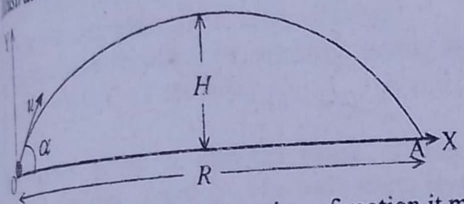
Chapter Two
PROJECTILES

a particle is projected directly upwards, the motion it scribes is purely kinematics. But if it is projected at angle to the horizontal, the motion described is parabolic, and this can be split into two sets of motion

i. Vertical motion under constant acceleration = g ms^{-1}
ii. Horizontal motion with uniform velocity (assuming air resistance is negligible)

Note: projectiles on inclined planes are outside the scope of our coverage

Illustration



The angle α is the initial direction of motion it makes with the horizontal (angle of projection).
u = the initial velocity of projection.
H = Greatest height reached above the point of projection.
R = distance at which it arrives on the horizontal through the point of projection.
The parabolic path described is called the trajectory.

Finding velocities at any time *t*

From $v = u + at$

Horizontally; $v_x = u\cos\alpha$

Vertically; $v_y = u\sin\alpha - gt$

Finding distances at any time *t*

From $s = ut + \frac{1}{2}at^2$

Horizontally, $x = (u\cos\alpha)t$

Vertically; $y = u\sin\alpha\, t - \frac{1}{2}gt^2$

By using vectorial approach;

The acceleration due to gravity acts vertically and therefore in vector form it is given as:

$a = -g\mathbf{j}$ (for upward motion)

If the particle is projected from the origin with velocity u ms^{-1}, at an angle α to the horizontal, then at any time *t*, the velocity acquired is given by

$v = \int(-g\mathbf{j})\,dt$

$v = -gt\mathbf{j} + c$

At time $t = 0$;

$v = u\cos\alpha\mathbf{i} + u\sin\alpha\mathbf{j}$

By substitution,

$c = u\cos\alpha\mathbf{i} + u\sin\alpha\mathbf{j}$

Hence $v = u\cos\alpha\mathbf{i} + (u\sin\alpha - gt)\mathbf{j}$

Where horizontal velocity, $v_x = u\cos\alpha\mathbf{i}$ and

Vertical velocity, $v_y = (u\sin\alpha - gt)\mathbf{j}$

The distance acquired at any time *t* is given by

$$\mathbf{r} = \int (u\cos\alpha\mathbf{i} + (u\sin\alpha - gt)\mathbf{j})\,dt$$

$$= (u\cos\alpha)t + \left(u\sin\alpha\, t - \frac{1}{2}gt^2\right) + c$$

At time $t = 0$, $\mathbf{r} = 0$

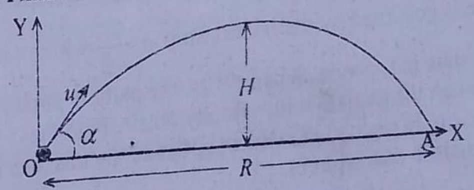
Hence by substitution, $c = 0$

Therefore $\mathbf{r} = (u\cos\alpha)t + \left(u\sin\alpha\, t - \frac{1}{2}gt^2\right)$

Where horizontal distance, $x = (u\cos\alpha)t$ and

vertical distance $y = \left(u\sin\alpha\, t - \frac{1}{2}gt^2\right)$

Time for Greatest Height



At maximum height, $v_y = 0$

From $v_y = u\sin\alpha - gt$

$\Rightarrow 0 = u\sin\alpha - gt$

$t = \frac{u\sin\alpha}{g}$

Maximum height (*H*)

From vertical motion, at any time *t*

$$y = \left(u\sin\alpha\, t - \frac{1}{2}gt^2\right)$$

substituting for $t = \frac{u\sin\alpha}{g}$,

$$H = \left(u\sin\alpha\left(\frac{u\sin\alpha}{g}\right) - \frac{1}{2}g\left(\frac{u\sin\alpha}{g}\right)^2\right)$$

$$H = \frac{u^2\sin^2\alpha}{g} - \frac{u^2\sin^2\alpha}{2g} = \frac{u^2\sin^2\alpha}{2g}$$

Alternatively;

From $v^2 = u^2 + 2as$;

$$v_y^{\,2} = (u\sin\alpha)^2 - 2gH$$

At maximum height, $v_y = 0$

$$(u\sin\alpha)^2 - 2gH = 0$$

$$H = \frac{u^2\sin^2\alpha}{2g}$$

Time of Flight (T)

This is the time taken by the particle for the complete motion i.e. time taken by the particle to cover horizontal distance OA

From vertical motion,

$$y = u\sin\alpha t - \frac{1}{2}gt^2$$

At A, $y = 0$ and $t = T$

$$(u\sin\alpha)T - \frac{1}{2}gT^2 = 0$$

$$T\left(u\sin\alpha - \frac{1}{2}gT\right) = 0$$

$$T = 0 \text{ or } u\sin\alpha - \frac{1}{2}gT = 0$$

$$T = \frac{2u\sin\alpha}{g}$$

$T = 0$ corresponds to time at the point of projection, O.

and

$T = \frac{2u\sin\alpha}{g}$ Corresponds to time at A

Hence time of flight $= \frac{2u\sin\alpha}{g}$ and it is twice time taken by the particle to reach maximum height.

Horizontal Range (R)

Here velocity is constant $= u\cos\alpha$

In time T, horizontal distance traveled (R) is

$$R = \text{velocity} \times \text{time}$$

$$= u\cos\alpha \times \frac{2u\sin\alpha}{g}$$

Hence Horizontal Range $= \frac{2u^2\sin\alpha\cos\alpha}{g}$

But $2\sin\alpha\cos\alpha = \sin 2\alpha$

Therefore $R = \frac{u^2\sin 2\alpha}{g}$

Maximum Range on a Horizontal Plane

From $R = \frac{u^2\sin 2\alpha}{g}$

R is maximum when $\sin 2\alpha$ is maximum

i.e. when $\sin 2\alpha = 1$

$$\therefore R_{max} = \frac{u^2 \times 1}{g} = \frac{u^2}{g}$$

This is occurs when $\sin 2\alpha = 1$

$$2\alpha = 90^0$$

$$\alpha = 45^0$$

Equation of Trajectory

This is an equation describing the parabolic motion of the projectile. It is expressed in terms of horizontal distance x and vertical distance y. from horizontal motion at any time t,

$$x = (u\cos\alpha)t$$

$$t = \frac{x}{u\cos\alpha}$$

Substituting for t into vertical motion at any time t;

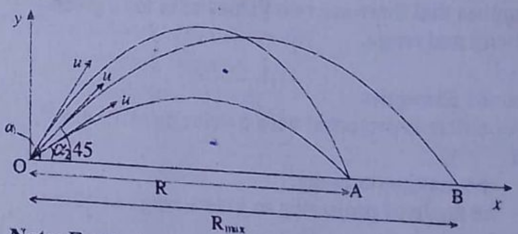
$$y = u\,\overset{\sin\alpha}{\cancel{\cos}}\,\alpha t - \frac{1}{2}gt^2$$

$$y = u\sin\cdot\frac{x}{u\cos\alpha} - \frac{1}{2}g\left(\frac{x}{u\cos\alpha}\right)^2$$

$$= x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha}$$

$$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2u^2}$$

To show that for given velocity of projection and range there are two angles of projection



Note: For a given velocity and range, there are two angles of projection rather than 45^0, which gives maximum range.

From the diagram above, projecting a particle above or below 45^0, produces horizontal range OA but projecting the particle at 45^0 produces maximum range OB

From $R = \frac{u^2\sin\alpha}{g}$, if R and u are fixed, then:

$$\Rightarrow \quad \frac{Rg}{u^2} = \text{constant},$$

$$\frac{Rg}{u^2} = k$$

$$\Rightarrow \quad \sin 2\alpha = k$$

There exists some angle θ such that

$\sin 2\theta = k$

$\sin 2\alpha = \sin 2\theta$

$2\alpha = 2\theta$

$\alpha = \theta$ ....(i)

$\sin 2\theta = \sin(180 - 2\theta)$

$\sin 2\alpha = \sin(\pi - 2\theta)$

$2\alpha = \pi - 2\theta$

$\alpha = \frac{\pi}{2} - \theta$

$\alpha = (90 - \theta)$ ....(ii)

...e there are two angles of projection when velocity ...nge are constant.

...natively

...the equation of trajectory given by

$y = x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha}$, we can rearrange it as

$$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2v^2}$$

$$y = x\tan\alpha - \frac{gx^2}{2v^2}\left(1+\tan^2\alpha\right)$$

$$y = x\tan\alpha - \frac{gx^2}{2v^2} - \frac{gx^2}{2v^2}\times\tan^2\alpha$$

$$\frac{gx^2}{2v^2}\tan^2\alpha - x\tan\alpha + \left(y + \frac{gx^2}{2v^2}\right) = 0$$

...the above equation is quadratic in terms of $\tan\alpha$, ...plies that there are two values of α for a given ...ity and range.

...ral Examples

...particle is projected with a velocity of 19.6ms^{-1},

...the maximum range:

...the angle of projection to give a range of 12m.

...tion

$$R_{max} = \frac{u^2}{g}$$

$$R_{max} = \frac{19.6^2}{9.8} = 39.2\text{m}$$

...hen $R = 12\text{m}$; $12 = \frac{u^2\sin 2\alpha}{g} = \frac{(19.6)^2\times\sin 2\alpha}{g}$

$$\sin 2\alpha = \frac{12\times 9.8}{19.6^2}$$

$\sin 2\alpha = 0.306122$

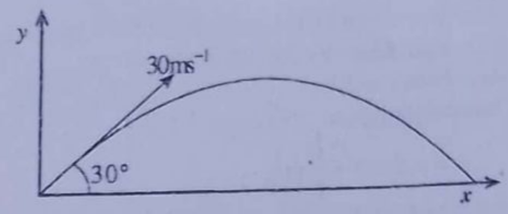
$$\alpha = \frac{1}{2}\sin^{-1}(0.306122) \quad = 8.55' \quad = 9^0$$

...ther angle is $= 90 - 8^055' = 81^05' = 81^0$

...particle is projected with a velocity of 30ms^{-1} at ...gle of elevation of 30^0. Find:

i. The greatest height reached
ii. The time of flight
iii. The horizontal range
iv. The velocity and direction of motion at a height of 4m.

Solution



(i) Given $u = 30\text{ms}^{-1}$, $\alpha = 30^0$

We know that $H = \frac{u^2\sin^2\theta}{2g}$

$$H = \frac{900\times\sin^2 30}{2g} = \frac{900}{8g} = 11.5\text{m}$$

(ii) $T = \frac{2u\sin\alpha}{g} = \frac{2\times 30\sin 30}{g} = \frac{60}{8g}$

$= 3.06'$ seconds

(iii) $R = \frac{2u^2\sin\alpha}{g} = \frac{900\times\sin 60}{9.8} = 79.53\text{m}$

(iv) Given $y = 4$

From $y = u\sin\alpha t - \frac{1}{2}gt^2$,

$$4 = 30t\sin 30 - \frac{1}{2}gt^2$$

$$4 = 15t - \frac{1}{2}gt^2$$

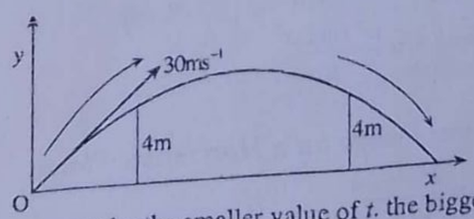
$$8 = 30t - gt^2$$

$$gt^2 - 30t + 8 = 0$$

$$t = \frac{30\pm\sqrt{900-32g}}{2g} = \frac{30\pm 24.216}{19.6}$$

$$t = \frac{30\pm 24.216}{19.6}$$

$t_1 = 2.766$ sec, $t_2 = 0.295$ s.



Here we take the smaller value of t. the bigger value of t is the time taken by particle to be at the height of 4m on its way downwards

So at height of 4m upwards;

$t = 0.295$ s.

So $v_x = u\cos\alpha$

$= 30\cos 30 = 15\sqrt{3}$

$v_y = U\sin\alpha - gt$

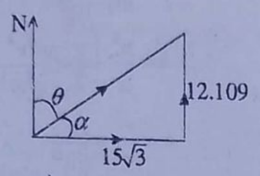
$= 30\sin 30 - 9.8 \times 0.295$

$= 15 - 9.8 \times 0.295 = 12.109$

Speed at that instant

$$= \sqrt{12.109^2 + (15\sqrt{3})^2} = 28.664\ \text{ms}^{-1}$$

Direction



$$\alpha = \tan^{-1}\left(\frac{12.109}{15\sqrt{3}}\right) = 24.99 \text{ to the horizontal}$$

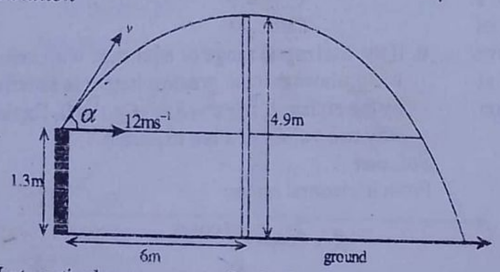
or $\theta = 90 - 24.99$

$\theta = 65^0$

Hence $\theta = \text{N}65^0\text{E}$

3. A ball is thrown with velocity whose horizontal component is 12ms^{-1} from a point 1.3m above the ground and 6m away from a vertical wall 4.9m high in such a way as to clear the wall. At what time will it reach the ground?

Solution



Let vertical component of velocity = v

Time taken to clear wall = time taken to go up height $(4.9 - 1.3)\text{m} = 3.6\text{m}$

or time taken to travel horizontal distance of 6m

$$t = \frac{\text{horizontal distance}}{\text{horizontal component of velocity}} = \frac{6}{12} = 0.5\,\text{s}$$

Height, $h = vt - \frac{1}{2}gt^2$

$$3.6 = v_3 \times \frac{1}{2} - \frac{1}{2} \times g \times x \times \frac{1}{4}$$

$$7.2 = v_3 - \frac{9.8}{4}$$

$$v_3 = 7.2 + \frac{9.8}{4}$$

$$= 9.65\text{ms}^{-1}$$

When it hits the ground;

$h = -1.3$ from point of projection.

$\Rightarrow \quad -1.3 = 9.65t - 4.9t^2$

$$4.9t^2 - 9.65t - 1.3 = 0$$

$$t = \frac{96.5 + \sqrt{96.5^2 + 4 \times 4.9 \times 13}}{9.8} = 2.1 \text{ seconds.}$$

4. A ball is thrown from a point A in a horizontal plane so as just to pass over a wall standing on the same plane, the horizontal component of the ball's velocity being equal to the velocity it would acquire in falling a distance equal to the horizontal distance of A from the wall. Prove that the ball pitches behind the wall at a distance from it equal to 4 times the height of the wall.

Solution

Let the horizontal distance from the wall be x,

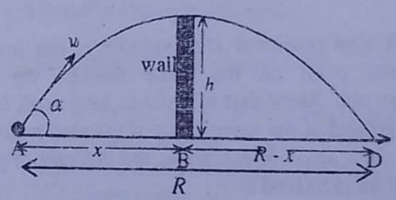
Let the vertical component of velocity acquired be $v\text{ms}^{-1}$

During free fall; $u = 0$, $a = g$

From $v^2 = u^2 + 2as$,

$v^2 = 2gx$

$v = \sqrt{2gx}$ = horizontal component of velocity since the vertical velocity is equal to the horizontal component of velocity.



To show that $R - x = 4h$

Time taken to clear the wall,

$$= \frac{\text{Horizontal distance AB}}{\text{Horizontal Velocity}}$$

$$t = \frac{x}{\sqrt{2gx}} \quad \text{..........................} \quad \text{(i)}$$

Height of wall, $h = vt - \frac{1}{2}gt^2$

$$\Rightarrow h = \frac{vx}{\sqrt{2gx}} - \frac{\frac{1}{2}gx^2}{2gx}$$

$$h = \frac{vx}{\sqrt{2gx}} - \frac{x}{4} \quad \text{..................} \quad \text{(ii)}$$

At point A and D, $h = 0$

$$vt = \frac{1}{2}gt^2 \Rightarrow t = \frac{2v}{g}$$

Hence time of flight, $T = \frac{2v}{g}$

Horizontal range $= R = v_x \cdot T$

$$R=\sqrt{2gx}\cdot\frac{2v}{g}$$

Distance behind the wall

$$R-x=\sqrt{2gx}\times\frac{2v}{g}-x \quad \text{.............................. (iii)}$$

Making v the subject from Eqn (ii)

$$h+\frac{x}{4}=\frac{vx}{\sqrt{2gx}}$$

$$v=\frac{\sqrt{2gx}}{x}\cdot\frac{(4h+x)}{4}$$

Substituting for v into Eqn (iii);

Distance $= BD = R-x$

$$=\sqrt{2gx}\times\frac{2}{g}\left(\frac{\sqrt{2gx}}{x}\cdot\frac{(4h+x)}{4}\right)-x$$

$$=\frac{2gx}{x}\cdot\frac{2}{g}\left(h+\frac{x}{4}\right)-x$$

$$=4\left(h+\frac{x}{4}\right)-x$$

$$=4h+x-x$$

$$=4h \quad \text{as required}$$

5. A shot projected with velocity v can just reach a certain point on the plane through the point of projection. Show that in order to hit a mark h m above the ground at the same point, if the shot is projected at the same angle of elevation, the velocity of projection must be increased to

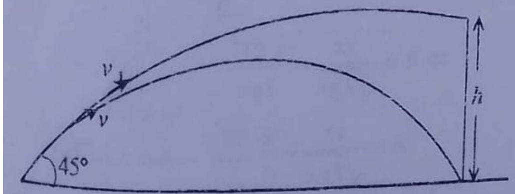
$$\frac{v^2}{(v^2-gh)^{1/2}}$$

Solution

Can just reach' refers to the maximum range

Angle of projection $=45^\circ$

$$R_{\max}=\frac{v^2}{g}$$



Let velocity of projection be increased to v_1

Horizontal component $= v_1\cos 45^0=\frac{v_1\sqrt{2}}{2}$

$$t=\frac{\text{horizontal distance}}{\text{Horizontal velocity}}=\frac{v^2}{g}\div\frac{v_1v_2}{2}$$

$$\frac{v^2}{g}\cdot\frac{2}{v_1\sqrt{2}}=\frac{v^2\sqrt{2}}{gv_1}$$

Heighth h above the ground is given by

$$h=v_1\sin 45t-\frac{1}{2}gt^2$$

$$=\frac{v_1}{\sqrt{2}}\cdot\frac{v^2\sqrt{2}}{gv_1}-\frac{1}{2}g\cdot\left(\frac{v^2\sqrt{2}}{gv_1}\right)^2$$

$$h=\frac{v^2}{g}-\frac{g}{2}\times\frac{2v^4}{g^2v_1^2}$$

$$h=\frac{v^2}{g}-\frac{v^4}{gv_1^2}$$

$$\frac{v^4}{gv_1^2}=\frac{v^2}{g}-h$$

$$\frac{v_1^4}{gv_1^2}=\frac{v^2-gh}{g}$$

$$v_1^2=\frac{v^4}{(v^2-gh)}$$

$$v_1=\frac{v^2}{(v^2-gh)^{1/2}} \quad \text{As required.}$$

6. If the horizontal range of a particle with velocity v is a, show that the greatest height is satisfied by the equation $16gx^2-8xv^2+ga^2=0$. Explain why two values of x are expected.

Solution

From horizontal range;

$$R=\frac{2v^2\sin\alpha\cos\alpha}{g}$$

$$\Rightarrow a=\frac{2v^2\sin\alpha\cos\alpha}{g} \quad \text{.......................... (1)}$$

$$\text{Greatest height, } H=\frac{v^2\sin^2\alpha}{2g}$$

$$x=\frac{v^2\sin^2\alpha}{2g} \quad \text{.................................. (2)}$$

$$\text{From (2) } \sin\alpha=\sqrt{\frac{2gx}{v^2}}$$

$$\sin\alpha=\frac{\sqrt{2gx}}{v} \quad \text{.......................... (3)}$$

$$\text{From (1) } \cos\alpha=\frac{ga}{(2\sin\alpha)v^2}$$

But $\sin\alpha = \frac{\sqrt{2gx}}{v^2}$, by substitution;

$$\Rightarrow \cos\alpha = \frac{ga}{2v^2 \cdot \frac{\sqrt{2gx}}{v}}$$

$$\cos\alpha = \frac{ga}{2v\sqrt{2gx}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$\cos^2\alpha + \sin^2\alpha = 1$

From Eqn (3) and (4)

$$\left(\frac{ga}{2v\sqrt{2gx}}\right)^2 + \left(\frac{\sqrt{2gx}}{v}\right)^2 = 1$$

$$\frac{ga^2}{8v^2x} + \frac{2gx}{v^2} = 1$$

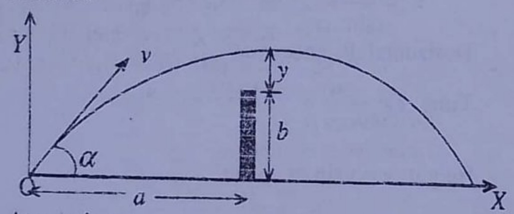
Multiplying through by $8v^2x$;

$$ga^2 + 16gx^2 = 8xv^2$$

$$ga^2 - 8v^2x + 16gxag = 0 \text{ As required}$$

7. A ball is projected from a point on the ground at distance a from the foot of the vertical wall of height b, the velocity of projection being v at an angle α to the horizontal. Find how high above the wall the ball just clears the wall. Prove that the greatest height reached is $\frac{\frac{1}{2}a^2\tan^2\alpha}{(a\tan\alpha - b)}$

Solution



At any time t

$$v_x = v\cos\alpha$$

$$v_y = v\sin\alpha - gt$$

Horizontal distance, $x = v\cos\alpha \cdot t$

Time taken to cover, horizontal distance a

$$t = \frac{a}{v\cos\alpha}$$

Height $h = v\sin\alpha t - \frac{1}{2}gt^2$

$$h = v\sin\alpha \cdot \frac{a}{v\cos\alpha} - \frac{\frac{1}{2}ga^2}{v^2\cos^2\alpha}$$

$$h = a\tan\alpha - \frac{ga^2}{2v^2\cos^2\alpha}$$

Distance above wall $y = h - b$

$$y = a\tan\alpha - \frac{ga^2}{2v^2\cos^2\alpha} - b$$

(ii) If the ball just clears the wall, $y = 0$

$$a\tan\alpha - \frac{ga^2}{2v^2\cos^2\alpha} - b = 0$$

$$\Rightarrow \quad a\tan\alpha - b = \frac{ga^2}{2v^2\cos^2\alpha}$$

$$v^2 = \frac{ga^2}{2(a\tan\alpha - b)\cos^2\alpha}$$

But greatest height, H is

$$H = \frac{v^2\sin^2\alpha}{2g}$$

Substituting for v^2;

$$H = \frac{ga^2}{2(a\tan\alpha - b)\cos^2\alpha} \times \frac{\sin^2\alpha}{2g}$$

$$= \frac{a^2}{4(a\tan\alpha - b)} \times \frac{\sin^2\alpha}{\cos^2\alpha}$$

$$H = \frac{a^2\tan^2\alpha}{4(a\tan\alpha - b)} \quad \textit{As required}$$

10. A particle is projected from a point O on the horizontal ground with velocity v at an angle of elevation α and passes through point P which is at a height y above the ground.

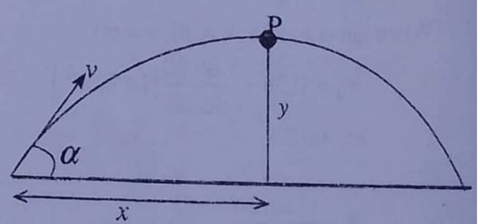
(i) Prove that $y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2v^2}$

(ii) If $x = 30$, $y = 10$, $v = 60$ Find the two possible values of $\tan\alpha$.

(iii) Find also the gradients at P of two possible trajectories.

(iv) Show that the ratio of the ranges on the horizontal ground in the two cases is 1: 8.

Solution



At time $t = 0$;

Horizontal velocity = $v\cos\alpha$

Vertical velocity = $v\sin\alpha$

Time taken to cover horizontal distance x is given by

$$t = \frac{x}{v\cos\alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Height, $y = v\sin\alpha t - \frac{1}{2}gt^2 \quad \ldots\ldots\ldots\ldots\ldots (2)$

Substituting fo t into Eqn (2)

$$y=\frac{v\sin\alpha x}{v\cos\alpha}-\frac{gx^2}{2v^2\cos^2\alpha}$$

$$y=\frac{x\sin\alpha}{\cos\alpha}-\frac{gx^2}{2v^2}\cdot\frac{1}{\cos^2\alpha}$$

$$y=x\tan\alpha-\frac{gx^2\sec^2\alpha}{2v^2}$$ As required

If $x=30$, $y=10$ and $v=60$

$$\Rightarrow 10=30\tan\alpha-\frac{9.8\times900\sec^2\alpha}{2\times60^2}$$

$$10=30\tan\alpha-\frac{9.8\times9(1+\tan^2\alpha)}{2\times36}$$

$$720=2160\tan\alpha-88.2-88.2\tan^2\alpha$$

$$7200=21600\tan\alpha-882-882\tan^2\alpha$$

$$\tan^2\alpha-1200\tan\alpha+449=0$$

$$\alpha=\frac{1200\pm\sqrt{1200^2-4\times49\times449}}{98}$$

$\alpha=24.11$ or $\tan\alpha=0.38$

$$y=x\tan\alpha-\frac{1}{2}\frac{gx^2\alpha}{v^2}$$

radient $= m=\frac{dy}{dx}$

$$\frac{y}{x}=\tan\alpha-\frac{gx}{v^2}\left(1+\tan^2\alpha\right)$$

P, $x=30$

hen $\tan\alpha=24.11$, $v=60$

$$m_1=24.11-\frac{9.8\times30}{3600}\left(1+24.11^2\right)$$

$m_2=23.44$

en $\tan\alpha=0.38$, $x=30$, $v=60$

$$m_2=0.38-\frac{9.8\times30}{3600}\left(1+0.38^2\right)$$

$m_2=0.29$

$$R=\frac{v^2\sin2\alpha}{g}$$

$\tan\alpha=24.11$

$\alpha=\tan^{-1}24.11$

$$R_1=\frac{60^2\sin(2\tan^{-1}(24.11))}{98}$$

$\tan\alpha=0.38$,

$$R_2=\frac{60^2\sin(2\tan^{-1}(0.38))}{98}$$

=

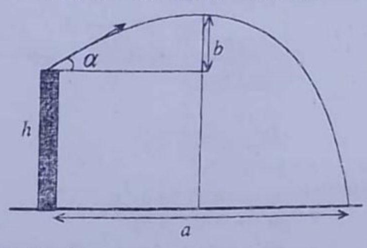
$$\frac{\sin(2\tan^{-1}(24.11))}{98}:\frac{60^2\sin(2\tan^{-1}(0.38))}{98}$$

2811 : 0.6641

$=1:8$

11 A gun is fired from the top of a cliff of height h and the shot attained a maximum height of $(h+b)$ above sea level and strikes the sea level a distance from the foot of the cliff. Prove that the angle of elevation of the gun is given by the equation

$$a^2\tan^2\alpha-4ab\tan\alpha-4bh=0$$



At a time $t=0$

$v_x=v\cos\alpha$

$v_y=v\sin\alpha$

Let height above horizontal at point of projection be y

Greatest height $=b$

From $v^2=u^2+2aS$

$0=v^2\sin^2\alpha-2gb$

$v^2\sin^2\alpha=2gb$

$$v^2=\frac{2gb}{\sin^2\alpha}$$

Horizontal distance $=a$

Time $t=\frac{a}{v\cos\alpha}$

Height $y=v\sin\alpha t-\frac{1}{2}gt^2$

When $t=\frac{a}{v\cos\alpha}$, $y=-h$

$$-h=v\sin\alpha\cdot\frac{a}{v\cos\alpha}-\frac{1}{2}g\cdot\left(\frac{a^2}{v^2\cos^2\alpha}\right)$$

$$-h=a\tan\alpha-\frac{ga^2}{2v^2\cos^2\alpha}$$

But $v^2=\frac{2gb}{\sin^2\alpha}$

$$-h=a\tan\alpha-\frac{ga^2\sin^2\alpha}{2\cos^2\alpha\cdot2gb}$$

$$-h=a\tan\alpha-\frac{a^2\tan^2\alpha}{4b}$$

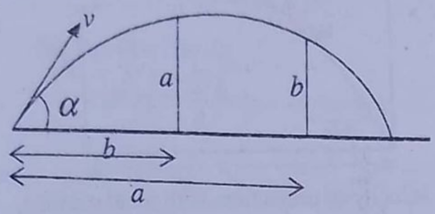
$-4bh=4ab\tan\alpha-a^2\tan^2\alpha$

$a^2\tan^2\alpha-4ab\tan\alpha-4bh=0$ as required.

12. A ball is projected so as just to clear two walls, the first of height a at a distance b from the point of projection and the second of height b at a

distance a from the point of projection. Show that the range on horizontal plane is $\frac{a^2+ab+b^2}{a+b}$ and that the angle of projection exceeds $\tan^{-1}3$

Solution



Required is to show that $R = \frac{a^2+ab+b^2}{a+b}$

And that $\alpha > \tan^{-1} 3$

Now at any time t, horizontal distance, $x = u\cos\alpha . \times t$

$$t = \frac{x}{u\cos\alpha}$$

When horizontal distance $= b$, $y = a$

Time, $t = \frac{b}{v\cos\alpha}$

From $y = v\sin\alpha t - \frac{1}{2}gt^2$

$$a = v\sin\alpha \cdot \frac{b}{v\cos\alpha} - \frac{gb^2}{2v^2\cos^2\alpha}$$

$$a = b\tan\alpha - \frac{gb^2}{2v^2\cos^2\alpha} \quad \text{.......................... (i)}$$

Similarly, for horizontal distance, $x = a$, $y = b$

$$t = \frac{a}{v\cos\alpha}$$

Vertically;

$$b = a\tan\alpha - \frac{ga^2}{2v^2\cos^2\alpha} \quad \text{................. (ii)}$$

Also when $y = 0$

Horizontal distance = range (R)

By substitution;

$$0 = R\tan\alpha - \frac{gR^2}{2v^2\cos^2\alpha} \quad \text{............................ (iii)}$$

From (iii), $\frac{g}{2v^2\cos^2\alpha} = \frac{\tan\alpha}{R}$

Substituting into (i) and (ii)

$$a = b\tan\alpha - \frac{b^2\tan\alpha}{R}$$

$$a = \left(b - \frac{b^2}{R}\right)\tan\alpha \quad \text{.................................... (iv)}$$

Similarly $b = \left(a - \frac{a^2}{R}\right)\tan\alpha$ ....(v)

Eqn (iv) ÷ Eqn (v);

$$\frac{a}{b} = \frac{(b - b^2/R)}{(a - a^2/R)}$$

$$a^2 - \frac{a^3}{R} = b^2 - \frac{b^3}{R}$$

$$a^2R - b^2R = a^3 - b^3$$

$$R = \frac{a^3 - b^3}{a^2 - b^2} = \frac{(a-b)(a^2+ab+b^2)}{(a+b)(a-b)}$$

$$R = \frac{a^2+ab+b^2}{a+b}$$

Substituting R into equation (iv) and by simplifying

$$\tan\alpha = \frac{a^2+ab+b^2}{ab}$$

For $a > b$, as shown from the sketch above

When you replace b by a

$$\frac{a^2+ab+b^2}{a+b} > \frac{a^2+a^2+a^2}{a^2}$$

$$\frac{a^2+ab+b^2}{a+b} > \frac{3a^2}{a^2}$$

$$\frac{a^2+ab+b^2}{a+b} > 3$$

But $\tan\alpha = \frac{a^2+ab+b^2}{a+b}$

$$\Rightarrow \tan\alpha > 3$$

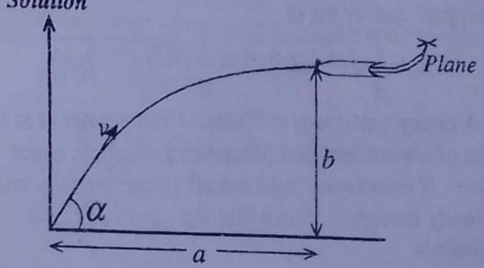
$\alpha > \tan^{-1}(3)$ *as required.*

14. A shell is fired from a gun with muzzle velocity v so as to hit a helicopter hovering at a height b and at a horizontal distance a from the gun. If the angle of projection of the shell to the horizontal is α, show that

$$ga^2\tan^2\alpha - 2v^2a\tan\alpha + ga^2 + 2v^2b = 0$$

By considering this as a quadratic equation in $\tan\alpha$, deduce that the furthest horizontal distance at which the helicopter can be hit if the height b and muzzle velocity v are given is $\frac{v}{g}\sqrt{v^2 - 2gb}$

Solution



With usual notation

$v_x = v\cos\alpha$
$v_y = v\sin\alpha$

At any time t,
Horizontal distance $= v\cos\alpha\ t$

$a = v\cos\alpha\ t$ .... (i)

$$t = \frac{a}{v\cos\alpha}$$

Vertical distance $= v_y t - \frac{1}{2}gt^2$

Since the helicopter should be hit, vertical distance $= b$

$$\Rightarrow b = u\sin\alpha t - \frac{1}{2}gt^2 \quad \text{.................. (ii)}$$

Substituting for t into Eqn (ii)

$$b = \frac{u\sin\alpha a}{u\cos\alpha} - \frac{\frac{1}{2}ga^2}{v^2\cos^2\alpha}$$

$$b = a\tan\alpha - \frac{ga^2}{2v^2\cos^2\alpha}$$

$$b = a\tan\alpha - \frac{ga^2\sec^2\alpha}{2v^2}$$

$2v^2 b = 2v^2 a\tan\alpha - ga^2(\tan^2\alpha + 1)$
$= 2v^2 a\tan^2\alpha - ga^2\tan^2\alpha - ga^2$
$ga^2\tan\alpha - 2v^2 a\tan\alpha + ga^2 + 2v^2 b = 0$

As required

The equation above has a real solution for $\tan\alpha$, if

$b^2 - 4ac \geq 0$

where $a = ga^2$, $b = -2v^2 a$,
$c = ga^2 + 2v^2 b$,

$4v^4a^2 \geq 4ga^2(ga^2 + 2v^2 b)$
$v^4 \geq g(ga^2 + 2v^2 b)$

$$\frac{v^4}{g} \geq ga + 2v^2 b$$

$$ga^2 + 2v^2 b \leq \frac{v^4}{g}$$

$$a^2 \leq \frac{v^4}{g^2} - \frac{2v^2 gb}{g^2}$$

$$a \leq \frac{\sqrt{v^4 - 2v^2 gb}}{g}$$

The maximum horizontal distance at which the helicopter can be hit is

$$a = \frac{1}{g}\sqrt{v^4 - 2v^2 gb} = \frac{1}{g}\sqrt{v^2(v^2 - 2gb)}$$

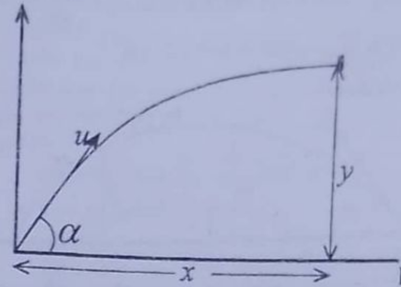
A heavy particle is projected from a point O at an angle of elevation α and describes a parabola under gravity. If coordinate axes are taken horizontally and vertically though 0. Prove that the equation of the parabola is

$$y = x\left(1 - \frac{x}{R}\right)\tan\alpha$$

Where R is the horizontal range. If the distance between the two points on the parabola which are of the same height h above the horizontal is $2a$

Show that $R(R - 4h\cot\alpha) = 4a^2$

Solution



Let u be velocity of projection. with usual notation,

At $t = 0$

$v_x = u\cos\alpha$
$v_y = u\sin\alpha$

At any time t, horizontally;

$x = u\cos\alpha t$ ....(i)

$$t = \frac{x}{u\cos\alpha}$$

For vertical distance

$$y = u\sin\alpha t - \frac{1}{2}gt^2 \quad \text{.......................... (ii)}$$

Substituting for t into Eqn (ii)

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} \quad \text{...............(iii)}$$

But range, $R = \dfrac{2u^2\cos\alpha\sin\alpha}{g}$

$$u^2 = \frac{Rg}{2\cos\alpha\sin\alpha}$$

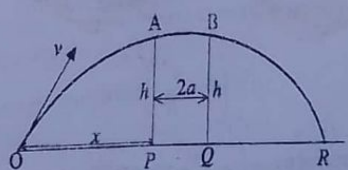
Substituting for u^2 into Eqn (iii)

$$y = x\tan\alpha - \frac{gx^2\tan\alpha}{2\cos^2\alpha}\cdot\frac{2\cos^2\alpha}{Rg}$$

$$= x\tan\alpha - \frac{x^2\tan\alpha}{R}$$

$$y = x\tan\alpha\left(1 - \frac{x}{R}\right)$$

$$y = x\left(1 - \frac{x}{R}\right)\tan\alpha \quad \text{As required}$$



$A = (x,\ h)$

$B = [(x+2a), h]$

The two points satisfy the parabola .

For point A, $h = x\tan\alpha\left(1-\frac{x}{R}\right)$ ....(i)

For point B,

$$h = (x+2a)\tan\alpha\left(1-\frac{(x+2a)}{R}\right) \text{.............(ii)}$$

But Eqn (i) = Eqn (ii)

$$x\tan\alpha\left(1-\frac{x}{R}\right) = (x+2a)\tan\alpha\left(1-\frac{(x+2a)}{R}\right)$$

$$x\left(1-\frac{x}{R}\right) = \frac{(x+2a)(R-(x+2a))}{R}$$

$$\frac{x(R-x)}{R} = \frac{(x+2a)(R-(x+2a))}{R}$$

$$xR - x^2 = R(x+2a) - (x+2a)^2$$

$$xR - x^2 = Rx + 2aR - (x^2 + 4ax + 4a^2)$$

$$0 = 2aR - 4ax - 4a^2$$

$$0 = R - 2x - 2a$$

$$x = \frac{R-2a}{2}$$

Substituting for x into equation (i)

$$h = \frac{R-2a}{2}\tan\alpha\left(1-\frac{\frac{(R-2a)}{2}}{R}\right)$$

$$h = \left(\frac{R-2a}{2}\right)\tan\alpha\left(\frac{2R-R+2a}{2R}\right)$$

$$h = \left(\frac{R-2a}{2}\right)\tan\alpha\left(\frac{R+2a}{2R}\right) \text{.................. (iii)}$$

From Eqn (iii)

$$\frac{2h}{R-2a} = \tan\alpha\left(\frac{R+2a}{2R}\right) \text{........................ (iv)}$$

Also $\frac{2hR}{R+2a} = \tan\alpha\left(\frac{R-2a}{2}\right)$ .... (v)

Adding Eqn (iv) and Eqn (v);

$$\frac{2h}{R-2a} + \frac{2hR}{R+2a} = \tan\alpha\left(\frac{R+2a}{2R} + \frac{R-2a}{2}\right)$$

$$\frac{2hR + 4ha + 2hR^2 - 4haR}{R^2 - 4a^2} = \tan\alpha\left(\frac{R+2a+R^2-2aR}{2R}\right)$$

$$2h\left(\frac{R+2a+R^2-2aR}{R^2-4a^2}\right) = \tan\alpha\left(\frac{R+2a+R^2-2aR}{2R}\right)$$

$$\frac{2h}{R^2-4a^2} = \frac{\tan\alpha}{2R}$$

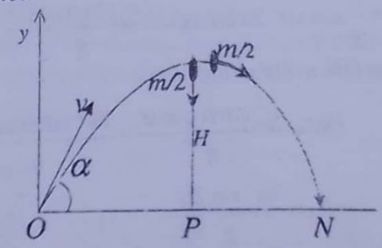
$$4hR\cot\alpha = R^2 - 4a^2$$

$$4a^2 = R(R - 4h\cot\alpha)$$

16.A shell of mass m is projected at an angle α to the horizontal at a speed u. At its highest point, the shell explodes into two equal masses, one of which falls directly downwards and the other continues to move until when it strikes the ground. Write down the horizontal and vertical components of the shell's velocity of projection.

Deduce that the maximum horizontal distance away from the point of projection that the second particle lands is given by; $\frac{3u^2}{g}$

Solution



With usual notation, at time $t = 0$

$$v_x = u\cos\alpha$$

$$v_y = u\sin\alpha$$

At anytime, t

$$v_x = u\cos\alpha$$

$$v_y = u\sin\alpha - gt$$

At maximum height, $v_y = 0$

$$0 = u\sin\alpha - gt$$

$$t = \frac{u\sin\alpha}{g}$$

At that time horizontal distance traveled is $x = OP$ from

$$x = u\cos\alpha . t$$

$$\Rightarrow t = \frac{u\sin\alpha}{g}$$

$$OP = \frac{u \cdot u\sin\alpha\cos\alpha}{g}$$

$$= \frac{u^2\sin\alpha\cos\alpha}{g}$$

$$OP = \frac{u^2\sin\alpha\cos\alpha}{g} \text{ ....(iii)}$$

At the highest point, when explosion occurs, the horizontal velocity changes. The new horizontal velocity can be found by using the law of conservation of momentum, which is defined as

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \text{....................(1)}$$

But $m_1 = m_2 = ½m$

$$u_1 = u_2 = u\cos\alpha$$

$v_1 = 0$ As the second portion falls vertically downwards.

By substitution,

$$mu\cos\alpha = ½\, mv_2$$

$$v_2 = 2\,u\cos\alpha$$

is is the horizontal velocity with which the first rtion continues up to point N.

ne $(t) = \frac{u}{g}\sin\alpha$ from P to N

$= \frac{u}{g}\sin\alpha \cdot v_2$

$= \frac{u}{g} u\sin\alpha \cdot 2u\cos\alpha = \frac{2u^2\sin\alpha\cos\alpha}{g}$

ice $ON = OP + PN$

$$ON = \frac{u^2\sin\alpha\cos\alpha}{g} + \frac{2u^2\sin\alpha\cos\alpha}{g}$$

$$= \frac{3u^2\sin 2\alpha}{g}$$

is the maximum when $\sin 2\alpha = 1$

ce ON maximum $= \frac{3u^2}{2g}$ As required

Examination Questions

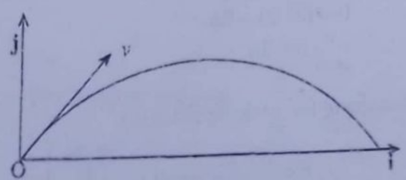
1.a) A Particle is projected upwards, from a point 0 with an initial velocity of $2\mathbf{i} + 5\mathbf{j}$. $\mathbf{i}$ and $\mathbf{j}$ being horizontal and vertical unit vectors respectively. Find in vector form the velocity and position of the particle at any time t.

b) A particle P is projected from a point A with initial velocity 30ms^{-1} at angle of elevation 30^0 to the horizontal. At the same instant a particle Q is projected in opposite direction with initial speed of 40ms^{-1} from a point at the same level with A and 120m from A. Given that the particles collide. Find

i) The angle of projection of Q

ii) The time when collision occurs. **1988 No 13**

Solution



i) Acceleration, $a = \frac{dv}{dt}$

Since the particle is projected upwards, $a = -g$

$$\Rightarrow \frac{dv}{dt} = -g\mathbf{j}$$

$$dv = -g\mathbf{j}dt$$

$$\int dv = \int -g\mathbf{j}dt$$

$$v = -gt\mathbf{j} + c$$

$$v = -gt\mathbf{j} + c$$

by substitution, at $t = 0$, $v = 2\mathbf{i} + 5\mathbf{j}$

$$2\mathbf{i} + 5\mathbf{j} = 0 + c$$

$$c = 2\mathbf{i} + 5\mathbf{j}$$

Substituting for c

$v = 2\mathbf{i} + (5 - gt)\mathbf{j}$

Also $v = \frac{dr}{dt}$

$$\frac{dr}{dt} = 2\mathbf{i} + (5 - gt)\mathbf{j}$$

$$dr = (2\mathbf{i} + (5 - gt)\mathbf{j})dt$$

$$\int dr = \int (2\mathbf{i} + (5 - gt)\mathbf{j})dt$$

$$\mathbf{r} = 2t\mathbf{i} + (5t - \tfrac{1}{2}gt^2)\mathbf{j} + c$$

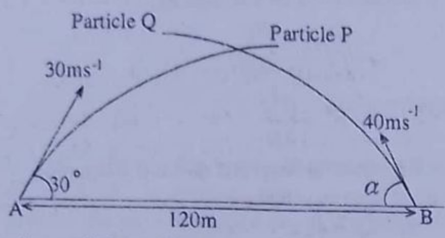
At time $t = 0, r = 0$

$$\Rightarrow 0 = 0 + 0 + c$$

$$c = 0$$

Substituting for c; $\mathbf{r} = 2t\mathbf{i} + (5t - \tfrac{1}{2}gt^2)\mathbf{j}$

b) Let the angle of projection Q be a with the horizontal



Since the particles collide

$r_P = r_Q$

we know that $\frac{dv}{dt} = -g\mathbf{j}$

$dv = -g\mathbf{j}\,dt$

$\int dv = -\int g\mathbf{j}\,dt$

$v = -gt\mathbf{j} + c$

For particle P

Let A be the origin (0, 0)

$v_0 = 30\cos30\mathbf{i} + 30\sin30\mathbf{j}$

$= 15\sqrt{3}\mathbf{i} + 15\mathbf{j}$

$v_p = -gt\mathbf{j} + c$

At time $t = 0$; $15\sqrt{3}\mathbf{i} + 15\mathbf{j} = 0 + c$

Hence $c = 15\sqrt{3}\mathbf{i} + 15\mathbf{j}$

By substitution;

$v_p = 15\sqrt{3}\mathbf{i} + (15 - gt)\mathbf{j}$

$\mathbf{r}_p = \int v_p\,dt$

$\mathbf{r}_p = 15\sqrt{3}t\mathbf{i} + (15t - \frac{1}{2}gt^2)\mathbf{j} + c$

At time $t = 0$, $r = 0$, Hence $c = 0$

By substitution;

$\mathbf{r}_p = 15\sqrt{3}t\mathbf{i} + (15t - \frac{1}{2}gt^2)\mathbf{j}$..........................(1)

For particle Q

$v_o = -40\cos\alpha\mathbf{i} + 40\sin\alpha\mathbf{j}$

$v_Q = -gt\mathbf{j} + c$

At time $t = 0$; $-40\cos\alpha\mathbf{i} + 40\sin\alpha\mathbf{j} = 0 + c$

Hence $c = -40\cos\alpha\mathbf{i} + 40\sin\alpha\mathbf{j}$

By substitution;

$\Rightarrow v_Q = -40\cos\alpha\mathbf{i} + (40\sin\alpha - gt)\mathbf{j}$

$r_Q = \int v_Q dt$

$r_Q = -40\cos\alpha t\mathbf{i} + (40\sin\alpha t - \frac{1}{2}gt^2)\mathbf{j} + c$

At time $t = 0$, $r = 120\mathbf{i}$, By substitution;

$c = 120\mathbf{i}$

Substituting for c

$r_Q = (120 - 40t\cos\alpha)\mathbf{i} + (40t\sin\alpha - \frac{1}{2}gt^2)\mathbf{j}$

For collision, $\mathbf{r}_p = \mathbf{r}_Q$

$15\sqrt{3}t\mathbf{i} + (15t - \frac{1}{2}gt^2)\mathbf{j} = (120 - 40t\cos\alpha)\mathbf{i} + (40t\sin\alpha - \frac{1}{2}gt^2)\mathbf{j}$

Equating coefficients of $\mathbf{j}$;

$15t - \frac{1}{2}gt^2 = 40t\sin\alpha - \frac{1}{2}gt^2$

$15 = 40\sin\alpha$

$\sin\alpha = \frac{15}{40} = \frac{3}{8}$

$\alpha = 24.5^0$

$\therefore$ The angle of projection of particle Q is $24.5°$ to the horizontal or $N68^0W$.

Equating the coefficients of $\mathbf{i}$

$15\sqrt{3}t = 120 - 40t\cos\alpha$

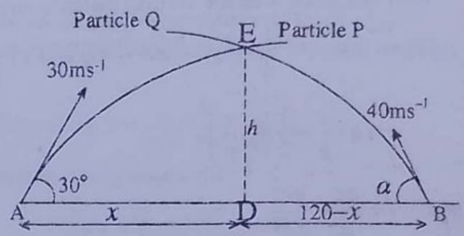
$t = \frac{120}{15\sqrt{3} + 40\cos24.5}$

$t = 1.9s$

The time of collision is 1.9seconds

Alternatively; *When the two particles collide, their distance h above, from the ground will be the same*

(b)



From vertical distance covered at any time t given by $y = u\sin\alpha t - \frac{1}{2}gt^2$;

For particle P, $h = 30\sin30t - \frac{1}{2}gt^2$ ....(i)

For particle Q, $h = 40\sin\alpha t - \frac{1}{2}gt^2$ ....(ii)

Equating equation (i) to (ii);

$30\sin30t - \frac{1}{2}gt^2 = 40\sin\alpha t - \frac{1}{2}gt^2$

$30\sin30t = 40\sin\alpha t$

$\sin\alpha = \frac{30\sin30}{40} = \frac{3}{8}$

$\alpha = \sin^{-1}\left(\frac{3}{8}\right) = 24.5°$

(ii) From horizontal motion, $x = v_x t$

For particle P; $x = 30\cos30t$ ....(iii)

For particle Q; $120 - x = 40\cos24.5t$ ....(iv)

Substituting Eqn(iii) into (iv); $120 - 30\cos30t = 40\cos24.5t$

$\frac{120}{30\cos30 + 40\cos24.5} = t$

$t = 1.9s$

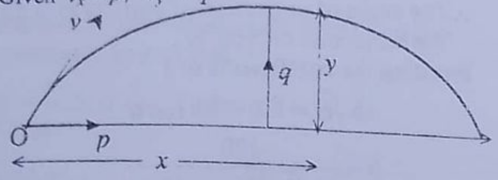
2. The horizontal and vertical components of the initial velocity of a particle projected from a point O on a horizontal plane are p and q respectively.

(a) Express the vertical distance y, traveled in terms of the horizontal distance x. Find the components of p and q.

(b) Find the greatest height, H, attained and the range, R, on the horizontal plane through 0. Hence show that $y = \frac{4Hx(R-x)}{R^2}$. Given that the particle passes through the point (20, 80) and H =100m find the velocity of projection. **(1990 No 5)**

Solution

a) Given $v_x = p$, $v_y = q$



If at any time t, the horizontal distance traveled is x

$x = v_x t$ but $v_x = p$

$t = \frac{x}{p}$

With this same time the vertical distance covered is y.

From $y = v_y t - \frac{1}{2}gt^2$, but $v_y = q$ and $t = \frac{x}{p}$

$y = q\cdot\frac{x}{p} - \frac{1}{2}g\left(\frac{x}{p}\right)^2$

$y = \frac{qx}{p} - \frac{gx^2}{2p^2}$

Hence the vertical distance travelled is $y = \frac{qx}{p} - \frac{gx^2}{2p^2}$

b) From $v^2 = u^2 + 2aS$ where v = vertical velocity

At the greatest height, $v = 0$

=> $0 = q^2 - 2gH$

$H = \frac{q^2}{2g}$ .... (1)

At point of projection or range, $y = 0$

$\Rightarrow$ From $y = qt - \frac{1}{2}qt^2$

$0 = qt - \frac{1}{2}gt^2$

$0 = \left(q - \frac{1}{2}gt\right)t$

$t = 0$ or $t = \frac{2q}{g}$

Time of flight, $T = \frac{2q}{g}$

Range (Horizontal distance)

= Horizontal velocity × time of flight

Range, $R = \frac{2pq}{g}$

$R = \frac{2pq}{g}$ ....(2)

What is required is to eliminate p and q

From Eqn (i), $q = \sqrt{2Hg}$

From Eqn (ii); $q = \frac{Rg}{2p}$

Equating the two equations:

$\sqrt{2Hg} = \frac{Rg}{2p}$

$p^2 = \frac{R^2 g}{8H}$

Substituting for p and q into the equation

$y = \frac{qx}{p} - \frac{gx^2}{2p^2}$

$y = \frac{Rgx}{2p^2} - \frac{gx^2}{2p^2}$

$y = \frac{Rgx - gx^2}{2(\frac{R^2 g}{8H})}$

$y = \frac{Rx - x^2}{\frac{R^2}{4H}} = \frac{4xH(R - x^2)}{R^2}$

From $H = \frac{q^2}{2g}$

When $H = 100$

$\Rightarrow 100 = \frac{q^2}{2g}$

$q = \sqrt{200g}$ .... (5)

From $y = \frac{qx}{p} - \frac{gx^2}{2p^2}$

$80 = \frac{20\sqrt{200g}}{p} - \frac{400g}{2p^2}$

$80p^2 - 20p\sqrt{200g} = 0$

$80p^2 - 20p\sqrt{200g} + 200g = 0$

This is a quadratic equation in p.

Now $80p^2 - 20p\sqrt{200g} + 200g = 0$

$4p^2 - p\sqrt{200g} + 10g = 0$

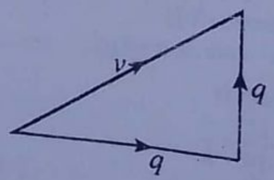
$4p^2 - \left(\sqrt{100\times 2g}\right)p + 10g = 0$

$4p^2 - 10\left(\sqrt{2g}\right)p + 10g = 0$

$2p^2 - 5\left(\sqrt{2g}\right)p + 5g = 0$

$p = \frac{5\sqrt{2g} + \sqrt{(50g - 40g)}}{4}$

$p = \frac{5\sqrt{2g} + \sqrt{10g}}{4}$ .... (6)



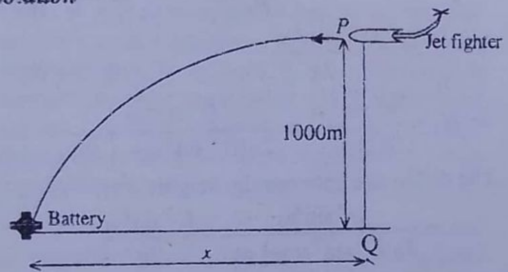
$$v=\sqrt{p^2+q^2}$$

$$v=\sqrt{\left(\frac{5\sqrt{2g}+\sqrt{10g}}{4}\right)^2+200g}=45\text{ms}^{-1}$$

3(a) A jet fighter is flying at a height of 1km at a speed of 1080kmh^{-1} to bomb an anti – aircraft battery on the ground, the pilot releases a bomb at a point P which is vertically above Q on the ground. How far is Q from the battery?

(b) Water gushes out of a nozzle of a pipe at a speed of 10ms^{-1}. The nozzle of the pipe is placed 30^0 to the horizontal and 1m below and outside the rim of the top of a tank. If the water just flows into the tank, calculate the maximum distance between the nozzle and the tank. **(1991 No 7)**

Solution



Initial horizontal velocity of bomb = velocity of air craft.

$$1080\text{kmh}^{-1}=\frac{1080\times1000}{3600}\text{ms}^{-1}$$

Horizontal velocity $=300\text{ms}^{-1}$.......................... (i)

Initial vertical velocity of bomb = 0

Time taken to hit the battery

= time taken to fall the 1000m

From $S=ut+\frac{1}{2}at^2$

$$1000=\frac{1}{2}gt^2$$

$$t=\sqrt{\frac{2000}{9.8}}\text{ Seconds.} \quad ....(ii)$$

Horizontal distance travelled = horizontal velocity × time

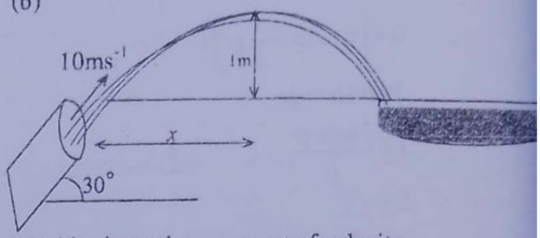
$$=300\times\sqrt{\frac{2000}{9.8}}$$

Distance of Q from battery $=300\sqrt{\frac{2000}{9.8}}$

$=4.285$km

Voc ~ 4286.26m

(b)



Initial horizontal component of velocity

$=10\cos30^0 = 5\sqrt{3}\text{ ms}^{-1}$

Initial vertical component $=10\sin30=5\text{ms}^{-1}$

Let horizontal distance between nozzle and tank = x

Time taken $t=\frac{x}{hor.\ Velocity}=\frac{x}{5\sqrt{3}}$

Height at that time = 1m

$$1=\frac{5x}{5\sqrt{3}}-\frac{gx^2}{2\times75}=\frac{x}{\sqrt{3}}-\frac{gx^2}{150}$$

$$1=\frac{\sqrt{3}x}{3}-\frac{gx^2}{150}$$

$$150=\left(50\sqrt{3}\right)x-9.8x^2$$

$$9.8x^2-\left(50\sqrt{3}\right)x+150=0$$

$$x=\frac{50\sqrt{3}+\sqrt{\left(50\sqrt{3}\right)^2-4\times9.8\times150}}{19.6}$$

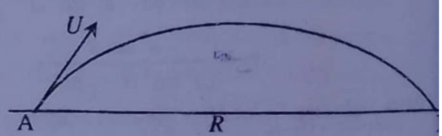
$$=\frac{50\sqrt{3}+\sqrt{1620}}{19.6}$$

$x=6.47$m

4. A particle projected from A with velocity u at an angle of elevation α to the horizontal hits a horizontal plane through A to B. Show that if the particle is to be projected from A with the same angle of elevation to the horizontal so as to hit a target at a height h, above B, the velocity of projection must be $\frac{u^2\sin\alpha}{(u^2\sin^2\alpha-\frac{1}{2}gh)^{1/2}}$

Determine the difference between the greatest heights attained by the particle in these cases. **(1992 No 5)**

Solution



At point A and B, vertical distance $y=0$

From $y=u\sin\alpha t-\frac{1}{2}gt^2$

$$0 = u\sin\alpha t - \frac{1}{2}gt^2$$

$$0 = \left(u\sin\alpha - \frac{1}{2}gt\right)t$$

$$2u\sin\alpha = gt$$

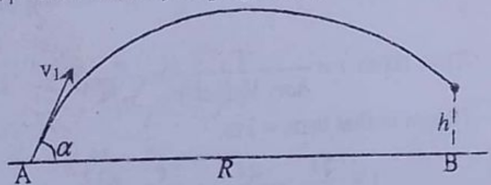
$$t = \frac{2u\sin\alpha}{g}$$

$$R = v_x T$$

$$R = \frac{2u\sin\alpha}{g} \times u\cos\alpha = \frac{2u^2\sin\alpha\cos\alpha}{g} \quad ....(i)$$

f the particle hits hm above point B, we have:

et v_1 = new velocity of projection;



$v_x = v_1\cos\ \alpha$

$v_y = v_1\sin\ \alpha$

$R = v_1\cos\ \alpha \times t$

$$t = \frac{R}{v_1\cos\alpha} \quad .... (ii)$$

ubstituting Eqn (i) into Eqn (ii)

$$t = \frac{2u^2\sin\alpha\cos\alpha}{gv_1\cos\alpha} = \frac{2u^2\sin\alpha}{gv_1}$$

ut for vertical motion;

$$h = v_1\sin\alpha t - \frac{1}{2}gt^2$$

ubstituting for t;

$$h = v_1\sin\alpha\left(\frac{2u^2\sin\alpha}{gv_1}\right) - \frac{1}{2}g\left(\frac{4u^4\sin^2\alpha}{g^2v_1^2}\right)$$

$$= \frac{2u^2\sin^2\alpha}{g} - \frac{2u^4\sin^{2\alpha}}{gv_1^2}$$

$$h = \frac{2u^2v_1^2\sin^2\alpha - 2u^4\sin^2\alpha}{gv_1^2}$$

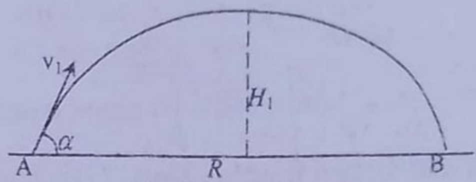
$$hgv_1^2 = 2u^2v_1^2\sin^2\alpha - 2u^4\sin^2\alpha$$

$$v_1^2\left(2u^2\sin^2\alpha - gh\right) = 2u^4\sin^2\alpha$$

$$v_1^2 = \frac{2u^4\sin^2\alpha}{2u^2\sin^2\alpha - gh} = \frac{u^4\sin^2\alpha}{u^2\sin^2\alpha - \frac{1}{2}gh}$$

$$v_1 = \frac{u^2\sin\alpha}{(u^2\sin^2\alpha - \frac{1}{2}gh)^{1/2}}$$

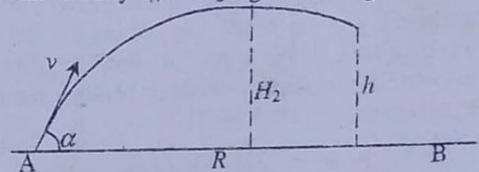
ith velocity U, let H_1 = greatest height



Using $H = \dfrac{u^2\sin^2\alpha}{2g}$

$$H_1 = \frac{u^2\sin^2\alpha}{2g} \quad .... (i)$$

With velocity v_1, let H_2 = greatest height



Using $H = \dfrac{U^2\sin^2\alpha}{2g}$

$$\Rightarrow H_2 = \frac{v_1^2\sin^2\alpha}{2g} = \frac{U^4\sin^4\alpha}{2g(U^2\sin^2\alpha - \frac{1}{2}gh)}$$

The difference between the heights = $H_2 - H_1$

$$= \frac{U^4\sin^4\alpha}{2g(U^2\sin^2\alpha - \frac{1}{2}gh)} - \frac{U^2\sin^2\alpha}{2g}$$

$$= \frac{U^4\sin^4\alpha - U^2\sin^2\alpha(U^2\sin^2\alpha - \frac{1}{2}gh)}{2g(U^2\sin^2\alpha - \frac{1}{2}gh)}$$

$$= \frac{U^4\sin^4\alpha - U^4\sin^4\alpha + \frac{1}{2}ghU^2\sin^2\alpha}{2g(U^2\sin^2\alpha - \frac{1}{2}gh)}$$

$$= \frac{hU^2\sin^2\alpha}{4(U^2\sin^2\alpha - \frac{1}{2}gh)}$$

5. A stone thrown upwards at an angle α to the horizontal with speed,$u$ms^{-1}, just clears a vertical wall 4m high and 10m from the point of projection when travelling horizontally. Find the angle of projection. **(March 1998 N0 15b)**

Answer: 38.66^0

6. A particle is projected with a speed of $10\sqrt{2g}$ ms^{-1} from the top of a cliff, 50m high. The particle hits the sea at a distance 100m from the vertical through the point of projection. Show that there are two possible directions of projection, which are perpendicular. Determine the time taken from the point of projection in each case. **(Nov 1998 No 10)**

Answer: 9.83s, 2.32s

7. A particle is projected from level ground towards a vertical pole, 4m high and 30m away from the point of projection. It just passes the pole in one second. Find:

(i) its initial speed and angle of projection

$V^2 = u^2 + 2gs$

(ii) the distance beyond the pole where the particle will fall. **(2002 No 16)**

Answer: (i) $39.292ms^{-1}$, 16.5^0 (ii) 24.42m

8. A particle is projected at an angle of 60^0 to the horizontal with a velocity of $20\ ms^{-1}$. Calculate the greatest height the particle attains. [Use $g = 10ms^{-2}$

Answer: 15m **(2004 No 4)**

9 (a) A particle is projected at an angle of elevation of 30^0 with a speed of 21 m/s. If the point of projection is 5m above the horizontal ground, find the horizontal distance that the particle travels before striking the ground. (Take $g = 10\ ms^{-3}$).

(b) A boy throws a ball at an initial speed of $40ms^{-1}$ at an angle of elevation, α. Show, taking g to be $10\ m/s^2$, that the times of flight corresponding to a horizontal range of 80m are positive roots of the equation $T^4 - 64T^2 + 256 = 0$ **(2006 No. 16)**

Answers: (a) 39.42842

10. (a) Derive the equation of the path of a projectile projected from origin O at angle α to the horizontal with initial speed Ums^{-1}.

(b) A particle projected from a point on a horizontal ground moves freely under gravity and hits the ground again at A. Taking O as the origin, the equation of the path of the particle is $60y = 20\sqrt{3x - x^2}$, where x and y are measured in metres.

Determine the:

(i) Initial direction and speed of projection

(ii) Distance OA

(Take g as $10ms^{-2}$). **(2008 No 11)**

Answer: a) $y = x\tan\alpha - \dfrac{gx^2(1+\tan^2\alpha)}{2u^2}$

b) 30^0, $20ms^{-1}$ (ii) $20\sqrt{3}$ metres

EXERCISE TWO

1. A particle is projected with a velocity of $10ms^{-1}$ at an angle of 45^0 to the horizontal. It hits the ground at a point, which is 3m below its point of projection. Find the time for which it is in the air and the horizontal distance covered by the particle in this time.

2. A particle is projected with a velocity of $70ms^{-1}$ at angle of 20^0 to the horizontal. Find the greatest height reached by the particle above its point of projection.

3. A ball is thrown from ground level so that it just clears a wall 3m high when it is moving horizontally. If the initial speed of the ball is $20ms^{-1}$, find the angle of projection.

4. Two seconds after projection from a point O, a projectile P passes through a point with position vector of $8\mathbf{i} - 2\mathbf{j}$. Find the initial velocity vector of P. Find also the position vector of P after 3 seconds.

5. A particle is projected from a point O with initial velocity vector $20\mathbf{i} + 30\mathbf{j}$. 2seconds later a second particle is projected from O with velocity vector $60\mathbf{i} + 50\mathbf{j}$. Prove that the particles collide 1second after the projection of the second particle.

6. A and B are two points on level ground, 60m apart. A particle is projected from A towards B with initial velocity $30ms^{-1}$ at 45^0 to the horizontal. At the same instant another particle is projected from B towards A with the same initial velocity. Find when the particles collide and the height above the level of AB at which they collide.

7.(a) A particle is projected from the origin O with velocity v at an angle of elevation α to the horizontal. Show that its height y above O when it has travelled a distance x horizontally is given by $y = \dfrac{x\tan\alpha - gx^2\sec^2\alpha}{2v^2}$

(b) A ball thrown from O with speed 1400cm/s is caught at a point P, which is 1000cm horizontally from O and 187.5cm above the level of O. Find the two possible angles of projection. If the ball is thrown from O with the same initial speed to pass through a point 562.5cm vertically above P. Show that there is only one possible angle of projection.

8. Two boys stand on horizontal ground at a distance a apart. One throws a ball from a height $2h$ with velocity v and the other catches it at height h. If θ is the inclination above the horizontal at which the first boy throws the ball, show that

$ga^2\tan^2\theta - 2v^2a\tan\theta + ga^2 - 2v^2h = 0$. When $a = 2\sqrt{2h}$ and $v^2 = 2gh$, calculate

a) the value of θ

b) the greatest height attained by the ball above the ground, in terms of h.

9. A particle is projected at an elevation α, where tanα = 3, from a point A on a horizontal plane distant 100m from the foot of a vertical tower of height 50m. The particle just clears the tower and lands at a point B on the horizontal plane. Determine the initial speed of the particle and the distance AB. Find also the greatest height reached by the particle above the plane. (Take g as $10ms^{-1}$)

10. An aircraft is flying with speed v in a direction inclined at an angle α above the horizontal. When the aircraft is at height h, a bomb is dropped. Show that the horizontal distance R, measured from the point vertically below the point at which the bomb is released to the point where the bomb hits the ground is given by $gR = \frac{1}{2}v^2\tan 2\alpha + v\left(2gh + v^2\sin^2\alpha\right)^{1/2}\cos\alpha$

11. A stone is thrown horizontally with speed u from the edge of a vertical cliff of height h. The stone hits the ground at a point which is a distance d horizontally from the base of the cliff. Show that $2hu^2 = gd^2$.

12. A and B are two points on level ground. A vertical tower of height $4h$ has its base at A and a vertical tower of height h has its base at B. When a stone is thrown horizontally with speed V from the top of the taller tower towards the smaller tower, it lands at a point X where $AX = \frac{3}{4}AB$. When a stone is thrown horizontally with speed u from the top of the smaller tower towards the taller tower, it also lands at the point X. Show that $3u = 2v$.

13. Two particles A and B are projected simultaneously, A from the top of a vertical cliff and B from the base. Particle A is projected horizontally with speed $3ums^{-1}$ and B is projected at angle θ above the horizontal with speed $5ums^{-1}$. The height of the cliff is 56m and the particles collide after 2second.

Find the horizontal and vertical distances from the point of collision to the base of the cliff and the values of u and θ.

14. A point O is vertically above a fixed point A on a horizontal plane. A particle P is projected from O with speed $5v$ at an angle $\cos^{-1}(3/5)$ above the horizontal and hits the plane at a point B a distance $\frac{48v^2}{g}$ from A.

(i) Show that the height O above A is $\frac{64v^2}{g}$

(ii) Find the distance of P from O when it is directly level with it.

A second particle is now projected with speed 24w from O at an angle α above the horizontal and it also hits the plane at B. Find an equation. Involving V, W and α. Given that one value of α is 45^0 find W in terms of v and show that the other value of α is such that $7\tan^2\alpha - 6\tan\alpha - 1 = 0$