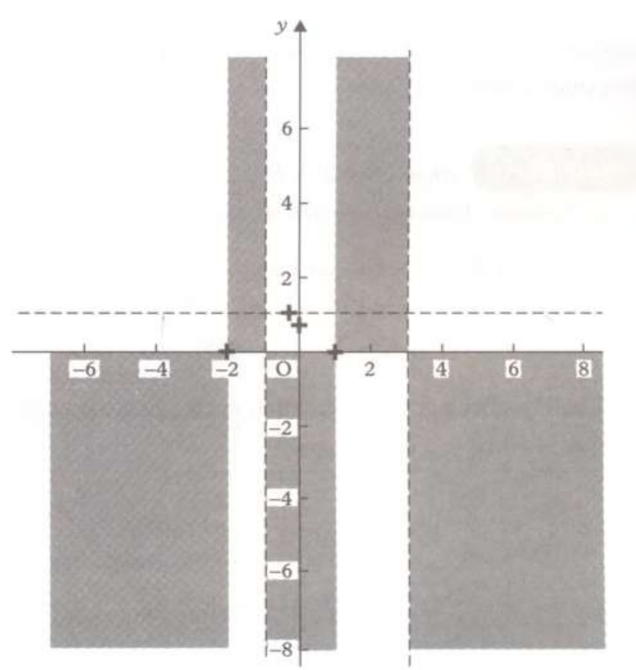


Figure 28.3

The graph is then sketched as in Fig. 28.4.

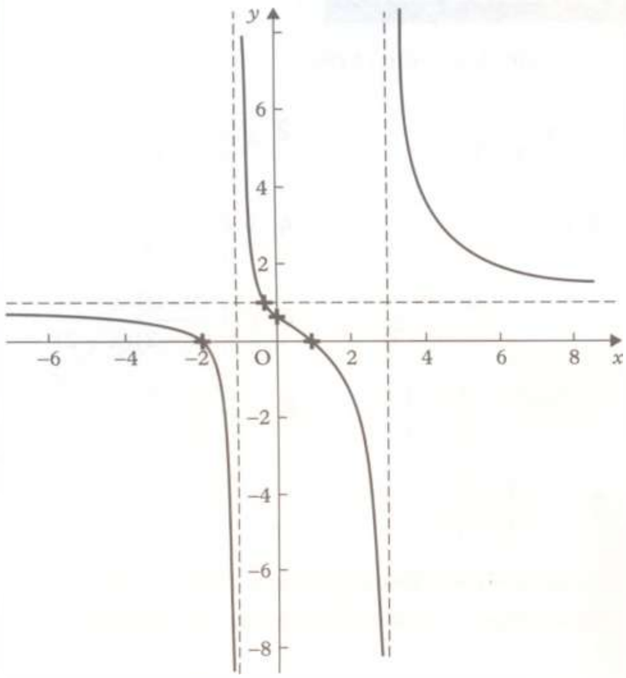


Figure 28.4

Asymptotes

In Fig. 28.4 the broken lines $x=-1$, $x=3$, $y=1$ are called **asymptotes**. Note that the curve approaches them more and more closely, without ever meeting them, as it recedes from the origin. It is possible, however, for the curve to cut an asymptote, as at $(-\frac{1}{3}, 1)$.

Be careful when finding from which sides the graph approaches the asymptotes. For $x=-1$ and $x=3$ this was ensured by examining the sign of y. For $y=1$ the point of intersection with the graph was found. Another method for $y=1$ is to take a second approximation for y, namely

$$\frac{x^2+x}{x^2-2x}$$

If $x>0$, the numerator is greater than the denominator, so that the graph approaches $y=1$ from above. On the other hand, when x is large and negative, $y<1$.

Example 7 *Prove that $(3x-9)/(x^2-x-2)$ cannot lie between two certain values. Illustrate graphically.*

Let $y=\dfrac{3x-9}{x^2-x-2}$.

Regard this equation as a quadratic which gives x in terms of y, then

$$(x^2-x-2)y=3x-9$$
$$\therefore\ yx^2-x(y+3)+9-2y=0 \qquad (1)$$

When x is *not* real,

$$\{-(y+3)\}^2-4y(9-2y)<0$$
$$\therefore\ 9y^2-30y+9<0$$
$$\therefore\ 3(3y-1)(y-3)<0$$
$$\therefore\ \tfrac{1}{3}<y<3$$

Therefore there are no real values of y between $\frac{1}{3}$ and 3.

Now $y=\dfrac{3(x-3)}{(x+1)(x-2)}$, and we may proceed as in Example 6.

- **a** If $y=0$, $x=3$. **b** If $x=0$, $y=4\frac{1}{2}$.
- **c** The lines $x=-1$ and $x=2$ are asymptotes. The function is discontinuous when $x=-1$ and $x=2$.
- **d** There is only one value of y for each value of x.
- **e** The sign of y is obtained:

	$x<-1$	$-1<x<2$	$2<x<3$	$3<x$
$x+1$	−	+	+	+
$x-2$	−	−	+	+
$x-3$	−	−	−	+
y	−	+	−	+

- **f** As $x\to\infty$, $y\to 0$.
- **g** The values of x corresponding to $y=\frac{1}{3}$ and $y=3$ are found from equation (1).

[Note that $y=\frac{1}{3}$ and $y=3$ make the discriminant 'b^2-4ac' $=0$, so that equation (1) above has equal roots. The sum of the roots is $(y+3)/y$, therefore $x=\frac{1}{2}(y+3)/y$.] When $y=\frac{1}{3}$, $x=5$; when $y=3$, $x=1$.

These findings are shown in Fig. 28.5 and the curve has been sketched in Fig. 28.6.

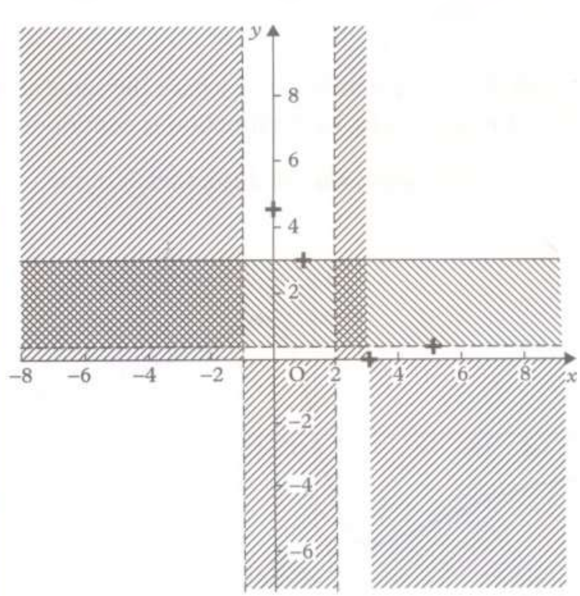


Figure 28.5

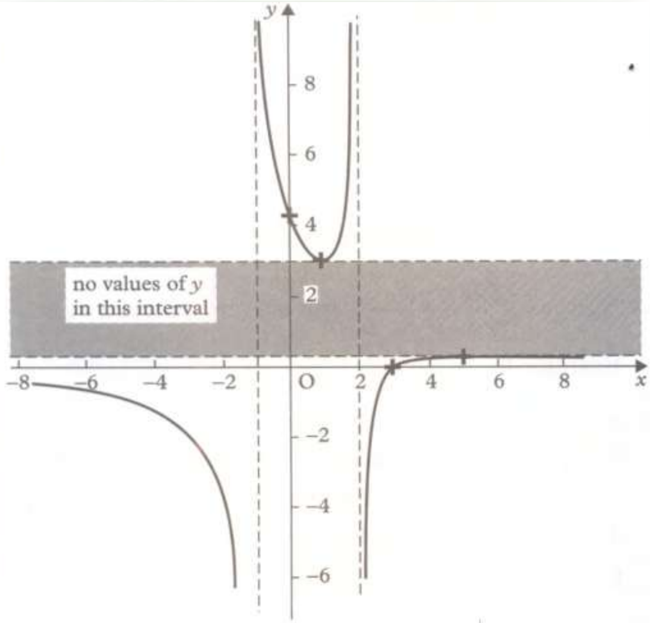


Figure 28.6

Example 8 *Sketch the curve $y = 2x/(x^2 + 1)$.*

a The curve cuts the axes only at (0, 0).

b As a quadratic in x, the equation is $x^2y - 2x + y = 0$.

For real values of x,

$$(-2)^2 - 4y^2 \geqslant 0$$
$$\therefore 4(1-y)(1+y) \geqslant 0$$
$$\therefore -1 \leqslant y \leqslant +1$$

c When $y = -1$, $x = -1$, and when $y = +1$, $x = +1$. Therefore $(-1, -1)$ is a minimum and $(1, 1)$ a maximum.

d As $x \to \infty$, $y \to 0$.

e Since $x^2 + 1$ is positive, x and y have the same sign; $x^2 + 1$ is never zero, so the function is continuous.

Using the above notes, Fig. 28.7 is a sketch of the curve.

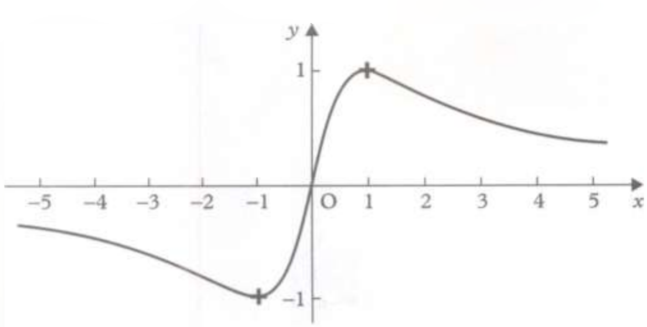


Figure 28.7

Exercise 28b

Sketch the following curves.

1 $y = \dfrac{x-2}{x+3}$ **2** $y = \dfrac{x}{x-1}$

3 $y = \dfrac{x}{x^2-1}$ **4** $y = \dfrac{x^2}{x^2-1}$

5 $y = \dfrac{2x-4}{(x-1)(x-3)}$ **6** $y = \dfrac{(x-1)(x+3)}{(x-2)(x+2)}$

7 $y = \dfrac{x^2-4x+1}{x^2-4x+4}$ **8** $y = \dfrac{(x-1)^2}{x(x-2)}$

9 $y = \dfrac{(x-2)^2}{x(x+2)}$

For each of the following curves, find the intervals within which y *cannot* lie. Illustrate graphically.

10 $y = \dfrac{4}{(x-1)(x-3)}$ **11** $y = \dfrac{3x-6}{x(x+6)}$

12 $y = \dfrac{1}{x^2+1}$ **13** $y = \dfrac{4x^2-3x}{x^2+1}$

14 $y = \dfrac{(x-3)(x-1)}{(x-2)^2}$ **15** $y = \dfrac{x^2+1}{x^2-x-2}$

Find the turning points of the following and sketch the curves.

16 $y = \dfrac{x^2-4x}{x^2-4x+3}$ **17** $y = \dfrac{x^2-x-2}{x^2-2x-8}$

18 $y = \dfrac{x^2-3x}{x^2+5x+4}$ **19** $y = \dfrac{2x^2-9x+4}{x^2-2x+1}$

28.3 Curve sketching

This section provides further aids to curve sketching. The most useful of these is symmetry.

Tests for symmetry

First consider the graph of $y = x^2$ (Fig. 28.8i), which is symmetrical about the y-axis. If the point (h, k) lies on the curve, we have $k = h^2$, and so the point $(-h, k)$ also lies on the curve. In general, if an equation is unaltered by replacing x by $-x$, the curve is symmetrical about the y-axis. The graphs of all *even* functions are symmetrical about the y-axis.

Similarly, if the equation of a curve is unaltered by replacing y by $-y$, there is symmetry about the x-axis (Fig. 28.8ii).

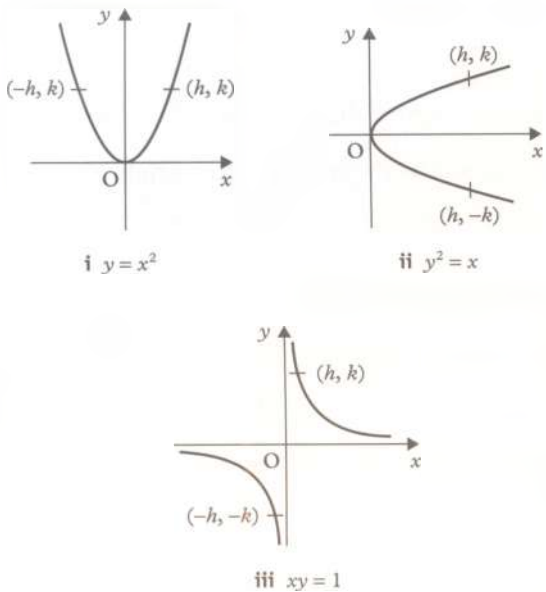


Figure 28.8

Fig. 28.8iii represents the curve $xy = 1$, which has *rotational* symmetry about a *point*, the origin. If (h, k) lies on the locus, so does $(-h, -k)$. In general, if an equation is unaltered when x and y are replaced by $-x$ and $-y$ respectively, the curve is said to have rotational symmetry about the origin. The graphs of all *odd* functions have rotational symmetry about the origin.

Questions

Q6 Which of the following show symmetry about **i** the y-axis, **ii** the x-axis, **iii** the origin?

- **a** $4x^2 + y^2 = 1$
- **b** $y^2 = x(x+1)$
- **c** $x^5 + y^5 = 5xy^2$
- **d** $x^2 - 3xy + y^2 = 1$
- **e** $y^2 = x^2(x+1)(x-1)$
- **f** $x^2y - x + y^3 = 0$
- **g** $y^2 = \cos x$
- **h** $\tan y = \sin x$

Q7 Some equations are unaltered by the following substitutions:

a $x = y,\quad y = x$ **b** $x = -y,\quad y = -x$

About what lines are the corresponding curves symmetrical?

Q8 Show that a curve which is symmetrical about the x- and y-axes has rotational symmetry about the origin.

Example 9 *Sketch the curve* $x^2 - y^2 = 1$.

- **a** The equation shows symmetry about both axes and the origin.
- **b** Since $y^2 = x^2 - 1$, y is not real when x is numerically less than 1.
- **c** When $y = 0$, $x = \pm 1$.
- **d** As x increases in magnitude, so does y.
- **e** On differentiation,

$$2y\frac{dy}{dx} = 2x$$

$$\therefore \frac{dy}{dx} = \frac{x}{\sqrt{(x^2-1)}}$$

$$\therefore \text{as } x \to \pm 1, \quad \frac{dy}{dx} \to \infty$$

and

$$\text{as } x \to \pm\infty, \quad \frac{dy}{dx} \to \pm 1$$

- **f** Since $y^2 = x^2 - 1$, when x, y are large, y^2 is nearly equal to x^2. Thus the curve approaches the lines $y = \pm x$.

We use the information in **a** to **f** to sketch the curve (Fig. 28.9).

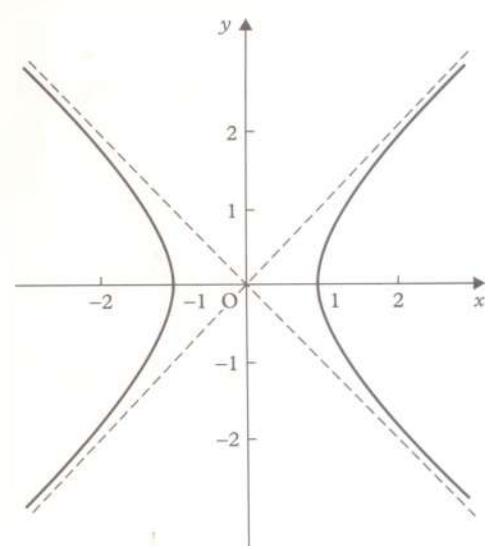


Figure 28.9

The form $y^2 = f(x)$

If an equation can be expressed in the form $y^2 = f(x)$, then its graph will have a number of special features. Since y^2 cannot be negative, x must be limited to values for which $f(x)$ is non-negative. For any such value we can write $y = \pm\sqrt{f(x)}$, so the graph will be symmetrical about the x-axis.

Example 10 *Sketch the graph of $y^2 = x(x-2)^2$.*

The factor $(x-2)^2$ is never negative so the sign of the R.H.S. is determined by the factor x. So x must be greater than, or equal to, zero to obtain real values of y. Also, y is zero at $x = 0$ and $x = 2$.

Consider $y^2 = x(x-2)^2$.

On differentiating, we obtain

$$2y\frac{dy}{dx} = (x-2)^2 + 2x(x-2)$$

$$\frac{dy}{dx} = \frac{(x-2)(3x-2)}{\pm 2x^{1/2}(x-2)}$$

$$= \pm\frac{3x-2}{2x^{1/2}}$$

From this we can see that $\frac{dy}{dx} = 0$ when $x = \frac{2}{3}$, and that

as $x \to 0$, $\quad \frac{dy}{dx} \to \infty$

as $x \to 2$, $\quad \frac{dy}{dx} \to \pm\sqrt{2}$

and

as $x \to \infty$, $\quad \frac{dy}{dx} \to \pm\infty$

The graph of $y^2 = x(x-2)^2$ is sketched in Fig. 28.10.

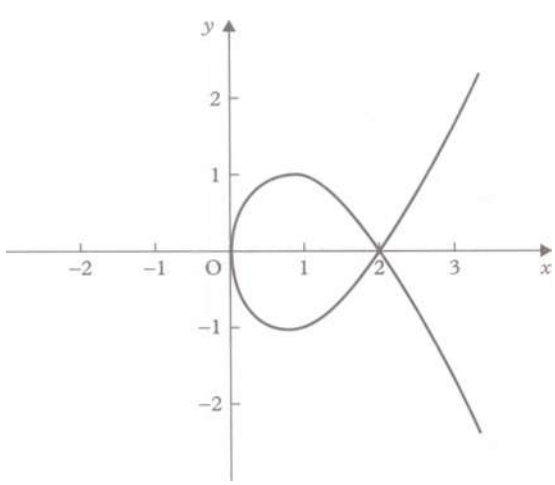


Figure 28.10

[Note that the equation $y^2 = x(x-2)^2$ cannot be regarded as a rule for expressing y as a *function* of x, because there are *two* values of y for each value of x. However, $y = +\sqrt{x}(x-2)$ and $y = -\sqrt{x}(x-2)$ could be regarded as two functions whose graphs could be combined to produce the graph of $y^2 = x(x-2)^2$.]

Question

Q9 Find the gradient of

a $\quad y^2 = x(x-2)(x-4)$

b $\quad y^2 = x^2(x+2)$

at the points where the graphs cut the x-axis.

Sketch the curves by the method of Example 10.

Simple changes of axes

The equation of a circle, centre C(a, b) and radius r, is

$$(x-a)^2 + (y-b)^2 = r^2 \qquad \text{(see §21.1 on page 234)}$$

and the equation of an equal circle, centre the origin, is

$$x^2 + y^2 = r^2$$

Therefore, if new axes CX and CY were taken parallel to Ox and Oy, the equation of the former would become

$$X^2 + Y^2 = r^2$$

This is equivalent to making the substitutions

$$X = x - a \qquad Y = y - b$$

or, as is often more convenient,

$$x = X + a \qquad y = Y + b$$

These relationships may easily be verified from a diagram.

Such a change of axes is sometimes helpful in curve sketching. Thus

$$(y-1)^2 = 4(x+2)$$

becomes $\quad Y^2 = 4X$

referred to parallel axes through $(-2, 1)$ and the curve is now easily drawn, as in Fig. 28.11.

Note that the equation $y = ax^2 + bx + c$ may be written

$$y + \frac{b^2}{4a} - c = a\left(x + \frac{b}{2a}\right)^2$$

Referred to parallel axes through $\left(-\frac{b}{2a}, -\frac{b^2-4ac}{4a}\right)$

the equation becomes $\quad Y = aX^2$

which is a parabola (see §10.1 on page 128 and §22.6 on page 249).

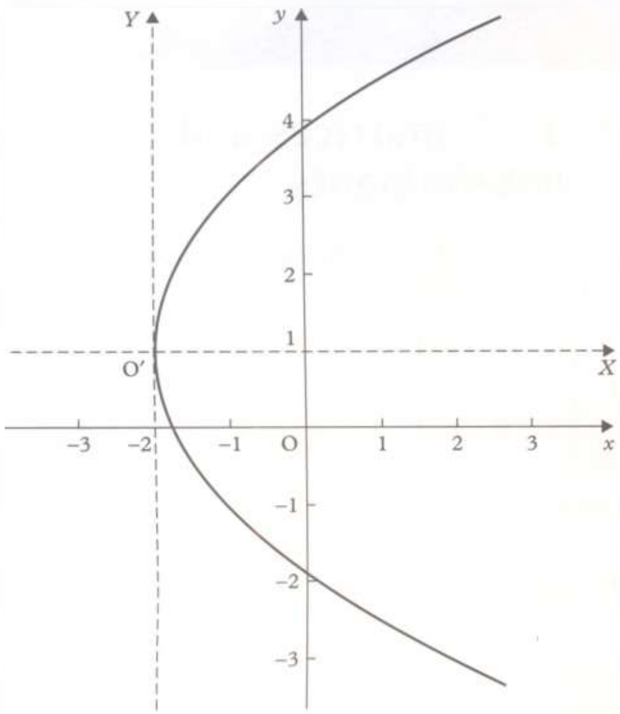


Figure 28.11

Example 11 *Sketch the function* $1 + 2\sin(\theta + \frac{1}{4}\pi)$ *for values of* θ *from 0 to* 2π.

Write $\quad y = 1 + 2\sin(\theta + \frac{1}{4}\pi)$

$\therefore y - 1 = 2\sin(\theta + \frac{1}{4}\pi)$

With the substitutions

$$\Theta = \theta + \tfrac{1}{4}\pi \qquad \text{and} \qquad Y = y - 1 \qquad (1)$$

the equation becomes

$$Y = 2\sin\Theta$$

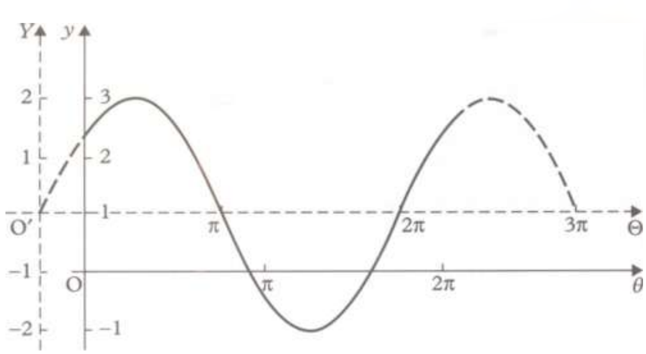


Figure 28.12

The graph of $Y = 2\sin\Theta$ has been sketched in **Fig. 28.12**. Writing $\theta = y = 0$ in equations (1), the origin of the θ, y axes is found to be $(\frac{1}{4}\pi, -1)$, referred to the Θ, Y axes. The θ, y axes were then drawn to pass through this point.

The form $y = 1/f(x)$

Example 12 *Sketch on the same axes the graphs of*

a $\;y = (x + 1)(2x - 3)$, **b** $\;y = 1/\{(x + 1)(2x - 3)\}$.

a The graph of $f(x) = (x + 1)(2x - 3)$ is a parabola meeting the x-axis at $(-1, 0)$ and $(1\frac{1}{2}, 0)$. As $x \to \pm\infty$, $y \to +\infty$. See the broken line in **Fig. 28.13**.

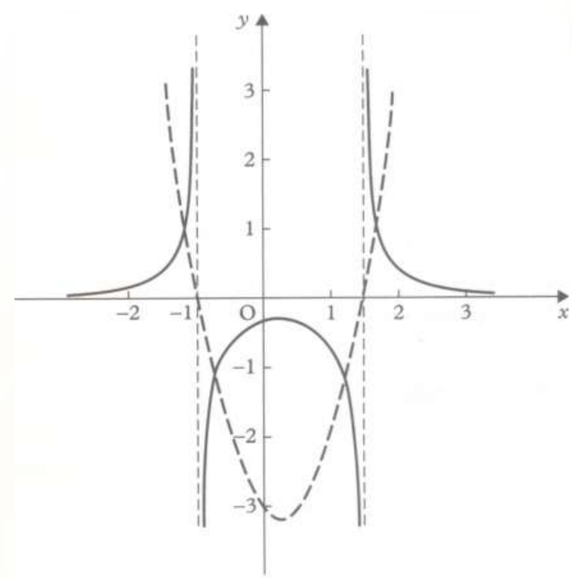


Figure 28.13

b The reciprocal of $f(x)$ is sketched as follows:

i the signs of $f(x)$ and $1/f(x)$ are the same,

ii as $f(x) \to \infty$, $1/f(x) \to 0$ and vice versa,

iii when $f(x) = \pm 1$, $1/f(x)$ has the same value.

The two graphs are shown in **Fig. 28.13**.

Exercise 28c

Sketch the following curves.

1 $2x^2 + y^2 = 4$

2 $2x^2 - y^2 = 4$

3 $x^2 y = 1$

4 $x^2 y^2 = 4$

5 $y^2 = x(x - 2)$

6 $y^2 = x^3(4 - x)$

7 $y^2 = x^2(2 - x)$

8 $y^2 = x^2(x^2 - 1)$

9 $y^2 = x^2(1 - x^2)$

10 $y^2 = x^2(2 - x)^3$

11 $y^2 = (x^2 - 1)(4 - x^2)$

12 $y = 1/\{(x - 2)(5 - x)\}$

13 $y = 1/(x^2 - 4x + 3)$

14 $xy(x^2 - 1) = 1$

15 $y = (x - 2)^2 + 1$

16 $(x - 1)(y + 2) = 1$

17 $y = 1 + \frac{1}{2}\cos(x + \frac{1}{6}\pi)$

18 $y = 3 - 2\sin(2x - \frac{1}{4}\pi)$

19 $y = 1/(1 + 2\sin x), \quad 0 \leqslant x \leqslant 4\pi$

20 $y = 1/(1 - 2\cos 2x), \quad 0 \leqslant x \leqslant 2\pi$

Chapter 29
Matrices and determinants (2)

Some standard notation

We have already met the determinant of a 2×2 matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (see Chapter 11), and we have seen that it plays an important role in finding the inverse of $\mathbf{M}$ (see §11.5 on page 141) and in finding the transformation which corresponds to $\mathbf{M}$ (see §11.6 on page 145).

In this chapter, we will meet 3×3 matrices and their determinants. Before doing this it is necessary to introduce some standard notation:

- **a** the determinant of a given matrix $\mathbf{M}$ is written $\det(\mathbf{M})$,
- **b** when writing out a determinant in full, the array of numbers is enclosed in a pair of vertical lines (round brackets are used for matrices),
- **c** the matrix formed by interchanging the rows and columns of a matrix $\mathbf{M}$ is called the **transpose** of $\mathbf{M}$ and it is written $\mathbf{M}^{\mathrm{T}}$.

So, if $\mathbf{M} = \begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix}$, we write

$$\det(\mathbf{M}) = \begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix} = 2 \times 7 - 3 \times 4 = 14 - 12 = 2$$

and

$$\mathbf{M}^{\mathrm{T}} = \begin{pmatrix} 2 & 4 \\ 3 & 7 \end{pmatrix}$$

Questions

Q1 Given that $\mathbf{M} = \begin{pmatrix} 3 & 4 \\ x & 8 \end{pmatrix}$, find x such that $\det(\mathbf{M}) = 0$.

Q2 Given that $\mathbf{A} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 7 \\ 3 & -1 \end{pmatrix}$, verify that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

***Q3** Prove that, for any 2×2 matrix $\mathbf{M}$, $\det(\mathbf{M}^{\mathrm{T}}) = \det(\mathbf{M})$.

29.1 3×3 matrices and determinants

In this chapter we will frequently refer to the following general 3×3 matrix $\mathbf{M}$:

$$\mathbf{M} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

Definition

$$\det(\mathbf{M}) = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

Notice that the three 2×2 matrices in this definition are obtained by deleting the row and column containing the letter by which each is multiplied. This definition can easily be extended to cover determinants of higher order.

When the 2×2 determinants are multiplied out, we obtain

$$\begin{aligned}\det(\mathbf{M}) &= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - b_2a_3) \\ &= a_1b_2c_3 - a_1b_3c_2 - b_1a_2c_3 + b_1a_3c_2 + c_1a_2b_3 - c_1b_2a_3\end{aligned}$$

For convenience, these terms should be re-arranged so that in each term the letters occur in alphabetical order. Putting the terms with + signs first, we have

$$\det(\mathbf{M}) = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_3b_2c_1 - a_2b_1c_3$$

When written this way, we can see that the positive terms have suffixes 1, 2, 3 that occur in *clockwise cyclic* order (see Fig. 29.1i). When the suffixes go anticlockwise (see Fig. 29.1ii) the terms are negative.

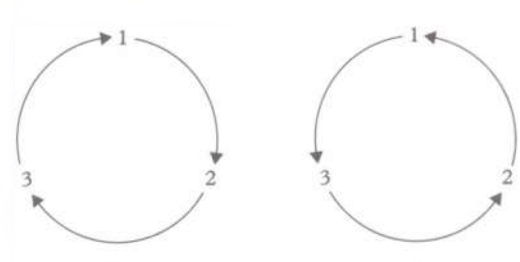


Figure 29.1

Example 1 *Evaluate* $\begin{vmatrix} 7 & 2 & 3 \\ 4 & 1 & 5 \\ 2 & 0 & 3 \end{vmatrix}$.

$$\begin{vmatrix} 7 & 2 & 3 \\ 4 & 1 & 5 \\ 2 & 0 & 3 \end{vmatrix} = 7\begin{vmatrix} 1 & 5 \\ 0 & 3 \end{vmatrix} - 2\begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix} + 3\begin{vmatrix} 4 & 1 \\ 2 & 0 \end{vmatrix}$$
$$= 7(3-0) - 2(12-10) + 3(0-2)$$
$$= 7 \times 3 - 2 \times 2 + 3 \times (-2)$$
$$= 21 - 4 - 6$$
$$= 11$$

Example 2 *Solve the equation* $\begin{vmatrix} x-3 & 1 & -1 \\ -7 & x+5 & -1 \\ -6 & 6 & x-2 \end{vmatrix} = 0.$

$$\begin{vmatrix} x-3 & 1 & -1 \\ -7 & x+5 & -1 \\ -6 & 6 & x-2 \end{vmatrix}$$
$$= (x-3)\begin{vmatrix} x+5 & -1 \\ 6 & x-2 \end{vmatrix} - \begin{vmatrix} -7 & -1 \\ -6 & x-2 \end{vmatrix} - \begin{vmatrix} -7 & x+5 \\ -6 & 6 \end{vmatrix}$$
$$= (x-3)(x^2+3x-10+6) - (-7x+14-6) - (-42+6x+30)$$
$$= (x-3)(x^2+3x-4) - (-7x+8) - (6x-12)$$
$$= x^3 - 13x + 12 + 7x - 8 - 6x + 12$$
$$= x^3 - 12x + 16$$

Hence we must solve the cubic equation

$$x^3 - 12x + 16 = 0$$

Factorising,

$$(x-2)(x^2+2x-8) = 0$$
$$\therefore (x-2)(x-2)(x+4) = 0$$
$$\therefore x = 2 \quad \text{or} \quad -4$$

Example 3 *Prove that*

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x).$$

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$
$$= \begin{vmatrix} y & z \\ y^2 & z^2 \end{vmatrix} - \begin{vmatrix} x & z \\ x^2 & z^2 \end{vmatrix} + \begin{vmatrix} x & y \\ x^2 & y^2 \end{vmatrix}$$
$$= (yz^2 - zy^2) - (xz^2 - zx^2) + (xy^2 - yx^2)$$
$$= yz^2 - zy^2 - xz^2 + zx^2 + xy^2 - yx^2$$
$$= z^2(y-x) - z(y^2-x^2) + xy(y-x)$$
$$= -z^2(x-y) + z(x+y)(x-y) - xy(x-y)$$
$$= (x-y)(-z^2 + zx + zy - xy)$$
$$= (x-y)\{-z(z-x) + y(z-x)\}$$
$$= (x-y)(z-x)(y-z)$$
$$= (x-y)(y-z)(z-x)$$

Exercise 29a

1 Evaluate:

a $\begin{vmatrix} 5 & 2 \\ 3 & 4 \end{vmatrix}$ b $\begin{vmatrix} 7 & 1 \\ 2 & -1 \end{vmatrix}$

c $\begin{vmatrix} 21 & 14 \\ 15 & 10 \end{vmatrix}$ d $\begin{vmatrix} 91 & 35 \\ 65 & 25 \end{vmatrix}$

2 Simplify:

a $\begin{vmatrix} x & y \\ -y & x \end{vmatrix}$ b $\begin{vmatrix} x^2 & xy \\ xy & y^2 \end{vmatrix}$

c $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ d $\begin{vmatrix} x+1 & 1 \\ -1 & x-1 \end{vmatrix}$

3 Prove that $\det(\mathbf{M}) = \det(\mathbf{M}^T)$, where $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

4 Solve:

a $\begin{vmatrix} x & x \\ 5 & 3x \end{vmatrix} = 0$ b $\begin{vmatrix} x-2 & 1 \\ 2 & x-3 \end{vmatrix} = 0$

***5** Given that $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$,

prove that $\det(\mathbf{MN}) = \det(\mathbf{M})\det(\mathbf{N})$.

6 If $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ prove that $\det(\mathbf{M}^{-1}) = \dfrac{1}{\det(\mathbf{M})}$.

7 Evaluate:

a $\begin{vmatrix} 2 & -1 & 0 \\ 3 & 2 & 0 \\ 4 & 7 & 3 \end{vmatrix}$ b $\begin{vmatrix} 3 & 4 & 2 \\ 1 & 2 & 0 \\ -2 & 3 & 5 \end{vmatrix}$

c $\begin{vmatrix} 3 & 6 & 9 \\ 4 & 8 & 12 \\ 5 & 7 & 13 \end{vmatrix}$ d $\begin{vmatrix} 5 & -1 & 1 \\ 5 & -1 & 1 \\ 6 & -1 & 6 \end{vmatrix}$

8 Solve the equation $\begin{vmatrix} x+3 & 5 & 6 \\ -1 & x-3 & -1 \\ 1 & 1 & x+4 \end{vmatrix} = 0.$

***9** Prove that for any 3×3 matrix $\det(\mathbf{M}) = \det(\mathbf{M}^T)$.

10 Given

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & -1 & 0 \\ 4 & 2 & 1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & -3 & 2 \\ 1 & 1 & -1 \end{pmatrix},$$

find $\mathbf{AB}$ and show that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

29.2 Cofactors

In the following sections we will refer to a standard determinant Δ, where

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Definition

The **cofactor** *of an element of a 3×3 determinant is the 2×2 determinant obtained by deleting the row and column containing that element and multiplying by $+1$ or -1 according to the pattern:*

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

We always show a cofactor by the capital letter corresponding to the element to which it belongs.

This definition is not as complicated as it seems when we write out the nine cofactors in full. They are:

$$A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \quad B_1 = -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} \quad C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$A_2 = -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} \quad B_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} \quad C_2 = -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$A_3 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \quad B_3 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \quad C_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Expanding the determinant in the usual way, we can see that

$$\Delta = a_1A_1 + b_1B_1 + c_1C_1.$$

Check that

$$a_2A_2 + b_2B_2 + c_2C_2 \quad \text{and} \quad a_3A_3 + b_3B_3 + c_3C_3$$

are also equal to Δ.

Expanding the determinant by columns we can see that $a_1A_1 + a_2A_2 + a_3A_3$, $b_1B_1 + b_2B_2 + b_3B_3$ and $c_1C_1 + c_2C_2 + c_3C_3$ are also all equal to Δ. In other words, whenever the elements of one row (or column) are combined with the cofactors of the *same* row (or column), the sum is equal to Δ. However, notice what happens when the elements of one row are combined with the cofactors of a *different* row. For example, consider

$$a_1A_2 + b_1B_2 + c_1C_2$$

(i.e. the elements of the first row, combined with the cofactors of the second row). Writing the cofactors as 2×2 determinants, we have

$$-a_1\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_1\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_1\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$= -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$= 0$ (because this determinant has a pair of identical rows)

Check that this will happen whenever the elements of a row (or column) are combined with the cofactors of a different row (or column). (These cofactors are sometimes called *alien* cofactors.)

Question

Q4 Evaluate the determinant below, and find the values of its nine cofactors. Verify that they satisfy the above relationships.

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{vmatrix}$$

The properties of cofactors can be used to produce an important method for solving linear equations, known as Cramer's rule.

Cramer's rule

Consider the following three linear equations, and the associated determinant as discussed above.

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Multiplying the first by A_1, the second by A_2 and the third by A_3 (where A_1, A_2, A_3 are the cofactors corresponding to a_1, a_2 and a_3 respectively) and adding, we obtain

$$\begin{aligned} &(a_1A_1 + a_2A_2 + a_3A_3)x \\ &+ (b_1A_1 + b_2A_2 + b_3A_3)y \\ &+ (c_1A_1 + c_2A_2 + c_3A_3)z \\ &= d_1A_1 + d_2A_2 + d_3A_3 \end{aligned}$$

Now, using the relationships established in the preceding section, we can see that the coefficient of x is Δ, and the coefficients of y and z are both zero. Hence, we have

$$\Delta x = d_1A_1 + d_2A_2 + d_3A_3$$

$$= \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

This determinant is the original determinant Δ, with the first column replaced by d_1, d_2 and d_3. It is convenient to abbreviate this determinant to Δ_1. (This abbreviation can be used more generally if we write Δ_i, meaning the determinant formed by replacing the ith column of Δ by d_1, d_2 and d_3.) The result we have just obtained can then be written $\Delta x = \Delta_1$. Proceeding in a similar fashion, it is fairly easy to show that $\Delta y = \Delta_2$ and $\Delta z = \Delta_3$. Hence the solutions to the three linear equations can be expressed very neatly as:

$$\Delta x = \Delta_1, \qquad \Delta y = \Delta_2, \qquad \Delta z = \Delta_3 \qquad (1)$$

and, provided $\Delta \neq 0$, we can divide through by it and obtain,

$$x = \Delta_1/\Delta, \qquad y = \Delta_2/\Delta, \qquad z = \Delta_3/\Delta$$

This is **Cramer's rule**.

Example 4 *Use Cramer's rule to solve*

$$\begin{aligned} x - 2y - 3z &= 0 \\ 3x + 5y + 2z &= 0 \\ 2x + 3y - z &= 2 \end{aligned}$$

Using the above notation,

$$\Delta = \begin{vmatrix} 1 & -2 & -3 \\ 3 & 5 & 2 \\ 2 & 3 & -1 \end{vmatrix}$$
$$= 1 \times (-11) + 2 \times (-7) - 3 \times (-1)$$
$$= -11 - 14 + 3 = -22$$

$$\Delta_1 = \begin{vmatrix} 0 & -2 & -3 \\ 0 & 5 & 2 \\ 2 & 3 & -1 \end{vmatrix} = 2 \begin{vmatrix} -2 & -3 \\ 5 & 2 \end{vmatrix} = 2 \times (+11) = 22$$

$$\Delta_2 = \begin{vmatrix} 1 & 0 & -3 \\ 3 & 0 & 2 \\ 2 & 2 & -1 \end{vmatrix} = -2 \begin{vmatrix} 1 & -3 \\ 3 & 2 \end{vmatrix} = -2 \times (+11) = -22$$

$$\Delta_3 = \begin{vmatrix} 1 & -2 & 0 \\ 3 & 5 & 0 \\ 2 & 3 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & -2 \\ 3 & 5 \end{vmatrix} = 2 \times (+11) = 22$$

Hence, by Cramer's rule, $x = \Delta_1/\Delta = 22/(-22) = -1$,
$y = \Delta_2/\Delta = -22/(-22) = +1$,
$z = \Delta_3/\Delta = 22/(-22) = -1$.

If Δ *is* equal to zero, we must *not* divide by it. In this case the equations (1) above read

$$0x = \Delta_1, \qquad 0y = \Delta_2, \qquad 0z = \Delta_3$$

These have no solution unless Δ_1, Δ_2 and Δ_3 are also zero. If these three determinants are zero, then it *is* possible to find a solution (see Example 5, below).

Example 5 *Solve the equations*

$$\begin{aligned} x + 2y + z &= 7 \\ 3x + y \phantom{{}+2z} &= -2 \\ 5x + 5y + 2z &= 12 \end{aligned}$$

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 5 & 5 & 2 \end{vmatrix}$$
$$= 1 \times (+2) - 2 \times (+6) + 1 \times (+10)$$
$$= 2 - 12 + 10$$
$$= 0$$

Also $\Delta_1 = \Delta_2 = \Delta_3 = 0$. (The detailed working is left to you.) Hence, in this case, Cramer's rule gives

$$0 \times x = 0, \qquad 0 \times y = 0, \qquad 0 \times z = 0$$

It is certainly *possible* for values of x, y and z to exist which satisfy these equations, but Cramer's rule is not applicable in this case. Therefore return to the original equations and eliminate z from the first and third equations to obtain:

$$\begin{aligned} 2x + 4y + 2z &= 14 \\ 5x + 5y + 2z &= 12 \end{aligned}$$

and subtracting, we have

$$3x + y = -2$$

This is identical to the second equation. (If the right-hand side had *not* been -2, we would have concluded, at this stage, that there was no solution. If, on the other hand the equations had been *distinct*, we could have solved them to find x and y. But Cramer's rule has already shown that this is not possible.)

Although we cannot solve $3x + y = -2$ and find a *unique* solution, we can let $x = t$, where t is any real number. With this value for x, there is no choice in the value of y; it must be $-2 - 3t$. Substituting these values for x and y into the first of the original equations:

$$z = 7 - t - 2(-2 - 3t)$$
$$= 11 + 5t$$

Hence, we have a *set* of solutions,

$$x = t \qquad y = -2 - 3t \qquad z = 11 + 5t$$

where t is any real number.

(Compare this with §15.11 on page 188.)

Question

Q5 Give a geometrical interpretation of the result of Example 5.

Example 6 *Solve the equations*

$$\begin{aligned} x + 2y + z &= 7 \\ 3x + y &= -2 \\ 5x + 5y + 2z &= 10 \end{aligned} \qquad \textit{if possible.}$$

(Notice that the first two equations are the same as those in Example 5, and in the third equation only the constant is different. As in Example 5, $\Delta = 0$, but this time $\Delta_1 \neq 0$.)

$$\begin{aligned} \Delta_1 &= \begin{vmatrix} 7 & 2 & 1 \\ -2 & 1 & 0 \\ 10 & 5 & 2 \end{vmatrix} \\ &= 7 \times (+2) - 2 \times (-4) + 1 \times (-20) \\ &= 14 + 8 - 20 \\ &= 2 \end{aligned}$$

Since $\Delta_1 \neq 0$, there is *no* solution.

Question

Q6 Give a geometrical interpretation of the result of Example 6.

Cramer's rule is valuable because it helps to explain the behaviour of simultaneous linear equations.
In practice, elimination is usually the most satisfactory method to use.

Exercise 29b

The questions in this exercise provide practice in using Cramer's rule.

Solve the simultaneous equations:

1 $\begin{aligned} x + y + z &= 6 \\ 2x + y - z &= 1 \\ x - y + z &= 2 \end{aligned}$

2 $\begin{aligned} 2x + 3y + z &= 1 \\ x - y + z &= 4 \\ 5x + y + 3z &= 10 \end{aligned}$

3 $\begin{aligned} 7x + y + z &= -1 \\ x - 3y + 2z &= 0 \\ x + 4y - 3z &= 4 \end{aligned}$

4 $\begin{aligned} 5x - 7y + 3z &= 0 \\ 2x - 3y + 5z &= -1 \\ 3x - 4y + 2z &= 1 \end{aligned}$

5 $\begin{aligned} 10x + 20y + 40z &= 1 \\ 3x + 7y + 10z &= 0 \\ 25x + 12y + 37z &= 0 \end{aligned}$

6 Find the condition for the simultaneous equations below to have no solution.

$$\begin{aligned} x + 5y + az &= 2 \\ 2x + y + 3z &= 1 \\ 7x + 8y + 8z &= k \end{aligned}$$

29.3 The inverse of a 3 × 3 matrix

As in previous sections, $\mathbf{M} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$, and the capital letters, A_1, A_2, A_3, etc., represent the cofactors corresponding to the elements a_1, a_2, a_3, etc.

Definition

The **adjoint** *of the matrix* **M** *is* $\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$.

The standard abbreviation for 'the adjoint of matrix **M**' is adj(**M**). Notice that adj(**M**) is the transpose of the matrix formed by replacing each element of **M** by its cofactor.

Questions

Q7 Given that $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$, find adj(**A**) and determine the matrix product **A** adj(**A**).

Q8 Repeat Q7, for $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$.

The product **M** adj(**M**) is always a **diagonal matrix**, that is, a matrix in which all the elements are zero, except those on the 'leading diagonal' (the diagonal which goes from the top left-hand corner to the bottom right-hand corner). To see the reason for this, check the following working in detail:

$$\mathbf{M}\,\mathrm{adj}(\mathbf{M}) = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{pmatrix}$$

Each term which is *on* the leading diagonal, consists of an element of one of the rows of **M** combined with its *own* cofactor. This, as we saw in §29.2 on page 302, is always equal to Δ, the determinant of **M**. Each term which is *not* on the leading diagonal consists of an element from a row of **M**, combined with an *alien* cofactor. This, as we also saw in §29.2 on page 302, is always zero.

Hence,

$$\mathbf{M}\,\mathrm{adj}(\mathbf{M}) = \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{pmatrix} = \Delta\mathbf{I}$$

where **I** is the 3×3 unit matrix (see §11.5 on page 141).

Check that $\mathrm{adj}(\mathbf{M})\,\mathbf{M}$ is also $\Delta\mathbf{I}$.

Question

Q9 Verify that the answers to Q7 and Q8 are equal to $\Delta\mathbf{I}$.

This enables us to solve a problem which we postponed in §11.5 on page 141. At that stage we found the inverse of a general 2×2 matrix, but did not attempt to find the inverse of any other square matrix. If, now we look at the product $\mathbf{M}\,\mathrm{adj}(\mathbf{M}) = \Delta\mathbf{I}$, we can see that the problem is almost solved. Only one more step is necessary, namely, to divide each side by Δ. We then have

$$\mathbf{M}\frac{\mathrm{adj}(\mathbf{M})}{\Delta} = \mathbf{I}$$

In other words $\mathbf{M}^{-1}$, the inverse of **M**, is $\dfrac{\mathrm{adj}(\mathbf{M})}{\Delta}$.

(Δ must not equal zero; if it does, there is no inverse matrix, i.e. **M** is a *singular* matrix.)

Example 7 *Find the inverse of* $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -2 & 3 \end{pmatrix}$.

$$\mathrm{adj}(\mathbf{A}) = \begin{pmatrix} 1 & -5 & -2 \\ -4 & 2 & 2 \\ -3 & 3 & 0 \end{pmatrix}$$

$$\mathbf{A}\,\mathrm{adj}(\mathbf{A}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & -5 & -2 \\ -4 & 2 & 2 \\ -3 & 3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

(Notice that det(**A**) is required, but it is not necessary to work it out separately, because we know it must be the −6 which appears on the leading diagonal.) Hence

$$\mathbf{A}^{-1} = \frac{1}{-6}\begin{pmatrix} 1 & -5 & -2 \\ -4 & 2 & 2 \\ -3 & 3 & 0 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -1 & 5 & 2 \\ 4 & -2 & -2 \\ 3 & -3 & 0 \end{pmatrix}$$

Example 8 *Find the inverse of the matrix* $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 1 & -2 \end{pmatrix}$.

Write the following equations in matrix form and hence find x, y and z:

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x + y + z &= 5 \\ 3x + y - 2z &= 1 \end{aligned}$$

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 1 & -2 \end{pmatrix}$, then $\mathrm{adj}(\mathbf{A}) = \begin{pmatrix} -3 & 7 & -1 \\ 7 & -11 & 5 \\ -1 & 5 & -3 \end{pmatrix}$,

and

$$\mathbf{A}\,\mathrm{adj}(\mathbf{A}) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 1 & -2 \end{pmatrix}\begin{pmatrix} -3 & 7 & -1 \\ 7 & -11 & 5 \\ -1 & 5 & -3 \end{pmatrix} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

(Note that this step serves two purposes: it checks the accuracy of the arithmetic up to this point, and it evaluates det(**A**). In this case det(**A**) = 8.) Hence

$$A^{-1} = \frac{1}{8}\begin{pmatrix} -3 & 7 & -1 \\ 7 & -11 & 5 \\ -1 & 5 & -3 \end{pmatrix}$$

The simultaneous equations can be written

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix}$$

The matrix of the coefficients is the matrix **A**, whose inverse we have found. Multiplying both sides by $\mathbf{A}^{-1}$, the left-hand side becomes $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, since

$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, and $\mathbf{I}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{8}\begin{pmatrix} -3 & 7 & -1 \\ 7 & -11 & 5 \\ -1 & 5 & -3 \end{pmatrix}\begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix} 16 \\ -8 \\ 16 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

Hence $x = 2$, $y = -1$ and $z = 2$.

Exercise 29c

Find, where possible, the inverses of the following matrices.

1 $\begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & 3 \\ 3 & 0 & 1 \end{pmatrix}$ **2** $\begin{pmatrix} 2 & 1 & 7 \\ 3 & 4 & 5 \\ 1 & -2 & 9 \end{pmatrix}$

3 $\begin{pmatrix} 1 & -10 & 7 \\ 1 & 4 & -3 \\ -1 & 2 & -1 \end{pmatrix}$ **4** $\begin{pmatrix} 2 & 1 & 0 \\ 3 & 7 & 5 \\ 1 & 17 & 15 \end{pmatrix}$

5 $\begin{pmatrix} 3 & 3 & -1 \\ -6 & 2 & 2 \\ -1 & -1 & 3 \end{pmatrix}$ **6** $\begin{pmatrix} 1 & p & q \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}$

7 $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{pmatrix}$

Solve the following simultaneous equations, using matrices (as in Example 8).

8
$$\begin{aligned} x - y + 2z &= 4 \\ x + 2y + 3z &= 2 \\ 3x + z &= 4 \end{aligned}$$

9
$$\begin{aligned} x - 10y + 7z &= 13 \\ x + 4y - 3z &= -3 \\ -x + 2y - z &= -3 \end{aligned}$$

10
$$\begin{aligned} 2x + y + z &= 4 \\ x - y - 2z &= 0 \\ 5x - 2y - 4z &= 3 \end{aligned}$$

11 Given that $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 1 & 3 & 0 \end{pmatrix}$,

find **a** $\mathbf{A}^{-1}$, **b** $\mathbf{B}^{-1}$, **c** $\mathbf{AB}$, **d** $(\mathbf{AB})^{-1}$, and verify that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

12 Repeat question 11 with

$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 5 & 2 \\ 2 & -1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 2 & 0 \\ 1 & 3 & 2 \\ 2 & 0 & 1 \end{pmatrix}$.

13 Given that **A** and **B** are non-singular square matrices of the same order, prove that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. [Hint: consider the product $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1})$.] (You may assume that matrix multiplication is associative.)

What is the corresponding result for $(\mathbf{ABC})^{-1}$?

Chapter 30

Coordinate geometry (4)

Conic sections

Introduction

In this chapter we study three curves: the parabola, ellipse, and hyperbola, which are all known as **conic sections** or **conics**. These get their name from the cross sections of a cone.

From the point of view of coordinate geometry a conic is the locus of a point which moves so that its distance from a fixed point bears a constant ratio to its distance from a fixed line (see Fig. 30.1).

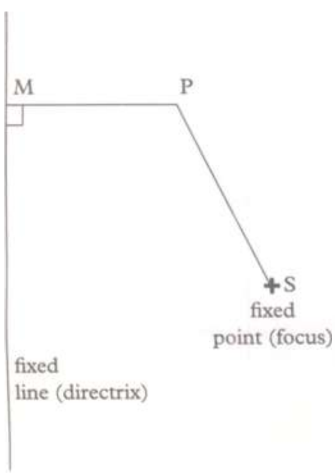


Figure 30.1

The fixed point S is called the **focus**. The fixed line is called the **directrix**. The constant ratio is called the **eccentricity** and is denoted by e. Thus, if P is a point on the locus, M is the foot of the perpendicular from P to the directrix and if

$$\frac{SP}{PM} = e$$

then the locus of P is a conic.

When: $e = 1$, the conic is a **parabola**,
$e < 1$, the conic is an **ellipse**,
$e > 1$, the conic is a **hyperbola**.

We shall first consider the parabola, which was first mentioned in Chapter 22.

30.1 The parabola

Given the focus S of the parabola and the directrix, we can take whichever axes we find most convenient. First note that the figure formed by the focus and directrix has an axis of symmetry through S perpendicular to the directrix. This we take as the x-axis, as shown in Fig. 30.2. If we now plot a few points, using the definition of the locus given in the last section,

$$\frac{SP}{PM} = 1$$

we obtain the curve in Fig. 30.2.

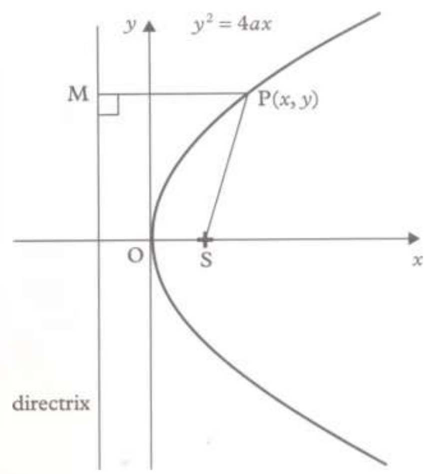


Figure 30.2

We take the y-axis through the point on the axis of symmetry mid-way between the focus and directrix. This point is called the **vertex** of the parabola. Let the distance from the vertex to the focus be a, then

the focus S is $(a, 0)$

and

the directrix is the line $x = -a$

If P(x, y) is any point on the parabola, and M is the foot of the perpendicular from P to the directrix,

$$SP^2 = (x - a)^2 + y^2$$

and

$$PM = x + a$$

But from the definition,

$$\frac{SP}{PM} = 1, \quad \text{so} \quad SP^2 = PM^2$$

$$\therefore (x-a)^2 + y^2 = (x+a)^2$$
$$\therefore x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$$
$$\therefore \boldsymbol{y^2 = 4ax}$$

which is the **standard equation of a parabola**.

Question

Q1 Find the equations of the parabolas:

a focus $(-a, 0)$, directrix $x = a$,

b focus $(0, b)$, directrix $y = -b$.

Example 1 *Find, in terms of a and m, the value of c which makes the line $y = mx + c$ a tangent to the parabola $y^2 = 4ax$. Also obtain the coordinates of the point of contact.*

$$y = mx + c$$

Multiply both sides by $4a$.

$$4ay = m \times 4ax + 4ac$$

Substituting from $y^2 = 4ax$ and collecting terms,

$$my^2 - 4ay + 4ac = 0 \qquad (1)$$

The line will be a tangent if this equation has equal roots (see §10.1 on page 128)

$$\therefore (-4a)^2 = 16mac$$

$$\therefore c = \frac{a}{m}$$

When the roots of equation (1) are equal, they are given by half the sum of the roots,

$$\therefore y = \frac{1}{2} \times \frac{4a}{m} = \frac{2a}{m}$$

Now

$$x = \frac{y^2}{4a} = \left(\frac{2a}{m}\right)^2 \times \frac{1}{4a} = \frac{a}{m^2}$$

Therefore the point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

It follows from Example 1 that the equation of a general tangent to $y^2 = 4ax$ may be written

$$y = mx + \frac{a}{m} \qquad (m \neq 0)$$

This provides a very useful way of representing a point on a parabola. Substituting $t = 1/m$, we see that the tangent

$$y = \frac{x}{t} + at$$

touches the parabola at $(at^2, 2at)$. Since it was a general tangent, we have shown that *any point on the parabola $y^2 = 4ax$ may be written $(at^2, 2at)$*. The equations $\boldsymbol{x = at^2}$, $\boldsymbol{y = 2at}$ are the **parametric equations** of the parabola $y^2 = 4ax$.

Question

Q2 Verify, by substitution, that $(at^2, 2at)$ always lies on the parabola $y^2 = 4ax$.

We have found the equation of the tangent at $(at^2, 2at)$, but a more direct method follows in Example 2.

Example 2 *Find the equation of the tangent to $y^2 = 4ax$ at $(at^2, 2at)$.*

To find the gradient at $(at^2, 2at)$,

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$$

But $y = 2at$, $x = at^2$,

$$\therefore \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$$

The equation of the tangent is obtained by the method shown in §22.1 on page 239, which will be used from now on.

$$x - ty = at^2 - t \times 2at$$

i.e.

$$\boldsymbol{x - ty + at^2 = 0}$$

Question

***Q3** Show that the equation of the normal to $y^2 = 4ax$ at $(at^2, 2at)$ is

$$\boldsymbol{tx + y - at^3 - 2at = 0}$$

Example 3 *Show that the equation of the tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is $yy_1 = 2a(x + x_1)$.*

Differentiating both sides of $y^2 = 4ax$ with respect to x, to find the gradient:

$$2y\frac{dy}{dx} = 4a$$

Therefore at (x_1, y_1), $\dfrac{dy}{dx} = \dfrac{4a}{2y_1} = \dfrac{2a}{y_1}$, and the tangent is

$$2ax - y_1y = 2ax_1 - y_1^2$$
$$\therefore y_1y = 2ax - 2ax_1 + y_1^2$$

Since (x_1, y_1) lies on the parabola, $y_1^2 = 4ax_1$.

$$\therefore y_1y = 2ax - 2ax_1 + 4ax_1$$
$$\therefore \boldsymbol{yy_1 = 2a(x + x_1)}$$

Question

Q4 Find the equation of the normal to $y^2 = 4ax$ at (x_1, y_1).

Example 4 *Find the equation of the chord joining the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$.*

The gradient of the chord is

$$\frac{2at_1 - 2at_2}{at_1^2 - at_2^2} = \frac{2a(t_1 - t_2)}{a(t_1 - t_2)(t_1 + t_2)}$$
$$= \frac{2}{t_1 + t_2}$$

Therefore the equation of the chord is

$$2x - (t_1 + t_2)y = 2at_1^2 - (t_1 + t_2) \times 2at_1$$
$$= -2at_1t_2$$

Therefore the equation of the chord is

$$\boldsymbol{2x - (t_1 + t_2)y + 2at_1t_2 = 0.}$$

Question

***Q5** As $t_2 \to t_1$, the chord approaches the tangent at t_1. Deduce the equation of the tangent from the equation of the chord.

Focal chords

Definitions

Any chord of a parabola passing through the focus is called a **focal chord**.
The axis of symmetry is usually called the **axis** *of the parabola. The focal chord perpendicular to the axis is called the* **latus rectum**.

To find the length of the latus rectum of the parabola $y^2 = 4ax$, substitute $x = a$;

$$y^2 = 4a^2$$
$$\therefore y = \pm 2a$$

Hence the length of the latus rectum is $4a$.

Work all the questions in Exercise 30a, using the parametric coordinates $(at^2, 2at)$ whenever the opportunity arises. The point $(at^2, 2at)$ is frequently abbreviated to the point t.

Exercise 30a

1 Find the coordinates of the point of intersection of the tangents at the points t_1, t_2 of the parabola $y^2 = 4ax$.

2 Points t_1, t_2 lie on the parabola $y^2 = 4ax$. Find a relation connecting t_1, t_2 if the line joining the points is a focal chord.

3 Prove that the tangents at the ends of a focal chord of a parabola are perpendicular.

4 Find the focus of the parabola $x^2 = 2y$.

5 Find the equation of a parabola whose focus is $(2, 0)$ and directrix $y = -2$.

6 Find the equation of the parabola whose focus is $(-1, 1)$ and directrix $x = y$.

7 Find the gradient of the normal to the parabola $y^2 = 4ax$ at $P(at^2, 2at)$ and the gradient of the chord joining P to $(at_1^2, 2at_1)$. Deduce the coordinates of the point where the normal at P cuts the parabola again.

8 Prove that the foot of the perpendicular from the focus of a parabola on to any tangent lies on the tangent at the vertex.

9 Find the points on the parabola $y^2 = 8x$ where **a** the tangent and **b** the normal are parallel to the line $2x + y = 1$.

10 The tangents at the end of a focal chord meet each other at P and the tangent at the vertex at Q and R. Show that the centroid of the triangle PQR lies on the line $3x + a = 0$.

11 Find the point of intersection of the normals at the points t_1, t_2 of the parabola $y^2 = 4ax$.

12 Prove that, in general, from any point (h, k) three normals can be drawn to a parabola.

13 If the normals from a point (h, k) meet the parabola $y^2 = 4ax$ at the three points t_1, t_2, t_3, show that $t_1 + t_2 + t_3 = 0$.

14 PQ is a variable chord of a parabola. If the chords joining the vertex A to P and Q are perpendicular, show that PQ meets the axis of the parabola in a fixed point R, and find the length of AR.

15 Find the equations of the tangents to the parabola $y^2 = 4ax$ from the point $(16a, 17a)$.

16 If the tangents at the end of a focal chord of a parabola meet the tangent at the vertex in C and D, prove that CD subtends a right angle at the focus.

Further examples on the parabola

Example 5 *Find the focus and directrix of the parabola $y^2 = 2a(x - 4a)$ and give the length of its latus rectum.*

The equation $y^2 = 2a(x - 4a)$ may be written in the form

$$Y^2 = 2aX$$

by the substitutions $y = Y$, $x - 4a = X$. We have thus taken new axes as shown in Fig. 30.3.

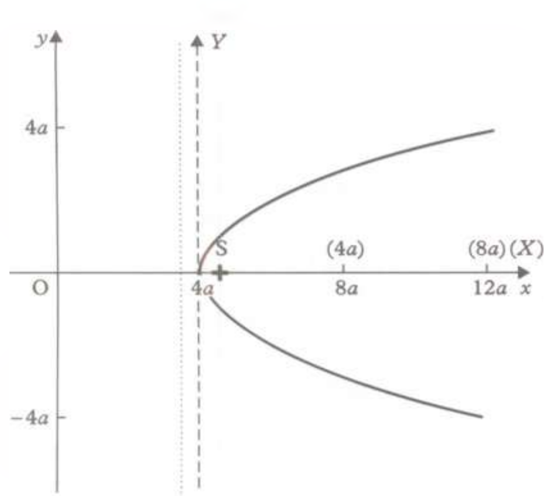


Figure 30.3

The parabola $y^2 = 4bx$ has focus $(b, 0)$, directrix $x = -b$ and latus rectum $4b$. Comparing this with $Y^2 = 2aX$, it follows that the latter has focus $(\frac{1}{2}a, 0)$, directrix $X = -\frac{1}{2}a$ and latus rectum $2a$. Therefore, with the original axes (see Fig. 30.3), the focus is $(9a/2, 0)$, the directrix $x = 7a/2$ and latus rectum $2a$.

Example 6 *Show that the equation $y = 5x - 2x^2$ represents a parabola and find the length of its latus rectum.*

Express the equation in the parabolic form $X^2 = -4aY$. The equation may be written as

$$x^2 - \frac{5}{2}x = -\frac{y}{2}$$

$$\therefore \left(x - \frac{5}{4}\right)^2 = \left(\frac{5}{4}\right)^2 - \frac{y}{2}$$

$$\therefore \left(x - \frac{5}{4}\right)^2 = -\frac{1}{2}\left(y - \frac{25}{8}\right)$$

This is now in the form $X^2 = -4aY$, giving the latus rectum as length $\frac{1}{2}$.

Question

Q6 Find the coordinates of the focus and the equation of the directrix in Example 6.

Example 7 *If the line $lx + my + n = 0$ touches the parabola $y^2 = 4ax$, find the equation connecting l, m, n, a.*

Since any tangent to $y^2 = 4ax$ may be written

$$x - ty + at^2 = 0 \qquad (1)$$

let it represent the same tangent as

$$lx + my + n = 0 \qquad (2)$$

Comparing coefficients,

$$\frac{1}{l} = -\frac{t}{m} = \frac{at^2}{n}$$

$$\therefore t = -\frac{m}{l} \quad \text{and} \quad t^2 = \frac{n}{al}$$

$$\therefore \frac{m^2}{l^2} = \frac{n}{al}$$

Therefore the equation is $am^2 = ln$.

The next two examples show the use of symmetrical relationships between t_1, t_2. When symmetry exists, the working becomes easier. Care should be taken to use symmetrical equations and expressions.

Example 8 *A chord of the parabola $y^2 = 4ax$ subtends a right angle at the vertex. Find the locus of the mid-point of the chord.*

Let the ends of the chord be $P_1(at_1^2, 2at_1)$, $P_2(at_2^2, 2at_2)$. Then the gradient of the line joining the vertex O(0, 0) to P_1 is

$$\frac{2at_1}{at_1^2} = \frac{2}{t_1}$$

Similarly the gradient of OP_2 is $\dfrac{2}{t_2}$.

P_1P_2 subtends a right angle at O if OP_1, OP_2, are perpendicular,

$$\therefore \frac{2}{t_1} \times \frac{2}{t_2} = -1$$

$$\therefore t_1t_2 = -4 \qquad (1)$$

The mid-point of P_1P_2 is given by

$$x = \frac{a(t_1^2 + t_2^2)}{2} \qquad (2)$$

$$y = a(t_1 + t_2) \qquad (3)$$

[Note that we have three equations, (1), (2), (3), from which to eliminate the two parameters t_1, t_2. Note, also, that these equations are symmetrical in t_1, t_2. Here, as is often the case, we use the following identity.]

$$(t_1 + t_2)^2 = t_1^2 + t_2^2 + 2t_1t_2$$

Substituting from equations (3), (2), (1):

$$\frac{y^2}{a^2} = \frac{2x}{a} - 8$$

Therefore the locus is $y^2 = 2a(x - 4a)$.

Example 9 *Show that the equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is*

$$y + tx = 2at + at^2$$

The normal at a point $P(ap^2, 2ap)$ meets the x-axis at G. Find the coordinates of the point G. H is the point on PG produced, such that PG = GH. Find the coordinates of H in terms of p and show that H lies on the parabola $y^2 = 4a(x - 4a)$.

The gradient of the parabola at the point $(at^2, 2at)$ is given by

$$\frac{dy}{dx} = \frac{dy}{dt} \bigg/ \frac{dx}{dt} = \frac{2a}{2at} = \frac{1}{t}$$

Hence the gradient of the normal is $-t$, and consequently the equation of the normal at the point $(at^2, 2at)$ is

$$y + tx = 2at + at^3$$

Fig. 30.4 shows the normal at the point $(ap^2, 2ap)$. Its equation is

$$y + px = 2ap + ap^3$$

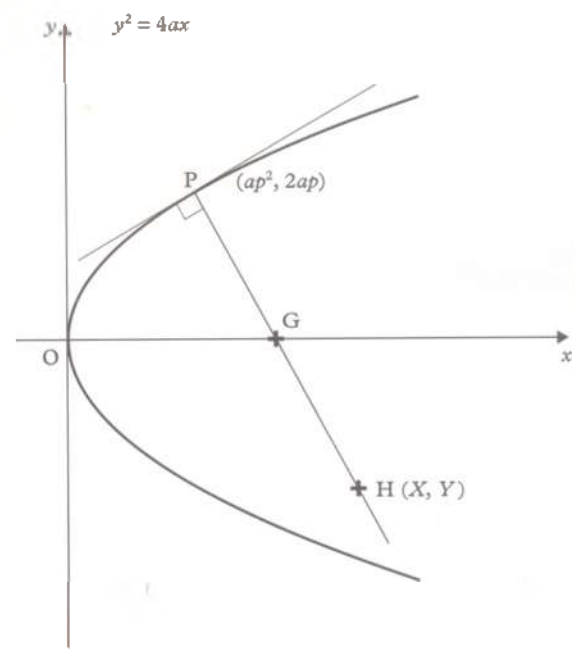


Figure 30.4

To obtain the x-coordinate of G put $y = 0$.
Therefore at G

$$px = 2ap + ap^3$$
i.e. $$x = 2a + ap^2$$

Hence G is the point $(2a + ap^2, 0)$.

Let H be the point (X, Y). Then since G is the mid-point of PH,

$$\tfrac{1}{2}(ap^2 + X) = 2a + ap^2$$
$$\therefore\ ap^2 + X = 4a + 2ap^2$$
$$\therefore\ X = 4a + ap^2 \qquad (1)$$

Similarly,

$$\tfrac{1}{2}(2ap + Y) = 0$$
$$\therefore\ 2ap + Y = 0$$
$$\therefore\ Y = -2ap \qquad (2)$$

Hence H is the point $(4a + ap^2, -2ap)$.

Eliminating p from equations (1) and (2) gives

$$X = 4a + a\left(-\frac{Y}{2a}\right)^2$$
$$= 4a + \frac{Y^2}{4a}$$
$$\therefore\ Y^2 = 4a(X - 4a)$$

Hence the point H lies on the curve $y^2 = 4a(x - 4a)$. This is the equation of a parabola, with its vertex at $(4a, 0)$.

Example 10 *A variable tangent is drawn to the parabola $y^2 = 4ax$. If the perpendicular from the vertex meets the tangent at P, find the locus of P.*

Let the variable tangent be

$$x - ty + at^2 = 0 \qquad (1)$$

Then the perpendicular from the vertex $(0, 0)$ is

$$tx + y = 0 \qquad (2)$$

$P(x, y)$ satisfies equations (1), (2). Therefore the locus of P is found by eliminating t from these equations. [Note that it is *not* necessary to solve them to find the coordinates of P in terms of t.]

From (2), $t = -y/x$. Substituting in (1),

$$x + \frac{y^2}{x} + a\frac{y^2}{x^2} = 0$$

So the locus of P is $x^3 + xy^2 + ay^2 = 0$.

Exercise 30b

1 Show that the equation $x^2 + 4x - 8y - 4 = 0$ represents a parabola whose focus is at $(-2, 1)$. Find the equation of the tangent at the vertex.

2 Prove that $x = 3t^2 + 1$ and $y = \frac{1}{2}(3t + 1)$ are the parametric equations of a parabola and find its vertex and the length of the latus rectum.

3 Find the focus of the parabola $y = 2x^2 + 3x - 5$.

4 Prove that the line $y = mx + \frac{3}{4}m + 1/m$ touches the parabola $y^2 = 4x + 3$ whatever the value of m.

5 If $ax + by + c = 0$ touches the parabola $x^2 = 4y$, find an equation connecting a, b, c.

6 A parabola, symmetrical about the axis of y, passes through the points $(1, 3)$ and $(2, 0)$. Find its equation and that of the tangent at $(1, 3)$.

7 Prove that the circles which are drawn on a focal chord of a parabola as diameter touch the directrix.

8 A variable chord of the parabola $y^2 = 4ax$ has a fixed gradient k. Find the locus of the mid-point.

9 A chord of the parabola $y^2 = 4ax$ is drawn to pass through the point $(-a, 0)$. Find the locus of the point of intersection of the tangents at the ends of the chord.

10 The difference of the ordinates of two points on the parabola $y^2 = 4ax$ is constant and equal to k. Find the locus of the point of intersection of the tangents at the two points.

11 Find the locus of the mid-points of focal chords of the parabola $y^2 = 4ax$.

12 The tangent at any point P of the parabola $y^2 = 4ax$ meets the tangent at the vertex at the point Q. S is the focus and SQ meets the line through P parallel to the tangent at the vertex at the point R. Find the locus of R.

13 Show that $y = ax^2 + bx + c$ is the equation of a parabola. Find its focus and directrix.

14 Two tangents to the parabola $y^2 = 4ax$ pass through the point (x_1, y_1). Find the equation of their chord of contact.

15 Find the points of contact on the parabola of the tangents common to the circle $(x-a)^2+y^2=4a^2$ and the parabola $y^2=4ax$. [Start by writing down the equation of the tangent at $(at^2, 2at)$.]

16 The normal at the point P of the parabola $y^2=4ax$ meets the curve again at Q. The circle on PQ as diameter goes through the vertex. Find the x-coordinate of P.

17 Prove that rays of light parallel to the axis of a parabolic mirror are reflected through the focus.

18 A variable chord of the parabola $y^2=4ax$ passes through the point (h, k). Find the locus of the orthocentre of the triangle formed by the chord and the tangents at the two ends.

19 A tangent to the parabola $y^2=4ax$ meets the parabola $y^2=8ax$ at P and Q. Find the locus of the mid-point of PQ.

20 Find the locus of the mid-point of a variable chord through the point $(a, 2a)$ of the parabola $y^2=4ax$.

30.2 The ellipse

An ellipse is defined on page 307. Given a fixed point S, the *focus*, and a fixed line, the *directrix*, if P is a point on the locus and M is the foot of the perpendicular from P to the directrix, then

$$\frac{\text{SP}}{\text{PM}}=e \qquad (e<1)$$

e is called the *eccentricity* of the ellipse.

Question

Q7 On a sheet of squared paper, rule the directrix along one line near the edge, take the focus 2.7 cm in and plot an ellipse with a pair of compasses, taking $e=4/5$. Measure the width of the ellipse parallel and perpendicular to the directrix.

The result of Q7 should be an enlargement of Fig. 30.5. It follows from the definition that an ellipse is symmetrical about the line through S perpendicular to the directrix, so we take the x-axis along this axis of symmetry.

Let the x-axis cut the ellipse in A′, A, as in Fig. 30.5.

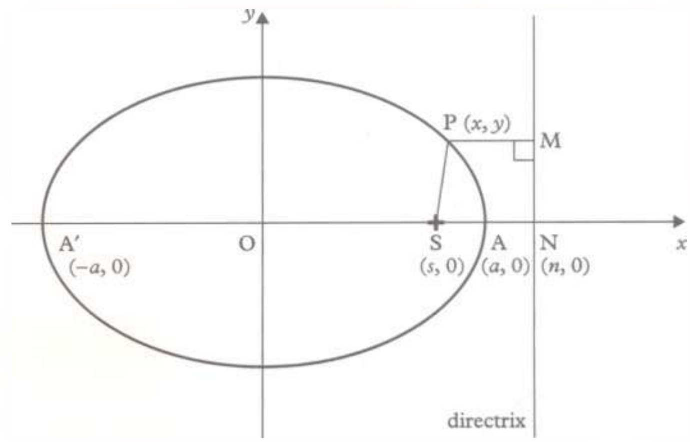


Figure 30.5

It appears that there is an axis of symmetry parallel to the directrix, so we shall take the y-axis passing through the mid-point of A′A, parallel to the directrix.

Let A be $(a, 0)$ so that A′ is $(-a, 0)$. Let S be $(s, 0)$ and let the x-axis cut the directrix at N$(n, 0)$. We shall now find s, n in terms of a, e.

A′ and A lie on the ellipse and so, by the definition of the locus,

$$\frac{\text{SA}'}{\text{A}'\text{N}}=e \quad \text{and} \quad \frac{\text{SA}}{\text{AN}}=e$$

Hence

$$a+s=e(n+a)$$
$$a-s=e(n-a)$$

Adding,

$$2a=2en \qquad \therefore\ n=\frac{a}{e}$$

Subtracting,

$$2s=2ae \qquad \therefore\ s=ae$$

Therefore S is the point $(ae, 0)$ and the equation of the directrix is $x=a/e$.

To find the equation of the ellipse, let P(x, y) be any point on the locus, then

$$\frac{\text{SP}}{\text{PM}}=e$$

$$\therefore\ \text{SP}^2=e^2\text{PM}^2$$

But $\text{SP}^2=(x-ae)^2+y^2$, and $\text{PM}=(a/e)-x$.

$$\therefore\ (x-ae)^2+y^2=e^2\left(\frac{a}{e}-x\right)^2$$
$$\therefore\ x^2-2aex+a^2e^2+y^2=a^2-2aex+e^2x^2$$
$$\therefore\ x^2(1-e^2)+y^2=a^2(1-e^2)$$
$$\therefore\ \frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$$

Therefore the **equation of the ellipse** is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{where} \qquad b^2=a^2(1-e^2)$$

Note that we have also found that the focus S is $(ae, 0)$ and the directrix is $x = a/e$. Notice that since the equation of the ellipse is unaltered by replacing x by $-x$, it follows that there is another focus $(-ae, 0)$ and another directrix $x = -a/e$. Hence

the foci are $(ae, 0)$ and $(-ae, 0)$

the directrices are $x=\frac{a}{e}$ and $x=-\frac{a}{e}$

(Note: *directrices* is the plural of *directrix*.)

The axes of symmetry meet at the **centre** of the ellipse. Any chord passing through the centre is called a **diameter**.

The diameter through the foci is the **major axis** and the perpendicular diameter is called the **minor axis**.

Questions

Q8 Show that the lengths of the axes are $2a$, $2b$.

Q9 Find the length of the semi-axes of the ellipse $x^2/16 + y^2/9 = 1$.

Q10 Find the eccentricity of the ellipse $x^2/25 + y^2/16 = 1/4$.

Q11 Find the foci of the ellipse $x^2 + 4y^2 = 9$.

Parametric coordinates for an ellipse

When dealing with an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

working is generally made easier by using a parameter. An equation in the form

$$(\quad)^2+(\quad)^2=1$$

suggests the identity

$$\cos^2\theta+\sin^2\theta=1$$

Thus, if we write $x = a\cos\theta$, $y = b\sin\theta$, the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

will always be satisfied. We therefore take as a general point on the ellipse

$(a\cos\theta, b\sin\theta)$

θ is called the **eccentric angle** of the point.

In §22.5 on page 245, we saw that the parameter θ for a circle in

$$x=a\cos\theta, \qquad y=a\sin\theta$$

could be interpreted in terms of an angle. This is not so simple for an ellipse but it can be done as follows.

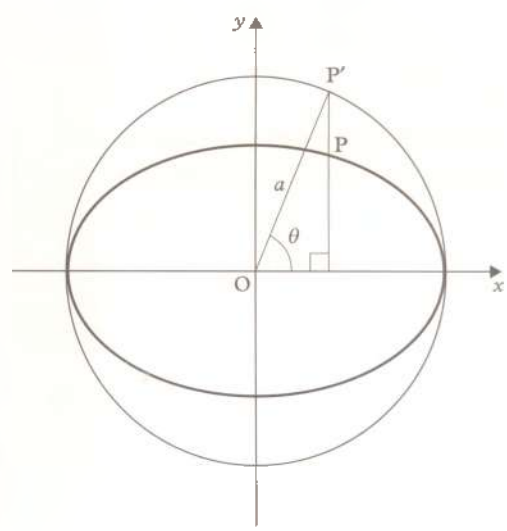


Figure 30.6

In Fig. 30.6, P is the point $(a\cos\theta, b\sin\theta)$ on the ellipse $x^2/a^2 + y^2/b^2 = 1$ and P′ is a point on the circle $x^2 + y^2 = a^2$ (called the auxiliary circle) such that OP′ makes an angle θ with Ox. Since P, P′ have the same x-coordinate, $a\cos\theta$, PP′ is perpendicular to the major axis of the ellipse. Therefore the eccentric angle θ of any point P is found as follows: draw the ordinate of P to meet the auxiliary circle at P′, join P′ to the origin, then OP′ makes an angle θ with the positive x-axis.

Question

Q12 Show how to obtain the y-coordinate of the point $(a\cos\theta, b\sin\theta)$ from a circle of radius b. Draw two concentric circles and hence plot an ellipse.

Example 11 *Find the equation of the tangent to the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *at* $(a\cos\theta, b\sin\theta)$.

The gradient $\frac{dy}{dx}=\frac{dy}{d\theta}\Big/\frac{dx}{d\theta}$.

$$x=a\cos\theta, \qquad y=b\sin\theta$$

$$\therefore \frac{dy}{dx}=\frac{b\cos\theta}{-a\sin\theta}.$$

Therefore the equation of the tangent is

$$b\cos\theta\, x+a\sin\theta\, y$$
$$=b\cos\theta\times a\cos\theta+a\sin\theta\times b\sin\theta$$
$$\therefore bx\cos\theta+ay\sin\theta=ab(\cos^2\theta+\sin^2\theta)$$

Therefore the tangent is

$$\mathbf{bx\cos\theta+ay\sin\theta-ab=0}$$

Question

***Q13** Show that the equation of the normal to the ellipse $x^2/a^2+y^2/b^2=1$ at $(a\cos\theta, b\sin\theta)$ is

$$\mathbf{ax\sin\theta-by\cos\theta-(a^2-b^2)\sin\theta\cos\theta=0}$$

If the general point on the curve is taken to be (x_1, y_1), it is frequently necessary to use the extra equation

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$$

This is why working is usually easier when parameters are used.

Question

***Q14** Show that the equation of the tangent at (x_1, y_1) to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is

$$\frac{\mathbf{xx_1}}{\mathbf{a^2}}+\frac{\mathbf{yy_1}}{\mathbf{b^2}}=1$$

Verify that this gives the equation found in Example 11 for the tangent at $(a\cos\theta, b\sin\theta)$.

Example 12 *Find the equation of the chord of the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

joining the points whose eccentric angles are θ, ϕ.

The ends of the chord are $(a\cos\theta, b\sin\theta)$, $(a\cos\phi, b\sin\phi)$, therefore the gradient of the chord is

$$\frac{b\sin\theta-b\sin\phi}{a\cos\theta-a\cos\phi}=\frac{2b\cos\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\theta-\phi)}{-2a\sin\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\theta-\phi)}$$
$$=-\frac{b\cos\frac{1}{2}(\theta+\phi)}{a\sin\frac{1}{2}(\theta+\phi)}$$

Therefore the equation of the chord is

$$b\cos\tfrac{1}{2}(\theta+\phi)x+a\sin\tfrac{1}{2}(\theta+\phi)y$$
$$=b\cos\tfrac{1}{2}(\theta+\phi)\times a\cos\theta$$
$$+a\sin\tfrac{1}{2}(\theta+\phi)\times b\sin\theta$$

$$\text{R.H.S.}=ab\{\cos\tfrac{1}{2}(\theta+\phi)\cos\theta+\sin\tfrac{1}{2}(\theta+\phi)\sin\theta\}$$
$$=ab\cos\{\tfrac{1}{2}(\theta+\phi)-\theta\}$$
$$=ab\cos\tfrac{1}{2}(\phi-\theta)$$
$$=ab\cos\tfrac{1}{2}(\theta-\phi)$$

Therefore the equation of the chord is

$$\mathbf{bx\cos\tfrac{1}{2}(\theta+\phi)+ay\sin\tfrac{1}{2}(\theta+\phi)}$$
$$\mathbf{-ab\cos\tfrac{1}{2}(\theta-\phi)=0}$$

Question

***Q15** Show, by putting $\phi=\theta$, that the equation of the chord approaches the equation of the tangent at θ as $\phi\to\theta$.

Example 13 *A tangent to the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

at the point P *meets the minor axis at* L. *If the normal at* P *meets the major axis at* M, *find the locus of the mid-point* of LM.

Let P be the point $(a\cos\theta, b\sin\theta)$, then the tangent at P has equation

$$bx\cos\theta+ay\sin\theta-ab=0$$

This meets the minor axis $x=0$ at L$\left(0, \frac{b}{\sin\theta}\right)$.

The normal at P is

$$ax \sin\theta - by \cos\theta - (a^2 - b^2)\sin\theta\cos\theta = 0$$

This meets the major axis $y = 0$ at

$$\mathrm{M}\left(\frac{a^2 - b^2}{a}\cos\theta, 0\right).$$

The mid-point of LM is given by

$$x = \frac{a^2 - b^2}{2a}\cos\theta \qquad y = \frac{b}{2\sin\theta}$$

θ can be eliminated from these equations by means of the identity

$$\cos^2\theta + \sin^2\theta = 1$$

Therefore the locus of the mid-point of LM is

$$\left(\frac{2ax}{a^2 - b^2}\right)^2 + \left(\frac{b}{2y}\right)^2 = 1$$

Exercise 30c

1 Find the foci and directrices of the ellipse
  a $4x^2 + 9y^2 = 36$ b $x^2 + 16y^2 = 25$

2 Write down the equation of the tangent to
  a $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ at $(3\cos\theta, 2\sin\theta)$
  b $9x^2 + 16y^2 = 25$ at $(1, 1)$

3 Find the equation of the normal to
  a $9x^2 + 16y^2 = 25$ at $(1, 1)$
  b $x^2 + 2y^2 = 9$ at $(1, -2)$

4 A point moves so that its distance from $(3, 2)$ is half its distance from the line $2x + 3y = 1$.
Why is the locus an ellipse?
Find the equation of the major axis.

5 P is any point on an ellipse; S and S′ are its foci. Prove directly from the focus-directrix definition of the ellipse that SP + S′P = $2a$, where $2a$ is the length of the major axis.

6 Find the relation between the eccentric angles of the points which are at the ends of a focal chord.

7 Prove that the chord joining points of an ellipse whose eccentric angles are $(\alpha + \beta)$, $(\alpha - \beta)$ is parallel to the tangent at the point whose eccentric angle is α.

8 Find the equation of the tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the end of the latus rectum which lies in the first quadrant.

9 The tangent at P to an ellipse meets a directrix at Q. Prove that lines joining the corresponding focus to P and Q are perpendicular.

10 Find the coordinates of the point of intersection of the tangents to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the points whose eccentric angles are θ, ϕ.

11 P is any point on an ellipse, and S and S′ are its foci. Prove that the normal at P bisects the angle S′PS.

12 Find the locus of the mid-point of the line joining the focus $(ae, 0)$ to any point on the ellipse $x^2/a^2 + y^2/b^2 = 1$.

13 The eccentric angles of two points P and Q differ by a constant k. Find the locus of the mid-point of PQ.

14 The normal at the point $(a\cos\theta, b\sin\theta)$ on the ellipse $x^2/a^2 + y^2/b^2 = 1$ meets the axes at L and M. Find the locus of the mid-point of LM.

15 A variable tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ meets the axes at R and S.
Find the locus of the mid-point of RS.

16 Prove that the tangents to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at points whose eccentric angles differ by a right angle meet on a concentric ellipse and find its equation.

17 Prove that perpendicular tangents to the ellipse $x^2/a^2 + y^2/b^2 = 1$ meet on the circle $x^2 + y^2 = a^2 + b^2$ (called the *director circle*).

18 The tangents to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at P and Q meet at the point (x_1, y_1).
Show that the equation of the chord of contact PQ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

[Use the results of question 10 and Example 12.]

Further examples on the ellipse

Example 14 *Find the condition that the line $y = mx + c$ should touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.*

The equation of any tangent to the ellipse may be written

$$bx\cos\theta + ay\sin\theta - ab = 0$$

Let this equation represent the same tangent as the given line, written as

$$mx - y + c = 0$$

Comparing coefficients,

$$\frac{b\cos\theta}{m} = \frac{a\sin\theta}{-1} = \frac{-ab}{c}$$

$$\therefore \cos\theta = -\frac{am}{c}, \qquad \sin\theta = \frac{b}{c}$$

But $\cos^2\theta + \sin^2\theta = 1$.

$$\therefore \frac{a^2m^2}{c^2} + \frac{b^2}{c^2} = 1$$

Therefore $y = mx + c$ touches the ellipse if

$$\boldsymbol{c^2 = a^2m^2 + b^2}$$

Question

Q16 Re-work Example 14 by eliminating y between the two equations, using the condition that the resulting quadratic in x should have equal roots.

Example 15 *Prove that perpendicular tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meet on a circle and find its equation.*

From Example 14 we see that the equation of a general tangent to the ellipse may be written

$$y = mx + (a^2m^2 + b^2)^{1/2}$$

$$\therefore (y - mx)^2 = a^2m^2 + b^2$$

$$\therefore m^2(x^2 - a^2) - 2xym + y^2 - b^2 = 0 \qquad (1)$$

If (x, y) is a point of intersection of two perpendicular tangents to the ellipse, we may regard equation (1) is a quadratic in m, the gradient of the tangents. Since the tangents are perpendicular the product of the roots of the equation is -1,

$$\therefore \frac{y^2 - b^2}{x^2 - a^2} = -1$$

$$\therefore y^2 - b^2 = a^2 - x^2$$

This is a circle with equation

$$x^2 + y^2 = a^2 + b^2$$

It is called the **director circle** of the ellipse.

Example 16 *A variable straight line with constant gradient m meets the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at Q and R. Find the locus of P, the mid-point of QR.

Let the equation of the line be

$$y = mx + c \qquad (1)$$

To find the coordinates of Q and R, we solve the equation of the line and the equation of the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2$$

simultaneously:

$$b^2x^2 + a^2(m^2x^2 + 2mxc + c^2) - a^2b^2 = 0$$

$$\therefore x^2(b^2 + a^2m^2) + 2a^2mcx + a^2c^2 - a^2b^2 = 0$$

The x-coordinates of Q and R, say x_1 and x_2, are the roots of this equation. But if P is the point (X, Y),

$$X = \tfrac{1}{2}(x_1 + x_2)$$

$$\therefore X = \tfrac{1}{2} \times \frac{-2a^2mc}{b^2 + a^2m^2} \qquad (2)$$

The coordinates of P satisfy equation (1), so

$$Y = mX + c \qquad (3)$$

Find the locus of P by eliminating c between equations (2), (3). Substituting

$$c = Y - mX$$

in equation (2) rearranged as

$$X(b^2 + a^2m^2) = -a^2mc$$

we obtain

$$X(b^2 + a^2m^2) = -a^2m(Y - mX)$$

Therefore the locus of P is

$$b^2x + a^2my = 0$$

which is a diameter of the ellipse.

Exercise 30d

1 Write down the equations of the tangents to
- **a** $x^2/4 + y^2/9 = 1$ with gradient 2
- **b** $x^2 + 3y^2 = 3$ with gradient -1
- **c** $4x^2 + 9y^2 = 144$ with gradient $\frac{1}{2}$

2 *Without* solving the equations completely, find the coordinates of the mid-points of the chords formed by the intersection of
- **a** $x - y - 1 = 0$ and $x^2/9 + y^2/4 = 1$
- **b** $10x - 5y + 6 = 0$ and $4x^2 + 5y^2 = 20$
- **c** $2x + 3y - 4 = 0$ and $y^2 = 8x$

3 Prove that the line $x - 2y + 10 = 0$ touches the ellipse $9x^2 + 64y^2 = 576$.

4 Find the equations of the tangents to the ellipse $x^2 + 4y^2 = 4$ which are perpendicular to the line $2x - 3y = 1$.

5 The line $y = x - c$ touches the ellipse $9x^2 + 16y^2 = 144$. Find the value of c and the coordinates of the point of contact.

6 Find the condition for the line $y = mx + c$ to cut the ellipse $x^2/a^2 + y^2/b^2 = 1$ in two distinct points.

7 The line $y = mx + c$ touches the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Prove that the foot of the perpendicular from a focus on to this line lies on the auxiliary circle $x^2 + y^2 = a^2$.

8 Find the locus of the foot of the perpendicular from the centre of the ellipse $x^2/a^2 + y^2/b^2 = 1$ on to any tangent.

9 Find the equation of the normal at the point (x_1, y_1) on the ellipse $x^2/a^2 + y^2/b^2 = 1$.

10 Find the coordinates of the mid-point of the chord formed by the intersection of
- **a** $y = mx + c$ and $b^2x^2 + a^2y^2 = a^2b^2$
- **b** $lx + my + n = 0$ and $y^2 = 4ax$

11 Find the equation of the diameter bisecting the chord $3x + 2y = 1$ of the ellipse $4x^2 + 9y^2 = 16$.

12 Find the equation of the line with gradient m passing through the focus $(ae, 0)$ of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

If the line meets the ellipse in P and Q, find the coordinates of the mid-point of PQ and show that they satisfy the equation

$$a^2my + b^2x = 0$$

Substitute the value of m obtained from this equation into the equation of PQ, and hence find the locus of the mid-point of PQ.

13 A variable line passes through the point $(a, 0)$. Find the locus of the mid-point of the chord formed by the intersection of this line and the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

14 Find the locus of points from which the tangents to the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2$$

are inclined at 45°.

15 Lines of gradient m are drawn to cut the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

Prove that the mid-points of the chords so formed lie on a straight line through the origin with gradient $-b^2/(a^2m)$. Deduce the equation of the chord whose mid-point is (h, k).

16 Show that a general tangent to the circle $x^2 + y^2 - a^2 = 0$ may be written

$$y = mx \pm a\sqrt{(1 + m^2)}$$

A variable tangent to the circle $x^2 + y^2 - a^2 = 0$ meets the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ $(b > a)$† at P and Q. Find the locus of the mid-point of PQ.

†I.e. the major axis lies along the y-axis.

17 A variable tangent to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ meets the parabola $y^2 = 4ax$ at L and M. Find the locus of the mid-point of LM.

18 The chord of contact of the point (x_1, y_1) with respect to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

cuts the axes at L and M. If the locus of the mid-point of LM is the circle $x^2 + y^2 = 1$, find the locus of (x_1, y_1). [Use the result of Exercise 30c on page 316.]

30.3 The hyperbola

In §30.2 on pages 314–315, key results were obtained for the ellipse. The working is so similar for the hyperbola that it is left to you to obtain the corresponding results. Starting with the focus–directrix definition with $e > 1$ work through the following questions.

Questions

Q17 On a sheet of squared paper, draw a vertical directrix along a line near the middle. Take the focus 4 cm from the directrix along the x-axis and plot part of a hyperbola (there are two branches) taking $e = 2$.

Q18 Show that, with suitable choice of axes, **the equation of a hyperbola** may be written

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $b^2 = a^2(e^2 - 1)$,
the foci are $(ae, 0)$ and $(-ae, 0)$,
and the directrices $x = \frac{a}{e}$ and $x = -\frac{a}{e}$.

Fig. 30.9 on page 321 shows the general shape of a hyperbola.

Q19 Show that any point on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ may be written

$$(a \sec\theta, b\tan\theta)$$

Q20 Show that at $(a\sec\theta, b\tan\theta)$ on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

the equation of the tangent is

$$bx - ay\sin\theta - ab\cos\theta = 0$$

and the equation of the normal is

$$ax\sin\theta + by - (a^2 + b^2)\tan\theta = 0$$

Q21 Show that the equation of the tangent at (x_1, y_1) to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

Show that the equation of the tangent in Q20 may be deduced from this.

Asymptotes to a hyperbola

Example 17 *Find c in terms of a, b, m if $y = mx + c$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.*

Solving the two equations simultaneously,

$$b^2x^2 - a^2y^2 = a^2b^2$$
$$\therefore\ b^2x^2 - a^2(m^2x^2 + 2mcx + c^2) - a^2b^2 = 0$$
$$\therefore\ x^2(b^2 - a^2m^2) - 2a^2mcx - a^2(b^2 + c^2) = 0 \qquad (1)$$

The line is a tangent if and only if this equation has equal roots,

i.e. if and only if

$$(-2a^2mc)^2 = -4(b^2 - a^2m^2)a^2(b^2 + c^2)$$
$$a^2m^2c^2 = -(b^2 - a^2m^2)(b^2 + c^2)$$
$$a^2m^2c^2 = -b^4 - b^2c^2 + a^2m^2b^2 + a^2m^2c^2$$
$$b^2c^2 = a^2m^2b^2 - b^4$$

Therefore $y = mx + c$ is a tangent to the hyperbola if and only if

$$\boldsymbol{c^2 = a^2m^2 - b^2}$$

[Compare this with the method of Example 14.]

In Example 17, since the roots are equal the value of x at the point of contact is given by half the sum of the roots of equation (1).

$$\therefore\ x = \frac{a^2mc}{b^2 - a^2m^2}$$
$$= \mp\frac{a^2m\sqrt{(a^2m^2 - b^2)}}{a^2m^2 - b^2}$$

Therefore, at the point of contact,

$$x = \mp\frac{a^2m}{\sqrt{(a^2m^2 - b^2)}}.$$

Hence as $m \to \pm b/a$, $x \to \infty$ and, since $c^2 = a^2m^2 - b^2$, $c \to 0$. Therefore

$$y = \pm\frac{b}{a}x$$

is the limit of a tangent to the hyperbola as the point of contact tends to infinity. Therefore

$$\boldsymbol{y = \pm\frac{b}{a}x}$$

represents the equations of the **asymptotes** of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

The rectangular hyperbola

There is a special hyperbola which has interesting properties. A **rectangular hyperbola** has asymptotes that are perpendicular. The asymptotes of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{are} \quad y = \pm\frac{b}{a}x.$$

These are perpendicular when

$$-\frac{b}{a} \times \frac{b}{a} = -1$$

that is, when $b = a$. Hence

$$\boldsymbol{x^2 - y^2 = a^2}$$

represents a rectangular hyperbola and its asymptotes are

$$x - y = 0, \qquad x + y = 0$$

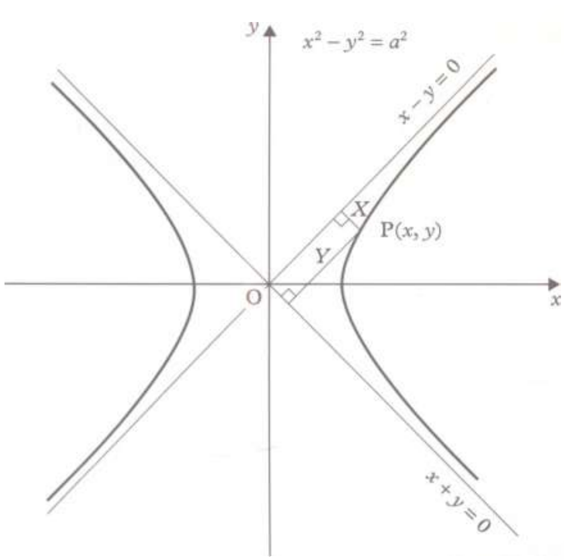


Figure 30.7

Since the asymptotes are perpendicular we can write the equation of the rectangular hyperbola in a very simple way. Let (x, y) be any point on the curve in Fig. 30.7, then

$$x^2 - y^2 = a^2$$

Note that this equation can be written

$$(x - y)(x + y) = a^2 \qquad (1)$$

If we rotate the plane through 45° the asymptotes will coincide with the axes (remember that under such transformations we always regard the axes as fixed). The two branches of the hyperbola will then occupy the first and third quadrants as in Fig. 30.8.

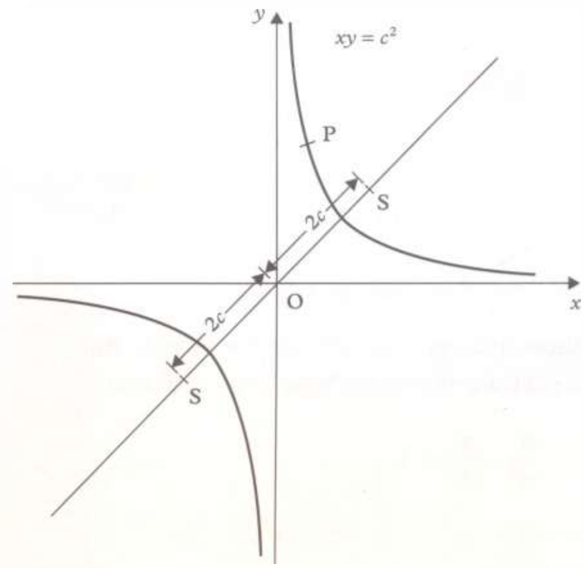


Figure 30.8

Using the matrix method described in §11.7 **d** on page 148, the point (x, y) will be mapped onto point (X, Y), where

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} x - y \\ x + y \end{pmatrix}$$

Hence $X = \frac{1}{\sqrt{2}}(x - y)$ and $Y = \frac{1}{\sqrt{2}}(x + y)$, i.e.,

$$(x - y) = \sqrt{2}X \quad \text{and} \quad (x + y) = \sqrt{2}Y$$

Substituting these expressions into equation (1) gives

$$2XY = a^2$$
$$\therefore\ XY = \tfrac{1}{2}a^2$$

Hence, with the rectangular hyperbola in its new position, any point on the curve with coordinates (x, y) satisfies the equation

$$\boldsymbol{xy = c^2} \qquad \text{where } c = \frac{1}{\sqrt{2}}a$$

The eccentricity of $x^2 - y^2 = a^2$ is given by $a^2 = a^2(e^2 - 1)$ from which we find that $e = \surd 2$ and hence the foci are $(\pm\surd 2a, 0)$. Since $a = \surd 2c$, the foci of $xy = c^2$ are on the major axis at a distance $2c$ from the centre (see Fig. 30.8). Therefore the coordinates of the foci of $xy = c^2$ are $(\surd 2c, \surd 2c)$ and $(-\surd 2c, -\surd 2c)$.

Now work through the following questions which lead to very important properties of the rectangular hyperbola.

Questions

Q22 Show that any point on the rectangular hyperbola $xy = c^2$ may be represented by

$$\left(ct, \frac{c}{t}\right)$$

Q23 Show that the gradient of the rectangular hyperbola at $(ct, c/t)$ is

$$-\frac{1}{t^2}$$

Show also that the equation of the tangent is

$$x + t^2y - 2ct = 0$$

and that the equation of the normal is

$$t^2x - y - ct^3 + \frac{c}{t} = 0$$

Q24 Show that the gradient of the chord joining the points $(ct_1, c/t_1)$, $(ct_2, c/t_2)$ on the hyperbola $xy = c^2$ is

$$-\frac{1}{t_1t_2}$$

and that the equation of the chord is

$$x + t_1t_2y - c(t_1 + t_2) = 0$$

Q25 Verify that the equation of the chord in Q24 becomes the equation of the tangent in Q23 when $t_1 = t_2 = t$.

Further examples on the hyperbola

The following examples do not illustrate any new principles. However they show that the same methods that were used for problems about the ellipse may also be used with the hyperbola.

Example 18 *A tangent to a hyperbola at P meets a directrix at Q. If S is the corresponding focus, prove that PQ subtends a right angle at S.* (Fig. 30.9.)

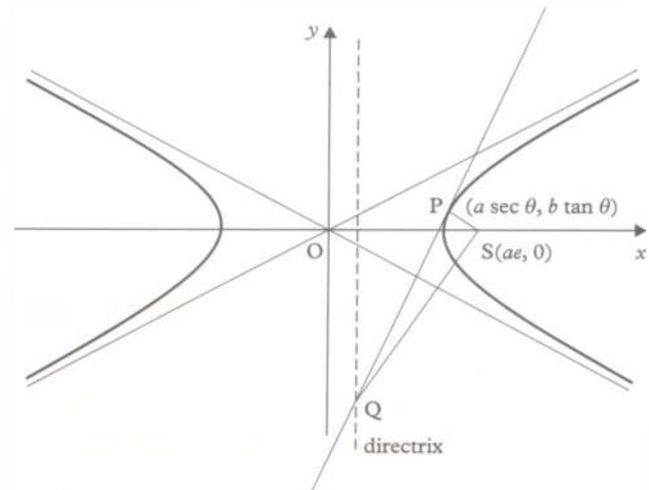


Figure 30.9

Let P be the point $(a\sec\theta, b\tan\theta)$ on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The tangent at P is

$$bx - ay\sin\theta - ab\cos\theta = 0.$$

This meets the directrix $x = a/e$ at a point given by

$$\frac{ba}{e} - ay\sin\theta - ab\cos\theta = 0.$$

$$\therefore\ y\sin\theta = \frac{b(1 - e\cos\theta)}{e}$$

$$\therefore\ \text{Q is the point}\left(\frac{a}{e}, \frac{b(1 - e\cos\theta)}{e\sin\theta}\right)$$

Therefore the gradient of QS

$$m_1 = \frac{\dfrac{b(1 - e\cos\theta)}{e\sin\theta}}{\dfrac{a}{e} - ae} = \frac{b(1 - e\cos\theta)}{a(1 - e^2)\sin\theta}$$

The gradient of PS

$$m_2 = \frac{b\tan\theta}{a\sec\theta - ae} = \frac{b\sin\theta}{a(1 - e\cos\theta)}$$

$$\therefore\ m_1m_2 = \frac{b(1 - e\cos\theta)}{a(1 - e^2)\sin\theta} \times \frac{b\sin\theta}{a(1 - e\cos\theta)}$$

$$= \frac{b^2}{a^2(1 - e^2)} = -1$$

since $b^2 = a^2(e^2 - 1)$. (See Q18 on page 319.)

Therefore SQ and SP are perpendicular and so PQ subtends a right angle at S.

Example 19 *S is a focus of the hyperbola*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and P is the foot of the perpendicular from S to a variable tangent. Find the locus of P.

$$y = mx \pm (a^2m^2 - b^2)^{1/2} \qquad (1)$$

is the equation of a general tangent to the hyperbola.

$$x + my - ae = 0 \qquad (2)$$

is the equation of the perpendicular from S to the tangent.

The coordinates of P satisfy equations (1) and (2). So we can find the locus by eliminating m between these equations.

From (1),

$$y^2 - 2mxy + m^2x^2 = a^2m^2 - b^2$$

From (2),

$$m^2y^2 + 2mxy + x^2 = a^2e^2$$

Adding,

$$y^2(1 + m^2) + x^2(1 + m^2) = a^2m^2 - b^2 + a^2e^2$$

$$\text{R.H.S.} = a^2m^2 + a^2 \quad \text{since} \quad b^2 = a^2(e^2 - 1)$$

$$\therefore y^2(1 + m^2) + x^2(1 + m^2) = a^2(1 + m^2)$$

Therefore the locus of P is $x^2 + y^2 = a^2$, which is the **auxiliary circle**.

Example 20 *PQ is a chord of the rectangular hyperbola $xy = c^2$ and R is its mid-point. If PQ has a constant length k, find the locus of R.*

Let P be $(cp, c/p)$ and Q be $(cq, c/q)$. Then, if R is (x, y),

$$x = \tfrac{1}{2}c(p + q) \qquad (1)$$

$$y = \frac{c(p + q)}{2pq} \qquad (2)$$

Since the length of PQ is k,

$$PQ^2 = (cp - cq)^2 + \left(\frac{c}{p} - \frac{c}{q}\right)^2 = k^2$$

$$\therefore c^2(p - q)^2 + c^2\frac{(q - p)^2}{p^2q^2} = k^2$$

$$\therefore c^2(p - q)^2(1 + p^2q^2) = k^2p^2q^2 \qquad (3)$$

From (1) and (2), $p + q = 2x/c$ and $pq = x/y$.

Now $(p - q)^2 \equiv (p + q)^2 - 4pq$, so that (3) becomes

$$c^2\{(p + q)^2 - 4pq\}(1 + p^2q^2) = k^2p^2q^2$$

Substituting for $p + q$ and pq,

$$c^2\left(\frac{4x^2}{c^2} - \frac{4x}{y}\right)\left(1 + \frac{x^2}{y^2}\right) = k^2\frac{x^2}{y^2}$$

$$\therefore \left(4x^2 - \frac{4c^2x}{y}\right)(y^2 + x^2) = k^2x^2$$

Therefore the locus of P is $4(xy - c^2)(x^2 + y^2) = k^2xy$.

Exercise 30e

1 P is any point on the rectangular hyperbola $xy = c^2$. Show that the line joining P to the centre, and the tangent at P, are equally inclined to the asymptotes.

2 P is any point on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ and Q is the point (a, b). Find the locus of the point dividing PQ in the ratio 2:1.

3 Prove that the product of the lengths of the perpendiculars from any point of a hyperbola to its asymptotes is constant.

4 The normal at any point of a hyperbola meets the axes at E and F. Find the locus of the mid-point of EF.

5 Find the coordinates of the point at which the normal at $(ct, c/t)$ meets the rectangular hyperbola $xy = c^2$ again.

6 Any tangent to the rectangular hyperbola $xy = c^2$ meets the asymptotes at L and M. Find the locus of the mid-point of LM.

7 The normal at any point on the hyperbola $xy = c^2$ meets the x-axis at A, and the tangent meets the y-axis at B. Find the locus of the mid-point of AB.

8 Find the equation of the chord of the hyperbola $xy = c^2$ whose mid-point is (x_1, y_1).

9 Show that, in general, four normals can be drawn from any point to the rectangular hyperbola $xy = c^2$.

10 The normal at any point P of the rectangular hyperbola $xy = c^2$ meets the y-axis at A, and the tangent meets the x-axis at B. Find the coordinates

of the fourth vertex Q of the rectangle APBQ in terms of t, the parameter of P.

11 Find the locus of the foot of the perpendicular from the origin on to a tangent to the rectangular hyperbola $xy = c^2$.

12 Find the condition that the line $lx + my + n = 0$ should touch the rectangular hyperbola $xy = c^2$.

13 Prove that the locus of the mid-points of parallel chords of the rectangular hyperbola $xy = c^2$ is a diameter.

14 Find the locus of the point of intersection of perpendicular tangents to the hyperbola $x^2/a^2 - y^2/b^2 = 1$.

15 Find the locus of the foot of the perpendicular from the origin to a tangent of the hyperbola $x^2/a^2 - y^2/b^2 = 1$.

16 PQ is a variable chord of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ with constant gradient m_1. Show that the locus of the mid-point of PQ is a diameter with gradient m_2 such that $m_1 m_2 = b^2/a^2$.

17 The chord AB of a hyperbola meets the asymptotes at M and N. Prove that AM = BN. Show that AB and MN have the same mid-point.

18 Find the equation of the chord joining the points $(a \sec\theta, b \tan\theta)$, $(a \sec\phi, b \tan\phi)$ on the hyperbola $x^2/a^2 - y^2/b^2 = 1$.

19 The tangent at any point P on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ meets the asymptotes at Q and Q′. Prove that PQ = PQ′.

20 P, Q, R are three points on a rectangular hyperbola such that PQ subtends a right angle at R. Show that PQ is perpendicular to the tangent at R.

Chapter 31

Series and approximations

Introduction

The expansion of functions of a variable as a series has considerable theoretical and practical importance. There are some problems that are most easily solved by means of series, for instance estimating the value of the constant e. Also there are problems in science and engineering which have no practicable solution except by series. The development of computers has considerably added to the importance of approximate numerical solutions to problems. So far, in Chapter 27, the function $(1 + x)^n$ has been expanded in a series. In this chapter we will consider the functions e^x, $\ln(1 + x)$, $\sin x$ and $\cos x$.

31.1 The exponential series

The fundamental property of the function e^x is that

$$\frac{d}{dx}(e^x) = e^x$$

If two assumptions are made:

a that e^x can be expanded as a series of ascending powers of x and

b that the nth derivative of such a series is the sum to infinity of the nth derivatives of the individual terms,

it is easy to find the coefficients of the terms in the series.

Suppose that

$$e^x = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n \ldots \qquad (1)$$

Differentiating (1) once, twice, and three times respectively,

$$e^x = a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1} + \ldots \qquad (2)$$

$$e^x = 2a_2 + 3 \times 2a_3x + \ldots + n(n-1)a_nx^{n-2} + \ldots \qquad (3)$$

$$e^x = 3 \times 2a_3 + \ldots + n(n-1)(n-2)a_nx^{n-3} + \ldots \qquad (4)$$

Differentiating (1) n times,

$$e^x = n!a_n + \ldots \qquad (5)$$

Substituting $x = 0$ in (1), (2), (3), (4), (5),

$$1 = a_0$$
$$1 = a_1$$
$$1 = 2a_2$$
$$1 = 3 \times 2a_3$$
$$1 = n!a_n$$

Substituting the values we have just found for a_0, a_1, a_2, a_3, a_n into equation (1),

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

This is the **exponential series** and is often denoted by **exp *x***. It is valid for all values of x (see below).

Questions

Q1 Write down the first four terms and the general terms in the expansions of:
a e^{-x}, **b** e^{x^2}, **c** e^{3x} in ascending powers of x,
d $e^{1/x}$, **e** e^{-1/x^2} in descending powers of x.

Q2 (Another method of proof.) Find the coefficients of the terms in the expansion of e^x by equating coefficients in equations (1) and (2) above.

Example 1 *Find the value of e correct to four places of decimals.*

Substituting $x = 1$ in the series for e^x,

$$e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{r!} + \ldots$$

The working is shown on the right, although you may prefer to use a calculator, if available. Each term in the series, after the first, is obtained from the previous one by dividing by 1, 2, 3, …, 9, … respectively. The working has been taken to five places of decimals. The value obtained for e is 2.7183, correct to four places of decimals.

1.00000
1.00000
0.50000
0.16667
0.04167
0.00833
0.00139
0.00020
0.00002
0.00000
2.71828

Although beyond the scope of this book, it can be shown that e is irrational. It can also be shown that e is *transcendental*, that is, e satisfies no algebraic equation in the form

$$a_0 + a_1x + \ldots + a_nx^n = 0$$

where the coefficients $a_0, a_1, \ldots, a_n$ are integers.

Example 2 *Find the first four terms in the expansions in ascending powers of x of* **a** e^{1-x^2}, **b** e^{x-x^2}, *giving the general term in* **a**.

a $e^{1-x^2} = e^1 \times e^{-x^2}$

$$= e\left\{1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \ldots + \frac{(-x^2)^r}{r!} + \ldots\right\}$$

$$\therefore e^{1-x^2} = e\{1 - x^2 + \tfrac{1}{2}x^4 - \tfrac{1}{6}x^6 + \ldots + (-1)^r x^{2r}/r! + \ldots\}$$

b $$e^{x-x^2} = 1 + (x - x^2) + \frac{(x-x^2)^2}{2!} + \frac{(x-x^2)^3}{3!} + \ldots$$

$$= 1 + x - x^2 + \tfrac{1}{2}x^2 - x^3 + \ldots + \tfrac{1}{6}x^3 + \ldots$$

$$\therefore e^{x-x^2} = 1 + x - \tfrac{1}{2}x^2 - \tfrac{5}{6}x^3 + \ldots$$

Example 3 *Find the sum to infinity of the series*

$$1 + \frac{3x}{1!} + \frac{5x^2}{2!} + \frac{7x^3}{3!} + \ldots$$

The general term is $\dfrac{1+2n}{n!}x^n$. We aim to find terms in the form $x^r/r!$, so the general term is split up as

$$\frac{x^n}{n!} + \frac{2n}{n!}x^n = \frac{x^n}{n!} + 2x \times \frac{x^{n-1}}{(n-1)!} \qquad (n \geqslant 1)$$

Therefore the series may be written:

$$1 + (x + 2x \times 1) + \left(\frac{x^2}{2!} + 2x \times x\right) + \left(\frac{x^3}{3!} + 2x \times \frac{x^2}{2!}\right) + \ldots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} +$$

$$\ldots + 2x \times 1 + 2x \times x + 2x \times \frac{x^2}{2!} + 2x \times \frac{x^3}{3!} + \ldots$$

$$= e^x + 2x\,e^x = (1 + 2x)\,e^x$$

Exercise 31a

1 Use the expansion $\exp x$ to find the values of
a $e^{0.1}$, **b** $1/e$, **c** $\sqrt{e}$,
giving your answers correct to 4 d.p.

In questions 2–10, expand the functions of x as far as the fourth non-zero terms and give the general terms.

2 e^{x^3} **3** $\sqrt[3]{(e^x)}$ **4** $(1/e^x)^2$

5 e^{2+x} **6** $1/\sqrt{(e^x)}$ **7** $(1+x)e^x$

8 $(1+2x)e^{-2x}$ **9** $\dfrac{e^{3x} \times e^{2x}}{e^x}$ **10** $\dfrac{e^{3x} + e^{2x}}{e^x}$

11 Find the greatest terms in the expansion of e^x when $x = 10$.

In questions 12–15 expand the functions in ascending powers of x as far as the term in x^3.

12 e^{x^2+2x} **13** e^{x^2-3x+1}

14 $\dfrac{e^x}{1+x}$ **15** $\dfrac{1-e^{-x}}{e^x - 1}$

16 Find the limits of the following functions as x approaches zero.

a $\dfrac{e^x - (1+x)}{e^{2x} - (1+2x)}$ **b** $\dfrac{e^{2x} - (1+4x)^{1/2}}{e^{-x} - (1-3x)^{1/3}}$

c $\dfrac{e^x + e^{-x} - 2}{e^{x^2} - 1}$

Find the sums to infinity of the following series.

17 $1 + \dfrac{2x}{1!} + \dfrac{3x^2}{2!} + \dfrac{4x^3}{3!} + \ldots$

18 $1 + \dfrac{3x}{2!} + \dfrac{9x^2}{3!} + \dfrac{27x^3}{4!} + \ldots$

19 $1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots$
[Start by writing down the series for e^x and e^{-x}.]

20 $x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots$

31.2 The logarithmic series

The sum to infinity of the geometric series $1 - u + u^2 - u^3 + \ldots$ is $1/(1+u)$ (see §13.8 on page 168). So we can write

$$\frac{1}{1+u} = 1 - u + u^2 - u^3 + \ldots$$

Assuming that the integral of the sum of an infinite series is the sum of the integrals of its terms, integrate between 0 and x:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

The nth term of the geometric series is $(-1)^{n-1}u^{n-1}$ so that the nth term of the logarithm series is $(-1)^{n-1}x^n/n$. Since the geometric series has a sum only if $|u| < 1$, we would expect that the logarithmic series would be valid when $|x| < 1$. However it can also be shown that the series has a sum when $x = 1$ (see Exercise 31b, question 25). Thus

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots$$

provided $-1 < x \leqslant 1$.

Note that if x is replaced by $-x$ in this series,

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots - \frac{x^n}{n} - \ldots$$

provided $-1 \leqslant x < 1$.

Example 4 *Expand as series in ascending powers of x:*

a $\ln(2+x)$ **b** $\ln(2+x)^3$
c $\ln(x^2-3x+2)$

and state the range of values of x for which each expansion is valid.

a $\ln(2+x) = \ln\{2(1+\frac{1}{2}x)\}$
$= \ln 2 + \ln(1+\frac{1}{2}x)$

$$\ln(2+x) = \ln 2 + \tfrac{1}{2}x - \frac{(\frac{1}{2}x)^2}{2} + \ldots + (-1)^{n-1}\frac{(\frac{1}{2}x)^n}{n} + \ldots$$

$$\therefore \ln(2+x) = \ln 2 + \frac{x}{2} - \frac{x^2}{8} + \ldots + (-1)^{n-1}\frac{x^n}{2^n \times n} + \ldots$$

The expansion is valid if $-1 < \frac{1}{2}x \leqslant 1$, i.e. if $-2 < x \leqslant 2$.

b $\ln(2+x)^3 = 3\ln(2+x)$.

Therefore, using the result of part **a**,

$$\ln(2+x)^3 = 3\ln 2 + \frac{3x}{2} - \frac{3x^2}{8} + \ldots + (-1)^{n-1}\frac{3x^n}{2^n \times n} + \ldots$$

The expansion is again valid if $-2 < x \leqslant 2$.

c $\ln(x^2-3x+2) = \ln\{(1-x)(2-x)\}$
$= \ln(1-x) + \ln(2-x)$

$$\ln(1-x) = -x - \frac{x^2}{2} - \ldots - \frac{x^n}{n} - \ldots$$

From **a**,

$$\ln(2-x) = \ln 2 - \frac{x}{2} - \frac{x^2}{8} - \ldots - \frac{x^n}{2^n \times n} - \ldots$$

Adding,

$$\ln(x^2-3x+2) = \ln 2 - \frac{3}{2}x - \frac{5}{8}x^2 - \ldots$$
$$- \frac{x^n}{n}\{1 + (\tfrac{1}{2})^n\} - \ldots$$

For the expansions to be valid, x must satisfy both $-1 \leqslant x < 1$ and $-2 \leqslant x < 2$, i.e. $-1 \leqslant x < 1$.

Question

Q3 Expand in ascending powers of x:

a $\ln(1+\frac{1}{4}x)$
b $\ln(3-x)$
c $\ln(x^2-2x+1)$

Give the first three terms and the general term. State the ranges of values of x for which the expansions are valid.

Example 5 *If $|x| > 1$, show that*

$$\ln(1+x) = \ln x + \frac{1}{x} - \frac{1}{2x^2} + \ldots + \frac{(-1)^{n-1}}{nx^n} + \ldots$$

[Since $|x| > 1$, we can express the series in terms of $\frac{1}{x}$, which is numerically *smaller* than 1.]

$\ln(1+x) = \ln\{x(1+1/x)\}$
$= \ln x + \ln(1+1/x)$

$$\therefore \ln(1+x) = \ln x + \frac{1}{x} - \frac{1}{2x^2} + \ldots + \frac{(-1)^{n-1}}{nx^n} + \ldots$$

Other series have been developed for the calculation of logarithms. One of these is shown below.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{x^{2n-1}}{2n-1} - \frac{x^{2n}}{2n} + \ldots$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots - \frac{x^{2n-1}}{2n-1} - \frac{x^{2n}}{2n} - \ldots$$

The expansions are valid if $-1 < x \leqslant 1$, $-1 \leqslant x < 1$, respectively, so for both to be valid, $-1 < x < 1$.

Subtracting,

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \ldots + \frac{x^{2n-1}}{2n-1} + \ldots\right)$$

Dividing by 2 and writing

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \ln\sqrt{\left(\frac{1+x}{1-x}\right)}$$

we obtain

$$\ln\sqrt{\left(\frac{1+x}{1-x}\right)} = x + \frac{x^3}{3} + \ldots + \frac{x^{2n-1}}{2n-1} + \ldots$$

provided $-1 < x < +1$.

The advantage of this series is seen when attempting to calculate, for example, $\ln 1.5$ by two methods.

a Substitute $x = \frac{1}{2}$ in

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \ldots + (-1)^{n-1}x^n/n + \ldots$$

$$\ln 1.5 \quad = \tfrac{1}{2} - \tfrac{1}{8} + \tfrac{1}{24} - \tfrac{1}{64} + \tfrac{1}{160} - \tfrac{1}{384} + \tfrac{1}{896} - \tfrac{1}{2048} + \ldots$$

b Substitute $x = \frac{1}{5}$ in

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left\{x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \ldots + x^{2n-1}/(2n-1) + \ldots\right\}.$$

$$\begin{aligned}\ln 1.5 \quad &= 2(\tfrac{1}{5} + \tfrac{1}{375} + \tfrac{1}{15625} + \ldots)\\ &= 0.4055 \qquad \text{to 4 d.p.}\end{aligned}$$

Clearly the value correct to 4 d.p. is obtained more rapidly by the second series.

Note that, using $\log_{10} 1.5 = \log_{10} e \times \ln 1.5$, we can calculate $\log_{10} 1.5$.
(The abbreviation $\lg x$ is sometimes used for $\log_{10} x$.
With this notation we could write $\lg 1.5 = \lg e \times \ln 1.5$.)

Example 6 *Find the first three terms in the expansion of*

$$\frac{\ln(1+x)}{\ln(1-x)}$$

in ascending powers of x.

Let $\dfrac{\ln(1+x)}{\ln(1-x)} = a_0 + a_1x + a_2x^2 + \ldots$, where a_0, a_1, a_2 are constants to be determined.

$$\therefore \ln(1+x) = \ln(1-x)(a_0 + a_1x + a_2x^2 + \ldots)$$

$$\begin{aligned}\therefore x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\\ = \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\right)(a_0 + a_1x + a_2x^2 + \ldots)\end{aligned}$$

Equating coefficients of x, x^2, x^3:

$$\begin{aligned}1 &= -a_0\\ -\tfrac{1}{2} &= -\tfrac{1}{2}a_0 - a_1\\ \tfrac{1}{3} &= -\tfrac{1}{3}a_0 - \tfrac{1}{2}a_1 - a_2\end{aligned}$$

from which we obtain

$$a_0 = -1, \qquad a_1 = 1, \qquad a_2 = -\tfrac{1}{2}$$

$$\therefore \frac{\ln(1+x)}{\ln(1-x)} = -1 + x - \frac{1}{2}x^2 + \ldots$$

Question

Q4 Write down the first three terms of the expansion of $\dfrac{\lg(1+x)}{\lg(1-x)}$ in ascending powers of x.

Example 7 *Find the sum to infinity of the series*

$$\frac{1}{1\times 2}\times\frac{1}{3} + \frac{1}{2\times 3}\times\frac{1}{3^2} + \frac{1}{3\times 4}\times\frac{1}{3^3} + \ldots$$

The general term is

$$\frac{1}{n(n+1)}\times\frac{1}{3^n}$$

This may be expressed in partial fractions as

$$\left(\frac{1}{n} - \frac{1}{n+1}\right)\frac{1}{3^n}$$

Therefore the series may be written

$$\begin{aligned}&1\times\frac{1}{3} + \frac{1}{2}\times\frac{1}{3^2} + \frac{1}{3}\times\frac{1}{3^3} + \ldots + \frac{1}{n}\times\frac{1}{3^n} + \ldots\\ &-\frac{1}{2}\times\frac{1}{3} - \frac{1}{3}\times\frac{1}{3^2} - \frac{1}{4}\times\frac{1}{3^3} - \ldots - \frac{1}{n+1}\times\frac{1}{3^n} - \ldots\\ &= -\ln(1-\tfrac{1}{3}) + S\\ &= -\ln\tfrac{2}{3} + S \qquad (1)\end{aligned}$$

where

$$S = -\frac{1}{2}\times\frac{1}{3} - \frac{1}{3}\times\frac{1}{3^2} - \frac{1}{4}\times\frac{1}{3^3} - \ldots - \frac{1}{n+1}\times\frac{1}{3^n} - \ldots$$

$$\begin{aligned}&\therefore -\frac{1}{3} + \frac{1}{3}S\\ &= -\frac{1}{3} - \frac{1}{2}\times\frac{1}{3^2} - \frac{1}{3}\times\frac{1}{3^3} - \ldots - \frac{1}{n+1}\times\frac{1}{3^{n+1}} - \ldots\\ &= \ln(1-\tfrac{1}{3})\end{aligned}$$

$$\therefore S = 3\ln\tfrac{2}{3} + 1$$

Therefore, from (1), the sum of the series is $2\ln\frac{2}{3} + 1$.

Exercise 31b

1 Expand the following functions in ascending powers of x, giving the first three or four terms, as indicated, and the general term. State the ranges of values of x for which the expansions are valid.

a $\ln(3+x)$, (4) **b** $\ln(1-\frac{1}{2}x)$, (4)

c $\ln(2-5x)$, (4) **d** $\ln(1-x^2)$, (4)

e $\ln\left(\dfrac{3+x}{3-x}\right)$, (3) **f** $\ln\left(\dfrac{4-3x}{4+3x}\right)$, (3)

Find the first three terms and the general terms in the expansions of the functions in questions 2–8. State the necessary restrictions on the values of x.

2 $\ln\left(\dfrac{2-x}{3-x}\right)$ **3** $\ln\dfrac{1}{3-4x-4x^2}$

4 $\ln\left\{\dfrac{(1+4x)^3}{(1+3x)^4}\right\}$ **5** $\ln\sqrt{(x^2+3x+2)}$

6 $\ln(1+x+x)^2$ [Hint: $(1-x^3)=(1-x)(1+x+x^2)$]

7 $\ln\{(1+x)^{1/x}\}$ **8** $\ln(1-x+x^2)$

Expand the following functions in ascending powers of x as far as the terms indicated. State the ranges of values of x for which the expansions are valid.

9 $\dfrac{\ln(1+x)}{1-x}$, (x^3) **10** $e^x\ln(1+x)$, (x^3)

11 $\dfrac{x+x^2}{\ln(1+x)}$, (x^2) **12** $\{\ln(1-x)\}^2$, (x^4)

13 By substituting $x=\frac{1}{3}$ in the expansion of $\ln\{(1+x)/(1-x)\}$ in ascending powers of x, find the value of ln 2 correct to 4 s.f. Taking $\ln 1.5 = 0.4055$, estimate the value of ln 3.

In questions 14–16, take $\ln 2 = 0.693\,147$ and $\ln 3 = 1.098\,612$.

14 Find ln 10 correct to four places of decimals by substituting $x=\frac{1}{9}$ in the expansion of $\ln(1+x)$. Deduce an approximate value of lg e.

15 Find the value of ln 7 by substituting $x=\frac{1}{8}$ in the expansion of $\ln\{(1+x)/(1-x)\}$. Give your answer correct to 4 d.p.

16 Find the value of lg 11 correct to 4 d.p. Use the expansion of $\ln\{(1+x)/(1-x)\}$ with $x=0.1$. Take $\lg e = 0.434\,29$.

17 Find the limits of the following functions as x approaches zero.

a $\ln\{(1-x^2)^{1/x^2}\}$ **b** $\dfrac{\ln(1+x)-x}{\ln(1-x)+x}$

c $\dfrac{\ln\{(1+x)^2\}+x^2-2x}{\ln(1-x^3)}$

d $\dfrac{\ln(1-x)+x\sqrt{(1+x)}}{\ln(1+x^2)}$

Find the sums to infinity of the following series.

18 $\dfrac{1}{3}-\dfrac{1}{2}\times\dfrac{1}{9}+\dfrac{1}{3}\times\dfrac{1}{27}-\dfrac{1}{4}\times\dfrac{1}{81}+\ldots$

19 $\dfrac{1}{2}+\dfrac{1}{2}\times\dfrac{1}{2^2}+\dfrac{1}{3}\times\dfrac{1}{2^3}+\dfrac{1}{4}\times\dfrac{1}{2^4}+\ldots$

20 $\dfrac{1}{4}+\dfrac{1}{3}\times\dfrac{1}{4^3}+\dfrac{1}{5}\times\dfrac{1}{4^5}+\dfrac{1}{7}\times\dfrac{1}{4^7}+\ldots$

21 $1-\dfrac{1}{2}\times\dfrac{2}{5}+\dfrac{1}{3}\times\dfrac{2^2}{5^2}-\dfrac{1}{4}\times\dfrac{2^3}{5^3}+\ldots$

22 $1+\dfrac{1}{3\times 2^2}+\dfrac{1}{5\times 2^4}+\dfrac{1}{7\times 2^6}+\ldots$

***23** Integrate the inequalities

$$\frac{1}{(1+t)^2}<\frac{1}{1+t}<1 \qquad (t>0)$$

from 0 to u and deduce that

$$\frac{u}{1+u}<\ln(1+u)<u \qquad (u>0)$$

Sketch the graph of $y=1/x$ and illustrate the latter inequalities graphically.

Also prove that, if $-1<u<0$,

$$\frac{u}{1+u}<\ln(1+u)<u$$

24 Sketch the graph of $y=1/x$ and show that, when n is a positive integer greater than 1,

$$\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}<\ln n<1+\frac{1}{2}+\ldots+\frac{1}{n-1}$$

25 Let s_n denote the sum of n terms of the series

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\ldots$$

By considering the terms of the series in pairs, show that s_{2n} increases as $n\to\infty$. By considering the terms of the series after 1 in pairs, show that s_{2n+1} is less than 1 and decreases as $n\to\infty$. Show that $|s_{2n}|-|s_{2n+1}|\to 0$ as $n\to\infty$. What can you conclude about s_n as $n\to\infty$?

31.3 Approximation

This section considers ways of expressing functions in an approximate form. Already we have used the binomial theorem to do this. In §31.1 on page 324 and §31.2 on page 325 we saw how to express e^x and $\ln(1+x)$ as series in x. We now establish a basic form of approximation, and apply it to numerical examples.

Linear approximation

Figure 31.1 represents part of the graph of $y = f(x)$. P is a fixed point $(a, f(a))$, PT is the tangent at P, and Q is a variable point on the curve given by $x = a + h$, where h is small. We shall establish an approximate relationship between $f(a)$ and $f(a+h)$.

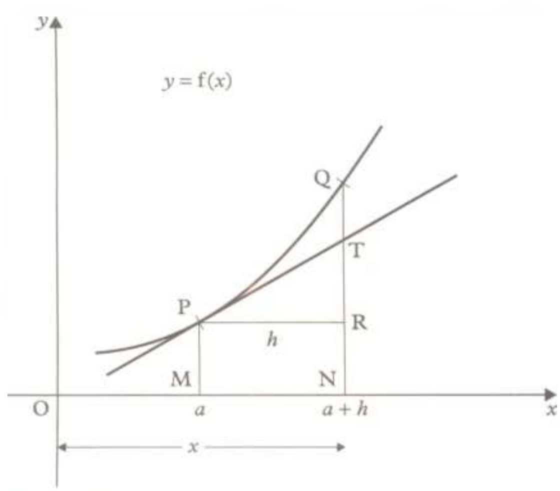


Figure 31.1

$$f(a+h) = NQ \approx NT \qquad \text{since } h \text{ is small}$$
$$NT = MP + RT = f(a) + PR \tan \angle RPT = f(a) + f'(a)h^*$$

Hence, if h is small,

$$\mathbf{f(a+h) \approx f(a) + f'(a)h}$$

This is called a **linear approximation** since we consider the straight line PT in place of the curve PQ. Expressing it another way, when $x \approx a$, the function $f(x)$ can be expressed in an approximate linear form, since

$$\mathbf{f(x) \approx f(a) + f'(a)(x-a)}$$

*The fact that $RQ \approx RT = f'(a)h$ was used in, §7.7 on page 103. There it was stated in the form $\delta y \approx \dfrac{dy}{dx}\delta x$.

Example 8 *Find an approximate value of* sin 31°.

[Here we wish to establish an approximate relationship between sin 30°, which we know, and sin 31°. Since we use the derivative of sin x we work in radians. Note that $1° = \pi/180$ radians.]

$$31° = \frac{\pi}{6} + \frac{\pi}{180} \text{ radians}$$

Since $f(a+h) \approx f(a) + f'(a)h$,

$$\sin\left(\frac{\pi}{6} + \frac{\pi}{180}\right) \approx \sin\frac{\pi}{6} + \cos\frac{\pi}{6} \times \frac{\pi}{180}$$
$$= 0.5 + \frac{\sqrt{3}}{2} \times \frac{\pi}{180}$$
$$\therefore \sin 31° \approx 0.515$$

Questions

Q5 Use the method of Example 8 to find a value of tan 45.6° to 5 s.f. Compare this with the value you find on a calculator.

Q6 Assuming $\cos^{-1} 0.8 \approx 36°\,52'$ and $52' \approx 0.0151$ radians, find an approximation for cos 36°.

Q7 If $x \approx \pi/6$, prove that $\cos x \approx \frac{1}{12}(\pi + 6\sqrt{3} - 6x)$, using the second form of linear approximation given above.

Q8 Use the method of Example 8 to obtain approximations for the following. Work to 4 d.p. and compare your answers with the values given by a calculator.

- **a** cosec 61.5° (take $\sqrt{3}$ as 1.7321)
- **b** cot 28.5° (take $\sqrt{3}$ as 1.7321)
- **c** $e^{1.08}$ (take e as 2.7183)
- **d** ln 2.001 (take ln 2 as 0.6931)

Q9 Use the fact that if $x \approx a$, $f(x) \approx f(a) + f'(a)(x-a)$ to prove that

- **a** if $x \approx 0$, $e^x \approx 1 + x$
- **b** if $x \approx \pi$, $\sin x \approx \pi - x$
- **c** if $x \approx 2$, $\dfrac{1}{(1+x)^2} \approx \dfrac{1}{27}(7-2x)$
- **d** if $x \approx \dfrac{\pi}{4}$, $\tan x \approx 1 - \dfrac{\pi}{2} + 2x$
- **e** if $x \approx 7$, $\sqrt{(2+x)} \approx \frac{1}{6}(x+11)$
- **f** if $x \approx 1$, $\ln x \approx x - 1$

Quadratic approximation

This section shows how the accuracy of approximation may be improved.

In §31.1 on page 324 it was established that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

If x is small, an approximation for e^x can be found by ignoring high powers of x. Thus the linear approximation obtained in Q9**a** is $1 + x$, the first two terms of the above series. Clearly a better approximation would be obtained by taking more terms.

$y = x + 1$ is the tangent to $y = e^x$ at $(0, 1)$. So when $x = 0$, the graphs of the function and of its linear approximation have equal ordinates and equal gradients. If we take a **quadratic approximation,** $f(x)$, we can further stipulate that the *rate of change of gradient* of $y = e^x$ and $y = f(x)$ are equal when $x = 0$. $y = f(x)$ is a parabola, and this gives a better approximation to the curve $y = e^x$ over a wider range of values of x.

Suppose

$$f(x) = c_0 + c_1 x + c_2 x^2$$

Then for small values of x,

$$e^x \approx c_0 + c_1 x + c_2 x^2$$

Differentiating twice,

$$e^x \approx \quad c_1 + 2c_2 x$$

and

$$e^x \approx \quad 2c_2$$

But when $x = 0$, these are *not* approximations but equalities,

$$\therefore\ c_0 = 1 \qquad c_1 = 1 \qquad c_2 = \tfrac{1}{2}$$

Therefore for small values of x

$$e^x \approx 1 + x + \frac{x^2}{2}$$

As expected, we have obtained the first three terms of the series for e^x.

Question

Q10 On the same axes sketch the graphs of e^x, $1 + x$, $1 + x + x^2/2$ from $x = 0$ to $x = 1$ at intervals of 0.1.

We now consider in Q11 the function $\ln x$ when $x \approx 1$, and investigate how we can obtain an improvement on the linear approximation $\ln x \approx x - 1$. (See Q9**f**.)

Question

Q11 Given that the graphs of $y = \ln x$ and $y = c_0 + c_1(x-1) + c_2(x-1)^2$ have the same ordinate, gradient, and rate of change of gradient when $x = 1$, prove that when $x \approx 1$,

$$\ln x \approx -\frac{3}{2} + 2x - \frac{x^2}{2}$$

The following table gives values of the function $\ln x$, and of the first and second approximations $x - 1$ and $-\frac{3}{2} + 2x - \frac{1}{2}x^2$, in the vicinity of $x = 1$. **Fig. 31.2** shows the graphs of these functions.

x	0.2	0.3	0.4	0.5	0.6
$x - 1$	−0.8	−0.7	−0.6	−0.5	−0.4
$-\frac{3}{2} + 2x - \frac{1}{2}x^2$	−1.12	−0.95	−0.78	−0.63	−0.48
$\ln x$	−1.61	−1.20	−0.92	−0.69	−0.51

x	0.7	0.8	0.9	1	1.1
$x - 1$	−0.3	−0.2	−0.1	0	0.1
$-\frac{3}{2} + 2x - \frac{1}{2}x^2$	−0.345	−0.22	−0.105	0	0.095
$\ln x$	−0.357	−0.223	−0.105		0.0953

x	1.2	1.3	1.4	1.5	1.6
$x - 1$	0.2	0.3	0.4	0.5	0.6
$-\frac{3}{2} + 2x - \frac{1}{2}x^2$	0.18	0.255	0.32	0.38	0.42
$\ln x$	0.182	0.262	0.34	0.41	0.47

x	1.7	1.8	1.9	2
$x - 1$	0.7	0.8	0.9	1
$-\frac{3}{2} + 2x - \frac{1}{2}x^2$	0.46	0.48	0.495	0.5
$\ln x$	0.53	0.59	0.64	0.69

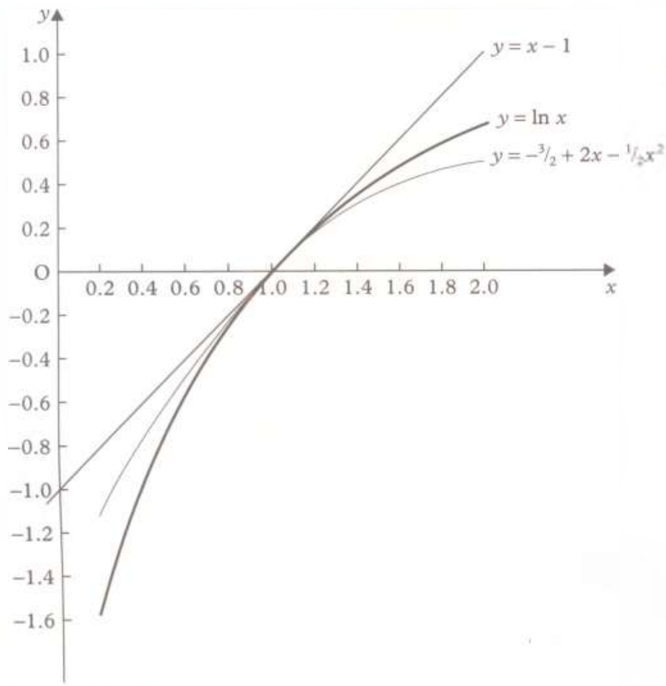


Figure 31.2

If you have access to a computer use it to plot these graphs. Notice that the approximation is very accurate for values of x which are near $x = 1$.

Questions

Q12 Given that the graphs of $y = f(x)$ and $y = c_0 + c_1(x-a) + c_2(x-a)^2$ have the same ordinate, gradient, and rate of change of gradient when $x = a$, find c_0, c_1, c_2, and hence give an approximation for f(x) when $x \approx a$.

Q13 If a is a constant and h is small, re-write the answer to Q12 to give an approximation for $f(a + h)$ in ascending powers of h as far as h^2.

31.4 Taylor's theorem

From Q11 and Q12 we may reasonably suppose that if we add terms of successively higher powers of $(x - a)$ to an approximation for f(x) when $x \approx a$ then we shall obtain better approximations to f(x).

In §31.1 on page 324 and §31.2 on page 325, e^x and $\ln(1 + x)$ are expressed as infinite series in ascending powers of x. We now assume that if f(x) is any function of x, and a is a constant, then provided that f(a) exists and that successive derivatives of f(x) all have finite values when $x = a$, f(x) can be expressed as an infinite series in ascending powers of $(x - a)$.* In what follows, we assume that it is possible to differentiate an infinite series term by term.

Let

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \dots$$

then

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots$$
$$f''(x) = 2!c_2 + 3\times 2c_3(x-a) + 4\times 3c_4(x-a)^2 + \dots$$
$$f'''(x) = 3!c_3 + 4\times 3\times 2c_4(x-a) + \dots$$
$$f''''(x) = 4!c_4 + \dots$$

and putting $x = a$ in each line, we find that

$$c_0 = f(a), \quad c_1 = f'(a), \quad c_2 = \frac{f''(a)}{2!}, \quad c_3 = \frac{f'''(a)}{3!}, \quad c_4 = \frac{f''''(a)}{4!}$$

Thus

$$\mathbf{f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f''''(a)}{4!}(x-a)^4 + \dots}$$

or if $x = a + h$,

$$\mathbf{f(x) = f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \frac{f'''(a)}{3!}h^3 + \frac{f''''(a)}{4!}h^4 + \dots}$$

This result is called **Taylor's theorem**.

Example 9 *Use Taylor's theorem to expand* $\sin(\pi/6 + h)$ *in ascending powers of h as far as the term in* h^4.

Let $f(x) = \sin x$

$$= \sin\left(\frac{\pi}{6} + h\right) \qquad f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$$

$$f'(x) = \cos x \qquad f'\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$f''(x) = -\sin x \qquad f''\left(\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2}$$

$$f'''(x) = -\cos x \qquad f'''\left(\frac{\pi}{6}\right) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$

$$f''''(x) = \sin x \qquad f''''\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$$

*We proved in §31.2 on page 325 that the expansion of $\ln(1 + x)$ in ascending powers of x is valid only if $-1 < x \leqslant +1$. We may therefore expect some limitations on the value of x in certain cases of the general expansions we are about to discuss.

By Taylor's theorem,

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \frac{f'''(a)}{3!}h^3 + \frac{f''''(a)}{4!}h^4 + \dots$$

$$\therefore \sin\left(\frac{\pi}{6}+h\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}h + \frac{(-\frac{1}{2})}{2!}h^2 + \frac{(-\sqrt{3}/2)}{3!}h^3 + \frac{\frac{1}{2}}{4!}h^4 + \dots$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}h - \frac{1}{4}h^2 - \frac{\sqrt{3}}{12}h^3 + \frac{1}{48}h^4 + \dots$$

Questions

Q14 Using only the first three terms of the expansion in Example 9, obtain a value for sin 31° to 5 s.f., taking $\sqrt{3}$ as 1.7321 and 1° as 0.01745 radians. Compare your answer with that of Example 1.

Q15 Use Taylor's theorem to express $\tan(\pi/4 + h)$ as a series in ascending powers of h as far as the term in h^3.

Q16 Use Taylor's theorem to find the first four terms in the expansion of $\cos x$ in ascending powers of $(x - \alpha)$, where $\alpha = \tan^{-1}\frac{4}{3}$.

31.5 Maclaurin's theorem

Bearing in mind the relationship $x = a + h$, where a is a constant and x and h are variable (see **Fig. 31.1**), there is a special case given by $a = 0$, when $x = h$, and Taylor's theorem on page 331 reduces to

$$\mathbf{f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \dots}$$

This is known as **Maclaurin's theorem**.

Example 10 *Use Maclaurin's theorem to expand* $\ln(1+x)$ *in ascending powers of* x *as far as the term in* x^5.

$f(x) = \ln(1+x)$	$f(0) = 0$
$f'(x) = (1+x)^{-1}$	$f'(0) = 1$
$f''(x) = -(1+x)^{-2}$	$f''(0) = -1$
$f'''(x) = 2(1+x)^{-3}$	$f'''(0) = 2!$
$f''''(x) = -3 \times 2(1+x)^{-4}$	$f''''(0) = -3!$
$f'''''(x) = 4 \times 3 \times 2(1+x)^{-5}$	$f'''''(0) = 4!$

By Maclaurin's theorem

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \frac{f'''''(0)}{5!}x^5 + \dots$$

$$\therefore \ln(1+x) = 0 + 1 \times x + \frac{(-1)}{2!}x^2 + \frac{2!}{3!}x^3 + \frac{(-3!)}{4!}x^4 + \frac{4!}{5!}x^5 + \dots$$

$$\therefore \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$$

Questions

Q17 Use Maclaurin's theorem

a to expand e^x in ascending powers of x as far as the x^5 term,

b to show that when x is small,
$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!},$$

c to find the first three terms of the expansion of $\cos x$ in ascending powers of x.

Q18 Express 17° 11′ in radians correct to 1 s.f. Use the approximation in Q13b to express sin 17° 11′ to 4 s.f. Check your answer against the value given on a calculator.

Power series

In Example 10 and Questions 17 and 18 we used Maclaurin's theorem to establish the following expansions:

a) $\mathbf{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots}$

b) $\mathbf{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots}$

c) $\mathbf{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots}$

d) $\mathbf{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots}$

These are often called **power series** and should be memorised.

Exercise 31c

1 Given that the graphs of $y = \ln x$ and $y = c_0 + c_1(x-2) + c_2(x-2)^2$ have the same value of y, gradient and rate of change of gradient when $x = 2$, determine c_0, c_1, c_2 and deduce an approximation for $\ln x$ when $x \approx 2$.

2 Obtain a quadratic approximation for $\sin x$ when $x \approx \alpha$.

3 Apply Taylor's theorem

a to expand $\ln x$ in ascending powers of $(x - \mathrm{e})$ as far as the term in $(x - \mathrm{e})^4$,

b to expand $\operatorname{cosec} x$ in ascending powers of $(x - \pi/2)$ as far as the term in $(x - \pi/2)^4$.

4 Use Taylor's theorem to expand $\cos(\pi/3 + h)$ in ascending powers of h up to the h^3 term. Taking $\sqrt{3}$ as 1.7321 and 5.5° as 0.095 99 radians, find the value of cos 54.5° to 3 d.p.

5 Given that the functions $f(x)$ and $c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots$ have the same value when $x = 0$, and equal successive derivatives when $x = 0$, find the first five terms of the Maclaurin expansion of $f(x)$ in ascending powers of x.

6 Write down the first four terms of the expansions of the following in ascending powers of x.

a e^{2x}

b $\ln(1 - x)$

c $\cos x^2$

d $\sin \dfrac{x}{2}$

7 By subtracting the expansion of $\ln(1-x)$ from that of $\ln(1+x)$ deduce that

$$\ln\sqrt{\left(\frac{1+x}{1-x}\right)} = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots$$

8 Find approximations for the following:

a $\mathrm{e}^{0.4}$ (correct to 5 s.f.)

b ln 1.2 (correct to 4 s.f.)

c cos 0.3 (correct to 3 s.f.)

d sin 0.2 (correct to 3 s.f.)

9 Apply Maclaurin's theorem directly (see Example 10) to obtain expansions for the following in ascending powers of x up to the given term.

a $\sin^2 x$, (x^4)

b $(1+x)^n$, (x^3)

c 2^x, (x^3)

d $\arccos x$, (x^3)

e $\mathrm{e}^x \sin x$, (x^5)

f $\ln\{x + \sqrt{(x^2+1)}\}$, (x^3)

10 If $f(x) = \mathrm{e}^x \sin x$ show that $f^n(x) = (\sqrt{2})^n \mathrm{e}^x \sin(x + n\pi/4)$. Use this with Maclaurin's theorem to find an expansion for $f(x)$ in ascending powers of x as far as the x^6 term.

11 Find the expansion of $\ln x$ in ascending powers of $(x - 4)$ up to the fourth term

a by writing $\ln x$ as $\ln\left\{4\left(1 + \dfrac{x-4}{4}\right)\right\}$ and applying the power series for $\ln(1+x)$,

b by applying Taylor's theorem.

Deduce an approximation for ln 4.02 correct to 4 d.p.

Chapter 32

Differentiation (4)

32.1 Logarithmic differentiation

The first three sections of this chapter extend your powers of differentiation and revise earlier work. We shall also discuss how to integrate certain functions.

Logarithmic differentiation can considerably simplify the differentiation of

a products (and quotients) of a number of functions,
b certain exponential functions.

But first it is advisable to revise some of the properties of logarithms and how to differentiate functions of y with respect to x.

Questions

Q1 $\ln(a^3\sqrt{b}/c^2) = \ln a^3 + \ln\sqrt{b} - \ln c^2,$
$= 3\ln a + \frac{1}{2}\ln b - 2\ln c.$
(See §25.3 on page 270.)

Write in a similar form:

a $\ln(a^2b)$ b $\ln(a^3/b^3)$
c $\ln\sqrt{(abc)}$ d $\ln(a\sqrt[3]{b}/c^3)$
e $\ln(1/c^4)$ f $\ln(a^b)$

Q2 $\log_{10} 10\,000 = \log_{10} 10^4 = 4$. Simplify in a similar manner:

a $\log_{10} 1000$ b $\log_{10}(1/100)$
c $\log_2(2^4)$ d $\ln(e^2)$
e $\ln(e^{2x})$ f $\ln(e^{3x^2})$

Q3 Differentiate with respect to x:

a $\ln x$ b $\ln(1+2x)$
c $\ln(1-x)$ d $\ln 4x^3$
e $\ln\sin x$ f $\ln\tan x$

When differentiating functions of y with respect to x we can, where necessary, use the chain rule,

$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$$

Thus if $z = y^4$,

$$\frac{dz}{dy} = 4y^3$$

$$\therefore \frac{dz}{dx} = 4y^3 \times \frac{dy}{dx}$$

Question

Q4 Differentiate with respect to x:

a $3y^2$ b y^3
c $\cos y$ d $\ln y$

Express, in your own words, a rule which will help you to differentiate any function of y with respect to x. Use this rule to differentiate with respect to x:

e $5y^4$ f $3/y^2$
g $\sqrt{y}$ h $\tan y$

Example 1 *Differentiate* $\dfrac{e^{x^2}\sqrt{(\sin x)}}{(2x+1)^3}$.

Let $y = \dfrac{e^{x^2}\sqrt{(\sin x)}}{(2x+1)^3}$.

$$\therefore \ln y = \ln(e^{x^2}) + \ln\sqrt{(\sin x)} - \ln(2x+1)^3$$
$$= x^2 + \tfrac{1}{2}\ln\sin x - 3\ln(2x+1)$$

Differentiating with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = 2x + \frac{\cos x}{2\sin x} - \frac{6}{2x+1}$$

$$\therefore \frac{dy}{dx} = \frac{e^{x^2}\sqrt{(\sin x)}}{(2x+1)^3}\left\{2x + \frac{\cos x}{2\sin x} - \frac{6}{2x+1}\right\}$$

(Often this is the most convenient form in which to use the derivative. However here we shall go on to simplify the expression in brackets.)

$$2x + \frac{\cos x}{2\sin x} - \frac{6}{2x+1} = \frac{4x^2+2x-6}{2x+1} + \frac{\cos x}{2\sin x}$$

$$\therefore \frac{dy}{dx} = \frac{e^{x^2}}{2\sqrt{(\sin x)}(2x+1)^4}\{(8x^2+4x-12)\sin x + (2x+1)\cos x\}$$

Question

Q5 Use the method of Example 1 to differentiate with respect to x:

a $\sqrt[3]{\dfrac{x+1}{x-1}}$ b $\dfrac{\sqrt{(x^2+1)}}{(2x-1)^2}$ c $\dfrac{x^2e^x}{(x-1)^3}$

Example 2 *Differentiate 10^x with respect to x.*

Let $y = 10^x$.

$\therefore \ln y = \ln 10^x$
$= x \ln 10$

Differentiating with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = \ln 10$$

$$\therefore \frac{dy}{dx} = 10^x \ln 10$$

Example 3 *Differentiate, with respect to x,* **a** 2^{x^2}, **b** x^x.

a Let $y = 2^{x^2}$.

$\therefore \ln y = \ln 2^{x^2}$
$= x^2 \ln 2$

Differentiating with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = 2x \ln 2$$

$$\therefore \frac{dy}{dx} = 2^{x^2} 2x \ln 2$$

b Let $y = x^x$.

$\therefore \ln y = \ln x^x$
$= x \ln x$

Differentiating with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{x} + 1 \times \ln x$$
$$= 1 + \ln x$$

$$\therefore \frac{dy}{dx} = (1 + \ln x)x^x$$

Questions

Q6 Example 2 shows that the derivative of 10^x is $10^x \ln 10$. Write down a function whose derivative is 10^x. What is $\int 10^x\,dx$?

Q7 Differentiate with respect to x:

a 2^x **b** 3^x **c** $(\frac{1}{2})^x$
d 10^{5x} **e** 10^{x^2}

Q8 From your answers to Q7, write down:

a $\int 2^x\,dx$ **b** $\int 3^x\,dx$
c $\int (\frac{1}{2})^x\,dx$ **d** $\int 10^{5x}\,dx$

32.2 Integration by trial

Q6 is an example of 'integration by trial'. This was discussed in §24.1 on page 259 and §25.5 on page 273. Its stages are shown in the next two examples.

Example 4 *Integrate 2^{-x} with respect to x.*

(*Stage* 1: make a guess.)
From the last section it is to be expected that the integral involves 2^{-x}.

Let $y = 2^{-x}$

$\therefore \ln y = -x \ln 2$

(*Stage* 2: differentiate.)

$$\therefore \frac{1}{y}\frac{dy}{dx} = -\ln 2$$

$$\therefore \frac{dy}{dx} = -2^{-x} \ln 2$$

(*Stage* 3: compare with the given functions.)
We have an extra constant factor of $-\ln 2$.

(*Stage* 4: change the guessed function to allow for the constant.)

$$\frac{d}{dx}\left(\frac{2^{-x}}{-\ln 2}\right) = 2^{-x}$$

$$\therefore \int 2^{-x}\,dx = -\frac{2^{-x}}{\ln 2} + c$$

Example 5 *Integrate $x \ln x$ with respect to x.*

(*Stage* 1: make a guess.)
When we *differentiate* a product, we differentiate each function in turn and multiply by the other. So, to integrate $x \ln x$, try integrating one factor and multiply by the other. As we do not know how to integrate $\ln x$, let us try $\frac{1}{2}x^2 \ln x$.

Let $y = \frac{1}{2}x^2 \ln x$.

(*Stage* 2: differentiate.)

$$\therefore \frac{dy}{dx} = x \ln x + \frac{1}{2}x^2 \times \frac{1}{x}$$

$$\therefore \frac{d}{dx}\left(\frac{1}{2}x^2 \ln x\right) = x \ln x + \frac{1}{2}x$$

(*Stage* 3: compare with the given function.)
We have an extra term of $\frac{1}{2}x$ on the right-hand side.

(*Stage* 4: change the guessed function allowing for the $\frac{1}{2}x$.)

$$\frac{d}{dx}\left(\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right) = x \ln x + \frac{1}{2}x - \frac{1}{2}x$$
$$= x \ln x$$

$\therefore \int x \ln x \, dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + c$

Question

Q9 Integrate by trial the following functions with respect to x.

- **a** $(3x+1)^{1/2}$ **b** $\sin x \cos^5 x$
- **c** $\dfrac{\sin x}{(1+\cos x)^3}$ **d** 5^x
- **e** 2^{2x} **f** $\ln x$

32.3 Inverse trigonometrical functions

The functions $\sin^{-1} x$, $\tan^{-1} x$ (or $\arcsin x$, $\arctan x$) were introduced in §18.6 on page 219. We now consider the problem of differentiating inverse trigonometrical functions. This is shown by examples but some revision may be advisable. Remember to use radians.

Questions

Q10 $y = \sin^{-1} x$ means 'y is the angle whose sine is x' so that $\sin y = x$. Similarly rewrite:

- **a** $y = \tan^{-1} x$ **b** $\sec^{-1} x = y$
- **c** $\cos^{-1} p = q$

Q11 Differentiate with respect to x:

- **a** y^2 **b** $\sin y$
- **c** $\tan y$ **d** $\sec y$

Example 6 *Differentiate with respect to x:*

a $\sin^{-1} x$ **b** $\tan^{-1}(x^2+1)$

a Let $y = \sin^{-1} x$.
$\therefore \sin y = x$

Differentiating with respect to x,

$$\cos y \frac{dy}{dx} = 1$$

$1 - \sin^2 y = 1 - x^2$.
$\therefore \cos^2 y = 1 - x^2$.

(y was our own introduction, so we must express $\frac{dy}{dx}$ in terms of x.)

$$\therefore \sqrt{(1-x^2)}\frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{(1-x^2)}}$$

b Let $y = \tan^{-1}(x^2+1)$.
$\therefore \tan y = x^2 + 1$

Differentiating with respect to x,

$$\sec^2 y \frac{dy}{dx} = 2x$$

$1 + \tan^2 y = 1 + (x^2+1)^2$.
$\therefore \sec^2 y = x^4 + 2x^2 + 2$.

(We must again express $\frac{dy}{dx}$ in terms of x.)

$$\therefore (x^4 + 2x^2 + 2)\frac{dy}{dx} = 2x$$

$$\therefore \frac{dy}{dx} = \frac{2x}{x^4 + 2x^2 + 2}$$

Question

Q12 Differentiate with respect to x:

- **a** $\cos^{-1} x$ **b** $\cot^{-1} x$
- **c** $\sin^{-1}(2x+1)$

Exercise 32a

1 Express in the form $p \ln a + q \ln b + r \ln c$:

- **a** $\ln (a^3b^4)$ **b** $\ln (a/b)$
- **c** $\ln \sqrt{(a^3/b)}$ **d** $\ln (a^2b/\sqrt{c})$
- **e** $\ln \sqrt{(ab/c)}$ **f** $\ln \{1/\sqrt{(abc)}\}$

2 Write the following in a form which does not use logarithm notation.

- **a** $\lg 100\,000$ **b** $\log_2 8$
- **c** $\ln e^4$ **d** $\ln \sqrt{e}$
- **e** $\ln e^{x^3}$ **f** $\ln (1/e^{2x})$

Differentiate questions 3–10 with respect to x, using logarithmic differentiation.

3 $\sqrt{\dfrac{(2x+3)^3}{1-2x}}$ **4** $\dfrac{e^{x/2}\sin x}{x^4}$

5 $\dfrac{1}{\sqrt{(x^2+1)}\sqrt[3]{(x^2-1)}}$ **6** $\dfrac{1}{x e^x \cos x}$

7 7^x 8 10^{3x}

9 $10^{-x/2}$ 10 $1/10^x$

Integrate questions 11–14 with respect to x.

11 5^x 12 8^x

13 $(\frac{1}{3})^x$ 14 3^{2x}

15 Show that $e^{\ln a} = a$ (see §25.3 on page 270). Write a^x in the form $e^{x \ln a}$ and hence find $\dfrac{d}{dx}(a^x)$.

16 Find $\int a^x\,dx$ by writing $a^x = e^{x \ln a}$.

Differentiate questions 17–22 with respect to x.

17 $\tan^{-1} x$ 18 $\sec^{-1} x$

19 $\sin^{-1}(x+1)$ 20 $\cos^{-1}(2x-1)$

21 $\tan^{-1}(1/x^2)$ 22 $2\cos^{-1} 5x$

23 Find: a $\dfrac{d}{dx}(\sin^{-1} x + \cos^{-1} x)$,

b $\dfrac{d}{dx}(\tan^{-1} x + \cot^{-1} x)$.

Explain these answers.

24 Find $\dfrac{d}{dx}(\sin^{-1} x)$ and hence write down:

a $\dfrac{d}{dx}(\sin^{-1} 2x)$ b $\dfrac{d}{dx}(\sin^{-1} x^2)$

c $\dfrac{d}{dx}\{\cos^{-1}\surd(1-x^2)\}$

Integrate questions 25–28 by trial with respect to x.

25 $\surd(4x+3)$ 26 $x(2x^2+1)^3$ 27 $\ln x$

28 $\sin^{-1} x$ [Find $\dfrac{d}{dx}(x \sin^{-1} x)$.]

Differentiate with respect to x.

29 x^{-x} 30 $x^{\sin x}$

32.4 Local maxima and minima

The first derivative test

We have already met the first derivative test in Chapter 5. The diagrams in Fig. 32.1 illustrate **i** a local maximum and **ii** a local minimum. (The + and – signs indicate the sign of the gradient.)

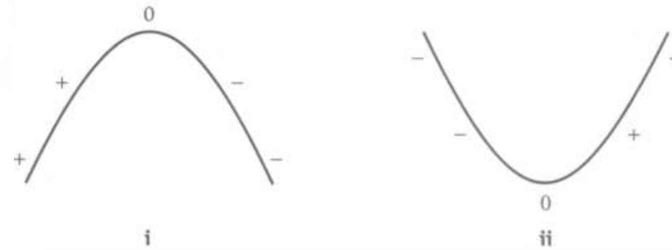


Figure 32.1

At a turning point, $\dfrac{dy}{dx} = 0$ *and it changes sign*. In the case of a maximum (see diagram **i**) it changes from + to – as x increases, whereas at a minimum it changes from – to + (see diagram **ii**).

If however $\dfrac{dy}{dx} = 0$ but does *not* change sign, then we have a **stationary point of inflexion** (see Fig. 32.2).

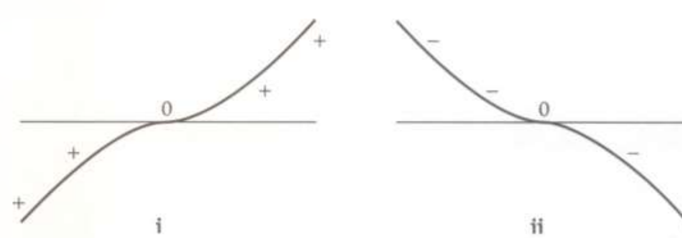


Figure 32.2

The first derivative test is very easy to apply as the next two examples show. (Remember to factorise the derived function where possible.)

Example 7 *Investigate the stationary points on the graph of*

$$y = x^2e^{-x}$$

and sketch the curve.

$$y = x^2e^{-x}$$

Differentiating:

$$\frac{dy}{dx} = 2xe^{-x} - x^2e^{-x}$$
$$= (2x - x^2)e^{-x}$$
$$= x(2-x)e^{-x}$$

$\dfrac{dy}{dx}$ is zero when $x = 0$ and when $x = 2$.

We know that e^{-x} is always positive, so the sign of the gradient is determined by the other factors.

By inspection, we can see that $\dfrac{dy}{dx}$ is negative when $x < 0$, and that between 0 and 2, $\frac{dy}{dx}$ is positive.

When $x > 2$, $\frac{dy}{dx}$ is negative again.

Therefore there is a local minimum at (0, 0), and a local maximum at $(2, 4e^{-2})$. The curve can now be sketched (see Fig. 32.3).

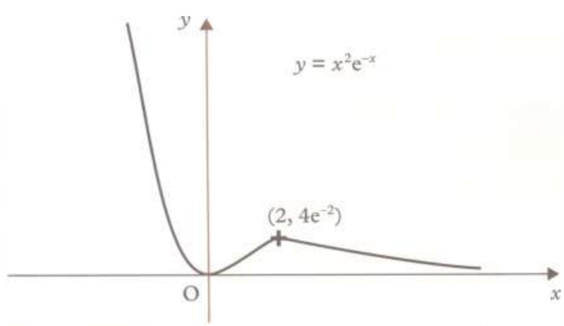


Figure 32.3

Example 8 *Investigate the stationary values of the function*

$$f(x) = x^3 - 3x^2 + 3x$$

and sketch the graph of $y = f(x)$.

In this case,

$$\begin{aligned} f'(x) &= 3x^2 - 6x + 3 \\ &= 3(x^2 - 2x + 1) \\ &= 3(x - 1)^2 \end{aligned}$$

We can see that $f'(1)$ is zero, but as $(x - 1)^2$ is a square, it can never be negative. In other words, the gradient of $y = f(x)$ is zero at $x = 1$, but everywhere else it is positive. Therefore there is a point of inflexion at (1, 1) (see Fig. 32.4).

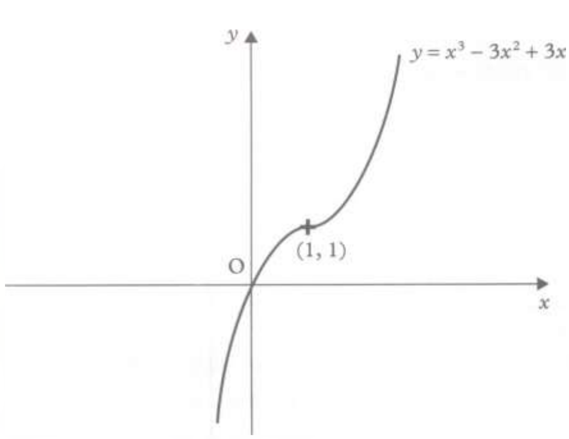


Figure 32.4

Question

Q13 Investigate the stationary values of the function xe^{-x} and sketch the graph of $y = xe^{-x}$.

The second derivative test

The second derivative test depends on the fact that if the gradient of a curve $y = f(x)$ is *increasing* with x, the rate of change of the gradient is *positive*; if the gradient is *decreasing*, its rate of change is *negative*. Alternatively, consider the graph of $\frac{dy}{dx}$ plotted against x. The gradient of *this* curve is given by $\frac{d^2y}{dx^2}$. If the ordinate, $\frac{dy}{dx}$, is increasing with x, the gradient, $\frac{d^2y}{dx^2}$, is positive. If $\frac{dy}{dx}$ is decreasing, $\frac{d^2y}{dx^2}$ is negative.

Looking back to Fig. 32.1, we see that at a local maximum, the gradient is decreasing (it is changing from a positive value, through zero, to a negative value as x increases). Therefore at such a point the derivative *of the gradient function* is negative. In other words, for $y = f(x)$ at $x = a$

if $\frac{dy}{dx}$ is zero and $\frac{d^2y}{dx^2}$ is *negative*,
then y has a (local) maximum at $x = a$;

on the other hand,

if $\frac{dy}{dx}$ is zero and $\frac{d^2y}{dx^2}$ is *positive*,
then y has a (local) minimum at $x = a$.

Fig. 32.5 shows (as a continuous curve) a graph representing $y = f'(x)$, with positive gradient at $x = a$, i.e. $f''(a) > 0$, and $f'(a) = 0$. The dotted curve represents the corresponding graph of $y = f(x)$, showing a minimum when $x = a$.

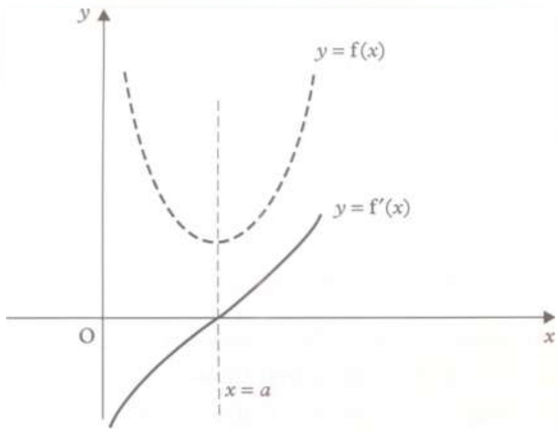


Figure 32.5

Question

Q14 Draw a diagram, like **Fig. 32.5**, showing the graphs of $y = f'(x)$ and $y = f(x)$, with $f''(a) < 0$ and $f'(a) = 0$.

Example 9 *Use the second derivative test to investigate the stationary values of the function xe^{-x}.*

Let $y = xe^{-x}$.

$$\therefore \frac{dy}{dx} = e^{-x} - xe^{-x}$$
$$= (1 - x)e^{-x}$$

From this we can see that there is a stationary value of $1/e$ when $x = 1$.

$$\frac{d^2y}{dx^2} = -e^{-x} - e^{-x} + xe^{-x}$$
$$= -2e^{-x} + xe^{-x}$$

When $x = 1$, $\frac{d^2y}{dx^2} = -2e^{-1} + e^{-1} = -e^{-1}$.

This is negative, so by the second derivative test, there is a local maximum of $1/e$ when $x = 1$.

It is important to notice that no conclusion can be drawn from the second derivative test when $\frac{d^2y}{dx^2}$ is zero. (There is *not* always a point of inflexion in this case. E.g. consider $y = x^4$: both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are zero at $x = 0$, but the function clearly has a minimum at this point.)

32.5 Points of inflexion

We have already met *stationary* points of inflexion, see **Fig. 32.2**. In **Fig. 32.2i**, the gradient is positive everywhere except at the stationary point, where it is zero, i.e. at this point the gradient has a minimum value. In **Fig. 32.2ii**, the gradient is negative everywhere except at the stationary point, where it is zero, so at this point the gradient has a maximum value. In general, a point of inflexion is a point where the gradient has a local maximum or minimum value.

Fig. 32.6 shows some points of inflexion for which $\frac{dy}{dx} \neq 0$.

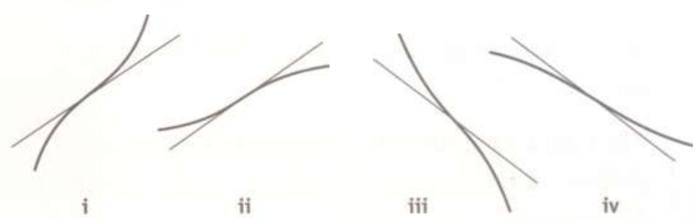


Figure 32.6

Looking at **Fig. 32.6i**, we see that the gradient is always positive. It is decreasing as it approaches the point of inflexion, and after that it increases again. I.e. the *gradient* has a *minimum* value at this point of inflexion. You should analyse the other diagrams similarly. (On a graph, a point of inflexion is easily recognised, because the graph 'crosses its own tangent' at such a point.)

Points of inflexion are easily located by applying the first derivative test to the gradient function, i.e.

at a point of inflexion $\frac{d^2y}{dx^2}$ is zero

and it changes sign

(In the case of $y = x^4$, mentioned on page 338, the second derivative, namely $12x^2$, does not change sign at $x = 0$.)

Example 10 *Find the points of inflexion of the function $y = \frac{48}{12 + x^2}$ and sketch its graph.*

We are given $y = \frac{48}{12 + x^2}$, therefore

$$\frac{dy}{dx} = \frac{-96x}{(12 + x^2)^2}$$

$\frac{dy}{dx}$ is zero when $x = 0$, and its sign changes from positive to negative as x passes through zero. Therefore there is a local maximum at this point. The maximum value of the function is 4.

To find the points of inflexion, differentiate again:

$$\frac{d^2y}{dx^2} = \frac{-96}{(12+x^2)^2} + \frac{96 \times 4x^2}{(12+x^2)^3}$$
$$= 96\left(\frac{-(12+x^2)+4x^2}{(12+x^2)^3}\right)$$
$$= 96\left(\frac{3x^2-12}{(12+x^2)^3}\right)$$
$$= \frac{288(x-2)(x+2)}{(12+x^2)^3}$$

$\frac{d^2y}{dx^2}$ is zero at $x = \pm 2$, and it changes sign at these points. Hence there are points of inflexion at $(-2, 3)$ and $(+2, 3)$.

When sketching this curve, notice that y is always positive and that it tends to zero as x tends to infinity. Also, this is an even function, so its graph is symmetrical about the y-axis (see **Fig. 32.7**).

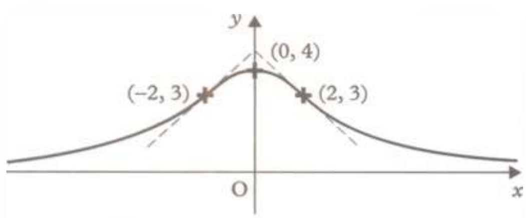


Figure 32.7

Question

Q15 Find the point of inflexion on $y = 2x^3 - 18x^2 + 12x + 80$.

Exercise 32b

Find the nature of the stationary points in questions 1–4.

1 $x(x-3)^2$ **2** $x + \frac{4}{x^2}$

3 $x - 2 + \frac{1}{x-3}$ **4** $x - \ln x$

Find the points of inflexion in questions 5 and 6.

5 $y = x^4 - 54x^2$ **6** $y = x^4 - 4x^3 + 6x^2 - 4x$

7 Sketch the graphs of questions 1–6.

In questions 8–11, find the maxima and minima of the functions of θ in the interval $0 \leqslant \theta \leqslant 2\pi$.

8 $\sin\theta + \frac{1}{2}\sin 2\theta$. Sketch the graph.

9 $(\sin\theta)/(1+\sin\theta)$. Sketch the graph.

10 $\ln\cos\theta - \cos\theta$.

11 $\cos\theta - \frac{1}{3}\cos 3\theta$. Sketch the graph.

12 Find the turning point of the function $x\,e^{-x}$ and determine its nature. Show that there is a point of inflexion when $x = 2$ and sketch the curve.

13 Find the turning point of the function $10\arctan x - \frac{1}{2}x^2$ and sketch the curve.

14 Show that the function $x\ln x$ has a minimum at $(1/e, -1/e)$. Given that $x\ln x \to 0$ as $x \to 0$, sketch the graph of the function.

15 Show that $e^x\cos x$ has turning points at intervals of π in x. Distinguish between maxima and minima and show that these values are in a geometrical progression with common ratio $-e^{\pi}$.

16 A right circular cylinder is inscribed in a sphere of given radius a. Show that the volume of the cylinder is $\pi h(a^2 - \frac{1}{4}h^2)$, where h is the height of the cylinder. Find the ratio of the height to the radius of the cylinder when its volume is greatest.

17 A right circular cylinder is inscribed in a sphere. Show that the height of the cylinder is equal to its diameter when its curved surface area is greatest.

18 A funnel is in the form of a right circular cone. If the funnel is to hold a given quantity of liquid, find the ratio of the height to the radius when the area of the curved surface is a minimum.

19 A right circular cone of vertical angle 2θ is inscribed in a sphere of radius a. Show that the area of the curved surface of the cone is $\pi a^2(\sin 3\theta + \sin\theta)$ and prove that its maximum area is $8\pi a^2/(3\sqrt{3})$.

20 An open box has a square horizontal cross-section. If the box is to hold a given amount of material and the internal surface area of the box is to be a minimum, find the ratio of its height to the length of its sides.

Chapter 33

Integration (4)

33.1 Integration by parts

We have seen the importance of recognising integrals such as

$\int x\,e^{x^2}\,dx = \frac{1}{2}e^{x^2} + c$ and

$\int 2x \cos(x^2+2)\,dx = \sin(x^2+2) + c$

However, when the integrand is the product of two functions of x but cannot be integrated by inspection, e.g. $\int x\,e^x\,dx$, $\int x\cos x\,dx$, we use a technique known as **integration by parts**. This is based on the idea of differentiating the product of two functions of x.

If u and v are two functions of x,

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

Integrating each side with respect to x,

$$uv = \int v\frac{du}{dx}\,dx + \int u\frac{dv}{dx}\,dx$$

$$\therefore \int \boldsymbol{u}\frac{\mathbf{d}\boldsymbol{v}}{\mathbf{d}\boldsymbol{x}}\,\mathbf{d}\boldsymbol{x} = \boldsymbol{uv} - \int \boldsymbol{v}\frac{\mathbf{d}\boldsymbol{u}}{\mathbf{d}\boldsymbol{x}}\,\mathbf{d}\boldsymbol{x}$$

Example 1 *Find $\int x\cos x\,dx$.*

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Let $u = x$

Let $\frac{dv}{dx} = \cos x$

$\therefore v = \sin x$

$\int x\cos x\,dx$

$= x\sin x - \int \sin x \times 1\,dx$

$= x\sin x + \cos x + c$

Integration by part works only if the factor chosen as $\frac{dv}{dx}$ can be integrated. Example 1 shows that its success depends upon the *correct choice of u*, since this determines whether $\int v\frac{du}{dx}\,dx$ is easier to integrate than the original integral.

Questions

Q1 Check the answer to Example 1 by differentiation.

Q2 Attempt Example 1 taking $\cos x$ as u.

Q3 Find the following integrals:

- **a** $\int x\sin x\,dx$
- **b** $\int x\cos 2x\,dx$
- **c** $\int x\ln x\,dx$
- **d** $\int x\,e^x\,dx$

Q4 Find $\frac{d}{dx}(e^{x^2})$, and deduce $\int x^3 e^{x^2}\,dx$.

The integral $\int \tan^{-1} x\,dx$ does not at first sight appear to be susceptible to the method under discussion. However, this is one of a small group of integrals which may be found by taking $\frac{dv}{dx}$ as 1.

Example 2 *Find $\int \tan^{-1} x\,dx$.*

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Let $u = \tan^{-1} x$

Let $\frac{dv}{dx} = 1$

$\therefore v = x$

$\int \tan^{-1} x \times 1\,dx$

$= \tan^{-1} x \times x - \int x \times \frac{1}{1+x^2}\,dx$

$= x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + c$

Questions

Q5 Find $\int \ln x\,dx$.

Q6 a Find $\frac{d}{dx}(\sin^{-1} x)$

b Find $\int \sin^{-1} x\,dx$

In some cases it is necessary to apply the method of integration by parts more than once, as shown in Example 3.

Example 3 *Find* $\int x^2 \sin x \, dx$.

$\int x^2 \sin x \, dx$ — Let $u = x^2$

$= x^2(-\cos x) - \int -\cos x \times 2x \, dx$ — Let $\dfrac{dv}{dx} = \sin x$

$= -x^2 \cos x + \int 2x \cos x \, dx$ — $\therefore v = -\cos x$

$\int 2x \cos x \, dx$ — Let $u = 2x$

$= 2x \sin x - \int \sin x \times 2 \, dx$ — Let $\dfrac{dv}{dx} = \cos x$

$= 2x \sin x + 2 \cos x + c$ — $v = \sin x$

$\therefore \int x^2 \sin x \, dx$

$= -x^2 \cos x + 2x \sin x + 2 \cos x + c$

$= 2x \sin x + (2 - x^2) \cos x + c$

Questions

Q7 Check the answer to Example 3 by differentiation.

Q8 Find the following integrals.

a $\int x^2 \cos x \, dx$

b $\int x^2 e^x \, dx$

Exercise 33a

1 Find the following integrals, and check by differentiation.

a $\int 2x \sin x \, dx$	**b** $\int \frac{1}{2}x \, e^x \, dx$
c $\int x \sin 2x \, dx$	**d** $\int x^2 \ln x \, dx$
e $\int x \cos(x + 2) \, dx$	**f** $\int x(1 + x)^7 \, dx$
g $\int x \, e^{2x} \, dx$	**h** $\int x \, e^{x^2} \, dx$
i $\displaystyle\int \frac{\ln x}{x^2} \, dx$	**j** $\int x \sec^2 x \, dx$
k $\int x^n \ln x \, dx$	**l** $\int x \, 3^x \, dx$

2 Find the following integrals, and check by differentiation.

a $\int \ln 2x \, dx$	**b** $\int \sin^{-1} 3x \, dx$
c $\int \ln y^2 \, dy$	**d** $\displaystyle\int \tan^{-1} \frac{\theta}{2} \, d\theta$
e $\int \cos^{-1} t \, dt$	**f** $\int e^{\sqrt{x}} \, dx$

3 Find the following integrals (see Q4 on p. 341).

a $\int x^5 \, e^{x^3} \, dx$	**b** $\int x \, e^{-x^2} \, dx$
c $\int x^3 \, e^{-x^2} \, dx$	**d** $\int x^3 \cos x^2 \, dx$
e $\int x^3 \sec^2 (x^2) \, dx$	

4 Find the following integrals.

a $\int x^2 \cos 3x \, dx$	**b** $\int x^3 \, e^x \, dx$
c $\int x^2 \sin x \cos x \, dx$	**d** $\int x^2 \, e^{-x} \, dx$
e $\int (x \cos x)^2 \, dx$	**f** $\int x (\ln x)^2 \, dx$

5 Find the following integrals.

a $\int x \sin x \cos x \, dx$	**b** $\displaystyle\int \frac{x}{e^x} \, dx$
c $\int x(1 + 2x)^5 \, dx$	**d** $\displaystyle\int \frac{\ln y}{y} \, dy$
e $\int u \tan^{-1} u \, du$	**f** $\int x \, e^{-x^2} \, dx$
g $\int x^3 \, e^{-x} \, dx$	**h** $\int x(1 - x^2)^6 \, dx$
i $\int t \sin^2 t \, dt$	**j** $\int v \, e^{3v} \, dv$

6 **a** Find $\int x \tan^2 x \, dx$.

b Show that $\int x \sin^{-1} x \, dx$
$= \frac{1}{4}(2x^2 - 1) \sin^{-1} x + \frac{1}{4}x\sqrt{(1 - x^2)} + c$.

7 Evaluate:

a $\displaystyle\int_0^{\pi/2} x \cos x \, dx$	**b** $\displaystyle\int_0^1 x^2 \, e^x \, dx$
c $\displaystyle\int_1^{e^2} \ln x \, dx$	**d** $\displaystyle\int_0^1 \sin^{-1} y \, dy$
e $\displaystyle\int_0^{\pi} t \sin^2 t \, dt$	**f** $\displaystyle\int_1^{10} x \log_{10} x \, dx$

33.2 Inverse trigonometrical functions

In §32.3 on page 336 we dealt with the differentiation of inverse trigonometrical functions. Inverse sine and tangent functions are very common in integration. Therefore we should be able to differentiate these functions on sight.

If $y = \sin^{-1} u$, where u is a function of x,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{\sqrt{(1 - u^2)}} \times \frac{du}{dx}$$

Thus $$\frac{d}{dx}\left\{3 \sin^{-1} \frac{x}{2}\right\} = 3\frac{1}{\sqrt{\left(1 - \dfrac{x^2}{4}\right)}} \times \frac{1}{2} = \frac{3}{\sqrt{(4 - x^2)}}$$

and $$\frac{d}{dx}\{2 \tan^{-1} 5x\} = 2 \times \frac{1}{1 + 25x^2} \times 5 = \frac{10}{1 + 25x^2}$$

Question

Q9 Write down, and simplify where necessary, the derivatives of the following functions.

a $\sin^{-1} 3x$ b $\tan^{-1} 2x$

c $\sin^{-1} \dfrac{x}{3}$ d $\cos^{-1} 2x$

e $\frac{1}{2}\tan^{-1} 3x$ f $3\tan^{-1}\dfrac{x}{2}$

g $\frac{1}{2}\sin^{-1}(x-1)$ h $2\tan^{-1}\left(\dfrac{x+1}{2}\right)$

You should now be able to write down certain integrals, which we have so far obtained by the change of variable $x = k\sin u$ or $x = k\tan u$.

For example, $\displaystyle\int \frac{2}{3+4x^2}\,dx$, written as $\displaystyle\int \frac{\frac{2}{3}}{1+\frac{4}{3}x^2}\,dx$, is in the form $k\tan^{-1}\dfrac{2x}{\surd 3}+c$. Now,

$$\frac{d}{dx}\left(k\tan^{-1}\frac{2x}{\surd 3}\right)=\frac{1}{1+\frac{4}{3}x^2}\times\frac{2k}{\surd 3}$$

Comparing this with the integrand we find $k = 1/\surd 3$,

$$\therefore \int \frac{2}{3+4x^2}\,dx=\frac{1}{\surd 3}\tan^{-1}\left(\frac{2x}{\surd 3}\right)+c$$

Question

Q10 Find the following integrals.

a $\displaystyle\int \frac{2}{4+x^2}\,dx$

b $\displaystyle\int \frac{3}{1+4x^2}\,dx$

c $\displaystyle\int \frac{4}{\surd(9-x^2)}\,dx$

d $\displaystyle\int \frac{1}{\surd(1-9x^2)}\,dx$

e $\displaystyle\int \frac{1}{2+25x^2}\,dx$

f $\displaystyle\int \frac{2}{\surd(3-4x^2)}\,dx$

g $\displaystyle\int \frac{1}{3-2x+x^2}\,dx$

h $\displaystyle\int \frac{5}{\surd\{9-(x+2)^2\}}\,dx$

33.3 Change of variable $t = \tan\dfrac{x}{2}$

We have not yet integrated the trigonometrical ratios $\sec x$ and $\operatorname{cosec} x$.

Now $$\operatorname{cosec} x=\frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\frac{\sec^2\frac{x}{2}}{2\tan\frac{x}{2}}$$

$\left(\text{dividing numerator and denominator by } \cos^2\dfrac{x}{2}\right)$.

Thus $$\int \operatorname{cosec} x\,dx=\int\frac{\frac{1}{2}\sec^2\frac{x}{2}}{\tan\frac{x}{2}}\,dx$$

$$\therefore \int \mathbf{cosec}\,\boldsymbol{x}\,\mathbf{d}\boldsymbol{x}=\mathbf{ln\ tan}\,\frac{\boldsymbol{x}}{\mathbf{2}}+\boldsymbol{c}^*$$

Furthermore,

$$\sin\left(x+\frac{\pi}{2}\right)=\sin x\cos\frac{\pi}{2}+\cos x\sin\frac{\pi}{2}=\cos x$$

$$\therefore \operatorname{cosec}\left(x+\frac{\pi}{2}\right)=\sec x$$

Thus

$$\int \mathbf{sec}\,\boldsymbol{x}\,\mathbf{d}\boldsymbol{x}=\int\operatorname{cosec}\left(x+\frac{\pi}{2}\right)dx$$
$$=\mathbf{ln\ tan}\left(\frac{\boldsymbol{x}}{\mathbf{2}}+\frac{\boldsymbol{\pi}}{\mathbf{4}}\right)+\boldsymbol{c}$$

The above working suggests a change of variable which is very important. If we write $\tan\dfrac{x}{2}$ as t,

since $\tan 2A=\dfrac{2\tan A}{1-\tan^2 A}$,

$$\mathbf{tan}\,\boldsymbol{x}=\frac{\mathbf{2}\boldsymbol{t}}{\mathbf{1}-\boldsymbol{t}^{\mathbf{2}}}$$

It is also possible (see §17.3 on page 206) to express $\sin x$ and $\cos x$ in terms of t.

$$\mathbf{sin}\,\boldsymbol{x}=\frac{\mathbf{2}\boldsymbol{t}}{\mathbf{1}+\boldsymbol{t}^{\mathbf{2}}}\quad\text{and}\quad\mathbf{cos}\,\boldsymbol{x}=\frac{\mathbf{1}-\boldsymbol{t}^{\mathbf{2}}}{\mathbf{1}+\boldsymbol{t}^{\mathbf{2}}}$$

*See footnote to p. 344.

Fig. 33.1 is a useful aid to remembering these identities.

Starting with $\tan x = \dfrac{2t}{1-t^2}$, it is clear that $AC = 1 + t^2$.

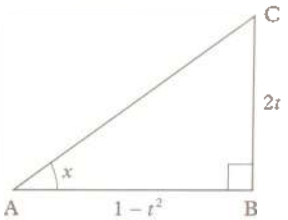


Figure 33.1

When we make the change of variable $t = \tan \dfrac{x}{2}$,

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$$

$$\frac{dx}{dt} = \frac{2}{\sec^2\frac{x}{2}} = \frac{2}{1+\tan^2\frac{x}{2}}$$

$$\therefore \frac{dx}{dt} = \frac{2}{1+t^2}$$

Questions

Q11 Find $\int \operatorname{cosec} x \, dx$ using the change of variable $t = \tan \dfrac{x}{2}$.

Q12 Find $\displaystyle\int \frac{\sin\theta}{1+\cos\theta}\, d\theta$

a by expressing the integrand in terms of ratios of $\dfrac{\theta}{2}$,

b by the change of variable $t = \tan\dfrac{\theta}{2}$.

Q13 Use the change of variable $t = \tan\dfrac{x}{2}$ to show that

$$\int \sec x \, dx = \ln \frac{1+t}{1-t} + c$$

Compare this form of the integrand with that obtained earlier and deduce that

$$\int \sec x \, dx = \ln(\sec x + \tan x) + c^*$$

The above change of variable is best thought of in more general terms as '$t = \tan$ (half angle)'. For example, when applied to $\int \operatorname{cosec} 4x \, dx$ it is $t = \tan 2x$;

then $1 = 2\sec^2 2x \dfrac{dx}{dt}$, giving $\dfrac{dx}{dt} = \dfrac{1}{2(1+t^2)}$. Take care when finding the correct numerical factor in the expression for $\dfrac{dx}{dt}$.

Example 4 *Find* $\displaystyle\int \frac{1}{3+5\cos\frac{1}{2}x}\, dx.$

$$\int \frac{1}{3+5\cos\frac{1}{2}x}\,\frac{dx}{dt}\, dt$$

$$= \int \frac{1}{3+5\times\frac{1-t^2}{1+t^2}} \times \frac{4}{1+t^2}\, dt$$

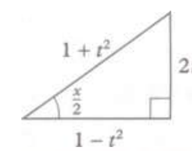
$$= \int \frac{4}{3(1+t^2)+5(1-t^2)}\, dt$$

$$= \int \frac{2}{4-t^2}\, dt$$

$$= \int \left\{\frac{1}{2(2+t)} + \frac{1}{2(2-t)}\right\} dt$$

$$= \tfrac{1}{2}\ln(2+t) - \tfrac{1}{2}\ln(2-t) + c$$

$$= \ln \frac{k\sqrt{(2+\tan\frac{1}{4}x)}}{\sqrt{(2-\tan\frac{1}{4}x)}}$$



Let $t = \tan\dfrac{x}{4}$

$1 = \frac{1}{4}\sec^2\dfrac{x}{4} \times \dfrac{dx}{dt}$

$\dfrac{dx}{dt} = \dfrac{4}{1+t^2}$

Questions

Q14 Find $\dfrac{dx}{dt}$ in terms of t if

a $t = \tan x$ **b** $t = \tan 4x$

c $t = \tan \frac{3}{2}x$

Q15 Find

a $\int \operatorname{cosec} 2x \, dx$ **b** $\displaystyle\int \frac{1}{1+\sin 3\theta}\, d\theta$

c $\displaystyle\int \frac{1}{\sqrt{(x^2-1)}}\, dx$ (use $x = \sec u$)

*If $\tan\dfrac{x}{2}$ is negative $\displaystyle\int \operatorname{cosec} x\, dx = \ln\left(-\tan\frac{x}{2}\right) + c$.

If $\sec x + \tan x$ is negative $\int \sec x \, dx = \ln(-\sec x - \tan x) + c$.
(See §25.6 on page 274.)

33.4 Change of variable $t = \tan x$

An integrand containing $\sin x$ and $\cos x$, particularly even powers of these, may often be expressed as a function of $\tan x$ and $\sec x$. In such a case the change of variable $t = \tan x$ is worth trying.

Example 5 *Find* $\displaystyle\int \frac{1}{1+\sin^2 x}\,dx.$

[In this case divide the numerator and denominator by $\cos^2 x$.]

Let $t = \tan x$, $\quad \dfrac{dx}{dt} = \dfrac{1}{1+t^2}$

$$\begin{aligned}
&\int \frac{1}{1+\sin^2 x}\,dx\\
&= \int \frac{\sec^2 x}{\sec^2 x + \tan^2 x}\,dx\\
&= \int \frac{\sec^2 x}{1+2\tan^2 x} \times \frac{dx}{dt}\,dt\\
&= \int \frac{1+t^2}{1+2t^2} \times \frac{1}{1+t^2}\,dt\\
&= \int \frac{1}{1+2t^2}\,dt\\
&= \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}t) + c\\
&= \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x) + c
\end{aligned}$$

Question

Q16 Find **a** $\displaystyle\int \frac{1}{1+\cos^2 x}\,dx$ **b** $\displaystyle\int \frac{2\tan x}{\cos 2x}\,dx$

33.5 Splitting the numerator

When a fractional integrand with a quadratic denominator cannot be written in simple partial fractions, it is often useful to express it as two fractions by splitting the numerator. To take a simple example (already met in Exercise 24d),

$$\begin{aligned}
\int \frac{1+x}{1+x^2}\,dx &= \int\left(\frac{1}{1+x^2} + \frac{x}{1+x^2}\right)dx\\
&= \tan^{-1} x + \ln\sqrt{(1+x^2)} + c
\end{aligned}$$

The key to a more general application of this method is to express the numerator in two parts, one of which is *a multiple of the derivative of the denominator.*

Example 6 *Find* $\displaystyle\int \frac{5x+7}{x^2+4x+8}\,dx.$

Since $\dfrac{d}{dx}(x^2+4x+8) = 2x+4$,

let $\quad 5x+7 \equiv A(2x+4)+B;\quad$ giving $A = \frac{5}{2}$, $B = -3$.

$$\begin{aligned}
\therefore \int \frac{5x+7}{x^2+4x+8}\,dx &= \int\left\{\frac{\frac{5}{2}(2x+4)}{x^2+4x+8} - \frac{3}{x^2+4x+8}\right\}dx\\
&= \frac{5}{2}\ln(x^2+4x+8) - 3\int \frac{1}{(x+2)^2+4}\,dx\\
&= \frac{5}{2}\ln(x^2+4x+8) - \frac{3}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + c
\end{aligned}$$

This method is also appropriate for integrands of the form

$$\frac{a\cos x + b\sin x}{\alpha\cos x + \beta\sin x}$$

since the numerator may be expressed as

A (derivative of denominator) + B (denominator)

Example 7 *Find* $\displaystyle\int \frac{2\cos x + 3\sin x}{\cos x + \sin x}\,dx.$

Let $2\cos x + 3\sin x \equiv A(-\sin x + \cos x) + B(\cos x + \sin x)$;
from which $A = -\frac{1}{2}$, $B = \frac{5}{2}$.

$$\begin{aligned}
&\therefore \int \frac{2\cos x + 3\sin x}{\cos x + \sin x}\,dx\\
&= \int\left\{\frac{-\frac{1}{2}(-\sin x + \cos x)}{\cos x + \sin x} + \frac{\frac{5}{2}(\cos x + \sin x)}{\cos x + \sin x}\right\}dx\\
&= -\tfrac{1}{2}\ln(\cos x + \sin x) + \tfrac{5}{2}x + c
\end{aligned}$$

Question

Q17 Find

a $\displaystyle\int \frac{2x+3}{x^2+2x+10}\,dx$

b $\displaystyle\int \frac{1-2x}{\sqrt{\{9-(x+2)^2\}}}\,dx$

c $\displaystyle\int \frac{\sin x}{\cos x + \sin x}\,dx$

d $\displaystyle\int \frac{2\cos x + 9\sin x}{3\cos x + \sin x}\,dx$

33.6 Improper integrals

Two types of integrals are discussed in this section. We shall consider them in terms of the area under a curve.

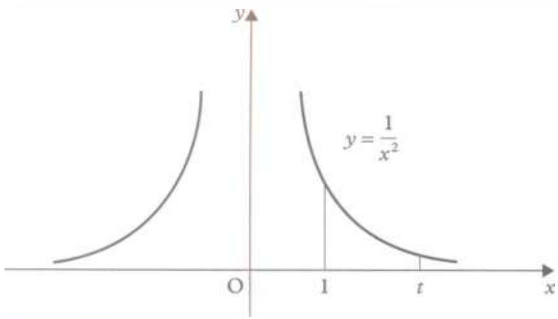


Figure 33.2

Fig. 33.2 shows part of the curve $y = 1/x^2$, of which the x-axis is an asymptote. The area under this curve from $x = 1$ to $x = t$ $(t > 1)$ is

$$\int_1^t \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^t = 1 - \frac{1}{t}$$

As $t \to \infty$, this area $\to 1$. Thus although the area 'enclosed' by $y = 1/x^2$, $x = 1$ and the x-axis is not really a finite enclosed area, it can be evaluated as the limiting value of the area $\int_1^t \frac{1}{x^2}\,dx$ as $t \to \infty$.

For brevity it is permissible to write

$$\int_1^\infty \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^\infty = 1$$

(Integrals like this are usually called **improper integrals of the first kind**.)

A similar situation arises when we consider 'the area under the curve $y = 1/\sqrt{(1-x^2)}$ from $x = 0$ to $x = 1$' (Fig. 33.3), since $x = 1$ is an asymptote to the curve. (Integrals like this are usually called **improper integrals of the second kind**.)

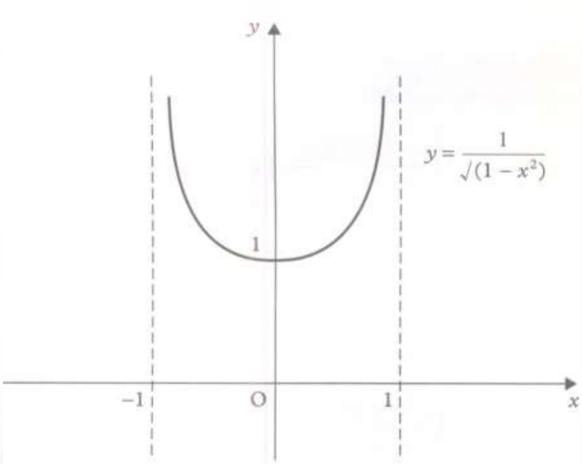


Figure 33.3

The area under this curve from $x = 0$ to $x = t$ $(0 < t < 1)$ is

$$\int_0^t \frac{1}{\sqrt{(1-x^2)}}\,dx = \left[\sin^{-1} x\right]_0^t = \sin^{-1} t$$

As $t \to 1$, this area $\to \pi/2$.

Thus, although the integrand $1/\sqrt{(1-x^2)}$ is meaningless when $x = 1$, the limiting process is implied when we write

$$\int_0^1 \frac{1}{\sqrt{(1-x^2)}}\,dx = \left[\sin^{-1} x\right]_0^1 = \frac{\pi}{2}$$

Questions

Q18 Evaluate:

a $\int_1^\infty \frac{1}{x^2}\,dx$, using the change of variable $x = \frac{1}{u}$,

b $\int_0^1 \frac{1}{\sqrt{(1-x^2)}}\,dx$, using the change of variable $x = \sin u$.

Q19 Evaluate the following integrals where possible, otherwise show that they are meaningless. Illustrate with a sketch.

a $\int_0^1 \frac{1}{x^2}\,dx$ **b** $\int_0^\infty \frac{1}{1+x^2}\,dx$

c $\int_1^\infty \frac{1}{x}\,dx$ **d** $\int_0^3 \frac{1}{(x-1)^2}\,dx$

e $\int_0^\infty e^{-x}\,dx$ **f** $\int_1^2 \frac{1}{\sqrt{(4-x^2)}}\,dx$

Exercise 33b

1 Differentiate the following with respect to x.

a $\sin^{-1} 2x$ **b** $\tan^{-1}(3x+1)$

c $\frac{1}{3}\cos^{-1} 2x$ **d** $2\sin^{-1}\left(\frac{x-1}{3}\right)$

e $\frac{1}{2}\tan^{-1}\frac{x}{2}$ **f** $\frac{2}{3}\sin^{-1}\frac{3x}{2}$

g $\cot^{-1} x$ **h** $\sec^{-1} x$

i $x^2\tan^{-1} x^2$ **j** $\cot^{-1} x + \tan^{-1} x$

2 Find the following integrals.

a $\displaystyle\int \frac{1}{9+x^2}\,dx$ **b** $\displaystyle\int \frac{3}{\sqrt{(4-y^2)}}\,dy$

c $\displaystyle\int \frac{2}{1+9u^2}\,du$ **d** $\displaystyle\int \frac{2}{\sqrt{(1-16x^2)}}\,dx$

e $\displaystyle\int \frac{2}{3+4t^2}\,dt$ **f** $\displaystyle\int \frac{1}{\sqrt{(5-4x^2)}}\,dx$

g $\displaystyle\int \frac{1}{2+3y^2}\,dy$ **h** $\displaystyle\int \frac{1}{3\sqrt{(3-6x^2)}}\,dx$

i $\displaystyle\int \frac{1}{2y^2-8y+17}\,dy$ **j** $\displaystyle\int \frac{2}{\sqrt{(1+6x-3x^2)}}\,dx$

3 Find the following integrals.

a $\displaystyle\int \operatorname{cosec}\frac{x}{2}\,dx$ **b** $\int \sec 2\theta\,d\theta$

c $\int \operatorname{cosec} 3x\,dx$ **d** $\int \sec 4\phi\,d\phi$

e $\int \sec x \operatorname{cosec} x\,dx$ **f** $\displaystyle\int \frac{1}{1+\cos y}\,dy$

g $\displaystyle\int \frac{1}{1+\sin 2x}\,dx$ **h** $\displaystyle\int \frac{\sin\theta}{1-\cos\theta}\,d\theta$

i $\displaystyle\int \frac{1}{4+5\cos x}\,dx$ **j** $\displaystyle\int \frac{1}{5+3\cos\frac{1}{2}\theta}\,d\theta$

4 Use the change of variable $\tan x = t$ to find the following integrals.

a $\displaystyle\int \frac{1}{1+2\sin^2 x}\,dx$ **b** $\displaystyle\int \frac{1}{\cos 2x - 3\sin^2 x}\,dx$

c $\displaystyle\int \frac{\sin^2 x}{1+\cos^2 x}\,dx$ **d** $\displaystyle\int \frac{1}{1-10\sin^2 x}\,dx$

5 Find the following integrals.

a $\displaystyle\int \frac{x+5}{x^2+3}\,dx$ **b** $\displaystyle\int \frac{y+4}{y^2+6y+9}\,dy$

c $\displaystyle\int \frac{3u+8}{u^2+2u+5}\,du$ **d** $\displaystyle\int \frac{3-7x}{\sqrt{(4x-x^2)}}\,dx$

e $\displaystyle\int \frac{\cos\theta}{\cos\theta+\sin\theta}\,d\theta$ **f** $\displaystyle\int \frac{3\cos x - 2\sin x}{\cos x + \sin x}\,dx$

6 Evaluate:

a $\displaystyle\int_3^\infty \frac{1}{(x-2)^2}\,dx$, using the change of variable $x-2=\dfrac{1}{u}$,

b $\displaystyle\int_0^{2/3} \frac{1}{\sqrt{(4-9x^2)}}\,dx$, using the change of variable $x=\dfrac{2}{3}\sin u$.

7 Evaluate the following integrals where possible, otherwise show that they are meaningless. Illustrate with a sketch.

a $\displaystyle\int_1^2 \frac{1}{x-1}\,dx$ **b** $\displaystyle\int_2^3 \frac{1}{\sqrt{(x-2)}}\,dx$

c $\displaystyle\int_0^3 \frac{1}{(x-3)^2}\,dx$ **d** $\displaystyle\int_1^4 \frac{1}{(x-2)^2}\,dx$

e $\displaystyle\int_3^\infty \frac{1}{(x-1)^2}\,dx$ **f** $\displaystyle\int_{-\infty}^0 e^x\,dx$

g $\displaystyle\int_0^{1/2} \ln x\,dx$ **h** $\displaystyle\int_{-\infty}^0 x\,e^x\,dx$

i $\displaystyle\int_1^{3/2} \frac{1}{\sqrt{(9-4x^2)}}\,dx$ **j** $\displaystyle\int_0^\infty \frac{1}{4+25x^2}\,dx$

8 The area enclosed by the x-axis, $x=1$, $x=t$, and the curve $y=1/x$ is rotated through 2π radians about the x-axis. What may be said about the volume of the solid so generated
a as $t\to\infty$, **b** as $t\to 0$?

***9** Find the area of the ellipse given by the parametric equations

$$x = 5\cos\theta, \qquad y = 3\sin\theta$$

$$\left(\text{Use the fact that } \int y\,dx = \int y = \frac{dx}{d\theta}\,d\theta.\right)$$

10 Find the area of the segment cut off by $x=8$ from the parabola given by the parametric equations $x=2t^2$, $y=4t$.

11 If $\displaystyle S \equiv \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta+\sin\theta}\,d\theta$, and

$$C \equiv \int_0^{\pi/2} \frac{\cos\theta}{\cos\theta+\sin\theta}\,d\theta,$$

prove that $S = C = \pi/4$.

33.7 Further integration by parts

The purpose of this section is to consolidate the method of integration by parts. It also introduces an interesting development in its application to certain integrals in which the original integral appears again.

Example 8 *Find* $\int e^{ax} \cos bx \, dx$.

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

Let $u = \cos bx$

Let $\frac{dv}{dx} = e^{ax}$

$\therefore v = \frac{1}{a} e^{ax}$

Let

$I = \int e^{ax} \cos bx \, dx$

$$= \frac{1}{a} e^{ax} \cos bx - \int \frac{1}{a} e^{ax}(-b \sin bx) \, dx$$

$$= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} \int e^{ax} \sin bx \, dx \qquad (1)$$

But

$$\int e^{ax} \sin bx \, dx$$

Let $u = \sin bx$

Let $\frac{dv}{dx} = e^{ax}$

$\therefore v = \frac{1}{a} e^{ax}$

$$= \frac{1}{a} e^{ax} \sin bx - \int \frac{1}{a} e^{ax} \, b \cos bx \, dx$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} I$$

Substituting in (1),

$$I = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} I$$

$$\therefore a^2 I = a\, e^{ax} \cos bx + b\, e^{ax} \sin bx - b^2 I$$

$$\therefore I(a^2 + b^2) = e^{ax}(a \cos bx + b \sin bx) + k$$

$$\therefore I = \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) + c$$

Questions

Q20 Find $\int e^{2x} \sin 3x \, dx$.

Q21 Find $\int e^{x} \cos 2x \, dx$,

- **a** taking e^x as u throughout,
- **b** taking $\cos 2x$ as u in the first step, and $\sin 2x$ as u in the second.
- **c** Can we usefully take $\cos 2x$ as u in the first step, and e^x as u in the second?

Exercise 33c

1 Use the method of Example 8 to find the following integrals.

a $\int e^{3x} \cos 2x \, dx$ **b** $\int e^{4x} \sin 3x \, dx$

c $\int e^{-t} \cos \frac{t}{2} \, dt$ **d** $\int e^{x} \sin (2x + 1) \, dx$

e $\int e^{2\theta} \cos^2 \theta \, d\theta$

2 Find $\int \sec^3 x \, dx$ by first proving it equal to $\frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x \, dx$.

3 Find the following integrals.

a $\int x^3 \ln x \, dx$ **b** $\int \tan^{-1} 2y \, dy$

c $\int \frac{x}{e^{x^2}} \, dx$ **d** $\int x \sin 3x \, dx$

e $\int x^2 \sin 2x \, dx$ **f** $\int e^{3x} \sin 2x \, dx$

g $\int \frac{1}{2} u^3 e^{u^2} \, du$ **h** $\int x(2x - 1)^5 \, dx$

i $\int x \ln \sqrt{(x-1)} \, dx$ **j** $\int \ln (3x) \, dx$

k $\int x^2 e^{2x} \, dx$ **l** $\int e^{-y} \cos \frac{y}{2} \, dy$

m $\int x^{-3} \ln x \, dx$ **n** $\int \sin^{-1} \frac{t}{3} \, dt$

o $\int \ln x^3 \, dx$ **p** $\int y^2 \cos^2 y \, dy$

q $\int x \cos x^2 \, dx$ **r** $\int x \ln (x^2) \, dx$

s $\int \theta^3 \sin (\theta^2) \, d\theta$ **t** $\int x^3 \cos 2x \, dx$

4 If $C = \int e^{ax} \cos bx \, dx$, and $S = \int e^{ax} \sin bx \, dx$, prove that

$$aC - bS = e^{ax} \cos bx \quad \text{and} \quad aS + bC = e^{ax} \sin bx$$

Hence find C and S.

5 Prove that $\int_0^{\infty} e^{-2x} \sin 3x \, dx = 3/13$.

6 Find the area enclosed by the x-axis and the curve $y = x(2 - x)^5$.

7 Prove that $\int \cos^4 x \, dx$

$$= \frac{1}{4} \sin x \cos^3 x + \frac{3}{4} \int \cos^2 x \, dx.$$

8 Find the area bounded by the x-axis and that part of the curve $y = e^{3x} \sin x$ from $x = 0$ to $x = \pi$.

33.8 Area of a sector*

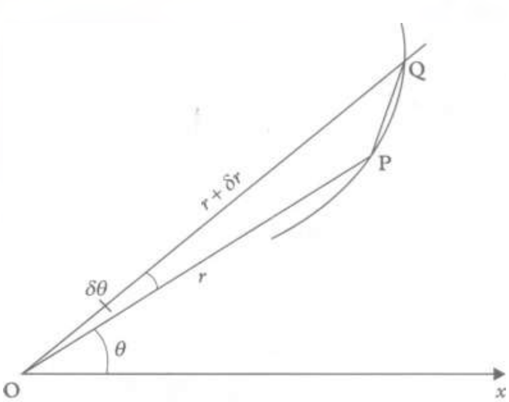


Figure 33.4

*You should work all the questions in this section.

In Fig. 33.4, the radius vectors OP, OQ are r, $r+\delta r$; the angles between them and the fixed line OX are θ, $\theta+\delta\theta$. If $\delta\theta$ is small, the **area of sector** OPQ is approximately equal to the area of triangle OPQ.

$\therefore$ sector OPQ $\approx \frac{1}{2}r(r+\delta r)\sin\delta\theta$

but

$$\sin\delta\theta = \delta\theta - \frac{(\delta\theta)^3}{3!} + \ldots$$

and $\delta r \sin\delta\theta$ is small compared with $\delta\theta$ so, correct to the term in $\delta\theta$,

$$\text{sector OPQ} = \tfrac{1}{2}r^2\,\delta\theta$$

(It is assumed that the difference between the sector OPQ and triangle OPQ is small compared to $\delta\theta$.)

Summing for all the elements in the sector concerned and proceeding to the limit,

$$\textbf{area of sector} = \int_\alpha^\beta \tfrac{1}{2}r^2\,\mathrm{d}\theta$$

where α, β are the values of θ corresponding to the bounding radius vectors of the sector.

Questions

Q22 Sketch the curve given by $r=a$. What does the integral $\int_\alpha^\beta \frac{1}{2}r^2\,\mathrm{d}\theta$ represent in this case?

Q23 Sketch the cardioid $r=a(1+\cos\theta)$ and find the area enclosed by it.

Q24 Find the area swept out by the radius vector of the equiangular spiral $r=a\,\mathrm{e}^{k\theta}$ as θ increases from $-\pi$ to π. Show this area on a sketch.

Q25 Sketch the trefoil $r=a\sin 3\theta$ and find the area of one of its loops.

Q26 The limaçon $r=1+2\cos\theta$ has a small loop contained within a larger one. See Fig. 22.4 on page 241. Find the area of the larger loop.

***Q27** If x and y are functions of a parameter t, show that

$$\int \tfrac{1}{2}r^2\,\mathrm{d}\theta = \int \tfrac{1}{2}\left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t$$

Questions

***Q28** The vertices of a triangle are O(0, 0), P(x, y), Q($x+\delta x$, $y+\delta y$). Show that the area of the triangle is $\frac{1}{2}(x\,\delta y - y\,\delta x)$. Hence show that the area of a sector may be found from the expression

$$\int_{t_1}^{t_2} \tfrac{1}{2}\left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t$$

where t_1, t_2 are the values of t corresponding to the bounding radius vectors of the sector.

Q29 Find the area enclosed by the ellipse $x=a\cos\theta$, $y=b\sin\theta$.

Q30 Find the area enclosed by the loop of the curve given by $x=t^2-4$, $y=t^3-4t$.

Q31 Find the area of one loop of the curve given by $x=\sin\theta$, $y=\sin 2\theta$. Why does the formula of Q28 give a negative answer?

Q32 Find the area between the cycloid $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$ and the portion of the x-axis between the points determined by $\theta=0$ and $\theta=2\pi$.

33.9 Length of an arc

The aim of this section is to find expressions for s, the arc length of an arc, in terms of the coordinates in which the equations of curves are commonly written. The most important of these are the cartesian coordinates x, y.

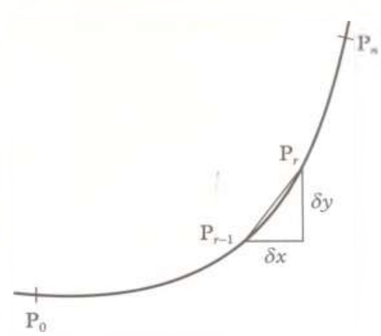


Figure 33.5

Suppose an arc of a curve is divided into n parts by points $P_0, P_1, \ldots, P_n$ (see Fig. 33.5). We shall assume that the

sum of the lengths of chords $\rightarrow$ length of arc as the lengths of the chords $\to 0$. If δx, δy are the increments in x, y from P_{r-1} to P_r,

$$P_{r-1}P_r^{\,2} = (\delta x)^2 + (\delta y)^2 \quad \text{(Pythagoras' theorem)} \qquad (1)$$

For an equation in the form $y = f(x)$, it will be convenient to work in terms of x. We therefore rewrite (1) as

$$P_{r-1}P_r^{\,2} = \left\{1 + \left(\frac{\delta y}{\delta x}\right)^2\right\}(\delta x)^2$$

$$\therefore P_{r-1}P_r = \sqrt{\left\{1 + \left(\frac{\delta y}{\delta x}\right)^2\right\}}\,\delta x$$

Summing for all the chords and proceeding to the limit,

$$\textbf{length of arc } s = \int_a^b \sqrt{\left\{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right\}}\,\mathrm{d}x$$

where a, b are the values of x corresponding to the ends of the arc. From this it follows that the arc length to a variable point on the curve is given by

$$s = \int_a^x \sqrt{\left\{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right\}}\,\mathrm{d}x \qquad (2)$$

$$\therefore \frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{\left\{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right\}}$$

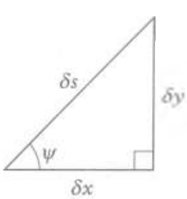


Figure 33.6

Do not try to memorise such formulae. Try instead to remember how to work them out when you need them. In this respect, it is easier to treat $P_{r-1}P_r$ as if it were δs, the increment in arc length. The triangle in **Fig. 33.6** helps us to remember the expression for arc length and to work out relationships such as $\frac{\mathrm{d}y}{\mathrm{d}s} = \sin\psi$, where ψ is the angle the tangent to the curve makes with the x-axis. We take the integrand of (2) to be positive; this fits in with the convention that the square root sign denotes the positive square root.

Questions

Q33 **a** Express $\tan\psi$ as a derivative.

b Express $\sec\psi$ as a derivative.

c Use the identity $\sec^2\psi = 1 + \tan^2\psi$ to express the derivative in **b** in terms of the derivative in **a**.

d Draw diagrams to show both ψ and the direction in which s is measured when $\frac{\mathrm{d}y}{\mathrm{d}x}$ is **i** positive, **ii** negative, and s is given by (2), above.

Q34 Find the length of the arc in the first quadrant of $y = 2x^{3/2}$ from $x = 0$ to $x = \frac{1}{3}$.

Q35 Find the length of the arc of $y = \ln\sec x$ from $x = -\frac{1}{6}\pi$ to $x = \frac{1}{6}\pi$.

Q36 Find the length of the arc of the parabola $y = x^2$ bounded by the line $y - 2 = 0$.

Arc length: parametric equations

Suppose we are given a curve in the form $x = f(t)$, $y = g(t)$ as, for example, $x = at^2$, $y = 2at$. It would then be more convenient, when finding the length of an arc, to have an integral with respect to t.

From (1) (opposite),

$$P_{r-1}P_r^{\,2} = \left\{\left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2\right\}(\delta t)^2$$

$$\therefore P_{r-1}P_r = \sqrt{\left\{\left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2\right\}}\,\delta t$$

Hence

$$s = \int_{t_1}^{t_2} \sqrt{\left\{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2\right\}}\,\mathrm{d}t$$

where t_1, t_2 are the values of t corresponding to the ends of the arc. Always make sure that the integrand is positive throughout the range of integration. This applies particularly to Q39.

Questions

Q37 Find the length of the arc from $\theta = 0$ to $\theta = \alpha$ of the curve given by $x = a\cos\theta$, $y = a\sin\theta$. What is this curve?

Q38 Find an expression for the distance measured along the curve from the origin to any point on the locus $x = at^2, y = at^3$.

Q39 Sketch the astroid given by $x = a\cos^3 t$, $y = a\sin^3 t$ and find the length of its circumference.

Q40 Sketch the arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ from $\theta = 0$ to $\theta = 2\pi$. Find its length.

Arc length: polar equations

Q41, below, introduces the formula for arc length in polar coordinates.

Question

***Q41** In **Fig. 33.7**, $P_{r-1}N$ is the perpendicular from P_{r-1} on to a neighbouring radius OP_r of a curve given in polar coordinates.

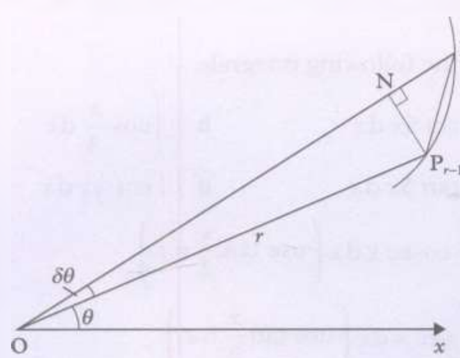


Figure 33.7

a Find approximations for NP_r, NP_{r-1} in terms of r, θ.

b Obtain the expression for arc length

$$s = \int_\alpha^\beta \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta$$

where α, β are the values of θ corresponding to the ends of the arc.

Questions

Q42 Find the length of the equiangular spiral $r = a\,e^{k\theta}$ from $\theta = 0$ to $\theta = 2\pi$.

Q43 Find the length of the spiral of Archimedes $r = a\theta$ from $\theta = 0$ to $\theta = \pi$.

Q44 What is the length of the circumference of the cardioid $r = a(1 + \cos\theta)$?
[Make sure the integrand is positive.]

Exercise 33d

1 Sketch the curve $r = 2 + \cos\theta$ and find its area.

2 Express the equation $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ in polar coordinates and find the sum of the areas enclosed by the loops.

3 P, Q are the points $(ca, c/a)$, $(c/a, ca)$ on the rectangular hyperbola $xy = c^2$. Find the area bounded by OP, OQ and the arc PQ.

4 $P(at^2, 2at)$ is a point on the parabola $y^2 = 4ax$ and S is the focus. Show that the area bounded by the parabola, its axis and the line PS is $\frac{1}{3}a^2(3t + t^3)$.

5 Find the area enclosed by the astroid $x = a\cos^3 t$, $y = a\sin^3 t$.

6 A circle of radius a rolls touching externally a fixed circle of radius $2a$. Show that the locus of a point on the smaller circle (a two-cusped epicycloid) may be written

$$x = 3a\cos\theta - a\cos 3\theta, \quad y = 3a\sin\theta - a\sin 3\theta$$

Find the length of this curve and also the area enclosed.

Exercise 33e (Revision of integration)

No list of 'standard integrals' is given in this book, since the authors consider that recognition of form is generally more important than learning formulae (see questions 13–15).

In this exercise,

Questions 1–7 summarise the main methods dealt with in Chapters 6, 8, 24, 25, 26, 33.

Questions 8–12 gather together the integrals of some trigonometrical functions and inverse functions, to enable you to assess your ability to handle these integrals.

Questions 13–15 are designed to develop discrimination in choice of method. These questions test essential skills, and recognition of form. You may confine your attention to these questions, together with some of the less obvious integrals in questions 8–12.

Find the integrals in questions 1–6.

1 **a** $\int x\sqrt{(x^2+1)}\,dx$ **b** $\int \frac{x^2+1}{\sqrt{(x^3+3x-4)}}\,dx$

c $\int \cos^5 u\,du$ **d** $\int \sec^6\theta\,d\theta$

e $\int \sec x\tan^5 x\,dx$ **f** $\int x\sin x^2\,dx$

g $\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}\,dx$ **h** $\int x(2x^2+3)^{-1}\,dx$

i $\int \frac{x}{e^{x^2}}\,dx$ **j** $\int \tan\frac{\theta}{2}\,d\theta$

2 *Change of variable*

a $\int x\sqrt{(2x-3)}\,dx$ **b** $\int 2x(3x-1)^7\,dx$

c $\int \frac{y(y-8)}{(y-4)^2}\,dy$ **d** $\int \frac{1}{\sqrt{(4-5y^2)}}\,dy$

e $\int \frac{1}{3+9u^2}\,du$ **f** $\int \frac{1}{u^2-6u+17}\,du$

g $\int \frac{1}{\sqrt{(7+4x-2x^2)}}\,dx$ **h** $\int \sqrt{(4-y^2)}\,dy$

i $\int \frac{1}{x\sqrt{(9x^2-1)}}\,dx$ **j** $\int \frac{1}{5+4\cos\theta}\,d\theta$

3 *Exponential and logarithmic functions*

a $\int e^{3x}\,dx$ **b** $\int 10^y\,dy$

c $\int \frac{x^2}{e^{x^3}}\,dx$ **d** $\int \frac{1}{3x}\,dx$

e $\int \frac{1}{3x+4}\,dx$ **f** $\int \frac{1}{3-2x}\,dx \quad (x>\frac{3}{2})$

g $\int \frac{1}{3x+9}\,dx$ **h** $\int \frac{1}{1-x^2}\,dx$

i $\int \ln x\,dx$

j $\int e^{\sqrt{x}}dx$ (write as $\int x^{1/2}x^{-1/2}e^{x^{1/2}}dx$)

4 *Partial fractions*

a $\int \frac{2}{9-x^2}\,dx$ **b** $\int \frac{1}{y(y-3)}\,dy$

c $\int \frac{1}{x^3-x^2}\,dx$ **d** $\int \frac{x}{(4-x)^2}\,dx$

e $\int \frac{2-x^2}{(x+1)^3}\,dx$ **f** $\int \frac{(x-2)^2}{x^3+1}\,dx$

5 *Integration by parts*

a $\int x\cos\frac{x}{2}\,dx$ **b** $\int \frac{x}{2}e^x\,dx$

c $\int y\,\mathrm{cosec}^2\,y\,dy$ **d** $\int 2y(1-3y)^6\,dy$

e $\int x\,3^x\,dx$ **f** $\int x\ln 2x\,dx$

g $\int \ln t\,dt$ **h** $\int \tan^{-1}3x\,dx$

i $\int 4^x\,dx$ **j** $\int x^3\sin x\,dx$

k Prove $\int \cos^4\frac{x}{2}\,dx = \frac{1}{2}\sin\frac{x}{2}\cos^3\frac{x}{2} + \frac{3}{4}\int\cos^2\frac{x}{2}\,dx$

6 *Splitting the numerator*

a $\int \frac{2x-1}{4x^2+3}\,dx$ **b** $\int \frac{1-4y}{\sqrt{(1+2y-y^2)}}\,dy$

c $\int \frac{\cos\theta}{2\cos\theta-\sin\theta}\,d\theta$ **d** $\int \frac{\cos x-2\sin x}{3\cos x+4\sin x}\,dx$

7 Evaluate the following.

a $\int_{1/3}^{2/3} \frac{1}{\sqrt{(4-9x^2)}}\,dx$ **b** $\int_1^{\sqrt{2}} \frac{1}{8+y^2}\,dy$

c $\int_5^{\infty} \frac{1}{(x-3)^2}\,dx$ **d** $\int_0^{\pi/2} \cos^{11}x\,dx$

e $\int_0^{\pi/2} \sin^{12}\theta\,d\theta$ **f** $\int_0^{\pi/8} \cos^6 4y\,dy$

g $\int_{-\pi/2}^{\pi/2} \sin^8 u\,du$ **h** $\int_0^{\pi} \cos^7 x\,dx$

i $\int_0^{\pi/2} \cos^9\theta\sin^{10}\theta\,d\theta$ **j** $\int_{-1}^{+1} \frac{1}{2x-3}\,dx$

8 Find the following integrals.

a $\int \sin 5x\,dx$ **b** $\int \cos\frac{x}{3}\,dx$

c $\int \tan 5x\,dx$ **d** $\int \cot\frac{1}{2}x\,dx$

e $\int \mathrm{cosec}\,x\,dx \left(\text{use } \tan\frac{x}{2}=t\right)$

f $\int \sec x\,dx \left(\text{use } \tan\frac{x}{2}=t\right)$

9 Find the following integrals.

a $\int \sec^2\frac{x}{3}\,dx$ **b** $\int \mathrm{cosec}^2\,4x\,dx$

c $\int \sin^2 x\,dx$ **d** $\int \cos^2 x\,dx$

e $\int \tan^2 x\,dx$ **f** $\int \cot^2 x\,dx$

10 Find the following integrals.

a $\int \sin^3 x\,dx$ ***b** $\int \cos^3 x\,dx$

c $\int \tan^3 x\,dx$ (use Pythagoras' theorem)

***d** $\int \cot^3 x\,dx$ (use Pythagoras' theorem) e $\int \sec^3 x \, dx$ (by reduction)

*f $\int \operatorname{cosec}^3 x \, dx$ (by reduction)

11 Find the following integrals (**a** and **b** by expressing the integrands in terms of cos $2x$, cos $4x$, or by reduction, the remainder by using Pythagoras' theorem).

a $\int \sin^4 x \, dx$ **b** $\int \cos^4 x \, dx$

c $\int \tan^4 x \, dx$ **d** $\int \operatorname{cosec}^4 x \, dx$

e $\int \sec^4 x \, dx$ **f** $\int \cot^4 x \, dx$

12 Find the following integrals using integration by parts (in **e** and **f** continue by using the change of variable $x = \sec u$).

a $\int \sin^{-1} x \, dx$ **b** $\int \cos^{-1} x \, dx$

c $\int \tan^{-1} x \, dx$ **d** $\int \cot^{-1} x \, dx$

e $\int \sec^{-1} x \, dx$ **f** $\int \operatorname{cosec}^{-1} x \, dx$

Find the integrals in questions 13–17.

13 a $\displaystyle\int \frac{1}{3+4x^2} \, dx$ **b** $\displaystyle\int \frac{x}{\surd(5+8x^2)} \, dx$

c $\displaystyle\int \frac{1}{\surd(1+x^2)} \, dx$ **d** $\displaystyle\int \frac{x}{2+3x^2} \, dx$

e $\int x\surd(3+x^2) \, dx$ **f** $\displaystyle\int \frac{x+1}{3+2x^2} \, dx$

g $\displaystyle\int \frac{x-2}{x^2-4x+7} \, dx$ **h** $\int \surd(2+x^2) \, dx$

i $\displaystyle\int \frac{3x-11}{x^2-4x+5} \, dx$ **j** $\int x\surd(2+3x) \, dx$

14 a $\displaystyle\int \frac{1}{\surd(4-5x^2)} \, dx$ **b** $\displaystyle\int \frac{x}{\surd(1-3x)} \, dx$

c $\displaystyle\int \frac{2}{9-x^2} \, dx$ **d** $\displaystyle\int \frac{3}{(16-x)^2} \, dx$

e $\int x\surd(6-x^2) \, dx$ **f** $\displaystyle\int \frac{3x}{4-x^2} \, dx$

g $\int \surd(4-x^2) \, dx$ **h** $\displaystyle\int \frac{x}{\surd(7-2x^2)} \, dx$

i $\displaystyle\int \frac{x-2}{\surd(3-4x^2)} \, dx$ **j** $\displaystyle\int \frac{1}{\surd(x^2-9)} \, dx$

15 a $\int \cos x^\circ \, dx$ **b** $\int x \sin 2x \cos 2x \, dx$

c $\displaystyle\int \sec\frac{\theta}{2} \operatorname{cosec}\frac{\theta}{2} \, d\theta$ **d** $\int \cos^6 x \sin^5 x \, dx$

e $\int y \sec^2 y \, dy$ **f** $\int x \sin x \, dx$

g $\int x \sin x^2 \, dx$ **h** $\int u^2 \cos u \, du$

i $\int \sin^2 y \cos^2 y \, dy$ **j** $\int \sin 5x \cos 2x \, dx$

*The change of variable $y = \frac{1}{2}\pi - x$ may be used.

Chapter 34
Differential equations

34.1 Introduction to differential equations

An equation containing *differential coefficients* such as $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, is called a **differential equation**.

A **solution** of a differential equation is an equation relating x and y containing *no* differential coefficients.

Given the differential equation $\frac{dy}{dx} = 3$, we obtain the **general solution** $y = 3x + c$, which is the equation of all straight lines of gradient 3. If the data also includes the fact that $y = 5$ when $x = 1$, we can determine that $c = 2$, and we obtain the *particular solution* $y = 3x + 2$.

In simple graphical terms,

a a **differential equation** defines some property common to a family of curves,

b the **general solution**, involving one or more arbitrary constants, is the equation of *any* member of the family,

c a **particular solution** is the equation of *one* member of the family.

Consider the differential equation $\frac{dy}{dx} = x$. We can easily solve this with our existing knowledge. Before we do so, consider for a moment what this differential equation tells us: it says that, for any value of x, the gradient is equal to x. This information is illustrated in Fig. 34.1.

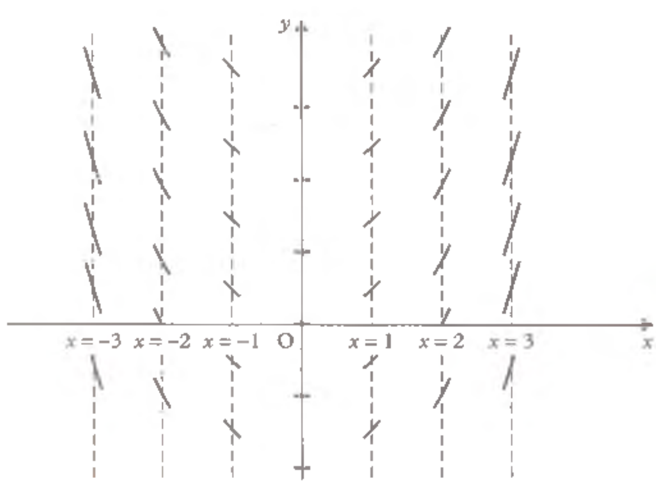


Figure 34.1

The solution of the differential equation is

$$y = \tfrac{1}{2}x^2 + c$$

(The constant of integration, c, is usually called 'an **arbitrary constant**'). Equations of this form represent parabolas with the y-axis as the axis of symmetry (see Fig. 34.2).

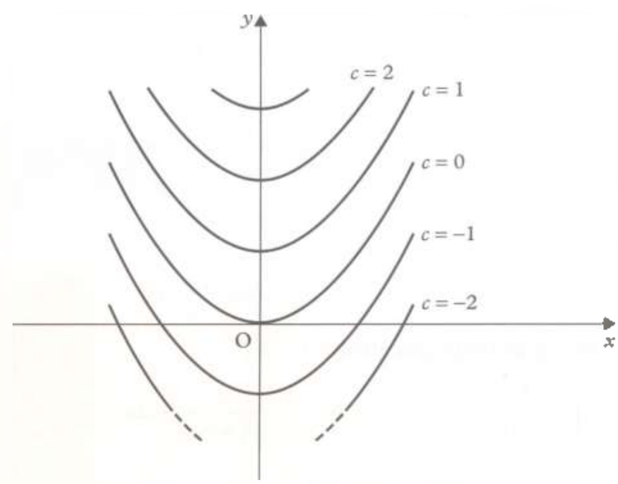


Figure 34.2

If we are given some further information, say when $x = 0$, $y = 2$, then we can find c and identify the particular parabola with this property, in this case $y = \tfrac{1}{2}x^2 + 2$.

Questions

Q1 Find the general solution of $\frac{d^2y}{dx^2} = 0$.

What is the particular solution given by $\frac{dy}{dx} = 3$, and $y = -2$ when $x = 1$?

Q2 Find the general solution of $\frac{dy}{dx} = 3x^2$.

Illustrate with a sketch.

Q3 For any circle centre the origin $\frac{dy}{dx} = -\frac{x}{y}$.

Solve this equation by writing it as $y\frac{dy}{dx} = -x$. $\left(\text{What is } \frac{d}{dx}(y^2)?\right)$

Q4 Find the general solution of $\frac{d^2s}{dt^2} = a$, where a is a constant. What does this become, given the initial conditions $s = 0$ and $\frac{ds}{dt} = u$ when $t = 0$?

Definition

The **order of a differential equation** *is determined by the highest differential coefficient present.*

Thus the equations in Q1 and Q4 are of the *second order*, whereas those in Q2 and Q3 are of the *first order*.

Since each step of integration introduces an arbitrary constant, it is generally true that the order of a differential equation indicates the number of arbitrary constants in the general solution.

This suggests that from an equation involving x, y, and n arbitrary constants there may be formed (by differentiating n times and eliminating the constants) a differential equation of order n.

Questions

Q5 Form differential equations by differentiating and eliminating the constants A, B from the following.

a $y = Ax + B$ **b** $y = Ax$

c $r = A\cos\theta$ **d** $xy = A$

e $y = A\,e^x$ **f** $y = e^{Bx}$

g $y = A\,e^{Bx}$ **h** $y = A\ln x$

i $x = \tan(Ay)$

Q6 Confirm the given general solution of each of the following differential equations.

a $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 0, \quad y = A\,e^{2x} + Be^{-x}$

b $\dfrac{d^2x}{dt^2} - 4\dfrac{dx}{dt} + 4x = 0, \quad x = e^{2t}(A + Bt)$

We now classify some of the simpler forms of differential equations.

34.2 First order differential equations

Separating the variables

The solutions of $\dfrac{dy}{dx} = f(x)$ and $\dfrac{dy}{dx} = f(y)$ $\left(\text{which may be written } \dfrac{dx}{dy} = \dfrac{1}{f(y)}\right)$ depend upon the integrals $\int f(x)\,dx$ and $\int \dfrac{1}{f(y)}\,dy$.

There are other differential equations that can be integrated directly provided they have been written in a suitable form.

Consider

$$\frac{dy}{dx} = xy \qquad (1)$$

We write this as

$$\frac{1}{y}\frac{dy}{dx} = x$$

Then integrating each side with respect to x,

$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int x\,dx$$

[But from §24.4 on page 261 we know that

$$\int f(y)\,dy = \int f(y)\frac{dy}{dx}\,dx.]$$

$$\therefore \int \frac{1}{y}\,dy = \int x\,dx \qquad (2)$$

$$\therefore \ln y + c = \tfrac{1}{2}x^2$$

$$\therefore \ln(ky) = \tfrac{1}{2}x^2, \quad \text{or} \quad y = A\,e^{x^2/2}$$

Note how the arbitrary constant of integration appears in different forms; we have written c as $\ln k$, and A as $1/k$.

Look back at (1) and (2) in the above working.
The symbols dx, dy have as yet no meaning in isolation; they have been used only in composite symbols such as $\dfrac{dy}{dx}$, $\dfrac{d}{dx}f(x)$, $\int f(x)\,dx$. However it is convenient to think of dx as an 'x-factor', and dy as a 'y-factor', and proceed direct from (1) to (2) by 'separating the variables' and adding the integral sign. The intervening lines provide the justification for this. See Example 1.

Example 1 *Solve* $x^2\dfrac{dy}{dx} = y(y-1)$.

$$x^2\frac{dy}{dx} = y(y-1)$$

Separating the variables,

$$\int \frac{1}{y(y-1)}\,dy = \int \frac{1}{x^2}\,dx$$

$$\therefore \int\left\{\frac{1}{y-1} - \frac{1}{y}\right\}dy = \int \frac{1}{x^2}\,dx$$

$$\therefore \ln\frac{k(y-1)}{y} = -\frac{1}{x}$$

or $$k(y-1) = y\,e^{-1/x}$$

Questions

Q7 Solve the following differential equations, and check solutions by differentiation and elimination of arbitrary constants.

a $\dfrac{dy}{dx} = \dfrac{x}{y}$ **b** $\dfrac{dy}{dx} = \dfrac{y}{x}$

c $\dfrac{dx}{dy} = xy$ **d** $x\dfrac{dy}{dx} = \tan y$

e $e^{-x}\dfrac{dy}{dx} = y^2 - 1$ **f** $\sqrt{(x^2+1)}\dfrac{dy}{dx} = \dfrac{x}{y}$

Q8 $v\dfrac{dv}{ds} = a$, where a is a constant. Solve this equation given that $v = u$ when $s = 0$.

Many physical problems are expressed in terms of differential equations (If you are studying applied mathematics or physics you have probably met some already). Solving, or at least attempting to solve, a differential equation is a very common task for scientists. The problem frequently arises in other disciplines, such as economics. What follows is an important application of the subject in physics.

It is known that radioactive substances decay at a rate which is proportional to the amount of the radioactive substance present. If we use x to represent the amount present at time t, we can express this in the form of a differential equation, namely,

$$\frac{dx}{dt} = -kx \quad \text{where } k \text{ is a positive constant.}$$

For different substances, the rate of decay is different; it is usual to quote the 'half-life' of the substance, that is, the time it takes for half of the original quantity to decay. For radium the half-life is about 1600 years. We shall now solve the differential equation, that is, express x as a function of t, and hence find the value of k. We shall then use the solution to find the percentage of a given sample of radium which would still exist after a lapse of 200 years in storage. [Remember, in the following working, to distinguish between the arbitrary constant of integration A (or x_0), and k, a constant which is determined by the half-life of radium.]

$$\frac{dx}{dt} = -kx$$

Separating the variables gives

$$\int \frac{1}{x}\,dx = \int -k\,dt$$

and integrating,

$$\begin{aligned} \ln x &= -kt + c \\ \therefore\ x &= e^{-kt+c} \\ &= e^c \times e^{-kt} \end{aligned}$$

and replacing e^c by A, we can write

$$\boldsymbol{x = Ae^{-kt}}$$

This is the general solution of the differential equation. (This particular differential equation is extremely common, and, unless specific instructions to the contrary are given, the solution $x = Ae^{-kt}$ may be quoted.)
We now continue with the solution:

When $t = 0$, $x = A$, in other words A is the original value of x. It is convenient to write this as x_0, so

$$x = x_0 e^{-kt}$$

Now, we are told that when $t = 1600$, $x = \tfrac{1}{2}x_0$, consequently

$$\begin{aligned} \tfrac{1}{2}x_0 &= x_0 e^{-1600k} \\ \therefore\ \tfrac{1}{2} &= e^{-1600k} \end{aligned}$$

i.e. $e^{1600k} = 2$

Taking natural logarithms,

$$1600k = \ln 2,$$

$$k = \frac{\ln 2}{1600}$$

Hence the solution can be expressed

$$x = x_0 e^{-(\ln 2/1600)t}$$

This, in turn, can be written

$$x = x_0 (e^{\ln 2})^{-t/1600}$$

But $e^{\ln 2} = 2$, (see §25.3 on page 270; this step is extremely common in this topic), hence

$$x = x_0 2^{-t/1600}$$

(We can verify by inspection that when $t = 1600$, $x = \tfrac{1}{2}x_0$. It is important to check your work like this whenever possible.)

Finally, when $t = 200$,

$$\begin{aligned} x &= x_0 2^{-1/8} \\ &= 0.917x_0 \end{aligned}$$

In other words, after 200 years, 91.7% of the original radioactive radium still exists.

Exercise 34a

1 By differentiating and eliminating the constants A and B from the following equations, form differential equations, and illustrate the geometrical significance of each.

- **a** $3x - 2y + A = 0$
- **b** $Ax + 2y + 1 = 0$
- **c** $Ax + By = 0$
- **d** $x^2 + y^2 = A$
- **e** $y = Ax^{-1}$
- **f** $y = A(x - 4)$

2 By differentiating and eliminating the constants A and B from the following equations, form differential equations.

- **a** $y = A\cos(3t + B)$
- **b** $y = A + B\,e^{3t}$
- **c** $y = A\,e^{3x} + B\,e^{-3x}$
- **d** $y = A\,e^{3x} + B\,e^{-2x}$
  (first multiply each side by e^{2x})
- **e** $y = e^{4x}(A + Bx)$
  $\left(\text{first show that } \dfrac{dy}{dx} = 4y + B\,e^{4x}\right)$

3 Obtain the equation of the straight line of gradient $\frac{3}{10}$, which passes through $(5, -2)$, by finding a particular solution of the differential equation $\dfrac{dy}{dx} = \frac{3}{10}$.

4 A family of parabolas has the differential equation $\dfrac{dy}{dx} = 2x - 3$. Find the equation of the member of the family which passes through $(4, 5)$.

5 Find the general solution of the differential equation $6t\dfrac{dt}{ds} + 1 = 0$, and the particular solution given by the condition $s = 0$ when $t = -2$.

6 Find the particular solutions of the differential equation

$$\operatorname{cosec} x\frac{dy}{dx} = e^{y}\operatorname{cosec} x + 3x$$

given by the conditions

- **a** $y = 0$ when $x = 0$,
- **b** $y = 3$ when $x = \pi/2$.

7 Find the general solutions of the following differential equations.

- **a** $\dfrac{dy}{dx} = y$
- **b** $\dfrac{1}{x}\dfrac{dy}{dx} = \sqrt{(x-1)}$
- **c** $(x+2)\dfrac{dy}{dx} = y$
- **d** $\dfrac{dy}{dx} = \sec^2 y$
- **e** $\dfrac{dv}{du} = v(v-1)$
- **f** $\ln x\dfrac{dx}{dy} = 1$
- **g** $\dfrac{dy}{dx} = \tan y$
- **h** $\tan^{-1} y\dfrac{dy}{dx} = 1$
- **i** $y\dfrac{dy}{dx} = x - 1$
- **j** $(x^2 - 1)\dfrac{dy}{dx} = y$
- **k** $\dfrac{d\theta}{dr} = \sin\theta$
- **l** $x^2\dfrac{dy}{dx} = y + 3$
- **m** $x\dfrac{dy}{dx} = y + xy$
- **n** $\dfrac{d\phi}{d\theta} = \tan\phi\tan\theta$
- **o** $\theta\dfrac{d\theta}{dr} = \cos^2\theta$
- **p** $\dfrac{y}{x}\dfrac{dy}{dx} = \ln x$
- **q** $2\sin\theta\dfrac{d\theta}{dr} = \cos\theta - \sin\theta$
- **r** $x\dfrac{dy}{dx} - 3 = 2\left(y + \dfrac{dy}{dx}\right)$
- **s** $e^{t}\dfrac{dx}{dt} = \sin t$
- **t** $e^{x}\dfrac{dy}{dx} + y^2 + 4 = 0$

8 Find the particular solutions of the following differential equations which satisfy the given conditions.

- **a** $(1 + \cos 2\theta)\dfrac{dy}{d\theta} = 2, \quad y = 1$ when $\theta = \pi/4$
- **b** $\dfrac{dy}{dx} = x(y - 2), \quad y = 5$ when $x = 0$
- **c** $(1 + x^2)\dfrac{dy}{dx} = 1 + y^2, \quad y = 3$ when $x = 2$
- **d** $\dfrac{dy}{dx} = \sqrt{(1 - y^2)}, \quad y = 0$ when $x = \pi/6$

9 According to Newton's law of cooling, the rate at which the temperature of a body falls is proportional to the amount by which its temperature exceeds that of its surroundings. Suppose the temperature of an object falls from 200° to 100° in 40 minutes, in a surrounding temperature of 10°. Prove that after t minutes, the temperature, T degrees, of the body is given by

$$T = 10 + 190e^{-kt} \quad \text{where } k = \tfrac{1}{40}\ln\left(\tfrac{19}{9}\right)$$

Calculate the time it takes to reach 50°.

10 A tank contains a solution of salt in water. Initially the tank contains 1000 litres of water with 10 kg of salt dissolved in it. The mixture is poured off at a rate of 20 litres per minute, and simultaneously pure water is added at a rate of 20 litres per minute. All the time the tank is stirred to keep the mixture uniform. Find the mass of salt in the tank after 5 minutes. The tank must be topped up by adding more salt when the mass of salt in the tank falls to 5 kg; after how many minutes will it need topping up?

34.3 First order exact equations

The variables in the equation

$$2xy\frac{dy}{dx} + y^2 = e^{2x}$$

may not be separated. However, the L.H.S. is $\frac{d}{dx}(xy^2)$ and the equation can be solved by integrating each side with respect to x. This is called an **exact equation** and the solution is

$$xy^2 = \tfrac{1}{2}e^{2x} + A$$

Question

Q9 Solve the following exact equations.

a $x^2\frac{dy}{dx} + 2xy = 1$

b $\frac{t^2}{x}\frac{dx}{dt} + 2t \ln x = 3\cos t$

c $x^2 \cos u\frac{du}{dx} + 2x\sin u = \frac{1}{x}$

d $e^y + x\,e^y\frac{dy}{dx} = 2$

Integrating factors

Some differential equations are **non-exact**. They may be made exact by multiplying each side by an **integrating factor**.

Example 2 *Solve* $xy\frac{dy}{dx} + y^2 = 3x$.

We cannot separate the variables. Can we find a function whose derivative is the L.H.S. as above? No. Then can we find a function whose derivative is $f(x) \times$ L.H.S.?

$$\frac{d}{dx}(xy^2) = y^2 + 2xy\frac{dy}{dx}; \quad \text{this is no good.}$$

$$\frac{d}{dx}(x^2y^2) = 2xy^2 + 2x^2y \times \frac{dy}{dx}$$
$$= 2x\left(y^2 + xy\frac{dy}{dx}\right)$$
$$= 2x \times \text{L.H.S.}$$

The required integrating factor is $2x$.

$$xy\frac{dy}{dx} + y^2 = 3x$$

Multiplying each side by $2x$,

$$2x^2y\frac{dy}{dx} + 2xy^2 = 6x^2$$
$$\therefore x^2y^2 = 2x^3 + A$$

Question

Q10 Find the integrating factors required to make the following differential equations into exact equations, and solve them.

a $x\frac{dy}{dx} + 2y = e^{x^2}$ **b** $x\,e^y\frac{dy}{dx} + 2e^y = x$

c $2x^2y\frac{dy}{dx} + xy^2 = 1$

d $r\sec^2\theta + 2\tan\theta\frac{dr}{d\theta} = 2r^{-1}$

34.4 First order linear equations

A differential equation is *linear in y* if it is of the form

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + P_{n-1}\frac{dy}{dx} + P_ny = Q$$

where $P_1, P_2, \ldots, P_n, Q$ are functions of x, or constants. This example is of the nth order.

Thus a *first order linear equation* is of the form

$$\frac{dy}{dx} + Py = Q$$

where P, Q are functions of x or constants. This type of differential equation deserves special attention because it's integrating factor, when required and if obtainable, is of a standard form.

Let us assume that the general first order linear equation given above can be made into an exact equation by using the integrating factor R, a function of x. If this is so,

$$R\frac{dy}{dx} + RPy = RQ \qquad (1)$$

is an exact equation. It is apparent from the first term that the L.H.S. of (1) is $\frac{d}{dx}(Ry) = R\frac{dy}{dx} + y\frac{dR}{dx}$.
Thus (1) may also be written

$$R\frac{dy}{dx} + y\frac{dR}{dx} = RQ \qquad (2)$$

Equating the second terms on the L.H.S. of (1) and (2),

$$y\frac{dR}{dx} = RPy$$

$$\therefore \frac{dR}{dx} = RP$$

Separating the variables,

$$\int \frac{1}{R}\,dR = \int P\,dx$$

$$\therefore \ln R = \int P\,dx$$

$$\therefore R = e^{\int P\,dx}$$

Thus the required **integrating factor is $e^{\int P\,dx}$**. The initial assumption that an integrating factor exists is therefore justified provided that it is possible to find $\int P\,dx$.

Example 3 *Solve the differential equation $\frac{dy}{dx} + 3y = e^{2x}$, given that $y = \frac{6}{5}$ when $x = 0$.*

The integrating factor is $e^{\int 3\,dx} = e^{3x}$. Multiplying each side of the given equation by e^{3x},

$$e^{3x}\frac{dy}{dx} + 3e^{3x}y = e^{5x}$$

$$\therefore e^{3x}y = \tfrac{1}{5}e^{5x} + A$$

Therefore the general solution is

$$y = \tfrac{1}{5}e^{2x} + A\,e^{-3x}$$

But $y = \frac{6}{5}$ when $x = 0$, $\quad\therefore \frac{6}{5} = \frac{1}{5} + A$, $\quad\therefore A = 1$.

Therefore the particular solution is

$$y = \tfrac{1}{5}e^{2x} + e^{-3x}$$

Example 4 *Solve $\frac{dy}{dx} + y\cot x = \cos x$.*

The integrating factor is

$$e^{\int \cot x\,dx} = e^{\ln \sin x} = \sin x$$

Multiplying each side of the given equation by $\sin x$,

$$\sin x\frac{dy}{dx} + y\cos x = \cos x \sin x$$

$$\therefore y\sin x = \tfrac{1}{2}\sin^2 x + A$$

Therefore the general solution is

$$y = \tfrac{1}{2}\sin x + A\operatorname{cosec} x$$

Questions

Q11 Find the general solution of $\frac{dy}{dx} + 2xy = x$.

What is the particular solution given by $y = -\frac{1}{2}$ when $x = 0$?

Q12 Show that the equation in Q10 **a** is of the type under discussion, and find the required integrating factor as $e^{\int P\,dx}$.

Q13 Solve:

a $\frac{dy}{dx} - y\tan x = x$ **b** $\frac{dy}{dx} + y + 3 = x$

34.5 First order homogeneous equations

In a **homogeneous differential equation** all the terms are of the same dimension. For example, suppose x and y measure units of length. The term x^2y is of dimension $(\text{length})^3$, or L^3;
the term $\frac{(x^2+y^2)^2}{x}$ is of dimension $\frac{L^4}{L} = L^3$.

The dimensions of some other terms are given below:

$$\frac{y}{x} \qquad \frac{L}{L} = L^0$$

$$\frac{y}{x^2} \qquad \frac{L}{L^2} = L^{-1}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0}\frac{\delta y}{\delta x} \qquad \frac{L}{L} = L^0$$

$$\frac{d^2y}{dx^2} = \lim_{\delta x \to 0} \frac{\delta\left(\frac{dy}{dx}\right)}{\delta x} \qquad \frac{L^0}{L} = L^{-1}$$

$$x\frac{dy}{dx} \qquad L \times L^0 = L^1$$

$$\frac{\left(\frac{y^2}{x}\right)^2}{\frac{dy}{dx}} \qquad \frac{L^2}{L^0} = L^2$$

Questions

Q14 Pick out the member of each of the following groups of terms and expressions which is not of the same dimensions as the rest:

a $xy\frac{dy}{dx}$, $y^3\frac{d^2y}{dx^2}$, $\left(\frac{y}{x}\right)^2$, x^2+y^2

b $(x+y)^2\frac{dy}{dx}$, $x^2\left(1+\frac{y}{x}\right)$, $\left(\frac{dy}{dx}\right)^2 xy$, $\frac{d^2y}{dx^2}+xy$

c $(y+2x)\frac{dy}{dx}$, $(y^2-x^2)\frac{d^2y}{dx^2}$, $x\surd(x^2+y^2)$, $2x+\frac{y^2}{x}$

Q15 Which of the following equations are homogeneous?

a $x^2\frac{dy}{dx}=y^2$ **b** $xy\frac{dy}{dx}=x^2+y^2$

c $x^2\frac{dy}{dx}=1+xy$ **d** $x^2\frac{d^2y}{dx^2}=y\frac{dy}{dx}$

e $(x^2-y^2)\frac{dy}{dx}=2xy$

f $(1+y^2)\frac{dy}{dx}=x^2$

A first order homogeneous equation is of the form

$$P\frac{dy}{dx}=Q$$

Since $\frac{dy}{dx}$ is of dimension 0, P and Q are homogeneous functions of x and y of the same dimensions, i.e. of the same degree. The significant point to note is that, if P and Q are of degree n, we may divide each side of the equation by x^n and thereby obtain

$$P'\frac{dy}{dx}=Q'$$

where P' and Q' are functions of y/x.

For example, the equation

$$xy\frac{dy}{dx}=x^2+y^2$$

when each side is divided by x^2, becomes

$$\frac{y}{x}\frac{dy}{dx}=1+\left(\frac{y}{x}\right)^2$$

This suggests the substitutions

$$\frac{\mathbf{y}}{\mathbf{x}}=\mathbf{u} \quad \text{and, since } y=ux, \quad \frac{\mathbf{dy}}{\mathbf{dx}}=\mathbf{u}+\mathbf{x}\frac{\mathbf{du}}{\mathbf{dx}}$$

Example 5 *Solve* $xy\frac{dy}{dx}=x^2+y^2$.

Dividing each side by x^2,

$$\frac{y}{x}\frac{dy}{dx}=1+\left(\frac{y}{x}\right)^2$$

Let $y=ux$, then $\frac{dy}{dx}=u+x\frac{du}{dx}$.

$$\therefore u\left(u+x\frac{du}{dx}\right)=1+u^2$$

$$\therefore ux\frac{du}{dx}=1$$

Separating the variables,

$$\int u\,du=\int\frac{1}{x}\,dx$$

$$\therefore \tfrac{1}{2}u^2=\ln(Bx)$$

$$\therefore \left(\frac{y}{x}\right)^2=2\ln(Bx)$$

$$\therefore \left(\frac{y}{x}\right)^2=\ln(Ax^2) \quad \text{where } A=B^2$$

Therefore the general solution is

$$y^2=x^2\ln(Ax^2)$$

Questions

Q16 Solve the following equations by the method of Example 5,

a $x^2\dfrac{dy}{dx} = y^2 + xy$ b $x\dfrac{dy}{dx} = x - y$

c $x^2\dfrac{dy}{dx} = 2y^2$

Q17 Solve the equation $x\dfrac{dy}{dx} = 2x + y$

a by the method of Example 5,
b by the method of Example 4.

Q18 Solve the equations in Q16 b and c *not* using the method of Example 5.

The above questions serve not only to illustrate the method under discussion but also to stress that the types of equations given in this chapter are not all mutually exclusive.

Exercise 34b

1 Solve the following exact differential equations.

a $y^2 + 2xy\dfrac{dy}{dx} = \dfrac{1}{x^2}$ b $xy^2 + x^2y\dfrac{dy}{dx} = \sec^2 2x$

c $\ln y + \dfrac{x}{y}\dfrac{dy}{dx} = \sec x \tan x$

d $(1 - 2x)\, e^y\dfrac{dy}{dx} - 2e^y = \sec^2 x$

e $2t\, e^s + t^2\, e^s\dfrac{ds}{dt} = \sin t + t\cos t$

f $e^u\, r^2 + 2r\, e^u\dfrac{dr}{du} = -\operatorname{cosec}^2 u$

2 Find, by inspection, the integrating factors required to make the following differential equations into exact equations, and solve them.

a $\sin y + \dfrac{1}{2}x\cos y\dfrac{dy}{dx} = 3$ b $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{e^x}{x}$

c $\dfrac{1}{x}\tan y + \sec^2 y\dfrac{dy}{dx} = 2e^{x^2}$

d $y\, e^x + y^2\, e^x\dfrac{dx}{dy} = 1$

3 Solve the following first order linear equations.

a $\dfrac{dy}{dx} + 2y = e^{-2x}\cos x$

b $\dfrac{1}{t}\dfrac{ds}{dt} = 1 - 2s$

c $\dfrac{dy}{dx} + (2x + 1)y - e^{-x^2} = 0$

d $\dfrac{dr}{d\theta} + 2r\cot\theta = \operatorname{cosec}^2\theta$

e $\dfrac{dr}{d\theta} + r\tan\theta = \cos\theta$ f $x\dfrac{dy}{dx} + 2y = \dfrac{\cos x}{x}$

g $x\dfrac{dy}{dx} - y = \dfrac{x}{x-1}$ h $2x\dfrac{dy}{dx} = x - y + 3$

i $\sin x\dfrac{dy}{dx} + y = \sin^2 x$

j $3y + (x - 2)\dfrac{dy}{dx} = \dfrac{2}{x-2}$

4 Solve the following homogeneous equations.

a $x^2\dfrac{dy}{dx} = 3x^2 + xy$ b $xy\dfrac{dy}{dx} = x^2 - y^2$

c $x^2\dfrac{dy}{dx} = x^2 + xy + y^2$ d $3x^2\dfrac{dy}{dx} = y^2$

e $(x^2 + y^2)\dfrac{dy}{dx} = xy$ f $(4x - y)\dfrac{dy}{dx} = 4x$

g $x\dfrac{dy}{dx} = y + \sqrt{(x^2 - y^2)}$ h $x\dfrac{dy}{dx} = x + 2y$

i $y\dfrac{dy}{dx} = 2x + y$ j $x^2\dfrac{dy}{dx} = x^2 + y^2$

***5** Solve the equation $\dfrac{dy}{dx} = \dfrac{x - y + 2}{x + y}$, reducing it to a homogeneous equation by the change of variables $x = X - 1$, $y = Y + 1$. (Note that this implies a change of origin to $(-1, 1)$ the point of intersection of the straight lines $x - y + 2 = 0$ and $x + y = 0$; see §28.3 on page 296. The new axes are parallel to the old so $\dfrac{dy}{dx} = \dfrac{dY}{dX}$.)

6 Solve the following equations by the method indicated in question 5.

a $\dfrac{dy}{dx} = \dfrac{y - 2}{x + y - 5}$ b $2y\dfrac{dy}{dx} = x + y - 3$

***7** State why the equation $\dfrac{dy}{dx} = \dfrac{y - x + 2}{y - x - 4}$ may not be reduced to a homogeneous equation by the method of question 5. Solve it by the change of variable $y - x = z$.

8 Solve the following equations by the method indicated in question 7.

a $\dfrac{dy}{dx} = \dfrac{2x+y-2}{2x+y+1}$ b $(x+y)\dfrac{dy}{dx} = x+y-2$

9 Solve the following differential equations.

a $(x+3)\dfrac{dy}{dx} - 2y = (x+3)^3$

b $x^2\dfrac{dy}{dx} = x^2 - xy + y^2$

c $\dfrac{dy}{dx} + (y+3)\cot x = e^{-2x}\operatorname{cosec} x$

d $\sin y + (x+3)\cos y\dfrac{dy}{dx} = \dfrac{1}{x^2}$

e $(x-4y+2)\dfrac{dy}{dx} = x+y-3$

f $\dfrac{dy}{dx} = y+2+e^{2x}(x+1)$

g $2y\ln y + x\dfrac{dy}{dx} = \dfrac{y}{x}\cot x$

h $2\tan\theta\dfrac{dr}{d\theta} + (2r+3)\tan^2\theta + 2r = 0$

i $x(y-x)\dfrac{dy}{dx} = y(x+y)$ j $\dfrac{dy}{dx} = \dfrac{x-y+1}{x-y+3}$

10 Find the particular solutions of the following differential equations which satisfy the given conditions.

a $(x+1)\dfrac{dy}{dx} - 3y = (x+1)^4$, $y = 16$ when $x = 1$

b $\dfrac{du}{d\theta} + u\cot\theta = 2\cos\theta$, $u = 3$ when $\theta = \dfrac{\pi}{2}$

c $(x+y)\dfrac{dy}{dx} = x-y$, $y = -2$ when $x = 3$

d $(x^2-y^2)\dfrac{dy}{dx} = xy$, $y = 2$ when $x = 4$

e $x - 1 + \dfrac{dx}{dt} = e^{-t}t^{-2}$, $x = 1$ when $t = 1$

34.6 Second order equations reducible to first order

In this section and the next, we consider some special second order differential equations which can be reduced to first order form. In §34.8 on page 367 we will study more general second order differential equations.

To the form $\dfrac{d^2y}{dx^2} = f(x)$ we may apply direct integration twice; likewise to an equation such as

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2x$$

which is exact, giving

$$x\frac{dy}{dx} = x^2 + A \quad \text{and} \quad y = \frac{1}{2}x^2 + A\ln x + B$$

Of wide application to other forms of second order equations is **the substitution** $\dfrac{dy}{dx} = p$, from which we obtain

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\times\frac{dy}{dx} = p\frac{dp}{dy}$$

Thus

a the equation $\dfrac{d^2y}{dx^2} = f(y)$ becomes $p\dfrac{dp}{dy} = f(y)$,

b an equation containing $\dfrac{d^2y}{dx^2}, \dfrac{dy}{dx}, y$ but *with x absent*, becomes a *first order* equation containing $p\dfrac{dp}{dy}, p, y$,

c an equation containing $\dfrac{d^2y}{dx^2}, \dfrac{dy}{dx}, x$ but *with y absent*, becomes a *first order* equation containing $\dfrac{dp}{dx}, p, x$.

Example 6 *Solve* $(1+x^2)\dfrac{d^2y}{dx^2} = 2x\dfrac{dy}{dx}$.

Let $\dfrac{dy}{dx} = p$, and since y is absent, write $\dfrac{d^2y}{dx^2}$ as $\dfrac{dp}{dx}$.

$$(1+x^2)\frac{dp}{dx} = 2xp$$

Separating the variables,

$$\int\frac{1}{p}\,dp = \int\frac{2x}{1+x^2}\,dx$$

$$\therefore \ln p = \ln\{C(1+x^2)\}$$

$$\therefore p = \frac{dy}{dx} = C + Cx^2$$

$$\therefore y = Cx + \tfrac{1}{3}Cx^3 + B$$

Therefore, writing $3A$ for C, the general solution is

$$y = Ax^3 + 3Ax + B \qquad (1)$$

The solution in Example 6 contains two arbitrary constants A and B. When we considered $\dfrac{dy}{dx} = 2x$ earlier, we saw that its solution $y = x^2 + c$ represented a set of curves (see Fig. 34.2), and that if we were given some further information, for instance a point through which one of the curves passes, we could find the value of c which gives the equation of this particular curve.

The solution (1) above represents a set of cubic curves. To identify one particular member of this set, we must be given sufficient information to find the values of both constants. This could be done either by giving *two* points through which the curve passes, or by giving *one* point and the gradient at a specified point. For example if the curve passes through $(0, -5)$ and $(1, 3)$, then substituting these coordinates into (1), we obtain

$$-5 = A \times 0^3 + 3 \times A \times 0 + B$$
$$\therefore B = -5$$

and

$$3 = A \times 1^3 + 3 \times A \times 1 + B$$
$$\therefore 4A + B = 36$$
$$4A = 8$$
$$A = 2$$

So the particular member of the set of curves represented by (1) which passes through the points $(0, -5)$ and $(1, 3)$ is

$$y = 2x^3 + 6x - 5$$

Questions

Q19 Solve:

a $x\dfrac{d^2y}{dx^2} = 2$ **b** $\dfrac{d^2y}{dx^2} = x \cos x$

c $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 9x^2$

d $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = \cos x$

Q20 Solve:

a $y\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$, giving the general solution.

b $(2x-1)\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = 0$, given that when $x = 0, y = 2$ and $\dfrac{dy}{dx} = 3$.

Q21 Write the differential equation $2\dfrac{dy}{dx} + x\dfrac{d^2y}{dx^2} = \dfrac{2}{x}$, by means of the substitution $\dfrac{dy}{dx} = p$, as a differential equation linear in p, and proceed as in §34.3 on page 358.

34.7 Simple harmonic motion

The substitutions mentioned in §34.6 above arise in a less abstract form in mechanics. With the usual notation, the velocity

$$v = \frac{dx}{dt} \qquad \left(\text{compare with } p = \frac{dy}{dx}\right)$$

and the acceleration is

$$\frac{d^2x}{dt^2} = \frac{\mathbf{d}v}{\mathbf{d}t} = \frac{dv}{dx} \times \frac{dx}{dt} = v\frac{\mathbf{d}v}{\mathbf{d}x}$$

You may already appreciate that in dealing with variable forces, Newton's Second Law of Motion may be usefully written

$$P = m\frac{dv}{dt}, \quad \text{if } P \text{ is a function of } t \qquad \text{or}$$

$$P = mv\frac{dv}{dx}, \quad \text{if } P \text{ is a function of } x$$

Simple Harmonic Motion (S.H.M.) is a particular case of motion under the action of a force that varies with displacement.

Definition

A body moves in Simple Harmonic Motion in a straight line when its acceleration is proportional to its distance from a given point on the line, and is directed always towards that point.

Before studying this section you should have some knowledge of this topic. We shall not confine our attention only to finding the general solution of the typical S.H.M. equation

$$\frac{d^2x}{dt^2} = -n^2x$$

We must discuss in some detail the constants which arise in the solution. The constant n in the above equation is determined by the physical situation which gives rise to S.H.M. For example, if a body of given mass hangs at rest from a spring attached to a fixed point, and is then displaced vertically and released, it will oscillate in S.H.M. In this case the mass of the body, and the natural length and elasticity of the spring, together determine the constant n, and the **periodic time** $2\pi/n$. However, do not confuse n with the two *arbitrary constants* which will arise in the general solution of the above second order differential equation.

a Quite independent of the periodic time is an *arbitrary* **amplitude** a, the maximum displacement from the centre of oscillation (dependent in the above example upon how far we displace the body from its equilibrium position before releasing it).

b The general solution of the S.H.M. equation will give the displacement x from the centre of oscillation at time t. Here is the second *arbitrary* choice, the instant at which we take t to be zero.

Example 7 *Find the general solution of the Simple Harmonic Motion equation* $\frac{d^2x}{dt^2} = -n^2x$.

[Since t is absent, write $\frac{d^2x}{dt^2}$ as $v\frac{dv}{dx}$.]

$$v\frac{dv}{dx} = -n^2x$$

$$\therefore \tfrac{1}{2}v^2 = -\tfrac{1}{2}n^2x^2 + c$$

[At this stage express the arbitrary c in terms of the arbitrary amplitude a.]

If the amplitude is a, $v = 0$ when $x = a$,

$$\therefore 0 = -\tfrac{1}{2}n^2a^2 + c \quad \text{whence} \quad c = \tfrac{1}{2}n^2a^2$$

$$\therefore \boldsymbol{v^2 = n^2(a^2 - x^2)}$$

We now consider separately the positive and negative velocities which occur in any position (other than the extreme positions when $x = \pm a$). Thus

$$\frac{dx}{dt} = +n\sqrt{(a^2 - x^2)} \quad \text{or} \quad \frac{dx}{dt} = -n\sqrt{(a^2 - x^2)}$$

Separating the variables,

$$+\int \frac{1}{\sqrt{(a^2 - x^2)}}\,dx = \int n\,dt \quad \text{or}$$

$$-\int \frac{1}{\sqrt{(a^2 - x^2)}}\,dx = \int n\,dt$$

[Here it is preferable to use the change of variable $x = a\cos u$ on the L.H.S. of each equation, rather than the more usual $x = a\sin u$. This helps us to handle more easily the remaining arbitrary constant.]

The solution of these equations may be written

$$-\cos^{-1}\frac{x}{a} = nt + \varepsilon' \quad \text{and} \quad \cos^{-1}\frac{x}{a} = nt + \varepsilon,$$

$$\therefore x = a\cos(-nt - \varepsilon') \quad \text{and} \quad x = a\cos(nt + \varepsilon).$$

But $\cos(-\theta) = \cos\theta$, so we may write

$$x = a\cos(nt + \varepsilon') \quad \text{and} \quad x = a\cos(nt + \varepsilon)$$

for motion in the

positive and *negative*

directions respectively.

At an extreme position when $x = a$, and $t = t_1$ say, the motion is changing from positive to negative direction, both solutions are valid, and

$$\cos(nt_1 + \varepsilon') = \cos(nt_1 + \varepsilon) = 1$$

$\therefore nt_1 + \varepsilon'$ and $nt_1 + \varepsilon$ are multiples of 2π

$\therefore \varepsilon' = \varepsilon + 2k\pi$ (where k is an integer or zero)

Hence $x = a\cos(nt + \varepsilon') = a\cos(nt + \varepsilon + 2k\pi)$ $= a\cos(nt + \varepsilon)$. Therefore the motion is fully defined by the general solution

$$\boldsymbol{x = a\cos(nt + \varepsilon)} \qquad (1)$$

where ε, the initial phase, is explained below Q25.

Questions

Q22 Write down the general solutions of the following equations.

a $\frac{d^2x}{dt^2} = -4x$ b $\frac{d^2y}{dx^2} + 9y = 0$

c $\frac{d^2y}{dx^2} = -16x$

Q23 In Example 7 what integrating factor will enable you to obtain the first order equation $\left(\frac{dx}{dt}\right)^2 = -n^2x^2 + k$?

Q24 A Simple Harmonic Motion of amplitude 2 cm has the equation $\frac{d^2x}{dt^2} = -\frac{9}{4}x$. Write down the solution of this equation given that $x = 2$ when $t = 0$. Find expressions for $\frac{dx}{dt}$

a in terms of x, b in terms of t.

Q25 What special form is taken by the general solution $x = a\cos(nt + \varepsilon)$ if the motion is timed

a from an extreme position (i.e. $x = a$ when $t = 0$),

b from the centre of oscillation?

You may be familiar with the fact that, if a radius OP of a circle centre O radius a rotates about O with constant angular velocity n rad/sec, and Q is the projection of P on a diameter AB, Q moves with S.H.M. along AB. See Fig. 34.3.

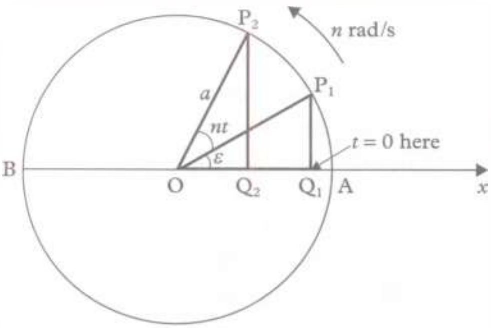


Figure 34.3

Take $t = 0$ at Q_1, where $\angle AOP_1 = \varepsilon$ radians, and suppose that Q moves directly to position Q_2 in time t. Then $\angle P_1OP_2 = nt$ radians, and $\angle AOP_2 = (nt + \varepsilon)$ radians. Thus if x is the displacement of Q from O at time t,

$$x = a\cos(nt + \varepsilon)$$

When $t = 0$, $x = a\cos\varepsilon$. This shows the significance of this constant ε, which is called the **initial phase**.

Questions

Q26 What does the general solution $x = a\cos(nt+\varepsilon)$ become if the initial phase is (a) 0, (b) $-\pi/2$? Illustrate each case with a sketch.

Q27 A Simple Harmonic Motion has amplitude 3 mm. If $t = 0$ when the body is +1.5 mm from the centre of oscillation, what is the initial phase?

The two arbitrary constants which appear in the general solution (1) on page 364 are the amplitude a, and the initial phase ε. However, the general solution is often given in a form in which these are not explicitly stated.

Expanding the R.H.S. of (1)

$$x = a\cos nt\cos\varepsilon - a\sin nt\sin\varepsilon \quad \text{or}$$
$$\boldsymbol{x = A\cos nt + B\sin nt} \qquad (2)$$

where $A = a\cos\varepsilon$ and $B = -a\sin\varepsilon$

In this form we see that the amplitude

$$a = \sqrt{(A^2 + B^2)}$$

and the initial phase

$$\varepsilon = \tan^{-1}\left(-\frac{B}{A}\right)$$

Reduction to the form $\dfrac{d^2x}{dt^2} = -n^2x$

When solving S.H.M. problems you may possibly choose to measure displacement from a point other than the centre of oscillation. Example 8 shows how an equation thus obtained may be reduced to the standard form.

Example 8 *Solve* $\dfrac{d^2x}{dt^2} + 9x - 18 = 0.$

This equation may be written

$$\frac{d^2x}{dt^2} = -9(x-2)$$

Let $x - 2 = u$, then $\dfrac{dx}{dt} = \dfrac{du}{dt}$ and $\dfrac{d^2x}{dt^2} = \dfrac{d^2u}{dt^2}$,

$$\therefore \frac{d^2u}{dt^2} = -9u$$

The general solution of this equation is

$$u = a\cos(3t + \varepsilon)$$

But $x = u + 2$,

$$\therefore x = a\cos(3t+\varepsilon) + 2$$

Question

Q28 Solve the following differential equations.

a $\dfrac{d^2y}{dx^2} + 4y + 4 = 0$

b $\dfrac{d^2\theta}{dt^2} + 2\theta - 6 = 0$

c $\dfrac{d^2x}{dt^2} + \frac{9}{4}t = -1$

Exercise 34c

1 Solve the following differential equations.

a $x\dfrac{d^2y}{dx^2} - 1 = 0$

b $\cos\theta\dfrac{d^2y}{d\theta^2} - \sin\theta\dfrac{dy}{d\theta} = \cos\theta$

c $e^x\dfrac{d^2y}{dx^2} = 2$

d $(2x+1)\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = 0$

e $\dfrac{1}{x}\dfrac{d^2y}{dx^2} - \dfrac{1}{x^2}\dfrac{dy}{dx} = e^x$

2 Use the substitution $\dfrac{dy}{dx} = p$ to solve the following differential equations.

a $\dfrac{d^2y}{dx^2} = 2\left(\dfrac{dy}{dx}\right)^2$ b $\dfrac{d^2y}{dx^2} + y\left(\dfrac{dy}{dx}\right)^3 = 0$

c $(1+x^2)\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} = 0$

3 Solve the differential equation $\dfrac{d^2s}{dt^2} + \dfrac{1}{10}\left(\dfrac{ds}{dt}\right)^2 = 0$, using the substitution $\dfrac{ds}{dt} = v$, and writing $\dfrac{d^2s}{dt^2}$

a as $\dfrac{dv}{dt}$, b as $v\dfrac{dv}{ds}$.

4 Use the substitution $\dfrac{dy}{dx} = p$ to write the following as differential equations linear in p, and proceed as in §34.4 on page 358.

a $\dfrac{d^2y}{dx^2} + \cot x\dfrac{dy}{dx} = 1$ b $\dfrac{d^2y}{dx^2} + \dfrac{1}{x+2}\dfrac{dy}{dx} = x$

c $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = x \ln x$

5 Find the solution of the differential equation

$$\frac{dy}{dx} \times \frac{d^2y}{dx^2} + x = 0$$

which satisfies the condition that y and $\dfrac{dy}{dx}$ are both zero when $x = 1$.

6 Write down the general solution of each of the following differential equations.

a $\dfrac{d^2s}{dt^2} = -25s$ b $\dfrac{d^2y}{dx^2} + \frac{9}{4}y = 0$

c $9\dfrac{d^2s}{d\theta^2} = -2\theta$ d $4\dfrac{d^2y}{dt^2} + 3y = 0$

7 Find the solution of the differential equation

$$16\frac{d^2s}{dt^2} + 9s = 0$$

given that $s = 4$ and $\dfrac{ds}{dt} = 0$ when $t = 0$.

8 A body moves in a straight line so that when it is x cm from a point O on the line its acceleration is $9x$ cm/s^2 towards O. Write down the differential equation which describes this motion. Solve the equation (see Example 7) given that the body is at rest when 2 cm from O, and its distance from O is $+\sqrt{3}$ cm at the instant from which time is measured.

9 A body moves in S.H.M. of amplitude 4 cm and has initial phase $-\pi/2$ s. It takes 1 s to travel 2 cm from the centre of oscillation, O. What was its initial position, and what is its periodic time?

10 A body moving in S.H.M. is timed from an extreme position, and is found to take 2 s to reach a point mid-way between the centre of oscillation and the other end of its path. State the initial phase, and calculate the periodic time.

11 A body moves in a straight line so that it is x m from a fixed point on the line at time t s, where $x = \cos 2t + \sin 2t$. Write this in the form $x = a\cos(nt + \varepsilon)$ and state the amplitude, initial phase, and periodic time of the motion.

12 Repeat question 11 for $x = 3\cos \frac{1}{2}t - 4\sin \frac{1}{2}t$, giving the initial phase correct to three significant figures.

13 The two simple harmonic motions defined by $x = a\cos(nt + \varepsilon_1)$ and $x = a\cos(nt + \varepsilon_2)$ are said to have a *phase difference* of $\varepsilon_1 - \varepsilon_2$. Find the phase difference between the following pairs of S.H.M.

a $x = a\cos nt$, $x = a\sin nt$

b $x = 2\cos\left(3t + \dfrac{\pi}{6}\right)$, $x = \sqrt{2}(\cos 3t - \sin 3t)$

c $x = \dfrac{3\sqrt{2}}{2}(\cos nt - \sin nt)$, $x = \dfrac{3\sqrt{2}}{2}(\cos nt + \sin nt)$

d $x = a\sin nt$, $x = -a\sin nt$

e $x = 5\cos nt - 5\sqrt{3}\sin nt$, $x = 5\sqrt{2}\cos nt + 5\sqrt{2}\sin nt$

14 Solve the following differential equations.

a $\dfrac{d^2y}{dx^2} = -4(y+3)$ b $2\dfrac{d^2\theta}{dt^2} + 9\theta = 3$

c $3\dfrac{d^2s}{dt^2} + 4t = 1$ d $\dfrac{d^2x}{dt^2} + 4x + 8 = 0$

given that in **d**, $x = -1$ when $t = 0$, and $x = -3$ when $t = \pi/4$.

15 Solve the following differential equations.

a $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 2 = 0$ b $\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx}$

c $\dfrac{d^2x}{dt^2} + x = 0$ d $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$

e $(3y-1)\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$

34.8 Second order linear differential equations

The auxiliary quadratic equation (A.Q.E.)

'Second order' means that the *second* derivative will appear, but not derivatives of higher order. 'Linear' means that none of the terms containing y will be squared, or cubed, or raised to any power except one. In other words this section considers equations of the form

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

where $a, b, c \in \mathbb{R}$, and $a \neq 0$.

We shall see that the nature of the solution will depend upon the relative magnitudes of the constants a, b and c. We will frequently to refer to the **auxiliary quadratic equation** (**A.Q.E.** for short):

$$am^2+bm+c=0$$

The A.Q.E. is also sometimes called the **characteristic equation**.

The general solution of the A.Q.E. is

$$m=\frac{-b\pm\surd(b^2-4ac)}{2a}$$

As we know (see §10.1 on page 128) there are three cases which can arise from this:

a if $b^2 > 4ac$, the A.Q.E. has two real, distinct roots,
b if $b^2 = 4ac$, the A.Q.E. has identical, real roots,
c if $b^2 < 4ac$, the A.Q.E. has a pair of conjugate complex or imaginary roots.

Each case gives rise to a distinct type of solution to the original differential equation, and we shall consider each of these in turn. But first, note that we have already solved one important type of second order differential equation, namely the S.H.M. equation

$$\frac{d^2y}{dx^2}+n^2y=0$$

The A.Q.E. for this is

$$m^2+n^2=0$$

which has roots

$$m=\pm in$$

In other words it is an example of case **c** above, and we already know from page 365 that one form of the general solution of this equation is

$$y=A\cos nx+B\sin nx$$

Type I — A.Q.E. with real, distinct roots

Suppose the roots of the A.Q.E. are α and β. It can then be written

$$m^2-(\alpha+\beta)m+\alpha\beta=0$$

and the corresponding differential equation is

$$\frac{d^2y}{dx^2}-(\alpha+\beta)\frac{dy}{dx}+\alpha\beta y=0$$

This differential equation can be rearranged so that it reads

$$\frac{d^2y}{dx^2}-\beta\frac{dy}{dx}-\alpha\left(\frac{dy}{dx}-\beta y\right)=0$$

and this, in turn, can be written

$$\frac{d}{dx}\left(\frac{dy}{dx}-\beta y\right)-\alpha\left(\frac{dy}{dx}-\beta y\right)=0$$

We can now simplify this equation by substituting

$$v=\frac{dy}{dx}-\beta y \qquad (1)$$

The differential equation can then be written

$$\frac{dv}{dx}-\alpha v=0 \qquad (2)$$

This is a *first order differential equation*, and we already have many ways of solving such equations.

The solution of (2) is

$v=Ce^{\alpha x}$ (see §34.2 on page 355)

where C is an arbitrary constant. Substituting this in equation (1) gives

$$\frac{dy}{dx}-\beta y=Ce^{\alpha x}$$

Once again we have a first order differential equation. This one can be solved by using the integrating factor $e^{\int-\beta dx}=e^{-\beta x}$. Multiplying through by this gives

$$e^{-\beta x}\frac{dy}{dx}-\beta e^{-\beta x}y=Ce^{(\alpha-\beta)x}$$

and integrating, this becomes

$$e^{-\beta x}y = \frac{C}{\alpha - \beta}e^{(\alpha-\beta)x} + B$$

where B is an arbitrary constant.
Multiplying through by $e^{\beta x}$, we have

$$y = \frac{C}{\alpha - \beta}e^{\alpha x} + Be^{\beta x}$$

Since C was an arbitrary constant, there is nothing to be gained from writing $C/(\alpha - \beta)$, and so we replace this by another arbitrary constant A. Hence the general solution is

$$\boldsymbol{y = Ae^{\alpha x} + Be^{\beta x}}$$

where α and β are the (real) roots of the Type I A.Q.E.

Example 9 *Solve the differential equation*

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 0$$

Find the values of the arbitrary constants such that when $x = 0$, $y = 5$ *and* $\frac{dy}{dx} = 23$.

The A.Q.E. is

$$m^2 + 2m - 15 = 0$$

Hence

$$(m + 5)(m - 3) = 0$$
$$\therefore m = -5 \quad \text{or} \quad +3$$

Since the roots are real and distinct, this is a 'Type I' differential equation and its general solution is

$$y = Ae^{-5x} + Be^{+3x}$$

When $x = 0$, $y = 5$, so

$$5 = A + B$$

Also, when $x = 0$, $\frac{dy}{dx} = 23$, and since $\frac{dy}{dx} = -5Ae^{-5x} + 3Be^{3x}$, we have

$$23 = -5A + 3B$$

Solving this pair of simultaneous equations for A and B gives $A = -1$, $B = 6$. Hence the solution which fits the given conditions is

$$y = -e^{-5x} + 6e^{3x}$$

In Example 9 we were given the values of y and $\frac{dy}{dx}$, when $x = 0$. When differential equations are formed in applied mathematics it is quite common for the independent variable to represent t, the time. Frequently the values of the dependent variable, and its derivative, will be given for $t = 0$, since these determine the initial state of the system. These values are usually called the *initial conditions*.

Question

Q29 Find the general solutions of the following differential equations.

a $\frac{d^2y}{dx^2} = y$

b $\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 20y = 0$

c $2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 3y = 0$

d $15\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + y = 0$

Do not assume that differential equations are always expressed in terms of x (for the independent variable) and y (for the dependent variable). In Q30, z and t are used.

Questions

Q30 Find the general solution of the differential equations.

a $\frac{d^2z}{dt^2} - 25z = 0$

b $6\frac{d^2z}{dt^2} - \frac{dz}{dt} - z = 0$

Q31 Find the solutions of the differential equations in Q30 which satisfy the initial conditions, $z = 0$, and $\frac{dz}{dt} = 10$, when $t = 0$.

Q32 Solve the differential equation $f''(x) - 6f'(x) + 5f(x) = 0$, given that $f(0) = 1$ and $f'(0) = 9$.

Type II — A.Q.E. with identical roots

In this type, the A.Q.E. can be written in the form

$$m^2 - 2pm + p^2 = 0$$

This can be factorised to give

$$(m-p)^2 = 0$$

and so the solution of the A.Q.E. is $m = p$. The corresponding differential equation is

$$\frac{d^2y}{dx^2} - 2p\frac{dy}{dx} + p^2y = 0.$$

Before we solve this, notice the following very useful transformation. If we write

$$y = e^{px}v$$

where v is a function of x, and p is a constant, then

$$v = e^{-px}y$$

Differentiating once,

$$\frac{dv}{dx} = e^{-px}\frac{dy}{dx} - pe^{-px}y$$

and differentiating again,

$$\frac{d^2v}{dx^2} = e^{-px}\frac{d^2y}{dx^2} - 2pe^{-px}\frac{dy}{dx} + p^2e^{-px}y$$

$$= e^{-px}\left(\frac{d^2y}{dx^2} - 2p\frac{dy}{dx} + p^2y\right)$$

Hence

$$\frac{d^2y}{dx^2} - 2p\frac{dy}{dx} + p^2y = e^{px}\frac{d^2v}{dx^2}$$

Using this transformation, the Type II second order differential equation

$$\frac{d^2y}{dx^2} - 2p\frac{dy}{dx} + p^2y = 0$$

can be written

$$e^{px}\frac{d^2v}{dx^2} = 0$$

$$\therefore \frac{d^2v}{dx^2} = 0$$

Integrating once gives

$$\frac{dv}{dx} = A$$

and integrating again,

$$v = Ax + B$$

where A and B are arbitrary constants. If we now replace v by $e^{-px}y$, we have

$$e^{-px}y = Ax + B$$

$$\therefore \boldsymbol{y = (Ax + B)e^{px}}$$

This is the general solution of the Type II equation.

Example 10 *Find the general solution of the differential equation*

$$4\frac{d^2u}{d\theta^2} - 12\frac{du}{d\theta} + 9u = 0$$

The A.Q.E. is

$$4m^2 - 12m + 9 = 0$$
$$(2m-3)^2 = 0$$
$$2m = 3$$
$$m = 3/2$$

The A.Q.E. has real identical roots, so this is a Type II differential equation. Its general solution is

$$u = e^{(3/2)\theta}(A\theta + B)$$

Questions

Q33 Solve the differential equations.

a $\dfrac{d^2V}{dt^2} + 6\dfrac{dV}{dt} + 9V = 0$

b $100\dfrac{d^2r}{dt^2} - 60\dfrac{dr}{dt} + 9r = 0$

Q34 Find the solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

given that when $x = 0$, $y = 0$ and when $x = 1$, $y = e$.

Type III — A.Q.E. with imaginary roots

Suppose the roots of the A.Q.E. are $p \pm iq$, where p and $q \in \mathbb{R}$. The sum of the roots is

$$(p + iq) + (p - iq) = 2p$$

and the product of the roots is

$$(p+\mathrm{i}q)(p-\mathrm{i}q)=p^2-\mathrm{i}^2q^2=p^2+q^2$$

so the A.Q.E. can be written

$$m^2-2pm+(p^2+q^2)=0$$

The corresponding differential equation is

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2p\frac{\mathrm{d}y}{\mathrm{d}x}+(p^2+q^2)y=0$$

Using the same transformation that we used in the previous section, i.e. putting $y=\mathrm{e}^{px}V$, we can write the equation

$$\mathrm{e}^{px}\frac{\mathrm{d}^2v}{\mathrm{d}x^2}+q^2\,\mathrm{e}^{px}v=0$$

Dividing through by e^{px}, we have

$$\frac{\mathrm{d}^2v}{\mathrm{d}x^2}+q^2v=0$$

This is an equation which you should recognise — it is the S.H.M. equation, and we know that its general solution can be written

$$v=A\cos qx+B\sin qx$$

Returning to the original variable y, gives

$$\mathrm{e}^{-px}y=A\cos qx+B\sin qx$$
$$\boldsymbol{y=\mathrm{e}^{px}(A\cos qx+B\sin qx)}$$

This is the general solution of the Type III differential equation.

Example 11 *Solve the differential equation*

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+6\frac{\mathrm{d}x}{\mathrm{d}t}+10x=0$$

given the initial conditions, $x=0$ *and* $\frac{\mathrm{d}x}{\mathrm{d}t}=1$, *when* $t=0$.

The A.Q.E. is $m^2+6m+10=0$.

Solving by the formula,

$$m=\frac{-6\pm\sqrt{(36-40)}}{2}$$
$$=\frac{-6\pm\sqrt{(-4)}}{2}=\tfrac{1}{2}(-6\pm2\mathrm{i})$$
$$=-3\pm\mathrm{i}$$

Since the roots are imaginary this is a Type III differential equation, and its general solution is

$$x=\mathrm{e}^{-3t}(A\cos t+B\sin t)$$

The initial conditions are $x=0$ and $\frac{\mathrm{d}x}{\mathrm{d}t}=1$, when $t=0$, so

$$0=(A\cos 0+B\sin 0)$$
$$\therefore A=0$$

Hence

$$x=\mathrm{e}^{-3t}B\sin t$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}t}=B\mathrm{e}^{-3t}\cos t-3B\mathrm{e}^{-3t}\sin t$$

Putting $t=0$, and $\frac{\mathrm{d}x}{\mathrm{d}t}=1$, gives

$$1=B\cos 0$$
$$\therefore B=1$$

Hence the solution which fits the initial conditions is

$$x=\mathrm{e}^{-3t}\sin t$$

Questions

Q35 Find the general solutions of the differential equations.

a $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\frac{\mathrm{d}y}{\mathrm{d}x}+50y=0$

b $\frac{\mathrm{d}^2V}{\mathrm{d}t^2}+6\frac{\mathrm{d}V}{\mathrm{d}t}+34V=0$

c $36\frac{\mathrm{d}^2r}{\mathrm{d}t^2}+r=0$

Q36 Find the solutions to the equations in Q35, which satisfy the conditions.

a when $x=0$, $y=0$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=35$

b when $t=0$, $V=1$ and $\frac{\mathrm{d}V}{\mathrm{d}t}=7$

c when $t=\pi$, $r=1$, and when $t=3\pi$, $r=0$

Summary

To solve a differential equation of the form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=0$$

write down its auxiliary quadratic equation

$$am^2 + bm + c = 0$$

and solve it. The general solution of the differential equation then takes one of the three forms shown in the following table.

type	nature of the roots of the A.Q.E.	general solution of the differential equation
I	real, distinct roots, $m = \alpha$ or β	$y = Ae^{\alpha x} + Be^{\beta x}$
II	real, identical roots, $m = p$	$y = e^{px}(Ax + B)$
III	imaginary roots, $m = p \pm iq$	$y = e^{px}(A\cos qx + B\sin qx)$

Since solving a second order differential inevitably entails integrating *twice*, the general solution of such an equation must include *two* arbitrary constants.

Example 12 *Solve the differential equations*

a $\dfrac{d^2y}{dx^2} - 4y = 0,$

b $\dfrac{d^2y}{dx^2} + 4y = 0,$

c $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 0.$

In each case find the solution for which $y = 0$ and $\dfrac{dy}{dx} = 2$, when $x = 0$ and sketch its graph.

(To save space, some of the simpler steps in the working are left to you.)

a $\dfrac{d^2y}{dx^2} - 4y = 0.$

The A.Q.E. is

$$m^2 - 4 = 0$$
$$\therefore m = \pm 2$$

Hence the general solution is

$$y = Ae^{2x} + Be^{-2x}$$

Putting in the initial conditions gives $A = \frac{1}{2}$, $B = -\frac{1}{2}$, hence the solution which fits the initial conditions is

$$y = \tfrac{1}{2}(e^{2x} - e^{-2x})$$

The sketch of this solution is shown in Fig. 34.4.

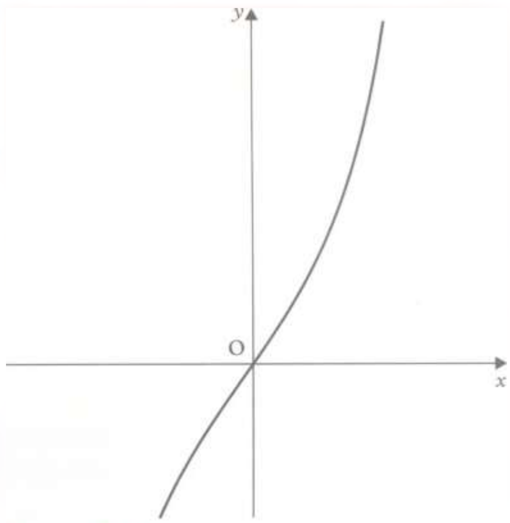


Figure 34.4

b $\dfrac{d^2y}{dx^2} + 4y = 0.$

This is the S.H.M. equation; its general solution is

$$y = A\sin 2x + B\cos 2x$$

and putting in the initial conditions gives

$$y = \sin 2x$$

The sketch of this solution is shown in Fig. 34.5.

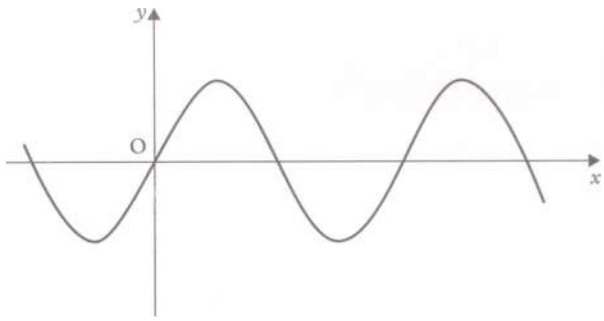


Figure 34.5

c $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 0.$

The A.Q.E. is $m^2 + 2m + 5 = 0$, and the solution of this is

$$m = -1 \pm 2i$$

Consequently the general solution is

$$y = e^{-x}(A\sin 2x + B\cos 2x)$$

Putting $x = 0, y = 0$, gives $B = 0$.

$$\therefore y = Ae^{-x}\sin 2x$$

and $\dfrac{dy}{dx} = -Ae^{-x}\sin 2x + 2Ae^{-x}\cos 2x$

Putting $x = 0$, $\dfrac{dy}{dx} = 2$ gives $2 = 2A$, hence $A = 1$.

Hence the particular solution required is

$$y = e^{-x}\sin 2x$$

The solution is sketched in **Fig. 34.6**. (When x is positive, the exponential factor will soon overwhelm $\sin 2x$, so y quickly becomes negligible, for example when $x = 5$, $y = -0.004$. For negative values of x, y still oscillates, but the amplitude of the oscillation increases enormously, due to the exponential factor. For example, when $x = -5$, $y = 80.7$; it is not possible to show this in **Fig. 34.6**.)

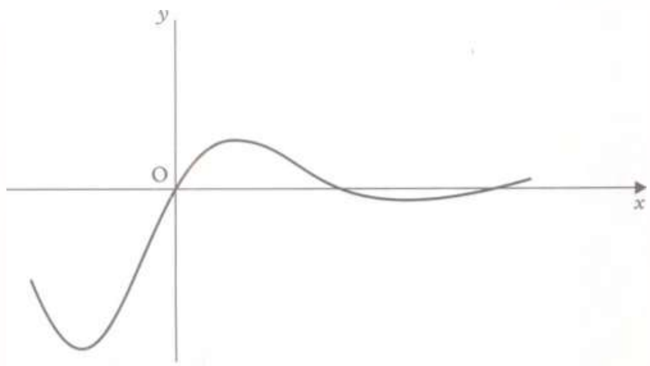


Figure 34.6

Exercise 34d

Find the general solutions of the following differential equations.

1 $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 10y = 0$

2 $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6y = 0$

3 $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 29y = 0$

4 $25\dfrac{d^2y}{dx^2} = y$

5 $\dfrac{d^2x}{dt^2} - 10\dfrac{dx}{dt} + 25x = 0$

6 $16\dfrac{d^2x}{dt^2} - 8\dfrac{dx}{dt} + x = 0$

7 $6\dfrac{d^2u}{dt^2} = 5\dfrac{du}{dt} - u$

8 $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} = 0$

9 $2\dfrac{d^2x}{dt^2} - \dfrac{dx}{dt} - 3x = 0$

10 $\dfrac{d^2r}{d\theta^2} = \dfrac{dr}{d\theta}$

11 $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 20y = 0$

12 $100\dfrac{d^2r}{dt^2} + r = 0$

13 $\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = 0$ [Hint: put $\dfrac{dy}{dx} = v$.]

14 $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 8$ [Hint: put $y = z + 2$.]

15 $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 2x = 1$ [Hint: put $x = u + \frac{1}{2}$.]

In questions 16–20, find the solutions which fit the conditions given.

16 $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} - 6y = 0$

When $x = 0$, $y = 5$ and $\dfrac{dy}{dx} = 16$.

17 $\dfrac{d^2u}{dt^2} + 9u = 0$

When $t = 0$, $u = 4$, and, when $t = \pi/6$, $u = 5$.

18 $\dfrac{d^2r}{dt^2} - 12\dfrac{dr}{dt} + 36r = 0$

When $t = 0$, $r = 1$ and when $t = 1$, $r = 0$.

19 $\dfrac{d^2z}{dt^2} + \frac{1}{4}z = 0$ When $t = 0$, $z = 4$ and $\dfrac{dz}{dt} = 0$.

20 $\dfrac{d^2u}{d\theta^2} = u$ When $\theta = 0$, $u = 1$ and $\dfrac{du}{d\theta} = 1$.

Chapter 35

Numerical methods

Introduction

In Chapter 23, we developed a method for finding an approximate solution to an equation when it was impossible to find an exact solution. In this chapter we will study two methods for estimating the area under a curve which can be used when integration is impossible. The first method depends on the formula for the area of a trapezium.

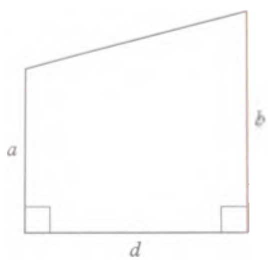


Figure 35.1

For the trapezium in Fig. 35.1, in which the lengths of the parallel sides are a and b, and where the distance between them is d, its area A is given by

$$A = \left(\frac{a+b}{2}\right)d$$

Consider the following problem. A cyclist travels along a straight road. He starts from rest and his speed in m/s measured at 2 second intervals is given in the table below.

time in seconds	0	2	4	6	8	10
speed in m/s	0	1.0	2.8	4.9	6.4	7.4

Find the distance travelled by the cyclist in the 4 seconds from $t = 6$ to $t = 10$. (See the graph in Fig. 35.2, which is not drawn to scale.)

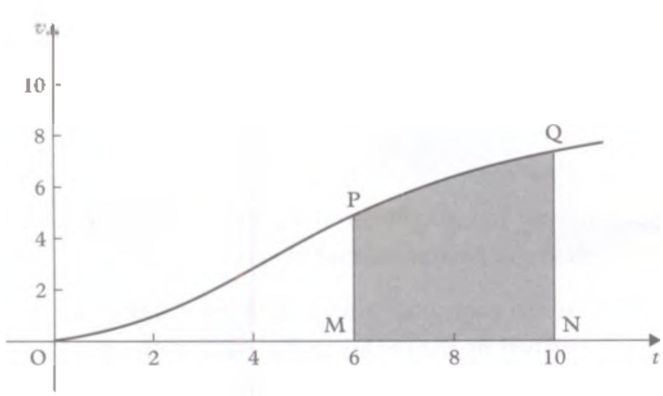


Figure 35.2

When we did problems like this before (see §6.2 on page 88), we used integration. But in this example we do not know the function whose graph is shown in Fig. 35.2. However, we can say that the distance required is represented by the area bounded by the lines MN, MP, NQ and the curve PQ. This area is almost the same as that of the trapezium PQNM.

In making this approximation, we have lost the area bounded by the curve PQ and the straight line PQ, but this is only a very small proportion of the total area. We calculate the area of the trapezium, using the formula above, i.e.

$$A \approx \left(\frac{4.9+7.4}{2}\right)\times 4$$
$$\approx 24.6$$

So the distance required is approximately 24.6 m.

In §35.1 below we will apply this method more generally.

Questions

Q1 Use the method above to estimate the distance travelled by the cyclist over the 2 second interval from $t = 4$ to $t = 6$.

Q2 Estimate the distance travelled by the cyclist from $t = 6$ to $t = 10$, by dividing the area into two trapeziums, each two units wide. Would you expect this answer to be better than the one in the text? Say why.

35.1 The trapezium rule

Suppose we wish to find the area under the curve in Fig. 35.3. We draw lines parallel to the y-axis at (equal) intervals of d units, and we form an estimate of the area required by calculating the areas of the trapeziums shown. In this diagram there are four trapeziums, but any convenient number may be used. In general the more intervals there are, the better the approximation.

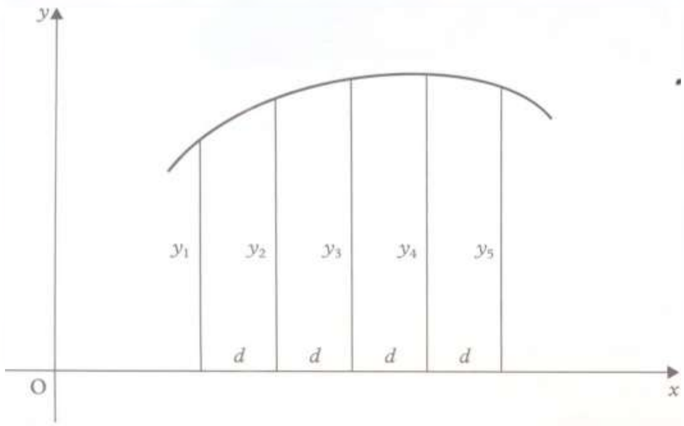


Figure 35.3

The area estimated by this method will be

$$\frac{y_1 + y_2}{2}d + \frac{y_2 + y_3}{2}d + \frac{y_3 + y_4}{2}d + \frac{y_4 + y_5}{2}d$$
$$= \tfrac{1}{2}d(y_1 + 2y_2 + 2y_3 + 2y_4 + y_5) \qquad (1)$$

This is the **trapezium rule** for five ordinates.*

Questions

Q3 Use the trapezium rule to estimate the area, from $x = 0.2$ to $x = 1$, under the curve given by

x	0.20	0.40	0.60	0.80	1.00
y	0.24	0.56	0.96	1.44	2.00

Given that the equation of the curve is $y = x^2 + x$, check your answer by integration.

Q4 Find expressions similar to (1) for
a eight, **b** nine ordinates.
Now express the trapezium rule in words.

Q5 Estimate the area under the curve given by the following table. Beware of the catch!

x	0	10	15	20	25
y	7	9	11	12	10

Another way of looking at the expression (1) above for the area is to take $a = 4d$ so that a is the total interval along the x-axis. In this case the area is estimated to be

$$a\left(\frac{y_1 + 2y_2 + 2y_3 + 2y_4 + y_5}{8}\right) \qquad (2)$$

where the expression in brackets appears as the average height of the curve, with a total of eight ordinates (y_2, y_3, y_4 counted twice) divided by 8.

Question

Q6 Obtain the expressions equivalent to (2) for **a** eight ordinates, **b** n ordinates.

Example 1 illustrates the accuracy of the trapezium rule. We shall compare the answer with that obtained by another rule later.

Example 1 *Use the trapezium rule to estimate the area under the curve $y = 1/x$ from $x = 1$ to $x = 2$. Then use integration to calculate the area.*

With six ordinates:

x	1.0	1.2	1.4	1.6	1.8	2.0
y	1	0.8333	0.7143	0.6250	0.5556	0.5

$$\begin{array}{rr}
y_1 = 1.0000 & y_2 = 0.8333 \\
y_6 = \underline{0.5000} & y_3 = 0.7143 \\
1.5000 & y_4 = 0.6250 \\
 & y_5 = \underline{0.5556} \\
 & 2.7282 \\
 & \underline{\times 2} \\
\underline{5.4564} \longleftarrow & 5.4564 \\
6.9564 & \\
\tfrac{1}{2}d = 0.1 &
\end{array}$$

$\therefore$ estimated area $= 0.696$, correct to 3 s.f.

By integration the area is

$$\int_1^2 \frac{1}{x}\,dx = \Big[\ln x\Big]_1^2$$
$$= \ln 2$$
$$= 0.693, \quad \text{correct to 3 s.f.}$$

Questions

Q7 Repeat the calculation of Example 1 but with eleven ordinates instead of six.

Q8 Use the trapezium rule to find the distance travelled by the cyclist in **Fig. 35.2** in the first ten seconds.

*Ordinate means y-coordinate, see §1.1.

If you have access to a computer write a program for evaluating definite integrals by the trapezium method. Since the computer will be doing the arithmetic, a large number of strips can be used and hence a high degree of accuracy can be achieved.

35.2 Simpson's rule

It is clear from Fig. 35.3 and Example 1 that the trapezium rule with a small number of strips is not very accurate. If, however, we join the tops of the ordinates by a smooth curve, we might get a better estimate. The question then arises as to what curve to use — and there are a number of possibilities. However, if we take three ordinates we can find a parabola in the form

$$y = ax^2 + bx + c$$

to pass through the three corresponding points.

Given a curve with three ordinates y_1, y_2, y_3 at *equal* intervals of d apart, take the y-axis along the middle ordinate and the x-axis through its foot as in Fig. 35.4.

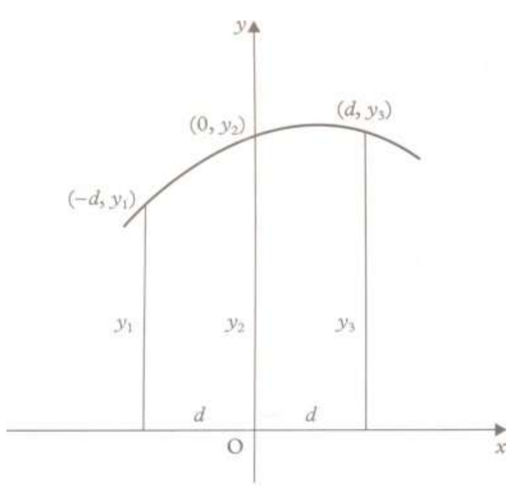


Figure 35.4

Let

$$y = ax^2 + bx + c$$

be the *parabola* through the points $(-d, y_1)$, $(0, y_2)$, (d, y_3). Its equation is therefore satisfied by their coordinates.

$$\begin{aligned} \therefore\ y_1 &= ad^2 - bd + c, \\ y_2 &= \qquad\qquad\ c, \\ y_3 &= ad^2 + bd + c. \end{aligned}$$

The area under the parabola is

$$\begin{aligned} \int_{-d}^{d} (ax^2 + bx + c)\,\mathrm{d}x &= \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_{-d}^{d} \\ &= \tfrac{2}{3}ad^3 + 2cd \end{aligned}$$

(Note that we do not need to find the equation of the parabola because we can express this area in terms of the data y_1, y_2, y_3, d.)

Now

$$\begin{aligned} y_1 + y_3 - 2y_2 &= 2ad^2 \\ \therefore\ y_1 + 4y_2 + y_3 &= 2ad^2 + 6c \\ \therefore\ \tfrac{1}{3}d(y_1 + 4y_2 + y_3) &= \tfrac{2}{3}ad^3 + 2cd \end{aligned}$$

So an approximation for the area under the given curve is

$$\boldsymbol{\tfrac{1}{3}d(y_1 + 4y_2 + y_3)}$$

This result is known as **Simpson's rule**.

Note: it makes very little difference to the proof exactly what points we are given originally. If, for instance, we are told that the curve passes through (x_1, y_1), (x_2, y_2), (x_3, y_3), where $x_2 = \frac{1}{2}(x_1 + x_3)$, we can at once take new axes, parallel to the given ones, with the new origin at $(x_2, 0)$. Let $d = x_3 - x_2 = x_2 - x_1$. The rest of the proof is as above.

In practice we usually require the area under a curve with more than three ordinates and so, provided there is an *odd* number of ordinates, we may apply Simpson's rule a number of times. Thus with seven ordinates (as in Fig. 35.5) the area is

$$\begin{aligned} &\tfrac{1}{3}d(y_1 + 4y_2 + y_3) + \tfrac{1}{3}d(y_3 + 4y_4 + y_5) + \tfrac{1}{3}d(y_5 + 4y_6 + y_7) \\ &= \tfrac{1}{3}d(y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 + 4y_6 + y_7) \end{aligned}$$

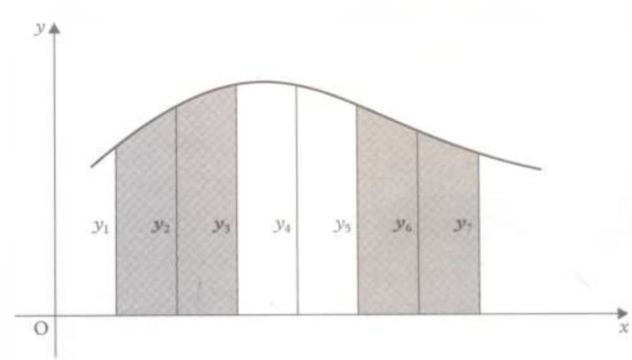


Figure 35.5

Question

Q9 Find similar expressions for the area with five ordinates. Now express Simpson's rule for an odd number of ordinates in words.

Example 2 below is the same as Example 1. From this you may compare the accuracy of Simpson's rule and the trapezium rule.

Example 2 *Use Simpson's rule to find an approximation for the area under the curve $y = 1/x$ between $x = 1$ and $x = 2$.*

With five ordinates:

x	1	1.25	1.5	1.75	2
y	1	0.8000	0.6667	0.5714	0.5

$y_1 = 1.0000$	$y_3 = 0.6667$	$y_2 = 0.8000$
$y_5 = 0.5000$	$\times 2$	$y_4 = 0.5714$
1.5000	1.3334	1.3714
1.3334		$\times 4$
5.4856		5.4856
8.3190		

$\frac{1}{3}d = \frac{1}{12}$

$\therefore$ the area $= \dfrac{8.3190}{12} = 0.693$, correct to 3 s.f.

This is closer to ln 2 than the result obtained with the trapezium rule using eleven ordinates (see Q7).

If the arithmetic in Example 2 is done on a calculator, the result is 0.693 253 97, correct to 8 d.p. Whereas the exact value for ln 2 is 0.693 147 18, correct to 8 d.p. Therefore Simpson's rule gave the first three significant figures correctly. Remember that the accuracy of the result depends on the method selected, the number of strips used and the shape of the graph.

Questions

Q10 Evaluate approximately $\displaystyle\int_1^2 \frac{1}{x}\,dx$ using Simpson's rule with eleven ordinates.

Q11 Repeat Q3, using Simpson's rule.

Exercise 35a

1 Evaluate $\displaystyle\int_0^{\pi/4} \tan x\,dx$,

a by integration,
b by using the trapezium rule with four strips,
c by Simpson's rule with four strips.

Comment on the accuracy of your answers.

2 Repeat question 1, parts **b** and **c**, for $\displaystyle\int_0^1 e^{x^2}\,dx$.

3 (x_1, y_1), (x_2, y_2), (x_3, y_3), where $x_2 = \frac{1}{2}(x_1 + x_3)$, are three points on the parabola $y = ax^2 + bx + c$. Prove that the area under the curve between the lines $x - x_1 = 0$, $x - x_3 = 0$ is equal to $\frac{1}{3}(x_2 - x_1)(y_1 + 4y_2 + y_3)$.
Use this formula to find the area between the parabola $y = x(10 - x)$ and the x-axis. Check your answer by integration.

4 Evaluate $\displaystyle\int_0^1 e^{-x^2}\,dx$ by Simpson's rule with ten intervals.

5 Estimate the area of a quadrant of a circle of radius 8 cm by dividing it into eight intervals and using **a** the trapezium rule and **b** Simpson's rule. Use the better of these results to find an approximate value of π.

6 The area in square centimetres of the cross-section of a solid 28 cm long at intervals of 3.5 cm is as follows:

0	11.5	15.3	16.3	16.2	13.4	9.3	4.9	0

Estimate the volume of the solid.

7 A jug of circular cross-section is 16 cm high inside and its internal diameter is measured at equal intervals from the bottom:

height (cm)	0	4	8	12	16
diameter (cm)	10.2	13.8	15.3	9.3	9.9

What volume of liquid will the jug hold if filled to the brim?

8 Using tables, or a calculator, where necessary, calculate the value of

$$\int_{0.1}^{0.5} e^{-x}\,dx$$

a by direct integration,

b by Simpson's rule, using five ordinates spaced at intervals of 1/10 unit.

(Give your answers to 4 d.p.)

9 Using Simpson's rule and taking unit intervals of x from $x = 8$ to $x = 12$, find approximately the area enclosed by the curve $y = \log_{10} x$, the lines $x = 8$ and $x = 12$, and the x-axis. Deduce the average value of $\log_{10} x$ between $x = 8$ and $x = 12$.

10 The coordinates of three points on the curve $y = A + Bx + Cx^2$ are (x_1, y_1), (x_2, y_2) and (x_3, y_3), where $x_2 = \frac{1}{2}(x_3 + x_1)$. Prove that the area under the curve between the lines $x = x_1$ and $x = x_3$ is equal to

$$\tfrac{1}{6}(x_3 - x_1)(y_1 + 4y_2 + y_3).$$

Using five ordinates, apply Simpson's rule to evaluate the integral $4\displaystyle\int_0^1 \frac{dx}{1+x^2}$ and thus find a value for π correct to 3 d.p.

11 The coordinates of three points on the curve $y = ax^3 + bx^2 + cx + d$ are (x_1, y_1), (x_2, y_2), (x_3, y_3). Prove that, if $x_2 - x_1 = x_3 - x_2 = h$, the area under the curve between the lines $x = x_1$, $x = x_3$ is $\frac{1}{3}h(y_1 + 4y_2 + y_3)$.

Find the area between the curve $y = x(x-2)^2$ and the x-axis by means of Simpson's rule with three ordinates. Use integration to check that your answer is exact.

12 Show that the area under the curve $y = 1/x$, from $x = n - 1$ to $x = n + 1$, is $\ln\{(n+1)/(n-1)\}$, provided $n > 1$.

By applying Simpson's rule to this area, show that, approximately,

$$\ln\frac{n+1}{n-1} = \frac{1}{3}\left(\frac{1}{n-1} + \frac{4}{n} + \frac{1}{n+1}\right)$$

and that the error in this approximation is $4/(15n^5)$, when higher powers of $1/n$ are neglected.

13 Use the binomial theorem to expand $(1 + x^3)^{10}$ in ascending powers of x, up to and including the term in x^9. Hence estimate I to 3 d.p., where

$$I = \int_0^{0.2} (1 + x^3)^{10}\, dx.$$

Make another estimate of I, again to 3 d.p., by using Simpson's rule with three ordinates.

14 Tabulate, to three decimal places, the values of the function

$$f(x) = \sqrt{(1 + x^2)}$$

for values of x from 0 to 0.8 at intervals of 0.1. Use these values to estimate

$$\int_0^{0.8} f(x)\, dx$$

a by the trapezium rule, using all the ordinates,

b by Simpson's rule, using only the ordinates at intervals of 0.2.

15 By considering suitable areas, or otherwise, show that, for any $n > 0$,

$$\tfrac{1}{2} \leqslant \int_0^1 (1 + x^n)^{-1}\, dx \leqslant 1$$

When $n = 4$, find to 3 s.f. a value for the integral, using Simpson's rule with five ordinates.

Chapter 36
Hyperbolic functions

36.1 Hyperbolic cosine and sine

We begin this chapter by defining two new functions, the **hyperbolic cosine** and the **hyperbolic sine**. No reason for adopting these definitions is given at present (more knowledge of complex numbers is needed for the reason to be fully appreciated. See §37.5 on page 392.) You will, however, soon find strong similarities between hyperbolic functions and the familiar trigonometrical functions which, to save confusion, are often referred to as the *circular functions*. Hyperbolic functions are being introduced because they will very quickly extend your powers of integration, and you may begin to need them in mechanics.
First we shall study the functions themselves.

Definitions

The hyperbolic cosine of x is defined as

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x})$$

and the hyperbolic sine of x

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x})$$

$\cosh x$ is pronounced as it is spelled; $\sinh x$ is usually pronounced 'sinch x' (or 'shine x').

First we sketch their graphs. Starting with the graph of e^x in Fig. 36.1i, that of e^{-x} has been shown dotted in the same figure. Cosh x is half the sum of these two functions (see Fig. 36.1ii) and sinh x is half the difference (see Fig. 36.1ii).

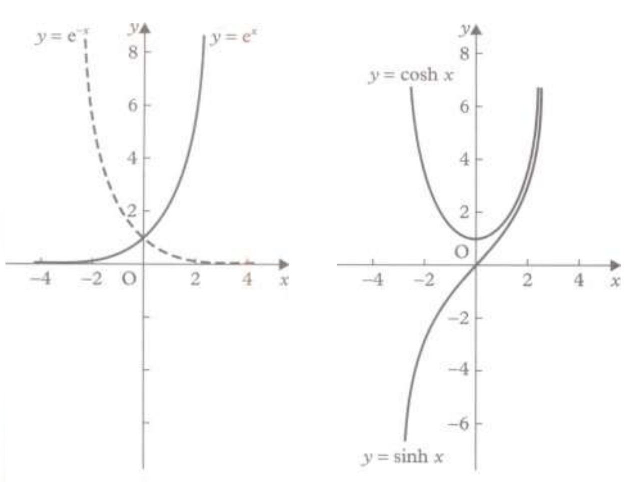


Figure 36.1

The two graphs are distinct in the first quadrant but they approach so close that, when they are sketched to this scale, the lines run together.

In general, the properties of hyperbolic functions are easily proved and this will be left for you to do in Exercise 36a. First we shall prove one important identity.

Example 1 *Prove the identity* $\cosh^2 x - \sinh^2 x = 1$.

From the definitions of $\cosh x$ and $\sinh x$,

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \{\tfrac{1}{2}(e^x + e^{-x})\}^2 - \{\tfrac{1}{2}(e^x - e^{-x})\}^2\\ &= \tfrac{1}{4}(e^{2x} + 2 + e^{-2x}) - \tfrac{1}{4}(e^{2x} - 2 + e^{-2x})\\ &= \tfrac{1}{4}e^{2x} + \tfrac{1}{2} + \tfrac{1}{4}e^{-2x} - \tfrac{1}{4}e^{2x} + \tfrac{1}{2} - \tfrac{1}{4}e^{-2x}\end{aligned}$$

$\therefore \cosh^2 x - \sinh^2 x = 1$

Definitions

The hyperbolic tangent, cotangent, secant, cosecant are defined as follows:

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{cosech} x = \frac{1}{\sinh x}$$

Exercise 36a

Most of the following properties of the hyperbolic functions can be deduced from the definitions.
Work all of questions 1–14.

1 From a sketch of $\cosh x$ and $\sinh x$ referred to the same axes, sketch the graph of $\tanh x$.

2 Prove that **a** $\cosh(-x) = \cosh x$,
b $\sinh(-x) = -\sinh x$.

3 Prove that $\cosh x > \sinh x$. [Show that $\cosh x - \sinh x > 0$.] Prove also that, when $x < 0$, $\cosh x > |\sinh x|$. Deduce the values between which $\tanh x$ lies.

4 From sketches of $\cosh x$ and $\sinh x$, sketch the graphs of sech x and cosech x.

5 Prove that $\cosh x \geqslant 1$. [See hint in question 3.]

6 Prove that

$$\cosh x + \sinh x = e^x$$
$$\cosh x - \sinh x = e^{-x}$$

Deduce the identity $\cosh^2 x - \sinh^2 x = 1$.

7 Prove that the point $(a \cosh t, b \sinh t)$ lies on one branch of the *hyperbola*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

[Hence the name *hyperbolic functions*. Use the result of question 6.]

8 Prove that $\sinh 2x = 2 \sinh x \cosh x$.

9 Prove that $\cosh 2x = \cosh^2 x + \sinh^2 x$
$= 2\cosh^2 x - 1$
$= 1 + 2\sinh^2 x$.

10 Use the results of questions 8 and 9 to show that

$$\tanh 2x = 2 \tanh x/(1 + \tanh^2 x).$$

11 Prove that

$$\operatorname{sech}^2 x = 1 - \tanh^2 x$$
$$\operatorname{cosech}^2 x = 1 - \coth^2 x$$

[Use the identity connecting $\cosh x$ and $\sinh x$.]

12 Prove that

$$\cosh (A + B) = \cosh A \cosh B + \sinh A \sinh B$$

Deduce a similar expression for $\cosh (A - B)$.

13 Prove that

$$\sinh (A + B) = \sinh A \cosh B + \cosh A \sinh B$$

Deduce a similar expression for $\sinh (A - B)$.

14 Use the results of questions 12 and 13 to find expressions for $\tanh (A + B)$, $\tanh (A - B)$ in terms of $\tanh A$, $\tanh B$.

15 Solve the equation $8 \cosh x + 17 \sinh x = 20$.

16 Find the condition that the equation $a \cosh x + b \sinh x = c$ should have equal roots.

17 If $a > b > 0$, prove that $b < \dfrac{a\,e^x + b\,e^{-x}}{e^x + e^{-x}} < a$.

18 If $|a| < |b|$, prove that the equation $a \cosh x + b \sinh x = 0$ has one and only one root.

19 Prove that $\sinh 3\theta = 3 \sinh \theta + 4 \sinh^3 \theta$.

20 Prove that $\cosh^2 x \sin^2 x - \sinh^2 x \cos^2 x$
$= \frac{1}{2}(1 - \cosh 2x \cos 2x)$.

Further properties of cosh x and sinh x

The domain of both $\cosh x$ and $\sinh x$ is $\mathbb{R}$. The range of $\sinh x$ is also $\mathbb{R}$, but the range of $\cosh x$ is $\{y: y \in \mathbb{R}, y \geqslant 1\}$, see **Fig. 36.1ii** and Exercise 36a, question 5. Notice also that, like the corresponding trigonometrical functions, $\cosh x$ is an even function and $\sinh x$ is odd; see Exercise 36a, question 2 and **Fig. 36.1ii**. (See §2.12 on page 53 for the definitions of odd and even functions.) Furthermore, $\sinh x$ is a one-to-one function (see §2.13 on page 55), but $\cosh x$ is not. This causes a slight difficulty when we come to the inverse hyperbolic functions in §36.3 on page 380.

Osborn's rule

You will have noticed similarities between the identities connecting hyperbolic functions and those connecting the corresponding circular functions. In fact the standard identities are in the same form except that certain signs are changed. **Osborn's rule** provides a simple way of remembering these changes of signs. The rule is to change the sign of any term containing the *square* of a sine (or cosecant, tangent, or cotangent, because these all include a sine by implication:

$$\operatorname{cosec} x = 1/\sin x,$$
$$\tan x = (\sin x)/(\cos x),$$
$$\cot x = (\cos x)/(\sin x).)$$

For instance,

$\sin 2x = 2 \sin x \cos x$	$\sinh 2x = 2 \sinh x \cosh x$
$\cos 2x = \cos^2 x - \sin^2 x$	$\cosh 2x = \cosh^2 x + \sinh^2 x$
$\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$	$\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$

Question

Q1 Write down the identities connecting hyperbolic functions corresponding to

a $\sin A + \sin B = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$

b $\cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$

c $\cos A - \cos B = -2 \sin \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)$

d $\sec^2 \theta = 1 + \tan^2 \theta$

e $\operatorname{cosec}^2 \theta = 1 + \cot^2 \theta$

f $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$

g $\tan 3\theta = (3 \tan \theta - \tan^3 \theta)/(1 - 3 \tan^2 \theta)$

Warning: Osborn's rule holds for only the standard trigonometrical identities. It does *not* hold for some derived formulae. For instance, application to

$$\frac{\cos 2A}{\cos A + \sin A} = \cos A - \sin A$$

and

$$\frac{\cos A}{\sin B} - \frac{\sin A}{\cos B} = \frac{2\cos(A+B)}{\sin 2B}$$

leads to incorrect results. Further it *cannot* be relied upon as an aid to remembering calculus formulae. See 36.2 below.

36.2 Derivatives of hyperbolic functions

The derivatives of $\cosh x$ and $\sinh x$ are most easily obtained by starting from the definitions of these functions.

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\{\tfrac{1}{2}(e^x + e^{-x})\}$$
$$= \tfrac{1}{2}(e^x - e^{-x})$$

$$\therefore \ \mathbf{\frac{d}{dx}(\cosh x) = \sinh x}$$

Similarly,

$$\mathbf{\frac{d}{dx}(\sinh x) = \cosh x}$$

The derivatives of the other hyperbolic functions are easily obtained by first expressing them in terms of $\cosh x$, $\sinh x$.

Question

Q2 Remembering that $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx}$, write down the derivatives of

a $\cosh 2x$
b $\sinh \frac{1}{2}x$
c $3\cosh \frac{1}{3}x$
d $\frac{1}{2}\sinh 4x$
e $\sinh^2 x$
f $\cosh^3 2x$

36.3 Inverse hyperbolic functions

Remember from previous work on inverse functions (§2.13 on page 55 and §18.6 on page 219), that only one-to-one functions can have inverses and that the graph of an inverse function is the reflection in the line $y = x$ of the graph of the original function.

The function $y = \sinh x$ *is* a one-to-one function so there is no difficulty over the existence of its inverse. The graphs of $y = \sinh x$ and $y = \sinh^{-1} x$ are shown in **Fig. 36.2**. Values of $\sinh^{-1} x$ could be estimated from the graph, but calculating them is more difficult. We shall return to this in §36.5 on page 383. In both $y = \sinh x$ and $y = \sinh^{-1} x$, the variables x and y can take any real values, i.e. the domain and range of both of these functions is $\mathbb{R}$.

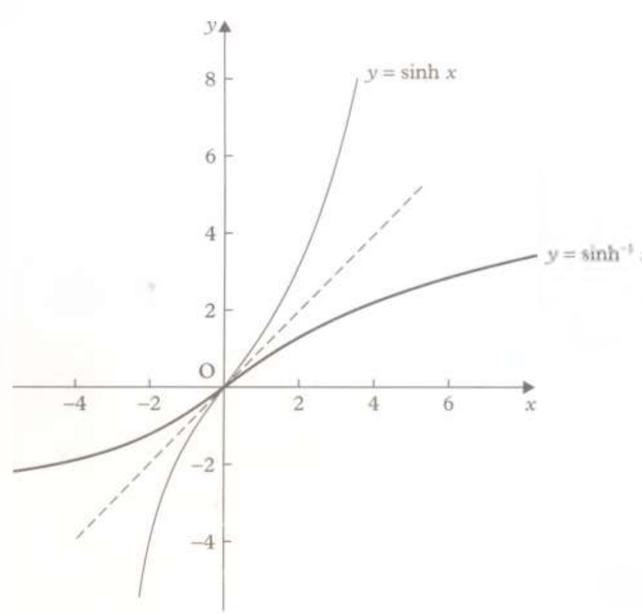


Figure 36.2

Unlike $\sinh x$, $\cosh x$ is *not* a one-to-one function. Indeed, for any value of x, $\cosh(-x) = \cosh x$. This causes a slight difficulty when we define the inverse function. The equation $\cosh x = 2$ has two approximate roots, $x \approx \pm 1.317$, but when we define the inverse function $\cosh^{-1} x$, it is essential that there should be a *unique* answer. (The same problem arises over $f(x) = x^2$ and its inverse $f^{-1}(x) = \sqrt{x}$.) However, this technical difficulty is overcome by restricting the domain of $\cosh x$ to *non-negative* real numbers. The graphs of $y = \cosh x$ and $y = \cosh^{-1} x$ are illustrated by the unbroken curves in **Fig. 36.3**.

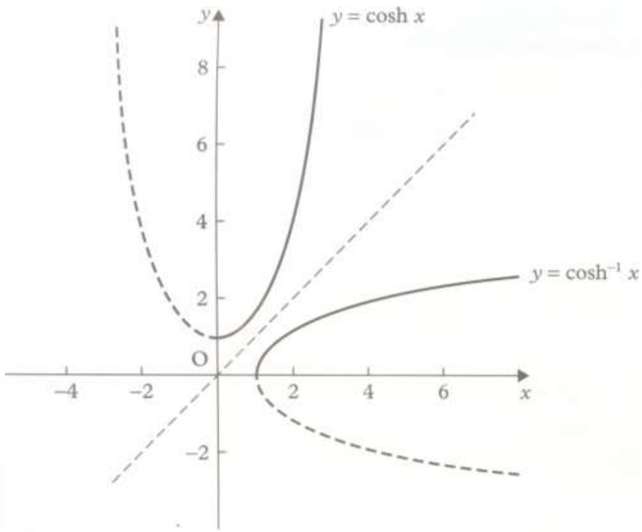


Figure 36.3

The domain of $\cosh^{-1} x$ is $\{x: x \in \mathbb{R}, x \geqslant 1\}$ and the corresponding range is $\{y: y \in \mathbb{R}, y \geqslant 0\}$.

Exercise 36b

1 Prove the following results by first expressing the functions concerned in terms of $\cosh x$ and $\sinh x$.

a $\dfrac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

b $\dfrac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$

c $\dfrac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

d $\dfrac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$

2 Write down $\int \cosh x \, dx$, $\int \sinh x \, dx$.

3 Write down the derivatives of

a $\cosh 3x$ b $\sinh 2x$ c $\cosh^2 x$

d $2\sinh^3 x$ e $3\tanh 2x$ f $\frac{1}{2}\operatorname{sech}^2 x$

g $\sinh^2 3x$ h $\sqrt{(\coth x)}$ i $2\tanh^2 \frac{1}{2}x$

*4 Sketch, on the same pair of axes, the graphs of $y = \tanh x$ and $y = \tanh^{-1} x$. State the domain and range of the inverse function, $y = \tanh^{-1} x$. State, giving a reason, whether $\tanh x$ is an odd or even function.

5 Differentiate the following functions with respect to x, simplifying your answers.

a $\ln \tanh x$ b $e^x \sinh x$ c $\dfrac{e^x - 1}{e^x + 1}$

6 Find:

a $\int \operatorname{sech}^2 2x \, dx$, b $\displaystyle\int \frac{\sinh x}{\cosh^2 x} \, dx$.

7 Find the minimum value of $5\cosh x + 3\sinh x$.

8 Prove that

a $\dfrac{d}{dx}(\cosh^{-1} x) = \dfrac{1}{\sqrt{(x^2 - 1)}}$

b $\dfrac{d}{dx}(\tanh^{-1} x) = \dfrac{1}{1 - x^2}$

and find an expression for $\dfrac{d}{dx}(\sinh^{-1} x)$.

[Hint: see the method of §32.3 on page 336.]

9 Prove that $\dfrac{d}{dx}\{\tan^{-1}(e^x)\} = \frac{1}{2}\operatorname{sech} x$.

10 Find $\dfrac{d}{dx}[\ln\{x + \sqrt{(1 + x^2)}\} - \sinh^{-1} x]$.

11 Find: a $\int \cosh 2x \sinh 3x \, dx$,
b $\int \cosh x \cosh 3x \, dx$.

12 Find the distance from the y-axis of the centroid of the area formed by $y = \sinh x$, $x - 1 = 0$ and the x-axis.

13 Find the equations of the tangent and normal to the hyperbola

$$b^2x^2 - a^2y^2 = a^2b^2$$

at the point $(a\cosh\theta, b\sinh\theta)$. If the tangent meets the y-axis at T and the normal meets the x-axis at N, find the locus of the mid-point of NT.

14 If $y = A\cosh 2x + B\sinh 3x$, find an equation connecting $\dfrac{d^2y}{dx^2}$, $\dfrac{dy}{dx}$, y which does not contain A, B.

*15 Prove that

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!} + \ldots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + \ldots$$

and obtain the first three non-zero terms of the expansion, in ascending powers of x, of $\tanh x$.

16 Expand $\tanh^{-1} x$ as a series of ascending powers of x. Express the sum of this series as a logarithm.

17 If $\tanh^{-1} x = y$, show that $x = (e^{2y} - 1)/(e^{2y} + 1)$. Hence express y as a logarithm.

18 Investigate the stationary values of $\cosh 3x - 12\cosh x$.

36.4 Integration

As mentioned at the beginning of this chapter, one reason for including hyperbolic functions in this book is to extend your powers of integration. So far, we have integrated functions in the forms

$$\frac{1}{\surd(a^2-x^2)} \quad \text{and} \quad \frac{1}{x^2+a^2}$$

but not those in the forms

$$\frac{1}{\surd(x^2-a^2)} \quad \text{and} \quad \frac{1}{\surd(x^2+a^2)}$$

We have seen (§24.7 on page 264) that the identity

$$\cos^2\theta + \sin^2\theta = 1$$

helps us to eliminate the square root sign in $1/\surd(a^2-x^2)$. We may therefore expect the identity

$$\cosh^2\theta - \sinh^2\theta = 1$$

to assist us with corresponding integrals.

Questions

Q3 What substitution using hyperbolic functions would eliminate the square root sign in the following?

a $\displaystyle\int \frac{1}{\surd(x^2-a^2)}\,\mathrm{d}x$

b $\displaystyle\int \frac{1}{\surd(x^2+a^2)}\,\mathrm{d}x$

***Q4** Show that

a $\displaystyle\int \frac{1}{\surd(a^2-x^2)}\,\mathrm{d}x = \sin^{-1}\frac{x}{a}+c$

$\displaystyle\int \frac{1}{\surd(a^2+x^2)}\,\mathrm{d}x = \sinh^{-1}\frac{x}{a}+c$

b $\displaystyle\int \frac{1}{\surd(x^2-a^2)}\,\mathrm{d}x = \cosh^{-1}\frac{x}{a}+c$

It is fairly easy to remember these results. Any confusion between them is easily avoided by thinking, 'What substitution would eliminate the square root.' In fact, by taking this thought and doing some side-work, you can dispense with memorising them as formulae. If you do memorise them, note that the results in Q6 are slightly and inconveniently different.

Questions

Q5 What substitution would enable us to write the denominator of the following integrands as a square?

a $\displaystyle\int \frac{1}{a^2+x^2}\,\mathrm{d}x$

b $\displaystyle\int \frac{1}{a^2-x^2}\,\mathrm{d}x$

***Q6** Show that

a $\displaystyle\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$

b $\displaystyle\int \frac{1}{a^2-x^2}\,\mathrm{d}x = \frac{1}{a}\tanh^{-1}\frac{x}{a}+c$

Q5 and Q6 raise a number of points of interest:

1. If you like to memorise formulae, note that these results contain a factor $1/a$ in front of the $\tan^{-1}(x/a)$ and $\tanh^{-1}(x/a)$. See also Q7.
2. $\displaystyle\int \frac{1}{a^2-x^2}\,\mathrm{d}x$ is usually found by first expressing $\displaystyle\frac{1}{a^2-x^2}$ in partial fractions. See Q8.
3. $x = a\tan\theta$, $x = a\tanh\theta$ are not the only possible substitutions by which to express a^2+x^2, a^2-x^2 respectively as squares, but the others are more complicated!

Questions

Q7 If x and a have dimensions of length L, write down the dimensions of both sides of the formulae below. [Since $\int y\,\mathrm{d}x$ is the limit of $\sum y\delta x$ and $\delta x \to 0$, take $\mathrm{d}x$ to have the same dimensions as δx, i.e. L.]

a $\displaystyle\int \frac{1}{\surd(a^2-x^2)}\,\mathrm{d}x = \sin^{-1}\frac{x}{a}+c$

b $\displaystyle\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$

Q8 Use partial fractions to show that

$$\int \frac{1}{a^2-x^2}\,\mathrm{d}x = \frac{1}{2a}\ln\frac{a+x}{a-x}+k$$

What conclusion can be drawn from a comparison of this result and the second result in Q6?

Question

***Q9** (A repeat of Exercise 36b, question 17.)
If $\tanh^{-1} x = y$, show that $x = (e^{2y} - 1)/(e^{2y} + 1)$.
Hence prove that

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x}$$

36.5 Inverse hyperbolic functions expressed in logarithms

The result of Q9 suggests that it may be possible to express $\cosh^{-1} x$ and $\sinh^{-1} x$ in terms of logarithms.

Let $y = \sinh^{-1} x$, then $\sinh y = x$

Also $\cosh^2 y = 1 + \sinh^2 y$ (1)

$\therefore \cosh y = \sqrt{(1 + x^2)}$ (2)

[$\cosh y > 0$, so the negative square root does not give a real value of y.]

Now $\cosh y + \sinh y = \frac{1}{2}(e^y + e^{-y}) + \frac{1}{2}(e^y - e^{-y})$
$= e^y$

But from (1), (2),

$$\begin{aligned} \cosh y + \sinh y &= x + \sqrt{(1 + x^2)} \\ \therefore e^y &= x + \sqrt{(1 + x^2)} \\ \therefore y &= \ln \{x + \sqrt{(1 + x^2)}\} \end{aligned}$$

That is,

$$\boldsymbol{\sinh^{-1} x = \ln \{x + \sqrt{(1 + x^2)}\}}$$

An expression for $\cosh^{-1} x$ is obtained in a similar manner.

Remember that $x \geqslant 1$ and $\cosh^{-1} x \geqslant 0$.

Let $y = \cosh^{-1} x$, then $\cosh y = x$

Now, $\sinh^2 y = \cosh^2 y - 1$
$= x^2 - 1$
$\therefore \sinh y = \sqrt{(x^2 - 1)}$

(The positive square root is used because y, and hence $\sinh y$, is positive.)

Now

$$\begin{aligned} e^y &= \cosh y + \sinh y \\ &= x + \sqrt{(x^2 - 1)} \\ \therefore y &= \ln \{x + \sqrt{(x^2 - 1)}\} \end{aligned}$$

That is

$$\boldsymbol{\cosh^{-1} x = \ln \{x + \sqrt{(x^2 - 1)}\}}$$

Question

Q10 Use the formulae in §36.5 above to find the values of
a $\sinh^{-1} 1$, **b** $\cosh^{-1} 2$, **c** $\sinh^{-1} 0.58$,
giving your answers correct to 4 d.p.

Once you have grasped the forms which require the substitution of a hyperbolic function, the integrations in Exercise 36c should present no new difficulty. Only in exceptional cases, e.g. in Example 5, is the treatment of hyperbolic functions completely different from the treatment of circular functions. If you are unable to integrate any particular function in Exercise 36c, refer back to Chapters 24 and 33. Examples 2–5 show how a knowledge of integrating with circular functions helps with the present work.

Example 2 *Find* $\displaystyle\int \frac{1}{\sqrt{(x^2 + 2x + 10)}}\,dx.$

First complete the square:

$$x^2 + 2x + 10 = (x + 1)^2 + 9$$

[The substitution $x + 1 = 3 \sinh \theta$ makes $(x + 1)^2 + 9 = 9 \cosh^2 \theta$.]

$$\begin{aligned} &\int \frac{1}{\sqrt{(x^2 + 2x + 10)}}\,dx \\ &= \int \frac{1}{\sqrt{\{(x+1)^2 + 9\}}} \frac{dx}{d\theta}\,d\theta \\ &= \int \frac{1}{3 \cosh \theta} 3 \cosh \theta\,d\theta \\ &= \int 1\,d\theta = \theta + c \end{aligned}$$

Let $x + 1 = 3 \sinh \theta$

$\therefore \dfrac{dx}{d\theta} = 3 \cosh \theta$

$$\therefore \int \frac{1}{\sqrt{(x^2 + 2x + 10)}}\,dx = \sinh^{-1} \frac{x+1}{3} + c$$

Example 3 *Evaluate* $\int_2^3 \cosh^{-1} x\,dx$.

[Integrate $\int \cos^{-1} x\,dx$ by parts as $\int 1 \times \cos^{-1} x\,dx$.]

$$\int_2^3 1 \times \cosh^{-1} x\,dx$$
$$= \Big[x \cosh^{-1} x\Big]_2^3 - \int_2^3 x \frac{1}{\sqrt{(x^2-1)}}\,dx$$
$$= 3\cosh^{-1} 3 - 2\cosh^{-1} 2 - \Big[\sqrt{(x^2-1)}\Big]_2^3$$
$$= 3\ln(3+\sqrt{8}) - 2\ln(2+\sqrt{3}) - (\sqrt{8}-\sqrt{3})$$
$$= 1.56, \quad \text{correct to 3 s.f.}$$

Example 4 *Find* $\int \sinh^3 \theta\,d\theta$.

$$\int \sinh^3\theta\,d\theta = \int (\cosh^2\theta - 1)\sinh\theta\,d\theta$$
$$= \tfrac{1}{3}\cosh^3\theta - \cosh\theta + c$$

Question

Q11 Find $\int \sinh^3\theta\,d\theta$ by means of an identity analogous to

$$\sin 3A = 3\sin A - 4\sin^3 A$$

Example 5 *Integrate* $\operatorname{sech} x$ *with respect to* x.

[The method of §33.3 on page 343 cannot be used here because $\cosh x$ is *not* $\sinh(x + \frac{1}{2}\pi)$. We might guess an integral but we can go back to the definition of $\cosh x$.]

$$\int \operatorname{sech} x\,dx$$
$$= \int \frac{2}{e^x + e^{-x}}\,dx$$
$$= \int \frac{2e^x}{e^{2x}+1}\frac{dx}{d\theta}\,d\theta$$
$$= \int \frac{2\sec^2\theta}{\sec^2\theta}\,d\theta$$
$$= 2\theta + c$$

Let $e^x = \tan\theta$

$e^x \dfrac{dx}{d\theta} = \sec^2\theta$

$e^{2x} + 1 = \tan^2\theta + 1 = \sec^2\theta$

$$\therefore \int \operatorname{sech} x\,dx = 2\tan^{-1}(e^x) + c$$

Exercise 36c

Integrate questions 1–8 with respect to x.

1 $\dfrac{1}{\sqrt{(x^2+9)}}$

2 $\dfrac{3}{\sqrt{\{1-(x-2)^2\}}}$

3 $\dfrac{2}{\sqrt{(4x^2-1)}}$

4 $\dfrac{1}{\sqrt{(4x+x^2)}}$

5 $\dfrac{1}{x^2+x+1}$

6 $\dfrac{1}{\sqrt{(x^2-6x+10)}}$

7 $\dfrac{1}{\sqrt{(4x^2+x)}}$

8 $\dfrac{1}{\sqrt{(3x-4x^2)}}$

Evaluate questions 9–12.

9 $\displaystyle\int_0^1 \frac{1}{\sqrt{(x^2+4)}}\,dx$

10 $\displaystyle\int_{1.5}^2 \frac{2}{\sqrt{(4x^2-9)}}\,dx$

11 $\displaystyle\int_1^2 \frac{1}{\sqrt{(x^2+4x+5)}}\,dx$

12 $\displaystyle\int_0^{\sqrt{3}} \frac{1}{\sqrt{(2x^2+3)}}\,dx$

Integrate questions 13–20 with respect to x.

13 $\cosh^2 x$

14 $\cosh^3 x$

15 $\sinh^4 x$

16 $\tanh^2 x$

17 $\tanh x$

18 $\coth^3 x$

19 $\tanh^4 x$

20 $\operatorname{cosech} x$

Chapter 37

Further complex numbers

Revision

Chapter 10 introduced **complex numbers**, i.e. the set of numbers

$$\mathbb{C} = \{x + iy: x, y \in \mathbb{R}\} \qquad \text{where } i^2 = -1$$

and the rules for adding, subtracting, multiplying and dividing them were explained. These were as follows: if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

$$z_1 \times z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$

$$= \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2}$$ (i.e. multiplying the top and the bottom by z_2^*, the complex conjugate of z_2)

$$= \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

These four rules show that complex numbers obey the same rules as real numbers, but wherever i^2 appears, it is replaced by -1. There is no need to commit these rules to memory.

§10.5 on page 134 contained a useful method for showing complex numbers, namely the **Argand diagram**. **Fig. 37.1** is an Argand diagram in which the vector $\overrightarrow{OP}$ represents the complex number $z = x + iy$. (Sometimes it is convenient to say the *point* P represents z.) The modulus of z, written $|z|$, and the argument, written arg (z), are given by

$$|z| = OP = \sqrt{(x^2 + y^2)}$$

$$\arg(z) = \theta = \arctan\left(\frac{y}{x}\right)$$

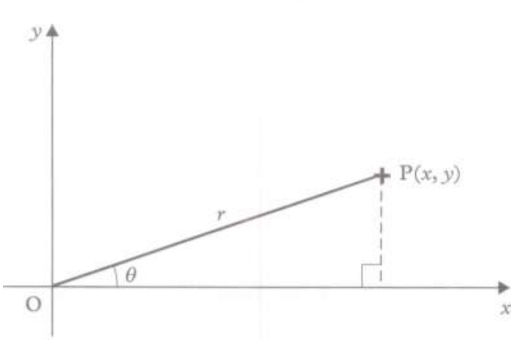


Figure 37.1

In this chapter, arg (z) is normally expressed in radians.

This leads to some useful alternatives to the rules for multiplying and dividing. If we write $|z| = r$ and arg $(z) = \theta$, then

$$|z_1z_2| = r_1r_2 = |z_1| \times |z_2|$$

and

$$\arg(z_1z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$$

Similarly,

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

and

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$$

The following special cases are important:

a $|z^2| = |z|^2$ and $\arg(z^2) = 2\arg(z)$

b $\left|\frac{1}{z}\right| = \frac{1}{|z|}$ and $\arg\left(\frac{1}{z}\right) = -\arg(z)$

Notice that there is a very close similarity between adding vectors and adding complex numbers. This is very useful when interpreting complex numbers geometrically. In particular, diagrams like **Fig. 37.2** are very common.

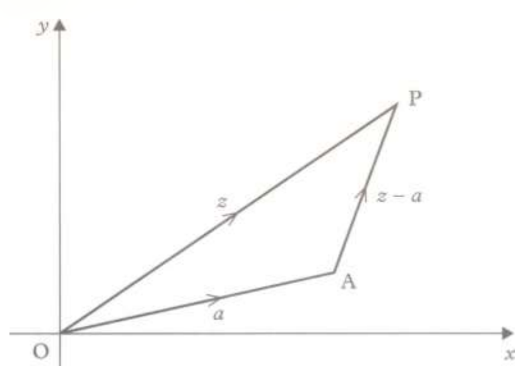


Figure 37.2

We could regard this diagram either as an Argand diagram illustrating the complex numbers z, a, and $z - a$, or as a vector triangle representing vectors **z**, **a**, and **z** − **a**. Consequently, a statement in complex numbers, such as

$$|z - a| = r$$

in which z represents a (variable) complex number $x + iy$, a is a (constant) complex number and r is a given real constant, tells us that the length AP is equal to r. In other words the variable point P lies on a circle, centre A, radius r.

Similarly the equation

$$|z - a| = |z - b|$$

where a and b are complex numbers represented by fixed points A and B, tells us that the variable point P which represents z, is equidistant from A and B.

You need to understand these elementary concepts thoroughly. The purpose of Exercise 37a is to revise this work.

Exercise 37a

1 Given that $z_1 = 3 + 4i$ and $z_2 = 1 + i$, find

a $z_1 + z_2$ **b** $z_1 - z_2$ **c** $z_1 z_2$
d z_1/z_2 **e** $z_1{}^2$ **f** $z_1{}^3$
g $1/z_1$ **h** $1/z_2$

2 Write down the modulus and argument of z_1 and z_2 in question 1. Find also the modulus and argument of the complex number in each part of question 1. (Leave surds in the moduli; express the arguments in radians, correct to 3 d.p.)

3 Solve the equation $z^2 - 4z + 53 = 0$, expressing the roots in the form $a + ib$, where $a, b \in \mathbb{R}$. Verify that the sum of the roots is 4 and their product is 53.

4 Draw Argand diagrams to illustrate the loci:

a $|z - 10| = 5$ **b** $\arg(z) = \pi/6$
c $|z - 1| = |z - i|$
d $\arg(z - a) = \pi/4$, where $a = 1 + i$

5 Writing $z = x + iy$, find the equations of the following loci in terms of x and y.

a $|z - 10| = 5$ **b** $|z - 1| = |z - i|$

6 Draw Argand diagrams to illustrate the regions

a $\text{Re}(z) > 0$ **b** $\text{Im}(z) > 0$
c $|z| < 3$ **d** $3 < |z| < 5$
e $|z - 3| > |z - 5|$ **f** $0 < \arg(z) < \pi/4$
g $0 < \arg(z - a) < \pi/2$, where $a = 1 + i$
h $4 < |z + a| < 5$, where $a = 1 + i$

7 Given that $z = r$ and $\arg(z) = \theta$, show that z can be expressed in the form

$$r(\cos\theta + i\sin\theta)$$

Verify that

$$z^2 = r^2(\cos 2\theta + i\sin 2\theta) \quad \text{and that}$$
$$z^3 = r^3(\cos 3\theta + i\sin 3\theta)$$

***8** Given that $z = \cos\theta + i\sin\theta$, prove by induction that

$$z^n = \cos n\theta + i\sin n\theta$$

where n is a positive integer.

9 The points A, B and C represent the complex numbers z_1, z_2 and z_3, respectively, and ABC is an isosceles triangle, with a right angle at A. Prove that

$$2z_1{}^2 + z_2{}^2 + z_3{}^2 = 2(z_3 z_1 + z_1 z_2)$$

10 The coordinates of the points A and B are (x_1, y_1) and (x_2, y_2), respectively, and O is the origin. Given that OAB is an equilateral triangle, prove that

$$x_2 + iy_2 = \pm\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)(x_1 + iy_1)$$

Hence, or otherwise, show that it is not possible to draw an equilateral triangle with its three vertices at lattice points. (A lattice point is a point whose coordinates are integers.)

37.1 Functions of a complex variable

In real numbers we have already studied many functions, e.g.

$$x \mapsto x^2$$

and we have either plotted or sketched their graphs ($y = x^2$ in this case). We shall now consider the corresponding problem in complex numbers. To keep the discussion fairly simple, we shall take the function

$$z \mapsto z^2$$

as an illustration. For convenience, in this book, we shall use $z = x + iy$ as a typical member of the domain (i.e. the set of complex numbers to which the rule is applied) and we shall use $w = u + iv$ as the corresponding point in the co-domain. We shall illustrate the effect of the function by drawing a *pair* of Argand diagrams, the left-hand one will always be the z-plane, i.e. the domain, and the right-hand one the w-plane, i.e. the co-domain.

First let us consider the effect of the function $z \mapsto z^2$ on a particular point, say $z = 2 + i$, then

$$w = (2 + i)^2$$
$$= 3 + 4i$$

Figure 37.3 shows P(2, 1) and its image P′(3, 4).

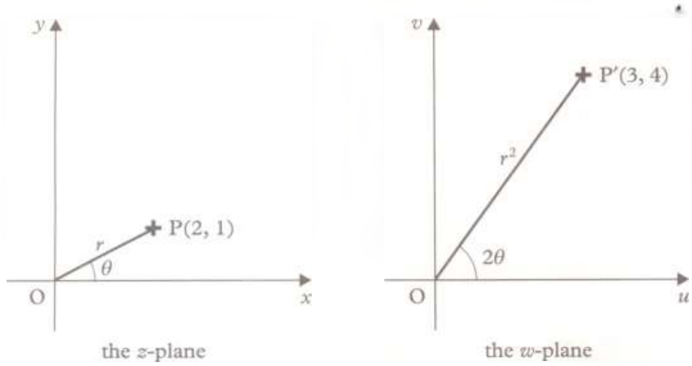


Figure 37.3

More generally if $|z| = r$ and $\arg(z) = \theta$ then, under this function, $|w| = r^2$ and $\arg(w) = 2\theta$.

Question

Q1 Verify that this last statement is true for the points P and P′ above.

We shall now consider the effect of this function on a particular set of points in the z-plane. Let us take, for example, the circle, centre O, radius 2 units, in which case, $|z| = 2$, and let $\arg(z) = \theta$. Then $|w| = 4$ and $\arg(w) = 2\theta$ (see Fig. 37.4).

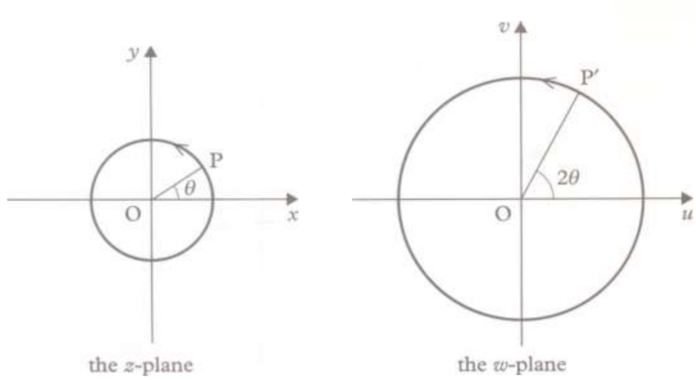


Figure 37.4

The image, in the w-plane, is also a circle, centre O, but its radius is 4 units and OP′ is inclined at an angle 2θ to the u-axis. If we imagine the radius OP rotating anti-clockwise at a constant speed, then OP′ would be rotating anti-clockwise at twice the speed. Therefore when P has completed one full circle P′ will have completed two full circles.

Example 1 *In the z-plane, the point P(x, y) moves around the semi-circle $|z| = 2$, where $0 \leqslant \arg(z) \leqslant \pi$. Describe the locus of its image P′(u, v) in the w-plane, where $w = 1/z$.*

Under the function $w = 1/z$,

$$|w| = \frac{1}{|z|} \quad \text{and} \quad \arg(w) = -\arg(z)$$

so P′ lies on the semi-circle, centre O, radius $\frac{1}{2}$. As P moves anti-clockwise, from $\theta = 0$ to $\theta = \pi$, the point P′ moves clockwise from $\theta = 0$ to $\theta = -\pi$ (see Fig. 37.5).

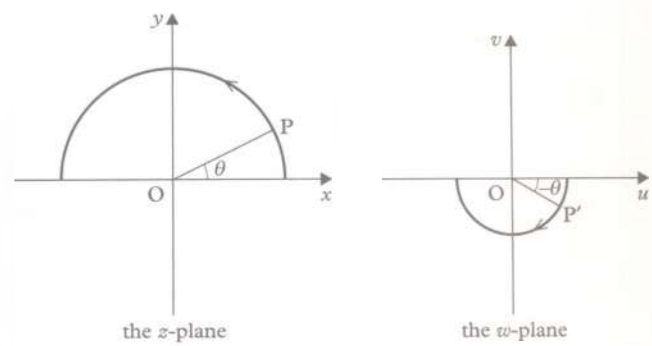


Figure 37.5

Example 2 *Given that $w = 1/z$, find the image of the straight line $x = 1$. Find the images of $(1, -1)$, $(1, 0)$ and $(1, 1)$. Describe the locus of P′, the image of P$(1, y)$ as y increases from $-\infty$ to $+\infty$.*

In this example $|z|$ is not constant and so the modulus–argument method of Example 1 is not the best approach. Instead we tackle the problem algebraically.

The image P′(u, v) of the point P$(1, y)$ is given by

$$\begin{aligned} u + \mathrm{i}v &= \frac{1}{1 + \mathrm{i}y} \\ &= \frac{1}{1 + \mathrm{i}y} \times \frac{1 - \mathrm{i}y}{1 - \mathrm{i}y} \\ &= \frac{1 - \mathrm{i}y}{1 + y^2} \end{aligned}$$

Hence, equating the real and imaginary parts,

$$u = \frac{1}{1 + y^2} \quad \text{and} \quad v = \frac{-y}{1 + y^2}$$

From these equations we can see that the images of $(1, -1)$, $(1, 0)$, and $(1, 1)$ are $(\frac{1}{2}, \frac{1}{2})$, $(1, 0)$ and $(\frac{1}{2}, -\frac{1}{2})$ respectively.

The equations for u and v, in terms of y, can be regarded as parametric equations, with y as the parameter. To find the equation relating u and v we must eliminate y. Dividing v by u, we obtain

$$\frac{v}{u} = -y$$

Using this to eliminate y, gives

$$u = \frac{1}{1+(-v/u)^2} = \frac{u^2}{u^2+v^2}$$

$$\therefore u(u^2+v^2) = u^2$$

Dividing by u (which is never zero),

$$u^2 + v^2 = u$$

This is the equation of a circle in the w-plane; we can find its centre and radius by completing the square,

$$(u - \tfrac{1}{2})^2 + v^2 = \tfrac{1}{4}$$

From this we see that the locus is a circle, centre $(\frac{1}{2}, 0)$, radius $\frac{1}{2}$ but excluding the point $(0, 0)$, since $u > 0$. (See **Fig. 37.6**; note in particular that as y increases, P′ moves clockwise.)

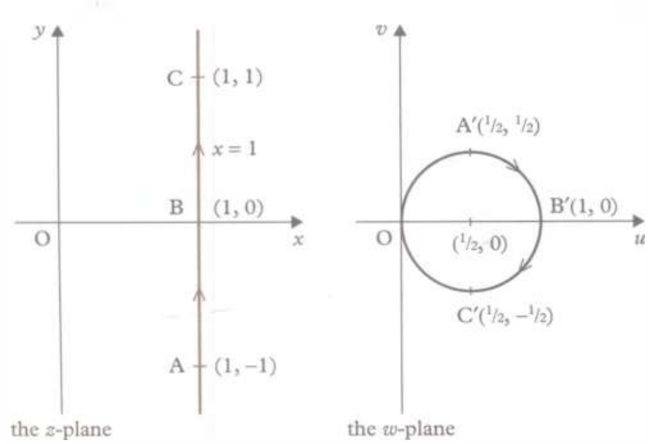


Figure 37.6

Exercise 37b

In questions 1 and 2, $a, b \in \mathbb{R}$ and $c = a + ib$. (If you find this difficult, try $a = 3$ and $b = 4$, and then return to the general case.)

1 Find the image of the unit square OIRJ, whose vertices are (0, 0), (1, 0), (1, 1) and (0, 1) respectively, under each of the following transformations.

a $w = az$ **b** $w = a + z$ **c** $w = c + z$
d $w = cz$ **e** $w = z^*$

Describe each of these transformations in words.

2 Show that the matrix transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

has the same effect as the complex number transformation $z \mapsto cz$.

In questions 3–6, draw Argand diagrams to show the effect of the given function on the region indicated.

3 $w = z + 1 + \mathrm{i}$; $0 \leqslant \arg(z) \leqslant \pi/4$

4 $w = (\cos \pi/4 + \mathrm{i}\sin \pi/4)z$; $0 \leqslant \arg(z) \leqslant \pi/4$

5 $w = z^2$; $1 \leqslant |z| \leqslant 2$

6 $w = z^2$; $0 \leqslant \arg(z) \leqslant \pi/4$

7 Find the range of the function $w = z^2$, given that the domain is

$\{z: |z| = 5, 0 \leqslant \arg(z) \leqslant \pi/6\}$

8 Given that $w = z^2$, find the images of

a the hyperbola $xy = 10$
b the line $y = x$
c the line $y = -x$

9 Find the image of the region $1 \leqslant x \leqslant 2$ under the transformation $z \mapsto 1/z$.

10 Given that $w = z^2$, find the images of the lines $x = k$ and $y = k$, where $k \in \mathbb{R}$.

Question

Q2 Find the image of $y = 1$ when $w = 1/z$.

37.2 de Moivre's theorem

In Exercise 37a, question 8, the outcome of the proof is known as **de Moivre's theorem**. The theorem states that, for any rational value of n, one value of $(\cos\theta + i\sin\theta)^n$ is given by

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

The reason for saying *one* value of $(\cos\theta + i\sin\theta)^n$ is that there is more than one value for expressions such as $(\cos\theta + i\sin\theta)^{3/2}$. (We return to this in §37.3 below.)
A proof of the theorem follows the stages below:

Stage 1. Prove that

$$(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = \cos(\theta+\phi) + i\sin(\theta+\phi).$$

[Expand the left-hand side.]

Stage 2. Use induction to prove the theorem for positive integral values of n.

Stage 3. Use the identity $(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = 1$ to show that

a $(\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta)$,

b if $n = -m$, where m is a positive integer,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

In accordance with the usual laws of algebra we take

$$(\cos\theta + i\sin\theta)^0 = 1$$

Stage 4. If $n = 1/q$, where q is an integer (positive or negative, but not zero), show that *one* value of $(\cos\theta + i\sin\theta)^n$ is $\cos\frac{1}{q}\theta + i\sin\frac{1}{q}\theta$ by finding the value of

$$\left(\cos\frac{1}{q}\theta + i\sin\frac{1}{q}\theta\right)^q.$$

Stage 5. If n is a rational number say $\frac{p}{q}$, where p, q are integers, we have by Stage 4 one value of

$$(\cos\theta + i\sin\theta)^{1/q} = \cos\frac{1}{q}\theta + i\sin\frac{1}{q}\theta$$

$$(\cos\theta + i\sin\theta)^{p/q} = \left(\cos\frac{1}{q}\theta + i\sin\frac{1}{q}\theta\right)^p$$

$$= \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$$

Questions

Q3 Express in the form $x + iy$:

a $(\cos\theta + i\sin\theta)^5$

b $1/(\cos\theta + i\sin\theta)^2$

c $(\cos\theta - i\sin\theta)^{-3}$

d $(\cos\theta + i\sin\theta)^2(\cos\theta + i\sin\theta)^3$

e $\dfrac{\cos\theta + i\sin\theta}{\cos\phi + i\sin\phi}$

f $\dfrac{\cos\theta + i\sin\theta}{\cos\phi - i\sin\phi}$

Q4 Find one value of:

a $\sqrt{(\cos 2\theta + i\sin 2\theta)}$

b $\sqrt[3]{(\cos 2\pi + i\sin 2\pi)}$

c $\sqrt{(\cos\theta + i\sin\theta)^3}$

37.3 Complex roots of unity

In §10.3 on page 131, we referred to the theorem that an equation of the nth degree has n roots. This means that the equation

$$z^3 - 1 = 0$$

has three roots. One of them is 1 but what are the others? Q5 gives one method of finding out.

Question

Q5 Use the identity $z^3 - 1 = (z-1)(z^2+z+1)$ to find the three cube roots of unity.

The method of Q5 would not work for, say, the roots of $z^7 - 1 = 0$. You may wonder what this has got to do with de Moivre's theorem. We can express 1 as a complex number in infinitely many ways:

$$\ldots \cos(-2\pi) + i\sin(-2\pi),\ \cos 0 + i\sin 0,$$
$$\cos 2\pi + i\sin 2\pi,\ \cos 4\pi + i\sin 4\pi,\ \ldots$$

or, in general,

$$\cos 2k\pi + i\sin 2k\pi \qquad \text{where } k \text{ is an integer}$$

By de Moivre's theorem, one value of

$$\sqrt[3]{(\cos\theta + i\sin\theta)} = (\cos\theta + i\sin\theta)^{1/3} = \cos\tfrac{1}{3}\theta + i\sin\tfrac{1}{3}\theta$$

Therefore values of $\sqrt[3]{1}$ are given by

$$\ldots \cos\left(-\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{3}\right),\ \cos\frac{0}{3} + \mathrm{i}\sin\frac{0}{3},$$
$$\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3},\ \cos\frac{4\pi}{3} + \mathrm{i}\sin\frac{4\pi}{3},\ \ldots$$

or, in general,

$$\cos\frac{2k\pi}{3} + \mathrm{i}\sin\frac{2k\pi}{3} \qquad \text{(see Fig. 37.7)}$$

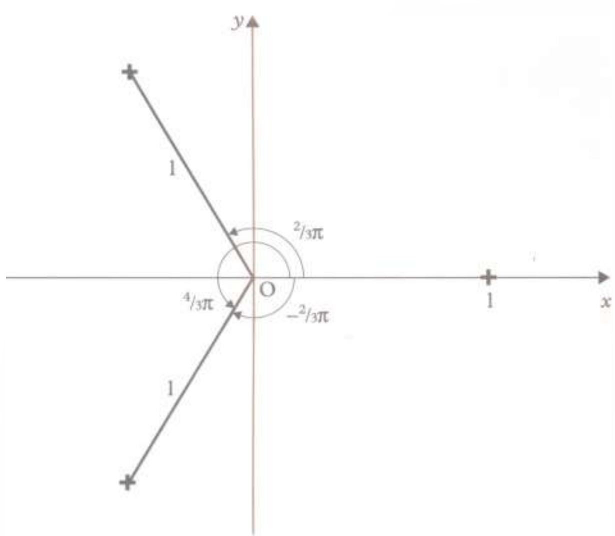


Figure 37.7

Questions

Q6 Show that the expression $\cos \frac{2}{3}k\pi + \mathrm{i}\sin \frac{2}{3}k\pi$ represents the same complex number when k is replaced by **a** $k + 3$, **b** $k + 3m$, $(k, m \in \mathbb{Z})$.

Q7 By writing $-1 = \cos \pi + \mathrm{i}\sin \pi$, and in two other ways, find the cube roots of -1.

Q8 Writing $\omega = \cos \frac{2}{3}\pi + \mathrm{i}\sin \frac{2}{3}\pi$, show that the cube roots of 1 are $1, \omega, \omega^2$.

Prove that $1 + \omega + \omega^2 = 0$ in two different ways. [For a hint, see Q5.]
Show also that ω is the square of ω^2.

37.4 Real and imaginary parts

One advantage of using complex numbers is that sometimes it is possible to do two pieces of working simultaneously. This is because: *if $a + \mathrm{i}b = c + \mathrm{i}d$, where a, b, c, d are real, then $a = c$ and $b = d$.* [I.e. by 'equating real and imaginary parts'. See §10.3 on page 131.]

Example 3 *Prove that* $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

By de Moivre's theorem,

$$\cos 3\theta + \mathrm{i}\sin 3\theta = (\cos\theta + \mathrm{i}\sin\theta)^3$$

The R.H.S. may be written

$$\cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta$$

$$\therefore \cos 3\theta + \mathrm{i}\sin 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta + \mathrm{i}(3\cos^2\theta\sin\theta - \sin^3\theta)$$

Equating real and imaginary parts,

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$
$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

By division,

$$\frac{\sin 3\theta}{\cos 3\theta} = \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}$$

Dividing numerator and denominator of the R.H.S. by $\cos^3\theta$,

$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

Question

Q9 Show that $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$.

Exercise 37c

Simplify the expressions in questions 1–8.

1 $(\cos\theta + \mathrm{i}\sin\theta)^3(\cos 2\theta + \mathrm{i}\sin 2\theta)$

2 $(\cos\theta + \mathrm{i}\sin\theta)^2(\cos\theta - \mathrm{i}\sin\theta)^{-2}$

3 $\dfrac{\cos 4\theta + \mathrm{i}\sin 4\theta}{(\cos\theta + \mathrm{i}\sin\theta)^3}$ **4** $\dfrac{(\cos 2\theta + \mathrm{i}\sin 2\theta)^3}{(\cos\theta + \mathrm{i}\sin\theta)^5}$

5 $\dfrac{\cos 5\theta + \mathrm{i}\sin 5\theta}{(\cos\theta - \mathrm{i}\sin\theta)^3}$ **6** $\dfrac{\cos\theta + \mathrm{i}\sin\theta}{\cos 2\theta - \mathrm{i}\sin 2\theta}$

7 $(\cos\phi + \mathrm{i}\sin\phi)^2(\cos\theta + \mathrm{i}\sin\theta)^3$

8 $(\cos 2\theta + \mathrm{i}\sin 2\theta)^3(\cos 3\phi - \mathrm{i}\sin 3\phi)^2$

9 By writing 1 in the form $\cos\theta + \mathrm{i}\sin\theta$, where $\theta = -2\pi, 0, 2\pi, 4\pi$, find the fourth roots of unity. What are the fourth roots of -1?

*10 By writing 1 in the form $\cos 2k\pi + i \sin 2k\pi$ $(k = -2, -1, 0, 1, 2)$, find the fifth roots of unity.
Show that, if z and z^* are conjugate complex numbers, the expansion of $(x - z^*)(x - z)$ is a quadratic in x with real coefficients. Hence write $x^5 - 1$ in the form $(x - 1)q_1q_2$, where q_1, q_2 are quadratic expressions with real coefficients. (See §10.3 on page 131 for a definition of z^*.)

11 Find the sixth roots of 1 and show the corresponding vectors on the Argand diagram. Deduce the real quadratic factors of $x^4 + x^2 + 1$.

12 Show that the nth roots of -1, together with the nth roots of 1, form a complete set of $2n$th roots of 1. How can the $2n$th roots of -1 be deduced from these? Illustrate your answer by showing the corresponding vectors on the Argand diagram for the case $n = 5$.

13 If α is a seventh root of unity, other than 1, show that the other roots are $\alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6, 1$. Show further that

$$1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 = 0$$

Do similar properties hold for all other nth roots of unity?

14 Find the real factors of
a $x^7 - 1$, b $x^5 + 1$, c $x^{2n} - 1$.

*15 If $z = \cos\theta + i\sin\theta$, show that

$$\frac{1}{z} = \cos\theta - i\sin\theta, \qquad z^3 = \cos 3\theta + i\sin 3\theta,$$

$$\frac{1}{z^3} = \cos 3\theta - i\sin 3\theta$$

Show further that $(z + 1/z)^3 = 8\cos^3\theta$, and, by expanding $(z + 1/z)^3$, prove that

$$2\cos 3\theta + 6\cos\theta = 8\cos^3\theta$$

Hence express $\cos 3\theta$ in terms of powers of $\cos\theta$.

16 With the notation of question 15, show that $(z - 1/z)^3 = -8i\sin^3\theta$ and hence express $\sin 3\theta$ in terms of $\sin\theta$.

17 Use the method of question 15 to prove that

$$\cos^4\theta = \tfrac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

and express $\cos 4\theta$ in terms of $\cos\theta$. [Expand $(z + 1/z)^4$.]

18 Prove that $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$.

19 Prove that
$\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$.

20 Show that $\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$.
[Use the method of Example 3.]

21 Find expressions in terms of $\tan\theta$ for
a $\tan 6\theta$, b $\tan 2n\theta$, c $\tan(2n+1)\theta$.

22 Show that $\tan 4\theta = \dfrac{4t - 4t^3}{1 - 6t^2 + t^4}$, where $t = \tan\theta$.
Hence find the roots of the equation $t^4 + 4t^3 - 6t^2 - 4t + 1 = 0$ correct to three significant figures.

23 Solve the equation $t^5 - 10t^4 - 10t^3 + 20t^2 + 5t - 2 = 0$ correct to three significant figures.

24 If $w = u + iv$, $z = x + iy$, and $w = z^3$, express u, v in terms of x, y.

25 Find $(a + ib) \div (c + id)$ by equating real and imaginary parts in the equation $(c + id)(p + iq) = a + ib$.

26 a Find the square roots of $-5 + 12i$ by equating real and imaginary parts of $(a + ib)^2 = -5 + 12i$.
b Find the square roots of i
i by the method above
ii by using de Moivre's theorem.

*27 Let $C = 1 + \cos\theta + \cos 2\theta + \ldots + \cos(n-1)\theta$

$S = \sin\theta + \sin 2\theta + \ldots + \sin(n-1)\theta$

$Z = C + iS, \qquad z = \cos\theta + i\sin\theta$

a Show that $Z = (1 - z^n)/(1 - z)$.
b Express Z with a real denominator.
c Deduce expressions for C and S by equating real and imaginary parts.

28 Sum the series

$$C = 1 + a\cos\theta + a^2\cos 2\theta + \ldots + a^n\cos n\theta$$

$$S = a\sin\theta + a^2\sin 2\theta + \ldots + a^n\sin n\theta$$

29 Examine the answers to question 27c, and 28. Show how the series in these two questions could be summed by multiplying both sides of the equations by some expression.

30 Find the sum to infinity of the series

$$\cos\theta\cos\theta + \cos^2\theta\cos 2\theta + \cos^3\theta\cos 3\theta + \ldots$$

37.5 e^z, $\cos z$, $\sin z$, where $z \in \mathbb{C}$

In this section we show how to extend the above functions to cover complex variables.

First consider the function e^x, where x is real. We have to find some property of the function which we can use to define what we mean by e^z where z is complex. An obvious way might be to go back to the definition of e^x. Unfortunately this does not lend itself to an extension to complex numbers, so we have to find some other property of e^x. Remember, from §37.1 on page 386 that, e^x can be expanded as a power series:

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$$

so we can readily give e^z a meaning by *defining* it by the series

$$e^z = 1 + z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots$$

We know that the series for e^x is convergent for all real values of x, and we shall assume that the series for e^z is convergent for complex values of z (but you should be aware that this does not follow automatically).

The definition of e^z by a series poses another problem: will the usual laws of indices still hold? In particular, we must satisfy ourselves that

$$e^w \times e^z = e^{w+z}$$

This is something which has to be deduced from our definition of e^z. That is, it is necessary to show that

$$\left(1 + w + \frac{w^2}{2!} + \ldots + \frac{w^n}{n!} + \ldots\right)\left(1 + z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots\right)$$
$$= 1 + (w + z) + \frac{(w+z)^2}{n!} + \ldots + \frac{(w+z)^n}{n!} + \ldots$$

The conditions under which infinite series may be multiplied are outside the scope of this book and so a proof will not be given here, but you should work Q10.

Question

Q10 Show that term-by-term multiplication of the first few terms of the series for e^w and e^z gives the first few terms of the series for e^{w+z}. [go as far as the 3rd term only.]

Also find the terms of degree n in the product and show that they reduce to $(w + z)^n/n!$

Now let us see what happens if we write $z = x + iy$ in the expression e^z.

$$e^z = e^{x+iy} = e^x \times e^{iy}$$

e^x is real (and familiar), so we shall examine the function e^{iy}.

$$e^{iy} = 1 + iy + \frac{i^2y^2}{2!} + \frac{i^3y^3}{3!} + \frac{i^4y^4}{4!} + \ldots + \frac{i^ny^n}{n!} + \ldots$$
$$= 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots$$
$$+ i\left(y - \frac{y^3}{3!} + \ldots\right)$$

But

$$1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots = \cos y \quad \text{and} \quad y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots = \sin y$$

so that

$$\mathbf{e^{iy} = \cos y + i \sin y}$$

Question

Q11 Use the results

$$(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = 1 \quad \text{and}$$
$$e^w \times e^z = e^{w+z}$$

to show that

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

Note. In most texts you will find the series

$$1 + z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots$$

written as $\exp z$.

Euler's formula

We are now in a position to turn to the problem of assigning meanings to the functions $\sin z$, $\cos z$, where z is complex.

From above, $e^{i\theta} = \cos\theta + i\sin\theta$, and from Q11, $e^{-i\theta} = \cos\theta - i\sin\theta$.

Hence

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Since e^z has been defined for complex z, we may use these last two equations to define $\cos z$ and $\sin z$:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad (1)$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad (2)$$

These results are sometimes called **Euler's formula**.

The hyperbolic functions $\cosh x$, $\sinh x$ were defined for real values of x in Chapter 36:

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}), \qquad \sinh x = \tfrac{1}{2}(e^x - e^{-x})$$

These definitions can be used to define the functions $\cosh z$, $\sinh z$ of a complex variable z:

$$\cosh z = \tfrac{1}{2}(e^z + e^{-z}), \qquad \sinh z = \tfrac{1}{2}(e^z - e^{-z})$$

Replacing z by iz,

$$\cosh iz = \tfrac{1}{2}(e^{iz} + e^{-iz}), \quad \sinh iz = \tfrac{1}{2}(e^{iz} - e^{-iz}) \qquad (3)$$

and so from (1), (2), (3) above we obtain the following relations connecting circular and hyperbolic functions:

$$\cosh iz = \cos z \qquad \sinh iz = i \sin z$$

Questions

Q12 Confirm these last relationships for real x by replacing z by x in equations (3) and expressing e^{ix}, e^{-ix} in terms of $\cos x$, $\sin x$.

Q13 Express $\cosh z$, $\sinh z$ in terms of the corresponding circular functions.

Q14 Use the series $e^w = 1 + w + \dfrac{w^2}{2!} + \ldots + \dfrac{w^n}{n!} + \ldots$ to show that

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$$

Q15 Deduce from equations (1) and (2) above that

a $\cos^2 z + \sin^2 z = 1$

b $\cos(-z) = \cos z$

c $\sin(-z) = -\sin z$

d $\cos(w + z) = \cos w \cos z - \sin w \sin z$

e $\sin(w + z) = \sin w \cos z + \cos w \sin z$

The other trigonometrical identities follow from these (if you cannot satisfy yourself about this, see Chapter 17, where they were proved for real numbers). Therefore trigonometrical functions of a complex variable may be manipulated by the same identities as those for a real variable.

Question

Q16 A function of a complex variable is said to be *periodic* with period p if $f(z + p) = f(z)$ for all z. Show that $\cos z$, $\sin z$ have period 2π.

Exercise 37d

1 Express in the form $a + ib$:

a $(x + iy)^6$

b $\dfrac{1}{3\cos\theta + 2i\sin\theta - 1}$

c $\sqrt{\dfrac{\cos\theta + i\sin\theta}{\cos 2\theta - i\sin 2\theta}}$

d $\dfrac{z+1}{z-1}$ where $z = x + iy$

2 Express in the form $a + ib$:

a $\dfrac{(\cos\theta + i\sin\theta)^3}{(\cos 2\theta + i\sin 2\theta)^2}$

b $\dfrac{(\cos 2\theta + i\sin 2\theta)^2}{(\cos\theta - i\sin\theta)^4}$

c $1 - \operatorname{cis}\theta + \operatorname{cis}2\theta - \ldots + (-1)^{n-1}\operatorname{cis}(n-1)\theta$, where $\operatorname{cis} r\theta = \cos r\theta + i\sin r\theta$.

3 P_1, P_2 are points on the Argand diagram corresponding to the complex numbers z_1, z_2. Show that the mid-point of P_1P_2 corresponds to $\frac{1}{2}(z_1 + z_2)$. Hence prove that the mid-points of the sides of a plane quadrilateral are the vertices of a parallelogram.

4 What is the locus given by $zz^* + 2(z + z^*) = 0$, where $z = x + iy$ and z^* is its conjugate? Express as an equation connecting z, z^* the condition that (x, y) should lie on the circle centre $(2, -1)$ radius 3.

5 a Prove that $\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$, and find an expression for $\sin^4\theta$ in terms of $\cos 2\theta$, $\cos 4\theta$.

b Express $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4)$ in terms of $\tan\theta_1$, $\tan\theta_2$, $\tan\theta_3$, $\tan\theta_4$.

6 **a** Express -1 in the form $\cos\theta + \mathrm{i}\sin\theta$. Hence obtain the three linear factors of $z^3 + 1$.

b Find the real factors of $x^5 - a^5$.

7 Show that if a quadratic equation is satisfied by a complex number then the conjugate complex number is also a root of the equation.

Hence find the quadratic equation satisfied by the complex number $2 - 3\mathrm{i}$.

Find the four roots of the equation $z^4 - 3z^3 + 4z^2 - 3z + 1 = 0$.

8 **a** Solve the equation $z^4 - 6z^2 + 25 = 0$.

b Given that one root of the equation $z^4 - 6z^3 + 23z^2 - 34z + 26 = 0$ is $1 + \mathrm{i}$, find the others.

9 If $z + \dfrac{1}{z} = -1$, prove that $z^5 + \dfrac{1}{z^5} = -1$ and find the value of $z^{11} + \dfrac{1}{z^{11}}$.

10 Prove that, if $a + \mathrm{i}b = c + \mathrm{i}d$ where a, b, c, d are real, then $a = c$, $b = d$.

Find the sum to infinity of $1 - \frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta - \frac{1}{8}\cos 3\theta + \dots$.

11 Find the sum of the series $1 + 2\cos 2\theta + 4\cos 4\theta + \dots + 2^n \cos 2n\theta$.

12 Prove by induction that, for positive integral values of n,

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$$

Hence show that the identity also holds for negative integral values of n.

Find the sum to infinity of the series

$$\sin\theta\sin\theta + \sin^2\theta\sin 2\theta + \sin^3\theta\sin 3\theta + \dots$$

In questions 13–15 x, y, u, v are real numbers, z, w are the complex numbers $x + \mathrm{i}y$, $u + \mathrm{i}v$.

13 If w, z are connected by the equation $w = z + 1$, find the locus of (u, v) when

a $x + 1 = 0$ **b** $x + y - 1 = 0$

c $|z + 1| = 1$ **d** $|z - 3| = 2$

14 If $w = 3z$, find the locus of (u, v) when

a $x - 2y = 0$ **b** $y - 1 = 0$

c $|z| = 1$ **d** $|z - 2| = 2$

15 The point (x, y) moves counter-clockwise once round each of the circles **a** $|z| = 1$, **b** $|z| = 2$. Describe the corresponding motions of (u, v) if $w = z^2$.

16 The point $P(x, y)$ represents a complex number $z = x + \mathrm{i}y$. Given that $|z + 1| = 2|z - 1|$, find, in terms of x and y, the equation of the locus of P and describe the locus in words.

17 Repeat question 16 for $|z - \mathrm{i}| = 3|z + \mathrm{i}|$.

18 **a** Given that $(2 + 3\mathrm{i})z = 4 - \mathrm{i}$, find the complex number z, giving your answer in the form $a + \mathrm{i}b$, where $a, b \in \mathbb{R}$.

b Find the modulus and argument of the complex number $5 - 3\mathrm{i}$.

The complex number w is represented in an Argand diagram by the point W.

Describe geometrically the locus of W in each of the following cases:

i $|w| = |5 - 3\mathrm{i}|$,

ii $\arg(w - 5 + 3\mathrm{i}) = \arg(5 - 3\mathrm{i}) + \frac{1}{2}\pi$.

19 Express z_1, where $z_1 = \dfrac{10 - \mathrm{i}\,2\sqrt{3}}{1 - \mathrm{i}\,3\sqrt{3}}$, in the form $p + \mathrm{i}q$, where p and q are real.

Given that $z_1 = r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, obtain values for r and θ. Hence determine $z_1{}^9$.

Sketch on an Argand diagram the locus of the points representing the complex number z such that $|z - z_1| = \sqrt{3}$.

20 The point P represents the complex number $z = x + \mathrm{i}y$ on an Argand diagram. Describe the locus of P if

a $|2z + 1 - 2\mathrm{i}| = 3$ **b** $\arg\left(\dfrac{z + \mathrm{i}}{z - \mathrm{i}}\right) = \frac{1}{4}\pi$

c $|z - \mathrm{i}| + |z + \mathrm{i}| = 4$

If $|z| = 1$, find the locus of the point representing the complex number $z + \dfrac{1}{z}$.

21 **a** Find the modulus and argument of each of the following complex numbers:

i $\dfrac{5 + \mathrm{i}}{2 + 3\mathrm{i}}$ **ii** $\dfrac{(5 + \mathrm{i})^4}{(2 + 3\mathrm{i})^4}$

iii $\dfrac{(\cos\pi/6 - \mathrm{i}\sin\pi/6)^4}{(\sin\pi/6 + \mathrm{i}\cos\pi/6)^3}$

b Find, in the form $re^{\mathrm{i}\theta}$, all the complex numbers z, such that

$$z^3 = \frac{5 + \mathrm{i}}{2 + 3\mathrm{i}}$$

22 The transformation $w = (z+1)^2 + 3$ maps the complex number $z = x + iy$ to the complex number $w = u + iv$. Show that as z moves along the y-axis from the origin to the point (0, 2) in the z-plane, w moves from the point (4, 0) to the point (0, 4) along a curve in the w-plane. Write down the equation of this curve.

23 **a** Given that the real part of $(z - 2i)/(z + 4)$ is zero, prove that, in the Argand diagram, the locus of z is a circle. Find the centre of the circle and show that the radius is $\sqrt{5}$.

b Find the image of the circle $|z| = 1$ under each of the transformations given in **i** and **ii** below. If the image is a circle, give its centre and radius. If the image is a straight line, give its equation (in any form).

i $w = \dfrac{2}{i + 2z}$ **ii** $w = \dfrac{2 + z}{i - z}$

24 Two complex numbers w and z are connected by the relation

$$w = 2\left(\frac{1+z}{1-z}\right)$$

Prove that, as the point P representing z in the Argand diagram follows the locus $i\lambda$ with λ decreasing from ∞ to 0, the locus of Q, representing w is a semi-circle C. Determine the centre and radius of this semi-circle.

Determine the locus C′ of the point Q′ corresponding to the point P′, as P′ describes the real axis in the positive direction from the origin.

Indicate on a diagram the loci C and C′ and the directions in which the loci are described.

25 The point P represents the complex number z and the point Q represents the complex number w, where $w = 1/(z - 1)$. Prove that if $|z| = 1$, then $|w| = |w + 1|$.

The point P moves anti-clockwise once round the circle C with centre the origin and radius 1,

a describe the locus of Q;

b given that P starts from $z = 1$, describe carefully the locus of a point R which represents the complex number $i/(z + i)$.

26 Express $\cos z$ and $\sin z$ in terms of cosh and sinh, and deduce the following identities:

a $\sinh 2z = 2 \sinh z \cosh z$

b $\cosh 2z = \cosh^2 z + \sinh^2 z$

c $\cosh (w - z) = \cosh w \cosh z - \sinh w \sinh z$

d $\cosh w + \cosh z = 2 \cosh \frac{1}{2}(w + z) \cosh \frac{1}{2}(w - z)$

27 Define $\cosh z$, $\sinh z$ when z is complex. Deduce from your definitions the identities:

a $\cosh^2 z - \sinh^2 z = 1$,

b $\sinh (w + z) = \sinh w \cosh z + \cosh w \sinh z$.

Express $\cosh z$, $\sinh z$ in terms of the circular cosine and sine, and deduce identities corresponding to the two above.

28 **a** Write down the values of $|z|$ and $\arg(z)$, where $z = x + iy$. Illustrate by means of an Argand diagram.

The numbers c and p are given, c being real and p being complex, with $p = a + ib$. z^* and p^* denote the conjugates of z and p respectively. Prove that, if

$$zz^* - p^*z - pz^* + c = 0$$

then the point on the Argand diagram which represents z lies on a certain circle whose centre and radius should be determined.

b Prove that, if

$$x + iy = \frac{1}{\lambda + i\mu}$$

then the points on the Argand diagram defined by making λ constant lie on a circle, and the points defined by making μ a constant also lie on a circle.

Prove also that, whatever be the values of the constants, the centres of the two systems of circles obtained lie on two fixed perpendicular lines.

29 The coordinates (x, y) of a point P are expressible in terms of real variables u and v by the formula

$$x + iy = (u + iv)^2$$

Prove that the locus of P is a parabola **a** when u varies and v is constant, and also **b** when v varies and u is constant. Prove also that all the parabolas have a common focus and a common axis.

Prove that through a given point (x_0, y_0) there pass two parabolas, one of each system $u =$ constant, $v =$ constant, which cut at right angles.

Vectors (2)

38.1 Scalar products and planes

In §15.11 on page 188, we found the equation of the plane through three given points. Another way of describing a plane is to specify a vector which is perpendicular to the plane, and give a point through which the plane passes.

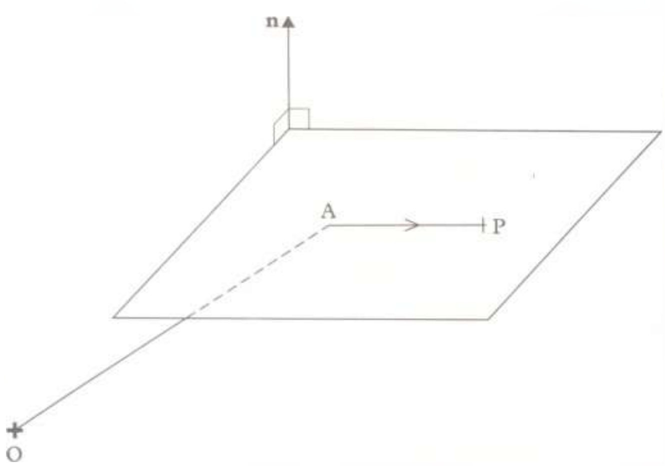


Figure 38.1

Suppose the plane is perpendicular to a given vector **n**, where $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Then every line in the plane is perpendicular to **n**. Let the given point through which the plane passes be $A(x_1, y_1, z_1)$ and let $P(x, y, z)$ be any point in the plane (Fig. 38.1), then AP is perpendicular to **n**. This can be expressed in terms of a **scalar product**, often called a **dot product**:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{pmatrix} = 0$$

Hence $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

$\therefore\ ax + by + cz = d$

where $d = ax_1 + by_1 + cz_1$. Notice that the coefficients a, b, c of x, y and z form the vector **n**. Therefore if we are given the equation of a plane, we can immediately write down the vector which is perpendicular to it.

Example 1 *Write down the unit vector which is normal to the plane $2x + 3y + 6z = 10$.*

Note: *normal to* means the same as *perpendicular to*.

The vector $\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$ is normal to the plane and the magnitude of this is $\sqrt{(4 + 9 + 36)} = \sqrt{49} = 7$.

So the required unit vector is $\frac{1}{7}\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$.

Example 2 *Find the equation of the plane through the point (1, 2, 3) which is perpendicular to the vector* $4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$.

Any plane which is perpendicular to the given vector will have an equation of the form

$4x + 5y + 6z = d$ where d is a constant.

To find the equation of the plane which passes through (1, 2, 3), we choose the value of d so that the equation is satisfied when $x = 1$, $y = 2$ and $z = 3$, i.e.

$4 \times 1 + 5 \times 2 + 6 \times 3 = d$

i.e. $d = 32$

Hence the equation we require is $4x + 5y + 6z = 32$.

Example 3 *Find the angle between the planes* $4x + 3y + 12z = 10$ *and* $8x - 6y = 14$.

The angle required is the angle between the normal vectors and these are $\mathbf{m} = 4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}$ and $\mathbf{n} = 8\mathbf{i} - 6\mathbf{j}$. We find the angle between **m** and **n** by expressing the scalar product **m.n** in two forms and equating them. Firstly,

$$\begin{aligned} \mathbf{m.n} &= (4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}).(8\mathbf{i} - 6\mathbf{j}) \\ &= 32 - 18 \\ &= 14 \end{aligned}$$

Alternatively, using $\mathbf{m.n} = mn \cos\theta$, where m and n are the magnitudes of the vectors **m** and **n**, and θ is the angle between them, we obtain

$$\begin{aligned} \mathbf{m.n} &= \sqrt{(16 + 9 + 144)} \times \sqrt{(64 + 36)} \times \cos\theta \\ &= \sqrt{169} \times \sqrt{100} \times \cos\theta \\ &= 130 \cos\theta \end{aligned}$$

Equating these two expressions for **m.n** gives

$$130 \cos \theta = 14$$

$$\therefore \cos \theta = \frac{14}{130}$$

$$\therefore \theta = 83.8°$$

The angle between the planes is 83.8°, correct to 1 d.p.

Example 4 *Find the distance of the point* A(25, 5, 7) *from the plane* $12x + 4y + 3z = 3$.

Let P be the point in the plane such that $\overline{AP}$ is perpendicular to the plane. Then the distance required is the length of the vector $\overline{AP}$.

We know that $12\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ is perpendicular to the plane, so let $\overline{AP} = t(12\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$. Then, since $\overline{OP} = \overline{OA} + \overline{AP}$,

$$\begin{aligned}\overline{OP} &= (25\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}) + t(12\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})\\ &= (25 + 12t)\mathbf{i} + (5 + 4t)\mathbf{j} + (7 + 3t)\mathbf{k}\end{aligned}$$

Hence P is the point $(25 + 12t, 5 + 4t, 7 + 3t)$. Since this point lies in the plane, its coordinates satisfy the equation of the plane. Therefore:

$$12(25 + 12t) + 4(5 + 4t) + 3(7 + 3t) = 3$$

$$\therefore 169t + 341 = 3$$

$$\therefore t = -2$$

Hence $\overline{AP} = -2(12\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) = -24\mathbf{i} - 8\mathbf{j} - 6\mathbf{k}$, and so

$$\begin{aligned}AP^2 &= 24^2 + 8^2 + 6^2\\ &= 576 + 64 + 36\\ &= 676\end{aligned}$$

$$\therefore AP = 26$$

The distance of point A from the plane is 26 units.

Exercise 38a

1 Find **p.q**, **p.r** and **p.(q + r)**, given that

a $\mathbf{p} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{q} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$, $\mathbf{r} = 4\mathbf{i} + \mathbf{j} - 8\mathbf{k}$

b $\mathbf{p} = 3\mathbf{i} + 7\mathbf{j} - 5\mathbf{k}$, $\mathbf{q} = 2\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$, $\mathbf{r} = 4\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$

2 Find the angle between the vectors $\mathbf{a} = 2\mathbf{i} + \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$.

3 Find the coordinates of the point N, where the perpendicular from (37, 9, 10) meets the plane $12x + 4y + 3z = 3$.

4 Find the reflection of the point (5, 7, 11) in the plane $2x + 3y + 5z = 10$.

5 Given that vectors **a** and **b** have equal magnitudes, prove that $(\mathbf{a} + \mathbf{b}).(\mathbf{a} - \mathbf{b}) = 0$. Interpret this result in terms of the parallelogram which has **a** and **b** as a pair of adjacent sides.

*6 In Fig. 38.2, **OA** = **a**, **OB** = **b** and the angle AOB is θ.

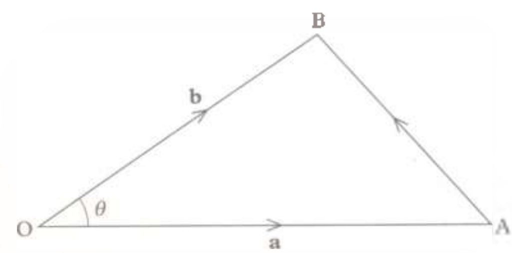


Figure 38.2

By considering the scalar product **AB.AB**, prove that

$$AB^2 = OA^2 + OB^2 - 2 \times OA \times OB \times \cos \theta$$

7 The points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ lie in the plane

$$ax + by + cz = d$$

Show that $a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0$. Hence prove that the vector $\overline{AB}$ is perpendicular to the vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

*8 The point O is the centre of the circumcircle of triangle ABC (the circumcircle is the circle which passes through the vertices of a triangle) and G is its centroid. H is a point on OG such that $\overline{OH} = 3\overline{OG}$. Prove that $\overline{AH}$ is perpendicular to $\overline{BC}$. Prove also that $\overline{BH}$ is perpendicular to $\overline{AC}$ and $\overline{CH}$ is perpendicular to $\overline{AB}$. (The point H is called the *orthocentre* of the triangle.)

9 In triangle OPQ angle POQ is a right angle. Point R lies on PQ and PR:RQ = 1:3. Express the position vector of R in terms of **p** and **q**, the position vectors of P and Q. Given that **OR** is perpendicular to **PQ**, prove that OP:OQ = 1:$\sqrt{3}$.

*10 In Fig. 38.3, **OA** and **OB** are unit vectors making angles α and β respectively with the x-axis. Express in terms of α and β: **a** OA, **b** OB, **c** the angle AOB.

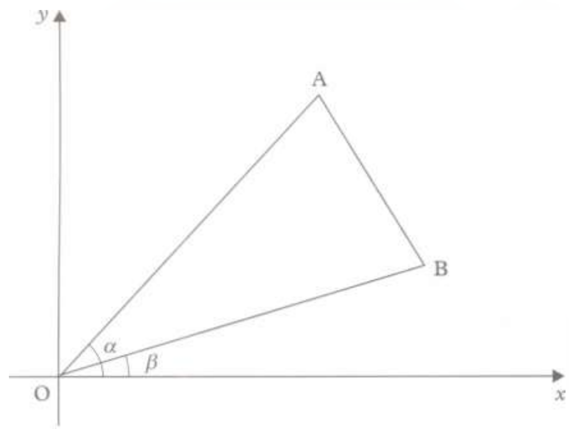


Figure 38.3

Write down the dot product **OA.OB** in two ways. Hence prove that

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

38.2 Vector product

As we have seen, the *scalar* product combines two vectors to produce a scalar result. The *vector* product, which we now examine, combines two vectors and produces a vector result. The vector product is useful in mechanics, for example when finding the moment of a force about a point (for further applications you should consult a suitable book on mechanics).

Definition

The **vector product** of two vectors **a** and **b**, which is written $\mathbf{a} \wedge \mathbf{b}$, is given by

$$\mathbf{a} \wedge \mathbf{b} = ab \sin\theta\, \hat{\mathbf{n}}$$

where a and b are the magnitudes of the vectors **a** and **b**,
θ is the angle between **a** and **b**, and
$\hat{\mathbf{n}}$ is the unit vector, perpendicular to both **a** and **b**, such that **a**, **b** and $\hat{\mathbf{n}}$, *taken in that order*, form a right-handed set (see Fig. 38.4).

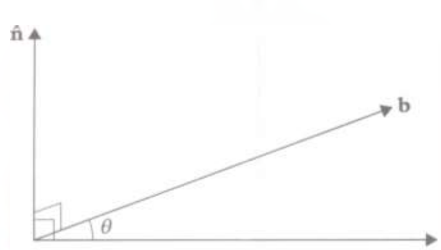


Figure 38.4

The notation $\mathbf{a} \times \mathbf{b}$ is frequently used instead of $\mathbf{a} \wedge \mathbf{b}$. When doing this, it is usual to call it **the cross product** of **a** and **b**. ($\mathbf{a} \wedge \mathbf{b}$ is usually read as '**a** *vec* **b**'.) For clarity we will use the notation $\mathbf{a} \wedge \mathbf{b}$ for vector product. However, you may prefer to use $\mathbf{a} \times \mathbf{b}$.

Several features of the vector product $\mathbf{a} \wedge \mathbf{b}$ should be noted immediately:

(a) $\mathbf{b} \wedge \mathbf{a} = -\mathbf{a} \wedge \mathbf{b}$. This follows from the fact that reversing the order of **a** and **b** in the right-handed set will have the effect of reversing the direction of the unit vector. Therefore vector multiplication is *not commutative*.

(b) Given that **i**, **j** and **k** represent the usual base vectors,

$$\mathbf{j} \wedge \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \wedge \mathbf{i} = \mathbf{j}, \qquad \mathbf{i} \wedge \mathbf{j} = \mathbf{k},$$

but

$$\mathbf{k} \wedge \mathbf{j} = -\mathbf{i}, \qquad \mathbf{i} \wedge \mathbf{k} = -\mathbf{j}, \qquad \mathbf{j} \wedge \mathbf{i} = -\mathbf{k}.$$

(c) For any vector **a**, $\mathbf{a} \wedge \mathbf{a} = \mathbf{0}$ (and thus $\mathbf{i} \wedge \mathbf{i} = \mathbf{j} \wedge \mathbf{j} = \mathbf{k} \wedge \mathbf{k} = 0$).

These features of the vector product should be contrasted with those of the scalar product $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a} = ab\cos\theta$, set out in §15.12 on page 190.

Question

Q1 Given that $\mathbf{a} = 5\mathbf{i}$ and $\mathbf{b} = \sqrt{3}\mathbf{i} + \mathbf{j}$, verify that $b = 2$ and that the angle between **a** and **b** is $\pi/6$. Hence, from the definition above, show that $\mathbf{a} \wedge \mathbf{b} = 5\mathbf{k}$. Verify that the same result is obtained by assuming that vector multiplication obeys the distributive law (i.e. that for any vectors **p**, **q**, **r**, $\mathbf{p} \wedge (\mathbf{q} + \mathbf{r}) = \mathbf{p} \wedge \mathbf{q} + \mathbf{p} \wedge \mathbf{r}$. This will be proved in §38.4 overleaf.)

38.3 Scalar triple product

For scalar multiplication, the triple product $\mathbf{a}.(\mathbf{b}.\mathbf{c})$ is meaningless, because $\mathbf{b}.\mathbf{c}$ is not a vector. However, because $\mathbf{b} \wedge \mathbf{c}$ *is* a vector, the triple product $\mathbf{a}.(\mathbf{b} \wedge \mathbf{c})$ can be found. We shall show that it is equal to the volume of the *parallelepiped*, whose six faces are the parallelograms formed by taking the vectors **a**, **b** and **c**, two at a time (see Fig. 38.5).

First notice that $\mathbf{b} \wedge \mathbf{c}$ is equal in magnitude to the area of the parallelogram formed by **b** and **c**. It will be convenient to call this area S. The direction of $\mathbf{b} \wedge \mathbf{c}$ is perpendicular to the plane of $\mathbf{b}$ and $\mathbf{c}$. See the unit vector $\hat{\mathbf{n}}$ in Fig. 38.5.

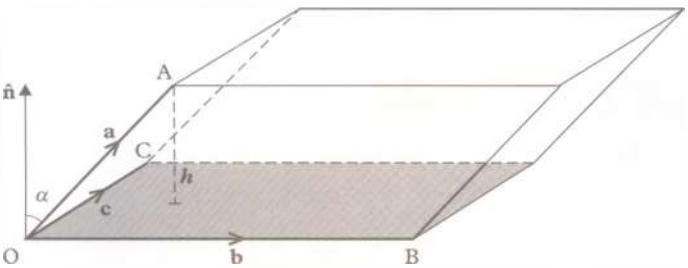


Figure 38.5

When we simplify $\mathbf{a}.(\mathbf{b} \wedge \mathbf{c})$, we obtain a scalar whose magnitude is

$$S\,a\cos\alpha$$

where α is the angle between $\hat{\mathbf{n}}$ and $\mathbf{a}$ (see Fig. 38.5). However, notice that $a\cos\alpha$ is h, the perpendicular height of the point A above the plane of $\mathbf{b}$ and $\mathbf{c}$. Thus we can write

$$\mathbf{a}.(\mathbf{b} \wedge \mathbf{c}) = Sh$$

However the product of the area of the base and the perpendicular height of the parallelepiped is equal to its volume. So we have shown that

$$\mathbf{a}.(\mathbf{b} \wedge \mathbf{c}) = \text{the volume of the parallelepiped}$$

Notice that $\mathbf{a}.(\mathbf{b} \wedge \mathbf{c}) = \mathbf{b}.(\mathbf{c} \wedge \mathbf{a}) = \mathbf{c}.(\mathbf{a} \wedge \mathbf{b})$, because they all represent the volume of the same parallelepiped.

Questions

Q2 Verify that the result in the preceding section is true for the cuboid formed by the vectors

$$\mathbf{a} = 4\mathbf{i}, \quad \mathbf{b} = 3\mathbf{j}, \quad \mathbf{c} = 2\mathbf{k}$$

Q3 Describe the circumstances in which $\mathbf{p}.(\mathbf{q} \wedge \mathbf{r})$ is **a** zero, **b** negative.

38.4 Distributive law

The **distributive law** enables us to remove brackets to obtain results such as

$$\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c}$$

Working in the reverse direction enables us to factorise the R.H.S. of such an expression. The distributive law will also enable us to expand products of vectors expressed in terms of the base vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

We prove the distributive law for scalar triple products by considering the vector

$$\mathbf{v} = \mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) - \mathbf{a} \wedge \mathbf{b} - \mathbf{a} \wedge \mathbf{c}$$

and showing that it is zero.

Scalar multiplying both sides by $\mathbf{v}$ gives

$$\mathbf{v}.\mathbf{v} = \mathbf{v}.\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) - \mathbf{v}.\mathbf{a} \wedge \mathbf{b} - \mathbf{v}.\mathbf{a} \wedge \mathbf{c}$$

(We can do this because we know that scalar multiplication *does* obey the distributive law. See §15.12 on page 190.)

From the last section we know that we can permute the factors in a scalar triple product, so this expression can be written

$$\mathbf{v}.\mathbf{v} = (\mathbf{b} + \mathbf{c}).\mathbf{v} \wedge \mathbf{a} - \mathbf{b}.\mathbf{v} \wedge \mathbf{a} - \mathbf{c}.\mathbf{v} \wedge \mathbf{a}$$

Now we use the distributive law for scalar multiplication to factorise the R.H.S.

$$\begin{aligned} \mathbf{v}.\mathbf{v} &= [(\mathbf{b} + \mathbf{c}) - \mathbf{b} - \mathbf{c}].(\mathbf{v} \wedge \mathbf{a}) \\ &= [\mathbf{b} + \mathbf{c} - \mathbf{b} - \mathbf{c}].(\mathbf{v} \wedge \mathbf{a}) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} v^2 &= 0 \\ \therefore \mathbf{v} &= \mathbf{0} \end{aligned}$$

Thus we have shown that

$$\begin{aligned} &\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) - \mathbf{a} \wedge \mathbf{b} - \mathbf{a} \wedge \mathbf{c} = \mathbf{0} \\ \therefore\ &\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c} \end{aligned}$$

which completes the proof of the distributive law.

Exercise 38b

1 Find $\mathbf{p} \wedge \mathbf{q}$, when
- **a** $\mathbf{p} = \mathbf{i} + \mathbf{j}, \quad \mathbf{q} = \mathbf{k}$
- **b** $\mathbf{p} = \mathbf{i} + \mathbf{j}, \quad \mathbf{q} = \mathbf{i} - \mathbf{j}$
- **c** $\mathbf{p} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}, \quad \mathbf{q} = 5\mathbf{i}$
- **d** $\mathbf{p} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \quad \mathbf{q} = 4\mathbf{i} - \mathbf{j} + 7\mathbf{k}$

2 Find $\mathbf{p}.(\mathbf{q} \wedge \mathbf{r})$, when
- **a** $\mathbf{p} = \mathbf{i}, \quad \mathbf{q} = \mathbf{i} + \mathbf{j}, \quad \mathbf{r} = \mathbf{k}$
- **b** $\mathbf{p} = \mathbf{i} + \mathbf{j}, \quad \mathbf{q} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \quad \mathbf{r} = 5\mathbf{j} - \mathbf{k}$
- **c** $\mathbf{p} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \quad \mathbf{q} = 2\mathbf{i} + 3\mathbf{j}, \quad \mathbf{r} = 4\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$
- **d** $\mathbf{p} = \mathbf{i} + \mathbf{j}, \quad \mathbf{q} = -\mathbf{i} + \mathbf{j}, \quad \mathbf{r} = -\mathbf{k}$

3 Using the vectors in question 2, find $(\mathbf{p} \wedge \mathbf{q}).\mathbf{r}$ in each part.

*4 Show that if $\mathbf{p} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{q} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$$\mathbf{p} \wedge \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Comment on the properties of the determinant and their relationship to the geometrical properties of $\mathbf{p}$ and $\mathbf{q}$, when $\mathbf{p} \wedge \mathbf{q} = \mathbf{0}$.

5 Use the determinant form of $\mathbf{p} \wedge \mathbf{q}$ in question 4 to find $\mathbf{p} \wedge \mathbf{q}$, when

a $\mathbf{p} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{q} = 5\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

b $\mathbf{p} = 7\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, $\mathbf{q} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$

(Many people find this a very convenient method for evaluating vector products.)

6 Using the vectors in question 2, find $\mathbf{p} \wedge (\mathbf{q} \wedge \mathbf{r})$ and in each part show that $\mathbf{p} \wedge (\mathbf{q} \wedge \mathbf{r}) = (\mathbf{p.r})\mathbf{q} - (\mathbf{p.q})\mathbf{r}$.

7 Using notation similar to that in question 4, show that

$$\mathbf{p}.(\mathbf{q} \wedge \mathbf{r}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Comment on the case $\mathbf{p}.(\mathbf{q} \wedge \mathbf{r}) = 0$.

(This form of the triple product can be used to prove that

$$\mathbf{p}.(\mathbf{q} \wedge \mathbf{r}) = \mathbf{q}.(\mathbf{r} \wedge \mathbf{p}) = \mathbf{r}.(\mathbf{p} \wedge \mathbf{q}))$$

8 Find a unit vector which is perpendicular to the vectors

$$\mathbf{i} + \mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$

9 Prove that if $\mathbf{p} + \mathbf{q} + \mathbf{r} = \mathbf{0}$, then $\mathbf{p} \wedge \mathbf{q} = \mathbf{q} \wedge \mathbf{r} = \mathbf{r} \wedge \mathbf{p}$.

10 If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are four given vectors and $\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c} = \mathbf{d}$, where $\lambda, \mu, \nu \in \mathbb{R}$, prove that

$$\lambda = \frac{\mathbf{d}.(\mathbf{b} \wedge \mathbf{c})}{\mathbf{a}.(\mathbf{b} \wedge \mathbf{c})}$$

Using this, and corresponding expressions for μ and ν, solve the equations

$$\begin{aligned} \lambda + 4\mu + 2\nu &= 0 \\ 2\lambda - \mu + \nu &= 0 \\ 8\lambda + 5\mu + 6\nu &= 6 \end{aligned}$$

38.5 Vector triple product

Since $\mathbf{b} \wedge \mathbf{c}$ is a vector, it should be possible to form a triple product $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$. Do question 4, below.

Question

Q4 Given that $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{j}$ and $\mathbf{c} = 3\mathbf{i} + 4\mathbf{j}$, show that

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = 3\mathbf{j}$$
$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = 3\mathbf{j} - 4\mathbf{i}$$

This shows that the triple products $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$ and $(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c}$ are *not* the same. We say that vector multiplication is *not associative*. This is not surprising if we consider the direction of the vector $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$. We know that $\mathbf{b} \wedge \mathbf{c}$ is perpendicular to the plane containing $\mathbf{b}$ and $\mathbf{c}$, and that $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$ is perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b} \wedge \mathbf{c}$. It follows that $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$ is parallel to the plane of $\mathbf{b}$ and $\mathbf{c}$. By similar reasoning $(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c}$ is parallel to the plane of $\mathbf{a}$ and $\mathbf{b}$. Therefore we would expect $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$ and $(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c}$, in general, to be different.

From the last paragraph we can deduce that $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$ is a linear combination of $\mathbf{b}$ and $\mathbf{c}$. In other words we can write

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = \lambda\mathbf{b} + \mu\mathbf{c}$$

where λ and μ are scalars.

Clearly the scalars λ and μ will depend on the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. We now show that $\lambda = \mathbf{a.c}$ and $\mu = -\mathbf{a.b}$, i.e. that

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a.c})\mathbf{b} - (\mathbf{a.b})\mathbf{c} \qquad (1)$$

There is no loss of generality if we fix the axes to suit our own convenience (provided they form a right-handed set of mutually perpendicular lines). For convenience we choose the z-axis as the direction of $\mathbf{c}$. There is no loss of generality if we choose the scale to suit ourselves, so we select a scale such that $\mathbf{c}$ is a unit vector. In other words we choose

$$\mathbf{c} = \mathbf{k} \qquad (2)$$

We are still free to choose *one* of the other axes (the remaining axis must complete the right-handed set). We shall choose for the y-axis a line perpendicular to the axis which we have already fixed, and in the plane containing $\mathbf{b}$ and $\mathbf{c}$. This enables us to write $\mathbf{b}$ as a linear combination of $\mathbf{j}$ and $\mathbf{k}$. We shall write

$$\mathbf{b} = \beta\mathbf{j} + \gamma\mathbf{k} \qquad (3)$$

We cannot have any choice over how we write **a** so this must be expressed in terms of all three base vectors, i.e.

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \qquad (4)$$

Now we are ready to start!
First we shall find $\mathbf{b} \wedge \mathbf{c}$:

$$\begin{aligned}\mathbf{b} \wedge \mathbf{c} &= (\beta\mathbf{j} + \gamma\mathbf{k}) \wedge \mathbf{k} \\ &= \beta\mathbf{i}\end{aligned}$$

Now consider $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$:

$$\begin{aligned}\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \wedge \beta\mathbf{i} \\ &= -\beta a_2\mathbf{k} + \beta a_3\mathbf{j}\end{aligned}$$

This is a simple looking result, but we have to transform it into a linear combination of **b** and **c**. We do this by writing

$$\begin{aligned}-\beta a_2\mathbf{k} + \beta a_3\mathbf{j} &= a_3(\beta\mathbf{j} + \gamma\mathbf{k}) + (-\beta a_2 - \gamma a_3)\mathbf{k} \\ &= a_3\mathbf{b} - (\beta a_2 + \gamma a_3)\mathbf{c}\end{aligned}$$

This is now in the required form $\lambda\mathbf{b} + \mu\mathbf{c}$, but we still have to express λ and μ in terms of *scalar* products, as in (1).

From (2) and (4)

$$\mathbf{a.c} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}).\mathbf{k} = a_3 = \lambda$$

From (3) and (4)

$$\mathbf{a.b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}).(\beta\mathbf{j} + \gamma\mathbf{k}) = \beta a_2 + \gamma a_3 = -\mu$$

$\therefore\ \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a.c})\mathbf{b} - (\mathbf{a.b})\mathbf{c}$

Question

Q5 Given that

$$\begin{aligned}\mathbf{a} &= \mathbf{i} + \mathbf{j} + \mathbf{k} \\ \mathbf{b} &= \mathbf{i} - \mathbf{j} - \mathbf{k} \\ \mathbf{c} &= \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\end{aligned}$$

find $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$ and verify that it is equal to $(\mathbf{a.c})\mathbf{b} - (\mathbf{a.b})\mathbf{c}$.

There is no need for a separate proof of the formula for the triple product $(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c}$. It can be deduced from the one above as follows:

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = -\mathbf{c} \wedge (\mathbf{a} \wedge \mathbf{b})$$

Now we only have to permute the letters in the formula we have already proved, i.e.

$$\begin{aligned}(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} &= -[(\mathbf{c.b})\mathbf{a} - (\mathbf{c.a})\mathbf{b}] \\ &= (\mathbf{c.a})\mathbf{b} - (\mathbf{c.b})\mathbf{a}\end{aligned}$$

Question

Q6 Prove that

$$(\mathbf{a} \wedge \mathbf{b}) \wedge (\mathbf{c} \wedge \mathbf{d}) = \mathbf{b}(\mathbf{a.c} \wedge \mathbf{d}) - \mathbf{a}(\mathbf{b.c} \wedge \mathbf{d}).$$

38.6 Perpendicular distance of a point from a line

In **Fig. 38.6** assume we are given the coordinates of the point P and the equation of the line l.

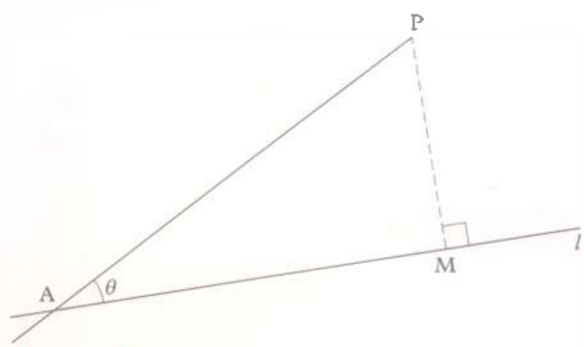


Figure 38.6

To calculate the distance of P from l, we need to calculate the length of PM, where M is the foot of the perpendicular from P to l.

Now

$$\begin{aligned}\text{PM} &= \text{AP}\sin\theta \\ &= |\mathbf{AP} \wedge \hat{\mathbf{u}}|\end{aligned}$$

where $\hat{\mathbf{u}}$ is a unit vector along the line l. So the perpendicular distance of P from l is given by

$$|(\mathbf{p} - \mathbf{a}) \wedge \hat{\mathbf{u}}| \qquad (1)$$

Given the equation of a line, it is always possible to find a point A on it, and a unit vector parallel to it. The next example illustrates how to use formula (1).

Example 5 *Find the perpendicular distance of the point* P(0, 14, 10) *from the line whose equation is* $\mathbf{r} = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \lambda(3\mathbf{i} + 4\mathbf{k})$.

By putting $\lambda = 0$, we can see that the point (1, 2, 3) lies on the line, so this will be the point A in (1) above. Also we know that the line is parallel to $(3\mathbf{i} + 4\mathbf{k})$, but this is not a *unit* vector, so we must divide by its magnitude, i.e. 5, thus taking $\hat{\mathbf{u}}$ to be $\frac{1}{5}(3\mathbf{i} + 4\mathbf{k})$. Using (1), the perpendicular distance required is the magnitude of $(\mathbf{p} - \mathbf{a}) \wedge \mathbf{u}$. In this example,

$$(\mathbf{p}-\mathbf{a}) = (14\mathbf{j}+10\mathbf{k}) - (\mathbf{i}+2\mathbf{j}+3\mathbf{k})$$
$$= -\mathbf{i}+12\mathbf{j}+7\mathbf{k}$$
$$\therefore (\mathbf{p}-\mathbf{a}) \wedge \mathbf{u} = \tfrac{1}{5}(-\mathbf{i}+12\mathbf{j}+7\mathbf{k}) \wedge (3\mathbf{i}+4\mathbf{k})$$
$$= \tfrac{1}{5}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 12 & 7 \\ 3 & 0 & 4 \end{vmatrix} \quad \text{(See Exercise 38b, question 4)}$$
$$= \tfrac{1}{5}(48\mathbf{i}+25\mathbf{j}-36\mathbf{k})$$

The magnitude of this (and hence the required perpendicular distance of P from the line) is equal to

$$\tfrac{1}{5} \times \sqrt{(48^2+25^2+36^2)}$$
$$= \tfrac{1}{5} \times 65$$
$$= 13 \text{ units}$$

(For another method, see Exercise 38c, questions 12 and 13.)

Question

Q7 Find the perpendicular distance of the point P(2, 3, 4) from the line

$$\mathbf{r} = (\mathbf{i}+15\mathbf{j}+11\mathbf{k}) + \lambda(4\mathbf{i}-12\mathbf{j}-3\mathbf{k})$$

Exercise 38c

1 Given that $\mathbf{p} = 2\mathbf{i}-3\mathbf{j}+\mathbf{k}$, $\mathbf{q} = \mathbf{i}-2\mathbf{j}+\mathbf{k}$, $\mathbf{r} = 3\mathbf{i}+2\mathbf{j}-\mathbf{k}$, find:

- **a** $\mathbf{p}.(\mathbf{q} \wedge \mathbf{r})$
- **b** $\mathbf{p} \wedge (\mathbf{q} \wedge \mathbf{r})$
- **c** $(\mathbf{p} \wedge \mathbf{q}) \wedge \mathbf{r}$

2 Find a vector which is perpendicular to both $\mathbf{p}$ and $\mathbf{q}$, where

$$\mathbf{p} = \mathbf{i}+3\mathbf{j}+5\mathbf{k} \quad \text{and} \quad \mathbf{q} = 4\mathbf{i}-\mathbf{j}+2\mathbf{k}$$

Hence write down the equation of the plane through the origin, parallel to $\mathbf{p}$ and $\mathbf{q}$.

3 Prove that

- **a** $\mathbf{p} \wedge (\mathbf{q} \wedge \mathbf{r}) + \mathbf{q} \wedge (\mathbf{r} \wedge \mathbf{p}) + \mathbf{r} \wedge (\mathbf{p} \wedge \mathbf{q}) = \mathbf{0}$.
- **b** $(\mathbf{p} \wedge \mathbf{q}).(\mathbf{r} \wedge \mathbf{s}) = \begin{vmatrix} \mathbf{p}.\mathbf{r} & \mathbf{p}.\mathbf{s} \\ \mathbf{q}.\mathbf{r} & \mathbf{q}.\mathbf{s} \end{vmatrix}$.

4 Prove that if the points A, B, C, whose position vectors are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, respectively, are coplanar, then $\mathbf{a}.(\mathbf{b} \wedge \mathbf{c}) = 0$. Hence or otherwise, find the equation of the plane through the points (6, 3, 0), (2, 2, 2) and (3, 3, 1). [Hint: if $P(x, y, z)$ is a general point in the plane, then the vectors **AP**, **AB** and **AC** are coplanar.]

5 Let $\mathbf{a}$ be the unit vector along a line l, and let $\mathbf{m} = \mathbf{r} \wedge \mathbf{a}$, where $\mathbf{r}$ is the position vector of any point P on l with respect to a fixed origin O. Show that $\mathbf{m}$ is independent of the choice of P.

If another line l' has corresponding vectors $\mathbf{a}'$ and $\mathbf{m}'$, prove that, if l and l' intersect, then $\mathbf{a}.\mathbf{m}' + \mathbf{a}'.\mathbf{m} = 0$.

6 The non-collinear points K, L and M have position vectors $\mathbf{k}$, $\mathbf{l}$ and $\mathbf{m}$ respectively with respect to an origin O, and $\mathbf{l} \wedge \mathbf{m} = \mathbf{p}$, $\mathbf{m} \wedge \mathbf{k} = \mathbf{q}$, $\mathbf{k} \wedge \mathbf{l} = \mathbf{r}$. Show, from the definitions of the scalar and vector products, that

- **a** $\mathbf{k}.\mathbf{q} = \mathbf{k}.\mathbf{r} = 0$,
- **b** the normal to the plane KLM is parallel to $\mathbf{p}+\mathbf{q}+\mathbf{r}$.

Given that K, L and M lie on a sphere of unit radius with centre O, and that the normal through O to the plane KLM meets the sphere at a point N, whose position vector with respect to O is $\mathbf{n}$, show that

$$(\mathbf{k}.\mathbf{p})\mathbf{n} = (\mathbf{p}+\mathbf{q}+\mathbf{r})\cos\theta$$

where θ is the angle KON.

7 The points A and B have coordinates (2, 1, 1) and (0, 5, 3) respectively. Find the equation of the line AB in terms of a parameter. If C is the point (5, −4, 2) find the coordinates of the point D on AB such that CD is perpendicular to AB.

Find the equation of the plane containing AB and perpendicular to the line CD.

8 The parametric equations of two planes are

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + s\begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

and

$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + u\begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} + v\begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}$$

- **a** Find the cosine of the acute angle between the planes.
- **b** The line of intersection is l. Find, in the form $\mathbf{r} = \mathbf{p} + \lambda\mathbf{q}$, the equation of l.

c Show that the length of the perpendicular from the point (1, 5, 1) to the line l is $\sqrt{2}$.

9 Show that if it is possible to find a vector $\mathbf{r}$ such that $\mathbf{r} \wedge \mathbf{p} = \mathbf{q}$, where $\mathbf{p}$ and $\mathbf{q}$ are given vectors, then $\mathbf{p}.\mathbf{q} = 0$.

Find the set of vectors $\mathbf{r}$ which satisfy $\mathbf{r} \wedge \mathbf{p} = \mathbf{q}$ in the following cases:

a $\mathbf{p} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}$

b $\mathbf{p} = \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 8 \\ 0 \\ 7 \end{pmatrix}$

10 a Find the image of the origin by reflection in the plane

$$x + 2y + 3z = 14$$

b Find the coordinates of the foot of the perpendicular from the origin to the line

$$\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{3}$$

c Find the equations of the spheres which touch the plane containing the y- and z-axes at O and also touch the plane

$$x + 2y + 2z = 1.$$

***11** Prove that the line through the point (x_1, y_1, z_1) perpendicular to the plane $ax + by + cz = d$ meets the plane at a point whose coordinates are $(x_1 + ta, y_1 + tb, z_1 + tc)$, where $t = \dfrac{d - (ax_1 + by_1 + cz_1)}{a^2 + b^2 + c^2}$. Hence show that the perpendicular distance from the point to the plane is $\left|\dfrac{d - (ax_1 + by_1 + cz_1)}{\sqrt{(a^2 + b^2 + c^2)}}\right|$.

12 Given that $\mathbf{a} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$, $\mathbf{n} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$ and $\mathbf{p} = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$ and that R is a point on the line $\mathbf{r} = \mathbf{a} + t\mathbf{n}$, express PR^2 in terms of t. Show that, as t varies, the least value of PR^2 is 37 and verify that, in this case, PR is perpendicular to the line.

13 Given that $\mathbf{a}$ is a constant vector which is perpendicular to a unit vector $\hat{\mathbf{u}}$, and that R is any point on the line $\mathbf{r} = \mathbf{a} + t\hat{\mathbf{u}}$, show that the distance to R from a fixed point P, whose position vector is $\mathbf{p}$, is given by

$$PR^2 = (\mathbf{a} - \mathbf{p}).(\mathbf{a} - \mathbf{p}) - 2t(\hat{\mathbf{u}}.\mathbf{p}) + t^2$$

Hence show that the least value of PR^2, as t varies, is $(\mathbf{a} - \mathbf{p}).(\mathbf{a} - \mathbf{p}) - (\hat{\mathbf{u}}.\mathbf{p})^2$. Prove that $\overline{PR}$ is then perpendicular to the given line.

14 Use the formula of §38.6 on page 401 to show that the least value of PR in question 12 is $\sqrt{37}$.

15 Use the formula in §38.6 on page 401 to show that the perpendicular distance of the point P from the line in question 13 is

$$\sqrt{\{(\mathbf{a} - \mathbf{p}).(\mathbf{a} - \mathbf{p}) - (\hat{\mathbf{u}}.\mathbf{p})^2\}}$$

Answers

Chapter P1

Exercise P1a, page 1

1 $4xh$ **2** $2x(x^2+3h^2)$ **3** $2h(3x^2+h^2)$
4 $3x-4x^3$ **5** $2t-\sqrt{t}-3$ **6** $(3+2t)(1+t)$
7 $\dfrac{x}{x-1}$ **8** $\dfrac{x+y}{xy-1}$

Exercise P1b, page 1

1 $(5x-2)(7x+3)$ **2** $2(x+7)(x-7)$
3 $(2x+y)(x-y)$ **4** $(x+a)(y+b)$ **5** $(x+3)(y-2)$
6 $(x+1)(x-1)(2x+3)$ **7** $(x+3)(2x^2+3x+3)$
8 $(x+1)(12x+5)$ **9** $20(x-2)$ **10** $2(x-2)^2(3x-1)$

Exercise P1c, page 2

1 $\dfrac{y-x}{xy}$ **2** $\dfrac{x^2+y^2}{xy}$ **3** $\dfrac{1+a}{a^2}$ **4** $\dfrac{a+b}{a^2b^2}$
5 $\dfrac{2x}{(x-h)(x+h)}$ **6** $\dfrac{-h(2x+h)}{x^2(x+h)^2}$ **7** $\dfrac{3x}{(1-x)(2+x)}$
8 $\dfrac{-(x^2-2x+4)}{(x^2+2)(2+x)}$ **9** $\dfrac{n+1}{n+2}$ **10** $\dfrac{x^2+3x+3}{(x+1)^2}$

Exercise P1d, page 2

1 $2/(T+t)$ **2** $ty=x+t^2$ **3** $-1/t$ **4** $-Tt$
5 $(N+1)(2N+1)(2N+3)$ **6** $\sqrt{(a+b)}$
7 $(ad+bc)/(bd+ac)$ **8** $3x^2+3xh+h^2$
9 $\dfrac{1}{\sqrt{(1+x^2)}\times(1+x^2)}=\dfrac{1}{(1+x^2)^{3/2}}$
10 $(1-t)/(1+t)$ [or $(t-1)/(t+1)$ whichever is positive]

Exercise P1e, page 2

1 6 **2** $-10, 25$ **3** 12 **4** 5, 25 **5** 7, 49
6 3, 4, 9 **7** $1, \frac{1}{4}$ **8** $1, \frac{1}{4}, 1$
9 10, 9 **10** $\frac{1}{2}, \frac{1}{9}, \frac{1}{4}$

Exercise P1f, page 3

1 $m=(y-c)/x$ **2** $e=(a-b)/a$ **3** $x=y^2/(4a)$
4 $y=\dfrac{(K-k)(x-h)}{H-h}+k$ **5** $c=4+3m$
6 $x=(b-1)/(a-1)$ **7** $l=T^2g/(4\pi^2)$ **8** $g=4\pi^2 l/T^2$
9 $m=\dfrac{2x+2y+1}{2(2y-x)}$ **10** $m=\dfrac{2x-3y+4}{3x-2y+2}$

Exercise P1g, page 4

1 3 **2** 2 **3** $\frac{1}{2}$ **4** 17 **5** $7, 1\frac{1}{2}$
6 $7, -2$ **7** $+2, -2$ **8** $c(2T+3t)/5$
9 $t/5, 3t$ **10** $1/t, -T$

Exercise P1h, page 5

1 $2, -1$ **2** $2, -3$ **3** $-7/19, 8/19$ **4** 0, 0; 4, 4
5 $16, 4; -1, -64$ **6** 0, 1; 2, 3 **7** $10c, 7c$
8 $tT, (t+T)$ **9** $t, 1/t; -1/t^3, -t^3$ **10** $5a, 3a; 4a, 0$

Exercise P1i, page 5

1 $0, +2, -2$ **2** 0, 0, 7 **3** $0, -4, 5$ **4** $+4, -4, +1, -1$
5 $+2/3, -2/3$ **6** $0, 0, -k$ **7** $a, (a+b), (a-b)$
8 0 **9** $+a, -a$ **10** $p+q$

Chapter P2

Q1 **a** 2 **b** 6 **c** a **d** ab **e** 18 **f** 80
g $4a$ **h** 6 **i** 35 **j** 16 **k** 36 **l** ab
Q2 **a** 3 **b** 3 **c** 9 **d** 2 **e** 8 **f** 243
g 16 **h** 8
Q3 '$0^0 = 1$' would have to be derived from '$0^n - 0^n = 0^0$', but division by 0, or by 0^n, is meaningless.
Q4 Bases: **a** 10 **b** 10 **c** 3 **d** 4 **e** 2
f $\frac{1}{2}$ **g** a
Logarithms: **a** 2 **b** 1.6021 **c** 2 **d** 3 **e** 0
f -3 **g** b
Q5 $x=\log_c a, y=\log_c b, x+y=\log_c (ab)$,
$x-y=\log_c (a/b)$
Q6 $x=\log_c a, nx=\log_c a^n$
Q7 **a** 10 **b** 100 **c** 0.1 **d** 1 **e** 0 **f** $\frac{1}{2}$
Q8 **a** a **b** a^2 **c** $1/a$ **d** 1 **e** 0 **f** $\frac{1}{2}$
Q9 **a** $\frac{2}{3}, -\frac{7}{3}$ **b** $-\frac{11}{5}, \frac{3}{5}$ **c** $-\frac{5}{2}, -\frac{1}{2}$ **d** $-\frac{1}{2}, -\frac{7}{2}$
Q10 **a** $x^2-7x+12=0$ **b** $x^2-3x-2=0$
c $8x^2+4x-3=0$ **d** $3x^2-2x=0$
Q11 $-3, \frac{7}{3}$ **Q12** Polynomials

Exercise P2a, page 7

1 **a** 5 **b** $\frac{1}{2}$ **c** 48 **d** $\frac{1}{2}$ **e** a/b **f** 15
g 21 **h** p/q **i** $1/(4p)$ **j** $9a/(2b)$
2 **a** $2\sqrt{2}$ **b** $2\sqrt{3}$ **c** $3\sqrt{3}$ **d** $5\sqrt{2}$ **e** $3\sqrt{5}$
f $11\sqrt{10}$ **g** $5\sqrt{3}$ **h** $4\sqrt{2}$ **i** $6\sqrt{2}$ **j** $7\sqrt{2}$
k $2\sqrt{15}$ **l** $16\sqrt{2}$
3 **a** $\sqrt{18}$ **b** $\sqrt{12}$ **c** $\sqrt{80}$ **d** $\sqrt{24}$ **e** $\sqrt{72}$ **f** $\sqrt{216}$
g $\sqrt{128}$ **h** $\sqrt{1000}$ **i** $\sqrt{\frac{1}{2}}$ **j** $\sqrt{\frac{1}{3}}$ **k** $\sqrt{\frac{1}{6}}$ **l** $\sqrt{\frac{2}{3}}$
4 **a** $\sqrt{5}/5$ **b** $\sqrt{7}/7$ **c** $-\sqrt{2}/2$ **d** $2\sqrt{3}/3$ **e** $\sqrt{6}/2$
f $\sqrt{2}/4$ **g** $-\sqrt{3}/2$ **h** $3\sqrt{6}/8$ **i** $\sqrt{2}-1$ **j** $2+\sqrt{3}$
k $(4+\sqrt{10})/6$ **l** $\sqrt{6}-2$ **m** $(\sqrt{5}+\sqrt{3})/2$
n $3\sqrt{6}+3\sqrt{5}$ **o** $3+2\sqrt{2}$ **p** $(3\sqrt{2}+2\sqrt{3})/6$

Exercise P2b, page 7

1 **a** $3\sqrt{2}$ **b** $6\sqrt{3}$ **c** $4\sqrt{7}$ **d** $5\sqrt{10}$ **e** $28\sqrt{2}$ **f** 0

2 **a** 25.5 **b** 2.26 **c** 3.15 **d** 19.5
e 0.354 **f** 0.260

3 **a** $\frac{6}{7}+\frac{2}{7}\sqrt{2}$ **b** $9+4\sqrt{5}$ **c** $-1+\sqrt{2}$ **d** $4-2\sqrt{3}$
e $-1-\sqrt{2}$ **f** $\frac{1}{2}+\frac{3}{4}\sqrt{2}$ **g** $\frac{2}{9}\sqrt{3}$ **h** $\frac{6}{25}\sqrt{5}$
i $\frac{8}{11}+\frac{5}{11}\sqrt{3}$ **j** $\frac{3}{2}+\frac{1}{2}\sqrt{5}$ **k** $\frac{3}{7}+\frac{5}{14}\sqrt{2}$ **l** 0

4 **a** $5+2\sqrt{6}$ **b** $\frac{1}{2}(5+\sqrt{3}+\sqrt{5}+\sqrt{15})$ **c** $-7+3\sqrt{6}$
d $4+\sqrt{10}$ **e** $3+2\sqrt{2}$ **f** $\sqrt{2}$

5 **a** $2-\sqrt{2}$ **b** $4(2+\sqrt{3})$ **c** $-(2+\sqrt{3})$ **d** $2+\sqrt{3}$
e $3+2\sqrt{2}$ **f** $6+4\sqrt{2}$

Exercise P2c, page 9

1 **a** 5 **b** 3 **c** 2 **d** 7 **e** $\frac{1}{2}$ **f** 1 **g** -2 **h** -1
i 16 **j** 9 **k** 125 **l** 343 **m** $\frac{1}{8}$
n $\frac{2}{3}$ **o** $1\frac{1}{2}$ **p** $\frac{2}{3}$

2 **a** 1 **b** $\frac{1}{3}$ **c** 1 **d** $\frac{1}{4}$ **e** $\frac{1}{8}$ **f** 2 **g** 9 **h** 1
i $\frac{1}{27}$ **j** $-\frac{1}{6}$ **k** 1 **l** $\frac{9}{4}$ **m** 4 **n** 3
o $4\frac{1}{2}$ **p** $\frac{5}{9}$

3 **a** $\frac{1}{2}$ **b** $\frac{1}{4}$ **c** $\frac{1}{2}$ **d** $\frac{1}{8}$ **e** $\frac{1}{9}$ **f** 2 **g** 2 **h** 9
i $1\frac{1}{2}$ **j** $1\frac{1}{2}$ **k** $1\frac{1}{2}$ **l** $\frac{16}{81}$

Exercise P2d, page 9

1 **a** 16 **b** 36 **c** 4 **d** 6 **e** $1\frac{1}{2}$ **f** $1\frac{1}{3}$ **g** $\frac{1}{2}$
h $\frac{1}{8}$ **i** $\frac{1}{16}$ **j** $\frac{1}{27}$ **k** $2\frac{3}{4}$ **l** 64
m $\frac{4}{9}$ **n** 1.1 **o** 125 **p** $\frac{1}{2}$

2 **a** $\frac{1}{2}$ **b** 1 **c** $\frac{1}{8}$ **d** 2 **e** 2 **f** 1

3 **a** 2^{-n} **b** 3^{n+1} **c** 4 **d** 3 **e** 12 **f** $10^{n/2}$

4 **a** $x^{-7/12}$ **b** 2 **c** $x^{n/2+3/2}$ **d** 1 **e** y^{-q} **f** 1

5 **a** $-\dfrac{1}{x^2(x^2+1)^{1/2}}$ **b** $\dfrac{x-2}{2x^2(1-x)^{1/2}}$
c $-\dfrac{1}{2x^{3/2}(1+x)^{1/2}}$ **d** $\dfrac{3+2x}{3(1+x)^{4/3}}$ **e** $\dfrac{1}{(1-x)\sqrt{(1-x^2)}}$

Exercise P2e, page 10

1 **a** $\log_2 16=4$ **b** $\log_3 27=3$ **c** $\log_5 125=3$
d $\log_{10} 1\,000\,000=6$ **e** $\log_{12} 1728=3$
f $\log_{16} 64=\frac{3}{2}$ **g** $\log_{10} 10\,000=4$
h $\log_4 1=0$ **i** $\log_{10} 0.01=-2$ **j** $\log_2 \frac{1}{2}=-1$
k $\log_9 27=\frac{3}{2}$ **l** $\log_8 \frac{1}{4}=-\frac{2}{3}$ **m** $\log_{1/3} 81=-4$
n $\log_e 1=0$ **o** $\log_{16} \frac{1}{2}=-\frac{1}{4}$ **p** $\log_{1/8} 1=0$
q $\log_{81} 27=\frac{3}{4}$ **r** $\log_{1/16} 4=-\frac{1}{2}$ **s** $\log_{-2/3} \frac{4}{9}=2$
t $\log_{-3}(-\frac{1}{3})=-1$ **u** $\log_a c=5$ **v** $\log_a b=3$
w $\log_p r=q$ **x** $\log_b a=c$

2 **a** $2^5=32$ **b** $3^2=9$ **c** $5^2=25$ **d** $10^5=100\,000$
e $2^7=128$ **f** $9^0=1$ **g** $3^{-2}=\frac{1}{9}$ **h** $4^{1/2}=2$
i $e^0=1$ **j** $27^{1/3}=3$ **k** $a^2=x$ **l** $3^b=a$
m $a^c=8$ **n** $x^y=z$ **o** $q^p=r$

3 **a** 6 **b** 2 **c** 7 **d** 2 **e** $\frac{1}{3}$ **f** 0 **g** $\frac{1}{3}$ **h** 2
i 3 **j** -1 **k** 3 **l** -1

Exercise P2f, page 11

1 **a** $\log a+\log b$ **b** $\log a-\log c$ **c** $-\log b$
d $2\log a+\frac{3}{2}\log b$ **e** $-4\log b$
f $\frac{1}{3}\log a+4\log b-3\log c$ **g** $\frac{1}{2}\log a$ **h** $\frac{1}{3}\log b$
i $\frac{1}{2}\log a+\frac{1}{2}\log b$ **j** $1+\lg a$ **k** $-2-2\lg b$
l $\frac{1}{2}\log a-\frac{1}{2}\log b$ **m** $\frac{1}{2}\log a+\frac{3}{2}\log b-\frac{1}{2}\log c$
n $\log b+\frac{1}{2}\log a-\frac{1}{3}\log c$ **o** $\frac{1}{2}+\frac{1}{2}\lg a-\frac{5}{2}\lg b-\frac{1}{2}\lg c$

2 **a** log 6 **b** log 2 **c** log 6 **d** log 2 **e** $\log(ac)$
f $\log(xy/z)$ **g** $\log(a^2/b)$ **h** $\log(a^2b^3/c)$
i $\log\sqrt{(x/y)}$ **j** $\log(p/\sqrt[3]{q})$ **k** $\lg(100\,a^3)$
l $\lg(10a/\sqrt{b})$ **m** $\lg(a^2/2000c)$ **n** $\lg(10x^3/\sqrt{y})$

3 **a** 3 **b** 2 **c** 2 **d** 1 **e** log 2 **f** log 7
g $\log\frac{1}{2}$ **h** 0 **i** 0 **j** 3 **k** 2 **l** $\frac{2}{3}$

4 **a** 2.322 **b** 0.6309 **c** 0.3155 **d** 1.161
e -2.585 **f** 6.838

5 **a** 3.170 **b** 0.7211 **c** 1.042 **d** 2.303
e 1.145 **f** -0.6309

7 **a** 3.119 **b** 1.297 **c** 23.14 **d** 0.7936
e 0.3674 **f** 0.000 759 7

Exercise P2g, page 14

1 **a** $\frac{11}{2}, \frac{3}{2}$ **b** $-\frac{1}{2}, -\frac{1}{2}$ **c** $\frac{7}{3}, -2$ **d** $-1, -1$
e $1, -3$ **f** $1, -5$ **g** $4, 1$ **h** $3, -2$

2 **a** $x^2-3x+4=0$ **b** $x^2+5x+6=0$
c $2x^2-3x-5=0$ **d** $3x^2+7x=0$ **e** $x^2-7=0$
f $5x^2-6x+4=0$ **g** $36x^2+12x+1=0$
h $10x^2+25x-16=0$

3 **a** $\frac{25}{4}$ **b** $\frac{3}{4}$ **c** $-\frac{5}{2}$ **d** $-\frac{25}{8}$

4 **a** $\frac{5}{9}$ **b** $\frac{16}{9}$ **c** $\frac{80}{27}$ **d** $\frac{80}{9}$

5 **a** 72 **b** 5 **c** $-\frac{9}{8}$ **d** -32

6 **a** $x^2-39x+49=0$ **b** $x^2-7x-1=0$
c $x^2+35x-343=0$

7 **a** $2x^2+4x+1=0$ **b** $x^2-4x+2=0$
c $x^2-6x+1=0$

8 $4x^2-49x+36=0$

9 $\frac{35}{4}$ **10** ± 6

12 **a** $-bc/a^2$ **b** $(b^2-2ac)/a^2$ **c** $b(3ac-b^2)/a^3$
d $-b/c$ **e** $(b^2-2ac)/(ac)$
f $(b^4-4ab^2c+2a^2c^2)/a^4$

13 **a** $ax^2-bx+c=0$ **b** $ax^2+(b-2a)x+a-b+c=0$
c $a^2x^2+(2ac-b^2)x+c^2=0$ **d** $cx^2-bx+a=0$
e $a^2x^2-(b^2-4ac)=0$
f $a^2x^2+3abx+(2b^2+ac)=0$

17 $2, -9, 9; 3, \frac{3}{2}$

18 **a** $ay^2+y(b-2a)+a-b+c=0, \alpha+1, \beta+1$
b $ay^4+by^2+c=0, \pm\sqrt{\alpha}, \pm\sqrt{\beta}$
c $a^2y^2+(2ac-b^2)y+c^2=0, \alpha^2, \beta^2$

19 **a** $ay^2+(b-4a)y+4a-2b+c=0$
b $cy^2+by+a=0$
c $ay^4-4ay^3+(6a+b)y^2-2(2a+b)y+a+b+c=0$

Exercise P2h, page 17

1 a $-12, -12, -6, 0, 0$; $(x-2)$, or $(x+2)$
b $-1, 0, -2, 19, -21$, $(x-1)$ c $0, 6, -2, 88, -24$, x
d $3, 0, 0, 3, 3$, $(x-1)$, or $(x+1)$
2 a 2 b 18 c -11 d -1 e 2 f $-2\frac{1}{2}$
3 a -3 b -10 c 2 d 4 e 4 f 2
4 $(x+3)(2x-1)$ 5 $(2x-1)(2x+3)(3x+1)$
6 a $(x-1)(x+2)(x-3)$ b $(x+1)(x-2)(x-3)$
c $(2x+1)(x-2)(x+2)$ d $(x+1)(x+2)(2x-1)$
e $(x+2)(x+3)(2x+1)$ f $(x^2+1)(2x-1)$
7 $a=3, b=2$ 8 $p=1, q=-3$ 9 $a=3, b=-1, c=-2$
10 $a=2, b=-1, c=-2$

Chapter P3

Exercise P3a, page 19

1 Different calculators may give different results for many of these key sequences. The important thing is to get to know how *your* calculator operates.
2 For non-scientific calculators with an eight-digit display: a 99 999 999 b 0.000 000 1
3 Looks like SHELL. 4 Looks like SOIL.
5 a

7^1	7
7^2	49
7^3	343
7^4	2401
7^5	16 807
7^6	117 649
7^7	823 543
7^8	5 764 801

b Final digits comprise the repeating sequence 7, 9, 3, 1.
c Final two digits comprise the repeating sequence 07, 49, 43, 01.

Exercise P3b, page 21

1 Final answers:
a 9 b 4 c 28 d 86 e 30 f 97 g 36
h 27 i 24.3 j 15.5
2 a i, ii, iv, v, vii and viii are incorrect.
b Corrections:
i 13 ii 67 iv 908 v 915 vii 28 viii 84
3 The outcomes are all negative:
a -5 b -10 c -33 d -37 e -67 f -216
g -655 h -56
4 The bill is correct. 5 $54.20

Exercise P3c, page 22

1 a 5896 b 73 c 27 d 2116 e 42 679
f 7 g 136 h 615
2 a i 61.1992, ii 61.20 b i 1282.3867, ii 1282.39
c i 333.423 52, ii 333.42 d i 1.850 898 2, ii 1.85
e i 8*, ii 8 f i 28*, ii 28
g i 983.952 99, ii 983.95 h i 63.6255, ii 63.63
*Some calculators may give rounding errors in these cases.
3 a i, iii, iv, v, vi and viii are incorrect.
b Corrections:
i 45 iii 5628 iv 46 v 33.333 333
vi 3.315 viii 5.065 110 6
4 1.509 or 1.500 000 0 depending on calculator.
5 a 11 111 111 (calculator display)
b 12 345 679 (calculator display)
6 31 536 000
7 b age × 365 c age × 365 × 24
d age × 365 × 24 × 60
e most calculators will not cope with this if your age is above 3
8 a $5496 per month 9 $26.25
b $181 per day (365 day year)
10 a 9.17 km b 9166.67 m c 152.78 m

Exercise P3d, page 24

1 b $68 + 17 + 45$ c $42 \div 3 + 22$ d $18 \times 5 + 63$
e $(171 - 15) \times 18$ g $(6 + 3) \times 487$
h $(31 - 14) \times 100$
2 a 229 b 560 c 70 d 21 e 784 f 7
g 22.361 077 h 23.32
3 a 26 b 12 c 12 d 435 e 3.994 867 4
f 282.76

Exercise P3e, page 24

1 a 4490 b 5780 c 96.0 d 613 000
e 45.8 f 2370
2 a 1.73 b 2.24 c 27.6 d 2.61
e 4.99 f 62.3
3 a 7334, foot (looks like 'heel' upside down)
b 345, female (looks like 'she' upside down)
c Numerals 0, 1, 3, 4, 5, 7 look like O, I, E, h, S, L when turned upside down. Make up 'words' from these letters (e.g. his, hose, lie, loss, shoes, sole), then find the corresponding numbers. After that, make up calculations that give these numbers. (Also, 8 looks a bit like B; this will give you more words!)

Exercise P3f, page 26

1 a 0.573 576 4 (0.574) b 0.219 846 2 (0.220)
c 4.437 35 (4.44) d 74.475 889° (74.5°)
e 58.211 669° (58.2°) f –E-

g 0.014 925 3 (0.0115) h 1.492 537 3 (1.49)
i 14 925.373 (14 900)

2 The cosine of any angle θ must lie in the range $-1 \leqslant \theta \leqslant 1$.

3 If you press n times, when n is even you get the original number, and the reciprocal when n is odd.

4 a y is the second number entered, i.e. power to which x (the first number entered) is raised
c i 343, ii 61.4656, iii 1 419 857

5 9.47 cm 6 48.8° to 3 s.f.

7 a 27.5 cm^2 b 61.4° c 20.3 m d 9.2 cm

Exercise P3g, page 31

1 Keep trying!

2 c The outcomes in the **Totals, Average** and **% Female** cells change automatically to reflect the new data. This is because the underlying formulae remain unchanged.

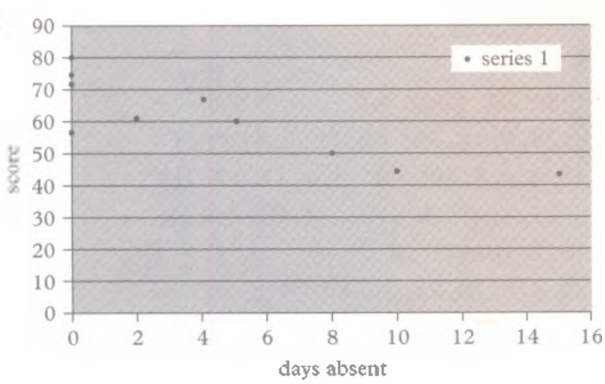
3 a

Microsoft Excel — NGM Ch12.xls (N9)

	A	B	C	D	E	F	G	H	I	J	K
1	0	4	0	0	2	15	8	0	10	5	
2	71	66	80	56	61	43	50	74	44	60	
3											

b 61

c



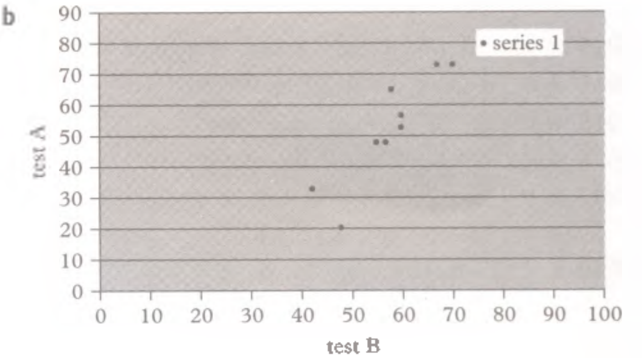
The graph shows the following trend: the more days absent, the lower the test score.

4 a

Microsoft Excel — NGM Ch12.xls (C2 =A2*4)

	A	B	C	D
1	TEST A (25)	TEST B (100)	TEST A (100)	
2	5	48	20	
3	8	42	32	
4	12	57	48	
5	12	55	48	
6	13	60	52	
7	14	60	56	
8	16	58	64	
9	18	70	72	
10	18	67	72	
11	20	68	80	
12				

b



c In general, the scores on Test B were better than those on Test A, indicating that the geography teacher was effective.

5 a, b, c,

Microsoft Excel — NGM Ch12.xls (F8)

	A	B	C	D	E	F
1	ITEM	Unit Cost (N)	No of Units	Cost (N)	VAT (5%)	Cost + VAT (N)
2						
3	A4 Ream	600	125	75000	3750	78750
4	Rulers	50	250	12500	625	13125
5	Exercise books	60	750	45000	2250	47250
6	Wallet folder	240	250	60000	3000	63000
7	Box file	300	325	97500	4875	102375
8						

d The formula would be = D3*0.12.

6

1	4	3	2
2	3	4	1
3	2	1	4
4	1	2	3

Chapter 1

Q1 (−3, 2), (2, −3), (0, 0)

Q3 a 13 b $\sqrt{41}$ c $\sqrt{\{(r-p)^2+(s-q)^2\}}$

Q4 a (5, 6) b (−1, 4) c $(-\frac{3}{2}, -\frac{5}{2})$
d $\left(\dfrac{p+r}{2}, \dfrac{q+s}{2}\right)$

Q5 a $\frac{9}{4}$ b $\frac{3}{2}$ c $-\frac{4}{5}$ d $-\frac{10}{11}$ e 0
f $(s-q)/(r-p)$ g −1 h b/a

Q6 $\frac{4}{3}, -\frac{3}{4}, -1$ Q7 $\frac{12}{5}, -\frac{5}{12}, -1$

Q8 a $-\frac{1}{3}$ b −4 c $\frac{1}{6}$ d $\frac{3}{2}$ e $-1/(2m)$
f a/b g $2/m$

Q9 a parallel b perpendicular c neither

Q10 5, 20, −1 Q11 2, 0, $-\frac{5}{2}$

Q12 a $(-\frac{1}{2}, 0), (1, 0)$ b $(0, -1)$

Q13 **a** yes **b** no **c** no **d** no **e** yes **f** yes
Q14 **a** 0 **b** 2 **c** 3 **d** $\frac{1}{2}$ **e** -1
Q15 **a** $y=\frac{1}{3}x$ **b** $y=-2x$ **c** $y=mx$
Q16 **a** $\frac{1}{4}$ **b** $-\frac{5}{4}$ **c** $\frac{3}{2}$ **d** $\frac{7}{4}$ **e** q/p
Q17 **a** $y=3x+2$ **b** $y=3x+4$ **c** $y=3x-1$
d $y=\frac{1}{5}x+2$ **e** $y=\frac{1}{5}x+4$
Q18 **a** $\frac{2}{3}, 2$ **b** $\frac{1}{4}, \frac{1}{2}$ **c** $-3, -6$ **d** $\frac{7}{3}, -\frac{5}{3}$
e $0, -4$ **f** $-l/m, -n/m$
Q19 **a** $y=0$ **b** $x=0$ **c** $x=4$ **d** $y=-7$
Q20 **a** $5x-2y-26=0$ **b** $5x+2y-1=0$
Q21 **a** $3x-2y-19=0$ **b** $12x+5y-1=0$
Q22 $y-y_1=m(x-x_1)$
Q23 **a** $(4\frac{1}{2}, 1)$ **b** $(1, 5)$ **c** $(0, c)$ **d** $(-a, c-a)$
Q24 No. They are parallel. **Q25** $(-\frac{3}{4}, 0)$, $(\frac{2}{3}, 0)$

Exercise 1a, page 34

1 **a** 4 **b** 5 **c** 6 **d** 13 **e** $\sqrt{74}$ **f** 10
2 **a** $(3, 2)$ **b** $(5, \frac{5}{2})$ **c** $(1, 3)$ **d** $(0, \frac{7}{2})$
e $(-\frac{1}{2}, -\frac{9}{2})$ **f** $(-6, -7)$
3 17 **4** $(-\frac{5}{2}, \frac{9}{2})$ **5** $(-\frac{3}{2}, -\frac{3}{2})$ **6** P, R, S
7 A, B, D; $\sqrt{50}$ **8** $13, 6\frac{1}{2}$

Exercise 1b, page 40

1 **a** $\frac{9}{5}$ **b** $-\frac{7}{3}$ **c** $-\frac{1}{14}$ **d** $\frac{1}{5}$
3 **a** 1 **b** -1 **c** $\sqrt{3}$ **d** $-1/\sqrt{3}$
4 **a** perpendicular **b** parallel **c** perpendicular
d parallel **e** perpendicular **f** neither
6 $\sqrt{50}$; $(3\frac{1}{2}, 4\frac{1}{2})$ **7** $\frac{1}{2}\sqrt{34}$; $(\frac{3}{4}, -1\frac{3}{4})$
8 $10, 1, 2, 26; \pm 2, \pm 4$
9 $-27, -1, 1, 27; -2, 0, 2$
10 **a** yes **b** no **c** no **d** yes **11** $-\frac{10}{3}, +5; \frac{25}{3}$
12 **a** $(4, 0), (-3, 0), (0, -12)$
b $(\frac{2}{3}, 0), (\frac{1}{2}, 0), (0, 2)$
c $(0, 9)$, and touches x-axis at $(3, 0)$
d $(9, 0)$, and cuts y-axis, touches x-axis at $(0, 0)$
e $(-1, 0), (0, 25)$ and touches x-axis at $(5, 0)$
f $(1, 0), (-1, 0), (3, 0), (-3, 0), (0, 9)$
13 **a** $y=x$ **b** $y=-x$ **c** $y=\frac{1}{2}x$ **d** $y=\frac{1}{2}x-4$
e $x=-5$ **f** $y=-\frac{2}{3}x+5$
14 **a** $y=11$ **b** $x=4$ **c** $y=6x-10$
d $y=-8x+2$ **e** $y=-\frac{2}{3}x-1$
15 $y=\frac{1}{8}x$ **16** $M(0, -\frac{3}{2})$; $S(5, -1)$
17 **a** $(b-q)/(a-p)$, 7 **b** $-\frac{3}{2}$

Exercise 1c, page 43

1 **a** $4x-y-1=0$ **b** $3x-y+11=0$
c $x-3y-17=0$ **d** $3x+4y-41=0$
e $3x-6y-4=0$ **f** $20x+12y+31=0$
2 **a** $3x-4y+21=0$ **b** $5x+4y-23=0$
c $3x+11y-35=0$ **d** $x-5y-19=0$
e $2x+3y-7=0$ **f** $2x-y+1=0$
3 **a** $(7, -7)$ **b** $(-\frac{3}{2}, -\frac{11}{2})$ **c** $(\frac{11}{7}, -\frac{13}{7})$ **d** $(4, -7)$
4 **a** $3x-4y+1=0$ **b** $5x-2y+16=0$
c $7x-y-28=0$ **d** $3x-4y-6=0$
5 $2x-5y+19=0$ **6** $26x+4y-21=0$
7 $7x-10y-70=0; 7x+10y=0$ **8** $2x-7y-3=0$
9 $\frac{8}{21}$ **10** $(2, -5)$ **11** $4x-3y-13=0; 5$
12 $x+4y-15=0$ **13** $7x-4y-43=0$
14 $(0, 0), (16, 64)$ **15** $\sqrt{512}=16\sqrt{2}$

Chapter 2

Q1 **a** F **b** T **c** F **d** F
Q2 **a** -2.5 **b** $-3, -5$ **c** $+5, -5$ **d** $+\sqrt{3}, -\sqrt{3}$
Q3 **a** $\{x: x \in \mathbb{R}, x \neq 3\}, \mathbb{R}$
b $\{x: x \in \mathbb{R}, x \leqslant 10\}, \{y: y \in \mathbb{R}, 0 \leqslant y\}$
c $\{x: x \in \mathbb{R}, |x| \leqslant 5\}, \{y: y \in \mathbb{R}, |y| \leqslant 5\}$
d $\{x: x \in \mathbb{R}, x \neq \pm 5\}, \{y: y \in \mathbb{R}, y < 0 \text{ or } y > \frac{1}{25}\}$
e $\mathbb{R}, \{y: y \in \mathbb{R}, 0 < y \leqslant \frac{1}{25}\}$
Q4 **a** many-to-one **b** one-to-one
c not a function **d** many-to-one
Q7 Odd **Q8** 2 **Q9** 9
Q10 7.389 **Q11** 1 **Q12** 0.5

Exercise 2a, page 47

1 **a** $\{1, 4, 9, 16, 25\}$ **b** $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$
c $\{2, 4, 6, 8, 10\}$ **d** $\{5, 9, 13, 17, 21\}$
2 **a** $\{0, 1, 4, 9\}$ **b** $\{-24, -6, 0, +6, +24\}$
c $\{0, 1, 16, 81\}$ **d** $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}\}$
3 **a** $\{1, 4, 9, 16, 25, 36, 49, 64, 81\}$
b $\{9, 16, 21, 24, 25\}$ **c** $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
d $\{\frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}, 4, 4\frac{1}{2}\}$
4 **a** F **b** T **c** T **d** T **5** $\{1, 2, 3\}$
6 **a** $\{5, 10, 15, \ldots 95\}, \{7, 14, 21, \ldots 98\}, \{35, 70\}$,
$\{5, 7, 10, 14, \ldots 95, 98\}$
b multiples of 35 **c** 19, 14, 2, 31
7 **a** $\{3, 6, 9, \ldots 18\}$ **b** $\{4, 8, 12, 16, 20\}$
c $\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20\}$
d $\{1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 18, 19\}$
e $\{1, 2, 5, 7, 10, 11, 13, 14, 17, 19\}$ **f** same as **e**
8 **a** $0.\dot{3}$ **b** $0.\dot{2}8571\dot{4}$ **c** $0.\dot{2}\dot{7}$
9 $\frac{7}{9}$ **10** **a** $\frac{4}{33}$ **b** $\frac{73}{111}$ **c** $\frac{3}{7}$

Exercise 2b, page 49

1 **a** 1 **b** 126 **c** $1\frac{27}{64}$ **d** -7
2 $\{1, 6, 11, 16, 21, 26\}$ **3** $\{y: y \in \mathbb{R}, y \geqslant 1\}$

4 $\{y: y \in \mathbb{R}, 0 < y \leqslant 1\}$ 5 $\{x: x \in \mathbb{R}, x < 25\}$
6 fg: $x \mapsto 5x^2 + 1$, gf: $x \mapsto (5x+1)^2$
7 a $(5+h)^2 = 25 + 10h + h^2$ b $10 + h$ 8 $2a + h$
9 a 8 b -1000 c $\frac{1}{8}$ d $125a^3$ e $a^3/27$
f $a^3 + 3a^2h + 3ah^2 + h^3$ g $6a^2h + 2h^3$ h $3a^2 + h^2$
10 a 56 b 91 c 2, F: $x \mapsto (x-21)/7$

Exercise 2c, page 52

18 a translation a to the right
b translation a vertically upwards
c 'stretch', $\times k$, parallel to the y-axis
d reflection in the x-axis
e reflection in the y-axis

Exercise 2d, page 57

1 a $x = 7$ b $x = -3$ c $x = -0.2$ d $x = (a-1)/5$
2 a $t = 7$ b $t = 5.5$ c $t = 4$ d $t = 1/a + 5$
3 a $f^{-1}(x) = 24 - 2x$ b $f^{-1}(x) = 3 + 2x$
c $f^{-1}(x) = \frac{1}{2}(5x - 1)$ d $f^{-1}(x) = (7 - 10x)/3$
4 a $f^{-1}(x) = 9x/5 + 32$ b $f^{-1}(x) = x/180 + 2$
c $f^{-1}(x) = x/2\pi$ d $f^{-1}(x) = \frac{3}{5}(x+9) - 7$
5 a $F^{-1}: t \mapsto \sqrt{(t-5)}, (t \geqslant 5)$ b $F^{-1}: t \mapsto t^2/25$
c $F^{-1}: t \mapsto \sqrt[3]{t} + 5$ d $F^{-1}: t \mapsto t^3 - 1$
6 a $g^{-1}: x \mapsto 1/x + 3, (x \neq 0)$
b $g^{-1}: x \mapsto (1/x - 1)/2, (x \neq 0)$
c $g^{-1}: x \mapsto 4 - 3/x, (x \neq 0)$
d $g^{-1}: x \mapsto x/(2-x), (x \neq 2)$

Exercise 2e, page 60

1 a 2.5 b 0 c ∞ d 0 2 a 6 b 10 c 75 d ∞
3 a $f(0) = 1$ b not possible c not possible d 3
4 a continuous b discontinuous
c discontinuous d continuous
5 ± 3, $f(3) = 4$, $f(-3) = -2$

Chapter 3

Q1 The circle. Q2 90°
Q3 PQ $\to 0$, QO $\to 0$, PQ/QO $= \frac{1}{3}$
Q4 3, $2\frac{1}{2}$, 2.1, 2.01; 2
Q5 Q $\to$ P; gradient of PQ $\to$ gradient of tangent at P; 2
Q6 4 Q7 $-3, -2, 1, 4$
Q8 a $6x$ b $10x$ c x d $2cx$ e $2x$ f $2x$
Q9 $4x^3$ Q10 $6x^2$
Q11 a $12x^2$ b $20x^3$ c $2ax$
d $4nx^{n-1}$ e $k(n+1)x^n$
Q12 a $3x^2 + 4x + 3$ b $16x^3 - 6x$ c $2ax + b$
Q13 a $12x^2 - 4x$ b $2x - 1$ c 5

Exercise 3a, page 68

1 $12x^{11}$ 2 $21x^6$ 3 5 4 5 5 0 6 $10x - 3$
7 $12x^3 - 6x^2 + 2x - 1$ 8 $8x^3 + x^2 - \frac{1}{2}x$
9 $3ax^2 + 2bx + c$ 10 $18x^2 - 8$
11 $15x^2 + 3x$ 12 -1 13 0 14 $12x^2 - 3$
15 $ax - 2b$ 16 $4x + 2$ 17 $6x - 3$ 18 $x^2 - 1$
19 $2x - 1$ 20 $6x$ 21 $x + \frac{7}{4}$ 22 $\frac{4}{3}x - \frac{1}{3}$ 23 x
24 1; 2 25 1; 1 26 3; -4 27 -5; 4 28 28; -36
29 9; -24 30 (4, 16) 31 $(-2, -8), (2, 8)$ 32 (0, 0)
33 $(\frac{3}{2}, -\frac{5}{4})$ 34 $(-1, 8), (1, 6)$ 35 $(2, -12)$
36 $(0, 1), (\frac{3}{2}, -\frac{11}{16})$ 37 $(-\frac{1}{3}, \frac{4}{27}), (1, 0)$
38 (1, 4), (3, 0)

Exercise 3b, page 69

1 a $4x - y - 4 = 0$ b $24x - y - 46 = 0$
c $x + y - 1 = 0$ d $8x + y - 5 = 0$
e $18x + y + 54 = 0$
2 a $x + 4y - 18 = 0$ b $x + 24y - 1204 = 0$
c $x - y + 1 = 0$ d $x - 8y - 25 = 0$ e $x - 18y + 3 = 0$
3 $9x - y - 27 = 0$; $x + 9y - 3 = 0$
4 $16x - y = 0$; $x + 16y = 0$ 5 $2x - y - 10 = 0$
6 $4x + y - 3 = 0$ 7 $y + 4 = 0$; $y - 23 = 0$
8 $y - 10 = 0$; $y + 17 = 0$

Chapter 4

Q1 6.1 m, 12.2 m/s
Q2 a 1.0 m, 10 m/s b $4.9(2h + h^2)$ m, $4.9(2 + h)$ m/s
Q3 9.8 m/s
Q4 a 24.5 m, 24.5 m/s b 11 m, 22 m/s
c 2.0 m, 20 m/s
d $4.9(4h + h^2)$ m, $4.9(4 + h)$ m/s; 19.6 m/s
Q5 a 6.9, 23.6, 50.1, 86.4 m below top
b 11.8, 21.6, 31.4, 41.2 m/s c 26.5 m/s
Q6 a 19.8, 29.6, 39.4, $10 + 9.8t$, m/s
b straight line through (0, 10) of gradient 9.8

Exercise 4a, page 72

1 a 10.5 m, 10.5 m/s
b 13, 15, $(15.4 - 4.9h)$ m/s c 15.4 m/s
2 $v = 24.5 - 9.8t$ a $t = 0$, 5 seconds
b 19.6, 29.4, 29.4, -29.4 m; 14.7, 4.9, -4.9, -34.3 m/s
c below ledge; falling d $t = 2.5$; 30.6 m e 2.4 m
3 $v = 3 + 2t$ a At O, 3 m/s b $t = 0$, or -3
c $t = -\frac{3}{2}$, $\frac{9}{4}$ m from O on the negative side d -3 m/s
4 a 0, 8, 9, 8, 0, -7 m; on AO produced
b 6, 2, -2, -6 m/s; moving in direction $\overline{\text{AO}}$
c $t = 3$; 9 m from O, on OA

5 a 11.59 a.m., 12.03 p.m.   b $\frac{5}{27}$, 1 km
c $\frac{8}{27}$ km/min = $17\frac{7}{9}$ km/h   d $\frac{1}{3}$ km/min = 20 km/h
6 a 11.57 a.m., 12.02 p.m.   b $\frac{9}{8}$, $\frac{11}{18}$ km
c $\frac{25}{72}$ km/min = $20\frac{5}{6}$ km/h   d $\frac{1}{2}$ km/min = 30 km/h
7 29.4 m/s

Exercise 4b, page 73

1 2.5 m/s^2 2 3 m/s^2
3 a 18 km/h per s   b 64 800 km/h^2
4 a 3.6 km/h per s   b 1 m/s^2 c 12 960 km/h^2
5 6.25 s   6 −1.5 m/s^2; −5   7 130 km/h

Exercise 4c, page 75

1 a +5.6 m, +0.7 m/s (up), −9.8 m/s^2 (decreasing speed)
b +1.4 m, −9.1 m/s (down), −9.8 m/s^2 (increasing speed)
c −12.6 m, −18.9 m/s (down), −9.8 m/s^2 (increasing speed)
2 24.9 m, 29.8 m/s, 9.8 m/s^2
3 a 31.5 m, −4.2 m/s   b $t = 2\frac{4}{7}$ c 32.4 m
d 2.5 m   e −9.8 m/s^2 (constant)
4 a 18, 54, 114 m/s^2 b 58 m/s^2
5 a $t = 2$ b $t = \frac{2}{3}$, $\frac{32}{27}$ m from O on OA; $t = 2$, at O
c $\frac{32}{27}$, $\frac{64}{27}$ m   d 3 m/s
e 1 m from O, on OA; towards O (−1 m/s); increasing ($a = -2$ m/s^2)
6 9 m from O on AO produced ($s = -9$); towards O (+15 m/s); decreasing ($a = -14$ m/s^2)
7 a After 0, 1, 2 s   b 2, −1, 2 m/s; −6, 0, +6 m/s^2
c 0 m/s   d 0 m/s^2

Chapter 5

Q1 a $2x - 4$ b $6x$ c $6x^2 - 10x$ d $2x - 2$
e $3x^2 - 4x - 3$
Q2 a (1, 2)   b $(-\frac{1}{3}, -5\frac{1}{3})$ c $(\frac{3}{4}, -\frac{1}{4})$
Q3 a $\frac{5}{3}$, highest   b $\frac{7}{6}$, lowest   c $-\frac{3}{2}$, lowest
d $-\frac{4}{3}$, highest
Q4 a $-4x^{-5}$ b $-6x^{-3}$ c $-6x^{-4}$ d $-\frac{3}{2}x^{-4}$
e $-mx^{-m-1}$ f $4x - 3 - 5x^{-2}$ g $1 - 3x^{-2} + 8x^{-3}$
Q5 A, E min., D, F max., B, C infl.; G max., I min., H infl.; K max., J, L, infl.
Q6 a neg., pos., decreasing   b pos., pos., increasing
c neg., zero, neither

Exercise 5a, page 78

1 a $6x - 2$ b $10x + 4$ c $2 - 4x$ d $6x + 1$ e $48x + 6$
2 a $(-2\frac{1}{2}, -8\frac{1}{4})$ b $(\frac{9}{14}, 7\frac{25}{28})$ c $(\frac{1}{3}, -\frac{1}{3})$ d $(-\frac{5}{8}, 7\frac{9}{16})$
3 a $2\frac{1}{2}$, lowest   b −6, lowest   c $\frac{4}{3}$, highest
d −25, highest
4 a $-2\frac{1}{4}$, least   b 4, greatest   c 16, greatest
d $-6\frac{1}{8}$, least
6 12.1 m, $1\frac{4}{7}$ s   7 50 m by 50 m   8 10 cm
9 250 m, 500 m, 125 000 m^2 10 50 m, 5 s
11 2 cm, 3 cm

Exercise 5b, page 82

1 a 0, infl.   b 0, y max.   c 2, y max.; 3, y min.
d −3, y max.; 5, y min.   e −6, y max.; −1, y min.
f 1, y min.; 3, y max.   g −3, y min.; 4, y max.
h −6, y min.; 1, y max.   i −5, y min.; 3, y max.
j $-\sqrt{\frac{27}{5}}$, y max.; 0, infl.; $\sqrt{\frac{27}{5}}$, y min.
k −2, y max.; 2, y min.
2 a $-\frac{16}{9}$, min.; $\frac{16}{9}$, max.   b 0, max.; −27, min.
c 0, max.; $-\frac{256}{27}$, min.   d −2, max. +2, min.
e $\frac{100}{27}$, max., −9 min.
3 a (−2, 16) max.; (2, −16) min.
b $(\frac{1}{3}, 2\frac{13}{27})$ max.; (3, −7) min.
c (0, 0) min.; (2, 4) max.   d $(\frac{1}{2}, 3)$ min.
e $(3\frac{1}{3}, 181\frac{13}{27})$ max.; (12, −144) min.
4 a 0, min.   b 3, infl.   c 0, infl.; $\frac{27}{16}$, max.
d 19, infl.; 3, min.
5 18 cm^3; $x = 1$ 6 $7\frac{11}{27}$ cm^3; $x = \frac{2}{3}$ 7 $\frac{2}{27}$ m^3
8 $\frac{2}{27\pi}$ m^3 9 $\sqrt[3]{(5/\pi)}$ cm; $2\sqrt[3]{(5/\pi)}$ cm   11 4 cm
12 6, 6, 3 cm

Exercise 5c, page 83

1 (0, 0), (3, 0); (0, 0) min., (2, 4) max.
2 (0, 0), (6, 0); (0, 0) max., (4, −32) min.
3 (0, 0), (1, 0); $(\frac{1}{3}, \frac{4}{27})$ max., (1, 0) min.
4 (−1, 0), (2, 0), (0, 2); (−1, 0) min., (1, 4) max.
5 (0, 0), (2, 0); (0, 0) min., (1, 1) max., (2, 0) min.
6 (0, 0), (8, 0); (0, 0) infl., (6, −432) min.
7 (±1, 0), (±3, 0), (0, 9); (0, 9) max., $(\pm\sqrt{5}, -16)$ min.
8 (0, 0), $(-\sqrt[3]{32}, 0)$; (−2, −48) min.
9 (0, 0), $(1\frac{1}{4}, 0)$; (0, 0) max., (1, −1) min.
10 (0, 0), $(\pm\sqrt{\frac{5}{3}}, 0)$; (−1, 2) max., (0, 0) infl., (1, −2) min.
11 (0, 0), $(-\sqrt[3]{\frac{5}{2}}, 0)$; (−1, 3) max., (0, 0) min.

Exercise 5d, page 85

2 $v = 6t^2 - 22t + 12$, $a = 12t - 22$
a 4 m from O on BO produced ($s = -4$)
b away   c 8 m/s ($v = -8$)
d decreasing   e 2 m/s^2 ($a = +2$)
3 a 3 m from O on OB ($s = +3$)   b away
c 4 m/s ($v = -4$)   d increasing   e 10 m/s^2 ($a = -10$)
4 After $\frac{11}{6}$ s; $s = -\frac{143}{54}$
5 a 100 m from O on OA ($s = +100$); approaching A at 40 m/s ($v = +40$); retarding at 14 m/s^2 ($a = -14$)
b $t = 3\frac{1}{3}$ to $t = 12$ c $t = 7\frac{2}{3}$

Chapter 6

Q1 a $2x+c$ b $mx+c$ c x^3+c d $\frac{3}{2}x^2+c$
e $\frac{3}{5}x^5+c$ f $3x+x^2+c$ g $\frac{1}{2}x^2-\frac{1}{3}x^3+c$
h $\frac{1}{2}ax^2+bx+c$

Q2 a $-\dfrac{1}{2}x^{-2}+c=\dfrac{-1}{2x^2}+c$ b $-\dfrac{1}{3}x^{-3}+c=\dfrac{-1}{3x^3}+c$
c $-2x^{-1}+c$ d $\dfrac{-x^{-(n-1)}}{n-1}+c$

Q3 $\dfrac{x^0}{0}+c$ is meaningless.

Q4 $y=4x+18$. A straight line of gradient 4 through $(-2, 10)$.

Q5 $v=15+9.81t$, $s=15t+4.905t^2$

Q6 +9.8 m/s (rising), −9.8, −29.4 m/s (falling); 14.7, 14.7 m (above start), −24.5 m (below).

Q7 a 9 b 42 c −6 d 35 **Q8** $12\frac{2}{3}$ m

Q9 a 13 m past O b 5 m past O c 7 m past O
d 100 m short of O

Q10 a 72 b 9 c 36 d 21

Q11 a $3\frac{1}{4}$ b 9 c 2 d −8 e −38 f $9\frac{1}{4}$

Q12 25 **Q13** a 9 b 81

Q14 $2\frac{2}{3}$ **Q15** $\frac{1}{2}$

Exercise 6a, page 87

1 a $\frac{1}{2}x+c$, $\frac{1}{6}x^3+c$, $\frac{1}{3}x^3+\frac{3}{2}x^2+c$, $\frac{4}{3}x^3+6x^2+9x+c$, $-\frac{1}{4}x^{-4}+c$, $\frac{2}{3}x^{-3}+c$
b $\frac{1}{2}at^2+c$, $\frac{1}{12}t^4+c$, $\frac{1}{3}t^3-\frac{1}{2}t^2-2t+c$, $-\dfrac{1}{n}t^{-n}+c$, $-t^{-1}+3t+t^2+c$
c $ay^{-1}+c$, $-ky^{-1}+c$, $\frac{1}{3}y^3-y+6y^{-1}+c$

2 a $y=ax^3+c$ b $s=\frac{3}{4}t^4+c$ c $s=ut+\frac{1}{2}at^2+c$
d $x=t+t^{-1}+c$ e $y=t+3t^{-1}-2t^{-2}+c$
f $A=-x^{-1}-x-\frac{2}{3}x^3+c$

3 $x-6y+34=0$ 4 $y=x^2+5x-25$

5 $y=x^3+1/x-8\frac{1}{2}$ 6 $(1, 0)$, $(3, 0)$

7 $(4, 0)$; $y=9\frac{13}{27}=\frac{256}{27}$ 8 $s=\frac{3}{2}t^2+8/t-8$

9 $A=c-3x^{-1}-\frac{1}{2}x^{-2}+x^{-3}+\frac{1}{4}x^{-4}$; $\frac{49}{64}$

Exercise 6b, page 89

1 $v=20+9.81t$; $s=20t+4.90t^2$

2 $v=-12+9.8t$; $s=-12t+4.9t^2$; −2.2 m/s (rising), +7.6, +17.4 m/s (falling), −7.1, −4.4 m (above ground level), +8.1 m (below).

3 a $s=3t+3$ b $s=2t^2-t-6$
c $s=t^3+\frac{5}{2}t^2-2t-13$
d $s=\frac{1}{3}t^3+5t+2t^{-1}-7$

4 a 32 b 328 c −21 d 16

5 a $s=2t^2+3t+c$, 14 m
b $s=\frac{1}{3}t^3-3t+c$, $3\frac{1}{3}$ m
c $s=\frac{1}{3}t^3-\frac{3}{2}t^2+2t+c$, $3\frac{5}{6}$ m
d $s=\frac{1}{2}t^2+3t+1/t+c$, $179\frac{19}{20}$ m

6 $v=\frac{1}{2}At^2+B$; $s=\frac{1}{6}At^3+Bt+c$

7 a $v=\frac{3}{2}t^2+3$, $s=\frac{1}{2}t^3+3t$
b $v=2t+\frac{1}{2}t^2$, $s=-3+t^2+\frac{1}{6}t^3$
c $v=-7\frac{1}{2}+10t-\frac{1}{2}t^2$, $s=-7\frac{1}{2}t+5t^2-\frac{1}{6}t^3$
d $v=\frac{1}{4}t^2+5$, $s=\frac{1}{12}t^3+5t+c$
e $v=\frac{1}{3}t^3+c$, $s=\frac{1}{12}t^4+ct+9\frac{11}{12}-c$

8 a $13\frac{1}{2}$ m past O b $2\frac{1}{2}$ m past O
c 8 m past O d $7\frac{1}{2}$ m

9 a $s=-5+6t-t^2$, 5 m b 13 m

10 a $1\frac{2}{7}$ s b 8.1 m c 7.7, 2.9 m

11 a 40 km b 20 km/h c 30 km/h

12 a $13\frac{1}{3}$ km/h b 20 km/h

13 35 m/s, 28 m/s² 14 $k=6$; 4800 m

15 a After 4 s, −64 m/s b 27 m c 16 m/s, 64 m/s

16 a $11.59\frac{1}{2}$ a.m., $12.01\frac{1}{2}$ p.m.
b $s=\frac{1}{48}(5+18t+12t^2-8t^3)$ c 20 km/h
d 30 km/h

Exercise 6c, page 94

1 a $3\frac{63}{64}$ b −2 c $10\frac{2}{3}$ d $36\frac{137}{144}$ 2 50

3 a 26 b $58\frac{1}{3}$ c $22\frac{7}{60}$ d $2\frac{1}{2}$

4 $5\frac{1}{3}$ 5 $-\frac{1}{6}$ 6 $-\frac{5}{12}$, $2\frac{2}{3}$

7 a $-2\frac{1}{3}$ b 4 c $2\frac{9}{20}$ d $-\frac{1}{2}$ 8 $1\frac{1}{3}$ 9 $4\frac{1}{2}$

10 a $(0, 0)$, $(4, 8)$, $5\frac{1}{3}$ b $(-2, 12)$, $(1, 3)$, $13\frac{1}{2}$
c $(-1, 0)$, $(3, 4)$, $10\frac{2}{3}$

11 a 96 b 60 c $1\frac{1}{2}$ 12 $833\frac{1}{3}$

Chapter 7

Q1 a $-4x^{-5}$ b $-6x^{-4}$ c $-\dfrac{2}{x^3}$ d $-\dfrac{4}{x^2}$ e $\dfrac{4}{x^3}$
f $-\dfrac{1}{x^4}$ g $\dfrac{4}{x^5}$ h $-\dfrac{3}{x^6}$ i $-\frac{1}{2}x^{-2}+c$
j $-2x^{-1}+c$ k $-\dfrac{1}{x}+c$ l $-\dfrac{1}{x^2}+c$
m $-\dfrac{1}{6x^2}+c$ n $-\dfrac{2}{15x^3}+c$

Q2 a $\frac{1}{2}x^{-1/2}$ b $-\frac{2}{3}x^{-4/3}$ c $\dfrac{1}{2\sqrt{x}}$ d $\dfrac{1}{3\sqrt[3]{x^2}}$
e $-\dfrac{1}{3\sqrt[3]{x^4}}$ f $\dfrac{2}{3\sqrt[3]{x^4}}$ g $3\sqrt{x}$ h $-\dfrac{1}{3\sqrt{x^3}}$
i $\frac{4}{3}x^{3/4}+c$ j $\frac{4}{5}x^{5/2}+c$ k $\frac{2}{3}\sqrt{x^3}+c$ l $\frac{3}{4}\sqrt[3]{x^4}+c$
m $2\sqrt{x}+c$ n $-\dfrac{2}{\sqrt{x}}+c$

Q3 a $2(x+4)$ b $3(x+2)^2$ c $6(3x+1)$
d $-4(5-2x)$ e $3(x+4)^2$ f $6x^2(x^3+1)$
g $6x(5+x^2)^2$ h $-(2/x^2)(2+1/x)$
i $-6x^2(1-x^3)$ j $\frac{3}{2}(\frac{1}{2}x-7)^2$

Q4 a $2x+3$ b $2x(2x^2+1)$ c $4(x-2)(x^2-x-1)$
d $2(x+1)(x+2)(2x+3)$

Q6 a and b $\dfrac{2(3x-1)}{(x+3)^3}$

Q7 a 1 b $\dfrac{dy}{dx}$ c $2x$ d $2y\dfrac{dy}{dx}$ e $y+x\dfrac{dy}{dx}$
f $2xy+x^2\dfrac{dy}{dx}$ g $y^2+2xy\dfrac{dy}{dx}$

Q8 $\dfrac{2x-6y+3}{6x-2y+2}$ **Q9** $\dfrac{2x+y}{3y^2-x}$

Q10 a $2x+\dfrac{2}{x^3}, 2-\dfrac{6}{x^4}$ c $-\dfrac{1}{(x-1)^2}, \dfrac{2}{(x-1)^3}$

Q11 $\dfrac{1}{t}, -\dfrac{1}{2at^3}$

Exercise 7a, page 97

1 a $4(2x+3)$ b $24(3x+4)^3$
c $-2(2x+5)^{-2}$ d $2(3x-1)^{-1/3}$
e $(3-2x)^{-3/2}$ f $12(3-4x)^{-4}$

2 a $\frac{1}{12}(3x+2)^4+c$ b $\frac{1}{6}(2x+3)^3+c$
c $-\frac{1}{3}(3x-4)^{-1}+c$ d $\frac{1}{3}(2x+3)^{3/2}+c$

3 a $\dfrac{-3}{(3x+2)^2}$ b $\dfrac{-4}{(2x+3)^3}$ c $\dfrac{-3}{2\sqrt{(3x+1)^3}}$
d $\dfrac{-4}{3(2x-1)^{5/3}}$

4 a $-\frac{1}{2}(2x-3)^{-1}+c$ b $\frac{2}{3}\sqrt{(3x+2)}+c$
c $2(2x-1)^{1/4}+c$

5 a $18x(3x^2+5)^2$ b $(18x^2+10)(3x^3+5x)$
c $\dfrac{14x}{3}(7x^2-4)^{-2/3}$ d $-(36x^2-8)(6x^3-4x)^{-3}$
e $-\frac{2}{3}(6x-5)(3x^2-5x)^{-5/3}$

6 a $\dfrac{-6x}{(3x^2+2)^2}$ b $\dfrac{-3x}{\sqrt{(2+x^2)^3}}$ c $\dfrac{1}{\sqrt{x}(1+\sqrt{x})^3}$
d $\dfrac{3}{x^2}\left(1-\dfrac{1}{x}\right)^2$ e $\dfrac{-2x}{3(x^2-1)^{4/3}}$

7 a $3(3\sqrt{x}-2x)^2\left(\dfrac{3}{2\sqrt{x}}-2\right)$ b $\dfrac{1}{\sqrt{x}(2-\sqrt{x})^2}$
c $\dfrac{1}{3}\left(2x^2-\dfrac{3}{x^2}\right)^{-2/3}\left(4x+\dfrac{6}{x^3}\right)$
d $\dfrac{1}{2}\left(x-\dfrac{1}{x}\right)^{-1/2}\left(1+\dfrac{1}{x^2}\right)$

8 a $\dfrac{-\frac{3}{2}\sqrt{x}}{(x^{3/2}-1)^2}$ b $\dfrac{1}{2x^{3/2}\sqrt{(x-1)}}$ c $\dfrac{-1}{6\sqrt{x}\sqrt[3]{(1-\sqrt{x})^2}}$
d $\dfrac{x^2-1}{x^2}$

9 a $\dfrac{-3(2x-7)}{(x^2-7x)^4}$ b $\dfrac{1-4x^{3/2}}{\sqrt{x}(x^2-\sqrt{x})^3}$ c $\dfrac{x}{(1-x^2)^{3/2}}$
d $\dfrac{1}{\sqrt{x}(1-\sqrt{x})^3}$

10 a $\dfrac{x^4+1}{x^2\sqrt{(x^4-1)}}$ b $\dfrac{-2(\sqrt{x}+1)}{\sqrt{x}(x+2\sqrt{x})^2}$
c $\dfrac{1}{3x^{3/2}(1-2/\sqrt{x})^{2/3}}$ d $\dfrac{1}{4x^{3/2}\sqrt{(1-1/\sqrt{x})}}$

Exercise 7b, page 99

1 1458 cm^3/s 2 16π cm^2/s 3 $\frac{2}{27}$ cm/s 4 $\frac{3}{2}$ cm/s
5 Decreasing, 8π cm^2/s 6 24 7 $1/(8\pi)$ cm/s
8 $1/(2\pi)$ cm/s 9 $\frac{4}{45}$ cm/s 10 a 6 cm b $\frac{1}{6}$ cm/min
11 30 12 $\frac{4}{15}$ 13 0.27 cm/s 14 4.8 litres/min

Exercise 7c, page 101

1 $x(5x+2)(x+1)^2$ 2 $(9x^2+1)(x^2+1)^3$
3 $2(2x-1)(x+1)^2$
4 $\dfrac{1}{(x+1)^2}$ 5 $\dfrac{-4x}{(1+x^2)^2}$ 6 $\dfrac{2(x-1)}{(x+1)^3}$
7 $2x(1+x^2)(1-3x^2)$ 8 $2x-\frac{3}{2}\sqrt{x}$
9 $-x(x+1)(x-1)^2(4+7x+7x^2)$
10 $\dfrac{2x^2-x+1}{\sqrt{(x^2+1)}}$ 11 $\dfrac{x(2+3x^2)}{\sqrt{(1+x^2)}}$ 12 $\dfrac{x(2+x^2)}{\sqrt{(1+x^2)^3}}$
13 $\dfrac{(x-1)(3x+1)}{2\sqrt{x^3}}$ 14 $\dfrac{x(-2x^3+2x^2+3x-4)}{\sqrt{(x^2-1)^3}}$
15 $\dfrac{2x+5}{2\sqrt{(x+3)}\sqrt{(x+2)}}$ 16 $\dfrac{1}{2\sqrt{\{x(x+1)^3\}}}$
17 $\dfrac{-1}{\sqrt{x}(1+\sqrt{x})^2}$ 18 $\dfrac{1}{2\sqrt{\{(1+x)(2+x)^3\}}}$
19 $\dfrac{(4x+5)\sqrt{(x+2)}}{2\sqrt{(x+1)}}$ 20 $\dfrac{(2x+5)\sqrt{(x+1)}}{2\sqrt{(x+2)^3}}$

Exercise 7d, page 103

1 $\pm\frac{1}{3}$ 2 $-1, \frac{11}{3}$ 3 $\frac{3}{7}$ 4 $-\frac{3}{2}$ 5 a $\dfrac{3t}{2}$ b $\frac{3}{2}\sqrt{x}$
6 (9, 3), (−1, 3) 7 a $\dfrac{-2y}{3x}$ b $\dfrac{y(2x-y)}{x(2y-x)}$
8 a $\dfrac{1}{t}$ b $\dfrac{t}{t+1}$ 9 $2t-t^2$ 10 $\dfrac{2(x-y-1)}{2x-2y-3}$
11 $\dfrac{9(t+2)^2}{4(t+3)^2}$ 12 $\dfrac{4y-3x}{3y-4x}$

Exercise 7e, page 104

1 8π cm^2 2 9% 3 a 2.000 83 b 5.01 4 $\frac{1}{2}x$
5 $\delta p/p=-\delta v/v$ 6 a $1\frac{1}{4}$ b $1\frac{1}{4}$ 7 1.6π cm^3
8 4% 9 2% 10 a 25.04 b 10.0166 11 $1\frac{1}{3}$%

Chapter 8

Q1 $60 + 50/n$; 60

Q2 **a** $\frac{3}{2}x^2 - 4x + c$ **b** $\frac{8}{3}x^3 + 3x^{-1} + c$ **c** $\frac{7}{8}x^{8/7} + c$
d $-t^2 + \frac{10}{3}t^{3/2} - 3t + c$

Q3 **a** $17\frac{5}{6}$ **b** $\frac{7}{48}$ **c** $-5\frac{3}{5}$

Q4 **a** **i** A cone, vertex C
ii two cones with common base, vertices A and C
b sphere **c** hemisphere
d ring internal dia. 4, external dia. 8 **e** cylinder

Q5 **a** $31\pi/5$ **b** $56\pi/15$ **Q6** **a** $(\frac{6}{5}, 0)$ **b** $(\frac{6}{5}, 3\sqrt{2}/8)$

Exercise 8a, page 107

4 **a** $\frac{3}{4}x^{4/3} + c$ **b** $\frac{4}{5}x^{5/4} + c$ **c** $\frac{5}{3}x^{6/5} + c$
d $\frac{3}{4}kx^{4/3} + c$ **e** $2x^{1/2} + c$ **f** $\frac{3}{2}x^{2/3} + c$ **g** $\frac{6}{5}x^{5/6} + c$
h $\frac{5}{2}x^{4/5} + c$ **i** $\frac{3}{5}x^{5/3} + c$ **j** $\frac{3}{10}x^{10/3} + c$ **k** $\frac{2}{5}x^{5/2} + c$
l $-3x^{-1/3} + c$ **m** $\dfrac{a}{a+1}x^{(a+1)/a} + c$ **n** $\dfrac{n}{n-1}x^{(n-1)/n} + c$
o $\frac{2}{7}x^{7/2} + \frac{4}{5}x^{5/2} - 2x^{3/2} + c$ **p** $\frac{2}{3}x^{3/2} + 4x^{1/2} + c$
q $\frac{1}{2}x^2 - \frac{2}{3}x^{3/2} - 6x + c$ **r** $\frac{2}{3}(x+2)^{3/2} + c$
s $\frac{1}{3}(x^2 - 3)^{3/2} + c$

5 **a** $-\frac{1}{2}$ **b** 21 **c** $12\frac{2}{3}$

Exercise 8b, page 111

1 **a** $\frac{1}{3}x^3 - \frac{3}{2}x^2 + c$
b $-2x^{-1} + x^{-2} + c$
c $\frac{1}{3}at^3 + bt - ct^{-1} + k$
d $\frac{1}{5}x^5 - \frac{3}{5}x^{5/3} + 2x + x^{-1} + c$
e $\frac{1}{3}y^3 + \frac{2}{3}y^{3/2} + y - 2y^{-1/2} + c$
f $\frac{3}{8}s^{8/3} + \frac{6}{5}s^{5/3} + \frac{3}{2}s^{2/3} + c$

2 **a** $26\frac{2}{3}$ **b** $25\frac{2}{3}$ **c** $1\frac{11}{15}$ **d** $3\frac{19}{24}$ **e** $21\frac{1}{3}$ **f** $24\frac{2}{3}$

3 **a** 12 **b** $-31\frac{1}{4}$ **c** $1\frac{7}{24}$

4 **a** 9 **b** $11\frac{1}{4}$ **c** 12 **d** $2(\sqrt{3} - \sqrt{2})$

5 **a** $4\frac{1}{2}$, on the negative side of the y-axis
b $4\frac{1}{2}$ **c** $1\frac{1}{3}$

6 $18\frac{2}{3}$ **7** -36 **8** $28\frac{4}{9}$ **9** **a** 4 **b** $\frac{3}{4}$

10 **a** $10\frac{2}{3}$ **b** $1\frac{1}{3}$ **c** $4\frac{1}{2}$ **d** $4\frac{1}{2}$ **e** $20\frac{5}{6}$ **f** $20\frac{5}{6}$

11 **a** $\frac{9}{8}$ **b** $\frac{1}{3}$ **c** $\frac{1}{24}$ **d** $\frac{16}{3}$ **e** $41\frac{2}{3}$ **f** $13\frac{1}{2}$

12 Other points of intersection $(2, \frac{1}{4})$ and $(\frac{1}{2}, 4)$; $1\frac{11}{16}$.

Exercise 8c, page 115

1 **a** 144π **b** $28\pi/15$ **c** 2π **d** $16\pi/15$
e $\pi/105$ **f** $3\pi/4$

2 **a** 18π **b** $9\pi/2$ **c** $96\frac{3}{5}\pi$ **d** $34\frac{2}{15}\pi$
e $3\pi/5$ **f** $3\pi/10$

3 **a** $8\pi/3$ **b** $256\pi/15$ **c** $8\pi/3$ **d** $16\pi/15$
e $128\pi/105$ **f** $7\pi/3$

4 **a** 5π **b** $64\pi/15$ **c** $32\pi/3$ **d** $\frac{1}{2}\pi$

5 $\frac{1}{3}\pi r^2 h$ **6** $\frac{3}{4}\pi r^3$ **7** $661\frac{1}{3}\pi\ \text{cm}^3$
8 $1296\pi\ \text{cm}^3$ **9** $57\frac{6}{7}\pi\ \text{cm}^3$
10 $27\frac{1}{2}\pi\ \text{cm}^3$ **11** $16\pi/15$
12 $37\pi/10$ **13** $37\frac{1}{3}\pi$
14 8π **15** $45\pi/2$

Chapter 9

Q1 **a** $8 < 14$ **b** $0 > -\frac{1}{3}$ **c** $x > 5$ **d** $y \leqslant 5$
e $17 \leqslant m$ **f** $17 \geqslant n$

Q2 **a** true **b** true **c** false **d** false **e** true
f true **g** false **h** false

Exercise 9a, page 117

1 **a** $x \geqslant 1$ **b** $x < 4$ **c** $x \leqslant 2$ **d** $x \leqslant 4$ **e** $x > -6$
f $x > -5$

2 **a** $x \geqslant 2$ **b** $x \leqslant 6$ **c** $p < 47$ **d** $x \geqslant 3$
e $x > 1$ **f** $x < 11$ **g** $x \geqslant -10$ **h** $y < -47$
i $x \geqslant 4$

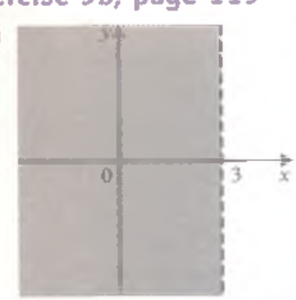
3 $x \geqslant \frac{-9}{8}$ or $-1\frac{1}{8}$

4 $x > -19$

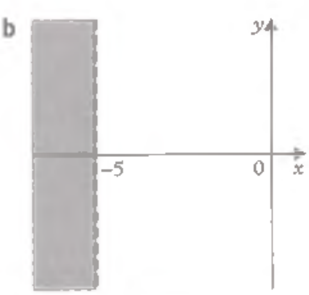
5 $x \leqslant 5$

Exercise 9b, page 119

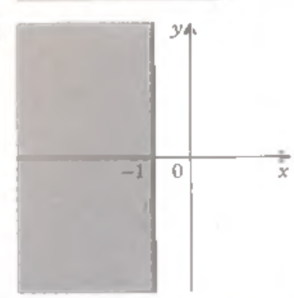
1 **a**



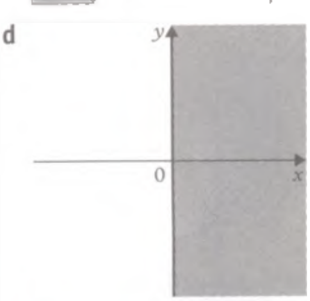
b



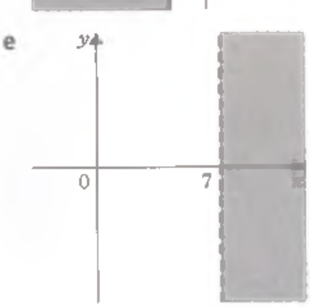
c



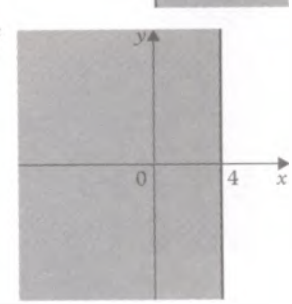
d



e



f



2 a $x \geqslant 4$ **b** $x < -2$ **c** $0 \leqslant x < 5$
d $-9 < x < -1$ **e** $-3 < x \leqslant 8$
f $0 \leqslant x \leqslant 10$

4 a 

b 

c 

d 

e 

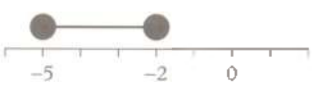
f 

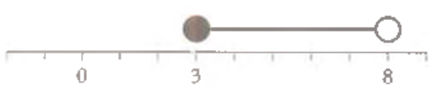
5 a $x \geqslant -\frac{7}{5}$

6 a $-3 \leqslant x < 1$

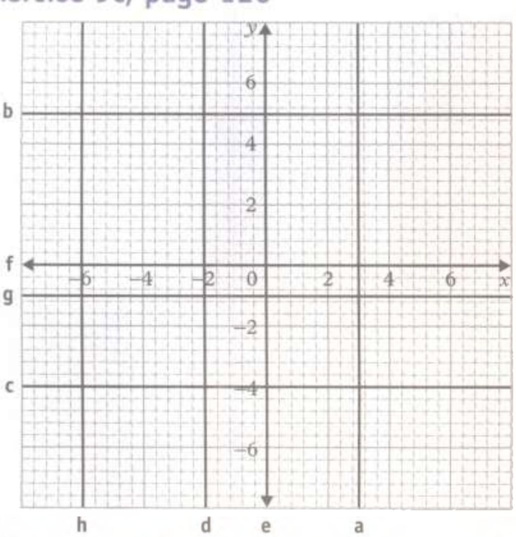
b 

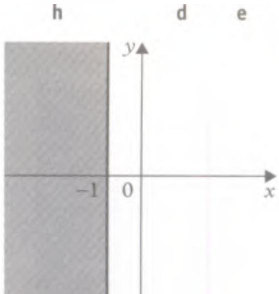
7 b $-4 \leqslant x < 3$

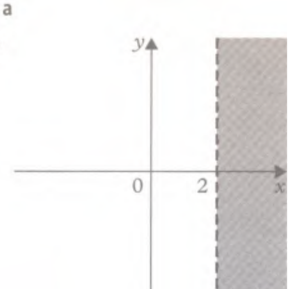
8 a 

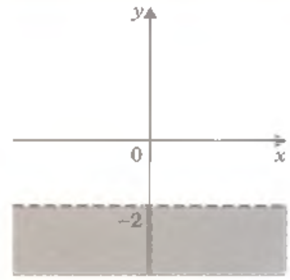
b 

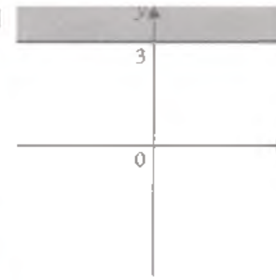
Exercise 9c, page 120

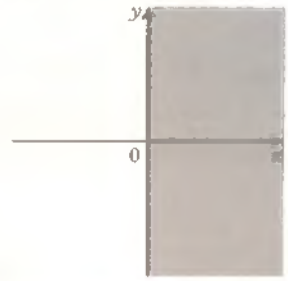
1 

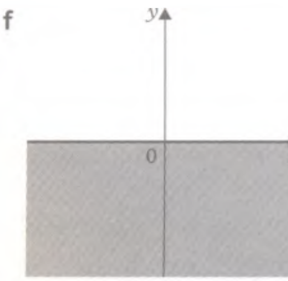
2 a 

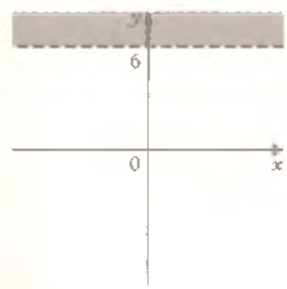
b 

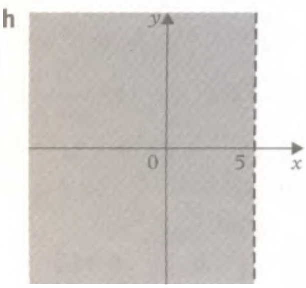
c 

d 

e 

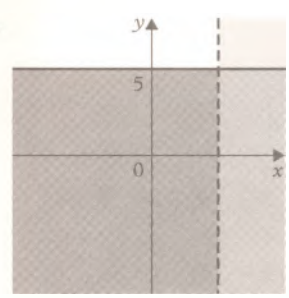
f 

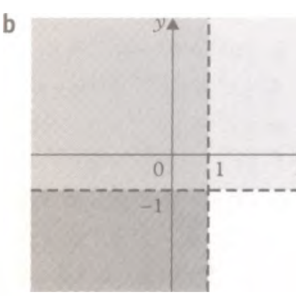
g 

h 

3 The points (x, y) such that
a $x \geqslant 1$ **b** $y > -1$ **c** $x < -2$ **d** $y \leqslant 5$

4 a $x < 5$ and $y \leqslant 7$ **b** $x \geqslant -3$ and $y > -4$

5 a 

b 

Exercise 9d, page 123

1 a $x = 3$ **b** $x + 2y \leqslant 5,\ x + y > 2,\ x < 3$
c (1, 2), (2, 1)

2 $x \geqslant 1,\ x + y < 5,\ 3y \geqslant x$ **3** $y < 2,\ y > 3x,\ x + y \geqslant -2$

5 a (1, 0), (2, 0), (3, 0), (2, 1)
b (−1, 1), (−2, 1), (−3, 1), (−1, 2), (−2, 2)
c (1, 1), (1, 0), (1, −1), (2, −1)
d (1, 0), (2, 0), (0, 2), (1, 1)
e (1, 1), (1, 2), (0, 1), (0, 2), (0, 3), (−1, 2), (−1, 3), (−2, 3)

6 a P(0, 4) **b** $y = -2x$ (or $2x + y = 0$)
c $x < 0,\ 2x + y > 0,\ 2y < x + 8$

Exercise 9e, page 126

1 14 buses, 11 minibuses

2 a two ways: 3 notebooks, 5 pencils or 4 notebooks, 3 pencils

b yes, the 2nd way: $0.60 change

3 a 93, either (72, 21) or (73, 20) b 60 cheap, 30 dear

4 a i (96, 34), ii (140, 20) b $130 000

5 a $4x + y > 20$, $4x + 3y > 30$

c 4 Feelgood pills and 5 Getbetta pills; $500

6 15 lorries (5 Landmasters; 10 Sandrovers)

7 a 37 (10, 27) b either (10, 27) or (11, 25) c $7.05

8 a 10 b 4 of A and 6 of B

9 a 20 b i (16, 38), ii either (20, 30) or (18, 35)

10 $x + y < 1000$, $y > 2x$, $x > 100$, $y < 800$ 333 cans of Kula, 667 cans of Sundown.

Chapter 10

Q1 a 37, i b 0, ii c −8, iii d 17, i Q2 $f(t) \leqslant 30$

Q4 a $\pm 8i$ b $\pm\sqrt{7}i$ c $\pm\frac{3}{2}i$ d $-3 \pm 5i$ Q5 $3 \pm 5i$

Q6 a $2 \pm 3i$ b $\pm 5i/3$ c $(1 \pm 5i)/2$ d $(3 \pm 5i)/34$

Q7 $\frac{5}{2} + \frac{1}{2}i$

Q9 a $[a + c, 0]$ b $[ac, 0]$ c $[a - c, 0]$ d $[a/c, 0]$

Q11 $-y + ix, -x - iy, y - ix$

Q12 a 5 b 1 c 1 d 1 e 3 f $\sqrt{2}$

Q13 a 45° b 0° c −90° d −45° e 60° f 120° g −20° h 70°

Exercise 10a, page 130

1 a $1\frac{1}{2}, 1$ b $3, -7$ c ± 2.5 d $0, -\frac{5}{7}$

2 a $\frac{3 \pm \sqrt{11}}{2}$ b $\frac{-6 \pm \sqrt{6}}{5}$ c $\frac{-7 \pm \sqrt{61}}{2}$ d $\frac{3 \pm \sqrt{89}}{4}$

3 a $\frac{7 \pm \sqrt{61}}{6}$ b $\frac{-3 \pm \sqrt{149}}{10}$ c $\frac{-13 \pm \sqrt{153}}{2}$

d $\frac{7 \pm \sqrt{73}}{6}$

4 a $\frac{15 \pm \sqrt{165}}{4}$ b $0, \frac{48}{11}$ c no real solution d $5, \frac{3}{7}$

5 a $2\left(x - \frac{3 + \sqrt{11}}{2}\right)\left(x - \frac{3 - \sqrt{11}}{2}\right)$

b $5(x + 1.2 + \sqrt{0.24})(x + 1.2 - \sqrt{0.24})$

c $\left(x + \frac{7 + \sqrt{61}}{2}\right)\left(x + \frac{7 - \sqrt{61}}{2}\right)$

d $-2\left(x - \frac{3 + \sqrt{89}}{4}\right)\left(x - \frac{3 - \sqrt{89}}{4}\right)$

8 $|k| < 12$ 10 a 3 b 5 c 10 d −17

Exercise 10b, page 133

1 a $-i$ b 1 c i d -1 e -1 f $-i$ g i

2 a $4 + 3i$ b 9 c $1 - 5i$ d $2i$

3 a $-7 + 22i$ b $8 + i$ c 2 d 25

e $u^2 + v^2$ f $2x^2 - 2y^2 + 5ixy$

g $-3q + 2ip$ h $p^2 + 4q^2$

4 a $-i$ b $\frac{2 + 3i}{13}$ c $\frac{4 + 7i}{5}$ d $\frac{9 + 40i}{41}$

e $\frac{x - iy}{x^2 + y^2}$ f $\frac{x + iy}{x^2 + y^2}$ g 4/13

5 a $-5 + 12i$ b $-9 - 40i$ c $x^2 - y^2 + 2ixy$

6 a $-2 + 2i$ b $-2 - 2i$ c $-\frac{1}{4}(1 + i)$

7 a $2 \pm 5i$ b $\pm\frac{1}{2}\sqrt{7}i$ c $\frac{-3 \pm \sqrt{31}i}{4}$ d $\frac{1}{2}(-1 \pm 2i)$

10 $\frac{1}{2}, -1 \pm 2i$

Exercise 10c, page 136

1 a $\sqrt{2}$, 45° b $\sqrt{13}$, 146.3° c $\sqrt{13}$, −146.3°

d 5, −53.1° e 5, 143.1° f 1, 60°

g 1, 120° h 1, 180°

2 a 1 b i c -1 d $-i$ e 1 f $\frac{1}{2}\sqrt{3} + \frac{1}{2}i$

g $\frac{1}{2}\sqrt{3} - \frac{1}{2}i$ h $-\frac{1}{2} + \frac{1}{2}\sqrt{3}i$ i $-\frac{1}{2} - \frac{1}{2}\sqrt{3}i$

j $-\frac{1}{2}\sqrt{3} + \frac{1}{2}i$

3 a 5, 53.1° b 13, 22.6° c $\frac{1}{5}$, −53.1°

d $\frac{1}{13}$, −22.6° e 65, 75.7° f 5, −53.1°

g 13, −22.6° h 65, −75.7° i 25, 106.3°

j 169, 45°

4 $2i, -2 + 2i, -4$; 45°, 90°, 135°, 180°

5 a $\frac{1}{2} + \frac{1}{2}\sqrt{3}i, i, -\frac{1}{2} + \frac{1}{2}\sqrt{3}i$; 30°, 60°, 90°, 120°

b $2 + 2\sqrt{3}i, 8i, -8 + 8\sqrt{3}i$; 30°, 60°, 90°, 120°

6 $(a^2 - b^2) + 2abi, \frac{a - ib}{a^2 + b^2}$

8 $(ac - bd) + i(ad + bc)$

Chapter 11

Q1 $\begin{pmatrix} 17 & 23 & 29 \\ 14 & 20 & 34 \end{pmatrix}$

Q2 a $\begin{pmatrix} 5 & 1 \\ 2 & 0 \\ 6 & 1 \end{pmatrix}$ b $\begin{pmatrix} 29 \\ 20 \end{pmatrix}$ c $(20 \quad 7)$ d $\begin{pmatrix} 6 & 5 \\ 8 & 5 \end{pmatrix}$

Q5 a $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ b $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ c $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

Q6 $\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$

Exercise 11a, page 140

1 a $\begin{pmatrix} 9 & 3 & 6 \\ 15 & 3 & 21 \end{pmatrix}$ **b** $\begin{pmatrix} 8 & -2 & 4 \\ 6 & 2 & 6 \end{pmatrix}$

c $\begin{pmatrix} 17 & 1 & 10 \\ 21 & 5 & 27 \end{pmatrix}$ **d** $\begin{pmatrix} 1 & 5 & 2 \\ 9 & 1 & 15 \end{pmatrix}$

2 $\mathbf{PS} = (3240 \quad 7500 \quad 10\,500 \quad 6600 \quad 9540 \quad 9900)$

3 a $(19 \quad 31)$ **b** $\begin{pmatrix} 14 & 19 & 24 \\ 17 & 22 & 27 \end{pmatrix}$

c not possible **d** $\begin{pmatrix} 38 \\ 68 \end{pmatrix}$

4 $\begin{pmatrix} \frac{1}{2} & -2 \\ 7 & 10 \end{pmatrix}, \begin{pmatrix} 1\frac{1}{2} & -\frac{1}{2} \\ 11 & 9 \end{pmatrix}$ **5** $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ **6** $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

7 $\begin{pmatrix} 6 \\ -8\frac{1}{2} \end{pmatrix}$ **8** $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ **9** $\begin{pmatrix} 3 \\ -7 \end{pmatrix}$

10 a $\begin{pmatrix} 11 & -1 & 29 \\ 29 & -1 & 62 \\ 1 & 1 & -7 \end{pmatrix}$ **b** $\begin{pmatrix} 19 & -2 \\ 9 & -4 \\ 24 & 0 \end{pmatrix}$

14 $\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$

Exercise 11b, page 144

1 a 1 **b** 14 **c** 30 **d** 1

2 a 0 **b** $\frac{2}{15}$ **c** $a^2 + b^2$ **d** $ad - bc$

3 b, c, d **4 a** 28 **b** ± 4 **c** 1, 4 **d** none

5 a $\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$ **c** $\begin{pmatrix} 7/20 & -11/20 \\ -2/20 & 6/20 \end{pmatrix}$

d $\frac{1}{x^2+1}\begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix}$

Exercise 11c, page 144

1 a $\begin{pmatrix} 3 & -4 \\ -5 & 7 \end{pmatrix}$ **b** $\frac{1}{2}\begin{pmatrix} 3 & -2 \\ -11 & 8 \end{pmatrix}$ **c** $\frac{1}{2}\begin{pmatrix} 6 & -2 \\ -3 & 3 \end{pmatrix}$

d not possible

2 a $\frac{1}{2}\begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}$ **b** $\frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$ **c** $\frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$

d $\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

3 $\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ **4** $\frac{1}{20}\begin{pmatrix} 4 & -2 \\ -8 & 9 \end{pmatrix}, \frac{1}{20}\begin{pmatrix} -2 \\ 19 \end{pmatrix}$

5 $\begin{pmatrix} -5 \\ 4 \end{pmatrix}$ **6** $\begin{pmatrix} -2 & -8 \\ 3 & 11 \end{pmatrix}$

7 $\frac{1}{25}\begin{pmatrix} 304 & 372 \\ 372 & 521 \end{pmatrix}$ **9** $\begin{pmatrix} 17 \\ -7 \\ 19 \end{pmatrix}$ **10** $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

11 $\frac{1}{14}\begin{pmatrix} 7 & -3 & -1 \\ 0 & 6 & 2 \\ 0 & 8 & -2 \end{pmatrix}$ **12** $\frac{1}{14}\begin{pmatrix} 7 & -3 & -1 \\ 0 & 6 & 2 \\ 0 & 8 & -2 \end{pmatrix}$

Exercise 11d, page 151

1 a Reflection in x-axis **b** reflection in y-axis

c rotation through 90° **d** reflection in $x + y = 0$

e shear parallel to x-axis

2 (6, 17), (22, 29), (9, 38); 6, 150

3 $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; rotation, through 90° clockwise **4** $\pi a^2, \pi ab$

5 a $\begin{pmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{pmatrix}$ **b** $\begin{pmatrix} 0 & 5 \\ 5 & 0 \end{pmatrix}$

6 Rotation and enlargement, $a^2 + b^2 = 1$.

7 Enlargement, with scale-factor $\sqrt{2}$ and reflection in the line $y = (\tan 22\frac{1}{2}°)x$; $\lambda = \sqrt{2}$, $m = \tan 22\frac{1}{2}°$.

8 Reflection in $y = (\tan \alpha)x$, where $\cos 2\alpha = 3/5$.

9 $\beta - \alpha$ **10** $\cos 2\theta = \cos^2\theta - \sin^2\theta$, $\sin 2\theta = 2 \sin\theta \cos\theta$

Chapter 12

Q1 a 56 **b** 210 **Q2** $\frac{n!}{(n-r)!\,r!}$

Exercise 12a, page 152

1 720 **2** 360 **3** 24, 120 **4** 243 **5** 72 **6** 24
7 27 000 **8** 120 **9** 900 **10** 120 **11** 719
12 48 **13** 5040 **14** 168 **15** 336, 144
16 3 628 800, 3 628 800 **17** 78 **18** 80
19 10 368 000 **20** 40 320, 384

Exercise 12b, page 154

1 a 6 **b** 24 **c** 120 **d** 90 **e** 210 **f** 1320
g 330 **h** $\frac{1}{28}$ **i** 4 **j** 20 **k** 120 **l** 2520

2 a $\frac{6!}{3!}$ **b** $\frac{10!}{8!}$ **c** $\frac{12!}{8!}$ **d** $\frac{n!}{(n-3)!}$ **e** $\frac{(n+2)!}{(n-1)!}$

f $\frac{10!}{8!\,2!}$ **g** $\frac{7!}{4!\,3!}$ **h** $\frac{52!}{49!\,3!}$ **i** $\frac{n!}{(n-2)!\,2!}$

j $\frac{(n+1)!}{(n-2)!\,3!}$ **k** $\frac{(2n)!}{(2n-2)!\,2!}$ **l** $\frac{n!}{(n-r)!}$

3 a $20! \times 22$ **b** $25! \times 25$ **c** $13! \times 12$ **d** $14! \times 19$
e $n!(n+2)$ **f** $(n-2)!(n-2)$ **g** $(n-1)!(n+2)$
h $n!(n+2)^2$

4 a $\dfrac{16!}{12!4!}$ b $\dfrac{22!}{14!8!}$ c $\dfrac{18!}{7!11!}$ d $\dfrac{37!}{19!18!}$
e $\dfrac{(n+1)!}{r!(n-r+1)!}$ f $\dfrac{(n+2)!}{r!(n-r+2)!}$

Exercise 12c, page 156

1 282 240 2 362 880, 40 320 3 6720, 1680
4 $24 \times 17!$, $48 \times 16!$ 5 $\frac{1}{60} \times 13!$ 6 181 440
7 $20 \times 10!$ 8 768 9 16 10 144 11 30 240
12 60 480 13 528 14 1 404 000 15 2400
16 11 520, 276 480 17 23 520 18 100
19 138 600 20 34 560, 31 680

Exercise 12d, page 158

1 a 45 b 15 c 35 d 126 e 70 f $\frac{1}{2}n(n-1)$
g $\frac{1}{6}n(n-1)(n-2)$ h $\frac{1}{2}n(n-1)$ i $\frac{1}{2}n(n+1)$
j $\frac{1}{2}n(n+1)$
2 78 3 70 4 252 5 126 6 30 7 252
8 286 9 792 10 200 11 495 12 840
13 182 14 420 15 11 550 16 34 650
17 25 200 18 2142 19 31 733

Chapter 13

Q1 a 9, 11 b 14, 17 c 16, 32 d $\frac{1}{48}, \frac{1}{96}$ e $5^3, 6^3$
f $\frac{5}{6}, \frac{6}{7}$ g 25, 36 h 720, 5040 i $\frac{5}{81}, \frac{6}{243}$
j $-4, -6$ k $1, -1$ l $\frac{1}{16}, -\frac{1}{32}$
Q2 i a 6, 8 b 8, 16 ii a $0, -6$ b $3, 1\frac{1}{2}$
Q3 a 34 b 16 Q4 $8, 12\frac{1}{2}, 10$ Q5 $2ac/(a+c)$
Q6 a $n(2n+1)$ b $\frac{1}{6}(n+1)(n+2)(2n+3)$
c $\frac{1}{4}(n-1)^2n^2$ d $n(2n-1)$
e $\frac{1}{3}n(2n+1)(4n+1)$ f $n^2(2n-1)^2$

Exercise 13a, page 160

1 a $1\frac{1}{2}$ b -3 c 0.1 e $\frac{1}{3}$ g n i $1\frac{1}{8}$
j -7 l -0.2
2 a 75, 147 b $-34, -82$ c $7\frac{1}{8}, \frac{1}{8}(5n-3)$
d $-148, 52-2n$ e $-13\frac{1}{2}, \frac{1}{2}(15-n)$ f $799, 3+4n$
3 a 23 b 13 c 31 d 21 e 91 f 13
g $2n$ h n i n j $(l-a)/d+1$
4 a 2601 b 632 c 420 d 288 e 250.5 f $60\frac{1}{2}$
g $121x$ h $\frac{1}{2}n(2a+n-1)$ i $\frac{1}{2}n\{2a+(n-1)d\}$
5 a 444 b -80 c 20 100 d -520
e $n(2n+4)$ f $\frac{1}{8}n(11-n)$
6 2, 13, 220 7 $33, -72$ 8 5 9 14, 4 10 7500
11 7650 12 $3\frac{1}{2}, \frac{1}{10}, 148\frac{1}{2}$ 14 60

Exercise 13b, page 161

1 a 3 b $\frac{1}{4}$ c -2 d -1 f a g 1.1 j 6
2 a $5 \times 2^{10}, 5 \times 2^{19}$ b $10(\frac{5}{2})^6, 10(\frac{5}{2})^{18}$
c $\frac{2}{3}(\frac{9}{8})^{11}, \frac{2}{3}(\frac{9}{8})^{n-1}$ d $3(-\frac{2}{3})^7, 3(-\frac{2}{3})^{n-1}$
e $\frac{2}{7}(-\frac{3}{2})^8, \frac{2}{7}(-\frac{3}{2})^{n-1}$ f $3(\frac{1}{2})^{18}, 3(\frac{1}{2})^{2n-1}$
3 a 9 b 8 c 7 d 8 e $n+1$ f n
4 a $2^{10}-2$ b $\frac{1}{2}(3^5-\frac{1}{27})$ c $0.03(2^7-1)$
d $-\frac{16}{405}\{(\frac{3}{2})^8-1\}$ e $5(2^{n+1}-1)$ f $a\left(\dfrac{1-r^n}{1-r}\right)$
5 a $2(3^{12}-1)$ b $\frac{45}{2}\{1-(\frac{1}{3})^{20}\}$ c $-\frac{1}{3}(2^{50}-1)$
d $16\{1+(\frac{1}{2})^{17}\}$ e $11(1.1^{23}-1)$ f $1-(\frac{1}{2})^{13}$
g $3(2^n-1)$ h $\frac{3}{4}\{1-(-\frac{1}{3})^n\}$
6 $2, 2\frac{1}{2}, 157\frac{1}{2}$ 7 $\pm 3, \pm\frac{2}{3}$ 8 $6, 13\frac{1}{2}$
9 £10 700 000 10 $6\frac{3}{4}$ 12 $\frac{5}{2}, -\frac{1}{3}$
13 $\sqrt{2}-1, 5\sqrt{2}-7$ 14 1023

Exercise 13c, page 164

1 2550 2 8 3 98 4 $\frac{3}{4}, -\frac{3}{2}, 3$ 5 16 400
6 432 7 $\frac{7}{2}, 2$ 8 17, -2, 10th 9 $1\frac{1}{2}, 2, 24$
10 3, 4; 3, 7, 11, 15, 19 11 $-2, 1, 4, 7, 10$ 12 $-3, -2$
13 18 14 18th, 655 360 15 14 16 $-9, 5$
17 2, 4, 6, 8, 10 18 5808 19 6, 8, 10
20 $2\frac{1}{2}, 5, 7\frac{1}{2}, 10$ 21 \$2270 22 \$19 100

Exercise 13e, page 167

1 a $1^3+2^3+3^3+4^3$ b $2^2+3^2+\ldots+n^2$
c $2+6+\ldots+(n^2+n)$ d $\dfrac{1}{1\times 2}+\dfrac{1}{2\times 3}+\dfrac{1}{3\times 4}$
e $2^2+2^3+2^4+2^5$ f $-1+4-9+16$
g $1+2^2+\ldots+n^n$ h $-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}$
i $n(n-1)+(n+1)n+(n+2)(n+1)$
j $\dfrac{n-2}{n-1}+\dfrac{n-1}{n}+\dfrac{n}{n+1}$
2 a $\sum_{1}^{n} m$ b $\sum_{1}^{n+1} m^4$ c $\sum_{1}^{5} \dfrac{1}{m}$ d $\sum_{2}^{5} 3^m$
e $\sum_{2}^{6} m(m+5)$ f $\sum_{1}^{5} \dfrac{m}{3^{m-1}}$ g $\sum_{1}^{5} \dfrac{m(2m+1)}{2(m+1)}$
h $\sum_{1}^{6} (-1)^m m$ i $\sum_{0}^{5} (-2)^m$ j $\sum (-1)^{m+1}m(2m+1)$
3 a $(n+1)(2n+1)$ b $\frac{1}{6}n(n-1)(2n-1)$
c $n^2(2n+1)^2$ d $n(n+2)$ e $\frac{1}{2}n(3n+1)$
f $n(2n+3)$ g $\frac{1}{6}n(2n^2+3n+7)$ h $\frac{1}{3}n(n+1)(n+2)$
i $\frac{1}{6}n(n+1)(2n+7)$ j $\frac{2}{3}n(n+1)(2n+1)$
k $\frac{1}{3}n(2n-1)(2n+1)$ l $\frac{1}{4}n(n+1)(n^2+n+2)$
m $\frac{1}{12}n(n+1)(n+2)(3n+1)$

Exercise 13f, page 168

1 a $1\frac{1}{2}$ **b** 24 **c** $\frac{1}{3}$ **d** $\frac{13}{99}$ **e** $\frac{5}{9}$ **f** $\frac{6}{11}$
g $\frac{2}{3}$ **h** $40\frac{1}{2}$

2 a $\frac{8}{9}$ **b** $\frac{4}{33}$ **c** $3\frac{2}{9}$ **d** $2\frac{23}{33}$ **e** $1\frac{1}{225}$ **f** $2\frac{317}{330}$

3 $\frac{2}{3}$ **4** $2, \frac{1}{2}, \frac{1}{4}$ **5** $\frac{2}{5}, 60; \frac{3}{5}, 40$

Chapter 14

Exercise 14a, page 170

1 a $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$
b $x^3 + 3x^2y + 3xy^2 + y^3$
c $x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$
d $1 - 4z + 6z^2 - 4z^3 + z^4$
e $16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4$
f $64z^3 + 48z^2 + 12z + 1$
g $a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$
h $a^3 - 6a^2b + 12ab^2 - 8b^3$
i $81x^4 - 108x^3y + 54x^2y^2 - 12xy^3 + y^4$
j $8x^3 + 4x^2 + \frac{2}{3}x + \frac{1}{27}$
k $x^5 - 5x^3 + 10x - 10x^{-1} + 5x^{-3} - x^{-5}$
l $\frac{1}{16}x^4 + x^2 + 6 + 16x^{-2} + 16x^{-4}$
m $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4$
$+ 21a^2b^5 + 7ab^6 + b^7$
n $a^{10} - 5a^8b^2 + 10a^6b^4 - 10a^4b^6 + 5a^2b^8 - b^{10}$
o $a^6 - 3a^4b^2 + 3a^2b^4 - b^6$

2 a 14 **b** 194 **c** $10\sqrt{2}$ **d** $160\sqrt{6}$ **e** 98 **f** $40\sqrt{2}$

3 $32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$, 32.080 08

4 $1 + x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \frac{1}{256}x^4$, 1.104

5 $64 - 192x + 240x^2 - 160x^3 + 60x^4 - 12x^5 + x^6$,
63.616 96, 5

Exercise 14b, page 172

1 a $448x^5$ **b** $1080u^3$ **c** $-3168t^7$ **d** $1320x^3y^8$

2 a $84x^3$ **b** $-14\,080x^3$ **c** $945x^4$ **d** $190x^2$

3 a $\frac{105}{512}$ **b** 540 **c** 6048 **d** 1386

4 a 120 **b** -9120 **c** 4320 **d** 5670

5 a $15x^2$ **b** 20 **6 a** 70 **b** $3\frac{3}{4}$

7 a 6 **b** 14 **c** -16

8 $3/(5x)$ **9** $8/(45x)$ **10** $b(r+1)/\{a(n-r)\}$

11 a $1 + 10x + 45x^2 + 120x^3$ **b** $1 + \frac{9}{2}x + 9x^2 + \frac{21}{2}x^3$
c $1 - 11x + 55x^2 - 165x^3$ **d** $1 + 12x + 66x^2 + 220x^3$
e $256 + 512x + 448x^2 + 224x^3$
f $128 - 224x + 168x^2 - 70x^3$

12 a 1.105 **b** 1029.13 **c** 0.965 **d** 253.96

13 a $1 + 3x + 6x^2 + 7x^3$
b $1 + 12x + 54x^2 + 100x^3$
c $1 - 4x + 2x^2 + 8x^3$
d $32 + 80x + 160x^2 + 200x^3$
e $1 - 8x + 36x^2 - 112x^3$
f $128 + 448x - 224x^2 - 2128x^3$
g $81 - 216x + 324x^2 - 312x^3$
h $81 + 108x + 54x^2 + 120x^3$

Exercise 14c, page 175

1 a 10 **b** 5 **c** $-1/8$ **d** $-15/128$

2 a $1 - 2x + 3x^2 - 4x^3, -1 < x < 1$
b $1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3, -1 < x < 1$
c $1 + \frac{1}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3, -1 < x < 1$
d $1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3, -\frac{1}{2} < x < \frac{1}{2}$
e $1 - \frac{3}{2}x + \frac{3}{2}x^2 - \frac{5}{4}x^3, -2 < x < 2$
f $1 + \frac{3}{2}x + \frac{27}{8}x^2 + \frac{135}{16}x^3, -\frac{1}{3} < x < \frac{1}{3}$
g $1 - 3x + 9x^2 - 27x^3, -\frac{1}{3} < x < \frac{1}{3}$
h $1 - \frac{1}{2}x^2, -1 < x < 1$
i $1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3, -1 < x < 1$
j $1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3, -\frac{1}{2} < x < \frac{1}{2}$
k $1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3, -2 < x < 2$
l $1 - 3x + \frac{3}{2}x^2 + \frac{1}{2}x^3, -\frac{1}{2} < x < \frac{1}{2}$
m $\frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3, -2 < x < 2$
n $\sqrt{2}(1 - \frac{1}{4}x - \frac{1}{32}x^2 - \frac{1}{128}x^3), -2 < x < 2$
o $\sqrt[3]{3}(1 + \frac{1}{9}x - \frac{1}{81}x^2 + \frac{5}{2187}x^3), -3 < x < 3$
p $\frac{1}{2}\sqrt{2}(1 - \frac{1}{4}x^2), -\sqrt{2} < x < \sqrt{2}$
q $\frac{1}{9} + \frac{2}{27}x + \frac{1}{27}x^2 + \frac{4}{243}x^3, -3 < x < 3$
r $\sqrt[3]{9}(1 + \frac{1}{9}x^3), -\sqrt[3]{3} < x < \sqrt[3]{3}$

3 a 1.000 500 **b** 0.9612 **c** 0.998 999
d 1.0099 **e** 1.0102

4 a $1 + 2x + 2x^2 + 2x^3$ **b** $2 - 3x + 4x^2 - 5x^3$
c $1 - \frac{3}{2}x + \frac{7}{8}x^2 - \frac{11}{16}x^3$ **d** $1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3$
e $-\frac{3}{2} + \frac{7}{4}x - \frac{7}{8}x^2 + \frac{7}{16}x^3$ **f** $1 - 2x + \frac{3}{2}x^2 - x^3$
g $3 + 4x + 7x^2 + 16x^3$

5 $1 - 4x - 8x^2 - 32x^3$, 4.7958

6 $1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3$, 3.332 22

7 $1 - 4x - 24x^2 - 224x^3$, 2.499 00, six

Chapter 15

Q1 a $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} 3 \\ -3 \end{pmatrix}$ **c** $\begin{pmatrix} -7 \\ -1 \end{pmatrix}$ **d** $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ **e** $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$

Q2 a $\sqrt{20}, 63.4°$ **b** $\sqrt{18}, -45°$
c $\sqrt{50}, -171.9°$ **d** $2, 90°$ **e** $3, 0°$

Q3 $(2, 4)$ **Q4** $(2, 3, 4)$

Q5 $\overline{AB} = \overline{DC} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}$ **Q6** $\frac{1}{13}\begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix}$

Q8 $2x + 3y + z = 5$ **Q9** $x + 2y + z = 8$ **Q10** -75

Q11 $x_1x_2 + y_1y_2 + z_1z_2 = 0$

Q12 101° **Q14** 76.7°, 72.1°, 22.6°

Exercise 15a, page 181

1 a $\begin{pmatrix}6\\10\end{pmatrix}$ b $\begin{pmatrix}12\\-18\end{pmatrix}$ c $\begin{pmatrix}-4\\6\end{pmatrix}$ d $\begin{pmatrix}2\\-3\end{pmatrix}$ e $\begin{pmatrix}7\\-1\end{pmatrix}$

f $\begin{pmatrix}18\\-8\end{pmatrix}$ g $\begin{pmatrix}-1\\11\end{pmatrix}$ h $\begin{pmatrix}1\\27\end{pmatrix}$

2 a 5, 53.1° b 13, 112.6°
c 10, −90° d $\sqrt{2}$, −45°

3 $\begin{pmatrix}8.66\\5\end{pmatrix}$ 4 $-4.33\mathbf{i} + 2.5\mathbf{j}$

5 a $(12, 11\frac{1}{2})$ b (21, 16) c (−21, −5)
6 a $\frac{1}{2}(\mathbf{c} - \mathbf{a})$ b $\mathbf{c} - \mathbf{a}$ c $\frac{1}{2}(\mathbf{c} - \mathbf{a})$
7 $\frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c})$

Exercise 15b, page 183

1 a $\begin{pmatrix}6\\2\end{pmatrix}$ b $\begin{pmatrix}4\frac{2}{3}\\3\end{pmatrix}$ c $\begin{pmatrix}34\\-19\end{pmatrix}$ d $\begin{pmatrix}5.2\\2.6\end{pmatrix}$

e $\begin{pmatrix}8.4\\0.2\end{pmatrix}$ f $\frac{1}{m+n}\begin{pmatrix}2n+10m\\5n-m\end{pmatrix}$

2 a $\begin{pmatrix}-4\\-6\end{pmatrix}$ b $\begin{pmatrix}-5\\-3\end{pmatrix}$ c $\begin{pmatrix}17\\-69\end{pmatrix}$ d $\begin{pmatrix}-4.6\\-4.2\end{pmatrix}$

e $\begin{pmatrix}-2.2\\-11.4\end{pmatrix}$ f $\frac{1}{m+n}\begin{pmatrix}-7n-m\\3n-15m\end{pmatrix}$

3 −2, 3; 1.5, −0.5 4 3, −2; $\frac{2}{3}, \frac{1}{3}$

5 $\begin{pmatrix}5\\-5\sqrt{3}\end{pmatrix}$, $\begin{pmatrix}20+5t\\15-5\sqrt{3}t\end{pmatrix}$, $\sqrt{3}$, $20+5\sqrt{3}$

6 2, −1 7 2:3
8 $-\frac{5}{12}\mathbf{b} + \frac{2}{3}\mathbf{c}$; $(\frac{3}{4} - \frac{5}{12}t)\mathbf{b} + \frac{2}{3}t\mathbf{c}$; $\overline{OM} = \frac{6}{5}\overline{OC}$; $-\frac{1}{6}$
9 $\frac{2}{3}\mathbf{b}$, $\frac{3}{5}\mathbf{a} + \frac{2}{5}\mathbf{b}$, $\frac{1}{3}$

Exercise 15c, page 190

1 a $\begin{pmatrix}2\\6\\2\end{pmatrix}$ b $\begin{pmatrix}-2\\0\\0\end{pmatrix}$ c $\begin{pmatrix}4\\3\\0\end{pmatrix}$ d $\begin{pmatrix}-3\\4\\-6\end{pmatrix}$ e $\begin{pmatrix}2k\\0\\-2k\end{pmatrix}$

2 a $\begin{pmatrix}2\\3\\3\end{pmatrix}$ b $\begin{pmatrix}4\\0\\4\end{pmatrix}$ c $\begin{pmatrix}4\\2.5\\3\end{pmatrix}$ d $\begin{pmatrix}3.5\\6\\4\end{pmatrix}$ e $\begin{pmatrix}2k\\2k\\2k\end{pmatrix}$

3 a $\begin{pmatrix}11\\30\\12\end{pmatrix}$ b $\begin{pmatrix}-5\\0\\4\end{pmatrix}$ c $\begin{pmatrix}22\\16\\3\end{pmatrix}$ d $\begin{pmatrix}-10\\24\\-23\end{pmatrix}$ e $\begin{pmatrix}11k\\2k\\-7k\end{pmatrix}$

4 $x + y + z = 3$ 5 $3x - 3y + z = 1$ 6 (7, 4, 9)
8 (4, 5, 10) 10 3:1

11 $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\2\\4\end{pmatrix} + t\begin{pmatrix}2\\-3\\1\end{pmatrix}$

Exercise 15d, page 193

1 a 0 b 12, 15 c 90° 2 44, 64.4° 5 $\frac{3}{2}\mathbf{b} - \frac{1}{2}\mathbf{c}$

Chapter 16

Q1 a sin 10° b −tan 60° c −cos 20° d −sin 50°
e cos 20° f −sin 35° g tan 40° h −cos 16°
i −cosec 50° j −tan 37° k −cos 50° l −sin 70°
m −tan 50° n cot 20° o cos 67° p sin 50°
q −sec 38° r −cot 24° s −cosec 53° t −sec 8°
Q3 360°, 180°
Q4 a $\frac{1}{2}$ b $\sqrt{3}/2$ c $1/\sqrt{2}$ d $1/\sqrt{3}$ e 2
f $2/\sqrt{3}$ g 1 h $\sqrt{2}$
Q5 a cot θ b cosec θ c −cosec θ d −tan θ
e sec θ f −cosec θ g −sin θ h sin θ
i −tan θ j −cos θ k −cos θ l cosec θ

Exercise 16a, page 197

1 a 0 b 0 c −1 d −1 e $\frac{1}{2}$ f $-\sqrt{3}/2$
g $-\sqrt{3}$ h $\sqrt{3}/2$ i $-\sqrt{3}/2$ j $1/\sqrt{2}$ k $-1/\sqrt{2}$
l $-1/\sqrt{2}$ m $-\sqrt{3}$ n 1 o $1/\sqrt{3}$
3 360° 4 180° 5 a 180° b 720° c 240°
d 360° e 360°
6 a 240° b 225° c none d 230°, 310° e 306.9°
f 300° g 240°, 360° h 270°, 330°
7 a 30°, 150°, 210°, 330° b 30°, 150°, 210°, 330°
c 15°, 75°, 195°, 255° d $67\frac{1}{2}°, 157\frac{1}{2}°, 247\frac{1}{2}°, 337\frac{1}{2}°$
e 10°, 110°, 130°, 230°, 250°, 350°
f 90°, 210°, 330° g 45°, 135°, 225°, 315°
h 35.3°, 144.7°, 215.3°, 324.7°
i 15°, 45°, 75°, … 345°
j 37.8°, 142.2°, 217.8°, 322.2°
k 11.6°, 48.4°, 191.6°, 228.4°
l 23.9°, 83.9°, 143.9°, 203.9°, 263.9°, 323.9°
8 a −180°, −45°, 0°, 135°, 180° b ±60°, ±90°
c 0°, ±180°, −19.5°, −160.5° d −150°, −30°, 90°
e ±120°, ±180° f ±60°, ±90°, ±120° g 0°, ±180°
h ±45°, ±135° i ±90°, 11.5°, 168.5°
j ±40.9°, ±139.1° k ±90°, 41.8°, 138.2°
l −104°, −45°, 76.0°, 135° m 23.6°, 156.4°
n ±109.5°
9 (Maxima first), a 1, 90°; −1, 270° b 3, 0°; −3, 180°
c 2, 0°; −2, 360° d $\frac{1}{2}$, 135°; $-\frac{1}{2}$, 45°
e 3, 270°; −1, 90° f 5, 0°; 1, 60° g 1, 270°; $\frac{1}{3}$, 90°
h 1, 0°; $\frac{1}{7}$, 180° i −1, 120°; 1, 0° j no max.; 0, 0°
k $\frac{1}{2}$, 90°; no min. l none m none
10 a, c, d, e, g
12 a 180° b 1080° c 60° d 360° e 540°

Exercise 16b, page 200

1 a $\cos\theta$ **b** $\tan\theta$ **c** $\cos\theta\cot\theta$
2 a $\sin\theta$ **b** $\tan\theta$ **c** $\operatorname{cosec}\theta\cot\theta$
3 a $\sec\theta$ **b** $\sec^2\theta\tan\theta$ **c** $\sin\theta$
4 a $\cot\theta$ **b** $\cos\theta$ **c** $\operatorname{cosec}\theta\tan^2\theta$
5 a $a^2\cos^2\theta$ **b** $\frac{1}{a}\sec\theta$ **c** $a\cos\theta\cot\theta$
6 a $b^2\operatorname{cosec}^2\theta$ **b** $b^2\cot\theta\operatorname{cosec}\theta$ **c** $\frac{1}{b}\sin\theta\cos\theta$
7 a $a^2\tan^2\theta$ **b** $\frac{1}{a}\cot\theta$ **c** $\sin\theta$
8 0°, 60°, 300°, 360° **9** 270°
10 45°, 63.4°, 225°, 243.4°
11 26.6°, 135°, 206.6°, 315° **12** 60°, 300°
13 30°, 41.8°, 138.2°, 150°
14 a $\pm\frac{4}{5}$ **b** $\pm\frac{3}{4}$ **15** $\frac{15}{17}, -\frac{8}{15}$ **16 a** $-\frac{25}{24}, -\frac{7}{25}$
32 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **33** $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ **34** $\frac{b^2}{y^2}-\frac{x^2}{a^2}=1$
35 $(x-1)^2+(y-1)^2=1$ **36** $x(y-b)=ac$
37 $\frac{a^2}{x^2}+\frac{b^2}{y^2}=1$ **38** $y^2(x-1)^2+y^2=1$ **39** $x^2+y^2=2$
40 $xy=1$ **47** $\frac{4}{(x+y)^2}-\frac{4}{(x-y)^2}=1$

Chapter 17

Q3 a 0°, 112.6°, 360° **b** 53.1°, 323.1°
c 48.4°, 205.3° **d** 119.6°, 346.7°

Exercise 17a, page 203

1 a $\frac{1}{4}\sqrt{2}(\sqrt{3}+1)$ **b** $\frac{1}{4}\sqrt{2}(\sqrt{3}+1)$ **c** $\frac{1}{4}\sqrt{2}(\sqrt{3}+1)$
d $\frac{1}{4}\sqrt{2}(1-\sqrt{3})$ **e** $-\frac{1}{4}\sqrt{2}(\sqrt{3}+1)$ **f** $\frac{1}{4}\sqrt{2}(\sqrt{3}-1)$
g $\frac{1}{4}\sqrt{2}(\sqrt{3}-1)$ **h** $\frac{1}{4}\sqrt{2}(\sqrt{3}-1)$
2 a $\frac{56}{65}$ **b** $\frac{33}{65}$ **c** $\frac{33}{56}$ **3 a** $\frac{63}{65}$ **b** $-\frac{63}{16}$ **c** $-\frac{33}{56}$
4 a $\frac{56}{65}$ **b** $\frac{56}{33}$ **c** $-\frac{63}{65}$ **5** $\frac{1}{3}$ **6** -2 **7** 45° **8** 135°
9 a $\cos(x+60°)=\sin(30°-x)$
b $\cos(45°-x)=\sin(45°+x)$ **c** $\tan(x+60°)$
d sin 26° **e** sec 39° **f** cos 15° = sin 105° = sin 75°
10 a $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{1}{3}\sqrt{3}$ **d** 0 **e** $\frac{1}{2}$ **f** $\frac{1}{2}\sqrt{2}$
g $\frac{1}{3}\sqrt{3}$ **h** $\frac{1}{2}\sqrt{6}$
11 2 **12** $\frac{12}{31}$ **13 a** $\frac{1}{3}$ **b** 1 **c** $-\frac{7}{4}$ **d** $2-\sqrt{3}$
16 a 9.9°, 189.9° **b** $157\frac{1}{2}°, 337\frac{1}{2}°$
c 49.1°, 229.1° **d** 56.5°, 236.5°

Exercise 17b, page 205

1 sin 34° **2** tan 60° **3** cos 84° **4** $\sin\theta$
5 cos 45° **6** $\tan\theta$ **7** cos 30° **8** $\sin 4A$
9 $\cos\theta$ **10** $\cos 6\theta$ **11** $\frac{1}{2}\tan 4\theta$ **12** $\frac{1}{2}\sin 2x$
15 2 cot 40° **14** $2\operatorname{cosec} 2\theta$ **15** $\cos\theta$

Exercise 17c, page 205

1 a $\frac{1}{2}$ **b** 1 **c** $-\frac{1}{2}\sqrt{3}$ **d** $-\frac{1}{2}\sqrt{2}$ **e** $\frac{1}{2}\sqrt{2}$
f $2\sqrt{3}$ **g** 1 **h** $2\sqrt{2}$
2 a $\pm\frac{24}{25}, \frac{7}{25}$ **b** $\pm\frac{120}{169}, \frac{119}{169}$ **c** $\pm\frac{1}{2}\sqrt{3}, -\frac{1}{2}$
3 a $-\frac{24}{7}$ **b** $\frac{240}{161}$ **c** $\pm\frac{120}{119}$
4 a $\pm\frac{3}{4}, \pm\frac{1}{4}\sqrt{7}$ **b** $\pm\frac{4}{5}, \pm\frac{3}{5}$ **c** $\pm\frac{12}{13}, \pm\frac{5}{13}$
5 a $\frac{1}{3}, -3$ **b** $\frac{1}{2}, -2$ **c** $-\frac{2}{3}, \frac{3}{2}$
6 $\sqrt{2}-1$ **7** 90°, 120°, 240°, 270°
8 0°, 180°, 360°; 60°, 300° **9** 30°, 150°; 270°
10 56.4°, 123.6°; 270° **11** 30°, 150°; 90°, 270°
12 0°, 180°, 360°; 85.2°, 274.8°
13 0°, 180°, 360°; 120°, 240°; 36.9°, 323.1°
14 0°, 180°, 360°; 30°, 150°, 210°, 330°
15 45°, 225°; 121.0°, 301.0°
16 18.4°, 161.6°, 198.4°, 341.6°
17 a $y=2x^2-1$ **b** $2y=3(2-x^2)$
c $y(1-x^2)=2x$ **d** $x^2y=8-x^2$

Exercise 17d, page 208

1 90°, 330° **2** 94.9°, 219.9° **3** 114.3°, 335.7°
4 204.6°, 351.7° **5** 72.6°, 319.3° **6** 76.7°, 209.6°
7 28.1°, 208.1°; 159.5°, 339.5°
8 0°, 180°, 360°; 45°, 225°
10 max. 2, 330°; min. −2, 150°
11 max. $\sqrt{13}$, 33.7°; min. $-\sqrt{13}$, −146.3° **12** 5, 53.1°
13 max. $\sqrt{5}$, 63.4°; min. $-\sqrt{5}$, −116.6°
14 $\sqrt{2}$, 45°; $-\sqrt{2}$, 225° **15** 5, 126.9°; −5, 306.9°
16 2, 60°; −2, 240° **17** 17, 298.1°; −17, 118.1°
18 $\sqrt{37}$, 170.5°; $-\sqrt{37}$, 350.5°
19 1, 240°; −1, 60° **20** 5, 53.1°; −5, 233.1°

Exercise 17e, page 208

1 $\cos(x+y)-\cos(x-y)$
2 $\cos(x+y)+\cos(x-y)$
3 $\cos 4\theta+\cos 2\theta$ **4** $\cos 2S-\cos 2T$
5 $\cos 2x-\cos 8x$ **6** $\cos 2x+\cos 2y$
7 $\cos A+\cos B$ **8** $\cos B-\cos C$
9 $\cos 2x$ **10** $\cos 4x+\cos 60°$

Exercise 17f, page 209

1 $\sin(x+y)+\sin(x-y)$ **2** $\sin(x+y)-\sin(x-y)$
3 $\sin 4\theta+\sin 2\theta$ **4** $\sin 2S+\sin 2T$ **5** $\sin 8x-\sin 2x$
6 $\sin 2x-\sin 2y$ **7** $\sin 2x-\sin 6x$ **8** $\sin A+\sin B$
9 $\sin A-\sin B$ **10** $\sin R-\sin S$

Exercise 17g, page 210

1 $2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$ **2** $2\sin 4x\cos x$
3 $2\cos(y+z)\sin(y-z)$ **4** $2\cos 6x\cos x$
5 $-2\sin\frac{3}{2}A\sin\frac{1}{2}A$ **6** $2\cos 3x\sin x$
7 $2\sin 4A\sin A$ **8** $2\sin 6\theta\cos\theta$ **9** $\sqrt{3}\sin x$
10 $\sqrt{2}\cos(y-35°)$ **11** $-2\cos 4\theta\sin\theta$ **12** $-\sin x$

13 $-2\sin x \sin \frac{1}{2}x$ 14 $2\sin 2x \cos 80°$
15 $2\cos(45° - \frac{1}{2}x + \frac{1}{2}y)\cos(45° - \frac{1}{2}x - \frac{1}{2}y)$
16 $2\cos(45° - \frac{1}{2}A + \frac{1}{2}B)\cos(45° - \frac{1}{2}A - \frac{1}{2}B)$
17 $2\sin(\frac{3}{2}x + 45°)\cos(\frac{3}{2}x - 45°)$
18 $2\sin(x + 45°)\cos(x - 45°)$
19 $2\cos(45° - \frac{1}{2}A + \frac{1}{2}B)\sin(45° - \frac{1}{2}A - \frac{1}{2}B)$
20 $2\cos(30° + \theta)\cos(30° - \theta)$

Exercise 17h, page 211

14 30°, 90°, 150°, 210°, 270°, 330°; 45°, 135°, 225°, 315°
15 0°, 120°, 240°, 360°; 72°, 144°, 216°, 288°
16 0°, 180°, 360°; 45°, 135°, 225°, 315°
17 0°, 72°, 144°, 180°, 216°, 288°, 360° 18 175°, 355°
19 45°, 135°, 225°, 315° 20 25°, 205

Chapter 18

Q1 54.1 Q2 6.95
Q3 a 6.49(5) b 72.4 c 32.2 d 43.8 e 76.3
f 123 g 32 600
Q4 a 6 deg/s b 1 rev/min
Q5 a 3000 deg/s b $2\frac{1}{7}$ deg/h
Q6 $120° + 360n°$, or $240° + 360n°$ Q7 $\frac{1}{4}\pi + n\pi$

Exercise 18a, page 215

1 a $A = 48°, b = 13.8, c = 15.4$
b $B = 56.1°, a = 6.53, c = 5.04$
c $C = 45.1°, a = 231, b = 213$
2 a $B = 95°, a = 1.40, c = 1.80$
b $B = 19.7°, b = 4.63, c = 8.29$
c $A = 32.7°, b = 244, c = 172$
3 a $B = 59.1°, A = 72.6°, a = 19.6$;
or $B = 120.9°, A = 10.9°, a = 3.87$
b $C = 26.7°, A = 24.3°, a = 4.18$
c $B = 55.5°, C = 96.25°, c = 17.9$;
or $B = 124.5°, C = 27.25°, c = 8.22$
4 a $A = 38.2°, B = 81.8°, C = 60°$
b $A = 54.6°, B = 78.1°, C = 47.2°$
c $A = 64.2°, B = 43.5°, C = 72.4°$
5 a $a = 13, B = 32.2°, C = 87.8°$
b $b = 11.7, A = 72.3°, C = 54.7°$
c $c = 7.60, A = 82.6°, B = 54.2°$
6 a $A = 29.5°, B = 38.0°, C = 112.4°$
b $A = 17.9°, B = 120°, C = 42.1°$
c $A = 35.8°, B = 49.3°, C = 94.9°$
7 a $A = 11.6°, B = 73°, C = 48.4°$
b $a = 17.4, B = 33.8°, C = 41.9°$
c $A = 31.2°, B = 44.6°, c = 58.0$

8 1.43 km 9 25.8 m 10 1.0°
11 347.3°, 3.64 n.mi. 12 200 m

Exercise 18b, page 217

1 a 90° b 45° c 60° d 120° e 30°
f 270° g 450° h 720° i 900° j 240°
k 630° l 135°
2 a 2π b $\frac{1}{2}\pi$ c $\frac{1}{4}\pi$ d $\frac{1}{12}\pi$ e $\frac{1}{3}\pi$ f $\frac{2}{3}\pi$
g $\frac{5}{3}\pi$ h $\frac{3}{2}\pi$ i 3π j $\frac{1}{6}\pi$ k $\frac{5}{6}\pi$ l $\frac{5}{2}\pi$
3 8 cm 4 9.6 cm 5 6 cm 6 $\frac{4}{5}$ rad
7 3 cm^2 8 4 rad 9 12 cm 10 4 cm^2

Exercise 18c, page 217

1 a $\frac{1}{8}\pi$ b 6π c $\pi/900$ d $5\pi/24$
2 a 72° b 5° c 105° d 630°
3 2.705 cm 4 $3/\pi$ 5 1.2 rad, 68.8° 6 6.43 cm
7 4.03 cm^2 8 a 151 cm^2 b 62.4 cm^2 c 88.4 cm^2
9 24.1 cm^2 10 22.4 cm^2 11 $\frac{1}{2}r^2(2\pi - \theta + \sin\theta)$

Exercise 18d, page 219

1 a $\frac{1}{60}$ rev/min b $\frac{1}{10}$ deg/s c $\pi/1800$ rad/s
2 a 1200 deg/s b $20\pi/3$ rad/s
3 a 1536 rev/min b 161 rad/s
4 0.262 rad/h 5 a 100π rad/s b 65π cm/s
6 a 3.89 rad/s b 15.6 cm/s
7 a 40π rad/s b 1.57 m/s
8 a 35.2 rad/s b 336 rev/min
9 1600 rev/min 10 128 rad/s
11 1.99×10^{-7} rad/s, 30 km/s 12 4.8 km/h

Exercise 18e, page 221

1 $45° + n360°$, or $135° + n360°$ 2 $n360°$
3 $60° + n180°$ 4 $270° + n360°$
5 $120° + n360°$, or $240° + n360°$ 6 $150° + n180°$
7 $\pi/3 + 2n\pi$, or $5\pi/3 + 2n\pi$ 8 $3\pi/4 + n\pi$
9 $\pi/12 + n\pi$, or $5\pi/12 + n\pi$
10 $\pi/6 + 2n\pi$, or $11\pi/6 + 2n\pi$; $5\pi/6 + 2n\pi$, or $7\pi/6 + 2n\pi$
11 $\pi/3$ 12 $\pi/4$ 13 $\pi/4$ 14 $-\pi/6$ 15 $5\pi/6$
16 $-\pi/4$ 17 $-\pi/2$ 18 π 19 0 20 $\pi/2$

Chapter 19

Q1 a $1\frac{1}{2}$ b 2 c $\frac{1}{2}$ d $\frac{1}{2}$ e $\sin\alpha$ f $\cos\alpha$
g $\frac{2}{9}$ h 2 i $\sec^2\alpha$
Q3 $2\cos\frac{1}{2}(A + B)\sin\frac{1}{2}(A - B)$
Q6 a $-3\sin 3x$ b $2\sin x\cos x = \sin 2x$
c $4\cos 2x$ d $-3\cos^2 x\sin x$

Exercise 19a, page 224

1 a $-2\sin 2x$ **b** $6\cos 6x$ **c** $-3\sin(3x-1)$
d $2\cos(2x-3)$ **e** $15\sin 5x$ **f** $8\cos 4x$
g $-6\cos\frac{3}{2}x$ **h** $\cos\frac{1}{2}(x+1)$ **i** $2x\cos x^2$

2 a $-\frac{1}{3}\cos 3x+c$ **b** $\frac{1}{3}\sin 3x+c$ **c** $-\frac{1}{2}\cos 4x+c$
d $\sin 2x+c$ **e** $\frac{1}{12}\cos 6x+c$ **f** $\frac{3}{2}\sin 4x+c$
g $-\frac{1}{2}\cos(2x+1)+c$ **h** $\frac{3}{2}\sin(2x-1)+c$
i $-\frac{4}{3}\cos\frac{1}{2}x+c$

3 a $2\sin x\cos x=\sin 2x$ **b** $-8\cos x\sin x=-4\sin 2x$
c $-3\cos^2 x\sin x$ **d** $6\sin^2 x\cos x$
e $-12\cos^3 x\sin x$ **f** $\dfrac{\cos x}{2\sqrt{(\sin x)}}$ **g** $\dfrac{-\sin x}{2\sqrt{(\cos x)}}$
h $-6\cos 3x\sin 3x=-3\sin 6x$
i $4\sin 2x\cos 2x=2\sin 4x$ **j** $-18\sin^2 3x\cos 3x$
k $24\sin^3 2x\cos 2x$ **l** $\dfrac{\cos 2x}{\sqrt{(\sin 2x)}}$

4 a $\cos x-x\sin x$ **b** $\sin 2x+2x\cos 2x$
c $x(2\sin x+x\cos x)$ **d** $\cos^2 x-\sin^2 x=\cos 2x$
e $(x\cos x-\sin x)/x^2$ **f** $-(2x\sin 2x+\cos 2x)/x^2$
g $(\sin x-x\cos x)/\sin^2 x$
h $x(2\cos x+x\sin x)/\cos^2 x$
i $\sec^2 x$ **j** $-\operatorname{cosec}^2 x$ **k** $\sec x\tan x$
l $-\operatorname{cosec} x\cot x$

5 a 1 m **b** 2 m/s^2 **c** 0.983 s
6 a $\frac{1}{3}\pi$ s **b** $-\frac{3}{2}\sqrt{3}$ cm/s **c** $-5, 3\frac{3}{4}, -3$ cm/s^2
7 a 5 **b** -20 **8 a** 0.841 **b** $\frac{5}{3}\sqrt{5}$ **c** $-\frac{17}{3}\sqrt{5}$
9 $\frac{2}{3}$ **10** 2π **11** $\frac{1}{3}\pi+\frac{1}{2}\sqrt{3}, \frac{1}{8}\pi^2+1$
12 $\sqrt{3}-\frac{1}{3}\pi, 2-\frac{1}{8}\pi^2$ **14** $\frac{1}{2}(1+\cos 2x)$ **17** $\frac{1}{2}$

Exercise 19b, page 226

1 a $2\sec^2 2x$ **b** $-3\operatorname{cosec}^2 3x$ **c** $6\sec 2x\tan 2x$
d $-\operatorname{cosec}\frac{1}{2}x\cot\frac{1}{2}x$ **e** $-2\sec^2(2x+1)$
f $\sec(3x-2)\tan(3x-2)$ **g** $6\operatorname{cosec}^2(3x+2)$
h $-2x\operatorname{cosec}^2 x^2$ **i** $(\sec^2\sqrt{x})/(2\sqrt{x})$

2 a $2\tan x\sec^2 x$ **b** $2\sec^2 x\tan x$
c $-6\cot^2 x\operatorname{cosec}^2 x$ **d** $-6\operatorname{cosec}^2 x\cot x$
e $-4\sec^2 2x\tan 2x$ **f** $-3\operatorname{cosec}^2 3x\cot 3x$
g $\sec^3 2x\tan 2x$ **h** $8\operatorname{cosec}^4 x\cot x$
i $(\sec^2 x)/(2\sqrt{\tan x})$

3 a $\tan x+x\sec^2 x$ **b** $\sec x(\sec^2 x+\tan^2 x)$
c $x(2\cot x-x\operatorname{cosec}^2 x)$ **d** $3\operatorname{cosec} x(1-x\cot x)$
e $-\operatorname{cosec} x(\operatorname{cosec}^2 x+\cot^2 x)$ **f** $(x\sec^2 x-\tan x)/x^2$
g $\sec x(x\tan x-2)/x^3$ **h** $x\sin x$ **i** $2x\sec^2 x\tan x$

4 a $\frac{1}{2}\tan 2x+c$ **b** $3\sec x+c$ **c** $2\cot\frac{1}{2}x+c$
d $-\frac{1}{6}\operatorname{cosec} 3x+c$ **e** $\sec^2 x+c$, *or* $\tan^2 x+c$
f $\tan x+c$ **g** $\sec x+c$ **h** $-\frac{1}{2}\cot 2x+c$
i $-\frac{1}{2}\operatorname{cosec} 2x+c$

5 $1-\frac{1}{4}\pi$ **6** 2π **7 a** $2\sqrt{3}$ **b** $5\sqrt{5}$ **c** $5\sqrt{3}$
9 $\cot^2 x=\operatorname{cosec}^2 x-1, -\cot x-x+c$

Chapter 20

Q2 $4x-6y-13=0$
Q4 a $y=x\pm\sqrt{7}$ **b** $y=\sqrt{3}x\pm\sqrt{13}$
Q5 a $(0, 0)$ **b** $(0, 0), (3, 6)$

Exercise 20a, page 228

1 $x^2+y^2=25$ **2** $x^2+y^2-6x-2y+6=0$
3 $4x-10y+29=0$ **4** $5x-3y-4=0$
5 $x+1, y^2=2x+1$ **6** $x^2=4y$
7 $2x^2+2y^2-x-1=0$
8 $3x^2+3y^2+36x-38y+159=0$
9 $3x^2-y^2=48$ **10** $3x^2+4y^2=48$
11 $x^2+y^2=9$ **12** $y^2=4ax$
13 $3x^2+4y^2=12$
14 $y=0$ **15** $2x+3y-13=0$

Exercise 20b, page 230

1 a $(y-2)(y+5)+(x+3)(x-4)=0$
b $(y-1)(y-4)+(x-\frac{1}{2})(x+\frac{3}{2})=0$
c $y(y-a)+x(x-a)=0$
d $(y-y_1)(y-y_2)+(x-x_1)(x-x_2)=0$
2 a $x^2+y^2=36$ **b** $4x^2+y^2=64$ **3** $x^2+y^2=16$
4 $xy=3y+4x$ **5** $xy=3$ **6** $y=6x^2+1$
7 $y^2=8x+4$ **9** $x^2+y^2=4$
10 $4x^2+4y^2-8x+3=0$
11 $2y=x^2+x+2$ **12** $2x+3y-13=0$
13 $xy=2x+3y$ **14** $x^2-4xy+5y^2=4$
15 $y^2-xy-y+2x=0$

Exercise 20c, page 232

1 a $4x-y-4=0, x+4y-18=0$
b $4x-y-2=0, x+4y-9=0$
c $y+2=0, x+1=0$
d $x+y+1=0, x-y-3=0$
e $6x+y+4=0, x-6y+50=0$
f $x+y-4=0, x-y=0$
g $2x-3y+1=0, 3x+2y-5=0$
2 a $(\frac{1}{2}, 2)$ **b** $(2, -2)$ **c** $(6, \frac{2}{3})$ **d** $(-\frac{5}{2}, -\frac{9}{4})$
3 $(1, 0), (3, 0); 2x+y-2=0, x-2y-1=0;$
$2x-y-6=0, x+2y-3=0$
4 $5x-y-11=0, 3x+y+3=0$
5 $x+2y-1=0, x-2y+1=0, (0, \frac{1}{2})$
6 $(0, 0), (1, 1); x=0, 2y-x-1=0; y=0, y-2x+1=0$
7 $4y-x+48=0, (48, 0)$
8 $9x-y-27=0, 9x-y+5=0$ **9** $x+y\pm 4=0$
10 $0, 2; y=0, y-4x+4=0$
11 $x+4y-4c=0$ **13** $3x-8y\pm 10=0$
14 $x-y\pm 4=0$ **15** $n^2=a^2l^2+b^2m^2$

Chapter 21

Q1 $g = -a, f = -b, c = (a^2 + b^2 - r^2)$
Q2 **a** 0 **b** no real length
Q3 $X^2 + Y^2 - 1, X^2 + Y^2 - 6X - 8Y + 21;$
$3x + 4y - 11 = 0$
Q4 $x + y = 0$

Exercise 21a, page 235

1 **a** $x^2 + y^2 - 4x - 6y + 12 = 0$ **b** $x^2 + y^2 + 6x - 8y = 0$
c $9x^2 + 9y^2 - 12x + 6y + 1 = 0$ **d** $x^2 + y^2 + 10y = 0$
e $x^2 + y^2 - 6x + 7 = 0$
f $144x^2 + 144y^2 + 72x - 96y - 47 = 0$
2 **a** $1, (-2, 3)$ **b** $2, (1, 2)$ **c** $\frac{3}{2}, (\frac{3}{2}, 0)$
d $\frac{7}{2}, (-\frac{3}{2}, 2)$ **e** $\frac{1}{4}\sqrt{2}, (-\frac{1}{4}, -\frac{1}{4})$ **f** $1, (\frac{1}{3}, \frac{1}{2})$
g $\sqrt{(a^2 + b^2)}, (a, b)$ **h** $\sqrt{(g^2 + f^2 - c)}, (-g, -f)$
3 **a**, **d** if $a > 0$, **f** if $b = 0$, **g** if $c < 0$
4 $x^2 + y^2 - 4x - 2y - 15 = 0$
5 $(5, 3), \sqrt{10}; x^2 + y^2 - 10x - 6y + 24 = 0$
6 $x^2 + y^2 - 4x + 6y + 4 = 0$ **7** 4, 6
8 The y-axis is a tangent.
9 $x^2 + y^2 \pm 8x - 10y + 16 = 0$
10 $x^2 + y^2 - 4x - 4y + 4 = 0$
11 $(2, 1), x^2 + y^2 - 4x - 2y - 45 = 0$
12 $x^2 + y^2 - 16x + 8y - 5 = 0$
13 **a** $x^2 + y^2 + 4x - 2y = 0$ **b** $x^2 + y^2 - 10x - 8y + 28 = 0$
c $x^2 + y^2 - 2x - 49 = 0$
14 $(4, 0), 2$ **15** $(4, 1), 3$

Exercise 21b, page 237

1 **a** $3x - y = 0$ **b** $x - 4y + 17 = 0$ **c** $4x + y - 11 = 0$
d $3x + y - 8 = 0$ **e** $4x + 9y + 5 = 0$
2 **a** $\sqrt{10}$ **b** $\sqrt{15}$ **c** $\sqrt{29}$ **d** $2\sqrt{7}$
e $\sqrt{(x_1{}^2 + y_1{}^2 - a^2)}$ **f** $\sqrt{c}$
3 5 **4** $x - y - 1 = 0, x + y - 5 = 0$
5 $(23, 0), (0, 7\frac{2}{3}), 88\frac{1}{6}$ **6** $\sqrt{13}$
7 $\sqrt{(X^2 + Y^2 - 4)}, 2x - 5 = 0$
8 $2x + 3y - 6 = 0$ **10** $(7, 4)$

Exercise 21c, page 238

2 $(0, 0)$ **4** $2x - 5 = 0$

Chapter 22

Q1 **a** 1 **b** $\frac{2}{13}$ **c** $\frac{1}{2}\sqrt{26}$ **d** $3\sqrt{2}$ **e** $\frac{16}{17}\sqrt{34}$
f $\frac{2}{13}\sqrt{13}$ **g** $\frac{3}{5}a$ **h** $\frac{4}{5}q$ **i** $\frac{1}{13}(12X - 5Y + 7)$
j $\frac{1}{17}(8x_1 - 15y_1)$

Exercise 22a, page 239

1 **a** $y - x = -1$ **b** $y + 2x = -1$ **c** $2y - x = -12$
d $3y + x = 13$ **e** $5y + 7x = -9$ **f** $4y - 3x = 7$
g $6y + 5x = -39$ **h** $3y - 4x = 23$ **i** $yt - x = at^2$
j $y + tx = at^3 + 2at$ **k** $y \sin\theta + x \cos\theta = a$
l $t^2y + x = 2ct$
2 **a** $2x - 3y = -2$ **b** $3x + 4y = 0$ **c** $6x - 5y = -43$
d $2x + 3y = 7$ **e** $y + tx = k + th$
f $bx - ay = bx_1 - ay_1$ **g** $y - t^2x = c/t - ct^3$
3 **a** $x/3 + y/2 = 1$ **b** $y/2 - x = 1$ **c** $2x + 5y = 1$
d $4y - 3x = 1$
4 **a** $x/3 + y/2 = 1$ **b** $y/5 - x = 1$ **c** $3y/2 - 2x = 1$
5 $p \sec\alpha, p \operatorname{cosec}\alpha, x \cos\alpha + y \sin\alpha = p$
6 **a** $y - 3x + 9 = 0$ **b** $2y + x = 0$ **c** $5y - 2x - 3 = 0$
d $4y + 3x + 12 = 0$ **e** $y - 6x + 16 = 0$
f $4y - 9x - 3 = 0$ **g** $y = 2$ **h** $2y + x - 4 = 0$
i $3y - 4x - 13 = 0$ **j** $6y + x - 19 = 0$
k $y + x - 1 = 0$ **l** $y - t^2x = k - t^2h$

Exercise 22b, page 242

2 **a** $r = a$ **b** $\theta = \alpha$ **c** $r = a \sec\theta$ **d** $r = a \operatorname{cosec}\theta$
e $r = a \cos\theta$ **f** $r = 2a \sin\theta$ **g** $a^2 = r^2 + c^2 - 2cr \cos\theta$
h $r = 2a/(1 + \cos\theta)$
5 **a** $r = a$ **b** $r^2 = a^2 \sec 2\theta$ **c** $\theta = 0$ **d** $r = 2a/(1 + \cos\theta)$
e $r = 2 \sin\theta$ **f** $r^2 = 2c^2 \operatorname{cosec} 2\theta$
6 **a** $x^2 + y^2 = 4$ **b** $(x^2 + y^2 - ax)^2 = a^2(x^2 + y^2)$
c $x^2 + y^2 - ax = 0$ **d** $x^4 + x^2y^2 = a^2y^2$
e $(x^2 + y^2)^3 = 4a^2(x + y)^4$ **f** $4xy = c^2$
g $x^2 + y^2 = (l - ex)^2$ **h** $y^2 = 4ax$
7 **a** $1, 60°$ **b** $2\sqrt{2}, -45°$ **c** $2, \tan^{-1}\frac{4}{3}$
d $2, \tan^{-1}(-\frac{12}{5})$ **e** $\frac{1}{5}\sqrt{10}, \tan^{-1} 3$
f $c/\sqrt{(a^2 + b^2)}, \tan^{-1}(b/a)$

Exercise 22c, page 245

1 **a** $4\frac{1}{5}$ **b** $2\frac{1}{13}$ **c** $\frac{10}{17}\sqrt{17}$ **d** 0 **e** $\frac{38}{29}\sqrt{29}$ **f** $1\frac{1}{5}$
g $\frac{6}{41}\sqrt{41}$ **h** $\frac{5}{13}\sqrt{13}$ **i** p **j** $\frac{1}{13}(5X - 12Y + 1)$
k $\frac{22}{17}c$ **l** $\frac{1}{5}(4y_1 - 3x_1 + 2)$
2 **a** $3x - y - 2 = 0, x + 3y - 4 = 0$
b $7x - 7y + 4 = 0, x + y - 2 = 0$
c $17x + 17y - 4 = 0, 7x - 7y - 4 = 0$
d $x + (1 \pm \sqrt{2})y - 1 = 0$
3 **a** $8x - 4y + 17 = 0$ **b** $8y + 1 = 0$ **c** $4x + 12y + 5 = 0$
4 $4x^2 - 4xy + y^2 - 20x + 30y + 65 = 0$ **5** $4x + 3y - 24 = 0$
6 $7x^2 - 2xy + 7y^2 - 40x - 40y + 48 = 0$
7 $4y - 3x - 15 = 0, 4y - 3x + 35 = 0$ **8** $2x - 3y \pm 13 = 0$
9 $x^2 + y^2 - 4x - 14y + 49 = 0$ **10** $n^2 = a^2(l^2 + m^2)$

Exercise 22d, page 247

2 **a** $\frac{4}{3}, \frac{4}{3}$ **b** $\pm\frac{3}{2}, \pm 3a$ **c** $-2, -\frac{1}{3}$ **d** $60°, (\sqrt{3}/2)b$
3 **a** $(y - 2)^2 = x - 1$ **b** $x^3 = y^2$ **c** $xy = 1$
d $2x + y - 5 = 0$ **e** $y^2 = 4ax$ **f** $xy^2 = 1$

g $5x+y-13=0$ h $4x^2-9y^2=144$
i $4x^2+9y^2=36$ j $9x^2-16y^2=144$

4 a $x=t^4, y=t^5$ b $x=t-2, y=t^2-2t$
c $x=\dfrac{2}{t^2-1}, y=\dfrac{2t}{t^2-1}$ d $x=\dfrac{1}{1-t^3}, y=\dfrac{t}{1-t^3}$
e $x=\dfrac{3t}{1+t^3}, y=\dfrac{3t^2}{1+t^3}$

5 $3x-2y+1=0$ 6 $-\frac{4}{13}, -\frac{4}{3}$ 7 $(\frac{1}{2}t^2, \frac{3}{2}t)$, $2y^2=9x$
8 $y^2=8ax$ 9 $x=y(2x-1)^2$ 10 $(1, 1), (-1, -1), \sqrt{2}$
11 $(a, 2a)$, inflexion; $(4a, -4a)$, minimum
12 a $(p+q)y-2x=2pq$ b $pqy-x+(p+q)=0$
c $(p^2+pq+q^2)y-x=pq(p+q)$
d $(pq-1)y-2pqx+2(p+q)=0$
13 $(a, 2a), \frac{3}{2}\sqrt{2}a$ 14 $-1, 2; (1, -2), (4, 4)$
15 $(1, \pm 2), (4, \pm 4)$

Exercise 22e, page 248

1 a $2y+3x-1=0, 2x-3y-5=0$
b $y+4x=3, 16y-4x=31$
c $x+y+a=0, x-y-3a=0$
d $y+x+2c=0, y-x=0$
e $2y+x+9=0, 2x-y+3=0$
f $3\sqrt{3}y+2x-12=0, 6\sqrt{3}x-4y-5\sqrt{3}=0$

2 a $ty-2x-t^3=0, 2y+tx=6t^2+t^4$
b $ty-x-at^2=0, y+tx=2at+at^3$
c $y-tx+t^4=0, ty+x=3t^5+4t^3$
d $t^2y+x-2ct=0, y-t^2x=c/t-ct^3$
e $bx\cos t+ay\sin t=ab$,
$ax\sin t-by\cos t=\frac{1}{2}(a^2-b^2)\sin 2t$
f $bx\sec t-ay\tan t=ab, ax\sin t+by=(a^2+b^2)\tan t$

3 a $(p+q)y-2x=2pq, py-x=p^2$
b $y+pq(p+q)x=p^2+pq+q^2, y+2p^3x=3p^2$
c $pqy+x=c(p+q), p^2y+x=2cp$
d $bx\cos\frac{1}{2}(p+q)+ay\sin\frac{1}{2}(p+q)=ab\cos\frac{1}{2}(p-q)$,
$bx\cos p+ay\sin p=ab$

4 $2x+y-12a=0, (9a, -6a)$ 5 $(-\frac{1}{8}c, -8c)$ 6 $(-\frac{1}{2}, 4)$
7 $yt-x=at^2; 2, 4; 2y-x=4a, 4y-x=16a$
8 $y+x=2c, 9y+x=6c$
9 $y+2x=12a, y-4x+72a=0$ 10 $(-c/t^3, -ct^3)$

Exercise 22f, page 249

4 $yt-x=at^2, y+tx=2at+at^3$
17 $2, -1; y+2x=12a, y-x+3a=0$
18 $y-x-a=0, 4y-x-16a=0$

Chapter 23

Q1 a 4.123 b 6.325 c 9.220 d 9.798

Exercise 23a, page 253

1 a 3.46 b 5.48 c 7.07 d 8.66
2 2.33, 2.29, 2.29; $x^3-12=0$ 3 2.71
4 $x_{r+1}=(x_r^2+1)/5$; 0.208, 0.209, 0.209;
$x_{r+1}=5-1/x_r$; $0, -\infty, 5$
5 0.1001 6 1.93
7 $x_{r+1}=8-10/x_r$; $-2, 13, 7.23$; the sequence does not appear to converge, however it eventually converges to 6.45, the other root.
8 a 2.17, 2.15, 2.15 b 2.5, 1.6, 3.91; 2.15 9 2.19
10 6.54, 0.459

Exercise 23b, page 256

Nos. 1, 2, 3, 5, 6, 8, 10 (i.e. *not* 4, 7, 9).

Exercise 23c, page 258

1 4.74	2 3.28	3 1.90	4 5.15	5 3.58
6 0.771	7 4.15	8 3.70	9 2.09	10 1.04

Chapter 24

Q1 a $16x(2x^2+3)^3$ b $\dfrac{x-1}{\sqrt{(x^2-2x+1)}}$
c $-4(2x-1)^{-3}$ d $4\cos(4x-7)$
e $3\tan^2 x\sec^2 x$ f $-6\cos 3x\sin 3x$

Q2 a $\frac{1}{6}(x^2+1)^3+c$ b $\frac{1}{10}(2x+1)^5+c$
c $\frac{1}{7}x^7+\frac{3}{5}x^5+x^3+x+c$ d $-\frac{1}{6}\cos 3x+c$
e $\frac{2}{9}(x^3+1)^{3/2}+c$ f $\frac{1}{2}\tan^2 x+c$

Q3 a $\frac{1}{3}\cos^3 x-\cos x+c$
b $\sin x-\frac{2}{3}\sin^3 x+\frac{1}{5}\sin^5 x+c$

Q4 a $\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+c$ b $\frac{1}{5}\cos^5 x-\frac{1}{3}\cos^3 x+c$

Q5 $\frac{1}{3}\sec^3 x-\sec x+c$ Q6 $\frac{1}{12}\sin^3 4x+c$

Q7 $-\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x+c$

Q8 a $\frac{1}{15}(2x+1)^{3/2}(3x-1)+c$
b $\frac{1}{15}(2x+1)^{3/2}(3x-1)+c$
c $\frac{1}{504}(3x-2)^7(21x+2)+c$

Q9 a 45°, $\pi/4$ rad b 30°, $\pi/6$ rad c 45°, $\pi/4$ rad
d 60°, $\pi/3$ rad e 30°, $\pi/6$ rad f 0°, 0 rad
g 60°, $\pi/3$ rad h 30°, $\pi/6$ rad i 30°, $\pi/6$ rad
j 60°, $\pi/3$ rad k 90°, $\pi/2$ rad

Q10 a $\pi/9$ b $7\pi/18$ c $5\pi/6$ d $5\pi/3$ e $9\pi/4$

Q11 a 57.3° b 1.7° c 71.6° d 41.0° e 36°

Q12 a 1.29 b 0.927 c 0.784

Q13 a $3\cos u, \sin^{-1}\dfrac{x}{3}$ b $\cos u, \sin^{-1}5x$
c $2\cos u, \sin^{-1}\dfrac{3x}{2}$ d $\sqrt{7}\cos u, \sin^{-1}\dfrac{x}{\sqrt{7}}$
e $\cos u, \sin^{-1}\sqrt{3}x$ f $\sqrt{3}\cos u, \sin^{-1}\sqrt{\frac{2}{3}}x$

Q14 **a** $\sin^{-1}\dfrac{x}{2}+c$ **b** $\dfrac{1}{\sqrt{3}}\sin^{-1}\sqrt{3}x+c$

c $\dfrac{1}{3}\sin^{-1}\dfrac{3x}{4}+c$

Q16 $\tan^{-1}x+c$

Q17 **a** $9\sec^2 u, \tan^{-1}\dfrac{x}{3}$ **b** $\sec^2 u, \tan^{-1}2x$

c $25\sec^2 u, \tan^{-1}\dfrac{3x}{5}$ **d** $3\sec^2 u, \tan^{-1}\dfrac{x}{\sqrt{3}}$

e $\sec^2 u, \tan^{-1}\sqrt{5}x$ **f** $7\sec^2 u, \tan^{-1}\sqrt{\tfrac{3}{7}}x$

Q18 **a** $\tfrac{1}{2}\tan^{-1}\dfrac{x}{2}+c$ **b** $\tfrac{1}{4}\tan^{-1}4x+c$

c $\dfrac{1}{2\sqrt{3}}\tan^{-1}\dfrac{2x}{\sqrt{3}}+c$

Q19 **a** 0.866 **b** 0.479 **c** 2.57 **d** –0.990 **e** 1.04

Q20 **a** $\tfrac{1}{2}\pi$ **b** $-\tfrac{1}{4}\pi$ **c** $\tfrac{1}{6}\pi$ **d** 1.11

Exercise 24a, page 260

1 **a** $30x(5x^2-1)^2$ **b** $\dfrac{2-8x}{(2x^2-x+3)^3}$ **c** $\tfrac{2}{3}x(x^2+4)^{-2/3}$

d $-5\,\mathrm{cosec}^2\,5x$ **e** $-5\sin(5x-1)$ **f** $\tfrac{1}{3}\sin\tfrac{2}{3}x$

g $\dfrac{1}{2\sqrt{x}}\sec^2\sqrt{x}$ **h** $4\sec^2 2x\tan 2x$

i $-\tfrac{1}{2}\cot x\sqrt{\mathrm{cosec}\,x}$

2 **a** $\tfrac{1}{12}(x^2-3)^6+c$ **b** $\tfrac{1}{18}(3x-1)^6+c$

c $\tfrac{1}{4}x^4+\tfrac{4}{3}x^3+2x^2+c$ **d** $-\tfrac{1}{2}(x^2+1)^{-1}+c$

e $-\tfrac{1}{4}(x^2+2x-5)^{-2}+c$ **f** $\tfrac{1}{3}(x^2-3x+7)^3+c$

g $-\tfrac{1}{4}(4x^2-7)^{-1}+c$ **h** $\tfrac{2}{9}(3x^2-5)^{3/2}+c$

i $\tfrac{1}{7}x^7+\tfrac{1}{2}x^4+x+c$ **j** $\tfrac{2}{3}\sqrt{(x^3-3x)}+c$

k $-\tfrac{1}{2}(2x^2-4x+1)^{-1/2}+c$ **l** $\tfrac{8}{7}x^7-\tfrac{12}{5}x^5+2x^3-x+c$

3 **a** $\sin 3x+c$ **b** $-\tfrac{1}{2}\cos(2x+3)+c$ **c** $\tfrac{1}{2}\sin^2 x+c$

d $\tfrac{1}{6}\sin 2x+c$ **e** $-\tfrac{1}{9}\cos^3 3x+c$ **f** $\tfrac{1}{3}\tan^3 x+c$

g $\tfrac{1}{5}\sec^5 x+c$ **h** $\tfrac{2}{3}\sin^{3/2}x+c$ **i** $-\tfrac{1}{2}\cot x^2+c$

j $2\sin\sqrt{x}+c$ **k** $-\tfrac{1}{3}\mathrm{cosec}^3\,x+c$

4 **a** $\sin x-\tfrac{1}{3}\sin^3 x+c$

b $2\sin\tfrac{1}{2}x-\tfrac{4}{3}\sin^3\tfrac{1}{2}x+\tfrac{2}{5}\sin^5\tfrac{1}{2}x+c$

c $\tfrac{1}{6}\cos^3 2x-\tfrac{1}{2}\cos 2x+c$

d $\tfrac{1}{2}\sin(2x+1)-\tfrac{1}{6}\sin^3(2x+1)+c$

e $-\tfrac{1}{3}\cos^3 x+\tfrac{2}{5}\cos^5 x-\tfrac{1}{7}\cos^7 x+c$

f $\tfrac{1}{4}\sin^4 x-\tfrac{1}{6}\sin^6 x+c$

g $\tan x+\tfrac{1}{3}\tan^3 x+c$

h $\mathrm{cosec}\,x-\tfrac{1}{3}\mathrm{cosec}^3\,x+c$

i $\sec x-\tfrac{2}{3}\sec^3 x+\tfrac{1}{5}\sec^5 x+c$

5 **a** $\tfrac{1}{4}\sec^4 x+c$ **b** $\tfrac{1}{2}\tan^2 x+\tfrac{1}{4}\tan^4 x+k$

6 $A+\tfrac{1}{2}\sin^2 x, B-\tfrac{1}{2}\cos^2 x, C-\tfrac{1}{4}\cos 2x$

7 **a** $\tfrac{1}{2}(1-\cos x)$ **b** $\tfrac{1}{2}(1+\cos 6x)$

8 **a** $\tfrac{1}{2}x+\tfrac{1}{4}\sin 2x+c$ **b** $\tfrac{1}{2}x-\tfrac{1}{2}\sin x+c$

c $\tfrac{1}{2}x+\tfrac{1}{12}\sin 6x+c$

9 $\tfrac{1}{4}-\tfrac{1}{2}\cos 2x+\tfrac{1}{4}\cos^2 2x, \tfrac{1}{2}(1+\cos 4x)$

10 $\tfrac{3}{8}x+\tfrac{1}{4}\sin 2x+\tfrac{1}{32}\sin 4x+c$

11 **a** $\tfrac{1}{2}x-\tfrac{1}{4}\sin 2x+c$ **b** $\tfrac{1}{2}x+\tfrac{3}{4}\sin\tfrac{2}{3}x+c$

c $\tfrac{3}{8}x-\tfrac{1}{8}\sin 4x+\tfrac{1}{64}\sin 8x+c$

d $\tfrac{3}{8}x+\tfrac{1}{2}\sin x+\tfrac{1}{16}\sin 2x+c$

13 **a** $2\sqrt{2}\sin\dfrac{x}{2}+c$ **b** $-\dfrac{1}{\sqrt{2}}\mathrm{cosec}\,x+c$

c $\tfrac{1}{2}\sin^4 x+c$ **d** $-\dfrac{8}{3}\cos^3\dfrac{x}{2}+c$

14 **a** $2\sin 2x\cos x$ **b** $\sin 5x+\sin x$

c $-\tfrac{1}{10}\cos 5x-\tfrac{1}{2}\cos x+c$

15 **a** $\tfrac{1}{4}\cos 2x-\tfrac{1}{8}\cos 4x+c$ **b** $\tfrac{1}{2}\sin 2x+\sin x+c$

c $\tfrac{1}{6}\sin 3x-\tfrac{1}{10}\sin 5x+c$

Exercise 24b, page 262

1 **a** $\tfrac{1}{20}(4x-1)^{3/2}(6x+1)+c$ **b** $\tfrac{2}{375}(5x+2)^{3/2}(15x-4)+c$

c $\tfrac{1}{224}(2x-1)^7(14x+1)+c$ **d** $\tfrac{2}{3}(x+4)\sqrt{(x-2)}+c$

e $\tfrac{1}{30}(x-1)^5(5x+13)+c$

f $\tfrac{1}{168}(x-2)^6(21x^2+156x+304)+c$

g $\dfrac{x^2-4x+8}{x-2}+c$ **h** $\tfrac{1}{3}(x-6)\sqrt{(2x+3)}+c$

3 **a** $\tfrac{2}{135}(3x-4)^{3/2}(9x+8)+c, \tfrac{1}{9}(3x^2-4)^{3/2}+c$

b $\tfrac{1}{14}(x^2+5)^7+c, \tfrac{1}{56}(x+5)^7(7x-5)+c$

c $\tfrac{2}{3}(x+2)\sqrt{(x-1)}+c, \sqrt{(x^2-1)}+c$

4 **a** $\tfrac{1}{6}(2x^2+1)^{3/2}+c$ **b** $-\tfrac{1}{2}(x^3-x+4)^{-2}+c$

c $\tfrac{2}{15}(2x-1)^{3/2}(3x+1)+c$ **d** $\tfrac{1}{2}\sin 2x-\tfrac{1}{6}\sin^3 2x+c$

e $-\tfrac{2}{3}(\cos x)^{3/2}+c$ **f** $-\tfrac{1}{3}\cot^3 x+c$

g $\tfrac{1}{16}(4x^2-1)^4+c$ **h** $\tfrac{1}{2}\sqrt{(2x^2-5)}+c$

i $-2(8+x)\sqrt{(4-x)}+c$ **j** $-2\cos\sqrt{x}+c$

Exercise 24c, page 263

1 **a** $\tfrac{26}{15}$ **b** $\tfrac{1}{30}$ **c** $\sqrt{3}-\tfrac{2}{3}$ **d** $-\tfrac{7}{20}$ **e** $\tfrac{67}{48}$

2 **a** $\tfrac{7}{36}$ **b** $\tfrac{8}{15}$ **c** $\tfrac{1}{6}(4-\sqrt{2})$

3 **a** $1-\tfrac{1}{2}\sqrt{3}$ **b** $\tfrac{256}{15}$ **c** $-\tfrac{1}{10}$ **d** $\tfrac{4}{3}$

e $\tfrac{23}{108900}$ **f** 24.3 **g** $\tfrac{4}{3}$ **h** $\tfrac{74}{27}$ **i** $\tfrac{2}{3}$

4 $2\sqrt{2}-\sqrt{3}$ **5** $\tfrac{9}{8}$ **6** $\tfrac{1}{4}\pi^2$ **7** $\tfrac{1}{8}\pi$

Exercise 24d, page 265

1 **a** 45°, $\pi/4$ rad **b** 45°, $\pi/4$ rad **c** 17.3°, $\pi\sqrt{3}/18$ rad

d 60°, $\pi/3$ rad **e** 52.0°, $\pi\sqrt{3}/6$ rad **f** 15°, $\pi/12$ rad

g $67\tfrac{1}{2}$°, $3\pi/8$ rad **h** 15°, $\pi/12$ rad

2 **a** 0.559 **b** 1.05 **c** 0.0992 **d** 4.11

3 **a** 114.6° **b** 4.6° **c** 78.0° **d** 30°

4 **a** 0.927 **b** 0.588 **c** 1.12

5 **a** $4\cos u, \sin^{-1}\dfrac{x}{4}$ **b** $\cos u, \sin^{-1}3x$

c $3\cos u, \sin^{-1}\tfrac{2}{3}x$ **d** $\sqrt{10}\cos u, \sin^{-1}\dfrac{x}{\sqrt{10}}$

e $\cos u, \sin^{-1}\sqrt{6}x$ **f** $\sqrt{5}\cos u, \sin^{-1}\sqrt{\tfrac{3}{5}}x$

6 **a** $\sin^{-1}\frac{x}{5}+c$ **b** $\frac{1}{2}\sin^{-1}2x+c$ **c** $\frac{1}{3}\sin^{-1}\frac{3x}{2}+c$

d $\sin^{-1}\frac{x}{\sqrt{3}}+c$ **e** $\frac{1}{\sqrt{7}}\sin^{-1}\sqrt{7}x+c$

f $\frac{1}{\sqrt{3}}\sin^{-1}\sqrt{\frac{3}{2}}x+c$

7 **a** $16\sec^2 u, \tan^{-1}\frac{x}{4}$ **b** $\sec^2 u, \tan^{-1}3x$

c $4\sec^2 u, \tan^{-1}\frac{\sqrt{3}}{2}x$ **d** $2\sec^2 u, \tan^{-1}\frac{x}{\sqrt{2}}$

e $\sec^2 u, \tan^{-1}\sqrt{3}x$ **f** $5\sec^2 u, \tan^{-1}\sqrt{\frac{2}{5}}x$

8 **a** $\frac{1}{5}\tan^{-1}\frac{x}{5}+c$ **b** $\frac{1}{6}\tan^{-1}6x+c$

c $\frac{1}{4\sqrt{3}}\tan^{-1}\frac{\sqrt{3}}{4}x+c$ **d** $\frac{1}{\sqrt{5}}\tan^{-1}\frac{x}{\sqrt{5}}+c$

e $\frac{1}{\sqrt{6}}\tan^{-1}\sqrt{6}x+c$ **f** $\frac{1}{\sqrt{30}}\tan^{-1}\sqrt{\frac{10}{3}}x+c$

9 **a** $\frac{1}{3\sqrt{2}}\tan^{-1}\frac{\sqrt{2}}{3}x+c$ **b** $\frac{3}{\sqrt{5}}\sin^{-1}\frac{\sqrt{5}}{2}x+c$

c $\frac{1}{\sqrt{2}}\sin^{-1}\sqrt{\frac{2}{3}}x+c$ **d** $\frac{2}{\sqrt{15}}\tan^{-1}\sqrt{\frac{5}{3}}x+c$

10 **a** $\frac{1}{6}\pi$ **b** $\frac{1}{4}\pi$ **c** π **d** $\frac{1}{12}\pi$ **e** $\frac{1}{18}\pi$ **f** $\frac{1}{6}\pi$

11 **a** **i** $\sin^{-1}\frac{x}{3}+c$,

ii $-\cos^{-1}\frac{x}{3}+c=-\cos^{-1}\frac{x}{3}+\frac{\pi}{2}+k=\sin^{-1}\frac{x}{3}+k$

b **i** $\frac{1}{3}\pi$, **ii** $\frac{1}{3}\pi$

12 **a** $\sin^{-1}\frac{x+1}{2}+c$ **b** $\frac{1}{3}\tan^{-1}\frac{x-3}{3}+c$

13 **a** $\frac{1}{5}\tan^{-1}\frac{x+3}{5}+c$ **b** $\sin^{-1}\frac{x-1}{2}+c$

c $\frac{1}{\sqrt{15}}\tan^{-1}\frac{(x-2)\sqrt{3}}{\sqrt{5}}+c$ **d** $\frac{1}{\sqrt{3}}\sin^{-1}\frac{x+1}{\sqrt{3}}+c$

14 **a** **i** $(x-3)^2+7$, **ii** $3(x-2)^2+2$, **iii** $2(x-1)^2+3$

b **i** $\frac{1}{2}\tan^{-1}\frac{x-1}{2}+c$, **ii** $\frac{1}{3\sqrt{2}}\tan^{-1}\frac{(x+1)\sqrt{2}}{3}+c$,

iii $\frac{1}{3}\tan^{-1}\frac{x-2}{3}+c$, **iv** $\frac{1}{2\sqrt{3}}\tan^{-1}\frac{2(x-1)}{\sqrt{3}}+c$

15 **a** **i** $4-(x+1)^2$, **ii** $9-(2-x)^2$, **iii** $7\frac{1}{2}-2(x-\frac{1}{2})^2$

b **i** $\sin^{-1}\frac{x+1}{2}+c$, **ii** $\frac{1}{2}\sin^{-1}\frac{2}{\sqrt{5}}(x-1)+c$,

iii $\sin^{-1}\frac{x-2}{4}+c$, **iv** $\frac{1}{\sqrt{2}}\sin^{-1}\frac{(x-3)\sqrt{2}}{3}+c$

16 **a** $\frac{1}{4}\pi$ **b** $\frac{1}{2}\pi$

17 **a** $3\sin^{-1}x+\sqrt{(1-x^2)}+c$ **b** $3\sin^{-1}\frac{x}{2}-2\sqrt{(4-x^2)}+c$

18 **a** $\frac{1}{2}\sin^{-1}x-\frac{1}{2}x\sqrt{(1-x^2)}+c$

b $\frac{1}{54}\tan^{-1}\frac{x}{3}+\frac{x}{18(9+x^2)}+c$ **c** $\frac{1}{2}\sin^{-1}\frac{x^2}{2}+c$

19 **a** $\frac{x}{\sqrt{(1-9x^2)}}+c$ **b** $-\frac{1}{x}\sqrt{(1-x^2)}+c$ **c** $\sec^{-1}x+c$

Chapter 25

Q1 The larger a, the larger the gradient.

Q2 The reflection of $y=2^x$ in the y-axis.

Q3 **a** $0.7\times 2^x, 1.1\times 3^x$ **b** 1.4, 1.9

Q4 0.7×2^x **Q5** 1.08

Q6 **a** $30x^2(2x^3+1)^4$ **b** $6x^2\cos(2x^3)$ **c** $6x^2 e^{2x^3}$

d $2y e^{y^2}\frac{dy}{dx}$ **e** $-2x e^{-x^2}$ **f** $\sec^2 x\times e^{\tan x}$

g $\frac{1}{2\sqrt{x}}e^{\sqrt{x}}$ **h** $\cos y\times e^{\sin y}\frac{dy}{dx}$

Q7 **a** $-\frac{1}{2}(x^2+1)^{-1}+c$ **b** $-\frac{1}{2}\cos x^2+c$ **c** $\frac{1}{2}e^{x^2}+c$

d $-e^{\cos x}+c$ **e** $6e^{x/3}+c$ **f** $\frac{3}{2}e^{2x}+c$

g $\frac{1}{12}e^{3x^2}+c$ **h** $-\frac{1}{2}e^{\cot 2x}+c$

Q8 **a** $\log_{10}(100a^2b^{-1/3})$ **b** $\log_c\frac{B(1+x)}{1-x}$

Q9 **a** $\log_c 2+\log_c a$ **b** $2\log_c a$ **c** $-\log_c a$

d $\log_c 2-\log_c a$ **e** $\frac{1}{2}\log_c a$ **f** $\log_c a-\log_c 2$

g $-2\log_c a$ **h** $-\log_c 2-\log_c a$

Q10 **a** $x=\frac{3}{2}$ **b** $x=6.02$ **Q11** **b** 4.61 **Q12** 0.693

Q13 **a** $\ln(xy/e)$ **b** 2

Q14 **a** x^2 **b** $1/x$ **c** $\sqrt{x}$ **d** $\sin x$ **e** $\frac{1}{2}x^2$ **f** $\frac{1}{2}x$

Q15 **a** $10x(x^2-2)^4$ **b** $-2x\operatorname{cosec}x^2\cot x^2$ **c** $2x e^{x^2}$

d $\frac{2x}{x^2-2}$ **e** $2\cot x$ **f** $2x\cot x^2$

Q16 $\frac{3x+2}{2x(x+1)}$

Q17 **a** $\frac{1}{x}$ **b** $\frac{1}{x}$ **c** $\frac{3}{3x+1}$ **d** $\frac{1}{y}\frac{dy}{dx}$ **e** $\frac{3}{x}$

f $\frac{3x^2}{x^3-2}$ **g** $\frac{3}{x-1}$ **h** $\frac{1}{t}\frac{dt}{dx}$ **i** $\cot x$

j $-3\tan 3x$ **k** $-3\tan x$ **l** $6\cot 3x$

m $\frac{x}{x^2-1}$ **n** $\frac{1+x}{x(1-x)}$

Q18 $4^x\ln 4$, $16\ln 4$ **Q19** **a** 0.6931 **b** 1.0986

Q20 **a** $10^x\ln 10$ **b** $2^{3x}\ln 64$

Q21 $5^x\ln 5, \frac{5^x}{\ln 5}+c$ **Q22** $x\,2^{x^2}\ln 4, \frac{2^{x^2}}{\ln 4}+c$

Q23 **a** $\frac{3^{2x}}{\ln 9}+c$ **b** $\frac{1}{3}e^{x^3}+c$ **c** $\frac{2^{\tan x}}{\ln 2}+c$

Q24 **a** $2\ln x+c$ **b** $\frac{1}{3}\ln x+c$ **c** $\frac{1}{3}\ln(3x-2)+c$

d $\frac{1}{3}\ln(x-2)+c$

Q25 **a** $\frac{1}{2}\ln(2x+3)+c$ **b** $\ln\frac{1}{1-x}+c$

Q26 $\ln 8$ **Q27** **a** $c=\ln A$ **b** $c=\ln(k\sqrt{2})$

Q28 a $\dfrac{1}{3(2-x^3)}+c$ b $\frac{1}{3}\sin x^3+c$ c $\frac{1}{3}e^{x^3}+c$
d $\ln\{k\sqrt[3]{(x^3-2)}\}$ e $\ln\{k\sqrt{(x^2-2x)}\}$
f $\ln\dfrac{k}{3-x^2}$ g $\ln(k\sin x)$

Q29 $x+\ln(x-1)+c$ **Q30** a $\ln\frac{3}{4}$ b $-\ln 2$

Q31 $-\ln 2$ **Q32** No **Q33** $\dfrac{1}{x-3}, \dfrac{1}{x-3}$

Q34 a $\ln\frac{3}{2}$ b $-\ln 5$ **Q35** $-\ln 3$

Exercise 25a, page 269

2 Yes. a Neither b even

3 Even. $0<y\leqslant 1$ 4 1.1×3^x

5 a $4e^x$ b $3e^{3x}$ c $2e^{2x+1}$ d $4x\,e^{2x^2}$ e $-2e^{-2x}$
f $3e^{3y}\dfrac{dy}{dx}$ g $2x\,e^{x^2+3}$ h $-2x^{-3}e^{x^{-2}}$ i $-5x^{-2}e^{5/x}$
j $\frac{1}{3}x^{-2/3}e^{x^{1/3}}$ k $2ax\,e^{ax^2+b}$ l $\dfrac{1}{2\sqrt{t}}e^{\sqrt{t}}\dfrac{dt}{dx}$

6 a $-e^{\cos x}\sin x$ b $e^{\sec x}\sec x\tan x$
c $e^{3\tan y}\,3\sec^2 y\dfrac{dy}{dx}$ d $2e^{\sin 2x}\cos 2x$
e $e^{-\cot x}\operatorname{cosec}^2 x$ f $-2e^{\operatorname{cosec}^2 x}\operatorname{cosec}^2 x\cot x$
g $-\dfrac{\sin x}{2\sqrt{\cos x}}e^{\sqrt{\cos x}}$ h $ab\,e^{a\sin bx}\cos bx$
i $3e^{\sin 3t}\cos 3t\dfrac{dt}{dx}$ j $2x\,e^{\tan x^2}\sec^2 x^2$

7 a $\dfrac{x}{\sqrt{(x^2+1)}}e^{\sqrt{(x^2+1)}}$ b $\dfrac{2x}{(1-x^2)^2}e^{(1-x^2)^{-1}}$
c $4e^{\sin^2 4x}\sin 8x$ d $2x\,e^{\tan(x^2+1)}\sec^2(x^2+1)$
e $6e^{\sec^2 3x}\sec^2 3x\tan 3x$ f $e^{-\operatorname{cosec} x}\operatorname{cosec} x\cot x$
g $2x^{-3}e^{-x^{-2}}$ h $(\sin x+x\cos x)\,e^{x\sin x}$
i $e^{xy}\left(y+x\dfrac{dy}{dx}\right)$ j e^{x+e^x}

8 a $e^x(x^2+2x)$ b $(x-1)\,e^x/x^2$ c $\frac{1}{2}e^{\sin x}(1+x\cos x)$
d $e^x\operatorname{cosec} x\,(2x-\cot x)$ e $e^x\operatorname{cosec} x\,(1-\cot x)$
f $-e^{-x}x^{-2}(x\sin x+x\cos x+\cos x)$ g $(1+x)\,e^{x+xe^x}$
h $e^{ax}\sec bx\,(a+b\tan bx)$ i $e^{ax}\operatorname{cosec} bx\,(a-b\cot bx)$
j $n\,e^x\tan^{n-1}e^x\times\sec^2 e^x$ k $2e^x\cos x$

9 a $6e^{x/2}+c$ b $-e^{-x}+c$ c $3e^{x/3}+c$ d $\frac{2}{3}e^{3x-1}+c$
e $\frac{1}{4}e^{x^2}+c$ f $-\frac{1}{3}e^{-x^3}+c$ g $-e^{\cos x}+c$ h $e^{\tan x}+c$
i $-e^{\cot x}+c$ j $-e^{1/x}+c$

10 $y=e^a(x-a+1), y-e^2x+e^2=0$ 11 $\frac{1}{2}\pi(e^2-1)$

12 $e^x(1+x), e^x(x-1)+c$

13 Minimum $-\dfrac{1}{e}$ when $x=-1$, $e^2y+x+4=0$

18 $25e^{4x}\cos(3x+2\alpha)$, $125e^{4x}\cos(3x+3\alpha)$

19 $13^n e^{5x}\sin(12x+n\beta)$

Exercise 25b, page 271

1 a $\log_{10}\dfrac{a^3}{50}$ b $\log_e\dfrac{x^2e^3}{3}$ c $\log_e\dfrac{(x-3)^4}{(x-2)^3}$
d $\log_e\{k\sqrt{(1-y^2)}\}$

2 a $\log_e a+\log_e 3$ b $3\log_e a$ c $\log_e a-\log_e 3$
d $-3\log_e a$ e $\log_e 3-\log_e a$ f $-\log_e 3-\log_e a$
g $\frac{1}{3}\log_e a$

3 a $\log_e\cos x-\log_e\sin x$
b $2\log_e\sin x-2\log_e\cos x$
c $\log_e(x-2)+\log_e(x+2)$
d $\frac{1}{2}\log_e(x+1)-\frac{1}{2}\log_e(x-1)$
e $\log_e 3+2\log_e\sin x$

4 a $a=100$ b $y=100$ or $y=\frac{1}{100}$

5 a $x=\frac{2}{5}$ b $x=1.26$

6 1.10 7 $x=1$ or $x=-2$

8 $a=10$ or $a=100$ 9 $x=+\frac{2}{9}$

10 $x=1, y=10$ or $x=10, y=1$

Exercise 25c, page 273

1 a $\dfrac{1}{x}$ b $\dfrac{4}{x}$ c $\dfrac{2}{2x-3}$ d $\dfrac{1}{y}\dfrac{dy}{dx}$ e $\dfrac{1}{x-1}$ f $\dfrac{4}{x}$
g $\dfrac{2x}{x^2-1}$ h $\dfrac{2}{x}$ i $\dfrac{6}{x}$ j $\dfrac{2}{x+1}$ k $\dfrac{3}{t}\dfrac{dt}{dx}$ l $-\dfrac{1}{x}$
m $\dfrac{1}{x}$ n $\dfrac{1}{2x}$ o $-\dfrac{1}{x}$ p $-\dfrac{1}{x}$ q $-\dfrac{2}{x}$ r $\dfrac{1}{x\ln 10}$
s $-\dfrac{3}{t}\dfrac{dt}{dx}$ t $\dfrac{1}{3x}$

2 a $-\tan x$ b $2\cot x$ c $6\operatorname{cosec} 6x$ d $-6\tan 2x$
e $-4\operatorname{cosec} 2x$ f $-4\tan 2x$ g $\operatorname{cosec} x$ h $\tan x$
i $\sec x$ j $-2x\cot x^2$ k $2\sec 2x$

3 a $\dfrac{1}{x^2-1}$ b $\dfrac{2x^2-1}{x(x^2-1)}$ c $\dfrac{3x-5}{2(x^2-1)}$ d $\dfrac{1}{\sqrt{(x^2-1)}}$

4 a $\dfrac{1}{t}\dfrac{dt}{dx}$ b $1+\ln x$ c $x+2x\ln x$ d $\dfrac{1}{x^2}(1-\ln x)$
e $\ln y+\dfrac{x}{y}\dfrac{dy}{dx}$ f $\dfrac{y}{x}+\dfrac{dy}{dx}\ln x$ g $\dfrac{1-2\ln x}{x^3}$
h $\dfrac{\ln x-1}{(\ln x)^2}$ i $\dfrac{2}{x}\ln x$ j $\dfrac{1}{x\ln x}$ k $\cos x$

5 a $5^x\ln 5$ b $x\,2^{x^2}\ln 4$ c $\frac{2}{3}3^{2x}\ln 3$ d 1

6 a $3^x\ln 3, \dfrac{3^x}{\ln 3}+c$ b $x\,2^{x^2}\ln 4, \dfrac{2^{x^2}}{\ln 4}+c$

7 a $\dfrac{1}{\ln 10}10^x+c$ b $\dfrac{2^{3x}}{\ln 8}+c$ c $\dfrac{3^{x^2}}{\ln 9}+c$
d $-\dfrac{2^{\cos x}}{\ln 2}+c$ 8 $1+\ln x, x(\ln x-1)+c$

9 $2^x(1+x\ln 2), \dfrac{x\,2^x}{\ln 2}-\dfrac{2^x}{(\ln 2)^2}+c$

10 a $1/(x-2)$ b $1/(x-2)$

12 a $\{y: y\geqslant 0\}$ b $\mathbb{R}$

Exercise 25d, page 277

1 a $\ln(kx^{1/4})$ b $\ln(kx^5)$ c $\ln\{k\sqrt{(2x-3)}\}$
d $\ln\{k\sqrt{(x+4)}\}$ e $\ln\{k(3-2x)^{-1/2}\}$
f $\ln\{k(1-x^2)^{-1/2}\}$ g $\ln\{k(x^2-1)^{3/2}\}$
h $\ln\{k(x^2+x-2)\}$ i $\ln\{k\sqrt[3]{(3x^2-9x+4)}\}$
j $x-\ln\{k(x+2)^2\}$ k $\frac{3}{2}x-\ln\{k(2x+3)^{9/4}\}$
l $-2x-\ln\{k(3-x)^6\}$ m $-x-\ln\{k(2-x)\}$
n $-2x-\ln\{k(x-4)^5\}$ o $\ln(k\sec x)$
p $\ln\left(k\sin^2\dfrac{x}{2}\right)$ q $\ln\{k\sqrt{\sin(2x+1)}\}$
r $\ln\left(k\cos^3\dfrac{x}{3}\right)$ s $\ln\{k(x-\sin^2 x)\}$
t $\ln\{k(\sin x+\cos x)\}$ u $\ln\{k(x+\tan x)\}$
2 b $\dfrac{2}{2x-1}, \dfrac{2}{2x-1}$ c $\frac{1}{2}\ln 3, -\frac{1}{2}\ln 5$
3 a $\ln\frac{2}{3}$ b $\ln 2$ 4 a $\dfrac{1}{3-x}, \dfrac{1}{3-x}$ b $-\ln 1.5$ c $\ln 1.5$
5 a $\ln 2$ b $\frac{1}{3}\ln 2$ c $-\ln 2$ d $\frac{1}{2}\ln 5-\ln 3$
e $\frac{1}{2}\ln 3$ f $-\frac{1}{2}\ln 2$ g $\ln 7$ h $2+\ln 4$ i $2+\ln 4$
j $-\frac{1}{2}-\ln\frac{3}{2}$ k $\frac{1}{2}\ln\frac{4}{3}$ l $\frac{1}{2}\ln 2$ m $\frac{1}{2}\ln 3$

Chapter 26

Q1 a $\dfrac{3-x}{1-x^2}$ b $\dfrac{(x+2)(x-1)}{(x^2+1)(x+1)}$ c $\dfrac{3x^2-x+4}{(x-1)^2(x+1)}$
Q2 a $\dfrac{1}{x-2}-\dfrac{1}{x+2}$ b $\dfrac{1}{2(1-x)}+\dfrac{1}{2(1+x)}$ c $\frac{1}{2}-\frac{1}{3}$
d $\dfrac{1}{n}-\dfrac{1}{n+1}$
Q3 a $12A-3B+4C=17, 6A-4B+3C=5,$
$10A-15B+6C=-1$ b $A=2, B=1, C=-1$
Q4 a $A=1, B=-3, C=4$ b $A=2, B=1, C=-3$
c $A=5, B=-1$
Q5 $A=3, B=-2, C=-1$
Q6 a $A=-\frac{2}{3}, B=\frac{2}{3}, C=1$ b No
Q7 a $\dfrac{1}{x-3}-\dfrac{1}{x+3}$ b $\dfrac{1}{2(2-x)}-\dfrac{1}{2(2+x)}$
c $\dfrac{1}{x-2}-\dfrac{2}{3x-5}$ d $\dfrac{1}{2(x+1)}-\dfrac{1}{x+2}+\dfrac{1}{2(x-3)}$
e $\dfrac{2}{1+2x}-\dfrac{1}{2-x}$
Q8 a $\dfrac{1}{1-x}+\dfrac{2+x}{4+x^2}$ b $\dfrac{1}{x+1}-\dfrac{2x-1}{2x^2+x+3}$
c $\dfrac{1}{x+1}+\dfrac{1}{x-2}-\dfrac{2}{x+2}$ d $\dfrac{1}{2-x}+\dfrac{x}{3+x^2}$
Q9 a $\dfrac{1}{x+3}-\dfrac{2}{(x+3)^2}$ b $\dfrac{5}{x-2}-\dfrac{3}{x-1}-\dfrac{4}{(x-1)^2}$
Q10 $A=3, B=-2, C=1, D=5$
Q11 a $x+\dfrac{x+2}{(x-1)(x+3)}$ b $3+\dfrac{x-1}{(x-2)(x+1)}$
Q12 a $1+\dfrac{2}{x+1}-\dfrac{1}{x-2}$ b $x-1-\dfrac{3}{4(x-2)}+\dfrac{3}{4(x+2)}$
Q14 a $\frac{1}{6}\ln(x-3)-\frac{1}{6}\ln(x+3)+c$
b $\frac{1}{2}\ln(2x-3)-\frac{5}{2}(2x-3)^{-1}+c$
Q15 $-\dfrac{1}{2}\ln(2-x)-\dfrac{1}{2}\ln(2+x)+c=\ln\dfrac{k}{\sqrt{(4-x^2)}}$
Q16 No Q17 a $\ln\frac{25}{32}$ b $\ln 3$

Exercise 26a, page 279

1 a $\dfrac{x-12}{(x+3)(x-2)}$ b $\dfrac{7-3x-5x^2}{(x+2)^2(3x-1)}$
c $\dfrac{2-4x-3x^2}{(2+3x^2)(1-x)}$ d $\dfrac{-x^3+6x^2-7x+6}{(x^2+1)(x-1)^2}$
2 a $\dfrac{1}{3-x}-\dfrac{1}{3+x}$ b $\dfrac{1}{2(a-b)}+\dfrac{1}{2(a+b)}$ c $\frac{1}{5}-\frac{1}{6}$
d $\dfrac{1}{1-p}+\dfrac{1}{p}$
3 a $A=3, B=7$ b $A=1, B=-1, C=2$
c $A=2, B=-1, C=-3$ d $A=1, B=-2, C=3$
4 a $A=1, B=-1$ b $A=2, B=1$
c $A=2, B=-1, C=3$ d $A=1, B=-2, C=3$
5 a $A=3, B=-\frac{1}{2}, C=-\frac{1}{2}$ b No
c $A=2, B=3, C=1$ d No
e $A=1, B=3, C=2, D=-1$
6 $A=1, B=1, C=1$ a $(x+1)(x^2-x+1)$
b $(x-2)(x^2+2x+4)$ c $(x+3)(x^2-3x+9)$
d $(2x-3)(4x^2+6x+9)$ e $(3x+5)(9x^2-15x+25)$
7 $x(x-1)(x-2)+3x(x-1)+x+1$
8 $a=60, b=25$ 9 $\alpha+\beta=-b/a, \alpha\beta=c/a$
10 $\alpha+\beta+\gamma=-q/p, \beta\gamma+\gamma\alpha+\alpha\beta=r/p, \alpha\beta\gamma=-s/p$

Exercise 26b, page 282

1 a $\dfrac{2}{x+3}-\dfrac{1}{x-4}$ b $\dfrac{1}{2(5-x)}-\dfrac{1}{2(5+x)}$
c $\dfrac{4}{x+1}+\dfrac{2}{x-2}-\dfrac{3}{x-3}$ d $\dfrac{3}{x-1}-\dfrac{1}{x}+\dfrac{2}{x+1}$
e $\dfrac{1}{x+2}+\dfrac{2}{2x+1}-\dfrac{2}{3x+2}$ f $2x-1+\dfrac{1}{x+3}-\dfrac{3}{x-2}$
2 a $\dfrac{2}{x-3}+\dfrac{3x-1}{x^2+4}$ b $\dfrac{2}{x+1}-\dfrac{1}{x^2+2}$ c $\dfrac{1}{x-1}+\dfrac{2x}{x^2+5}$
d $\dfrac{3}{2x-3}+\dfrac{1-3x}{2x^2+1}$ e $\dfrac{3}{x-3}-\dfrac{2}{x+3}-\dfrac{1}{x+5}$
f $2+\dfrac{5}{x-3}+\dfrac{x}{x^2+1}$

3 a $\frac{1}{x-2}-\frac{3}{(x-2)^2}$ **b** $\frac{1}{x-1}-\frac{1}{x+2}+\frac{2}{(x+2)^2}$

c $\frac{23}{4(3x+1)}-\frac{1}{4(x+1)}-\frac{7}{2(x+1)^2}$

d $x+\frac{1}{x+2}+\frac{2}{x-1}+\frac{1}{(x-1)^2}$

4 a $\frac{1}{x-2}+\frac{2}{x+1}-\frac{3}{(x+1)^2}+\frac{1}{(x+1)^3}$

b $\frac{3}{x-1}-\frac{1}{(x-1)^2}-\frac{3}{x+1}-\frac{2}{(x+1)^2}$

5 a $x+2+\frac{1}{x-3}-\frac{2}{x+3}$ **b** $3-\frac{2}{x-1}-\frac{1}{x+2}$

c $2x-\frac{6}{x-2}+\frac{12}{x^2+3}$ **d** $x-2-\frac{6}{x+1}+\frac{2}{(x+1)^2}+\frac{3}{x}$

6 $\frac{1}{6(x+2)}-\frac{7}{2x}+\frac{10}{3(x-1)}$ **7** $\frac{3}{2x^2}-\frac{3}{4x}+\frac{3}{4(x+2)}$

8 $2x+4-\frac{1}{3(x-2)}-\frac{5x+61}{3(x^2+5)}$ **9** $\frac{5}{3+x}+\frac{2}{4-x}-\frac{3}{4+x}$

10 $\frac{1}{x-1}-\frac{x}{x^2+x+1}$ **11** $\frac{2}{(2x+1)^2}-\frac{5}{2x+1}+\frac{3}{x-3}$

12 $\frac{13}{29(2x-5)}-\frac{5}{29(3x+7)}$ **13** $\frac{3}{x-1}-\frac{3}{x}-\frac{2}{x^2}-\frac{1}{(x-1)^2}$

14 $\frac{2}{5(5x-2)}-\frac{10x+1}{5(25x^2+10x+4)}$

15 $\frac{2}{x}-\frac{2x}{x^2+3}+\frac{2-5x}{(x^2+3)^2}$ **16** $\frac{1}{x^2+2}-\frac{1}{x^2+3}$

17 $\frac{\sqrt{3}}{36(x-\sqrt{3})}-\frac{\sqrt{3}}{36(x+\sqrt{3})}-\frac{1}{6(x^2+3)}$

Exercise 26c, page 284

1 $\frac{1}{n}-\frac{1}{n+2}$ **2** $\frac{2}{n-1}-\frac{3}{n}+\frac{1}{n+1}$

3 a $\frac{n+4}{n(n+1)(n+2)}$ **b** $\frac{3}{2}-\frac{n+3}{(n+1)(n+2)}$ **c** $1\frac{1}{2}$ **4** 2

5 a $\frac{11}{18}-\frac{3n^2+12n+11}{3(n+1)(n+2)(n+3)}$ **b** $\frac{n}{4(n+1)}$

c $\frac{n}{9(n+1)}$ **d** $\frac{3}{16}-\frac{2n+3}{8(n+1)(n+2)}$

e $\frac{11}{96}-\frac{1}{8(n+1)}-\frac{1}{8(n+2)}+\frac{1}{8(n+3)}+\frac{1}{8(n+4)}$

f $\frac{1}{6}-\frac{n+2}{(n+3)(n+4)}$

6 a $\frac{2n}{2n+1}$ **b** $\frac{1}{12}-\frac{1}{4(2n+1)(2n+3)}$

c $\frac{5}{24}-\frac{4n+5}{8(2n+1)(2n+3)}$

7 a $\frac{1}{2}\ln\frac{k(x-2)}{x}$ **b** $\frac{1}{17}\ln\frac{k(5x-2)}{x+3}$

c $\ln\frac{kx}{3x+1}-\frac{2}{x}$ **d** $\ln\frac{k}{\sqrt{(16-x^2)}}$ **e** $\frac{1}{6}\ln\frac{k(x-5)}{x+1}$

f $\ln\{k(x^2-4x-5)^{1/2}\}$ **g** $\ln\{k(x+2)\sqrt{(x^2+3)}\}$

h $\ln\frac{k(3+x)^2(2-x)}{(4-x)^3}$

i $2\ln\{k(2x+1)\}-\frac{1}{2}\ln\{(x-3)(x+3)^3\}$

j $\frac{1}{3}\ln\frac{k(x-2)}{x+1}-\frac{4}{x-2}$ **k** $\ln\{k(2x+1)^{1/2}\}-\frac{4}{3}\tan^{-1}\frac{x}{3}$

l $\ln\{k(3x+2)^{1/3}\}-\frac{1}{6}\ln(9x^2-6x+4)$

m $\frac{1}{2}x^2+3x+\ln\frac{k(x-5)^2}{x+2}$

n $\frac{1}{4}\ln\{k(x-3)\}-\frac{1}{8}\ln(1+4x^2)-\frac{3}{2}\tan^{-1}2x$

8 a $\tan^{-1}x+c$ **b** $\ln\{k(1+x^2)^{1/2}\}$

c $\tan^{-1}x+\ln\{k(1+x^2)^{1/2}\}$ **d** $\ln\left(k\sqrt{\frac{1+x}{1-x}}\right)$

e $\ln\frac{k}{\sqrt{(1-x^2)}}$ **f** $c-\sqrt{(1-x^2)}$ **g** $\sin^{-1}x+c$

h $\sin^{-1}x-\sqrt{(1-x^2)}+c$ **i** $-\ln\{k(1-x)\}$

j $-\ln\{k(x-1)\}$ **k** $x-\ln(1+x)+c$ **l** $\frac{1}{1-x}+c$

m $\frac{1}{1-x}+\ln(1-x)+c$

9 a $\ln\frac{4}{3}\approx 0.288$ **b** $\frac{1}{2}\ln 2+\frac{1}{4}\pi\approx 1.13$

c $\ln\frac{45}{64}\approx -0.352$ **d** $-3\ln 2-\frac{1}{2}\ln 3\approx -2.63$

10 $\pi/2$, $(2+2\ln 2, 0)$

Chapter 27

Q1 a $(-1)^r(r+1)x^r$ **b** 3^rx^r **c** $(\frac{1}{2})^{r+1}(r+1)(r+2)x^r$

d $\frac{1}{6}(-1)^r(r+1)(r+2)(r+3)x^r$

Q2 a $\frac{20!(21-2r)}{(21-r)!r!}x^r$ **b** $\frac{10!(33-5r)(-1)^rx^r}{(11-r)!r!}$

Q3 a $(-1)^{r-1}(3r+1)x^r$

Q4

a $\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}-\ldots+\frac{(-1)^{r+1}}{x^r}+\ldots, |x|>1$

b $\frac{1}{x^2}-\frac{4}{x^3}+\frac{12}{x^4}-\ldots+\frac{(r-1)(-1)^r2^{r-2}}{x^r}+\ldots, |x|>2$

c $\frac{1}{9x^2}-\frac{2}{27x^3}+\frac{1}{27x^4}-\ldots+\frac{(-1)^r(r-1)}{3^rx^r}+\ldots, |x|>\frac{1}{3}$

d $\frac{3}{x^2}+\frac{9}{x^3}+\frac{21}{x^4}+\ldots+\frac{3(2^{r-1}-1)}{x^r}+\ldots, |x|>2$

e $\frac{1}{x}-\frac{1}{x^3}+\frac{1}{x^5}-\ldots+\frac{(-1)^r}{x^{2r+1}}+\ldots, |x|>1$

Q5 $792\times 4^7\times 3^5$

Q6 a ${}^{n+2}C_{r+1}=c_{r+1}+2c_r+c_{r-1}$, $(1\leqslant r\leqslant n-1)$

b ${}^{n+2}C_{r+2}=c_{r+2}+2c_{r+1}+c_r$, $(0\leqslant r\leqslant n-2)$

Exercise 27a, page 288

1 **a** $1-3x+9x^2-\dots+(-1)^r3^rx^r+\dots, |x|<\frac13$

b $1+2x+4x^2+\dots+2^rx^r+\dots, |x|<\frac12$

c $1-2x+3x^2-\dots+(-1)^r(r+1)x^r+\dots, |x|<1$

d $1+x+\frac34x^2+\dots+\frac{(r+1)}{2^r}x^r+\dots, |x|<2$

e $1-3x+6x^2-\dots+\frac12(r+1)(r+2)(-1)^rx^r+\dots, |x|<1$

f $\frac12-\frac14x+\frac18x^2-\dots+(-1)^rx^r/2^{r+1}+\dots, |x|<2$

g $\frac19+\frac{2}{27}x+\frac{1}{27}x^2+\dots+(r+1)x^r/3^{r+2}+\dots, |x|<3$

h $\frac18+\frac{9}{16}x+\frac{27}{16}x^2+\dots+(r+1)(r+2)3^rx^r/2^{r+4}+\dots,$ $|x|<\frac23$

i $1+\frac12x-\frac18x^2+\dots+(-1)^{r-1}\frac{1\times3\dots(2r-3)}{2^rr!}x^r+\dots,$ $|x|<1$

2 **a** $\frac{1}{1-x}+\frac{2}{1+2x}$,

$3-3x+9x^2+\dots+\{1-(-2)^{r+1}\}x^r+\dots, |x|<\frac12$

b $\frac{1}{1+x}-\frac{1}{2+x}$,

$\frac12-\frac34x+\frac78x^2+\dots+(-1)^r\{1-(\frac12)^{r+1}\}x^r+\dots, |x|<1$

c $\frac{1}{x+1}-\frac{2}{(x+1)^2}$,

$-1+3x-5x^2+\dots+(-1)^{r+1}(2r+1)x^r+\dots, |x|<1$

d $\frac{3}{1-3x}+\frac{2}{1+2x}$,

$5+5x+35x^2+\dots+\{3^{r+1}+(-1)^r2^{r+1}\}x^r+\dots, |x|<\frac13$

e $\frac{1}{x-2}+\frac{5}{(x-2)^2}$,

$\frac34+x+\frac{13}{16}x^2+\dots+(5r+3)x^r/2^{r+2}+\dots, |x|<2$

f $\frac{3}{2(x-1)}-\frac{1}{2(x+1)}$,

$-2-x-2x^2+\dots-\{3+(-1)^r\}x^r/2+\dots, |x|<1$

3 **a** $1-x^2+x^4-\dots+(-1)^rx^{2r}+\dots, |x|<1$

b $x+x^3+x^5+\dots+x^{2r+1}+\dots, |x|<1$

c $1-2x+2x^2-\dots+2(-1)^rx^r+\dots, |x|<1$

d $1-9x+35x^2-\dots+\frac{10!(-1)^r(11-2r)}{(11-r)!r!}x^r+\dots,$ all x

e $\frac43-\frac{16}{9}x+\frac{52}{27}x^2-\dots+(-1)^r2\left(1-\frac{1}{3^{r+1}}\right)x^r+\dots,$ $|x|<1$

f $-\frac53-\frac{28}{9}x-\frac{110}{27}x^2-\dots-(6-13\times2^r/3^{r+1})x^r+\dots,$ $|x|<1$

g $-\frac72+\frac{19}{4}x-\frac{57}{8}x^2+\dots-\{(\frac12)^{r+1}+(2r+3)(-1)^r\}x^r+\dots,$ $|x|<1$

4 **a** $\frac1x-\frac2{x^2}+\frac4{x^3}-\dots+\frac{(-1)^{r-1}2^{r-1}}{x^r}+\dots, |x|>2$

b $-\frac1{x^3}-\frac9{x^4}-\frac{54}{x^5}-\dots-\frac12(r-2)(r-1)\frac{3^{r-3}}{x^r}+\dots, |x|>3$

c $\frac1{4x^2}+\frac1{4x^3}+\frac3{16x^4}+\dots+\frac{r-1}{2^rx^r}+\dots, |x|>\frac12$

d $1+\frac1x-\frac1{x^2}+\dots+\frac{(-1)^{r+1}}{x^r}+\dots, |x|>1$

e $\frac1x-\frac5{x^2}+\frac{16}{x^3}-\dots+(-1)^{r-1}(3r-1)\frac{2^{r-2}}{x^r}+\dots, |x|>2$

f $\frac1{x^2}+\frac5{x^3}+\frac{19}{x^4}+\dots+\frac{3^{r-1}-2^{r-1}}{x^r}+\dots, |x|>3$

g $\frac2x+\frac6{x^3}-\frac{12}{x^4}+\dots+\frac{3\{1+(-1)^{r-1}3^{r-2}\}}{2x^r}+\dots, |x|>3$

h $-\left\{\frac2x+\frac2{x^3}+\frac2{x^5}+\dots+\frac2{x^{2r+1}}+\dots\right\}, |x|>1$

i $-\frac1{x^3}-\frac1{x^4}-\frac1{x^7}-\frac1{x^8}-\dots-\frac1{x^{4r-1}}-\frac1{x^{4r}}+\dots, |x|>1$

5 $x^{1/2}-x^{-1/2}-\frac12x^{-3/2}$; 1.4142 **6** 1.259 92

7 2.009 93 **8** 3.014 963 **9** 0.009 920

10 0.2425 **11** $1+2x+\frac52x^2$ **12** $1-x-x^2+3x^3$

13 $1-4x+6x^2+4x^3$ **14** $1+2x-\frac12x^2$

15 $1-\frac23x+\frac{14}{9}x^2-\frac{130}{81}x^3$ **16** $-1-\frac12x+\frac14x^2$

17 $x-x^2-\frac23x^3$ **18** ±5% **19** $\frac{86\,400}{86\,400-x}$ s, 17 s

20 21.6 s **21** $2x+y-z$ **22** $\frac12(p-q-r)$ **23** $\frac23$

24 $1/\sqrt{(1+2x)}, |x|<\frac12$ **25** $1/(1-2x)^2, |x|<\frac12$

26 $\frac{9}{16}$ **27** $\sqrt[3]{\frac32}$ **28** $1/(1+2x)^3, |x|<\frac12$

29 $(1-2x)^{1/2}, |x|<\frac12$ **30** $\sqrt[3]{4}$

Exercise 27b, page 291

1 **a** 15 360 **b** 20 412 **c** $792\times4^7\times3^5=2^{17}\times3^7\times11$

d $15\,504\times2^5\times5^{15}$ or

$38\,760\times2^6\times5^{14}=2^9\times3\times5^{15}\times17\times19$

e $330\times2^4/3^4=2^5\times5\times11/3^3$

f $126\times3^5\times2^4=2^5\times3^7\times7$ **g** $11^{11}/12^{12}$ **h** $3\times5^5/7^7$

2 **a** $66\times2^2\times9^{10}=2^3\times3^{21}\times11$ **b** $210(\frac12)^4(\frac23)^6=\frac{280}{243}$

c $56(\frac43)^3(\frac52)^5=2^4\times5^5\times7/3^3$ **d** $3^{18}\times2^{-8}\times5^{-10}$

Chapter 28

Q1 **a** Yes **b** No **Q2** **a** No **b** Yes **Q3** No

Q4 Not necessarily.

Q6 **a** i, ii, iii **b** ii **c** iii **d** iii **e** i, ii, iii **f** iii **g** i, ii, iii **h** iii

Q7 **a** $x-y=0$ **b** $x+y=0$

Q9 **a** As $x\to0, 2, 4, \frac{dy}{dx}\to\infty$.

b When $x=0, \frac{dy}{dx}=\pm\sqrt2$; as $x\to-2, \frac{dy}{dx}\to\infty$.

Exercise 28a, page 293

1 a $x<\frac{1}{2}, x>2$ **b** $\frac{1}{2}<x<2$
2 a $x<-2, x>-\frac{1}{2}$ **b** $-2<x<-\frac{1}{2}$
3 $-2<x<3$ **4** $x>2\frac{1}{2}, x<-2\frac{1}{3}$ **5** $-3<x<2\frac{1}{2}$
6 $x<-2, x>2\frac{1}{2}$ **7** $x<-1\frac{1}{4}, x>1$
8 $\frac{3}{5}<x<7$ **9** $1<x<2, 3\frac{1}{4}<x<3\frac{2}{3}$
10 $x<-1, 1<x<2, x>3$ **11** $\frac{3}{4}$ **12** -1
13 $\frac{7}{8}$ **14** $-\frac{11}{12}$ **15** $x<0, x>1; |y|>\frac{1}{2}$
16 $|x|>2, |y|>\sqrt{3}$ **17** $x<-1, 0<x<1$
18 Discontinuities when $x=2, 3$. **19** $y<-\frac{1}{2}, y>4\frac{1}{2}$
20 $x\geqslant 0$ **21** $x\leqslant 0$ **22** $|x|\leqslant\sqrt{2}$
25 $(2x-3y)^2+(x-2)^2>0$, unless $x=2, y=1\frac{1}{3}$

Exercise 28b, page 296

In Nos. 10–15, y *cannot* lie in the following intervals:
10 $-4<y<0$ **11** $\frac{1}{6}<y<1\frac{1}{2}$ **12** $y<0, y>1$
13 $y<-\frac{1}{2}, y>4\frac{1}{2}$ **14** $y>1$
15 $\frac{2}{9}(1-\sqrt{10})<y<\frac{2}{9}(1+\sqrt{10})$ **16** $(2, 4)$ min.
17 $(-6-2\sqrt{10}, (11+2\sqrt{10})/18)$ min.,
$(-6+2\sqrt{10}, (11-2\sqrt{10})/18)$ max.
18 $((-1-\sqrt{7})/2, (-23-8\sqrt{7})/9)$ max.,
$((-1+\sqrt{7})/2, (-23+8\sqrt{7})/9)$ min. **19** $(-\frac{1}{5}, 4\frac{1}{12})$ max.

Chapter 29

Q1 6 **Q4** -20, $\begin{matrix}+1 & -3 & -5\\ -5 & -5 & +5\\ -3 & +9 & -5\end{matrix}$

Q7 $\begin{pmatrix}4 & -1 & -1\\ -8 & 2 & 2\\ 4 & -1 & -1\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$

Q8 $\begin{pmatrix}-1 & 2 & -3\\ -2 & 1 & 0\\ 2 & -1 & 3\end{pmatrix}, \begin{pmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{pmatrix}$

Exercise 29a, page 301

1 a 14 **b** -9 **c** 0 **d** 0
2 a x^2+y^2 **b** 0 **c** 1 **d** x^2 **4 a** 0, 5/3 **b** 1, 4
7 a 21 **b** 24 **c** 0 **d** 0 **8** $2, -3\pm\sqrt{6}$
10 $\mathbf{AB}=\begin{pmatrix}10 & -4 & 2\\ 3 & 3 & 0\\ 11 & -5 & 7\end{pmatrix}$, $\det(AB)=198$

Exercise 29b, page 304

1 $(1, 2, 3)$ **2** $(1, -1, 2)$ **3** $(1, -3, -5)$ **4** $(7, 5, 0)$
5 $(-0.1, -0.1, +0.1)$ **6** $a=-1, k\neq 5$

Exercise 29c, page 306

1 $\frac{1}{18}\begin{pmatrix}-2 & -1 & 7\\ -8 & 5 & 1\\ 6 & 3 & -3\end{pmatrix}$ **2** No inverse **3** $\frac{1}{2}\begin{pmatrix}1 & 2 & 1\\ 2 & 3 & 5\\ 3 & 4 & 7\end{pmatrix}$

4 No inverse **5** $\frac{1}{8}\begin{pmatrix}1 & -1 & 1\\ 2 & 1 & 0\\ 1 & 0 & 3\end{pmatrix}$

6 $\begin{pmatrix}1 & -p & pr-q\\ 0 & 1 & -r\\ 0 & 0 & 1\end{pmatrix}$ **7** $\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\alpha & \sin\alpha\\ 0 & \sin\alpha & -\cos\alpha\end{pmatrix}$

8 $1, -1, 1$ **9** $2, 1, 3$ **10** $1, 3, -1$

11 a $\begin{pmatrix}-1 & -2 & 2\\ 2 & 5 & -4\\ 1 & 1 & -1\end{pmatrix}$ **b** $\frac{1}{6}\begin{pmatrix}3 & -3 & 3\\ -1 & 1 & 1\\ 2 & 4 & -2\end{pmatrix}$

c $\begin{pmatrix}3 & 5 & 1\\ 2 & 0 & 3\\ 4 & 2 & 4\end{pmatrix}$ **d** $\frac{1}{6}\begin{pmatrix}-6 & -18 & 15\\ 4 & 8 & -7\\ 4 & 14 & -10\end{pmatrix}$

12 a $\frac{1}{17}\begin{pmatrix}7 & -1 & 2\\ 3 & 2 & -4\\ -11 & 4 & 9\end{pmatrix}$ **b** $\frac{1}{3}\begin{pmatrix}3 & -2 & 4\\ 3 & -1 & 2\\ -6 & 4 & -5\end{pmatrix}$

c $\begin{pmatrix}-1 & 7 & 2\\ 8 & 17 & 12\\ -1 & 1 & -1\end{pmatrix}$ **d** $\frac{1}{51}\begin{pmatrix}-29 & 9 & 50\\ -4 & 3 & 28\\ 25 & -6 & -73\end{pmatrix}$

Chapter 30

Q1 a $y^2=-4ax$ **b** $x^2=4by$
Q4 $y_1x+2ay-y_1(x_1+2a)=0$
Q5 $x-ty+at^2=0$ **Q6** $(\frac{5}{4}, 3), y-\frac{13}{4}=0$
Q7 7.2 cm, 12 cm
Q9 4, 3 **Q10** $\frac{3}{5}$ **Q11** $(\pm 3\sqrt{3}/2, 0)$

Exercise 30a, page 309

1 $(at_1t_2, a(t_1+t_2))$ **2** $t_1t_2=-1$ **4** $(0, \frac{1}{2})$
5 $(x-2)^2=4(y+1)$ **6** $(x+y)^2+4(x-y+1)=0$
7 $-t, 2/(t+t_1), (a(t+2/t)^2, -2a(t+2/t))$
9 a $(\frac{1}{2}, -2)$ **b** $(8, 8)$
11 $(a(t_1{}^2+t_1t_2+t_2{}^2+2), -at_1t_2(t_1+t_2))$
14 $4a$ **15** $x-y+a=0, x-16y+256a=0$

Exercise 30b, page 312

1 $y+1=0$ **2** $(1, \frac{1}{2}), \frac{3}{4}$ **3** $(-\frac{3}{4}, -6)$ **5** $a^2=bc$
6 $y=4-x^2, 2x+y-5=0$ **8** $y=2a/k$ **9** $x=a$
10 $y^2-4ax=\frac{1}{4}k^2$ **11** $y^2=2a(x-a)$
12 $x(x-a)^2=ay^2$

13 $\left(-\frac{b}{2a}, -\frac{b^2-4ac}{4a}+\frac{1}{4a}\right)$, $y=-\frac{b^2-4ac}{4a}-\frac{1}{4a}$

14 $yy_1=2a(x+x_1)$ 15 $(3a, \pm 2\sqrt{3}a)$ 16 $2a$

18 $2x(h-x-2a)+ky=0$

19 $3y^2=16ax$ 20 $y^2-2ax-2ay+2a^2=0$

Exercise 30c, page 316

1 a $(\pm\sqrt{5}, 0)$, $x=\pm 9\sqrt{5}/5$ b $(\pm 5\sqrt{15}/4, 0)$, $x=\pm 4\sqrt{15}/3$

2 a $2x\cos\theta+3y\sin\theta-6=0$ b $9x+16y-25=0$

3 a $16x-9y-7=0$ b $4x+y-2=0$

4 $3x-2y-5=0$ 6 $\frac{\cos\frac{1}{2}(\theta-\phi)}{\cos\frac{1}{2}(\theta+\phi)}=\pm e$

8 $ex+y-a=0$

10 $\left(a\frac{\cos\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}, b\frac{\sin\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}\right)$

12 $\frac{(2x-ae)^2}{a^2}+\frac{4y^2}{b^2}=1$ 13 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\cos^2\frac{1}{2}k$

14 $4a^2x^2+4b^2y^2=(a^2-b^2)^2$ 15 $a^2y^2+b^2x^2=4x^2y^2$

16 $b^2x^2+a^2y^2=2a^2b^2$

Exercise 30d, page 318

1 a $2x-y\pm 5=0$ b $x+y\pm 2=0$ c $x-2y\pm 10=0$

2 a $(\frac{9}{13}, -\frac{4}{13})$ b $(-\frac{1}{2}, \frac{1}{5})$ c $(11, -6)$

4 $3x+2y\pm 2\sqrt{10}=0$ 5 ± 5, $(16/5, -9/5)$, $(-16/5, 9/5)$

6 $c^2<a^2m^2+b^2$ 8 $(x^2+y^2)^2=a^2x^2+b^2y^2$

9 $a^2y_1x-b^2x_1y-x_1y_1(a^2-b^2)=0$

10 a $\left(\frac{-a^2mc}{a^2m^2+b^2}, \frac{b^2c}{a^2m^2+b^2}\right)$ b $\left(\frac{2am^2-nl}{l^2}, \frac{2am}{l}\right)$

11 $8x-27y=0$

12 $mx-y-mae=0$, $\left(\frac{a^3em^2}{b^2+a^2m^2}, -\frac{ab^2em}{b^2+a^2m^2}\right)$,

$a^2y^2+b^2x^2-ab^2ex=0$

13 $a^2y^2+b^2x(x-a)=0$

14 $(a^2+b^2-x^2-y^2)^2=4(b^2x^2+a^2y^2-a^2b^2)$

15 $b^2hx+a^2ky-(b^2h^2+a^2k^2)=0$

16 $(a^2y^2+b^2x^2)^2=a^2(a^4y^2+b^4x^2)$

17 $(y^2-2ax)^2=4a^4+b^2y^2$

18 $a^4y^2+b^4x^2=4x^2y^2$

Exercise 30e, page 322

2 $9b^2x^2-9a^2y^2-12ab^2x+12a^2by-a^2b^2=0$

4 $(a^2+b^2)^2=4(a^2x^2-b^2y^2)$ 5 $(-c/t^3, -ct^3)$

6 $xy=c^2$ 7 $2xyc^2=c^4-y^4$

8 $y_1x+x_1y-2x_1y_1=0$

10 $(ct, -ct^3)$ 11 $(x^2+y^2)^2=4c^2xy$

12 $n^2=4lmc^2$ 14 $x^2+y^2=a^2-b^2$

15 $(x^2+y^2)^2=a^2x^2-b^2y^2$

18 $bx\cos\frac{1}{2}(\theta-\phi)-ay\sin\frac{1}{2}(\theta+\phi)-ab\cos\frac{1}{2}(\theta+\phi)=0$

Chapter 31

Q1 a $1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\dots+(-1)^n\frac{x^n}{n!}+\dots$

b $1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\dots+\frac{x^{2n}}{n!}+\dots$

c $1+3x+\frac{9x^2}{2}+\frac{9x^3}{2}+\dots+\frac{3^nx^n}{n!}+\dots$

d $1+\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{3!x^3}+\dots+\frac{1}{n!x^n}+\dots$

e $1-\frac{1}{x^2}+\frac{1}{2x^4}-\frac{1}{3!x^6}+\dots+\frac{(-1)^n}{n!x^{2n}}+\dots$

Q3 a $\frac{1}{4}x-\frac{1}{32}x^2+\frac{1}{192}x^3-\dots+(-1)^{n-1}\frac{x^n}{4^nn}+\dots$,

$-4<x\leqslant 4$

b $\ln 3-\frac{1}{3}x-\frac{1}{18}x^2-\dots-\frac{x^n}{3^nn}-\dots$, $-3\leqslant x<3$

c $-2x-x^2-\frac{2}{3}x^3-\dots-\frac{2x^n}{n}-\dots$, $-1\leqslant x<1$

Q4 $-1+x-\frac{1}{2}x^2$ Q5 1.0209 Q6 0.809

Q12 $f(x)\approx f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2$

Q13 $f(a+h)\approx f(a)+f'(a)h+\frac{f''(a)}{2}h^2$

Q15 $1+2h+2h^2+\frac{8}{3}h^3$

Q16 $\frac{3}{5}-\frac{4}{5}(x-\alpha)-\frac{3}{10}(x-\alpha)^2+\frac{2}{15}(x-\alpha)^3$

Q17 a $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}$ c $1-\frac{x^2}{2!}+\frac{x^4}{4!}$

Exercise 31a, page 325

1 a 1.1052 b 0.3679 c 1.6487

2 $1+x^3+\frac{1}{2}x^6+\frac{1}{6}x^9+\dots+x^{3n}/n!+\dots$

3 $1+\frac{1}{3}x+\frac{1}{18}x^2+\frac{1}{162}x^3+\dots+x^n/(n!3^n)+\dots$

4 $1-2x+2x^2-\frac{4}{3}x^3+\dots+(-1)^n2^nx^n/n!+\dots$

5 $e^2\{1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\dots+x^n/n!+\dots\}$

6 $1-\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{48}x^3+\dots+(-1)^nx^n(n!2^n)+\dots$

7 $1+2x+\frac{3}{2}x^2+\frac{2}{3}x^3+\dots+(n+1)x^n/n!+\dots$

8 $1-2x^2+\frac{8}{3}x^3-2x^4+\dots+(-1)^{n-1}(n-1)2^nx^n/n!+\dots$

9 $1+4x+8x^2+\frac{32}{3}x^3+\dots+4^nx^n/n!+\dots$

10 $2+3x+\frac{5}{2}x^2+\frac{3}{2}x^3+\dots+(1+2^n)x^n/n!+\dots$

11 $10^9/9!$, $10^{10}/10!$ 12 $1+2x+3x^2+\frac{10}{3}x^3$

13 $e(1-3x+\frac{11}{2}x^2-\frac{15}{2}x^3)$ 14 $1+\frac{1}{2}x^2-\frac{1}{3}x^3$

15 $1-x+\frac{1}{2}x^2-\frac{1}{6}x^3$ 16 a $\frac{1}{4}$ b $2\frac{2}{3}$ c 1 17 $(1+x)e^x$

18 $(e^{3x}-1)/(3x)$ 19 $\frac{1}{2}(e^x+e^{-x})$ 20 $\frac{1}{2}(e^x-e^{-x})$

Exercise 31b, page 328

1 a $\ln 3+\frac{1}{3}x-\frac{1}{18}x^2+\frac{1}{81}x^3-\dots+(-1)^{n-1}\frac{x^n}{3^n\times n}+\dots$,

$-3<x\leqslant 3$

b $-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{24}x^3-\frac{1}{64}x^4-\ldots-\frac{x^n}{2^n\times n}\ldots,$
$-2\leqslant x<2$

c $\ln 2-\frac{5}{2}x-\frac{25}{8}x^2-\frac{125}{24}x^3-\ldots-\frac{5^nx^n}{2^n\times n}-\ldots,$
$-\frac{2}{5}\leqslant x<\frac{2}{5}$

d $-x^2-\frac{1}{2}x^4-\frac{1}{3}x^6-\frac{1}{4}x^8\ldots-\frac{x^{2n}}{n}-\ldots,\ -1<x<1$

e $\frac{2}{3}x+\frac{2}{81}x^3+\frac{2}{1215}x^5+\ldots+\frac{2x^{2n-1}}{(2n-1)3^{2n-1}}+\ldots,$
$-3<x<3$

f $-\frac{3x}{2}-\frac{9}{32}x^3-\frac{243}{2560}x^5-\ldots-2\times\frac{3^{2n-1}x^{2n-1}}{(2n-1)4^{2n-1}}-\ldots,$
$-\frac{4}{3}<x<\frac{4}{3}$

2 $\ln\frac{2}{3}-\frac{1}{6}x-\frac{5}{72}x^2-\ldots-\left(\frac{1}{2^n}-\frac{1}{3^n}\right)\frac{x^n}{n}-\ldots,\ -2\leqslant x<2$

3 $-\ln 3+\frac{4}{3}x+\frac{20}{9}x^2+\ldots+2^n\{1+(-1)^n(\frac{1}{3})^n\}x^n/n+\ldots,$
$-\frac{1}{2}\leqslant x<\frac{1}{2}$

4 $-6x^2+28x^3-111x^4+\ldots+(-1)^{n-1}(4^{n-1}-3^{n-1})12x^n/n+\ldots,$
$-\frac{1}{4}<x\leqslant\frac{1}{4}$

5 $\frac{1}{2}\ln 2+\frac{3}{4}x-\frac{5}{16}x^2+\ldots+(-1)^{n-1}\{1+(\frac{1}{2})^n\}x^n/(2n)+\ldots,$
$-1<x\leqslant 1$

6 $x+\frac{1}{2}x^2-\frac{2}{3}x^3+\ldots-\frac{2}{3n}x^{3n}+\frac{x^{3n+1}}{3n+1}+\frac{x^{3n+2}}{3n+2}-\ldots,$
$-1\leqslant x<1$

7 $1-\frac{1}{2}x+\frac{1}{3}x^2-\ldots+(-1)^nx^n/(n+1)+\ldots,\ -1<x\leqslant 1$

8 $-x+\frac{1}{2}x^2+\frac{2}{3}x^3+\ldots-(-1)^{3n}\frac{2x^{3n}}{3n}$
$+(-1)^{3n+1}\frac{x^{3n+1}}{3n+1}+(-1)^{3n+2}\frac{x^{3n+2}}{3n+2}+\ldots,\ -1<x\leqslant 1$

9 $x+\frac{1}{2}x^2+\frac{5}{6}x^3,\ |x|<1$ **10** $x+\frac{1}{2}x^2+\frac{1}{3}x^3,\ -1<x\leqslant 1$

11 $1+\frac{3}{2}x+\frac{5}{12}x^2,\ -1<x\leqslant 1$ **12** $x^2+x^3+\frac{11}{12}x^4,\ -1\leqslant x<1$

13 0.693 1, 1.099 **14** 2.302 6, 0.434 3 **15** 1.945 9

16 1.041 4 **17** **a** −1 **b** 1 **c** $-\frac{2}{3}$ **d** 0 **18** $\ln\frac{4}{3}$

19 ln 2 **20** $\frac{1}{2}\ln\frac{5}{3}$ **21** $\frac{5}{2}\ln\frac{7}{5}$ **22** ln 3

25 s_n tends to a limit between 0 and 1.

Exercise 31c, page 333

1 $\ln 2-\frac{3}{2}+x-\frac{1}{8}x^2$

2 $\sin\alpha+(\cos\alpha)(x-\alpha)-\frac{\sin\alpha}{2}(x-\alpha)^2$

3 **a** $1+\frac{1}{e}(x-e)-\frac{1}{2e^2}(x-e)^2+\frac{1}{3e^3}(x-e)^3-\frac{1}{4e^4}(x-e)^4$
b $1+\frac{1}{2}(x-\pi/2)^2+\frac{5}{24}(x-\pi/2)^4$ **4** 0.581

5 $f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f''''(0)}{4!}x^4+\ldots$

6 **a** $1+2x+2x^2+\frac{4}{3}x^3+\ldots$ **b** $-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots$

c $1-\frac{x^4}{2!}+\frac{x^8}{4!}-\frac{x^{12}}{6!}+\ldots$

d $\frac{x}{2}-\frac{x^3}{48}+\frac{x^5}{3840}-\frac{x^7}{645\,120}+\ldots$

8 **a** 1.4918 **b** 0.1823 **c** 0.955 **d** 0.199

9 **a** $x^2-\frac{1}{3}x^4$ **b** $1+nx+\frac{n(n-1)}{2}x^2+\frac{n(n-1)(n-2)}{6}x^3$
c $1+x\ln 2+\frac{(x\ln 2)^2}{2!}+\frac{(x\ln 2)^3}{3!}$ **d** $\pi/2-x+x^3/6$
e $x+x^2+\frac{1}{3}x^3-\frac{1}{30}x^5$ **f** $x-\frac{1}{6}x^3$

10 $x+x^2+\frac{1}{3}x^3-\frac{1}{30}x^5-\frac{1}{90}x^6$

11 $\ln 4+\frac{1}{4}(x-4)-\frac{1}{32}(x-4)^2+\frac{1}{192}(x-4)^3$; 1.3913

Chapter 32

Q1 **a** $2\ln a+\ln b$ **b** $3\ln a-3\ln b$
c $\frac{1}{2}\ln a+\frac{1}{2}\ln b+\frac{1}{2}\ln c$ **d** $\ln a+\frac{1}{3}\ln b-3\ln c$
e $-4\ln c$ **f** $b\ln a$

Q2 **a** 3 **b** −2 **c** 4 **d** 2 **e** $2x$ **f** $3x^2$

Q3 **a** $1/x$ **b** $2/(1+2x)$ **c** $-1/(1-x)$ **d** $3/x$
e $\cot x$ **f** $\sec x\operatorname{cosec} x=2\operatorname{cosec} 2x$

Q4 **a** $6y\frac{dy}{dx}$ **b** $3y^2\frac{dy}{dx}$ **c** $-\sin y\frac{dy}{dx}$ **d** $\frac{1}{y}\frac{dy}{dx}$
e $20y^3\frac{dy}{dx}$ **f** $-\frac{6}{y^3}\frac{dy}{dx}$ **g** $\frac{1}{2\sqrt{y}}\frac{dy}{dx}$ **h** $\sec^2 y\frac{dy}{dx}$

Q5 **a** $-\frac{2}{3}(x+1)^{-2/3}(x-1)^{-4/3}$ **b** $-\frac{2x^2+x+4}{(2x-1)^3\sqrt{(x^2+1)}}$
c $\frac{x\,e^x(x^2-2x-2)}{(x-1)^4}$

Q6 $\frac{10^x}{\ln 10},\ \frac{10^x}{\ln 10}+c$

Q7 **a** $2^x\ln 2$ **b** $3^x\ln 3$ **c** $-(\frac{1}{2})^x\ln 2$ **d** $5(\ln 10)10^{5x}$
e $2x10^{x^2}\ln 10$

Q8 **a** $\frac{2^x}{\ln 2}+c$ **b** $\frac{3^x}{\ln 3}+c$ **c** $-\frac{(\frac{1}{2})^x}{\ln 2}+c$
d $\frac{10^{5x}}{5\ln 10}+c$

Q9 **a** $\frac{2}{9}(3x+1)^{3/2}+c$ **b** $-\frac{1}{6}\cos^6 x+c$
c $\frac{1}{2}(1+\cos x)^{-2}+c$ **d** $5^x/\ln 5+c$
e $2^{2x}/(2\ln 2)+c$ **f** $x\ln x-x+c$

Q10 **a** $\tan y=x$ **b** $x=\sec y$ **c** $p=\cos q$

Q11 **a** $2y\frac{dy}{dx}$ **b** $\cos y\frac{dy}{dx}$ **c** $\sec^2 y\frac{dy}{dx}$
d $\sec y\tan y\frac{dy}{dx}$

Q12 **a** $-1/\sqrt{(1-x^2)}$ **b** $-1/(1+x^2)$ **c** $1/\sqrt{(-x-x^2)}$

Q13 Maximum at (1, 1/e). **Q15** (3, 8)

Exercise 32a, page 336

1 **a** $3\ln a + 4\ln b$ **b** $\ln a - \ln b$ **c** $\frac{3}{2}\ln a - \frac{1}{2}\ln b$
d $2\ln a + \ln b - \frac{1}{2}\ln c$ **e** $\frac{1}{2}\ln a + \frac{1}{2}\ln b - \frac{1}{2}\ln c$
f $-\frac{1}{2}\ln a - \frac{1}{2}\ln b - \frac{1}{2}\ln c$

2 **a** 5 **b** 3 **c** 4 **d** $\frac{1}{2}$ **e** x^3 **f** $-2x$

3 $2(3-2x)\sqrt{\dfrac{2x+3}{(1-2x)^3}}$

4 $\dfrac{e^{x/2}}{2x^5}(x\sin x - 8\sin x + 2x\cos x)$

5 $\dfrac{x-5x^3}{3\surd(x^2+1)^3\sqrt[3]{(x^2-1)^4}}$ **6** $\dfrac{x\tan x - x - 1}{x^2 e^x \cos x}$

7 $7^x \ln 7$ **8** $10^{3x}\, 3\ln 10$ **9** $-\frac{1}{2}\,10^{-x/2}\ln 10$

10 $-\dfrac{\ln 10}{10^x}$ **11** $\dfrac{5^x}{\ln 5} + c$ **12** $\dfrac{8^x}{\ln 8} + c$

13 $-\dfrac{(\frac{1}{3})^x}{\ln 3} + c$ **14** $\dfrac{3^{2x}}{2\ln 3} + c$ **15** $a^x \ln a$ **16** $\dfrac{a^x}{\ln a} + c$

17 $\dfrac{1}{1+x^2}$ **18** $\dfrac{1}{x\surd(x^2-1)}$ **19** $\dfrac{1}{\surd(-x^2-2x)}$

20 $-\dfrac{1}{\surd(x-x^2)}$ **21** $-\dfrac{2x}{x^4+1}$ **22** $\dfrac{-10}{\surd(1-25x^2)}$

23 **a** 0 **b** 0. The angles are complementary.

24 **a** $\dfrac{2}{\surd(1-4x^2)}$ **b** $\dfrac{2x}{\surd(1-x^4)}$ **c** $\dfrac{1}{\surd(1-x^2)}$

25 $\frac{1}{6}(4x+3)^{3/2} + c$ **26** $\frac{1}{16}(2x^2+1)^4 + c$

27 $x\ln x - x + c$ **28** $x\sin^{-1}x + \surd(1-x^2) + c$

29 $-(1+\ln x)x^{-x}$ **30** $\left(\cos x \ln x + \dfrac{1}{x}\sin x\right)x^{\sin x}$

Exercise 32b, page 340

1 (1, 4) max., (3, 0) min. **2** (2, 3) min.

3 (2, −1) max., (4, 3) min. **4** (1, 1) min.

5 (±3, −405) **6** None

8 $(\frac{1}{3}\pi, \frac{3}{4}\surd 3)$ max., $(\pi, 0)$ infl., $(\frac{5}{3}\pi, -\frac{3}{4}\surd 3)$ min.

9 $(\frac{1}{2}\pi, \frac{1}{2})$ max. **10** $(0, -1)$ max., $(2\pi, -1)$ max.

11 $(0, \frac{2}{3})$ min., $(\frac{1}{4}\pi, \frac{2}{3}\surd 2)$ max.,
$(\frac{3}{4}\pi, -\frac{2}{3}\surd 2)$ min., $(\pi, -\frac{2}{3})$ max., $(\frac{5}{4}\pi, -\frac{2}{3}\surd 2)$ min.,
$(\frac{7}{4}\pi, \frac{2}{3}\surd 2)$ max., $(2\pi, \frac{2}{3})$ min.

12 $(1, 1/e)$ max. **13** $(2, 10\tan^{-1}2 - 2)$ max.

15 $\left(2n\pi + \dfrac{1}{4}\pi, \dfrac{1}{\surd 2}e^{2n\pi+\pi/4}\right)$ max.,
$\left(2n\pi + \dfrac{5}{4}\pi, -\dfrac{1}{\surd 2}e^{2n\pi+5\pi/4}\right)$ min.

16 $\surd 2:1$ **18** $\surd 2:1$ **20** 1:2

Chapter 33

Q3 **a** $\sin x - x\cos x + c$ **b** $\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$
c $\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + c$ **d** $xe^x - e^x + c$

Q4 $2xe^{x^2}$, $\frac{1}{2}e^{x^2}(x^2-1) + c$ **Q5** $x\ln x - x + c$

Q6 **a** $(1-x^2)^{-1/2}$ **b** $x\sin^{-1}x + \surd(1-x^2) + c$

Q8 **a** $(x^2-2)\sin x + 2x\cos x + c$
b $e^x(x^2-2x+2) + c$

Q9 **a** $3(1-9x^2)^{-1/2}$ **b** $2(1+4x^2)^{-1}$ **c** $(9-x^2)^{-1/2}$
d $-2(1-4x^2)^{-1/2}$ **e** $\frac{3}{2}(1+9x^2)^{-1}$ **f** $6(4+x^2)^{-1}$
g $\frac{1}{2}(2x-x^2)^{-1/2}$ **h** $4(5+2x+x^2)^{-1}$

Q10 **a** $\tan^{-1}\dfrac{x}{2} + c$ **b** $\frac{3}{2}\tan^{-1}2x + c$ **c** $4\sin^{-1}\dfrac{x}{3} + c$

d $\frac{1}{3}\sin^{-1}3x + c$ **e** $\dfrac{1}{5\surd 2}\tan^{-1}\dfrac{5x}{\surd 2} + c$

f $\sin^{-1}\dfrac{2x}{\surd 3} + c$ **g** $\dfrac{1}{\surd 2}\tan^{-1}\dfrac{x-1}{\surd 2} + c$

h $5\sin^{-1}\dfrac{x+2}{3} + c$

Q11 $\ln\tan\frac{1}{2}x + c$ **Q12** $-2\ln\cos\frac{1}{2}\theta + c$

Q14 **a** $(1+t^2)^{-1}$ **b** $(4+4t^2)^{-1}$ **c** $2(3+3t^2)^{-1}$

Q15 **a** $\frac{1}{2}\ln\tan x + c$ **b** $-\frac{2}{3}(1+\tan\frac{3}{2}\theta)^{-1} + c$
c $\ln\{x + \surd(x^2-1)\} + c$

Q16 **a** $\dfrac{1}{\surd 2}\tan^{-1}\left(\dfrac{1}{\surd 2}\tan x\right) + c$
b $-\ln(1-\tan^2 x) + c$

Q17 **a** $\ln(x^2+2x+10) + \frac{1}{3}\tan^{-1}\dfrac{x+1}{3} + c$

b $5\sin^{-1}\dfrac{x+2}{3} + 2\surd(5-4x-x^2) + c$

c $\frac{1}{2}x - \frac{1}{2}\ln(\sin x + \cos x) + c$
d $\frac{3}{2}x - \frac{5}{2}\ln(3\cos x + \sin x) + c$

Q18 **a** 1 **b** $\frac{1}{2}\pi$ **Q19** **b** $\frac{1}{2}\pi$ **e** 1 **f** $\pi/3$

Q20 $\frac{1}{13}e^{2x}(2\sin 3x - 3\cos 3x) + c$

Q21 **a** and **b** $\frac{1}{5}e^x(\cos 2x + 2\sin 2x) + c$, no.

Q22 Area of the sector. **Q23** $\frac{3}{2}\pi a^2$

Q24 $\dfrac{a^2}{2k}\sinh 2k\pi$ **Q25** $\frac{1}{12}\pi a^2$ **Q26** $2\pi + \frac{3}{2}\surd 3$

Q29 πab **Q30** $17\frac{1}{15}$ **Q31** $\frac{4}{3}$ **Q32** $3\pi a^2$

Q33 **a** $\dfrac{dy}{dx}$ **b** $\dfrac{ds}{dx}$ **c** $\dfrac{ds}{dx} = \sqrt{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}}$

Q34 $\frac{14}{27}$ **Q35** $\ln 3$

Q36 $\frac{1}{2}\sinh^{-1}(2\surd 2) + 3\surd 2$ **Q37** $a\alpha$

Q38 $\dfrac{a}{27}\{(9t^2+4)^{3/2} - 8\}$ **Q39** $6a$

Q40 $8a$ **Q41** **a** δr, $r\delta\theta$ **Q42** $a\surd(1+k^2)(e^{2k\pi}-1)/k$

Q43 $\frac{1}{2}a\{\sinh^{-1}\pi + \pi\surd(1+\pi^2)\}$ **Q44** $8a$

Exercise 33a, page 342

1 a $2\sin x - 2x\cos x + c$ **b** $\frac12(x-1)e^x + c$
c $\frac14\sin 2x - \frac12 x\cos 2x + c$ **d** $\frac19 x^3(3\ln x - 1)+c$
e $x\sin(x+2)+\cos(x+2)+c$
f $\frac1{72}(1+x)^8(8x-1)+c$ **g** $\frac12 xe^{2x} - \frac14 e^{2x}+c$
h $\frac12 e^{x^2}+c$ **i** $-\frac1x(\ln x+1)+c$ **j** $x\tan x + \ln\cos x + c$
k $(n+1)^{-2}x^{n+1}\{(n+1)\ln x - 1\}+c$
l $(\ln 3)^{-2}\times 3^x(x\ln 3 - 1)+c$

2 a $x\ln 2x - x + c$ **b** $x\sin^{-1}3x + \frac13\sqrt{(1-9x^2)}+c$
c $2y(\ln y - 1)+c$ **d** $\theta\tan^{-1}\frac{\theta}{2} - \ln(4+\theta^2)+c$
e $t\cos^{-1}t - \sqrt{(1-t^2)}+c$ **f** $2e^{\sqrt x}(\sqrt x - 1)+c$

3 a $\frac13 e^{x^3}(x^3-1)+c$ **b** $-\frac12 e^{-x^2}+c$ **c** $-\frac12 e^{-x^2}(1+x^2)+c$
d $\frac12 x^2\sin x^2 + \frac12\cos x^2 + c$ **e** $\frac12 x^2\tan x^2 + \frac12\ln\cos x^2 + c$

4 a $\frac13 x^2\sin 3x + \frac29 x\cos 3x - \frac2{27}\sin 3x + c$
b $e^x(x^3-3x^2+6x-6)+c$
c $\frac18\cos 2x(1-2x^2)+\frac14 x\sin 2x + c$
d $-e^{-x}(x^2+2x+2)+c$
e $\frac16 x^3 + \frac18(2x^2-1)\sin 2x + \frac14 x\cos 2x + c$
f $\frac14 x^2\{1 - 2\ln x + 2(\ln x)^2\}+c$

5 a $\frac18\sin 2x - \frac14 x\cos 2x + c$ **b** $-e^{-x}(1+x)+c$
c $\frac1{168}(1+2x)^6(12x-1)+c$ **d** $\frac12(\ln y)^2+c$
e $\frac12(1+u^2)\tan^{-1}u - \frac12 u + c$ **f** $-\frac12 e^{-x^2}+c$
g $-e^{-x}(x^3+3x^2+6x+6)+c$ **h** $-\frac1{14}(1-x^2)^7+c$
i $\frac14 t^2 - \frac14 t\sin 2t - \frac18\cos 2t + c$ **j** $\frac19 e^{3v}(3v-1)+c$

6 a $x\tan x + \ln\cos x - \frac12 x^2 + c$

7 a $\frac12\pi - 1 \approx 0.571$ **b** $e - 2 \approx 0.718$ **c** $e^2+1\approx 8.39$
d $\frac12\pi - 1 \approx 0.571$ **e** $\pi^2/4 \approx 2.47$
f $50 - 99/(4\ln 10)\approx 39.2$

Exercise 33b, page 346

1 a $2(1-4x^2)^{-1/2}$ **b** $3(2+6x+9x^2)^{-1}$
c $-\frac23(1-4x^2)^{-1/2}$ **d** $2(8+2x-x^2)^{-1/2}$ **e** $(x^2+4)^{-1}$
f $2(4-9x^2)^{-1/2}$ **g** $-(1+x^2)^{-1}$ **h** $\frac{1}{x\sqrt{(x^2-1)}}$
i $2x^3(1+x^4)^{-1}+2x\tan^{-1}x^2$ **j** 0

2 a $\frac13\tan^{-1}\frac x3 + c$ **b** $3\sin^{-1}\frac y2 + c$ **c** $\frac23\tan^{-1}3u + c$
d $\frac12\sin^{-1}4x+c$ **e** $\frac1{\sqrt3}\tan^{-1}\frac{2t}{\sqrt3}+c$
f $\frac12\sin^{-1}\frac{2x}{\sqrt5}+c$ **g** $\frac1{\sqrt6}\tan^{-1}\frac{\sqrt3 y}{\sqrt2}+c$
h $\frac1{3\sqrt6}\sin^{-1}\sqrt2 x + c$ **i** $\frac1{3\sqrt2}\tan^{-1}\frac{(y-2)\sqrt2}{3}+c$
j $\frac2{\sqrt3}\sin^{-1}\frac{(x-1)\sqrt3}{2}+c$

3 a $2\ln\tan\frac x4 + c$ **b** $\frac12\ln(\sec 2\theta + \tan 2\theta)+c$
c $\frac13\ln\tan\frac32 x + c$ **d** $\frac14\ln(\sec 4\phi+\tan 4\phi)+c$
e $\ln\tan x + c$ **f** $\tan\frac12 y + c$
g $-(1+\tan x)^{-1}+c$ **h** $\ln(1-\cos\theta)+c$
i $\frac13\ln(3+\tan\frac12 x) - \frac13\ln(3-\tan\frac12 x)+c$
j $\tan^{-1}(\frac12\tan\frac14\theta)+c$

4 a $\frac1{\sqrt3}\tan^{-1}(\sqrt3\tan x)+c$
b $\frac14\ln(1+2\tan x) - \frac14\ln(1-2\tan x)+c$
c $\sqrt2\tan^{-1}\left(\frac1{\sqrt2}\tan x\right) - x + c$
d $\frac16\ln(1+3\tan x) - \frac16\ln(1-3\tan x)+c$

5 a $\frac12\ln(x^2+3)+\frac5{\sqrt3}\tan^{-1}\frac x{\sqrt3}+c$
b $\ln(y+3)-(y+3)^{-1}+c$
c $\frac32\ln(u^2+2u+5)+\frac52\tan^{-1}\frac{u+1}2+c$
d $7\sqrt{(4x-x^2)} - 11\sin^{-1}\frac{x-2}{2}+c$
e $\frac12\theta+\frac12\ln(\sin\theta+\cos\theta)+c$
f $\frac12 x + \frac52\ln(\sin x+\cos x)+c$

6 a 1 **b** $\pi/6$

7 b 2 **e** $\frac12$ **f** 1 **g** $-\frac12(\ln 2+1)\approx -0.847$ **h** -1
i $\frac14\pi - \frac12\sin^{-1}\frac23 \approx 0.4205$ **j** $\frac1{20}\pi$

8 a tends to π **b** tends to infinity

9 15π **10** 256/3

Exercise 33c, page 348

1 a $\frac1{13}e^{3x}(3\cos 2x + 2\sin 2x)+c$
b $\frac1{25}e^{4x}(4\sin 3x - 3\cos 3x)+c$
c $\frac25 e^{-t}(\sin\frac12 t - 2\cos\frac12 t)+c$
d $\frac15 e^x\{\sin(2x+1) - 2\cos(2x+1)\}+c$
e $\frac18 e^{2\theta}(2+\cos 2\theta+\sin 2\theta)+c$

2 $\frac12\tan x\sec x + \frac12\ln(\sec x+\tan x)+c$

3 a $\frac1{16}x^4(4\ln x - 1)+c$
b $y\tan^{-1}2y - \frac14\ln(1+4y^2)+c$ **c** $-\frac12 e^{-x^2}+c$
d $\frac19(\sin 3x - 3x\cos 3x)+c$
e $\frac14\cos 2x(1-2x^2)+\frac12 x\sin 2x + c$
f $\frac1{13}e^{3x}(3\sin 2x - 2\cos 2x)+c$
g $\frac14 e^{u^2}(u^2-1)+c$
h $\frac1{168}(2x-1)^6(12x+1)+c$
i $\frac14(x^2-1)\ln(x-1) - \frac18 x^2 - \frac14 x + c$
j $x(\ln 3x - 1)+c$ **k** $\frac14 e^{2x}(2x^2-2x+1)+c$
l $\frac25 e^{-y}(\sin\frac12 y - 2\cos\frac12 y)+c$ **m** $-\frac14 x^{-2}(1+2\ln x)+c$
n $t\sin^{-1}\frac t3 + \sqrt{(9-t^2)}+c$ **o** $3x(\ln x - 1)+c$
p $\frac16 y^3 + \frac18(2y^2-1)\sin 2y + \frac14 y\cos 2y + c$
q $\frac12\sin x^2 + c$ **r** $\frac12 x^2(\ln x^2 - 1)+c$
s $\frac12\sin\theta^2 - \frac12\theta^2\cos\theta^2 + c$
t $\frac14(2x^3-3x)\sin 2x + \frac38(2x^2-1)\cos 2x + c$

4 $C = \dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$,

$S = \dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$ 6 $\frac{64}{21}$ 8 $\frac{1}{10}(1+e^{3\pi})$

Exercise 33d, page 351

1 $9\pi/2$ 2 a^2 3 $2c^2\ln a$ 5 $\frac{3}{8}\pi a^2$ 6 $24a$, $12\pi a^2$

Exercise 33e, page 351

1 a $\frac{1}{3}\sqrt{(x^2+1)^3}+c$ b $\frac{2}{3}\sqrt{(x^3+3x-4)}+c$
c $\sin u - \frac{2}{3}\sin^3 u + \frac{1}{5}\sin^5 u + c$
d $\tan\theta + \frac{2}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta + c$
e $\frac{1}{5}\sec^5 x - \frac{2}{3}\sec^3 x + \sec x + c$ f $-\frac{1}{2}\cos x^2 + c$
g $2\tan\sqrt{x}+c$ h $\frac{1}{4}\ln(2x^2+3)+c$ i $-\frac{1}{2}e^{-x^2}+c$
j $\ln\sec^2\dfrac{\theta}{2}+c$

2 a $\frac{1}{5}(x+1)\sqrt{(2x-3)^3}+c$ b $\frac{1}{324}(24x+1)(3x-1)^8+c$
c $y+16(y-4)^{-1}+c$ d $\dfrac{1}{\sqrt{5}}\sin^{-1}\dfrac{\sqrt{5}y}{2}+c$
e $\dfrac{1}{3\sqrt{3}}\tan^{-1}\sqrt{3}u+c$ f $\dfrac{1}{2\sqrt{2}}\tan^{-1}\dfrac{u-3}{2\sqrt{2}}+c$
g $\dfrac{1}{\sqrt{2}}\sin^{-1}\dfrac{(x-1)\sqrt{2}}{3}+c$
h $\frac{1}{2}y\sqrt{(4-y^2)}+2\sin^{-1}\dfrac{y}{2}+c$ i $\sec^{-1}3x+c$
j $\frac{2}{3}\tan^{-1}(\frac{1}{3}\tan\frac{1}{2}\theta)+c$

3 a $\frac{1}{3}e^{3x}+c$ b $(\ln 10)^{-1}10^y+c$ c $-\frac{1}{3}e^{-x^3}+c$
d $\frac{1}{3}\ln x+c$ e $\frac{1}{3}\ln(3x+4)+c$ f $-\frac{1}{2}\ln(2x-3)+c$
g $\frac{1}{3}\ln(x+3)+c$ h $\frac{1}{2}\ln\dfrac{1+x}{1-x}+c$ i $x(\ln x-1)+c$
j $2e^{\sqrt{x}}(\sqrt{x}-1)+c$

4 a $\frac{1}{6}\ln\dfrac{3+x}{3-x}+c$ b $\frac{1}{3}\ln\dfrac{y-3}{y}+c$
c $\dfrac{1}{x}+\ln\dfrac{x-1}{x}+c$ d $4(4-x)^{-1}+\ln(4-x)+c$
e $\dfrac{4x+5}{2(x+1)^2}-\ln(x+1)+c$ f $\ln\dfrac{(x+1)^3}{x^2-x+1}+c$

5 a $2x\sin\frac{1}{2}x+4\cos\frac{1}{2}x+c$ b $\frac{1}{2}e^x(x-1)+c$
c $\ln\sin y - y\cot y + c$ d $-\frac{1}{252}(21y+1)(1-3y)^7+c$
e $(\ln 3)^{-2}3^x(x\ln 3-1)+c$ f $\frac{1}{4}x^2(2\ln 2x-1)+c$
g $t(\ln t-1)+c$ h $x\tan^{-1}3x-\frac{1}{6}\ln(1+9x^2)+c$
i $4^x(\ln 4)^{-1}+c$ j $x(6-x^2)\cos x+3(x^2-2)\sin x+c$

6 a $\frac{1}{4}\ln(4x^2+3)-\dfrac{1}{2\sqrt{3}}\tan^{-1}\dfrac{2x}{\sqrt{3}}+c$
b $4\sqrt{(1+2y-y^2)}-3\sin^{-1}\dfrac{y-1}{\sqrt{2}}+c$
c $\frac{2}{5}\theta-\frac{1}{5}\ln(2\cos\theta-\sin\theta)+c$
d $\frac{2}{5}\ln(4\sin x+3\cos x)-\frac{1}{5}x+c$

7 a $\pi/9$ b $\dfrac{1}{2\sqrt{2}}\tan^{-1}\dfrac{18-10\sqrt{2}}{31}$ c $\frac{1}{2}$ d $\frac{256}{693}$
e $231\pi/2048$ f $5\pi/128$ g $35\pi/128$
h 0 i 128/230 945 j $\frac{1}{2}\ln\frac{1}{5}$

8 a $-\frac{1}{5}\cos 5x+c$ b $3\sin\frac{1}{3}x+c$ c $\frac{1}{5}\ln\sec 5x+c$
d $2\ln\sin\frac{1}{2}x+c$ e $\ln\tan\frac{1}{2}x+c$
f $\ln(\sec x+\tan x)+c$ or $\ln\tan(\frac{1}{2}x+\frac{1}{4}\pi)+c$

9 a $3\tan\frac{1}{3}x+c$ b $-\frac{1}{4}\cot 4x+c$ c $\frac{1}{2}x-\frac{1}{4}\sin 2x+c$
d $\frac{1}{2}x+\frac{1}{4}\sin 2x+c$ e $\tan x-x+c$ f $-\cot x-x+c$

10 a $\frac{1}{3}\cos^3 x-\cos x+c$ b $\sin x-\frac{1}{3}\sin^3 x+c$
c $\frac{1}{2}\tan^2 x+\ln\cos x+c$ d $-\frac{1}{2}\cot^2 x-\ln\sin x+c$
e $\frac{1}{2}\tan x\sec x+\ln\sqrt{(\sec x+\tan x)}+c$
f $\frac{1}{2}\ln\tan\frac{1}{2}x-\frac{1}{2}\cot x\operatorname{cosec} x+c$

11 a $\frac{1}{32}(12x-8\sin 2x+\sin 4x)+c$
b $\frac{1}{32}(12x+8\sin 2x+\sin 4x)+c$
c $x-\tan x+\frac{1}{3}\tan^3 x+c$ d $-\frac{1}{3}\cot^3 x-\cot x+c$
e $\frac{1}{3}\tan^3 x+\tan x+c$ f $x+\cot x-\frac{1}{3}\cot^3 x+c$

12 a $x\sin^{-1}x+\sqrt{(1-x^2)}+c$
b $x\cos^{-1}x-\sqrt{(1-x^2)}+c$
c $x\tan^{-1}x-\frac{1}{2}\ln(1+x^2)+c$
d $x\cot^{-1}x+\frac{1}{2}\ln(1+x^2)+c$
e $x\sec^{-1}x-\ln\{x+\sqrt{(x^2-1)}\}+c$
f $x\operatorname{cosec}^{-1}x+\ln\{x+\sqrt{(x^2-1)}\}+c$

13 a $\dfrac{1}{2\sqrt{3}}\tan^{-1}\dfrac{2x}{\sqrt{3}}+c$ b $\frac{1}{8}\sqrt{(5+8x^2)}+c$
*c $\ln\{x+\sqrt{(1+x^2)}\}+c$ d $\frac{1}{6}\ln(2+3x^2)+c$
e $\frac{1}{3}\sqrt{(3+x^2)^3}+c$ f $\frac{1}{4}\ln(3+2x^2)+\dfrac{1}{\sqrt{6}}\tan^{-1}\dfrac{x\sqrt{2}}{\sqrt{3}}+c$
g $\frac{1}{2}\ln(x^2-4x+7)+c$
h $\frac{1}{2}x\sqrt{(x^2+2)}+\ln\{x+\sqrt{(x^2+2)}\}+c$
i $\frac{3}{2}\ln(x^2-4x+5)-5\tan^{-1}(x-2)+c$
j $\frac{2}{135}(9x-4)\sqrt{(2+3x)^3}+c$

14 a $\dfrac{1}{\sqrt{5}}\sin^{-1}\dfrac{\sqrt{5}x}{2}+c$ b $-\frac{2}{27}(3x+2)\sqrt{(1-3x)}+c$
c $\frac{1}{6}\ln\dfrac{3+x}{3-x}+c$ d $3(16-x)^{-1}+c$
e $-\frac{1}{3}\sqrt{(6-x^2)^3}+c$ f $-\frac{3}{2}\ln(4-x^2)+c$
g $\frac{1}{2}x\sqrt{(4-x^2)}+2\sin^{-1}\dfrac{x}{2}+c$ h $-\frac{1}{2}\sqrt{(7-2x^2)}+c$
i $-\frac{1}{4}\sqrt{(3-4x^2)}-\sin^{-1}\dfrac{2x}{\sqrt{3}}+c$
*j $\ln\{x+\sqrt{(x^2-9)}\}+c$

15 a $\dfrac{180}{\pi}\sin x^\circ+c$ b $\frac{1}{32}\sin 4x-\frac{1}{8}x\cos 4x+c$
c $2\ln\tan\frac{1}{2}\theta+c$
d $-\frac{1}{7}\cos^7 x+\frac{2}{9}\cos^9 x-\frac{1}{11}\cos^{11}x+c$
e $y\tan y+\ln\cos y+c$ f $\sin x-x\cos x+c$
g $-\frac{1}{2}\cos x^2+c$ h $(u^2-2)\sin u+2u\cos u+c$
i $\frac{1}{8}y-\frac{1}{32}\sin 4y+c$ j $-\frac{1}{42}(3\cos 7x+7\cos 3x)+c$

*See also pp. 382, 383.

Chapter 34

Q1 $y = Ax + B, y = 3x - 5$ Q2 $y = x^3 + A$
Q3 $x^2 + y^2 = A$ Q4 $s = \frac{1}{2}at^2 + At + B; s = ut + \frac{1}{2}at^2$
Q5 a $\frac{d^2y}{dx^2} = 0$ b $\frac{dy}{dx} = \frac{y}{x}$ c $\frac{dr}{d\theta} + r\tan\theta = 0$
d $\frac{dy}{dx} = -\frac{y}{x}$ e $\frac{dy}{dx} = y$ f $x\frac{dy}{dx} = y\ln y$
g $y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$ h $x\frac{dy}{dx}\ln x = y$
i $(1+x^2)\frac{dy}{dx}\tan^{-1}x = y$
Q7 a $x^2 - y^2 + A = 0$ b $y = Ax$ c $x = Ae^{y^2/2}$
d $x = A\sin y$ e $\ln\sqrt{\frac{y-1}{y+1}} = e^x + A$
f $y^2 = 2\sqrt{(x^2+1)} + A$
Q8 $v^2 = u^2 + 2as$
Q9 a $x^2y = x + A$ b $t^2\ln x = 3\sin t + A$
c $x^2\sin u = \ln(kx)$ d $xe^y = 2x + A$
Q10 a $x, x^2y = \frac{1}{2}e^{x^2} + A$ b $x, x^2e^y = \frac{1}{3}x^3 + A$
c $\frac{1}{x}, xy^2 = \ln(kx)$ d $r, r^2\tan\theta = 2\theta + A$
Q11 $y = \frac{1}{2} + Ae^{-x^2}; y = \frac{1}{2} - e^{-x^2}$ Q12 x^2
Q13 a $y = 1 + x\tan x + A\sec x$ b $y = x - 4 + Ae^{-x}$
Q14 a $\left(\frac{y}{x}\right)^2$ b $\frac{d^2y}{dx^2} + xy$ c $x\sqrt{(x^2+y^2)}$
Q15 a, b, d, e
Q16 a $xe^{x/y} = A$ b $x^2 - 2xy = A$ c $x - 2y + Axy = 0$
Q17 $y = x\ln(Ax^2)$
Q19 a $y = 2x\ln x + Ax + B$
b $y = 2\sin x - x\cos x + Ax + B$
c $y = x^3 + A\ln x + B$ d $y^2 = Ax + B - 2\cos x$
Q20 a $y = Ae^{Bx}$ or $y = C$ b $y = -3x^2 + 3x + 2$
Q21 $y = \ln x^2 + Ax^{-1} + B$
Q22 a $x = a\cos(2t + \varepsilon)$ b $y = a\cos(3x + \varepsilon)$
c $y = -\frac{8}{3}x^3 + Ax + B$
Q23 $2\frac{dx}{dt}$
Q24 $x = 2\cos(\frac{3}{2}t)$ a $\frac{dx}{dt} = \frac{3}{2}\sqrt{(4-x^2)}$
b $\frac{dx}{dt} = -3\sin(\frac{3}{2}t)$
Q25 a $x = a\cos nt$ b $x = a\sin nt$
Q26 a $x = a\cos nt$ b $x = a\sin nt$ Q27 $\pm\pi/3$
Q28 a $y = a\cos(2x+\varepsilon) - 1$ b $\theta = a\cos(\sqrt{2}t + \varepsilon) + 3$
c $x = A + Bt - \frac{1}{2}t^2 - \frac{3}{8}t^3$
Q29 a $y = Ae^x + Be^{-x}$ b $y = Ae^{2x} + Be^{10x}$
c $y = Ae^{-x/2} + Be^{3x}$ d $y = Ae^{x/3} + Be^{x/5}$

Q30 a $z = Ae^{5t} + Be^{-5t}$ b $z = Ae^{t/2} + Be^{-t/3}$
Q31 a $z = e^{5t} - e^{-5t}$ b $z = 12(e^{t/2} - e^{-t/3})$
Q32 $f(x) = 2e^{5x} - e^x$
Q33 a $V = (At + B)e^{-3t}$ b $r = (At + B)e^{3t/10}$
Q34 $y = xe^x$
Q35 a $y = e^x(A\cos 7x + B\sin 7x)$
b $V = e^{-3t}(A\cos 5t + B\sin 5t)$
c $r = A\cos\frac{t}{6} + B\sin\frac{t}{6}$
Q36 a $y = 5e^x\sin 7x$ b $V = e^{-3t}(\cos 5t + 2\sin 5t)$
c $r = 2\sin\frac{t}{6}$

Exercise 34a, page 357

1 a $\frac{dy}{dx} = \frac{3}{2}$ b $\frac{dy}{dx} = \frac{y+\frac{1}{2}}{x}$ c $\frac{dy}{dx} = \frac{y}{x}$
d $\frac{dy}{dx} = -\frac{x}{y}$ e $\frac{dy}{dx} = -\frac{y}{x}$ f $\frac{dy}{dx} = \frac{y}{x-4}$
2 a $\frac{d^2y}{dt^2} = -9y$ b $\frac{d^2y}{dt^2} = 3\frac{dy}{dt}$ c $\frac{d^2y}{dx^2} = 9y$
d $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$ e $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$
3 $3x - 10y - 35 = 0$ 4 $y = x^2 - 3x + 1$
5 $s = A - 3t^2, s = 12 - 3t^2$
6 a $y = e^x - 3x\cos x + 3\sin x - 1$
b $y = e^x - 3x\cos x + 3\sin x - e^{\pi/2}$
7 a $y = Ae^x$ b $y = \frac{2}{15}(x-1)(3x+2)\sqrt{(x-1)} + A$
c $y = A(x+2)$ d $x = \frac{1}{2}y + \frac{1}{4}\sin 2y + A$
e $v - 1 = Ave^u$ f $y = x\ln x - x + A$ g $\sin y = Ae^x$
h $x = y\tan^{-1}y - \ln\sqrt{(1+y^2)} + A$ i $y^2 = x^2 - 2x + A$
j $y = A\sqrt{\frac{x-1}{x+1}}$ k $r = \ln\left(A\tan\frac{\theta}{2}\right)$ l $y + 3 = Ae^{-1/x}$
m $y = Axe^x$ n $\cos\theta\sin\phi = A$
o $r = \theta\tan\theta + \ln(A\cos\theta)$ p $2y^2 = x^2(\ln x^2 - 1) + A$
q $r = -\theta - \ln(\cos\theta - \sin\theta) + A$ r $2y + 3 = A(x-2)^2$
s $x = A - \frac{1}{2}e^{-t}(\cos t + \sin t)$ t $y = 2\tan(2e^{-x} + A)$
8 a $y = \tan\theta$ b $(y-2)^2 = 9e^{x^2}$ c $y = \frac{7x+1}{7-x}$
d $y = \sin(x - \frac{1}{6}\pi)$
9 83.4 minutes 10 9.05 kg, 34.7 minutes

Exercise 34b, page 361

1 a $y^2 = \frac{A}{x} - \frac{1}{x^2}$ b $y^2 = \frac{1}{x^2}(\tan 2x + A)$
c $x\ln y = \sec x + A$ d $(1-2x)e^y = \tan x + A$
e $t^2e^s = t\sin t + A$ f $r^2e^u = \cot u + A$
2 a $x^2\sin y = 3x^2 + A$ b $xy = e^x + A$
c $x\tan y = e^{x^2} + A$ d $ye^x = \ln(Ay)$

3 a $y = e^{-2x}(\sin x + A)$ **b** $s = \frac{1}{2} + Ae^{-t^2}$
c $y = e^{-x^2}(1 + Ae^{-x})$ **d** $r = (\theta + A)\,\mathrm{cosec}^2\,\theta$
e $r = (\theta + A)\cos\theta$ **f** $y = x^{-2}(\sin x + A)$
g $y = x\ln\dfrac{A(x-1)}{x}$ **h** $y = \dfrac{x}{3} + 3 + Ax^{-1/2}$
i $y = (x - \sin x + A)\cot\frac{1}{2}x$
j $y = (x-2)^{-1} + A(x-2)^{-3}$

4 a $x^3 = Ae^{y/x}$ **b** $x^2(x^2 - 2y^2) = A$
c $\tan^{-1}\dfrac{y}{x} = \ln(Ax)$ **d** $3x - y + Axy = 0$
e $y = Ae^{x^2/(2y^2)}$ **f** $2x = (2x - y)\ln\{A(2x-y)\}$
g $\sin^{-1}\dfrac{y}{x} = \ln(Ax)$ **h** $y = x(Ax - 1)$
i $(x+y)(2x-y)^2 = A$ **j** $\dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{2y-x}{x\sqrt{3}}\right) = \ln(Ax)$

5 $x^2 - y^2 - 2xy + 4x = A$

6 a $y - 2 = Ae^{(x-3)/(y-2)}$
b $(x - y - 3)^2(x + 2y - 3) = A$

7 $x^2 + y^2 - 2xy - 4x - 8y + A = 0$

8 a $x - y + A = \ln(2x + y)$ **b** $x + y - 1 = Ae^{x-y}$

9 a $y = (x + A)(x+3)^2$ **b** $x = (x - y)\ln(Ax)$
c $(y+3)\sin x = A - \frac{1}{2}e^{-2x}$ **d** $\sin y = \dfrac{Ax - 1}{x(x+3)}$
e $\tan^{-1}\dfrac{2(y-1)}{x-2} = \ln\{A(x^2 + 4y^2 - 4x - 8y + 8)\}$
f $y + 2 = xe^{2x} + Ae^{x}$ **g** $y^{x^2} = A\sin x$
h $(2r+3)\tan\theta = 3\theta + A$ **i** $xy = Ae^{y/x}$
j $x^2 + y^2 - 2xy + 2x - 6y + A = 0$

10 a $y = (x+1)^4$ **b** $u = \sin\theta + 2\,\mathrm{cosec}\,\theta$
c $x^2 - 2xy - y^2 = 17$ **d** $4y^2 - x^2 = 2y^2\ln\dfrac{y}{2}$
e $e^{t}(x-1) = 1 - 1/t$

Exercise 34c, page 365

1 a $y = x\ln x + Ax + B$
b $y = \ln\sec\theta + A\ln(\sec\theta + \tan\theta) + B$
c $y = 2e^{-x} + Ax + B$ **d** $y = A\ln\{B(2x+1)\}$
e $y = e^{x}(x-1) + Ax^2 + B$

2 a $A + Bx = e^{-2y}$ **b** $6x = y^3 + Ay + B$ or $y = C$
c $y = A\tan^{-1}x + B$

3 $s = \ln\{A(t+B)^{10}\}$

4 a $y = \ln\mathrm{cosec}\,x + A\ln\tan\dfrac{x}{2} + B$
b $\frac{1}{9}x^3 + \frac{1}{6}x^2 - \frac{2}{3}x + A\ln(x+2) + B$
c $36y = 6x^2\ln x - 5x^2 + Ax^{-1} + B$

5 $y + \pi/4 = \frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{(1-x^2)}$

6 a $s = a\cos(5t + \varepsilon)$ **b** $y = a\cos(\frac{3}{2}x + \varepsilon)$
c $s = A + B\theta - \frac{1}{27}\theta^3$ **d** $y = a\cos\left(\dfrac{13}{2}t + \varepsilon\right)$

7 $s = 4\cos\dfrac{3t}{4}$ **8** $\dfrac{d^2x}{dt^2} = -9x$, $x = 2\cos\left(3t \pm \dfrac{\pi}{6}\right)$

9 At O; 12 s **10** 0, 6 s

11 $x = \sqrt{2}\cos(2t - \pi/4)$, $\sqrt{2}$ m, $-\pi/4$ s, π s

12 5 m, 0.927 s, 4π s

13 a $\pi/2$ **b** $\pi/12$ **c** $\pi/2$ **d** π **e** $7\pi/12$

14 a $y = a\cos(2x + \varepsilon) - 3$ **b** $\theta = a\cos\left(\dfrac{3}{12}t + \varepsilon\right) + \dfrac{1}{3}$
c $s = A + Bt + \frac{1}{6}t^2 - \frac{2}{9}t^3$ **d** $x = \sqrt{2}\cos\left(2t + \dfrac{\pi}{4}\right) - 2$

15 a $y = A + Be^{2x} - x$ **b** $y = A + Be^{x}$ or $y = C$
c $x = a\cos(t + \varepsilon)$ **d** $y = A\ln(Bx)$ or $y = C$
e $(3y-1)^{2/3} = Ax + B$ or $y = C$

Exercise 34d, page 372

1 $y = Ae^{2x} + Be^{5x}$ **2** $y = Ae^{3x} + Be^{-2x}$
3 $y = e^{2x}(A\cos 5x + B\sin 5x)$ **4** $y = Ae^{x/5} + Be^{-x/5}$
5 $x = (At + B)e^{5t}$ **6** $x = (At + B)e^{t/4}$
7 $u = Ae^{t/2} + Be^{t/3}$ **8** $y = A + Be^{-5x}$
9 $x = Ae^{3t/2} + Be^{-t}$ **10** $r = A + Be^{\theta}$
11 $y = e^{-2x}(A\cos 4x + B\sin 4x)$
12 $r = A\cos\frac{1}{10}t + B\sin\frac{1}{10}t$ **13** $y = Ae^{-x} + Be^{-2x} + C$
14 $y = (Ax + B)e^{2x} + 2$ **15** $x = e^{-t}(A\cos t + B\sin t) + \frac{1}{2}$
16 $y = 3e^{6x} + 2e^{-x}$ **17** $u = 5\sin 3t + 4\cos 3t$
18 $r = (1-t)e^{6t}$ **19** $z = 4\cos\frac{1}{2}t$ **20** $u = e^{\theta}$

Chapter 35

Q1 7.7 m **Q2** 25.1 m
Q3 0.816, 304/375 ≈ 0.811
Q4 a $\frac{1}{2}d(y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + 2y_6 + 2y_7 + y_8)$
b $\frac{1}{2}d(y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + 2y_6 + 2y_7 + 2y_8 + y_9)$
Q5 240, to nearest 10 (First two ordinates are further apart than the others).
Q6 a $a(y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + 2y_6 + 2y_7 + y_8)/14$
b $a(y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n)/(2n-2)$
Q7 0.694 **Q8** 37.6 m
Q9 $\frac{1}{3}d(y_1 + 4y_2 + 2y_3 + 4y_4 + y_5)$
Q10 0.6931 **Q11** 304/375

Exercise 35a, page 376

1 a 0.347 **b** 0.350 **c** 0.347 **2 b** 1.49 **c** 1.46
3 $166\frac{2}{3}$ **4** 0.7468
5 a 49.4 cm^2 **b** 49.9 cm^2; 3.12 **6** 310 cm^3
7 1.86 litres **8 a** 0.2983 **b** 0.2983
9 3.988, 0.997 **10** 3.142 **11** $1\frac{1}{3}$

13 $1+10x^3+45x^6+120x^9$; 0.204; 0.204
14 **a** 0.879 **b** 0.879 **15** 0.867

Chapter 36

Q1 **a** $\sinh A+\sinh B=2\sinh\frac{1}{2}(A+B)\cosh\frac{1}{2}(A-B)$
b $\cosh A+\cosh B=2\cosh\frac{1}{2}(A+B)\cosh\frac{1}{2}(A-B)$
c $\cosh A-\cosh B=2\sinh\frac{1}{2}(A+B)\sinh\frac{1}{2}(A-B)$
d $\operatorname{sech}^2\theta=1-\tanh^2\theta$ **e** $\operatorname{cosech}^2\theta=\coth^2\theta-1$
f $\cosh 3\theta=4\cosh^3\theta-3\cosh\theta$
g $\tanh 3\theta=(3\tanh\theta+\tanh^3\theta)/(1+3\tanh^2\theta)$
Q2 **a** $2\sinh 2x$ **b** $\frac{1}{2}\cosh\frac{1}{2}x$ **c** $\sinh\frac{1}{3}x$
d $2\cosh 4x$ **e** $2\sinh x\cosh x=\sinh 2x$
f $6\cosh^2 2x\sinh 2x$
Q3 **a** $x=a\cosh\theta$ **b** $x=a\sinh\theta$
Q5 **a** $x=a\tan\theta$ (or $x=a\sinh\theta$)
b $x=a\tanh\theta$ (or $x=a\sin\theta$ or $x=a\cos\theta$)
Q7 **a** L^0 **b** L^{-1}
Q8 The two expressions differ by a constant (possibly zero).
Q10 **a** 0.8813 **b** 1.3169 **c** 0.5515
Q11 $\frac{1}{12}\cosh 3\theta-\frac{3}{4}\cosh\theta+c$

Exercise 36a, page 378

3 $-1<\tanh x<1$
14 $\dfrac{\tanh A+\tanh B}{1+\tanh A\tanh B}$, $\dfrac{\tanh A-\tanh B}{1-\tanh A\tanh B}$
15 ln 1.8 **16** $a^2=b^2+c^2$

Exercise 36b, page 381

2 $\sinh x+c$, $\cosh x+c$
3 **a** $3\sinh 3x$ **b** $2\cosh 2x$ **c** $\sinh 2x$
d $6\sinh^2 x\cosh x$ **e** $6\operatorname{sech}^2 2x$
f $-\operatorname{sech}^2 x\tanh x$ **g** $3\sinh 6x$
h $-\frac{1}{2}\operatorname{cosech}^2 x\sqrt{(\tanh x)}$ **i** $2\tanh\frac{1}{2}x\operatorname{sech}^2\frac{1}{2}x$
4 Domain $=\{x:x\in\mathbb{R},-1<x<+1\}$, range $=\mathbb{R}$; $\tanh x$ is odd.
5 **a** $2\operatorname{cosech} 2x$ **b** e^{2x} **c** $\frac{1}{2}\operatorname{sech}^2\frac{1}{2}x$
6 **a** $\frac{1}{2}\tanh 2x+c$ **b** $-\operatorname{sech} x+c$
7 4 **8** $1/\sqrt{(1+x^2)}$ **10** 0
11 **a** $\frac{1}{10}\cosh 5x+\frac{1}{2}\cosh x+c$
b $\frac{1}{8}\sinh 4x+\frac{1}{4}\sinh 2x+c$
12 $2/(e-1)^2$
13 $bx\cosh\theta-ay\sinh\theta-ab=0$,
$ax\sinh\theta+by\cosh\theta-(a^2+b^2)\sinh\theta\cosh\theta=0$,
$\dfrac{4a^2x^2}{(a^2+b^2)^2}-\dfrac{b^2}{4y^2}=1$

14 $\left(\dfrac{d^2y}{dx^2}-4y\right)(3\coth 3x\coth 2x-2)$
$=5\left(\coth 2x\dfrac{dy}{dx}-2y\right)$
15 $x-\dfrac{x^3}{3}+\dfrac{2x^5}{15}$
16 $x+\dfrac{x^3}{3}+\dfrac{x^5}{5}+\ldots+\dfrac{x^{2n+1}}{2n+1}+\ldots$, $\ln\sqrt{\dfrac{1+x}{1-x}}$
17 $\ln\sqrt{\dfrac{1+x}{1-x}}$ **18** $(\pm\sinh^{-1}\frac{1}{2},-5\sqrt{5})$ min., $(0,-11)$ max.

Exercise 36c, page 384

1 $\sinh^{-1}\frac{1}{3}x+c$ **2** $3\sin^{-1}(x-2)+c$ **3** $\cosh^{-1}(2x)+c$
4 $\cosh^{-1}\dfrac{x+2}{2}+c$ **5** $\frac{2}{3}\sqrt{3}\tan^{-1}\dfrac{2x+1}{\sqrt{3}}+c$
6 $\sinh^{-1}(x-3)+c$ **7** $\frac{1}{2}\cosh^{-1}(8x+1)+c$
8 $\frac{1}{2}\sin^{-1}\dfrac{8x-3}{3}+c$ **9** $\sinh^{-1}\frac{1}{2}\approx 0.481$
10 $\cosh^{-1}\frac{4}{3}\approx 0.795$ **11** $\sinh^{-1}4-\sinh^{-1}3\approx 0.2763$
12 $\frac{1}{2}\sqrt{2}\sinh^{-1}\sqrt{2}\approx 0.810$ **13** $\frac{1}{2}x+\frac{1}{4}\sinh 2x+c$
14 $\frac{1}{3}\sinh^3 x+\sinh x+c=\frac{1}{12}\sinh 3x+\frac{3}{4}\sinh x+c$
15 $(12x-8\sinh 2x+\sinh 4x)/32+c$ **16** $x-\tanh x+c$
17 $\ln\cosh x+c$ **18** $\ln\sinh x-\frac{1}{2}\operatorname{cosech}^2 x+c$
19 $x-\tanh x-\frac{1}{3}\tanh^3 x+c$
20 $\ln\{(e^x-1)/(e^x+1)\}+c=\ln\tanh\frac{1}{2}x+c$

Chapter 37

Q2 $u^2+(v-\frac{1}{2})^2=\frac{1}{4}$
Q3 **a** $\cos 5\theta+i\sin 5\theta$ **b** $\cos 2\theta-i\sin 2\theta$
c $\cos 3\theta+i\sin 3\theta$ **d** $\cos 5\theta+i\sin 5\theta$
e $\cos(\theta-\phi)+i\sin(\theta-\phi)$
f $\cos(\theta+\phi)+i\sin(\theta+\phi)$
Q4 **a** $\cos\theta+i\sin\theta$ **b** $\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi=-\frac{1}{2}+\frac{1}{2}\sqrt{3}\,i$
c $\cos\frac{3}{2}\theta+i\sin\frac{3}{2}\theta$
Q5 $1,-\frac{1}{2}\pm\frac{1}{2}\sqrt{3}\,i$ **Q7** $-1,\frac{1}{2}\pm\frac{1}{2}\sqrt{3}\,i$
Q10 $\displaystyle\sum_{r=0}^{n}\frac{w^r z^{n-r}}{r!(n-r)!}$
Q13 $\cosh z=\cos iz$, $\sinh z=-i\sin iz$

Exercise 37a, page 386

1 **a** $4+5i$ **b** $2+3i$ **c** $-1+7i$ **d** $3\frac{1}{2}+\frac{1}{2}i$ **e** $-7+24i$
f $-117+44i$ **g** $(3-4i)/25$ **h** $(1-i)/2$
2 5, 0.927; $\sqrt{2}$, 0.785 **a** $\sqrt{41}$, 0.896 **b** $\sqrt{13}$, 0.983
c $\sqrt{50}$, 1.713 **d** $\sqrt{12.5}$, 0.142 **e** 25, 1.855
f 125, 2.782 **g** 0.2, −0.927 **h** $1/\sqrt{2}$, −0.785
3 $2\pm 7i$ **5** **a** $(x-10)^2+y^2=25$ **b** $x=y$

Exercise 37b, page 388

1 a enlargement $\times a$ b translation $\begin{pmatrix} a \\ 0 \end{pmatrix}$

c translation $\begin{pmatrix} a \\ b \end{pmatrix}$

d rotation through arctan (b/a) and enlargement $\times |c|$

e reflection in real axis

5 $1 \leqslant |w| \leqslant 4$ 6 $0 \leqslant \arg(w) \leqslant \pi/2$

7 $\{w : |w| = 25, 0 \leqslant \arg(w) \leqslant \pi/3\}$

8 a $v = 20$ b the positive v-axis c the negative v-axis

9 The region between the circles $(u - \frac{1}{2})^2 + v^2 = \frac{1}{4}$ and $(u - \frac{1}{4})^2 + v^2 = \frac{1}{16}$.

10 $x = k \mapsto$ the parabola $v^2 = 4k^2(k^2 - u)$
$y = k \mapsto$ the parabola $v^2 = 4k^2(k^2 + u)$

Exercise 37c, page 390

1 $\cos 5\theta + i \sin 5\theta$ 2 $\cos 4\theta + i \sin 4\theta$

3 $\cos\theta + i \sin\theta$ 4 $\cos\theta + i\sin\theta$ 5 $\cos 8\theta + i \sin 8\theta$

6 $\cos 3\theta + i \sin 3\theta$ 7 $\cos(2\phi + 3\theta) + i \sin(2\phi + 3\theta)$

8 $\cos(6\theta - 6\phi) + i \sin(6\theta - 6\phi)$

9 $1, -1, i, -i; \frac{1}{2}\sqrt{2}(1 \pm i), \frac{1}{2}\sqrt{2}(-1 \pm i)$

10 $1, \cos\frac{2}{5}\pi \pm i \sin\frac{2}{5}\pi, \cos\frac{4}{5}\pi \pm i\sin\frac{4}{5}\pi$;
$(x^5 - 1) = (x - 1)(x^2 - 2x\cos\frac{2}{5}\pi + 1)(x^2 - 2x\cos\frac{4}{5}\pi + 1)$

11 $1, -1, \frac{1}{2}(1 \pm \sqrt{3}\,i), \frac{1}{2}(-1 \pm \sqrt{3}\,i); (x^2 + x + 1)(x^2 - x + 1)$

12 Rotate the radius vectors through an angle of $\pi/2n$.

13 When n is a prime number. If n is odd but not prime, the first property will hold for some roots but not for others. The second holds for all n.

14 a $(x - 1)(x^2 - 2x\cos\frac{2}{7}\pi + 1)(x^2 - 2x\cos\frac{4}{7}\pi + 1)$
$(x^2 - 2x\cos\frac{6}{7}\pi + 1)$

b $(x + 1)(x^2 - 2x\cos\frac{1}{5}\pi + 1)(x^2 - 2x\cos\frac{3}{5}\pi + 1)$

c $(x - 1)(x + 1)$

$$\left(x^2 - 2x\cos\frac{\pi}{n} + 1\right)\ldots\left(x^2 - 2x\cos\frac{(n-1)\pi}{n} - 1\right)$$

15 $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ 16 $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

17 $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

21 $\tan 6\theta = \dfrac{6t - 20t^3 + 6t^5}{1 - 15t^2 + 15t^4 - t^6}$,

$$\tan 2n\theta = \frac{\binom{2n}{1}t - \binom{2n}{3}t^3 + \ldots + (-1)^{n-1}\binom{2n}{2n-1}t^{2n-1}}{1 - \binom{2n}{2}t^2 + \ldots + (-1)^n t^{2n}},$$

$$\tan(2n+1)\theta = \frac{\binom{2n+1}{1}t - \binom{2n+1}{3}t^3 + \ldots + (-1)^n t^{2n+1}}{1 - \binom{2n+1}{2}t^2 + \ldots + (-1)^n\binom{2n+1}{2n}t^{2n}},$$

where $t = \tan\theta$

22 $-5.03, -0.668, 0.199, 1.50$

23 $-1.69, -0.431, 0.225, 1.14, 10.8$

24 $u = x^3 - 3xy^2, v = 3x^2y - y^3$ 25 $\dfrac{ac + bd + i(bc - ad)}{c^2 + d^2}$

26 a $\pm(2 + 3i)$ b $\pm\dfrac{1}{\sqrt{2}}(1 + i)$

27 b $Z = \dfrac{(1 - z^n)(1 - z^*)}{4\sin^2\frac{1}{2}\theta}$

c $C = \dfrac{\sin\frac{1}{2}n\theta\cos\frac{1}{2}(n-1)\theta}{\sin\frac{1}{2}\theta}, S = \dfrac{\sin\frac{1}{2}n\theta\sin\frac{1}{2}(n-1)\theta}{\sin\frac{1}{2}\theta}$

28 $C = \dfrac{1 - a\cos\theta - a^{n+1}\cos(n+1)\theta + a^{n+2}\cos n\theta}{1 - 2a\cos\theta + a^2}$,

$S = \dfrac{a\sin\theta - a^{n+1}\sin(n+1)\theta + a^{n+2}\sin n\theta}{1 - 2a\cos\theta + a^2}$

29 Multiply by $\sin\frac{1}{2}\theta$ and $1 - 2a\cos\theta + a^2$ respectively.

30 0

Exercise 37d, page 393

1 a $x^6 - 15x^4y^2 + 15x^2y^4 - y^6 + i(6x^5y - 20x^3y^3 + 6xy^5)$

b $\dfrac{3\cos\theta - 1}{5\cos^2\theta - 6\cos\theta + 5} + i\dfrac{-2\sin\theta}{5\cos^2\theta - 6\cos\theta + 5}$

c $\cos\frac{3}{2}\theta + i\sin\frac{3}{2}\theta$

d $\dfrac{x^2 + y^2 - 1}{x^2 - 2x + 1 + y^2} + i\dfrac{-2y}{x^2 - 2x + 1 + y^2}$

2 a $\cos\theta - i\sin\theta$ b $\cos 8\theta + i\sin 8\theta$

c $\dfrac{\operatorname{cis}(-\frac{1}{2}\theta) + (-1)^{n-1}\operatorname{cis}(n - \frac{1}{2})\theta}{2\cos\frac{1}{2}\theta}$

4 The circle $x^2 + y^2 + 4x = 0$;
$zz^* - 2(z + z^*) - i(z - z^*) - 4 = 0$.

5 a $\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$

b $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4)$
$$= \frac{\sum\tan\theta_1 - \sum\tan\theta_1\tan\theta_2\tan\theta_3}{1 - \sum(\tan\theta_1\tan\theta_2) + \tan\theta_1\tan\theta_2\tan\theta_3\tan\theta_4}$$

6 a $\cos\pi + i\sin\pi$; $(z + 1)(z - \frac{1}{2} - \frac{1}{2}\sqrt{3}\,i)(z - \frac{1}{2} + \frac{1}{2}\sqrt{3}\,i)$

b $(x - a)(x^2 - 2ax\cos\frac{2}{5}\pi + a^2)(x^2 - 2ax\cos\frac{4}{5}\pi + a^2)$

7 $z^2 - 4z + 13 = 0$; $1, 1, \frac{1}{2}(1 \pm \sqrt{3}\,i)$

8 a $-2 \pm i, 2 \pm i$ b $1 - i, 2 \pm 3i$ 9 -1

10 $\dfrac{4 + 2\cos\theta}{5 + 4\cos\theta}$

11 $\dfrac{1 - 2\cos 2\theta - 2^{n+1}\cos 2(n+1)\theta + 2^{n+2}\cos 2n\theta}{5 - 4\cos 2\theta}$

12 $\dfrac{\sin^2\theta}{1 + \sin^2\theta - \sin 2\theta}$

13 a $u = 0$ b $u + v - 2 = 0$
c $|w| = 1$ d $|w - 4| = 2$

14 a $u - 2v = 0$ b $v - 3 = 0$
c $|w| = 3$ d $|w - 6| = 6$

15 a (u, v) moves round $|w| = 1$ twice in a counter-clockwise sense

b (u, v) moves round $|w| = 4$ twice in a counter-clockwise sense

16 Circle, centre (5/3, 0), radius 4/3.

17 Circle, centre (0, −5/4), radius 3/4.

18 **a** $(5 - 14i)/13$ **b** $\sqrt{34}, -0.54$

i circle, centre O, radius $\sqrt{34}$

ii $3v = 5u - 34$ (half-line, from (5, −3) with gradient 5/3)

19 $1 + \sqrt{3}i$; 2, $\pi/3$; −512

20 **a** Circle, centre $(-\frac{1}{2}, 1)$, radius 3/2

b circle, centre (1, 0), radius $\sqrt{2}$

c $4x^2 + 3y^2 = 12$; $y = 0$

21 **a** **i** $\sqrt{2}, -\pi/4$, **ii** $4, \pi$, **iii** $1, \pi/3$

b $2^{1/6}$ cis $(2n\pi/3 - \pi/12)$, $n = 0, 1, 2$

22 $v^2 = 4(4 - u)$

23 **a** (−2, 1) **b** **i** circle, centre (0, 2/3), radius 4/3,

ii line $4v - 2u + 3 = 0$

24 Centre (0, 0), radius 2; $v = 0$.

25 **a** $u = -\frac{1}{2}$, v increasing **b** $u = \frac{1}{2}$, v decreasing

26 $\cos z = \cosh iz$, $\sin z = -i \sinh iz$

27 $\cosh z = (e^z + e^{-z})/2$, $\sinh z = (e^z - e^{-z})/2$, $\cosh z = \cos iz$, $\sinh z = -i \sin iz$, $\cos^2 z + \sin^2 z = 1$, $\sin(w + z) = \sin w \cos z + \cos w \sin z$

28 $\sqrt{(x^2 + y^2)}$, $\tan^{-1}(y/x)$ if z lies in 1st or 4th quadrants, $\tan^{-1}(y/x) + \pi$ if z lies in 2nd quadrant, $\tan^{-1}(y/x) - \pi$ if z lies in 3rd quadrant; (a, b), $\sqrt{(a^2 + b^2 - c)}$.

Chapter 38

Q3 **a** O, A, B, C coplanar

b when **a**, **b**, **c** form a *left*-handed set

Q5 $7\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$ **Q7** 5

Exercise 38a, page 397

1 **a** 19, 2, 21 **b** 33, −34, −1 **2** 45.9°

3 (1, −3, 1) **4** (−3, −5, −9)

5 The diagonals of a rhombus are perpendicular.

9 $\mathbf{r} = \frac{1}{4}\mathbf{p} + \frac{3}{4}\mathbf{q}$

10 **a** $\cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j}$ **b** $\cos\beta\,\mathbf{i} + \sin\beta\,\mathbf{j}$ **c** $\alpha - \beta$

Exercise 38b, page 399

1 **a** $\mathbf{i} - \mathbf{j}$ **b** $-2\mathbf{k}$ **c** $5\mathbf{j} - 15\mathbf{k}$ **d** $17\mathbf{i} + 5\mathbf{j} - 9\mathbf{k}$

2 **a** 1 **b** −16 **c** 0 **d** −2

3 **a** 1 **b** −16 **c** 0 **d** −2

5 **a** $-7\mathbf{i} + 7\mathbf{j} - 7\mathbf{k}$ **b** $3\mathbf{i} - 9\mathbf{j} - 15\mathbf{k}$

6 **a** $-\mathbf{k}$ **b** $5\mathbf{i} - 5\mathbf{j} + 18\mathbf{k}$ **c** $2\mathbf{i} + 8\mathbf{j} - 10\mathbf{k}$ **d** 0

8 $\dfrac{1}{\sqrt{6}}(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ **10** 4, 2, −6

Exercise 38c, page 402

1 **a** −4 **b** $-28\mathbf{i} - 16\mathbf{j} + 8\mathbf{k}$ **c** $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$

2 $11\mathbf{i} + 18\mathbf{j} - 13\mathbf{k}$; $11x + 18y - 13z = 0$

4 $x + 2y + 3z = 12$

7 $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$; D(4, −3, −1); $x - y + 3z = 4$

8 **a** $16/\sqrt{310}$ **b** $\mathbf{r} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

9 **a** $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ **b** no solution

10 **a** (2, 4, 6) **b** (4/7, 1/7, −2/7)

c $x^2 + y^2 + z^2 = \frac{1}{2}x$, $x^2 + y^2 + z^2 = -x$

12 $25t^2 - 30t + 46$

Index

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