CURVE SKETCHING

When the equation of the curve is unfamiliar, the shape of the curve requires valid and extensive considerations. It's impossible to study every shape of the curve but we can discuss those which occur most frequently.

(1) Intercepts: These are points where the curve crosses the coordinate axes

(2) Stationary points: These are points on the curve at which the gradient of the curve is zero $\left(\frac{dy}{dx} = 0\right)$. In curve sketching the absence of stationary points is as important as their presence.

(3) Linear asymptotes of rational functions:

Given the curve $y = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \ldots a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots b_1 x + b_0}$

The vertical asymptotes of y are the line $x = c$ where c is the set of roots of the equations

$$b_m x^m + b_{m-1}x^{m-1} + \ldots b_1 x + b_0 = 0$$

Note: For the vertical asymptotes, $y \longrightarrow \pm\infty$ (denominator = 0)

(i) If $n < m$, then y has a horizontal asymptote $y = 0$

(ii) if $m = n$ then y has a horizontal asymptote $y = \dfrac{a_n}{b_n}$

(iii) if $n > m$, then y has no horizontal asymptote

Given the curve $y = f(x)$ to find the horizontal asymptote (if any), we divide the numerator and the denominator by the highest power of x that appears in the denominator and then letting $x \longrightarrow \pm\infty$

Example

Find the vertical and horizontal asymptote of the curve $y = \dfrac{x}{x-2}$

Solution

$$y = \frac{x}{x-2}$$

For the vertical asymptote, the denominator is zero ($y \longrightarrow \pm\infty$)

$$x - 2 = 0$$

$x = 2$ is a vertical asymptote

Horizontal asymptote

The degree of the numerator and denominators are the same

For the horizontal asymptote
$\mathbf{y = \dfrac{\text{leading co-efficient of the numerator}}{\text{leading co-efficient of the denominator}}}$

For the curve $y = \frac{x}{x-2}$ $y = \frac{1}{1} = 1$

The horizontal asymptote of the curve $y = \frac{x}{x-2}$ is $y = \frac{1}{1} = 1$

Alternatively: To obtain the horizontal asymptote of $y = \frac{x}{x-2}$, we divide the numerator and denominator by the highest power of x that appears in the denominator and then letting $x \longrightarrow \pm\infty$

$$y = \frac{x}{x-2}$$

$$y = \frac{\frac{x}{x}}{\frac{x}{x} - \frac{2}{x}}$$

$$y = \frac{1}{1 - \frac{2}{x}}$$

As $x \longrightarrow \pm\infty$, $y = 1$

$\Rightarrow y = 1$ is a horizontal asymptote of the curve

Example II

Find the vertical and horizontal asymptotes of the curve $y = \dfrac{3x^2 - 2x - 1}{2x^2 + 3x - 2}$

Solution

$$y = \frac{3x^2 - 2x - 1}{2x^2 + 3x - 2}$$

For the vertical asymptote $y \longrightarrow \pm\infty$

$$2x^2 + 3x - 2 = 0$$

$$(2x - 1)(x + 2) = 0$$

$$x = \frac{1}{2} \quad and \quad x = -2$$

Horizontal asymptote

$$y = \frac{\text{leading co-efficient of the numerator}}{\text{leading co-efficient of the denominator}}$$

$\pm\infty$)

$$y = \frac{3}{2}$$

Which can also be obtained from $y=\frac{3x^2-2x-1}{2x^2+3x-2}$

$$y=\frac{\frac{3x^2}{x^2}-\frac{2x}{x^2}-\frac{1}{x^2}}{\frac{2x^2}{x^2}+\frac{3x}{x^2}-\frac{2}{x^2}}$$

$$y=\frac{3-\frac{2}{x}-\frac{1}{x^2}}{2+\frac{3}{x}-\frac{2}{x^2}}$$

As $x\longrightarrow\pm\infty$, $y\longrightarrow\frac{3}{2}$

$y=\frac{3}{2}$ is a horizontal asymptote of $y=\frac{3x^2-2x-1}{2x^2+3x-2}$

Example II

Find the vertical and horizontal asymptote of the curve $y=\frac{x^2-4x+4}{9x^2-9x+2}$

Solution

Horizontal asymptote

$$y=\frac{x^2-4x+4}{9x^2-9x+2}$$
$$=\frac{\frac{x^2}{x^2}-\frac{4x}{x^2}+\frac{4}{x^2}}{\frac{9x^2}{x^2}-\frac{9x}{x^2}+\frac{2}{x^2}}$$
$$=\frac{1-\frac{4}{x}+\frac{4}{x^2}}{9-\frac{9}{x}+\frac{2}{x^2}}$$

as $x\longrightarrow\pm\infty$ $y\longrightarrow\frac{1}{9}$

$\Rightarrow y=\frac{1}{9}$ is a horizontal asymptote

$$y=\frac{x^2-4x+4}{9x^2-9x+2}$$
$$\Rightarrow y=\frac{x^2-4x+4}{(3x-1)(3x-2)}$$

For the vertical asymptote $y\longrightarrow\pm\infty$

$$\Rightarrow(3x-1)(3x-2)=0$$
$$x=\frac{1}{3},x=\frac{2}{3}$$

$x=\frac{1}{3}$ and $x=\frac{2}{3}$ are the vertical asymptotes.

$$y=\frac{x^2-4x+4}{(3x-1)(3x-2)}$$

Example IV

Find the vertical asymptotes of the curve $y=\frac{4x^2}{x^2+8}$

Solution

$$y=\frac{4x^2}{x^2+8}$$

For the vertical asymptotes $y\longrightarrow\pm\infty$

$$x^2+8=0$$

The curve $y=\frac{4x^2}{x^2+8}$ has no vertical asymptotes (since there are no real values of x for which $x^2+8=0$.

For the horizontal asymptote, $x\longrightarrow\pm\infty$

$$y=\frac{4x^2}{x^2+8}$$
$$y=\frac{\frac{4x^2}{x^2}}{\frac{x^2}{x^2}+\frac{8}{x^2}}$$
$$y=\frac{4}{1+\frac{8}{x^2}}$$

As $x\longrightarrow\pm\infty$, $y=4$

$y=4$ is a horizontal asymptote of curve $y=\frac{4x^2}{x^2+8}$

Example IV

Find the asymptotes of the curve $y=\frac{x+1}{x(x+4)}$

Solution

For the vertical asymptotes $y\longrightarrow\pm\infty$

$$x(x+4)=0$$

$x=0$ and $x=-4$ are the vertical asymptote of the curve $y=\frac{x+1}{x(x+4)}$

Horizontal asymptote: $y=\frac{x+1}{x(x+4)}$

$$y=\frac{x+1}{x^2+4x}$$

Note: The degree of the numerator is less than the denominator (the rational fraction is proper)
The curve has a horizontal asymptote of $y=0$.

Alternatively:

$$y=\frac{x+1}{x(x+4)}$$
$$y=\frac{x+1}{x^2+4x}$$

$$y = \frac{\frac{x}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{4}{x^2}}$$

$$y = \frac{\frac{1}{x} + \frac{1}{x^2}}{1 + \frac{4}{x^2}}$$

As $x \longrightarrow \pm\infty$

As $y \longrightarrow 0$

$y = 0$ is a horizontal asymptote of the curve $y = \frac{x+1}{x(x+1)}$

Slant Asymptotes

If $y = \frac{P(x)}{Q(x)}$ is a rational fraction in which the degree of the numerator is greater than the degree of the denominator, we use long division to find the slanting asymptote of the curve.

$$y = (ax + b) + \frac{R(x)}{Q(x)}$$

Where the degree of R is less than the degree of Q and $a \neq 0$. This means that as $x \longrightarrow \pm\infty$ $\frac{R(x)}{Q(x)} \longrightarrow 0$; so the graph $y = \frac{P(x)}{Q(x)}$ approaches the graph $y = (ax + b)$. In this situation, $y = (ax + b)$ is a slant asymptote.

4. Empty Sections of a Curve

These are regions where no part of the curve lies. They can be adopted using the following method.

$$y = \frac{3x - 9}{x^2 - x - 2}$$

$$y = \frac{3x - 9}{(x - 2)(x + 1)}$$

To find whether the curve lies above or below the x-axis we equate the numerator and denominator to zero

$$y = \frac{3x - 9 = 0}{(x - 2)(x + 1) = 0}$$

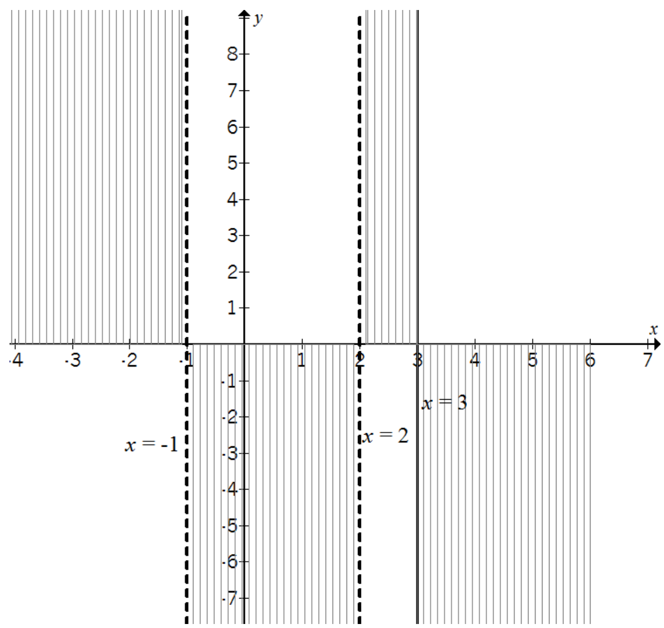
$$3x - 9 = 0$$
$$x = 3$$
$$(x - 2)(x + 1) = 0$$
$$x = 2, x = -1$$

$\Rightarrow x = -1, x = 2, x = 3$

	$x < -1$	$-1 < x < 2$	$2 < x < 3$	$x > 3$
$3x - 9$	$-ve$	$-ve$	$-ve$	$+ve$
$(x - 2)(x + 1)$	$+ve$	$-ve$	$+ve$	$+ve$
y	$-ve$	$+ve$	$-ve$	$+ve$

When y is positive, the curve lies above the x-axis and when y is negative, the curve lies below the x-axis.

The empty section of the curve can be represented.



Region of restriction

It is a region of the graph where the curve doesn't lie. We normally have a region of restriction if the curve has a maximum and minimum point

For example, if a curve $y = f(x)$ has a maximum point (2, 1) and a minimum point (-1, 3).

Its implies that there is no curve in the interval $1 < y < 3$. If a curve has a maximum point $(5, \frac{1}{3})$ and a minimum point (1, 3). It implies that there is no curve in the interval $\frac{1}{3} < y < 3$.

We can now find the region of restriction for the curve $y = \frac{3x-9}{(x-2)(x+1)}$

$$y = \frac{3x - 9}{(x - 2)(x + 1)}$$
$$y[(x - 2)(x + 1)] = 3x - 9$$
$$y[x^2 - x - 2] = 3x - 9$$
$$yx^2 - yx - 2y = 3x - 9$$
$$yx^2 - (3 + y)x + 9 - 2y = 0$$

$$B^2 - 4AC \geq 0$$

For real values of x (for the curve to be defined)

$$\Rightarrow (3+y)^2 - 4y(9-2y) > 0$$
$$9 + 6y + y^2 - 36y + 8y^2 > 0$$
$$9y^2 - 30y + 9 > 0$$
$$3y^2 - 10y + 3 > 0$$
$$3y^2 - 10y + 3 > 0$$

Factors are -1, 9 and the product 9

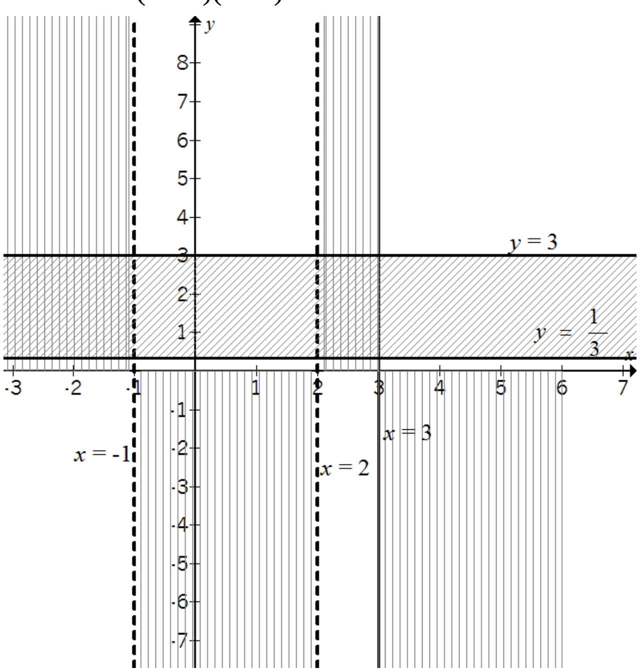
$$3y^2 - y - 9y + 3 > 0$$
$$y(3y-1) - 3(3y-1) > 0$$
$$(y-3)(3y-1) > 0$$
$$y = 3, y = \frac{1}{3}$$

	$y < \frac{1}{3}$	$\frac{1}{3} < y < 3$	$y > 3$
$(y-3)$	-ve	-ve	+ve
$(3y-1)$	-ve	+ve	+ve
$(y-3)(3y-1)$	+ve	−ve	+ve

For real values of x, $(y-3)(3y-1)$ must be positive

There is no curve on the interval $\frac{1}{3} < y < 3$.

We can now represent the empty section for the curve $y = \dfrac{3x-9}{(x-2)(x+1)}$



Steps involved when sketching curves of rational functions:

> **(1) Find *x* and *y* intercepts**
> **(2) Investigate the nature of stationary points**
> **(3) Find the asymptotes of the curve vertical slanting or horizontal.**
> **(4) Determine where the curve lies either above or below the *x*- axis**
> **(5) Determine the region where the curve has got restrictions**
> **(6) Sketch the curve**

Example I

Sketch the curve $y = \dfrac{3x-9}{(x-2)(x+1)}$

Solution

$$y = \frac{3x-9}{(x-2)(x+1)}$$

Intercepts:

For the x- axis $y = 0$

$$\Rightarrow \frac{3x-9}{(x-2)(x+1)} = 0$$
$$x = 3$$

The curve cuts the y – axis at (3, 0)

For y – axis, $x = 0$

$$y = \frac{3\times 0 - 9}{(0-2)(0+1)}$$
$$y = 4.5$$

The curve crosses the y – axis at (0, 4.5)

Stationary points of the curve:

$$y = \frac{3x-9}{(x^2-x-2)}$$

$$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2-x-2)(3) - (3x-9)(2x-1)}{(x^2-x-2)^2}$$

$$\frac{dy}{dx} = \frac{(3x^2-3x-6) - (6x^2-3x-18x+9)}{(x^2-x-2)^2}$$

$$\frac{dy}{dx} = \frac{-3x^2+18x-15}{(x^2-x-2)^2}$$

At stationary point $\frac{dy}{dx} = 0$

$$\frac{-3x^2+18x-15}{(x^2-x-2)^2} = 0$$

$$-3x^2 + 18x - 15 = 0$$
$$x^2 - 6x + 5 = 0$$
$$(x-5)(x-1) = 0$$
$$x = 5 \; and \; x = 1$$

For $x = 1$

$$y = \frac{3x-9}{x^2-x-2}$$
$$y = \frac{3 \times 1 - 9}{1^2 - 1 - 2}$$

For $x = 1$, $y = \frac{-6}{-2}$

$$y = 3$$

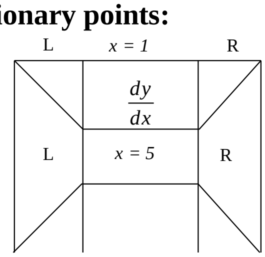
(1, 3) is a stationary point

For $x = 5$, $y = \frac{3 \times 5 - 9}{5^2 - 5 - 2}$

$$y = \frac{6}{18}$$
$$y = \frac{1}{3}$$

(5, $\frac{1}{3}$) is a stationary point

Nature of stationary points:



(1, 3) is a minimum point of the curve and (5, $\frac{1}{3}$) is a maximum point of the curve.

Since the curve has a maximum and minimum point, it implies that there is no curve in the region $\frac{1}{3} < y < 3$ (***to be proved at a later stage***)

Asymptotes of the curve $y = \frac{3x-9}{(x^2-x-2)}$

For the curve, $y = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots b_1 x + b_0}$

If $n < m$, the curve has a horizontal asymptote of $y = 0$

Since $y = \frac{3x-9}{(x^2-x-2)}$ is a proper fraction, it implies that it has a horizontal asymptote of $y = 0$

Alternatively,

$$y = \frac{3x-9}{(x^2-x-2)}$$
$$y = \frac{\frac{3x}{x^2} - \frac{9}{x^2}}{\frac{x^2}{x^2} - \frac{x}{x^2} - \frac{2}{x^2}}$$
$$y = \frac{\frac{3}{x} - \frac{9}{x^2}}{1 - \frac{1}{x} - \frac{2}{x^2}}$$

As $x \longrightarrow \pm\infty$, $y \longrightarrow 0$

$y = 0$ is a horizontal asymptote of the curve

$$y = \frac{3x-9}{(x^2-x-2)}$$
$$y = \frac{3x-9}{(x-2)(x+1)}$$

For the vertical asymptote $y \longrightarrow \pm\infty$

$$\Rightarrow (x-2)(x+1) = 0$$

$x = 2$ and $x = -1$ are vertical asymptotes of the curve

$$y = \frac{3x-9}{(x^2-x-2)}$$

Empty Sections

$$y = \frac{3x - 9 = 0}{(x-2)(x+1) = 0}$$
$$x = 3, x = 2, x = -1$$

	$x < -1$	$-1<x<2$	$2<x<3$	$x > 3$
$3x - 9$	-ve	-ve	-ve	+ve
$(x-2)(x+1)$	+ve	$-ve$	+ve	+ve
y	-ve	+ve	-ve	+ve

The negative value of y shows the curve lies below the x-axis. The positive value shows that the curve lies above the x-axis.

Region of restriction $y = \frac{3x-9}{(x-2)(x+1)}$

$$y(x-2)(x+1) = 3x - 9$$
$$y(x^2 - x - 2) = 3x - 9$$
$$yx^2 - yx - 2y = 3x + 9$$
$$yx^2 - (y+3)x + 9 - 2y = 0$$
$$B^2 - 4AC > 0$$

(For real values of x)

$$(y+3)^2 - 4y(9-2y) > 0$$
$$y^2 + 6y + 9 - 36y + 8y^2 > 0$$

$$6y + y^2 - 36y + 8y^2 > 0$$
$$9y^2 - 30y + 9 > 0$$
$$3y^2 - 10y + 3 > 0$$
$$3y^2 - 10y + 3 > 0$$

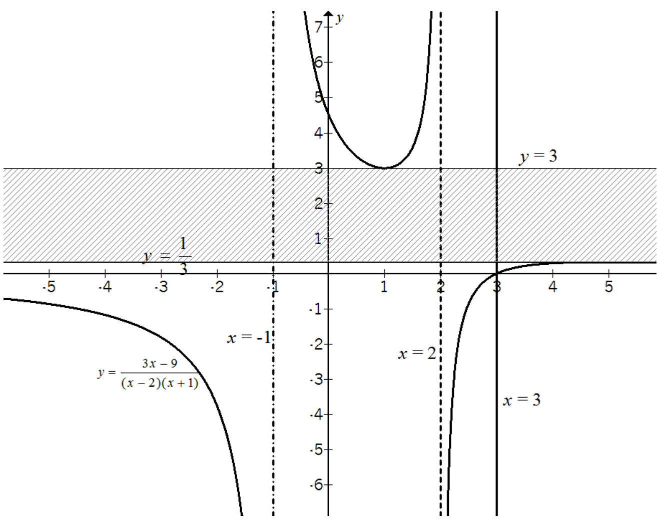
Factors are -1, 9 and the product 9

$$3y^2 - y - 9y + 3 > 0$$
$$y(3y - 1) - 3(3y - 1) > 0$$
$$(y - 3)(3y - 1) > 0$$
$$y = 3, y = \frac{1}{3}$$

For the boundary conditions;

	$y < \frac{1}{3}$	$\frac{1}{3} < y < 3$	$y > 3$
$(y-3)$	$-ve$	$-ve$	$+ve$
$(3y-1)$	$-ve$	$+ve$	$+ve$
$(y-3)(3y-1)$	$+ve$	$-ve$	$+ve$

We can now sketch the curve $y = \frac{3x-9}{(x-2)(x+1)}$



Example II

Sketch the curve $y = \frac{x(x-3)}{(x-1)(x-4)}$

Solution

$$y = \frac{x(x-3)}{(x-1)(x-4)}$$

Intercepts

For the x-axis y = 0

$$x(x - 3) = 0$$
$$x = 0, x = 3$$

The curve cuts the x- axis at $(0, 0)$ *and* $(3, 0)$

For the y- axis, $x = 0$

$$y = 0$$

The curve cuts the y- axis at (0, 0)

Stationary points of the curve $y = \frac{x^2 - 3x}{x^2 - 5x + 4}$

$$\frac{dy}{dx} = \frac{(x^2 - 5x + 4)(2x - 3) - (x^2 - 3x)(2x - 5)}{(x^2 - 5x + 4)^2}$$

$$\frac{dy}{dx} = \frac{-2x^2 - 8x - 12}{(x^2 - 5x + 4)^2}$$

At stationary point, $\frac{dy}{dx} = 0$

$$-2x^2 - 8x - 12 = 0$$
$$x^2 + 4x + 6 = 0$$
$$x = \frac{-4 \pm \sqrt{16 - 4(1)6}}{2}$$
$$x = \frac{-4 \pm \sqrt{-8}}{2}$$

The curve has no stationary points

Asymptotes:

$$y = \frac{x(x-3)}{(x-1)(x-4)}$$

For the vertical asymptotes $y \longrightarrow \pm\infty$

$$(x - 1)(x - 4) = 0$$

$x = 1$ *and* $x = 4$ are vertical asymptotes of the curve $y = \frac{x(x-3)}{(x-1)(x-4)}$

$$y = \frac{x^2 - 3x}{x^2 - 5x + 4}$$

$$\begin{array}{r|l} & \quad 1 \\ \hline x^2-5x+4 & x^2 - 3x \\ & \underline{x^2 - 5x + 4} \\ & \quad 2x - 4 \end{array}$$

$$y = 1 + \frac{2x - 4}{x^2 - 5x + 4}$$

$$y = 1 + \frac{\frac{2x}{x^2} - \frac{4}{x^2}}{\frac{x^2}{x^2} - \frac{5x}{x^2} + \frac{4}{x^2}}$$

$$y = 1 + \frac{\frac{2}{x} - \frac{4}{x^2}}{1 - \frac{5}{x} + \frac{4}{x^2}}$$

As $x \longrightarrow \pm\infty$, $y \longrightarrow 1$

$y = 1$ is a horizontal asymptote of the curve

Does the curve cross the horizontal asymptote?

$y = \frac{x^2-3x}{x^2-5x+4}$, $y = 1$ is a horizontal asymptote

$$1 = \frac{x^2 - 3x}{x^2 - 5x + 4}$$

$$x^2 - 5x + 4 = x^2 - 3x$$
$$2x = 4$$
$$x = 2$$
$$\therefore (2, 1)$$

The curve crosses the horizontal asymptote at (2, 1)

Empty sections:

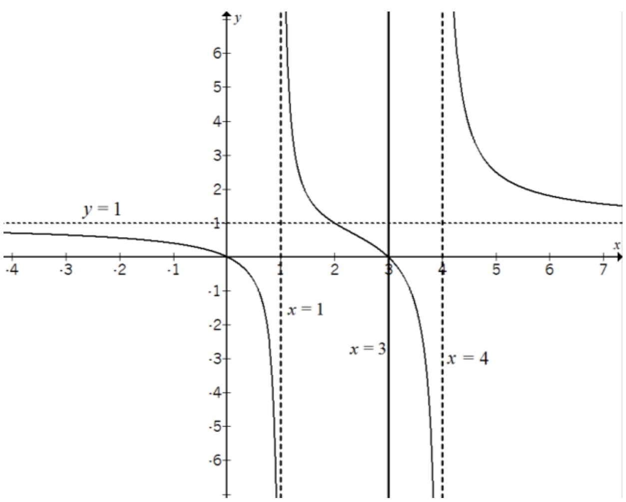
$$y = \frac{x(x-3)}{(x-1)(x-4)}$$
$$x = 0, x = 3, x = 1, x = 4$$

	$X < 0$	$0<x<1$	$1<x<3$	$3<x<4$	$X > 4$
$x(x-3)$	+ve	−ve	−ve	+ve	+ve
$(x-1)(x-4)$	+ve	+ve	−ve	−ve	+ve
y	+ve	−ve	+ve	−ve	+ve

The positive sign indicates that the curve lies above the x-axis and the negative sign shows that the curve lies above the *x*-axis.

We can now sketch the curve $y = \dfrac{x^2 - 3x}{x^2 - 5x + 4}$

Or $y = \dfrac{x(x-3)}{(x-1)(x-4)}$



Example III

Given that $y = \dfrac{4x-10}{x^2-4}$

(a) find the range of values where the curve doesn't lie
(b) hence determine the stationary points of the curve
(c) state the equations of the three asymptotes of the curve , sketch the curve

Solution

$$y = \frac{4x-10}{x^2-4}$$
$$y(x^2-4) = 4x - 10$$
$$yx^2 - 4y = 4x - 10$$
$$yx^2 - 4x + 10 - 4y = 0$$

For the real values of x

$$B^2 - 4AC > 0$$
$$(-4)^2 - 4(y)(10-4y) > 0$$
$$(-4)^2 - 4y(10-4y) > 0$$
$$16 - 40y + 16y^2 > 0$$
$$4y^2 - 10y + 4 > 0$$
$$2y^2 - 5y + 2 > 0$$
$$2(y-2)(2y-1) > 0$$
$$(y-2)(2y-1) > 0$$

For the boundary conditions

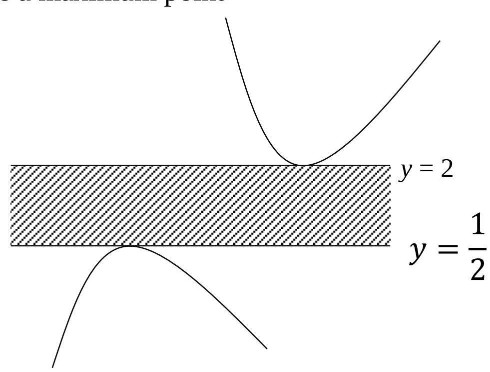
$$y = 2, y = \frac{1}{2}$$

	$y < \frac{1}{2}$	$\frac{1}{2}<y<2$	$y > 2$
$y - 2$	−ve	−ve	+ve
$2y - 1$	−ve	+ve	+ve
$(y-2)(2y-1)$	+ve	−ve	+ve

There is no curve in the region $\frac{1}{2} < y < 2$

When a curve comes from up and reaches the line y = 2, it turns upwards and when the curve comes from downwards and reaches the line $y = \frac{1}{2}$ it moves downwards

At $y = 2$ we have a minimum point and at $y = \frac{1}{2}$, we have a maximum point



From

$$y = \frac{4x-10}{x^2-4}$$
$$2 = \frac{4x-10}{x^2-4}$$
$$2x^2 - 8 = 4x - 10$$
$$2x^2 - 4x + 2 = 0$$
$$x^2 - 2x + 1 = 0$$
$$(x-1)^2 = 0$$
$$x = 1$$

If $x = 1$, $y = 2$

(1, 2) is a point of minima

$$y = \frac{4x-10}{x^2-4}$$
$$\frac{1}{2} = \frac{4x-10}{x^2-4}$$
$$x^2-4 = 8x-20$$
$$x^2-8x+16 = 0$$
$$(x-4)^2 = 0$$
$$x = 4$$

$(4, \frac{1}{2})$ is a point of maxima

Asymptotes:

$$y = \frac{4x-10}{x^2-4}$$
$$y = \frac{\frac{4x}{x^2}-\frac{10}{x^2}}{\frac{x^2}{x^2}-\frac{4}{x^2}}$$
$$y = \frac{\frac{4}{x}-\frac{10}{x^2}}{1-\frac{4}{x^2}}$$

$x \longrightarrow \pm\infty$, $y = 0$

$y = 0$ is a horizontal asymptote of the curve

$$y = \frac{4x-10}{x^2-4}$$

For the vertical asymptote, $y \longrightarrow \pm\infty$

$$\Rightarrow x^2-4 = 0$$
$$x^2 = 4$$
$$x = \pm 2$$

$x = 2$ and $x = -2$ are vertical asymptotes of the curve

$$y = \frac{4x-10}{x^2-4}$$

Whether the curve lies either above or below the x- axis:

$$y = \frac{4x-10}{x^2-4}$$
$$4x-10 = 0$$
$$x = 2.5$$
$$x^2-4 = 0$$
$$x = \pm 2$$

	$x < -2$	$-2 < x < 2$	$2 < x < 2.5$	$x > 2.5$
$4x-10$	$-ve$	$-ve$	$-ve$	$+ve$
x^2-4	$+ve$	$-ve$	$+ve$	$+ve$
y	$-ve$	$+ve$	$-ve$	$+ve$

The positive sign indicates that the curve lies above the x-axis and the negative sign shows that the curve lies above the *x*-axis.

***Intercepts*:**

For the x- intercepts $y = 0$.

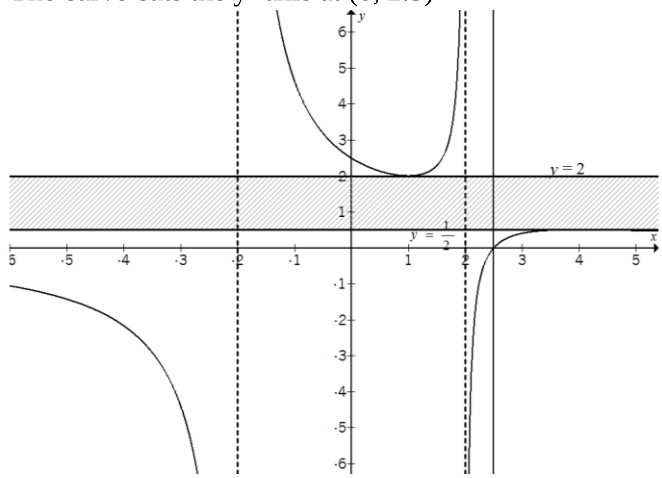
$$0 = 4x-10$$
$$x = 2.5$$

The curve cuts the *x*-axis at (2.5, 0)

For the y- intercept, $x = 0$

$$y = \frac{-10}{-4}$$
$$y = 2.5$$

The curve cuts the y- axis at (0, 2.5)



Example IV

Sketch the curve $y = \dfrac{2x^2-8}{2x-5}$

Solution

$$y = \frac{2x^2-8}{2x-5}$$

***Intercepts*:**

For the *x*- intercepts, $y = 0$

$$0 = \frac{2x^2-8}{2x-5}$$
$$2x^2-8 = 0$$
$$x^2 = 4$$
$$x = \pm 2$$

The curve cuts the *x*- axis at (2, 0) and (-2, 0)

For the y- intercepts, $x = 0$

$$y = \frac{-8}{-5} = 1.6$$

The curve cuts the y- axis at (0, 1.6)

Asymptotes:

$$y = \frac{2x^2-8}{2x-5}$$

For the vertical asymptote, $y \longrightarrow \pm\infty$

$$2x-5 = 0$$
$$x = 2.5$$

$x = 2.5$ is a vertical asymptote of the curve

$$y = \frac{2x^2 - 8}{2x - 5}$$

$$\begin{array}{r|l} & x + \frac{5}{2} \\ \hline 2x - 5 & 2x^2 - 8 \\ & \underline{2x^2 - 5x} \\ & \quad 5x - 8 \\ & \quad \underline{5x - 25/2} \\ & \qquad 9/2 \end{array}$$

$$y = \left(x + \frac{5}{2}\right) + \frac{9/2}{2x - 5}$$

$$y = (x + \frac{5}{2}) + \frac{9/2x}{2 - \frac{5}{x}}$$

As $x \longrightarrow \pm\infty$, $y = x + \frac{5}{2}$

$y = x + \frac{5}{2}$ is a slanting asymptote of the curve $y = \frac{2x^2 - 8}{2x - 5}$

Stationary points:

$$\frac{dy}{dx} = \frac{(2x-5).\ 4x - (2x^2 - 8).2}{(2x-5)^2}$$

$$\frac{dy}{dx} = \frac{4x^2 - 20x + 16}{(2x-5)^2}$$

At stationary point $\frac{dy}{dx} = 0$

$$4x^2 - 20x + 16 = 0$$
$$x^2 - 5x + 4 = 0$$
$$(x-1)(x-4) = 0$$
$$x = 1 \ and \ x = 4$$

If $x = 4$, $y = 8$

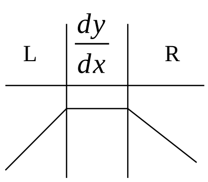
If $x = 1$, $y = 2$

(4, 8) and (1, 2) are stationary points of the curve

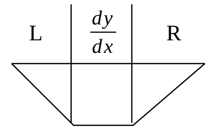
$$y = \frac{2x^2 - 8}{2x - 5}$$

Nature of stationary points

$x = 1$



$x = 4$



Since the curve has a maximum point at (1, 2) and minimum point at (4, 8)

$\Rightarrow$ There is no curve in the region $2 < y < 8$ where the curve lies

Where the curve lies:

$$y = \frac{2x^2 - 8 = 0}{2x - 5 = 0}$$

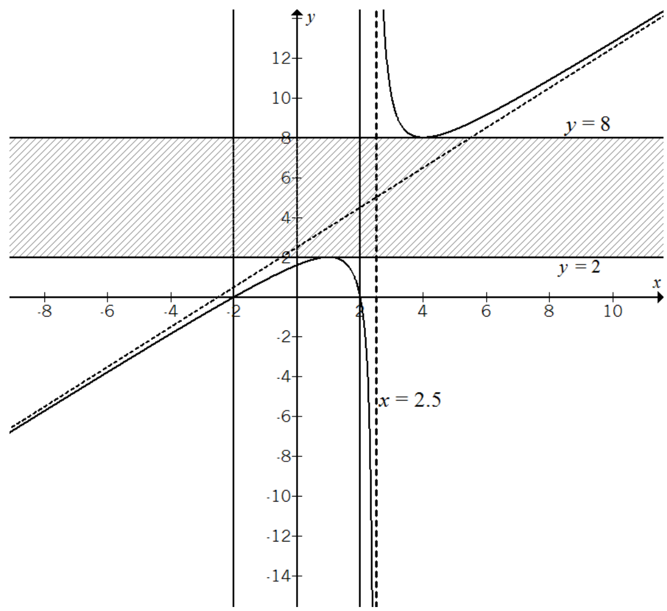
$2x^2 - 8 = 0$

$\Rightarrow x = \pm 2$

$2x - 5 = 0$

$\Rightarrow x = 2.5$

	$x < -2$	$-2 < x < 2$	$2 < x < 2.5$	$x > 2.5$
$2x^2 - 8$	+ve	−ve	+ve	+ve
$2x - 5$	−ve	−ve	−ve	−ve
y	−ve	+ve	−ve	+ve



Example V

Find the Cartesian equation of the curve

$$x = \frac{1+t}{1-t}$$

$$y = \frac{2t^2}{1-t}$$

Hence sketch the curve $y = f(x)$ where $f(x)$ is the Cartesian equation of the curve

Solution

$$x = \frac{1+t}{1-t}, \quad y = \frac{2t^2}{1-t}$$

From $x = \dfrac{1+t}{1-t}$

$$x(1-t) = 1+t$$
$$x - xt = 1+t$$
$$x - 1 = xt + t$$
$$x - 1 = t(x+1)$$
$$t = \frac{x-1}{x+1}$$
$$y = \frac{2t^2}{1-t}$$
$$y = \frac{2\left(\frac{x-1}{x+1}\right)^2}{1-\left(\frac{x-1}{x+1}\right)}$$
$$y = \frac{\frac{2(x^2-2x+1)}{(x+1)^2}}{\left(\frac{(x+1)-(x-1)}{x+1}\right)}$$
$$y = \frac{\frac{2(x^2-2x+1)}{(x+1)^2}}{\frac{2}{x+1}}$$
$$y = \frac{x^2-2x+1}{x+1}$$

$y = \frac{x^2-2x+1}{x+1}$ is the Cartesian equation of the curve

Intercepts:

For x- intercepts $y = 0$

$$\frac{x^2-2x+1}{x+1} = 0$$
$$(x-1)^2 = 0$$
$$x = 1$$

The curve cuts the x- axis at (1, 0)

For the y- intercept $x = 0$

$$y = \frac{1}{1}$$

The curve cuts the y – axis at (0, 1)

Asymptotes :

$$y = \frac{x^2-2x+1}{x+1}$$

For the vertical asymptote $y \longrightarrow \pm\infty$

$$x + 1 = 0$$
$$x = -1$$

$x = -1$ is the vertical asymptote of the curve $y = \frac{x^2-2x+1}{x+1}$

$$\begin{array}{r|l} & x-3 \\ \hline x+1 & x^2-2x+1 \\ & \underline{x^2+x} \\ & \quad -3x+1 \\ & \quad \underline{-3x-3} \\ & \qquad 4 \end{array}$$

$$y = (x-3) + \frac{4}{x+1}$$
$$y = (x-3) + \frac{\frac{4}{x}}{1+\frac{1}{x}}$$

As $x \longrightarrow \pm\infty$, $y \longrightarrow x-3$

$y = (x-3)$ is the slanting asymptote

Stationary point:

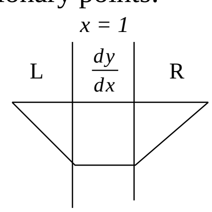
$$\frac{dy}{dx} = \frac{(x+1)(2x-2)-(x^2-2x+1)1}{(x+1)^2}$$
$$\frac{dy}{dx} = \frac{x^2+2x-3}{(x+1)^2}$$
$$x^2+2x-3 = 0$$
$$(x+3)(x-1) = 0$$
$$x = -3, x = 1$$

If $x = -3$, $y = -8$

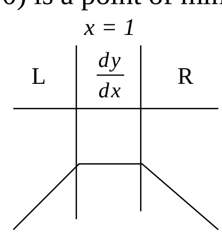
If $x = 1$, $y = 0$

(-3, -8) and (1, 0) are stationary points

Nature of the stationary points:



(1, 0) is a point of minima



(-3, -8) is a point of maxima

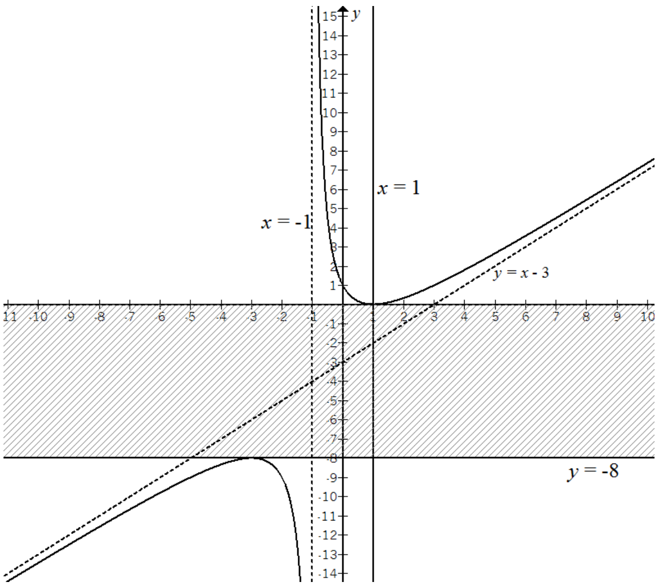
There is no curve in the interval $-8 < y < 0$

Where the curve lies:

$$y = \frac{x^2-2x+1=0}{x+1=0}$$

$$x = 1$$
$$x = -1$$

	$x < -1$	$-1 < x < 1$	$x > 1$
$x^2 - 2x + 1$	+ve	+ve	+ve
$x + 1$	−ve	+ve	+ve
y	−ve	+ve	+ve



Example V

Sketch the curve $y = \dfrac{x^2 - 6x + 5}{2x - 1}$

Solution

$$y = \frac{x^2 - 6x + 5}{2x - 1}$$

For the x- intercept; $y = 0$

$$0 = \frac{x^2 - 6x + 5}{2x - 1}$$
$$x^2 - 6x + 5 = 0$$
$$(x - 1)(x - 5) = 0$$
$$x = 1 \text{ and } x = 5$$
$$\Rightarrow (1, 0) \text{ and } (5, 0)$$

The curve cuts the x- axis at(1, 0) and (5, 0)

For the y- axis, $x = 0$

$$y = \frac{5}{-1}$$

The curve cuts the y–axis at (0, -5)

Asymptotes

$$y = \frac{x^2 - 6x + 5}{2x - 1}$$

For the vertical asymptote $y \longrightarrow \pm\infty$

$$2x - 1 = 0$$
$$x = \frac{1}{2}$$

$x = \frac{1}{2}$ is the vertical asymptote of the curve $y = \frac{x^2-6x+5}{2x-1}$

$$\begin{array}{r|l} & \frac{x}{2} - \frac{11}{4} \\ \hline 2x - 1 & x^2 - 6x + 5 \\ & x^2 - \frac{x}{2} \\ \hline - & -\frac{11x}{2} + 5 \\ & -\frac{11x}{2} + \frac{11}{4} \\ \hline & \frac{9}{4} \end{array}$$

$$y = \left(\frac{x}{2} - \frac{11}{4}\right) + \frac{\frac{9}{4}}{2x + 1}$$
$$y = \left(\frac{x}{2} - \frac{11}{4}\right) + \frac{\frac{9}{4x}}{2 + \frac{1}{x}}$$

As $x \longrightarrow \pm\infty$, $y \longrightarrow \dfrac{x}{2} - \dfrac{11}{4}$

$\dfrac{x}{2} - \dfrac{11}{4}$ is the slanting asymptote of the curve

Stationary points.

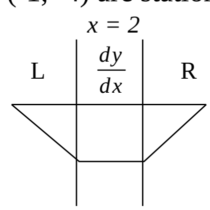
$$\frac{dy}{dx} = \frac{(2x-1)(2x-6) - (x^2 - 6x + 5)(2)}{(2x-1)^2}$$
$$\frac{dy}{dx} = \frac{2x^2 - 2x - 4}{(2x - 1)^2}$$

At a stationary point

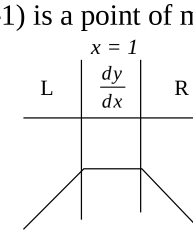
$$\frac{dy}{dx} = 0$$
$$2x^2 - 2x - 4 = 0$$
$$x^2 - x - 2 = 0$$
$$(x - 2)(x + 1) = 0$$
$$x = 2, x = -1$$

If $x = 2$, $y = -1$ and if $x = -1$, $y = -4$

(2, -1) and (-1, -4) are stationary points.



(2, -1) is a point of minima



(-1, -4) is a point of maxima.

There is no curve in the interval -4 < y < -1

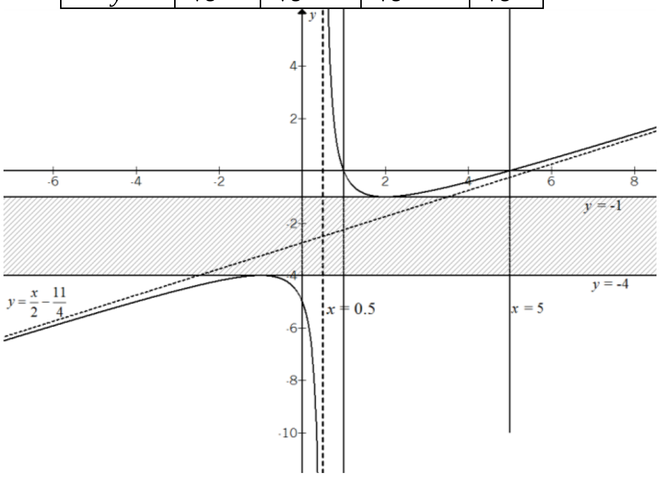
Where the curve lies:

$$y = \frac{x^2 - 6x + 5 = 0}{2x - 1 = 0}$$

$$x = 1, x = 5$$

$$x = \frac{1}{2}$$

	$x < \frac{1}{2}$	$\frac{1}{2} < x < 1$	$1 < x < 5$	$x > 5$
$x^2 – 6x + 5$	+ve	+ve	–ve	+ve
$2x$ - 1	–ve	+ve	+ve	+ve
y	–ve	+ve	–ve	+ve



Reciprocal Curves

Consider the curves whose equations are $y_1 = f(x)$ *and* $y_2 = \frac{1}{f(x)}$. When the graph of the function f(x) is familiar, the following simple properties provide the means to adapt the known graph of f(x) in order to sketch the graph of $\frac{1}{f(x)}$

If the graph of f(x) is known, to sketch the graph $\frac{1}{f(x)}$ the following steps are involved

(1) For a given value of x, f(x) and $\frac{1}{f(x)}$ have the same sign when $f(x)$ lies above the x–axis, $\frac{1}{f(x)}$ lies above the x–axis and when f(x) lies below the x–axis also $\frac{1}{f(x)}$ also lies below the x-axis)

(2) If $f(x) \longrightarrow 0$, $\frac{1}{f(x)} \longrightarrow \pm\infty$ so that x–intercepts became the vertical asymptotes of the curve $\frac{1}{f(x)}$

(3) If $f(x)$ has a maximum turning point at a given value if x, $\frac{1}{f(x)}$ has minimum turning point at that given value of x and vice versa

Example I

Sketch the graph of $f(x) = 4 + 3x - x^2$ hence sketch the graph of $\frac{1}{f(x)}$

Solution

$$y = f(x)$$

$$y = 4 + 3x - x^2$$

$$\frac{dy}{dx} = 3 - 2x$$

At a stationary point, $\frac{dy}{dx} = 0$

$$3 - 2x = 0$$

$$x = 1.5$$

When x = 1.5, y = 6.25

(1.5, 6.25) is a turning point

$$\frac{d^2y}{dx^2} = -2$$

$\Rightarrow$ (1.5, 6.25) is a point of maxima

***Intercepts*:**

For x–axis y = 0

$$0 = 4 + 3x - x^2$$

$$x^2 - 3x - 4 = 0$$

$$x = 4 \text{ and } x = -1$$

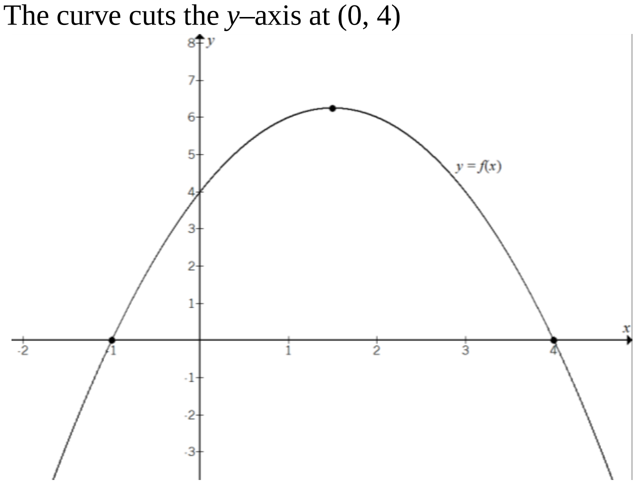
The curve cuts the x–axis at (4, 0) and (-1, 0)

$$y = 4 + 3x - x^2$$

For y – intercept, x = 0

$\Rightarrow y = 4 + 0 - 0^2 = 4$

The curve cuts the y–axis at (0, 4)



From the above information we can now sketch the graph of $\frac{1}{f(x)}$, using the known graph of $f(x)$. Using the following properties

(1) For a given value of x, $f(x)$ and $\frac{1}{f(x)}$ have the same sign. For x <-1, $f(x)$ lies below the x – axis implying that $\frac{1}{f(x)}$ also lies below the x – axis

For $-1 < x < 4$ f(x) lies above the x–axis

$\Rightarrow \frac{1}{f(x)}$ also lies above the x- axis

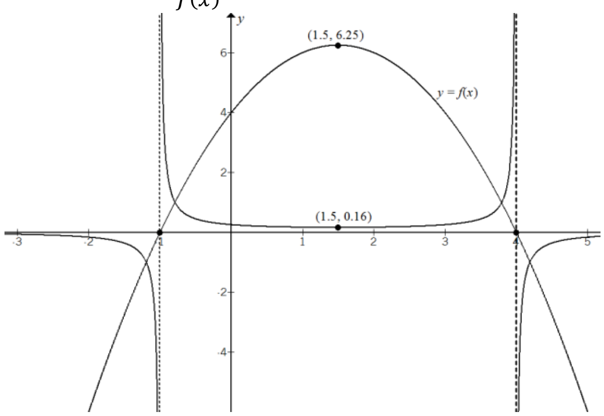
For x > 4, $f(x)$ lies below the x– axis implying that $\frac{1}{f(x)}$ also lies below the x– axis.

At x = 1.5, $f(x)$ has a maximum point at (1.5, 6.25)

$\Rightarrow \frac{1}{f(x)}$ has minimum point at (1.5, 0.16)

If $f(x) \longrightarrow 0$, $\frac{1}{f(x)} \longrightarrow \pm\infty$

x = 4 and x = -1 are the vertical asymptotes of the curve $y = \frac{1}{f(x)}$



Example II

Sketch the graph of $f(x) = x^2(x+2)$ hence sketch the graph of $g(x) = \frac{1}{f(x)}$

Solution

$$f(x) = x^2(x+2)$$

Let $y = f(x)$

$$\frac{dy}{dx} = 3x^2 + 4x$$

At a stationary point

$$\frac{dy}{dx} = 0$$

$$3x^2 + 4x = 0$$

$$x(3x+4) = 0$$

$$x = 0, x = \frac{-4}{3}$$

If $x = 0$, $y = 0$

If $x = \frac{-4}{3}$, $y = \frac{32}{27}$

$\left(\frac{-4}{3}, \frac{32}{27}\right)$ is a stationary point

Intercept:

For the x–intercept, $y = 0$

$$x^2(x+2) = 0$$

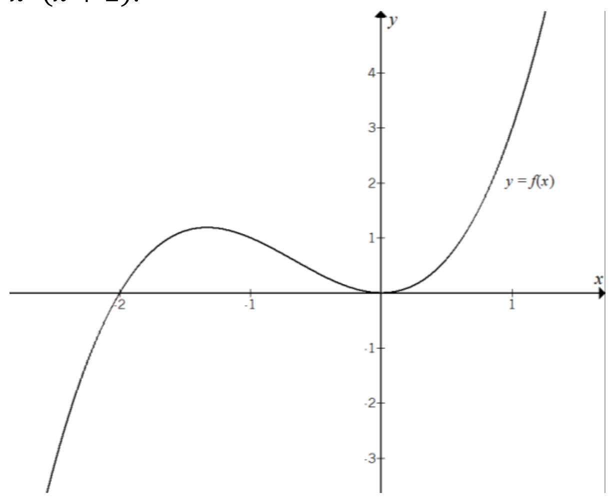
$$x = 0, x = -2$$

The curve cuts the x – axis at (0, 0) and (-2, 0)

For the y – axis, $x = 0$ and $y = 0$

The curve cuts the y–axis at (0, 0)

We can now sketch the graph of $y = f(x) = x^2(x+2)$.



For $x < -2, f(x)$ lies below the x–axis implying that $\frac{1}{f(x)}$ also lies below the x – axis

For $-2 < x < 0$, f(x) lies above the x–axis implying that $\frac{1}{f(x)}$ also lies above the x – axis

For $x > 0, f(x)$ lies above the x – axis implying that $\frac{1}{f(x)}$ also lies above the x – axis

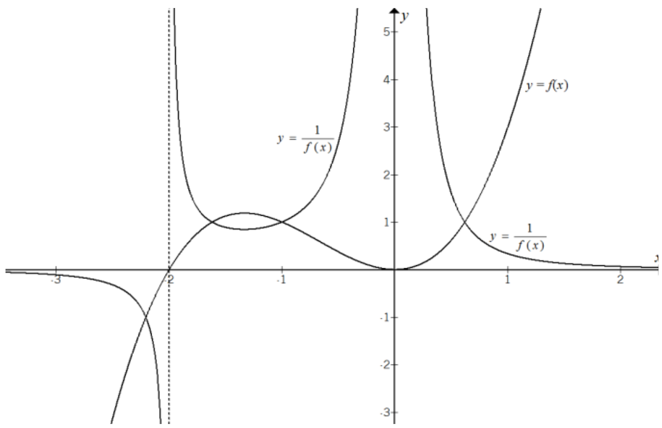
At $x = \frac{-4}{3}, f(x)$ has a maximum point at $\left(\frac{-4}{3}, \frac{32}{27}\right)$

At $x = \frac{-4}{3}$, $\frac{1}{f(x)}$ has a minimum point at $\left(\frac{-4}{3}, \frac{27}{32}\right)$

At (0, 0), $f(x)$ has a minimum point. $\frac{1}{f(x)}$ has maximum point $\left(0, \frac{1}{0}\right)$, $\Rightarrow \frac{1}{f(x)}$ has only one turning with a minimum at $\left(\frac{-4}{3}, \frac{27}{32}\right)$ because the maximum point $\left(0, \frac{1}{0}\right)$ is not defined.

When $f(x) \longrightarrow 0$, $\frac{1}{f(x)} \longrightarrow \pm\infty$

x = 0 and x = -2 are vertical asymptotes of the curve $y = \frac{1}{f(x)}$



Example

Show that $f(x)=\frac{x(x-5)}{(x-3)(x+2)}$ has no turning points. Sketch the curve $y=f(x)$. If $g(x)=\frac{1}{f(x)}$

Sketch the graph of $f(x)$ *and* $g(x)$ on the same axes

Solution

$$y=f(x)$$
$$y=\frac{x^2-5x}{x^2-x-6}$$
$$\frac{dy}{dx}=\frac{(x^2-x-6)(2x-5)-(x^2-5x)(2x-1)}{(x^2-x-6)^2}$$
$$4x^2-12x+30=0$$
$$2x^2-6x+15=0$$
$$x=\frac{6\pm\sqrt{6^2-4\times2\times15}}{2\times2}$$
$$x=\frac{6\pm\sqrt{-48}}{4}$$

The curve has no turning point

Intercepts:

For the x-intercept, y = 0

$$\frac{x(x-5)}{(x-3)(x+2)}=0$$
$$x=0, x=5$$

The curve $f(x)$ cuts the x – axis at (0, 0) and (5, 0)

For the y – axis, x = 0

$$y=\frac{0}{-6},\qquad y=0$$

The curve $f(x)$ cuts the y – axis at (0, 0)

Asymptotes

$$y=\frac{x(x-5)}{(x-3)(x+2)}$$
$$y=\frac{x^2-5x}{x^2-x-6}$$
$$y=1-\frac{6-4x}{x^2-x-6}\quad \text{(By long division)}$$
$$y=1-\frac{\frac{6}{x^2}-\frac{4}{x}}{1-\frac{1}{x}-\frac{6}{x^2}}$$

As $x\longrightarrow\pm\infty$, $y\longrightarrow1$

y = 1 is a horizontal asymptote of the curve $y=f(x)$

For vertical asymptote $f(x)\longrightarrow\pm\infty$

$$\Rightarrow x^2-x-6=0$$

x = 3 and x = -2 are vertical asymptotes of $f(x)$

Does the curve $f(x)$ cut the horizontal asymptote?

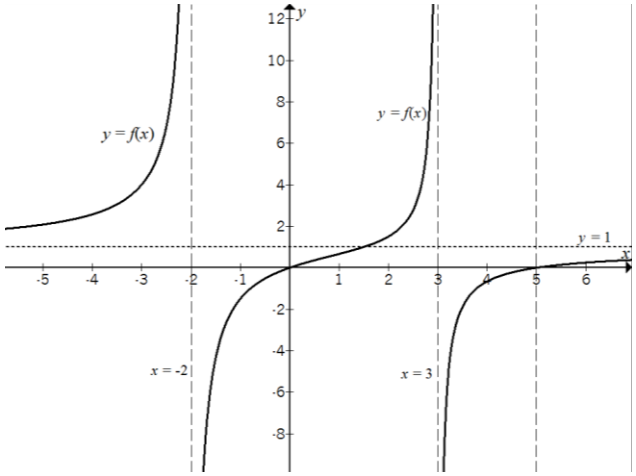
$$y=\frac{x^2-5x}{x^2-x-6}$$
$$1=\frac{x^2-5x}{x^2-x-6}$$
$$x^2-x-6=x^2-5x$$
$$4x=6$$
$$x=1.5$$

The curve $f(x)$ cut the horizontal asymptote at (1.5, 1)

Where does the curve of $y=f(x)$ lie.

$$y=\frac{x(x-5)}{(x-3)(x+2)}=f(x)$$
$$\frac{x(x-5)=0}{(x-3)(x+2)=0}$$
$$x=0, x=5, x=3, x=-2$$

	x<-2	-2<x<0	0<x<3	3<x<5	x>5
x(x-5)	+ve	+ve	–ve	–ve	+ve
(x-3)(x+2)	+ve	–ve	–ve	+ve	+ve
y	+ve	–ve	+ve	–ve	+ve



We can now sketch the graph of $\frac{1}{f(x)}$ using the known graph of $f(x)$

For $x < -2$ the graph of $f(x)$ lies above the x – axis implying that $g(x)$lies above the x – axis

For $-2 < x < 0$

$f(x)$ lies below the x–axis implying that $g(x)$ also lies below the x–axis

For $0 < x < 3, f(x)$ lies above the x–axis implying that $g(x)$ also lies above the x – axis

For $3 < x < 5, f(x)$ lies below the x – axis, $g(x)$also lies below the x – axis

For $x > 5, f(x)$ lies above the x – axis also $g(x)$ lies above the x – axis

When $f(x) \longrightarrow 0$

$$g(x) = \frac{1}{f(x)} \longrightarrow \pm\infty$$

$$g(x) = \frac{(x-3)(x+2)}{x(x-5)}$$

$\Rightarrow x = 0$ and $x = 5$ are vertical asymptotes of $\frac{1}{f(x)}$