P425/1
PURE MATHEMATICS
PAPER 1
JUNE MOCK 2013

SECTION A (40 MARKS)

1. The roots of the quadratic equation, $3x^2 - 12x + 6 = 0$ are α and β. Find the quadratic whose roots are $\frac{\alpha}{\beta^3}$ and $\frac{\beta}{\alpha^3}$ ***(5 marks)***

$$3x^2 - 12x + 6 = 0$$
$$\alpha + \beta = 4$$
$$\alpha\beta = 2$$
$$\frac{\alpha}{\beta^3} + \frac{\beta}{\alpha^3} = \frac{\alpha^4 + \beta^4}{(\alpha\beta)^3}$$
$$= \frac{(\alpha+\beta)^4 - 4\alpha^3\beta - 6\alpha^2\beta^2 - 4\alpha\beta^3}{(\alpha\beta)^3}$$
$$= \frac{4^4 - 6\cdot 2^2 - 4\cdot 2\cdot(4^2 - 2\cdot 2)}{2^3}$$
$$= 17$$
$$\frac{\alpha\beta}{(\alpha\beta)^3} = \frac{1}{(\alpha\beta)^2}$$
$$= \frac{1}{4}$$

$$from \qquad x^2 - (sum)x + product = 0$$
$$\Rightarrow x^2 - 17x + \frac{1}{4} = 0$$
$$\Rightarrow 4x^2 - 68x + 1 = 0$$

2. Solve for θ: $\cot^2\theta = 5(\cos ec\theta + 1), \quad for \quad 0^0 \le \theta \le 360^0$.

$\cot^2\theta = 5(coec\theta + 1)$

$\cos ec^2\theta - 1 = 5\cos ec\theta + 5$

$\cos ec^2\theta - 5\cos ec\theta - 6 = 0$

$(\cos ec\theta + 1)(\cos ec\theta - 6) = 0$

either $\cos ec\theta = -1$

$\Rightarrow \sin\theta = -1$ ***(5 marks)***

$\Rightarrow \theta = \sin^{-1} - 1$

$\Rightarrow \theta = 270^0$

or $\cos ec\theta = 6$

$\Rightarrow \theta = 9.6^0, \quad 170^0$

3. Show that $\frac{d}{dx}\log_e\left[\frac{(1+x)e^{-2x}}{1-x}\right]^{\frac{1}{2}} = \frac{x^2}{1-x^2}$ ***(5 marks)***

$$\frac{d}{dx}\log_e\left[\frac{(1+x)e^{-2x}}{1-x}\right]^{\frac{1}{2}} = \frac{1}{2}\frac{d}{dx}\left[\log_e(1+x) - 2x - \log_e(1-x)\right]$$

$$= \frac{1}{2}\left(\frac{1}{1+x} - 2 + \frac{1}{1-x}\right)$$

$$= \frac{1-x-2+2x^2+1+x}{2(1-x^2)}$$

$$= \frac{x^2}{1-x^2}$$

4. Solve the inequality; $\frac{x^2+4x+5}{x+3} \le 1$ ***(5 marks)***

$$\frac{x^2+4x+5}{x+3} \le 1$$

$$\frac{x^2+4x+5}{x+3} - 1 \le 0$$

$$\frac{x^2+3x+2}{x+3} \le 0$$

$$\frac{(x+1)(x+2)}{x+3} \le 0$$

	X < -3	-3 < x < -2	-2 < x < -1	X > -1
X+1	_	_	_	+
X+2	_	_	+	+
X+3	_	+	+	+
$\frac{(x+1)(x+2)}{x+3}$	_	+	_	+

Solution is $x \le -3$ and $-2 \le x \le -1$

5. Find the values(s) of x such that $\log_3 x + \log_9 x + \log_{81} x = \frac{21}{4}$ ***(5 marks)***

$$\log_3 + \log_9 x + \log_{81} x = \frac{21}{4}$$
$$\log_3 x + \frac{\log_3 x}{\log_3 9} + \frac{\log_3 x}{\log_3 81} = \frac{21}{4}$$
$$\log_3 x + \frac{1}{2}\log_3 x + \frac{1}{4}\log_3 x = \frac{21}{4}$$
$$\log_3 x = \frac{21 \times 4}{4 \times 7}$$
$$\log_3 x = 3$$
$$x = 27$$

6. Prove by induction ***(5 marks)***

$$1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + ... + n(2^{n-1}) = 1 + (n-1)2^n$$

$for \quad n = 1$

$L.H.S = 1 \qquad R.H.S = 1, \qquad it \ \ holds$

$for \quad n = 2$

$L.H.S = 1 + 2 = 5 \qquad R.H.S = 1 + (2-1)2^2 = 5, \qquad it \ \ is \ \ true$

Suppose the statement holds for $n = k$, then,

$1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + ... + k(2^{k-1}) = 1 + (k-1)2^k$

$for \ \ n = k + 1$

$1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + ... + k(2^{k-1}) + (k+1)(2^k) = 1 + (k-1)2^k + (k+1)2^k$

$= 1 + [(k-1) + k + 1]2^k$

$= 1 + 2k \cdot 2^k$

$= 1 + [(k+1) - 1]2^{k+1} \qquad it \ \ holds \ \ for \ \ n = k + 1$

Since the statement is true for n=1, 2 , … k and d k+1, it is true for all positive integers.

7. Find **(5 marks)**

$$\int x\tan^{-1}3x dx$$

$$let \qquad u=\tan^{-1}3x$$

$$\tan u = 3x$$

$$\sec^2 u\frac{du}{dx}=3$$

$$\frac{du}{dx}=\frac{3}{1+9x^2}$$

$$\frac{dv}{dx}=x$$

$$v=\frac{x^2}{2}$$

$$\int x\tan^{-1}3x dx=\frac{x^2}{2}\tan^{-1}3x-\frac{1}{6}\int\frac{9x^2}{1+9x^2}dx$$

$$\begin{array}{r|l} & \quad\quad\quad 1 \\ \hline 1+9x^2 & 9x^2 \\ & -(1+9x^2) \\ \hline & \quad\quad -1 \end{array}$$

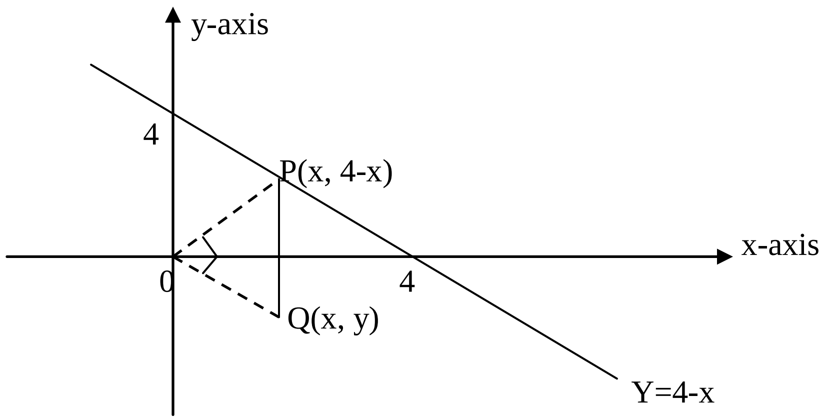
$$\int\frac{9x^2}{1+9x^2}dx=\int dx-\int\frac{1}{1+9x^2}dx$$

$$=x-\frac{1}{3}\tan^{-1}3x+d$$

$$\therefore\int x\tan^{-1}3x dx=\frac{x^2}{2}\tan^{-1}3x-\frac{1}{6}x+\frac{1}{18}\tan^{-1}3x+c$$

8. P is a point on the line $x + y = 4$, and Q is a point such that PQ is parallel to the $y - axis$, and angle POQ is 90^0, where O is the origin (0, 0). Show that the $y -$coordinate of Q, as P varies, is $\frac{x^2}{x-4}$.

(5 marks)



$$\overline{OP}^2 + \overline{OQ}^2 = \overline{PQ}^2$$

$$x^2 + (4-x)^2 + x^2 + y^2 = (x-x)^2 + (y-(4-x))^2$$

$$x^2 + (4-x)^2 + x^2 + y^2 = y^2 - 2y(4-x) + (4-x)^2$$

$$2x^2 = 2y(x-4)$$

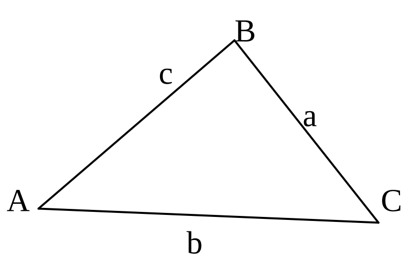
$$y = \frac{x^2}{x-4}$$

SECTION B (60 MARKS)

9. Prove that in any triangle ABC,

$\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\left(\frac{C}{2}\right)$. Hence solve a triangle in which $a = 9,\ b = 5.5\ and\ C = 57^0$

(12 marks)



$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$$

$$a=\frac{b\sin A}{\sin B}$$

$$\frac{a-b}{a+b}\cot\frac{c}{2}=\left(\frac{\frac{b\sin A}{\sin B}-b}{\frac{b\sin A}{\sin B}+b}\right)\cdot\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}$$

$A+B+C=180 \qquad \sin\tfrac{C}{2}=\cos\tfrac{A+B}{2}, \quad \cos\tfrac{C}{2}=\sin\tfrac{A+B}{2}$

$$=\frac{(\sin A-\sin B)\sin\frac{A+B}{2}}{(\sin A+\sin B)\cos\frac{A+B}{2}}$$

$$=\frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}\sin\frac{A+B}{2}}{a\sin\frac{A+B}{2}\cos\frac{A-B}{2}\cos\frac{A+B}{2}}$$

$$=\frac{\sin\frac{A-B}{2}}{\cos\frac{A-B}{2}}$$

$$=\tan\tfrac{A-B}{2}$$

$a=9, \quad b=5.5 \quad and \quad C=57^0$

$c^2=\sqrt{a^2+b^2-2ab\cos C}$

$c=\sqrt{9^2+5.5^2-2\times 9\times 5.5\cos 57^0}$

$c=7.6$

Since $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{c}{2}$

$$\tan\tfrac{A-B}{2}=\frac{9-5.5}{9+5.5}\cot\tfrac{c}{2}$$

$$\frac{A-B}{2}=\tan^{-1}0.4457$$

$A-B=47.9^0 --------(i)$

$A+B+C=180^0$

$A+B=180^0-57^0$

$A+B=123^0 --------(ii)$

$(i)+(ii)$

$2A=170.9^0$

$A=85.5^0$

$B=180^0-C-A$

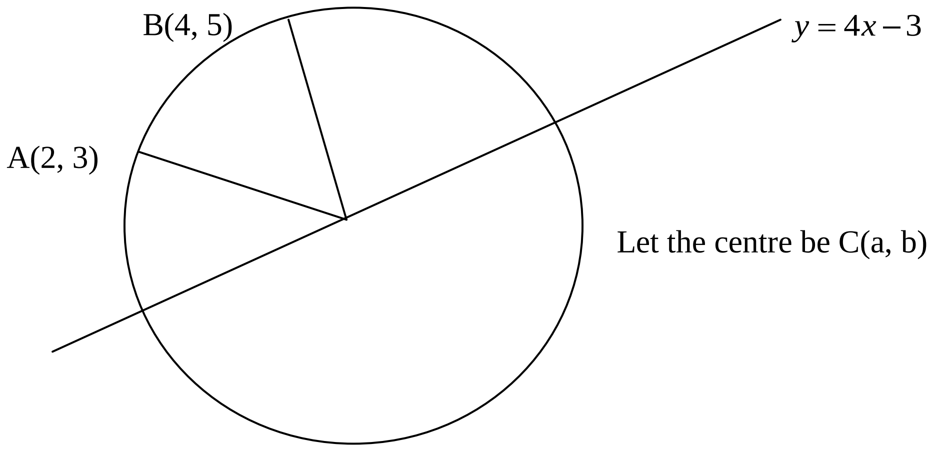
$B=180^0-57^0-85.5^0$

$B=37.5$

10. a) Find the equation of a circle passing through (2, 3) and (4, 5), that has its centre on the line $y-4x+3=0.$ State its centre and radius.

b) Determine the equation of the normal to the circle, in (a) above, at (2, 3)

(12 marks)



Let the centre be C(a, b)

$\overline{AC} = \overline{BC}$

$\sqrt{(2-a)^2 + (3-b)^2} = \sqrt{(4-a)^2 + (5-b)^2}$

$4 - 4a + a^2 + 9 - 6b + b^2 = 16 - 8a + a^2 + 25 - 10b + b^2$

$2a + 2b + 14$

$a + b = 7 ------------(i)$

$C(a,\ b)$ *lies on* $y = 4x - 3$

$\Rightarrow b = 4a - 3$

from (i)

$a + (4a - 3) = 7$

$a = 2$

$b = 4 \times 2 - 3$

$b = 5$

$\overline{AC} = \sqrt{(2-2)^2 + (5-3)^2}$

$= 2$

equation is $(x-2)^2 + (y-5)^2 = 2$

centre is $C(2,\ 5)$

radius is 2 *units*

(b) Normal at (2, 3)

$(x-2)^2 + (y-5)^2 = 2$

$2(x-2)dx + 2(y-5)dy = 0$

$\frac{dy}{dx} = \frac{y-5}{2-x}$

At $(2,\ 3)$, $\frac{dy}{dx} = \infty$, Gradient of the normal is $=\frac{-1}{\infty}=0$

Gradient = $\frac{y_2-y_1}{x_2-x_1}$

$$0=\frac{y-3}{x-1}$$

$$y=3$$

11. a) Expand $\sqrt{\left(\frac{1+x}{1-x}\right)}$ up to the term in x^2.

b) In a geometrical progression (G.P), the sum of the sum of the second and third terms is 12. The sixth term is nine times the fourth term. Find the first three terms of the G.P if it has only positive terms.

c) Jane deposited shs. 10,000 in a bank at the beginning of every year for eight years. How much did she receive at the end of that period if she was paid a compound interest of 14% per annum?

(12 marks)

$$\sqrt{\left(\frac{1+x}{1-x}\right)}=(1+x)^{\frac{1}{2}}(1-x)^{\frac{-1}{2}}$$

From the binomial theorem,

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}+\ldots$$

$$(1+x)^{\frac{1}{2}}=1+\tfrac{1}{2}x+\frac{\frac{1}{2}\cdot\frac{-1}{2}\cdot x^2}{2}+\ldots$$

$$=1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\ldots$$

$$(1-x)^{-\frac{1}{2}}=1+\left(-\tfrac{1}{2}\right)(-x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-x)^2}{2!}+\ldots$$

$$=1+\tfrac{1}{2}x+\tfrac{3}{8}x^2+\ldots$$

$$\sqrt{\left(\frac{1+x}{1-x}\right)}=\left(1+\tfrac{1}{2}x-\tfrac{3}{8}x^2+\ldots\right)\left(1+\tfrac{1}{2}x+\tfrac{3}{8}x^2+\ldots\right)$$

$$=1+\tfrac{1}{2}x+\tfrac{3}{8}x^2+\tfrac{1}{2}x+\tfrac{1}{4}x^2-\tfrac{1}{4}x^2+\ldots$$

$$=1+x+\tfrac{1}{2}x^2$$

12. a) If $y = e^{\tan^{-1} x}$, show that $(1+x^2)\dfrac{d^2y}{dx^2} + (2x-1)\dfrac{dy}{dx} = 0.$

b) Using the method of calculus, find $\sqrt[3]{8.02}$ [correct to 3d.p]

c) Determine the area enclosed by the curve $y = 4x - x^2$ and the line $y = 0$

(12 marks)

$$y = e^{\tan^{-1} x}$$

$$let \quad u = \tan^{-1} x$$

$$\tan u = x$$

$$\sec^2 u du = dx$$

$$(1+\tan^2 u)du = dx$$

$$\frac{du}{dx} = \frac{1}{1+x^2}$$

$$y = e^u$$

$$\frac{dy}{du} = e^{\tan^{-1} x}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{e^{\tan^{-1} x}}{1+x^2}$$

$$\frac{d^2y}{dx^2} = \frac{(1+x^2)\cdot \frac{e^{\tan^{-1} x}}{(1+x^2)} - e^{\tan^{-1} x}(2x)}{(1+x^2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{e^{\tan^{-1} x}(1-2x)}{(1+x^2)^2}$$

$$(1+x^2)\frac{d^2y}{dx^2} + \frac{e^{\tan^{-1} x}(2x-1)}{1+x^2} = 0$$

$$(1+x^2)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} = 0$$

13. a) show that $\int_0^1 \frac{x^2+6}{(x^2+4)(x^2+9)}dx = \frac{\pi}{20}$

b) Differentiate $\sin^2 x$ with respect to x from first principles.

(12 marks)

$$\int_0^1 \frac{x^2+6}{(x^2+4)(x^2+9)}dx \qquad let \quad \frac{x^2+6}{(x^2+4)(x^2+9)} \equiv \frac{A}{x^2+4} + \frac{B}{x^2+9}$$

$$x^2+6 \equiv A(x^2+9)+B(x^2+4)$$

$$let \quad x=0$$

$$9A+4B=6 ----------------(i)$$

$$let \quad x=1$$

$$10A+5B=7 ---------------(ii)$$

$$(i)\times 10-(ii)\times 9$$

$$-5B=-3$$

$$B=\frac{3}{5}$$

$$from \quad (ii)$$

$$10A+5\times\frac{3}{5}=7$$

$$A=\frac{2}{5}$$

$$\int_0^1 \frac{x^2+6}{(x^2+4)(x^2+9)}dx = \frac{2}{5}\int_0^1 \frac{1}{x^2+4}dx + \frac{3}{5}\int_0^1 \frac{1}{x^2+9}dx$$

$$=\left[\frac{2}{5}\cdot\frac{1}{2}\tan^{-1}\frac{x}{2}+\frac{3}{5}\cdot\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^1$$

$$=\frac{1}{5}\left(\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}-0\right)$$

$$=\frac{\pi}{5}\left(0.1476+0.1024\right)$$

$$=\frac{\pi}{20}$$

14. Given that $y=\frac{x^2+3}{x-1}$,

a) Show that for real values of x, y cannot lie between -2 and 6

b) Determine the turning points.

c) State the asymptotes of the curve

d) Sketch the curve. ***(12 marks)***

$$y=\frac{x^2+3}{x-1}$$

$$yx-y=x^2+3$$

$$x^2-yx+y+3=0$$

$$for \quad imaginary \quad roots, \quad b^2-4ac<0$$

$$y^2-4(y+3)<0$$

$$y^2-4y-12<0$$

$$(y-6)(y+2)<0$$

	Y<-2	-2<y<6	y>6
y-6	–	–	+
Y+2	–	+	+
(y-6)(y+2)	+	–	+

$\therefore \quad -2<y<6$ Satisfies the condition hence for real values of x, y cannot lie between -2 and 6.

b)

$$y = \frac{x^2+3}{x-1}$$

$$\frac{dy}{dx} = \frac{(x-1)2x-(x^2+3)}{(x-1)^2}$$

$$At \quad turning \quad po\text{int}\,s, \quad \frac{dy}{dx}=0$$

$$(x-1)2x-(x^2+3)=0$$

$$x^2-2x-3=0$$

$$(x-3)(x+1)=0$$

$x=3$ $\qquad$ $x=-1$

$y=6$ $\qquad$ $y=-2$

$$\begin{array}{cccc} & L & 3 & R \\ \sin g \quad of \quad \dfrac{dy}{dx} & - & 0 & + \\ & \backslash & \_ & / \end{array} \qquad \begin{array}{cccc} & L & -1 & R \\ \sin g \quad of \quad \dfrac{dy}{dx} & + & 0 & - \\ & / & \text{—} & \backslash \end{array}$$

$(3,\ 6)$ *is a* min*imum* *po*int *and* $(-1,\ -2)$ *is a* max*imum* *po*int

c)

$$y = \frac{x^2+3}{x-1}$$

$$\begin{array}{r l} & \phantom{x^2+{}}x+1 \\ x-1 \,\big) & x^2+3 \\ - & x^2-x \\ \hline & x+3 \\ - & x-1 \\ \hline & 4 \end{array}$$

$$\therefore \quad y=(x+1)+\frac{4}{x-1}$$

As $x \to \pm\infty$, $\quad y \to x+1$ $\quad$ hence $y = x+1$ $\quad$ is an oblique asymptote.

As $x \to 1$, $\quad y \to \pm\infty$ $\quad$ hence $x=1$ $\quad$ is a vertical asymptote.

d) Intercepts.

$for \quad x = 0$

$y = -3 \qquad (0,\ -3)$

$for \quad y = 0$

$x^2 + 3 = 0$

$\Rightarrow the\ curve\ does\ not\ cut\ the\ x-axis$

	X < -1	X > -1
X^2+3	+	+
X-1	–	+
$y = \frac{x^2+3}{x-1}$	–	+

15. a) Obtain the Cartesian equation of the locus given by $\left|\frac{z-3}{z-i}\right| = 1$, where z is the complex number $x + iy$.

b) i) If z =1 is a root of the equation $z^3 - 5z^2 + 9z - 5 = 0$, find the other roots.

ii) Find the modulus and argument of $\frac{\left[\left(\sqrt{3}\right)(\cos\theta + i\sin\theta)\right]^4}{\cos 2\theta - i\sin 2\theta}$

(12 marks)

$$\left|\frac{z-3}{z-i}\right| = 1 \qquad , z = x + iy$$

$$\frac{|z-3|}{|z-i|} = 1$$

$$|z-3| = |z-i|$$

$$|x+iy-3| = |x+iy-i|$$

$$(x-3)^2 + y^2 = x^2 + (y-1)^2$$

$$x^2 - 6x + 9 + y^2 = x^2 + y^2 - 2y + 1$$

$$y = 3x - 4$$

16. a) Part of the line $x-3y+3=0$ is a chord of the rectangular hyperbola $x^2-y^2=5$. Find the length of the chord.

b) Find the equation of the tangent at the point $P\left(ct,\ \frac{c}{t}\right)$ on the rectangular hyperbola $xy=c^2$ and prove that the equation of the normal at *P* is $y=t^2x+\frac{c}{t}-ct^3$.

A point $N(X,\ Y)$ on the normal is such that $\overline{ON}=\overline{NP}$, where O is the origin. Show that $4t^3X-3ct^4+c=0$.

$x-3y+3=0$

$x=3y-3$

$x^2-y^2=5$

$(3y-3)^2-y^2=5$

$4y^2-9y+4=0$

$(4y-1)(y-2)=0$

$y=\frac{1}{4} \qquad or \qquad y=2$

$x=-\frac{9}{4} \qquad\qquad x=3$

$L=\sqrt{(3+\frac{9}{4})^2+(2-\frac{1}{4})^2}$

$L=5.534\ units$

b)

$$P\left(ct,\ \frac{c}{t}\right)$$

$$xy = c^2$$

$$y + x\frac{dy}{dx} = 0$$

$$\frac{c}{t} + ct\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{1}{t^2}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$-\frac{1}{t^2} = \frac{y - \frac{c}{t}}{x - ct}$$

$$t^2 y + x - 2ct = 0$$

$$m_1 m_2 = -1$$

$$if \quad m_1 = -\frac{1}{t^2}, \quad m_2 = t^2$$

$$t^2 = \frac{y - \frac{c}{t}}{x - ct}$$

$$y - \tfrac{c}{t} = xt^2 - ct^3$$

$$y = t^2 x + \tfrac{c}{t} - ct^3$$

c)

$$N(X,\ Y), \qquad O(0,\ 0)$$

$$\overline{ON} = \overline{NP}, \qquad Y = Xt^2 + \frac{c}{t} - ct^3, \qquad L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\left[(X-0)^2 + (Y-0)^2\right]^{\frac{1}{2}} = \left[(X-ct)^2 + (Y-\tfrac{c}{t})^2\right]^{\frac{1}{2}}$$

$$X^2 + Y^2 = X^2 - 2Xct + c^2t^2 + Y^2 - 2Y\frac{c}{t} - \frac{c^2}{t^2}$$

$$2Xt^3 + 2Yt - c - ct^4 = 0$$

$$2Xt^3 + 2t(Xt^2 + \frac{c}{t} - ct^3) - c - ct^4 = 0$$

$$2Xt^3 + 2Xt^3 + 2c - 2ct^4 - c - ct^4 = 0$$

$$4t^3 X - 3ct^4 + c = 0$$