CHAPTER 1

ALGEBRA 1 TOPICS

INDICES

Definition: An index is the power or an exponent of the base.

e.g

(i) a^n, a – base, n – power or index

(ii) 2^3 , 2 – base, 3 – an index.

An index can be an integer or a fraction.

Laws of indices

(a) $b^m \times b^n = b^{m+n}$

(b) $b^m \div b^n = b^{m-n}$

(c) $(b^m)^n = b^{mxn}$, where b, m and n are real numbers and b # 0.Also.

(d) $a^o = 1$

(e) $a^{-m} = \frac{1}{a^m}$

(f) $a^{1/n} = \sqrt[n]{a}$

Examples

Find the values of the following expressions.

(a) $8^{4/3} \times 4^{-2}$

(b) $\frac{9^{1/2} \times 8^{1/3}}{2^{1/2}}$

(c) $(1/2)^{-2}$

(d) $(x^{1/3} - x^{-1/3})(x^{2/3} + 1 + x^{-2/3})$

Solution

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(a) $\underline{8^{4/3} x 4^{-2}}$

$= 2^{(3)(4/3)} x 2^{(2)(-2)}$

$= 2^4 x 2^{-4}$

$= 2^{(4)+(-4)}$

$= 2^o$

$= 1$

(b) $\dfrac{9^{½} x 8^{1/3}}{2^{½}}$

$= 3^{(2)(½) x} 2^{(3)(1/3)}/2^{1/2}$

$= 3^1 x 2^{(1-½)}$

$= 3 \sqrt{2}$

Because 2 ½ = √ 2

(d) $(x^{1/3} - x^{-1/3})(x^{2/3} + 1 + x^{-2/3})$

$= x - x^{1/3} + x^{1/3} - x^{-1/3} + x^{-1/3} - x^{-1}$

$= x - \dfrac{1}{x}$

(e) $x^{½}(x^{½} - x^{-½})$

$= x^{½+½} - x^{½+-½}$

$= x^1 - x^o$

$= x - 1$

(c) $(½)^{-2}$

$= 1/(1/2)^2$

$= \dfrac{1}{(¼)}$

$= 4$

Qn.2 Solve the equation.

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$2^{2x} - 3(2^x) + 2 = 0$

Solution

$(2^x)^2 - 3(2^x) + 2 = 0$

$2^x = \frac{3 \pm \sqrt{(9 - 4 \times 2)}}{2}$

$= \frac{3 \pm \sqrt{1}}{2}$

$= \frac{3 \pm 1}{2}$

either $2^x = 2 \quad \Rightarrow x = 1$

or

$2^x = 1 \ \Rightarrow x = 0$

Exercise 1(a)

Solve the following equations.

(a) $2^x = 32$

(b) $5^{2x} \times 5^{1-x} = 625$

(c) $2^{(2x+1)} + 5(2^x) - 3 = 0$

(d) $3^{2y} - 4(3^y) + 3 = 0$

(e) $2^{2x} - 2^{(x+2)} + 4 = 0$

(f) $2^{2x} - 2^x - 2 = 0$

(g) $2^{(2x+1)} - 5(2^x) + 2 = 0$

(h) $2^{(2x+1)} - 9(2^x) + 4 = 0$

2. (i) $3^{2x} - 4(3^x) + 3 = 0$

(ii) $3^{(2x+2)} - 10(3^x) + 1 = 0$

(iii) $3^{(2x+1)} + 5(3^x) - 2 = 0$

Answers

a. 5
b. 3
c. -1
d. 0,1
e. 1
f. 1
g. -1,1
h. -1,2

i. (0,1)
ii. (0,-21)
iii. (-1)

LOGARITHMS

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A logarithm is another term to mean an **INDEX** or **Power** of a given positive number base.

Alternatively: The logarithm of a positive quantity y to a given base x is defined as the index of the power to which the base x must be raised to make it equal to the given quantity y.

$\Rightarrow$ $x^n = y$ i.e. n is the logarithm of y to the base x.

LAWS OF LOGARITHMS

1. $\log_x(AB) = \log_x A + \log_x B$
2. $\log_x(A/B) = \log_x A - \log_x B$
3. $\log_x 1 = 0$ (the logarithm of unity is zero)
4. $\log_x x = 1$ (the logarithm of the base itself is unity)
5. $\log_x A^{1/t} = \frac{1}{t} \log_x A$
6. If $\log_x B = y$

   then $B = x^y$

Proof (proving of laws of logarithms)

1. $\log_x(AXB) = \log_x A + \log_x B$

Let $A = x^t$, $B = x^n$ ……..

$\Rightarrow AB = x^t.x^n$

$AB = x^{t+n}$

taking logs to base x

$\log_x AB = \log_x x^{(t+n)}$

$= (t+n) \log_x x$

$\log_x AB = t+n$ ………..

From

But $\log_x A = \log_x x^t = t \log_x x = t$

Also $\log_x B = \log_x x^n = n \log_x x = n$

Thus $\log_x AB = \log_x A + \log_x B$

Note: let $y = x^n$

$\text{Log}_x y = \log_x x^n = n \log_x x$

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$= n$

$\Rightarrow n = \log_x y$

$\Rightarrow y = x^{\log_x y}$

2. $\log_x^{(\frac{A}{B})} = \log_x^A - \log_x^B$

Let $A = x^t$, $B = x^n$ …….

$\Rightarrow \frac{A}{B} = \frac{x^t}{x^n} = x^{t-n}$

Taking logs to base x

$\log_x A/B = \log_x x^{(t-n)}$

$= (t-n) \log_x x$

$\log_x A/B = t-n$ ……….

From

But $\log_x A = \log_x x^t = t \log_x x = t$

Also $\log_x B = \log_x x^n = n \log_x x = n$

<u>Thus $\log_x A/B = \log_x A - \log_x B$</u>

3. <u>$\log_x 1 = 0$</u>

Let $y = \log_x N$

Thus

$N = x^{\log_x N}$

If $y = 0$

$\Rightarrow X^o = N = 1$

$0 = \log_x N$

Therefore $\text{Log}_x 1 = 0$

The logarithm of unity is zero.

4. $\log_x x = 1$

Let $\log_x^x = N$

$\Rightarrow X^N = x$

But $N = 1$, then $X^1 = x$

Thus Thus $\log_x^x = 1$

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N.B The logarithm of the base itself is unit.

5. $\log_x A^{1/t} = 1/t \log A$

Let $A = X^{\log}xA$

$A^{1/t} = (X^{\log_x A})^{1/t}$

$A1/t = X^{1/t \log_x A}$

Taking logs to base x

logxA1/t = logxX(1/t logxA)

logxA1/t = 1/t logxA logxX

logxA1/t = (1/t logxA).1

Thus $\log_x A1/t = 1/t \log_x A$

6 $\log_x B = y$
let $B = x^y$
taking logs to base x
$\log_x B = \log_x x^y$
$\log_x B = y \log_x x$
$\log_x B = y.1$
$y = \log_x B$
thus, for
$\log_x B = y$

CHANGE FROM ONE BASE TO ANOTHER

1. let $\log_a x = p$

$\Rightarrow x = a^p$

Taking logarithms to base x

$\text{Log}_x x = \log_x a^p$

$1 = p\log_x a$

$\Rightarrow P = \frac{1}{\log_x a}$

$\Rightarrow \boxed{\text{Log}_a x = \frac{1}{\log_x a}}$

2. let $\log_b a = N$

$a = b^N$

Taking logarithms to base C

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$\text{Log}_c a = \log_c b^N$

$\text{Log}_c a = N\log_c b$

$\Rightarrow N = \dfrac{\log_c a}{\log_c b}$

Thus $\boxed{\text{Log}_b a = \dfrac{\log_c a}{\log_c b}}$

3. let $\log_8 2 = x$

$2 = 8^x$

Taking logarithms to base 10

$\log_{10} 2 = \log_{10} 8^x$

$\log_{10} 2 = x\log_{10} 8$

$\Rightarrow x = \dfrac{\log_{10} 2}{\text{Log}_{10} 8}$

Thus $\boxed{\text{Log}_8{}^2 = \dfrac{\log_{10} 2}{\log_{10} 8}}$

4. let $\log_2 N = t$

$N = 2^t$

Taking logarithms to base x

$\log_X N = \log_x 2^t$

$\log_x N = t \log_x 2$

$\Rightarrow t = \dfrac{\log_x N}{\text{Log}_x 2}$

Thus $\boxed{\text{Log}_2 N = \dfrac{\log_x N}{\log_x 2}}$

Examples

1. Simplify $2\log b5 + 3\log_b 7$?

Solution

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$\log_b 5^2 + \log_b 7^3$

$= \log_b(5^2 \text{ x } 7^3)$

$= \log_b 25 \text{ x } 343)$

$= \log_b 8575$

2. Simplify $\frac{1}{3}\log_8 27 - \log_8 \frac{3}{5}$

Solution

$= \log_8 27^{1/3} - \log_8 3/5$

$= \log_8 3 - \log_8 3/5$

$= \log_8 (3 \div 3/5)$

$= \log_8 3 \text{ x } \frac{3}{5}$

$= \log_8 5$

3. Simplify

$\log b\sqrt{b^4 + 1)} - ½ \log_b (1 + 1/b^4)$

Solution

$\log_b(b^4 + 1)^{½} - \log_b (1 + {}_b¼)^{½}$

$= \log_b(b^4 + 1)^{½} - \log_b \left(\frac{b^4 + 1}{b^4}\right)^{½}$

$= \log_b\left((b^4+1)^{½} \div \frac{b^4 + 1}{b^4}^{½}\right)$

$= \log_b\left((b^4 + 1) ½ \text{ x } \left(\frac{b^4}{b^4 + 1}\right) ½\right)$

$= \log_b (b^4) ½$

$= \log_b b^2$

$= 2$

4. Express as a single logarithm.

(a) $\log_7 14 - \log_7 21 + \log_7 6$

Solution:

$(\log_7 14 - \log_7 21) + \log_7 6$

$= \log_7\left(\frac{14}{21}\right) + \log_7 6$

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$= \log_7\left(\frac{14}{21} \times 6\right)$

$= \log_7 4$

(b) $\frac{3}{2}\log_5 9 - 2\log_5 6$

Solution

$\log_5 9^{3/2} - \log_5 6^2$

$= \log_5 3^3 - \log_5 6^2$

$= \log_5(3^3 \div 6^2)$

$= \log_5 \left(\frac{27}{36}\right)$

$= \log_5(\,¾\,)$

EXERCISE

Simplify		**Answers**
1.	$2 \log_4 a + \log_4 b - 3 \log_4 c$	$\mathbf{\log_4\left(\frac{a^2b}{c^3}\right)}$
2.	$\log_9 \frac{75}{16} - 2 \log_9 \frac{5}{9} + \log_9 \frac{32}{243}$	$\mathbf{(\log_9{}^2)}$
3.	$2 \log_3 \frac{2}{3} - \log_3 \frac{8}{9}$	$\mathbf{(\log_3\ (½)}$
4.	$\frac{3\log_{10}2 - \log_{10} 4}{\log_{10}4 - \log_{10} 2}$	**1**
5.	$2 \log_a 5 - 2 \log_a 15 + 3 \log_a 3 - \log_a 6$	$\mathbf{-\log_a 2}$
6.	$2\log_a 5 + \log_a 4 - 2\log_a 10$	**0**
7.	$½ \log_c 25 - 2\log_c 3 + 2\log_c 6$	$\mathbf{\log_c 20}$
8.	$\frac{\log_a 81}{\log_a 27}$	**2**
9.	$3 \log_2 \left[\frac{5}{3}\right] - 2 \log_2 \left(\frac{10}{9}\right) + \log_2 \left(\frac{1}{30}\right)$	**ˉ3**

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10. $\log_4\left[(a^2+1)^2-(a^2-1)^2\right]-\log_2 2a$ **0**

MORE EXAMPLES: SOLVE THE FOLLOWING EQUATIONS.

1. $2(2^{2x})-5(2^x)+2=0$

Solution

$2(2^x)^2-5(2^x)+2=0$

let $2^x=t$

$2t^2-5t+2=0$

$t=\frac{5\pm\sqrt{25-16}}{4}$

$t=\frac{5\pm 3}{4}$

Either t = 2 or ½

$\Rightarrow 2^x=2$

$\Rightarrow x=1$

Or $2^x=½$

$\Rightarrow x={}^{-}1$

2. $(5^x)(5^{x-1})=10$

Solution

$(5^x)(5^x.5^{-1})=10$

$(5^x)(5^x)\times\frac{1}{5}=10$

$(5^x)^2=50$

$5^x=7.0711$

$x\log_{10}5=\log_{10}7.0711$

$x=\frac{\log_{10}7.0711}{\log_{10}5}$

$=1.2153$

3. $\log_2 x+\log x2=2$

Solution

Change of base

$\log_2 x+\frac{\log_2 2}{\log_2 x}=2$

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$$\log_2 x + \frac{1}{\text{Log}2x} = 2$$

$(\log_2 x)^2 + 1 = 2(\log_2 x)$

$(\log_2 x)^2 - 2\log_2 x + 1 = 0$

(quadratic in $\log_2 x$)

$$\log_2 x = \frac{2 \pm \sqrt{4-4}}{2}$$

$$\log_2 x = \frac{2}{2} = 1$$

$\log_2 x = 1$

$x = 2^1$

$= 2$

4. $2(10^{2x}) - 5(10^x) + 2 = 0$

$2(10^x)^2 - 5(10^x) + 2 = 0$

Quadratic in 10^x

$$\Rightarrow 10^x = \frac{5 + \sqrt{25-16}}{4}$$

$$10^x = \frac{5+3}{4}$$

either $10^x = 2$ or $10^x = ½$

from $10^x = 2$

$\log_{10} 10^x = \log_{10} 2$

$x = \log_{10} 2$

$= 0.3010$

and $10^x = ½$

$$x = \frac{\log_{10} 0.5}{1}$$

$= -0.3010$

Thus $x = \underline{+}\, 0.3010$

5. $2^{2y+8} - 32(2^y) + 1 = 0$

$2^{2y}.2^8 - 32(2^y) + 1 = 0$

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$256\,(2^y)^2 - 32\,(2^y) + 1 = 0$

Quadratic in 2^y

$$\Rightarrow 2^y = \frac{32 + \sqrt{32^2 - 4 \times 256}}{2 \times 256}$$

$$2^y = \frac{32 + 0}{512}$$

$$2^y = \frac{1}{16}$$

$2^y = 4^{-2}$

$2^y = 2^{-4}$

$\Rightarrow y = {}^{-}4$

6. $\log_2 x = \log_4(8x - 16)$

Solution

Change of base to 2

$$\log_2 x = \frac{\log_2 (8x - 16)}{\log_2 4}$$

$$\log_2 x = \frac{\log_2(8x - 16)}{2}$$

$\log_2 x = \log_2 (8x - 16)$

$\text{Log}_2 x^2 = \log_2 (8x - 16)$

$\Rightarrow x^2 = 8x - 16$

$\Rightarrow x^2 - 8x + 16 = 0$

$(x - 4)^2 = 0$

$x = 4$ repeated

7. If loga $N = X$ and $\log_b N = Y$

Show that $\log_{ab} N = \dfrac{XY}{X + Y}$

Solution

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$\log_{ab}N = \frac{\log_a N}{\log_a(ab)}$

$= \frac{\log_a N}{\text{Log}_a a + \log_a b}$

From: $\log_{ab}N = \frac{\log_a N}{1 + \log_a b}$

From $\log_b N = Y$

$N = b^Y$

Taking logarithms to base a

$\log_a N = \log_a b^Y$

$\log_a N = Y\log_a b$

$\Rightarrow \frac{\log_a N}{Y} = \log_a b$

$\Rightarrow \log_a b = \frac{X}{Y}$

From: $\log_{ab} N = \frac{X}{1 + {}^{x}/{}_{y}}$

$\log = \frac{YX}{Y + X}$

Thus $\log_{ab}N = \frac{XY}{X+Y}$

8. Prove that $\log_a b \times \log_b c \times \log_c a = 1$

Solution

$\text{Log}_a b \times \frac{\log_a c}{\log_a b} \times \frac{\log_a a}{\log_a c}$

$= \log_a a$

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$= 1$

9. If $\log_{10}y = 2 - \log_{10}X^{2/3}$, express y as a function of x not involving logarithms, hence show that if x =8, then y = 25.

Solution

$\text{Log}_{10}y = \log_{10}100 - \log_{10}x^{2/3}$

$$\text{Log}_{10}y = \log_{10}\left(\frac{100}{X^{2/3}}\right)$$

$$\Rightarrow y = \frac{100}{X^{2/3}}$$

$$\Rightarrow y = 100x^{-2/3}$$

If x = 8

$Y = 100x8^{-2/3}$

${}^{100}/_{8}{}^{2/3}$

$$= \frac{100}{((8)^{1/3})^2}$$

$$= \frac{100}{(2)^2}$$

$$= \frac{100}{4}$$

Y= 25

Thus when x = 8 then y = 25

10. Solve the simultaneous equations

$10^x .4^y = 1, \quad 8^x = 10^{(y+1)}$

Solution

$8^x = 10^y.10^1$

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$8^x = 10(10^y)$ and $10^x . 10^y = 1$

Taking logs to base 10

$\log_{10}(10^x.4^b) = \log_{10}1$ and $\log_{10}8^x = \log_{10}(10)10^y$

$\text{Log}_{10}10^x + \log_{10}4^y = 0$ and $\log_{10}8^x = \log_{10} 10 + \log_{10}10^y$

$x \log_{10}10 + y \log_{10} 4 = 0$ and $x\log_{10}8 = 1 + y \log_{10}10$

$x + y \log_{10} 4 = 0$ and $x \log_{10} 8 = 1 + y$

$x + 0.6021y = 0$ and $0.9031x = 1 + y$

$x = -0.6021y$

$\Rightarrow 0.9031(-0.6021y) = 1+ y$

$-0.5438y = 1 + y$

$-1.5438 y = 1$

$y = {}^{-}0.6478$

Thus $x = -0.6021 \text{ x } -0.6478 = 0.3900$

REVISION EXERCISE (1 C)

1. Solve for a

$\frac{3}{2} \log_{10}a^3 - \log_{10}\sqrt{a} - 2\log_{10}a = 4$ **Ans 100**

2. Determine the positive number x which satisfies the equation. **Ans 3**
$\text{Log}_3 x = \log_9 (x+6)$

3. Solve the simultaneous equations for positive values of x and y

$xy = 16$

$\log_x y = 3$ **Ans x = 2, y = 8**

4. Show that $\log_{16} (xy) = \frac{1}{2} \log_4 x + \frac{1}{2} \log_4 y$

Hence or otherwise, solve the simultaneous equations

$\log_{16} (xy) = 7.5$

$\frac{\log_x x}{\log_4 y} = -8$ **65536,0.25**

5. Find the real values of x

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For which $\log_3 x - \log_x 9 = 4^o$ **Ans 9, 1/3**

6. If $\log_a t = \theta$ and $\log_c t = \phi$ where t # 1

prove that

$$\frac{\theta - \phi}{\theta + \phi} = \frac{\log_b c - \log_b c}{\log_b c + \log_b a}$$

7. If $\log_8 C^2 = p$; $1 + \log_2 c = q$ and $q - p = 4$, find c **Ans 512**

8. Solve the equation:

$\text{Log}_2 x^4 + \log_2 4x = 12$ **Ans 4**

9. $2 \log_4 x + 3 \log_x 4 = 7$ Ans 2, 64

10. $3 \log_8 x = 2\log_x 8 + 5$ **½ ,64**

COMMON LOGARITHMS

There are two types of logarithms.

(1) Common logarithms (2) Natural logarithms

Common logarithms are logarithms to the base ten i.e $\log_{10}x$

Natural logarithms are logarithms to base e .

They are also called Napierian logarithms after a mathematician called John Napier

e.g $\log_e x$ or written as Inx. The value of e ≈2.718289(5dec.)

Here in this topic we are concerned with common logarithms.

The logarithms to base 10 consist of the sum of the relevant power of 10 together with a decimal value representing the logarithm of some number between 1 and 10.

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e.g

$\log_{10} (t \times 10^y)$

$= \log_{10} t + \log_{10} 10^y$

$= \log_{10} t + y \log_{10} 10$

$= y + \log_{10} t$(since $\log_{10} 10 = 1$)

Numerically

$384 = 3.84 \times 10^2$

$\log_{10} 384 = \log_{10} (3.84 \times 10^2)$

$= \log_{10} 3.84 + \log_{10} 10^2$

$= 2 + \log_{10} 3.84$

$= 2 + 0.5843$

The power of 10 which is 2 is called the characteristic and 0.5843 is called the mantissa.

Examples

1. Evaluate $\log_5 7$

   let $x = \log_5 7$

   $7 = 5^x$

   Takings to base 10

   $\log_{10} 7 = \log_{10} 5^x$

   $\log_{10} 7 = x \log_{10} 5$

   $x = \dfrac{\log_{10} 7}{\text{Log}_{10} 5}$

   $= \dfrac{0.8451}{0.69897}$ (using a calculator)

   $= 1.2091$ (4dec)

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Solve for x

2. $3^{x+1} = 16^{x-1}$

$= 2^{(4x-4)}$

Taking logarithms to base 10

$\log_{10} 3^{x+1} = \log_{10} 2^{(4x-4)}$

$(x+1) \log_{10} 3 = (4x-4) \log_{10} 2$

$x \log_{10} 3 + \log_{10} 3 = 4x \log_{10} 2 - 4\log_{10} 4$

$4x \log_{10} 2 - x\log_{10} 3 = \log_{10} 3 + 4\log_{10} 4$

$x (4\log_{10} 2 - \log_{10} 3) = \log_{10} 3 + \log_{10} 4^4$

$x (\log_{10} 2^4 - \log_{10} 3) = \log_{10} 3 + \log_{10} 256$

$x (\log_{10} \frac{16}{3}) = \log_{10} (3 \times 256)$

$x \log_{10} (\frac{16}{3}) = \log_{10} 768$

$\Rightarrow x = \frac{\log_{10} 768}{\log_{10}(16/3)}$

$= \frac{2.88536}{0.72700}$

X = 3.9689

REVISION EXERCISE ON COMMON LOGARITHMS

1. Solve for x: $2^{2x} - 5(2^x) + 6 = 0$ **Ans: X = 1**
X = 1.5850

2. Solve for y: $5^{(2y+1)} = 3$ **Ans y = -0.1587**

3. Solve the equation $3^{(t+1)} = 4^{(2t-1)}$ **Ans: t = 1.484(3dp)**

4. Solve for x and y
$2\log_{10} y = \log_{10} 2 + \log_{10} x$ and $2^y = 4^x$ **Ans y = 1, x = ½**

5. Solve for n **n = -0.1038**
$4^{2n+1} = 3$

6. Solve the simultaneous equations.
nm = 80

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$\log_{10} n - 2\log_{10} m = 1$ **Ans n = 40, m = 2**

7. Write down the value of b if
$\log_b 0.01 = \log_{0.1} 100$ **b = 10**

8. Solve for the unknown
$[2t^2 = 16^{t-1}]$ **t = 2,2**

9. Evaluate (i) $\log_{25} 3.142$ (ii) $\left(\frac{1}{12}\right)^{1.405}$

0.3557 **0.0305**

10. Determine value of x and y: $2^{x+y} = 6^y$ and $3^x = 6(2^y)$ the value of x and y

2.71.1.71

Expressions such as √4, √25 have exact numerical values.
But expressions such as √2, √3 do not have exact values. We can never find an exact quantity equal to √2. Numbers like √2, √3, √5, are called irrational numbers, and it is often convenient to leave them in the form √2,

SURDS

A set of numbers of the form √**n** where **n** is a member of positive integers and have no exact values and are left in the √n form are called **SURDS**.
An expression involving surds is called **surd expression**.

EXAMPLES

SIMPLIFYING SURDS

(a) $\sqrt{80} = \sqrt{2 \times 2 \times 2 \times 2 \times 5}$
$= \sqrt{2^2 \times 2^2 \times 5}$
$= \sqrt{2^2} \times \sqrt{2^2} \times \sqrt{5}$
$= 2 \times 2 \times \sqrt{5}$
$= 4\sqrt{5}$

(b) $\sqrt{12} - \sqrt{27} + 2\sqrt{3}$
$= \sqrt{3 \times 4} - \sqrt{3 \times 3 \times 3} + 2\sqrt{3}$
$= \sqrt{3} \times \sqrt{4} - \sqrt{3^2} \times \sqrt{3} + 2\sqrt{3}$
$= 2\sqrt{3} - 3\sqrt{3} + 2\sqrt{3}$
$= \sqrt{3}\,(2 - 3 + 2)$

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$= \sqrt{3}\ (1)$
$= \sqrt{3}$

(c) $2\sqrt{18} \times 3\sqrt{2}$
$= 2\sqrt{2x3x3} \times 3\sqrt{2}$
$= 2\sqrt{2} \times \sqrt{3^2} \times 3\sqrt{2}$
$= 2\sqrt{2} \times 3 \times 3\sqrt{2}$
$= 2 \times 3\ \sqrt{2} \times 3\ \sqrt{2}$
$= 6 \times 3 \times \sqrt{2} \times \sqrt{2}$
$= 18 \times 2$
$= 36$

(d) $6 - 9\sqrt{3} + 4\sqrt{3} - 6(\sqrt{3})^2$
$= 6 - 9\sqrt{3} + 4\sqrt{3} - 6\ (3)$
$= 6 - 9\sqrt{3} + 4\sqrt{3} - 18$
$= 6 - 18 + \sqrt{3}\ (4 - 9)$
$= -12 + \sqrt{3}\ (-5)$
$-12 - 5\ \sqrt{3}$

(e) $\sqrt{32} \times \sqrt{15} \div \sqrt{24}$
$= 4\sqrt{2} \times \sqrt{3} \times \sqrt{5} \div 2 \times \sqrt{3} \times \sqrt{2}$
$= 2\sqrt{2} \times \sqrt{3} \div \sqrt{3} \times \sqrt{2}$
$= 2\sqrt{2} \times \sqrt{3} \times \sqrt{5} \div \sqrt{3} \times \sqrt{2}$

EXERCISE (IE)
Simplify the following surd expressions

1. $\sqrt{125}$
Ans $5\sqrt{5}$

2. $\sqrt{32} - \sqrt{18}$
Ans $\sqrt{2}$

3. $\sqrt{84} \times \sqrt{140} \div \sqrt{120}$
Ans $7\sqrt{2}$

4. $(\sqrt{3} - 1)\ (\sqrt{2} + 1)$
Ans $\sqrt{6} + \sqrt{3} - \sqrt{2} - 1$

5. $(3\sqrt{3} - 2)\ (3\sqrt{3} + 2)$
And 23

6. $\sqrt{50} + \sqrt{98} - \sqrt{32} - \sqrt{72} + \sqrt{8}$
Ans $10\sqrt{2}$

7. $\sqrt{252} \div \sqrt{63}$
Ans 2

8. $(5 - 2\sqrt{7})\ (5 + 2\sqrt{7})$
Ans -3

<u>RATIONALIZATION</u>

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The removing of surds from the denominator is a process called **Rationalization of the denomination**. The denominator becomes a rational number.

EXAMPLES

Rationalize the following surd expressions to the simplest expressions.

1. $\frac{2}{\sqrt{12}}$

   Here we multiply in both the numerator and denominator by $\sqrt{12}$

   $\Rightarrow \frac{2}{\sqrt{12}} = \frac{2\sqrt{12}}{\sqrt{12} \times \sqrt{12}} = \frac{2\sqrt{12}}{12} = \frac{1}{6}\sqrt{12}$

   $= \frac{1}{3}\sqrt{3}$

2. $\left(\frac{1+\sqrt{3}}{2+\sqrt{3}}\right)$. Here we multiply to the numerator and denominator by $2 - \sqrt{3}$ so that the denominator becomes of the type $(a - b)(a + b) = a2 - b2$ (difference of two squares.

   Thus $\frac{1+\sqrt{3}}{2+\sqrt{3}}$ become $\left(\frac{1+\sqrt{3}}{2+\sqrt{3}}\right)\left(\frac{2-\sqrt{3}}{2-\sqrt{3}}\right)$

   $= \frac{1(2-\sqrt{3}) + \sqrt{3}(2-\sqrt{3})}{2^2 - (\sqrt{3})^2}$

   $= \frac{2 - \sqrt{3} + 2\sqrt{3} - (\sqrt{3})^2}{4 - 3}$

   $= \frac{2 - \sqrt{3} + 2\sqrt{3} - 3}{1}$

   $= {}^{-}1 + \sqrt{3}$

3. $\frac{1 + 2\sqrt{2}}{5 - 3\sqrt{2}} = \frac{(1 + 2\sqrt{2})(5 + 3\sqrt{2})}{(5 - 3\sqrt{2})(5 + 3\sqrt{2})}$

   $= \frac{1(5 + 3\sqrt{2}) + 2\sqrt{2}(5 + 3\sqrt{2})}{(5)^2 - (3\sqrt{2})^2}$

   $= \frac{(5 + 3\sqrt{2}) + 10\sqrt{2} + 6(\sqrt{2})^2}{25 - 18}$

   $= \frac{5 + 3\sqrt{2} + 10\sqrt{2} + 12}{7}$

   $= \frac{17 + 13\sqrt{2}}{7}$

4. $\frac{\sqrt{3}}{\sqrt{3} - 1} + \frac{\sqrt{3}}{\sqrt{3}+1}$

   Ans: $\frac{\sqrt{3}}{\sqrt{3}-1} + \frac{\sqrt{3}}{\sqrt{3}+1} = \frac{\sqrt{3}(\sqrt{3} + 1) + \sqrt{3}(\sqrt{3} - 1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$

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$$= \frac{3 + \sqrt{3} + 3 - \sqrt{3}}{(\sqrt{3})^2 - (1)^2}$$

$$= \frac{6}{3 - 1}$$

$$= \frac{6}{2}$$

$$= 3$$

EXERCISE ON RATIONALIZATION (IF)

Rationalize the following surd expressions

1. $\sqrt{(1-x)} + \frac{1}{\sqrt{(1-x)}}$ **Ans:** $\mathbf{\frac{\sqrt{(1-x)}}{(1-x)}}$

2. $\frac{1}{1+\sqrt{x}} + \frac{1}{1-\sqrt{x}} + \frac{1}{1-x}$ **Ans:** $\mathbf{\frac{3}{1-x}}$

3. $(\sqrt{x} + \frac{1}{\sqrt{x}})(\sqrt{x} - 1 - \frac{1}{\sqrt{x}})$

   Ans: $\mathbf{\frac{x^2 - (x+1)\sqrt{x} - 1}{X}}$

4. $\frac{3\sqrt{20} - \sqrt{60}}{2\sqrt{3}}$ **Ans:** $\mathbf{\sqrt{5}(\sqrt{3} - 1)}$

5. $\frac{2\sqrt{3} - 2}{2\sqrt{3} + 2}$ **Ans:** $\mathbf{2 - \sqrt{3}}$

DETERMINATION OF SQUARE ROOTS OF SURDS
SURDS

Note (1) To determine the square root of $a + \sqrt{b}$, we suppose that

$(a + \sqrt{b})^{1/2} = \pm (\sqrt{x} + \sqrt{y})$

Because the square root of a rational quatity cannot be partly rational and partly irrational.

$\Rightarrow a + \sqrt{b} = (\pm (\sqrt{x} + \sqrt{y}))^2$

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$= x + 2\sqrt{xy} + y$
$\Rightarrow a = x + y$ and $\sqrt{b} = 2\sqrt{xy}$
Thus $a = x + y$ and **$b = 4xy$**

2. To determine the square root of $a - \sqrt{b}$, we let $(a - \sqrt{b})^{½} = \pm (\sqrt{x} - \sqrt{y})$

EXAMPLES

1. Find the square root of
$11 + 2\sqrt{18}$

Solution

Let $(11 + 2\sqrt{18})^{½} = \pm (\sqrt{x} + \sqrt{y})$

Squaring both sides

$11 + 2\sqrt{18} = \quad (\pm (x + \sqrt{y})\}^2$

$11 + 2\sqrt{18} = \quad x + y + 2\sqrt{xy}$

$\Rightarrow x + y = 11$ and $2\sqrt{18} = 2\sqrt{xy}$

i.e $x + y = 11$ and $\sqrt{xy} = \sqrt{18}$

$x+y = 11$ and $xy = 18$

$xy = 18$ and $x+y = 11$

$y = \frac{18}{x}$

$x + \frac{18}{x} = 11$

$x^2 + 18 = 11x$

$x^2 - 11x + 18 = 0$

$(x - 2)(x - 9) = 0$

$x = 2$ or 9

if $x = 2$, $\quad y = 9$

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if $x = 9$, $y = 2$

Hence $x = 2$ or $x = 9$.
The corresponding values of y are 2 and 9.
Thus

$11 + 2\sqrt{18} = \pm(\sqrt{2} + \sqrt{9})$
$= \pm(\sqrt{2} + 3)$

It makes no difference to the final result whether we take the solution $x = 9$, $y = 2$ or the alternative one $x = 2$ and $y = 9$.

2. Find the square root of

$12 - 3\sqrt{12}$

let $(12 - 3\sqrt{12})^{½} = +(\sqrt{a} - \sqrt{b})$

Squaring both sides

$12 - 3\sqrt{12} = \{+(\sqrt{a} - \sqrt{b})\}^2$

$12 - 3\sqrt{12} = a + b - 2\sqrt{ab}$

$\Rightarrow a + b = 12$

And $-2\sqrt{ab} = -3\sqrt{12}$

$\Rightarrow a + b = 12$

And $4ab = 9 \times 12$

$\Rightarrow a + b = 12$ …………..

$ab = 27$

let $a = \frac{27}{b}$

from

$\Rightarrow \frac{27}{b} + b = 12$

$27 + b^2 = 12b$

$b^2 - 12b + 27 = 0$

$(b - 3)(b - 9) = 0$

$b = 3$ or $a = 9$

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Thus the square root of $(12 - 3\sqrt{12})$ is $\pm (3 + \sqrt{3})$

EXERCISE (1g)

1. Determine $(5 + 2\sqrt{6})^{½}$ **$\pm (\sqrt{3} + \sqrt{2})$**

2. Find $(18 - 12\sqrt{2})^{½}$ **$\pm (\sqrt{12} - \sqrt{6}$**

3. Find x and y such that

   $3 + \sqrt{2} = (x + y\sqrt{2})(6 - \sqrt{2})^2$ $\quad \frac{69}{578}, \frac{37}{578}$

EQUATIONS INVOLVING SQUARE ROOTS (SURD EQUATIONS)

The general method of solving such equations is to overcome the difficulty of the surd by isolating it on one side of the equation and then squaring throughout.

In the above method, there is squaring twice, thus introducing some unnecessary solutions. Thus the solutions obtained should be rechecked in the original equation before being offered as the final answers.

Examples

1. Solve the equation

   $\sqrt{(-x + 3)} - \sqrt{(7 + x)} - 2 = 0$

   Solution

   $\sqrt{(-x + 3)} = 2 + \sqrt{(7 + x)}$

   $\Rightarrow (-x + 3) = 4 + 7 + x + 4\sqrt{(7 + x)}$

   $-x + 3 - 4 - 7 - x = 4\sqrt{(7 + x)}$

   $-x - 4 = 2\sqrt{(7 + x)}$

   $\{-(x + 4)\}^2 = 4(7 + x)$

   $x^2 + 8x + 16 = 28 + 4x$

   $x^2 + 4x - 12 = 0$

   $(x + 6)(x - 2) = 0$

   Either

   $x = {}^-6$ or 2

   Checking: using $\sqrt{(-x + 3)} - \sqrt{(7 + x)} - 2 = 0$

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If $x = {}^{-}6$

$\sqrt{(-(-6)+3} - \sqrt{(7-6)} - 2$

$3 - 1 - 2 = 0$ (correct) in $x = {}^{-}6$

if $x = 2$

$\sqrt{-(2)+3} \quad - \quad \sqrt{7+2} - 2$

$1 - 3 - 2 = {}^{-}4$ (not correct)

Thus the only root of the given equation is $x = {}^{-}6$

2. Solve the equation

$\sqrt{(12+x)} - \sqrt{(13-x)} = 1$

$\sqrt{(12+x)} = 1 + \sqrt{(13-x)}$

$(12+x) = 1 + 13 - x + 2\sqrt{(13-x)}$

$12 + x - 1 - 13 + x = 2\sqrt{(13-x)}$

$2x - 2 = 2\sqrt{(13-x)}$

$(x-1)^2 = 13 - x$

$x^2 - 2x + 1 = 13 - x$

$x^2 - x - 12 = 0$

$(x-4)(x+3) = 0$

Either $x = 4$ or $x = -3$

Checking: Using $\sqrt{(12+x)} - \sqrt{(13-x)} = 1$

If $x = 4$

$\sqrt{(12+4)} - \sqrt{(13-4)} = 4 - 3 = 1$ (correct)

If $x = -3$

$\sqrt{(12-3)} - \sqrt{13+3)} = 3 - 4 = -1$. (not correct)

Therefore the only value of x is 4.

REVISION

EXERCISE – (IH)

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1. $\sqrt{(2x + 7)} - \sqrt{x} = 2$ 1,9

2. $\sqrt{(t + 2)} - \sqrt{(t - 3)} - 1 = 0$ **7**

3. $\sqrt{(2b + 1)} - \sqrt{b} = \sqrt{(b - 3)}$ **4**

4. $\sqrt{(x + 6)} - \sqrt{(x+1)} = 1$ **3**

5. $\sqrt{(3x+1)} - \sqrt{(x-1)} = 2$ **(1,5)**

6. $\sqrt{(9 - 4x)} - \sqrt{(5 - x)} = 2$ **-4**

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