44 • C3 • Relation between dy/dx and dx/dy

44.1 Relation between dy/dx and dx/dy

It is very tempting to think that $\frac{dy}{dx}$ is a fraction, and treat it as such.

Strictly speaking $\frac{dy}{dx}$ is a function, and should not be confused with a fraction, although in practise it often appears to behave like one.

Deriving the link between $\frac{dy}{dx}$ and $\frac{dx}{dy}$ is as follows:

From C1 recall that a derivative of a function is defined as:

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

Now $\frac{\delta y}{\delta x}$ **is** a fraction, hence:

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{1}{\frac{\delta x}{\delta y}}$$

As $\delta x \to 0$ then $\delta y \to 0$

$$\frac{dy}{dx} = \frac{1}{\lim\limits_{\delta y \to 0} \frac{\delta x}{\delta y}} = \frac{1}{\frac{dx}{dy}}$$

Hence $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

or $$\frac{dy}{dx} \times \frac{dx}{dy} = 1$$

E.g. Consider:

$$y = ax + b \qquad \Rightarrow \qquad \frac{dy}{dx} = a$$

Rearrange to make x the subject:

$$x = \frac{y - b}{a}$$

$$x = \frac{y}{a} - \frac{b}{a} \qquad \Rightarrow \qquad \frac{dx}{dy} = \frac{1}{a}$$

$$\therefore \quad \frac{dy}{dx} \times \frac{dx}{dy} = a \times \frac{1}{a}$$

$$= 1$$

44.2 Finding the Differential of $x = g(y)$

Don't assume that every differential has to start with a $\frac{dy}{dx}$

44.2.1 Example:

1 Find the gradient of $x = y^3 + 6y$ at the point (7, 1).

Note that a gradient is given as $\frac{dy}{dx}$.

Solution:

$$\frac{dx}{dy} = 3y^2 + 6$$

At $y = 1$ $$\frac{dx}{dy} = 3 \times 1 + 6 = 9$$

Recall: $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$\therefore$ $$\frac{dy}{dx} = \frac{1}{9}$$

2 Find the differential of $y = \ln x$

Solution:

$$y = \ln x$$

$$e^y = x$$

$$x = e^y$$

$$\frac{dx}{dy} = e^y$$

But $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$\frac{d[\ln x]}{dx} = \frac{1}{x}$$

44.3 Finding the Differential of an Inverse Function

This relationship $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ can be used to find the differential of an inverse function. Recall that for a one to one function there is an inverse relationship. We can treat either x or y as the dependent variable.
We can, therefore, write:

$$y = f(x) \qquad \Rightarrow \qquad x = f^{-1}(y)$$

The differential of the function and its inverse is linked by the relationship:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

The advantage of this relationship is that you don't need to know the exact inverse function.

44.3.1 Example:

1 Find the gradient of the inverse function at the point (2, 1), where the function is defined as $f(x) = x^4 + 3x^2 - 2$.

Solution:

$$y = f(x) \qquad \Rightarrow \qquad x = f^{-1}(y)$$

$$y = x^4 + 3x^2 - 2$$

$$x = y^4 + 3y^2 - 2$$

Rearranging to make y the subject is not required since:

$$\frac{dx}{dy} = 4y^3 + 6y$$

$$\therefore \qquad \frac{dy}{dx} = \frac{1}{4y^3 + 6y}$$

$$\frac{dy}{dx} = \frac{1}{4 + 6} = \frac{1}{10}$$

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45 • C3 • Differentiation: The Chain Rule

45.1 Composite Functions Revised

Recall that a composite function, otherwise known as a ‘function of a function’, is formed by applying one function, then immediately applying another function to the result of the first function.

In simple terms:

$$\text{Input } x \xrightarrow{f} \text{output } f(x)$$

$$\text{Input } f(x) \xrightarrow{g} \text{output } g\left[f(x)\right] \text{ or } gf(x)$$

In other words, apply f to x first, then g to $f(x)$. You read the result, $gf(x)$, from right to left.

E.g.

If $y = (x + 3)^3$ then y is said to be a function of x.

If we make $u = x + 3$ then $y = u^3$ and so y is a function of u and u is a function of x. In function notation we would write:

$$F(x) = gf(x)$$

where $g(u) = u^3$ and $f(x) = x + 3$

Reading a function of a function:

$(x^2 - 4)^3$ is a cubic function g, of a quadratic function f

$\sqrt{(1 - x)^3}$ is a square root function g, of a cubic function f

45.2 Intro to the Chain Rule

We have seen from earlier modules that in order to differentiate a polynomial such as $(2x + 3)^3$ we can use the Binomial theorem to expand the brackets and differentiate each term individually. However, a problem arises if we want to differentiate something like $(2x + 3)^{42}$. Using the Binomial theorem would be tedious to say the least, but as always in mathematics, there is generally an easier way.

The answer to this and many other problems involving composite functions is the **chain rule**, which is given as:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

where y is a function of u and u is a function of x.

In function terminology we write

$$F'(x) = g'(f(x)) \times f'(x)$$

where $F(x) = gf(x)$ and $F(x) = g(u)$ and $u = f(x)$

In other words, if $g(u)$ is the outside function and $f(x)$ is the inside function we differentiate the outside function and multiply by the differential of the inside function.

45.3 Applying the Chain Rule

Typical examples of composite functions that can be differentiated by the chain rule are:

$$(x^2 - 4)^3 \qquad \sqrt{(1 - x^3)} \qquad e^{2x+5} \qquad \ln(3x^2 - 2) \qquad \frac{1}{(x^2 - 4)^3}$$

45.3.1 Example:

1 Find $\frac{dy}{dx}$ when $y = (x^2 - 4)^5$

Solution:

$$y = (x^2 - 4)^5$$

$$\Rightarrow \quad u = x^2 - 4 \qquad \Rightarrow \qquad y = u^5$$

$$\therefore \quad \frac{du}{dx} = 2x \qquad \qquad \frac{dy}{du} = 5u^4$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 5u^4 \times 2x$$

$$\therefore \quad \frac{dy}{dx} = 10x(x^2 - 4)^4$$

Alternative Solution:

$$\frac{dy}{dx} = \frac{d}{du}(u^5) \times \frac{d}{dx}(x^2 - 4) = 5(x^2 - 4)^4 \times 2x \qquad \text{etc}$$

2 Find $\frac{dy}{dx}$ when $y = \sqrt{(1 - x^3)}$

Solution:

$$y = \sqrt{(1 - x^3)} \qquad \Rightarrow \qquad y = (1 - x^3)^{\frac{1}{2}}$$

$$\Rightarrow \quad u = 1 - x^3 \qquad \Rightarrow \qquad y = u^{\frac{1}{2}}$$

$$\therefore \quad \frac{du}{dx} = -3x^2 \qquad \qquad \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \times (-3x^2)$$

$$\therefore \quad \frac{dy}{dx} = -\frac{3}{2}x^2(1 - x^3)^{-\frac{1}{2}}$$

$$= -\frac{3x^2}{2\sqrt{(1 - x^3)}}$$

Alternative Solution:

$$\frac{dy}{dx} = \frac{d}{du}(u^{\frac{1}{2}}) \times \frac{d}{dx}(1 - x^3) = \frac{1}{2}(1 - x^3)^{-\frac{1}{2}} \times (-3x^2) \qquad \text{etc}$$

3 Find $\frac{dy}{dx}$ when $y = ln(3x^2 - 2)$

Solution:

$$y = ln(3x^2 - 2)$$

$$\Rightarrow \quad u = 3x^2 - 2 \quad \Rightarrow \quad y = ln\, u$$

$$\therefore \frac{du}{dx} = 6x \qquad \frac{dy}{du} = \frac{1}{u}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times 6x$$

$$\therefore \frac{dy}{dx} = \frac{6x}{3x^2 - 2}$$

4 Find $\frac{dy}{dx}$ when $y = e^{2x+5}$

Solution:

$$y = e^{2x+5}$$

$$\Rightarrow \quad u = 2x + 5 \quad \Rightarrow \quad y = e^u$$

$$\therefore \frac{du}{dx} = 2 \qquad \frac{dy}{du} = e^u$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \times 2$$

$$\therefore \frac{dy}{dx} = 2e^{2x+5}$$

5 Take the parametric curve defined by $x = 2t^2$ & $y = 4t$. Point P has the co-ordinates, $(2p^2, \ 4p)$.
Find the gradient at point P:

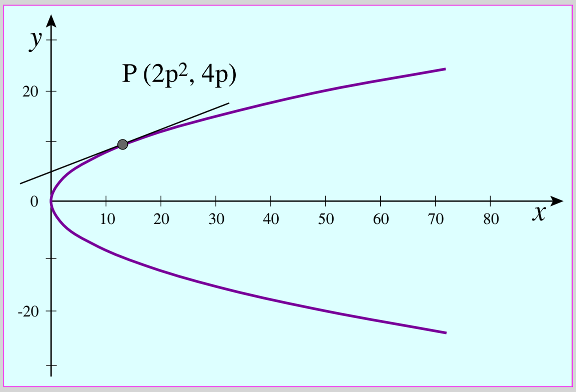
Solution:

Draw a sketch!!!!!!

$$t = \frac{y}{4} \quad \Rightarrow$$

$$x = 2\left(\frac{y}{4}\right)^2 \quad \Rightarrow$$

$$y^2 = 8x$$



$$x = 2t^2 \qquad y = 4t$$

$$\therefore \frac{dx}{dt} = 4t \qquad \frac{dy}{dt} = 4$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 4 \times \frac{1}{4t} = \frac{1}{t}$$

At point $P(2p^2, \ 4p)$; $y = 4p \quad \Rightarrow \quad \therefore 4p = 4t \quad \Rightarrow \quad p = t$

The gradient at point $P = \frac{1}{p}$

45.4 Using the Chain Rule Directly

After some practise, it is possible to use the chain rule with out formally writing down each stage. We can express this by writing the rule as:

If $\quad y = u^n$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{d\,[u^n]}{du} \times \frac{du}{dx}$$

$$= n\,u^{n-1}\left[\frac{du}{dx}\right]$$

45.4.1 Example:

$$y = (6x + 8)^4 \quad \Rightarrow \quad \frac{dy}{dx} = 4(6x + 8)^3 \times 6$$

$$y = (ax + b)^n \quad \Rightarrow \quad \frac{dy}{dx} = n(ax + b)^3 \times a$$

$$y = ln(4x - 1) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{4x - 1} \times 4 = \frac{4}{4x - 1}$$

$$y = \sqrt{4e^{3x} + 2} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\left[4e^{3x} + 2\right]^{-\frac{1}{2}} \times 12e^{3x} = \frac{6e^{3x}}{\sqrt{4e^{3x} + 2}}$$

$$y = \frac{1}{x^4 + 2} \quad \Rightarrow \quad \frac{dy}{dx} = -1\left[x^4 + 2\right]^{-2} \times 4x^3 = -\frac{4x^3}{(x^4 + 2)^2}$$

$$y = ln\,kx \quad \Rightarrow \quad \frac{d}{dx} = \frac{1}{kx} \times k = \frac{1}{x}$$

$$y = ln\,(ax + b) \quad \Rightarrow \quad \frac{d}{dx} = \frac{1}{ax + b} \times a = \frac{a}{ax + b}$$

45.5 Related Rates of Change

See also 50 • C3 • Differentiation: Rates of Change

The Chain Rule is a powerful way of connecting the rates of change of two dependent variables.

Consider a sphere, in which the volume is increasing at a given rate. Since the volume of the sphere is connected to the radius, how can the rate of increase in the radius be calculated?

If we are given the rate of increase in the volume, we have a value for $\frac{dV}{dt}$. The volume is connected to the radius via the function:

$$V = \frac{4}{3}\pi r^3 \qquad \text{and hence} \qquad \frac{dV}{dr} = 4\pi r^2$$

The required rate of increase in the radius is given by $\frac{dr}{dt}$. We can connect all these related rates of change using the chain rule such that:

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \times \frac{dr}{dt}$$

If the volume is increasing at $980\ cm^3\ per\ hour$ and $r = 7\ cm$ at time t, then:

$$980 = 4\pi \times 49 \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{980}{196\pi} = \frac{5}{\pi} = 1{\cdot}59\ cm/hour$$

45.5.1 Example:

1 Let A be the surface area of a spherical balloon. What is the rate of increase in the surface area of the balloon when the radius r is 6 cm, and the radius is increasing at 0·08 cm/sec?

Solution:

We want to find $\frac{dA}{dt}$, and we know that $A = 4\pi r^2$

$$\frac{dA}{dr} = 8\pi r$$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

$$\frac{dA}{dt} = 8\pi \times 6 \times 0{\cdot}08$$

$$\frac{dA}{dt} = 3{\cdot}84\pi\ \text{cm}^2/sec$$

2 An ice cube of side, $6x$ cm, melts at a constant rate of $0{\cdot}8$ cm^3/min.
Find the rate at which x and the surface area A changes with time, when $x = 2$

Solution:

We want to find $\frac{dx}{dt}$ & $\frac{dA}{dt}$

The volume of the cube is $V = (6x)^3 \quad \Rightarrow \; 216x^3$

The surface area of the cube is $A = 6\,(6x)^2 \quad \Rightarrow \; 216x^2$

Now: $$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt} = -0{\cdot}8$$ (Negative as it is a decreasing value)

$$-\,0{\cdot}8 = 3 \times 216x^2 \times \frac{dx}{dt}$$

$$\frac{dx}{dt} = -\frac{0{\cdot}8}{3 \times 216 \times 4} = -\,0{\cdot}000308 \text{ cm/min}$$

Also: $$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$$

$$\frac{dA}{dt} = 2 \times 216x \times \left(-\frac{0{\cdot}8}{3 \times 216 \times 4}\right)$$

$$\frac{dA}{dt} = 2 \times 216 \times 2 \times \left(-\frac{0{\cdot}8}{3 \times 216 \times 4}\right)$$

$$\frac{dA}{dt} = \left(-\frac{0{\cdot}8}{3}\right) = -\,0{\cdot}267 \text{ cm}^2\text{/min}$$

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45.6 Deriving the Chain rule

Start with a composite function of $y = gf(x)$ where $y = g(u)$ and $u = f(x)$. An increase in x by a small amount δx means a corresponding increase in u, by a small amount δu and hence y by δy.

$$\text{Now} \qquad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

Since $\delta y, \ \delta u, \ \delta x$ can be handled algebraically, we have:

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$$

$$\therefore \qquad \frac{dy}{dx} = \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x} \right)$$

As $\delta x \to 0, \ \delta u \to 0$

$$\therefore \qquad \frac{dy}{dx} = \lim_{\delta u \to 0} \left(\frac{\delta y}{\delta u} \right) \times \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} \right)$$

$$\therefore \qquad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

45.7 Chain Rule Digest

Used to differentiate composite functions.

If y is a function of u and u is a function of x then:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

In function terminology:

$$F'(x) = g'(f(x)) \times f'(x)$$

where $F(x) = gf(x)$ and $F(x) = g(u)$ and $u = f(x)$

$$\frac{dy}{dx} = \frac{d[g(u)]}{du} \times \frac{du}{dx}$$

$$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} f'(x)$$

$$\frac{d}{dx}k[f(x)]^n = kn[f(x)]^{n-1} f'(x)$$

$$\frac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$$

$$\frac{d}{dx}\left[e^{f(x)}\right] = \frac{d[f(x)]}{du} \times e^{f(x)} \qquad \Rightarrow \qquad = f'(x)\,e^{f(x)}$$

$$\frac{d}{dx}\left[ke^{f(x)}\right] = k\frac{d[f(x)]}{du} \times e^{f(x)} \qquad \Rightarrow \qquad = kf'(x)\,e^{f(x)}$$

$$\frac{d}{dx}\left[e^x\right] = e^x$$

$$\frac{d}{dx}\left[e^{kx}\right] = ke^{kx}$$

$$\frac{d}{dx}\left[e^{ax+b}\right] = ae^{ax+b}$$

$$\frac{d}{dx}\left[ln\, f(x)\right] = \frac{f'(x)}{f(x)}$$

$$\frac{d}{dx}\left[k\, ln\, f(x)\right] = k\frac{f'(x)}{f(x)}$$

$$\frac{d}{dx}\left[ln\, x\right] = \frac{1}{x}$$

$$\frac{d}{dx}\left[ln\, kx\right] = \frac{1}{x}$$

$$\frac{d}{dx}\left[ln\,(ax + b)\right] = \frac{a}{ax + b}$$

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46 • C3 • Differentiation: Product Rule

46.1 Differentiation: Product Rule

Assume y to be a function of x such that $y = f(x)$. Then consider y to be a product of two functions u and v, which themselves are also functions of x.

We now have:

$$y = uv \qquad \text{where } u \text{ and } v \text{ are functions of } x$$

In this situation, where y is a product of two functions we use the **Product Rule**, thus:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\text{where} \quad \frac{du}{dx} \quad \text{is } u \text{ differentiated w.r.t } x$$

$$\& \quad \frac{dv}{dx} \quad \text{is } v \text{ differentiated w.r.t } x$$

In function notation the rule is $\quad y = f(x)g(x) \qquad \dfrac{dy}{dx} = f(x)g'(x) + f'(x)g(x)$

or $\quad (uv)' = uv' + vu'$

Note: other text books sometimes have the product rule laid out slightly differently. Use whatever you find comfortable learning. e.g.

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

46.2 Deriving the Product Rule

Starting with $y = uv$ and increasing x by a small amount δx, with corresponding increases in y, u and v, we have:

$$y + \delta y = (u + \delta u)(v + \delta v)$$

Substituting $y = uv$ $\qquad uv + \delta y = uv + u\delta v + v\delta u + \delta u\delta v$

Subtracting uv from both sides $\qquad \delta y = u\delta v + v\delta u + \delta u\delta v$

Divide by δx $\qquad \dfrac{\delta y}{\delta x} = u\dfrac{\delta v}{\delta x} + v\dfrac{\delta u}{\delta x} + \dfrac{\delta v}{\delta x}\delta u$

As $\delta x \to 0 \quad \dfrac{\delta y}{\delta x} \to \dfrac{dy}{dx}, \quad \dfrac{\delta u}{\delta x} \to \dfrac{du}{dx}, \quad \dfrac{\delta v}{\delta x} \to \dfrac{dv}{dx}, \quad \delta u \to 0$

More formerly $\quad \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x} = \dfrac{dy}{dx}, \quad \lim\limits_{\delta x \to 0} \dfrac{\delta u}{\delta x} = \dfrac{du}{dx} \quad \lim\limits_{\delta x \to 0} \dfrac{\delta v}{\delta x} = \dfrac{dv}{dx}$ and $\lim\limits_{\delta x \to 0} \delta u = 0$

$$\therefore \quad \frac{dy}{dx} = u \lim_{\delta x \to 0} \frac{\delta v}{\delta x} + v \lim_{\delta x \to 0} \frac{\delta u}{\delta x} + \lim_{\delta x \to 0} \frac{\delta v}{\delta x} \lim_{\delta x \to 0} \delta u$$

$$\therefore \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

46.3 Product Rule: Worked Examples

46.3.1 Example:

1 Differentiate $y = 2x(x-1)^4$ and find the stationary points.

Solution:

$$u = 2x \qquad v = (x-1)^4$$

$$\frac{du}{dx} = 2 \qquad \frac{dv}{dx} = 4(x-1)^3$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\therefore \quad \frac{dy}{dx} = 2x \times 4(x-1)^3 + (x-1)^4 \times 2$$

$$= 8x(x-1)^3 + 2(x-1)^4$$

$$= 2(x-1)^3\left[4x + (x-1)\right]$$

$$= 2(x-1)^3(5x-1)$$

Stationary points when $\frac{dy}{dx} = 0$

$$2(x-1)^3(5x-1) = 0$$

$$x = 1 \quad \textit{and} \quad x = \frac{1}{5}$$

2 Differentiate $y = x^2(x^2+7)^2$

Solution:

$$u = x^2 \qquad v = (x^2+7)^2$$

$$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = 2(x^2+7) \times 2x \quad \Rightarrow 4x(x^2+7) \quad \text{(chain rule)}$$

$$\therefore \quad \frac{dy}{dx} = x^2 \times 4x(x^2+7) + (x^2+7)^2 \times 2x$$

$$= 4x^3(x^2+7) + 2x(x^2+7)^2$$

$$= 2x(x^2+7)\left[2x^2 + (x^2+7)\right]$$

$$= 2x(x^2+7)(3x^2+7)$$

3 Differentiate $y = xe^x$

Solution:

$$u = x \qquad v = e^x$$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = e^x$$

$$\therefore \quad \frac{dy}{dx} = x \times e^x + e^x \times 1$$

$$= e^x(x+1)$$

4 Differentiate $y = (x^2 + 4)(x^5 + 7)^4$

Solution:

$$u = (x^2 + 4) \qquad v = (x^5 + 7)^4$$

$$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = 4(x^5 + 7)^3 \times 5x^4 \quad \Rightarrow 20x^4(x^5 + 7)^3$$

$$\therefore \quad \frac{dy}{dx} = (x^2 + 4) \times 20x^4(x^5 + 7)^3 + (x^5 + 7)^4 \times 2x$$

$$= 20x^4(x^5 + 7)^3(x^2 + 4) + 2x(x^5 + 7)^4$$

$$= 2x(x^5 + 7)^3\left[10x^3(x^2 + 4) + (x^5 + 7)\right]$$

$$= 2x(x^5 + 7)^3(11x^5 + 40x^3 + 7)$$

5 Differentiate $y = (x + 4)(x^2 - 1)^{\frac{1}{2}}$

Solution:

$$u = (x + 4) \qquad v = (x^2 - 1)^{\frac{1}{2}}$$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \frac{1}{2}(x^2 - 1)^{-\frac{1}{2}} \times 2x \quad \Rightarrow x(x^2 - 1)^{-\frac{1}{2}}$$

$$\therefore \quad \frac{dy}{dx} = (x + 4) \times x(x^2 - 1)^{-\frac{1}{2}} + (x^2 - 1)^{\frac{1}{2}} \times 1$$

$$= x(x + 4)(x^2 - 1)^{-\frac{1}{2}} + (x^2 - 1)^{\frac{1}{2}}$$

$$= (x^2 - 1)^{-\frac{1}{2}}\left[x(x + 4) + (x^2 - 1)^1\right]$$

$$= (x^2 - 1)^{-\frac{1}{2}}(2x^2 + 4x - 1)$$

$$= \frac{(2x^2 + 4x - 1)}{(x^2 - 1)^{\frac{1}{2}}} = \frac{2x^2 + 4x - 1}{\sqrt{x^2 - 1}}$$

6 Differentiate $y = x^4 \sin x$

Solution:

$$u = x^4 \qquad v = \sin x$$

$$\frac{du}{dx} = 4x^3 \qquad \frac{dv}{dx} = \cos x$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\therefore \quad \frac{dy}{dx} = x^4 \times \cos x + \sin x \times 4x^3$$

$$= x^4 \cos x + 4x^3 \sin x$$

$$= x^3(x \cos x + 4 \sin x)$$

7 Differentiate $y = \cos x \sin x$

Solution:

$$u = \cos x \qquad v = \sin x$$

$$\frac{du}{dx} = -\sin x \qquad \frac{dv}{dx} = \cos x$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\therefore \quad \frac{dy}{dx} = \cos x \times \cos x + \sin x \times (-\sin x)$$

$$= \cos^2 x - \sin^2 x$$

$$= \cos 2x$$

8 Differentiate $y = a\,e^{bx} \sin ax$

Solution:

$$u = a\,e^{bx} \qquad v = \sin ax$$

$$\frac{du}{dx} = ab\,e^{bx} \qquad \frac{dv}{dx} = a\cos ax$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\therefore \quad \frac{dy}{dx} = a\,e^{bx} \times a\cos ax + \sin ax \times ab\,e^{bx}$$

$$= a^2 e^{bx} \cos ax + ab\,e^{bx} \sin ax$$

$$= a\,e^{bx} (a\cos ax + b\sin ax)$$

46.4 Topical Tips

Leave answers in factorised form. It is then easier to find the stationary points on a curve.

47 • C3 • Differentiation: Quotient Rule

47.1 Differentiation: Quotient Rule

Assume y to be a function of x such that $y = f(x)$. Then consider y to be a quotient of two functions u and v, which themselves are also functions of x. We now have:

$$y = \frac{u}{v} \qquad \text{where } u \text{ and } v \text{ are functions of } x$$

In this situation, where y is a quotient of two functions we use the **Quotient Rule**, thus:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

Alternative forms of the equation as given in the exam formulae book:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$$

47.2 Quotient Rule Derivation

Starting with $y = \frac{u}{v}$ and the product rule we have:

$$\begin{aligned}
y &= \frac{u}{v} \\
u &= yv \\
\frac{du}{dx} &= y\frac{dv}{dx} + v\frac{dy}{dx} \\
v\frac{dy}{dx} &= \frac{du}{dx} - y\frac{dv}{dx} \\
v\frac{dy}{dx} &= \frac{du}{dx} - \frac{u}{v}\frac{dv}{dx} \\
\frac{dy}{dx} &= \frac{1}{v}\left[\frac{du}{dx} - \frac{u}{v}\frac{dv}{dx}\right] \\
&= \frac{1}{v}\left[\frac{v}{v}\frac{du}{dx} - \frac{u}{v}\frac{dv}{dx}\right] \\
&= \frac{1}{v^2}\left[v\frac{du}{dx} - u\frac{dv}{dx}\right] \\
\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}
\end{aligned}$$

47.3 Quotient Rule: Worked Examples

47.3.1 Example:

1 Differentiate $y = \dfrac{x}{x+1}$

Solution:

$$u = x \qquad v = x + 1 \qquad \text{Recall:} \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 1$$

$$\therefore \quad \frac{dy}{dx} = \frac{(x+1) \times 1 - x \times 1}{(x+1)^2}$$

$$= \frac{x + 1 - x}{(x+1)^2}$$

$$= \frac{1}{(x+1)^2}$$

2 Differentiate $y = \dfrac{x+2}{x^2+3}$

Solution:

$$u = x + 2 \qquad v = x^2 + 3$$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 2x$$

$$\therefore \quad \frac{dy}{dx} = \frac{(x^2+3) \times 1 - (x+2) \times 2x}{(x^2+3)^2}$$

$$= \frac{(x^2+3) - 2x(x+2)}{(x^2+3)^2}$$

$$= \frac{x^2 + 3 - 2x^2 - 4x}{(x^2+3)^2}$$

$$= \frac{3 - x^2 - 4x}{(x^2+3)^2}$$

3 Differentiate $y = \dfrac{3x}{e^{4x}}$

Solution:

$$u = 3x \qquad v = e^{4x}$$

$$\frac{du}{dx} = 3 \qquad \frac{dv}{dx} = 4\,e^{4x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{e^{4x} \times 3 - 3x \times 4\,e^{4x}}{(e^{4x})^2}$$

$$= \frac{3e^{4x} - 12x\,e^{4x}}{(e^{4x})^2} \qquad \Rightarrow \qquad \frac{3e^{4x}(1 - 4x)}{(e^{4x})^2}$$

$$= \frac{3(1-4x)}{e^{4x}}$$

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4 Differentiate $y = \sqrt{\dfrac{x + 1}{x^2 + 1}}$

Solution:

$$y = \frac{(x + 1)^{\frac{1}{2}}}{(x^2 + 1)^{\frac{1}{2}}}$$

$$u = (x + 1)^{\frac{1}{2}} \qquad v = \left(x^2 + 1\right)^{\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{1}{2}(x + 1)^{-\frac{1}{2}} \qquad \frac{dv}{dx} = \frac{1}{2}\left(x^2 + 1\right)^{-\frac{1}{2}} \times 2x \qquad \Rightarrow\ x\left(x^2 + 1\right)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{(x^2 + 1)^{\frac{1}{2}} \times \frac{1}{2}(x + 1)^{-\frac{1}{2}} - (x + 1)^{\frac{1}{2}} \times x(x^2 + 1)^{-\frac{1}{2}}}{\left[(x^2 + 1)^{\frac{1}{2}}\right]^2}$$

$$= \frac{\frac{1}{2}(x^2 + 1)^{\frac{1}{2}}(x + 1)^{-\frac{1}{2}} - x(x + 1)^{\frac{1}{2}}(x^2 + 1)^{-\frac{1}{2}}}{(x^2 + 1)}$$

$$= \frac{\frac{1}{2}(x + 1)^{-\frac{1}{2}}(x^2 + 1)^{-\frac{1}{2}}\left[(x^2 + 1) - 2x(x + 1)\right]}{(x^2 + 1)}$$

$$= \frac{\left(x^2 + 1\right) - 2x^2 - 2x}{2(x^2 + 1)^{\frac{1}{2}}(x + 1)^{\frac{1}{2}}(x^2 + 1)}$$

$$= \frac{1 - 2x - x^2}{2(x^2 + 1)^{\frac{3}{2}}(x + 1)^{\frac{1}{2}}}$$

5 Differentiate $y = \dfrac{\ln x}{x + 1}$ and find the gradient at $x = e$

Solution:

$$u = \ln x \qquad v = x + 1 \qquad \text{Recall: } \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{du}{dx} = \frac{1}{x} \qquad \frac{dv}{dx} = 1$$

$$\therefore \quad \frac{dy}{dx} = \frac{(x + 1) \times \frac{1}{x} - \ln x \times 1}{(x + 1)^2}$$

$$= \frac{x + 1 - x\ln x}{x(x + 1)^2}$$

$$= \frac{e + 1 - e}{e(e + 1)^2}$$

$$= \frac{1}{e(e + 1)^2}$$

47.4 Topical Tips

Some quotients can be simplified such that the product or the chain rule can be used which are probably easier to handle.

E.g.

$$y = \frac{5}{(3x+2)^2} \Rightarrow 5(3x+2)^{-2} \quad \Rightarrow \quad \frac{dy}{dx} = -30(3x+2)^{-3} \quad \text{chain rule}$$

$$y = \frac{1-x}{1+x} \Rightarrow (1-x)(1+x)^{-1} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-2}{(1+x)^2} \quad \text{product rule}$$

Note how the quotient rule is given in the formulae book:

$$y = \frac{f(x)}{g(x)}$$

$$\frac{dy}{dx} = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$$

Compare with our derivation:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

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48 • C3 • Differentiation: Exponential Functions

48.1 Differentiation of e^x

Recall from Exponential & Log Functions that the value of e is chosen such that the gradient function of $y = e^x$ is the same as the original function and that when $x = 0$ the gradient of $y = e^x$ is 1.

Hence:

$$y = e^x \qquad \frac{dy}{dx} = e^x$$

$$y = e^{kx} \qquad \frac{dy}{dx} = ke^{kx}$$

$$y = e^{f(x)} \qquad \frac{dy}{dx} = f'(x)\,e^{f(x)}$$

48.1.1 Example:

1 Differentiate $y = 5e^{3x} + 2e^{-4x}$

Solution:

$$\frac{dy}{dx} = 5 \times 3e^{3x} + 2 \times (-4)\,e^{-4x}$$

$$\therefore \quad \frac{dy}{dx} = 15e^{3x} - 8e^{-4x}$$

2 Differentiate $y = \frac{1}{3}e^{9x}$ and find the equation of the tangent at $x = 0$

Solution:

$$\frac{dy}{dx} = \frac{1}{3} \times 9e^{9x} = 3e^{9x}$$

$\therefore$ When $x = 0$, $\quad \dfrac{dy}{dx} = 3$

and $\quad y = \dfrac{1}{3}e^0 = \dfrac{1}{3}$

Now equation of a straight line is $y = mx + c$

$\therefore$ Equation of the tangent is $\quad y = 3x + \dfrac{1}{3}$

3 Differentiate $y = e^{x^3}$

Solution:

$u = x^3 \qquad \frac{du}{dx} = 3x^2$

$y = e^u \qquad \frac{dy}{du} = e^u$

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$\frac{dy}{dx} = 3x^2 \times e^u = 3x^2\, e^u$$

$$\therefore \quad \frac{dy}{dx} = 3x^2\, e^{x^3}$$

4 Differentiate $y = e^{(x-1)^2}$

Solution:

$t = x - 1 \qquad \frac{dt}{dx} = 1$

$u = (t)^2 \qquad \frac{du}{dt} = 2t$

$y = e^u \qquad \frac{dy}{du} = e^u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = e^u \times 2t \times 1 = 2t\, e^u$$

$$\therefore \quad \frac{dy}{dx} = 2(x-1)\, e^{(x-1)^2}$$

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49 • C3 • Differentiation: Log Functions

49.1 Differentiation of ln x

Recall that $\ln x$ is the reciprocal function of e^x and that $y = e^x$ is a reflection of $y = \ln x$ in the line $y = x$

$$y = \ln x \qquad \frac{dy}{dx} = \frac{1}{x}$$

$$y = \ln f(x) \qquad \frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

Note that if: $y = \ln kx \Rightarrow y = \ln k + \ln x$

$\therefore \quad \frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}$

This can be shown thus:

Recall that: $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ (1)

If $y = \ln x$ then $x = e^y$

Differntiate w.r.t to y $\frac{dx}{dy} = e^y$

Hence $\frac{dx}{dy} = x$

From (1) $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{x}$

49.2 Worked Examples

49.2.1 Example:

1 Differentiate $y = \ln x^2$

Solution:

$$u = x^2 \qquad \frac{du}{dx} = 2x$$

$$y = \ln u \qquad \frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = 2x \times \frac{1}{u} = \frac{2x}{u}$$

$$\therefore \quad \frac{dy}{dx} = \frac{2x}{x^2} = \frac{2}{x}$$

2 Differentiate $y = ln\left(x^2\sqrt{2x^3 + 3}\right)$

Solution:

Let $z = x^2\left(2x^3 + 3\right)^{\frac{1}{2}}$ and $z = uv$

Where $u = x^2$ and $v = \left(2x^3 + 3\right)^{\frac{1}{2}}$

$$\therefore \quad \frac{du}{dx} = 2x \qquad \text{and} \qquad \frac{dv}{dx} = \frac{1}{2}\left(2x^3 + 3\right)^{-\frac{1}{2}} \times 6x^2 \Rightarrow 3x^2\left(2x^3 + 3\right)^{-\frac{1}{2}}$$

$$\therefore \quad \frac{dz}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= x^2 \times 3x^2\left(2x^3 + 3\right)^{-\frac{1}{2}} + \left(2x^3 + 3\right)^{\frac{1}{2}} \times 2x$$

$$= 3x^4\left(2x^3 + 3\right)^{-\frac{1}{2}} + 2x\left(2x^3 + 3\right)^{\frac{1}{2}}$$

$$\text{but} \quad \frac{dy}{dx} = \frac{3x^4\left(2x^3 + 3\right)^{-\frac{1}{2}} + 2x\left(2x^3 + 3\right)^{\frac{1}{2}}}{x^2\left(2x^3 + 3\right)^{\frac{1}{2}}} = \frac{3x^4\left(2x^3 + 3\right)^{-\frac{1}{2}}}{x^2\left(2x^3 + 3\right)^{\frac{1}{2}}} + \frac{2x\left(2x^3 + 3\right)^{\frac{1}{2}}}{x^2\left(2x^3 + 3\right)^{\frac{1}{2}}}$$

$$= \frac{3x^2}{\left(2x^3 + 3\right)} + \frac{2}{x}$$

3 Differentiate $y = e^{x\,ln\,2}$

Solution:

$$u = x\,ln\,2 \qquad \frac{du}{dx} = ln\,2$$

$$y = e^u \qquad \frac{dy}{du} = e^u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = ln\,2 \times e^u = ln(2)\,e^u$$

$$\therefore \quad \frac{dy}{dx} = e^{x\,ln\,2}\,ln(2)$$

4 Differentiate $y = e^{(x-1)^2}$

Solution:

$$t = x - 1 \qquad \frac{dt}{dx} = 1$$

$$u = (t)^2 \qquad \frac{du}{dt} = 2t$$

$$y = e^u \qquad \frac{dy}{du} = e^u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = e^u \times 2t \times 1 = 2t\,e^u$$

$$\therefore \quad \frac{dy}{dx} = 2(x - 1)\,e^{(x-1)^2}$$

50 • C3 • Differentiation: Rates of Change

50.1 Connected Rates of Change

Differentiation is all about rates of change. In other words, how much does y change with respect to x. Thinking back to the definition of a straight line, the gradient of a line is given by the change in y co-ordinates divided by the change in x co-ordinates. So it should come as no surprise that differentiation also gives the gradient of a curve at any given point.

Perhaps the most obvious example of rates of change is that of changing distance with time which we call speed. This can be taken further, and if the rate of change of speed with respect to time is measured we get acceleration.

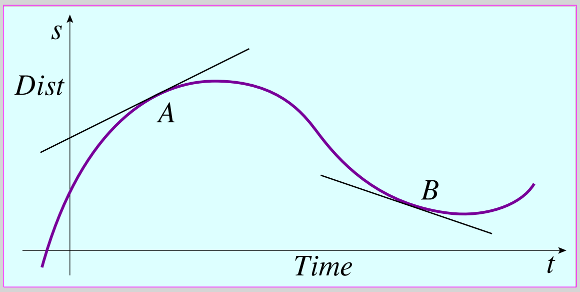
In terms of differentiation this can be written as:

$$\frac{ds}{dt} = v \qquad \text{where } s = \text{distance, } t = \text{time \& } v = \text{velocity}$$

$$\frac{dv}{dt} = a \qquad \text{where } s = \text{distance, } t = \text{time \& } a = \text{acceleration}$$

$$\frac{dv}{dt} = \frac{d}{dt}(v) = \frac{d}{dt}\cdot\frac{ds}{dt} = \frac{d^2s}{dt^2}$$

The gradient at A is the rate at which distance is changing w.r.t time. i.e. speed. A +ve slope means speed is increasing and a −ve slope means it is decreasing.



50.2 Rate of Change Problems

- One of the primary uses of differential calculus
- Rates of change generally relate to a change w.r.t time
- Rate of increase is +ve
- Rate of decrease is −ve
- Often problems involve two variables changing with time - hence the chain rule is required:

$$\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt}$$

- This means that y is a function of u and u is a function of t
- When answering these problems, state:
    - What has been given
    - What is required
    - Find the connection between variables
    - Make sure units are compatible
- Recall these formulae:
    - Volume of sphere $\frac{4}{3}\pi r^3$
    - Surface area of sphere $4\pi r^2$
    - Volume of a cone $\frac{1}{3}\pi r^2 h$

50.2.1 Example:

1 An objects speed varies according to the equation $y = 4 sin\, 2\theta$ and θ increases at a constant rate of 3 radians / sec. Find the rate at which y is changing when $\theta = \frac{15\pi}{18}$

Given: $y = 4\, sin\, 2\theta;\quad \frac{d\theta}{dt} = 3$

Required: $\frac{dy}{dt}$ when $\theta = \frac{15\pi}{18}$

Connection: $\frac{dy}{dt} = \frac{d\theta}{dt} \times \frac{dy}{d\theta}$

$$\frac{dy}{d\theta} = 8\, cos\, 2\theta$$

$$\therefore \quad \frac{dy}{dt} = 3 \times 8\, cos\, 2\theta = 24\, cos\, 2\theta$$

When $\theta = \frac{15\pi}{18}$ $\quad \frac{dy}{dt} = 24\, cos\left(2 \times \frac{15\pi}{18}\right) = 24 \times \frac{1}{2} = 12\, units\ /\ sec$

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3 A spherical balloon (specially designed for exams) is being inflated. When the diameter is 10 cm, its volume is increasing at 200 cm^3 / sec. What rate is the surface area increasing.

Given: Volume of sphere: $V = \frac{4}{3}\pi r^3$; $\frac{dV}{dt} = 200$

Surface area of sphere: $A = 4\pi r^2$

Required: $\frac{dA}{dt}$ when $r = 5$

Connection: $\frac{dA}{dt} = \frac{dV}{dt} \times \frac{dA}{dV}$

To find $\frac{dA}{dV}$ will require a connection between Volume and Area which is the radius.

Using the chain rule to connect all the variables: $\frac{dA}{dV} = \frac{dr}{dV} \times \frac{dA}{dr}$

Extending the first chain connection we get:

$$\frac{dA}{dt} = \frac{dV}{dt} \times \frac{dr}{dV} \times \frac{dA}{dr}$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\therefore \quad \frac{dr}{dV} = \frac{1}{4\pi r^2}$$

$$\frac{dA}{dr} = 8\pi r$$

$$\frac{dA}{dt} = 200 \times \frac{1}{4\pi r^2} \times 8\pi r$$

$$\frac{dA}{dt} = \frac{400}{r}$$

When $r = 5$ $\quad \frac{dA}{dt} = \frac{400}{5} = 80\ cm^2/\ sec$

4 The same balloon has its volume increased by 4 m^3/ sec. Find the rate at which the radius changes when $r = 4$ cm.

Given: Volume of sphere: $V = \frac{4}{3}\pi r^3$; $\frac{dV}{dt} = 4$

Required: $\frac{dr}{dt}$ when $r = 4$

Connection: $\frac{dr}{dt} = \frac{dV}{dt} \times \frac{dr}{dV}$

$$\frac{dV}{dr} = 4\pi r^2 \quad \therefore \quad \frac{dr}{dV} = \frac{1}{4\pi r^2}$$

$$\frac{dr}{dt} = \frac{dV}{dt} \times \frac{dr}{dV}$$

$$\frac{dr}{dt} = 4 \times \frac{1}{4\pi r^2} = \frac{1}{\pi r^2}$$

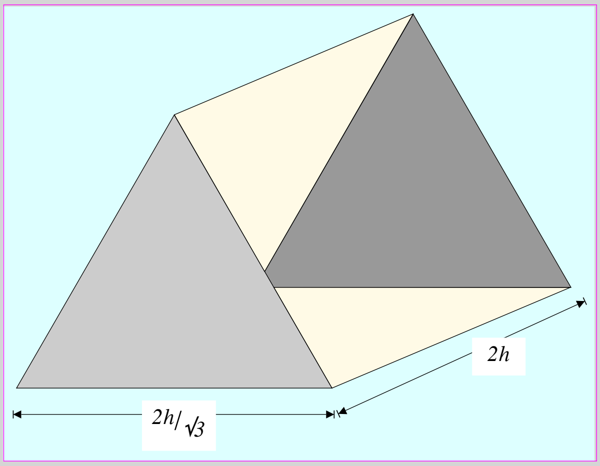
When $r = 4$ $\quad \frac{dr}{dt} = \frac{1}{16\pi}\ cm/\ sec$

5 A prism, with a regular triangular base has length $2h$ and each side of the triangle measures $\frac{2h}{\sqrt{3}}$ cm. If h is increasing at 2 m/sec what is the rate of increase in the volume when $h = 9$?

Given: Volume of prism: $V = \frac{1}{2}bhl$; $\frac{dh}{dt} = 2$

Required: $\frac{dV}{dt}$ when $h = 9$

Connection: $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$



Volume of prism: $V = \frac{1}{2} \times \frac{2h}{\sqrt{3}} \times h \times 2h = \frac{2h^3}{\sqrt{3}} \qquad \therefore \frac{dV}{dh} = \frac{6h^2}{\sqrt{3}}$

$$\frac{dV}{dt} = \frac{6h^2}{\sqrt{3}} \times 2 = \frac{12h^2}{\sqrt{3}}$$

When $h = 9$ $\quad \frac{dV}{dt} = \frac{12 \times 9^2}{\sqrt{3}} = 561{\cdot}2$ (4 *sf*)

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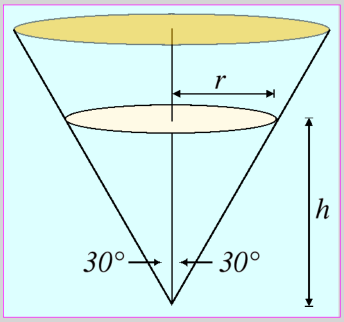
6 A conical vessel with a semi vertical angle of 30° is collecting fluid at the rate of 2 cm^3/ sec. At what rate is the fluid rising when the depth of the fluid is 6 cm, and what rate is the surface area of the fluid increasing?

Given: Volume of cone: $V = \frac{1}{3}\pi r^2 h$; $\frac{dV}{dt} = 2$

Radius of fluid: $r = h \tan 30 = \frac{h}{\sqrt{3}}$

Required, part 1: $\frac{dh}{dt}$ when $h = 6$

Connection, part 1: $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$



Volume of cone in terms of h: $V = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h = \frac{1}{9}\pi h^3$

$$\frac{dV}{dh} = \frac{3}{9}\pi h^2 = \frac{\pi h^2}{3}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$\frac{dh}{dt} = \frac{3}{\pi h^2} \times 2$$

When : $h = 6$ $\quad \frac{dh}{dt} = \frac{6}{\pi 36} = \frac{1}{6\pi}\ cm/sec$

Required, part 2: $\frac{dA}{dt}$ when $h = 6$

Connection, part 2: $\frac{dA}{dt} = \frac{dA}{dh} \times \frac{dh}{dt}$

Area of fluid surface: $A = \pi r^2 = \pi\left(\frac{h}{\sqrt{3}}\right)^2 = \frac{\pi h^2}{3}$

$$\frac{dA}{dh} = \frac{2\pi h}{3}$$

$$\therefore \quad \frac{dA}{dt} = \frac{2\pi h}{3} \times \frac{6}{\pi h^2} = \frac{4}{h}$$

When : $h = 6$ $\quad \frac{dA}{dt} = \frac{4}{6} = \frac{2}{3} = 0{\cdot}333\ cm^2/\ sec$

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51 • C3 • Integration: Exponential Functions

51.1 Integrating e^x

Recall that:

$$\frac{d}{dx}e^x = e^x$$

and since integration is the reverse of differentiation, (i.e. integrate the RHS) we derive:

$$\int e^x dx = e^x + C$$

Similarly:

$$\frac{d}{dx}ae^x = ae^x \qquad \text{and} \qquad \frac{d}{dx}e^{(ax+b)} = ae^{(ax+b)}$$

$$\int ae^x dx = ae^x + C \qquad \text{and} \qquad \int e^{(ax+b)}dx = \frac{1}{a}e^{(ax+b)} + C$$

Note: to integrate an exponential with a different base that is not e, then the base must be converted to base e. A good reason to use base e at all times for calculus!

51.2 Integrating $1/x$

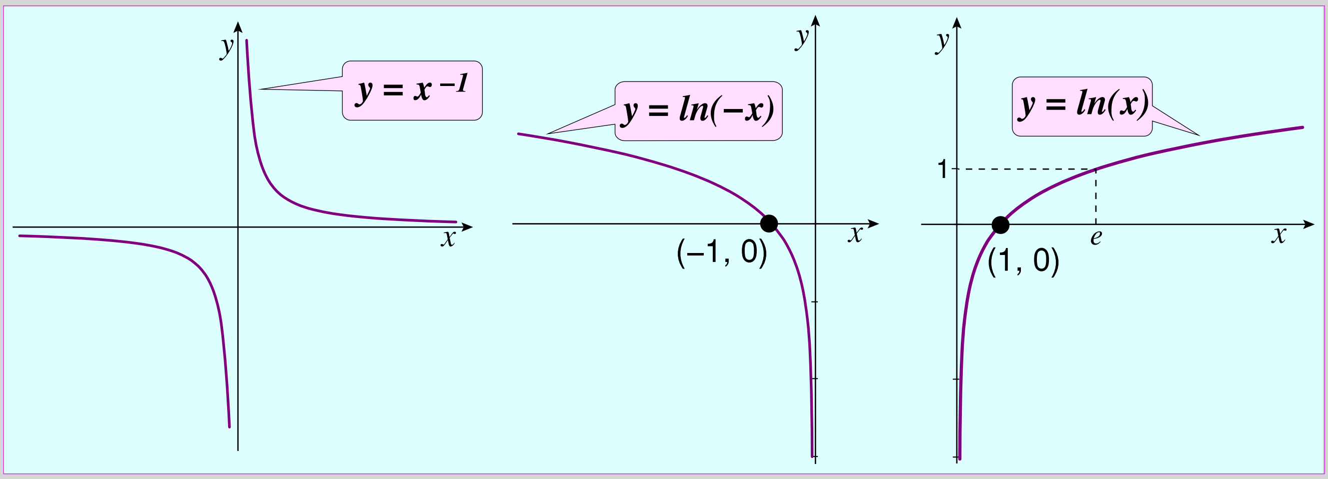
If you try to use the standard method of integration on a reciprocal function you end up in a mess, such as:

$$\int \frac{1}{x}dx = \int x^{-1}dx = \frac{1}{-1+1}x^{-1+1} + C = \frac{1}{0}x^0 + C \ ?\ ?\ ?\ ?\ ?\ ?$$

Now recall the work on differentiating $ln\,x$:

$$\frac{d}{dx}ln\,x = \frac{1}{x} \qquad \text{valid for} \qquad x > 0$$

and review the graphs for $ln\,x$ and $\frac{1}{x}$:



Graphs of 1/x , ln(−x) & ln x

Since $ln\,x$ is only valid for positive values of x (see graph above) and taking the reverse of the differential of $ln\,x$, (i.e. integrate the RHS) and provided $x > 0$ we derive:

$$\frac{d}{dx} ln\,x = \frac{1}{x} \qquad \Leftrightarrow \qquad \int \frac{1}{x}\,dx = ln\,x + C \qquad \text{valid for } x > 0$$

However, from the graph of $y = x^{-1}$ we can see that solutions exist for negative values of x, so it must be possible to integrate $y = \frac{1}{x}$ for all values of x except for $x = 0$. The problem is dealing with $x < 0$.

From the graph, we can see that $ln\,(-x)$ is defined for negative values of x and so using the chain rule it can be shown that:

$$\frac{d}{dx} ln\,kx = \frac{1}{x} \qquad \text{where } k \text{ is a constant}$$

$$\text{If } k = -1 \qquad \text{then} \qquad \frac{d}{dx} ln\,(-x) = \frac{-1}{-x} = \frac{1}{x}$$

Hence:

$$\frac{d}{dx} ln\,(-x) = \frac{1}{x} \qquad \Leftrightarrow \qquad \int \frac{1}{x}\,dx = ln\,(-x) + C \qquad \text{valid for } x < 0$$

Combining these two results using modulus notation we have:

$$\int \frac{1}{x}\,dx = ln\,|x| + C \qquad \text{provided} \qquad x \neq 0$$

With the restriction of $x \neq 0$, you cannot find the area under a curve with limits that span $x \neq 0$.

51.3 Integrating other Reciprocal Functions

Similar arguments can be made for reciprocals of the form $\frac{1}{ax+b}$.

Recall that:

$$\frac{d}{dx} ln\,(ax + b) = \frac{a}{ax + b} \qquad \therefore \qquad \int \frac{1}{ax + b}\,dx = \frac{1}{a} ln\,|ax + b| + C$$

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52 • C3 • Integration: By Inspection

52.1 Integration by Inspection

Recall that integration is the reverse of differentiation such that:

$$\frac{d}{dx}f(x) = f'(x) \qquad \Leftrightarrow \qquad \int f'(x)\,dx = f(x) + C$$

This reversal of the process leads to a number of standard integrals (many of which can be found in the appendix).

Recognising and using standard integrals is often called

52.2 Integration of $(ax+b)^n$ by Inspection

Recall that using the chain rule:

$$\frac{d}{dx}(ax+b)^n = an(ax+b)^{n-1}$$

and since integration is the reverse of differentiation, (i.e. integrate the RHS) we can derive the following standard integral:

$$\int (ax+b)^n = \frac{1}{a(n+1)}(ax+b)^{n+1} + C \qquad n \neq -1$$

52.2.1 Example:

1 If $y = (2x-1)^6$ then $\dfrac{dy}{dx} = 12(2x-1)^5$

$$\therefore \qquad \int (2x-1)^5\,dx = \frac{1}{12}(2x-1)^6 + C$$

$$\therefore \text{ Formula:} \qquad \int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + C$$

2 Find the integral of $\sqrt{2-7x}$

Solution:

$$\int \sqrt{2-7x}dx = \int (2-7x)^{\frac{1}{2}}\,dx = \frac{1}{-7\times\frac{3}{2}}(2-7x)^{\frac{3}{2}} + C$$

$$= -\frac{1}{\frac{21}{2}}(2-7x)^{\frac{3}{2}} + C$$

$$= -\frac{2}{21}(2-7x)^{\frac{3}{2}} + C$$

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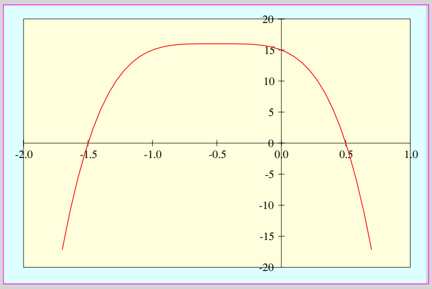
3 Find the area between the curve $y = 16 - (2x+1)^4$ and the x axis:



The curve crosses the x-axis at 2 points when:

$$16 - (2x+1)^4 = 0$$

$$\therefore (2x+1)^4 = 16 \qquad \Rightarrow (2x+1) = \pm 2$$

$$\therefore x = \tfrac{1}{2} \text{ or } x = -\tfrac{3}{2}$$

$$\text{Area} = \int_{-\frac{3}{2}}^{\frac{1}{2}} 16 - (2x+1)^4\, dx = \left[16x - \frac{(2x+1)^5}{10}\right]_{-\frac{3}{2}}^{\frac{1}{2}}$$

$$= \left[\frac{16}{2} - \frac{(1+1)^5}{10}\right] - \left[-\frac{16\times 3}{2} - \frac{(-3+1)^5}{10}\right]$$

$$= \left[8 - \frac{(2)^5}{10}\right] - \left[-24 - \frac{(-2)^5}{10}\right]$$

$$= \left[8 - \frac{32}{10}\right] - \left[-24 - \frac{(-32)}{10}\right]$$

$$= 4.8 - (-20.8) = 25.6$$

52.3 Integration of $(ax+b)^{-1}$ by Inspection

The standard integral also applies to $(ax+b)^n$ for all values of n, except $n = 1$, which is a special case.

$$\int (ax+b)^{-n}\,dx = \frac{1}{a(-n+1)}(ax+b)^{-n+1} + C \qquad \text{Not valid for } n = 1$$

The standard integral for $(ax+b)^{-1}$ is:

$$\int (ax+b)^{-1}\,dx = \frac{1}{a}\,ln(ax+b) + C$$

52.3.1 Example:

1 Find the integral of $\dfrac{1}{(3x-5)}$

$$\int (3x-5)^{-1}\,dx = \frac{1}{3}ln(3x-5) + C$$

2 Find the integral of $\dfrac{1}{(3x-5)^3}$

$$\int (3x-5)^{-3}\,dx = \frac{1}{3\times(-2)}(3x-5)^{-2} + C$$

$$= -\frac{1}{6}(3x-5)^{-2} + C$$

$$= -\frac{1}{6(3x-5)^2} + C$$

53 • C3 • Integration: Linear Substitutions

See also the C4 topic on Substitution. Integration by Substitution

53.1 Integration by Substitution Intro

Although integrating something like $\int 3x(2x - 1)^3\, dx$ could be solved by laboriously multiplying out the brackets and terms, an easier way is to use substitution, which is the integrals version of the chain rule.

Sometimes this is known as changing the variable.

We let u equal an expression in the integral and change all instances of x to u, (since we cannot integrate mixed variables).

$$\int (ax + b)^n\, dx = \int u^n dx = \int u^n \frac{dx}{du}\, du$$

Substitution method as follows:

- Choose the expression to be substituted and make equal to u
- Differentiate to find $\frac{du}{dx}$ and hence $\frac{dx}{du}$
- Substitute the new variable into the original integral
- Integrate w.r.t u
- Write the answer in terms of x.

53.2 Integration of $(ax+b)^n$ by Substitution

Although these types can be done by inspection, substitution can also be used if required.

53.2.1 Example:

1

$$\int (2x - 1)^5\, dx$$

Let $u = 2x - 1 \qquad \frac{du}{dx} = 2 \qquad \frac{dx}{du} = \frac{1}{2}$

$$\therefore \int (2x - 1)^5\, dx = \int (u)^5 \frac{dx}{du}\, du = \int u^5 \frac{1}{2}\, du$$

$$= \frac{1}{2 \times 6} u^6 + C$$

$$= \frac{1}{12} (2x - 1)^6 + C$$

2 Find the integral of $\sqrt{2 - 7x}$

$$\int \sqrt{2 - 7x}\, dx = \int (2 - 7x)^{\frac{1}{2}}\, dx$$

Let $u = 2 - 7x \qquad \frac{du}{dx} = -7 \qquad \frac{dx}{du} = -\frac{1}{7}$

$$\therefore \int (2 - 7x)^{\frac{1}{2}}\, dx = \int (u)^{\frac{1}{2}} \frac{dx}{du}\, du = \int (u)^{\frac{1}{2}} \left(-\frac{1}{7}\right) du$$

$$= \left(-\frac{1}{7} \times \frac{2}{3}\right) (u)^{\frac{3}{2}} + C$$

$$= -\frac{2}{21} (2 - 7x)^{\frac{3}{2}} + C$$

3 Find the integral of $\dfrac{1}{(3x - 5)^3}$

Solution:

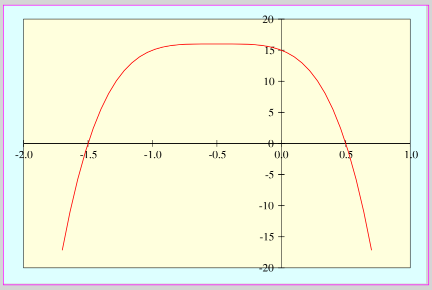
Let $u = 3x - 5 \qquad \dfrac{du}{dx} = 3 \qquad \dfrac{dx}{du} = \dfrac{1}{3}$

$$\therefore \int (3x - 5)^{-3}\,dx = \int (u)^{-3}\,\frac{dx}{du}\,du = \int (u)^{-3}\left(\frac{1}{3}\right)du$$

$$= \frac{1}{3}\left(-\frac{1}{2}\right)(u)^{-2}\,du$$

$$= -\frac{1}{6\,(3x - 5)^2} + C$$

4 Find the area between the curve $y = 16 - (2x + 1)^4$ and the x axis:



The curve crosses the x-axis at 2 points when:

$16 - (2x + 1)^4 = 0$

$\therefore (2x + 1)^4 = 16 \qquad \Rightarrow (2x + 1) = \pm 2$

$\therefore x = \frac{1}{2}$ or $x = -\frac{3}{2}$

$$\text{Area } A = \int_{-\frac{3}{2}}^{\frac{1}{2}} 16 - (2x + 1)^4\,dx$$

Let $u = 2x + 1 \qquad \dfrac{du}{dx} = 2 \qquad \dfrac{dx}{du} = \dfrac{1}{2}$

Change the limits to be in terms of u:

$x = \dfrac{1}{2},\ u = 2 \qquad x = -\dfrac{3}{2},\ u = -2$

$$A = \int_{-2}^{2} 16 - (u)^4\,\frac{dx}{du}\,du$$

$$A = \frac{1}{2}\int_{-2}^{2} 16 - (u)^4\,du$$

$$= \frac{1}{2}\left[16u - \frac{(u)^5}{5}\right]_{-2}^{2}$$

$$= \left[8u - \frac{(u)^5}{10}\right]_{-2}^{2}$$

$$= \left[16 - \frac{(2)^5}{10}\right] - \left[-16 - \frac{(-2)^5}{10}\right]$$

$$= 12.8 - (-12.8) = 25.6$$

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53.3 Integration Worked Examples

53.3.1 Example:

1 $\int 4x(6x+5)^4\,dx$

Solution:

Let $u = 6x + 5 \qquad \frac{du}{dx} = 6 \qquad \frac{dx}{du} = \frac{1}{6}$

$\therefore \quad x = \frac{u-5}{6}$

$$\therefore \int 4x(6x+5)^4\,dx = \int 4x(u)^4 \frac{dx}{du}\,du = \int 4x(u)^4 \frac{1}{6}\,du$$

$$= \frac{4}{6}\int x\,u^4\,du$$

$$= \frac{2}{3}\int \left(\frac{u-5}{6}\right)u^4\,du$$

$$= \frac{2}{18}\int (u-5)\,u^4\,du$$

$$= \frac{1}{9}\int \left(u^5 - 5u^4\right)du$$

$$= \frac{1}{9}\left[\frac{u^6}{6} - \frac{5u^5}{5}\right] + C$$

$$= \frac{1}{9}\left[\frac{u^6}{6} - u^5\right] + C$$

$$= \frac{1}{9}\left[\frac{u^6}{6} - \frac{6}{6}u^5\right] + C$$

$$= \frac{1}{54}\left[u^6 - 6u^5\right] + C$$

$$= \frac{1}{54}u^5\left[u - 6\right] + C$$

$$= \frac{1}{54}(6x+5)^5\left[6x + 5 - 6\right] + C$$

$$= \frac{1}{54}(6x+5)^5(6x-1) + C$$

2 $\int x\sqrt{2-x^2}\,dx = \int x\left(2-x^2\right)^{\frac{1}{2}}dx$

Let $u = 2 - x^2 \qquad \frac{du}{dx} = -2x \qquad \frac{dx}{du} = -\frac{1}{2x}$

$$\int x\left(2-x^2\right)^{\frac{1}{2}}dx = \int x(u)^{\frac{1}{2}}\frac{dx}{du}\,du = \int x(u)^{\frac{1}{2}}\left(-\frac{1}{2x}\right)du$$

$$= -\frac{1}{2}\int (u)^{\frac{1}{2}}\,du$$

$$= \left(-\frac{1}{2}\times\frac{2}{3}\right)(u)^{\frac{3}{2}} + C$$

$$= -\frac{1}{3}\left(2-x^2\right)^{\frac{3}{2}} + C$$

3 $\displaystyle\int (x+1)(3x-4)^4\,dx$

Solution:

Let $u = 3x - 4$ $\qquad \dfrac{du}{dx} = 3 \qquad \dfrac{dx}{du} = \dfrac{1}{3}$

$\therefore \quad x = \dfrac{u+4}{3} \quad \Rightarrow \quad x + 1 = \dfrac{u+4}{3} + 1 \Rightarrow \quad x + 1 = \dfrac{u+7}{3}$

$$\int (x+1)(3x-4)^4\,dx = \int \left(\frac{u+7}{3}\right)(u)^4 \frac{dx}{du}\,du = \int \left(\frac{u+7}{3}\right)(u)^4 \frac{1}{3}\,du$$

$$= \frac{1}{9}\int (u+7)u^4\,du$$

$$= \frac{1}{9}\int \left(u^5 + 7u^4\right)du$$

$$= \frac{1}{9}\left[\frac{u^6}{6} - \frac{7u^5}{5}\right] + C$$

$$= \frac{1}{9}\left[\frac{5u^6}{30} - \frac{42u^5}{30}\right] + C$$

$$= \frac{1}{270}u^5\left(u - 42\right) + C$$

$$= \frac{1}{270}(3x-4)^5\left(3x - 4 - 42\right) + C$$

$$= \frac{1}{270}(3x-4)^5\left(3x - 46\right) + C$$

4 Find the integral of $\dfrac{1}{\sqrt{x}\left(3+\sqrt{x}\right)}$ using $u = \sqrt{x}$ as the substitution.

Solution:

Let $u = \sqrt{x} = x^{\frac{1}{2}}$ $\qquad \dfrac{du}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}} \qquad \dfrac{dx}{du} = 2\sqrt{x} = 2u$

$$\int \frac{1}{\sqrt{x}\left(3+\sqrt{x}\right)}\,dx = \int \left(\frac{1}{u(3+u)}\right)\frac{dx}{du}\,du = \int \left(\frac{1}{u(3+u)}\right)2u\,du$$

$$= \int \frac{2u}{u(3+u)}\,du = 2\int \frac{1}{3+u}\,du$$

$$= 2\,ln(3+u) + C$$

$$= 2\,ln\left(3+\sqrt{x}\right) + C$$

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5 $\int (6x + 3)(6x - 3)^5 \, dx$

Solution:

Let $u = 6x - 3$ $\quad \dfrac{du}{dx} = 6 \quad \dfrac{dx}{du} = \dfrac{1}{6}$

$\therefore \quad x = \dfrac{u + 3}{6} \quad \Rightarrow \quad 6x + 3 = 6\left(\dfrac{u + 3}{6}\right) + 3 \quad \Rightarrow \quad 6x + 3 = u + 6$

$$\int (6x + 3)(6x - 3)^5 \, dx = \int (u + 6)(u)^5 \frac{dx}{du} \, du = \int (u + 6)(u)^5 \frac{1}{6} \, du$$

$$= \frac{1}{6} \int (u + 6)(u)^5 \, du$$

$$= \frac{1}{6} \int (u^6 + 6u^5) \, du$$

$$= \frac{1}{6}\left[\frac{u^7}{7} + \frac{6u^6}{6}\right] + C$$

$$= \frac{u^6}{6}\left[\frac{u}{7} + 1\right] + C$$

$$= \frac{u^6}{42}[u + 7] + C$$

$$= \frac{1}{42}(6x - 3)^6 (6x - 3 + 7) + C$$

$$= \frac{1}{42}(6x - 3)^6 (6x + 4) + C$$

$$= \frac{2}{42}(3x + 2)(6x - 3)^6 + C$$

$$= \frac{1}{21}(3x + 2)(6x - 3)^6 + C$$

6 Find the integral of $\dfrac{e^x}{(2e^x + 3)}$

Solution:

Let $\quad u = 2e^x + 3 \qquad \dfrac{du}{dx} = 2e^x \qquad \dfrac{dx}{du} = \dfrac{1}{2e^x}$

$e^x = \dfrac{u - 3}{2} \qquad \Rightarrow \qquad 2e^x = u - 3 \qquad \therefore \quad \dfrac{dx}{du} = \dfrac{1}{u - 3}$

$$\int \frac{e^x}{(2e^x + 3)}\, dx = \int \left(\frac{u - 3}{2}\right) \times \frac{1}{u} \frac{dx}{du}\, du = \int \left(\frac{u - 3}{2}\right) \frac{1}{u} \left(\frac{1}{u - 3}\right) du$$

$$= \int \frac{1}{2u}\, du$$

$$= \frac{1}{2} \ln(u) + C$$

$$= 2 \ln\left(2e^x + 3\right) + C$$

7 Find the integral of $\dfrac{1}{6x + 3}$ between $x = 0$ & $x = 1$

Solution:

Let $\quad u = 6x + 3 \qquad \dfrac{du}{dx} = 6 \qquad \dfrac{dx}{du} = \dfrac{1}{6}$

$$\int_0^1 \frac{1}{6x + 3} dx = \int_{x=0}^{x=1} \frac{1}{u} \frac{dx}{du}\, du = \int_{x=0}^{x=1} \frac{1}{6u}\, du$$

$$= \frac{1}{6} \int_{x=0}^{x=1} \frac{1}{u}\, du$$

$$= \frac{1}{6} \left[\ln(u)\right]_{x=0}^{x=1}$$

$$= \frac{1}{6} \left[\ln(6x + 3)\right]_{x=0}^{x=1}$$

$$= \frac{1}{6} \left(\ln(6 + 3) - \ln(0 + 3)\right)$$

$$= \frac{1}{6} \left(\ln(9) - \ln(3)\right)$$

$$= \frac{1}{6} \left(\ln \frac{9}{3}\right) = \frac{1}{6} \ln 3$$

Alternatively - change the limits to be in terms of u:

$x = 1 \qquad \Rightarrow \qquad u = 9$

$x = 0 \qquad \Rightarrow \qquad u = 3$

$$= \frac{1}{6} \int_3^9 \frac{1}{u}\, du$$

$$= \frac{1}{6} \left[\ln(u)\right]_3^9$$

$$= \frac{1}{6} \left(\ln(9) - \ln(3)\right)$$

$$= \frac{1}{6} \ln 3$$

8 $\displaystyle\int 4x\left(x^2 - 5\right)^4 \, dx$

Solution:

Let $u = x^2 - 5 \qquad \dfrac{du}{dx} = 2x \qquad \dfrac{dx}{du} = \dfrac{1}{2x}$

$$\therefore \int 4x\left(x^2 - 5\right)^4 \, dx = \int 4x\,(u)^4 \, \frac{dx}{du}\, du = \int 4x\,(u)^4 \, \frac{1}{2x}\, du$$

$$= \int 2\,(u)^4 \, du$$

$$= \frac{2}{5}\,(u)^5 + C$$

$$= \frac{2}{5}\,(x^2 - 5)^5 + C$$

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53.4 Derivation of Substitution Method

The argument goes something like this:

Given that:

$$x = g(u) \qquad \& \qquad f(x) = f[g(u)]$$

and $\quad y = \int f(x)\,dx \qquad\qquad (1)$

Differentiating both sides of (1): $\quad \dfrac{dy}{dx} = f(x) \qquad\qquad (2)$

From the chain rule: $\quad \dfrac{dy}{du} = \dfrac{dy}{dx} \times \dfrac{dx}{du}$

From (2) $\quad \dfrac{dy}{du} = f(x) \times \dfrac{dx}{du}$

Integrating both sides w.r.t u: $\quad y = \displaystyle\int f(x)\,\frac{dx}{du}\,du$

But $f(x) = f[g(u)] \qquad \therefore \quad y = \displaystyle\int f[g(u)]\,\frac{dx}{du}\,du$

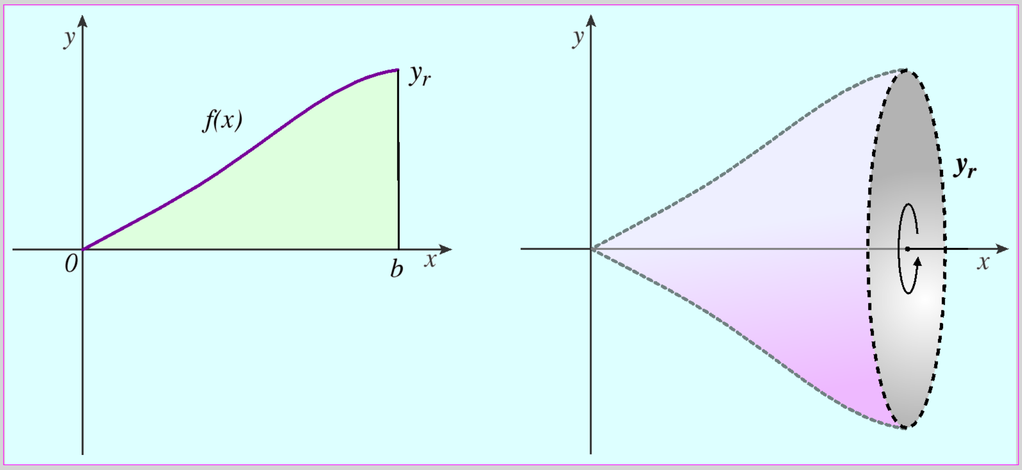
From (1) $\quad \displaystyle\int f(x)\,dx = \int f[g(u)]\,\frac{dx}{du}\,du$

Note that in integrating $f(x)$ w.r.t. x, dx is replaced by $\frac{dx}{du}\,du$ and the rest of the integral is expressed in terms of u.

54 • C3 • Integration: Volume of Revolution

54.1 Intro to the Solid of Revolution

Integration gives us a convenient method for finding the area under a curve. Now consider the effect of rotating that area through 2π radians about the x-axis. This will produce a 'solid of revolution', as the example below illustrates. It then becomes possible to calculate the volume of this solid shape.

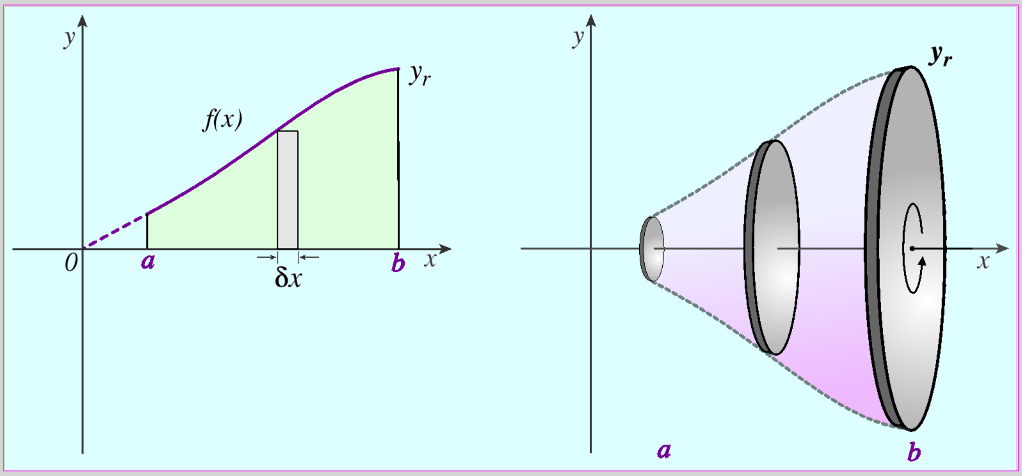


Solid of Revolution

54.2 Volume of Revolution about the x-axis

In a similar method to that of finding the area under a curve, we will put the solid shape though a bacon slicer, and produce a very large number of very thin slices. By assuming that each slice is a perfect cylinder of thickness δx the volume of each slice can be found. Summing all these slices together will give us the volume of the solid, or the 'Volume of Revolution'.

For the rotation of a curve $y = f(x)$ about the y-axis we have:



Volume of Revolution

Recall that the volume of a cylinder is $\pi r^2 d$, where r is the radius and d is the depth of the cylinder.

The volume of a thin slice, δV, is given by:

$$\delta V \approx \pi y^2 \delta x$$

Hence, the total volume of revolution about the x-axis is approximated by adding these slices together:

$$V \approx \sum \delta V \approx \sum_{i=1}^{n} \pi y^2 \delta x$$

Accuracy improves as δx becomes ever smaller and tends towards zero, hence the volume is the limit of the sum of all the slices as $\delta x \rightarrow 0$.

$$V = \lim_{\delta x \to 0} \sum_{i=1}^{n} \pi y^2 \delta x$$

Since $y = f(x)$ we can write:

$$V = \lim_{\delta x \to 0} \sum_{i=1}^{n} \pi [f(x)]^2 \delta x$$

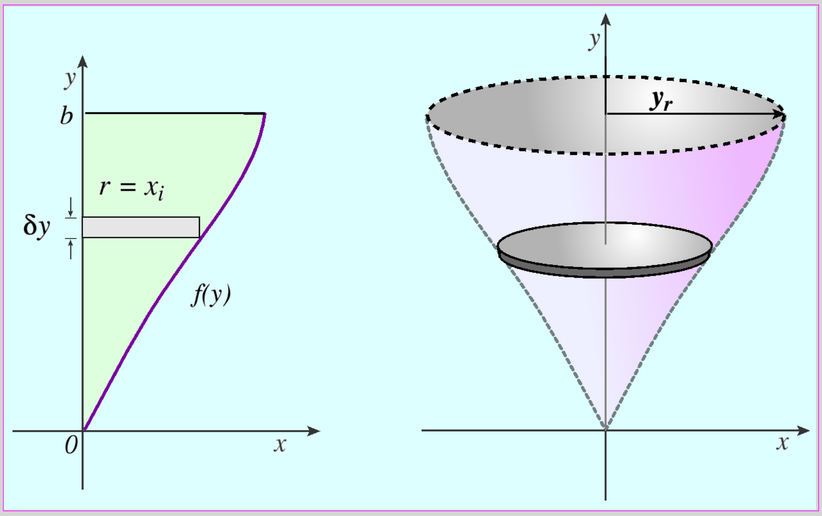
So the volume between the points $x = a$ and $x = b$ is given by:

$$\therefore \quad V = \int_a^b \pi [f(x)]^2 \, dx \equiv \int_a^b \pi y^2 dx$$

Note that since integration is done w.r.t x, then the limits are for $x = a$, & $x = b$.

54.3 Volume of Revolution about the y-axis

A similar argument can be made for the rotation of a curve $x = g(y)$ about the y-axis.



The volume of a slice is given by:

$$\delta V \approx \pi x^2 \delta y$$

Hence total volume of revolution about the y-axis is approximated by:

$$V \approx \sum \delta V \approx \sum_{i=1}^{n} \pi x^2 \delta y$$

Hence:

$$V = \lim_{\delta y \to 0} \sum_{i=1}^{n} \pi x^2 \delta y$$

$$V = \lim_{\delta y \to 0} \sum_{i=1}^{n} \pi [f(y)]^2 \delta y$$

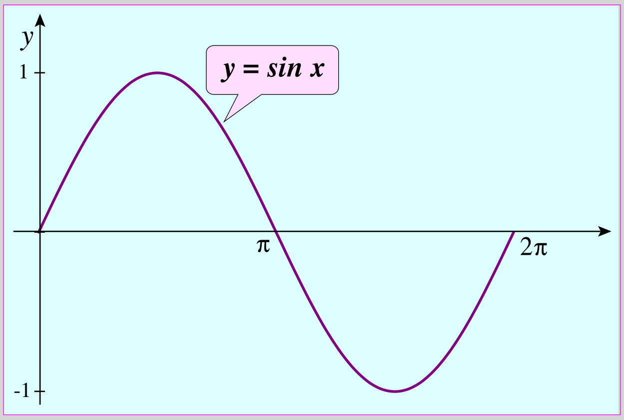
$$\therefore \quad V = \int_a^b \pi [f(y)]^2 \, dy$$

Note that since integration is done w.r.t y, then the limits are for $y = a$, & $y = b$.

54.4 Volume of Revolution Worked Examples

54.4.1 Example:

1



Find the volume of the solid generated by rotating the area under the curve of $y = \sin x$ when rotated through 2π radians about the x-axis, and between the y-axis and the line $x = \pi$.

Solution:

$$V = \int_a^b \pi y^2 dx$$

$$\therefore \quad V = \int_0^\pi \pi \sin^2 x \, dx$$

Now: $\quad 2\sin^2 x = 1 - \cos 2x$

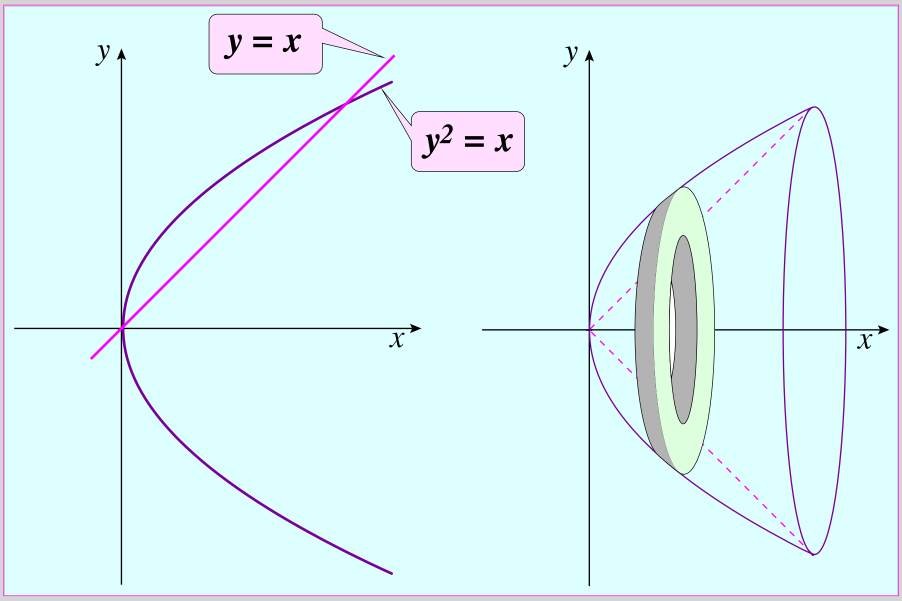
$$\therefore \quad V = \frac{\pi}{2}\int_0^\pi 1 - \cos 2x \; dx$$

$$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^\pi$$

$$= \frac{\pi}{2}\left[(\pi - 0) - \frac{1}{2}(0 - 0)\right]$$

$$= \frac{\pi^2}{2}\ units^3$$

2



Find the volume of the solid generated by rotating the area between the curve $y^2 = x$ and the line $y = x$ through 2π radians, about the x-axis.

Solution:

In general, the solid of rotation of similar shapes is the difference between the solids of rotation of the two separate curves or lines.

Note that the limits are found from the intersection of the straight line and the curve. The intersection points are easy found to be (0, 0) and (1, 1).

$$V = \int_a^b \pi \left(y_1{}^2 - y_2{}^2\right) dx$$

In this case: $y_1{}^2 = x$ and $y_2 = x$

$$\therefore \qquad V = \int_0^1 \pi \left(x - x^2\right) dx$$

$$= \pi \int_0^1 \left(x - x^2\right) dx$$

$$= \pi \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1$$

$$= \pi \left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= \pi \left[\frac{3}{6} - \frac{2}{6}\right] = \frac{1}{6}\pi \ units^3$$

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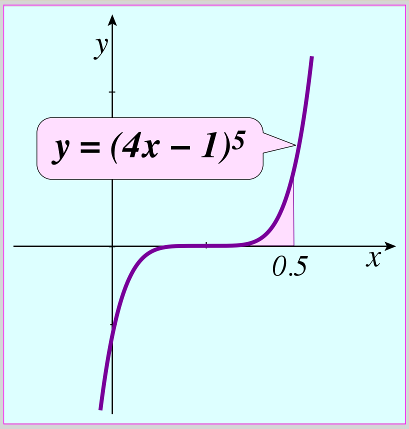
3



The shaded region is rotated about the x-axis, find the volume of the solid.

Solution:

The limits of the shaded region are found when $y = 0$ and $x = 0{\cdot}5$ (given)

When $y = 0$ then $(4x - 1)^5 = 0$

$$4x - 1 = 0$$

$$4x = 1$$

$$x = 0{\cdot}25$$

To find the volume:

$$V = \int_a^b \pi y^2 dx$$

$$\therefore \quad V = \int_{0\cdot25}^{0\cdot5} \pi\left[(4x - 1)^5\right]^2 dx$$

let $u = 4x - 1$ and $\frac{du}{dx} = 4$, $\quad \therefore \quad dx = \frac{du}{4}$

Changing the limits: $x = 0{\cdot}5 \quad u = 4 \times 0{\cdot}5 - 1 = 1$

$x = 0{\cdot}25 \quad u = 4 \times 0{\cdot}25 - 1 = 0$

$$\therefore \quad V = \int_0^1 \pi\left(u^5\right)^2 \frac{du}{4}$$

$$= \frac{\pi}{4}\int_0^1 u^{10}\, du$$

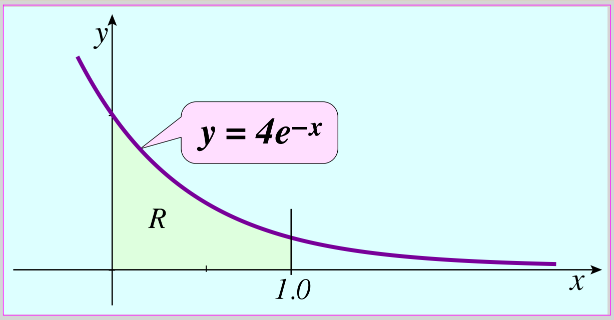
$$= \frac{\pi}{4}\left[\frac{1}{11}u^{11}\right]_0^1$$

$$= \frac{\pi}{44}\left[u^{11}\right]_0^1$$

$$= \frac{\pi}{44}\left[1 - 0\right]$$

$$= \frac{\pi}{44}\ units^3$$

4



The shaded region R enclosed by the curve $y = 4e^{-x}$ is rotated about the x-axis. Find the volume of the solid when the curve is bounded by the lines $x = 0$, $x = 1$ and $y = 0$.

Solution:

To find the volume:

$$V = \int_a^b \pi y^2 dx$$

$$\therefore \quad V = \int_0^1 \pi \left[4e^{-x}\right]^2 dx$$

$$= \int_0^1 16\pi\, e^{-2x} dx$$

$$= 16\pi \int_0^1 e^{-2x} dx$$

$$= 16\pi \left[-\frac{1}{2}e^{-2x}\right]_0^1$$

$$= \frac{16}{2}\pi \left[\left(-e^{-2}\right) - (-1)\right]$$

$$= 8\pi \left[\left(-e^{-2}\right) + 1\right]$$

$$= 8\pi\left(1 - e^{-2}\right)$$

54.5 Volume of Revolution Digest

Volume of Revolution about the x-axis:

$$V = \int_a^b \pi y^2 dx = \int_a^b \pi\left[f(x)\right]^2 dx$$

$$V = \pi \int_a^b y^2 dx$$

Note that since integration is done w.r.t x, then the limits are for $x = a$, & $x = b$.

Volume of Revolution about the y-axis:

$$V = \int_a^b \pi x^2 dy = \int_a^b \pi\left[f(y)\right]^2 dy$$

$$V = \pi \int_a^b x^2 dy$$

Note that since integration is done w.r.t y, then the limits are for $y = a$, & $y = b$.

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Module C4

Core 4 Basic Info

Algebra and graphs; Differentiation and integration; Differential equations; Vectors.

The C4 exam is 1 hour 30 minutes long and is in two sections, and worth 72 marks (75 AQA).
(That's about a minute per mark allowing some time for over run and checking at the end)
Section A (36 marks) 5 – 7 short questions worth at most 8 marks each.
Section B (36 marks) 2 questions worth about 18 marks each.

OCR Grade Boundaries.

These vary from exam to exam, but in general, for C4, the approximate raw mark boundaries are:

Grade	100%	*A* ∗	*A*	*B*	*C*
Raw marks	72	62 ± 3	55 ± 3	48 ± 3	41 ± 3
UMS %	100%	90%	80%	70%	60%

The raw marks are converted to a unified marking scheme and the UMS boundary figures are the same for all exams.

C4 Contents

Plus other minor editoral alterations and corrections. * means latest items to be updated

C4 Brief Syllabus

1 Algebra and Graphs

- divide a polynomial, (degree $\leqslant 4$), by a linear or quadratic polynomial, & give quotient & any remainder
- express rational functions as partial fractions, and carry out decomposition, where the denominator is no more complicated than $(ax + b)(cx + d)(ex + f)$ or $(ax + b)(cx + d)^2$, and not top heavy.
- use the expansion of $(1 + x)^n$ where n is a rational number and $x <1$ (finding a general term is not included, but adapting the standard series to expand, e.g. $(2 - \frac{1}{2}x)^{-1}$ is included)
- understand the use of a pair of parametric equations to define a curve, and use a given parametric representation of a curve in simple cases
- convert the equation of a curve between parametric and Cartesian forms.

2 Differentiation and Integration

- use the derivatives of $\sin x$, $\cos x$, $\tan x$ together with sums, differences and constant multiples
- find and use the first derivative of a function which is defined parametrically or implicitly
- extend the idea of 'reverse differentiation' to include the integration of trig functions (e.g. $\sin x$, $\sec^2 2x$)
- use trig identities (e.g. double angle formulae) in the integration of functions such as $\cos^2 x$
- integrate rational functions by decomposition into partial fractions
- recognise an integrand of the form $\frac{kf'(x)}{f(x)}$, and integrate, for example $\frac{x}{x^2+1}$
- recognise when an integrand can be regarded as a product, and use integration by parts to integrate, for example, $x \sin 2x$, $x^2 e^2$, $\ln x$ (understand the relationship between integration by parts and differentiation of a product)
- use a given substitution to simplify and evaluate either a definite or an indefinite integral
  (understand the relationship between integration by substitution and the chain rule).

3 First Order Differential Equations

- derive a differential equation from a simple statement involving rates of change (with a constant of proportionality if required)
- find by integration a general form of solution for a differential equation in which the variables are separable
- use an initial condition to find a particular solution of a differential equation
- interpret the solution of a differential equation in the context of a problem being modelled by the equation.

4 Vectors

- use of standard notations for vectors
- carry out addition and subtraction of vectors and multiplication of a vector by a scalar, and interpret these operations in geometrical terms
- use unit vectors, position vectors and displacement vectors
- calculate the magnitude of a vector, and identify the magnitude of a displacement vector $\overrightarrow{AB}$ as being the distance between the points A and B
- calculate the scalar product of two vectors (in either two or three dimensions), and use the scalar product to determine the angle between two directions and to solve problems concerning perpendicularity of vectors
- understand the significance of all the symbols used when the equation of a straight line is expressed in the form $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$
- determine whether two lines are parallel, intersect or are skew
- find the angle between two lines, and the point of intersection of two lines when it exists.

C4 Assumed Basic Knowledge

Knowledge of C1, C2 and C3 is assumed, and you may be asked to demonstrate this knowledge in C4.
You should know the following formulae, (which are NOT included in the Formulae Book).

1 Differentiation and Integration

Function $f(x)$	Differential $\frac{dy}{dx} = f'(x)$
$\sin kx$	$k \cos kx$
$\cos kx$	$-k \sin kx$
$\tan kx$	$k \sec^2 kx$

Function $f(x)$	Integral $\int f(x)\,dx$
$\sin kx$	$-\frac{1}{k} \cos kx + c$
$\cos kx$	$\frac{1}{k} \sin kx + c$
$\tan kx$	$\frac{1}{k} \ln \lvert \sec kx \rvert + c$

x in radians!

$$\int f'(g(x))\, g'(x)\; dx = f(g(x)) + c$$

2 Vectors

$$\lvert x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \rvert = \sqrt{x^2 + y^2 + z^2}$$

$$(a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}) \bullet (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) = ax + by + cz$$

$$\boldsymbol{p} \bullet \boldsymbol{q} = \lvert \boldsymbol{p} \rvert \lvert \boldsymbol{q} \rvert \cos\theta$$

$$\boldsymbol{p} \bullet \boldsymbol{q} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz = \left(\sqrt{a^2 + b^2 + c^2}\right)\left(\sqrt{x^2 + y^2 + z^2}\right)\cos\theta$$

$$\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{p}$$

3 Trig

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$= 1 - 2\sin^2 A$$

$$= 2\cos^2 A - 1$$

$$\tan 2A \equiv \frac{2 \tan A}{1 - \tan^2 A}$$

$$a \sin x \pm b \cos x \equiv R \sin(x \pm \alpha)$$

$$a \cos x \pm b \sin x \equiv R \cos(x \mp \alpha) \qquad \text{(watch the signs)}$$

$$R = \sqrt{a^2 + b^2} \qquad R\cos\alpha = a \qquad R\sin\alpha = b$$

$$\tan\alpha = \frac{b}{a} \qquad 0 < a < \frac{\pi}{2}$$

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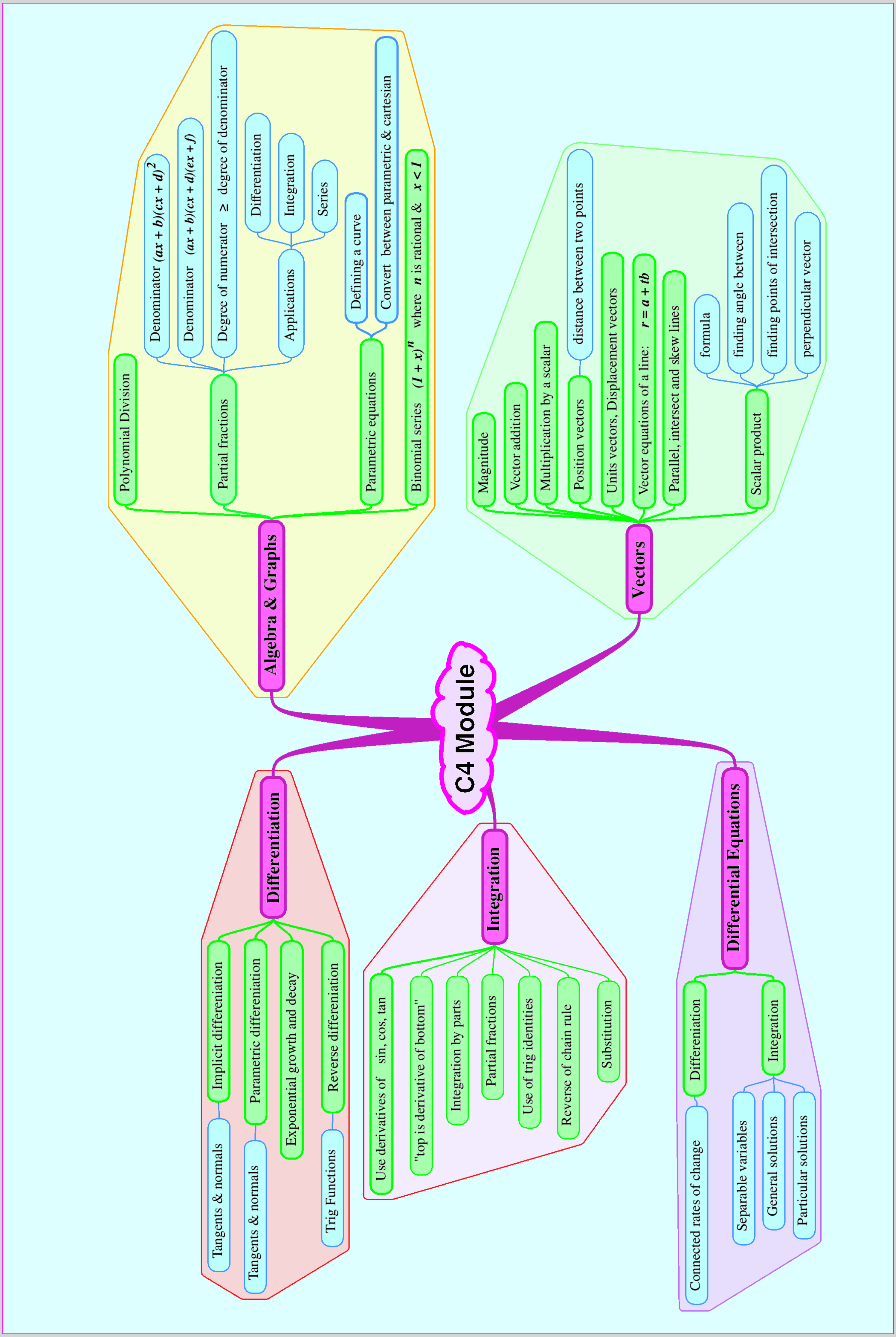
C4 Module
Algebra & Graphs
Polynomial Division
Partial fractions
Denominator $(ax + b)(cx + d)^2$
Denominator $(ax + b)(cx + d)(ex + f)$
Degree of numerator ≥ degree of denominator
Applications
Differentiation
Integration
Series
Parametric equations
Defining a curve
Convert between parametric & cartesian
Binomial series $(1 + x)^n$ where n is rational & $|x| < 1$
Vectors
Magnitude
Vector addition
Multiplication by a scalar
Position vectors
distance between two points
Units vectors, Displacement vectors
Vector equations of a line: $r = a + tb$
Parallel, intersect and skew lines
Scalar product
formula
finding angle between
finding points of intersection
perpendicular vector
Differentiation
Implicit differeniation
Tangents & normals
Parametric differeniation
Tangents & normals
Exponential growth and decay
Reverse differeniation
Trig Functions
Integration
Use derivatives of sin, cos, tan
"top is derivative of bottom"
Integration by parts
Partial fractions
Use of trig identities
Reverse of chain rule
Substitution
Differential Equations
Differeniation
Connected rates of change
Integration
Separable variables
General solutions
Particular solutions

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56 • C4 • Differentiating Trig Functions

56.1 Defining other Trig Functions

This depends on 3 ideas:

- Definitions of $\tan x$, $\cot x$, $\sec x$ & $\operatorname{cosec} x$ in terms of $\sin x$ & $\cos x$
- The differential of $\sin x$ & $\cos x$
- Product and Quotient rules of differentiation.

From previous module: (*Note the coloured letters in bold - an easy way to remember them*).

$$\sec x \equiv \frac{1}{\cos x} \qquad \operatorname{cosec} x \equiv \frac{1}{\sin x} \qquad \cot x \equiv \frac{1}{\tan x}$$

Function $f(x)$	Differential $\frac{dy}{dx} = f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$

true for x in radians

Product and Quotient rules:

Product rule: if $y = uv$ then $\dfrac{dy}{dx} = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$

Quotient rule: if $y = \dfrac{u}{v}$ then $\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Chain rule: $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

We can use these results to find the differentials of the other trig functions:

1 ***tan x***

$$y = \tan x$$

$$y = \frac{\sin x}{\cos x} \Rightarrow \frac{u}{v}$$

Quotient rule: if $y = \dfrac{u}{v}$ then $\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{\cos x \times \cos x - \sin x \times -\sin x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

2 ***sec x***

$$y = \sec x = \frac{1}{\cos x} = (\cos x)^{-1}$$

$$u = \cos x \qquad \frac{du}{dx} = -\sin x$$

$$y = u^{-1} \qquad \frac{dy}{du} = -u^{-2}$$

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$\frac{dy}{dx} = -\sin x \times (-u^{-2}) = \frac{\sin x}{\cos^2 x} = \tan x \sec x$$

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3 **cosec *x***

$$y = cosec\,x = \frac{1}{\sin x} = (\sin x)^{-1}$$

$$u = \sin x \qquad \frac{du}{dx} = \cos x$$

$$y = u^{-1} \qquad \frac{dy}{du} = -u^{-2}$$

use the Chain rule: $$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$\frac{dy}{dx} = \cos x \times (-u^{-2})$$

$$\frac{dy}{dx} = -\frac{\cos x}{\sin^2 x} = -\frac{1}{\tan x \sin x} = -\cot x\, cosec\, x$$

or use the Quotient rule:

$$u = 1 \qquad \frac{du}{dx} = 0$$

$$v = \sin x \qquad \frac{dv}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{\sin x \times 0 - 1 \times \cos x}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\cot x\, cosec\, x$$

4 **cot *x***

$$y = \cot x = \frac{1}{\tan x} = (\tan x)^{-1}$$

$$u = \tan x \qquad \frac{du}{dx} = \sec x$$

$$v = u^{-1} \qquad \frac{dv}{dx} = -u^{-2} = -\frac{\sec x}{\tan^2 x}$$

$$\frac{dy}{dx} = \frac{-1 - \tan x}{\tan^2 x} = \frac{-1}{\tan^2 x} \times \frac{-\tan x}{\tan^2 x}$$

$$\frac{dy}{dx} = -\cot^2 x - 1 = -cosec^2 x$$

Summary so far:

Function $f(x)$	Differential $\frac{dy}{dx} = f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-cosec\,x$
$cosec\,x$	$-cosec\,x \cot x$
$\sec x$	$\sec x \tan x$

Function $f(x)$	Differential $\frac{dy}{dx} = f'(x)$
$\sin kx$	$k \cos kx$
$\cos kx$	$-k \sin kx$
$\tan kx$	$k \sec^2 kx$

56.2 Worked Trig Examples

56.2.1 Example: Differentiate the following:

1

$y = x^3 \sin x$ [product rule]

$u = x^3 \qquad v = \sin x$

$\frac{du}{dx} = 3x^2 \qquad \frac{dv}{dx} = \cos x$

$\frac{dy}{dx} = x^3 \times \cos x + \sin x \times 3x^2$

$\frac{dy}{dx} = x^2 (x \cos x + 3 \sin x)$

2

$y = \frac{1}{x} \cos x \Rightarrow \frac{\cos x}{x}$ [quotient rule]

Let: $u = \cos x \qquad v = x$

$\frac{du}{dx} = -\sin x \qquad \frac{dv}{dx} = 1$

$\frac{dy}{dx} = \frac{x \times (-\sin x) - \cos x}{x^2}$

$\frac{dy}{dx} = \frac{-x \sin x - \cos x}{x^2} = -\frac{x \sin x + \cos x}{x^2}$

3

$y = \cos^4 x$ [chain rule]

$u = \cos x \qquad y = u^4$

$\frac{du}{dx} = -\sin x \qquad \frac{dy}{du} = 4u^3$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = 4u^3 \times (-\sin x)$

$\frac{dy}{dx} = 4 \cos^3 x (-\sin x) = -4 \cos^3 x \sin x$

4

$y = \cos^4 x$ [quick method - diff out - diff in]

$y = (\cos x)^4$

$\frac{dy}{dx} = 4 (\cos x)^3 \times (-\sin x)$ [differentiate outside bracket - differentiate inside bracket]

$\frac{dy}{dx} = -4 \cos^3 x \sin x$

5

$y = \ln \sec x$ [chain rule]

$u = \sec x \qquad y = \ln u$

$\frac{du}{dx} = \sec x \tan x \qquad \frac{dy}{du} = \frac{1}{u}$

$\frac{dy}{dx} = \sec x \tan x \times \frac{1}{\sec x} = \tan x$

6

$$y = \sin\left(3x - \frac{\pi}{4}\right)$$ [chain rule]

$$u = 3x - \frac{\pi}{4} \qquad y = \sin u$$

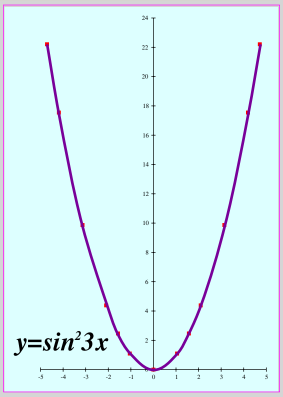
$$\frac{du}{dx} = 3 \qquad \frac{dy}{du} = \cos u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \cos(u) \times 3$$

$$\frac{dy}{dx} = 3\cos\left(3x - \frac{\pi}{4}\right)$$

7



$$y = \sin^2 3x$$

$$u = 3x \qquad y = \sin^2 u$$

$$v = \sin u$$

$$\therefore y = v^2$$

$$\frac{du}{dx} = 3 \qquad \frac{dy}{dv} = 2v \qquad \frac{dv}{du} = \cos u$$ [extended chain rule]

$$\frac{dy}{dx} = \frac{dy}{dv} \times \frac{du}{dx} \times \frac{dv}{du}$$

$$\frac{dy}{dx} = 2v \times 3 \times \cos u$$

$$\frac{dy}{dx} = 2\sin u \times 3 \times \cos u$$

$$\frac{dy}{dx} = 6\cos 3x \times \sin 3x$$

$$\frac{dy}{dx} = 3\sin 6x$$ double angle formula

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8 Alternative approach to above problem

$$y = \sin^2 3x = (\sin 3x)^2 \qquad \text{[chain rule]}$$

$$u = \sin 3x \qquad y = u^2$$

$$\frac{du}{dx} = 3\cos 3x \qquad \frac{dy}{du} = 2u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = 2u \times 3\cos 3x$$

$$\frac{dy}{dx} = 6\sin 3x\,.\,\cos 3x$$

$$\frac{dy}{dx} = 3\sin 6x \qquad \text{[double angle formula]}$$

9

$$y = \sin^5 x \cos^3 x$$

$$u = \sin^5 x \qquad v = \cos^3 x$$

Use chain rule on u

$$\text{If } z = \sin x$$

$$u = z^5$$

$$\frac{du}{dz} = 5z^4 \qquad \frac{dz}{dx} = \cos x$$

$$\frac{du}{dx} = \frac{du}{dz} \times \frac{dz}{dx}$$

$$\frac{du}{dx} = 5z^4 \times \cos x$$

$$\frac{du}{dx} = 5\sin^4 x \cos x$$

Use chain rule on v

$$\text{If } w = \cos x$$

$$v = w^3$$

$$\frac{dv}{dw} = 3w^2 \qquad \frac{dw}{dx} = -\sin x$$

$$\frac{dv}{dx} = \frac{dv}{dw} \times \frac{dw}{dx}$$

$$\frac{dv}{dx} = 3w^2 \times (-\sin x)$$

$$\frac{dv}{dx} = -3\cos^2 x \sin x$$

Use product rule:

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = \cos^3 x \times 5\sin^4 x \times \cos x - \sin^5 x \times 3\cos^2 x \times \sin x$$

$$\frac{dy}{dx} = 5\cos^4 x \times \sin^4 x - 3\sin^6 x \times \cos^2 x$$

$$\frac{dy}{dx} = \sin^4 x \cos^2 x\,(5\cos^2 x - 3\sin^2 x)$$

10

$y = \ln \sqrt{\sin x}$ [chain rule]

$= \ln (\sin x)^{\frac{1}{2}}$

$= \frac{1}{2} \ln (\sin x)$ [log laws]

Let $u = \sin x$

$\therefore \quad y = \frac{1}{2} \ln u$

$\frac{dy}{du} = \frac{1}{2} \times \frac{1}{u} \qquad \frac{du}{dx} = \cos x$

$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{u} \times \cos x = \frac{\cos x}{2 \sin x}$

11

$y = 4x^6 \sin x$ [product rule]

Let $u = 4x^6 \qquad v = \sin x$

$\therefore \frac{du}{dx} = 24x^5 \qquad \frac{dv}{dx} = \cos x$

$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$\frac{dy}{dx} = \sin x \times 24x^5 + 4x^6 \times \cos x$

$\frac{dy}{dx} = 4x^5 (6 \sin x + x \cos x)$

12

$y = \tan^3 x - 3 \tan x$ [chain rule]

$= (\tan x)^3 - 3 \tan x$

Let $u = \tan x \qquad \therefore \frac{du}{dx} = \sec^2 x$

$y = u^3 - 3u$

$\frac{dy}{du} = 3u^2 - 3$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = (3u^2 - 3) \times \sec^2 x$

$\frac{dy}{dx} = (3\tan^2 x - 3) \times \sec^2 x$

$\frac{dy}{dx} = 3\sec^2 x (\tan^2 x - 1)$

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13

$$y = \frac{\cos 3x}{e^{3x}}$$ [quotient rule]

Let $u = \cos 3x \qquad v = e^{3x}$

$$\therefore \frac{du}{dx} = -3\sin 3x \qquad \frac{dv}{dx} = 3e^{3x}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{e^{3x}(-3\sin 3x) - \cos 3x \times 3e^{3x}}{(e^{3x})^2}$$

$$= \frac{-3\sin 3x - \cos 3x}{(e^{3x})}$$

$$= -\frac{3(\sin 3x + \cos 3x)}{e^{3x}}$$

14

$y = \operatorname{cosec} 3x$ [quick method - diff out - diff in]

$y = \operatorname{cosec}(3x)$

$$\frac{dy}{dx} = -\operatorname{cosec}(3x)\cot(3x) \times 3$$ [differentiate outside bracket - differentiate inside bracket]

$$\frac{dy}{dx} = -3\operatorname{cosec} 3x \cot 3x$$

15

$y = \cot^2 3x$ [quick method - diff out - diff in]

$y = (\cot(3x))^2$

$$\frac{dy}{dx} = 2(\cot(3x))^1 \times (-\operatorname{cosec}^2 3x) \times 3$$

[differentiate outside bracket - differentiate inside bracket – used twice]

$$\frac{dy}{dx} = -6\cot 3x \operatorname{cosec}^2 3x$$

16 Find the smallest value of θ for which the curve $y = 2\theta - 3\sin\theta$ has a gradient of 0.5

$y = 2\theta - 3\sin\theta$

$$\frac{dy}{d\theta} = 2 - 3\cos\theta$$

When $\frac{dy}{d\theta} = 0{\cdot}5 \quad \Rightarrow \quad 2 - 3\cos\theta = 0{\cdot}5$

$\therefore \cos\theta = 0{\cdot}5$

$\therefore$ smallest +ve value of $\theta = \frac{\pi}{3}$

Note: the answer is given in radians, differentiation and integration are valid only if angles are measured in radians.

17

$$y = \frac{\sin^4 3x}{6x}$$ [quotient rule & chain rule]

$$y = \frac{(\sin(3x))^4}{6x}$$

Let $z = \sin(3x)$ $\quad \frac{dz}{dx} = 3\cos(3x)$

$u = (z)^4$ $\quad v = 6x$

$\therefore$ $\frac{du}{dz} = 4z^3$ $\quad \frac{dv}{dx} = 6$

$$\frac{dz}{dx} = 3\cos(3x)$$

$$\frac{du}{dx} = \frac{dz}{dx} \times \frac{du}{dz} = 3\cos(3x) \times 4z^3 = 12\cos(3x)\sin^3(3x)$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{6x[12\cos(3x)\sin^3(3x)] - \sin^4(3x) \times 6}{(6x)^2}$$

$$\frac{dy}{dx} = \frac{6x[4(\sin(3x))^3 \times 3\cos(3x)] - \sin^4(3x) \times 6}{(6x)^2}$$

[or differentiate outside bracket - differentiate inside bracket]

$$= \frac{72x\sin^3(3x)\cos(3x) - 6\sin^4(3x)}{36x^2}$$

$$= \frac{\sin^3(3x)[12\cos(3x) - \sin(3x)]}{6x^2}$$

18

$$y = \sin^2 x\cos 3x$$

Need product rule and chain rule:

$$y = (\sin x)^2 \times \cos(3x)$$

$$\frac{dy}{dx} = \sin^2 x \times -3\sin(3x) + \cos 3x \times 2(\sin x)\cos x$$

$$\frac{dy}{dx} = \sin x[2\cos 3x\cos x - 3\sin x\sin 3x]$$

56.3 Differentiation of Log Functions

These can be used to find the integrals by reversing the process.

1

$$y = \ln|\sin x| \qquad \text{[chain rule]}$$

Let $u = \sin x \quad \therefore \frac{du}{dx} = \cos x$

$y = \ln|u| \quad \therefore \frac{dy}{du} = \frac{1}{u}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{u} \times \cos x$$

OR

If $y = \ln|f(x)| \Rightarrow \frac{dy}{dx} = \frac{1}{f(x)} \times f'(x)$

$$\frac{dy}{dx} = \frac{1}{\sin x} \times \frac{d}{dx}(\sin x)$$

$$\frac{dy}{dx} = \frac{1}{\sin x} \times \cos x$$

$$\frac{dy}{dx} = \cot x$$

2

$$y = \ln|\sec x| \qquad \text{[chain rule]}$$

$$\frac{dy}{dx} = \frac{1}{\sec x} \times \frac{d}{dx}(\sec x)$$

$$\frac{dy}{dx} = \frac{1}{\sec x} \times \sec x \tan x$$

$$\frac{dy}{dx} = \tan x$$

3

$$y = \ln|\sec x + \tan x| \qquad \text{[chain rule]}$$

$$\frac{dy}{dx} = \frac{1}{\sec x + \tan x} \times \frac{d}{dx}(\sec x + \tan x)$$

$$\frac{dy}{dx} = \frac{1}{\sec x + \tan x} \times \sec x \tan x + \sec^2 x$$

$$\frac{dy}{dx} = \frac{1}{\sec x + \tan x} \times \sec x\,(\tan x + \sec x)$$

$$\frac{dy}{dx} = \sec x$$

4 $y = -\ln\left| \operatorname{cosec} x + \cot x \right|$ [chain rule]

$$\frac{dy}{dx} = -\frac{1}{\operatorname{cosec} x + \cot x} \times \frac{d}{dx}(\operatorname{cosec} x + \cot x)$$

$$\frac{dy}{dx} = -\frac{1}{\operatorname{cosec} x + \cot x} \times (-\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x$$

$$\frac{dy}{dx} = -\frac{1}{\operatorname{cosec} x + \cot x} \times (-\operatorname{cosec} x\ (\cot x + \operatorname{cosec} x)$$

$$\frac{dy}{dx} = -\frac{-\operatorname{cosec} x\ (\cot x + \operatorname{cosec} x)}{\operatorname{cosec} x + \cot x}$$

$$\frac{dy}{dx} = \operatorname{cosec} x$$

57 • C4 • Integrating Trig Functions

57.1 Intro

Integrating trig functions is mainly a matter of recognising the standard derivative and reversing it to find the standard integral. You need a very good working knowledge of the trig identities and be able to use the chain rule. Although this chapter has been divided up into a number of smaller sections to aid recognition of the different types of integral, most of the methods used are similar to each other.

57.2 Integrals of *sin x, cos x and sec² x*

From the standard derivative of the basic trig functions, the integral can be found by reversing the process. Thus:

$$\frac{d}{dx}(\sin x) = \cos x \quad \Rightarrow \quad \int \cos x \, dx = \sin x + c$$

$$\frac{d}{dx}(\cos x) = -\sin x \quad \Rightarrow \quad \int \sin x \, dx = -\cos x + c$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \Rightarrow \quad \int \sec^2 x \, dx = \tan x + c$$

Only valid for x in radians

57.3 Using Reverse Differentiation:

In a similar manner the following can be found:

Function $y = f(x)$	Integral $\int f(x)\,dx$	
$\sin x$	$-\cos x + c$	* *
$\cos x$	$\sin x + c$	* *
$\sin kx$	$-\frac{1}{k}\cos kx +$	* *
$\cos kx$	$\frac{1}{k}\sin kx +$	* *
$\sec^2 kx$	$\frac{1}{k}\tan kx + c$	*
$\sec x \tan x$	$\sec x + c$	
$\operatorname{cosec} x \cot x$	$-\operatorname{cosec} x + c$	
$\operatorname{cosec}^2 x$	$-\cot x + c$	
$\cot x$	$\ln\lvert \sin x \rvert$	

Valid for *x* in radians

E.g.

$$\int \operatorname{cosec} 2x \cot 2x \, dx = -\frac{1}{2}\operatorname{cosec} 2x + c$$

57.3.2 Example:

1

$$\int tan\,(2x - \pi)\; dx$$

$$= \frac{1}{2} ln \left| sec\;(2x - \pi) \right| + c$$

2

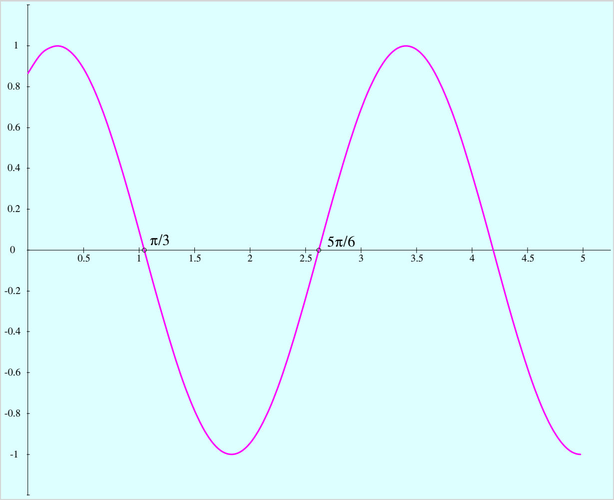
$$\int tan^2 x\; dx$$

use $\quad 1 + tan^2 x = sec^2 x \quad$ Avoids use of *ln* in the answer …

$$\therefore \quad \int (sec^2 x - 1)\; dx = tan\, x - x + c$$

3

Find the area under the curve $y = sin\,(2x + \frac{1}{3}\pi)$ from $x = 0$ to the first point at which the graph cuts the positive x-axis.



To find limits of function:

Axis is cut when: $sin\left(2x + \frac{1}{3}\pi\right) = 0$

$$\therefore \quad 2x + \frac{1}{3}\pi = 0,\; \pi,\; etc$$

$$2x = -\frac{\pi}{3},\; \frac{2\pi}{3},\; etc$$

$\therefore \quad x = -\frac{\pi}{6}$ (which can be ignored as it is outside the range required)

or $\quad x = \frac{2\pi}{6} = \frac{\pi}{3}$

$$\int_0^{\frac{\pi}{3}} sin\left(2x + \frac{\pi}{3}\right) dx = \left[-\frac{1}{2} cos\left(2x + \frac{\pi}{3}\right)\right]_0^{\frac{\pi}{3}}$$

$$= \left[-\frac{1}{2} cos\left(\frac{2\pi}{3} + \frac{\pi}{3}\right)\right] - \left[-\frac{1}{2} cos\left(\frac{\pi}{3}\right)\right] \qquad [cos\,\pi = -1]$$

$$= \left(\frac{1}{2}\right) - \left(-\frac{1}{4}\right) = \frac{3}{4} \qquad [cos\,\pi/3 = \tfrac{1}{2}]$$

57.4 Integrals of *tan x* and *cot x*

To find the integrals recognise the standard integral type:

$$\int \frac{k f'(x)}{f(x)}\, dx = k\, ln|\, f(x)\,| + c$$

Derive $\int tan\, x$

$$tan\, x = \frac{sin\, x}{cos\, x}$$

$$\begin{aligned}
\int tan\, x &= \int \frac{sin\, x}{cos\, x}\, dx \\
&= -\int \frac{-sin\, x}{cos\, x}\, dx \\
&= -ln|\, cos\, x\,| + c \\
&= ln|(cos\, x)^{-1}| + c \\
&= ln\left|\frac{1}{cos\, x}\right| + c \\
&= ln|\, sec\, x\,| + c
\end{aligned}$$

$$\int tan\, x = -ln|\, cos\, x\,| + c = ln|\, sec\, x\,| + c$$

$$\int tan\, x\, dx = -ln|\, cos\, x\,| + c = ln|\, sec\, x\,| + c$$

For the general case:

$$\int tan\, ax\, dx = \frac{1}{a} ln|\, sec\, ax\,| + c$$

This is often asked for in the exam.

Similarly it can be shown that:

$$\begin{aligned}
\int cot\, x &= \int \frac{cos\, x}{sin\, x}\, dx \\
&= ln|\, sin\, x\,| + c
\end{aligned}$$

$$\int cot\, ax\, dx = \frac{1}{a} ln\, |\, sin\, x\,| + c$$

NB the modulus sign means you can’t take the natural log of a negative number.

57.5 Recognising the Opposite of the Chain Rule

Reversing the derivatives (found using the chain rule), the following can be derived:

$$\frac{d}{dx} \sin(ax+b) = a\cos(ax+b) \quad \Rightarrow \quad \int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$$

$$\frac{d}{dx} \cos(ax+b) = -a\sin(ax+b) \quad \Rightarrow \quad \int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$$

$$\frac{d}{dx} \tan(ax+b) = a\sec^2(ax+b) \quad \Rightarrow \quad \int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c$$

57.5.1 Example:

1 Show that $\int_0^{\frac{\pi}{4}} \sec^2\left(2x - \frac{\pi}{4}\right) = 1$

Solution:

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \sec^2\left(2x - \frac{\pi}{4}\right) &= \left[\frac{1}{2}\tan\left(2x - \frac{\pi}{4}\right)\right]_0^{\frac{\pi}{4}} \\
&= \frac{1}{2}\tan\left(2\frac{\pi}{4} - \frac{\pi}{4}\right) - \frac{1}{2}\tan\left(0 - \frac{\pi}{4}\right) \\
&= \frac{1}{2}\left[\tan\left(\frac{\pi}{4}\right) - \tan\left(-\frac{\pi}{4}\right)\right] \\
&= \frac{1}{2}[1 + 1] = 1
\end{aligned}$$

2

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57.6 Integrating with Trig Identities

This covers many of the sub topics in this chapter. You really, really need to know these, the most useful of which are:

Pythag:

$$\cos^2 A + \sin^2 A \equiv 1$$

$$1 + \tan^2 A \equiv \sec^2 A$$

Double angle

$$\cos 2A \equiv 2\cos^2 A - 1 \qquad \therefore \qquad \cos^2 A \equiv \tfrac{1}{2}(1 + \cos 2A)$$

$$\cos 2A \equiv 1 - 2\sin^2 A \qquad \therefore \qquad \sin^2 A \equiv \tfrac{1}{2}(1 - \cos 2A)$$

$$\sin 2A \equiv 2\sin A\cos A$$

Addition or compound angle formulae

$$\sin(A + B) \equiv \sin A\cos B + \cos A\sin B$$

$$\sin(A - B) \equiv \sin A\cos B - \cos A\sin B$$

$$\cos(A + B) \equiv \cos A\cos B - \sin A\sin B$$

$$\cos(A - B) \equiv \cos A\cos B + \sin A\sin B \qquad * * * *$$

From the Addition or compound angle formulae

$$2\sin A\cos B \equiv \sin(A - B) + \sin(A + B)$$

$$2\cos A\cos B \equiv \cos(A - B) + \cos(A + B)$$

$$2\sin A\sin B \equiv \cos(A - B) - \cos(A + B)$$

$$2\sin A\cos A \equiv \sin 2A$$

$$\therefore \qquad \sin A\cos B \equiv \tfrac{1}{2}\big(\sin(A - B) + \sin(A + B)\big)$$

$$\therefore \qquad \cos A\cos B \equiv \tfrac{1}{2}\big(\cos(A - B) + \cos(A + B)\big)$$

$$\therefore \qquad \sin A\sin B \equiv \tfrac{1}{2}\big(\cos(A - B) - \cos(A + B)\big)$$

$$\therefore \qquad \sin A\cos A \equiv \tfrac{1}{2}\sin 2A$$

Factor formulae

$$\sin A + \sin B = 2\sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$$

$$\sin A - \sin B = 2\cos\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right) \qquad * * * *$$

57.6.1 Example:

1

$$\int cos^2 3x\,dx$$

$$cos^2 A \equiv \tfrac{1}{2}(1 + cos\,2A) \qquad \text{[Double angle]}$$

$$\int cos^2 3x\,dx = \tfrac{1}{2}\int (1 + cos\,6x)\,dx$$

$$= \tfrac{1}{2}(x + \tfrac{1}{6}sin\,6x) + c$$

2

$$\int sin\,3x\,cos\,3x\,dx$$

$$2\,sin\,A\,cos\,B \equiv sin\,(A - B) + sin\,(A + B) \qquad \text{[Compound angle]}$$

$$\int sin\,3x\,cos\,3x\,dx = \tfrac{1}{2}\int sin\,(3x - 3x) + sin\,(3x + 3x)\,dx$$

$$= \tfrac{1}{2}\int sin\,(6x)\,dx$$

$$= \frac{1}{2}\left(-\frac{1}{6}cos\,6x\right) + c$$

$$= -\frac{1}{12}cos\,6x + c$$

3

$$\int_0^{4\pi} sin^2\left(\tfrac{1}{2}x\right)\,dx$$

$$\int_0^{4\pi} sin^2\left(\tfrac{1}{2}x\right)\,dx = \tfrac{1}{2}\int_0^{4\pi}(1 - cos\,\tfrac{2x}{2})\,dx \qquad \text{[Double angle]}$$

$$= \tfrac{1}{2}\left[x - sin\,x\right]_0^{4\pi}$$

$$= \tfrac{1}{2}\left[(4\pi - sin\,4\pi) - (0 - sin\,0)\right]$$

$$= \tfrac{1}{2}(4\pi - 0)$$

$$= 2\pi$$

4

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57.7 Integrals of Type: *cos A cos B, sin A cos B & sin A sin B*

This type of problem covers the most common questions. Use the addition (compound angle) trig identities.

57.7.1 Example:

1 Integrate $\sin 3x \cos 4x$

Use formula: $2 \sin A \cos B \equiv \sin(A - B) + \sin(A + B)$

Let: $A = 3x$, $B = 4x$

$$\therefore \quad 2 \sin 3x \cos 4x = \sin(3x - 4x) + \sin(3x + 4x)$$

$$= \sin(-x) + \sin 7x$$

$$\therefore \quad \sin 3x \cos 4x = \tfrac{1}{2}\left(\sin(-x) + \sin 7x\right)$$

$$\int \sin 3x \cos 4x \, dx = \int \tfrac{1}{2}\left(\sin(-x) + \sin 7x\right) dx$$

$$= \tfrac{1}{2}\left(\cos x - \tfrac{1}{7}\cos 7x\right) + c$$

$$= \frac{1}{2}\cos x - \frac{1}{14}\cos 7x + c$$

2 Integrate $\sin 4x \cos 4x$

Use formula: $\sin A \cos A \equiv \tfrac{1}{2}\left(\sin(2A)\right)$

Let: $A = 4x$

$$\int \sin 4x \cos 4x = \tfrac{1}{2}\int \sin 8x \, dx$$

$$= \frac{1}{2}\left[-\frac{1}{8}\cos 8x\right] + c = -\frac{1}{16}\cos 8x$$

57.8 Integrating EVEN powers of $\sin x$ & $\cos x$

For this we need to adapt the double angle cosine identities:

$$\cos 2A \equiv 2\cos^2 A - 1$$

$$\cos 2A \equiv 1 - 2\sin^2 A$$

$$\therefore \quad \cos^2 A \equiv \tfrac{1}{2}(1 + \cos 2A)$$

$$\therefore \quad \sin^2 A \equiv \tfrac{1}{2}(1 - \cos 2A)$$

This technique can be used for any even power of $\sin x$ or $\cos x$, and also $\sin^2(ax + b)$ etc.

57.8.1 Example:

1

Find: $\int \sin^2 x \, dx$

Recognise: $\sin^2 A = \tfrac{1}{2}(1 - \cos 2A)$

$$\therefore \quad \int \sin^2 x \, dx = \tfrac{1}{2}\int (1 - \cos 2x)\, dx$$

$$= \tfrac{1}{2}\left(x - \tfrac{1}{2}\sin 2x\right) + c$$

2

Find: $\int \cos^2 x \, dx$

Recognise: $\cos^2 A = \tfrac{1}{2}(1 + \cos 2A)$

$$\int \cos^2 x \, dx = \tfrac{1}{2}\int (1 + \cos 2x)\, dx$$

$$= \tfrac{1}{2}\left(x + \tfrac{1}{2}\sin 2x\right) + c$$

3

Find: $\int_0^{\frac{\pi}{4}} \sin^2 2x \, dx$

now $\sin^2 A = \tfrac{1}{2}(1 - \cos 2A)$ $\quad$ Let $A = 2x$, $\quad \therefore \sin^2 2x = \tfrac{1}{2}(1 - \cos 4x)$

$$\therefore \quad \int_0^{\frac{\pi}{4}} \sin^2 2x \, dx = \tfrac{1}{2}\int_0^{\frac{\pi}{4}} (1 - \cos 4x)\, dx$$

$$= \frac{1}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{4} - \sin\frac{4\pi}{4}\right) - (0 - 0)\right]$$

$$= \frac{\pi}{8}$$

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4

Find: $\int cos^4 x\, dx$

$$\int cos^4 x\, dx = \int cos^2 x\, cos^2 x\, dx$$

$$= \int \frac{1}{2}(1 + cos\,2x) \times \frac{1}{2}(1 + cos\,2x)\ dx$$

$$= \frac{1}{4}\int (1 + cos\,2x)(1 + cos\,2x)\ dx$$

$$= \frac{1}{4}\int (1 + 2cos\,2x + cos^2\,2x)\ dx$$

$$= \frac{1}{4}\int 1 + 2cos\,2x + \frac{1}{2}\left(1 + cos\,4x\right) dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} + 2cos\,2x + \frac{1}{2}cos\,4x\right)\ dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x + sin\,2x + \frac{1}{8}sin\,4x\right] + c$$

$$= \frac{3}{8}x + \frac{1}{4}sin\,2x + \frac{1}{32}sin\,4x + c$$

5

Find: $\int sin^2(2x + 3)\, dx$

Recognise: $sin^2 A = \frac{1}{2}(1 - cos\,2A)$

$$\int sin^2(2x + 3)\, dx = \tfrac{1}{2}\int (1 - cos\,2(2x + 3))\, dx$$

$$= \tfrac{1}{2}\int (1 - cos\,(4x + 6))\, dx$$

Recall: $\int cos\,(ax + b)\, dx = \frac{1}{a}sin\,(ax + b) + c$

$$\int sin^2(2x + 3)\, dx = \tfrac{1}{2}\left[x - \frac{1}{4}cos\,(4x + 6)\right] + c$$

57.9 Integrals of Type: $\cos^n A \sin A$, $\sin^n A \cos A$

Another example of applying the reverse of the differentiation and the chain rule:

From the chain rule, the derivative required is

$$\frac{d}{dx}(\sin^n x) = n \sin^{n-1} x \cos x$$

In reverse

$$\int \sin^n x \cos x \, dx = \frac{1}{n+1} \sin^{n+1} + c$$

Similarly:

$$\int \cos^n x \sin x \, dx = -\frac{1}{n+1} \cos^{n+1} + c$$

57.9.1 Example:

1

$$\int \sin^4 x \cos x \, dx = \frac{1}{5} \sin^5 x + c$$

2

$$\int \cos^7 x \sin x \, dx = -\frac{1}{8} \cos^8 x + c$$

3 Three ways of integrating $\sin x \cos x$:

$$\int \sin x \cos x \, dx = \frac{1}{2} \sin^2 x + c \qquad \text{See } \sin^n x \cos x \text{ above}$$

$$= -\frac{1}{2} \cos^2 x + c \qquad \text{See } \cos^n x \sin x \text{ above}$$

$$= \int \frac{1}{2} \sin 2x \, dx \qquad \text{See addition formula}$$

$$= -\frac{1}{4} \cos 2x + c$$

57.10 Integrating ODD powers of $\sin x$ & $\cos x$

This technique is entirely different - change all but one of the *sin*/*cos* functions to the opposite by using the pythag identity:

$$\cos^2 x + \sin^2 x = 1$$

Hence:

$$\sin^2 x = 1 - \cos^2 x$$
$$\cos^2 x = 1 - \sin^2 x$$

57.10.1 Example:

1

Find: $\int \sin^3 x\, dx$

$$\int \sin^3 x\, dx = \int \sin x \sin^2 x\ dx$$

$$\int \sin x \sin^2 x\ dx = \int \sin x\ (1 - \cos^2 x)\, dx$$

$$= \int (\sin x - \cos^2 x \sin x)\ dx$$

Recognise standard type $\int \cos^n x \sin x\, dx$ [from previous section]

$$= -\cos x + \frac{1}{3}\cos^3 x + c$$

2

Find: $\int \sin^5 x\, dx$

$$\int (\sin x \sin^2 x \sin^2 x)\ dx = \int \sin x (1 - \cos^2 x)(1 - \cos^2 x)\ dx$$

$$= \int \sin x\ (1 - 2\cos^2 x + \cos^4 x)\, dx$$

$$= \int (\sin x - 2\cos^2 x \sin x + \cos^4 x \sin x)\ dx$$

Recognise standard type $\int \cos^n x \sin x\ dx$

$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c$$

57.11 Integrals of Type: *sec x, cosec x & cot x*

From the standard derivative of these functions, the integral can be found by reversing the process. Thus:

$$\frac{d}{dx}(sec\,x) = sec\,x\,tan\,x \qquad \Rightarrow \qquad \int sec\,x\,tan\,x\,dx = sec\,x + c$$

$$\frac{d}{dx}(cosec\,x) = -cosec\,x\,cot\,x \qquad \Rightarrow \qquad \int cosec\,x\,cot\,x\,dx = -cosec\,x + c$$

$$\frac{d}{dx}(cot\,x) = -cosec^2\,x \qquad \Rightarrow \qquad \int cosec^2\,x\,dx = -cot\,x + c$$

57.11.1 Example:

1

$$\int \frac{cos\,3x}{sin^2\,3x}$$

Rewrite integral as: $\displaystyle\int \frac{cos\,3x}{sin^2\,3x} = \int \frac{1}{sin\,3x} \times \frac{cos\,3x}{sin\,3x}\,dx$

$$= \int cosec\,3x\,cot\,3x\,dx$$

$$= -\frac{1}{3}cosec\,3x + c$$

2

$$\int cot^2\,x\,dx$$

Recognise identity: $1 + cot^2\,x = cosec^2\,x$

$$\int cot^2\,x\,dx = \int (cosec^2\,x - 1)\,dx$$

$$= -cot\,x - x + c$$

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57.12 Integrals of Type: $sec^n x \tan x$, $\tan^n x \sec^2 x$

From the standard derivative of these functions, the integral can be found by reversing the process. Thus:

$$\frac{d}{dx}(\sec x) = \sec x \tan x \quad \Rightarrow \quad \int \sec x \tan x \, dx = \sec x + c$$

and
$$\frac{d}{dx}(\sec^n x) = n \sec^{n-1} x (\sec^n \tan x)$$
$$= n \sec^n x \tan x$$

Reversing the derivative gives $\Rightarrow \int \sec^n x \tan x \, dx = \frac{1}{n} \sec^n x + c$

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \Rightarrow \quad \int \sec^2 x \, dx = \tan x + c$$

and
$$\frac{d}{dx}(\tan^{n+1} x) = (n+1) \tan^n x \sec^2 x$$

Reversing the derivative gives $\Rightarrow \int \tan^n x \sec^2 x \, dx = \frac{1}{n+1} \tan^{n+1} x + c$

57.12.1 Example:

1 Find: $\int \tan^2 x \sec^2 x \, dx$

$$\int \tan^2 x \sec^2 x \, dx = \frac{1}{3} \tan^3 x + c$$

2 Find: $\int \tan^2 x \, dx$

$$\int \tan^2 x \, dx = \int (\sec^2 - 1) \, dx$$
$$= \tan x - x + c$$

3 Find: $\int \tan^3 x \, dx$

$$\int \tan^3 x \, dx = \int \tan x \tan^2 x \, dx$$
$$= \int \tan x \, (\sec^2 x - 1) \, dx$$
$$= \int (\tan x \sec^2 x - \tan x) \, dx$$
$$= \tfrac{1}{2} \tan^2 x + \ln(\cos x) + c$$

4 Alternatively

$$\int \tan^3 x \, dx = \int (\tan x \sec^2 x - \tan x) \, dx$$
$$= \tfrac{1}{2} \sec^2 x + \ln(\cos x) + c$$

57.13 Standard Trig Integrals (radians only)

$$\frac{d}{dx}(\sin x) = \cos x \quad \Rightarrow \quad \int \cos x\,dx = \sin x + c$$

$$\frac{d}{dx}(\cos x) = -\sin x \quad \Rightarrow \quad \int \sin x\,dx = -\cos x + c$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \Rightarrow \quad \int \sec^2 x\,dx = \tan x + c$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \quad \Rightarrow \quad \int \sec x \tan x\,dx = \sec x + c$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x \quad \Rightarrow \quad \int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \quad \Rightarrow \quad \int \operatorname{cosec}^2 x\,dx = -\cot x + c$$

$$\frac{d}{dx}(\sin(ax+b)) = \cos(ax+b) \quad \Rightarrow \quad \int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$$

$$\frac{d}{dx}(\cos(ax+b)) = -\sin(ax+b) \quad \Rightarrow \quad \int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$$

$$\frac{d}{dx}(\tan(ax+b)) = \sec^2(ax+b) \quad \Rightarrow \quad \int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c$$

$$\frac{d}{dx}(\sin f(x)) = f'(x)\cos f(x) \quad \Rightarrow \quad \int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\frac{d}{dx}(\cos f(x)) = -f'(x)\sin f(x) \quad \Rightarrow \quad \int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\frac{d}{dx}(\tan f(x)) = f'(x)\sec^2 f(x) \quad \Rightarrow \quad \int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int \tan x\,dx = -\ln|\cos x| + c = \ln|\sec x| + c$$

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + c$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + c$$

$$\int \operatorname{cosec} x\,dx = -\ln|\operatorname{cosec} x + \cot x| + c$$

$$\int \sin^n x \cos x\,dx = \frac{1}{n+1}\sin^{n+1} x + c$$

$$\int \cos^n x \sin x\,dx = -\frac{1}{n+1}\cos^{n+1} x + c$$

$$\int \sec^n x \tan x\,dx = \frac{1}{n}\sec^n x + c$$

$$\int \tan^n x \sec^2 x\,dx = \frac{1}{n+1}\tan^{n+1} x + c$$

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58 • C4 • Integration by Inspection

OCR C4 / AQA C3

58.1 Intro to Integration by Inspection

This covers two forms of integration which involve a function combined with its differential, either as a product or a quotient. These include:

- Integrals of the form $\int \frac{k\,f'(x)}{f(x)}dx$
- Integrals of the form $\int k\,f'(x)\,[\,f(x)]^n\,dx$
- Integrals of the form $\int k\,f'(x)\,e^{f(x)}\,dx$

Integration of these types is often called ‘integration by inspection’ or ‘integration by recognition’, because once proficient in using this method, you should be able to just write down the answer by ‘inspecting’ the function.

It is derived from reversing the ‘function of a function’ rule for differentiation, i.e. the chain rule.

The key to using this method is recognising that one part of the integrand is the differential (or scalar multiple) of the other part.

There are several methods of integrating fractions and products, depending of the form of the original function, and recognition of this form will save a good deal of calculations. A common alternative to this method is ‘integration by substitution’.

58.2 Method of Integration by Inspection

The basic method for any of these types is the same:

- Guess — at a suitable integral by inspecting the function
- Test — your guess by differentiating
- Reverse — if $\frac{d}{dx}(guess) = z$, then $\int z\,dx = (guess) + c$, since differentiation & integration are inverse processes
- Adapt — compare your $\int z\,dx$ with original question and adapt the answer accordingly.
  Note that any adjustment must be a number only, not a function of x.
  (This step not required if $f'(x)$ is the exact differential of $f(x)$)

58.3 Integration by Inspection — Quotients

Integrals of the form $\int \frac{k\,f'(x)}{f(x)}dx$ are basically fractions with a function in the denominator and a multiple of its differential in the numerator, assuming the function is rational.

E.g.

$$\int \frac{4x}{x^2 + 1}dx \Rightarrow \frac{2 \times \text{differential of the denominator}}{\text{a function with a differential of } 2x}$$

$$\int \frac{4\sin x}{\cos x + 1}dx \Rightarrow \frac{-4 \times \text{differential of the denominator}}{\text{a function with a differential of } -\sin x}$$

From C3 work, using the chain rule, recall that:

If $y = \ln x$ then $\frac{dy}{dx} = \frac{1}{x}$

and if $y = \ln f(x)$ then $\frac{dy}{dx} = \frac{1}{f(x)} \times f'(x)$

Reversing the differential by integrating we get:

$$\int \frac{k\,f'(x)}{f(x)}dx \;\Rightarrow\; k\,ln\left|\,f(x)\,\right| + c$$

Note that the modulus sign indicates that you cannot take the natural log of a negative number.

Following our method, our first guess should, therefore, be: $(guess) \;=\; ln\,\left|denominator\right|$.

Note that the numerator has to be an exact derivative of the denominator and not just a derivative of a function inside the denominator.

E.g.

$$\int \frac{x}{\sqrt{x+2}}\,dx \;\neq\; ln\left|\sqrt{x+2}\right| + c$$

In this case use substitution to evaluate the integral.

Recall the following standard integrals and differential:

$$\int \frac{1}{x}\,dx \;=\; ln\left|x\right| + c$$

$$\int \frac{1}{ax+b}\,dx \;=\; \frac{1}{a}\,ln\left|ax+b\right| + c$$

$$\int \frac{k}{ax+b}\,dx \;=\; \frac{k}{a}\,ln\left|ax+b\right| + c$$

$$\frac{d}{dx}\left(\left[f(x)\right]^{n}\right) \;=\; n\,f'(x)\left[f(x)\right]^{n-1} \qquad \text{[chain rule]}$$

58.3.3 Example:

1

$$\int \frac{x^2}{1+x^3}dx$$

Guess: $ln\,\left|1+x^3\right|$

Test: $\frac{d}{dx}\left[ln\,\left|1+x^3\right|\right] \;=\; \frac{1}{1+x^3} \times 3x^2 \;=\; \frac{3x^2}{1+x^3}$

Reverse: $\int \frac{3x^2}{1+x^3}dx \;=\; ln\,\left|1+x^3\right| + c$

Adapt: $\int \frac{x^2}{1+x^3}dx \;=\; \frac{1}{3}ln\,\left|1+x^3\right| + c$

Note: Adjustment has to be a number only.

2

$$\int \frac{2e^x}{e^x+4}dx$$

Guess: $ln\,\left|e^x+4\right|$

Test: $\frac{d}{dx}\left[ln\,\left|e^x+4\right|\right] \;=\; \frac{1}{e^x+4} \times e^x \;=\; \frac{e^x}{e^x+4}$

Reverse: $\int \frac{e^x}{e^x+4}dx \;=\; ln\,\left|e^x+4\right| + c$

Adapt: $\int \frac{2e^x}{e^x+4}dx \;=\; 2\,ln\,\left|e^x+4\right| + c$

$$= \; ln\,\left(e^x+4\right)^2 + c \qquad \text{Squared term is +ve}$$

3 $\displaystyle\int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx$

Guess: $\ln|\sin x + \cos x|$

Test: $\dfrac{d}{dx}\left[\ln|\sin x + \cos x|\right] = \dfrac{1}{\sin x + \cos x} \times (\cos x - \sin x) = \dfrac{\cos x - \sin x}{\sin x + \cos x}$

Reverse: $\displaystyle\int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx = \ln|\sin x + \cos x| + c$

Adapt: Not required because the numerator is the exact differential of the denominator.

4 $\displaystyle\int \frac{2x}{x^2 + 9}\,dx$

Of the form $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$

$$\therefore \quad \int \frac{2x}{x^2 + 9}\,dx = \ln|x^2 + 9| + c$$

$$= \ln(x^2 + 9) + c$$

Note: for all real values of x, $(x^2 + 9) > 0$, hence modulus sign not required.

5 $\displaystyle\int \tan x\,dx$ Often comes up in the exam!

Think $\tan x = \dfrac{\sin x}{\cos x}$ and $\dfrac{d}{dx}(\cos x) = -\sin x$

Guess: $\ln|\cos x|$

Test: $\dfrac{d}{dx}\left[\ln|\cos x|\right] = \dfrac{1}{\cos x} \times (-\sin x) = \dfrac{-\sin x}{\cos x}$

Reverse: $\displaystyle\int \frac{-\sin x}{\cos x}\,dx = \ln|\cos x| + c$

Adapt: $\displaystyle\int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| + c$

$$\therefore \quad \int \tan x\,dx = -\ln|\cos x| + c$$

$$= \ln|\cos x|^{-1} + c$$

$$= \ln\left|\frac{1}{\cos x}\right| + c$$

$$= \ln|\sec x| + c$$

6 $\displaystyle\int \cot 2x\,dx$

Think $\cot 2x = \dfrac{1}{\tan 2x} = \dfrac{\cos 2x}{\sin 2x}$ and $\dfrac{d}{dx}(\sin 2x) = 2\cos 2x$

Guess: $\ln|\sin 2x|$

Test: $\dfrac{d}{dx}\left[\ln|\sin 2x|\right] = \dfrac{1}{\sin 2x} \times (2\cos 2x) = \dfrac{2\cos 2x}{\sin 2x}$

Reverse: $\displaystyle\int \frac{2\cos 2x}{\sin 2x}\,dx = \ln|\sin 2x| + c$

Adapt: $\displaystyle\int \frac{\cos 2x}{\sin 2x}\,dx = \frac{1}{2}\ln|\sin 2x| + c$

$$\int \cot 2x\,dx = \frac{1}{2}\ln|\sin 2x| + c$$

7

$$\int \frac{x^3}{x^4+9}\,dx$$

Guess: $\quad ln\left|x^4+9\right|$

Test: $\quad \frac{d}{dx} ln\left|x^4+9\right| = \frac{1}{x^4+9} \times 4x^3 = \frac{4x^3}{x^4+9}$

Reverse: $\quad \int \frac{4x^3}{x^4+9}\,dx = ln\left|x^4+9\right| + c$

Adapt: $\quad \int \frac{x^3}{x^4+9}\,dx = \frac{1}{4} ln\left|x^4+9\right| + c$

$$= \frac{1}{4} ln\left(x^4+9\right) + c \qquad x \text{ term is +ve}$$

58.4 Integration by Inspection — Products

Integrals of the form $\int k\,f'(x)\,(f(x))^n\,dx$ and $\int k\,f'(x)\,e^{f(x)}\,dx$ involves a function raised to a power or e raised to the power of the function, multiplied by a multiple of the differential of *f(x)*. Note that many of these examples can also be solved by other methods like substitution.

E.g.

$$\int x\left(x^2+1\right)^2 dx \qquad f(x) = x^2+1 \quad \Rightarrow \quad f'(x) = 2x$$

$$\int x^2\left(3x^3+1\right)^4 dx \qquad f(x) = 3x^3+1 \quad \Rightarrow \quad f'(x) = 9x^2$$

$$\int x\,e^{x^2} \qquad f(x) = x^2 \quad \Rightarrow \quad f'(x) = 2x$$

$$\int 3x^4\,e^{x^5+6} dx \qquad f(x) = x^5+6 \quad \Rightarrow \quad f'(x) = 5x^4$$

Some quotients have to be treated as a product:

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \int x\left(x^2+1\right)^{-\frac{1}{2}} dx \quad : f(x) = x^2+1 \quad \Rightarrow \quad f'(x) = 2x$$

From earlier work with the chain rule, recall that:

If $y = \left[f(x)\right]^n$ then $\frac{dy}{dx} = n f'(x)\left[f(x)\right]^{n-1}$

If $y = e^{f(x)}$ then $\frac{dy}{dx} = f'(x)\,e^{f(x)}$

Reversing the differentials by integrating we get:

$$\int f'(x)\left[f(x)\right]^n dx \Rightarrow \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

$$\int f'(x)\,e^{f(x)}\,dx \Rightarrow e^{f(x)} + c$$

58.4.2 Example:

1 $\int x\,(x^2+1)^2\,dx$

Guess: $(x^2+1)^{2+1} \Rightarrow (x^2+1)^3$

Test: $\dfrac{d}{dx}\left[(x^2+1)^3\right] = 3(x^2+1)^2 \times 2x = 6x(x^2+1)^2$

Reverse: $\int 6x(x^2+1)^2\,dx = (x^2+1)^3 + c$

Adapt: $\int x\,(x^2+1)^2\,dx = \dfrac{1}{6}(x^2+1)^3 + c$

2 $\int \cos x \sin^3 x\,dx \Rightarrow \int \cos x\,(\sin x)^3\,dx$

Guess: $(\sin x)^4$

Test: $\dfrac{d}{dx}\left[(\sin x)^4\right] = 4(\sin x)^3 \times \cos x = 4\cos x\,(\sin x)^3$

Reverse: $\int 4\cos x\,(\sin x)^3\,dx = (\sin x)^4 + c$

Adapt: $\int \cos x \sin^3 x\,dx = \dfrac{1}{4}\sin^4 x + c$

3 $\int x^2\sqrt{(x^3+5)}\,dx \Rightarrow \int x^2(x^3+5)^{\frac{1}{2}}\,dx$

Guess: $(x^3+5)^{\frac{3}{2}}$

Test: $\dfrac{d}{dx}\left[(x^3+5)^{\frac{3}{2}}\right] = \dfrac{3}{2}(x^3+5)^{\frac{1}{2}} \times 3x^2 = \dfrac{9}{2}x^2(x^3+5)^{\frac{1}{2}}$

Reverse: $\int \dfrac{9}{2}x^2(x^3+5)^{\frac{1}{2}}\,dx = (x^3+5)^{\frac{3}{2}} + c$

Adapt: $\int x^2(x^3+5)^{\frac{1}{2}}\,dx = \dfrac{2}{9}(x^3+5)^{\frac{3}{2}} + c$

4 $\int x\,e^{x^2}dx$

Guess: e^{x^2}

Test: $\dfrac{d}{dx}\left[e^{x^2}\right] = e^{x^2} \times 2x = 2x\,e^{x^2}$

Reverse: $\int 2x\,e^{x^2}dx = e^{x^2} + c$

Adapt: $\int x\,e^{x^2}dx = \dfrac{1}{2}e^{x^2} + c$

5 $\int \cos x\,e^{\sin x}\,dx$

Guess: $e^{\sin x}$

Test: $\dfrac{d}{dx}\left[e^{\sin x}\right] = e^{\sin x} \times \cos x = \cos x\,e^{\sin x}$

Reverse: $\int \cos x\,e^{\sin x} = e^{\sin x} + c$

Adapt: not required

6 $\displaystyle\int \frac{x}{(3x^2 - 4)^5}\,dx \Rightarrow \int x\left(3x^2 - 4\right)^{-5} dx$

Guess: $\left(3x^2 - 4\right)^{-4}$

Test: $\dfrac{d}{dx}\left[\left(3x^2 - 4\right)^{-4}\right] = -4 \times 6x\left(3x^2 - 4\right)^{-5} = -24x\left(3x^2 - 4\right)^{-5}$

Reverse: $\displaystyle\int -24x\left(3x^2 - 4\right)^{-5} dx = \left(3x^2 - 4\right)^{-4} + c$

Adapt: $\displaystyle\int x\left(3x^2 - 4\right)^{-5} dx = -\frac{1}{24}\left(3x^2 - 4\right)^{-4} + c$

7 After a while it becomes easier to write the answer down, but always check the possible answer by differentiating.

$\displaystyle\int e^x\left(6e^x - 5\right)^2 dx$

Note: $f(x) = 6e^x - 5 \quad \Rightarrow \quad f'(x) = 6e^x$

Adapt: $\displaystyle\int e^x\left(6e^x - 5\right)^2 dx = \frac{1}{6}\int 6e^x\left(6e^x - 5\right)^2 dx$

Inspect: $\displaystyle\frac{1}{6}\int 6e^x\left(6e^x - 5\right)^2 dx = \frac{1}{6} \times \frac{1}{3}\left(6e^x - 5\right)^3$

$\displaystyle = \frac{1}{18}\left(6e^x - 5\right)^3$

Test: $\dfrac{d}{dx}\left[\dfrac{1}{18}\left(6e^x - 5\right)^3\right] = \dfrac{1}{18} \times 6e^x \times 3\left(6e^x - 5\right)^2 = e^x\left(6e^x - 5\right)^2$

58.5 Integration by Inspection Digest

$$\frac{d}{dx}\left[\ln f(x)\right] = \frac{1}{f(x)} \times (f'(x))$$

$$\int \frac{k f'(x)}{f(x)} dx = k \ln\left|f(x)\right| + c$$

$$\int f'(x) \cos f(x)\; dx = \sin f(x) + c$$

$$\int \sin^n x \cos x\, dx = \frac{1}{n+1}\sin^{n+1} + c$$

$$\frac{d}{dx}\left[(f(x))^n\right] = n f'(x)\left[f(x)\right]^{n-1}$$

$$\int f'(x)\left[f(x)\right]^n dx \Rightarrow \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

$$\frac{d}{dx}\left[e^{f(x)}\right] = f'(x)\, e^{f(x)}$$

$$\int e^x\, dx = e^x + c$$

$$\int f'(x)\, e^{f(x)} dx \Rightarrow e^{f(x)} + c$$

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59 • C4 • Integration by Parts

OCR C4 / AQA C3

This is the equivalent of the product rule for integration. It is usually used when the product we want to integrate is not of the form $f'(x)(f(x))^n$ and so cannot be integrated with this standard method, or by recognition or by substitution.

Integrating by Parts is particularly useful for integrating the product of two types of function, such as a polynomial with a trig, exponential or log function, (e.g. $x\ sinx$, $x^2 e^x$, $ln\,x$).

59.1 Rearranging the Product rule:

The rule for Integrating by Parts comes from integrating the product rule.

Product rule: $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating w.r.t x to get: $$\int \frac{d}{dx}(uv)\ dx = \int u\frac{dv}{dx}\ dx + \int v\frac{du}{dx}\ dx$$

$$uv = \int u\frac{dv}{dx}\ dx + \int v\frac{du}{dx}\ d$$

Rearranging: $$\int u\frac{dv}{dx}\ dx = uv - \int v\frac{du}{dx}\ dx$$

$$\boxed{\int u\,\frac{dv}{dx}\ dx = uv - \int v\,\frac{du}{dx}\ dx}$$

59.2 Choice of u & dv/dx

Care must be taken over the choice of u *and* dv/dx.

The aim is to ensure that it is simpler to integrate $v\frac{du}{dx}$ than the original $u\frac{dv}{dx}$. So we choose u to be easy to differentiate and when differentiated to become simpler. Choose dv to be easy to integrate.

Normally, u is assigned to any polynomial in x, and if any exponential function is involved, assign this to $\frac{dv}{dx}$. However, if $ln\,x$ is involved make this u, as it is easier to differentiate the *ln* function than to integrate it.

59.3 Method

- Let u = the bit of the product which will differentiate to a constant, even if it takes 2 or 3 turns, such as polynomials in x, (e.g. x^3 differentiates to $3x^2 \rightarrow 6x \rightarrow 6$)
- If this is not possible or there is a difficult part to integrate let this be u. e.g. $ln\,x$.
- Differentiate to find $\frac{du}{dx}$.
- Let the other part of the product be $\frac{dv}{dx}$, like e^{ax} which is easy to integrate.
- Integrate to find v.
- Substitute into the rule and finish off.
- Add the constant of integration at the end.
- Sometimes integrating by parts needs to be applied more than once (see special examples). Do not confuse the use of u in the second round of integration.
- This is the method used to integrate $ln\,x$.

59.4 Evaluating the Definite Integral by Parts

Use this for substituting the limits:

$$\int_a^b u\frac{dv}{dx}\,dx = \Big[uv\Big]_a^b - \int_a^b v\frac{du}{dx}dx$$

59.5 Handling the Constant of Integration

The method listed above suggests adding the constant of integration at the end of the calculation. Why is this?

The best way to explain this is to show an example of adding a constant after each integration, and you can see that the first one cancels out during the calculation.

Example :

Find: $\int x \sin x \, dx$

Solution:

Let: $u = x$ & $\dfrac{dv}{dx} = \sin x$

$\dfrac{du}{dx} = 1$ $\qquad v = \displaystyle\int \frac{dv}{dx} = -\cos x + k$

where k is the constant from the first integration and c is the constant from the second integrations.

Recall: $\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

$$\int x \cos x \, dx = x(-\cos x + k) - \int(-\cos x + k) \times 1 \, dx$$

$$= -x\cos x + kx + \int \cos x \, dx - \int k \, dx$$

$$= -x\cos x + kx + \sin x - kx + c$$

$$= -x\cos x + \sin x + c$$

59.6 Integration by Parts: Worked examples

59.6.1 Example:

1

Find: $\int x \cos x \, dx$

Solution:

Let: $u = x$ & $\frac{dv}{dx} = \cos x$ Note: $u = x$ becomes simpler when differentiated.

$$\frac{du}{dx} = 1 \qquad v = \int \frac{dv}{dx} = \int \cos x = \sin x$$

$$\int x \cos x \, dx = x \sin x - \int \sin x \times 1 \, dx$$

$$= x \sin x + \cos x + c$$

Alternative (longer) Solution:

Let: $u = \cos x$ & $\frac{dv}{dx} = x$

$$\frac{du}{dx} = -\sin x \qquad v = \frac{x^2}{2}$$

$$\int x \cos x \, dx = \sin x . \frac{x^2}{2} - \int \frac{x^2}{2} (-\sin x) \, dx$$

$$= \frac{x^2}{2} \sin x + \int \frac{x^2}{2} \sin x \, dx \qquad etc\ etc$$

As you can see, this gives a more involved solution, that has to have another round of integration by parts. This emphasises the importance of choosing u wisely. In this case it would be prudent to start again with $u = x$.

2

Find: $\int x \sec^2 x \, dx$

Solution:

Let: $u = x$ & $\frac{dv}{dx} = \sec^2 x$

$$\frac{du}{dx} = 1 \qquad v = \int \frac{dv}{dx} = \tan x$$ Standard tables

$$\int x \sec^2 x \, dx = x \tan x - \int \tan x \times 1 \, dx$$

$$= x \sin x + \ln(\cos x) + c$$ Standard tables

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3

Find: $\int (4x + 2)\ \sin 4x\, dx$

Solution:

Let: $u = 4x + 2 \qquad \& \qquad \frac{dv}{dx} = \sin 4x$

$\frac{du}{dx} = 4 \qquad\qquad v = -\frac{1}{4}\cos 4x$

$$\int (4x + 2)\ \sin 4x\, dx = (4x + 2)(-\frac{1}{4}\cos 4x) - \int -\frac{1}{4}\cos 4x \,.\, 4\, dx$$

$$= -\frac{1}{4}(4x + 2)\ \cos 4x + \int \cos 4x\ dx$$

$$= -\frac{1}{4}(4x + 2)\ \cos 4x + \frac{1}{4}\sin 4x + c$$

4

Find: $\int x^2 \sin x\, dx$

Solution:

Let: $u = x^2 \qquad \& \qquad \frac{dv}{dx} = \sin x$

$\frac{du}{dx} = 2x \qquad\qquad v = \int \frac{dv}{dx} = -\cos x$

$$\int x^2 \sin x\, dx = x^2(-\cos x) - \int -\cos x \,.\, 2x\, dx$$

$$= -x^2 \cos x + \int 2x \cos x\ dx$$

Now integrate by parts again and then one final integration to give…

Now let $u = 2x \qquad \& \qquad \frac{dv}{dx} = \cos x$

$\frac{du}{dx} = 2 \qquad\qquad v = \sin x$

$$\int x^2 \sin x\, dx = -x^2 \cos x + \left[2x \sin x - \int \sin x \times 2\ dx\right]$$

$$= -x^2 \cos x + 2x \sin x - \int 2 \sin x\, dx$$

$$= -x^2 \cos x + 2x \sin x - 2\ (-\cos x) + c$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + c$$

$$= 2 \cos x - x^2 \cos x + 2x \sin x + c$$

$$= (2 - x^2)\cos x + 2x \sin x + c$$

Note: Integrating any function of the form $x^n \sin x$ or $x^n \cos x$, will require n rounds of integration by parts.

5 Find: $\int_0^{\pi} x^2 \cos x \, dx$

Solution:

Let: $u = x^2$ & $\frac{dv}{dx} = \cos x$

$\frac{du}{dx} = 2x$ $v = \int \frac{dv}{dx} = \sin x$

$$\int_0^{\pi} x^2 \cos x \, dx = \left[x^2 \sin x\right]_0^{\pi} - \int_0^{\pi} \sin x \, . \, 2x \, dx$$

$$= [0 - 0] - \int_0^{\pi} 2x \sin x \, dx$$

Now integrate by parts again, and then one final integration to give…

Now let: $u = 2x$ & $\frac{dv}{dx} = \sin x$

$\frac{du}{dx} = 2$ $v = -\cos x$

$$\int_0^{\pi} x^2 \cos x \, dx = 0 - \left\{\left[2x\,(-\cos x)\right]_0^{\pi} - \int_0^{\pi} -\cos x \, . \, 2 \, dx\right\}$$

$$= 0 - \left\{\left[-2x \cos x\right]_0^{\pi} + \int_0^{\pi} 2 \cos x \, dx\right\}$$

$$= -\left\{[2\pi - 0] + \int_0^{\pi} 2 \cos x \, dx\right\}$$

$$= -2\pi - \int_0^{\pi} 2 \cos x \, dx$$

$$= -2\pi - \left[2 \sin x\right]_0^{\pi}$$

$$= -2\pi - [0 - 0] = -2\pi$$

6 Find: $\int 2x \sin(3x - 1) \, dx$

Solution:

Let: $u = 2x$ & $\frac{dv}{dx} = \sin(3x - 1)$

$\frac{du}{dx} = 2$ $v = -\frac{1}{3}\cos(3x - 1)$

$$\int 2x \sin(3x - 1) \, dx = 2x\left(-\frac{1}{3}\cos(3x - 1)\right) - \int -\frac{1}{3}\cos(3x - 1) \, . \, 2 \, dx$$

$$= -\frac{2}{3} x \cos(3x - 1) + \frac{2}{3}\int \cos(3x - 1) \, dx$$

$$= -\frac{2}{3} x \cos(3x - 1) + \frac{2}{3} \times \frac{1}{3} \sin(3x - 1) + c$$

$$= -\frac{2}{3} x \cos(3x - 1) + \frac{2}{9} \sin(3x - 1) + c$$

$$= \frac{2}{9} \sin(3x - 1) - \frac{2}{3} x \cos(3x - 1) + c$$

7 Solve by parts $\int x(2x+3)^5\,dx$

This can be solved by inspection, but is included here for completeness.

Solution:

Let: $u = x \qquad \dfrac{dv}{dx} = (2x+3)^5$

$\dfrac{du}{dx} = 1$

Recall: $\displaystyle\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c$

$$\therefore v = \int (2x+3)^5\,dx = \frac{1}{2(6)}(2x+3)^6 + c$$

$$= \frac{1}{12}(2x+3)^6 + c$$

Recall: $\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

$$\int x(2x+3)^5\,dx = x.\frac{1}{12}(2x+3)^6 - \int \frac{1}{12}(2x+3)^6 \times 1\,dx$$

$$= \frac{x}{12}(2x+3)^6 - \frac{1}{12}\int (2x+3)^6\,dx$$

$$= \frac{x}{12}(2x+3)^6 - \frac{1}{12} \times \frac{1}{2\times 7}(2x+3)^7 + c$$

$$= \frac{x}{12}(2x+3)^6 - \frac{1}{12} \times \frac{1}{14}(2x+3)^7 + c$$

$$= \frac{1}{12}(2x+3)^6\left[x - \frac{1}{14}(2x+3)\right] + c$$

$$= \frac{1}{12}(2x+3)^6\left[\frac{14x}{14} - \frac{(2x+3)}{14}\right] + c$$

$$= \frac{1}{12}(2x+3)^6\left[\frac{14x - 2x - 3)}{14}\right] + c$$

$$= \frac{1}{12}(2x+3)^6\left[\frac{12x - 3}{14}\right] + c$$

$$= \frac{3}{12}(2x+3)^6\left[\frac{4x - 1}{14}\right] + c$$

$$= \frac{1}{56}(2x+3)^6[4x-1] + c$$

$$= \frac{1}{56}(2x+3)^6(4x-1) + c$$

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8 Find: $\int x\, e^{3x}\, dx$

Solution:

Let: $u = x$ & $\frac{dv}{dx} = e^{3x}$

$\frac{du}{dx} = 1$ $v = \frac{1}{3}e^{3x}$

$$\int x\, e^{3x}\, dx = x \,.\, \frac{1}{3}e^{3x} - \int \frac{1}{3}e^{3x} \times 1\, dx$$

$$\int x\, e^{3x}\, dx = \frac{1}{3}x\, e^{3x} - \frac{1}{9}e^{3x} + c$$

$$= \frac{1}{3}\, e^{3x}\left(x - \frac{1}{3}\right) + c$$

$$= \frac{1}{9}\, e^{3x}(3x - 1) + c$$

9 Find: $\int x^2\, e^{4x}\, dx$

Solution:

Let: $u = x^2$ & $\frac{dv}{dx} = e^{4x}$

$\frac{du}{dx} = 2x$ $v = \frac{1}{4}e^{4x}$

$$\int x^2\, e^{4x}\, dx = x^2 \,.\, \frac{1}{4}e^{4x} - \int \frac{1}{4}e^{4x} \,.\, 2x\, dx$$

$$= \frac{1}{4}x^2\, e^{4x} - \frac{1}{2}\int x\, e^{4x}\, dx$$

$$= \frac{1}{4}x^2\, e^{4x} - \frac{1}{2}\int u\, \frac{dv}{dx}\, dx$$

Now integrate by parts again and then one final integration to give…

Now let: $u = x$ & $\frac{dv}{dx} = e^{4x}$

$\frac{du}{dx} = 1$ $v = \frac{1}{4}e^{4x}$

$$\therefore \quad \int x\, e^{4x}\, dx = x \,.\, \frac{1}{4}e^{4x} - \int \frac{1}{4}e^{4x}\, dx$$

$$= \frac{1}{4}\, x\, e^{4x} - \frac{1}{16}e^{4x}$$

Substituting back into the original...

$$\therefore \quad \int x^2\, e^{4x}\, dx = \frac{1}{4}x^2\, e^{4x} - \frac{1}{2}\left(\frac{1}{4}\, x\, e^{4x} - \frac{1}{16}e^{4x}\right) + c$$

$$= \frac{1}{4}x^2\, e^{4x} - \frac{1}{8}\, x\, e^{4x} + \frac{1}{32}e^{4x} + c$$

$$= e^{4x}\left(\frac{1}{4}x^2 - \frac{1}{8}x + \frac{1}{32}\right) + c$$

$$\int x^2\, e^{4x}\, dx = \frac{1}{32}\, e^{4x}\left(8x^2 - 4x + 1\right) + c$$

10 Infinite integral example. Find: $\int_0^\infty x\,e^{-ax}\,dx$

Solution:

Let: $u = x \quad \& \quad \frac{dv}{dx} = e^{-ax}$

$\frac{du}{dx} = 1 \qquad v = -\frac{1}{a}e^{-ax}$

$$\int_0^\infty x\,e^{-ax}\,dx = \left[-\frac{x}{a}e^{-ax}\right]_0^\infty - \int_0^\infty 1 \times \left(-\frac{1}{a}e^{-ax}\right) dx$$

$$= \left[-\frac{x}{a}e^{-ax}\right]_0^\infty + \frac{1}{a}\int_0^\infty e^{-ax}\,dx$$

$$= \left[-\frac{x}{a}e^{-ax} + \frac{1}{a} \times \left(-\frac{1}{a}e^{-ax}\right)\right]_0^\infty$$

$$= \left[-\frac{x}{a}e^{-ax} - \frac{1}{a^2}e^{-ax}\right]_0^\infty$$

$$= \left[-\frac{x}{ae^{ax}} - \frac{1}{a^2e^{ax}}\right]_0^\infty$$

As $x \to \infty, \quad \frac{x}{ae^{ax}} \to 0 \quad$ and $\quad \frac{1}{a^2e^{ax}} \to 0$

$$\therefore \int_0^\infty x\,e^{-ax}\,dx = [0 - 0] - \left[0 - \frac{1}{a^2}\right]$$

$$= \frac{1}{a^2}$$

Alternatively, you can evaluate the bracketed part early, thus:

$$\int_0^\infty x\,e^{-ax}\,dx = \left[-\frac{x}{a}\,e^{-ax}\right]_0^\infty + \frac{1}{a}\int_0^\infty e^{-ax}\,dx$$

$$= \left[-\frac{x}{ae^{ax}}\right]_0^\infty + \frac{1}{a}\int_0^\infty e^{-ax}\,dx$$

$$= [0 - 0] + \frac{1}{a}\int_0^\infty e^{-ax}\,dx$$

$$\int_0^\infty x\,e^{-ax}\,dx = \frac{1}{a}\int_0^\infty e^{-ax}\,dx$$

$$= \left[-\frac{1}{a^2e^{ax}}\right]_0^\infty$$

$$= [0] - \left[-\frac{1}{a^2}\right]$$

$$\int_0^\infty x\,e^{-ax}\,dx = \frac{1}{a^2}$$

59.7 Integration by Parts: $ln\,x$

So far we have found no means of integrating $ln\,x$, but now, by regarding $ln\,x$ as the product $ln\,x \times 1$, we can now apply integration by parts. In this case make $u = ln\,x$ as $ln\,x$ is hard to integrate and we know how to differentiate it.

The 'trick' of multiplying by 1 can be used elsewhere, especially for integrating inverse trig functions.

59.7.1 Example:

1 **Integrating** $ln\,x$

$$\int ln\,x \times 1\,dx$$ Multiply by 1 to give a product to work with.

Let: $u = ln\,x$ & $\frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{x}$ $v = x$

$$\int ln\,x \times 1\,dx = x\,ln\,x - \int x\frac{1}{x}\,dx$$

$$= x\,ln\,x - \int dx$$

$$= x\,ln\,x - x + c$$

$$\int ln\,x = x\,(ln\,x - 1) + c$$

2 Find: $\int x^4\,ln\,x\,dx$

Solution:

Following the guidelines on choice of u & dv, then we would let $u = x$ and $\frac{dv}{dx} = ln\,x$

However, $ln\,x$ is difficult to integrate, so choose $u = ln\,x$

Let: $u = ln\,x$ & $\frac{dv}{dx} = x^4$

$\frac{du}{dx} = \frac{1}{x}$ $v = \frac{1}{5}x^5$

$$\int x^4\,ln\,x\,dx = ln\,x\,.\,\frac{1}{5}x^5 - \int \frac{1}{5}x^5 \cdot \frac{1}{x}\,dx$$

$$= \frac{1}{5}x^5\,ln\,x - \frac{1}{5}\int x^4\,dx$$

$$= \frac{1}{5}x^5\,ln\,x - \frac{1}{5} \times \frac{1}{5}x^5 + c$$

$$\int x^4\,ln\,x\,dx = \frac{1}{5}x^5\,ln\,x - \frac{1}{25}x^5 + c$$

$$= \frac{1}{5}x^5\left(ln\,x - \frac{1}{5}\right) + c$$

$$= \frac{1}{25}x^5\,(5\,ln\,x - 1) + c$$

3 Evaluate: $\int_2^8 x \ln x \, dx$

Solution:

As above, choose $u = \ln x$

Let: $u = \ln x$ & $\frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{x}$ $\quad v = \frac{x^2}{2}$

$$\int_2^8 x \ln x \, dx = \left[\frac{x^2}{2} \ln x\right]_2^8 - \int_2^8 \frac{x^2}{2} \cdot \frac{1}{x} \, dx$$

$$= \left[\frac{x^2}{2} \ln x\right]_2^8 - \int_2^8 \frac{x}{2} \, dx$$

$$= \left[\frac{x^2}{2} \ln x - \frac{x^2}{4}\right]_2^8$$

$$= (32 \ln 8 - 16) - (2 \ln 2 - 1)$$

$$= 32 \ln 8 - 2 \ln 2 - 15$$

$$= 32 \ln 2^3 - 2 \ln 2 - 15$$

$$= 96 \ln 2 - 2 \ln 2 - 15$$

$$\int_2^8 x \ln x \, dx = 94 \ln 2 - 15$$

4 Find: $\int \sqrt{x} \ln x \, dx$

Solution:

Let: $u = \ln x$ & $\frac{dv}{dx} = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{x}$ $\quad v = \int \sqrt{x} = \frac{2}{3} x^{\frac{3}{2}}$

$$\int \sqrt{x} \ln x \, dx = \ln x \, . \frac{2}{3} x^{\frac{3}{2}} - \int \frac{2}{3} x^{\frac{3}{2}} \cdot \frac{1}{x} \, dx$$

$$= \frac{2}{3} x^{\frac{3}{2}} \ln x - \frac{2}{3} \int x^{\frac{1}{2}} \, dx$$

$$= \frac{2}{3} x^{\frac{3}{2}} \ln x - \frac{2}{3} \times \frac{2}{3} x^{\frac{3}{2}} + c$$

$$= \frac{2}{9} \sqrt{x^3} \, (3 \ln x - 2) + c$$

59.8 Integration by Parts: Special Cases

These next examples use the integration by parts twice, which generates a term that is the same as the original integral. This term can then be moved to the LHS, to give the final result by division.

Generally used for integrals of the form $e^{ax} \sin bx$ or $e^{ax} \cos bx$. In this form, the choice of u & dv does not matter.

59.8.1 Example:

1 Find: $\displaystyle\int \frac{\ln x}{x}\, dx$

Solution:

Let: $u = \ln x$ & $\dfrac{dv}{dx} = \dfrac{1}{x}$

$\dfrac{du}{dx} = \dfrac{1}{x}$ $\quad v = \ln x$

$$\int \frac{\ln x}{x}\, dx = \ln x \,.\, \ln x - \int \ln x \,.\, \frac{1}{x}\, dx$$

$$= (\ln x)^2 - \int \frac{\ln x}{x}\, dx$$

$$2\int \frac{\ln x}{x}\, dx = (\ln x)^2$$

$$\therefore \int \frac{\ln x}{x}\, dx = \frac{1}{2}(\ln x)^2 + c$$

Note: $(\ln x)^2$ is not the same as $\ln x^2$

2 Find: $\displaystyle\int e^x \sin x\, dx$

Solution 1:

Let: $u = \sin x$ & $\dfrac{dv}{dx} = e^x$

$\dfrac{du}{dx} = \cos x$ $\quad v = e^x$

$$\int e^x \sin x\, dx = \sin x \,.\, e^x - \int e^x \cos x\, dx$$

Now integrate by parts again, which changes $\cos x$ to $\sin x$ to give…

$u = \cos x$ & $\dfrac{dv}{dx} = e^x$

$\dfrac{du}{dx} = -\sin x$ $\quad v = e^x$

$$\int e^x \sin x\, dx = e^x \sin x - \left[\cos x \,.\, e^x - \int e^x(-\sin x)\, dx\right]$$

$$= e^x \sin x - \left[e^x \cos x + \int e^x \sin x\, dx\right]$$

$$\int e^x \sin x\, dx = e^x \sin x - e^x \cos x - \int e^x \sin x\, dx$$

$$\therefore 2\int e^x \sin x\, dx = e^x(\sin x - \cos x) + c$$

$$\therefore \int e^x \sin x\, dx = \frac{1}{2}e^x(\sin x - \cos x) + c$$

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Solution 2:

Let: $u = e^x$ & $\frac{dv}{dx} = \sin x$

$\frac{du}{dx} = e^x$ $v = -\cos x$

$$\int e^x \sin x \, dx = e^x (-\cos x) - \int -\cos x \, . \, e^x \, dx$$

$$\int e^x \sin x \, dx = -e^x \cos x + \int \cos x \, . \, e^x \, dx$$

Now integrate by parts again to give…

$u = e^x$ & $\frac{dv}{dx} = \cos x$

$\frac{du}{dx} = e^x$ $v = \sin x$

$$\int e^x \sin x \, dx = -e^x \cos x + \left[e^x . \sin x - \int \sin x \, . \, e^x \, dx \right]$$

$$= -e^x \sin x + e^x \cos x - \int e^x \sin x \, dx$$

$$2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x + c$$

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + c$$

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3 Find: $\int e^x \cos x \, dx$

Solution:

Let: $u = \cos x$ & $\dfrac{dv}{dx} = e^x$

$\dfrac{du}{dx} = -\sin x$ $v = e^x$

$$\int e^x \cos x \, dx = \cos x \,.\, e^x - \int e^x \,(-\sin x)\, dx$$

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx$$

Now integrate by parts again and then one final integration to give…

$u = \sin x$ & $\dfrac{dv}{dx} = e^x$

$\therefore \dfrac{du}{dx} = \cos x$ $v = e^x$

$$\int e^x \cos x \, dx = e^x \cos x + \left[\sin x \,.\, e^x - \int e^x \cos x \, dx\right]$$

$$\int e^x \cos x \, dx = e^x (\cos x + \sin x) - \int e^x \cos x \, dx$$

$$2\int e^x \cos x \, dx = e^x (\cos x + \sin x) + c$$

$$\therefore \int e^x \cos x \, dx = \frac{1}{2} e^x (\cos x + \sin x) + c$$

4 Find: $\int e^{2x} \sin 4x \, dx$

Solution:

Let: $u = \sin 4x$ & $\frac{dv}{dx} = e^{2x}$

$\frac{du}{dx} = 4 \cos 4x$ $v = \frac{1}{2} e^{2x}$

$$\int e^{2x} \sin 4x \, dx = \sin 4x \,.\, \frac{1}{2} e^{2x} - \int \frac{1}{2} e^{2x} \,.\, 4 \cos 4x \, dx$$

$$\int e^{2x} \sin 4x \, dx = \frac{1}{2} e^{2x} \sin 4x - 2 \int e^{2x} \cos 4x \, dx$$

Now integrate by parts again and then one final integration to give…

$u = \cos 4x$ & $\frac{dv}{dx} = e^{2x}$

$\therefore$ $\frac{du}{dx} = -4 \sin 4x$ $v = \frac{1}{2} e^{2x}$

$$\int e^{2x} \sin 4x \, dx = \frac{1}{2} e^{2x} \sin 4x - 2\left[\cos 4x \,.\, \frac{1}{2} e^{2x} - \int \frac{1}{2} e^{2x} \,.\, (-4 \sin 4x) \, dx\right]$$

$$\int e^{2x} \sin 4x \, dx = \frac{1}{2} e^{2x} \sin 4x - 2\left[\frac{1}{2} e^{2x} \cos 4x + 2 \int e^{2x} \sin 4x \, dx\right]$$

$$\int e^{2x} \sin 4x \, dx = \frac{1}{2} e^{2x} \sin 4x - e^{2x} \cos 4x - 4 \int e^{2x} \sin 4x \, dx$$

$$5 \int e^{2x} \sin 4x \, dx = \frac{1}{2} e^{2x} \sin 4x - e^{2x} \cos 4x + c$$

$$= \frac{1}{2} e^{2x} \sin 4x - \frac{2}{2} e^{2x} \cos 4x + c$$

$$5 \int e^{2x} \sin 4x \, dx = \frac{1}{2} e^{2x} (\sin 4x - 2 \cos 4x) + c$$

$$\therefore \int e^{2x} \sin 4x \, dx = \frac{1}{10} e^{2x} (\sin 4x - 2 \cos 4x) + c$$

59.9 Integration by Parts Digest

$$\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx$$

$$\int_a^b u \frac{dv}{dx} \, dx = \left[uv\right]_a^b - \int_a^b v \frac{du}{dx} dx$$

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60 • C4 • Integration by Substitution

60.1 Intro to Integration by Substitution

Also known as integration by change of variable. This is the nearest to the chain rule that integration can get. It is used to perform integrations that cannot be done by other methods, and is also an alternative method to some other methods. It is worth checking if the integration can be done by inspection, which may be simpler.

Substitution is often used to define some standard integrals.

The object is to substitute some inner part of the function by a second variable u, and change all the instances of x to be in terms of u, including dx.

The basic argument for Integration by Substitution is:

$$\text{If} \qquad y = \int f(x)\ dx$$

$$\frac{dy}{dx} = f(x)$$

From the chain rule, if u is a function of x

$$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$$

$$\frac{dy}{du} = f(x)\frac{dx}{du}$$

$$\int \frac{dy}{du}du = \int f(x)\frac{dx}{du}\ du$$

$$y = \int f(x)\ \frac{dx}{du}\ du$$

$$\therefore \qquad \int f(x)\ dx = \int f(x)\ \frac{dx}{du}\ du$$

60.2 Substitution Method

- Used for integrating products and quotients,
- Let u = part of the expression, usually the messy part in brackets or the denominator of a fraction,
- If necessary, express any other parts of the function in terms of u,
- Differentiate u to find $\frac{du}{dx}$,
- Re-arrange $\frac{du}{dx}$ to find dx in terms of du as we need to replace dx if we are to integrate an expression w.r.t u, i.e. we need to find $dx = (z)\,du$,
- Substitute the expressions, found above, for x and dx, back into the original integral and integrate in terms of u. It should be reasonable to integrate, or allow the use of standard integrals,
- If the integration is a definite integral, change the x limits to limits based on u,
- Put your x's back in again at the end, and finish up,
- If the substitution is not obvious, then it should be given to you in the exam,
- There is often more that one substitution that could be chosen, practise makes perfect,
- All integrals that can be done by inspection, can also be done by substitution.

60.3 Required Knowledge

From C3 module recall:

$$\int (ax + b)^n = \frac{1}{a(n + 1)} (ax + b)^{n+1} + c$$

$$\int \frac{1}{ax + b} dx = \frac{1}{a} ln \left| ax + b \right| + c$$

$$\int e^{(ax+b)} dx = \frac{1}{a} e^{(ax+b)} + c$$

60.4 Substitution: Worked Examples

60.4.1 Example:

Examples 1 & 2 are based on the form of: $\int f(x)\ dx = \int f(x) \frac{dx}{du} du$

1 Use substitution to find: $\int (5x - 3)^3 dx$

Solution:

Let: $u = 5x - 3$

$$\frac{du}{dx} = 5 \qquad \Rightarrow \frac{dx}{du} = \frac{1}{5}$$

$$\int f(x)\ dx = \int f(x) \frac{dx}{du} du$$

Substituting:

$$\int (5x - 3)^3 dx = \int (u)^3 \frac{1}{5} du \Rightarrow \frac{1}{5} \int (u)^3\ du$$

$$= \frac{1}{5} \times \frac{1}{4} u^4 + c$$

$$= \frac{1}{20} (5x - 3)^4 + c$$

2 Use substitution to find: $\int \frac{1}{4x + 2} dx$

Solution:

Let: $u = 4x + 2$

$$\frac{du}{dx} = 4 \qquad \Rightarrow \frac{dx}{du} = \frac{1}{4}$$

Substituting

$$\int \frac{1}{4x + 2} dx = \int \frac{1}{u} \frac{1}{4} du = \frac{1}{4} \int \frac{1}{u} du$$

$$= \frac{1}{4} ln\, u + c$$

$$= \frac{1}{4} ln (4x + 2) + c$$

This is a standard result:

$$\int \frac{1}{ax + b} dx = \frac{1}{a} ln (ax + b) + c$$

The integration process can be streamlined somewhat if we find dx in terms of u and du, rather that find $\frac{dx}{du}$ specifically each time, as in the following examples.

3 Use substitution to find:

$$\int \frac{1}{x + \sqrt{x}}\, dx$$

Solution:

Let: $u = \sqrt{x}, \qquad \Rightarrow \qquad u = x^{\frac{1}{2}}$

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \qquad \Rightarrow \qquad \frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$du = \frac{1}{2\sqrt{x}}\, dx \qquad \Rightarrow \qquad dx = 2\sqrt{x}\, du$$

but we still have x involved, so substitute for x $\qquad \therefore\ dx = 2u\, du$

Substituting into the original:

$$\int \frac{1}{x + \sqrt{x}}\, dx = \int \frac{1}{u^2 + u} 2u\, du \Rightarrow \int \frac{2u}{u(u + 1)}\, du \Rightarrow \int \frac{2}{(u + 1)}\, du$$

$$= 2\, ln\left| u + 1 \right| + c$$

$$= 2\, ln\left| \sqrt{x} + 1 \right| + c$$

4 Use substitution to find:

$$\int 3x\sqrt{1 + x^2}\, dx$$

Solution:

Let: $u = x^2$

$$\frac{du}{dx} = 2x \qquad \Rightarrow \qquad dx = \frac{du}{2x}$$

Substituting:

$$\int 3x\sqrt{1 + x^2}\, dx = 3\int x(1 + u)^{\frac{1}{2}} \frac{du}{2x} \Rightarrow \frac{3}{2}\int (1 + u)^{\frac{1}{2}}\, du$$

$$= \frac{3}{2} \times \frac{2}{3}(1 + u)^{\frac{3}{2}} + c = (1 + u)^{\frac{3}{2}} + c$$

$$= \left(1 + x^2\right)^{\frac{3}{2}} + c$$

Alternative solution:

Let: $u = 1 + x^2 \qquad \Rightarrow \quad x^2 = u - 1 \qquad \Rightarrow \quad x = (u - 1)^{\frac{1}{2}}$

$$\frac{du}{dx} = 2x \qquad \Rightarrow \qquad dx = \frac{du}{2x} = \frac{du}{2(u - 1)^{\frac{1}{2}}}$$

Substituting:

$$\int 3x\sqrt{1 + x^2}\, dx = 3\int (u - 1)^{\frac{1}{2}}(u)^{\frac{1}{2}} \frac{du}{2(u - 1)^{\frac{1}{2}}} \Rightarrow \frac{3}{2}\int (u)^{\frac{1}{2}}\, du$$

$$= \frac{3}{2} \times \frac{2}{3}(u)^{\frac{3}{2}} + c = (u)^{\frac{3}{2}} + c$$

$$= \left(1 + x^2\right)^{\frac{3}{2}} + c$$

5 Use substitution to find: $\int 3x\left(1 + x^2\right)^5 dx$

Solution:

Let: $u = (1 + x^2)$ $\qquad \dfrac{du}{dx} = 2x \qquad \Rightarrow \qquad dx = \dfrac{du}{2x}$

Substituting:

$$\int 3x\left(1 + x^2\right)^5 dx = 3\int x\,(u)^5 \frac{du}{2x} \Rightarrow \frac{3}{2}\int (u)^5\, du$$

$$= \frac{3}{2} \times \frac{1}{6}u^6 + c = \frac{1}{4}u^6 + c$$

$$= ¼\left(1 + x^2\right)^6 + c$$

6 Use substitution to find: $\int \dfrac{6x}{\sqrt{2x + 1}}\, dx$

Solution:

Let: $u = (2x + 1)$ $\qquad \dfrac{du}{dx} = 2 \qquad \Rightarrow \qquad dx = \dfrac{du}{2}$

Substituting

$$\int \frac{6x}{\sqrt{2x + 1}}\, dx = \int \frac{6x}{\sqrt{u}} \frac{du}{2} \Rightarrow \int \frac{3x}{u^{\frac{1}{2}}}\, du$$

We have an x left, so go back to the substitution and find x

$$u = 2x + 1 \qquad \Rightarrow \qquad x = \frac{u - 1}{2}$$

$$\therefore \int \frac{3x}{u^{\frac{1}{2}}}\, du = 3\int x\, \frac{1}{u^{\frac{1}{2}}}\, du = 3\int \frac{u - 1}{2} \times \frac{1}{u^{\frac{1}{2}}} \cdot du$$

$$= \frac{3}{2}\int \frac{u - 1}{u^{\frac{1}{2}}}\, du = \frac{3}{2}\int \left[\frac{u}{u^{\frac{1}{2}}} - \frac{1}{u^{\frac{1}{2}}}\right] du$$

$$= \frac{3}{2}\int \left[u^{+\frac{1}{2}} - u^{-\frac{1}{2}}\right] du$$

$$= \frac{3}{2}\left[\frac{2u^{\frac{3}{2}}}{3} - 2u^{\frac{1}{2}}\right] + c$$

$$= u^{\frac{3}{2}} - 3u^{\frac{1}{2}} + c \qquad = u^{\frac{1}{2}}\left[u - 3\right] + c$$

$$= (2x + 1)^{\frac{1}{2}}(2x + 1 - 3) + c$$

$$= 2\,(x - 1)\,(2x + 1)^{\frac{1}{2}} + c$$

7 Find: $\qquad \int 2x\, e^{x^2} dx$

Solution:

Let: $u = x^2$ $\qquad \dfrac{du}{dx} = 2x \qquad \Rightarrow dx = \dfrac{du}{2x}$

$$\int 2x\, e^{x^2} dx = \int 2x\, e^u \frac{du}{2x}$$

$$= \int e^u\, du = e^u + c$$

$$= e^{x^2} + c$$

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8 Find:
$$\int \frac{e^x}{(1 - e^x)^2}\, dx$$

Note e^x is a derivative of $1 - e^x$ not $(1 - e^x)^2$, so use substitution.

Solution:

Let: $u = 1 - e^x$

$$\frac{du}{dx} = -e^x \qquad \Rightarrow \; dx = -\frac{du}{e^x}$$

Substituting

$$\int \frac{e^x}{(1 - e^x)^2}\, dx = -\int \frac{e^x}{(u)^2}\, \frac{du}{e^x} \;\Rightarrow\; -\int \frac{1}{u^2}\, du$$
$$= -\int u^{-2} du$$
$$= u^{-1} + c$$
$$= \frac{1}{1 - e^x} + c$$

9 Use substitution to find:

$$\int (x + 5)(3x - 1)^5\, dx$$

Solution:

Let: $u = 3x - 1$

$$\frac{du}{dx} = 3 \qquad \Rightarrow \qquad dx = \frac{du}{3}$$

Substituting

$$\int (x + 5)(3x - 1)^5\, dx = \int (x + 5)(u)^5 \cdot \frac{1}{3}\, du$$

We have an x left, so go back to the substitution and find x

$$u = 3x - 1 \qquad \Rightarrow \qquad x = \frac{u + 1}{3}$$

$$\int (x + 5)(3x - 1)^5\, dx = \frac{1}{3}\int \left(\frac{u + 1}{3} + 5\right) u^5\, du$$
$$= \frac{1}{3}\int \left(\frac{u + 1 + 15}{3}\right) u^5\, du$$
$$= \frac{1}{9}\int (u + 16)\, u^5\, du = \frac{1}{9}\int \left(u^6 + 16u^5\right) du$$
$$= \frac{1}{9}\left[\frac{u^7}{7} + \frac{16u^6}{6}\right] + c = \frac{1}{9}\left[\frac{u^7}{7} + \frac{8u^6}{3}\right] + c$$
$$= \frac{1}{9}\left[\frac{3u^7 + 56u^6}{21}\right] + c = \frac{1}{189}\left(3u^7 + 56u^6\right) + c$$
$$= \frac{u^6}{189}(3u + 56) + c$$
$$= \frac{(3x - 1)^6}{189}\left[3(3x - 1) + 56\right] + c$$
$$= \frac{(3x - 1)^6}{189}(9x + 53) + c$$

10 Use substitution to find:

$$\int \sqrt{4 - x^2}\, dx \qquad \left[\text{Has the form } \sqrt{a^2 - b^2x^2} \text{ use } x = a/b \sin u\right]$$

Solution:

Let: $x = 2 \sin u$

$\dfrac{dx}{du} = 2 \cos u \qquad \therefore \; dx = 2 \cos u\, du$

Substituting

$$\int \sqrt{4 - x^2}\, dx = \int \sqrt{4 - (2 \sin u)^2} \times 2 \cos u\, du$$

$$= \int \sqrt{4 - 4\sin^2 u} \times 2 \cos u\, du = \int \sqrt{4(1 - \sin^2 u)} \times 2 \cos u\, du$$

but $\cos^2 u = 1 - \sin^2 u$

$$= \int \sqrt{4 \cos^2 u} \times 2 \cos u\, du = \int 2 \cos u \times 2 \cos u\, du$$

$$= 4 \int \cos^2 u\, du$$

but: $\quad 2 \cos^2 u = 1 + \cos 2u$

$$= 4 \int \frac{1}{2} (1 + \cos 2u)\, du$$

$$= 2 \int (1 + \cos 2u)\, du$$

$$= 2\left[u + \frac{1}{2}\sin 2u\right] + c$$

$$= 2u + \sin 2u + c$$

Substituting back:

Given: $\quad x = 2\sin u$

Identity: $\quad \sin^2 u = 1 - \cos^2 u$

Identity: $\quad \sin 2u = 2 \sin u \cos u$

$\therefore \quad$ Need to find $\sin u$ & $\cos u$

$\therefore \quad \sin u = \dfrac{x}{2} \qquad \& \qquad \sin^2 u = \dfrac{x^2}{4}$

$\sin^{-1}\left(\dfrac{x}{2}\right) = u$

$\cos^2 u = 1 - \sin^2 u$

$\cos^2 u = 1 - \dfrac{x^2}{4} \qquad = \dfrac{4 - x^2}{4}$

$\cos u = \dfrac{1}{2}\sqrt{4 - x^2}$

Substituting: $\qquad = 2\sin^{-1}\left(\dfrac{x}{2}\right) + 2\left(\dfrac{x}{2}\right) \times \dfrac{1}{2}\sqrt{4 - x^2} + c$

$$\therefore \quad \int \sqrt{4 - x^2}\, dx = 2\sin^{-1}\left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)\sqrt{4 - x^2} + c$$

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60.5 Definite Integration using Substitutions

Because of the substitution, you must also change the limits into the new variable, so we can then evaluate the integral as soon as we have done the integration. This saves you having to put the x's back in at the end and using the original limits.

60.5.1 Example:

1 Use substitution to find:

$$\int_0^1 \sqrt{4 - x^2}\, dx \qquad \left[\text{Has form } \sqrt{a^2 - b^2x^2} \text{ : use } x = a/b \sin u\right]$$

Solution:

Let: $x = 2 \sin u$

$$\frac{dx}{du} = 2 \cos u \qquad \therefore dx = 2 \cos u\, du$$

Limits:

x	$2 \sin u$	$\sin u$	u
1	1	$\frac{1}{2}$	$\frac{\pi}{6}$
0	0	0	0

From previous example

$$\int \sqrt{4 - x^2}\, dx = 2u + \sin 2u + c$$

$$\int_0^1 \sqrt{4 - x^2}\, dx = \Big[2u + \sin 2u\Big]_0^{u = \frac{\pi}{6}} = \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) - 0$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2} = \frac{\pi + \sqrt{3}}{6}$$

2 Use substitution to find:

$$\int_0^1 \frac{1}{1 + x^2}\, dx$$

Solution:

Let: $x = \tan u$

$$\frac{dx}{du} = \sec^2 u \qquad \therefore dx = \sec^2 u\, du$$

Limits:

x	$\tan u$	u
1	1	0
0	0	$\frac{\pi}{4}$

Substituting:

$$\int_0^1 \frac{1}{1 + x^2}\, dx = \int_0^{x=1} \frac{1}{1 + \tan^2 u} \sec^2 u\, du$$

$$= \int_0^{x=1} 1\, du \qquad \text{Since: } 1 + \tan^2 u = \sec^2 u$$

$$= \Big[u\Big]_{u=0}^{u = \frac{\pi}{4}} = \frac{\pi}{4}$$

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3

$$\int_0^2 x(2x - 1)^6 \, dx$$

Solution:

Let: $u = 2x - 1 \qquad \Rightarrow x = \frac{1}{2}(u + 1)$

$\frac{du}{dx} = 2 \qquad \Rightarrow dx = \frac{1}{2}\, du$

Limits:

x	$u = 2x - 1$
2	3
1	−1

Substituting:

$$\int_0^2 x(2x - 1)^6 \, dx = \int_{u=-1}^{u=3} \tfrac{1}{2}(u + 1)(u^6)\tfrac{1}{2}\, du$$

$$= \frac{1}{4}\int_{-1}^{3} u^7 + u^6 \, du$$

$$= \frac{1}{4}\left[\frac{1}{8}u^8 + \frac{1}{7}u^7\right]_{-1}^{3}$$

$$= \frac{1}{4}\left[\frac{1}{8}u^8 + \frac{1}{7}u^7\right]_{-1}^{3}$$

$$= \frac{1}{4}\left[\frac{1}{8}3^8 + \frac{1}{7}3^7\right] - \frac{1}{4}\left[\frac{1}{8}(-1)^8 + \frac{1}{7}(-1)^7\right]$$

$$= \tfrac{1}{4}(1132{\cdot}57) = 283{\cdot}14$$

4

$$\int_{-1}^{2} x^2\sqrt{(x^3 + 1)}\, dx$$

Solution:

Let: $u = x^3 + 1 \qquad \Rightarrow x = \frac{1}{2}(u + 1)$

$\frac{du}{dx} = 3x^2 \qquad \Rightarrow dx = \frac{1}{3x^2}\, du$

Limits:

x	$u = x^3 + 1$
2	9
−1	0

Substituting:

$$\int_{-1}^{2} x^2\sqrt{(x^3 + 1)}\, dx = \int_{u=0}^{u=9} x^2 u^{\frac{1}{2}} \frac{1}{3x^2}\, du$$

$$= \frac{1}{3}\int_0^9 u^{\frac{1}{2}}\, du$$

$$= \frac{1}{3}\left[\frac{1}{\frac{3}{2}}u^{\frac{3}{2}}\right]_0^9 = \frac{1}{3}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^9$$

$$= \frac{1}{3}\left[\frac{2}{3}9^{\frac{3}{2}}\right] - 0 = \frac{2}{9}\left[9^{\frac{3}{2}}\right]$$

$$= 6$$

60.6 Reverse Substitution

This is where we have to recognise the substitution by ourselves, by recognising the reverse chain rule.

60.6.1 Example:

1 Use substitution to find: $\displaystyle\int \frac{1}{x}\ln x\,dx$

Solution:

Let: $x = e^u$

$$\frac{dx}{du} = e^u \qquad \therefore dx = e^u\,du$$

Substituting

$$\int \frac{1}{x}\ln x\,dx = \int \frac{1}{e^u}\ln e^u \times e^u\,du \Rightarrow \int \ln e^u\,du$$

but: $\quad u\ln e = u \qquad \therefore = \displaystyle\int u\,du = \frac{u^2}{2} + c$

$$x = e^u \qquad \therefore u = \ln x$$

$$\int \frac{1}{x}\ln x\,dx = \frac{(\ln x)^2}{2} + c$$

2 Find: $\displaystyle\int \frac{6x}{\sqrt{1 + x^2}}\,dx$

Solution:

Let: $u = 1 + x^2$

$$\frac{du}{dx} = 2x \qquad \Rightarrow \qquad dx = \frac{du}{2x}$$

$$\int \frac{6x}{\sqrt{1 + x^2}}\,dx = \int \frac{6x}{\sqrt{u}} \times \frac{du}{2x} = \int \frac{3}{\sqrt{u}}\,du$$

$$= \int 3u^{-\frac{1}{2}}\,du = \frac{3u^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 6u^{\frac{1}{2}} + c$$

$$= 6\left(1 + x^2\right)^{\frac{1}{2}} + c \qquad \Rightarrow 6\sqrt{1 + x^2} + c$$

3 Use reverse substitution to find: $\displaystyle\int x^2\sqrt{1 + x^3}\,dx$

Solution:

Let: $u = 1 + x^3$

$$\frac{du}{dx} = 3x^2 \qquad \Rightarrow \qquad dx = \frac{du}{3x^2}$$

$$\int x^2\sqrt{1 + x^3}\,dx = \int x^2\sqrt{u} \times \frac{du}{3x^2} = \frac{1}{3}\int u^{\frac{1}{2}}\,du$$

$$= \frac{1}{3}\left[\frac{2}{3}u^{\frac{3}{2}}\right] + c$$

$$= \frac{2}{9}\left[\left(1 + x^3\right)^{\frac{3}{2}}\right] + c$$

4 Consider:

$$\int \frac{7x}{(1 + 2x^2)^3}\, dx$$

Solution:

Let: $u = 1 + 2x^2$

$$\frac{du}{dx} = 4x \qquad \Rightarrow \qquad dx = \frac{du}{4x}$$

$$\int \frac{7x}{(1 + 2x^2)^3}\, dx = \int \frac{7x}{(u)^3} \cdot \frac{du}{4x} = \int \frac{7}{4u^3}\, du$$

$$= \int \frac{7}{4} u^{-3}\, du = \frac{7}{4} \int u^{-3}\, du$$

$$= \frac{7}{4}\left[\frac{1}{-2} u^{-2}\right] + c = -\frac{7}{8} u^{-2} + c \Rightarrow -\frac{7}{8u^2} + c$$

$$= -\frac{7}{8}\left(1 + 2x^2\right)^{-2} + c \Rightarrow -\frac{7}{8(1 + 2x^2)^2} + c$$

In the following two questions, note that we have a fraction, of which the top is the differential of the denominator, or a multiple thereof.

$$\boxed{\int \frac{f'(x)}{f(x)}\, dx = \ln\left| f(x) \right| + c}$$

5 Try:

$$\int \frac{\cos x - \sin x}{\sin x + \cos x}\, dx$$

Solution:

Let: $u = \sin x + \cos x$

$$\frac{du}{dx} = \cos x - \sin x \qquad \Rightarrow \qquad dx = \frac{du}{\cos x - \sin x}$$

$$\int \frac{\cos x - \sin x}{\sin x + \cos x}\, dx = \int \frac{\cos x - \sin x}{u} \times \frac{du}{\cos x - \sin x}$$

$$= \int \frac{1}{u}\, du$$

$$= \ln u + c$$

$$= \ln\left|\sin x + \cos x\right| + c$$

6 Try:

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$$

Solution:

Let: $u = e^x + e^{-x}$

$$\frac{du}{dx} = e^x - e^{-x} \qquad \Rightarrow \qquad dx = \frac{du}{e^x - e^{-x}}$$

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int \frac{e^x - e^{-x}}{u} \times \frac{du}{e^x - e^{-x}} = \int \frac{1}{u} du$$

$$= \ln u + c$$

$$= \ln\left(e^x + e^{-x}\right) + c$$

$\left(e^x + e^{-x}\right)$ is always +ve

7 Try:

$$\int \frac{\sec^2 x}{\tan^3 x}\,dx$$

Solution:

Note that $\sec^2 x$ is the derivative of $\tan x$ not $\tan^3 x$

Let: $u = \tan x$

$$\frac{du}{dx} = \sec^2 x \qquad \Rightarrow \qquad dx = \frac{du}{\sec^2 x}$$

$$\int \frac{\sec^2 x}{\tan^3 x}\,dx = \int \frac{\sec^2 x}{u^3} \times \frac{du}{\sec^2 x}$$

$$= \int \frac{1}{u^3} du$$

$$= \int u^{-3} du$$

$$= -\frac{1}{2} u^{-2} + c$$

$$= -\frac{1}{2\,u^2} + c$$

$$= -\frac{1}{2\tan^2 x} + c$$

The common trig functions that are of the form $\int \frac{f'(x)}{f(x)}\,dx$ are:

Function $y = f(x)$	Integral $\int f(x)\,dx$	
$\tan x$	$\ln\lvert\sec x\rvert + c$	*
$\cot x$	$\ln\lvert\sin x\rvert + c$	*
$\operatorname{cosec} x$	$-\ln\lvert\operatorname{cosec} x + \cot x\rvert + c$	*
$\sec x$	$\ln\lvert\sec x + \tan x\rvert + c$	*

60.7 Harder Integration by Substitution

If the integrand contains $a^2 + b^2x^2$ use $x = \frac{a}{b} \tan u$

If the integrand contains $a^2 - b^2x^2$ use $x = \frac{a}{b} \sin u$

N.B. integrand = the bit to be integrated.

60.7.1 Example:

1 $\displaystyle\int \frac{1}{25 + 16x^2}\, dx$

i.e. $a = 5,\ b = 4$

Let: $x = \frac{5}{4} \tan u \quad \Rightarrow \quad \frac{dx}{du} = \frac{5}{4} \sec^2 u$

$\therefore \quad dx = \frac{5}{4} \sec^2 u\, du$

$$\begin{aligned}
\int \frac{1}{25 + 16x^2}\, dx &= \int \frac{1}{25 + 16\left(\frac{5}{4} \tan u\right)^2} \times \frac{5}{4} \sec^2 u\, du \\
&= \int \frac{1}{25 + 25 \tan^2 u} \times \frac{5}{4} \sec^2 u\, du \\
&= \int \frac{1}{25\left(1 + \tan^2 u\right)} \times \frac{5}{4} \sec^2 u\, du \\
&= \int \frac{1}{25 \sec^2 u} \times \frac{5}{4} \sec^2 u\, du \\
&= \int \frac{1}{5} \times \frac{1}{4}\, du = \int \frac{1}{20}\, du \\
&= \frac{1}{20} u + c \qquad \text{Note: } \tan u = \frac{4x}{5} \\
&= \frac{1}{20} \tan^{-1}\left(\frac{4x}{5}\right) + c \qquad \text{Note: } u = \tan^{-1}\left(\frac{4x}{5}\right)
\end{aligned}$$

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2

$$\int_0^1 \sqrt{(1 - x^2)}\, dx$$

Let: $x = \sin u \quad \Rightarrow \quad \frac{dx}{du} = \cos u \quad \therefore dx = \cos u\, du$

Limits: $\Rightarrow \quad u = \sin^{-1} x$

x	u
1	$\sin^{-1} x = \frac{\pi}{2}$
0	$\sin^{-1} 0 = 0$

$$\int_0^1 \sqrt{(1 - x^2)}\, dx = \int_0^{u = \frac{\pi}{2}} \sqrt{(1 - \sin^2 u)} \times \cos u\, du$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{(\cos^2 u)} \times \cos u\, du$$

$$= \int_0^{\frac{\pi}{2}} \cos^2 u\, du$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2u)\, du$$

$$= \frac{1}{2}\left[u + \frac{1}{2}\sin 2u\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{1}{2}\sin \pi\right) - \left(0 + \frac{1}{2}\sin 0\right)\right]$$

$$= \frac{\pi}{4}$$

60.8 Options for Substitution

Substitution allows a wide range of functions to be integrated, but it is not always obvious which one should be used. The following table attempts to give some clues as to which to choose as the appropriate substitution.

For :	Try :
$(ax + b)^n$	$u = ax + b$
$\sqrt[n]{(ax + b)}$	$u^n = ax + b$
$a - bx^2$	$x = \sqrt{\frac{a}{b}} \sin u$
$a + bx^2$	$x = \sqrt{\frac{a}{b}} \tan u$
$bx^2 - a$	$x = \sqrt{\frac{a}{b}} \sec u$
e^x	$u = e^x \;:\; x = \ln u$
$\ln(ax + b)$	$ax + b = e^u \;:\; x = \frac{1}{a}e^u - \frac{b}{a}$

60.9 Some Generic Solutions

1 Use substitution to find: $\int x(ax + b)^n \, dx$

Solution:

Let: $u = ax + b \qquad \frac{du}{dx} = a \qquad \Rightarrow dx = \frac{du}{a}$

$ax = u - b \qquad x = \frac{u - b}{a}$

Substituting:

$$\int x(ax + b)^n \, dx = \int \frac{u - b}{a} (u)^n \frac{du}{a}$$

$$= \frac{1}{a^2} \int (u - b) u^n \, du$$

$$= \frac{1}{a^2} \int (u^{n+1} - bu^n) \, du$$

$$= \frac{1}{a^2} \left[\frac{u^{n+2}}{n + 2} - \frac{bu^{n+1}}{n + 1} \right] + c$$

$$= \frac{1}{a^2} \left[\frac{(ax + b)^{n+2}}{n + 2} - \frac{b(ax + b)^{n+1}}{n + 1} \right] + c$$

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61 • C4 • Partial Fractions

61.1 Intro to Partial Fractions

$$\text{If:} \qquad \frac{3}{2x+1} + \frac{2}{x-2} = \frac{7x-8}{(2x+1)(x-2)}$$

then we ought to be able to convert

$$\frac{7x-8}{(2x+1)(x-2)} \text{ back into its partial fractions of: } \frac{3}{2x+1} + \frac{2}{x-2}$$

The process is often called decomposition of a fraction. To do this, we create an identity that is valid for all values of x and then find the missing constants of the partial fractions.

To decompose a fraction we need to start with a proper fraction. Improper fractions (see later) have to be converted into a whole number part with a proper fraction remainder. Later on, partial fractions will be useful in integration, differentiation and the binomial theorem.

There are four different types of decomposition based on the sort of factors in the denominator. These are:

- Linear factors in the denominator:

$$\frac{x}{(ax+b)(cx+d)} \equiv \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$

- Squared terms in the denominator (includes quadratics that will not factorise easily)

$$\frac{x}{(ax+b)(cx^2+d)} \equiv \frac{A}{(ax+b)} + \frac{Bx+C}{(cx^2+d)}$$

- Repeated Linear factors in the form:

$$\frac{x}{(ax+b)(cx+d)^3} \equiv \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2} + \frac{D}{(cx+d)^3}$$

- Improper (top heavy) fractions in the form:

$$\frac{x^{n+m}}{ax^n + bx + d}$$

To solve for the unknown constants, A, B & C *etc*., we can use one or more of the following four methods:

- Equating coefficients
- Substitution in the numerator
- Separating the unknown by multiplication and substituting
- Cover up method (only useful for linear factors)

61.2 Type 1: Linear Factors in the Denominator

This the simplest of them all. The denominator factorises into two or more different linear factors of the form $(ax+b)$, $(cx+d)$ etc. Recognise that each of these linear factors is a root of the expression in the denominator. We set up the partial fractions on the RHS and each root must have its own 'unknown constant' assigned to it.

$$\text{e.g.} \qquad \frac{7x-8}{2x^2-5x+2} = \frac{7x-8}{(2x-1)(x-2)} = \frac{A}{(2x-1)} + \frac{B}{(x-2)}$$

$$\text{e.g.} \qquad \frac{8x}{(2x-1)(x-2)(x+4)} = \frac{A}{(2x-1)} + \frac{B}{(x-2)} + \frac{C}{(x+4)}$$

61.3 Solving by Equating Coefficients

Taking the first example from above:

61.3.1 Example:

The first task is to factorise the denominator:

$$\frac{7x - 8}{2x^2 - 5x + 2} = \frac{7x - 8}{(2x - 1)(x - 2)}$$

Then set up the identity with the correct number of partial fractions:

$$\frac{7x - 8}{(2x - 1)(x - 2)} \equiv \frac{A}{2x - 1} + \frac{B}{x - 2}$$

Add the fractions on the RHS to give:

$$\frac{7x - 8}{(2x - 1)(x - 2)} \equiv \frac{A}{2x - 1} + \frac{B}{x - 2} \equiv \frac{A(x - 2) + B(2x - 1)}{(2x - 1)(x - 2)}$$

$$\therefore \quad 7x - 8 \equiv A(x - 2) + B(2x - 1)$$

$$7x - 8 \equiv Ax - 2A + 2Bx - B$$

Equate the terms in: x : $7x = Ax + 2Bx$ $\therefore 7 = A + 2B$ $\Rightarrow$ $A = 7 - 2B$

Equate the Constants : $-8 = -2A - B$ $\therefore 2A + B = 8$

Substituting or using simultaneous equations:

$$\therefore \quad 2(7 - 2B) + B = 8$$

$$14 - 4B + B = 8 \quad \Rightarrow \quad -3B = -6$$

$$3B = 6 \quad \therefore \quad B = 2 \quad \& \quad A = 3$$

$$\therefore \quad \frac{7x - 8}{(2x - 1)(x - 2)} = \frac{3}{2x - 1} + \frac{2}{x - 2}$$

61.4 Solving by Substitution in the Numerator

Using the same example as above:

61.4.1 Example:

$$\frac{7x - 8}{(2x - 1)(x - 2)} \equiv \frac{A}{2x - 1} + \frac{B}{x - 2} \Rightarrow \frac{A(x - 2) + B(2x - 1)}{(2x - 1)(x - 2)}$$

$$\therefore \quad 7x - 8 \equiv A(x - 2) + B(2x - 1)$$

Find B by choosing $x = 2$ (to make the A term zero)

$$14 - 8 = B(4 - 1)$$

$$3B = 6 \quad \therefore \quad B = 2$$

Find A by choosing $x = \frac{1}{2}$ (to make 2nd (B) term zero)

$$\frac{7}{2} - 8 = A\left(\frac{1}{2} - 2\right)$$

$$\therefore \quad -\frac{9}{2} = -\frac{3}{2}A \quad \therefore A = 3$$

$$\therefore \quad \frac{7x - 8}{(2x - 1)(x - 2)} = \frac{3}{2x - 1} + \frac{2}{x - 2}$$

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61.5 Solving by Separating an Unknown

A variation on the substitution method which involves multiplying by one of the factors, and then using substitution. In some cases this can be used if the other two methods don't work.

61.5.1 Example:

$$\frac{4x}{x^2-4} \equiv \frac{A}{x+2} + \frac{B}{x-2}$$

Multiply both sides by one of the factors, say $(x+2)$

$$\frac{4x(x+2)}{x^2-4} \equiv \frac{A(x+2)}{x+2} + \frac{B(x+2)}{x-2}$$

Cancel common terms:

$$\frac{4x}{x-2} \equiv A + \frac{B(x+2)}{x-2}$$

$$\therefore \qquad A = \frac{4x}{x-2} - \frac{B(x+2)}{x-2}$$

Now substitute a value for x such that the B term is zero:

$$\text{If} \qquad x = -2 \;:\; A = \frac{-8}{-4} - 0 = 2$$

Now multiply both sides by one of the other factors, $(x-2)$ in this case:

$$\frac{4x(x-2)}{x^2-4} \equiv \frac{A(x-2)}{x+2} + \frac{B(x-2)}{x-2}$$

Cancel common terms:

$$\frac{4x}{x+2} \equiv \frac{A(x-2)}{x+2} + B$$

$$\therefore B = \frac{4x}{x+2} + \frac{A(x-2)}{x+2}$$

Now substitute a value for x such that the A term is zero:

$$\text{If} \qquad x = 2 \;:\; B = \frac{8}{4} = 2$$

Hence:

$$\frac{4x}{x^2-4} \equiv \frac{2}{x+2} + \frac{2}{x-2}$$

Test this by substituting a value for x on both sides. Don't use the values chosen above, as we need to check it is valid for all values of x.

$$\text{If} \qquad x = 1 \;: \qquad \frac{4}{1-4} \equiv \frac{2}{1+2} + \frac{2}{1-2}$$

$$-\frac{4}{3} \equiv \frac{2}{3} - \frac{2}{1}$$

$$\equiv \frac{2}{3} - \frac{6}{3} = -\frac{4}{3}$$

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61.6 Type 2: Squared Terms in the Denominator

This is where we have a squared term in the denominator that cannot be factorised into linear factors. It can be of the form of $(ax^2 + b)$ or it may be a traditional quadratic, such as $x^2 + bx + c$, that cannot be factorised. In either case we have to take this into account. The general form of the partial fractions are:

$$\frac{x}{(ax + b)(cx^2 + d)} \equiv \frac{A}{(ax + b)} + \frac{Bx + C}{(cx^2 + d)}$$

Note that the numerator is always one degree less than the denominator.

61.6.1 Example:

1

$$\frac{4x}{(x + 1)(x^2 - 3)} \equiv \frac{A}{x + 1} + \frac{Bx + C}{x^2 - 3} \equiv \frac{A(x^2 - 3) + (Bx + C)(x + 1)}{(x + 1)(x^2 - 3)}$$

$\therefore \quad 4x \equiv A(x^2 - 3) + (Bx + C)(x + 1)$

To eliminate the $(Bx + C)$ term, let $x = -1$

$\therefore \quad -4 = A(1 - 3) + 0 \qquad \therefore A = 2$

Equate the terms in x

$4x \equiv A(x^2 - 3) + (Bx + C)(x + 1)$

$4x \equiv Ax^2 - 3A + Bx^2 + Bx + Cx + C$

$4 = B + C$

Equate the constants terms :

$0 = -3A + C$

But $\quad A = 2$

$\therefore \quad C = 6$

$\therefore \quad B = 4 - C = 4 - 6 = -2$

Hence:

$$\frac{4x}{(x + 1)(x^2 - 3)} = \frac{2}{x + 1} + \frac{6 - 2x}{x^2 - 3}$$

Check result by substituting any value for x, except -1 used above. So let $x = 1$

$$\frac{4}{(1 + 1)(1 - 3)} = \frac{2}{1 + 1} + \frac{6 - 2}{1 - 3}$$

$$\frac{4}{(2)(-2)} = \frac{2}{2} + \frac{4}{-2}$$

$$-1 = 1 - 2 = -1$$

2 A trick question - know your factors (difference of squares)!

$$\frac{4x}{x^2 - 4} \equiv \frac{A}{x + 2} + \frac{B}{x - 2} \equiv \frac{A(x - 2) + B(x + 2)}{(x + 2)(x - 2)}$$

$\therefore 4x \equiv A(x - 2) + B(x + 2)$

If $x = 2$: $\qquad 8 = A(0) + B(4) \qquad B = 2$

If $x = -2$: $\quad -8 = A(-4) + B(0) \qquad A = 2$

$$\therefore \quad \frac{4x}{x^2 - 4} \equiv \frac{2}{x + 2} + \frac{2}{x - 2}$$

61.7 Type 3: Repeated Linear Factors in the Denominator

A factor raised to a power such as $(x + 2)^3$ gives rise to repeated factors of $(x + 2)$. Handling these repeated factors requires a partial fraction for each power of the factor, up to the highest power of the factor.

Thus, a cubed factor requires three fractions using descending powers of the factor.

e.g. $$\frac{x}{(x+2)^3} = \frac{A}{(x+2)^3} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)}$$

Similarly, factors of $(x + 2)^4$ would be split into fractions with $(x + 2)^4$, $(x + 2)^3$, $(x + 2)^2$, $(x + 2)$

e.g. $$\frac{x}{(x+2)^4} = \frac{A}{(x+2)^4} + \frac{B}{(x+2)^3} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)}$$

The general rule is that the number of unknowns on the RHS must equal the degree of the denominators polynomial on the left. In the example below, the degree of the expression in the denominator is four. Hence:

e.g. $$\frac{x}{(x+1)(x+4)(x+2)^2} = \frac{A}{(x+1)} + \frac{B}{(x+4)} + \frac{C}{(x+2)} + \frac{D}{(x+2)^2}$$

The reasoning behind the use of different powers of a factor requires an explanation that is really beyond the scope of these notes. Suffice it to say that anything else does not provide a result that is true for **all values of *x***, which is what we require. In addition, we need the same number of equations as there are unknowns in order to find a unique answer.

An alternative way to view this problem, is to treat the problem in the same way as having a squared term in the denominator. For example:

$$\frac{x}{(x+2)^2} = \frac{Ax+B}{(x+2)^2}$$

However, the whole point of partial fractions is to simplify the original expression as far as possible, ready for further work such as differentiation or integration. In the exam, repeated linear factors need to be solved as discussed above.

61.7.1 Example:

1

$$\frac{x}{(x+1)(x+2)^2}$$

$$\frac{x}{(x+1)(x+2)^2} \equiv \frac{A}{x+1} + \frac{B}{(x+2)^2} + \frac{C}{x+2}$$

$$\therefore \quad x \equiv A(x+2)^2 + B(x+1) + C(x+1)(x+2)$$

$x = -2 \qquad -B = -2 \qquad \Rightarrow B = 2$

$x = -1 \qquad A = -1 \qquad \Rightarrow A = -1$

Look at x^2 term: $\quad A + C = 0 \qquad \therefore C = 1$

$$\frac{x}{(x+1)(x+2)^2} = -\frac{1}{x+1} + \frac{2}{(x+2)^2} + \frac{1}{x+2}$$

2 Same problem, but treated as a squared term (interest only).

$$\frac{x}{(x+1)(x+2)^2}$$

$$\frac{x}{(x+1)(x+2)^2} \equiv \frac{A}{x+1} + \frac{Bx+C}{(x+2)^2}$$

$$\therefore \quad x \equiv A(x+2)^2 + (Bx+C)(x+1)$$

$$x = -1$$

$$-1 = A(1)^2 + 0 \qquad \Rightarrow \quad A = -1$$

$$x = -2$$

$$-2 = 0 + (-2B + C)(-1)$$

$$-2 = 2B - C$$

Equate constant terms

$$0 = 4A + C \qquad \Rightarrow \qquad C = 4$$

$$\therefore \quad 2B = C - 2 \qquad \Rightarrow \qquad B = 1$$

$$\frac{x}{(x+1)(x+2)^2} \equiv -\frac{1}{x+1} + \frac{x+4}{(x+2)^2}$$

3

$$\frac{x^2+7x+5}{(x+2)^3} \equiv \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$$

Compare numerators:

$$x^2 + 7x + 5 \equiv A(x+2)^2 + B(x+2) + C$$

Let: $x = -2$ $\quad 4 - 14 + 5 = C \qquad C = -5$

Compare coefficients: x^2 $\quad 1 = A$

Compare coefficients: constants $\quad 5 = 4A + 2B + C$

$$5 = 4 + 2B - 5$$

$$B = 3$$

$$\therefore \quad \frac{x^2+7x+5}{(x+2)^3} \equiv \frac{1}{x+2} + \frac{3}{(x+2)^2} - \frac{5}{(x+2)^3}$$

61.8 Solving by the Cover Up Method

One final method of solving partial fractions, which was first described by the scientist Oliver Heaviside, is the cover up method. The restriction with this method is that it can only be used on the highest power of any given linear factor. (This will make more sense after the second example). It makes a convenient way of finding the constants and is less prone to mistakes.

61.8.1 Example:

1

$$\frac{6x - 8}{(x - 1)(x - 2)} \equiv \frac{A}{x - 1} + \frac{B}{x - 2}$$

To find A, we 'cover up' its corresponding factor $(x - 1)$ and then set $x = 1$

$$\frac{6x - 8}{(\square\square\square\square)(x - 2)} = \frac{A}{(\square\square\square\square)}$$

$$\frac{6 - 8}{1 - 2} = \frac{-2}{-1} = 2 = A$$

Similarly, to find B, we 'cover up' its corresponding factor $(x - 2)$ and then set $x = 2$

$$\frac{6x - 8}{(x - 1)(\square\square\square\square)} = \frac{B}{(\square\square\square\square)}$$

$$\frac{12 - 8}{2 - 1} = \frac{4}{1} = 4 = B$$

Hence:

$$\frac{6x - 8}{(x - 1)(x - 2)} \equiv \frac{2}{x - 1} + \frac{4}{x - 2}$$

Why does this work? If we did it the long way by multiplying by one factor, say $(x - 1)$, we get:

$$\frac{(6x - 8)(x - 1)}{(x - 1)(x - 2)} \equiv \frac{A(x - 1)}{x - 1} + \frac{B\,(x - 1)}{x - 2}$$

Cancelling terms we get:

$$\frac{(6x - 8)}{(x - 2)} \equiv A + \frac{B\,(x - 1)}{x - 2}$$

When $x = 1$ the B term becomes zero, so we have:

$$\frac{(6x - 8)}{(x - 2)} \equiv A$$

So the Cover Up Method is just a short cut method for multiplying out by one of the factors.

2 The cover up method can be used on the linear parts of other more complex partial fractions. This speeds up the process, and simplifies the subsequent calculations.
For example, in the problem earlier, we had this to solve:

$$\frac{4x}{(x+1)(x^2-3)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2-3}$$

To find A: cover up $(x+1)$ and set $x = -1$

$$\frac{4x}{(\square\square\square\square)(x^2-3)} \equiv \frac{A}{(\square\square\square\square)}$$

$$\frac{-4}{(1-3)} \equiv A$$

$$\frac{-4}{-2} \equiv A \qquad \therefore A = 2$$

The other constants can now be found using the other methods.

3 The cover up method can also be used to partly solve problems with repeated linear factors. The proviso is that only the highest power of the repeated factor can be covered up.

$$\frac{6}{(x+2)(x-1)^2} \equiv \frac{A}{x+2} + \frac{B}{(x-1)^2} + \frac{C}{x-1}$$

To find A: cover up $(x+2)$ and set $x = -2$

$$\frac{6}{(\square\square\square\square)(x-1)^2} \equiv \frac{A}{(\square\square\square\square)}$$

$$\frac{6}{(-2-1)^2} \equiv A$$

$$\frac{6}{9} \equiv A \qquad \therefore A = \frac{2}{3}$$

To find B: cover up $(x-1)^2$ and set $x = 1$

$$\frac{6}{(x+2)(\square\square\square\square)} \equiv \frac{B}{(\square\square\square\square)}$$

$$\frac{6}{3} \equiv B \qquad \therefore B = 2$$

The cover up method cannot be used to find C, so one of the other methods is required.

To find C set $x = 0$

$$\frac{6}{(x+2)(x-1)^2} \equiv \frac{2}{3(x+2)} + \frac{2}{(x-1)^2} + \frac{C}{x-1}$$

$$\frac{6}{(2)(-1)^2} \equiv \frac{2}{3(2)} + \frac{2}{(-1)^2} + \frac{C}{-1}$$

$$\frac{6}{2} = \frac{2}{6} + \frac{2}{1} - \frac{C}{1}$$

$$3 = \frac{1}{3} + 2 - C$$

$$C = -\frac{2}{3}$$

$$\frac{6}{(x+2)(x-1)^2} = \frac{2}{3(x+2)} + \frac{2}{(x-1)^2} - \frac{2}{3(x-1)}$$

61.9 Partial Fractions Worked Examples

61.9.1 Example:

1

$$\frac{16}{x^3 - 4x} \equiv \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}$$

But $\quad \dfrac{16}{x^3 - 4x} = \dfrac{16}{x(x^2 - 4)} = \dfrac{16}{x(x+2)(x-2)}$

$$\therefore \quad \frac{16}{x(x+2)(x-2)} \equiv \frac{A(x-2)(x+2) + Bx(x-2) + Cx(x+2)}{x(x+2)(x-2)}$$

$$16 = A(x-2)(x+2) + Bx(x-2) + Cx(x+2)$$

Let $x = 0 \qquad 16 = A(-2)(2) = -4A \qquad A = -4$

Let $x = -2 \qquad 16 = B(-2)(-4) = +8B \qquad B = 2$

Let $x = 2 \qquad 16 = C(2)(4) = 8C \qquad C = 2$

$$\therefore \quad \frac{16}{x^3 - 4x} = -\frac{4}{x} + \frac{2}{x+2} + \frac{2}{x-2}$$

2

Express $\dfrac{13x - 6}{x(3x-2)}$ as partial fractions.

$$\frac{13x - 6}{x(3x-2)} = \frac{A}{x} + \frac{B}{3x-2} \qquad \Rightarrow \qquad \frac{A(3x-2) + B(x)}{x(3x-1)}$$

$$\therefore \quad 13x - 6 \equiv A(3x-2) + B(x)$$

Choose values of x

$x = 0 \qquad \therefore -6 = -2A \qquad \Rightarrow A = 3$

$x = \dfrac{2}{3} \qquad \therefore \dfrac{26}{3} - 6 = \dfrac{2}{3}B \qquad \Rightarrow B = 4$

Ans : $\quad = \dfrac{3}{x} + \dfrac{4}{3x-2}$

3

$$\frac{12x}{(x+1)(2x+3)(x-3)} = \frac{A}{x+1} + \frac{B}{2x+3} + \frac{C}{x-3}$$

$$= \frac{A(2x+3)(x-3) + B(x+1)(x-3) + C(x+1)(2x+3)}{(x+1)(2x+3)(x-3)}$$

$$\therefore 12x \equiv A(2x+3)(x-3) + B(x+1)(x-3) + C(x+1)(2x+3)$$

Choose values of x

$x = 3 \qquad \therefore 36 = C(3+1)(2 \times 3 + 3) \qquad \Rightarrow 36C = 36 \qquad \Rightarrow C = 1$

$x = -1 \qquad \therefore -12 = A(-2+3)(-1-3) \qquad \Rightarrow -4A = -12 \qquad \Rightarrow A = 3$

$x = -\dfrac{3}{2} \qquad \therefore -12 \times \dfrac{3}{2} = B\left(-\dfrac{3}{2} + 1\right)\left(-\dfrac{3}{2} - 3\right) \qquad \Rightarrow \dfrac{9}{4}B = -18 \qquad \Rightarrow B = -8$

Ans : $\quad = \dfrac{3}{x+1} - \dfrac{8}{2x+3} + \dfrac{1}{x-3}$

61.10 Improper (Top Heavy) Fractions

An algebraic fraction is top heavy if the highest power of x in the numerator is greater to or equal to the highest power in the denominator. The examples below illustrate two methods of finding the unknowns. You can of course do a long division to find the whole number and remainder. Then work the partial fractions on the remainder.

61.10.1 Example:

1

$$\frac{x^2}{(x-1)(x+2)}$$

$$\equiv A + \frac{B}{x-1} + \frac{C}{x+2}$$

Note: A in not divided by another term because the fraction is a top heavy one and dividing out a top heavy fraction will give a whole number plus a remainder.

$$x^2 \equiv A(x-1)(x+2) + B(x+2) + C(x-1)$$

$x = 1 \qquad 1 = 3B \qquad \Rightarrow \quad B = \frac{1}{3}$

$x = -2 \qquad 4 = -3C \qquad \Rightarrow \quad C = -\frac{4}{3}$

$A = 1$ (coefficient of x^2)

$$\therefore \quad \frac{x^2}{(x-1)(x+2)} = 1 + \frac{1}{3(x-1)} - \frac{4}{3(x+2)}$$

2

e.g. $\dfrac{3x^2 + 6x + 2}{(2x+3)(x+2)^2}$ $\quad \leftarrow$ this is NOT top heavy

$$\equiv \frac{A}{2x+3} + \frac{B}{(x+2)^2} + \frac{C}{x+2}$$

$$\therefore \quad 3x^2 + 6x + 2 \equiv A(x+2)^2 + B(2x+3) + C(2x+3)(x+2)$$

$x = -2 \qquad 2 = -B \qquad \Rightarrow \quad B = -2$

etc…

3

e.g. $\dfrac{3x^2 + 6x + 2}{(2x+3)(x+2)}$ $\quad \leftarrow$ this IS top heavy

$$\equiv A + \frac{B}{2x+3} + \frac{C}{x+2}$$

$$\therefore \quad 3x^2 + 6x + 2 \equiv A(2x+3)(x+2) + B(x+2) + C(2x+3)$$

$x = -2 \qquad 2 = -C$

etc…

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4 Here is an alternative method, which splits the numerator into parts that can be divided exactly by the denominator, giving the whole number part immediately.

$$\frac{x^2+3x-11}{(x+2)(x-3)} = \frac{x^2+3x-11}{x^2-x-6}$$

$$= \frac{x^2-x-6+4x-5}{x^2-x-6}$$

$$\frac{x^2+3x-11}{(x+2)(x-3)} = \frac{x^2-x-6}{x^2-x-6} + \frac{4x-5}{x^2-x-6}$$

$$= 1 + \frac{4x-5}{x^2-x-6}$$

$$= 1 + \frac{A}{x+2} + \frac{B}{x-3}$$

The partial fraction required is based on the remainder and is now:

$$\frac{4x-5}{x^2-x-6} = \frac{A}{x+2} + \frac{B}{x-3}$$

which can be solved in the normal manner.

61.11 Using Partial Fractions

Some examples of using partial fractions for differentiation and integration. Partial fractions can also be used for series expansions.

61.11.1 Example:

1 Differentiate the following function: $f(x) = \dfrac{x+9}{2x^2+x-6}$

$$f(x) = \frac{x+9}{(2x-3)(x+2)} = \frac{A}{(2x-3)} + \frac{B}{(x+2)}$$

$$x + 9 = A(x+2) + B(2x-3)$$

Let $x = -2$: $\quad 7 = -7B \quad B = -1$

Let $x = \dfrac{3}{2}$: $\quad \dfrac{3}{2} + 9 = A\left(\dfrac{3}{2} + 2\right) \Rightarrow 3 + 18 = A(3+4) \qquad A = 3$

$$\therefore \quad f(x) = \frac{3}{(2x-3)} - \frac{1}{(x+2)}$$

Recall: If $\quad y = [f(x)]^n \quad \Rightarrow \quad \dfrac{dy}{dx} = n f'(x)[f(x)]^{n-1}$

$$f(x) = 3(2x-3)^{-1} - (x+2)^{-1}$$

$$f'(x) = 3(-1)2(2x-3)^{-2} - (-1)(x+2)^{-2}$$

$$= -6(2x-3)^{-2} + (x+2)^{-2}$$

$$f'(x) = \frac{1}{(x+2)^2} - \frac{6}{(2x-3)^2}$$

We could have used the quotient rule, but this method is sometimes easier.

61.12 Topical Tips

- The number of unknown constants on the RHS should equal the degree of the polynomial in the denominator:
  e.g.

$$\frac{x^2 + 7x + 5}{(x+2)^3} \equiv \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$$

- The denominator on the LHS is a degree 3 polynomial, so the number of constants on the RHS = 3
- A rational function is one in which both numerator and denominator are both polynomials.

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62.1 Using Partial Fractions in Integration

The ideal format for integrating a fraction is :

$$\int \frac{1}{ax + b} dx = \frac{1}{a} = ln\left|ax + b\right| + c$$

Partial fractions gives us the tool to tackle fractions that are not in this ideal form.

62.2 Worked Examples in Integrating Partial Fractions

62.2.1 Example:

1

Find $\int \frac{1}{(x^2 - 1)} dx$

$$\frac{1}{(x^2 - 1)} = \frac{A}{(x + 1)} + \frac{B}{(x - 1)} = \frac{A(x - 1) + B(x + 1)}{(x + 1)(x - 1)}$$

$$\therefore \quad 1 = A(x - 1) + B(x + 1)$$

Let $x = 1 \quad \Rightarrow \quad 1 = 2B \quad \therefore B = \frac{1}{2}$

Let $x = -1 \quad \Rightarrow \quad 1 = -2A \quad \therefore A = -\frac{1}{2}$

$$\int \frac{1}{(x^2 - 1)} dx = \int \frac{1}{2(x - 1)} - \frac{1}{2(x + 1)} dx$$

$$= \frac{1}{2}\int \frac{1}{(x - 1)} dx - \frac{1}{2}\int \frac{1}{(x + 1)} dx$$

$$= \frac{1}{2} ln\left|x - 1\right| - \frac{1}{2} ln\left|x + 1\right| + c$$

$$= \frac{1}{2} ln \frac{\left|x - 1\right|}{\left|x + 1\right|} + c$$

2

Find $\int \frac{5(x + 1)}{(x - 1)(x + 4)} dx$

$$\frac{5(x + 1)}{(x - 1)(x + 4)} = \frac{A}{(x - 1)} + \frac{B}{(x + 4)} = \frac{A(x + 4) + B(x - 1)}{(x - 1)(x + 4)}$$

$$\therefore \quad 5(x + 1) = A(x + 4) + B(x - 1)$$

Let $x = -4 \quad \Rightarrow \quad -15 = -5B \quad \therefore B = 3$

Let $x = 1 \quad \Rightarrow \quad 10 = 5A \quad \therefore A = 2$

$$\int \frac{5(x + 1)}{(x - 1)(x + 4)} dx = \int \frac{2}{(x - 1)} dx + \int \frac{3}{(x + 4)} dx$$

$$= 2\int \frac{1}{(x - 1)} + 3\int \frac{1}{(x + 4)} dx$$

$$= 2\,ln\left|x - 1\right| + 3\,ln\left|x + 4\right| + c$$

3

Calculate the value of $\displaystyle\int_1^4 \frac{1}{x(x-5)}dx$

$$\frac{1}{x(x-5)} \equiv \frac{A}{x} + \frac{B}{x-5} = \frac{A(x-5) + B(x)}{x(x-5)}$$

$$\therefore \quad 1 \equiv A(x-5) + Bx$$

Let $x = 5$ $\quad 5B = 1 \quad \Rightarrow \quad B = \frac{1}{5}$

Let $x = 0$ $\quad -5A = 1 \quad \Rightarrow \quad A = -\frac{1}{5}$

$$\therefore \quad \frac{1}{x(x-5)} = -\frac{1}{5x} + \frac{1}{5(x-5)}$$

$$\int_1^4 \frac{1}{x(x-5)}dx = \frac{1}{5}\int_1^4 \left(-\frac{1}{x}\right) + \frac{1}{(x-5)}\,dx$$

$$= \frac{1}{5}\Big[-ln\,|x| + ln|x-5|\Big]_1^4$$

$$= \frac{1}{5}\Big[\big(-ln\,|4| + ln|4-5|\big) - \big(-ln\,|1| + ln|1-5|\big)\Big]$$

$$= \frac{1}{5}\Big[\big(-ln\,|4| + ln\,|1|\,\big) + ln\,|1| - ln\,|4|\Big] \qquad \text{(but: } ln\,1 = 0\text{)}$$

$$= \frac{1}{5}(-2\,ln\,4) = -\frac{2}{5}\,ln\,4 = \frac{2}{5}\,ln\left(\frac{1}{4}\right) = \frac{1}{5}\,ln\left(\frac{1}{16}\right)$$

4

Calculate the value of $\displaystyle\int_0^\infty \frac{1}{(x+1)(2x+3)}dx$

$$\frac{1}{(x+1)(2x+3)} \equiv \frac{A}{(x+1)} + \frac{B}{(2x+3)} \equiv \frac{A(2x+3) + B(x+1)}{(x+1)(2x+3)}$$

$$\therefore 1 \equiv A(2x+3) + B(x+1)$$

$x = -\frac{3}{2}$ $\quad -\frac{1}{2}B = 1 \quad \Rightarrow \quad B = -2$

$x = -1$ $\quad -2A + 3 = 1 \quad \Rightarrow \quad A = 1$

$$\therefore \qquad = \frac{1}{(x+1)} - \frac{2}{(2x+3)}$$

$$\int_0^\infty \frac{1}{(x+1)(2x+3)}dx = \int_0^\infty \frac{1}{(x+1)} - \frac{2}{(2x+3)}dx$$

$$= \left[ln\,(x+1) - \frac{2}{2}\,ln\,(2x+3)\right]_0^\infty$$

$$= \left[ln\left(\frac{x+1}{2x+3}\right)\right]_0^\infty = \left[ln\left(\frac{x}{2x+3} + \frac{1}{2x+3}\right)\right]_0^\infty$$

Substitute 0 into this bit... $\left[ln\left(\frac{x+1}{2x+3}\right)\right]_0^\infty = ln\left(\frac{1}{3}\right)$

Rearrange & substitute ∞ into this bit... $= \left[ln\left(\frac{1}{2+\frac{3}{x}} + \frac{1}{2x+3}\right)\right]_0^\infty = ln\left(\frac{1}{2}\right)$

$$\therefore \qquad = \left[ln\left(\frac{1}{2}\right) - ln\left(\frac{1}{3}\right)\right] \qquad = ln\left(\frac{3}{2}\right)$$

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