12.2 The area under a curve

Suppose A_1 is the area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ (see Figure 1). We say that A_1 is the area 'under' the curve from $x = a$ to $x = b$. One way to estimate this area would be to divide it into strips. Since each of these strips approximates to a rectangle (Figure 2), we can then sum the areas of these rectangles. This would give an approximate value for A_1; the more rectangles we use, the greater is the accuracy. Consider one such rectangle, width δx (Figure 3).

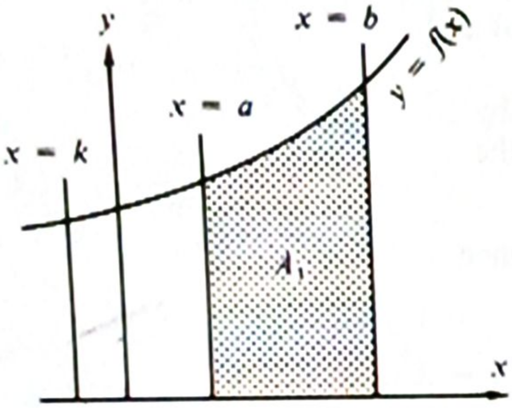


Fig. 1

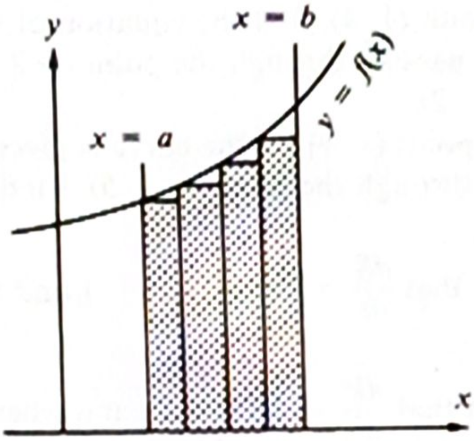


Fig. 2

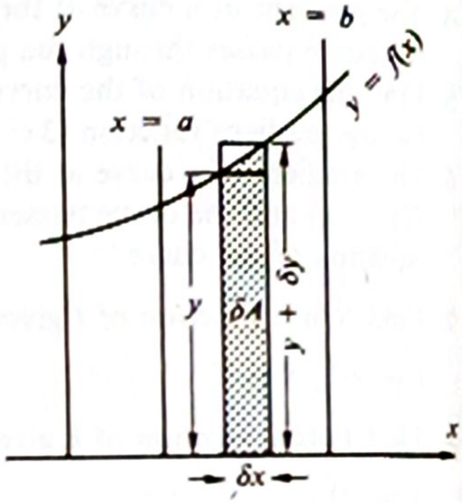


Fig. 3

In Figure 3, if δA is the shaded area, $y\,\delta x < \delta A < (y + \delta y)\,\delta x$

Thus $\quad y < \dfrac{\delta A}{\delta x} < y + \delta y$

Now as $\delta x \to 0$ (i.e. we increase the number of rectangles)

$\dfrac{\delta A}{\delta x} \to \dfrac{dA}{dx}$ and $\delta y \to 0$

Thus $\quad \dfrac{dA}{dx} = y \quad$ or $\quad A = \displaystyle\int y\,dx$

This integration will give an area function $A(x)$ and will involve a constant of integration c. As we substitute a value for x into the function $A(x)$, say $x = b$, we will obtain an answer for the area under the curve from a right-hand boundary of $x = b$ to some left-hand boundary. The position of the left-hand boundary will determine the value of c, the constant of integration. Suppose we take $x = k$ as the left-hand boundary, then

$$A(a) = \text{Area from } x = k \text{ to } x = a.$$
$$A(b) = \text{Area from } x = k \text{ to } x = b.$$
$$\therefore \quad A(b) - A(a) = \text{Area from } x = a \text{ to } x = b.$$

As $A = \displaystyle\int y\,dx$, we write $A(b) - A(a)$ as $\displaystyle\int_a^b y\,dx$.

$$\therefore \quad A_1 = \int_a^b y\,dx.$$

$y\,\delta x < \delta A < y\,\delta x + \delta y\delta x$

As $\delta x \to 0 \quad \delta y \to 0$ and so $\delta y\delta x$ becomes negligible compared with $y\,\delta x$.

Thus as $\delta x \to 0$, $\delta A \to y\,\delta x$.

But $\quad A_1 = \displaystyle\sum_{x=a}^{x=b} \delta A$

$$\therefore \quad A_1 = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\,\delta x$$

The area under the curve can therefore be found as the limit of a sum or by integration. Thus integration is a process of summation and

$$\boxed{A_1 = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\,\delta x = \int_a^b y\,dx, \quad \text{where} \quad y = f(x).}$$

Definite integrals

Note that $\int_a^b f(x)dx$ is known as a *definite* integral because the limits of integration, i.e. $x = a$ and $x = b$, are known.

Suppose $\qquad \int f(x)dx = F(x) + c$

then
$$\int_{x=a}^{x=b} f(x)dx = (F(b) + c) - (F(a) + c)$$
$$= F(b) - F(a)$$

We usually write this: $\qquad \int_{x=a}^{x=b} f(x)dx = \Big[F(x)\Big]_a^b$
$$= F(b) - F(a)$$

We see that the constants of integration cancel out so that in the case of a definite integral there is no need to give an arbitrary constant in the result.

Example 5

Evaluate the following definite integrals: (a) $\int_{-1}^{1} (2x - 3)dx$ (b) $\int_{1/4}^{1/2} \frac{1}{x^3}dx$

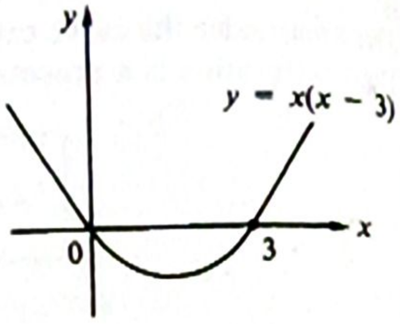
(a)
$$\int_{-1}^{1} (2x - 3)dx$$
$$= \Big[x^2 - 3x\Big]_{-1}^{1}$$
$$= [1^2 - 3(1)] - [(-1)^2 - 3(-1)]$$
$$= -2 - 4$$
$$= -6$$

(b)
$$\int_{1/4}^{1/2} \frac{1}{x^3}dx = \int_{1/4}^{1/2} x^{-3}dx$$
$$= \left[\frac{x^{-2}}{-2}\right]_{1/4}^{1/2}$$
$$= \left[-\frac{1}{2x^2}\right]_{1/4}^{1/2}$$
$$= \left(-\frac{1}{2(\frac{1}{2})^2}\right) - \left(-\frac{1}{2(\frac{1}{4})^2}\right)$$
$$= -2 + 8$$
$$= +6$$

Calculation of the area under a curve

When we calculate the area under a curve, the important first step is to make a sketch of the curve. We must then remember that an area lying 'above' the x-axis will have a positive value, whereas areas lying 'below' the x-axis will be negative. In some cases the required area may lie both 'above' and 'below' the x-axis and particular care is needed in these situations.

Example 6



Find the area between the curve $y = x(x - 3)$ and the x-axis.

First, make a sketch of the curve $y = x(x - 3)$
In this case the required area lies 'below' the x-axis.

Using $A = \int y\,dx$ and substituting for y from the equation of the curve, as we cannot integrate y with respect to x.

$$\therefore \quad A = \int_0^3 x(x - 3)dx$$

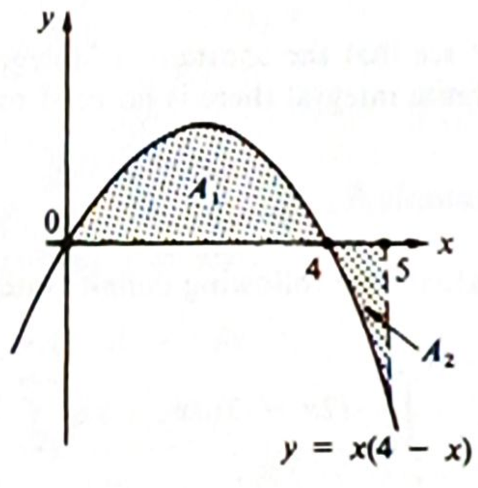
$$= \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_0^3$$

$$= \left(9 - \frac{27}{2}\right) - (0) = -4\tfrac{1}{2}$$

The area has a negative sign, as was anticipated, and the numerical value is $4\frac{1}{2}$ sq. units.

Example 7

Find the area between the curve $y = x(4 - x)$ and the x-axis from $x = 0$ to $x = 5$.
First, make a sketch of the curve $y = x(4 - x)$
The sketch shows that the required area is in two parts; one part lies above the x-axis and therefore has a positive area, the other part lies below the x-axis and has a negative area.



Using $A = \int y\,dx$ and calculating the two areas separately,

$$A_1 = \int_0^4 x(4 - x)dx \qquad A_2 = \int_4^5 x(4 - x)dx$$

$$= \left[2x^2 - \frac{x^3}{3}\right]_0^4 \qquad = \left[2x^2 - \frac{x^3}{3}\right]_4^5$$

$$= \left(32 - \frac{64}{3}\right) - (0) \qquad = \left(50 - \frac{125}{3}\right) - \left(32 - \frac{64}{3}\right)$$

$$= +\frac{32}{3} = +10\tfrac{2}{3} \qquad = -2\tfrac{1}{3}$$

The total area under the curve between $x = 0$ and $x = 5$ is given by the sum of the *numerical* values of these two areas:

$$\text{required area} = 10\tfrac{2}{3} + 2\tfrac{1}{3} = 13 \text{ sq. units.}$$

Note In the last example, it is possible to calculate $\int_0^5 x(4 - x)dx$, but this would not give the correct answer for the required area. Instead we would obtain an answer of $10\frac{2}{3} - 2\frac{1}{3}$ i.e. $8\frac{1}{3}$, as the following working shows:

$$\int_0^5 x(4 - x)dx = \left[2x^2 - \frac{x^3}{3}\right]_0^5$$

$$= \left(50 - \frac{125}{3}\right) - (0) = 8\tfrac{1}{3}$$

Discontinuous functions

Although we may be able to evaluate $\int_{x=a}^{x=b} f(x)\,dx$, this does not mean that the value obtained has any geometrical significance. In order that the definite integral has a meaning we must ensure that $f(x)$ is defined and continuous for this range of values of x, $a \leqslant x \leqslant b$.
The following examples illustrate this point:

(i) $\int_{-1}^{+1} \frac{1}{x}\,dx$ has no meaning since $\frac{1}{x}$ is not defined for $x = 0$.

(ii) $\int_{0}^{3} \frac{1}{x-2}\,dx$ has no meaning since $\frac{1}{x-2}$ is not defined for $x = 2$.

(iii) $\int_{0}^{2a} \frac{1}{x(x-a)}\,dx$ has no meaning since $\frac{1}{x(x-a)}$ is not defined for $x = 0$ or for $x = a$.

(iv) $\int_{-2}^{+2} \sqrt{(x+1)}\,dx$ has no meaning since $\sqrt{(x+1)}$ is not defined for $-2 \leqslant x < -1$.

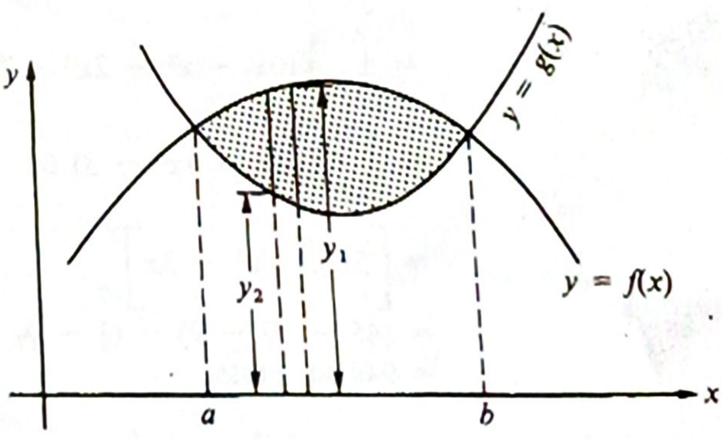
Area defined by two curves

An area can be defined by two curves and in this case it is essential to make a sketch and to determine the points of intersection of the two curves.
Suppose the curves $y = f(x)$ and $y = g(x)$ intersect at the points where $x = a$ and $x = b$.
The area between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$ is given by

$$\int_a^b f(x)\,dx.$$

The area between the curve $y = g(x)$ and the x-axis from $x = a$ to $x = b$ is given by

$$\int_a^b g(x)\,dx.$$



The shaded area between the two curve is then $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$ and

this may be written as $\int_a^b [f(x) - g(x)]\,dx$.

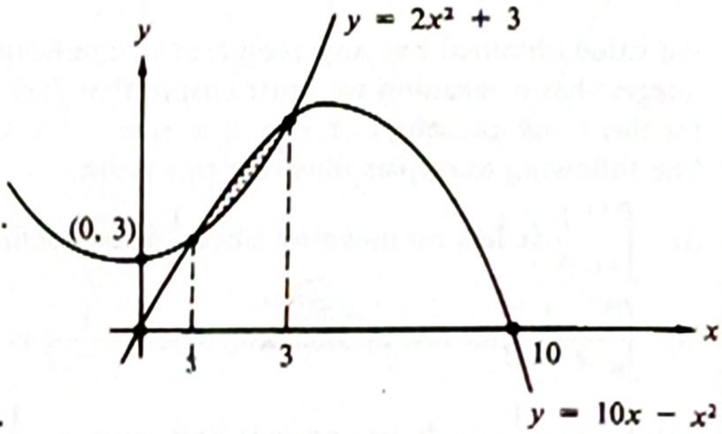
Alternatively, the second form of this solution can be obtained directly by considering a strip of width δx, drawn parallel to the y-axis. The length of this strip is $y_1 - y_2$, where $y_1 = f(x)$ and $y_2 = g(x)$, and the area of the strip is then $[f(x) - g(x)]\,\delta x$ and the result follows.

Example 8

Find the area enclosed between the curves $y = 2x^2 + 3$ and $y = 10x - x^2$.

First, make a sketch of the two curves, noting that the curves will intersect at the points where

$$2x^2 + 3 = 10x - x^2$$
$$3x^2 - 10x + 3 = 0$$
$$(3x - 1)(x - 3) = 0 \quad \text{i.e. at} \quad x = \tfrac{1}{3} \quad \text{and} \quad x = 3.$$



The curve $y = 2x^2 + 3$ intersects the y-axis at (0, 3) and does not cut the x-axis.
The curve $y = 10x - x^2$ intersects the axes at (0, 0) and at (10, 0).
The information is sufficient for a sketch to be made.
The area enclosed by $y = 10x - x^2$, the ordinates $x = \frac{1}{3}$ and $x = 3$ and the x-axis is

$$\int_{1/3}^{3} (10x - x^2)dx.$$

The area enclosed by $y = 2x^2 + 3$, the ordinates $x = \frac{1}{3}$ and $x = 3$ and the x-axis is

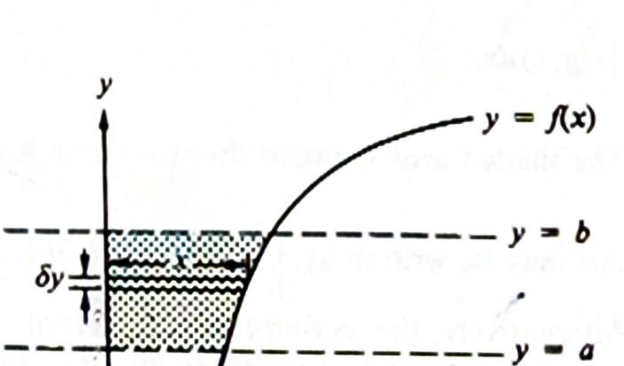
$$\int_{1/3}^{3} (2x^2 + 3)dx.$$

The shaded area is $\displaystyle\int_{1/3}^{3} (10x - x^2)dx - \int_{1/3}^{3} (2x^2 + 3)dx$

$$= \int_{1/3}^{3} (10x - x^2 - 2x^2 - 3)dx$$
$$= \int_{1/3}^{3} (10x - 3x^2 - 3)dx$$
$$= \Big[5x^2 - x^3 - 3x\Big]_{1/3}^{3}$$
$$= (45 - 27 - 9) - (\tfrac{5}{9} - \tfrac{1}{27} - 1)$$
$$= 9\tfrac{13}{27} \text{ sq. units.}$$

Area between a curve and the y-axis

Suppose we wish to find the area between some curve $y = f(x)$ and the y-axis, from $y = a$ to $y = b$.
Considering a strip of length x and width δy, drawn parallel to the x-axis, we see that

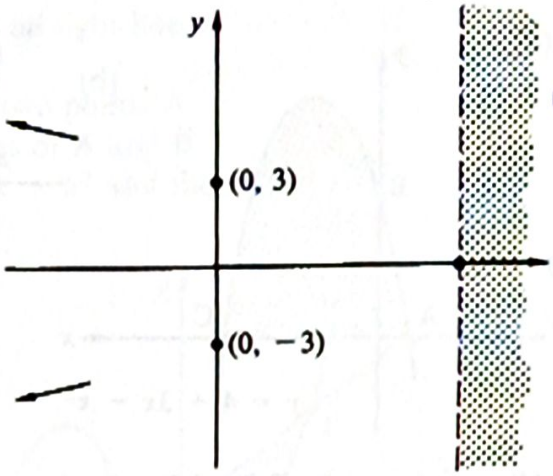


$$A = \lim_{\delta y \to 0} \sum_{y=a}^{y=b} x\,\delta y$$

$\int^b$

Example 9

Find the area enclosed between the curve $y^2 = 9 - x$ and the y-axis.



First we make a sketch of the curve $y^2 = 9 - x$.

Symmetry The equation is unchanged if y is replaced by $(-y)$. Hence the curve is symmetrical about the x-axis.

y-axis Cuts y-axis at $(0, 3)$ and at $(0, -3)$.

x-axis Cuts x-axis at $(9, 0)$.

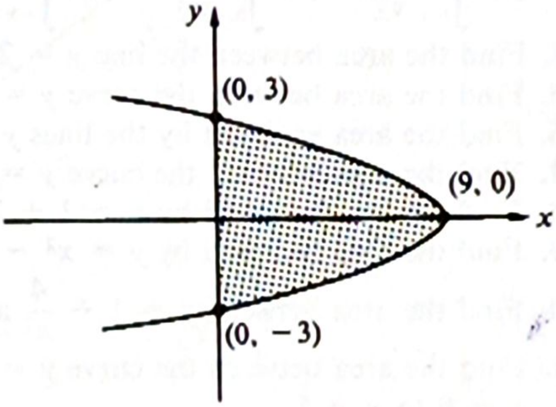
$x \to \pm\infty$ $y = \pm\sqrt{(9 - x)}$

$\therefore$ as $x \to +\infty$, y is undefined.

As $x \to -\infty$, $y = \pm\sqrt{(9 + \infty)}$

i.e. as $x \to -\infty$, $y \to \pm\infty$ slowly by comparison with x.

y undefined y is undefined for $x > 9$ because $(9 - x)$ will be negative. Thus the sketch can be completed and the required area shown shaded:



$$\text{Required area} = \int_{y=-3}^{y=3} x\,dy$$

Now we cannot integrate x with respect to y, so we substitute for x,

$$\therefore \quad A = \int_{y=-3}^{y=3} (9 - y^2)dy$$

which gives $A = 36$ sq. units

Exercise 12B

1. Evaluate the following definite integrals.

(a) $\displaystyle\int_1^5 2x\,dx$ (b) $\displaystyle\int_0^2 3x^2\,dx$

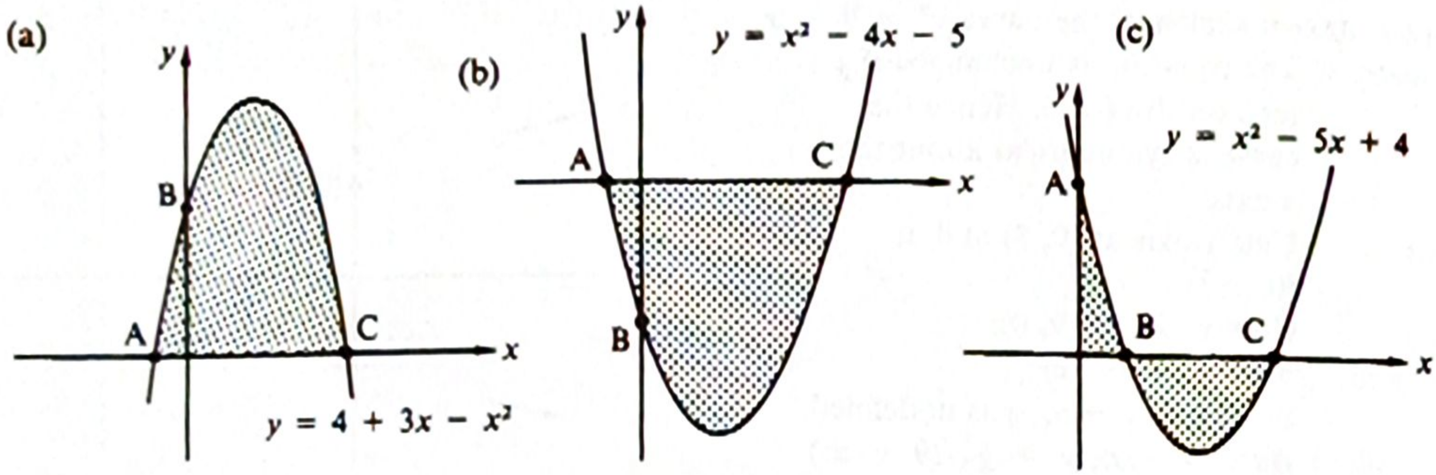
(c) $\displaystyle\int_{-1}^4 (6 - 2x)dx$ (d) $\displaystyle\int_{-1}^1 (1 + x)dx$

(e) $\displaystyle\int_{-1}^3 (3x - 2)dx$ (f) $\displaystyle\int_{-4}^0 (x^2 + x + 1)dx$

(g) $\displaystyle\int_1^2 \frac{1}{x^2}dx$ (h) $\displaystyle\int_4^9 \frac{1}{\sqrt{x}}dx$

(i) $\displaystyle\int_1^4 (x^3 - 2x - 3\sqrt{x})dx$ (j) $\displaystyle\int_1^4 \left(\frac{x^4 - x^3 + \sqrt{x} - 1}{x^2}\right)dx$

2. For each of the following, find the coordinate of points A, B and C and find the shaded area.



3. State why each of the following has no meaning

(a) $\int_{-3}^{4} \frac{1}{x}dx$ (b) $\int_{0}^{4} \frac{1}{x^2}dx$ (c) $\int_{-3}^{3} \frac{1}{x-1}dx$ (d) $\int_{-5}^{0} \frac{1}{x^2-1}dx$ (e) $\int_{-2}^{2} \sqrt{x}\,dx$.

4. Find the area between the line $y = 2x + 3$ and the x-axis from $x = 4$ to $x = 6$.
5. Find the area between the curve $y = x^3$ and the x-axis from $x = 1$ to $x = 2$.
6. Find the area enclosed by the lines $y = x^2 + 2$, the x-axis, $x = 1$ and $x = 3$.
7. Find the area between the curve $y = 10 + 3x - x^2$ and the x-axis from $x = -1$ to $x = 2$.
8. Find the area enclosed by $y = 3 + 2x - x^2$ and the x-axis.
9. Find the area enclosed by $y = x^2 - 6x$ and the x-axis.
10. Find the area between $y = 1 + \frac{4}{x^2}$ and the x-axis from $x = 1$ to $x = 2$.
11. Find the area between the curve $y = x^2 - 6x + 5$ and the x-axis from $x = 0$ to $x = 5$.
12. Find the area between the curve $y = 4 - x^2$ and the x-axis from $x = 0$ to $x = 3$.
13. Find the total area enclosed between $y = (x^2 - 1)(x - 3)$ and the x-axis.
14. Find the total area between the curve $y = \frac{4}{x^2} - 1$ and the x-axis from $x = 1$ to $x = 3$.
15. Using area $= \int_{y=a}^{y=b} x\,dy$, find the following shaded areas: