P425/1
PURE
MATHEMATICS
Paper 1
July /Aug. 2022
3 hours

UGANDA TEACHERS' EDUCATION CONSULT (UTEC)

Uganda Advanced Certificate of Education

PURE MATHEMATICS

Paper 1

3 hours

INSTRUCTIONS TO CANDIDATES:

*Answer **all** questions in section A and any **five** from section **B**.*

All necessary working must be shown clearly.

Silent non – programmable scientific calculators and mathematical tables may be used.

*Any extra question(s) attempted in section **B** will **not** be marked.*

Turn Over

SECTION A (40 MARKS)

Answer ALL questions in this section

1. Solve the equation: $1 - cos2\theta = sin\theta \; for \; 90^o \leq \theta \leq 180^o$. ***(05 marks)***

2. Given that $\frac{50}{(2 + i)^2} = a + bi,$ find the real numbers a and b.

(05 marks)

3. Show that the stationary points of the curve $y^2 = x^2 + 2xy + 8$ lie on the line $y = -x$. State the $x -$ coordinates of the points. ***(05 marks)***

4. The acute angle between the lines $2y - x - 3 = 0$ and $y = px + 3$ is 45^o. Find the possible values of **P**. ***(05 marks)***

5. Calculate the distance of the origin O $(0, 0, 0)$ from each of the planes $3x - 4y + 12Z + 13 = 0$ and $3x - 4y + 12Z - 39 = 0$; hence deduce the distance between the planes. ***(05 marks)***

6. Evaluate $\int_0^{1/2} \frac{4x}{4 - x^2} \, dx$ ***(05 marks)***

7. The first, second and fourth terms of an arithmetic progression form a geometrical progression. Find the common ratio of the G.P. ***(05 marks)***

8. The volume **V** of a cone varies such that the height , h, of the cylinder is twice its base radius.

   (a) Show that $V = \frac{\pi}{12} h^3$

   (b) Find the rate at which **V** changes with height, at the instant when $h = 4cm$.

(05 marks)

SECTION B (60 MARKS)

9. A curve is represented by the parametric equations ;
$x = t^2 - 2t - 3$
$y = t^2 + 2t - 3$ where t is the parameter.

Find: (a) $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *in terms of* t, hence find and determine the nature of the stationary points of the curve. ***(08 marks)***

(b) the equation of the tangent to the curve at the point where the curve cuts the positive $y-axis$. ***(04 marks)***

10. A circle whose centre is $C\,(1, 6)$ touches the line $y = {}^{3}/_{4}\,x - 1$ at point **A**.

Find the; (a) equation of the circle. ***(06 marks)***

(b) coordinates of point A. ***(06 marks)***

11. (a) Solve the equation: $3sin\theta - 4cos\theta = 4 \;\; for \;\; 0^o \leq \theta \leq 360^o$ ***(06 marks)***

(b) Prove that; $\frac{sin3A - sinA}{sin5A + sin3A} = \frac{1}{4}Sec^2A$ ***(06 marks)***

12. (a) Expand $\sqrt{4 - 3x}$ up to the term in x^3, hence find the error made in using $x = 1$ in the expansion. ***(07 marks)***

(b) Evaluate $\sqrt{61}$ to 4 decimal places. ***(05 marks)***

13. The tangent to the curve $y = x^3$ at the point $A\,(-1, -1)$ meets the curve again at point **B**.

(a) Find the;

(i) equation of the tangent at **A**

(ii) coordinates of point **B**. ***(07 marks)***

(b) Calculate the area bounded by the line **AB** and the curve. ***(05 marks)***

14. The line $r = \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ meets the plane $3x + 2y + Z = 29$ in point A. Find the:

(a) coordinates of point A *(07 marks)*

(b) acute angle between the line and the plane *(05 marks)*

15. (a) Given $Z_1 = -1 - i\sqrt{3}$, $Z_2 = -1 + i$, $Z_2 = 4i$; find the principal argument of $\frac{Z_1^3 Z_2^2}{Z_3}$. *(06 marks)*

(b) Find, in Cartesian form, the cube roots of -8. *(06 marks)*

16. (a) Solve: $Sin\, x \frac{dy}{dx} + y \cos x = \tan 3x$ *(05 marks)*

(b) The price P of a litre of petrol increases at a rate which is directly proportional to the price. If the price doubles every 10 days; find the percentage increase in the price after 20 days. *(07 marks)*

END

P425/1 PURE MATHS MARKING GUIDE. UTEC | Comments

N		Comments
1	$1-\cos 2\theta = \sin\theta$ $\Leftrightarrow 2\sin^2\theta - \sin\theta = 0$ M1 $\Leftrightarrow \sin\theta(2\sin\theta - 1) = 0$ M1 $\Rightarrow \sin\theta = 0$ or $\sin\theta = \frac{1}{2}$ $\Rightarrow \theta = 180°$ A1 $\Rightarrow \theta = 30°, 150°$ A1 A1	
2	$(2+i)^2 = 4+4i+i^2 = 3+4i$ B1 $\Rightarrow \frac{50}{(2+i)^2} \Leftrightarrow \frac{50(3-4i)}{(3+4i)(3-4i)}$ M1 $\Leftrightarrow 2(3-4i) = 6+(-8)i$ A1 $\Rightarrow a = 6$ A1, $b = -8$ A1	
3	$y^2 = x^2+2xy+8 \Rightarrow 2y\frac{dy}{dx} = 2x+2\left(y+x\frac{dy}{dx}\right) = 0$ M1 i.e. $(y-x)\frac{dy}{dx} = x+y \therefore \frac{dy}{dx} = \frac{x+y}{y-x}$ B1 Now: $\frac{dy}{dx} = 0 \Rightarrow x+y=0$ or $y=-x$. A1 using $y=-x$ in $y^2=x^2+2xy+8$ $\Rightarrow x^2 = x^2+8-2x^2$ $\Leftrightarrow x^2 = 4 \therefore x = 2$ A1 or $x = -2$. A1	

SOLUTIONS

$2y - x - 3 = 0 \iff y = \frac{1}{2}x + \frac{3}{2} \Rightarrow$ grad. $m_1 = \frac{1}{2}$ B_1

$y = px + 3 \Rightarrow$ grad. $m_2 = p$

Using $\tan\theta = \pm\left(\frac{m_2 - m_1}{1 + m_1 m_2}\right)$

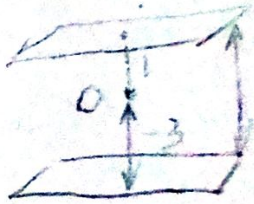
$\Rightarrow \tan 45^\circ = \pm\left(\frac{p - \frac{1}{2}}{1 + \frac{1}{2}p}\right)$ M_1

$\frac{1}{1} = \frac{\pm(2p-1)}{p+2} \iff \pm(2p-1) = p+2$ B_1

$\Rightarrow p = 3$ A_1 or $p = -\frac{1}{3}$ A_1

Distance from 1st plane: $d_1 = \frac{3(0) - 4(0) + 12(0) + 13}{\sqrt{3^2 + (-4)^2 + 12^2}}$

$= 1$ unit



Distance from 2nd plane: $d_2 = \frac{3(0) - 4(0) + 12(0) - 39}{\sqrt{3^2 + (-4)^2 + 12^2}}$

$= -3$ units.

required distance $= 1 + |-3|$

$= 4$ units.

$\int_0^{1/2} \frac{4x}{4 - x^2}\,dx \iff -2\int_0^{1/2} \frac{-2x}{4 - x^2}\,dx$ M_1

$= -2\left[\ln(4 - x^2)\right]_0^{1/2}$ M_1 B_1

$= -2\left(\ln\frac{15}{4} - \ln 4\right)$ B_1

$= 2\ln\left(\frac{16}{15}\right) \approx 0.12908$ (A

Solutions

SECTION B

(a) $x = t^2 - 2t - 3 \Rightarrow \frac{dx}{dt} = 2t - 2$ (M1)

$y = t^2 + 2t - 3 \Rightarrow \frac{dy}{dt} = 2t + 2$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t+1}{t-1}$ (M1) (A1)

$\frac{d^2y}{dx^2} = \frac{(t-1).1 - 1(t+1)}{(t-1)^2} \cdot \frac{1}{2(t-1)}$ (M1)

$= \frac{-1}{(t-1)^3}$ (A1)

$\frac{dy}{dx} = 0 \Rightarrow t = -1 \Rightarrow x = 0, y = -4$

The stationary point is $(0, -4)$ (A1)

At $t = -1$, $\frac{d^2y}{dx^2} = \frac{1}{8} > 0 \Rightarrow (0,4)$ is a minimum turning point. (M1) (B1)

(b) At the y-axis, $x = 0 \Rightarrow t^2 - 2t - 3 = 0$

$(t-3)(t+1) = 0 \Rightarrow t = 3, -1$ (M1)

When $t = -1$, y is negative; but $t = 3$, $y = 12$ (positive) (B1)

At $t = 3$, $\frac{dy}{dx} = 2 \Rightarrow$ equation of tangent at $(0, 12)$ (M1)

is $y = 2x + 12$. (using $y = mx + c$)

(A1)

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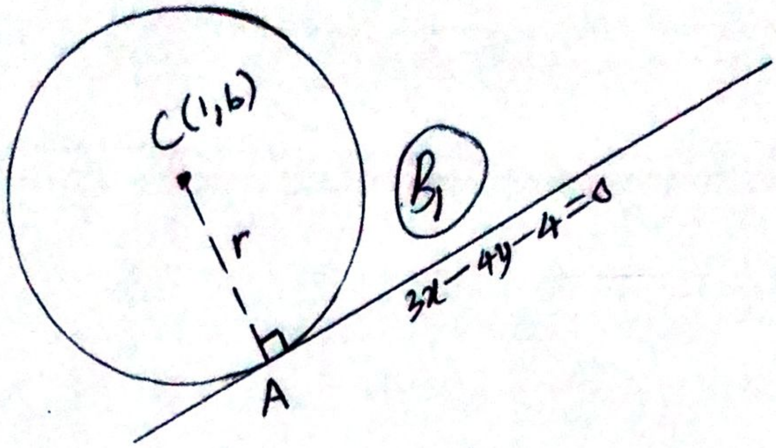
SOLNS. | COMMENTS



$y=\frac{3}{4}x-1$

$\Leftrightarrow 4y=3x-4$

$\Leftrightarrow 3x-4y-4=0$ (M1)

(a) radius, $r=\frac{|3(1)-4(6)-4|}{\sqrt{3^2+(-4)^2}}$ M1B1

$= 5$ units. (A1)

Equation of the circle: $(x-1)^2+(y-6)^2=25$ (A1)

(b) Gradient of tangent $=\frac{3}{4} \Rightarrow$ grad. of normal AC $=-\frac{4}{3}$ (M1)

Eqn of AC: $\frac{y-6}{x-1}=-\frac{4}{3} \Leftrightarrow 4x+3y=22$ (M1)

At point A; solve: $4x+3y=22$ ---(1)

with: $3x-4y=4$ ---(2) simultaneously.

$4\times$eqn(1): $16x+12y=88$ (M1)

$3\times$eqn(2): $9x-12y=12$

$\Rightarrow 25x=100 \therefore x=4$ (B1)

$\Rightarrow y=2$ (B1) $\Rightarrow A(4,2)$ (A1)

SOLUTIONS

1. (a) Using $t = \tan\frac{1}{2}\theta \Rightarrow \sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$

$\Rightarrow 3\left(\frac{2t}{1+t^2}\right) - \frac{4(1-t^2)}{1+t^2} = 4$ (M1)

$\Leftrightarrow 6t - 4 + 4t^2 = 4 + 4t^2$

$6t = 8 \therefore t = \frac{4}{3}$ (B1) or $t = \infty$ (B1) (since the t^2-term disappears)

i.e, $\tan\frac{1}{2}\theta = 4/3 \Rightarrow \frac{1}{2}\theta = 53.13° \therefore \theta = 106.26°$ (A1)

$\tan\frac{1}{2}\theta = \infty \Rightarrow \frac{1}{2}\theta = 90°$ (M1) $\therefore \theta = 180°$ (A1)

Alternatively

Use the R-formula ; and follow thru'

Let $3\sin\theta - 4\cos\theta = R\sin(\theta - \alpha)$

$= R\sin\alpha\cos\alpha - R\sin\alpha\cos\alpha$

$R\cos\alpha = 3,\ R\sin\alpha = 4$ (A1)

$\alpha = \tan^{-1}(4/3) = 53.13°$ (A1)

$R = 5$ B1

$5\sin(\theta - 53.13°) = 4$

$\theta - 53.13° = 53.13°,\ 126.87°$ (M1)

$\theta = \{106.26°, 180°\}$

~~A1~~ (A1)

(b) $L.H.S = \dfrac{2\cos\frac{3A+A}{2}\sin\frac{3A-A}{2}}{2\sin\frac{5A+3A}{2}\cos\frac{5A-3A}{2}}$ (M1) (M1)

$= \dfrac{\cos 2A \sin A}{\sin 4A \cos A}$; $\sin 2A = 2\sin A\cos A$

$\sin 4A = 2\sin 2A\cos 2A$

$= \dfrac{\cos 2A \sin A}{2\sin 2A \cos 2A \cos A}$ (M1)

$= \dfrac{(\cos 2A)\sin A}{4(\cos 2A)\sin A\cos^2 A}$ (M1)

$= \dfrac{1}{4\cos^2 A} = \frac{1}{4}\left(\frac{1}{\cos^2 A}\right)$ (B1)

$= \frac{1}{4}\sec^2 A.$ (A1)

$= R.H.S.$

SOLUTIONS

Comments.

$(4-3x)^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(1-\frac{3x}{4}\right)^{\frac{1}{2}}$ B1 M1

$= 2\left\{1 + \frac{1}{2}\left(\frac{-3x}{4}\right)^{1} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{-3x}{4}\right)^{2} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(\frac{-3x}{4}\right)^{3} + \dots\right\}$ M1 B1

$2\left(1 - 3/8 - 9/256 - 27/1024\right)$

$= 2 - \frac{3}{4}x - \frac{9}{64}x^2 - \frac{27}{512}x^3 + \dots$ A1

$2 - \frac{3}{4}x - \frac{9}{128}$

When $x=1$, Exact Value $= \sqrt{4-3\times1}$

$= 1$ B1

Approximate value $\simeq 2 - 3/4 - 9/64 - \frac{27}{512}$

$\simeq 1.056641$ B1

Hence the error $= 1.056641 - 1$

$\simeq 0.0566$ (4 dps) A1

(b) $\sqrt{61} = \sqrt{64-3}$

$= \sqrt{64(1-3/64)}$

$= 8\left[1 - \frac{3}{4}\left(\frac{1}{16}\right)\right]^{1/2}$ M1 B set $x = \frac{1}{16}$ in the expansion

$= 8\left\{1 - \frac{3}{8}\left(\frac{1}{16}\right) - \frac{9}{32}\left(\frac{1}{16}\right)^2 - \frac{27}{256}\left(\frac{1}{16}\right)^3 + \dots\right\}$ B1

$\simeq 2(1 - 0.0234375 - 0.00109863 - 0.000025749)$ B1

$\simeq 8 \times 0.975438118$

$\simeq 7.803505$ A1

$\simeq 7.8035$ (4 dps) B1

$x = 64$

$y = \sqrt{x}$, $y = 8$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ B1

$\frac{dy}{dx} = \frac{1}{2\sqrt{64}} = \frac{1}{16}$

$\delta y = \frac{dy}{dx}\delta x$ M1

$\delta x = -3$

$61 - 64 = -3$

$\delta y = -\frac{3}{16}$ M1

$y + \delta y = 8 - \frac{3}{16}$

$= \frac{125}{16}$ B1

Solutions

(a) Using $x = 5+2\lambda$, $y = -2-\lambda$, $z = 4+3\lambda$ in (B1)

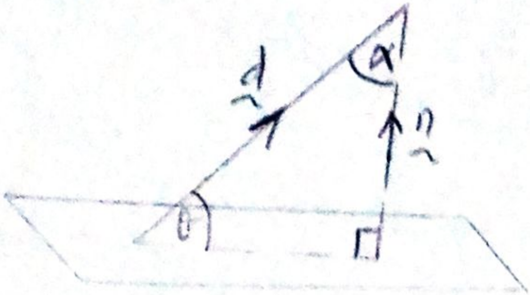
$3x+2y+z=29$ we've: $15+6\lambda-4-\cancel{2}\lambda+4+3\lambda=29$ (M1 B1)

$\Rightarrow 7\lambda = 14 \therefore \lambda = 2$ (B1)

$\Rightarrow x = 5+4$, $y=-2-2$; $z=4+6$ (B1)

$\Rightarrow A(9,-4,10)$ (A1)

(b)



$\underset{\sim}{d} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ (B1)

$\underset{\sim}{n} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

$\underset{\sim}{d} \cdot \underset{\sim}{n} = |\underset{\sim}{d}||\underset{\sim}{n}| \cos\alpha$; but $\alpha = 90^\circ - \theta$

$\Rightarrow \cos\alpha = \cos(90^\circ-\theta)$

$= \sin\theta$; where θ is the required angle.

$\Leftrightarrow \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \sqrt{4+1+9} \times \sqrt{9+4+1} \sin\theta$ (M1 B1)

$6-2+3 = \sqrt{14} \times \sqrt{14} \sin\theta$ (B1)

$7 = 14 \sin\theta$ (B1)

$\sin\theta = \frac{1}{2} \therefore \theta = \sin^{-1}\frac{1}{2}$

$= 30^\circ$. (A1)

SOLUTIONS

COMMENTS

(i) $y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2$ (M1)

At $A(-1,-1)$; grad. $= 3$ (B1); eqn of tangent: $\frac{y-(-1)}{x-(-1)} = 3$ (M1)

or $y = 3x+2$. (A1)

(ii) At point B, $x^3 = 3x+2 \Leftrightarrow x^3 - 3x - 2 = 0$

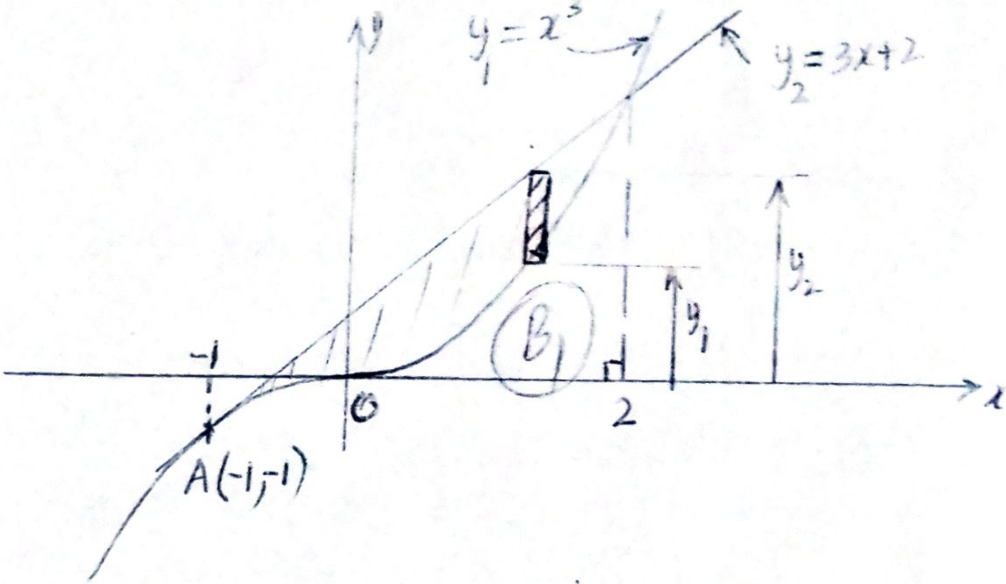
Or $x = -1$ $\Rightarrow x+1$ is a factor, perform long division to find the other factor

$x=-1$ is a repeated root $\Rightarrow (-1)+(-1)+x = 0$ (M1)

$\Rightarrow x = 2 \Rightarrow y = 8$ (B1)

$\therefore B(2,8)$. (A1)

(b)



$\Delta A = (y_2 - y_1)\Delta x$

$\Rightarrow$ Area $= \int_{-1}^{2} (y_2 - y_1)\,dx$ (M1)

$= \int_{-1}^{2} (3x+2-x^3)\,dx$

$= \left[\frac{3}{2}x^2 + 2x - \frac{x^4}{4}\right]_{-1}^{2}$ (M1)

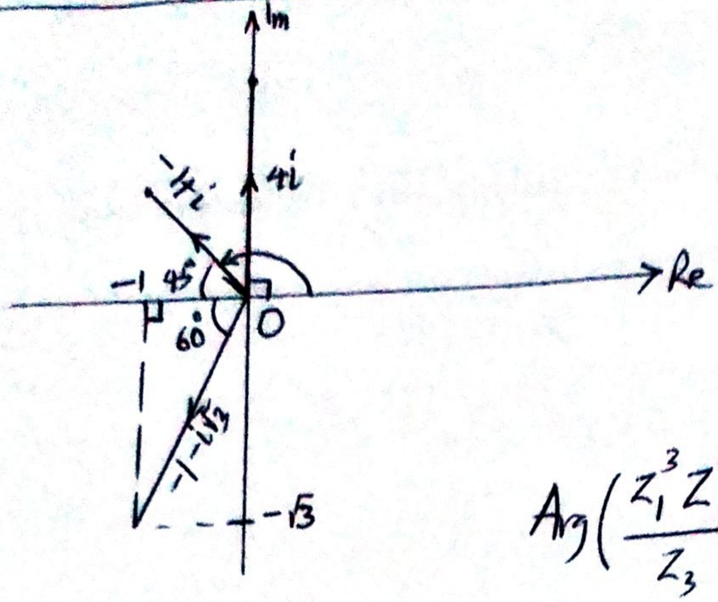
$= (6+4-4) - \left(\frac{3}{2} - 2 - \frac{1}{4}\right)$ (B1)

$= 27/4$ sq. units

$= 6\frac{3}{4}$ sq. units (A1)

SOLUTIONS

(a)



$Arg(4i) = 90°$ (B_1)

$Arg(-1+i) = 135°$ (B_1)

$Arg(-1-i\sqrt{3}) = -120°$ (B_1)

$$Arg\left(\frac{Z_1^3 Z_2^2}{Z_3}\right) = 3Arg(Z_1) + 2Arg(Z_2) - Arg(Z_3)$$

$= 3\times -120° + 2\times 135° - 90°$ (M_1)

$= -360° + 270° - 90°$

$= 0° - 90° - 90°$ (B_1)

$= -180°$ (not principal).

$\Rightarrow$ principal Argument is $180°$ (A_1)

(b) Let $Z = \sqrt[3]{-8} \Leftrightarrow Z^3 + 8 = 0$ (B_1)

$\Leftrightarrow (Z+2)(Z^2 - 2Z + 4) = 0$ (M_1)

$\Rightarrow Z + 2 = 0 \therefore Z = -2$ (A_1) and $Z^2 - 2Z + 4 = 0$

$$Z = \frac{2 \pm \sqrt{4-16}}{2} \quad (M_1)$$

$\Rightarrow Z = 1 + i\sqrt{3}$ (A_1)

and $Z = 1 - i\sqrt{3}$. (A_1)

SOLUTIONS | Comments

$\sin x \frac{dy}{dx} + y\cos x = \tan 3x$

$\Rightarrow \frac{d}{dx}(y\sin x) = \tan 3x$ (M1 B1)

$\Rightarrow y\sin x = \int \tan 3x \, dx$

$= \frac{1}{3}\int \frac{\sec 3x \tan 3x}{\sec 3x} dx$ (M1 B1)

$\therefore y\sin x = \frac{1}{3}\ln(\sec 3x) + c.$ (A1)

(b) $\frac{dP}{dt} \propto P \Rightarrow \frac{dP}{dt} = kP$ (M1)

$\Rightarrow \int \frac{1}{P} dP = \int k \, dt$

$\ln P = kt + c$ (B1)

Let $P = P_0$ at $t = 0$, $c = \ln P_0 \Rightarrow \ln P = kt + \ln P_0$

set $P = 2P_0$ at $t = 10$, $\Rightarrow \ln 2P_0 = 10k + \ln P_0$ (M1)

$\Rightarrow k = \frac{\ln 2}{10}$ (B1)

Set $t = 20 \Rightarrow \ln P = \frac{20\ln 2}{10} + \ln P_0$

$\ln P = \ln 4P_0 \Rightarrow P = 4P_0$ (B1)

%-age increase $= \frac{4P_0 - P_0}{P_0} \times 100$ (M1)

$= 300\%$ (A1)