3. DISCRETE PROBABILITY DISTRIBUTION.

Introduction:

A **discrete probability distribution** shall be understood as a *probability distribution* characterized by a **probability mass function**. Thus, the distribution of a **random variable** *X* is discrete, and *X* is then called a **discrete random variable**, if

$$\sum_{i=1}^{n}(X = x) \;=\; 1$$

as x runs through the set of all possible values of X. It follows that such a random variable can assume only a **finite or countably infinite** number of values.

In cases more frequently considered, this set of possible values is a topologically discrete set in the sense that all its points are **isolated points**. But there are discrete random variables for which this countable set is **dense** on the real line (for example, a distribution over **rational numbers).**

Among the most well-known discrete probability distributions that are used for statistical modeling are the Poisson distribution, the Bernoulli distribution, the binomial distribution, the geometric distribution, and the negative binomial distribution. In addition, the discrete uniform distribution is commonly used in computer programs that make equal-probability random selections between a number of choices.

Mathematically:

A discrete random variable X is defined as a probability function P(X = x), which gives the probability that X takes on values of x as a probability denoted by P(X = x) or P_i

Where $\sum_{i=1}^{n} P_i = 1$

(i) **EXPECTATION OR EXPECTED VALUE (E(X)) OR MEAN OF DISCRETE RANDOM VARIABLE**

Mean (μ) = E (X)

$$= \sum_{\text{all } x}^{n} xP(X = x)$$

If we replace P(X = x) with Pi and for which X takes on the value xi for i = 1, 2, … S, n then,

Mean(μ) = E(X)

$$= \sum_{i=1}^{n} P_i X_i \quad \textbf{for } i = 1, 2, , , , n$$

(ii) VARIANCE AND STANDARD DEVIATION OF A DISCRETE RANDOM VARIABLE

$$\text{Var (X)} = \sum_{\text{all } x} P_x (X - \mu)^2$$

$$= E(X^2) - [E(X)]^2$$

$$= E(X^2) - \mu^2$$

Standard deviation $= \sqrt{\text{Var}X}$

$$= \sqrt{E(X^2) - \mu^2}$$

Where $E(X^2) = \sum_{\text{all } x} x^2 p(X = x)$

$$= \sum_{i=1}^{n} P_i x_i^2 \quad \text{for } i = 1, 2, \ldots, n$$

(iii) MEDIAN (M) OF A DISCRETE RANDOM VARIABLE

Cumulate the probabilities from the top downwards until a value of 0.5 or slightly above 0.5 is obtained, also cumulate the probabilities from the bottom upwards until a value of 0.5 or slightly above 0.5 is obtained, for the values obtained read off the corresponding X values. If they are different find the average of the two. This explained as below.

$$\sum_{i=1}^{M} P_i \quad \geq \quad \frac{1}{2} \quad \leq \quad \sum_{i=M}^{N} P_i$$

If the two different values that satisfy the inequality are X_m and X_{m+1} then the median (m) $= \dfrac{X_m + X_{m+1}}{2}$

(iv) MODE OF A DISCRETE RANDOM VARIABLE

The mode of a discrete random variable is the value of X with the highest probability.

(v) DISCRETE UNIFORM DISTRIBUTION OR RECTANGULAR DISTRIBUTION

A discrete random variable X, taking on values 1, 2, 3,…k such that:

$$P(X = x) = \begin{cases} \dfrac{1}{k} & \text{for } x = 1, 2, 3, \ldots, k \\ 0 & \text{otherwise} \end{cases}$$

follows a rectangular or discrete uniform distribution.

Note:

$E(a) = a$

$E(ax) = a\,E(x)$,

$E(ax + b) = a\,E(x) + b$,

$\text{Var}(ax) = a^2\,\text{Var}(x)$

$\text{Var}(ax + b) = a^2\,\text{Var}(x)$

$E(x + y) = E(x) + E(y)$,

$E(x - y) = E(x) - E(y)$,

$E(ax + by) = a\,E(x) + b\,E(y)E(ax - by) = a\,E(x) - b\,E(y)$,

$\text{Var}(x + y) = \text{Var}(x - y) = \text{Var}(x) + \text{Var}(y)$

$\text{Var}(ax + by) = \text{Var}(ax - by) = a^2\,\text{Var}(x) + b^2\,\text{Var}(y)$

Example 1. A random variable x has probability function P(0) = 0.1. P(1) = 0.3, P(2) = 0.4 and P(4) = 0.2. Determine mean and standard deviation:

Solution.

X_i	0	1	2	4

P_i	0.1	0.3	0.4	0.2

Mean $(\mu) = \sum X_i P_i$

$= 0 \text{ x } 0.1 + 1 \text{ x } 0.3 + 2 \text{ x } 0.4 + 4\text{x}0.2$

Mean (μ) = 1.9

Standard deviation $= \sqrt{E(X^2) - \mu^2}$

$E(X^2) = \sum X_i^2 P_i = 02 \text{ x } 0.1 + 12 \text{ x } 0.3 + 22 \text{ x } 0.4 + 42 \text{ x } 0.2$

$= 5.1$

Standard deviation $= \sqrt{5.1 - (1.9)^2}$

$=$ **1·2207**

Example 2: The number of times a machine breaks down every month is a discrete random variable X with probability distribution.

$$\mathbf{P(x = x) =} \begin{cases} k\left(\frac{1}{4}\right)^x & x = 0, 1, 2, 3, ... \\ 0 & \text{otherwise} \end{cases}$$

Where k is a constant.

Determine the probability that the machine will break down not more than two times a month.

Solution:

$$k\left[\left(\frac{1}{4}\right)^0 + \left(\frac{1}{4}\right)^1 + \left(\frac{1}{4}\right)^2 + \cdots + \left(\frac{1}{4}\right)^n\right] = 1$$

$$k\left[1 + \frac{1}{4} + \frac{1}{4^2} + ... + \frac{1}{4^n}\right] = 1 \quad .... \quad *$$

It's a geometric progression (G.P) with a =1 and common ratio r = $\frac{1}{4}$

Sum of a G.P. $= \frac{a}{1-r}$

$$= \frac{1}{1 - 1/4}$$

$$= \frac{1}{3/4} = \frac{4}{3} \quad .... \quad **$$

Substitute ** into *

$$\frac{4k}{3} = 1$$

$$\therefore k = \frac{3}{4}$$

Probability that the machine will break down not more than two times.

$P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= \frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3}$$

$$= \frac{48 + 12 + 3}{4^3}$$

$$= \frac{63}{64} = \mathbf{0.9844}$$

Example 3. **A discrete random variable X has the following probability distribution.**

X	1	2	3	4	5
P(X = x)	**k**	**2k**	**3k**	**4k**	**5k**

(i) Determine the value of k.
(ii) Evaluate P(2 < X < 4)
(iii) Calculate mean, median, mode and standard deviation.

Solution.

(i) $\sum P_i = 1$

$k + 2k + 3k + 4k + 5k = 1$

$15k = 1$

$\therefore k = \frac{1}{15}$

X	1	2	3	4	5
P(X = x)	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

(ii) $P(2 < X < 4) = P(X = 3)$

$= \frac{3}{15}$

a. Mean (μ) $= \sum P_i X_i$

$= \frac{1}{15} \times 1 + \frac{2}{15} \times 2 + \frac{3}{15} \times 3 + \frac{4}{15} \times 4 + \frac{5}{15} \times 5$

$= \frac{55}{15} = \frac{11}{3}$

$=$ **3.6667**

X	P_i	Cumulative probability
1	$\frac{1}{15}$	$\frac{1}{15}$
2	$\frac{2}{15}$	$\frac{3}{15}$
3 Location of median	$\frac{3}{15}$	$\frac{6}{15}$
4	$\frac{4}{15}$	$\frac{10}{15}$ $\frac{9}{15}$
5	$\frac{5}{15}$	$\frac{5}{15}$

Median = 4

The highest frequency $= \frac{5}{15}$

$\therefore$ **Mode = 5**

Standard deviation $= \sqrt{\sum X_i^2 P_i - (\mu)^2}$

$$= \sqrt{1^2 \times \frac{1}{15} + 2^2 \times \frac{2}{15} + 3^2 \times \frac{3}{15} + 4^2 \times \frac{4}{15} + 5^2 \times \frac{5}{15} - \left(\frac{11}{3}\right)^2}$$

$= \sqrt{1.5556}$

$= \mathbf{1.2471}$

Example 5. **A discrete random variable can assume values 0, 1, 2, 3, only. Given that (P(X $\leq$ 2) = 0.9, P(X $\leq$ 1) = 0.5 and E(X) = 1.4 determine:**

- **(i)** **P(X =1)**
- **(ii)** **P(X = 0)**
- **(iii)** **Median and mode**
- **(iv)** **Standard deviation**

Solution:

$P_0 + P_1 + P_2 + P_3 = 1$ (i)

$P_0 + P_1 + P_2 = 0.9$ (ii)

Substitute (ii) into (i)

$0.9 + P_3 = 1$

:. $P(X = 3) = 0.1$

$P_0 + P_1 = 0.5$ (iv)

Substitute (iv) into (ii)

$0.5 + P_2 = 0.9$

$P_2 = 0.4$

$\mathbf{P(X = 4) = 0.4}$

$E(X) = X_oP_o + X_1P_1 + X_2P_2 + X_3P_3$

$= 1.4$

$0 \text{ x } P_0 + P_1 + 2 \text{ x } 0.4\ 3 \text{ x } 0.1 = 1.4$

$P_1 = 0.3$

:. $\mathbf{P(X = 1) = 0.3}$

(ii) $P_0 + P_1 + P_2 + P_3 = 1$

$P_0 + 0.3 + 0.4 + 0.1 = 1$

$P_0 = 0.2$

$\mathbf{P(x = 0) = 0.2}$

(iii)

X	P	Cumulative probability
0	0.2	0.2
1	0.3	0.5
2	0.4	0.6
3	0.5	0.1

Location of median

Median $= \frac{1+2}{2} = \frac{3}{2}$

$= 1.5$

The highest probability is 0.5

Mode = 2

(iv) Standard deviation $= \sqrt{\sum X^2P_i - (\mu)^2}$

$= \sqrt{(0^2 \text{ x } 0.2 + 1^2 \text{ x } 0.3 + 2^2 \text{ x } 0.5 + 3^2 \text{ x } 0.1 - (1.4)^2}$

$= \sqrt{3.2 - 1.96} = \mathbf{1.1136}$

Example 6.Find the variance of the sum of the scores when an ordinary die is thrown 10 times.
Solution:
To find the variance of the sum of the scores when an ordinary die is thrown 10 times, we have;

$$\text{Var}(x_1 + x_2 + x_3 + \ldots\ldots\ldots + x_{10})$$

But since it the same die, that is repeatedly thrown,

$$x_1 = x_2 = \ldots\ldots\ldots x_{10} = x$$

$$\text{Var}(x_1 + x_2 + \ldots + x_{10}) = \text{Var}(x_1) + \text{Var}(x_2) + \ldots\ldots\ldots + \text{Var}(x_{10})$$
$$= \text{Var}(x) + \text{Var}(x) + \ldots\ldots\ldots + \text{Var}(x_{10})$$
$$= 10\,\text{Var}(x)$$

x	1	2	3	4	5	6
x2	1	4	9	16	25	36
P(x = x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\text{Var}(x) = E(x^2) - [E(x)]^2$$

Now

$$E(x) = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right)$$
$$= \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6}$$
$$= \frac{21}{6} = \frac{7}{2}$$

$$E(x^2) = 1\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 9\left(\frac{1}{6}\right) + 16\left(\frac{1}{6}\right) + 25\left(\frac{1}{6}\right) + 36\left(\frac{1}{6}\right)$$
$$= \frac{91}{6}$$

$$\text{Var}(x) = \frac{91}{6} - \left(\frac{7}{2}\right)^2$$

The required variance is therefore given by;

$$10\,\text{Var}(x) = 10\left(\frac{91}{6} - \frac{49}{4}\right) = 29\frac{1}{6}$$

Thus, the variance of the sum of the scores when an ordinary die is thrown is $29\frac{1}{6}$

Example 7. A committee of 3 is to be chosen from 4 girls and 7 boys. Find the expected number of girls on the committee, if the members of the committee are chosen at random.

Solution.
This is a discrete random variable.
The number of ways of choosing 3 committee members from a total of 11 people is

$$\binom{11}{3} = \frac{11!}{3!8!} = 165$$

The probability that the committee contains only x girls is

$$P(X = x) = \frac{\binom{4}{x}\binom{7}{3-x}}{\binom{11}{3}} = \frac{\binom{4}{x}\binom{7}{3-x}}{165}$$

When $x = 0$

$$P(X=0) = \frac{\binom{4}{0}\binom{7}{3}}{165} = \frac{35}{165}$$

When x = 1

$$P(X=1) = \frac{\binom{4}{1}\binom{7}{2}}{165} = \frac{84}{165}$$

When x = 2

$$P(X=2) = \frac{\binom{4}{2}\binom{7}{1}}{165} = \frac{42}{165}$$

When x = 3

$$P(X=3) = \frac{\binom{4}{3}\binom{7}{0}}{165} = \frac{4}{165}$$

The probability distribution function is then given below

x	0	1	2	3
P(X = x)	$\frac{35}{165}$	$\frac{84}{165}$	$\frac{42}{165}$	$\frac{4}{165}$

The expected number of girls E(X) is

$$E(X) = \sum_{x=0}^{3} XP(X=x)$$

$$= 0\left(\frac{35}{165}\right) + 1\left(\frac{84}{165}\right) + 2\left(\frac{42}{165}\right) + 3\left(\frac{4}{165}\right)$$

$$\mathbf{E(X)} = \frac{12}{11}.$$

Example 8. A random variable X has the probability function.

$$\mathbf{f(x)} = \begin{cases} k2^x & ; \; x = 0,1,2,\ldots,6 \\ 0 & , \; \text{else where} \end{cases}$$

Determine: **(i)** **the value of k**
(ii) **E(X)**
(iii) **P(X < 4/ X > 1)**

Solution:

(i) $k\{2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^4 + 2^6 = 1$

$k\{1 + 2 + 4 + 8 + 16 + 32 + 64\} = 1$

$127k = 1$

$$\mathbf{k} = \frac{1}{127}$$

(ii) $E(X) = \sum P_i X_i$

$$= 0\times\frac{1}{127} + 1\times\frac{2}{127} + 2\times\frac{4}{127} + 3\times\frac{8}{127} + 4\times\frac{16}{127} + 5\times\frac{32}{127} + 6\times\frac{64}{127}$$

$$= \frac{642}{127}$$

$= \mathbf{5.0551}$

(iii) Let A = x < 4 B = x > 1

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = \frac{4}{127} + \frac{8}{127}$$
$$= \frac{12}{127}$$
$$P(B) = 1 - \frac{2}{127}$$
$$= \frac{125}{127}$$
$$= \frac{12/127}{125/127}$$
$$= \frac{12}{127} \qquad = \qquad \mathbf{0.096}$$

Example 9. **A random variable X has probability density function P(X = x) given below.**

$$\mathbf{P(X = x) =} \begin{cases} \dfrac{x}{2k} & , \ x = 1, 2, 3, \ldots n \\ 0 & , \ \text{else where} \end{cases}$$

Given that E(X) = 3, Find:

- **(i)** **the value of k**
- **(ii)** **the value of n**
- **(iii)** **the median and mode**
- **(iv)** **variance**

Solution:

X	1	2	3	…..	n
P(X = x)	$\frac{1}{2K}$	$\frac{2}{2K}$	$\frac{3}{2K}$		$\frac{n}{2k}$

From $\sum P_i = 1$

$$\frac{1}{2k} + \frac{2}{2k} + \frac{3}{2k} + \ldots + \frac{n}{2k} = 1$$
$$\frac{1}{2k}(1 + 2 + 3 + \ldots + n) = 1$$
$$\sum_{1}^{n} n = \frac{n}{2}(n + 1)$$
$$\frac{n(n+1)}{4k} = 1 \quad ,$$
$$k = \frac{n(n+1)}{4} \qquad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$E(X) = \sum P_i X_i$$
$$= \frac{1}{2k} + \frac{2^2}{2k} + \frac{3^2}{2k} + \ldots + \frac{n^2}{2k}$$

$$\sum^{n} n^2 = \frac{n}{6}(n + 1)(2n + 1) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots * $$

$$E(X) = \frac{n}{2k}(1 + 2^2 + 3^2 + \ldots + n^2) = 3 \qquad \ldots. **$$

Substitute * into **

$$\frac{n}{12k}(n+1)(2n+1) = 3$$

$$\frac{n(n+1)(2n+1)}{36} = k \quad \ldots\ldots\ldots\ldots(ii)$$

Equate equation (i) and (ii)

$$\frac{n(n+1)}{4} = \frac{n(n+1)(2n+1)}{36}$$

$$\frac{36}{4} = 2n + 1$$

$$2n + 1 = 9$$

$$2n = 8$$

$$\mathbf{n = 4}$$

Using equation (i)

$$k = \frac{n(n+1)}{4}$$

$$= \frac{4 \times 5}{4}$$

$$= 5$$

(i) $\mathbf{k = 5}$

(ii)

X	1	2	3	4
P(X =x)	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$

X	P_i	Cumulative probability
1	$\frac{1}{10}$	$\frac{1}{10}$
2	$\frac{2}{10}$	$\frac{3}{10}$
3	$\frac{3}{10}$	$\frac{6}{10}$ $\frac{7}{10}$
4	$\frac{4}{10}$	$\frac{4}{10}$

Location of median

Median = 3

The highest probability $= \frac{4}{10}$

:. Mode = 4

$\text{Var}(X) = \sum X_i^2 P_i - (E(x))^2$

$= 1^2 \times \frac{1}{10} + 2^2 \times \frac{2}{10} + 3^2 \times \frac{3}{10} + 4^2 \times \frac{4}{10}$

$= 10 - (3)^2 = 1$

:. Var(X) = 1

Example 10. **A discrete random variable X, has the following probability distribution.**

X	**1**	**2**	**3**	**4**
P(X)	$\frac{1}{16}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{16}$

Find: (i) the mean

(ii) median and mode

(iii) variance of x

Solution:

(i) Mean $E(X) = \sum xP(X)$

$= 1 \times \frac{3}{16} + 2 \times \frac{8}{16} + 3 \times \frac{4}{16} + 4 \times \frac{1}{16}$

$= \frac{35}{16}$

$=$ **2.25**

(ii)

x	P	Cumulative probability
1	$\frac{3}{16}$	$\frac{3}{16}$
2	$\frac{8}{16}$	$\frac{11}{16}$ $\frac{12}{16}$
3	$\frac{4}{16}$	$\frac{4}{16}$
4	$\frac{1}{16}$	$\frac{1}{16}$

Location of median

:. Median = 2

Highest probability $= \frac{8}{16}$

:. Mode = 2

(iii) $\text{Var}(X) = \sum x^2 P(X) - (E(X))^2$

$= \left[1 \times \frac{3}{16} + 4 \times \frac{8}{16} + 9 \times \frac{4}{16} + 16 \times \frac{1}{16}\right] - \left(\frac{35}{16}\right)^2$

$= \frac{87}{16} - \left(\frac{35}{16}\right)^2$

$= \frac{167}{256}$

$=$ **0.6523**

Example 11: A discrete random variable X is represented by the probability function

$$\mathbf{P(X = x) =} \begin{cases} \dfrac{1+x}{kx} & ; \quad \text{for } x\, 1, 2, 3, \ldots, 6 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Find **(i) the value of k**
(ii) Expectation of X
(iii) Mode and median
(iv) P(X $\geq$ 3 / X $\leq$ 4)

Solution

(i)

X	1	2	3	4	5	6
$P(X = x) = \dfrac{1+x}{kx}$	$\dfrac{2}{k}$	$\dfrac{3}{k}$	$\dfrac{4}{k}$	$\dfrac{5}{k}$	$\dfrac{6}{k}$	$\dfrac{7}{k}$

From $\sum P_i \quad = 1$

$$\frac{2}{k} + \frac{3}{k} + \frac{4}{k} + \frac{5}{k} + \frac{6}{k} + \frac{7}{k} = 1$$

$$\frac{1}{k}\left(\frac{120+90+80+75+72+70}{60}\right) = 1$$

$$\frac{507}{60k} = 1$$

$$k = \frac{507}{60}$$

X	1	2	3	4	5	6
P(X = x)	$\dfrac{120}{507}$	$\dfrac{90}{507}$	$\dfrac{90}{507}$	$\dfrac{75}{507}$	$\dfrac{72}{507}$	$\dfrac{70}{507}$

(ii) mean, (E(X)) $= \dfrac{120+180+240+300+360+420}{507}$

$$= \frac{1620}{507}$$

E(X) = 3.1953

(iii) the highest probability is $\dfrac{120}{507}$,

The mode is 1

X	P_i	Cumulative probability
1	$\dfrac{120}{507}$	$\dfrac{120}{507}$
2	$\dfrac{210}{507}$	$\dfrac{210}{507}$
3	$\dfrac{290}{507}$	$\dfrac{290}{507}$ $\qquad$ $\dfrac{297}{507}$
4	$\dfrac{75}{507}$	$\dfrac{217}{507}$
5	$\dfrac{72}{507}$	$\dfrac{142}{507}$

1	$\frac{120}{507}$	

← Location of median

Median = 3

(iv) let A = $X \geq 3$ B = $X \leq 4$

X	P_i	$W_i = 3$	$Z_i = 2x + 4$

$P(A/B) = \frac{P(A \cap B}{P(B)}$

$P(A \cap B) = P(X = 3) + P(X = 4)$

$$= \frac{80}{507} + \frac{70}{507}$$

$$= \frac{155}{507}$$

$$P(B) = 1 - (P(X = 6))$$

$$= 1 - \left(\frac{72}{507} + \frac{70}{507}\right)$$

$$= 1 - \frac{142}{507}$$

$$= \frac{365}{507}$$

$$P(X \geq 3 / X \leq 4) = \frac{155/507}{365/507}$$

$$= \frac{155}{365}$$

$$= \mathbf{0.4247}$$

Example

12. The table below shows a random variable X with the following probability distribution.

X	**1**	**2**	**3**	**4**	**5**
P(X = x)	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

(i) Construct tables for the distributions W and Z, such that W = 3x and Z = 2x + 4.
(ii) Find the expectations of W and Z
(iii) Calculate the variance of Z.

Solution:

(i)

1	$\frac{1}{5}$	3	6
2	$\frac{1}{5}$	6	8
3	$\frac{1}{5}$	9	10
4	$\frac{1}{5}$	12	12
5	$\frac{1}{5}$	15	14

(ii) E(W) $= \sum W_iP_i$

$= \frac{1}{5}x3 + \frac{1}{5}x6 + \frac{1}{5}x9 + \frac{1}{5}x12 + \frac{1}{5}x15$

$= \frac{3+6+9+12+15}{5}$

$= \frac{45}{5} = 9$

E(W) = 9

E(Z) $= \sum Z_iP_i$

$= \frac{1}{5} x\ 6 + \frac{1}{5} x\ 8 + \frac{1}{5} x\ 10 + \frac{1}{5} x\ 12 + \frac{1}{5} x\ 14$

$= \frac{6+8+10+12+14}{5}$

$= \frac{50}{5}$

= **10**

Example 13. A random variable X has a probability density function f (X) given as

X	**-1**	**0**	**1**
P(X = x)	**a**	$\frac{1}{2}$	**b**

Where a and b are the probability of P(x = -1) and P(x = 1) respectively.

Given that E(X) = $\frac{1}{6}$**, Determine**

(i) **the value of a and b**

(ii) **the variance and standard deviation**

(iii) **P(x > -1)**

Solution

(i) $\sum P_i = 1$

$a + \frac{1}{2} + b = 1$

$\therefore a + b = \frac{1}{2}$

or $2a + 2b = 1$ (i) x3

$6a + 6b = 3$ (ii)

$E(X) = \sum P_iX_i = \frac{1}{6}$

$-a + 0 + b = \frac{1}{6}$

$$6b - 6a = 1 \qquad \text{(iii)}$$
$$6a + 6b = 3 \qquad \text{(from (ii))}$$
$$12b = 4 \qquad \text{(adding (ii) and (iii))}$$
$$b = \frac{4}{12}$$
$$= \frac{1}{3}$$
$$a = \frac{1}{2} - \frac{1}{3}$$
$$= \frac{3-2}{6}$$
$$= \frac{1}{6}$$

:. a **=** $\frac{1}{6}$ and **b** **=** $\frac{1}{3}$

X	-1	0	1
P(X = x)	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

(ii) $\quad \text{Var}(x) = \sum X^2 P_i - (E(X))^2$

$$= (-1)^2 \text{ x } \frac{1}{6} + 0^2 \text{ x } \frac{3}{6} + 1^2 \text{ x} \frac{1}{2} - \left(\frac{1}{6}\right)^2$$
$$= \frac{3}{6} - \frac{1}{36}$$
$$= \frac{17}{36}$$

= 0.4722

Standard deviation $= \sqrt{\text{Var(X)}}$

$$= \sqrt{0.4722}$$

= 0.6872

(iii) $\quad P(x > -1) = P(X = 0) + P(X= 1)$

$$= \frac{3}{6} + \frac{2}{6}$$
$$= \frac{5}{6}$$

= 0.83

Example 14: The probability mass function P(X) of the random variable X is given by:

$$\textbf{P(X)} = \begin{cases} kx + d & ; \ x = -2, -1, 0, 1, 2 \\ 0 & , \text{ elsewhere} \end{cases}$$

Where k and d are constants.

(i) **If P(x = 2) = 2P(x = -2), determine the values of k and d.**

(ii) **Compute the mean and variance of x.**

(iii) **What is the probability that x $\neq$ 0?**

Solution:

(i) $\sum_{i=1}^{n} P_i = 1$

At $\quad x = -2 \qquad P(-2) = -2k + d$

$\quad x = -1 \qquad P(-1) = -k + d$

$x = 0 \quad P(0) = d$

$x = 1 \quad P(1) = k + d$

$x = 2 \quad P(2) = 2k + d$

$\Rightarrow P(-2) + P(-1) + P(0) + P(1) + P(2) = 1$

$-2k + d - k + d + k + d + 2k + d = 1$

$5d = 1$

$d = \frac{1}{5}$

$P(X = 2) = 2P(X = -2)$

$-2k + d = 2(-2k + d)$

$6k = d$

$6k = \frac{1}{5}$

$\mathbf{k} = \frac{1}{30}$

$$P(X) = \begin{cases} \frac{x}{30} + \frac{1}{5} & x = -2, -1, 0, 1, 2 \\ 0 & , \text{ elsewhere} \end{cases}$$

(ii) Mean $E(X) = \sum_{i=1}^{n} X_i P_i$ substitute the values of x in P(x)

X	-2	-1	0	1	2
P	0.1333	0.1666	0.2	0.2333	0.2666

E(X) = -2 x 0.1333 – 1 x 0.1666 +0.0.2 +1 1 x 0.2333 + 2 x 0.2666

= **0.3333**

(iii) Either : $P(X \neq 0) = P(X = -2) + P(X = -1) + P(X = 1) + P(X = 2)$

$= 0.1333 + 0.1666 + 0.2333 + 0.2666$

$= 0.7998$

$\approx$ **0.8**

Or : $P(X \neq 0) = 1 - P(x = 0)$

$= 1 - 0.2$

= **0.8**

Example 15; A discrete random variable X has a probability density function.

$P(X = x) = k\,|x|$ where x takes the values -3,-2, -1, 0, 1, 2, 3. Find;

(a) the value of the constant K

(b) E(X)

(c) E(X^2)

(d) Standard deviation

Solution

(a)

X_i	-3	-2	-1	0	1	2	3
P_i	3k	2k	k	0	k	2k	3k

$\sum_{i=1}^{n} X_i P_i = 1$

$3k + 2k + k + k + 2k + 3k = 1$

$12k = 1$

$\mathbf{k} = \frac{1}{12}$

X_i	-3	-2	-1	0	1	2	3

P_i	$\frac{3}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	0	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{3}{12}$

(b) $E(X) = \sum_{i=1}^{n} X_i P_i$

$E(X) = -3 \times \frac{3}{12} - 2 \times \frac{2}{12} - \frac{1}{12} \times 1 + 0 + 1 \times \frac{1}{12} + 2 \times \frac{2}{12} + 3 \times \frac{3}{12}$

$E(X) = \frac{9-4-1+0+1+4+9}{12}$

$= \frac{0}{12}$

$E(X) = 0$

(c) $E(X^2) = \sum_{i=1}^{n} X^2{}_i P_i$

$E(X^2) = (-3)^2 x \frac{3}{12} + (-2)^2 x \frac{2}{12} + (-1)^2 x \frac{1}{12} + 0 + 1^2 x \frac{1}{12} + 2^2 x \frac{2}{12} + 3^2 x \frac{3}{12}$

$= \frac{27}{12} + \frac{8}{12} + \frac{1}{12} + \frac{1}{12} + \frac{8}{12} + \frac{27}{12}$

$=$ **6**

(d) Standard deviation $= \sqrt{\text{Variance}}$

Variance $= E(X^2) - (E(X))^2$

$= 6 - (0)^2$

$= 6$

Standard deviation $= \sqrt{6}$

$=$ **2.4445**

Example 16. The probability distribution for the number of heads that show up when a coin is tossed 3 times is given by $P(X = x) = \left\{\frac{1}{k}\binom{3}{x}\right., x = 0, 1,2, 3,.$

Find:

(i) The value of k,

(ii) E(X).

Solution:

We have

$P(X = x) = \left\{\frac{1}{k}\binom{3}{x}\right., x = 0,1,2,3.$

This is a discrete random variable.

'It should be noted that

$\binom{n}{x} = \frac{n!}{(n-x)!x!}$

Hence when

$X = 0, \quad P(x = 0) = \frac{1}{k}\left(\frac{3!}{3!0!}\right) = \frac{1}{k}$

$X = 1, \quad P(x = 1) = \frac{1}{k}\left(\frac{3!}{2!1!}\right) = \frac{3}{k}$

$X = 2, \quad P(x = 2) = \frac{1}{k}\left(\frac{3!}{1!2!}\right) = \frac{3}{k}$

$X = 3, \quad P(x = 3) \quad = \frac{1}{k}\left(\frac{3!}{0!3!}\right) = \frac{1}{k}$

In table form,

X	0	1	2	3
P (x = x)	$\frac{1}{k}$	$\frac{3}{k}$	$\frac{3}{k}$	$\frac{1}{k}$

(i) To find the value of k

Now $\sum_{x=0}^{3} P(X = x) = 1$

$\Rightarrow \frac{1}{k} + \frac{3}{k} + \frac{3}{k} + \frac{1}{k} = 1$

$\frac{8}{k} = 1$

$\therefore$ **k = 8 #**

(ii)Table is now in form

X	0	1	2	3
P(x = x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Now

$$E(x) = \sum_{x=0}^{3} xP(x = x)$$

$$= 0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right)$$

$$= \frac{3}{8} + \frac{6}{8} + \frac{3}{8}$$

$$= \frac{12}{8} = \frac{3}{2} = 1.5$$

$\therefore \quad E(x) = 1.5$ #

Example 17. A balanced coin is tossed three times and the number of times X a 'Head' appear is recorded. Complete the following table.

n	**0**	**1**	**2**	**3**
Event	**(TTT)**		**HHT,HTT,THH**	
P(X=n)	**1/8**			

Determine the average of the expected number of heads to appear.

Solution:

n	0	1	2	3
Event	TTT	HTT THT TTH	HHT THH HTH	HHH
P(X = n)	1/8	3/8	3/8	1/8

Expected number of heads to appear

$$E(X) = \sum n\, P(X = n)$$
$$= (0 \times 1/8) + (1 \times 3/8) + (2 \times 3/8) + (3 \times 1/8)$$
$$= 0 + 3/8 + 6/8 + 3/8$$
$$= \frac{12}{8} = 3/2 = 1.5$$
$$E(X) = 1.5$$

Hence the average of the expected number of heads to appear is 1.5

EXERCISE 3

1.. A discrete random variable X has distribution function F(X) where

$$F(X) = 1 - \left(1 - \frac{x}{4}\right)^x \qquad \text{for } x = 1, 2, 3, 4$$

(a)Show that $F(3) = \frac{63}{64}$ and $F(2) = \frac{3}{4}$

(b)Obtain a probability distribution of x

(c)Find E(X) and Var(X).

(d)Find P(X > E(X))

2. The random variable X takes integer values only and has p d f

$P(X = x) = kx \qquad x = 1, 2, 3, 4, 5$

$P(X = x) = k(10 - x) \qquad x = 6, 7, 8, 9$

Find a) the value of the constant k.
b) E(X)
c) Var(X)
d) E(2x – 3)
e) Var(2x – 3)

3. If X is a random variable on "a biased die" and the probability density function of X is as shown.

X	1	2	3	4	5	6
P(X = x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{5}$	y	$\frac{1}{5}$	$\frac{1}{6}$

Find (a)the value of y
(b)E(X)
(c)$E(X^2)$
(d)Var(X)
(e)Var(4X)

4. A random variable R takes the integer value r with probability.

$P(r) = Kr^2 \qquad r = 1, 2, 3$

$P(r) = K(7 - r)^2 \qquad r = 4, 5, 6$

$P(r) = 0 \qquad$ otherwise

Find; (a) the value of y
(b) The mean
(c) The variance

5. A discrete random variable X can take only the values 0, 1, 2, or 3 and its probability distribution is given by P(X = 0) = k, P(X = 1) = 3k, P(X =2) = 4k and P(X = 3) = 5k, where K is a constant. Find;

(a) The value of K.

(b) Mean and variance of X.

6. A discrete random variable X has probability function given by;

$$P(X) = \begin{cases} \left(\frac{1}{2}\right)^2 & x = 1, 2, 3, 4, 5 \\ c & x = 6 \\ 0 & \text{otherwise} \end{cases}$$

Where C is a constant.

Determine

(i)the value of C

(ii)the mode

(iii)the arithmetic mean

7. The discrete random variable X has a probability density function given by;

$$P(X = x) = \frac{3x+1}{22} \quad \text{for } x = 0, 1, 2, 3.$$

Find (a) E(X)

(b) E(X^2)

(c) E(3x - 2)

(d) E($2x^2$ + 4x - 3)

8. A discrete random variable X has a probability density function.

X	0	1	2	3
P(X = x)	a	a^2	$a^2 + a$	$3a^2 + 2a$

Determine; (a) the constant a

(b)E(X)

9. A curiously shaped six-faced die produces a score X, for which the probability distribution is given in the following table.

R	1	2	3	4	5	6
P(X = r)	C	$\frac{c}{2}$	$\frac{c}{3}$	$\frac{c}{4}$	$\frac{c}{5}$	$\frac{c}{6}$

(i)Show that the constant c is $\frac{20}{49}$

(ii)Find the mean and variance

(iii)The die is thrown twice. Show that the probability of obtaining equal scores is approximately $\frac{1}{4}$

10. A discrete random variable has a probability density function of

P(x = r) = k(n – r) for r = 1, 2, 3 …, n where K is a constant.

show that

(i) k is $\frac{2}{n(n-1)}$

(ii) E(X) = $\frac{1}{3}$(n + 1)

(iii) Var(X) = $\frac{1}{18}(n + 1)(n - 2)$

11. A disc is drawn from a bag containing 10 disc numbered 0, 1, 2, …., 9. The random variable X is defined as the square of the number drawn. Find:

(i)E(X)

(ii)Var(X)

12. A bag contains one 50p coins, three 20p coins, seven 10p coins and several 5p coins. Given when one coin is selected at random the expectation is 10p. Find.

(i) The number of 5p coins

(ii) Find also the expectation when two coins are selected at random without replacement.

13. A random variable R takes values 1, 2, …, n with equal probabilities. Determine;

(i) The expectation μ of R

(ii) Show that the variance δ^2 is given by $12\delta^2 = n^2 - 1$

(iii)find P($|R - \mu|$) > δ) in the case n = 100

14. A cubical die is biased in such away that the probability of scoring n where n = 1, 2, 3, 4, 5, 6 is proportional to n.

Determine; (i) the mean value

(iii)Variance of the score obtained in a single score.

(iii)The mean and variance if the score is doubled.

EXERCISE 3

1.. (b)

X	1	2	3	4
P(X = x)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{15}{64}$	$\frac{1}{16}$

(c) $2\frac{1}{16}$ (d) $\frac{1}{4}$

2. (a) 0.04 (b) 5 (c) 4 (d) 7 (e) 16)

3. (a) $\frac{1}{10}$ (b) $3\frac{1}{2}$ (c) $15\frac{7}{30}$ (d) $2\frac{59}{60}$ (e) $47\frac{11}{15}$)

4. (a) $\frac{1}{28}$ (b) 3.4 (c) 1.25)

5. (a) $\frac{1}{31}$ (b) $2\frac{12}{13}$)

6.((i) $\frac{1}{32}$ (ii) 1 (iii)$1\frac{31}{32}$)

7. (a) $\frac{24}{11}$ (b) $\frac{61}{11}$ (c) $\frac{50}{11}$ (d) $16\frac{9}{11}$

8. (a) 0.2 (b) 2.08

9. (a) $\frac{120}{49}$ (b) 2.57)

10.Proof.

11. (a) 28.5 (b) 721.05

12. (a) 14 (b) 20p

13. ((i) ½(n + 1) (ii) 0.42)

14. (i) $4\frac{1}{3}$ (b) $2\frac{2}{9}$ (c) $8\frac{2}{3}$ (d) $8\frac{8}{9}$

4.CONTINUOUS PROBABILITY DISTRIBUTION.

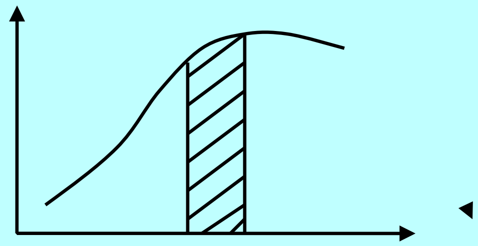
Introduction:

In probability theory, a **probability density function** (**pdf**), or **density** of a continuous random variable, is a function that describes the relative likelihood for this random variable to take on a given value. The probability for the random variable to fall within a particular region is given by the integral of this variable's density over the region. The probability density function is nonnegative everywhere, and its integral over the entire space is equal to one.

The terms "**probability distribution function**" and "**probability function"** have also sometimes been used to denote the probability density function. However, this use is not standard among statisticians. In other sources, "probability distribution function" may be used when the probability distribution is defined as a function over general sets of values, or it may refer to the cumulative distribution function, or it may be a probability mass function rather than the density. Further confusion of terminology exists because *density function* has also been used for what is here called the "probability mass function".

Mathematically:

A continuous random variable may be represented by the graph of a continuous function f (x) taking a value within a particular interval (a, b) equal to the area under the curve between a and b.



$$\text{Area} = \int_a^b f(x)\,dx.$$

PROBABILITY DENSITY FUNCTION OF A CONTINUOUS RANDOM VARIABLE.

It is a function whose integral from x = a to x = b where b ≥ a, given the probability that x takes a value in the interval (a, b)

$$P(a \le x \le b) = \int_a^b f(x)dx$$

CONDITIONS FOR PROBABILITY DENSITY FUNCTION OF A CONTINUOUS RANDOM VARIABLE.

There are mainly two, namely;

(i) $f(x) \ge o$, for all values of x.

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$, the interval is from $+\infty$ to $-\infty$ in order to cover all cases.

If x takes on positive values only in the interval (a, b) and is zero otherwise

$$\int_a^b f(x)dx = 1$$

MEAN OR EXPECTATION OR EXPECTED VALUE OF CONTINUOUS RANDOM VARIABLE

If the probability density function is f(x)

Mean $(\mu) \quad = \quad \int_{-\infty}^{\infty} xf(x)dx$

Or

if X takes only the positive values in the interval (a, b) then:

Expectation E(X) $\quad = \quad \int_{a}^{b} xf(x)dx$

VARIANCE AND STANDARD DEVIATION OF A CONTINUOUS RANDOM VARIABLE

The variance (δ^2) of a continuous random variable X whose probability density function is f(x) and mean μ is

$$\begin{aligned}
Var(x) = \sigma 2 &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx. \\
&= \int_{-\infty}^{\infty} x^2 f(x)dx - (\mu)^2 \\
&= E(x^2) - (\mu)^2 \\
&= E(x^2) - (E(x))^2
\end{aligned}$$

$$\begin{aligned}
\text{Standard deviation}(\sigma) &= \sqrt{\int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx} \\
&= \sqrt{\int_{-\infty}^{\infty} x^2 f(x)dx - (\mu)^2} \\
&= \sqrt{E(x^2) - (\mu)^2} \\
&= \sqrt{E(x^2) - (E(x))^2}
\end{aligned}$$

MEDIAN OF A CONTINUOUS RANDOM VARIABLE

(i) If a continuous random variable has probability density function f(x), in the interval (a, b) then the median (M) is such that

$$\int_{a}^{M} f(x)dx = \frac{1}{2}$$

OR

$$\int_{M}^{b} f(x)dx = \frac{1}{2}$$

(ii) If a continuous random variable has a probability density function f(x), in two intervals such that

$$f(x) = \begin{cases} f(x_1) \ ; & a \le x < b \\ f(x_2); & b \le x \le c \\ o \ ; & \text{elsewhere} \end{cases}$$

First find the probability $f(x_1)$ in the interval (a, b) by integrating $f(x_1)$

$$\int_{a}^{b} f(x_1)dx = y$$

(a) If the value obtained (y) is less than ½ then the median lies in the interval (b, c) and is given by

$$\int_{a}^{b} f(x_1)dx + \int_{b}^{M} f(x_2)dx = \tfrac{1}{2}$$

(b) If the value obtained (y) is greater then ½ then the median lies in the interval (a, b) and is given by

$$\int_a^M f(x_1)dx = \frac{1}{2}$$

MODE OF A CONTINUOUS RANDOM VARIABLE

The mode of a continuous random variable X with probability density function f(x) is the value of x for which f(x) has a relative maximum. Differentiating the probability density function will yield the mode. The second derivative show us the nature of the values obtained in the first derivative. It is a relative maximum if the value of the second derivative is a negative.

CUMULATIVE DISTRIBUTION FUNCTION (F(X)) (DISTRIBUTION FUNCTION) OF A CONTINUOUS RANDOM VARIABLE.

(i) If a continuous random variable X with probability density function f(x) has the interval (a, b) and zero elsewhere.

$$f(x) = \begin{cases} f(x_1) & ;\ a \le x \le b \\ 0 & ;\ \text{elsewhere} \end{cases}$$

Then the cumulative distribution function (F(X)) is

$$F(X) = 0 \qquad x \le a$$

$$F(X) = \int_a^x f(t)dt \qquad a \le x \le b$$

$$F(X) = 1, \qquad x \ge b$$

$$\therefore \quad F(X) = \begin{cases} 0 & x \le a \\ \int_a^x f(t)dt & a \le x \le b \\ 1 & x \ge b \end{cases}$$

(ii) If a continuous random variable X with probability density function f(x), such that

$$f(x) = \begin{cases} f(x_1) & ;\ a \le x \le b \\ f(x_2) & ;\ b \le x \le c \\ 0 & ;\ \text{elsewhere} \end{cases}$$

Then the cumulative distribution function F(X) is given by;

$$F(X) = 0 \qquad x \le a$$

$$F(X) = \int_a^x f(t)dt \qquad \text{for } a \le x < b$$

Substitute the upper limit x = b into $F(x_1)$. If it us equal to d.

$$F(x_2) = d + \int_b^x f(t_2)dt \qquad \text{for } b \le x < c$$

If the upper limit x = c is substituted into $F(x_2)$ it should give a value equal to 1.

$$F(X) = 1 \qquad \text{for } x \ge c$$

$$:.F(X) = \begin{cases} 0 & x \le a \\ F(x_1) & a \le x < b \\ F(x_2) & b \le x \le c \\ 1 & x \ge c \end{cases}$$

RECTANGULAR (CONTINUOUS UNIFORM) DISTRUTION.

A continuous random variable X has a uniform distribution over the internal (α, β) if the probability density function f(x) is given by;

$$f(x) = \begin{cases} \dfrac{1}{\beta-\alpha} & ;\ \alpha \le x \le \beta \\ 0 & ;\ \text{otherwise} \end{cases}$$

Checking its validity

$$\int_{\alpha}^{\beta} \frac{1}{\beta-\alpha} dx = \left| \frac{x}{\beta-\alpha} \right|_{\alpha}^{\beta}$$

$$= \frac{\beta}{\beta-\alpha} - \frac{\alpha}{\beta-\alpha}$$

$$= \frac{\beta-\alpha}{\beta-\alpha} = 1$$

EXPECTION OR EXPECTED VALUE E(X)) OR MEAN (μ) OF A RECTANGULAR DISTRIBUTION.

$$E(x) = \int_{\alpha}^{\beta} x\, f(x)\, dx$$

$$= \int_{\alpha}^{\beta} \frac{x}{\beta-\alpha} dx$$

$$= \frac{1}{2} \left[\frac{x^2}{\beta-\alpha} \right]_{\alpha}^{\beta}$$

$$= \frac{\beta^2}{2(\beta-\alpha)} - \frac{\alpha^2}{2(\beta-\alpha)}$$

$$= \frac{\beta^2-\alpha^2}{2(\beta-\alpha)}$$

$$= \frac{(\beta-\alpha)(\beta+\alpha)}{2(\beta-\alpha)}$$

$$E(X) = \tfrac{1}{2}(\alpha+\beta)$$

MEDIAN OF A RECTANGULAR DISTRIBUTION

$$\text{Median}\,(\mu) = \int_{\alpha}^{M} f(x)\, dx = 0.5$$

$$= \int_{\alpha}^{M} \left(\frac{1}{\beta-\alpha} \right) dx = \tfrac{1}{2}$$

$$\left[\frac{x}{\beta-\alpha} \right]_{\alpha}^{M} = \tfrac{1}{2}$$

$$\therefore \quad \frac{M}{\beta-\alpha} - \frac{\alpha}{\beta-\alpha} = \frac{M-\alpha}{\beta-\alpha} = \tfrac{1}{2}$$

$$\text{Median}(M) = \tfrac{1}{2}(\alpha+\beta)$$

VARIANCE AND STANDARD DEVIATION OF A RECTANGULAR DISTRIBUTION

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$E(X) = \tfrac{1}{2}(\alpha+\beta)$$

$$(E(X))^2 = \tfrac{1}{4}(\alpha^2 + 2\alpha-\beta+\beta^2)$$

$$E(X^2) = \int_\alpha^\beta \frac{x^2}{\beta-\alpha}dx$$

$$= \frac{1}{3(\beta-\alpha)} \left|x^3\right|_\alpha^\beta$$

$$= \frac{\beta^3-\alpha^3}{3(\beta-\alpha)} \qquad \text{But } \beta^3-\alpha^3 = (\beta-\alpha)(\alpha^2-\alpha\beta+\beta^2)$$

$$E(X^2) = \frac{(\beta-\alpha)(\alpha^2+\alpha\beta+\beta^2)}{3(\beta-\alpha)}$$

$$= \frac{1}{3}(\alpha^2+\alpha\beta+\beta^2)$$

$$\text{Var}(x) = \frac{1}{3}(\alpha^2+\alpha\beta+\beta^2) - \frac{1}{4}(\alpha^2+2\alpha\beta+\beta^2)$$

$$= \frac{4\alpha^2+4\alpha\beta+4\beta^2-3\alpha^2-6\alpha\beta-3\beta^2}{12}$$

$$= \frac{\alpha^2-2\alpha\beta+\beta^2}{12}$$

$$\text{Var}(X) = \frac{1}{2}(\beta-\alpha)^2$$

Standard deviation $= \sqrt{\frac{1}{12}(\beta-\alpha)^2}$

Example of a rectangular distribution is

$$f(x) = \begin{cases} \tfrac{1}{2} & ;\ 0 \le x \le 2 \\ \tfrac{1}{2} & ;\ 2 \le x \le 4 \\ 0 & ;\ \text{elsewhere} \end{cases}$$

EXPONENTIAL DISTRIBUTION

If x has a p.d.f. given as $f(x) = \lambda e^{-\lambda x,}\ x \ge 0$
then

$$E(X) = \frac{1}{\lambda} \text{ and}$$

$$\text{Var}(X) = \frac{1}{\lambda^2},$$

if M = Median it is obtained from

$$e^{-\lambda m} = \tfrac{1}{2}$$

$$P(X < a) = 1 - e^{-\lambda a}$$

$P(X > a) \quad = \quad e^{-\lambda a}$.

If X has a p.d.f. given as $f(x) = \frac{1}{\alpha} e^{-x/\alpha}, \quad x \geq 0$

Then $E(X) = \alpha$ and

$F(X) = 1 - e^{-x/\alpha}$

Example 1:
A probability density function is given as

$$\mathbf{f(x)} = \begin{cases} kx(4 - x^2) & ; \; 0 \leq x \leq 2 \\ 0 & ; \; \text{elsewhere} \end{cases}$$

Find the (i) value of k
(ii) median
(iii) mean
(iv) standard deviation

Solution:

(i) $k\int_0^2 (4x - x^3)\, dx = 1$

$$k \left| 2x^2 - \frac{x^4}{4} \right|_0^2 = 1$$

$$4k = 1$$

$$k = \frac{1}{4}$$

$$= 0.25$$

(ii) Let m = median

$$\frac{1}{4}\int_0^m (4x - x^3) dx = \frac{1}{2}$$

$$\frac{1}{4}\left| 2x^2 - \frac{x^4}{4} \right|_0^M = \frac{1}{2}$$

$$\frac{M^2}{2} - \frac{M^4}{16} = \frac{1}{2}$$

$$8M^2 - M^4 = 8$$

$$:. \quad M^4 - 8M^2 + 8 = 0$$

Let $M^2 = y$

$$\Rightarrow \quad y^2 - 8y + 8 = 0$$

$$y = \frac{8 \pm \sqrt{8^2 - 4 x 1 x 8}}{2}$$

$$= \frac{4 \pm \sqrt{32}}{2}$$

$$= 4 \pm 2\sqrt{2}$$

:. $M^2 = 4 + 2\sqrt{2}$ or $M^2 = 4 - 2\sqrt{2}$

$M = 1.082$ or $M = 2.6131$

$M = 2.6131$ is impossible **ie M cannot be greater than 2 !**

:. **The median = 1.082**

(iii) $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$= \frac{1}{4}\int_0^2 (4x^2 - x^4)\, dx$$

$$= \frac{1}{4}\left|\frac{4x^3}{3} - \frac{x^5}{5}\right|_0^2$$

$$= 8\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$= \frac{16}{15}$$

$$= 1.07$$

Standard deviation $= \sqrt{E(X^2) - (E(X))^2}$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$E(X^2) = \frac{1}{4}\int_0^2 4x^3 - x^5\ dx$$

$$= \frac{1}{4}\left(x^4 - \frac{x^6}{6}\right)\Big|_0^2$$

$$= 4\left(1 - \frac{4}{6}\right)$$

$$= \frac{4}{3}$$

$$= 1.33$$

Standard deviation $= \sqrt{\frac{4}{3} - -\left(\frac{16}{15}\right)^2}$

$$= \sqrt{\frac{44}{225}}$$

$$= \mathbf{0.4422}$$

Example 2. A continuous random variable f(x) is given as

$$\mathbf{f(x)} = \begin{cases} 12x^2(1-x) & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find (i) Mean
(ii) Mode
(iii) Variance

Solution

(i) mean (μ) $= \int_{-\infty}^{\infty} x^2 f(x)\, dx$

$$\mu = 12\int_0^1 (x^3 - X^4)\, dx$$

$$= 12\left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Big|_0^1$$

$$\mu = 12\left[\frac{1}{4} - \frac{1}{5}\right]$$

$$= 12\ \frac{(5-4)}{5}$$

$$= \frac{12}{20}$$

$$= \frac{3}{5} = \mathbf{0.6}$$

(ii) f(x) $= 12x^2 - 12x^3$

$$f^1(x) = 24x - 36x^2 = 0$$

$$\Rightarrow 12x(2 - 3x) = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad x = \frac{2}{3}$$

$$f^{11}(x) = 24 - 72x$$

At $x = 0$

$$f^{11}(0) = 24 \text{ positive therefore a minimum point at } x = \frac{2}{3}$$

$$f^{11}\left(\frac{2}{3}\right) = 24 - 72 \times \frac{2}{3}$$

$$= -24 \quad \text{(negative) therefore it is a relative maximum point.}$$

:. **The mode** $= \frac{2}{3}$

(iii) $Var(x) = E(x^2) - (E(X))^2$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$E(X^2) = 12\int_0^1 \left(x^4 - x^5\right) dx$$

$$= 12\left|\left(\frac{x^5}{5} - \frac{x^6}{6}\right)\right|_0^1$$

$$= 12\left(\frac{1}{5} - \frac{1}{6}\right) = \frac{13}{30} = 0.4$$

$$Var(X) = 0.4 - (0.6)^2$$

$$= \mathbf{0.04}$$

Example 3. A random variable X has a probability density function

$$\mathbf{f(x)} = \begin{cases} kx(6-x)^2 & ; \ 0 \le x \le 6 \\ 0 & ; \ \text{elsewhere} \end{cases}$$

Determine (i) arithmetic mean

(ii) Mode

(iii) Variance and standard deviation

Solution:

(i) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$k\int_6^6 x(x^2 - 12x + 36) = 1$$

$$k\int_6^6 (x^3 - 12x^2 + 36x)\, dx = 1$$

$$k\left|\left(\frac{x^4}{4} - 4x^3 + 18x^2\right)\right|_0^6 = 1$$

$$k(324 - 864 + 648) = 1$$

$$108k = 1$$

$$k = \frac{1}{108}$$

$$\text{Mean } (\mu) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$\mu = \frac{1}{108}\int_0^6 (x^4 - 12x^3 + 36x^2)\, dx$$

$$= \frac{1}{108}\left|\left(\frac{x^5}{5} - 3x^4 + 12x^3\right)\right|_0^6$$

$$\mu = \frac{1}{108}\left(\frac{7776}{5} - \frac{19440}{5} + \frac{12960}{5}\right)$$

$$= \frac{1296}{540}$$

$$= \mathbf{2.4}$$

(ii) $f(x) = \frac{1}{108}\left(x^3 - 12x^2 + 36x\right)$

$f^1(x) = \frac{1}{108}\left(3x^2 - 24x + 36\right)$

$f^1(x) = \frac{1}{108}(3x^2 - 24x + 36) = 0$

$\Rightarrow \quad x^2 - 8x + 12 = 0$

$(x - 2)(x = 6) = 0$

$x = 2 \quad \text{or } x = 6$

$f^{11}(x) = \frac{6x}{108} - \frac{24}{108}$

$= \frac{1}{108}(6x - 24)$

At $x = 2$,

$f^{11}(2) = \frac{1}{108}(12 - 24)$

$= -\frac{12}{108}$ (negative therefore relative Maximum hence the mode).

Mode = 2

(iii) $E(X^2) = \int_{-\infty}^{\infty} x^2 \ f(x)\, dx$

$E(X^2) = \frac{1}{108}\int_0^6 (x^5 - 12x^4 + 36x^3)\, dx$

$= \frac{1}{108}\left|\left(\frac{x^6}{6} - \frac{12x^5}{5} + 9x^4\right)\right|_0^6$

$= \frac{1}{108 \times 30}\left(5 x 6^6 - 12 x 6^6 - 9 x 30 x 6^4\right)$

$= \frac{23328}{3240} = 7.2$

$Var(x) = 7.2 - (2.4)^2 = 1.44$

Standard deviation $= \sqrt{Var(x)}$

$= \sqrt{1.44} = \mathbf{1.2}$

Example 4: **The probability density function of a random variable X is given by;**

$$\mathbf{f(X)} = \begin{cases} k(x+2) & ; \ -1 \le x \le 0 \\ 2k(1-x) & ; \ 0 \le x \le 1 \\ 0 & ; \ \text{elsewhere} \end{cases}$$

(i) Sketch the function

(ii) find k and the mean of x

(iii) find the probability p(0 < x < ½ /x >0)

Solution

(i) For $f(x) = k(x + 2)$,

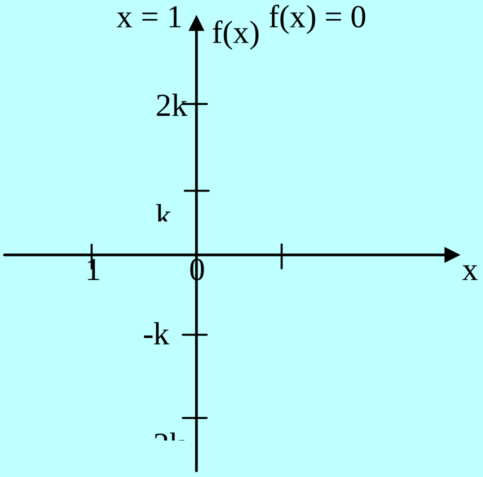
at $x = 1$, $f(x) = k$

at $x = 0$, $f(x) = 2k$

For $f(x) = 2k(1-x)$

at $x = 0$, $f(x) = 2k$

$x = 1$ $f(x) = 0$



(ii) $$\int_{-1}^{1} f(x)\ dx = 1$$

$$\int_{-1}^{0} (x+2)\ dx + 2k \int_{0}^{1} (1-x)\,dx = 1$$

$$k\left[\frac{x^2}{2}+2x\right]_{-1}^{0} + 2k\left[x-\frac{x^2}{2}\right]_{-1}^{0} = 1$$

$$\frac{3}{2}k + k = 1$$

$$\frac{5}{2}k = 1$$

$$\mathbf{k} = \frac{2}{5}$$

$$\text{Mean } (\mu) = \frac{2}{5}\int_{-1}^{0}(x^2+2x)\,dx + \frac{4}{5}\int_{0}^{1}(x-x^2)\,dx$$

$$= \frac{2}{5}\left[\frac{x^3}{2}+x^2\right]_{-1}^{0} + \frac{4}{5}\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{0}^{1}$$

$$= \frac{4}{15} + \frac{4}{30} = \frac{-2}{15}$$

$$\therefore \textbf{Mean } (\mu) = \frac{-2}{15}$$

(iii) $P((0 < x < ½) / x > 0) = P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

$\therefore A = 0 < x < ½,$

$B = x > 0$

A is a subset of B

$\therefore P(A \cap B) = P(0 < x < ½)$

$P(B) = P(x > 0)$

$$P((0 < x < ½) / x > 0) = \frac{P(0<x<\frac{1}{2})}{P(x>0)}$$

$$= \frac{\frac{4}{5}\int_{0}^{\frac{1}{2}}(1-x)dx}{\frac{4}{5}\int_{0}^{1}(1-x)dx}$$

$$= \frac{\left| x - \frac{x^2}{2} \right|_0^{\frac{1}{2}}}{\left| x - \frac{x^2}{2} \right|_0^1}$$

$$= \frac{\left(\frac{1}{2} - \frac{1}{8}\right)}{\left(1 - \frac{1}{2}\right)} = \frac{\frac{3}{8}}{\frac{1}{2}}$$

$$= \frac{3}{4} = \mathbf{0.75}$$

Example 5. (a) The probability density function of a random variable X is defined as follows.

$$\mathbf{f(x)} = \begin{cases} x(x-1)(x-2) & ; 0 \le x \le 1 \\ k & ; 1 \le x \le 3 \\ 0 & ; \text{otherwise} \end{cases}$$

Where k is a constant.

Calculate;

(i) the value of k and the expectation

(ii) the probability that X is less or equal to the mean

(b) A random sample X has the distribution function.

$$\mathbf{f(x)} = \begin{cases} 0 & ; x \le 0 \\ kx^3 & ; 0 \le x \le 2 \\ 1 & ; x \le 2 \end{cases}$$

Where k is a constant.

Find; (i) the mean

(ii) median

(iii) variance of x

Solution.

(i) $\int_0^1 x(x-1)(x-2)\,dx + \int_1^3 k\,dx = 1$

$$\int_0^1 \left(x^3 - 3x^2 + 2x\right) dx + k\,[x]_1^3 = 1$$

$$\left| \frac{x^4}{4} - x^3 + x^2 \right|_0^1 + 2k = 1$$

$$\frac{1}{4} - 1 + 1 + 2k = 1$$

$$2k = \frac{3}{4}$$

$$k = \frac{3}{8}$$

Expectation E(X) $= \int_{-\infty}^{\infty} x\, f(x)\,dx$

$$E(X) = \int_0^1 (x^4 - 3x^3 + 2x^2)\,dx + \frac{3}{8}\int_1^3 x\,dx$$

$$= \left|\left(\frac{x^5}{5} - \frac{3x^4}{4} - + \frac{2x^3}{3}\right)\right|_0^1 + \frac{3}{16}\left|x^2\right|_1^3$$

$$E(x) = \frac{1}{5} - \frac{3}{4} + \frac{2}{3} + \frac{27}{16} - \frac{3}{16} = \mathbf{1.6167}$$

$:. E(X) = 1.6167$

(ii) $P((X) \le E(X)) = P(X \le 1.6167)$

$$= \int_0^1 (x^3 - 3x^2 + 2x)\,dx + \frac{3}{8}\left|x^2\right|_1^{1.6167}$$

$$= \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 + \frac{3}{8} \text{ x } 0.6167$$

$$= \frac{1}{4} + \frac{3}{8} \text{ x } 0.6167$$

$$= \mathbf{0.4812}$$

(b)(ii) $k\left[x^3\right]_0^2 = 1$

$$8k = 1$$

$$k = \frac{1}{8}$$

Let M = Median

$$\int_{-\infty}^{M} f(x)\,dx = ½$$

$$\text{Or } \left|F(x)\right|_{-\infty}^{m} = ½,$$

$$:. \frac{1}{8}\left|x^3\right|_0^M = \frac{1}{2}$$

$$\frac{M^3}{8} = \frac{1}{2}$$

$$M^3 = 4$$

$$M = \sqrt[3]{4}$$

$$= \mathbf{1.5874}$$

(i) $F(x) = \frac{1}{8}x^3$

$$f(x) = \frac{d}{dx}(F(X))$$

$$= \frac{3}{8}x^2$$

Mean (μ) $= \int_0^2 x\ f(x)\,dx$

$$= \frac{3}{8}\int_0^2 x^3\,dx$$

$$\mu = \frac{3}{32}\left[x^4\right]_0^2$$

$$= \frac{3 \times 2^4}{2^5}$$

$$= \frac{3}{2}$$

$$= \mathbf{1.5}$$

(iii) Variance (σ^2) $= E(X^2) - (\mu)^2$

$$= \frac{3}{2}\int_0^2 x^4\,dx - \left(\frac{3}{2}\right)^2$$

$$= \frac{3}{40}\left[x^5\right]_0^2 - \frac{9}{4}$$

$= \frac{96}{40} - \frac{9}{4}$

$= \frac{96-90}{40}$

$= \frac{6}{40}$

Var(X) $= \frac{3}{20} =$ **0.15**

Example 6; **A continuous random variable X has a probability density function;**

f(x) = kx(3 – x) **for 0 ≤ x ≤ 2**
f(x) = k(4 – 4) **for 2≤ x ≤ 4**
0 **elsewhere**

Find **(i) the value of k.**
(ii) The mean
(iii) F(x), the cumulative distribution function.
(ii) **P(1 ≤ x ≤ 3)**

Solution:

(i) $k\int_0^2 (3x - x^2)\,dx + k\int_2^4 (4 - x)\,dx = 1$

$k\left\{\left|\frac{3x^2}{2} - \frac{x^3}{3}\right|_0^2 + \left|4x - \frac{x^2}{2}\right|_2^4\right\} = 1$

$k\left\{\left[6 - \frac{8}{3}\right] + \left[(16-8)-(8-2)\right]\right\} = 1$

$\therefore k = \frac{3}{16}$

(ii) The mean

$E(x) = \frac{3}{16}\left[\int_0^2 (3x^2 - x^3)\,dx + \int_2^4 (4x - x^2)\,dx\right]$

$= \frac{3}{16}\left[\left[x^3 - \frac{x^4}{4}\right]_0^2 + \left[2x^2 - \frac{x^3}{3}\right]_2^4\right]$

$= \frac{3}{16}\left[(8-4) + \left(32 - \frac{64}{3}\right) - \left(8 - \frac{8}{3}\right)\right]$

$= \frac{3}{16} \times \frac{28}{3}$

$= \frac{17}{8}$

$= 1\frac{3}{4} =$ **1.75**

(iii) $F(x) = 0,$ $\mathbf{x \le 0}$

$\mathbf{0 \le x \le 2}$

$F(x) = \frac{3}{16}\int_0^x (3t - t^2)\,dt$

$= \frac{3}{16}\left[\frac{3t^2}{2} - \frac{t^3}{3}\right]_0^x$

$F(x) = \frac{3}{16}\left(\frac{3x^2}{2} - \frac{x^3}{3}\right)$

$$= \frac{9x^2}{32} - \frac{x^3}{16}$$

$$= \frac{1}{32}\left(9x^2 - 2x^3\right)$$

For $\frac{3}{16}\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_0^2 = \frac{3}{16}\left(\frac{36-16}{6}\right)$

$$= \frac{20 \times 3}{16 \times 6}$$

$$= \frac{10}{16}$$

$2 \le x \le 4$

$$F(x) = \frac{10}{16} + \frac{3}{16}\int_2^x (4-t)\,dt$$

$$F(x) = \frac{10}{16} + \frac{3}{16}\left[4t - \frac{t^2}{2}\right]_2^x$$

$$= \frac{10}{16} + \frac{3}{16}\left[\left(4x - \frac{x^2}{2}\right) - 6\right]$$

$$= \frac{10}{16} + \frac{3}{16}\left(4x - \frac{x^2}{2}\right) - \frac{18}{16}$$

$$= \frac{12}{16}x - \frac{3x^2}{32} - \frac{8}{16}$$

$$= \frac{1}{32}(24x - 3x^2 - 16)$$

For $\frac{1}{32}\left(24x - 3x^2 - 16\right)\Big|^4 = \frac{1}{32} \times 32 = 1$

$$F(x) = 1 \quad x \ge 4$$

$$\therefore \quad F(x) = \begin{cases} 0 & x \le 0 \\ \frac{1}{32}\left(9x^2 - 2x^3\right) & x \le x \le 2 \\ \frac{1}{32}\left(24x - 3x^2 - 16\right) & 2 \le x \le 4 \\ 1 & x \le 4 \end{cases}$$

(iv) $P(1 \le x \le 3) = F(3) - F(1)$

$$= \left[\frac{1}{32}(72 - 27 - 16) - \frac{1}{32}(9-2)\right]$$

$$= \frac{29}{32} - \frac{7}{32}$$

$$= \frac{22}{32}$$

$$= \frac{11}{16}$$

$$= \mathbf{0.6875}$$

Example 7; The time taken to travel from Mbarara to Kabwohe and Kabwohe to Ishaka is represented in t hours and has the probability density function;

$$\mathbf{f(t)} = \begin{cases} 10ct^2 & ; \quad 0 \le t \le 0.6 \\ 9c(1-t) & ; \quad 0.6 \le t \le 1.0 \\ 0 & ; \text{otherwise} \end{cases}$$

Where c is a constant

(a) Find the value of c and sketch the graph.
(b) Write down the most likely time
(c) Find the expected time.
(d) Determine the probability that the time will be
i. More than 48 minutes
ii. Between 24 and 48 minutes.

Solution

(a) $$\int_{\text{allt}} f(t)\,dt = 1$$

$$\int_0^{0.6} t^2\,dt + 9c\int_{0.6}^{1.6} (1-t)\,dt = 1$$

$$\frac{10c}{3}\left[t^3\right]_0^{0.6} + 9c\left[t - \frac{t^2}{2}\right]_{0.6}^{1.0} = 1$$

$$0.72c + 0.72c = 1$$

$$1.44c = 1$$

$$c = \frac{1}{1.44}$$

$$= \frac{100}{1.44} = \frac{25}{36}$$

for $f(t) = 10ct^2$

at $t = 0$ $\quad f(0) = 0$

at $t = 0.6$ $\quad f(0.6) = \frac{250}{36} \times 0.6^2$

$= 2.5$

for $f(t) = \frac{225}{36}(1-t)$

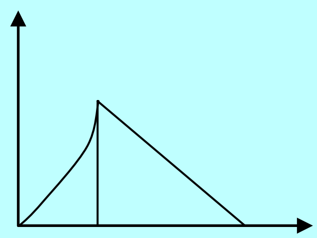
at $t = 0.6$, $\quad f(0.6) = \frac{225}{36} \times 0.4$

$= 2.5$

at $t = 1$, $\quad f(1) = \frac{225}{36} \times 0$

$= 0$

SKETCH OF THE DISTRIBUTION:



(b) the most likely time (mode) = 0.6 hours
= 36 minutes

(c) $E(t) = \int_{\text{allt}} t\,f(t)\,dt.$

$= \quad 10c\int_0^{0.6} t^3\ dt + 9c\int_{0.6}^{1} t - t^2\ dt$

$= \quad \frac{10c}{4}\left[t^4\right]_0^{0.6} + 9c\left[\frac{t^2}{2} - \frac{t^3}{3}\right]_{0.6}^{1}$

$= \quad 0.225 + 0.366$

$= \quad 0.591$ hours

$= \quad 35.5$ minutes

(d) (i) 48 minutes $= \quad 0.8$ hours

$P(t > 0.8) \quad = \quad 9c\int_{0.8}^{1.0} (1 - t)\ dt$

$= \quad 9c\left[t - \frac{t^2}{2}\right]_{0.8}^{1.0}$

$= \quad$ **0.125**

(ii) 24 minutes =0.4 hours

$P(0.4 < t < 0.8) \quad = \quad 1 - P(t>0.8)\ P(t<0.4)$

$P(t < 0.4) \quad = \quad 10c\int_0^{0.4} t^2\ dt$

$= \quad \frac{10c}{3}\left[t^3\right]_0^{0.4}$

$= \quad 0.1481$

$P(0.4<t<0.8) \quad = \quad 1 - 0.125 \quad = \quad$ **0.1481**

Example 8; **The output of 9 machines in a factory are independent random variables each with probability function given by;**

$$\mathbf{F(x)} = \begin{cases} ax & ;\ 0 \le x \le 10 \\ a(20 - x) & ;\ 10 \le x \le 20 \\ 0 & ;\ \text{otherwise} \end{cases}$$

Find (i) the value of a

(ii) the expected value and variance of the output of each machine. Hence or otherwise find the expected value and variance of the total ouput from all machines.

Solution;

(i) $\quad \int_0^{10} x\ dx + a\int_{10}^{20} (20 - x)\ dx \quad = \quad 1$

$a\left[\frac{x^2}{2}\right]_0^{10} + 9\left[20x - \frac{x^2}{2}\right]_{10}^{20} \quad = \quad 1$

$50a + a\ [(400 - 200) - (200 - 50)] \quad = \quad 1$

$50a + 50a \quad = \quad 1$

$100a \quad = \quad 1$

$:.\ a \quad = \quad \frac{1}{100}$

(ii) $\quad E(X) \quad = \quad a\int_0^{10} x^2\ dx + a\int_{10}^{20} (20x - x2)\ dx$

$= \quad a\left[\frac{x^3}{3}\right]_0^{10} + a\left[10x^2 - \frac{x^3}{3}\right]_{10}^{20}$

$= \quad \frac{1000a}{3} + a\left[\left(4000 - \frac{8000}{3}\right) - \left(1000 - \frac{1000}{3}\right)\right]$

$= \quad \frac{1000a}{3} + a\left[\left(1000 - \frac{7000}{3}\right)\right]$

$$= \frac{1000}{3} \text{ x } \frac{1}{100} + \frac{1}{100} \text{ x } \frac{2000}{3}$$

$$= \frac{10}{3} + \frac{20}{3}$$

$$= \frac{20+10}{3}$$

$$= \frac{30}{3} = 10$$

$$\mathbf{E(X) = 10}$$

$$\text{Var}(x) = E(X^2) - (E(X))^2$$

$$= a\int_0^{10} x^3\,dx + a\int_{10}^{20} (20x^2 \;\; x^3)\,dx$$

$$= a\left|\frac{x^4}{4}\right|_0^{10} + 9\left|\frac{20x^3}{3} - \frac{x^4}{4}\right|_{10}^{20}$$

$$= 2500a + a\left[\left(\frac{160{,}000}{3} - \frac{160{,}000}{4}\right) - \left(\frac{20{,}000}{3} - \frac{10{,}000}{4}\right)\right]$$

$$= 2500a + \left(\frac{160{,}000}{12} - \frac{50{,}000}{12}\right)$$

$$= 2500a + \frac{110{,}000a}{12}$$

$$= 2500 \text{ x } \frac{1}{100} + \frac{110{,}000}{12} \text{ x } \frac{1}{100}$$

$$= 25 + \frac{1{,}100}{12}$$

$$= \frac{700}{6} = \frac{350}{3}$$

Example 9; A probability density function is given as

$$f(x) = \begin{cases} kx(4 - x^2) & :0 \le x \le 2 \\ 0 & \text{Elsewhere} \end{cases}$$

find the (i) value of k ,

(ii) median,

(iii) mean,

(iv) standard deviation.

Solution:

(i) The value of *k* is got from the general relation that

$$\int_{-\infty}^{+\infty} f(x)dx = 1$$

$$\Rightarrow \quad \int_0^2 kx(4 - x^2)dx = 1$$

$$k\int_0^2 (4x - x^3)dx = 1$$

$$k\left\{2x^2 - \tfrac{1}{4}x^4\right\}_0^2 = 1$$

$$k[2x2 - ¼ \text{ x } 4]_0^2 = 1$$

$$k\left[\left(2\times 2^2 - \frac{1}{4}\times 2^{4^0}\right) - (0)\right] = 1$$

$$k\left[8-\frac{16}{4}\right) = 1$$

$$4k = 1$$

$$k = 1/4$$

$\therefore$ $k = 1/4$ #.

(i) The median is given by

$$\int_0^m f(x)dx = \tfrac{1}{2}$$

Where m = median.

$$\frac{1}{4}\int_0^m (4x - x^3)dx = \tfrac{1}{2}$$

$$\frac{1}{4}\left\{2x^2 - \tfrac{1}{4}x^4\right\}_0^m = \tfrac{1}{2}$$

2m2 - ¼ m4 = 2

$\Rightarrow$ 8m2 – m4 = 8

Let m2 = p

$\therefore$ 8p - p2 = 8

p2 - 8p + 8 = 0

$$(p-4)^2 + 8 - 16 = 0$$

(p - 4)2 = 8

$$p - 4 = \underline{\pm\sqrt{8}}$$

$$= \pm 2\sqrt{2}$$

$\therefore$ $p = 4 + 2\sqrt{2}$ **or** $4 - 2\sqrt{2}$

$\therefore$ $p = 6.83$ or $p = 1.17.$

But $p = m^2$

$m = \sqrt{6.83}$ or $m = \sqrt{1.17.}$

$= 2.61$ or $m = 1.08$

Since $f(x)$ is such that

$0 \le x \le 2$, it follows that the median is between 0 and 2.

Therefore = 2.61

is false.

Hence median (m) = **1.08** #.

The mean for f (x) is given by

$$\overline{x} = \int_0^2 x f(x)dx.$$

$$= \frac{1}{4}\int_0^2 x(4x - x^3)dx$$

$$= \tfrac{1}{4}\int_0^2 (4x^2 - x^4)\, dx.$$

$$= \frac{1}{4}\left\{\tfrac{4}{3}x^3 - \tfrac{1}{5}x^5\right\}_0^2$$

$$= \tfrac{1}{4}\left\{\tfrac{4}{3}\times 2^3 - \tfrac{1}{5}\times 2^5\right\}$$

$$= \frac{1}{4}\left\{\frac{32}{3}-\frac{32}{5}\right.$$

$$= \frac{1}{4}\left\{\frac{160-96}{15}\right\}$$

$$= \frac{1}{4}\times\frac{64}{15}$$

$$= 16/15$$

$$\therefore \quad E(x) = \overline{x} = \frac{16}{15}$$

(i) Standard deviation is given by:

$$\sigma = \sqrt{E(x^2)-\{E(x)\}^2}$$

But $E(x^2) = \int_0^2 x^2 f(x)\partial x$

$$= \frac{1}{4}\int_0^2 (4x^3-x^5)dx$$

$$= \frac{1}{4}\left[\frac{4}{4}x^4-\frac{1}{6}x^6\right]_0^2$$

$$= \frac{1}{4}\left[16-\frac{64}{6}\right]$$

$$= 4-\frac{8}{3}$$

$$E(x^2) = \frac{16}{3}$$

$$\sigma = \sqrt{\frac{16}{3}-\left(\frac{16}{15}\right)^2}$$

$$\sigma = 2.05$$

$\therefore$ **Standard deviation is 2.05**

Example 10; A continuous random variable X is define by the p.d.f

$$f(x) = \begin{cases} k\left(x-\frac{1}{a}\right), & 0<x<3 \\ 0, & \text{Elsewhere} \end{cases}$$

Given that P(x > 1) = 0.8, find the

(i) values of α and k,

(ii) (ii) probability that X lies between 0.5 and 2. (iii) mean of X

. ***Solution:***

$$f(x) = \begin{cases} k\left(x-\frac{1}{a}\right), & 0<x<3 \\ 0, & \text{elsewhere} \end{cases}$$

Given $P(X > 1) = 0.8$

Using the given statement,

$$\int_1^3 f(x)dx = 0.8$$

$$\int_1^3 k\left(x-\frac{3}{a}\right)dx = 0.8$$

$$k\left[\frac{1}{2}x^2 - \frac{x}{a}\right]_1^3 = 0.8$$

This simplifies to

$$k\left[4 - \frac{2}{a}\right] = 0.8 \ldots\ldots\ldots(1)$$

Using the usual probability analogy;

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_0^3 k(x - {}^1/_a)\,dx = 1$$

$$k\left[\frac{1}{2}x^2 - \frac{x}{a}\right]_0^3 = 1$$

$$k\left[\frac{9}{2} - \frac{3}{a}\right] = 1\ldots\ldots\ldots\ldots.(ii)$$

Dividing i by ii

$$\Rightarrow \frac{4 - {}^2/_a}{4.5 - {}^3/_a} = \frac{0.8}{1}$$

$$4 - {}^2/a = 0.8(4.5 - \frac{3}{a})$$

Solving for a yields

$$a = -1$$

from

$$k\,(4 - 2/a) = 0.8$$
$$k\,(4 - 2/-1) = 0.8$$
$$k\,(4 + 2) = 0.8$$
$$k = \frac{0.8}{6}$$
$$\therefore \quad k = \frac{2}{15}$$

(ii) We find

$$\int_{0.5}^{2.5} f(x)\,dx = \int_{0..5}^{2.5} k(x - \frac{1}{a})\,dx$$

$$= \frac{2}{15}\left[\frac{1}{2}x^2 + x\right]_{0.5}^{2.5}$$

$$\frac{2}{15}[5.625 - 0.625] = \frac{2}{15}\times 5$$

$$\therefore \mathbf{P(0.5 \le x \le 2.5)} = \frac{2}{3}$$

(iii) Mean of x = E(x)

$$= \int_0^3 x\,f(x)\,dx$$

$$= {}^2/_{15}\int_0^3 x(x+1)\,dx$$

$$= \frac{2}{15}\left[\frac{1}{3}x^3 + \frac{1}{2}x^2\right]_0^3$$

$$= \frac{2}{15}[9 + 4.5]$$

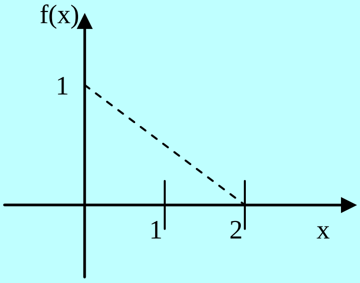
$$= \frac{27}{15} = \frac{9}{5}$$

$\therefore$ **Mean of x** $= \dfrac{9}{15}$

Example 11(a) A random variable x takes on the valkues of the interval 0 < x < 2 and has a probability density function given by

f(x) = $\begin{cases} a & ; \ 0 \langle x \leq 1\frac{1}{2}, \\ \frac{a}{2}(2-x); & 1\frac{1}{2}\langle x \leq 2, \\ 0 \ ; & \text{Elsewhere.} \end{cases}$

Find (i) the value if a,
(ii) P(x <1.6) (5 marks)
(b) The probability density function f(x) of the random variable X takes on the form shown in the diagram below



Determine the expression for f(x) . hence obtain the
(i) expression for the cumulative probability density function of X
(ii) mean and the variance of X (7 marks)
Solution:
We are given

$$f(x) \begin{cases} a & ; 0 < x \leq 1^1/_2 \\ \frac{a}{2}(2-x); & 1\frac{1}{2} < x \leq 2 \\ 0 & ; \text{Else where} \end{cases}$$

(i) To find a
We have

$$\int_{a11x} f(x)\,dx = 1$$

$$\int_0^{1^1/_2} a\,dx + \int_{1^1/_2}^2 \frac{a}{2}(2-x)\,dx = 1$$

$$a[x]_0^{3/2} + \frac{a}{2}\left[2x - \frac{x^2}{2}\right]_{3/2}^2 = 1$$

$$a\left[\frac{3}{2}\right] + \frac{a}{2}\left[(4-2) - (3 - \frac{-9}{8})\right] = 1$$

$$\frac{39}{2} + \frac{a}{2}\left[\frac{1}{8}\right] = 1$$

$$\frac{3a}{2} + \frac{9}{16} = 1$$

$$24a + a = 16$$

$$25a = 16$$

$$\therefore \mathbf{a} = \frac{16}{24} \quad \#$$

(ii) p(x<1.6) $= \int_0^{1.5} {}^{16}/_{25}\,dx + \int_{1.5}^{1.6} {}^{16}/_{50}(2-x)\,dx$

$$= \quad 16/25\ [x]_0^{1.5} + \frac{8}{25}\left[2x - \frac{x^2}{2}\right]_{1.5}^{1.6}$$

$$= \quad \frac{16}{25}\ \text{x}\ 1.5 + 8/25\ [(3.2 - 1.28) - (3 - 1.125)]$$

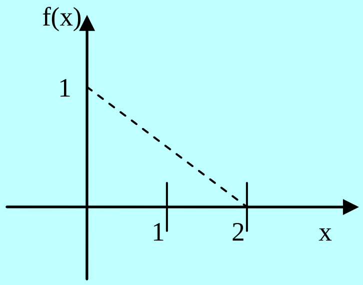
$= \quad 0.96 + 8/25\ (0.045)$

$= \quad 0.96 + 0.0144$

$= \quad 0.9744$

$\therefore$ P (x<1.6) $=$ **0.97** (2 dec. places)

b)



Using $\quad y \quad = \quad mx + c$

For two points A and B co-ordinates

(0,1) and (2,0) respectively .

$$m \quad = \quad \left(\frac{1-0}{0-2}\right) = \frac{1}{-2} = \frac{-1}{2}$$

and $\quad c \quad = \quad 1. \quad$ [f (x) int ercept]

$$\therefore \quad f(x) \quad = \quad -\tfrac{1}{2}x + 1$$

$$\text{Hence } f(x) \quad = \quad \begin{cases} {}^{-1}/_{2}x + 1 & ; 0 \le x \le 2 \\ 0 & ; \text{ else where} \end{cases}$$

(i) To find the expression for the cumulative probability density function f(x)

For $\quad 0 \le t \le 2, \quad$ t any value

$$f(t) \quad = \quad \int_0^t \left(-\frac{1}{2}x + 1\right) dx$$

$$= \quad \left[{}^{-1}/_{4}\,x^2 + x\right]_0^t$$

$$= \quad -\frac{1}{4}t2 + t, \quad 0 \le t \le 2$$

it can be noted that

$F(2) \quad = \quad 1$

$$ie \quad -\frac{1}{4}\text{x } 22 + 2 \quad = \quad -\frac{4}{4} + 2 \quad = \quad -1 + 2 = 1$$

Thus rewriting in terms of x

$$F(x) = \begin{cases} 0 & ; x \le 0 \\ x - \frac{1}{4}x^2 & ; 0 \le 2 \\ 1 & ; x \ge 2 \end{cases}$$

Mean of x is

$$E(x) \quad = \quad \int x + f(x)\, dx$$

$$= \quad \int_0^2 x\,({}^{-1}/_{2}\,x + 1) dx$$

$$= \int_0^2 (x - \frac{x^2}{2})dx$$

$$= \left[\frac{1}{2}x^2 - \frac{x^3}{6}\right]_0^2$$

$$= (2- \frac{1}{8}) \quad = \quad \frac{2}{3}$$

$\therefore E(x) = \frac{2}{3}$ #

Variance of x is

Var (x) = E(X2) – [E(X)]2

But

$$E(x2) = \int_{a11x} x^2 f(x)dx$$

$$= \int_0^2 (x^2 - \tfrac{x}{2}^3)dx$$

$$= \left[\frac{1}{3}x^3 - \frac{x^4}{8}\right]_0^2$$

$$= (\frac{8}{3} - 2) \quad = \quad \frac{2}{3}$$

Thus

$$Var(X) = \frac{2}{3} - \left(\frac{2}{3}\right)^2$$

$$= \frac{2}{3} - \frac{4}{9} = \frac{6-4}{9} = \frac{2}{9}$$

$\therefore Var(x) = \frac{2}{9}$ #

Example 12 A random variable X has a probability density function given.

$$f(x) = \begin{cases} kx & 0 \le x \le 1 \\ k(4-x^2); & 1 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

(i) Find the constant k.
(ii) Determine E(X) and Var (X).
(iii) Find the cumulative distribution, F(x) and sketch it.

Solution:

(i) $k\int_0^1 x\,dx + k\int_1^2 (4 - x2)\,dx = 1$

$$k\left\{\left|\frac{x^2}{2}\right|_0^1 + \left|4x - \frac{x^3}{3}\right|_1^2\right\} = 1$$

$$k\left\{\frac{1}{2} + \left(8 - \frac{8}{3}\right) - \left(4 - \frac{1}{3}\right)\right\} = 1$$

$$\therefore \quad k = \frac{6}{13}$$

(iii) $E(X) = \frac{6}{13}\left[\int_0^1 x2\,dx + \int_1^2 (4x - x3)\,dx\right.$

$$= \frac{6}{13}\left[\left|\frac{x^3}{3}\right|_0^1 + \left(2x^2 - \frac{x^4}{4}\right)_1^2\right]$$

$$= \frac{6}{13}\left[\frac{1}{3} + (8-4) - (2-\tfrac{1}{4})\right] = \frac{6}{13} \times \frac{31}{12}$$

$$E(X) = \frac{31}{26} = \mathbf{1.1923}$$

$$Var(X) = E(X2) - (E(X))2$$

$$E(X2) = \frac{6}{13}\left[\int_0^1 x^3\,dx + \int_1^2 (4x^2 - x^4)\,dx\right]$$

$$= \frac{6}{13}\left[\frac{x^4}{4}\right]_0^1 + \left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_1^2$$

$$= \frac{6}{13}\left[\frac{1}{4} + \frac{(64-17)}{15}\right] = \frac{203}{130} \approx 1.5615$$

$$Var(X) = 1.5615 - (1.1923)2 = 1.1399$$

$$= \mathbf{1.14}$$

(iii) $F(x) = 0 \quad x \le 0$

$$F(x) = \frac{6}{13}\int_0^x t\;dt = \frac{3x^2}{13}, \quad 0 \le x \le 1$$

$$F(1) = \frac{3x^2}{13}\Big|_0^1 = \frac{3}{13}$$

$$F(x) = \frac{3}{13} + \int_1^x (4 - t^2)\,dt = \frac{6}{13}\left(4x - \frac{x^3}{3}\right) - \frac{19}{13}$$

$$F(x) = \frac{1}{13}(70x - 6x^3 - 57) \qquad 1 \le x \le 2$$

$$F(x) = \frac{1}{13}(144 - 48 - 57) = \frac{39}{39} = 1$$

$F(x) = 1 \quad x \ge 2$

$$F(x) = \begin{cases} 0 & ;\ x \le 0 \\ \frac{3x^2}{13} & ;\ 0 \le x \le 1 \\ \frac{1}{39}(72x - 6x^3 - 57 & ;\ 1 \le x \le 2 \\ 1 & ;\ x \ge 2 \end{cases}$$

For $F(x) = \frac{3x^2}{13}$ at $x = 0$ $\quad F(0) = 0$

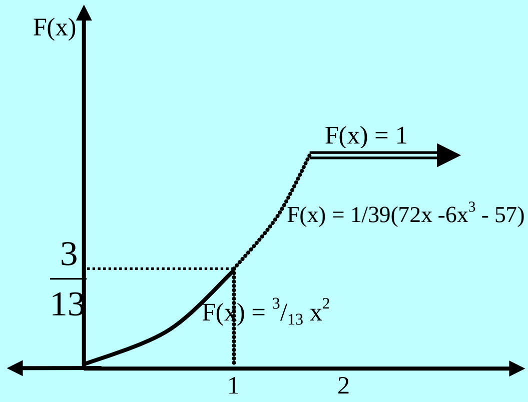
At $x = 1$ $\quad F(1) = \frac{3}{13}$

For $F(x) = \frac{1}{39}\left(72x - 6x^3 - 57\right)$

At $x = 1$ $\quad F(1) = \frac{1}{39}\left(72 - 6 - 57\right)$

$$= \frac{9}{39} = \frac{3}{39}$$

At $x = 2$ $\quad F(2) = 1$



Example 13: A continuous random variable X has a probability density function.

$$\mathbf{f(x)} = \begin{cases} 3c(x^2+3) & ; \quad -3 \le x \le 0 \\ 3c(x+3 & ; \quad 0 \le x \le 3 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

Determine

(i) the constant c

(ii) the expectation (E(X)) and variance var(X).

(iii) the cumulative distribution function.

Solution

(i)

$$3c\int_{-3}^{0}(x^2+3)\,dx + 3c\int_{0}^{3}(x+3)\,dx = 1$$

$$3c\left|\frac{x^3}{3}+3x\right|_{-3}^{0} + \left|\frac{x^2}{2}+3x\right|_{0}^{3} = 1$$

$$3c\left|(0-\left(\frac{27}{3}\right)\right| + 3c\left|\left(\frac{9}{2}+\frac{9}{2}\right)-0\right| = 1$$

$$18 \text{ x } 3c + 3c \text{ x } \frac{27}{2} = 1$$

$$\frac{189}{2}c = 1$$

$$c = \frac{2}{189}$$

(ii)

$$E(x) = \int_{\text{all } x} x\ f(x)\,dx$$

$$= 3c\left[\int_{-3}^{0}(x^3+3x)\,dx + \int_{0}^{3}(x^2+3x)\,dx\right]$$

$$= 3c\left\{\left[\left(\frac{x^4}{4}+\frac{3x^2}{2}\right)\Big|_{-3}^{0}\right] + \left(\frac{x^3}{3}+\frac{3x^2}{2}\right)\Big|_{0}^{3}\right\}$$

$$= 3c\left\{\left[0-\left(\frac{(-3)^4}{4}+\frac{3(-3)^2}{2}\right)_{-3}^{0}\right] + \left(9+\frac{27}{2}\right)-0]\right\}$$

$$= 3c\left[\frac{-(81+54)}{4}+\frac{45}{2}\right]$$

$$= 3c\left[\frac{90-135}{4}\right] = \frac{135c}{4}$$

E(X) = $\frac{-135}{4} \times \frac{2}{189}$ = $\frac{-135}{378}$

E(x) = $\frac{135}{378}$ = **- 0.3571**

Var(x) = E(x2) – (E(X))2

$E(X^2) = 3c\left[\int_{-3}^{0}(x^4+3x^2)\,dx+\int_{0}^{3}(x^3+3x^2)\,dx\right]$

$= 3c\left[\left[\left(\frac{x^5}{5}+x^3\right)\right]_{-3}^{0}+\left|\left(\frac{x^4}{4}+x^3\right)\right|_{0}^{3}\right]$

$= 3c\left[\left(0-\left(\frac{(-3)^5}{5}+(-3)^3\right)+\left(\frac{3^4}{3}+3^3\right)-0\right]$

$= 3c\left[\frac{(243+945)}{20}+\frac{(81+108)}{4}\right]$

$= 3c\left[\frac{1512+945}{20}\right]$

$= \frac{7371 \text{ x } 2}{20 \text{ x } 189}$ = **3.9**

Var(x) = 3.9 – (-0.3571)2

= **3.7725**

(iii) F(X) = 0, $x \leq -3$

F(x) = $\frac{6}{189}\int_{-3}^{x}(t^2+3)\ dt$

$= \frac{6}{189}\left|\left(\frac{t^3}{3}+3t\right)\right|_{-3}^{x}$

$= \frac{6}{189}\left[\left(\frac{x^3}{3}+3x\right)-(-9-9)\right]$

$= \frac{6}{189}\left[\frac{x^3}{3}+3x+18\right]$

F(X) = $\frac{2}{189}$ (x3 + 9x + 54), $-3 \leq x \leq 0$

F(0) = $\frac{2}{189}[x^3 + 9x + 54]$

$= \frac{2}{189} \text{ x } 54 = \frac{108}{189}$

F(x) = $\frac{108}{189} + \frac{6}{189}\int_{0}^{x}(t+3)\ dt = \frac{108}{189} + \frac{6}{189}\left[\left(\frac{t^2}{2}+3t\right)\right]_{0}^{x}$

F(X) = $\frac{1}{189}$(3x2 + 18x + 108), $= 0 \leq x \leq 3$

F(3) = $\frac{1}{189}$(27 + 54 + 108)

$= \frac{189}{189}$ = 1

F(X) = 1 $x \geq 3$

$$F(x) = \begin{cases} 0 & x \le -3 \\ \dfrac{2}{189}\left(x^3 + 9x + 54\right) & -3 \le x \le 0 \\ \dfrac{1}{189}\left(3x^2 + 18x + 108\right) & 0 \le x \le 3 \\ 1 & x \ge 1 \end{cases}$$

Example 14; A random variable X has a probability density function

$$\textbf{f(x)} \quad = \quad \begin{cases} \dfrac{2}{3a}(x+a) & ; \quad a \le x \le 0 \\ \dfrac{1}{3a}(2a-x) & ; \quad 0 \le x \le 2 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

Where a is constant.

Determine ;

(i) The value of a

(ii) The median of x

(iii) P [(x ≤1.5) / (x 0)]

(iv) The cumulative distribution function F(X)

Solution

(i) $$\int_{-\infty}^{\infty} f(x)\ dx \quad = \quad 1$$

$$\frac{2}{3a}\int_{-a}^{0}(x+a)\ dx + \frac{1}{3a}\int_{0}^{2a}(2a-x)\,dx = \quad 1$$

$$\frac{1}{3a}\left[2\left(\frac{x^2}{2}+ax\right)\Bigg|_{-a}^{0} + \left(2ax - \frac{x^2}{2}\right)\Bigg|_{0}^{2a}\right] = \quad 1$$

$$\frac{1}{3a}\left[2\left(\frac{a^2}{2} - a^2\right) + \left(4a^2 - 2a^2\right)\right] = \quad 1$$

$$\frac{1}{3a}(a2 + 2a2) = \quad 1$$

$$\frac{3a^2}{3a} = \quad 1$$

$$a = \quad 1$$

$$f(x) \quad = \quad \begin{cases} \dfrac{2}{3}(x+1) & ; \quad -1 \le x \le 0 \\ \dfrac{1}{3}(2-x) & ; \quad 0 \le x \le 2 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

(ii) $$\frac{2}{3}\int_{-1}^{0}(x+1)\ dt = \quad \frac{2}{3}\left(\frac{x^2}{2}+x\right)\Bigg|_{1}^{0}$$

$$= \quad \frac{1}{3}$$

Median is located between 0 and 2,

$$\frac{1}{3} + \frac{1}{3}\int_{0}^{m}(2-x)\ dx \quad = \quad \frac{1}{2}$$

$$\left(2x - \frac{x^2}{2}\right)\Bigg|_0^m = \frac{1}{6} \text{ x } 3$$

$$2m - \frac{m^2}{2} = \frac{1}{2}$$

$$:.\ m2 - 4m + 1 = 0$$

$$m = \frac{4 \pm \sqrt{12}}{2}$$

$$= \pm\sqrt{3}$$

$$2 + \sqrt{3} = 3.7321 \text{ is outside the range.}$$

$$:.\ \text{Median} = 2 - \sqrt{3} = \mathbf{0.2679.}$$

(iii) $P((x < 1.5) / (x > 0))$

let A $= x < 1.5$ B $= x > 0$

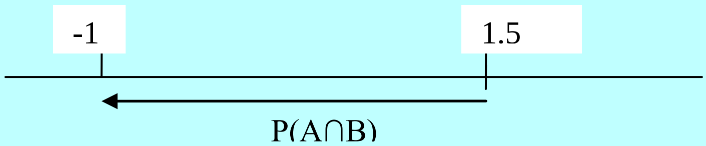
$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B) = P(x > 0)$$

$$= \frac{1}{3}\int_0^2 (2 - x)\ dt$$

$$= \frac{1}{3}\left[2x - \frac{x^2}{2}\right]_0^2$$

$$= \frac{1}{3}(4 - 2) = \frac{2}{3}$$



$$P(AnB) = P(0 \leq x \leq 1.5)$$

$$= \frac{1}{3}\int_0^{1.5} (2 - x\ dx$$

$$= \frac{1}{3}\left[2x - \frac{x^2}{2}\right]_0^{1.5}$$

$$= \frac{1}{3}\left(3 - \frac{9}{8}\right) = \frac{5}{8}$$

$$P(AnB) = \frac{5/8}{2/3}$$

$$= \frac{5}{8} \text{ x } \frac{3}{2}$$

$$= \frac{15}{16}$$

$$= \mathbf{0.9375}$$

(iv) $F(X) = 0 \quad x \leq -1$

$$F(X) = \frac{2}{3}\int_{-1}^{x} (t + 1)\ dt$$

$$= \frac{2}{3}\left[\frac{t^2}{2} + t\right]_{-1}^{x}$$

$$= \quad \frac{2}{3}\left(\frac{x^2}{2}+x+\frac{1}{2}\right)$$

$$F(0) \quad = \quad \frac{1}{3}(x2 + 2x + 1)\Big|^{0} \quad = \quad \frac{1}{3}$$

$$F(x) \quad = \quad \frac{1}{3} + \frac{1}{3}\int_0^x (2-t)\ dt$$

$$= \quad \frac{1}{3} + \frac{1}{3}\left[2t-\frac{t^2}{2}\right]_0^x$$

$$= \quad \frac{1}{3} + \frac{1}{3}\left(2x-\frac{x^2}{2}\right)$$

$$= \quad \frac{1}{6}(2 + 4x - x2)$$

$$F(x) \quad = \quad \frac{1}{6}(2 + 4x - x^2), \qquad 0 \le x \le 2$$

$$F(2) \quad = \quad \frac{1}{6}\ (2 + 4x - x2\Big|^{2}$$

$$= \quad \frac{1}{6}(2 + 4 + - 4)$$

$$= \quad \frac{6}{6}$$

$$= \quad 1$$

$$F(x) \quad = \quad 1, \qquad x \ge 2$$

$$F(x) \quad = \quad \begin{cases} 0 & ; \quad x \le 1 \\ \frac{1}{3}(x^2+2x+1) & ; \quad 1 \le x \le 0 \\ \frac{1}{6}(2+4x-x^2) & ; \quad o \le x \le 0 \\ 1 & ; \quad x \ge 2 \end{cases}$$

Example 15: (a) The random variable X has the probability density

$$\mathbf{f(X)} \quad = \begin{cases} cx & ; \quad 0 \le x \le 1 \\ c(2-x) & ; \quad 1 \le x \le 2 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

where c constant, find the median and mode of the probability distribution.

(b) Which of the two functions f(x) and g(x) below in C1 and C2 are positive constants, can be used a probability density functions over the specified ranges.

i. f(x) = C1 sin x ; $0 \le x \le 3\pi$

ii. g(x) = C2e-2x ; $0 < x < \ln 10$

Determine the cumulative probability function for the case where p.d.f. exists.

Solution:

(a) (i)

$$c\int_0^1 x\ dx + c\int_1^2 (2-x)\ dx \quad = \quad 1$$

$$\frac{c}{2}\left|x^2\right|_0^1 + \frac{c}{2}\left|4x-x^2\right|_1^2 \quad = \quad 1$$

$$\frac{c}{2} + \frac{c}{2}\left|(8-4)-(4-1)\right| \quad = \quad 1$$

$$\frac{c}{2} + \frac{c}{2} \quad = \quad 1$$

$$c \quad = \quad 1$$

$$f(x) = \begin{cases} x & ;\ 0 \le x \le 1 \\ 2 - x & ;\ 1 \le x \le 2 \\ 0 & ;\ \text{elsewhere} \end{cases}$$

(ii) $\int_0^1 x\ dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

:. **Median = 1**

Mode can be found by sketching f(x).

For f(x) = x at x = 0

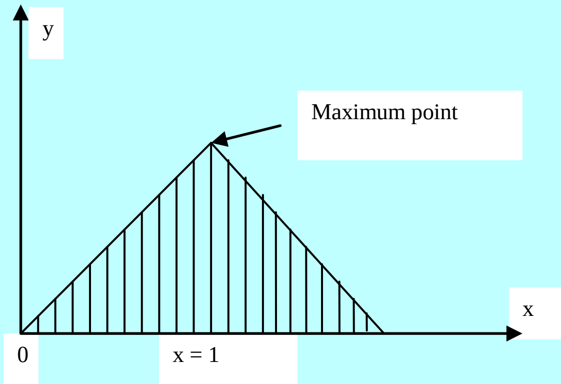
F(0) = 0

At x = 1 f(1) = 1

For f(x) = 2 – x

At x = 1 f(1) = 1

At x = 2 f(2) = 0



Mode = 1

(b) (i) f(x) = C_1 Sin x

$C1 \int_0^{3\pi} \sin x\, dx = 1$

$C1 \left| -\cos x \right|_0^{3\pi} = 1$

$C1 \left| (-\cos 540^0) - (-\cos 0^0) \right|_0^{3\pi} = 1$

C1 (1 + 1) = 1

2C1 = 1

$C1 = \frac{1}{2}$

$$f(x) = \begin{cases} \frac{1}{2}\sin x & ;\ 0 \le x \le 3\pi \\ 0 & ;\ \text{elsewhere} \end{cases}$$

The interval for g(x) is not well demarcated since its 0 < x < ln 10. Therefore its p.d.f. does not exist. It exists for $x \ge 0$.

F(x) = 0, $x \le 0$

F(x) = ½ $\int_0^x \sin t\, dt$

= ½ $\left|\cos t\right|_0^x$

= ½ cos x

F(x) = 1, $x \ge 3\pi$

$$F(x) = \begin{cases} 0 & ; \quad x \le 0 \\ -\frac{1}{2}\cos x & ; \quad 0 \le x \le 3\pi \\ 1 & ; \quad x \le 3\pi \end{cases}$$

Example 16: **A continuous random variable x has probability density function given by:**

F(x) = k(1 + cos x) ; $0 \le x \le \pi$

F(x) = 0 ; otherwise

(i)Show the k = $\frac{1}{\pi}$

(ii) Find, to four decimal places, the mean μ of the distribution.

HINT $\left(\text{Use} \int_0^{\pi} x \cos x \, dx = -2\right)$

(iv) Find the distribution function of X for all values of x.
Hence determine P(x ≤ μ).

Solution (i)

$$k\int_0^{\pi}(1+\cos x)\ dx. = 1$$

$$k\ \left|x+\sin x\right|_0^{\pi} = 1$$

$$k\ \left|(\pi+0)-(0+0)\right| = 1$$

$$\pi k = 1$$

$$k = \frac{1}{\pi} \quad \text{hence shown}$$

(ii) Mean (μ) $= \int_{-\infty}^{\infty} x\ f(x)\ dx.$

$$\mu = \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_0^{\pi} x\cos dx\right]$$

$$= \frac{1}{\pi}\left[\left|\frac{x^2}{2}\right|_0^{\pi} - 2\right]$$

$$= \frac{1}{\pi}\left[\left|\frac{\pi^2}{2} - 0\right| - 2\right]$$

$$= \frac{1}{\pi}.\frac{\pi^2}{2} - \frac{2}{\pi}$$

$$= \frac{\pi}{2} - \frac{2}{\pi}$$

$$= 1.5708 - 0.6366$$

Mean (μ) = 0.9342

(iii) F(x) = 0 $\quad x \le 0$

$$F(x) = \int_0^{x}\left(\frac{1}{\pi}+\frac{1}{\pi}\cos t\right)\ dt$$

$$= \left|\frac{t}{\pi}+\frac{t}{\pi}\sin t\right|_0^{x}$$

$$F(x) = \frac{1}{\pi}(x + \sin x) \qquad 0 \le x \le \pi$$

$$F(x) = 1 \qquad x \ge \pi$$

$$F(x) = \begin{cases} 0 & ; \quad x \le 0 \\ \frac{1}{\pi}(x+\sin x) & ; \quad 0 \le x \le \pi \\ 1 & ; \quad x \ge \pi \end{cases}$$

$$P(x \le \pi) = P(x \le 0.9342)$$
$$= \frac{1}{\pi}\left|x - \sin x\right|_0^{0.9342}$$
$$= 0.3183\,[(0.9342 - \sin 0.9342) - (0 - 0)]$$
$$\mathbf{P(x \le \mu) = 0.0414}$$

Example 17:
A continuous random variable X has a probability density function f(x) = 5e-5x, x > 0 find:
(a) P(x > 0.5)
(b) E(x)
(c) The standard deviation of x.
(d) The median
(e) The mode
(f) P(x < E(x))
Solution:

(a) $P(X > x) = e^{-\lambda x}$

$P(X > 0.5) = e^{-0.5\times 5} = \mathbf{0.0821}$

(b) $f(x) = \lambda e^{-\lambda x}$

$E(x) = \frac{1}{\lambda}$

:. $f(x) = 5e^{-5x}$

$E(x) = \frac{1}{5} = \mathbf{0.2}$

(c) $f(x) = \lambda e^{-\lambda x}$

$Var(x) = \frac{1}{\lambda^2}$

Standard deviation $= \sqrt{\frac{1}{\lambda^2}}$

$f(x) = 5e^{-5x}$

$Var(x) = \frac{1}{5^2}$

Standard deviation $= \sqrt{\frac{1}{5^2}} = \mathbf{0.2}$

(d) $f(x) = \lambda e^{-\lambda x}$ F(m) = 0.5

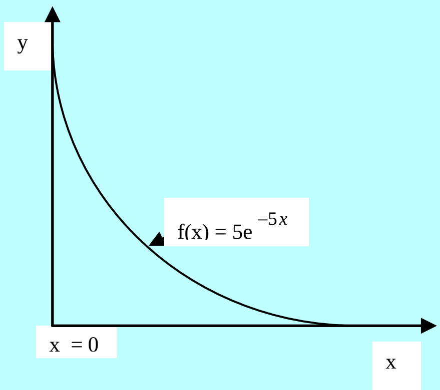
$1 - e^{-\lambda m} = 0.5$

$-\lambda m = \ln 0.5$

$m = \frac{1}{\lambda}$ in 0.5 $= -0.2 \ln 0.5$

$= \mathbf{0.1386}$

(e) The mode is the maximum value on the sketch.



Mode = 0

(f) P(x < E(x)) = P(x < 0.2).

= $1 - e^{-\lambda x}$

$\lambda = 5$, x = 0.2

P(x < E(x)) = $1 - e^{-5\times 2}$

= **0.6321**

Example 18: **Each batch of a chemical used in drug manufacture is tested for impurities. The percentage of impurity X, where X is a random variable with probability density function given by:**

$$\mathbf{f(x)} \quad = \quad \begin{cases} bx & ; \quad 0 \le x \le 1 \\ \frac{b}{3}(4 - x) & ; \quad 1 \le x \le 4 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

where b is a constant.

(i) Sketch the graph of f(x).

(ii) Show that b $= \frac{1}{2}$

(iii) Determine for all values of x, the distribution function.

(iv) P(| x - μ | ≤ 0.05)

Solution

(i) for f(x) = bx

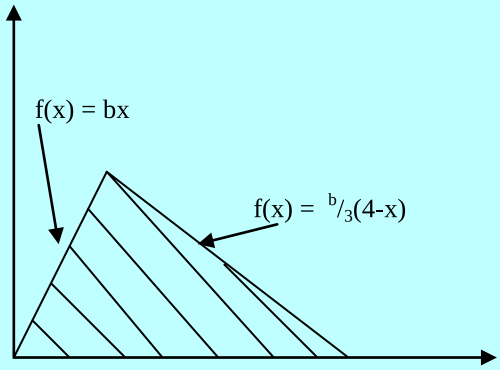
at x = 0 f(0) = 0

x = 1 f(1) = b

For f(x) = $\frac{b}{3}(4 - x)$

At x = 1 f(1) = b

x = 4 f(4) = 0



(i) $b\int_0^1 x\,dx + \frac{b}{3}\int_1^4 (4-x)\,dx = 1$

$$b\left|\frac{x^2}{2}\right|_0^1 + \frac{b}{3}\left|4x - \frac{x^2}{2}\right|_1^4 = 1$$

$$\frac{b}{2} + \frac{9b}{6} = 1$$

$$\frac{b}{2} + \frac{3b}{2} = 1$$

$$\frac{4b}{2} + 2b = 1$$

$$\therefore \mathbf{b} = \frac{b}{3}$$

$$f(x) = \begin{cases} \frac{x}{2} & ; \ 0 \le x \le 1 \\ \frac{1}{6}(4-x) & ; \ 1 \le x \le 4 \\ 0 & ; \ \text{elsewhere} \end{cases}$$

(iii) $f(x) = 0, \quad x \le 0$

$$F(x) = \frac{1}{2}\int_0^x t\,dx$$

$$= \frac{1}{4}\left|t^2\right|_0^x = \frac{x^2}{4} \qquad 0 \le x \le 1$$

$$F(1) = \frac{1^2}{4} = \frac{1}{4}$$

$$F(x) = \frac{1}{4} + \frac{1}{6}\int_1^x (4-t)\,dt$$

$$= \frac{1}{4} + \frac{1}{6}\left|\left(4t - \frac{t^2}{2}\right)\right|_1^x$$

$$F(x) = \frac{1}{4} + \frac{1}{6}\left(4x - \frac{x^2}{2} - 4 + \frac{1}{2}\right)$$

$$= \frac{1}{4} + \frac{1}{12}(8x - x2 - 7)$$

$$F(4) = \frac{1}{2}(32 - 16 - 4)$$

$$= \frac{12}{12} = 1$$

$F(x) = 1, \quad x \ge 4$

$$F(x) = \begin{cases} 0 & ; \ x \le 0 \\ \frac{x^2}{4} & ; \ 0 \le x \le 1 \\ \frac{1}{12}(8x - x^2 - 4) & ; \ 1 \le x \le 4 \\ 1 & ; \ x \ge 4 \end{cases}$$

(iv) E(x) or μ $= \int_{-\infty}^{\infty} x\,f(x)\,dx$

$$\mu = \frac{1}{2}\int_0^1 x^2\,dx + \frac{1}{6}\int_1^4 (4x - x^2)\,dx$$

$$= \frac{1}{2}\left|\frac{x^3}{3}\right|_0^1 + \frac{1}{6}\left|2x^2 - \frac{x^3}{3}\right|_1^4$$

$$= \frac{1}{6} + \frac{1}{6}\left(32 - \frac{64}{3}\right) - \left[\ 2 - \frac{1}{3}\ \right]$$

$$= \frac{1}{6} + \frac{1}{6}$$

$$= \frac{1}{6} + \frac{1}{6} \cdot \frac{27}{3}$$

$$= \frac{1}{6} + \frac{9}{6} = \frac{10}{6}$$

$$\mu = \frac{10}{6} = \frac{5}{3} = \mathbf{1.6667}$$

$P(|X - \mu| \leq 0.05) = P(-0.05 \leq x - 1.6667 \leq 0.05$

$= P(1.6167 \leq x \leq 1.7167)$

$= F(1.7167) - F(1.6167)$

$F(1.7167) = \frac{1}{12}(8x - x2 - 4)$

$= \frac{1}{12}(8 \text{ x } 1.71672 - 4)$

$= 6.7865$

$F(1.6167) = \frac{1}{12}(8x - x2 - 4)$

$= \frac{1}{12}(\ 8 \text{ x}1.6167 - 1.61672 - 4)$

$= 6.3199$

$P(1.6167 \leq X \leq 1.7167) = 6.7865 - 6.3199$

$= \mathbf{0.4667}$

EXERCISE 11

1..The probability density functions is given by

$$f(x) = \begin{cases} \frac{x}{6} & ; \quad 0 \leq x \leq 3 \\ \frac{1}{2}(4 - x) & ; \quad 3 \leq x \leq 4 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

(a) Sketch the graph f(x)

(b) Calculate the probability that x occurs in the interval (1, 2)

(c) Calculate the probability that x > 2.

(d) Obtain the cumulative probability function and hence or otherwise find the median of the distribution.

2.. Paraffin is delivered to a market every Monday morning. At this market the weekly demand of paraffin in thousands of liters is a continuous random variable X distributed with probability function of the form.

$$f(x) = \begin{cases} ax^2(b-x) & ; \quad 0 \le x \le 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

(i) Given that the mean weekly demand is 600 litres, determine the values of a and b.

(ii) If storage tanks at this market are filled to capacity of 900 litres every Monday, what is the probability that in any given week the market will be unable to meet the demand for paraffin?

3. (a) A probability density function is given by

$$f(x) = \begin{cases} \dfrac{x+1}{4} & ; \quad 0 \le x \le a \\ 0 & ; \quad \text{otherwise,} \end{cases}$$

Where a is a positive constant.

Determine:

(a) The value of a

(b) The cumulative probability function.

(c) P(x < ½)

(d) P(x > 1)

4. A continuous random variable X has the probability density function defined by

$$f(x) = \begin{cases} \dfrac{1}{3}cx & ; \quad 0 \le x < 3 \\ c & ; \quad 3 \le x \le 4 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Where c is a positive constant

Determine:

(i) The value of c

(ii) The mean of x

(iii) The value of a, for there to be a probability of 0.85 that a randomly observed value of x will exceed a.

5. The random variable X has probability density function given by

$$f(x) = \begin{cases} dx & ; \quad 0 \le x \le 1 \\ d & ; \quad 1 \le x \le 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Where d is a constant

(i) show that $d = \dfrac{2}{3}$

(ii) determine E(X) and E(X2)

(iii) show that the median m of x is 1.25 and find $P(|x-m| > ½)$

6. The continuous random variable X has probability density function f(x) defined by;

$$f(x) = \begin{cases} \dfrac{c}{x^4} & ; \quad x \le -1 \\ c(2-x^2) & ; \quad -1 \le x \le 1 \\ \dfrac{c}{x^4} & ; \quad x \ge 1 \end{cases}$$

Where c is a constant

(i) show that c = ½

(ii) Sketch the graph f(x).

(iii) determine the cumulative distribution function f(x)

(iv) Determine the expected value of X and the variance X.

7. The probability density function is given by:

$$f(x) = \begin{cases} a\sin \pi x & ; \ 0 \le x \le 1 \\ 0 & ; \ \text{elsewhere} \end{cases}$$

where a is a positive constant.

Determine:

(i) the constant a

(ii) The cumulative probability function.

(iii) $P(x < \frac{1}{3})$

(iv) $P\left(\frac{1}{2} < x < \frac{2}{3}\right)$

8. The continuous random variable X has cumulative distribution function F(x) where

$$F(x) = \begin{cases} 0 & ; \ x \le 1 \\ \dfrac{(x-1)^2}{12} & ; \ 1 \le x \le 3 \\ \dfrac{(14x - x^2 - 25)}{24} & ; \ 3 \le x \le 7 \\ 1 & ; \ x \ge 7 \end{cases}$$

(a) find the sketch of f(x)

(b) Find E(x). and Var(x)

(c) Find the median m.

(d) Find $P(2.8 \le x \le 5.2)$.

9. The probability density function of a distribution is given by:

$$f(x) = \begin{cases} k(1-x^2) & ; \ -1 \le x \le 1 \\ 0 & ; \ \text{elsewhere} \end{cases}$$

where k is a constant

(a) determine the value k

(b) $P\left(-\frac{1}{2} \le x \le \frac{1}{2}\right)$

(c) $P\left(x \ge \frac{1}{3}\right)$

(d) Obtain the expression for the cumulative probability function of the distribution.

10. The continuous random variable X has probability density function given by:

$$f(x) = \begin{cases} c(1+x^2) & ; -1 \le x \le 1 \\ 0 & ; \ \text{otherwise} \end{cases}$$

where C is a constant.

(i) determine the value of c

(ii) Determine E(X) and Var(X).

(iii) If A is event X > ½ and B is the event X > ¾. Find

a. P(B)

b. P(B/A)

11. The probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} ke^{-2x} & ; \ x \ge 0 \\ 0 & ; \ \text{elsewhere} \end{cases}$$

where k is a constant

(a) Prove that k = 2.

(b) Calculate E(X) and Var(X)

(c) Prove that the median (m) = ½ ln2

12. A random variable X has cumulative distribution function given below

$$F(X) = \begin{cases} 0 & ; \quad x \le 0 \\ \dfrac{x^2}{4} & ; \quad 0 \le x \le 1 \\ \alpha x + \beta & ; \quad 1 \le x \le 2 \\ \dfrac{1}{4}(5-x)(x-1) & ; \quad 2 \le x \le 3 \\ 1 & ; \quad x \ge 3 \end{cases}$$

Where $\propto$ and β are constants.

(a) Determine the values of $\propto$ and β.

(b) $P(1.5 \le x \le 2.5)$

(c) The probability density function

(d) E(X) and Var(X)

13. The continuous random variable X has probability density function f(x) where.

$$f(x) = \begin{cases} ax^2 + bx + c & ; \quad -1 \ge x \le 2 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

Where a , b c, are constants, given that E(x) = $\frac{5}{4}$

(i) Determine the values of a, b, and c.

(ii) the value of x which is such that $P(X \le x1) = \frac{1}{8}$ and show that x1= $\frac{1}{2}$

14. A random variable X has the probability density function f(x) given by;

$$f(x) = \begin{cases} Ce^{-2x} & ; \quad 0 < x \le \infty \\ 0 & ; \quad \text{otherwerwise} \end{cases}$$

where C is a constant. Determine

(i) the constant C

(ii) E(X) and Var(X)

(iii) Median

15. A random variable X has cumulative (distribution) function f(x) where.

$$F(x) = \begin{cases} 0 & ; \quad x < -1 \\ kx + k & ; \quad -1 \le x < 0 \\ 2kx + k & ; \quad 0 \le x < 1 \\ 3k & ; \quad \le x \end{cases}$$

Determine

(a) The value k.

(b) The frequency function f(x) of X

(c) The expected value μ of X.

(d) The standard deviation S of X.

(e) The probability that $|X - \mu|$ exceeds $\frac{1}{3}$.

4. ANSWERS

1 (a)

(b) $\frac{1}{4}$ (c) $\frac{3}{4}$

(d) $F(x) = \begin{cases} 0 & ; \ x \le 0 \\ \frac{x^2}{12} & ; \ 0 \le x \le 3 \\ 2x - \frac{x^2}{4} - 3 & ; \ 3 \le x \le 4 \\ 1 & ; \ x \ge 4 \end{cases}$

Median = $\sqrt{6}$)

2 (i) a = 12, b) = 1 (i) 0.0523

3. (a) a = 2 (b) $F(x) = \begin{cases} 0 & ; \ x \le 1 \\ \frac{x^2}{8} + \frac{x}{4} & ; 1 \le x \le 2 \\ 1 & ; \ x \ge 2 \end{cases}$

(c) $\frac{5}{32}$ (d) $\frac{5}{8}$

4. ((a) $\frac{2}{5}$ (b) $\frac{13}{5}$ (c) $\frac{3}{2}$

5. ((i) $1\frac{2}{9}$, $\frac{31}{18}$ (iii) $\frac{17}{48}$)

6. (iii) $F(x) = \begin{cases} \frac{-x^3}{12} & ; \ x \le -1 \\ \frac{(6+6x-x^3)}{12} & ; \ -1 \le x \le 1 \\ \frac{1-x^3}{12} & ; \ x \ge 1 \end{cases}$

(iv) E(x) = 0, Var(X) = $\frac{11}{15}$)

7. (i) $\frac{\pi}{2}$ (ii) $F(x) = \begin{cases} 0 & ; \ x \le 0 \\ \frac{1}{2}(1-\cos \pi x) & ; \ 0 \le x \le 1 \\ 1 & ; \ x \ge 1 \end{cases}$

(iii) $\frac{1}{4}$ (iv) $\frac{1}{4}$

8. a) $f(x) = \begin{cases} \frac{x-1}{6} & ; \ 0 \le x \le 1 \\ \frac{7-x}{12} & ; \ 1 \le x \le 2 \\ 0 & ; \ \text{otherwise} \end{cases}$

(b) $\frac{11}{3}$, $\frac{14}{9}$ (c) 3.54 (d) 0.595

9 (a) $k = \frac{3}{4}$ (b) $\frac{11}{16}$ (c) $\frac{7}{27}$

(d) $F(x) = \begin{cases} 0 & ; \ x \le -1 \\ \frac{1}{4}(2+3x-x^3) & ; \ -1 \le x \le 1 \\ 1 & ; \ x \ge 1 \end{cases}$

10.i) $\frac{3}{8}$ (ii) E(x) = 0, Var(x) = $\frac{2}{5}$ (iii) (a) $\frac{85}{512}$ (b) $\frac{85}{152}$

11. [(b) E(x) = ½ , Var(X) = ¼]

12. ((a) $\propto = ½$; $\beta = \frac{-1}{4}$ (b) $\frac{7}{16}$

(c) $f(x) = \begin{cases} \frac{x}{2} & ; \ 0 \le x \le 1 \\ \frac{1}{2} & ; \ 1 \le x \le 2 \\ \frac{-x}{2} + \frac{3}{2} & ; \ 2 \le x \le 3 \\ 0 & ; \ \text{elsewhere} \end{cases}$

(d) $E(X) = \frac{3}{2}$ $Var(X) = \frac{5}{12}$)

13. ((i) $a = \frac{1}{9}$ (b) $= \frac{2}{9}$ (c) $= \frac{1}{9}$)

14. (i) 2, (ii) $E(X) = \frac{1}{2}$, $Var(x) = \frac{1}{4}$ (d) $\frac{1}{2}\ln 2$

15. (a) $\frac{1}{3}$ (b) $f(x) = \begin{cases} k & ; \ -1 \le x \le 0 \\ 2k & ; \ 0 \le x \le 1 \\ 0 & ; \ \text{elswhere} \end{cases}$

(c) $\frac{1}{6}$ (d) 0.553 (e) $\frac{11}{18}$

5.BINOMIAL DISTRIBUTION

Introdution:

In **probability theory** and **statistics,** the **binomial distribution** is the **discrete probability distribution** of the number of successes in a sequence of n **independent** yes/no experiments, each of which yields success with **probability** p. Such a success/failure experiment is also called a **Bernoulli experiment** or **Bernoulli trial;** when n = 1, the binomial distribution is a **Bernoulli distribution.** The binomial distribution is the basis for the popular **binomial test** of **statistical significance**.

The binomial distribution is frequently used to model the number of successes in a sample of size n drawn **with replacement** from a population of size N.If the sampling is carried out without replacement, the draws are not independent and so the resulting distribution is a **hypergeometric distribution,** not a binomial one. However, for N much larger than n, the binomial distribution is a good approximation, and widely used.

Probability mass function

In general, if the random variable K follows the binomial distribution with parameters n and p, we write K ~ B(n, p). The probability of getting exactly k successes in n trials is given by the **probability mass function:**

$$f(k;n,p) \quad = \quad P(K = k) \quad = \quad \binom{n}{k} p^k(1-p)^{n-k}$$

for k = 0, 1, 2, ..., n, where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$

is the **binomial coefficient** (hence the name of the distribution) "**n choose k**", also denoted C(n, k), ${}_nC_k$, or ${}^nC_k$. The formula can be understood as follows: we want k successes (p^k) and n − k failures $(1 - p)^{n-k}$. However, the k successes can occur anywhere among the n trials, and there are C(n, k) different ways of distributing k successes in a sequence of n trials.

The Binomial distribution has two possible outcomes called failure and success. The trials are independent with the probability of success remaining constant. It is denoted by p and that of failure is denoted by q, p + q = 1

:. q = 1 - p.

CHARACTERISTICS OF A BIONIMAL DISTRIBUTION

(a) This distribution should have a number of repeated trials.
(b) The trials are independent.
(c) Each trial results into two possible outcomes, a success and failure.

(d) The probability of success (P) is a constant.

THE BINOMIAL DISTRIBUTION EXPRESSION.

If a random variable X follows a binomial distribution then it is corresponding probability, distribution denoted by P(X = x) or f(x) is given by

$$P(X = x) = \binom{n}{x} p^x q^{n-x} \text{ for } x = 0, 1, 2, \ldots.., n.$$

Where $\binom{n}{x} = \frac{n!}{(n-x)!x!}$

Where n = number of independent trials.
p = probability of success.
x = number of successes in n trials.
q = probability of failure where q = 1 – p.

Equation (i) can be written as;
P(X = x) = b(x, n, p,). For x = 0,1,2, …, n.
If X is distributed in this way
X ~ bin (n, y) where n is the number of independent trial statistical and P is the probability of a successful outcome in one trial.

PROBLEMS REALTED TO BINOMAL DISTRIBUTION

They are mainly categorized into two parts.

(a) Where the probabilities are read off from the statistical tables especially those with probabilities of success as 0.05, 0.1, 0.5, 0.25, 0.3, 0.35, 0.4, 0.45, 0-.5, 0.55, 0.60, 0.65, …, 0.8, 0.85, 0.9, 0.95, 1

(b) Where the probabilities need to be calculated using equation
(i) $P(X = x) = \binom{n}{x} p^x q^{n-x}$ or x = 0, 1, 2, … such probabilities are $\frac{1}{3}, \frac{2}{3}$, etc.

TYPES OF TABLES FOR BINOMIAL DISTRIBUTION

There are twp types pf Binomial tables, these include:
(a) That for individual Binomial probabilities.
(b) The cumulative Binomial probabilities or Binomial probabilities sums.

INDIVIDUAL BINOMIAL PROBABILITES TABLE

Using those tables, locate the value of n, x and p he probability of success. The format of the table is as follows;

B(x, n, p) or $P(X = x) = \binom{n}{x} p^x q^{n-x}$

n	X	0.01	0.05	0.10	0.15	0.20	0.25	0.3	0.35	0.40
2	0	0.9801	0.9025	0.8100	0.7225	0.6400	0.5625	0.4900	0.4225	0.3600
	1	0.0198	0.0950	0.1800	0.2550	0.3200	0.3750	0.4200	0.4550	0.4800
	2	0.0001	0.0025	0.0000	0.0225	0.0400	0.0625	0.0900	0.1225	0.1600

For b(0, 2, 0.2) = 0.6400
b(2, 2, 0.4)

(b) **CUMULATIVE BINOMIAL PROBABILITIES**

These give the sum of probabilities from the lowest value of x, form x = 0 up to the specified value of x = n. To obtain individual probability form Binomial probability sums we must subtract two neighboring values.

$$P(X = r) = P(X \leq r) - P(x \leq r - 1)$$

$$= \sum_{x=0}^{r} b(x, n, p) - \sum_{x=0}^{r=1} b(x, n, p)$$

The format of the Binomial probability sum tables $\sum_{0}^{r} b(x, n, p)$

n	r	0.01	0.05	0.10	0.15	0.20	0.25	0.30	0.35
2	0	0.9801	0.9025	0.8100	0.7225	0.6400	0.5625	0.4900	0.4225
	1	0.9999	0.9975	0.9900	0.9775	0.9600	0.9375	0.9100	0.8775
	2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

n	r	0.40	0.45	0.50
2	0	0.3600	0.3025	0.2500
	1	0.8400	0.7975	0.7500
	2	1.0000	1.0000	1.0000

It should be noted that there is need to understand the difference between the two types of tables. Or else use of wrong tables will lead to wrong answers. In some tables the probability of success P ranges from 0 to 0.5 times the values of probabilities of success beyond 0.5 are not given. There is need to use the formulae for conversion.

(i) $b(x, n, p) = b(n - x, n, q)$
where $q = 1 - p$.

eg, Let n = 7, x = 0, P = 0.7

$$b(0, 7, 0.7) = b(7 - 0, 7, 1 - 0.7)$$
$$= b(7, 7, 0.3)$$
$$= \mathbf{0.0002}$$

(ii) $\sum_{0}^{r} b(x, n, p) = 1 - \sum_{x=0}^{n-r-1} b(x, n, q)$ where $q = 1 - p$ and $r < n$.

Eg, If x = 7, n = 9, p = 0.8

$$\sum_{0}^{7} b(7, 9, 0.8) = 1 - \sum_{0}^{1} b(7, 9, 0.2)$$

$$= 1 - 1 = 0$$

(iii) P(At most r successes in n trials) = $P(X \leq r) = \sum_{0}^{r} b(x, n, p)$

(iv) P(At least r successes in n trials) = $P(X \geq r)$

$= 1 - P(X < r)$

$= 1 - \sum_{0}^{r-1} b(x, n, p)$

$= \sum_{x=r}^{n} b(x, n, p)$

(v) P(At least r successes) = P(At least (n – r) failures.

$\sum_{0}^{r} b(x, n, p) = 1 - \sum_{0}^{n-r-1} b(x, n, q)$

$\sum^{n-r-1} b(x, n, q) = 1 - \sum^{r} b(x, n, p)$

:. P(At least (r – 1) success) = P(At most (n – r – 1) failures.

MEAN, VARIANCE AND STANDARD DEVIATION OF A BINOMIAL DISTRIBUTION.

(a) MEAN (μ) (EXPECTATION E(X) OF A BINOMIAL DISTRIBUTION.

The mean or expected value E(X) of a binomial distribution is given by:

$$E(X) = \sum_{ux} x\,P(X = x)$$

$$= np$$

(b) VARIANCE (VAR(X) OF A BINOMIAL DISTRIBUTION

Variance of a binomial distribution is given by

$$Var(X) = E(X^2) - (E(X))^2$$

Where E(X) = npq where q = 1 – p

(c) STANDARD DEVIATION OF A BINOMIAL DISTRIBUTION

Standard deviation $= \sqrt{\text{Variance}}$

$= \sqrt{npq}$

Example 1.

The probability that Bob wins a tennis game is $\frac{2}{3}$. He plays 8 games. What is the probability that he wins

(i) At least 7 games?

(ii) Exactly 5 games?

(iii) Mean and standard deviation of winning a game.

Solution; (i) $n = 8$, $p = \frac{2}{3}$, $q = 1 - p = 1 - \frac{2}{3} = \frac{1}{3}$

Probabilities cannot be obtained from tables.

$P(X \geq 7) = b(X = 7) + b(X = 8)$.

$$= \binom{8}{7}\left(\frac{2}{3}\right)^7 \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^8$$

$$= \frac{8 \text{ x } 2^7}{3^8} + \frac{2^8}{3^8}$$

$$= 5\left(\frac{2}{3}\right)^8$$

$$= \frac{1280}{6561}$$

$$= \mathbf{0.1951}$$

(ii) $P(X = 5) = \binom{8}{5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^3$

$$= \frac{8!}{5! \quad 3!}\,\frac{(2)^5}{(3)^8}$$

$$= 7\text{x}\left(\frac{2}{3}\right)^8$$

$$= \frac{1792}{6561}$$

$$= 0.2731$$

(iii) Mean = nP = 8 x $\frac{2}{3} = \frac{16}{3}$ = 5.3333

Standard deviation $= \sqrt{npq}$

$$= \sqrt{8 \text{ x } \tfrac{2}{3} \text{ x } \tfrac{1}{3}} = \mathbf{1.3333}$$

Example 2. A fair coin is tossed 6 times. Determine the probability of;

(i) **Getting exactly 2 heads.**

(ii) **Getting exactly 4 heads.**

(iii) **At least 3 tails.**

(iv) **At most 2 tails.**

Solutions

(i) n, = 6, P = $\frac{1}{2}$, q = $\frac{1}{2}$

EITHER $P(X = 2) = {}^6C_2 \, (\tfrac{1}{2})^4$

$$= \frac{6!}{2! \quad 4!}\left(\frac{1}{2}\right)^6$$

$$= \frac{15}{64}$$

0R $P(X = 4)$ = b(4, 6, 0.5) **= 0.2344** from tables.

(iv) n = 6, P = $\frac{1}{2}$, $\frac{1}{2}$

EITHER.

P(at least 3 tails) = $P(X \geq 3) = P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6)$

$$\{P(X=0) + P(X = 1) + P(X=2)\} = \left\{{}^6C_0 \left(\tfrac{1}{2}\right)^0 \tfrac{1}{2}^6 + {}^6C_1\left(\tfrac{1}{2}\right)\left\{\left(\tfrac{1}{2}\right)^6 + {}^6C_2\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^4\right\}\right.$$

$$= \left\{\left(\frac{1}{2}\right)^6 + \left(\frac{1}{6}\right)^6 + \left(\frac{1}{6}\right)^6\right\}$$

$= \quad 1 - \left(\frac{1}{2}\right)^6 \times 22$

$= \quad 1 - \frac{22}{64} \quad = \quad \frac{42}{64}$ **= 0.6562**

OR $P(X \geq 3)$ = $1 - (P(X = 0) + P(X = 0) + P(X = 1) + P(X = 2))$.

= $1 - (b(0, 6,0.5) + b(1, 6, 0.5) + b(2, 6, 0.5)$

= $1 - (0.0156 + 0.0938 + 0.2344)$

= 0.6562

(v) $n = 6, P = \frac{1}{2}$

EITHER

P(At most 2 tails) = $P(X \leq 2)$

= $P(X = 0) + P(X = 1) + P(X = 2)$.

= ${}^6C_0 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^6 + {}^6C_1 \left(\tfrac{1}{2}\right)^5 + {}^6C_2 \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^4$

= $1 \times \left(\frac{1}{2}\right)^6 + 6 \times \left(\frac{1}{2}\right)^6 + 15 \left(\frac{1}{6}\right)^6$

= $22 \times \left(\frac{1}{2}\right)^6$

= $\frac{22}{64}$

= **0.3438**

OR $P(X \leq 2)$ = $P(X = 0) + P(X = 1) + P(X = 12)$

= $b(0, 6, 0.5) + b(1, 6, 0.5) + b(2, 6, 0.5)$

= $0.0156 + 0.0938 + 0.2344$

= 0.3438

Example 3;

The packets of Omo sold in a shop of four categories namely, small is such that the ration of small; medium; large; giant is equal to 4;2;1;1. The coast of the packets are in the ratio, small; medium; large; giant = 350; 500; 1400 respectively.

(a) 30 packets are sold randomly on a particular day, the total cost of the sales being S shillings. Calculate.

i. The expected value of S

ii. The standard deviation of S.

(b) Ten packets are picked at random. Determine the probability that six are medium size packets.

Solution;

(a) Let $\overline{S}$, M, L and G be small, medium, large and giant respectively.

Then $P(\overline{S}) = \frac{1}{2}$, $P(m) = \frac{1}{4}$, $P(L) = \frac{1}{8}$, $P(G) = \frac{1}{8}$.

(i) E(S) = E(30x) = 30E(x).

E(X) = $\sum_{i=1}^{n} P_i X_i$

E(S) = $30\left[350 \text{ x } \frac{1}{2} + 500 \text{ x } \frac{1}{4} + 800 \text{ x } \frac{1}{8} + 1400\text{x } \frac{1}{8}\right]$

= 30 [175 + 125 x+100 + 175]

= 30 x 575 = 17250/=

E(S) = 17250/=

(ii) Standard deviation (S) = $[\text{Var(s)}]^{½} = [\text{Var}(30x]^{½}$

= $[30^2 \text{ Var}(x)]$ ½

E(x) = 575

$= 30[(350^2 x \frac{1}{2} + 500^2 x \frac{1}{4} + 800^2 x \frac{1}{8} + 1400^2 x \frac{1}{8}) - (575)^2]^{½}$

= 30 x 343.6932

Standard deviation (S) = 10310 /=

(b) n = 10, $P(\overline{M}) = \frac{1}{4} = 0.25$,

$P(M) = 1 - \frac{1}{4} = \frac{3}{4} = 0.75$

$P(X = 6) = \binom{10}{6}(0.25)^6 (0.75)^4$

$= \frac{10!}{6!\ 4!}(0.000244 (0.3164)$

= 210 x 0.0000772

≈ **0.0162**

Example 4. The probability of winning is $4/5$. Ten games are played.
What is the

(i) mean number of success.
(ii)variance .
(iii)probability of at least 8 successes in the games?

Solution:

This can be identified as a binomial probability distribution

with probability of success = $4/5$

sample space (n) = 10

(i) mean number of success

= E(x) = np

$= 10 \times \frac{4}{5} = 8$

(ii) Variance = npq

$= 10 \times \frac{4}{5} \times \frac{1}{5} = 1.6$

Note: q is the probability of failure.
ie q = 1 – p.

(iii) Probability of atleast 8 successes

= p (x = 8)+ p(x = 9)+ p(x = 10)

$= \binom{10}{8}\left(\frac{4}{5}\right)^8\left(\frac{1}{5}\right)^2 + \binom{10}{9}\left(\frac{4}{5}\right)^9\left(\frac{1}{5}\right)^1 + \binom{10}{10}\left(\frac{4}{5}\right)^{10}\left(\frac{1}{5}\right)^0$

= 0.301989888 + 0.268435456 + 0.107374182

= 0.6777799526

$\therefore$ $P(x \geq 8)$ = 0.6778

Example 5:

Mutebi sat a test of 10 objective questions with a choice determine the probability that:

(i) He got at least 2 answers correct.
(ii) He got at most 3 answers wrong.
(iii) He got between 2 and 4 answers correct inclusive
(iv) Exactly 5 answers wrong.

Solution:

(i) P(at least 2 correct) = $P(x \geq 2)$ $n = 10$, $p = \frac{1}{5}$ $q = \frac{4}{5}$

$$
\begin{aligned}
P(x \geq 2) &= P(x = 2) + P(x = 3) + \ldots + P(x = 10). \\
&= 1 - P(x \leq 1) \\
&= 1 - (P(X = 1) + P(x = 0)). \\
&= 1 - b(1, 10, 0.2) + b(0, 10, 0.2)) \\
&= 1 - 0.3758 \\
&= \mathbf{0.6242}
\end{aligned}
$$

(ii) P(at most 3 wrong) = $P(x \leq 3)$.

$n = 10$, $p = \frac{4}{5}$, $q = \frac{1}{5}$

$$
\begin{aligned}
P(X \leq 3) &= P(x = 0) + P(x = 1) + P(x = 2) + P(x = 3) \\
&= b(0, 10, 0.8) + b(1, 10, 0.8) + b(2, 10, 0.8) + b(3, 10\ 0.8) \\
b(x, n, p) &= b(n - x, x, q). \\
&= P(x \leq 3) \\
&= (10, 10, 0.2) + b(9, 10, 0.2) + b(8, 10, 0.2) + b(7, 10, 0.2) \\
&= 0.0000 + 0.0000 + 0.0001 + 0.0008 \\
&= \mathbf{0.0009}
\end{aligned}
$$

(iii) $P(2 \leq x \leq 4)$ = P(between 2 and 4 inclusive).

$n = 10$, $P = \frac{1}{5}$, $q = \frac{4}{5}$

$$
\begin{aligned}
P(2 \leq x \leq 4) &= P(x = 2) + P(x = 3) + P(x = 4) \\
&= b(2, 10, 0.2) + b(3, 10, 0.2) + b(4, 10, 0.2) \\
&= 0.3020 + 0.2013 + 0.0881 \\
&= \mathbf{0.5914}
\end{aligned}
$$

(iv) $n = 10$, $p = \frac{4}{5}$, $q = \frac{1}{5}$

$$
\begin{aligned}
P(\text{exactly 5 wrong}) &= P(x = 5) \\
&= b(5, 10, 0.2) \\
&= \mathbf{0.0226}
\end{aligned}
$$

Example 5:

The probability that a house wife buys Nomi detergent is 0.65. find the probability that in a sample of 8 housewives who have each bought a packet of Nomi detergent;

(a) exactly 3 have bought Nomi detergent.

(b) more than 5 have bought Nomi detergent.

Solution:

Let us consider housewife buying 'Nomi detergent' as success.

Then $p = 0.65$ and $q = 1 - p = 0.35$

Let x be the random variable, 'The number of housewives who buy Nomi detergent'.

Then, $x \cong \text{Bin}(x \text{ in}, p)$ with

$n = 8$ and $p = 0.65$

$$
\begin{aligned}
P(X = x) &= {}^{n}C_{x}\, p^{x} q^{n - x} \\
&= 8c_{x}\,(0.65)^{x}\,(0.35)^{8 - x},\ x = 0, 1, ..8
\end{aligned}
$$

(i)We are asked,

$$P(X = 3) = {}^{8}C_{3}\ (0.65)^{3}\,(0.35)^{5}$$

$= \frac{8!}{5!3!}$ [0.274625] [0-.005252]

$= \frac{40320}{120 \times 6}$ x [0.274625] [0.005252]

$\therefore$ P(X = 3) = 0.081

(ii) More than 5 have bought Nomi detergent is;

$P(x > 5) = 1 - P(x \leq 5)$

But using B in (x; n, p)

Bin (5; 8, 0.65) from tables

Bin (5; 8, 0.65) = 0.5664

$\therefore$ $P(x > 5) = 1 - P(x \leq 5)$

$= 1 - 0.5664$

$\cong$ **0.43**

The probability that more than 5 housewives have bought Nomi detergent is 0.43.

Example 6: A cross word puzzle is published in the New Vision each day of the week, except Sunday. A woman is able to complete on average 8 out of 10 of the cross word puzzles.

(i) Find the expected value and the standard deviation of the number of completed cross word puzzles in a given week..

(ii) Show that the probability that she will complete at least 5 in a given week is 0.655 (to 3 significant figures).

(iii) Find the probability that in 4 weeks she completes 4 or less in only one of the 4 weeks. (Correct to 3 decimal points.)

Solution:

(i) $n = 6, p = \frac{8}{10}, q = \frac{2}{10}$

E(X) = np

$= 6 \text{ x } \frac{8}{10}$

$= 4.8$

Standard deviation $= \sqrt{npq}$

$= \sqrt{6 \text{ x } \frac{8}{9} \text{ x } \frac{2}{10}}$

$=$ **0.9798**

(ii) $P(x \geq 5) = P(x = 5) + P(x = 6)$

$= b(5, 6, 0.8) + b(6, 6, 0.8).$

$= b(1, 6, 0.2) + b(0, 6, 0.2).$

$= 0.2621 + 0.3932$

$= 0.6553$

$\approx$ **0.655**

(iii) $P(x \leq 4) = 1 - P(x \geq 5)$

$= 1 - 0.6553$

$= 0.3447$

P(complete 4 or less in one week) $= {}^4C_1(0.3447)^1(0.6553)^3$

$=$ **0.388**

Example 7: The probability of winning a game is $\frac{4}{5}$. Ten games are played. What is;

(i) mean number of successes?

(ii) **Variance?**
(iii) **Probability of at least 8 successes in the ten games?**
(iv) **Probability of at most 3 failures in the 10 games?**

Solution

(1) n = 10 , $p = \frac{4}{5}$, $q = \frac{1}{5}$

$$E(x) = np = 10 \times \frac{4}{5} = \mathbf{8}.$$

(ii) $$Var(X) = npq = 10 \times \frac{4}{5} \times \frac{1}{5} = \mathbf{1.6}$$

(iii) P(at least 8 successes) = $P(X \geq 8)$.

$n = 10$, $p = \frac{4}{5} = 0.8$ $q = \frac{1}{5} = 0.2$

$P(X \geq 8)$ = $P(x = 8) + P(x = 9) + P(x = 10)$.
$b(x, n, p)$ = $b(n - x, n, q)$. for conversion of equation (i)
= $b(2, 10, 0.2) + b(1, 10, 0.2) + b(0, 10, 0.2$.
= $0.3020 + 0.2684 + 0.1074$
= **0.6778**

(iv).. P(at most 3 failures) = $P(x \leq 3)$
$n = 10$, $P = 0.2$, $q = 0.8$
$P(X \leq 3)$ = $P(X = 0) + P(x + 1) + P(x + 2) + P(X + 3)$
= $b(0, 10, 0.2) + b(1, 10, 0.2) + b(2, 10, 0.2) + b(3, 10, 0.2)$
= $0.1074 + 0.2684 + 0.3020 + 0.2013$
= **0.8791**

Example 8:

The probability that Bob wins a tennis game is $\frac{2}{3}$ He plays 8 games, what is the probability that he wins
(i) at least 7 games,
(ii) exactly 5 games?

Solution:

This can be identified as a Binomial probability where

Prob(Bob wins) = $\frac{2}{3}$ (i.e. success)

Games played (n) = 8

This can be shown as

$$X \approx \text{Bin}\left(8, \tfrac{2}{3}\right)$$

This is a biriomial probability with
(i) Prob (Bob wins at least 7 games)
= $P(X = 7) + P(X = 8)$.
= $\binom{8}{7} p^7 q^1 + \binom{8}{8} p^8 q^0$

$$= \quad 8\times\left(\frac{2}{3}\right)^7\left(\frac{1}{3}\right)^1 + 1\left(\frac{2}{3}\right)^8\left(\frac{1}{3}\right)^0$$

$= \quad 0.156073769 \; + \; 0.039018$

$= \quad 0.1951$ [4 dec. places] #

(ii) P (Bob wins exactly 5 games)

$= \quad P(X = 5)$

$$= \quad \binom{8}{5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^3 \qquad \textbf{Note: } \binom{8}{5} = 8C_1 = \frac{8!}{4!5!}$$

$= \quad 56 \text{ x } 0.131687242 \text{ x } 0.037037032$

$=$ **0.273129 #.**

Example 9:

(a) A pupil has ten-multiple choice questions to answer, there are four alternative answers to choose from . if a pupil answers the questions randomly , find the probability.

(i) that at least four answer are correct.

(ii) of the most likely number of correct answers.

Solution:

We consider 'choosing the right answer' as success

Then $P = P(\text{success}) \;=\; \frac{1}{4}$ and $\quad q = \frac{3}{4}$.

Let X be the random variable

'the number of correct answers'

Then

$X \sim B\text{ in }(x;n,p)$

Where $\quad n = 10, \quad p = \frac{1}{4}$

(i) We find $P(X \geq 4)$

Now $\quad P(X \geq 4) \quad = \quad 1 - P(X < 4)$

$= \quad 1 - P(X \leq 3)$

$= \quad 1 - \sum_{x=0}^{3} B\text{ in }(x;10,\frac{1}{4})$

$= \quad 1 - 0.7759$

$\therefore \quad P(X \geq 4) \quad$ **= 0.2241**

(ii) The most likely number of correct answers is taken to be the mean .

Mean , $E(x) \;=\; np$

$= \quad 10 \times \frac{1}{4}$

$= \quad 2.5 = 3$

So $\quad P(x = 3) \quad = \quad \binom{10}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^7$

= 0.2503 #

Example 10:

The probability that Bob wins a tennis game is $\frac{2}{3}$ He plays 8 games,what is the probability that he wins

(i) at least 7 games,

(ii) exactly 5 games?

Solution:

This can be identified as a Binomial probability where

Prob(Bob wins) = $\frac{2}{3}$ (i.e. success)

Games played (n) = 8

This can be shown as

$$x \approx \text{Bin}\left(8, \tfrac{2}{3}\right)$$

(i) Prob (Bob wins at least 7 games)

$= P(X = 7) + P(X = 8).$

$= \binom{8}{7} p^7 q^1 + \binom{8}{8} p^8 q^0$

$= 8 \times \left(\frac{2}{3}\right)^7 \left(\frac{1}{3}\right)^1 + 1\left(\frac{2}{3}\right)^8 \left(\frac{1}{3}\right)^0$

$= 0.156073769 + 0.039018$

= 0.1951 #

(ii) P (Bob wins exactly 5 games)

$= P(X = 5)$

$= \binom{8}{5}\left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^3$

Note: $\binom{8}{5} = 8C_1 = \frac{8!}{4!5!}$

$= 56 \times 0.131687242 \times 0.037037032$

= 0.273129 #.

Example 11:

A coin is biased so that a head is twice as likely to occur as a tail. if the coin is tossed 15 times , determine the,

(i) expected number of heads,

(ii) probability of getting at most 2 tails .

Solution:

Let T be the event for obtaining a tail,

$P(H) = 2P(T)$

$\Rightarrow 2P(T) + P(T) = 1$

$P(T) = {}^1/_3$. $P(H) = {}^2/_3$

$n = 15$

(i) Expected number of heads = np

$= 15 \times {}^2/_3$

$= 10$

(ii) Probability of getting at most 2 tails

$P(x \le 2) = P(x = 0) + (x = 1) + (x = 2)$

Pro. of success, $P = P(T) = {}^1/_3$

$P(x = 0) = \binom{15}{0} P^x q^{n-x}$

$\Rightarrow P(x = 0) = \binom{15}{0}\left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{15}$

$= \left(\frac{2}{3}\right)^{15} = 0.00228$

$P(X = 1) = \binom{15}{1}\left(\frac{1}{3}\right)^{1}\left(\frac{2}{3}\right)^{14}$

$= 15 \text{ x } \frac{1}{3} \text{x} 0.003425$

$= 5 \text{ x } 0.003425$

$= 0.017125$

$P(x = 2) = \binom{15}{2}\left(\frac{1}{3}\right)^{2}\left(\frac{2}{3}\right)^{13}$

$= 105 \text{ x } 0.111 \text{ x } (0.00514)$

$= 0.06$

$\Rightarrow P(x \geq 2) = 0.$

Example 12: A pupil has ten multiple choice questions to answer. There are four alternative answers to choose from. If a pupil answers the question randomly, find;

(i) The probability that at least four answers are correct.
(ii) The probability that at most 3 answers are wrong.
(iii) The probability that at most 3 answers are wrong.
(iv) The probability of between 1 and 4 inclusive, wrong answers.
(v) Expected number and variance of wrong answers.

Solution: (i) n = 10, $p = \frac{1}{4} = 0.25$, $q = \frac{3}{4} = 0.75$

EITHER P(At least 4 correct) = $P(X \geq 4)$.

$P(X \geq 4) = P(X = 4) + P(X = 5) + P(X = 6) + \ldots + P(X = 10)$

$= 1 \ (P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$

$= 1 – (0.0563 + 0.1877 + 0.2816 + 0.2503)$

$= 1 – 0.7759$

$= \mathbf{0.2241}$

Or $P(X \geq 4) = 1 - \sum_{x=0}^{3} b(3, 10, 0.25)$

$= 1 – 0.7759$

$= \mathbf{0.2241}$

(ii) Most likely number of correct answers = np

$= 10 \text{x} \frac{4}{5} = \mathbf{2.5}$

P(X = 2) = 0.2816. P(X = 3) = 0.2503

:. 2 is the most likely number of correct answer since it has the highest probability.

(iii) n = 10, P = 0.25, q = 0.75

P(At most 3 correct) = $P(X \leq 3)$

EITHER $P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$

$= b(0, 10, 0.25) + b(1, 10, 0.25) + b(2, 10, 0.25 + b(3, 10, 0.25)$

$= 0.0563 + 0.1877 + 0.2816 + 0.7759$

$= 0.7759$

OR $P(X \leq 3) = \sum_{x=0}^{3} (3, 10, 0.25)$

$= \mathbf{0.7759}$

(iv) P(between 1 and 4 inclusive) = $P(2 \leq x \leq 4)$.

$n = 10, \quad P = 0.75, \quad q = 0.25$

$P(1 \leq x \leq 4) = P(x = 1) + P(x = 2) + P(x = 3) + P(x = 4)$

$= b(1, 10, 0.75) + b(2. 10, 0.75) + b(3, 10, 0.75) + b(4, 10, 0.75)$

Since P = 0.75 is not in the tables use

$b(x, n, p) = b(n – x, n, q)$.

$= b(9, 10, 0.25) + b(8, 10, 0.25) + b(7, 10, 0.25) + b(6, 10, 0.25)$.

$= 0.0000 + 0.0004 + 0.00031 + 0.0162$

$=$ **0.0197**

(iv)$E(x) = np$ and $Var(x) = npq$.

$n = 10, p = 0.75, \quad q = 0.25$

$E(X) = 10 \times 0.75$

$=$ **7.5.**

$Var(x) = 10 \times 0.75 \times 0.25$

$=$ **1.875**

Example 13:

A kraal of ten cows are exposed to Nagana, medical records indicate that the probability that any cow exposed to such a disease dies from it is $\frac{3}{10}$. Determine the probability that;

(i) 6 cows die of Nagana
(ii) 4 cows recover from Nagana
(iii) Between 5 and 7 inclusive die of it.
(iv) Expected number of cows to survive and the variance.
(v) At least 7 die of Nagana.s
(vi) At most 3 survive Nagana.

Solution: (i) $n = 10, \quad p = 0.3, \quad q = 0.7, \quad x = 6$

$P(x = 6) = b(6, 10, 0.3)$

$= 0.0368$

(ii) $n = 10, p = 0.7, \quad q = 0.3, \quad x = 4$

$P(x = 4) = b(4, 10, 0.7) =$ **0.0368**

(iii) P(between 5 and 7 inclusive die) = $P(5 \leq x \leq 7)$s

$n = 10, \quad P = 0.3, \quad q = 0.7,$

$P(5 \leq x \leq 7) = P(X = 5) P(x = 6) P(x = 7)$

$= b(5, 10, 0.3) + b(6, 10, 0.3) + b(7, 10, 0.3)$

$= 0.1029 + 0.0368 + 0.0090$

$= 0.1487$

(iv) $E(X) = np \qquad Var(X) = npq$

$P = 0.7, \quad q = 0.3, \quad n = 10.$

$E(X) = 0.7 \times 10 =$ **7**

$Var(x) = 10 \times 0.7 \times 0.3 =$ **2.1**

(iv) P(at least 7 die) = $P(X \geq 7)$

$n = 10, \quad p = 0.3, \quad q = 0.70$

$P(X > 7) = P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$

$= b(7, 10, 0.3) + P(8, 10, 0.3) + P(9, 10, 0.3) + P(10, 10, 0.3)$

$= 0.0090 + 0.0014 + 0.0001 + 0.0000$

$= 0.01505$

(v) P(At most 3 survive) = $P(X \le 3)$.

n = 10, P = 0.7, q = 0.3.

$$\begin{aligned}
P(X \le 3) &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) \\
&= b(0, 10, 0.7) + b(1, 10, 0.7) + b(2, 10, 0.7) + b(3, 10, 0.7) \\
&= b(10, 10, 0.3) + b(9, 10, 0.3) + b(8, 10, 0.3) + b(7, 10, 0.3) \\
&= 0.0000 + 0.0001 + 0.0014 + 0.0090 + 0.0105 \\
&= \mathbf{0.0210}
\end{aligned}$$

Example 14:

Joan played 12 chess games. The probability that she wins a game is $\frac{3}{4}$.

Find the probability that she will win:

(i) exactly 8 games,

(ii) more than 10 games

Solution:

This is a Binomial probability distribution.

We consider 'Joan winning a game' as 'success'

Then $P = P(\text{success}) = \frac{3}{4} = 0.75$

and Q = 0.25

Let x be the random variable 'the number of games played).

So X ~ B in (x; n, p)

~ B in $(x;12,\frac{3}{4})$

(i) $$\begin{aligned}
P(x = 8) &= \text{Bin}(8;12,\tfrac{3}{4}) \\
&= \binom{12}{8}\left(\frac{3}{4}\right)^8\left(\frac{1}{4}\right)^4 \\
&= 495 \times 0.000391066 \\
&= 0.1936 \text{ [4 dec. places]}
\end{aligned}$$

(ii) $$\begin{aligned}
P(x > 10) &= P(x = 11) + P(x = 12) \\
&= \text{Bin}(11;12,\tfrac{3}{4}) + \text{Bin}(11;12,\tfrac{3}{4}) \\
&= \binom{12}{11}\left(\frac{3}{4}\right)^{11}\left(\frac{1}{4}\right)^1 + \binom{12}{12}\left(\frac{3}{4}\right)^{12}\left(\frac{1}{4}\right)^0 \\
&= 12 \times 0.010558784 + 1 \times 0.03167635 \\
&= \mathbf{0.1584}
\end{aligned}$$

Example 15: A biased coin is tossed five times; the coin is such that the ratio of the head to the tail is 1:2. Determine:

(i) The probability of exactly three tails.

(ii) Expected number of heads and the variance.

(iii)At least 4 heads.

(iv)At most 2 tails

(v)

Solution: (i) P(exactly three tails) = P(X =3).

Let H = head and T = tail.

$P(H) = \frac{1}{3}$, $P(T) = \frac{2}{3}$,

:. n = 5, $P = \frac{2}{3}$, $q = \frac{1}{3}$, x = 3.

Since statistical tables cannot be used we use the combination method.

$$P(X = 3) = {}^5C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2$$

$$= \frac{5!}{3!\ 2!}\cdot\frac{8}{243}$$

$$= 0.3292.$$

(ii) $p = \frac{1}{3}$, $q = \frac{2}{3}$, n = 5,

$$E(x) = np$$

$$= 5 \text{ x } \frac{1}{3}$$

$$= 1.6667$$

$$Var(x) = nPq$$

$$= 5 \text{ x } \frac{1}{3} \text{ x } \frac{2}{3} = \mathbf{1.1111}$$

(iii) P(at least 4 heads) = $P(X \geq 4)$.

$p = \frac{1}{3}$, $q = \frac{2}{3}$, n = 5

$$P(X \geq 4) = P(x = 4) + P(X = 5).$$

$$= {}^5C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right) + {}^5C_4\left(\frac{1}{3}\right)^5$$

$$= \frac{5! \text{ x } 2!}{4!\ 1!\ 243} + \frac{5! \text{ x } 1}{5!\ 0!\ 243}$$

$$= \frac{10}{243} + \frac{1}{243} = \frac{11}{243}$$

$$= 0.0453$$

$$\mathbf{P(X \geq 4) = 0.0453}$$

(v)P(at most 2 tails) = $P(X \leq 2)$

$P = \frac{2}{3}$, $q = \frac{1}{3}$, n = 5

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2).$$

$$= {}^5C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^5 + {}^5C_1\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^4 + {}^5C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^3$$

$$= \frac{5! \text{ x } 1}{5!\ 0!\ 243} + \frac{5! \text{ x } 2}{4!\ 1!\ 243} + \frac{5! \text{ x } 4}{2!\ 3!\ 243}$$

$$= \frac{1}{243} + \frac{10}{243} + \frac{40}{243} = \frac{51}{243}$$

$$\mathbf{P(X \leq 2) = 0.2099}$$

Example 16: A biased die is thrown 3 times and the number of fours is noted. The procedure is performed 180 times in all and the results are shown in the table.

Number of fours	**0**	**1**	**2**	**3**

Frequency	**50**	**69**	**36**	**25**

(a)What is the mean of this distribution?
(b)What is the probability of obtaining a four when the die is thrown?
(c)Calculate the theoretical probabilities of obtaining 0, 1, 2, 3, four, using the binomial distribution.
(d)Calculate the corresponding theoretical frequencies.

Solution :

(a) Mean $(\bar{x}) = \dfrac{\sum fx}{\sum f}$

Mean $(\bar{x}) = \dfrac{0\times 50 + 1\times 69 + 2 \text{ x } 36 + 3 \text{ x } 25}{50 + 69 + 36 + 25}$

$= \dfrac{216}{180}$

$=$ **1.2**

(b) Mean $(\bar{x}) = np$ $\quad \bar{x} = 1.2, \quad n = 3$

$1.2 = 3p$

$p = \dfrac{1.2}{3} =$ **0.4**

(c) P(X = x) for x = 0, 1, 2, 3. Using

$P(X = x) = {}^3C_x\,(0.4)^x(0.6)^{3-x}$

$P(X = 0) = b(0, 3, 0.4) = 0.2160$

$P(X = 1) = b(1, 3, 0.4) = 0.4320$

$P(X = 2) = b(2, 3, 0.4) = 0.2880$

$P(X = 3) = b(3, 3, 0.4) = 0.0640$

(d) $0.216 \text{ x } 180 = 39$

$0.432 \text{ x } 180 = 78$

$0.288 \text{ x } 180 = 52$

$0.064 \text{ x } 180 = 12$

Theoretical binomial frequencies rounded to the nearest integer.

Number of fours	0	1	2	3
frequency	39	78	52	12

Example 17:

60% of the tourists who travel to Mombasa buy a Newspaper from vendors before boarding a train. The train is full and each compartment holds 8 passengers.

(a) What is the probability that all the tourists in a compartment have bought a Newspaper?

(b) What is the probability that none of the tourists in the compartment has bought Newspapers?

(c) What is the probability that exactly three of the tourists in a compartment have bought the Newspaper?

(d) What is the most likely number of tourists in a compartment to have bought the Newspaper?

(e) If there are 40 compartments on the train, in how many of them would you expect there to be exactly three copies of the newspaper?

(f) The train is so full that in each carriage ten tourists are standing in the corridor. What is the probability that the third tourist I pass in the corridor of a carriage of the first I meet who has bought the newspaper?
(g) What is the mean number of buyers of the newspaper standing in the corridor?

Solution: Let P = buying a Newspaper = 0.6

$q = 1 - P = 1 - 0.6 = 0.4$

$n = 8, \quad P = 0.6, \quad q = 0.4$

$P(X = x) = {}^{n}C_{x}\, P^{x} q^{n-x} \qquad x = 0, 1, \ldots, 8.$

$= {}^{8}C_{x}\, (0.6)^{x}\, (0.4)^{8-x}$

(a) $P(X = 8) = {}^{8}C_{8}\, (0.6)^{8}\, (0.4)^{0}$

= b(8, 8, 0.6)

= b(0, 8 0.4)

= 0.0168

(b) $P(X = 0) = {}^{8}C_{0}\, (0.6)^{0}\, (0.4)^{8}$

= b(3, 8, 0.6)

= b(8, 8, 0.4)

= 0.0007.

(c) $P(X = 3) = {}^{8}C_{3}(0.6)^{3}\, (0.4)^{5}$

= b(3, 8, 0.6)

= b(5, 8, 0.4)

= 0.1239.

(d) E(X) nP = 8 x 0.6 = 4.8

Consider x = 4, 5, 6.

P(X = 4) = b(4, 8, 0.6) = b(4, 8, 0.4) = 0.2322

P(X = 5) = b(5, 8, 0.6) = b(3, 8, 0.4) = 0.2787

P(X = 6) = b(6, 8, 0.6) = b(2, 8, 0.4) = 0.2090

∴ The most likely number of passengers is **5**

(e) P(X = 3) = 0.1239

n = 40, P = 0.1239, q = 0.8761

Let 2 represent the number of compartments

E(2) = nP

= 40 x 1239

= 4.956

(f) Let T = Tourist has a copy of a Newspaper.

$\overline{T}$ = Tourist has a copy of a newspaper.

P(third Tourist is the first to get) = $P(\overline{T}\,\overline{T}\,T)$

P(T) = 0.6 $\qquad P(\overline{T}) = 0.4$

$P(\overline{T}\,\overline{T}\,T)$ = 0.4 x 0.4 x 0.6

= 0.096

(g) Let C = number of people in the corridor to have bought a newspaper.

E(C) = 10 x 0.6

= **6**

EXERCISE 5

1.A hundred years ago the occupational disease in an industry was such that the working men had a 20% chance of suffering from.

a.If six workmen were selected at random what is the probability that two or less of them contracted the diseases?

b.How many workmen could have been selected at random before the probability that at least one of the them contracted the disease become greater than 0.9?

2.If there is a probability of 0.2 of failure to get through in any attempt to make a telephone call, calculate:

i. The most probable number of failures in ten attempts.

ii. The probability of three or more failure in ten attempts.

3.If 20% of the bolts produced by a machine are defective, determine the probability that out of 4 bolts chosen at random:

a.One is defective.

b.Zero is defective.

c.At most 2 bolts will be defective.

4.Out of 2000 families with 4 children each, how many would you expect to have:

a.At least 1 boy

b.2 boys

c.1 or 2 girls.

d.No girls.

5.An insurance salesman sells policies to 5 men, all of identical age and in good health. According to the actuarial tables the probability that a man of this particular age will be alive in 30 years is $\frac{2}{3}$.

Find the probability that in 30 years

a.All 5 men will be alive

b.At least 3 men will be alive.

c.Only 2 men will be alive.

d.At least 1 man will be alive.

6.Out of 800 families with 5 children each, how many would you expect to have:

a.3 boys

b.5 girls

c.Either 2 or 3 boys

7.The probability that a certain type of vacuum tube will shatter during a themal test is 0.15 if 25 such tubes are tested what is the probability that;

a.4 or more will shatter?

b.Between 16 and 30 inclusive will survice?

8.Samples, each of 8 articles, are taken at random from a large consignment in which 20% of the article are defective, if a sample of 100 of 8 articles are to be examined determine:

a.Number of defective articles which is most likely to occur in a single sample and the probability of this number.

b.The number of samples in which you would expect to find 3 or more defective articles.

9.In a large batch of items from a production line the probability that an item is faulty is P. 400 samples, each of size 5, are taken and the number of faulty items in each batch is noted. From the frequencies of 0, 1, 2, 3, 4, 5 faulty items per batch for a theoretical binomial distribution having the same mean.

Number of faulty items	0	1	2	3	4	5
Frequency	297	90	10	2	1	0

10. The probability of a pupil arriving at school late on any given day is $\frac{1}{10}$, what is the probability of his being punctual for a whole week that consists of 5 school days? Calculate the mean and variance of the number of days he will be late in a school term consisting of 14 weeks, with 70 school days. Also calculate the expected number of completely punctual weeks in the team.

11. A box contains a large number of screws. The screws are very similar in a appearance, but are in fact of 3 different types A, B, and C which are present in equal numbers. For a given job only screws of type A are suitable. If 4 screws are chosen at random, find the probability that:
a. Exactly two are suitable.
b. At least two are suitable.
If 20 screws are chosen at random, find the expected value and variance of the number of suitable screws.

12. Doctor Musinguzi estimates that his treatment of a particular type of dental illness is successful in $66\frac{2}{3}$% of cases. Determine:
a. The probability that he will treat successfully exactly 4 out of 6 patients currently in his care.
b. The probability of success in at least 2 out of the 6 cases.

13. In a large batch of plastic moldings 25% have faults. If a random sample of 10 moldings is inspected. Determines;
a. The probability that the sample contains:
i. 3 faulty molding.
ii. No more than 2 faulty moldings
b. the mean and variance of the number of faulty moldings.

14. One in eight of the radios assembled by a machine are imperfect. Eight radios are chosen at random. Determine the probability that:
a. One is imperfect
b. At least 6 are imperfect

15. The components produced by a particular machine are tested by taking samples containing 5 components and noting the number of rejects in each. The given table shows results of 100 samples.

Number of rejects (X)	0	1	2	3	4	5
Frequency (f)	10	27	31	20	9	3

(a) Calculate the mean and variance of the number of rejects.
(b) Estimate the probability P that a component selected at random is a reject.
(c) By assuming that the number of rejects in a sample of 5 components has binomial distribution with n = 5 and P, write down the theoretical mean and variance.
(d) Obtain the expected frequency distribution of 100 samples.

EXERCISE 5

1.**(a) 0.901** **(b) 11)**
2.**(i) 2** **(ii) 0.3222)**

3.**(a) 0.4096 (b) 0.4096 (c) 0.9728)**

4.**(a) 1875 (b) 750 (c) 1250 (d) 125)**

5.**(a)** $\frac{32}{243}$ **(b)** $\frac{192}{243}$ **(c)** $\frac{40}{243}$ **(d)** $\frac{242}{243}$ **)**

6.**(a) 250 (b) 25 (c) 500)**

7.**(a) 0.529 (b) 0.316)**

8 **.(a) 0.336 (ii) 20)**

9. **(a) 0.06 (b) 293 (c) 94, 12, 1, 0, 0**

10.**(a) 0.505; 7,6.3; 12.6)**

11.**(a)** $\frac{8}{27}$ **(b)** $\frac{11}{27}$**;** $\frac{20}{3}$**,** $\frac{40}{9}$**)**

12.**(i) 0.329 (ii) 0.982)**

13.**(i) (a) 0.2502 (b) 0.526 (ii) 2.5, 1.875)**

14.**(a) 0.395 (b) 0.0000852**

15 **(a) 2, 1.5 (b)** $\frac{2}{5}$ **(c) 2, 1.2**

(d)

Number of rejects	**0**	**1**	**2**	**3**	**4**	**5**
Frequency	**8**	**26**	**35**	**23**	**8**	**1**