

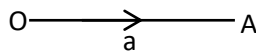
## Vectors

A vector is a quantity with both magnitude and direction.

Examples include displacement, velocity, acceleration, force, momentum etc.

### Representation of vectors

A vector is represented by a line segment or a small letter



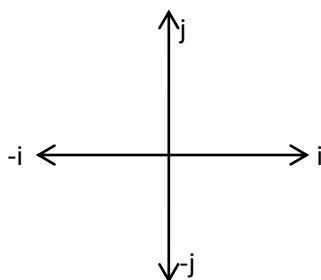
The above vector can be represented as  $\overrightarrow{OA}$ ,  $\underline{a}$  etc. which can be used interchangeably.

### Vectors in two dimensions

These are the representation of magnitude and directions of quantities in x – y plane.

x – direction is represented by  $\mathbf{i}$  or  $-\mathbf{i}$  while

y – direction is represented by  $\mathbf{j}$  or  $-\mathbf{j}$ .

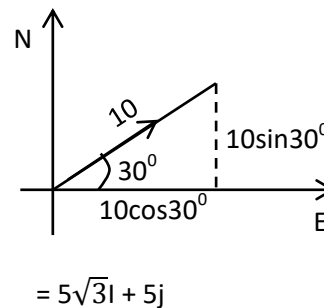


In the figure above the unit vectors in the x – y plane are  $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and are  $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Illustration

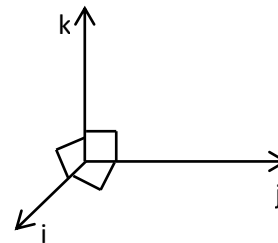
- (i) The velocity of a body moving eastward at  $5\text{kms}^{-1}$  is represented by  $5\mathbf{i}$ .
- (ii) The velocity of a body moving northwards at  $5\text{kms}^{-1}$  is represented by  $5\mathbf{j}$ .

- (iii) The velocity of a body moving westward at  $8\text{kms}^{-1}$  is represented by  $-8\mathbf{i}$
- (iv) The velocity of a body moving southward at  $6\text{kms}^{-1}$  is represented by  $-6\mathbf{j}$ .
- (v) A body moving at  $10\text{ms}^{-1}$  in the direction  $\text{N}60^\circ\text{E}$  is represented as



### Vectors in three dimensions

These represent magnitudes and directions in x, y and z planes and are represented by  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$



Where  $\mathbf{i}$  and  $\mathbf{j}$  represent direction in x – y plane (or east-north directions on ground) while  $\mathbf{k}$  represent direction in z- plane (vertical plane)

In summary the unit vectors in the x, y and z planes are

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

For example, the vector of a body that moves 10m due East, 8m due north and 12m vertically is represented as

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$$10i + 8j + 12k \text{ or } \begin{pmatrix} 10 \\ 8 \\ 12 \end{pmatrix}$$

## Basic concepts

### Position vector

If a point P in a two dimensional geometry has Cartesian coordinates (x, y), the position vector of P is given by  $OP = p = \begin{pmatrix} x \\ y \end{pmatrix}$  or

$$OP = p = xi + yj$$

If P has coordinates (x, y, z) in a three dimensional geometry, its position vector is given by

$$OP = p = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ or } xi + yj + zk$$

### Displacement vector

If points P and Q have coordinates  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  respectively, the displacement vector PQ is denoted by either PQ,  $\underline{PQ}$  or  $\overrightarrow{PQ}$  where

$$PQ = OQ - OP$$

$$= \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$= \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$$

### Example 1

Given the following pair of points, find their respective displacement vectors, P

- (i) P (3, 10) and Q (1, 1)

#### Solution

$$PQ = OQ - OP$$

$$= \begin{pmatrix} 3 \\ 10 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$$

- (ii) P(4, 0, 2) and Q(2, 4, 1)

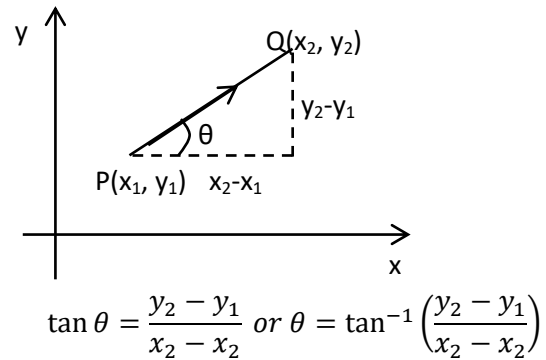
#### Solution

$$PQ = OQ - OP$$

$$= \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$$

### Direction of displacement vector

Direction of displacement vector in 2- D geometry is given by



### Example 2

Find the direction of the displacement PQ with the horizontal, given the following points

- (i) P(2, 4) and Q (6, 8)

#### Solution

$$\tan \theta = \frac{8-4}{6-2} = 1$$

$$\theta = \tan^{-1} 1 = 45^\circ$$

- (ii) P(1, 1) and Q(3, 5)

#### Solution

$$\tan \theta = \frac{5-1}{3-1} = 2$$

$$\theta = \tan^{-1} 2 = 63.4^\circ$$

### Modulus of a vector

Modulus of a vector is the same as magnitude of a vector.

- (i) For  $P = xi + yj$

$$\text{Modulus of } P, = |P| = \sqrt{x^2 + y^2}$$

- (ii) For  $P = xi + yj + zk$

$$\text{Modulus of } P, = |P| = \sqrt{x^2 + y^2 + z^2}$$

### Example 3

Find the modulus of the following vectors

- (i)  $P = 3i + 4j$

#### Solution

$$|P| = \sqrt{3^2 + 4^2} = 5$$

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(ii)  $P = 3i + 4j + 5k$

**Solution**

$$|P| = \sqrt{3^2 + 4^2 + 5^2} = 7.071$$

**Unit vector**

This is a vector whose magnitude or length is equal to one.

**Example 4**

Show that the vector  $P = \frac{3}{5}i + \frac{4}{5}j$  is a unit vector

$$|P| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = 1$$

**The unit vector parallel to a given vector**

The unit vector parallel to a vector P or a vector in direction of P is denoted by  $\hat{P}$  where

$$\hat{P} = \frac{P}{|P|}$$

**Example 5**

Find the unit vectors parallel to each of the following vectors.

(i)  $p = 6i + 8j$

**Solution**

$$\begin{aligned}\hat{p} &= \frac{6i+8j}{|6i+8j|} \\ &= \frac{6i+8j}{\sqrt{6^2+8^2}} \\ &= \frac{6i+8j}{\sqrt{100}} \\ &= \frac{6i+8j}{10} = \frac{3i+4j}{5} \\ &= \frac{3}{5}i + \frac{4}{5}j\end{aligned}$$

(ii)  $q = 3i + 4j + 5k$

**Solution**

$$\begin{aligned}\hat{q} &= \frac{3i+4j+5k}{|3i+4j+5k|} \\ &= \frac{3i+4j+5k}{\sqrt{3^2+4^2+5^2}} \\ &= \frac{3i+4j+5k}{\sqrt{50}} \\ &= \frac{3i+4j+5k}{5\sqrt{2}} \\ &= \frac{3}{5\sqrt{2}}i + \frac{4}{5\sqrt{2}}j + \frac{5}{5\sqrt{2}}k \\ &= \frac{3\sqrt{2}}{5}i + \frac{4\sqrt{2}}{5}j + \frac{5\sqrt{2}}{5}k\end{aligned}$$

Revision exercise 1

1. Find the magnitude of each of the following vectors

(a)  $3i + 4j$  [5]

(b)  $6i + 8j$  [10]

(c)  $\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$   $[5\sqrt{2}]$

(d)  $\begin{pmatrix} 5 \\ 8 \\ 10 \end{pmatrix}$  [13.75]

2. Find the value of q in each of the following

(a)  $|3i + qj| = 5$  [4]

(b)  $|2i + qj + 4k| = 6$  [4]

(c)  $|qi + 4j + 4k| = 2\sqrt{17}$  [6]

3. Find the direction  $\theta$  to the horizontal of each of the following vectors.

(a)  $P = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  [45]

(b)  $q = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$  [53.13]

4. Find a unit vector in the direction of each of the following vectors

(a)  $p = 8i + 6j$   $\left[\frac{4}{5}i + \frac{3}{5}j\right]$

(b)  $q = 5i + 8j$   $\left[\frac{5}{\sqrt{89}}i + \frac{8}{\sqrt{89}}j\right]$

(c)  $r = \begin{pmatrix} 7 \\ -9 \end{pmatrix}$   $\left[\frac{7}{\sqrt{130}}i - \frac{9}{\sqrt{130}}j\right]$

(d)  $3i - 2j + 5k$   $\left[\frac{3}{\sqrt{38}}i - \frac{2}{\sqrt{38}}j + \frac{5}{\sqrt{38}}k\right]$

(e)  $i + 3j + 2k$   $\left[\frac{1}{\sqrt{14}}i + \frac{3}{\sqrt{14}}j + \frac{2}{\sqrt{14}}k\right]$

(f)  $\begin{pmatrix} 3 \\ -12 \\ 4 \end{pmatrix}$   $\begin{pmatrix} \frac{3}{13} \\ -\frac{12}{13} \\ \frac{4}{13} \end{pmatrix}$

5. Find a vector of magnitude  $\sqrt{7}$  in the

direction of the vector  $\begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$ .  $\left[\frac{\sqrt{5}}{5} \begin{pmatrix} 5 \\ -3\sqrt{3} \\ 3 \end{pmatrix}\right]$

6. Find  $\overrightarrow{PR}$  in each case given that

(a)  $\overrightarrow{PQ} = 2i - 4j + 5k$  and  $\overrightarrow{QR} = 3i + 6j - 2k$   
 $[5i + 2j + 3k]$

(b)  $\overrightarrow{PQ} = \begin{pmatrix} 3 \\ 6 \\ -8 \end{pmatrix}$  and  $\overrightarrow{QR} = \begin{pmatrix} 5 \\ -5 \\ 0 \end{pmatrix}$ .  $\left[\begin{pmatrix} 8 \\ 1 \\ -8 \end{pmatrix}\right]$

7. (a) Given that  $\overrightarrow{PQ} = 5i - 7j - 2k$  and

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$$\overrightarrow{PR} = 2i + 3j - 2k, \text{ find } \overrightarrow{QR} \text{? } [-3i + 10j]$$

(b) Given that  $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$  and  $\overrightarrow{PR} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ ,  
find  $\overrightarrow{QR}$ ?  $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

8. Given that  $\overrightarrow{PQ} = ai + 6j + 4k$ ,  
 $\overrightarrow{QR} = 4i + bj + -2k$ , and  $\overrightarrow{PR} = -3i + ck$ , find the  
possible values of the constants a, b, c.  
[a = -7, b = -6, c = 1]

$$3p = 3 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix}$$

(ii)  $3p + 2q$   
Solution

$$3p + 2q = 3 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix} + \begin{pmatrix} -4 \\ 6 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 12 \\ 1 \end{pmatrix}$$

## Vector algebra

### Addition and subtraction of vectors

When adding or subtracting two or more vectors, corresponding elements are added or subtracted.

#### Example 6

1. Given that  $p = 2i + 3k$  and  $q = 3i + 6j + 5k$  find  
(i)  $p + q = (2+3)i + (0+6)j + (3+5)k$   
 $= 5i + 6j + 8k$   
(ii)  $p - q = (2-3)i + (0-6)j + (3-5)k$   
 $= -i - 6j - 2k$
2. Given that  $p = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix}$  and  $q = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ , find  
(i)  $p + q = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ 12 \end{pmatrix}$   
(ii)  $p - q = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

### Multiplication or division of vectors by a scalar

When a vector is multiplied or divided by a scalar the size of the vector changes but the direction remains unchanged

#### Example 7

Given the vectors  $p = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$  and  $q = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$   
find

(i)  $3p$

### Coplanar vectors

The vectors p, q and r are said to be coplanar when there exist scalars say  $\alpha$  and  $\beta$  such that  
 $r = \alpha p + \beta q$

#### Example 8

- (a) Given  $p = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ ,  $q = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$  and  $r = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ , find  
scalars  $\alpha$  and  $\beta$  such that  $r = \alpha p + \beta q$

#### Solution

$$\alpha \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$3\alpha + \beta = 4 \dots\dots\dots(i)$$

$$4\alpha + 2\beta = 0 \dots\dots\dots(ii)$$

Solving equation (i) and (ii) simultaneously,  
we obtain  $\alpha = \frac{4}{5}$  and  $\beta = \frac{-8}{5}$

- (b) Given  $p = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ ,  $q = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$  and  $r = \begin{pmatrix} 0 \\ 7 \\ 5 \end{pmatrix}$ ,

find scalars  $\alpha$ ,  $\beta$  and  $\gamma$  such that  
 $r = \alpha p + \beta q$

#### Solution

$$2\alpha - \beta = 0 \dots\dots\dots(i)$$

$$3\alpha + 2\beta = 7 \dots\dots\dots(ii)$$

$$\alpha + 2\beta = 5 \dots\dots\dots(iii)$$

Solving equation (i), (ii) and (iii)  
simultaneously, we obtain  $\alpha = 1$  and  $\beta = 2$

### Equal vectors

Two or more vectors are said to be equal when they have the same magnitude and direction.

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## Example 9

Given that vectors  $p = \alpha i + 2j + (4 - \beta)k$  and  $q = (2 - \beta)i + 2j + 8k$  are equal find the values of  $\alpha$  and  $\beta$ .

**Solution**

$p$  and  $q$  are equal

$$\alpha i + 2j + (4 - \beta)k = (2 - \beta)i + 2j + 8k$$

$$\Rightarrow 4 - \beta = 8$$

$$\beta = -4$$

$$\alpha = 2 - \beta = 2 - (-4) = 6$$

$$\text{Hence } \alpha = 6 \text{ and } \beta = -4$$

## Parallel vectors

Vectors  $p$  and  $q$  are parallel when one of them is a scalar multiple of another i.e.  $p = kq$  where  $k$  is a constant.

### To show that a given points are collinear

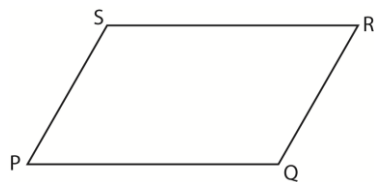
If points  $P$ ,  $Q$  and  $R$  are collinear, then

- (i)  $PQ$  and  $PR$  or  $QR$  are parallel.
- (ii)  $PQ = kQR$  where  $k$  is a constant and there is a common point on LHS and RHS in this case  $Q$ .

## Example 10

- a. PQRS is a parallelogram with coordinates  $P(2,4)$ ,  $Q(-1, 5)$  and  $R(4, 8)$ . Find the coordinate of  $S$ .

**Solution**



$$PS = QR$$

$$OS - OP = OR - OQ$$

$$OS = OR - OQ + OP$$

$$= \begin{pmatrix} 4 \\ 8 \end{pmatrix} - \begin{pmatrix} -1 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$$

$$\text{Hence } S(7,7)$$

- b. The position vectors of  $P$ ,  $Q$  and  $R$  are

$$OP = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, OQ = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, \text{ and}$$

$OR = \begin{pmatrix} 7 \\ 10 \\ -7 \end{pmatrix}$ , prove that  $P$ ,  $Q$  and  $R$  are collinear.

For collinear points  $PQ = kQR$

$$PQ = OQ - OP$$

$$= \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix}$$

$$PR = OR - OP$$

$$= \begin{pmatrix} 7 \\ 10 \\ -7 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \\ -9 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix}$$

Substituting for  $PQ$  and  $QR$

$$\Rightarrow PQ = 2QR$$

Hence  $QR$  is parallel to  $PQ$ , since  $Q$  is common to both sides of the equation, then  $P$ ,  $Q$  and  $R$  are collinear

- c. Given points  $P(2, 1, 0)$ ,  $Q(5, 2, 4)$  and  $R(14, 5, 16)$ ; show that the points are collinear

**Solution**

$$PQ = kQR$$

$$PQ = OQ - OP$$

$$= \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

$$PR = OR - OP$$

$$= \begin{pmatrix} 14 \\ 5 \\ 16 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ 4 \\ 16 \end{pmatrix} = 4 \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

Substituting for  $PQ$  and  $QR$

$$\Rightarrow PQ = 3QR$$

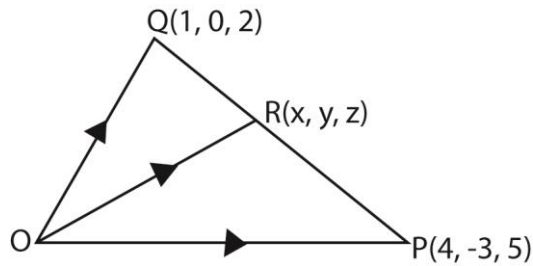
Hence  $QR$  is parallel to  $PQ$ , since  $Q$  is common to both sides of the equation, then  $P$ ,  $Q$  and  $R$  are collinear

- d. Given that  $OP = \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}$  and  $OQ = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$  find

the coordinates of point  $R$  such that  $PR:PQ = 1:2$  and points  $P$ ,  $Q$  and  $R$  are collinear.

**Solution**

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$$\vec{OR} = \frac{1}{2}\vec{OP}$$

$$\vec{OR} - \vec{OP} = \frac{1}{2}(\vec{OQ} - \vec{OP})$$

$$\vec{OR} = \frac{1}{2}(\vec{OQ} - \vec{OP}) + \vec{OP} = \frac{1}{2}(\vec{OQ} + \vec{OP})$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2} \left[ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} \right] = \begin{pmatrix} 2.5 \\ -1.5 \\ 3.5 \end{pmatrix}$$

Hence coordinates of R(2.5, -1.5, 3.5)

## The ratio theorem (section formula)

Consider the division of a line PR by a point Q as shown below:

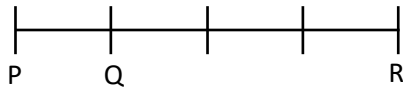


Fig: 1

In the Fig: 1 above, point Q divides line PR internally; PQ:QR = 1:3

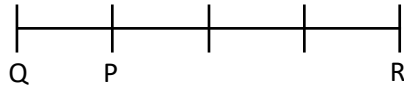


Fig: 2

In Fig: 2, point Q divides PR externally PQ:QR = -1:4 or 1: -4 (depending on the direction considered).

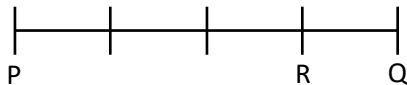


Fig: 3

In fig: 3, Point Q divides PR externally.

PQ :QR = 4: -1 or QP : RQ = 4:1

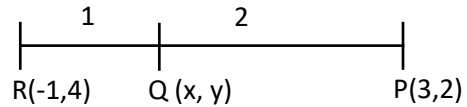
Note PQ and QR are in opposite direction.

## Example 11

(a) P(3, 2) and R(-1, 4) are two points on the line. A point Q divides PR in the ratio

(i) 2:1, (ii) 4: -1, (iii) 1: -4. Find the coordinates of Q in each case

## Solution



Here Q divides the line internally

$$PQ:QR = 2:1$$

$$\frac{PQ}{QR} = \frac{2}{1}$$

$$PQ = 2QR$$

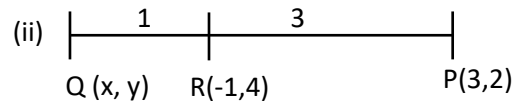
$$OQ - OP = 2(OR - OQ)$$

$$3OQ = 2OR + OP$$

$$= 2 \begin{pmatrix} -1 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} \\ \frac{10}{3} \end{pmatrix}$$

$$\text{Hence } Q\left(\frac{1}{3}, \frac{10}{3}\right)$$



Here Q divides the line externally

$$PQ:QR = 4: -1 \text{ (}\overrightarrow{PQ} \text{ as positive)}$$

$$\frac{PQ}{QR} = \frac{4}{-1}$$

$$-PQ = 4QR$$

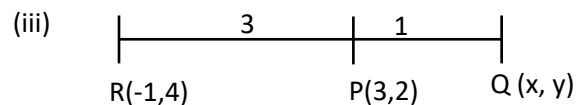
$$-(OQ - OP) = 4(OR - OQ)$$

$$3OQ = 4OR - OP$$

$$= 4 \begin{pmatrix} -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -7 \\ 14 \end{pmatrix}$$

$$OQ = \begin{pmatrix} -7/3 \\ 14/3 \end{pmatrix}$$

$$\text{Hence } Q\left(-\frac{7}{3}, \frac{14}{3}\right)$$



Here Q divides the line externally

$$PQ:QR = 1:-4 \text{ (taking PQ positive)}$$

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$$\frac{PQ}{QR} = \frac{1}{-4}$$

$$-4PQ = PR$$

$$-4(OQ - OP) = (OR - OQ)$$

$$3OQ = 4OP - OR$$

$$= 4 \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 13 \\ 4 \end{pmatrix}$$

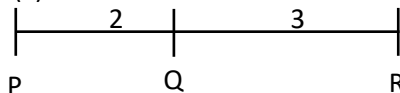
$$OQ = \begin{pmatrix} 13/3 \\ 4/3 \end{pmatrix}$$

$$\text{Hence } Q\left(\frac{13}{3}, \frac{4}{3}\right)$$

- (b) Two points P and Q are such that P(0, 1, 4) and Q(2, 6, 0). A point Q divides a line PQ in ratio 2:3. Find the position vector of Q if it divides PQ

(i) Internally

(ii) Solution



Here Q divides the line internally

$$PQ:QR = 2:3$$

$$\frac{PQ}{QR} = \frac{2}{3}$$

$$3PQ = 2QR$$

$$3(OQ - OP) = 2(OR - OQ)$$

$$3OQ - 3OP = 2OR - 2OQ$$

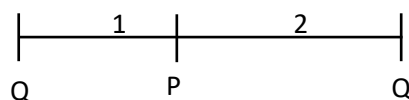
$$5OQ = 2OR + 3OP$$

$$= 2 \begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 15 \\ 12 \end{pmatrix}$$

$$OQ = \begin{pmatrix} 4/5 \\ 3 \\ 12/5 \end{pmatrix}$$

$$\text{Hence } Q\left(\frac{4}{5}, 3, \frac{12}{5}\right)$$

(ii) Externally



Here Q divides the line externally

$$PQ:QR = -2:3$$

$$\frac{PQ}{QR} = \frac{-2}{3}$$

$$3PQ = -2QR$$

$$3(OQ - OP) = -2(OR - OQ)$$

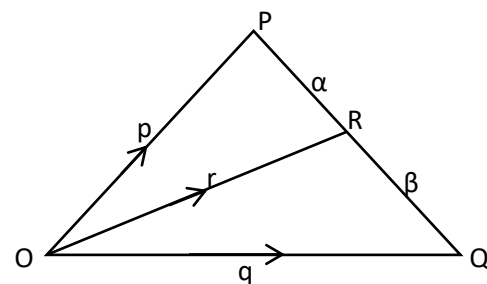
$$3OQ - 3OP = -2OR + 2OQ$$

$$OQ = 3OP + 2OR$$

$$= 3 \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -9 \\ 12 \end{pmatrix}$$

$$\text{Hence } Q(-4, -9, 12)$$

Note: in general, given that Q divides PR in ratio  $\alpha:\beta$



$$OR = OP + PR$$

$$= p + \frac{\alpha}{\alpha+\beta} PQ$$

$$= p + \frac{\alpha}{\alpha+\beta} (-PO + OQ)$$

$$= p + \frac{\alpha}{\alpha+\beta} (-p + q)$$

$$= \frac{p(\alpha+\beta) + \alpha(-p+q)}{\alpha+\beta}$$

$$= \frac{p(\alpha+\beta) - \alpha p + \alpha q}{\alpha+\beta}$$

$$= \frac{\beta}{\alpha+\beta} p + \frac{\alpha}{\alpha+\beta} q$$

## Finding the constants of equality

Suppose that  $r = \lambda a + kb$  and  $r = ma + nb$

$$\Rightarrow \lambda a + kb = ma + nb$$

Equating corresponding unit vectors

$$\lambda = m \text{ and } k = n$$

## Example 12

- (a) Find the position vector of Q if it divides PR in the ratio (i) 1:5 and (ii) 1:-4, given that  $OR = r$ ,  $OP = p$  and  $OQ = q$ .

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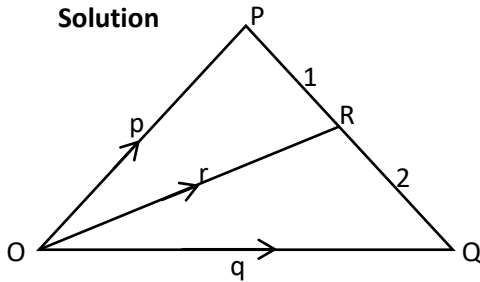
## Solution

$$(i) \quad r = \frac{5}{1+5}p + \frac{1}{1+5}q = \frac{5}{6}p + \frac{1}{6}q$$

$$(ii) \quad r = \frac{-4}{1+(-4)}p + \frac{1}{1+(-4)}q = \frac{4}{3}p - \frac{1}{3}q$$

- (b) OPQ is a triangle with vector  $OP = p$ ,  $OQ = q$ . Express in terms of  $p$  and  $q$  the position vector of  $R$ , where  $R$  divides  $PQ$  in ratio 1:2.

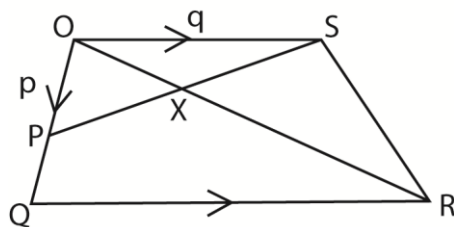
## Solution



$$OR = \frac{2}{1+2}p + \frac{1}{1+2}q = \frac{2}{3}p + \frac{1}{3}q$$

- (c) The diagram below shows a quadrilateral OSRQ,  $OS = q$ ,  $OP = p$  and  $SX = kSP$ .  
(i) Express vectors  $SP$  and  $OX$  in terms of  $p$  and  $q$

## Solution



$$SP = SO + OP$$

$$= -q + p$$

$$= p - q$$

$$OX = OS + kSP$$

$$= q + k(p - q)$$

$$= kp + q(1-k)$$

- (ii)  $OQ = 3p$  and  $QR = 2OS$  and  $OX = \lambda OR$ . Find  $k$  and  $\lambda$ .

## Solution

$$OX = \lambda OR$$

$$= \lambda(OQ + QR)$$

$$= \lambda(3p + 2q)$$

$$= 3\lambda p + 2\lambda q$$

Equating corresponding unit vectors

For  $p$

$$k = 3\lambda \dots\dots\dots(1)$$

For  $Q$

$$(1 - k) = 2\lambda$$

$$k = 1 - 2\lambda \dots\dots\dots(2)$$

Equations (1) and (2)

$$3\lambda = 1 - 2\lambda$$

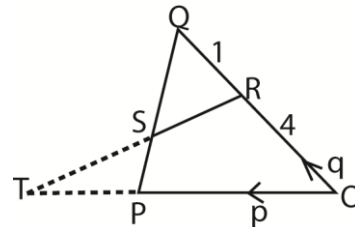
$$5\lambda = 1$$

$$\lambda = \frac{1}{5}$$

From eqn. (1)

$$k = 3 \times \frac{1}{5} = \frac{3}{5}$$

- (d) Given that  $OP = p$  and  $OQ = q$ , point  $R$  is on  $OQ$  such that  $OR : RQ = 4:1$ . Point  $S$  is on  $QP$  such that  $QP : SA = 2:3$  and  $RS$  and  $OP$  are both produced, they to meet at point  $T$ .



Find

- (i)  $OR$  and  $OS$  in terms of  $p$  and  $q$ .

## Solution

$$OR = \frac{4}{5}OQ = \frac{4}{5}q$$

$$OS = OQ + QS$$

$$= q + \frac{2}{5}QP$$

$$= q + \frac{2}{5}(p - q)$$

$$= \frac{1}{5}(2p + 3q)$$

- (ii)  $OT$  in terms of  $p$ .

## Solution

Let  $OT = \alpha OP$  and  $RT = \beta RS$

From  $\triangle OTR$

$$OT = OR + RT$$

$$\alpha OP + OR + \beta RS$$

$$RS = RO + OS$$

$$= -\frac{4}{5}q + \frac{1}{5}(2p + 3q)$$

$$RS = \frac{1}{5}(2p - q)$$

$$\alpha OP = OR + RT$$

$$\alpha p = \frac{4}{5}q + \beta RS$$

$$= \frac{4}{5}q + \frac{\beta}{5}(2p - q)$$



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$$\alpha p = \frac{2\beta}{5}p + \left(\frac{4}{5} - \frac{\beta}{5}\right)q$$

Comparing coefficients

$$\text{For } q: \frac{4}{5} - \frac{\beta}{5} = 0$$

$$\beta = 4$$

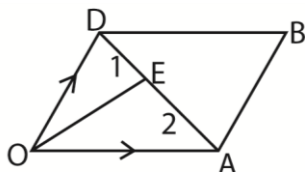
$$\text{For } p, OT = \frac{2\beta}{5} = \frac{2}{5} \times 4 = \frac{8}{5}$$

$$\therefore OT = \frac{8}{5}p$$

- (e) OABCD is a parallelogram, find the position vectors of E and F such that E divides DA in the ratio 1:2 and F divides it externally in ratio 1:2.

$$\text{Given that } OA = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \text{ and } OB = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

**Solution**



- (i)  $OA = DB$

$$\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = OB - OD$$

$$\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} - OD$$

$$OD = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix}$$

$$DE = OD + DE$$

$$= \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} + \frac{1}{3}(DA)$$

$$= \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} + \frac{1}{3}(OA - OD)$$

$$= \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} + \frac{1}{3} \left[ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} -1 \\ -6 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{3} \\ \frac{2}{3} \\ \frac{-4}{3} \end{pmatrix}$$

- (ii) When F divides DA externally, either 1 or 2 must be negative but not both. From the ratio 1: -2.

$$\Rightarrow DF : FA = 1 : -2$$

$$\frac{DF}{FA} = \frac{1}{-2}$$

$$-2DF = FA$$

$$-2(OF - OD) = OA - OF$$

$$OF = 2OD - OA$$

$$= 2 \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

$$OF = \begin{pmatrix} 3 \\ 10 \\ -8 \end{pmatrix}$$

## Exercise 2

1. Points P, Q, R have respective position

$$\text{vectors } \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \\ 4 \end{pmatrix} \text{ and } \begin{pmatrix} 11 \\ 7 \\ 10 \end{pmatrix}.$$

(a) Vectors PQ and QR  $\left[ \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}; \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} \right]$

- (b) Deduce that P, Q and R are collinear and find the ratio PQ:QR. [1:2]

2. The A, B and C have coordinates (1, -5, 6), (3, -2, 10) and (7, 4, 18) respectively. Show that A, B, C are collinear.

3. Show that the points P(5, 4, -5), Q(3, 8, -1) and R(0, 14, 2) are collinear.

4. Given A(2, 13, -5), B(3, x, -3) and C(6, -7, y) are collinear, find the values of x and y [8,3]

5. OABC is a parallelogram with  $\overrightarrow{OA} = a$  and  $\overrightarrow{OC} = c$ . S is the point on AB such that AS:SB = 3:1 and T is a point on BC such that BT:TC = 1:3

- (a) Express each of the following in terms of a and c.

(i)  $\overrightarrow{AC}$   $[c - a]$

(ii)  $\overrightarrow{SB}$

(iii)  $\overrightarrow{BT}$   $\left[-\frac{1}{4}a\right]$

(iv)  $\overrightarrow{ST}$   $[1:4]$

- (b) State the value of the ratio ST:AC

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6. Triangle OAB has  $OA = a$  and  $OB = b$ . C is a point on OA such that  $OC = \frac{2}{3}a$ . D is the midpoint of AB. When CD is produced it meets OB at E, such that  $DE = nCD$  and  $BE = kb$ . Express DE in terms of
- (a)  $n$ ,  $a$  and  $b$   $\left[\frac{5}{6}na - \frac{1}{2}nb\right]$   
 (b)  $k$ ,  $a$  and  $b$   $\left[\frac{1}{2}a + \frac{(2k-1)}{2}b\right]$   
 (c) hence find the values of  $n$  and  $k$ .  
 $\left[n = \frac{3}{5} \text{ and } k = \frac{1}{5}\right]$
7. Three non-collinear points A, B, and C have position vectors  $a$ ,  $b$ , and  $c$  respectively with respect to an origin O. The points M on AC is such that  $AM:MC = 2:1$  and point N on AB is such that  $AN:NB = 2:1$ .
- (a) Find in terms of  $a$ ,  $b$ ,  $c$  the vectors
- (i)  $BM$   $\left[\frac{1}{3}a - b + \frac{2}{3}c\right]$   
 (ii)  $CN$   $\left[\frac{1}{3}a + \frac{2}{3}b - c\right]$
- (b) The lines BM and CN intersect at L. Given that  $BL = rBM$  and  $CL = tCN$ , where  $r$  and  $t$  are scalars; express in terms of  $a$ ,  $b$ ,  $c$ ,  $r$  and  $t$ ;
- (i)  $BL$   $\left[\frac{1}{3}ra - rb + \frac{2rc}{3}\right]$   
 (ii)  $CL$   $\left[\frac{1}{3}ta + \frac{2}{3}tb - tc\right]$
- (c) Hence by using triangle BLC, or otherwise, find  $r$  and  $t$   $\left[r = \frac{3}{5} \text{ and } t = \frac{3}{5}\right]$
8. In the rectangle OABC,  $OA = a$  and  $OC = c$ . R is a point on AB such that  $AR:RB = 1:2$  and S is a point on BC such that  $BS:SC = 3:1$ . AS meets OR at P.
- (i) Find an expression of OP in terms  $a$  and  $c$   $\left[\frac{4}{5}a + \frac{4}{15}c\right]$   
 (ii) Show that  $OP:PR = 4:1$ .  
 (iii) Find the value of the ratio  $AP:PS$   $[4:1]$
9. In a triangle OAB,  $OA = a$  and  $OB = b$ , M is the midpoint of AB and N is a point on OB such that  $ON:NB = 1:4$ . OM meets AN at P.
- (a) Find an expression of OP in terms of  $a$  and  $b$ .  $\left[\frac{1}{6}(a + b)\right]$   
 (b) Find the ratio of  $AP:PN$   $[5:1]$
10. In a trapezium OABC,  $OA = a$ ,  $OC = c$  and  $CB = 3a$ . T is a point on BC such that  $BC:TC = 1:2$ . OT meets AC at P

- (a) Find an expression for OP in terms of  $a$  and  $c$ .  $\left[\frac{2}{3}a + \frac{1}{3}c\right]$   
 (b) Deduce that P is a point of trisection of both AC and OT

11. In a rectangle OABC, M is a midpoint of OA and N is a midpoint of AB. OB meets MC at P and NC at Q. show that  $OP = PQ = QB$ .
12. In the parallel gram OABC, P is a point on OA such that  $OP:PA = 1:2$  and Q is a point on AB such that  $AQ:QB = 1:3$ , OB meets PC at K and QC at M show that  $OK:KM:MB = 7:9:12$

## The scalar or dot products

The dot product of vectors  $p$  and  $q$  inclined at an angle  $\theta$  to each other is defined as  
 $p \cdot q = |p| \cdot |q| \cos \theta$ ,  $0 \leq \theta \leq \pi$

### Properties of scalar product.

- (a)  $i \cdot i = |i| \cdot |i| \cos 0^\circ = 1$  (the angle between  $i$  and  $i$  is zero)  
 (b)  $i \cdot j = |i| \cdot |j| \cos 90^\circ = 0$  ( $i$  and  $j$  are perpendicular)

Thus  $i \cdot i = j \cdot j = k \cdot k = 1$  and  $i \cdot j = i \cdot k = j \cdot k = 0$

Hence the dot product of two vectors is a scalar quantity.

Note that a dot product is used to show that the two vectors are perpendicular.

- (c)  $|p \cdot p| = |p|^2$   
 (d)  $p \cdot (q + r) = p \cdot q + p \cdot r$  (distribution law)  
 (e)  $p \cdot (kq) = (kp) \cdot q = k(p \cdot q)$  where  $k$  is constant.

### Example 13

- (a) Given that  $p = i - 2k$  and  $q = 3i - 3j + k$ ; find
- (i)  $p \cdot q$   
**Solution**  
 $p \cdot q = (i - 2k) \cdot (3i - 3j + k)$   
 $= 3 - 2 = 1$
- (ii) the angle between  $p$  and  $q$  corrected to the nearest degree.

**Solution**

$p \cdot q =$

$$\sqrt{1^2 + (-2)^2} \cdot \sqrt{3^2 + (-3)^2 + 1^2} \cos \theta$$

$$1 = \sqrt{5} \cdot \sqrt{19} \cos \theta$$

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$$\theta = \cos^{-1} \left( \frac{1}{\sqrt{95}} \right) = 84.11^\circ$$

∴ the angle between p and q corrected to the nearest degree is  $84^\circ$ .

- (b) Show that the following vectors are perpendicular.

(i)  $p = 2i + 6j + 4k$  and  $q = (-2i - 2j + 4k)$

**Solution**

$$p \cdot q = (2i + 6j + 4k) \cdot (-2i - 2j + 4k) \\ = -4 - 12 + 16 = 0$$

(hence perpendicular)

(ii)  $a = 3i - 4j + k$  and  $b = 2i + 3j + 6k$

**Solution**

$$a \cdot b = (3i - 4j + k) \cdot (2i + 3j + 6k) \\ = 6 - 12 + 6 = 0$$

(hence perpendicular)

- (c) (i) Find the values of the scalar x if the vectors  $p = 2xi + 7j - k$  and  $q = 3xi + xj + 3k$

**Solution**

$$p \cdot q = |p| \cdot |q| \cos \theta$$

If p and q are perpendicular then  $p \cdot q = 0$

$$p \cdot q = (2xi + 7j - k) \cdot (3xi + xj + 3k)$$

$$6x^2 + 21x - 3 = 0$$

$$(3x - 1)(2x + 3) = 0$$

$$\text{Either } x = \frac{1}{3} \text{ or } x = -\frac{3}{2}$$

- (ii) If the angle between the vector  $p = xi + 2j$  and  $q = 3i + j$  is  $45^\circ$ , find two possible values of x.

**Solution**

$$(xi + 2j) \cdot (3i + j) = (\sqrt{x^2 + 2^2}) \cdot (\sqrt{3^2 + 1^2}) \cos 45^\circ$$

$$3x + 2 = \frac{\sqrt{2}}{2} (\sqrt{x^2 + 4}) \cdot (\sqrt{10})$$

$$\Rightarrow x^2 + 3x - 4 = 0$$

By solving the equation  $x = -4$  or  $x = 1$

- (d) Find the angle between the vectors

$$p = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \text{ and } q = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$$

**Solution**

$$\begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} = \left| \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \right| \cdot \left| \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} \right| \cos \theta$$

$$-4 + 9 = \sqrt{2^2 + 3^2 + 7^2} \cdot \sqrt{(-2)^2 + 3^2 + 0^2} \cos \theta$$

$$5 = (\sqrt{62} \times \sqrt{13}) \cos \theta$$

$$\theta = \cos^{-1} \left( \frac{5}{\sqrt{806}} \right) = 79.86^\circ$$

## The vector (cross) product

Given two non – zero vectors p and q, their vector (cross) product is denoted by  $p \times q$  or  $p \wedge q$  is defined  $p \times q = [p] [q] \sin \theta \cdot \mu$  where  $\theta$  is the angle between p and q and  $\mu$  is the opposite unit vector to the given vectors. And  $0 \leq \theta \leq \pi$

The cross product is synonymous to determinant of a  $3 \times 3$  matrix.

## Properties of vector (cross) product

$$\begin{aligned} \text{(a) } i \times j &= |i| |j| \sin 90^\circ \cdot k = k \\ &= 1 \times 1 \times k = k \\ i \times k &= |i| |k| \sin 90^\circ \cdot j = j \\ &= 1 \times 1 \times j = j \\ j \times k &= |j| |k| \sin 90^\circ \cdot i = i \\ &= 1 \times 1 \times i = i \end{aligned}$$

$$\begin{aligned} \text{(b) } i \times i &= |i| |i| \sin 0^\circ = 0 \\ j \times j &= |j| |j| \sin 0^\circ = 0 \\ k \times k &= |k| |k| \sin 0^\circ = 0 \\ &= 1 \times 1 \times i = i \end{aligned}$$

Hence the cross product of two vectors is a vector quantity.

Note we use the cross product to show that two vectors are parallel.

- (c)  $p \times q = -(q \times p)$   
 (d) for any three vectors p, q, and r  
 $p(q \times r) = p \times q + p \times r$   
 (e) The cross product is perpendicular to either of the two vectors crossed.

Suppose we have vectors  $p = (p_1i + p_2j + p_3k)$  and  $q = (q_1i + q_2j + q_3k)$ , the cross product of p and q is

$$\begin{aligned} p \times q &= \begin{vmatrix} i & j & k \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix} \\ &= \begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix} i - \begin{vmatrix} p_1 & p_3 \\ q_1 & q_3 \end{vmatrix} j + \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} k \\ &= (p_2q_3 - p_3q_2)i - (p_1q_3 - p_3q_1)j + (p_1q_2 - p_2q_1)k \end{aligned}$$

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## Example 14

- (a) Given  $p = 3i - 2j + k$  and  $q = 4i + 3j - 2k$ , find  $p \times q$  and  $q \times p$ .

**Solution**

$$\begin{aligned} p \times q &= \begin{vmatrix} i & -j & k \\ 3 & -2 & 1 \\ 4 & 3 & -2 \end{vmatrix} \\ &= \begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} i - \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} j + \begin{vmatrix} 3 & -2 \\ 4 & 3 \end{vmatrix} k \\ &= (-2 \times -2 - 1 \times 3)i - (3 \times -2 - 1 \times 4)j + (3 \times 3 - -2 \times 4)k \\ &= (4 - 3)i - (-6 - 4)j + (9 + 8)k \\ &= i + 10j + 17k \end{aligned}$$

Or using matrix approach

$$\begin{aligned} p \times q &= \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} -2 \times -2 - 3 \times 1 \\ -(3 \times -2 - 4 \times 1) \\ 3 \times 3 - 4 \times -2 \end{pmatrix} \\ &= \begin{pmatrix} 4 - 3 \\ -(-6 - 4) \\ 9 + 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \\ 17 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} q \times p &= -(p \times q) \\ &= -(i + 10j + 17k) \\ &= -i - 10j - 17k \end{aligned}$$

$$\text{Or } p \times q = - \begin{pmatrix} 1 \\ 10 \\ 17 \end{pmatrix} = \begin{pmatrix} -1 \\ -10 \\ -17 \end{pmatrix}$$

- (b) Show that the cross product of vectors

$p = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$  and  $q = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$  is perpendicular to the vectors.

$$\begin{aligned} p \times q &= \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 3 \times -2 - 0 \times 5 \\ 2 \times -2 - -1 \times 5 \\ 2 \times 0 - -1 \times 3 \end{pmatrix} \\ &= \begin{pmatrix} -6 - 0 \\ -(-4 + 5) \\ 0 + 3 \end{pmatrix} = \begin{pmatrix} -6 \\ -1 \\ 3 \end{pmatrix} \end{aligned}$$

$$\text{Now } \begin{pmatrix} -6 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = -12 - 3 + 15 = 0$$

and

$$\text{and } \begin{pmatrix} -6 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} = 6 - 6 = 0$$

Hence the product is perpendicular

- (c) Find the vector perpendicular to  $p = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$  and  $q = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$

**Solution**

**Approach 1**

$$p \times q = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \\ 10 \end{pmatrix} = 5 \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

Hence the perpendicular vector is  $\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$

Note the  $\begin{pmatrix} -10 \\ 5 \\ 10 \end{pmatrix}$  and  $\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$  are parallel.

**Approach 2**

Let the perpendicular vector be  $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = 0$$

$$3p + 4q + r = 0 \dots\dots\dots(ii)$$

$$\text{And } \begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} = 0$$

$$-p + 2q - r = 0 \dots\dots\dots(ii)$$

$$\text{Eqn. (i)} - 2\text{Eqn (ii)}$$

$$5p + 5r = 0$$

$$p + r = 0$$

$$\text{Let } p = \lambda, \text{ then } r = -\lambda$$

Substituting for  $p$  and  $r$  in equation (i)

$$3\lambda + 4b - \lambda = 0$$

$$b = -\frac{1}{2}\lambda$$

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$$\therefore \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \lambda \\ -\frac{1}{2}\lambda \\ -\lambda \end{pmatrix} = -\frac{1}{2}\lambda \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

Hence the perpendicular vector is  $\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$

- (d) Given points P(1, 1, 2), Q(3, 7, 8) and R(4, 10, 11)

Show that PQ is parallel to QR.

**Solution**

$$PQ = OQ - OP$$

$$= (3i + 7j + 8k) - (i + j + 2k)$$

$$= (2i + 6j + 6k)$$

$$QR = OR - OQ$$

$$= (4i + 10j + 11k) - (3i + 7j + 8k)$$

$$= (i + 3j + 3k)$$

$$PQ \times QR = \begin{vmatrix} i & -j & k \\ 2 & 6 & 6 \\ 1 & 3 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 6 & 6 \\ 3 & 3 \end{vmatrix} i - \begin{vmatrix} 2 & 6 \\ 1 & 3 \end{vmatrix} j + \begin{vmatrix} 2 & 6 \\ 1 & 3 \end{vmatrix} k$$

$$= (6 \times 3 - 6 \times 3)i - (2 \times 3 - 6 \times 1)j + (2 \times 3 - 1 \times 6)k$$

$$= (18 - 18)i - (6 - 6)j + (6 - 6)k$$

$$0i - 0j - 0k = 0$$

- (e) If vectors  $p = 2i - 3j + k$  and  $q = ai - 6j + bk$  are parallel, find the values of  $a$  and  $b$

**Solution**

$$p \times q = \begin{vmatrix} i & -j & k \\ 2 & -3 & 1 \\ a & -6 & b \end{vmatrix}$$

$$= \begin{vmatrix} -3 & 1 \\ -6 & b \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ a & b \end{vmatrix} j + \begin{vmatrix} 2 & -3 \\ a & -6 \end{vmatrix} k$$

$$= (-3b + 6)i - (2b - a)j + (-12 + 3a)k$$

$$\Rightarrow -3b + 6 = 0; b = 2$$

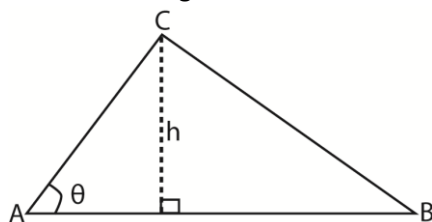
$$\Rightarrow 2b - a = 0; a = 2 \times 2 = 4$$

Hence the value of  $a = 4$  and  $b = 2$

## Application of dot and cross product of vectors

### (1) The triangle

- (i) The area of a triangle



$$\text{Area of triangle ABC} = \frac{1}{2} \times \overline{AB} \times h$$

$$= \frac{1}{2} \times \overline{AB} \times AC \sin \theta$$

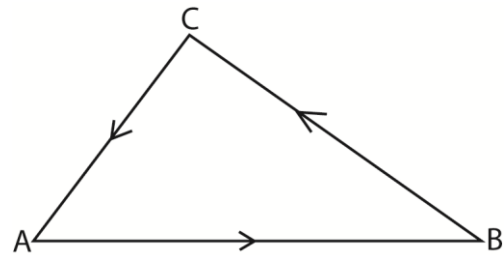
$$\text{But } |AB \times AC| = |AB||AC|\sin \theta$$

$$\text{Hence the area of the triangle} = \frac{1}{2} |AB \times AC|$$

In general, the area of a triangle ABC

$$= \frac{1}{2} |AB \times AC| = \frac{1}{2} |BA \times CC| = \frac{1}{2} |CB \times CA|$$

- (ii) To show that given vertices for a triangle



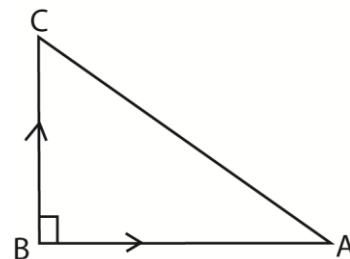
If ABC is a triangle, then it must be a closed polygon.

$$\text{i.e. } \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$$

$$\Rightarrow (OB - OA) + (OC - OB) + (OA - OC) = 0$$

- (iii) To show that a given triangle is right angled triangle.

Suppose  $\angle ABC = 90^\circ$

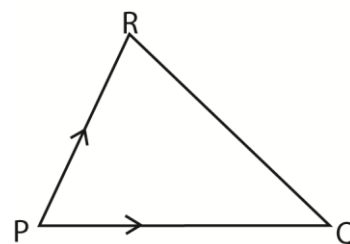


$$\Rightarrow BA \cdot BC = 0 \text{ (dot product of BA and BC)}$$

### Example 15

- (a) The vertices of a triangle PQR have position vectors  $p = i + 2j + k$ ,  $q = i + 3k$  and  $r = -1 + 2j - k$ . Determine the area of the triangle PQR.

**Solution**



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$$PQ = OQ - OP$$

$$= \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$$

$$PR = OR - OP$$

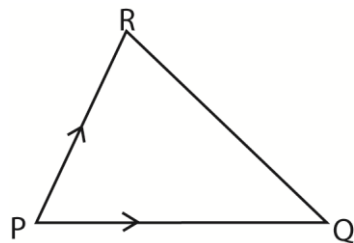
$$= \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$$

$$PQ \times PR = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} \times \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ -4 \end{pmatrix}$$

$$\begin{aligned} \text{Area of PQR} &= \frac{1}{2} |PQ \times PR| \\ &= \frac{1}{2} \sqrt{4^2 + (-4)^2 + (-4)^2} \\ &= 2\sqrt{3} \text{ sq. units.} \end{aligned}$$

- (b) Find the area of a triangle PQR with vertices P(0, 1, 3), Q(1, 5, 7) and R(4, -2, 4)

**Solution**



$$PQ = OQ - OP$$

$$= \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$$

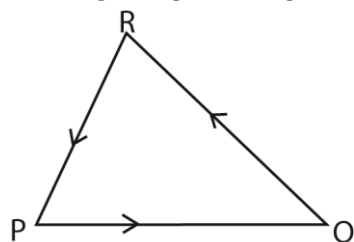
$$PR = OR - OP$$

$$= \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$$

$$PQ \times PR = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} \times \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 16 \\ 15 \\ -19 \end{pmatrix}$$

$$\begin{aligned} \text{Area of PQR} &= \frac{1}{2} |PQ \times PR| \\ &= \frac{1}{2} \sqrt{16^2 + 15^2 + (-19)^2} \\ &= 29 \text{ sq. units.} \end{aligned}$$

- (c) Show that the points P(13, -2, 0), Q(7, 1, -3) and R(2, -1, 5) are vertices of a triangle and it is a right angled triangle. Find its area



$$PQ = OQ - OP$$

$$= \begin{pmatrix} 7 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} 13 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \\ -3 \end{pmatrix}$$

$$QR = OR - OQ$$

$$= \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} - \begin{pmatrix} 7 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \\ 8 \end{pmatrix}$$

$$PR = OR - OP$$

$$= \begin{pmatrix} 13 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 11 \\ -1 \\ -5 \end{pmatrix}$$

$$\begin{aligned} PQ + QR + RP &= \begin{pmatrix} -6 \\ 3 \\ -3 \end{pmatrix} + \begin{pmatrix} -5 \\ -2 \\ 8 \end{pmatrix} + \begin{pmatrix} 11 \\ -1 \\ -5 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

Hence PQR is a triangle

To show that PQR is a right angled triangle

$$PQ \cdot QR = \begin{pmatrix} -6 \\ 3 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ -2 \\ 8 \end{pmatrix} = 30 - 6 - 20 = 0$$

Hence PQR is a right angled triangle

$$\text{Area PQR} = \frac{1}{2} (\overline{QP} \times \overline{QR})$$

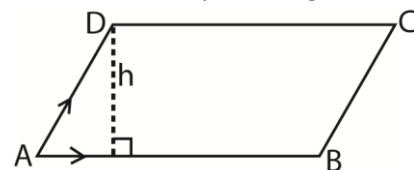
$$\overline{QP} = \sqrt{6^2 + (-3)^2 + 3^2} = \sqrt{54}$$

$$\overline{QR} = \sqrt{(-5)^2 + (-2)^2 + 8^2} = \sqrt{93}$$

$$\text{Area PQR} = \frac{1}{2} (\sqrt{54} \times \sqrt{93}) = 35.4 \text{ sq. units}$$

## (2) The parallelogram

- (i) The area of the parallelogram



Taking  $\angle DAB = \theta$

$$\begin{aligned} \text{Area of parallelogram} &= \text{base} \times \text{height} \\ &= |AB| |AD| \cos \theta \\ &= |AB \times AD| \end{aligned}$$

- (ii) Properties of parallelogram

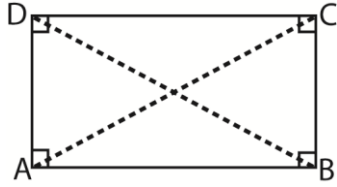
- Two sides are parallel and equal, i.e.,  $AB = DC$  and  $AD = BC$
- The diagonals are not perpendicular and not equal
- Opposite angles are equal i.e.

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$$\angle DAB = \angle DCB \text{ and } \angle ADC = \angle ABC$$

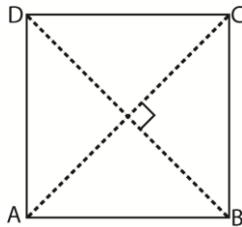
- The sides are not perpendicular i.e.,  
 $\angle DAB = \angle DCB \neq 90^\circ$  and  
 $\angle ADC = \angle ABC \neq 90^\circ$

(iii) Properties of a rectangle



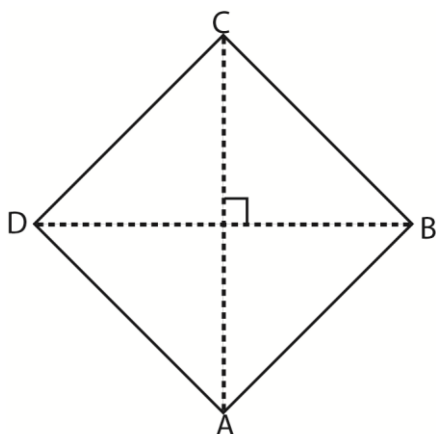
- Two sides are parallel and equal, i.e.,  
 $AB = DC$  and  $AD = BC$
- Diagonals are equal and perpendicular
- All angles are equal to  $90^\circ$  i.e.,  
 $\angle DAB = \angle ABC = \angle BCA = \angle CDA = 90^\circ$ .

(iv) Properties of a square



- All sides are parallel and equal, i.e.,  
 $AB = DC = AD = BC$
- Diagonals are equal and perpendicular
- All angles are equal to  $90^\circ$  i.e.,  
 $\angle DAB = \angle ABC = \angle BCA = \angle CDA = 90^\circ$ .

(v) Properties of a rhombus



- All sides are parallel and equal, i.e.,  
 $AB = DC = AD = BC$
- Diagonals are equal and perpendicular

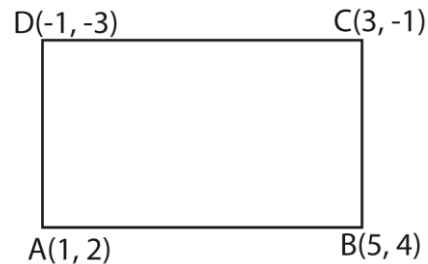
Opposite angles are equal but not equal to  $90^\circ$  i.e.

$$\angle DAB = \angle DCB \neq 90^\circ \text{ and } \angle ADC = \angle ABC \neq 90^\circ$$

## Example 16

- (a) A quadrilateral ABCD has coordinates A(1, 2), B(5, 4), C(3, -1) and D(-1, -3). Show whether ABCD is a rectangle or parallelogram

**Solution**



For both a rectangle and parallelogram,  
 $AB = DC$  and  $AD = BC$

$$AB = OB - OA$$

$$= \begin{pmatrix} 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$DC = OC - OD$$

$$= \begin{pmatrix} 3 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\Rightarrow AB = DC$$

$$AD = OD - OA$$

$$= \begin{pmatrix} -1 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$$

$$DC = OC - OD$$

$$= \begin{pmatrix} 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$$

$$\Rightarrow AD = DC$$

Hence ABCD is either a rectangle or parallelogram.

For a rectangle  $\angle DAB = \angle ABC = 90^\circ$

$$\Rightarrow AD \cdot AB = 0$$

$$\begin{pmatrix} -2 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \end{pmatrix} = -8 - 10 = -18$$

Hence ABCD is not a rectangle.

For a parallelogram,  $\angle DAB = \angle BCD$  and  $\angle ABC = \angle ADC$

$$AD \cdot AB = |AD||AB|\cos\theta$$

$$-18 = \sqrt{4^2 + 2^2}\sqrt{-2^2 + -5^2}\cos\theta$$

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$$-18 = \sqrt{20 \times 29} \cos \theta$$

$$\theta = \cos^{-1} \left( \frac{-18}{\sqrt{580}} \right) = 138.4^\circ$$

$$CB \cdot CD = |CB| |CD| \cos \theta$$

$$\begin{pmatrix} 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ -2 \end{pmatrix} = \sqrt{2^2 + 5^2} \sqrt{-4^2 + -2^2} \cos \theta$$

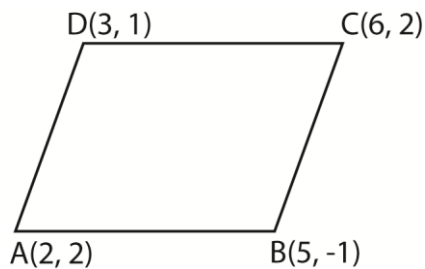
$$-8 - 10 = \sqrt{29 \cdot 20} \cos \theta$$

$$-18 = \sqrt{20 \times 29} \cos \theta$$

$$\theta = \cos^{-1} \left( \frac{-18}{\sqrt{580}} \right) = 138.4^\circ$$

Since  $\angle DAB = \angle BCD$ ; ABCD is a parallelogram.

- (b) ABCD is a quadrilateral with A(2, -2), B(5, -1), C(6, 2) and D(3, 1). Show whether the quadrilateral is a square or a rhombus.



For square or rhombus

$$AB = BC = CD = AD$$

$$AB = OB - OA$$

$$= \begin{pmatrix} 5 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$|AB| = \sqrt{3^2 + 1^2} = \sqrt{10}$$

$$BC = OC - OB$$

$$= \begin{pmatrix} 6 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$|BC| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$CD = OD - OC$$

$$= \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$$

$$|AB| = \sqrt{-3^2 + -1^2} = \sqrt{10}$$

$$AD = OD - OA$$

$$= \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$|AD| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

Since AD is parallel to BC and have the same magnitude; ABCD is either a square or rhombus.

For both a square and rhombus, the diagonals are perpendicular

$$\Rightarrow AC \cdot BD = 0$$

$$AC = OC - OA$$

$$= \begin{pmatrix} 6 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$BD = OD - OB$$

$$= \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$AC \cdot BD = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 2 \end{pmatrix} = -8 + 8 = 0$$

Hence ABCD is either a square or rhombus

For a square  $\angle DAB = \angle ABC = 90^\circ$

$$\Rightarrow AB \cdot AD = 0$$

$$\text{Now } \begin{pmatrix} 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 3 + 3 = 6$$

Hence ABCD is not a square.

For a rhombus  $\angle ADC = \angle BCD \neq 90^\circ$  and  $\angle ABC = \angle ADC \neq 90^\circ$

$$AD \cdot AB = |AD| |AB| \cos \theta$$

$$6 = \sqrt{3^2 + 1^2} \sqrt{1^2 + 3^2} \cos \theta$$

$$6 = \sqrt{10} \sqrt{10} \cos \theta$$

$$\theta = \cos^{-1} \frac{6}{10} = 53.13^\circ$$

$$CB \cdot CD = |AD| |AB| \cos \theta$$

$$\begin{pmatrix} -1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -1 \end{pmatrix} = \sqrt{-1^2 + -3^2} \sqrt{-3^2 + -1^2} \cos \theta$$

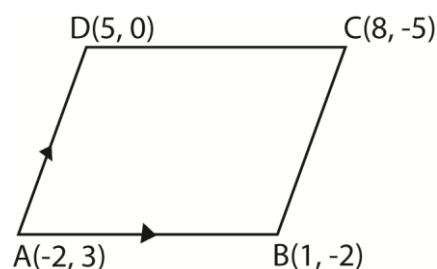
$$6 = \sqrt{10} \sqrt{10} \cos \theta$$

$$\theta = \cos^{-1} \frac{6}{10} = 53.13^\circ$$

Since  $\angle DAB = \angle BCD \neq 90^\circ$ ; ABCD is a rhombus.

- (c) A parallelogram ABCD has vertices A(-2, 3), B(1, -2), C(8, -5) and D(5, 0). Find the area of the parallelogram

Solution





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$$\text{Area of ABCD} = |AB \times AD|$$

$$AB = OB - OA$$

$$= \begin{pmatrix} 1 \\ -2 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

$$AD = OD - OA$$

$$= \begin{pmatrix} 5 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$$

$$AB \times AD = \begin{pmatrix} 3 \\ -5 \\ 0 \end{pmatrix} \times \begin{pmatrix} 7 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 26 \end{pmatrix}$$

$$|AB \times AD| = \sqrt{26^2} = 26$$

The area of ABCD = 26 s. units.

## Exercise 3

- Given that  $p = 4i + 5j$ ,  $q = \alpha i - 8j$  and  $r = i + \beta j$ .
  - Find the value of constants  $\alpha$  given that  $p$  and  $q$  are perpendicular. [10]
  - Find the value of constant  $\beta$  given that  $p$  and  $r$  are parallel  $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$
- Given that  $p = 6i - j$ ,  $q = \alpha i + 2j$  and  $r = 2i + \beta j$ .
  - Find the value of a constant  $\alpha$  given that  $p$  and  $q$  are parallel [-12]
  - Find the value of constant  $\beta$  given that  $p$  and  $r$  are perpendicular. [12]
- Given that  $\begin{pmatrix} \alpha \\ 2 + \alpha \\ 3 \end{pmatrix}$  and  $\begin{pmatrix} -1 \\ 3 \\ 4 - \alpha \end{pmatrix}$  are perpendicular vectors, find the value of  $\alpha$ . [18]
- Find the possible values of the constant  $\alpha$ , given the vectors  $\alpha i + 8j + (3\alpha + 1)k$  and  $(\alpha + 1)i + (\alpha - 1)j - 2k$  are perpendicular. [2 or -5]
- Given that the vectors  $\begin{pmatrix} t \\ 4 \\ 2t + 1 \end{pmatrix}$  and  $\begin{pmatrix} t + 2 \\ 1 - t \\ -1 \end{pmatrix}$  are perpendicular, find the possible values of  $t$  [5 or -1]
- Three point P, Q, and R have position vectors  $p = 7i + 10j$ ;  $q = 3i + 12j$  and  $r = -i + 4j$  respectively.
  - Write down vectors PQ and RQ and show that they are perpendicular.

$$[-4i + 2j; 4i, 8j; PQ.RQ=0]$$

- Using a scalar product or otherwise find and RQ [26.6°]
  - Find the position vectors of S, the mid-point of PR. [-4i, -3j]
- The points A, B, C have position vectors  $A = 2i + j - k$ ,  $b = 3i + 4j - 2k$ ,  $c = 5i - j + 2k$  respectively, relatively to fixed point O.
    - Evaluate the scalar product  $(a - b) \cdot (c - b)$ . Hence calculate the size of angle ABC. [17. 40.2°]
    - Given that ABCD is parallelogram
      - Determine the position vector of D [-4i, -2j + 3k]
      - Calculate the area of ABCD, [14.4]
    - The point E lies on BA produced so that  $\overrightarrow{BE} = 3\overrightarrow{BA}$ . Write down the position vector of E. the line CE cuts the line AD at X; find the position vector of X.  $[-2i + k; \frac{10}{3}i, \frac{7}{3}j, \frac{5}{3}k]$
  - The point A and B have position vectors  $i + 2j + 2k$  and  $4i + 3k$  respectively, relative an origin O.
    - Find the length of OA and OB. [3, 5]
    - Find the scalar product of OA and OB. Hence find angle OAB. [48.2°]
    - Find the area of the triangle AOB, giving your answer correct to 2 decimal places. [5.59]
    - The point C divides AB in ratio  $\alpha : 1 - \alpha$ 
      - Find an expression for OC.  $[(1 + \alpha)i + (2 + \alpha)j + 2(1 - \alpha)k]$
      - Show that  $OC^2 = 14\alpha^2 + 2\alpha + 9$
      - Find the position vectors of the two point on AB whose distance from O is  $\sqrt{21}$ .  $[-2i + j + 4k; \frac{25}{7}i + \frac{29}{7}j + \frac{2}{2}k]$
      - Show that the perpendicular distance of O from AB is approximately 2.99

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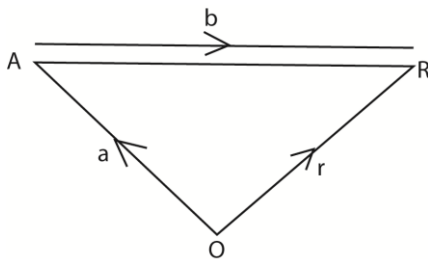
## Lines

### Equation of a line

An equation of line can be expressed in any of the three forms

- (i) Vectors
- (ii) Parametric form
- (iii) Cartesian form

### Finding equations of a line given one point on the line and the vector parallel to the line (direction vector)



In the figure above A is the point on the line with position vector  $OA = a$  and  $b$  is the vector parallel to the line

Taking  $R(x, y, z)$  as general point on the line AR is parallel to  $b$

$$\begin{aligned} \Rightarrow AR &= \lambda b \text{ where } \lambda \text{ is a constant} \\ OR - OA &= \lambda b \\ OR &= OA + \lambda b \\ \Rightarrow r &= a + \lambda b \end{aligned}$$

This is the vector equation of the line

$$\text{Suppose that } a = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \text{ and } b = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

Substituting for  $r$ ,  $a$  and  $b$  into the equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$x = x_0 + \lambda x_1$$

$$y = y_0 + \lambda y_1$$

$$z = z_0 + \lambda z_1$$

These are parametric equation of a line

By making  $\lambda$  the subject from the parametric equations,

$$\frac{x-x_0}{x_1} = \frac{y-y_0}{y_1} = \frac{z-z_0}{z_1}$$

This is the Cartesian equation of the line

Different values of  $\lambda$  define positions of R.

If  $\lambda < 0$ , the point R is on the left point A

If  $\lambda = 0$ , the point in question is A

If  $\lambda > 0$ , the point R is on the right of point A

### Example 17

- (a) Find the vector, parametric and Cartesian equation of the line passing through the point  $A(1,2,3)$  and is parallel to the vector  $2i - j + k$ .

#### Solution

The position vector of A is  $a = i + 2j + 3k$  and the parallel vector  $b = 2i - j + k$ .

Using  $r = a + \lambda b$

$$\Rightarrow r = i + 2j + 3k + \lambda(2i - j + k)$$

Or

$$r = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \text{ this a vector equation}$$

Substituting for  $r = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$x = 1 + 2\lambda$$

$$y = 2 - \lambda$$

$$z = 3 + \lambda \text{ this parametric equation}$$

From these equation

$$\begin{pmatrix} x-1 \\ y-2 \\ z-3 \end{pmatrix} = \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \frac{x-1}{2} = \frac{y-2}{-1} = \frac{z-3}{1} \text{ is the Cartesian equation}$$

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Note

Given the Cartesian equation of a line, the point through which the line passes (1, 2, 3)

and the vector parallel to this line  $\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$  can be deduced easily, for example the line

$$\frac{x-2}{-3} = \frac{y-1}{2} = \frac{z+4}{1} \text{ passes through}$$

$$(-2, 1, 4) \text{ and is parallel to } \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

Hence from the general Cartesian equation of the line,  $\frac{x-x_0}{x_1} = \frac{y-y_0}{y_1} = \frac{z-z_0}{z_1}$ , the parallel vector is the values of the denominator and for the vector equation, it is the coefficient of the constant.

- (b) Find the Cartesian equation of the line that passes the point

(i) P(2, 0, -1) and is parallel to  $\begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix}$

(ii) M(3, 2, 1) and is parallel to  $5i + 7k$ .

Solution

Using general Cartesian equation

$$\frac{x-x_0}{x_1} = \frac{y-y_0}{y_1} = \frac{z-z_0}{z_1}$$

(i)  $\frac{x-2}{-1} = \frac{y-0}{4} = \frac{z+1}{-2}$

$$\frac{x-2}{-1} = \frac{y}{4} = \frac{z+1}{-2}$$

(ii)  $\frac{x-3}{5} = \frac{y-2}{0} = \frac{z-1}{7}$

$$\frac{x-3}{5} = \frac{z-1}{7}; y = 2$$

- (c) Find the vector equation of the straight line that passes through point (2, 3) and perpendicular to the line  $r = 3i + 2j + \lambda(i - 2j)$ .

**Solution**

Using  $r = a + \lambda b$ , substituting for  $a = 2i + 3j$  and  $b = a_1i + b_1j$ , we have

$$r = 2i + 3j + \lambda(a_1i + b_1j)$$

since the lines are perpendicular, this means that their parallel vectors are also perpendicular

$$\Rightarrow (a_1i + b_1j)(i - 2j) = 0$$

$$a_1 - 2b_1 = 0$$

$$a_1 = 2b_1$$

$$\Rightarrow b = 2b_1i + b_1j$$

$$b = b_1(2i + j)$$

Hence the required line will be parallel to any vector of the form  $\lambda(2i + j)$ . So taking  $2i + j$  as one of such vector, the required equation is  $r = 2i + 3j + \lambda(2i + j)$

- (a) Defining a line given two points lying on the line

**Finding equations of a line given two points lying on the line**

Suppose the line passes through point A and B whose position vectors are a and b.

For a general point R(x, y, z)

AR is parallel to AB

$$AR = \lambda AB \text{ for any value of } \lambda.$$

$$OR - OA = \lambda(OB - OA)$$

$$OR = OA + \lambda(OB - OA)$$

$$r = a + \lambda(b - a)$$

**Example 18**

- (a) Find the equation of the line passing through the points A(3, 0, -2) and B(4, -2, 1)

**Solution**

$$AB = OB - OA = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

Using  $r = a + \lambda AB$

$$r = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \text{ the vector equation}$$

Substituting for r

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

$$x = 3 + \lambda$$

$$y = -2\lambda$$

# ST. THOMAS AQUINAS

$$z = -2 + 3\lambda \text{ parametric equation}$$

Making  $\lambda$  the subject

$$\frac{x-3}{1} = \frac{y}{-2} = \frac{z+2}{3}: \text{Cartesian equation}$$

- (b) Find the equation of the line passing through points A and B whose position vectors are  $i + 2j - 5k$  and  $2i - 5j + 8k$

**Solution**

$$AB = OB - OA = \begin{pmatrix} 2 \\ -5 \\ 8 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 \\ -7 \\ 13 \end{pmatrix}$$

Using  $r = a + \lambda AB$

$$r = \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -7 \\ 13 \end{pmatrix}: \text{the vector equation}$$

Substituting for  $r$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -7 \\ 13 \end{pmatrix}$$

$$x = 1 + \lambda$$

$$y = 2 - 7\lambda \quad \text{Parametric equation}$$

$$z = -5 + 13\lambda$$

Making  $\lambda$  the subject

$$\frac{x-1}{1} = \frac{y-2}{-7} = \frac{z+5}{13} \text{ Cartesian equation}$$

Note: in a situation where three points lying on a line are given such as ABC, the vector equation of the line is given by

$$r = a + \lambda(BC)$$

- (c) Find the equation of the line passing through points A(1, 2, 5), B(2, 1, 0) and C(5, 3, 2).

**Solution**

$$BC = OC - OB = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

$$r = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}: \text{vector equation}$$

Substituting for  $r$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}:$$

$$x = 1 + 3\lambda$$

$$y = 2 + 2\lambda \quad \text{Parametric equation}$$

$$z = 5 + 2\lambda$$

Making  $\lambda$  the subject

$$\frac{x-1}{3} = \frac{y-2}{2} = \frac{z-5}{2} \text{ Cartesian equation}$$

**To show that a given point lies on the line**

Suppose that a point (a, b, c) lies on

$\frac{x-x_0}{x_1} = \frac{y-y_0}{y_1} = \frac{z-z_0}{z_1}$ . The when we substitute this point into the equation, we obtain a constant for the values of a, b, c.

**Example 19**

- (a) Show that a point with coordinates (4, -1, 12) lies on the line  $r = 2i + 3j + 4k + \lambda(i - 2j + 4k)$

**Solution**

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

Substituting (x, y, z) = (4, -1, 12)

$$\begin{pmatrix} 4 \\ -1 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 4-2 \\ -1-3 \\ 12-4 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \Rightarrow \begin{pmatrix} \lambda \\ \lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

Since the value of  $\lambda$  is constant, the point lies on the line.

- (b) Show that a point with position vector  $i - 9j + k$  lies on the line  $r = 3i + 3k - k + \lambda(i + 6j - k)$

**Solution**

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 6 \\ -1 \end{pmatrix}$$

Substituting (x, y, z) = (1, -9, 1)

$$\begin{pmatrix} 1 \\ -9 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 6 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1-3 \\ -9-3 \\ 1+1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 6 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} \lambda \\ \lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}$$

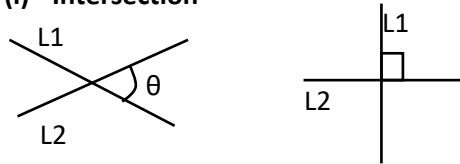
Since the value of  $\lambda$  is constant, the point lies on the line.

# ST. THOMAS AQUINAS

## Relationship between two lines

There are three types of relationship between lines

### (i) Intersection

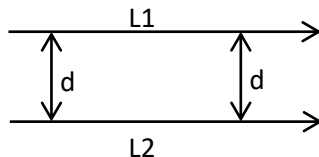


(a) intersecting at  $\theta$  (b) intersecting at  $90^\circ$

If two lines  $r_1$  and  $r_2$  meet, then at the point of intersection,  $r_1 = r_2$

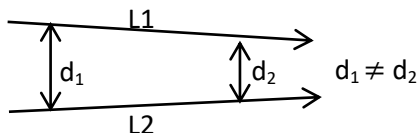
### (ii) Parallel lines

These are non-intersecting lines that are equidistant from one another.



### (iii) Skew lines

These are non-intersecting lines that are not equidistant from one another (not parallel)



## Example 20

### (a) Show that lines

$$r_1 = \begin{pmatrix} -2 \\ 8 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \text{ and } r_2 = \begin{pmatrix} 8 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} \text{ intersect and find the point of intersection.}$$

Solution

At the point of intersection  $r_1 = r_2$

$$\begin{pmatrix} -2 \\ 8 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$$

$$-2 + \mu = 8 - 4\lambda \text{ i.e. } \mu + 4\lambda = 10 \dots\dots\dots(i)$$

$$8 + 3\mu = -1 + \lambda \text{ i.e. } 3\mu - \lambda = -9 \dots\dots\dots(ii)$$

$$-1 - 2\mu = 3 \text{ i.e. } \mu = -2$$

Substituting for  $\mu$  in equation (i)

$$\lambda = 3$$

Checking with eqn. (ii)  $3(-2) - 3 = -9$  i.e. there is consistency.

Using  $\mu = -2$ , the point of intersection is

$$\begin{pmatrix} -2 \\ 8 \\ -1 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 5 \end{pmatrix}$$

Or point of intersection is  $(-4, 2, 5)$

### (b) Show that lines

$$\frac{x}{1} = \frac{y+2}{2} = \frac{z-5}{-1} \text{ and } \frac{x-1}{-1} = \frac{y+3}{-3} = \frac{z-4}{1}$$

intersect and find the point of intersection.

**Solution**

$$\frac{x}{1} = \frac{y+2}{2} \text{ i.e. } 2x - y = 2 \dots\dots\dots(i)$$

Also

$$\frac{x-1}{-1} = \frac{y+3}{-3} \text{ i.e. } -3x + y = -6 \dots\dots\dots(ii)$$

$$\text{Eqn. (i) + Eqn. (ii) } -x = -4 \text{ or } x = 4$$

Substituting for  $x$  in equation (i)

$$2 \times 4 - y = 2; y = 6$$

Finding the value of  $z$

$$\frac{x}{1} = \frac{z-5}{-1} \Rightarrow -x = z - 5$$

Substituting for  $x$

$$-4 = z - 5; z = 1$$

$$\text{Checking for consistency using } \frac{y+3}{-3} = \frac{z-4}{1}$$

$$\Rightarrow \frac{6+3}{-3} = \frac{1-4}{1} = -3 \text{ (consistent)}$$

Given that  $r_1 = a_1 + \lambda_1 b_1$  and  $r_2 = a_2 + \lambda_2 b_2$  intersect; then the shortest distance between the lines is zero.

$$\text{Or } (a_1 - a_2) \cdot (b_1 \times b_2) = 0$$

### (c) Show that the following lines are perpendicular

$$(i) \frac{x-1}{2} = \frac{y}{1} = \frac{z-4}{4} \text{ and } \frac{x}{3} = \frac{y+2}{-2} = \frac{z}{-1}$$

$$\text{The first line is parallel to } b_1 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$$

$$\text{The 2nd line is parallel to } b_2 = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$$

$$b_1 \cdot b_2 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} = 6 - 2 - 4 = 0$$

$\therefore$  the lines are perpendicular because  $b_1 \cdot b_2 = 0$

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(ii)  $r_1 = i - j + \lambda_1(i + 2j - k)$  and  
 $r_2 = 2i + j - k + \lambda_2(-2i - 4j + 2k)$

Solution

The 1<sup>st</sup> line is parallel to  $b_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

The 2<sup>nd</sup> line is parallel to  $b_2 = \begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

Substituting for  $b_1$  into  $b_2$ ;  $b_2 = -2b_1$

Hence the two lines are parallel.

**The shortest (perpendicular) distance from a given point to the line.**

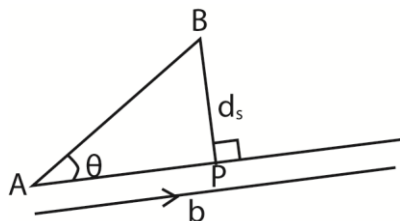
There are several methods of calculating this; here two methods are discussed in examples below

## Example 21

- (a) Find the perpendicular distance from point,  $A(1, 1, 3)$  to the line  $\frac{x+4}{2} = \frac{y+1}{3} = \frac{z-1}{3}$ .

**Solution**

### Approach 1



In the figure above

A is a point on the line

P is a point at which the perpendicular from B meets the line

B is the vector parallel to the line

$\theta$  is the angle between AB and the line (or the parallel vector)

Required is the distance  $d_s = |BP|$

From the figure,  $|BP| = |AB| \sin \theta$  ..... (i)

But by definition:  $|AB \times b| = |AB||b| \sin \theta$

$\Rightarrow |AB| \sin \theta = \frac{|AB \times b|}{|b|}$  .....(ii)

Combining equations (i) and (ii)

$$|BP| = \frac{|AB \times b|}{|b|}$$

The shortest distance,  $d_s = \frac{|AB \times b|}{|b|}$

From the given line in question A(-4, -1, 1)

and  $b = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$

$$AB = OB - OA = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -4 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}$$

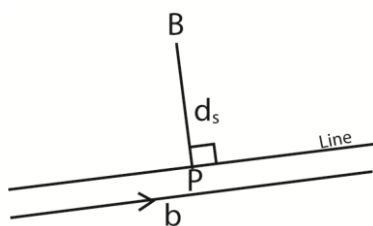
$$AB \times b = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -11 \\ 11 \end{pmatrix}$$

$$|AB \times b| = \sqrt{[(-11)^2 + 11^2]} = 11\sqrt{2} \text{ and}$$

$$|b| = \sqrt{2^2 + 3^2 + 3^2} = \sqrt{22}$$

$$d_s = \frac{|AB \times b|}{|b|} = \frac{11\sqrt{2}}{\sqrt{22}} = \sqrt{11} \text{ units}$$

### Approach 2



P is a point at which the perpendicular line meets the line

The vector equation of the line is

$$r = \begin{pmatrix} -4 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

Let  $\lambda = \lambda_1$  at Q i.e.  $p = \begin{pmatrix} -4 \\ -1 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$

$$= \begin{pmatrix} -4 + 2\lambda_1 \\ -1 + 3\lambda_1 \\ 1 + 3\lambda_1 \end{pmatrix}$$

Since PB is perpendicular to the line, it is also perpendicular to b.

$$BP \cdot \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} = 0 \text{ i.e. } (p - B) \cdot \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} = 0$$

$$\left\{ \begin{pmatrix} -4 + 2\lambda_1 \\ -1 + 3\lambda_1 \\ 1 + 3\lambda_1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \cdot \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} = 0$$

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$$-10 + 4\lambda_1 - 6 + 9\lambda_1 - 6 + 9\lambda_1 = 0$$

$$\lambda_1 = 1$$

$$BP = \begin{pmatrix} -5 + 2 \\ -2 + 3 \\ -2 + 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

$$|BP| = d_s = \sqrt{-3^2 + 1^2 + 1^2} = \sqrt{11} \text{ units}$$

- (b) Find the perpendicular distance from the point A(4, -3, 10) to the line L with vector

$$\text{equation } r = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

**Solution**

**Method 1**

$$\text{The shortest distance } d_s = \frac{|AB \times b|}{|b|}$$

From the given line in question, B(1, 2, 3)

$$\text{and } b = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$BA = OA - OB$$

$$= \begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 7 \end{pmatrix}$$

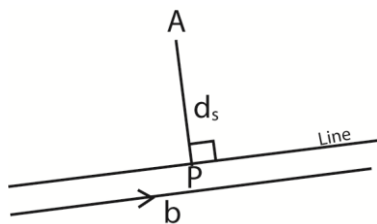
$$BA \times b = \begin{pmatrix} 3 \\ -5 \\ 7 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 15 \\ 12 \end{pmatrix}$$

$$|BA| = \sqrt{-3^2 + 15^2 + 12^2} = \sqrt{378}$$

$$|b| = \sqrt{3^2 + -1^2 + 2^2} = \sqrt{14}$$

$$d_s = \frac{|AB \times b|}{|b|} = \frac{\sqrt{378}}{\sqrt{14}} = \sqrt{27} = 3\sqrt{3}$$

**Method 2**



$$r = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$\text{Let } \lambda = \lambda_1 \text{ at P i.e. } p = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \\ = \begin{pmatrix} 1 + 3\lambda_1 \\ 2 - \lambda_1 \\ 3 + 2\lambda_1 \end{pmatrix}$$

Since AP is perpendicular to the line, it is also perpendicular to b

$$AP \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 0$$

$$\left\{ \begin{pmatrix} 1 + 3\lambda_1 \\ 2 - \lambda_1 \\ 3 + 2\lambda_1 \end{pmatrix} - \begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix} \right\} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 0$$

$$\begin{pmatrix} -3 + 3\lambda_1 \\ 5 - \lambda_1 \\ -7 + 2\lambda_1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 0$$

$$-9 + 9\lambda_1 - 5 + \lambda_1 - 14 + 4\lambda_1 = 0$$

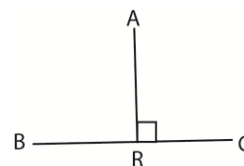
$$\lambda_1 = 2$$

$$AP = \begin{pmatrix} -3 + 6 \\ 5 - 2 \\ -7 + 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$$

$$|AP| = \sqrt{3^2 + 3^2 + -3^2} = \sqrt{27} = 3\sqrt{3} \text{ units}$$

- (c) The points A, B, C have position vectors  $a = 3i - j + 4k$ ,  $b = j - 4k$ ,  $c = 6i + 4j + 5k$  respectively. Find the position vector of the point R on BC such that AR is perpendicular to BC. Hence find the perpendicular distance of A from the line BC.

**Solution**



$$BR = \lambda BC$$

$$OR - OB = \lambda(OC - OB)$$

$$OR = OB + \lambda(OC - OB)$$

$$r = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + \lambda \left[ \begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ 3 \\ 9 \end{pmatrix}$$

$$R = \begin{pmatrix} 6\lambda \\ 1 + 3\lambda \\ -4 + 9\lambda \end{pmatrix}$$

$$\text{Perpendicular distance} = |AR|$$

$$\Rightarrow AR \cdot BC = 0$$

$$AR = OR - OA$$

$$= \begin{pmatrix} 6\lambda \\ 1 + 3\lambda \\ -4 + 9\lambda \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 + 6\lambda \\ 2 + 3\lambda \\ -8 + 9\lambda \end{pmatrix}$$

$$\text{Hence } \begin{pmatrix} -3 + 6\lambda \\ 2 + 3\lambda \\ -8 + 9\lambda \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 3 \\ 9 \end{pmatrix}$$

$$-18 + 36\lambda + 6 + 9\lambda - 72 + 81\lambda = 0$$

$$126\lambda = 84$$

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$$\lambda = \frac{84}{126} = \frac{2}{3}$$

substituting for  $\lambda$  into  $AR = \begin{pmatrix} -3 + 6\lambda \\ 2 + 3\lambda \\ -8 + 9\lambda \end{pmatrix}$  we

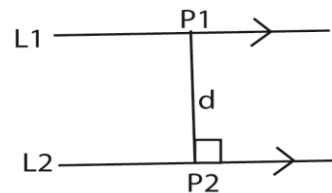
$$\text{get } AR = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$$

$$|AR| = \sqrt{1^2 + 4^2 + (-2)^2} = \sqrt{21} \text{ units}$$

Distance between two lines

Note that:

- (i) If  $r_1$  and  $r_2$  are parallel, the distance between the two lines is the length of any line segment  $P_1P_2$  with  $P_1$  on  $r_1$  and  $P_2$  on  $r_2$  perpendicular to both lines



The perpendicular distance,  $d = |P_1P_2|$

But  $P_1P_2 = OP_2 - OP_1$

- $\Rightarrow P_1P_2 \cdot b = 0$ , where  $b$  is a parallel vector to the lines.

This enables us to find the value of either  $\mu_1 - \mu_2$  or  $\mu_2 - \mu_1$  which is substituted to find  $d = |P_1P_2|$

- (ii) If  $r_1$  and  $r_2$  are not parallel (skew lines), there are unique points  $P_1$  on  $r_1$  and  $P_2$  on  $r_2$  such that the length of the segment  $P_1P_2$  is the shortest possible distance. The length  $P_1P_2$  is the distance between the two lines which is the common perpendicular to both lines  $r_1$  and  $r_2$ .

The distance,  $d$ , between skew lines

$r_1 = a_1 + \mu_1 b_1$  and  $r_2 = a_2 + \mu_2 b_2$  is normally taken to be the shortest distance

$d = [(a_1 - a_2) \cdot \hat{n}]$ , where  $\hat{n} = \frac{n}{|n|}$  and

$n = b_1 \times b_2$ .

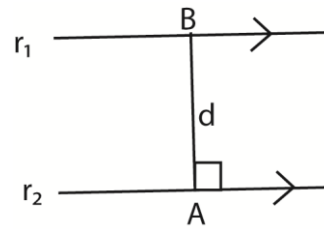
## Example 22

Determine the shortest distance between the following pairs of lines

(a)  $r_1 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$  and

$$r_2 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

**Solution**



$$r_1 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

So we have  $A(2+\lambda, -\lambda, 3+2\lambda)$

$$r_2 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Thus  $B(1+\mu, -1-\mu, 4+2\mu)$

$AB = OB - OA$

$$= \begin{pmatrix} 1+\mu \\ -1-\mu \\ 4+2\mu \end{pmatrix} - \begin{pmatrix} 2+\lambda \\ -\lambda \\ 3+2\lambda \end{pmatrix}$$

$$= \begin{pmatrix} -1+\mu-\lambda \\ -1-\mu+\lambda \\ 1+2\mu-2\lambda \end{pmatrix}$$

Now  $AB \cdot b = 0$

$$\begin{pmatrix} -1+\mu-\lambda \\ -1-\mu+\lambda \\ 1+2\mu-2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 0$$

$$-1+\mu-\lambda+1+\mu-\lambda+2+4\mu-4\lambda=0$$

$$\lambda - \mu = \frac{1}{3}$$

By substitution

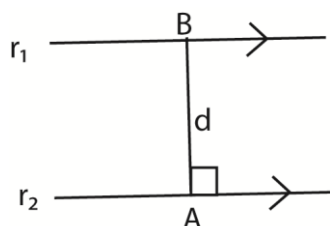
$$AB = -\frac{4}{3}i - \frac{2}{3}j + \frac{1}{3}k$$

$$d = |AB| = \sqrt{\frac{16}{9} + \frac{4}{9} + \frac{1}{9}} = \frac{\sqrt{21}}{3} \text{ units}$$



(b)  $\frac{x-2}{1} = \frac{y-1}{-1} = \frac{z-3}{2}$  and  $\frac{x+1}{1} = \frac{y-3}{-1} = \frac{z-1}{2}$

Solution



Let  $\frac{x-2}{1} = \frac{y-1}{-1} = \frac{z-3}{2} = \lambda$  and  $\frac{x+1}{1} = \frac{y-3}{-1} = \frac{z-1}{2} = \mu$

So we have

$A(2+\lambda, 1-\lambda, 3+2\lambda)$  and  $B(-1-\mu, 3+\mu, 1+2\mu)$

$AB = OB - OA$

$$= \begin{pmatrix} -1+\mu \\ 3-\mu \\ 1+2\mu \end{pmatrix} - \begin{pmatrix} 2+\lambda \\ 1-\lambda \\ 3+2\lambda \end{pmatrix} = \begin{pmatrix} -3+\mu-\lambda \\ 2-\mu+\lambda \\ -2+2\mu-2\lambda \end{pmatrix}$$

Now  $AB \cdot b = 0$

$$\begin{pmatrix} -3+\mu-\lambda \\ 2-\mu+\lambda \\ -2+2\mu-2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 0$$

$$-3+\mu-\lambda-2+\mu-\lambda-4+4\mu-4\lambda = 0$$

$$\mu - \lambda = \frac{9}{6} = \frac{3}{2}$$

By substitution;

$$AB = -\frac{3}{2}i - \frac{1}{2}j + k$$

$$d = |AB| = \sqrt{\frac{9}{4} + \frac{1}{4} + 1} = \frac{\sqrt{14}}{2} \text{ units}$$

(c)  $r_1 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$  and

$$r_2 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Solution

$$a_1 - a_2 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix} \text{ and}$$

$$b_1 \times b_2 = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$$

$$|n| = \sqrt{4^2 + 2^2 + 2^2} = \sqrt{24}$$

$$\hat{n} = \frac{1}{\sqrt{24}} \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$$

$$(a_1 - a_2) \cdot \hat{n} = \frac{1}{\sqrt{24}} \begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{24}} (-4 - 8 + 4) = -\frac{8}{\sqrt{24}}$$

$$d = |(a_1 - a_2) \cdot \hat{n}| = \left| -\frac{8}{\sqrt{24}} \right| = \frac{\sqrt{24}}{3}$$

$\therefore$  The distance apart is  $\frac{\sqrt{24}}{3}$  units

(d)  $r_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$  and

$$r_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Solution

$$a_1 - a_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} \text{ and}$$

$$b_1 \times b_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$|n| = \sqrt{-1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\hat{n} = \frac{1}{\sqrt{3}} (-i + j + k)$$

$$(a_1 - a_2) \cdot \hat{n} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

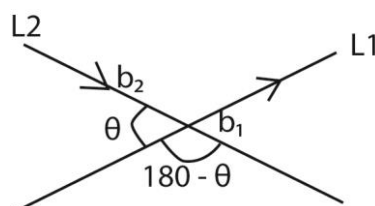
$$= \frac{1}{\sqrt{3}} (1 - 3 - 1) = -\frac{3}{\sqrt{3}}$$

$$d = |(a_1 - a_2) \cdot \hat{n}| = \left| -\frac{3}{\sqrt{3}} \right| = \sqrt{3}$$

$\therefore$  The distance apart is  $\sqrt{3}$  units

## Angle between two lines

The angle between two lines is equivalent to the angle between their parallel vectors.



In the illustration above, there are two angles:  $\theta$  and  $180 - \theta$  i.e. is obtuse.

### Example 23

(a) Determine the acute angle between each of the pairs of the lines

(i)  $r_1 = 2i + j - k + \lambda(2i + 3j + 6k)$  and

$$r_2 = i + 2j - 3k + \mu(2i - 2j + k)$$

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## Solution

$r_1$  is parallel to  $b_1 = 2i + 3j + 6k$  and

$r_2$  is parallel to  $b_2 = 2i - 2j + k$

Using  $b_1 \cdot b_2 = |b_1||b_2|\cos\theta$

$$(2i + 3j + 6k) \cdot (2i - 2j + k)$$

$$= (\sqrt{2^2 + 3^2 + 6^2} \cdot \sqrt{2^2 + (-2)^2 + 1^2}) \cos\theta$$

$$4 - 6 + 6 = (7)(3)\cos\theta$$

$$\cos^{-1}\left(\frac{4}{21}\right) = 79^\circ$$

$$(ii) \frac{x-2}{-4} = \frac{y-3}{3} = \frac{z+1}{-1} \text{ and } \frac{x-3}{2} = \frac{y-1}{6} = \frac{z+1}{-5}$$

## Solution

$r_1$  is parallel to  $b_1 = \begin{pmatrix} -4 \\ 3 \\ -1 \end{pmatrix}$  and

$r_2$  is parallel to  $b_2 = \begin{pmatrix} 2 \\ 6 \\ -5 \end{pmatrix}$

Using  $b_1 \cdot b_2 = |b_1||b_2|\cos\theta$

$$\begin{pmatrix} -4 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 6 \\ -5 \end{pmatrix}$$

$$= (\sqrt{-4^2 + 3^2 + (-1)^2} \cdot \sqrt{2^2 + 6^2 + (-5)^2}) \cos\theta$$

$$-8 + 18 + 5 = (\sqrt{26})(\sqrt{65})\cos\theta$$

$$\cos^{-1}\left(\frac{15}{\sqrt{1690}}\right) = 68.6^\circ$$

In general, angle  $\theta$  between the lines

$$r_1 = a_1 + \mu b_1 \text{ and } r_2 = a_2 + \lambda b_2 \text{ is}$$

$$\theta = \arccos\left(\frac{b_1 \cdot b_2}{|b_1||b_2|}\right)$$

(b) Given the equation of two lines are

$$y = m_1x + c_1 \text{ and } y = m_2x + c_2. \text{ Show that}$$

(i) Their vector equations are respectively

$$\begin{pmatrix} 0 \\ c_1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ m_1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ c_2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ m_2 \end{pmatrix}$$

where  $\mu$  and  $\lambda$  are constant

Solution

$$\Rightarrow \frac{x}{1} = \frac{y-c_1}{m_1} \text{ and } \frac{x}{1} = \frac{y-c_2}{m_2}$$

$$\text{i.e. } \begin{pmatrix} 0 \\ c_1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ m_1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ c_2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ m_2 \end{pmatrix}$$

(ii) The angle,  $\theta$ , between them is

$$\tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$$

Solution

$$\begin{pmatrix} 1 \\ m_1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ m_2 \end{pmatrix}$$

$$= \sqrt{(1 + m_1^2)} \cdot \sqrt{(1 + m_2^2)} \cos\theta$$

$$(1 + m_1 m_2)^2 = (1 + m_1^2 + m_2^2 + m_1^2 m_2^2) \cos^2\theta$$

$$\sec^2\theta = 1 + \tan^2\theta = \frac{1 + m_1^2 + m_2^2 + m_1^2 m_2^2}{(1 + m_1 m_2)^2}$$

$$\tan^2\theta = \frac{1 + m_1^2 + m_2^2 + m_1^2 m_2^2}{(1 + m_1 m_2)^2} - 1 = \frac{(m_1 - m_2)^2}{(1 + m_1 m_2)^2}$$

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\theta = \tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$$

## Exercise 4

1. Find the vector equation for the line passing through

(a) (4, 3) and is parallel to vector  $i - 2j$

$$\left[ r = \begin{pmatrix} 4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right]$$

(b) (5, -1, 3) and parallel to vector  $4i - 3j + k$

$$\left[ r = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} \right]$$

2. Find a vector equation for the line joining the following point

(a) (2, 6) and (5, -2)  $\left[ r = \begin{pmatrix} 2 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -8 \end{pmatrix} \right]$

(b) (-1, 2, -3) and (6, 3, 0)

$$\left[ r = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix} \right]$$

3. (a) Point A and B have coordinates (4, 1) and (2, -5) respectively. Find the vector equation for the line which passes through the point A and perpendicular to point AB

$$\left[ r = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -1 \end{pmatrix} \right]$$

(b) Point P and Q have coordinates (3, 5) and (-3, -7) respectively. Find vector equation for the line which passes through the point P which is perpendicular to PQ

$$\left[ r = \begin{pmatrix} 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right]$$

4. Find a vector equation for perpendicular bisector of the points

(a) (6, 3) and (2, -5)  $\left[ r = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right]$

(b) (7, -1) and (3, -3)  $\left[ r = \begin{pmatrix} 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right]$

5. Points P, Q and R have position vectors  $4i - 4j$ ,  $2i + 2j$  and  $8i + 6j$  respectively.

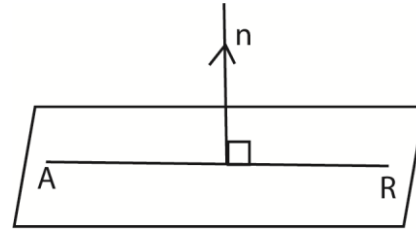
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- (a) Find a vector equation for  $L_1$  which is perpendicular bisector to points P and Q.  $[L_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \end{pmatrix}]$
- (b) Find a vector equation for  $L_2$  which is perpendicular bisector to points P and Q.  $[L_2 = \begin{pmatrix} 5 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \end{pmatrix}]$
- (c) Hence find the position vector of the point  $[\frac{59}{11}, \frac{4}{11}]$
6. Two lines  $L_1$  and  $L_2$  have equations  
 $L_1: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$  and  
 $L_2: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$
- (a) Show that  $L_1$  and  $L_2$  are concurrent (meet at a common point) and find the position vector of their point of intersection.  $[2i + 5j + 9k]$
- (b) Find the angle between  $L_1$  and  $L_2$ .  $[15.6^\circ]$
7. Points P, Q, and R have coordinates  $(-1, 1)$ ,  $(4, 6)$  and  $(7, 3)$  respectively.
- (a) Show that the perpendicular distance from the point R to the line PQ is  $3\sqrt{2}$ .
- (b) Deduce the area of the triangle PQR is 15 sq. units
8. Point A, B and C have position vectors  $-i + 3j + 5k$ ,  $5i - 6j - 4k$  and  $4i + 7j + 5k$  respectively. P is the point ON AB such that  $AP = \lambda AB$ . Find
- (a) AB
- (b) CP
- (c) The perpendicular distance from the point C to the line AB  $[3\sqrt{3}]$  {m v
9. Two lines  $L_1$  and  $L_2$  have vector equation  
 $r_1 = (2 - 3\lambda)i + (1 + \lambda)j + 4\lambda k$  and  
 $r_2 = (-1 + 3\mu)i + 3j + (4 - \mu)k$ . Find
- (a) The position vector of their common point of intersection.  $[r = -4i + 3j + 8k]$
- (b) The angle between the lines  $[143.7^\circ]$

## The Plane

### Equation of a plane

- (I) **Determining the equation of the plane given a vector perpendicular to the plane and one point contained in the plane.**



In the figure, A is a point in the plane

$n$  is the perpendicular vector to the plane and R is the general point in the plane

Since  $n$  is perpendicular to AR, then

$$n \cdot AR = 0$$

$$\text{i.e. } n \cdot (r - a) = 0 \dots\dots\dots(i)$$

$$\text{Let } n = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, A(x_1, y_1, z_1) \text{ and } R(x, y, z)$$

Substituting these into equation (i)

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{pmatrix} = 0$$

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

$$ax + by + cz - (ax_1 + by_1 + cz_1) = 0$$

$$\text{let } d = ax_1 + by_1 + cz_1$$

$$\Rightarrow ax + by + cz = d$$

Hence the Cartesian equation of the plane is  $ax + by + cz = d$  where  $d$  is a constant and the coefficient of  $x$ ,  $y$  and  $z$  form the perpendicular or normal vector.

Note: the above equation may be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = d$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = d \text{ i.e. } n = d$$

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This is called the scalar product of the vector equation of the plane

## Example 24

- (a) Find the vector normal to the plane  
 $3x - 2y + z = 7$

### Solution

The coefficient of x, y and z is  $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

Hence the normal vector is  $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

- (b) Find the equation of the plane that is normal to  $5\mathbf{i} - \mathbf{j} + 2\mathbf{k}$  and passes through A(4, 1, -3)

### Solution

Either:

$$\begin{pmatrix} x-4 \\ y-1 \\ z+3 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} = 0$$

$$5x - 20 - y + 1 + 2z + 6 = 0$$

$$5x - y + 2z = 13$$

Or:

Using the general equation  $ax + by + cz = d$

$$5x - y + 2z = (5 \times 4) - 1 + (2 \times -3)$$

$$5x - y + 2z = 13$$

- (c) Find the equation of the plane that is normal to  $4\mathbf{i} + 6\mathbf{j} + 5\mathbf{k}$  and passes through the point with position vector  $\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ .

### Solution

Either:

$$\begin{pmatrix} x-1 \\ y-3 \\ z-1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 6 \\ 5 \end{pmatrix} = 0$$

$$4x - 4 + 6y - 18 + 5z - 5 = 0$$

$$4x + 6y + 5z = 27$$

Or:

Using the general equation  $ax + by + cz = d$

$$4x + 6y + 5z = (4 \times 1) + 6 \times 3 + (5 \times 1)$$

$$4x + 6y + 5z = 27$$

## (II) Determining the equation of the plane given three non-collinear points.

Several methods are employed including the four outlined in the following example.

## Example 25

- (a) Find the equation of the plane containing the points A(1, 1, 1), B(5, 0, 0) and C(3, 2, 1)

### Solution

#### Method 1

Let the equation of the plane be

$$ax + by + cz = d$$

As the three points lie in the same plane, their coordinates satisfy the above equation

Substituting for A, B and C coordinates in the general equation

$$\text{For A(1, 1, 1): } a + b + c = d \dots\dots\dots(i)$$

$$\text{For B(5, 0, 0): } 5a = d \dots\dots\dots(ii)$$

$$\text{For C(3, 2, 1): } 3a + 2b + c \dots\dots\dots(iii)$$

Solving for a, b, and c in terms of d:

$$\text{From Eqn. (ii) } a = \frac{1}{5}d$$

Substituting for a into eqn. (i)

$$\frac{1}{5}d + b + c = d \Rightarrow b + c = \frac{4}{5}d \dots\dots\dots(iv)$$

Substituting for a into eqn. (iii)

$$3\left(\frac{1}{5}d\right) + 2b + c = d \Rightarrow 2b + c = \frac{2}{5}d \dots\dots\dots(v)$$

$$\text{Eqn. (v) - eqn. (iv): } b = -\frac{2}{5}d$$

Substituting b into eqn. (iv)

$$-\frac{2}{5}d + c = \frac{4}{5}d \Rightarrow c = \frac{6}{5}d$$

Substituting a, b, and c into the equation

$$ax + by + cz = d$$

$$\Rightarrow \frac{1}{5}dx - \frac{2}{5}dy + \frac{6}{5}dz = d$$

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Multiplying through by  $\frac{5}{d}$

$x - 2y + 6z = 5$  is the equation of the plane

## Method 2

One of the normal vectors of the plane is

$$AB \cdot AC = 0$$

$$\text{Where } AB = OB - OA = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$$

$$\text{And } AC = OC - OA = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{Now, } AB \times AC = \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix}$$

If  $R(x, y, z)$  is the general point in the plane, then  $AR$  is normal to  $AB \times AC$ .

$$(r-a) \cdot (AB \times AC) = 0$$

$$\begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix} = 0$$

$$x - 1 - 2y + 2 + 6z - 6 = 0$$

$$x - 2y + 6z = 5$$

## Method 3

Let  $R$  be the general point in the plane

The  $AR = \mu AB + \lambda AC$  for scalars  $\mu$  and  $\lambda$ .

$$r - a = \mu AB + \lambda AC$$

$$r = a + \mu AB + \lambda AC$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Equating the coefficients of  $x, y$  and  $z$

$$x = 1 + 4\mu + \lambda \dots\dots\dots (i)$$

$$y = 1 - \mu + \lambda \dots\dots\dots (ii)$$

$$z = 1 - \mu \dots\dots\dots (iii)$$

$$\text{From eqn. (iii): } \mu = 1 - z$$

Substituting for  $\mu$

$$\lambda = y - 1 + (1 - z) = y - z$$

Substituting  $\mu$  and  $\lambda$  in equation (i)

$$x = 1 + 4(1 - z) + 2(y - z)$$

$$= 1 + 4 - 4z + 2y - 2z$$

$$x - 2y + 6z = 5$$

Note that:

- If the plane passes through the origin, then its equation is  $r = \mu b + \lambda c$
- The plane  $r = a + \mu b + \lambda c$  passes through point  $a$  with position vector  $a$  and is parallel to  $b$  and  $c$ .
- If the vectors  $a, b$  and  $c$  are coplanar, then the sum of the coefficients of  $a, b$  and  $c$  must be zero.

## Method 4

This involves finding the determinant of a  $3 \times 3$  matrix. Taking  $A$  as the principal point, we have

$$AB \times AC = \begin{vmatrix} x-1 & y-1 & z-1 \\ 4 & -1 & -1 \\ 2 & 1 & 0 \end{vmatrix} = 0$$

$$\Rightarrow (x-1)1 - (y-1)2 + (z-1)6 = 0$$

$$x - 1 - 2y + 2 + 6z - 6 = 0$$

$$x - 2y + 6z = 5$$

- (d) Find the equation of the plane containing the points  $A(1, 2, 5)$ ,  $B(1, 0, 4)$  and  $C(5, 2, 1)$

## Solution

Using the determinant method

$$AB = OB - OA$$

$$= \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}$$

$$AC = OC - OA$$

$$= \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -4 \end{pmatrix}$$

$$AB \times AC = \begin{vmatrix} x-1 & y-2 & z-5 \\ 0 & -3 & -1 \\ 4 & 0 & -4 \end{vmatrix} = 0$$

$$\Rightarrow (x-1)8 - (y-2)4 + (z-5)8 = 0$$

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$$(x-1)2 - (y-2) + (z-5)2 = 0$$

$$2x - y + 2z = 10$$

## (III) Determining the equation of the plane given one point and a line in the plane.

Here more points are obtained from the equation and the problem worked out as in (III).

### Example 26

- (a) Find the equation of the plane through the point (1, 0, 1) and containing the line  $\frac{x}{1} = \frac{y}{-1} = \frac{z}{2}$ .

**Solution**

The vector equation for the line is

$$r = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Let 1<sup>st</sup> given point be P(1, 0, -1):

Taking  $\mu = 0$ : the 2<sup>nd</sup> point is O(0, 0, 0)

Taking  $\mu = 1$ : the 3<sup>rd</sup> point Q(1, -1, 2)

$$\text{Thus } OP = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } OQ = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

The normal vector is

$$OP \times OQ = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix}$$

The equation of the plane is

$$\begin{pmatrix} x-0 \\ y-0 \\ z-0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix} = 0$$

$$x + 3y + z = 0$$

Using determinant method

$$OP \times OQ = \begin{vmatrix} x & y & z \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$x(-1) - y(3) + z(-1) = 0$$

$$x + 3y + z = 0$$

$$x + 3y + z = 0$$

## (IV) Determining the equation of the plane given two lines in the plane.

This can be tackled in two ways

### Example 27

Find the equation of the plane containing the lines

$$\frac{x-3}{5} = \frac{y+1}{2} = \frac{z-3}{1} \text{ and}$$

$$\frac{x-3}{2} = \frac{y+1}{4} = \frac{z-3}{3}$$

### Solution

#### Method 1

The corresponding vector equations of the above lines are as follows

$$r_1 = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} \text{ and}$$

$$r_2 = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} \text{ respectively}$$

Taking  $\mu = 0$ , the 1<sup>st</sup> point is A(3, -1, 3)

Taking  $\mu = 1$ , the 2<sup>nd</sup> point is B(8, 1, 4)

Taking  $\lambda = 1$ , the 3<sup>rd</sup> point is C(5, 3, 6)

So with three points obtained, the above methods can be used.

$$\text{Now } AB = OB - OA = \begin{pmatrix} 8 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$$

$$\text{And } AC = OC - OA = \begin{pmatrix} 5 \\ 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$$

The normal vector

$$n = AB \times AC = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -13 \\ 16 \end{pmatrix}$$

Taking the point (3, -1, 3) which lies on the 1<sup>st</sup> line: the equation of the plane is

$$\begin{pmatrix} x-3 \\ y+1 \\ z-3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -13 \\ 16 \end{pmatrix} = 0$$

$$2x - 6 - 13y - 13 + 16z - 48 = 0$$

$$2x - 13y + 16z = 67$$

#### Method 2

The parallel vectors of the given lines are

$$\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} \text{ respectively.}$$

$$\text{The normal vector, } n = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -13 \\ 16 \end{pmatrix}$$

Taking the point (3, -1, 3) as before

$$\begin{pmatrix} x-3 \\ y+1 \\ z-3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -13 \\ 16 \end{pmatrix} = 0$$

$$2x - 13y + 16z = 67$$

## (V) Determining the equation of the plane given one point in the plane and a perpendicular line.

### Example 28

# ST. THOMAS AQUINAS

Find the equation of the plane perpendicular to the line  $\frac{x-1}{2} = \frac{y}{-1} = \frac{z}{3}$  and passing through point B(1, -3, 2)

**Solution**

The parallel vector to the line is  $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$

This means that this vector is also perpendicular to the plane

The equation of the plane is

$$\begin{pmatrix} x-1 \\ y+3 \\ z-2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 0$$

$$2x - 2 - y - 3 + 3z - 6 = 0$$

$$2x - y + 3z = 11$$

## (VI) Determining the equation of the plane given two points in the plane.

### Example 29

Find the equation of the plane containing the points A(1, 2, -1) and B(4, -3, 2)

**Solution**

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$$

The normal vector is

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ -11 \end{pmatrix}$$

the equation of the line is thus

$$\begin{pmatrix} x-1 \\ y-2 \\ z+1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -6 \\ -11 \end{pmatrix} = 0$$

$$x - 6y - 11z = 0$$

## (VII) Determining the equation of the plane given two parallel lines.

### Example 30

Find the equation of the plane passing through (1, 0, -1) and parallel to the line

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \text{ and }$$

$$\mathbf{r}_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

**Solution**

The normal vector,

$$\mathbf{n} = \mathbf{b}_1 \times \mathbf{b}_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -6 \\ 7 \end{pmatrix}$$

Equation of the plane

$$-x - 6y + 7z = -1(1) - 6(0) + 7(-1)$$

$$x + 6y - 7z = 8$$

## (VIII) Determining the equation of the plane given a line in the plane and a parallel vector.

### Example 31

Find the equation of the plane containing

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \text{ and parallel to } \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

**Solution**

The normal vector

$$\mathbf{n} = \mathbf{b}_1 \times \mathbf{b}_2 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -7 \\ -1 \\ 3 \end{pmatrix}$$

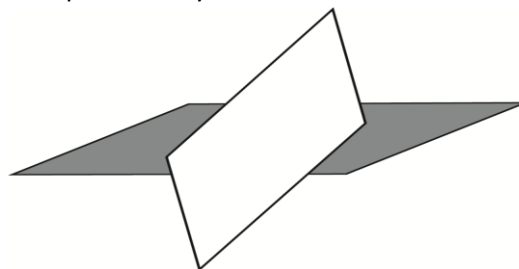
Equation of the plane is

$$-7x - y + 3z = -7(1) - 2 + 3(1) = -6$$

$$7x + y - 3z = 6$$

## Intersection of two planes

Two plane always on a line



Solving the two equations of the lines simultaneously gives equation of this line

### Example 32

Find the Cartesian equation of the lines of intersection of the following planes

$$(a) \quad 3x - 5y + z = 8 \text{ and } 2x - 3y + z = 3$$

**Solution**

#### Method 1

Note: solving for three unknown from two equations is quite hard, so we express them in terms of a constant say  $\lambda$

$$\text{Let } 3x - 5y + z = 8 \dots\dots\dots (i)$$

$$\text{and } 2x - 3y + z = 3 \dots\dots\dots (ii)$$

$$\text{Eqn. (i) - eqn. (ii)}$$

$$x - 2y = 5$$

$$\text{Let } x = \lambda \Rightarrow \lambda - 2y = 5 \text{ i.e. } y = \frac{1}{2}(\lambda - 5)$$

Substituting for x and y into eqn. (ii)

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$$2\lambda - \frac{3}{2}(\lambda - 5) + z = 3 \Rightarrow z = \frac{1}{2}(-9 - \lambda)$$

$$\text{So } x = \lambda, y = \frac{1}{2}(\lambda - 5), z = \frac{1}{2}(-9 - \lambda)$$

To eliminate fractions let  $\lambda = 1 + 2\mu$

$$x = 1 + 2\mu, y = -2 + \mu, z = -5 - \mu$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \text{ the vector}$$

equation

$$\frac{x-1}{2} = \frac{y+2}{1} = \frac{z+5}{-1} \text{ Cartesian equation}$$

## Method 2

The parallel vector

$$\begin{aligned} b = n_1 \times n_2 &= \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \\ &= - \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \end{aligned}$$

$$\text{The equation of the line is } r = a + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

From the equation  $3x - 5y + z$  and

$2x - 3y + z = 3$ , subtracting

$$\Rightarrow x - 2y = 5$$

when  $x = 1$ ,  $1 - 2y = 5$  i.e.  $y = -2$

substituting in the first equation

$$3(1) - 5(-2) + z = 8 \text{ i.e. } z = -5$$

$$r = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \text{ vector equation}$$

OR

Substituting for  $r$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$$\frac{x-1}{2} = \frac{y+2}{1} = \frac{z+5}{-1} \text{ Cartesian equation}$$

$$(b) \quad 3x + 4y + 2z = 3 \text{ and } 2x - 3y - z = 1$$

$$\text{Let } 3x + 4y + 2z = 3 \dots\dots\dots(i)$$

$$\text{and } 2x - 3y - z = 1 \dots\dots\dots(ii)$$

$$2\text{en. (i)} - \text{eqn. (ii)}$$

$$17y + 7z = 3$$

$$\text{Let } y = \lambda, 17\lambda + 7z = 3 \Rightarrow z = \frac{3-17\lambda}{7}$$

Substituting for  $y$  and  $z$  into eqn. (i)

$$3x + 4\lambda + \frac{2}{7}(3 - 17\lambda) = 3 \Rightarrow x = \frac{1}{7}(5 + 2\lambda)$$

$$x = \frac{1}{7}(5 + 2\lambda), y = \lambda, z = \frac{3-17\lambda}{7}$$

Let  $\lambda = 1 + 7\mu$  ( to eliminate fractions)

$$\text{Then } x = 1 + 2\mu, y = 1 + 7\mu \text{ and } z = -2 - 17\mu$$

The Cartesian equation is

$$\frac{x-1}{2} = \frac{y-1}{7} = \frac{z+2}{-17}$$

## Method 2

Parallel vector

$$b = n_1 \times n_2 = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ -17 \end{pmatrix}$$

$$\text{The equation of the line is } r = a + \lambda \begin{pmatrix} 2 \\ 7 \\ -17 \end{pmatrix}$$

$$\text{Let } 3x + 4y + 2z = 3 \dots\dots\dots(i)$$

$$\text{and } 2x - 3y - z = 1 \dots\dots\dots(ii)$$

$$2\text{en. (i)} - \text{eqn. (ii)}$$

$$17y + 7z = 3$$

$$\text{Let } y = 1, 17 + 7z = 3 \Rightarrow z = \frac{3-17}{7} = -2$$

Substituting for  $y$  and  $z$  into eqn. (i)

$$3x + 4 + 2(-2) = 3 \Rightarrow x = 1$$

$$r = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 7 \\ -17 \end{pmatrix} \text{ vector equation}$$

Or

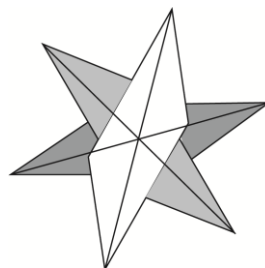
Substituting for  $r$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 7 \\ -17 \end{pmatrix}$$

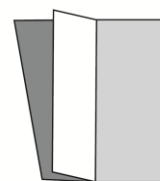
$$\frac{x-1}{2} = \frac{y-1}{7} = \frac{z+2}{-17}, \text{ Cartesian equation}$$

## Intersection of three planes

Three planes may intersect at a point or on a line (if they meet)



Three planes meeting at a point



Three planes meeting on a line (book pages)

## (I) Intersection of three planes at point

### Example 33

Find the point of intersection of the three planes

$$x + 2y - z = 2, 3x - y + z = 3 \text{ and } 2x + y - 3z = 3$$



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## Solution

Form simultaneous equation

$$x + 2y - z = 2 \dots\dots\dots(i)$$

$$3x - y + z = 3 \dots\dots\dots(ii)$$

$$2x + y - 3z = 3 \dots\dots\dots(iii)$$

Solving simultaneously the point of intersection is (1, 1, 1)

## (II) Intersection of three planes at point

If a plane and a line meet, they do so at a particular point.

### Example 33

- (a) Find the point where the line

$$\frac{x-3}{-1} = \frac{y-1}{2} = \frac{z+3}{4} \text{ meets the plane } 3x - y + 2z = 8$$

#### Solution

Expressing the equation of the line in parametric form

$$\text{Let } \frac{x-3}{-1} = \frac{y-1}{2} = \frac{z+3}{4} = \lambda$$

$$\text{Then } x = 3 - \lambda, y = 1 + 2\lambda \text{ and } z = -3 + 4\lambda$$

Substituting for parametric equations into the equation of the plane

$$3(3 - \lambda) - (1 + 2\lambda) + 2(-3 + 4\lambda) = 8 \Rightarrow \lambda = 2$$

Substituting for  $\lambda$  into parametric equations

$$x = 3 - 2 = 1, y = 1 + 2(2) = 5 \text{ and}$$

$$z = -3 + 4(2) = 5$$

Hence the point of intersection (x, y, z) is (1, 5, 5)

- (b) Find the position vector of a point where

$$\text{the line } r = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} \text{ meets the}$$

$$\text{plane } r \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 15$$

#### Solution

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$$

The parametric equations of the line are

$$x = 2 + 5\lambda, y = -1 + 3\lambda \text{ and } z = 3 + 2\lambda$$

The equation of the plane is  $x + 2y - z = 15$

Substituting parametric equations into the equation of the plane

$$2 + 5\lambda + (-1 + 3\lambda) - (3 + 2\lambda) = 15$$

$$\lambda = 2$$

Substituting  $\lambda$  into parametric equations

$$x = 2 + 5(2) = 12$$

$$y = (-1 + 3(2)) = 5$$

$$z = (3 + 2(2)) = 7$$

Hence the point of intersection (x, y, z) is (12, 5, 7)

## Perpendicular distance from a point to the plane

### A. Perpendicular distance from the origin to the plane

Rewriting the equation  $r \cdot n = d$  in the form

$r \cdot \hat{n} = d_1$  where  $\hat{n}$  is the unit normal to the plane

Or

By using the general formula, the perpendicular distance  $d_p$  from a plane  $ax + by + cz + d = 0$  to the point  $(x_1, y_1, z_1)$  is given by the expression

$$d_p = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

### Example 34

- (a) Find the distance from the origin to the plane  $4x + 8y - z = 18$

#### Solution

The normal vector  $n = \begin{pmatrix} 4 \\ 8 \\ -1 \end{pmatrix}$  and

$$|n| = \sqrt{4^2 + 8^2 + (-1)^2} = 9$$

$$\text{Now } \hat{n} = \frac{1}{9} \begin{pmatrix} 4 \\ 8 \\ -1 \end{pmatrix}$$

$$\Rightarrow r \cdot \frac{1}{9} \begin{pmatrix} 4 \\ 8 \\ -1 \end{pmatrix} = \frac{18}{9} = 2$$

Or

By using the general formula, rewrite the equation of the plane as

$$4x + 8y - z - 18 = 0, a = 4, b = 8, c = -1 \text{ and } d = -18$$

At the origin  $(x_1, y_1, z_1) = (0, 0, 0)$

$$d_p = \frac{|4(0) + 8(0) - 1(0) - 18|}{\sqrt{4^2 + 8^2 + (-1)^2}} = \frac{18}{9} = 2 \text{ units}$$

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- (b) Find the perpendicular distance from the origin to the plane  $r \cdot (2i - 14j + 5k) = 10$

**Solution**

The normal vector  $n = \begin{pmatrix} 2 \\ -14 \\ 5 \end{pmatrix}$  and

$$|n| = \sqrt{2^2 + (-14)^2 + 5^2} = 15$$

$$\text{Now } \hat{n} = \frac{1}{15} \begin{pmatrix} 2 \\ -14 \\ 5 \end{pmatrix}$$

$$\Rightarrow r \cdot \frac{1}{15} \begin{pmatrix} 2 \\ -14 \\ 5 \end{pmatrix} = \frac{10}{15} = \frac{2}{3}$$

Or

By using the general formula, rewrite the equation of the plane as

$$2x - 14y + 5z - 10 = 0, a = 2, b = -14, c = 5 \text{ and } d = -10$$

At the origin  $(x_1, y_1, z_1) = (0, 0, 0)$

$$d_p = \frac{|2(0) - 14(0) + 5(0) - 10|}{\sqrt{2^2 + (-14)^2 + (-1)^2}} = \frac{10}{15} = \frac{2}{3} \text{ units}$$

- B. Perpendicular distance for a given point rather than origin to a plane**

Several methods are employed

## Example 35

- (a) Determine the distance from the line  $12x - 3y - 4z = 39$  to the point  $(5, 3, 1)$

**Solution**

**Method 1**

The perpendicular distance  $d_p$  from a plane  $ax + by + cz + d = 0$  to the point  $(x_1, y_1, z_1)$  is given by the expression

$$d_p = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

By substitution

$$d_p = \frac{|12(5) - 3(3) - 4(1) + 39|}{\sqrt{12^2 + (-3)^2 + (-4)^2}} = 2 \text{ units}$$

Note that the equation of the plane should be rewritten in the form  $f(x, y, z) = 0$  before applying the formula.

## Method 2

The parallel plane containing the point given is obtained and the absolute difference of the resulting length of the plane from the origin computed.

Equation of the plane:  $12x - 3y - 4z = 39$  ..(i)

Equation of parallel plane  $12x - 3y - 4z = D$  for any constant D.

Since this parallel contains the point

$$(5, -3, 1): 12(5) - 3(-3) - 4(1) = 65 = D$$

The parallel plane:  $12x - 3y - 4z = 65$  .....(ii)

In both planes, the normal vector

$$n = 12i - 3j - 4k$$

$$|n| = \sqrt{12^2 + (-3)^2 + (-4)^2} = 13$$

Dividing equation by 13:

$$\frac{12}{13}x - \frac{3}{13}y - \frac{4}{13}z = \frac{65}{13} = 5$$

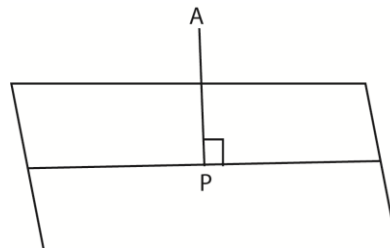
The distance between two planes is

$$|5 - 3| = 2$$

∴ the distance from point  $(5, -3, 1)$  to the plane  $12x - 3y - 4z = 39$  is 2 units.

## Method 3

$AP = \lambda n$  (AP is parallel to n)



Given the equation of the plane

$$12x - 3y - 4z = 39$$

$$\text{Let } n = \begin{pmatrix} 12 \\ -3 \\ -4 \end{pmatrix} \text{ and } A(5, -3, 1)$$

Substitute in  $AP = \lambda n$

$$p - a = \lambda n$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 12 \\ -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 5 - 12\lambda \\ -3 - 3\lambda \\ 1 - 4\lambda \end{pmatrix}$$

$$x = 5 - 12\lambda, y = -3 - 3\lambda, z = 1 - 4\lambda$$

Substitute these in the equation of the plane

$$12x - 3y - 4z = 39$$

$$12(5 - 12\lambda) - 3(-3 - 3\lambda) - 4(1 - 4\lambda) = 39$$

$$\lambda = -\frac{2}{13}$$

$$AP = -\frac{2}{13} \begin{pmatrix} 12 \\ -3 \\ -4 \end{pmatrix}$$

$$|AP| = \frac{2}{13} \sqrt{12^2 + (-3)^2 + (-4)^2} = 2 \text{ units}$$

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## Angle between two planes

The angle say  $\theta$  between the planes  $r \cdot n_1 = d_1$  and  $r \cdot n_2 = d_2$  is the angle between the normal vectors of the two planes. This is given by

$$\theta = \cos^{-1} \left( \frac{n_1 \cdot n_2}{|n_1||n_2|} \right)$$

### Example 36

Determine the angle between the planes

$$4x + 3y + 12z = 10 \text{ and } 4x - 3y = 7$$

### Solution

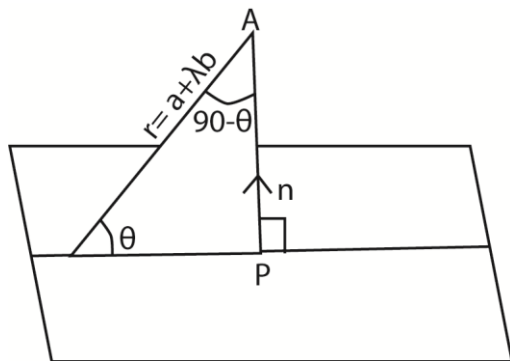
The normal  $n_1 = \begin{pmatrix} 4 \\ 3 \\ 12 \end{pmatrix}$  and  $n_2 = \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}$  respectively.

$$\begin{pmatrix} 4 \\ 3 \\ 12 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix} = \sqrt{4^2 + 3^2 + 12^2} \sqrt{4^2 + (-3)^2} \cos \theta$$

$$16 - 9 = \sqrt{169} \sqrt{25} \cos \theta$$

$$\theta = \cos^{-1} \left( \frac{7}{65} \right) = 83.8^\circ$$

## Angle between a line and a plane



The angle between a line and a plane is the angle between the normal vector to the plane and the parallel vector to the line.

Given a line  $r = a + \lambda b$  and the plane  $r \cdot n = d$ , the angle  $\theta$  between them can be computed from the dot product of vectors as

$$b \cdot n = |b||n| \cos(90 - \theta)$$

$$= |b||n| \sin \theta$$

$$\theta = \sin^{-1} \theta \left( \frac{b \cdot n}{|b||n|} \right)$$

### Example 37

- (a) Find the acute angle between the line  $\frac{x+1}{4} = \frac{y-2}{1} = \frac{z-3}{-1}$  and the plane  $3x - 5y + 4z = 5$

### Solution

The line is parallel to  $b = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$  and the

normal vector to the plane is  $n = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$

$$\begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$$

$$= \sqrt{4^2 + 1^2 + (-1)^2} \sqrt{3^2 + (-5)^2 + 4^2} \sin \theta$$

$$3 = \sqrt{900} \sin \theta$$

$$\theta = \sin^{-1} \frac{3}{30} = 5.7^\circ$$

- (b) Find the angle between the line

$$r = \begin{pmatrix} -4 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ 2 \\ -4 \end{pmatrix} \text{ and the plane}$$

$$4x + 3y - 3z = -1$$

### Solution

The line is parallel to  $b = \begin{pmatrix} 8 \\ 2 \\ -4 \end{pmatrix}$  and the normal

vector to the plane is  $n = \begin{pmatrix} 4 \\ 3 \\ -3 \end{pmatrix}$

$$\begin{pmatrix} 8 \\ 2 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ -3 \end{pmatrix}$$

$$= \sqrt{8^2 + 2^2 + (-4)^2} \sqrt{4^2 + 3^2 + (-3)^2} \sin \theta$$

$$50 = \sqrt{2856} \sin \theta$$

$$\theta = \sin^{-1} \frac{50}{\sqrt{2856}} = 69.3^\circ$$

### Exercise 5

- Find the equation of the plane containing points  $P(1, 1, 1)$ ,  $Q(1, 2, 0)$  and  $R(-1, 2, 1)$ .  
[ $x + 2y + 2z = 5$ ]
- Find the equation of the plane containing point  $(4, -2, 3)$  and parallel to the plane  $3x - 7z = 12$

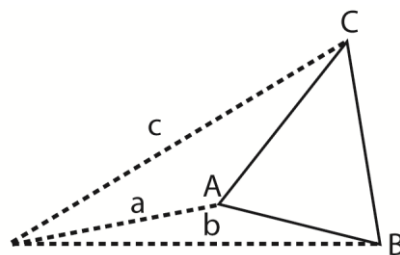
# ST. THOMAS AQUINAS

- $[3x - 7z = -9]$
- Show that the point with position vector  $7i - 5j - 4k$  lies in the plane  $r = 4i + 3j + 2k + \lambda(i - j - k) + \mu(2i + 3j + k)$ . Find the point at which the line  $x = y - 1 = 2z$  intersects the plane  $4x - y + 3z = 8$   $[(2, 3, 1)]$
  - Find the parametric equation for the line through the point  $(0, 1, 2)$  that is parallel to the plane  $x + y + z = 2$  and perpendicular to the line  $x = 1 + t, y = 1 - t, z = 2t$ .  
 $[x = 3t, y = 1 - t, z = 2 - 2t]$
  - Find the distance between parallel planes  $z = x + 2y + 1$  and  $3x + 6y - 3z = 4$   $\left[\frac{7\sqrt{6}}{18}\right]$
  - Two planes are given by their parametric equation:  $x = r + s, y = 3s, z = 2r$  and  $x = 1 + r + s, y = 2 + r, z = -3 + s$ . Find the Cartesian equation of the intersection point.  $[6x - 2y - 3z = 0]$
  - The equation of a plane P is given by  $r \cdot \begin{pmatrix} 2 \\ 6 \\ 9 \end{pmatrix} = 33$ , where  $r$  is position vector of P  
Find the perpendicular distance from the origin to the plane  $[3 \text{ units}]$
  - The line through point  $(1, -2, 3)$  and parallel to the line  $\frac{x}{3} = \frac{y+1}{-1} = \frac{z}{1} = z + 1$  meets the plane  $x + 2y + 2z = 8$  at Q. Find the coordinates of Q.  $\left[\left(6, -\frac{11}{3}, \frac{14}{3}\right)\right]$
  - (a) Find the angle between the plane  $x + 4y - z = 72$  and the line  $r = 9i + 6j + 8k$ .  $[34.5^\circ]$
  - Obtain the equation of the plane that passes through  $(1, -2, 2)$  and perpendicular to the line  $\frac{x-9}{4} = \frac{y-6}{-1} = \frac{z-8}{1}$   $[4x - y + z = 8]$
  - Find the parametric equations of the line of intersection of the planes  $x + y + z = 4$  and  $x - y + 2z = 0$   
 $[x = 3 + t, y = 2t, z = 1 + 3t]$
  - Find the points of intersection of the three planes  $2x - y + 3z = 4, 3x - 2y + 6z = 4$  and  $7x - 4y + 5z = 11$ .  $[(5, 6, 0)]$
  - Find the Cartesian equation of the plane with parametric vector equation  
 $r = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$   
 $[x + 2y - 3z = 0]$
  - Find the Cartesian equation of the plane containing the position vector  $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$  and parallel to the vectors  $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$  and  $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ .  
 $[3y - z = 10]$
  - Find the Cartesian equation of the plane containing the points with position vectors  $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$  and  $\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$   
 $[3x + 2y + z = 6]$
  - Find the perpendicular distance from the plane  $r \cdot (2i - 14j + 5k) = 10$  to the origin  $\left[\frac{2}{3}\right]$
  - Find the position vector of the point where the line  $r = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$  meets the plane  $r \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 15$   $\left[\begin{pmatrix} 12 \\ 5 \\ 5 \end{pmatrix}\right]$
  - Two line have vector equations  
 $r = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$  and  
 $r = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ . Find the position vector of the point of intersection of the two lines and the Cartesian equation of the plane containing the two lines.  
 $\left[\begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}, 5x - y + 3z = 19\right]$

## Example 38 (mixed questions)

- The position vector of point A is  $2i + 3j + k$ , of B is  $5j + 4k$  and of C is  $i + 2j + 12k$ . Show that ABC is a triangle.

### Solution



$$a = 2i + 3j + k$$

$$b = 5j + 4k$$

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$$c = i + 2j + 12k$$

Two conditions must be fulfilled:

## 1<sup>st</sup> condition

For a triangle to be,  $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$$

$$\begin{aligned} &= (\overrightarrow{OB} - \overrightarrow{OA}) + (\overrightarrow{OC} - \overrightarrow{OB}) + (\overrightarrow{OA} - \overrightarrow{OC}) \\ &= \begin{pmatrix} 0 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 12 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 12 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \\ 8 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -11 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

## Second condition

We work out for any angle and if it is not  $0^\circ$  or  $180^\circ$ , then we conclude that ABC is a triangle

Now finding angle A

From dot product of vectors

$$AB \cdot AC = |AB||AC|\cos A$$

$$\cos A = \frac{AB \cdot AC}{|AB||AC|}$$

$$\begin{aligned} AB \cdot AC &= (-2i + 2j + 3k) \cdot (-i - j + 11k) \\ &= 2 - 2 + 33 = 33 \end{aligned}$$

$$\begin{aligned} |AB| &= \sqrt{(-2)^2 + 2^2 + 3^2} \\ &= \sqrt{4 + 4 + 9} = \sqrt{17} \end{aligned}$$

$$\begin{aligned} |AC| &= \sqrt{(-1)^2 + 1^2 + 11^2} \\ &= \sqrt{1 + 1 + 121} = \sqrt{123} \end{aligned}$$

$$A = \cos^{-1} \left( \frac{33}{\sqrt{17} \times \sqrt{123}} \right) = 43.8^\circ$$

Since A is not  $0^\circ$  or  $180^\circ$ , hence ABC is a triangle

NB. The above two conditions **must** be clearly shown in order for the candidate to get all the marks.

2. (a) Find the point of intersection of the lines

$$\frac{x-3}{4} = \frac{y-7}{4} = \frac{z+3}{-5} \text{ and } \frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$$

**Solution**

$$\text{Let } \frac{x-3}{4} = \frac{y-7}{4} = \frac{z+3}{-5} = \mu$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 4 \\ -5 \end{pmatrix} \dots\dots\dots (i)$$

And

$$\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3} = \lambda$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix} \dots\dots\dots (ii)$$

Equating eqn. (i) and eqn. (ii)

$$\begin{pmatrix} 5 \\ 7 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 4 \\ -5 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$$

Equating corresponding unit vectors

$$5 + 4\mu = 8 + 7\lambda$$

$$4\mu - 7\lambda = 3 \dots\dots\dots (iii)$$

$$7 + 4\mu = 4 + \lambda$$

$$4\mu - \lambda = -3 \dots\dots\dots (iv)$$

Eqn. (iii) – eqn.(iv)

$$-6\lambda = 6$$

$$\lambda = -1$$

Substituting  $\lambda$  in eqn. (ii)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 5 \end{pmatrix} + -1 \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8-7 \\ 4-1 \\ 5-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

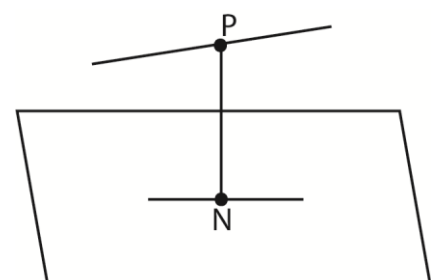
$$\therefore (x, y, z) = (1, 3, 2)$$

(b) The equations of a line and a plane are

$$\frac{x-2}{1} = \frac{y-2}{2} = \frac{z+3}{2} \text{ and } 2x + y + 4z = 9$$

respectively. P is a point on the line where  $x = 3$ , N is the foot of the perpendicular from P to the plane. Find the coordinates of N.

**Solution**



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Line equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$x = 2 + \lambda$$

$$\text{When } x = 3$$

$$3 = 2 + \lambda; \lambda = 1$$

$$\Rightarrow \text{OP} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

$$\therefore \text{P}(3, 4, 5)$$

$$\text{Plane equation: } 2x + y + 4z = 9$$

$$r \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = 9$$

$$\therefore n = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$$

$$\text{NP} = n$$

$$\text{NP} = \text{OP} - \text{ON}$$

$$\text{ON} = \text{OP} - \text{NP}$$

$$= \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

$$\therefore \text{N}(1, 3, 1)$$

3. (a) Find the Cartesian equation of the plane through the points whose position vectors are  $2i + 2j + 3k$ ,  $3i + j + 2k$  and  $-2j + 4k$ . (06marks)

**Solution**

**Method 1**

$$\text{Let OA} = 2i + 2j + 3k$$

$$\text{OB} = 3i + j + 2k$$

$$\text{OC} = -2j + 4k$$

Let R be the general point in the plane

$$\text{Then AR} = \mu(\text{AB}) + \lambda \text{AC}$$

$$\text{OR} = \text{OA} + \mu(\text{OB} - \text{OA}) + \lambda(\text{OC} - \text{OA})$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} + \mu \left[ \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \right] +$$

$$\lambda \left[ \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \right]$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -4 \\ 1 \end{pmatrix}$$

$$x = 2 + \mu - 2\lambda \dots\dots\dots (i)$$

$$y = 2 - \mu - 4\lambda \dots\dots\dots (ii)$$

$$z = 3 - \mu + \lambda \dots\dots\dots (iii)$$

Eqn (i) + eqn. (ii)

$$(x + y) = 4 - 6\lambda \dots\dots\dots (iv)$$

Eqn. (i) and eqn. (iii)

$$x + z = 5 - \lambda$$

$$\lambda = -x - z + 5$$

Substituting for  $\lambda$  into eqn. (iv)

$$x + y = 4 - 6(-x - z + 5)$$

$$5x - y + 6z - 26 = 0$$

**Method 2**

Let the equation of the plane be

$$ax + by + cz = d$$

Substituting point (2, 2, 3) in equation

$$2a + 2b + 3c = d \dots\dots\dots (i)$$

Substituting point (3, 1, 2) in equation

$$3a + b + 2c = d \dots\dots\dots (ii)$$

Substituting point (0, -2, 4) in equation

$$-2b + 4c = d \dots\dots\dots (iii)$$

We have to solve for a, b, c and d

$$3\text{Eqn. (i)} - 2\text{Eqn. (ii)}$$

$$6a + 6b + 9c = 3d$$

$$- 6a + 2b + 4c = 2d$$

$$\hline 4b + 5c = d \dots\dots\dots (iv)$$

$$2\text{eqn. (iii)} + \text{eqn. (iv)}$$

$$-4b + 8c = 2d$$

$$+4b + 4c = d$$

$$\hline 13c = 3d$$

$$c = \frac{3}{13}d$$

From eqn. (iv)

$$4b + \frac{15}{13}d = d; 4b = d - \frac{15}{13}d = \frac{-1}{13}d$$

From eqn. (i)

$$2a - \frac{2}{26}d + \frac{9}{13}d = d$$

$$2a = d + \frac{2}{26}d - \frac{9}{13}d = \frac{10}{26}d$$

$$a = \frac{5}{26}d$$

Substituting for a, b, c in the equation of the plane

$$\frac{5}{26}dx - \frac{1}{26}dy + \frac{3}{13}d = d$$

$$\text{Multiplying through by } \frac{26}{d}$$

$$5x - y + 6z = 26$$

- (b) Determine the angle between the plane in (a) and the line  $\frac{x-2}{2} = \frac{y}{-4} = \frac{z-5}{1}$ . (06marks)

Let n = normal vector to the plane

b = parallel vector to the plane

$$\Rightarrow b = 2i - 4j + k$$

$$n = 5i - j + 6k$$

Let  $\theta$  = angle between the line and the plane

$$b \cdot n = |b||n|\sin\theta$$

$$\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -1 \\ 6 \end{pmatrix} = (\sqrt{2^2 + (-4)^2 + 1^2} \cdot \sqrt{5^2 + (-1)^2 + 6^2}) \sin\theta$$

$$10 + 4 + 6 = (\sqrt{21} \cdot \sqrt{62}) \sin\theta$$

$$= \sqrt{1302} \sin\theta$$

$$\sin\theta = \frac{20}{\sqrt{1302}}; \theta = 33.66^\circ \text{ (2D)}$$

4. Three points A(2, -1, 0), B(-2, 5, -4) and C are on a straight line such that  $3AB = 2AC$ . Find the coordinates of C.

**Solution**

**Method 1**

$$3(AB) = 2AC$$

$$\frac{3}{2}AB = AC$$

$$\frac{3}{2}(OB - OA) = OC - OA$$

$$\frac{3}{2} \left[ \begin{pmatrix} -2 \\ 5 \\ -4 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$\frac{3}{2} \begin{pmatrix} -4 \\ 6 \\ -4 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -6 \\ 9 \\ -6 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

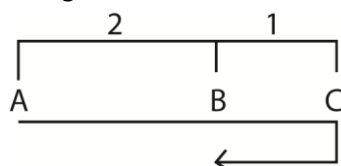
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6 \\ 9 \\ -6 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \\ -6 \end{pmatrix}$$

Hence coordinates of C are (-4, 8, -6)

**Method 2**

Using ratio theorem



C divides externally in the ratio 3: -1

$$OC = \frac{3(OB) - 1(OA)}{3 + (-1)}$$

$$OC = \frac{1}{2} \left\{ 3 \begin{pmatrix} -2 \\ 5 \\ -4 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}$$

$$= \frac{1}{2} \begin{pmatrix} -8 \\ 16 \\ 12 \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \\ -6 \end{pmatrix}$$

Hence C(-4, 8, -6)

**Method 3**

B divides AC internally in ratio of 2:1

$$\begin{pmatrix} -2 \\ 5 \\ -4 \end{pmatrix} = \frac{1}{3} \left\{ 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}$$

$$3 \begin{pmatrix} -2 \\ 5 \\ -4 \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2} \left\{ \begin{pmatrix} -6 \\ 15 \\ 12 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -8 \\ 16 \\ -12 \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \\ -6 \end{pmatrix}$$

Hence C(-4, 8, -6)

5. (a) Line A is the intersection of two planes whose equations are

$3x - y + z = 2$  and  $x + 5y + 2z = 6$ . Find the equation of the line.

$$3x - y + z = 2 \dots\dots\dots (i)$$

$$x + 5y + 2z = 6 \dots\dots\dots (ii)$$

5eqn. (i) + eqn. (ii)

$$\begin{array}{r} 15x - 5y + 5z = 10 \\ + \quad x + 5y + 2z = 6 \\ \hline 16x + 7z = 16 \end{array}$$

Let  $x = \lambda$

$$16\lambda + 7z = 16$$

$$z = \frac{1}{7}(16 - 16\lambda)$$

Substituting for x and z in equation (i)

$$3\lambda - y + \frac{1}{7}(16 - 16\lambda) = 2$$

$$21\lambda - 7y + 16 - 16\lambda = 14$$

$$y = \frac{1}{7}(2 + 5\lambda)$$

$$\text{let } \lambda = 1 + 7\mu$$

$$\Rightarrow x = 1 + 7\mu$$

$$y = \frac{1}{7}(2 + 5(1 + 7\mu))$$

$$= \frac{1}{7}(2 + 5 + 35\mu)$$

$$= 1 + 5\mu$$

$$z = \frac{1}{7}(16 - 16(1 + 7\mu))$$

$$= \frac{1}{7}(16 - 16 - 16 \times 7\mu)$$

$$= -16\mu$$

$$\underline{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ 5 \\ -16 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ 5 \\ -16 \end{pmatrix}$$

$$\frac{x-1}{7} = \frac{y-1}{5} = \frac{-z}{16}$$

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- (b) Given that line B is perpendicular to the plane  $3x - y + z = 2$  and passes through the point  $C(1, 1, 0)$ , find the

(i) Cartesian equation of line B

**Solution**

Normal to the plane  $b = 3i - j + k$

$$\begin{aligned} r &= a + \lambda b \\ &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \\ \frac{x-1}{3} &= \frac{y-1}{-1} = \frac{z}{1} \end{aligned}$$

- (ii) angle between line B and line A in (a) above

**Solution**

Let  $b_1 = 7i + 5j - 16k$  and  $b_2 = 3i - j + k$  and  $\theta =$  angle between line A and line B

$$b_1 \cdot b_2 = |b_1||b_2|\cos\theta$$

$$\begin{aligned} b_1 \cdot b_2 &= (7i + 5j - 16k) \cdot (3i - j + k) \\ &= 21 - 5 - 16 = 0 \end{aligned}$$

$$|b_1||b_2|\cos\theta = 0$$

$$\cos\theta = 0$$

$$\theta = \cos^{-1} 0 = 90^\circ$$

6. (a) The points A and B have position vectors  $a$  and  $b$ . A point C with vector position  $c$  lies on AB such that  $\frac{AC}{AB} = \lambda$ .

Show that  $c = (1 - \lambda)a + \lambda b$ . (04marks)

**Solution**

$$\frac{\overrightarrow{AC}}{\overrightarrow{AB}} = \lambda$$

$$\overrightarrow{AC} = \lambda \overrightarrow{AB}$$

$$\overrightarrow{OC} - \overrightarrow{OA} = \lambda(\overrightarrow{OB} - \overrightarrow{OA})$$

$$c - a = \lambda(b - a)$$

$$c = a + \lambda(b - a)$$

$$= (1 - \lambda)a + \lambda b$$

- (b) The vector equation of two lines are;

$$r_1 = 2i + j + \lambda(i + j + 2k) \text{ and}$$

$$r_2 = 2i + 2j + tk + \mu(i + 2j + k)$$

where  $i, j$  and  $k$  are unit vectors and  $\lambda, \mu$  and  $t$  are constants. Given that the two lines intersect, find

(i) the value of  $t$ .

$$x = 2 + \lambda = 2 + \mu \dots\dots\dots (i)$$

$$y = 1 + \lambda = 2 + 2\mu \dots\dots\dots (ii)$$

$$z = 2\lambda = t + \lambda \dots\dots\dots (iii)$$

From eqn. (i)

$$2 + \lambda = 2 + \mu$$

$$\lambda = \mu$$

From eqn. (ii)

$$1 + \lambda = 2 + 2\mu$$

$$1 + \mu = 2 + 2\mu$$

$$\mu = \lambda = -1$$

From eqn. (iii)

$$2\lambda = t + \lambda$$

$$2(-1) = t - 1$$

$$t = -1$$

- (ii) the coordinates of the point of intersection. (08marks)

$$x = 2 + \lambda = 2 - 1 = 1$$

$$y = 1 + \lambda = 1 - 1 = 0$$

$$z = 2\lambda = 2(-1) = -2$$

$$\therefore (x, y, z) = (1, 0, -2)$$

7. Determine the angle between the lines

$$\frac{x+4}{8} = \frac{y-2}{2} = \frac{z+1}{-4} \text{ and the plane } 4x + 3y - 3z + 1 = 0 \text{ (05marks)}$$

$$\begin{pmatrix} 8 \\ 2 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ -3 \end{pmatrix} =$$

$$\sqrt{8^2 + 2^2 + (-4)^2} \cdot \sqrt{4^2 + 3^2 + (-3)^2} \sin \theta$$

$$32 + 6 + 12 = \sqrt{84} \times \sqrt{34} \sin \theta$$

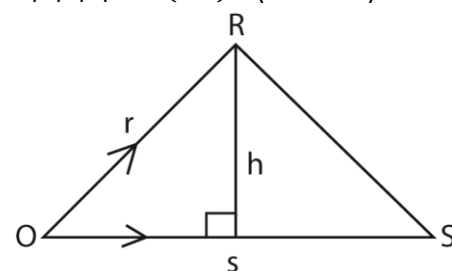
$$\sin \theta = \frac{50}{\sqrt{2856}}$$

$$\theta = 69.33^\circ$$

8. The position vectors of the vertices of a triangle are  $O, r$  and  $s$ , where  $O$  is the origin.

Show that its area ( $A$ ) is given by  $4A^2$

$$= |r|^2 |s|^2 - (r \cdot s)^2. \text{ (06marks)}$$



$$r \cdot s = |r||s| \cos \theta$$

$$(r \cdot s)^2 = |r|^2 |s|^2 \cos^2 \theta$$

$$\sin^2 \theta = 1 - \frac{(r \cdot s)^2}{|r|^2 |s|^2} = \frac{|r|^2 |s|^2 - (r \cdot s)^2}{|r|^2 |s|^2}$$

$$A = \frac{1}{2} |r||s| \sin \theta$$

$$2A = |r||s| \sin \theta$$

$$4A^2 = |r|^2 |s|^2 \sin^2 \theta$$



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$$4A^2 = |r|^2 |s|^2 \cdot \frac{|r|^2 |s|^2 - (r \cdot s)^2}{|r|^2 |s|^2}$$

$$4A^2 = |r|^2 |s|^2 - (r \cdot s)^2$$

Hence, find the area of a triangle when

$$r = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ and } s = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \text{ (06marks)}$$

$$|r|^2 = 2^2 + 3^2 = 13$$

$$|s|^2 = 1^2 + 4^2 = 17$$

$$r \cdot s = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} = (2 \times 1) + (3 \times 4) = 14$$

$$\therefore 4A^2 = 13 \times 17 - 14^2 = 25$$

$$A = \sqrt{\frac{25}{4}} = 2.5 \text{ units}$$

9. Given the plane  $4x + 3y - 3z - 4 = 0$

- (a) Show that the point A(1,1,1) lies on the plane (02marks)

**Solution**

Substitute A(1, 1, 1) into the equation of plane

$$4 \times 1 + 3 \times 1 - 3 \times 1 - 4 = 0$$

Hence the point lies on the plane

- (b) Find the perpendicular distance from the plane to the point B(1, 5, 1) (03marks)

$$d = \frac{|4 \times 1 + 3 \times 5 - 3 \times 1 - 4|}{\sqrt{4^2 + 3^2 + (-3)^2}} = 2.058$$

10. (a) Determine the perpendicular distance of the point (4, 6) from the line  $2x + 4y - 3 = 0$  (03marks)

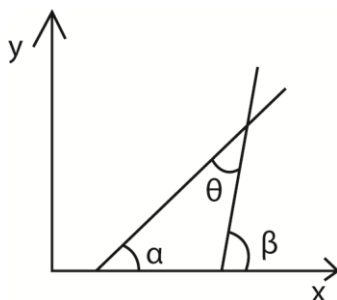
**Solution**

$$\text{Perpendicular distance, } d = \frac{|2(4) + 4(6) - 3|}{\sqrt{2^2 + 4^2}} =$$

$$\frac{29}{\sqrt{20}} = 6.4846$$

- (b) Show that the angle  $\theta$ , between two lines with gradient  $\lambda_1$  and  $\lambda_2$  is given by  $\theta = \tan^{-1} \left( \frac{\lambda_1 - \lambda_2}{1 + \lambda_1 \lambda_2} \right)$ . Hence find the acute angle between the lines  $x + y + 7 = 0$  and  $\sqrt{3}x - y + 5 = 0$  (09 marks)

**Solution**



$$\tan \alpha = \lambda_2, \tan \beta = \lambda_1$$

$$\alpha + \theta = \beta; \theta = \beta - \alpha$$

$$\tan \theta = \tan (\beta - \alpha)$$

$$= \frac{\tan \beta - \tan \alpha}{1 + \tan \alpha \tan \beta}$$

$$= \left( \frac{\lambda_1 - \lambda_2}{1 + \lambda_1 \lambda_2} \right)$$

$$\therefore \theta = \tan^{-1} \left( \frac{\lambda_1 - \lambda_2}{1 + \lambda_1 \lambda_2} \right)$$

$$\text{But } \lambda_1 = -1 \text{ and } \lambda_2 = \sqrt{3}$$

$$\theta = \tan^{-1} \left( \frac{-1 - \sqrt{3}}{1 + (-1 \times \sqrt{3})} \right) = 75^\circ$$

## Topical revision questions

- (a) The position vectors of points A, B and C are  $2i - j + 5k$ ,  $i - 2j + k$  and  $3i + j - 2k$  respectively. Give that L and M are the midpoint of AC and CB. Show that LM is parallel to BA.

(b) Show that the points with position vectors  $4i - 8j - 13k$ ,  $5i - 2j - 3k$  and  $5i + 4j + 10k$  are vertices of a triangle
- (a) The position vector of a body of mass 12 .5kg is  $8t^2i + 6tj$  meters at a given time t. determine the

  - Velocity after 4s.  $[64i + 6j]$
  - The force acting on the body  $[200N \text{ horizontally}]$

(b) The vector OA is represented by displacement vector a and OB by b. Point R divides AB in the ratio  $\lambda : \mu$ . Find the position vector of R in terms of vectors a and b and the scalars  $\lambda$  and  $\mu$ .

$$\left[ r = \frac{\mu}{\mu + \lambda} a + \frac{\lambda}{\mu + \lambda} b \right]$$

(c) If the points P, Q and R have position vector p, q and r respectively, and M is the midpoint of QR, show that the position vector of N is a point on PM that  $PN : NM = 2 : 1$  is  $\frac{1}{3}(p + q + r)$ .
- (a) Determine a unit vector perpendicular to the plane containing the points A(0, 2, -4), B(2, 0, 2) and C(-8, 4, 0)  $\sqrt{230}$

(b) Find the equation of the plane  $[5x + 14y + 3z = 16]$

(c) Show that the point (5, -4, 3) lies on the plane [does on lie on the line]

(d) Write down the equation in form of  $r = a + \mu b$  of the perpendicular through

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- the point  $P(3, 4, 2)$  to the plane  
 $[r = 3i + 4j + 2k + \mu(4i + 14j + 3k)]$
- (e) If the perpendicular meets the plane at N. determine NP [4.022units]
4. (a) A and B are points whose position vectors are  $a = 2i + k$  and  $b = i - j + 3k$  respectively. Determine the position vector of the point P that divides AB in the ratio 4:1  
 $\left[\frac{1}{5}(6i - 4j + 16k)\right]$   
 (b) Given that  $a = i - 3j + 3k$  and  $b = -i - 3j + 2k$  determine  
 (i) The equation of the plane containing a and b  $[-3x + 5y + 6z = 0]$   
 (ii) The angle the line  $\frac{x-4}{4} = \frac{y}{3} = \frac{z-1}{2}$  makes with the plane in (i) above. [19.446°]
5. A vector XY of magnitude a units makes an angle of  $\alpha$  with the horizontal. Another vector YZ beginning from the end Y, inclined at an angle  $\beta$  to the same horizontal axis is of magnitude b units. If  $\theta$  is the angle between the positive directions of the two vectors, where  $\theta = \beta - \alpha$  is acute, show that the resultant vector XZ has a magnitude xz equal to  $\sqrt{a^2 + b^2 + 2ab\cos\theta}$  units and is inclined at an angle  $\alpha + \sin^{-1}\left(\frac{b\sin\theta}{xz}\right)$  to the horizontal. Hence or otherwise calculate the magnitude and direction of the resultant vector of vectors XY and YZ, inclined at 30° and 75° to the horizontal and magnitude 9 and 6 units respectively. [47.7°]
6. (a) The position vector of points A and B with respect to the origin O are  
 $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$  and  $\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$  respectively. Determine the equation of the line AB  
 $\left[r = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}\right]$   
 (b) Find the equation of the plane OPQ where O is the origin and P and Q are points whose position vectors are  
 $\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$  respectively  $\left[r \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = 0 \text{ or } -2x + z = 0\right]$
- (c) (i) Given that R is a point at which AB meets the plane OPQ, find the coordinates of R  $[(7, -7, 14)]$   
 (ii) Show that the point S(1, -1, 2) lies on OR.
7. The points A, B, and C have position vectors  $(-2i + 3j)$ ,  $(i - 2j)$ , and  $(8i - 5j)$  respectively.  
 (a) Find the vector equation of line AC  
 $\left[r = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 10 \\ -8 \end{pmatrix}\right]$   
 (b) Determine the coordinates of D if ABCD is a parallelogram [5, 0]  
 (c) Write down the vector equation of the line through which point B perpendicular to AC and find where it meets AC  $\left[\frac{93}{41}, -\frac{17}{41}\right]$
8. (a) In the triangle ABC, P is the point on BC such that  $BP : PC = \lambda : \mu$ . Show that  $(\lambda + \mu)AP = \lambda AC + \mu AB$   
 (b) Three non collinear points A, B, and C have position vectors a, b, and c respectively with respect to O. The point M on AC is such that  $AM:MC = 2:1$  and the point N on AB is such that  $AN:NB = 2:1$ .  
 (i) Show that  $BM = \frac{1}{3}a - b + \frac{2}{3}c$ , and find a similar expression for CN  
 $\left[CN = \frac{1}{3}a + \frac{2}{3}b - c\right]$   
 (ii) The line BM and CN intersect at L. Given that  $BL = rBM$  and  $CL = sCN$ , where r and s are scalars, express BL and CL in terms of r, s, a, b, and c.  
 $\left[BL = \frac{1}{3}sa - rb + \frac{2}{3}rc;\right]$   
 $\left[CL = \frac{1}{3}sa - \frac{2}{3}sb - sc;\right]$   
 (iii) Hence by using triangle BLC or otherwise find r and s  $\left[r = \frac{2}{5}, s = \frac{3}{5}\right]$
9. Find the distance of the point  $(-2, 0, 6)$  from the plane  $2x - y + 3z = 21$  [1.8708 units]
10. ABCD is a quadrilateral with  $A(2, -2)$ ,  $B(5, -1)$ ,  $C(6, 2)$  and  $D(3, 1)$ . Show that the quadrilateral is a rhombus.
11. The points  $P(4, -6, 1)$ ,  $Q(2, 8, 4)$  and  $R(3, 7, 14)$  lie in the same plane. Find the angle between PQ and QR. [84.5°]

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12. (a) Given that  $OP = \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}$  and  $OQ = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ , find the coordinates of the point R such that  $PR:PQ = 1:2$  and the points P, Q and R are collinear. [R(2.5, -1.5, 3.5)]
- (b) Show that the vector  $5i - 2j + k$  is perpendicular to the line  $r = i - 4j + \lambda(2i + 3j - 4k)$ .
- (c) Find the equation of the plane through the point with position vector  $5i - 2j + 3k$  perpendicular to the vector  $3i + 4j - k$ .  $[3x + 4y - z = 4]$
13. Calculate the area of a triangle with vertices (-1, 3), (5, 2), (4, -1) [7.6811 sq. units]
14. PQRS is a quadrilateral with vertices P(1, -2), Q(4, -1), R(5, 2) and S(2, 1). [show that PQ is parallel to SR and PS is parallel to QR and that  $|PQ| = |SP| = |QR| = |PS|$  and PR and QS are perpendicular]
15. The vector equation of lines P and Q are given as  $r_p = t(4i + 3j)$  and  $r_q = 2i + 12j + 5(i - j)$ . Use the dot product to find the angle between P and Q.  $[8.13^\circ]$
16. The vector equations of two lines are  $r_1 = \begin{pmatrix} 5 \\ -6 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$  and  $r_2 = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix}$ . Determine the point where  $r_1$  meets  $r_2$ . [8, -5]
17. The equation of three planes P, Q and R are  $2x - y + 3z = 3$ ,  $3x + y + 2z = 7$  and  $x + 7y - 5z = 13$  respectively. Determine where the three planes intersect.  $[(-2, 5, 4)]$
18. (a) Find in Cartesian form the equation of the line passing through the points A(1, 2, 5), B(1, 0, 4) and C(5, 2, 1). [since  $AB = BC$ , points A, B and C are not collinear]
- (b) Find the angle between the line  $\frac{x+4}{8} = \frac{y-2}{2} = \frac{z+1}{-4}$  and the plane  $4x + 3y - 3z + 1$ .  $[69.32^\circ]$
19. Show that the equation of the line through points (1, 2, 1) and (4, -2, 2) is given as  $\frac{x-1}{3} = \frac{y-2}{-4} = \frac{z-1}{1}$
20. (a) Show that the equation of the plane through points with position vector  $-2i + 4k$  perpendicular to vector  $i + 3j - 2k$  is  $x + 3y - 2z + 10 = 0$
- (b)(i) Show that the vector  $2i - 5j + 3.5k$  is perpendicular to the plane  $r = 2i - j + \lambda(4i + 3j + 2k)$ .
- (ii) Calculate the angle between the vectors  $3i - 2j + k$  and the line in b(i) above.  $[66.6^\circ]$
21. Find the point of intersection of the line  $\frac{x}{5} = \frac{y+2}{2} = \frac{z-1}{4}$  with the plane  $3x + 4y + 2z - 25 = 0$  [(5, 0, 5)]
22. (a) Find the Cartesian equation of the plane through A(0, 3, -4), B(2, -1, 2) and C(7, 4, -1). Show that Q(10, 13, -10) lies in the same plane
- (b) Express the equation of the plane in (a) in the scalar product form.
- $$\left[ r \cdot \begin{pmatrix} 3 \\ -6 \\ -5 \end{pmatrix} = 2 \right]$$
- (c) Find the area of ABC in (a) [25.1 Sq. Units]
23. The vertices of a triangle are P(2, -1, 5), Q(7, 1, -3) and R (12, -2, 0). Show that  $\angle PQR = 90^\circ$ . Find the coordinates of S if PQRS is a rectangle [(8, -4, 8)]
24. (a) Find the equation of the perpendicular line from Point A =  $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$  onto the line  $r = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ . What is the distance of A from r.
- $$\left[ OP = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} -\frac{4}{9} \\ \frac{14}{9} \\ -\frac{8}{9} \end{pmatrix}; 1.795 \text{ units} \right]$$
- (b) Find the angle contained between the line OR and x - y plane, where  $OR = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$  [41.81°]
25. Given that vectors  $OA = (3, -2, 5)$  and  $OB = (9, 1, -1)$ , find the position vector of point C such that C divides AB internally in the ratio 5:-3
- $$\left[ xi + yj + zk = 18i + \frac{11}{2}j - 10k \right]$$
26. (a) In a triangle ABC, the altitudes from B and C meet the opposite sides at E and F

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- respectively. BE and CF intersect at O. Taking O as the origin, Use the dot product to prove that AO is perpendicular to BC.
- (b) Prove that  $\angle ABC = 90^\circ$  given that A is (0, 5, -3), B(2, 3, -4) and C(1, -1, 2). Find the coordinates of D if ABCD is a rectangle. [(-1, 1, 3)]
27. Find the equation of the plane through the point (1, 2, 3) and perpendicular to the vector  $r = 4i + 5j + k$ . [ $4x - 5y + z = 17$ ]
28. (a) Find the equation of the line through A(2, 2, 5) and B(1, 2, 3).  

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$$
- (b) If the line in (a) above meets the line  $\frac{x-1}{1} = \frac{y-2}{0} = \frac{z-1}{3}$  at P, find the  
 (i) Coordinate of P [3, 2, 7]  
 (ii) Angle between the lines [ $171.9^\circ$ ]
29. Given that the vector  $a i - 2j + k$  and  $2a i + aj - 4k$  are perpendicular, find the values of a. [-1, 2]
30. (a) Determine the coordinates of the point of intersection of the line  $\frac{x+1}{2} = \frac{y-3}{5} = \frac{z+2}{-1}$  and the plane  $x + y + z = 12$  [3, 13, -4]  
 (b) Find the angle between the line  $\frac{x+1}{2} = \frac{y-3}{5} = \frac{z+1}{-1}$  and the plane  $x + y + z = 12$  [ $50.7685^\circ$  or  $39.2515^\circ$ ]
31. Find the point of intersection of the plane  $11x - 3y + 7z = 8$  and the line  $r = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$  where  $\mu$  is a scalar [-4, -1, 7]
32. (a) Given the vector  $a = 3i - 2j + k$  and  $b = i - 2j + 2k$ , find  
 (i) the acute angle between the vectors. [ $36.7^\circ$ ]  
 (ii) vector  $c$  such that it is perpendicular to both vectors  $a$  and  $b$ . [ $c \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$ ]
- (b) Given that  $OA = a$  and  $OB = b$ , Point R is on OB such that  $OR : RB = 4:1$ . Point P is on BA such that  $BP:PA = 2:3$  and when RP and OA are both produced they meet at point Q. Find  
 (i) OR and OP in terms of  $a$  and  $b$ .
- $$\left[ OR = \frac{4}{5}b, OP = \frac{1}{5}(2a + 3b) \right]$$
- (ii) OQ in terms of  $a$ . [ $\frac{8}{5}a$ ]
33. A point P has coordinates (1, -2, 3) and a certain plane has equation  $x + 2y + 2z = 8$ . The line through P parallel to the line  $\frac{x}{3} = \frac{y+1}{-1} = z + 1$  meets the plane at a point Q. Find the coordinates of Q [ $(6, \frac{-11}{3}, \frac{14}{3})$ ]
34. Given that the position vectors of A, B, and C are  $OA = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ ,  $OB = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$  and  $OC = \begin{pmatrix} 7 \\ -2 \\ 2 \end{pmatrix}$   
 (a) Prove that A, B and C are collinear  
 (b) Find the angle between OA and OB [ $106.1^\circ$ ]  
 (c) If OABD is parallelogram, find the position vectors of E and F such that E divides DA in ratio 1:2 and F divides it externally in ratio 1:2.  

$$\left[ E = \begin{pmatrix} \frac{5}{3} \\ 2 \\ -\frac{4}{3} \end{pmatrix}, F = \begin{pmatrix} 3 \\ 10 \\ -8 \end{pmatrix} \right]$$
35. Given the vectors  $a = i - 3j + 3k$  and  $b = -i - 3j + 2k$ ; find the  
 (i) Acute angle between vectors  $a$  and  $b$  [ $30.86^\circ$ ]  
 (ii) Equation of the plane containing  $a$  and  $b$  [ $-3x + 5y + 6z = 0$ ]
36. The position vectors of A and B are  $OA = 2i - 4j - k$  and  $OB = 5i - 2j + 3k$  respectively. The line AB is produced to meet the plane  $2x + 6y - 3z = -5$  at point C. Find the  
 (a) coordinates of C [(8, 0, 7)]  
 (b) angle between AB and the plane [ $9.169^\circ$ ]
37. The points P(2, 3), Q(-11, 8) and R(-4, -5) are vertices of a parallelogram PQRS which has PR as the diagonal. Find the coordinates of the vertex S. [S(9, -10)]
38. (a) Find the angle between the planes  $x - 2y + z = 0$  and  $x - y = 1$  [ $30^\circ$ ]  
 (b) Two lines are given by the parametric equation:  $-i + 2j + k + t(i - 2j + 3k)$  and

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$-3i - + pj + 7k + s(i - j + 2k)$ . If the lines intersect, find

(i) values of  $t$ ,  $s$  and  $p$ .

$$[t = 10, s = 12, p = -6]$$

(ii) coordinates of the points of intersection  $[(9, 18, 31)]$

39. given the points  $A(-3, 3, 4)$ ,  $B(5, 7, 2)$  and  $C(1, 1, 4)$ , find the vector equation of a line which joins the mid-point of  $AB$  and  $BC$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

40. (a) The equation of the plane  $R$  is

$$r \cdot \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} = 16. \text{ Where } r \text{ is the position}$$

vector of  $R$ . Find the perpendicular distance of the plane from the origin  $[2.971 \text{ units}]$

(b) Find the Cartesian equation of the plane through the point  $P(1, 0, -2)$  and  $Q(3, -1, 1)$  and parallel to the line with a vector equation

$$r = 2i + (2\alpha - 1)j + (5 - \alpha)k$$

$$[-5x + 2y + 4z + 13 = 0]$$

41. Find the equation of the line through point  $(2, 3)$  and perpendicular to line  $x + 2y + 5 = 0$   
 $[y = 2x - 1]$

42. Show that the points  $A$ ,  $B$  and  $C$  with position vectors  $3i + 3j + k$ ,  $8i + 7j + 4k$  and  $11i + 4j + 5k$  respectively are vertices of a triangle.

43. (a) Find the angle between the lines

$$x = \frac{y-1}{2} = \frac{z-2}{3} \text{ and } \frac{x}{2} = \frac{y+1}{3} = \frac{z+2}{4}. [8.53^\circ]$$

(b) Find in vector form the equation of the line of intersection of two planes

$$2x + 3y - 2z = 4 \text{ and } x - y + 2z = 5$$

$$\begin{bmatrix} r = \begin{pmatrix} 0 \\ \frac{13}{5} \\ \frac{19}{5} \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \text{ or} \\ r = \begin{pmatrix} \frac{19}{5} \\ \frac{-6}{5} \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \text{ or} \\ r = \begin{pmatrix} \frac{13}{5} \\ 0 \\ \frac{-6}{5} \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \end{bmatrix}$$

44. A line passes through the point  $A(4, 6, 3)$  and  $B(1, 3, 3)$ .

- (a) Find the vector equation of the line

$$\begin{bmatrix} r = \begin{pmatrix} 4 \\ 6 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix} \end{bmatrix}$$

(b) Show that the point  $C(2, 4, 3)$  lies on the line in (a) above.

45. Triangle  $OAB$  has  $OA = a$  and  $OB = b$ .  $C$  is a point on  $OA$  such that  $OC = \frac{2}{3}a$ .  $D$  is the mid-point of  $AB$ . When  $CD$  is produced it meets  $OB$  produced at  $E$ , such that  $DE = nCD$  and  $BE = kb$ . Express  $DE$  in terms of

(a)  $n$ ,  $a$  and  $b$   $\left[ \frac{5n}{6}a + \frac{n}{2}b \right]$

(b)  $k$ ,  $a$  and  $b$   $\left[ \frac{1}{2}a + \frac{2k-1}{2}b \right]$

Hence find the values of  $n$  and  $k$ .

$$\left[ n = \frac{3}{5}, k = \frac{1}{5} \right]$$