

25/10/2021  
Monday

# LINEAR SYSTEMS

## System of Equations

A set of equations with the same variables (two or more equations graphed on the same coordinate plane).

## Solution of the system.

An ordered pair that is a solution to all equations is a solution to the equation.

terminologies

- a - one solution
- b - no solution
- c - an infinite number of solutions

## Consistent system.

A system that has at least one solution:

a) independent - has exactly one solution.

b) dependent - an infinite number of solutions.

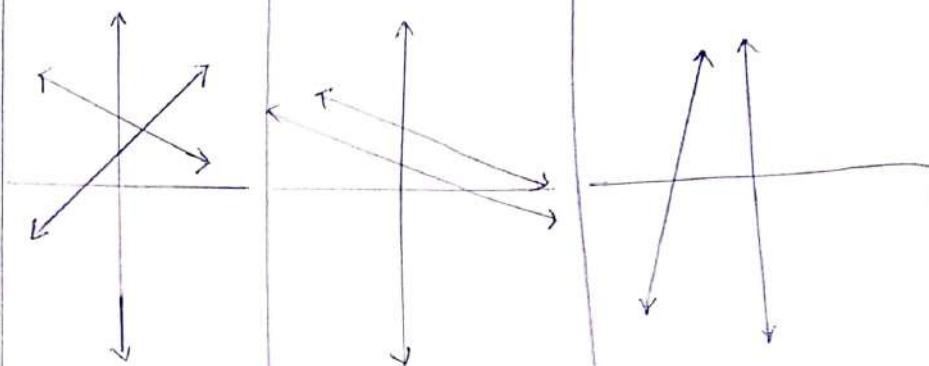
## Inconsistent System.

A system that has no solution.

# Illustrative display of the Solutions.

Number of Solutions Solutions are where they intersect.	one solution. Consistent and independent.	no solution inconsistent	infinitely many solutions. Consistent and dependent.
Definitions.			

Graph.



To solve a system of equations by graphing simply graph both equations on the same coordinate plane and find where they intersect.

Solving Linear systems.

## II Graphing

- Graph one equation.
- Graph the other equation on the same plane

- Find the point or points, or

$$y = 2x - 7 \quad -\textcircled{1}$$

$$y = \frac{2}{3}x + 1 \quad = \textcircled{11}$$

\textcircled{1} into \textcircled{11}.

$$2x - 7 = \frac{2}{3}x + 1$$

$$\frac{2}{1}x - \frac{2}{3}x = 1 + 7$$

$$\frac{6x - 2x}{3} = 8$$

$$\frac{4x}{3} = \frac{8}{1}$$

$$4x = 8$$

$$x = 6$$

$$(6, 5)$$

$$y = 2x -$$

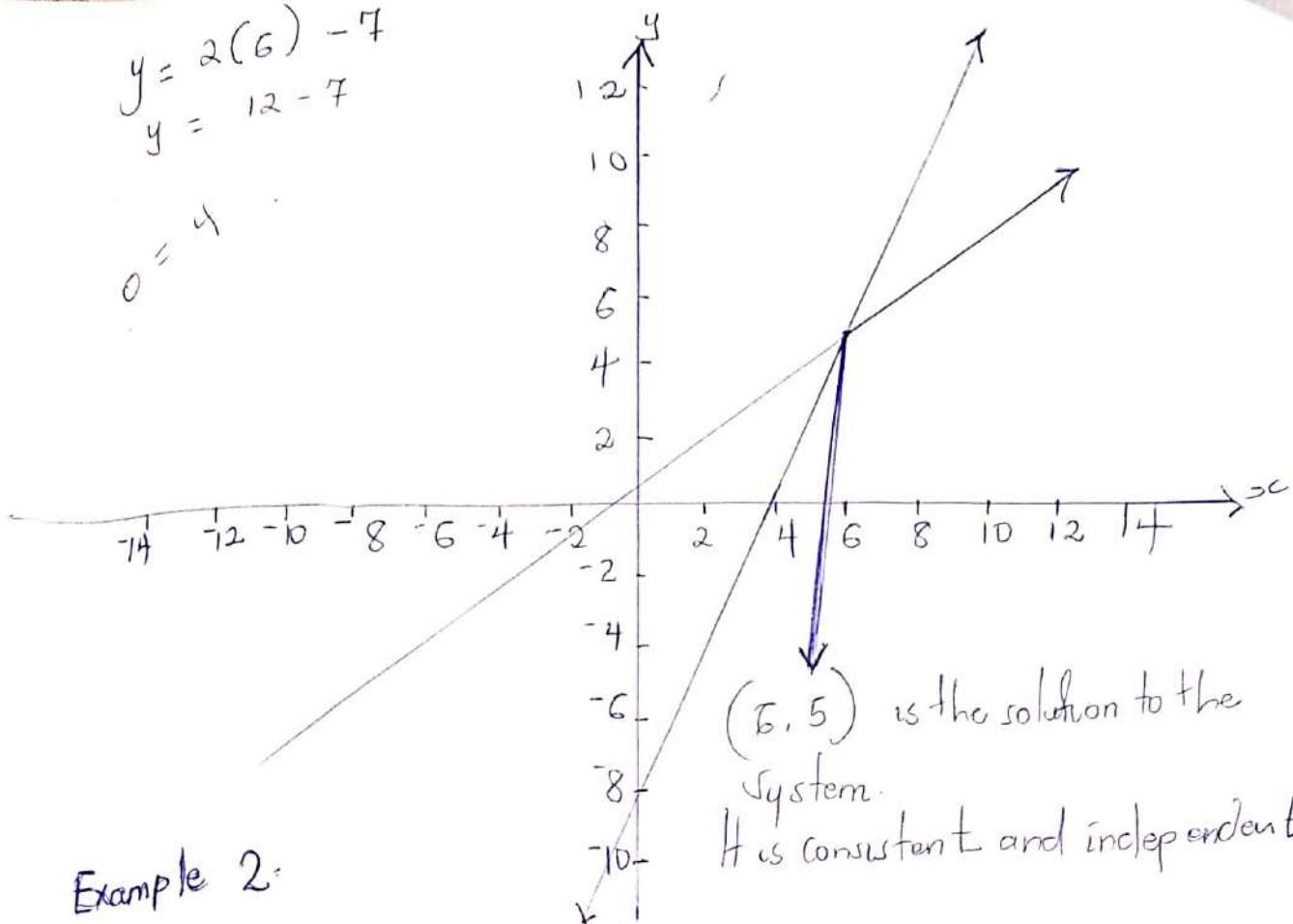
$$y = \frac{2}{3}x$$

$$y =$$

$$y = 2(x) - 7$$

$$y = 12 - 7$$

$$0 = x$$



Example 2:

$$x - 27 = 9y \quad \text{---(i)}$$

$$18 = x - 9y \quad \text{---(ii)}$$

$$\frac{x - 27}{9} = \frac{9y}{9} \quad \text{Eqn (i)}$$

$$\frac{1}{9}x - 3 = y$$

Eqn (ii)

$$\frac{18}{9} = \frac{x}{9} - \frac{9y}{9}$$

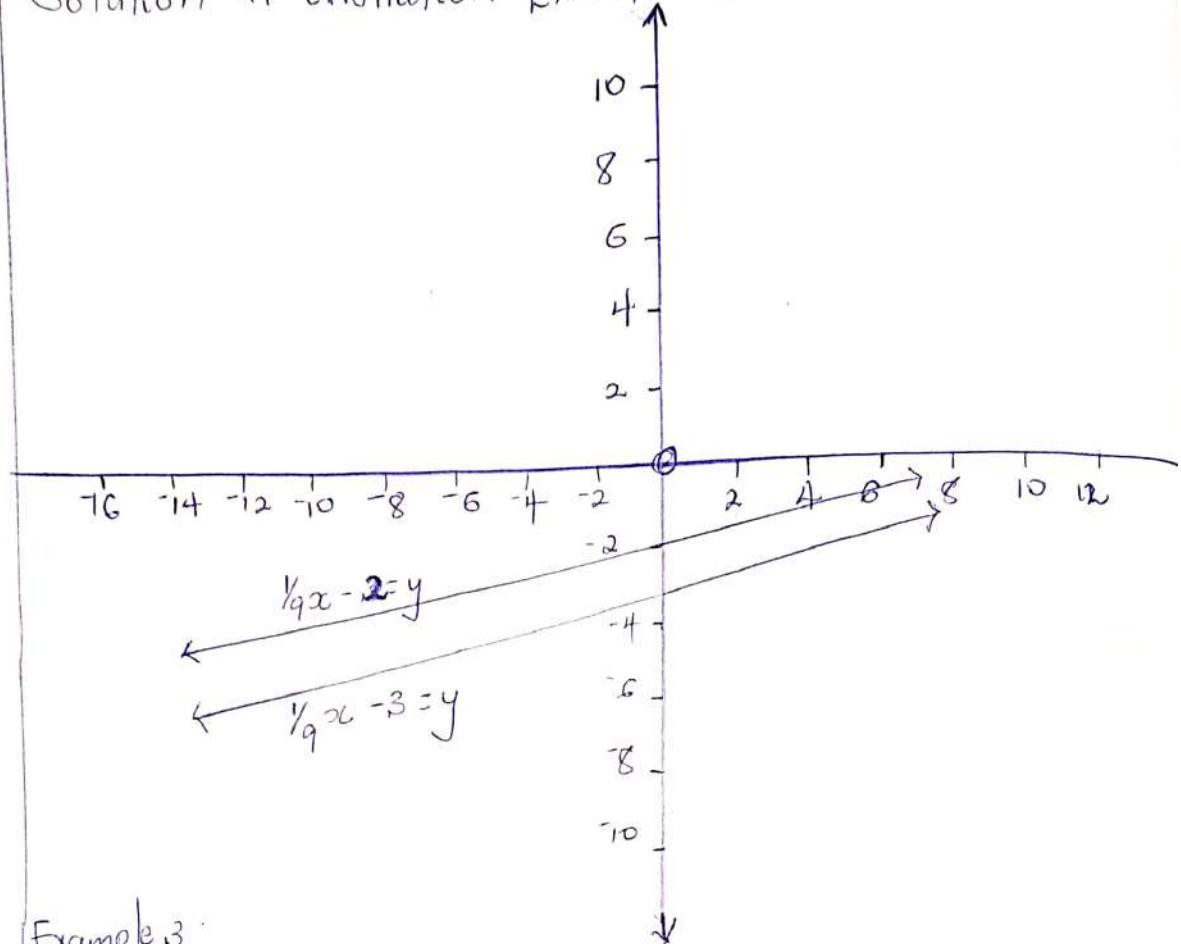
$$\frac{1}{9}x - 2 = y$$

The lines are parallel to each other because they have the same gradient:

∴ There is no solution to this system.  
It is inconsistent.

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## Solution in illustration Example 2.



Example 3

$$2x + 3y = 15 \quad \text{--- (1)}$$

$$y = -\frac{2}{3}x + 5 \quad \text{--- (2)}$$

Eqn (1).

$$\frac{2x + 3y}{3} = \frac{15}{3}$$

$$\frac{2}{3}x + y = 5$$

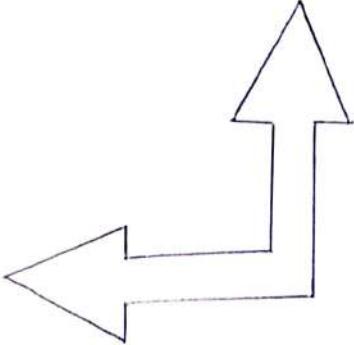
$$y = -\frac{2}{3}x + 5$$

$$\begin{aligned} 3y &= -2x + 15 \\ \frac{3y}{3} &= \frac{-2x}{3} + \frac{15}{3} \\ y &= -\frac{2}{3}x + 5 \end{aligned}$$

They are the same equation so they would graph into the same line

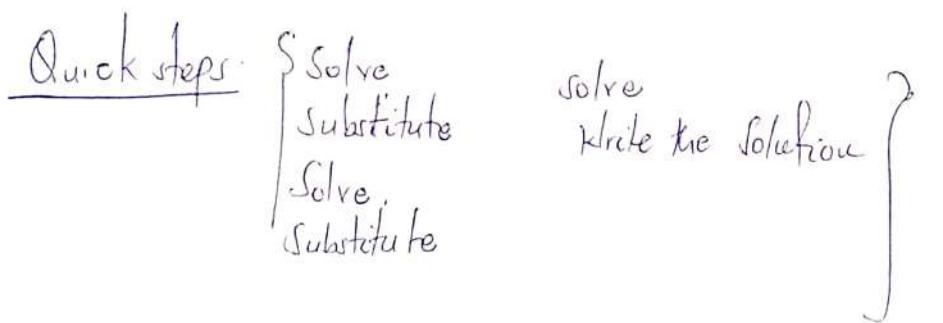
There are infinitely many solutions. Then the system is said to be consistent and dependent.

## Illustration.



## ② Substitution.

- If possible, solve at least one equation for one Variable.
- Substitute the result into the other equation to replace one of the Variables.
- Solve the equation.
- Substitute the value just found into the first equation.
- Solve for the other Variable.
- Write the solution as an ordered pair.



Example I.

$$8x + 5y = 2 \quad \text{--- (1)}$$

$$-2x + y = 4 \quad \text{--- (2)}$$

$$8x + 5y = 2$$

$$8x + 5(4+2x) = 2$$

$$8x + 20 + 10x = 2$$

$$\frac{18x}{18} = \frac{-18}{18}$$

$$\underline{\underline{x = -1}}$$

using Eqn (1)

$$y = 4 + 2x \quad \text{--- (3)}$$

Substitute into eqn (3).

$$y = 4 + 2(-1)$$

$$y = 4 - 2$$

$$\underline{\underline{y = 2}}$$

The solution is  $(-1, 2)$   
It is consistent and independent.

Example 2

$$-2x + 2y = -4 \quad \text{--- (1)}$$

$$-x = -y - 4 \quad \text{--- (2)}$$

Solution for (2) :

$$\frac{-x}{-1} = \frac{-y}{-1} - \frac{4}{-1}$$

$$x = y + 4 \quad \text{--- (3)}$$

$$y = 4 - x \quad \text{--- (4)}$$

Substitute Eqn (3) into Eqn (1),

$$-2x + 2y = -4$$

$$-2x + 2(4 - x) = -4$$

$$-2x + 8 - 2x = -4$$

$$\frac{-4x}{-4} = \frac{-12}{-4}$$

$$x = 3$$

Substitute Eqn (3) into Eqn (1),

$$-2x + 2y = -4$$

$$-2(y + 4) + 2y = -4$$

$$-2y - 8 + 2y = -4$$

$$-8 = -4$$

This is a false statement, therefore the system has no solution. The lines are parallel and are inconsistent.

Example 3:

$$2x + y = 5 \quad \text{---(i)}$$

$$-6x - 3y = -15 \quad \text{---(ii)}$$

using Eqn (i)

$$y = -2x + 5$$

$$-6x - 3(-2x + 5) = -15$$

$$-6x + 6x - 15 = -15$$

$$\underline{-15 = -15}$$

→ This is a true statement, therefore the system has infinitely many solutions.  
It is consistent and dependent.

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### § Elimination:

- Write both equations so that like terms are aligned vertically
- Multiply one or both equations by a constant to get two equations that contain at least one set of exactly opposite terms.

c. Add the equations: eliminating one variable.

d. Solve for the remaining variable.

e. Substitute the value from (d) into one of the equations and solve for the other variable.

f. Write the solution as an ordered pair.

### Example 1

$$6x + 14y = 6 \quad \text{---(1)}$$

$$-4x - 7y = -11 \quad \text{---(2)}$$

$$\begin{array}{r} 6x + 14y = 6 \\ -4x - 7y = -11 \\ \hline \end{array}$$

$$\begin{array}{r} 6x + 14y = 6 \\ -8x - 14y = -22 \\ \hline -2x = -16 \\ -2 \qquad \qquad -2 \end{array}$$

$$x = 8 \quad \text{Substitute 8 into Eqn (1)}$$

$$6x + 14y = 6$$

$$6(8) + 14y = 6$$

$$48 + 14y = 6$$

$$\frac{14y}{14} = \frac{-42}{14}$$

$$y = -3$$

The solution is  $(8, -3)$

This is consistent and independent.

### Example 11

$$-8x + 2y = -10 \quad \text{---(1)}$$

$$-4x + y = -2 \quad \text{---(2)}$$

$$\begin{array}{r} -8x + 2y = -10 \\ -4x + y = -2 \\ \hline \end{array}$$

$$-8x + 2y = -10$$

$$8x - 2y = 4$$

$$0 = -6$$

This is a false statement and has no solution.

The lines are parallel. It is inconsistent.

Example 3

$$6x + 8y = -28 \quad \text{--- (1)}$$

$$-3x - 4y = 14 \quad \text{--- (2)}$$

$$\begin{array}{r} | \\ \begin{array}{r} 6x + 8y = -28 \\ + 2(-3x - 4y) = 14 \\ \hline \end{array} \end{array}$$

$$6x + 8y = -28$$

$$-6x - 8y = 28$$

$$0 = 0$$

This is a true statement and has infinitely many solutions.

The equations are the exact same line. It is consistent and dependent.

## GAUSS' ELIMINATION METHOD

A system of equation

Is a collection of two or more equations with the same set of unknown variables that are considered simultaneously.

Example:

The following set of equations is a system of equations.

$$\begin{aligned}x - 2y + 3z &= 9 \\-x + 3y &= -4 \\2x - 5y + 5z &= 17\end{aligned}$$

An augmented matrix.

Is a rectangular array of numbers that represents a system of equations.

Ex. Turn the following system of equations into an augmented matrix.

Becomes:

$$\left[ \begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array} \right]$$

Gaussian Elimination

Is a method for solving systems of equations in matrix form.

Goal: turn matrix into row-echelon form.

$$\left[ \begin{array}{ccc|c} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{array} \right]$$

Once in this form, we can say that  $L = I$  and use back substitution to solve for  $y$  and  $x$ .

Use the elementary row operations and follow these steps:-

- 1) Get a 1 in the 1st column, 1st row.
- 2) Use the 1 to get 0's in the remainder of the first column.
- 3) Get a 1 in the second column, second row.
- 4) Use the 1 to get 0's in the remainder of the second column.
- 5) Get a 1 in the third column, third row.

### 3 Elementary Row Operations.

① Exchange two rows.

(Written  $R_i \leftrightarrow R_j$ )

② Multiply a row by a non-zero constant.

(Written  $\# R_i - R_i$  or  $\# R_i \rightarrow NR_j$ )

③ Add a multiple of a row to another row.

(Written  $\# R_i + R_j \rightarrow R_j$  or  
 $\# R_i + R_j \rightarrow NR_j$ ).

ISE

$$\left[ \begin{array}{cc|cc} f & 1 & a & \\ 0 & 1 & b & d \\ 0 & 0 & c & e \\ 0 & 0 & 1 & f \\ \hline & & 4th & 5th \end{array} \right]$$

3rd  
2nd  
4th  
5th.

## Points to note in Gaussian method

- Multiplying a row with a zero is not allowed.
- Adding a multiple to itself is not allowed.
- Swapping a row with itself is not allowed.

Solve the following using Gaussian methods.

Eqn ①

$$x+y=3 \quad -\text{①}$$

$$2x+y=3 \quad -\text{②}$$

$$x+y=3 \dots R_1$$

$$2x+y=3 \dots R_2$$

$$\left| \begin{array}{l} x+y=3 \\ \end{array} \right|$$

$$-2R_1 + R_2 \rightarrow R_2$$

$$\left| \begin{array}{l} x+y=3 \\ -y=-3 \\ \end{array} \right|$$

$$\frac{-y}{-1} = \frac{-3}{-1}$$

$$y=3$$

$$x+y=3$$

$$x+3=3$$

$$x=0$$

The solution is  $(0, 3)$

It has only one solution.

It is independent and Consistent.

$$\begin{array}{l} x+y=3 \\ 2x+y=3 \end{array} \dots R_1 \quad \dots R_2$$

$$-|R_1 + R_2 - R_2| \quad \left| \begin{array}{l} x+y=3 \\ -2x=0 \end{array} \right.$$

$$-2x=0$$

$$x=0$$

$$x+y=3$$

$$0+y=3$$

$$y=3$$

The solution is  $(0, 3)$ .

It has only one solution.

It is independent and consistent.

Example 2.

$$x-3y=1 \quad -R_1$$

$$-2x+2y-4z=2 \quad -R_2$$

$$x+3y-2z=3 \quad -R_3$$

$$| \begin{array}{l} x-3y=1 \end{array} |$$

$$2R_1 + R_2 = R_2 \longrightarrow | \begin{array}{l} -4y-4z=4 \end{array} |$$

$$-R_1 + R_3 = R_3 \longrightarrow | \begin{array}{l} 6y-2z=2 \end{array} |$$

$$\cdot \frac{1}{4}R_2 = R_2$$

$$| \begin{array}{l} -y-z=1 \end{array} |$$

$$6R_2 + R_3 \rightarrow$$

$$| \begin{array}{l} 6y-2z=2 \end{array} |$$

$$-6z=6$$

$$-2z=2$$

$$\begin{array}{r} -8z=8 \\ -8 \quad -8 \end{array}$$

$$z=-1$$

$$z = -1$$

$$-y - z = 1$$

$$-y - (-1) = 1$$

$$-y + 1 = 1$$

$$\frac{-y}{-1} = \frac{0}{-1}$$

$$y = 0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$x \Rightarrow x - 3y = 1$$

$$x - 3(0) = 1$$

$$x = 1$$

$$x = 1$$

The solution set  $(1, 0, -1)$

Example 3.

Solve the following set of equations.

$$x - 2y + 3z = 9$$

$$-x + 3y = -4$$

$$2x - 5y + 5z = 17$$

$$\left[ \begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array} \right]$$

$$R_1 + R_2 \rightarrow R_2$$

$$\left[ \begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & -1 \end{array} \right]$$

$$2R_1 + R_3 \rightarrow R_3$$

$$\left[ \begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array} \right]$$

$$\frac{1}{2} R_3 \rightarrow R_3 \quad \left[ \begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

New equation .

$$x + -2y + 3z = 9 \quad \text{--- (1)}$$

$$y + 3z = 5 \quad \text{--- (2)}$$

$$z = 2 \quad \text{--- (3)}$$

$$y = -1$$

$$x + -2(-1) + 3(2) = 9$$

$$x + 2 + 6 = 9$$

$$x = 1$$

---

$\overset{x}{\text{The solution set}} \quad (1, -1, 2)$

$$\begin{array}{l}
 \text{a) } \begin{array}{l}
 3x + y - 6z = 10 \\
 2x + y - 5z = -8 \\
 6x - 3y + 3z = 0
 \end{array} \quad \begin{array}{l}
 -\textcircled{1} \quad R_1 \\
 -\textcircled{2} \quad R_2 \\
 -\textcircled{3} \cdot R_3
 \end{array}
 \end{array}$$

Use Gauss's method to solve the linear systems below

$$\frac{1}{3}R_1 = R_1$$

$$x + \frac{1}{3}y - 2z = \frac{-10}{3} \quad -R_1$$

$$2x + y - 5z = -8 \quad -R_2$$

$$6x - 3y + 3z = 0 \quad -R_3$$

$$\left[ \begin{array}{ccc|c}
 1 & \frac{1}{3} & -2 & -\frac{10}{3} \\
 2 & 1 & -5 & -8 \\
 6 & -3 & 3 & 0
 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c}
 1 & \frac{1}{3} & -2 & -\frac{10}{3} \\
 0 & \frac{2}{3} & -1 & -\frac{14}{3} \\
 0 & -5 & 15 & 20
 \end{array} \right] \quad \begin{array}{l}
 2R_1 + R_2 = R_2 \\
 5R_1 + R_3 = R_3
 \end{array}$$

$$\left[ \begin{array}{ccc|c}
 1 & \frac{1}{3} & -2 & -\frac{10}{3} \\
 0 & 1 & -3 & -4 \\
 0 & -5 & 15 & 20
 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c}
 1 & \frac{1}{3} & -2 & -\frac{10}{3} \\
 0 & 1 & -3 & -4 \\
 0 & 0 & 0 & 0
 \end{array} \right]$$

# GAUSS JORDAN REDUCTION AND REDUCED ROW ECHELON FORM

## Gauss - Jordan elimination

Gauss - Jordan elimination is another method for solving of equations in matrix form.

It is really a continuation of Gaussian elimination.

Goal: Turn matrix into reduced row-echelon form.

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array} \right]$$

Once it is in this form,

we can say  $x = a$ ,  $y = b$  and  $z = c$  or  
 $(x, y, z) = (a, b, c)$

Use same row operations as before

The steps are slightly different because we need zeros above the diagonal line of 1's as well as below. We can either complete Gaussian elimination and then work on the 0's above the 1's, or

work on the zeros above as we move through the rows, as demonstrated below.

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array} \right] \text{ or } \left[ \begin{array}{ccc|c} 0 & 1 & 0 & a \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & c \end{array} \right]$$

1st  
2nd.  
3rd.  
4th. 4th.  
5th. 5th.  
6th.

Solve the following systems using Gauss-Jordan elimination method.

$$x + y + z = 5$$

$$2x + 3y + 5z = 8$$

$$4x + 5z = 2$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array} \right] \xrightarrow[-2R_1, R_2]{-4R_1+R_3} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array} \right]$$

$$\xrightarrow{+R_2+R_3} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array} \right]$$

$$\xrightarrow[\frac{1}{13}R_3]{\quad} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

$$\xrightarrow{-1R_3+R_1} \left[ \begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & +1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

$$\xrightarrow{-1R_2+R_1} \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

From this final matrix, we can read the solution of the system.

$$x = 3, y = 4, z = -2$$

$$\text{Solution} = \{3, 4, -2\}$$

$$\begin{aligned}2. \quad & x + 2y - 3z = 2 \\& 6x + 3y - 9z = 6 \\& 7x + 14y - 21z = 13\end{aligned}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 7 & 14 & -21 & 13 \end{array} \right]$$

$$\begin{aligned}-6R_1 + R_2 & \left[ \begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & -9 & 9 & -6 \\ 7 & 14 & -21 & 13 \end{array} \right] \\-7R_1 + R_3 & \left[ \begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & -9 & 9 & -6 \\ 0 & 0 & 0 & -1 \end{array} \right]\end{aligned}$$

We obtain a row whose elements are all zeros except the last one on the right. Therefore, we conclude that the system of equations is inconsistent i.e has no solution.

### 3. Example III

$$\begin{aligned}A + B + 2C &= 1 \\2A - B + D &= -2 \\A - B - C - 2D &= 4 \\2A - B + 2C - D &= 0\end{aligned}$$

$$\begin{aligned}\left[ \begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 2 & -1 & 0 & 1 & -2 \\ 1 & -1 & -1 & 2 & 4 \\ 2 & -1 & 2 & -1 & 0 \end{array} \right] \\-2R_1 + R_2 \left[ \begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 1 & -1 & -1 & 2 & 4 \\ 2 & -1 & 2 & -1 & 0 \end{array} \right] \\-1R_1 + R_3 \left[ \begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 2 & -1 & 2 & -1 & 0 \end{array} \right] \\-2R_1 + R_4 \left[ \begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & -3 & -2 & -1 & -2 \end{array} \right]\end{aligned}$$

$$-\frac{1}{3}R_2 \left[ \begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & \frac{4}{3} & -\frac{1}{3} & \frac{4}{3} \\ 0 & -2 & -3 & -2 & 3 \\ 0 & -3 & -2 & -1 & -2 \end{array} \right]$$

$$\left[ \begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & \frac{4}{3} & -\frac{1}{3} & \frac{4}{3} \\ 0 & 0 & -\frac{1}{3} & -\frac{8}{3} & \frac{1}{3} \\ 0 & 0 & +2 & -\frac{7}{2} & 2 \end{array} \right] \begin{matrix} 2R_2 + R_3 \\ 3R_2 + R_4 \end{matrix}$$

$$-3R_3 \left[ \begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & \frac{4}{3} & -\frac{1}{3} & \frac{4}{3} \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 2 & -2 & 2 \end{array} \right]$$

$$-2R_3 + R_4 \left[ \begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & \frac{4}{3} & -\frac{1}{3} & \frac{4}{3} \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 0 & -18 & 36 \end{array} \right]$$

$$-\frac{1}{18} \left[ \begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & \frac{4}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right] \begin{matrix} \\ \\ \\ -2 \end{matrix}$$

$$\begin{matrix} 1/3R_4 + R_2 \\ -8R_4 + R_3 \end{matrix} \left[ \begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & \frac{4}{3} & 0 & \frac{4}{3} \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array} \right]$$

$$-2R_3 + R_1 \left[ \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array} \right]$$

$$-R_3 + R_1 \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array} \right]$$

$$A = 1 \quad B = 2 \quad C = -1 \quad D = -2$$

$$\text{Solution Set} = \{1, 2, -1, -2\}$$

# ECHELON FORM

In each row of linear system, the first variable with a non-zero coefficient is the row's leading variable.

Specifically, a matrix is in row echelon form if:

All rows consisting of only zeros are at the bottom.

The leading coefficient (also called the pivot) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.

Row Echelon form.

Property 2.

If a column contains a leading entry then all entries below that leading entry are zero.

$$\left[ \begin{array}{cccc|cc} 1 & 0 & 2 & 3 & 4 & 7 \\ 0 & 1 & 5 & 6 & 7 & \\ 0 & 0 & 0 & 1 & 8 & \\ 0 & 0 & 0 & 0 & , & \end{array} \right]$$

Show that the solution to the following linear system is

$$\left| \begin{array}{l} \frac{3-z}{2}, \frac{1-3z}{2}, z \\ \hline \end{array} \right|_{\text{ZER}} \left. \begin{array}{l} \text{Ans} \\ \text{Ans} \end{array} \right\}$$

$$2x + z = 3 \quad (1)$$

$$x - y - z = 1$$

$$3x - y = 4$$

Swapping  $R_1 \leftrightarrow R_2$ .

$$x - y - z = 1$$

$$2x + z = 3$$

$$3x - y = 4$$

$$\left| \begin{array}{ccc|c} x & -y & -z & 1 \\ 2 & 0 & 1 & 3 \\ 3 & -1 & 0 & 4 \end{array} \right.$$

$$\begin{array}{l} -2R_1 + R_2 \\ -3R_1 + R_3 \end{array} \left| \begin{array}{ccc|c} x & -y & -z & 1 \\ 0 & +2y & +3z & 1 \\ 0 & +2y & +3z & 1 \end{array} \right.$$

$$\begin{array}{l} -R_2 + R_3 \end{array} \left| \begin{array}{ccc|c} x & -y & -z & 1 \\ 0 & +2y & +3z & 1 \\ 0 & 0 & 0 & 0 \end{array} \right. \begin{array}{l} \text{Redundancy} \\ \text{Redundancy} \end{array}$$

$$2y + 3z = 1$$

$$\frac{3z}{3} = \frac{1 - 2y}{3}$$

$$\frac{2y}{2} = \frac{1 - 3z}{2}$$

$$z = \frac{1 - 2y}{3}$$

$$y = \frac{1 + 3z}{2}$$

$$x - y - z = 1$$

$$\frac{x}{1} - \frac{1 + 3z}{2} - \frac{z}{1} = \frac{1}{1}$$

$$2x + 1 + 3z - 2z = 2$$

$$2x + z = 1$$

$$\frac{2x}{2} = \frac{1 - z}{2}$$

$$x = \frac{1 - z}{2} \quad \text{XX}$$

$$2x + z = 1$$

$$x = \frac{3 - z}{2}$$

$$x - y - z = 1$$

$$\frac{3 - z}{2} - \frac{(1 + 3z)}{2} - z = 1$$

$$\frac{3 - z - 1 - 3z}{2} - z = 1$$

$$\frac{-4z - 2}{2} - z = 1$$

$$-2z - 1 - z = 1$$

$$\frac{-3z}{3} = 0$$

$$z = 0 \quad \text{XX}$$

$$\text{Solution Set} = \left\{ x, y, z \right\} = \left\{ \frac{3 - z}{2}, \frac{1 - 3z}{2}, z \mid z \in \mathbb{R} \right\}$$

Example 2.

$$x + y + z + w = 1$$

$$y - z + w = -1$$

$$3x + 6z - 6w = 6$$

$$-y + z - w = 1$$

$$\begin{cases} x + y + z + w = 1 \\ 0 + y - z + w = -1 \\ -3x + 3y + 3z - 9w = 3 \\ 0 - y + z - w = 1 \end{cases}$$

$$\begin{cases} x + y + z + w = 1 \\ 0 + y - z + w = -1 \\ 0 \quad 0 \quad 0 - 6w = 0 \\ 0 \quad 0 \quad 0 \quad 0 = 0 \end{cases}$$

$$-6w = 0$$

$$w = 0$$

$$y - z + w = -1$$

$$y - z = -1$$

$$\underline{\underline{y = z - 1}}$$

$$x + y + z + w = 1$$

$$x + z - 1 + z + w = 1$$

$$x + 2z = 1 + 1$$

$$\underline{\underline{x = 2 - 2z}}$$

$$\text{Solution set} = \left\{ z \in \mathbb{R} \mid z = -1, 0 \right\}$$

Solve the following. Inconsistent System Examples

$$2x - 2z = 6$$

$$y + z = 1$$

$$2x + y - z = 7$$

$$3y + 3z = 0$$

$$\left( \begin{array}{ccc|c} 2 & 0 & -2 & 6 \\ 0 & 1 & 1 & 1 \\ 2 & 1 & -1 & 7 \\ 0 & 0 & 3 & 0 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 2 & 0 & -2 & 6 \\ 2 & 1 & -1 & 7 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 0 \end{array} \right)$$

$$\frac{1}{2}R_1 \left( \begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 2 & 1 & -1 & 7 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 0 \end{array} \right)$$

$$2R_1 + R_2 \left( \begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 0 \end{array} \right)$$

## Addendum-

- A linear system with a unique solution has a solution with one element.

A linear system with no solution has a solution set that is empty. However, describing a solution set with many solutions can be a challenge.

- Any linear system solution set can be described as

$$\circ \vec{P} + \overrightarrow{c_1 \beta_1} + \overrightarrow{c_2 \beta_2} + \dots + \overrightarrow{c_k \beta_k} : \text{CSR}$$

◦ where  $\vec{P}$  - any particular solution.

◦  $\beta_i$  - number of free variables that a system has after a Gaussian reduction.

## Homogeneous System

- A linear equation is said to be homogeneous if it has a constant of zero i.e. it can be expressed in this form.

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = 0$$

- A linear system comprising of a homogeneous equations is referred to as a homogeneous system.

Solve the homogenous equivalent of the system.

$$x + 2y = 0$$

$$2x + z = 0$$

$$3x + 2y + z - w = 0$$

Swapping  $R_3 \leftrightarrow R_1$

$$3x + 2y + z - w = 0$$

$$x + 2y = 0$$

$$2x + z = 0$$

$$\left( \begin{array}{cccc|c} 3 & 2 & 1 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \end{array} \right)$$

$$\begin{array}{l} -R_1 + 3R_2 \\ -R_1 + 3R_3 \end{array} \left( \begin{array}{cccc|c} 3 & 2 & 1 & -1 & 0 \\ 0 & 4 & -1 & 1 & 0 \\ 0 & -4 & 1 & 2 & 0 \end{array} \right)$$

$$R_2 + R_3 \cdot \left( \begin{array}{cccc|c} 3 & 2 & 1 & -1 & 0 \\ 0 & 4 & -1 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array} \right)$$

$$3w = 0$$

$$w = 0$$

$$4y - 1z + w = 0$$

$$4y - z = 0$$

$$y = \frac{z}{4}$$

$$3x + 2y + z - k = 0$$

$$3x + 2\left(\frac{z}{4}\right) + z - 0 = 0$$

$$3x + \frac{z}{4} + z = 0$$

$$3x = -\frac{3z}{2}$$

$$x = -\frac{z}{2}$$

$$\underline{\left\{-\frac{z}{2}, \frac{z}{4}, z, 0 \mid z \in \mathbb{R}\right\}}.$$

Example 2-

$$3x + 2y + z - w = 4$$

$$x + 2y = 1$$

$$2x + z = 2$$

$$\left( \begin{array}{cccc|c} 3 & 2 & 1 & -1 & 4 \\ 1 & 2 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 2 \end{array} \right)$$

$$\begin{aligned} & -R_1 + 3R_2 \rightarrow \\ & \left( \begin{array}{cccc|c} 3 & 2 & 1 & -1 & 4 \\ 0 & 4 & -1 & 1 & -1 \\ 2 & 0 & 1 & 2 & 2 \end{array} \right) \end{aligned}$$

$$R_2 + R_3 \left( \begin{array}{cccc|c} 3 & 2 & 1 & -1 & 4 \\ 0 & 4 & -1 & 1 & -1 \\ 0 & 0 & 0 & 3 & -3 \end{array} \right)$$

$$3w = -3$$

$$w = -1$$

$$4y - z + w = -1$$

$$4y - z - 1 = -1$$

$$4y = z$$

$$y = z/4$$

$$3x + 2y + z - w = 4$$

$$3x + 2\left(\frac{z}{4}\right) + z - -1 = 4$$

$$3x + \frac{z}{2} + z + 1 = 4$$

$$3x + \frac{z}{2} + z = 3$$

$$6x + z + 2z = 6$$

$$6x + 3z = 6$$

$$\frac{6x}{6} = \frac{6 - 3z}{6}$$

$$x = \frac{2 - z}{2}$$

Solution set is  $\left\{ \frac{2-z}{2}, \frac{z}{4}, z, -1 \right\}$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} \frac{2-z}{2} \\ \frac{z}{4} \\ z \\ -1 \end{pmatrix} \quad \text{for } z.$$

Solution in the form

$$\begin{pmatrix} 1-z/2 \\ 1/4z \\ z \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} -1/2 \\ 1/4 \\ 1 \\ 0 \end{pmatrix} z$$

$$\vec{P} + c_1 \vec{B}_1 + c_2 \vec{B}_2 + \dots + c_k \vec{B}_k$$

Solution:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + z \begin{pmatrix} -1/2 \\ 1/4 \\ 1 \\ 0 \end{pmatrix} \mid z \in \mathbb{R} \right\}$$

Example 2.

$$x + y + z + w = 1$$

$$y - z + w = -1$$

$$3x + 6z - 6w = 6$$

$$-y + z - w = 1$$

Solution

$$\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \mid z \in \mathbb{R} \right\}$$

In the form  $\vec{P} + z^k \sum c_i \vec{B}_i$

$$\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \mid z \in \mathbb{R} \right\}$$

## ROW EQUIVALENCE

Two matrices are said to be row equivalent if there is a sequence of elementary row operations that transforms one matrix into the other.

A linear system is also said to be row equivalent if the Row Reduced Echelon form (RREF) of a matrix is unique. If you can obtain an reduced echelon matrix from a system, then it is said to be row equivalent.

No matter what sequence of row operations you use, each matrix is row equivalent to one and only one reduced echelon matrix.

Example -

Are matrices A and B row equivalent?

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 4 \\ 5 & 8 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\begin{matrix} \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} & \xrightarrow[-3R_1+R_2]{R_2+R_1} & \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} & \xrightarrow{-\frac{1}{2}R_2} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = C \end{matrix}$$

$$B = \begin{pmatrix} 2 & 4 \\ 5 & 8 \end{pmatrix}$$

$$\begin{matrix} \begin{pmatrix} 1 & 2 \\ 5 & 8 \end{pmatrix} & \xrightarrow[\frac{1}{2}R_1]{5R_1+R_2} & \left\{ \begin{array}{l} \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} \\ \end{array} \right. & \xrightarrow[R_1+R_2]{R_1+R_2} & \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} & \xrightarrow[-\frac{1}{2}R_2]{R_2+R_1} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = D \end{matrix}$$

∴ They are row equivalent.

Example 2.

$$\begin{pmatrix} 0 & 0 & 4 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 \end{pmatrix}$$

$$R_2 \leftrightarrow R_1$$

$$R_3 \leftrightarrow R_4$$

$$\begin{pmatrix} 0 & 3 & 0 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{3}R_1} \begin{pmatrix} 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 4 & 1 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-4R_1 + R_3 \rightarrow \begin{pmatrix} 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 4 & 1 \\ 0 & 0 & -2 & -\frac{11}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{1}{4}R_2 \rightarrow \begin{pmatrix} 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & -2 & -\frac{11}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-2R_2 + R_3 \rightarrow \begin{pmatrix} 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & -\frac{11}{6} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-\frac{1}{6}R_3 \left( \begin{array}{cccc} 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\begin{array}{l} -\frac{1}{3}R_3 + R_1 \\ -\frac{1}{4}R_3 + R_2 \end{array} \left( \begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

This is RREF

∴ It is row equivalent.

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Tuesday .

Dr. Johnson Ssekakubo .

# VECTOR SPACES | linear Space.

is a system consisting of a set of generalized Vectors and a field of scalars, having the same rules for vector addition and Scalar multiplication as physical Vectors and Scalars.

Let  $V$  be a non empty set of goods or objects on which operations of addition and multiplication by scalars are defined.

If the following axioms are satisfied by all objects  $u, v, w$  in  $V$ , all scalars  $k_1, k_2$  then  $V$  is called a Vector space and the objects in  $V$  are called Vectors.

- vector space*
- 1) Addition conditions :  $u+v \in V$  (vector addition)
  - 2)  $u+v = v+u \rightarrow$  (associative)
  - 3)  $u+(v+w) = (u+v)+w$  (distributive)
  - 4) There is an object  $0$  in  $V$ , called a zero Vector for  $V$ , such that  $0+u=u$  for all  $u$  in  $V$   $\forall u \in V, u+(-u)=0$
  - 5) For each  $u$  in  $V$ , there is an object  $-u$  in  $V$ , called a negative of  $u$ , such that  $u+(-u)=0$  (additive inverse)
  - 6) If  $k$  is any scalar and  $u$  is any object in  $V$  then  $ku$  is in  $V$   $\{ Ku \in V \}$  scalar multiplication
  - 7)  $k(u+v) = ku + kv$
  - 8)  $(k+l)u = ku + lu$
  - 9)  $k(lu) = (kl)u$
  - 10)  $|u| = 1$   $1 =$  unit vector.

A Vector space (linear space),  $V$ , is a set that satisfies the following  
 & (forall)  $u, v, w \in V$  and  $c, d \in \mathbb{R}$   
 $c, d$  are scalars .

## Example of Vector Space :

Determine whether the set of  $V$  of all pair of real numbers  $(x_1, y_1)$  with the operations  $(x_1, y_1) + (x_2, y_2) = (x_1+x_2+1, y_1+y_2+1)$  and  $k(x_1, y_1) = (kx_1, ky_1)$  is a vectorspace.

Solution:

$$u = (x_1, y_1),$$

$$v = (x_2, y_2)$$

$w = (x_3, y_3)$  are objects in  $V$  and  $k_1, k_2$  are some scalars.

$$\textcircled{1} \quad u+v = (x_1, y_1) + (x_2, y_2) \\ = (x_1+x_2+1, y_1+y_2+1)$$

Since  $x_1+x_2$

Since  $x_1+x_2+1, y_1+y_2+1$  are also real numbers  
Therefore,  $u+v$  is also an object in  $V$ .  $(u+v \in V)$

$$\textcircled{2} \quad u+v = (x_1+x_2+1, y_1+y_2+1) \\ = (x_2+x_1+1, y_2+y_1+1) \quad (u+v = v+u) \\ = v+u$$

Therefore, vector addition is commutative.

$$\textcircled{3} \quad u+(v+w) \\ (x_1, y_1) + \left[ (x_2, y_2) + (x_3, y_3) \right] \\ (x_1, y_1) + \left[ x_2+x_3+1, y_2+y_3+1 \right] \\ \left[ x_1 + (x_2+x_3+1)+1, y_1 + (y_2+y_3+1)+1 \right] \\ \left[ (x_1+x_2+1)+x_3+1, (y_1+y_2+1)+y_3+1 \right] \\ \left[ x_1+x_2+1, y_1+y_2+1 \right] + (x_3+y_3) \\ = (u+v) + w.$$

Hence the vector addition is associative.

4 Let  $(a, b)$  be an object in  $V$  such that  $(a, b) + u = 0$   
 $(a, b) + (x_1, y_1) = (x_1, y_1)$   
 $(a+x_1+1, b+y_1+1) = (x_1, y_1)$   
 $a = -1, b = -1$   
Hence,  $(-1, -1)$  is zero vector in  $V$ .

5 Let  $(a, b)$  be an object in  $V$  such that  $(a, b) + u = (-1, -1)$   
 $(a, b) + (x_1, y_1) = (-1, -1)$   
 $(x_1+a+1, y_1+b+1) = (-1, -1)$   
 $x_1+a+1 = -1, y_1+b+1 = -1$   
 $a = -x_1-2, b = -y_1-2$   
Hence,  $(-x_1-2, -y_1-2)$  is the negative of  $u$  in  $V$ .

$$6 k_1 u = k_1(x_1, y_1) \\ = k_1 x_1, k_1 y_1$$

Since  $k_1 x_1, k_1 y_1$  are real numbers.

Therefore,  $V$  is closed under scalar multiplication.

$$7 k_1(u+v) = k_1(x_1 + x_2 + 1, y_1 + y_2 + 1) \\ = (k_1 x_1 + k_1 x_2 + k_1, k_1 y_1 + k_1 y_2 + k_1) \\ \neq k_1 u + k_1 v$$

$V$  is not distributive under scalar multiplication.

Hence,  $V$  is not a vector space.

## SUB SPACES:

For any Vector space, a subspace is a subset that is itself a vectorspace under the vector addition and scalar multiplication operations.

Definition:

$(V, +, \cdot)$ : a Vectorspace

$$W \subseteq V \left\{ \begin{array}{l} W \neq \emptyset \\ \text{an non empty subset} \end{array} \right.$$

$(W, +, \cdot)$ : a vectorspace under the operations of addition and scalar multiplication defined in  $V$ )

$\Rightarrow W$  is a subspace of  $V$ .

Prove that  $W = \left\{ \begin{pmatrix} x \\ 2x+3z \\ z \end{pmatrix} : x, z \in \mathbb{R} \right\}$  is a subspace of  $\mathbb{R}^3$

Solution

$$\begin{pmatrix} x \\ 2x+3z \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}x + \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}z ; x, z \in \mathbb{R}$$

$\uparrow$                      $\uparrow$

$$\text{let } u = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \text{ and } v = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

$$u+v = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} \in W$$

$$\begin{aligned} xu, x \in \mathbb{R} &= x \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} x \\ 2x \\ 0 \end{pmatrix} \in W \end{aligned}$$

Since Vector addition and scalar multiplication rules are satisfied,  $W$  is a subspace of  $\mathbb{R}^3$

Show that  $V = \{(x, -3x) \mid x \in \mathbb{R}\}$  is a subspace of  $\mathbb{R}^2$

Solution:

let  $u, w \in V$

$$u = (a, -3a)$$

$$w = (b, -3b)$$

Vector addition:

$$u+w = [(a+b), (-3a+3b)]$$

$$u+w = [(a+b), -3(a+b)]$$

$\forall$  is a set of coordinates  $x, y$  such that  $y = -3x$

If  $x = a+b$ ,  $y = -3(a+b)$

$u+w \in V$

Scalar multiplication

let  $r \in \mathbb{R}$ , then  $r u \in V$

$$\Rightarrow r(a, -3a) = (ra, r(-3a))$$

$$(ra, -3(ra))$$

$$\begin{matrix} \downarrow \\ x \end{matrix} \quad \begin{matrix} \downarrow \\ r \end{matrix}$$

Since  $V$  is defined on Vector addition and scalar multiplication  
then  $V$  is a subspace of  $\mathbb{R}^2$ .

## Span of Set of Vectors

A spanning set of the vector space is defined as a set of all the vectors in the set  $S = \{v_1, v_2, \dots, v_r\}$  is called span of  $S$  and denoted by  $\text{Span}\{v_1, v_2, \dots, v_r\}$ .

→ If  $S = \{v_1, v_2, \dots, v_r\}$  is a set of vectors in a vector space  $V$  then,

- (1) The span is a subspace of  $V$
- (2) The span  $S$  is the smallest subspace of  $V$  that contains the set  $S$ .

If  $S_1$  and  $S_2$  are two sets of vectors in  $V$  then,  $\text{Span}$

$$S_1 = \text{Span } S_2$$

If and only if each vector in  $S_1$  is a linear combination of those in  $S_1$  and  $S_2$  and vice versa.

Method to check span of  $V$

① choose an arbitrary vector  $b$  in  $V$

② Express  $b$  as a linear combination of  $v_1, v_2, \dots, v_r$

$$b = k_1 v_1 + k_2 v_2 + \dots + k_r v_r$$

③ if the system of the equation is consistent for all choice of  $b$  then vectors

$v_1, v_2, \dots, v_r$  span  $V$ . If it is inconsistent for some choice of  $b$ , vectors do not span  $V$ .

Consider a set of vectors -  $\{v_1, v_2, \dots, v_k\}$

The set of all linear combinations of these vectors space is called a span.

Determine whether following vector span the  $\mathbb{R}^3$

$$v_1 (2, 2, 2)$$

$$v_2 (0, 0, 1)$$

$$v_3 (0, 1, 1)$$

Let  $b = (b_1, b_2, b_3)$  be an arbitrary vector in  $\mathbb{R}^3$  and can be expressed as a linear combination of given vectors.

$$b = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$(b_1, b_2, b_3) = (2u_1, 2u_1 + u_3, 2u_1 + 3u_2 + u_3)$$

$$2k_1 = b_1$$

$$2k_1 + k_3 = b_2$$

$$2k_1 + 3k_2 + k_3 = b_3$$

Coefficient matrix =  $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix}$

let  $A = -6 \neq 0$

Hence given vectors span  $\mathbb{R}^3$

Q2

Given  $w = \left\{ \begin{pmatrix} x \\ 2x+z \\ z \end{pmatrix}, x, z \in \mathbb{R} \right\}$ ,

Find  $[w]$  or

Span  $[w]$

Solution

$$w = \begin{pmatrix} x \\ 2x+z \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\text{Span}(w) = \left\{ x \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, x, z \in \mathbb{R} \right\}$$

### Linear Combination.

If vector  $v$  is a vectorspace  $V$  is called a linear combination of vectors  $u_1, u_2, \dots, u_k$  in  $V$  if  $v$  can be written in the form

$$v = c_1 u_1 + c_2 u_2 + \dots + c_k u_k$$

$c_1, c_2, \dots, c_k$  : Scalars

If  $S = \{w_1, w_2, w_3, \dots, w_l\}$  is a nonempty set of vectors in a vectorspace  $V$ , then:

- a) The set  $L$  of all possible linear combinations of the vectors in  $S$  is a subspace of  $V$ .
- b) The set  $L$  in part (a) is the "smallest" subspace of  $V$  that contains all of the vectors in  $S$  in the sense that any other subspace that contains those vectors contains  $L$ .

Example:

Every vector  $v = (a, b, c)$  in  $\mathbb{R}^3$  is expressible as a linear combination of the standard basis vectors

$$\begin{aligned} i &= (1, 0, 0), j = (0, 1, 0), k = (0, 0, 1) \text{ since} \\ v = (a, b, c) &= a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) \\ &= ai + bj + ck \end{aligned}$$

Example:

Consider the vectors  $u = (1, 2, -1)$  and  $v = (6, 4, 2)$  in  $\mathbb{R}^3$ .

Show that  $w = (9, 3, 7)$  is a linear combination of  $u$  and  $v$  and that  $w = (4, -1, 8)$  is not a linear

combination of  $u$  and  $v$ .

$$\begin{aligned} w &= k_1 u + k_2 v \\ (4, -1, 8) &= k_1 (1, 2, -1) + k_2 (6, 4, 2) \\ (4, -1, 8) &= (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2) \\ k_1 + 6k_2 &= 4 \end{aligned}$$

$$2k_1 + 4k_3 = -1$$

$$2k_1 + 2k_2 = 3$$

finding linear combination

$$V_1(1, 3, 1) \quad V_2 = (0, 1, 2) \quad V_3(1, 0, 1)$$

Rowe  $k_1(1, 3, 1)$  is not linear combination of  $V_1, V_2, V_3$

Solution

$$kl = c_1 V_1 + c_2 V_2 + c_3 V_3$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \xrightarrow{\text{Gauss Seidel}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \\ 7 \end{bmatrix}$$

This system has no solution ( $\neq 0 \neq 1$ )

$$\Rightarrow kl \neq c_1 V_1 + c_2 V_2 + c_3 V_3$$

Basis of a Vector Space

(Generating sets) linearly independent sets.

Vector space.

$$S = \{v_1, v_2, \dots, v_n\} \subseteq V$$

| Spans  $V$  (i.e.,  $\text{span}(S) = V$ )  
| linearly independent

$\Rightarrow S$  is called a basis for  $V$

let  $V$  denote a vector space and  $S = \{u_1, u_2, \dots, u_n\}$  a

subset of  $V$ .

$S$  is called a basis for  $V$  if the following is true:

i)  $S$  spans  $V$

ii)  $S$  is linearly independent -

A subspace is linearly independent if none of its elements is a linear combination of the others.  
ie given  $\{v_1, v_2\}$ ,  $v_1$  and  $v_2$  are linearly independent if  $v_1 \neq kv_2$ ;  $k \in \mathbb{R}$ .

A set of vectors  $u_i$  is linearly independent iff (if and only if).

$\sum c_i u_i = 0$  has a trivial solution. Otherwise it is linearly dependent i.e.  $c_i \neq 0$

$$\Rightarrow c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0$$

$\uparrow$        $\uparrow$        $\uparrow$   
 $\downarrow$        $\downarrow$        $\downarrow$

In other words, a set of vectors  $v_1, v_2, \dots, v_n \in V$  are linearly dependent if there exists a set of scalars  $c_i$  not all zero such that

$$\sum c_i v_i = 0$$

Examples -

Determine if the vectors below are linearly dependent or independent.

i)  $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

ii)  $u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$

iii)  $w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$

4) Given  $v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$  and  $v_2 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ , show that  $\{v_1, v_2\}$  is a basis for  $\mathbb{R}^2$

*linearly independent*

Solution.

1)  $v_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$v_1$  and  $v_2$  are linearly independent if  $c_1 v_1 + c_2 v_2 = 0$   
has a trivial solution.

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} c_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 + c_2 \\ c_1 + 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \left\{ \begin{array}{l} c_1 \\ c_2 \end{array} \right\} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[ \begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right]$$

$$\xrightarrow{-R_1+R_2} \left[ \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -1 & 0 \end{array} \right]$$

$$\xrightarrow{-R_2} \left[ \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array} \right]$$

$$c_2 = 0$$

$$c_1 + c_2 = 0$$

$$c_1 + 0 = 0$$

$$c_1 = 0$$

$\therefore v_1$  and  $v_2$  are linearly independent.

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$$

$u_1, u_2, u_3$  are linearly independent if  $c_1u_1 + c_2u_2 + c_3u_3 = 0$

homogeneous solution

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} c_2 \\ c_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -2c_3 \\ 3c_3 \\ -c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 + c_2 + -2c_3 \\ 0 + c_2 + 3c_3 \\ c_1 + 0 - c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 0 & -1 & 0 \end{array} \right] \left[ \begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array} \right] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 0 & -1 & 0 \end{array} \right]$$

$$-R_1 + R_3 \quad \left[ \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -1 & 3 & 0 \end{array} \right]$$

$$R_2 + R_3 \quad \left[ \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 6 & 0 \end{array} \right]$$

$$\cancel{\frac{1}{6}R_3} \quad \left[ \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$C_3 = 0$$

$$C_2 + 3C_3 = 0$$

$$C_2 = 0$$

$$C_1 + C_2 - 2C_3 = 0$$

$$C_1 = 0$$

Vectors  $u_1, u_2$  and  $u_3$  are linearly independent.

$w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  &  $w_2 = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$   
 $w_1$  and  $w_2$  are linearly independent if  $c_1 w_1 + c_2 w_2 = 0$

then a trivial solution

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} -2c_2 \\ 2c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 - 2c_2 \\ c_1 - 2c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{c} \cancel{\begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}} \left( \begin{array}{c|c} 1 & -2 \\ 0 & 0 \end{array} \right) \begin{array}{l} \text{-R}_1 + \text{R}_2 \\ \Rightarrow \end{array} \end{array} \begin{array}{l} c_1 - 2c_2 = 0 \\ c_1 = 2c_2 \end{array}$$

$w_1$  and  $w_2$  are linearly dependent.

# Matrix Algebra

Is a rectangular array of numbers.

A matrix is a simple mathematical structure that holds numerical information in a rectangular or tabular form.

→ rows  
↓ columns.

Matrix are always in upper case i.e A, B, C

They are always in [ ]

$$\text{eg } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2 \times 2} \quad B = \begin{bmatrix} 1 & 2 \\ 4 & 6 \\ 1 & -1 \end{bmatrix}_{3 \times 2} \quad C = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}_{3 \times 1}$$

Terminologies used.

i) Order of a matrix | Dimension | Size

This referring to the no of rows and columns in a matrix.

written as Rows X Columns.

ii) Column matrix.

This is a matrix with only one column . it is also known as the column vector

$$\text{eg } C = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}_{3 \times 1}$$

iii) Row matrix.

This is a matrix with only one Row .

$$\text{eg } [1 \ 3 \ 4]_{1 \times 3}$$

iv) Scalar matrix.

This is a matrix that contains only one entry

$$\text{eg } D = [a]_{1 \times 1}$$

Equal matrices

These are matrices that are equal in size and have corresponding entrances eg

$$P = \begin{pmatrix} 0 & 2 \\ 5 & 1 \end{pmatrix} \quad Q = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix} \quad S = \begin{pmatrix} 0 & 2 \\ 5 & 1 \end{pmatrix}$$

$$\therefore P = S$$

Square matrix

This is a matrix with equal number of rows and columns.

$$P = \begin{pmatrix} 0 & 2 \\ 5 & 1 \end{pmatrix} \quad Q = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix} \quad S = \begin{pmatrix} 0 & 2 \\ 5 & 1 \end{pmatrix}$$

$2 \times 2 \qquad 2 \times 2 \qquad 2 \times 2$

Diagonal matrix

This is a square matrix whose elements  $a_{ij}$  in that are in the major diagonal are non zero's and all others are zero's.

major diagonal.

Identity matrix / unit matrix

This is a square matrix whose elements in the major diagonal are ones (1) and the rest are zero's.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \dots & \dots & \dots \end{bmatrix}$$

Transpose of a matrix

This is a matrix obtained after interchanging the rows of a matrix and the columns of the same matrix.

$$P^T \begin{bmatrix} 0 & 5 \\ 2 & 1 \end{bmatrix} \text{ using } P \begin{bmatrix} 0 & 2 \\ 5 & 1 \end{bmatrix}$$

Zero / null matrix.

This is a matrix with only zeros e.g.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Symmetric matrix.

This is a square matrix whose transpose is the same as the original matrix.

$$W = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} W^T = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

In short conclusion all diagonal matrix are the same of as symmetric matrix.

$$\begin{matrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

Read about

Triangular

Singular and non singular.

## Properties of Matrices

1. Matrix  $A + B = B + A$

2.  $A + B + C = A + (B + C) = (A + B) + C$

3.  $AB = BA$  if and only if one of the matrix is a scalar matrix.

And if one of the matrix is an identity rule.

4.  $A \cdot B \cdot C = A(BC) = AB(C)$

If and only if  $A_{m \times n}, B_{n \times q}, C_{q \times p}$

5.  $(A + B)C = AC + BC$

6. If we have a transpose of a product of matrices it is the same as

$$(AB)^T = A^T \cdot B^T$$

7. If we have an matrix  $A \cdot A^{-1} = I_n$  (Identity inverse)

$$A \cdot A^{-1} \times A \cdot A^{-1} = I_n$$

8. If we have an inverse of  $A$

$$(A^{-1})^{-1} = A$$

9.  $(A^{-1})^T = (A^T)^{-1}$

10.  $(AB)^{-1} = A^{-1} \times B^{-1}$

11.  $(A^T)^T = A$

12.  $(A+B)^T = A^T + B^T$

13.  $AB \neq BA$  if properties above don't hold.  
eg  $A_{m \times n}^{2 \times 2}$   $B_{n \times q}^{2 \times 2}$

14.  $A + 0 = A$

15.  $(A + (-A)) = 0$

16.  $k(A+B) = kA + kB$  ie  $k$  is a scalar.

## OPERATIONS.

- Addition & subtraction
- Scalar multiplication
- Multiplication of matrices.

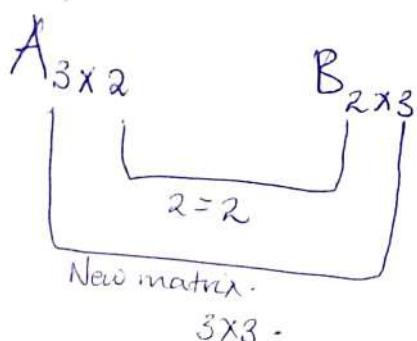
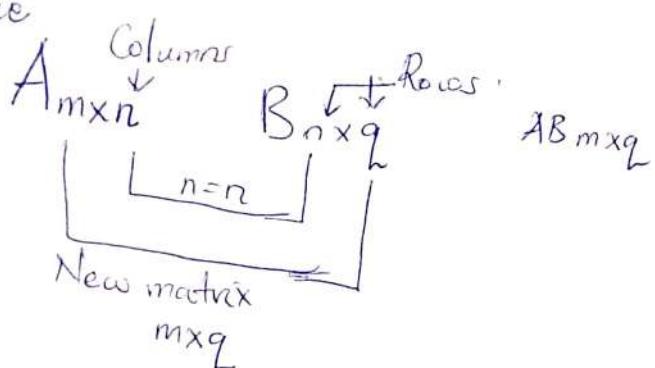
iii)  $\frac{1}{2}B$

$$= \begin{pmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

Matrix multiplication.

For multiplication to be possible it is essential that the number of columns in the 1st matrix should be equal to number of rows in the 2nd matrix

i.e



Answer  $AB_{3 \times 3}$

Given  $A = \begin{pmatrix} -2 & 4 \\ 5 & 3 \end{pmatrix}$

$E$   
 $B = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$

$$AB = \begin{pmatrix} -6+28 & -4+5 \\ 15+21 & 10+15 \end{pmatrix}$$

$$AB = \begin{pmatrix} 22 & 1 \\ 36 & 25 \end{pmatrix}$$

$$BA = \begin{pmatrix} (-6+28) \\ (-14+25) \end{pmatrix}$$

$$\begin{pmatrix} 32 \\ 15 \end{pmatrix} \quad \begin{pmatrix} -2 \\ 5 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} (-6+10) & (12+6) \\ (-14+25) & (28+15) \end{pmatrix}$$

$$BA = \begin{pmatrix} 4 & 18 \\ 11 & 43 \end{pmatrix}$$

$$AB \neq BA$$

$$A = \begin{pmatrix} -2 & 4 \\ 5 & 3 \\ 4 & -1 \end{pmatrix} \quad \xi^T \quad B = \begin{pmatrix} -2 & 4 \\ 5 & 3 \\ 4 & -1 \end{pmatrix}$$

$BA$

$$\begin{pmatrix} -2 & 4 \\ 5 & 3 \\ 4 & -1 \end{pmatrix} \quad \begin{pmatrix} -2 & 4 \\ 5 & 3 \\ 4 & -1 \end{pmatrix}$$

$$\begin{matrix} 2 \times 2 & 3 \times 2 \\ \text{---} & \text{---} \\ BA \text{ Not possible.} & \end{matrix}$$

$$AB = \begin{pmatrix} -2 & 4 \\ 5 & 3 \\ 4 & -1 \end{pmatrix} \quad \begin{pmatrix} -2 & 4 \\ 5 & 3 \end{pmatrix}$$

$$\begin{pmatrix} (4+20) & (-8+12) \\ (-10+25) & (20+9) \\ (-8-5) & (16-3) \end{pmatrix}$$

$$AB = \begin{pmatrix} 24 & 4 \\ 5 & 29 \\ -13 & 13 \end{pmatrix}$$

In any matrix product  $AB$  if the 1<sup>st</sup> matrix  $A$  has  $(m)$  rows and  $(n)$  columns and Second matrix  $B$  has  $(n)$  rows and  $(p)$  columns,

Then the product  $AB$  has  $m$  rows and  $p$  columns.

If the elements lies in the  $i$ th row and  $j$ th column of  $AB$  is the same of the products of the Corresponding elements of the  $i$ th row of  $A$  and

write column of B

$$CB \quad C = \begin{bmatrix} 2 & 1 \\ 4 & 4 \\ 3 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

$B \times 3$        $3 \times 2$        $2 \times 2$

$$CB = \begin{bmatrix} (2+3) & (4+5) \\ (4+12) & (8+20) \\ (3+6) & (6+10) \end{bmatrix} \quad CB = \begin{bmatrix} 5 & 9 \\ 16 & 28 \\ 9 & 16 \end{bmatrix}$$

Free Variable -

is a variable that can take on any value

3x3

$$\left[ \begin{array}{ccc|c} 3 & -6 & 2 & -3 \\ 5 & 9 & 4 & 1 \\ 2 & 4 & 2 & 0 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -\frac{15}{11} \end{array} \right]$$

$$z = -\frac{15}{11}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 5 & 9 & 4 & 1 \\ 3 & -6 & 2 & -3 \end{array} \right]$$

$$y+z = -1$$

$$y + -\frac{15}{11} = -1$$

$$y = -1 + \frac{15}{11}$$

$$y = \frac{-11+15}{11}$$

$$= \frac{4}{11}$$

$$\begin{aligned} & -5R_1 + R_2 = R_2 \\ & -3R_1 + R_3 = R_3 \end{aligned} \quad \left[ \begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & -12 & -1 & -3 \end{array} \right]$$

$$\begin{aligned} & 12R_2 + R_3 = R_3 \\ & \end{aligned} \quad \left[ \begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 11 & -15 \end{array} \right]$$

$$x + 2y + z = 0$$

$$x + \frac{8-15}{11} = 0$$

$$x = \frac{7}{11}$$

negthi Column of B

$$CB \quad C = \begin{bmatrix} 2 & 1 \\ 4 & 4 \\ 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

$$CB = \begin{bmatrix} (2+3) & (1+5) \\ (4+12) & (8+20) \\ (3+6) & (6+10) \end{bmatrix} \quad CB = \begin{bmatrix} 5 & 9 \\ 16 & 28 \\ 9 & 16 \end{bmatrix}$$

Free Variable -

is a variable that can take on any value

3x3

$$\left[ \begin{array}{ccc|c} 3 & -6 & 2 & -3 \\ 5 & 9 & 4 & 1 \\ 2 & 4 & 2 & 0 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -\frac{15}{11} \end{array} \right]$$

$$z = -\frac{15}{11}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 5 & 9 & 4 & 1 \\ 3 & -6 & 2 & -3 \end{array} \right]$$

$$y + z = -1$$

$$y + -\frac{15}{11} = -1$$

$$y = -1 + \frac{15}{11}$$

$$\begin{aligned} -5R_1 + R_2 &= R_2 \\ \left[ \begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array} \right] \\ -3R_1 + R_3 &= R_3 \\ \left[ \begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & -12 & -1 & -3 \end{array} \right] \end{aligned}$$

$$y = \frac{-11+15}{11}$$

$$= \frac{4}{11}$$

$$\begin{aligned} 12R_2 + R_3 &= R_3 \\ \left[ \begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 11 & -15 \end{array} \right] \end{aligned}$$

$$x + 2y + z = 0$$

$$x + \frac{8-15}{11} = 0$$

$$x = \gamma_{11}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -15/11 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 0 & 15/11 \\ 0 & 1 & 0 & 4/11 \\ 0 & 0 & 1 & -15/11 \end{array} \right] \quad \begin{matrix} -R_3 + R_1 = R_1 \\ -R_3 + R_2 = R_2 \end{matrix}$$

$$\left[ \begin{array}{ccc|c} 0 & 0 & 0 & 7/11 \\ 0 & 1 & 0 & 4/11 \\ 0 & 0 & 1 & -15/11 \end{array} \right] \quad -2R_2 + R_1 = R_1$$

A1

b).

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 3 & 2 & 1 & 3 \\ 1 & -2 & -5 & 1 \end{array} \right]$$

$$A \xrightarrow[-R_1+R_2]{3R_1+R_2} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & -2 & -6 \\ 0 & -3 & -6 & -2 \end{array} \right]$$

$$\xrightarrow{-R_2} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 6 \\ 0 & -3 & -6 & 2 \end{array} \right]$$

$$\xrightarrow[3R_2+R_3]{ } \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 20 \end{array} \right]$$

$$0 \neq 20$$

## Determinant of a matrix.

Given  $A$ , the determinant is denoted as  $\det A$  or  $|A|$  or  $\Delta A$  and can only be obtained for square matrices.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A| = ad - cb$$

$$\text{given } A = [a]$$

$$|A| = a$$

A square matrix is one which has the same number of rows and columns.

## Higher order determinants.

To obtain the determinants of a  $3 \times 3$ ,  $4 \times 4$ ,  $5 \times 5$ ,  $6 \times 6$  matrix etc., we use the Laplace expression which involves obtaining minors and cofactors.

### Minors

A minor of a matrix is denoted as  $(M_{ij})$  which is a determinant of a matrix obtained by deleting the  $i$ th row and  $j$ th column of a certain matrix.

e.g Given  $A$

$$\underset{m_i}{\underset{\text{3x4}}{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

Given  $B$

$$\underset{\text{3x3}}{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$\text{Submatrix } M_{11} = \left| \begin{bmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{bmatrix} \right|$$

$$\text{Determinant of submatrix} = (b_{22} \times b_{33}) - (b_{32} \times b_{23})$$

### Example ①

Given  $A = \begin{bmatrix} 2 & 3 & 7 \\ -4 & 0 & 6 \\ 1 & 5 & 0 \end{bmatrix}$ , find  $\det A$ .

$$M_{11} = \left| \begin{bmatrix} 0 & 6 \\ 5 & 0 \end{bmatrix} \right| = (0 \times 0) - (5 \times 6) = -30$$

$$M_{12} = \left| \begin{bmatrix} -4 & 6 \\ 1 & 0 \end{bmatrix} \right| = (0 \times -4) - (1 \times 6) = -6$$

$$M_{13} = \left| \begin{bmatrix} -4 & 0 \\ 1 & 5 \end{bmatrix} \right| = (-4 \times 5) - (0 \times 1) = -20$$

$$M_{21} = \left| \begin{pmatrix} 3 & 7 \\ 5 & 0 \end{pmatrix} \right| = (0 \times 3) - (5 \times 7) = -35$$

$$M_{22} = \left| \begin{pmatrix} 2 & 7 \\ 1 & 0 \end{pmatrix} \right| = (2 \times 0) - (1 \times 1) = -1$$

$$M_{23} = \left| \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} \right| = (5 \times 2) - (1 \times 3) = 7$$

$$M_{31} = \left| \begin{pmatrix} 3 & 7 \\ 0 & 6 \end{pmatrix} \right| = (3 \times 6) - (0 \times 7) = 18$$

$$M_{32} = \left| \begin{pmatrix} 2 & 7 \\ -4 & 6 \end{pmatrix} \right| = (2 \times 6) - (7 \times -4) = 40$$

$$M_{33} = \left| \begin{pmatrix} 2 & 3 \\ -4 & 0 \end{pmatrix} \right| = (0 \times 2) - (-4 \times 3) = 12$$

$$|A| = a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + a_{13} \cdot C_{13}$$

$$\text{or } a_{21} \cdot C_{21} + a_{22} \cdot C_{22} + a_{23} \cdot C_{23}$$

$$\text{or } a_{31} \cdot C_{31} + a_{32} \cdot C_{32} + a_{33} \cdot C_{33}$$

## Cofactors

A cofactor is a signed minor.  
denoted as  $c_{ij}$

or is a signed determinant of a submatrix.

$$c_{ij} = (-1)^{i+j} \cdot M_{ij}$$

Given example:

$$c_{11} = (-1)^{(1+1)} \cdot -30 = -30$$

$$c_{12} = (-1)^{(1+2)} \cdot 6 = 6$$

$$c_{13} = (-1)^{(1+3)} \cdot -20 = -20$$

$$c_{21} = (-1)^{(2+1)} \cdot -35 = 35$$

$$c_{22} = (-1)^{(2+2)} \cdot -7 = -7$$

$$c_{23} = (-1)^{(2+3)} \cdot 7 = -7$$

$$c_{31} = (-1)^{(3+1)} \cdot 18 = 18$$

$$c_{32} = (-1)^{(3+2)} \cdot -40 = -40$$

$$c_{33} = (-1)^{(3+3)} \cdot 12 = 12$$

$$\therefore \det A = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$$

$$= 2(-30) + 3(6) + 7(-20)$$

$$= -60 + 18 + -140$$

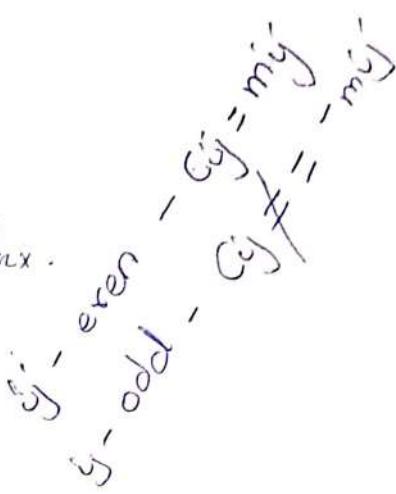
$$= -182$$

$$\text{or } \det A = a_{31}c_{31} + a_{32}c_{32} + a_{33}c_{33}$$

$$= 1(18) + 5(-40) + 0(12)$$

$$= 18 - 200 + 0$$

$$= -182$$



cofactor matrix

$$\begin{pmatrix} -30 & 6 & -20 \\ 35 & -7 & -7 \\ 18 & -40 & 12 \end{pmatrix}$$

$$\begin{pmatrix} -30 & 35 & 18 \\ 6 & -7 & -40 \\ -20 & -7 & 12 \end{pmatrix}$$

Inverse of a matrix :  $(A^{-1})$

Denoted as  $A^{-1}$ , can only be defined if  $A$  is a square matrix and it satisfies

$$A \cdot A^{-1} \times A^{-1} \cdot A = I_n$$

Procedures of determining  $A^{-1}$

- ① Determine all the minors.
- ② Find the cofactors.
- ③ Find the determinant of the matrix.
- ④ Put the cofactors together and form a cofactor matrix  $C$ .
- ⑤ Find  $C^T$  (the transpose of the cofactor matrix)
- ⑥ Determine the inverse by :-  
Adjunct matrix

$$A^{-1} = \frac{1}{|A|} \cdot \text{transpose of } C$$

Using the previous example  $A$  -

$$C = \begin{bmatrix} -30 & 6 & -20 \\ 35 & -7 & -7 \\ 18 & -40 & 12 \end{bmatrix}$$

$$C^T = \begin{bmatrix} -30 & 35 & 18 \\ 6 & -7 & -40 \\ -20 & -7 & 12 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \cdot C^T$$

$$A^{-1} = \frac{1}{-182} \cdot \begin{bmatrix} -30 & 35 & 18 \\ 6 & -7 & -40 \\ -20 & -7 & 12 \end{bmatrix}$$

$$\checkmark A^{-1} = \begin{bmatrix} \frac{30}{182} & \frac{35}{182} & \frac{18}{182} \\ \frac{-6}{182} & \frac{7}{182} & \frac{-40}{182} \\ \frac{-20}{182} & \frac{-7}{182} & \frac{12}{182} \end{bmatrix}$$

Prove for the inverse -

$$AA^{-1} = In$$

$$\left[ \begin{array}{ccc} 2 & 3 & 7 \\ -4 & 0 & 6 \\ 1 & 5 & 0 \end{array} \right] \left[ \begin{array}{ccc} \frac{30}{182} & \frac{35}{182} & \frac{-15}{182} \\ \frac{-6}{182} & \frac{7}{182} & \frac{40}{182} \\ \frac{20}{182} & \frac{7}{182} & \frac{-12}{182} \end{array} \right]$$

$$\left( 2 \times \frac{30}{182} \right) + \left( 3 \times \frac{-6}{182} \right) + \left( 7 \times \frac{20}{182} \right)$$

$$\frac{60 - 18 + 140}{182}$$

$$= \frac{182}{182}$$

$$= \underline{\underline{1}}$$

$$\left( -4 \times \frac{-18}{182} \right) + \left( 0 \times \frac{40}{182} \right) + \left( 6 \times \frac{-12}{182} \right)$$

$$\frac{72 + 0 + -72}{182}$$

$$= \underline{\underline{0}}$$

$$\left( 1 \times \frac{30}{182} \right) + \left( 5 \times \frac{7}{182} \right) + \left( 0 \times \frac{20}{182} \right)$$

$$\frac{30 - 30 + 0}{182}$$

$$= \underline{\underline{0}}$$

$$\left( 1 \times \frac{-35}{182} \right) + \left( 5 \times \frac{7}{182} \right) + \left( 0 \times \frac{1}{182} \right)$$

$$\frac{-35 + 35 + 0}{182}$$

$$= \underline{\underline{0}}$$

$$\left( 1 \times \frac{-18}{182} \right) + \left( 5 \times \frac{40}{182} \right) + \left( 0 \times \frac{6}{182} \right)$$

$$\frac{-18 + 200 + 0}{182}$$

$$= \underline{\underline{1}}$$

$$\left( 2 \times \frac{-18}{182} \right) + \left( 3 \times \frac{40}{182} \right) + \left( 7 \times \frac{-12}{182} \right)$$

$$\frac{-36 + 120 + -84}{182}$$

$$\frac{0}{182}$$

$$= \underline{\underline{0}}$$

$$\left( -4 \times \frac{30}{182} \right) + \left( 0 \times \frac{6}{182} \right) + \left( 6 \times \frac{20}{182} \right)$$

$$\frac{-120 + 0 + 120}{182}$$

$$\frac{0}{182}$$

$$= \underline{\underline{0}}$$

$$\left( 4 \times \frac{-35}{182} \right) + \left( 0 \times \frac{7}{182} \right) + \left( 6 \times \frac{7}{182} \right)$$

$$\frac{140 + 0 + 42}{182}$$

$$\frac{0}{182}$$

$$= \underline{\underline{1}}$$

$$\left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

Identity matrix

Example 3

Find the inverse of  $B = \begin{bmatrix} 3 & 5 \\ -2 & 1 \end{bmatrix}$

Note: Change the elements in main diagonal.

$$|B| = (3 \times 1) - (5 \times -2)$$

$$= 3 + 10$$

$$= 13$$

$$\text{Adjoint } B = \begin{bmatrix} 1 & -5 \\ -2 & 3 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} \cdot \begin{bmatrix} 1 & -5 \\ -2 & 3 \end{bmatrix}$$

$$\frac{1}{13} \begin{bmatrix} 1 & -5 \\ -2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{13} & -\frac{5}{13} \\ -\frac{2}{13} & \frac{3}{13} \end{bmatrix}$$

Prove the inverse.

$$B \cdot B^{-1} = I_2$$

$$\begin{bmatrix} 3 & 5 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{13} & -\frac{5}{13} \\ -\frac{2}{13} & \frac{3}{13} \end{bmatrix}$$

$$(3 \times \frac{1}{13}) + (5 \times -\frac{2}{13})$$

$$\frac{3+10}{13}$$

$$= 1$$

$$(3 \times -\frac{5}{13}) + (5 \times \frac{3}{13})$$

$$\frac{-15+15}{13}$$

$$= 0$$

$$(-2 \times \frac{1}{13}) + (1 \times \frac{3}{13})$$

$$\frac{-2+3}{13}$$

$$= 0$$

③ change sign in minor!

$$(-2 \times -\frac{5}{13}) + (1 \times \frac{3}{13})$$

$$\frac{10+3}{13}$$

$$= 1$$

$$BB^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$\boxed{A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \text{ find } A^{-1}}$$

$$|A| = (4 \times 2) - (3 \times 3)$$

$$|A| = -1$$

$$\text{Adjoint } A = \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{|A|} & \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} \end{bmatrix}$$

$$\frac{1}{-1} \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -1/2 & 3 \\ 3 & -4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$$

### Example 4

Given  $A = \begin{bmatrix} 5 & 7 & 9 \\ 4 & 3 & 8 \\ 7 & 5 & 6 \end{bmatrix}$ , find  $A^{-1}$

Given Solution.

minors:

$$m_{11} \left| \begin{pmatrix} 3 & 8 \\ 5 & 6 \end{pmatrix} \right| = 18 - 40 = -22$$

$$m_{12} \left| \begin{pmatrix} 4 & 8 \\ 1 & 6 \end{pmatrix} \right| = 24 - 56 = -32$$

$$m_{13} \left| \begin{pmatrix} 4 & 3 \\ 1 & 5 \end{pmatrix} \right| = 20 - 21 = -1$$

$$m_{21} \left| \begin{pmatrix} 1 & 9 \\ 5 & 6 \end{pmatrix} \right| = 42 - 45 = -3$$

$$m_{22} \left| \begin{pmatrix} 5 & 9 \\ 1 & 6 \end{pmatrix} \right| = 30 - 63 = -33$$

$$m_{23} \left| \begin{pmatrix} 5 & 7 \\ 1 & 5 \end{pmatrix} \right| = 25 - 49 = -24$$

$$M_{31} \left| \begin{pmatrix} 7 & 9 \\ 3 & 8 \end{pmatrix} \right| = 56 - 21 = 29$$

$$M_{32} \left| \begin{pmatrix} 5 & 9 \\ 4 & 8 \end{pmatrix} \right| = 40 - 36 = 4$$

$$M_{33} \left| \begin{pmatrix} 5 & 7 \\ 4 & 3 \end{pmatrix} \right| = 15 - 28 = -13$$

Cofactors:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$C_{11} = (-1)^{1+1} \cdot (-22) = 22$$

$$C_{12} = (-1)^{1+2} \cdot (-32) = 32$$

$$C_{13} = (-1)^{1+3} \cdot (-1) = 1$$

$$C_{21} = (-1)^{2+1} \cdot (-3) = 3$$

$$C_{22} = (-1)^{2+2} \cdot (-33) = -33$$

$$C_{23} = (-1)^{2+3} \cdot (-24) = 24$$

$$C_{31} = (-1)^{3+1} \cdot (29) = 29$$

$$C_{32} = (-1)^{3+2} \cdot (4) = -4$$

$$C_{33} = (-1)^{3+3} \cdot (-13) = -13$$

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= (5 \times 22) + (7 \times 32) + (9 \times 1) \\ &= 110 + 224 + -9 \\ &= 325 \end{aligned}$$

$$|A| = 105$$

Cofactor matrix  $\begin{bmatrix} 22 & 32 & 1 \\ 3 & -33 & 24 \\ 29 & -4 & -13 \end{bmatrix}$

$$\begin{bmatrix} 22 & 32 & 1 \\ 3 & -33 & 24 \\ 29 & -4 & -13 \end{bmatrix}$$

$$C^T = \begin{bmatrix} -22 & 3 & 29 \\ 32 & -33 & -4 \\ -1 & 24 & -13 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \cdot C^T$$

$$\frac{1}{105} \cdot \begin{bmatrix} -22 & 3 & 29 \\ 32 & -33 & -4 \\ -1 & 24 & -13 \end{bmatrix}$$

$$A^{-1} = \underline{\begin{bmatrix} -\frac{22}{105} & \frac{3}{105} & \frac{29}{105} \\ \frac{32}{105} & -\frac{33}{105} & -\frac{4}{105} \\ -\frac{1}{105} & \frac{24}{105} & -\frac{13}{105} \end{bmatrix}}$$

Prove for the inverse.

$$A \cdot A^{-1} = I_n$$

$$\begin{bmatrix} 5 & 7 & 9 \\ 4 & 3 & 8 \\ 7 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} -\frac{22}{105} & \frac{3}{105} & \frac{29}{105} \\ \frac{32}{105} & -\frac{33}{105} & -\frac{4}{105} \\ -\frac{1}{105} & \frac{24}{105} & -\frac{13}{105} \end{bmatrix}$$

$$(5 \times -\frac{22}{105}) + (7 \times \frac{3}{105}) + (9 \times \frac{29}{105})$$

$$\frac{-110 + 21 + 261}{105}$$

$$= 1$$

$$(5 \times \frac{3}{105}) + (7 \times -\frac{33}{105}) + (9 \times \frac{24}{105})$$

$$\frac{15 + -231 + 216}{105}$$

$$= 0$$

$$(5 \times \frac{29}{105}) + (7 \times -\frac{4}{105}) + (9 \times -\frac{13}{105})$$

$$= 0$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# APPLICATION OF MATRIX ALGEBRA

Storing information.

No of employees in an organisation

Transport tasks.

Branch	A	B	C
Kta	20	30	10
Area	15	16	10
Gurgaon	10	20	17

[2]	Peon
[3]	Technicians
[1]	Operations manager

① Business applications.

Applications of matrix algebra.

② Solving linear equations.

③ Input and output analysis | industry analysis.

④ Eigen vectors and given values.

① BUSINESS APPLICATIONS | MANAGEMENT.

In business mgmt matrix algebra can be used to determine prices, profits, costs, revenue and any other business situations.

Example 1

The quarterly sales of books pens and pencils for the years 2020 and 2021 are given below.

Year 2020 (A)				Year 2021 (B)				
Books	Q1	Q2	Q3	Q4	Books	Q1	Q2	
Books	20	25	22	20	Pens	10	15	20
Pens	10	20	18	10	Pencils	5	20	18
Pencils	15	20	15	15	Pencils	8	30	15

Find the total quarterly sales of books, pens and pencils of the two years.

$$A+B = \begin{bmatrix} 30 & 40 & 42 & 40 \\ 15 & 40 & 36 & 20 \\ 23 & 50 & 30 & 25 \end{bmatrix}$$

Nelly  
P Q R

### Example ③

Nelly, Marvin and Becky purchased biscuits of different brands P, Q and R. Nelly purchase 10 packets of P, 7 packets of Q and 3 packets of R. Marvin purchased 4 packets of P, 8 packets of Q and 10 packets of R. Becky purchased 4 packets of P, 7 packets of Q and 8 packets of R. If brand P costs Shs 600, Q Shs 400 and R Shs 500 each, using matrix operation, find the amount of money spent by these persons individually.

Solution

Let Q denote matrix for biscuit quantities  
and P denote matrix for biscuit prices.

$$Q = \begin{matrix} \text{Nelly} \\ \text{Marvin} \\ \text{Becky} \end{matrix} \begin{bmatrix} P & Q & R \\ 10 & 7 & 3 \\ 4 & 8 & 10 \\ 4 & 7 & 8 \end{bmatrix} \quad P = \begin{matrix} \text{Price of P} \\ \text{Price of Q} \\ \text{Price of R} \end{matrix} \begin{bmatrix} 600 \\ 400 \\ 500 \end{bmatrix}$$

$$Q \times P = \begin{bmatrix} 10 & 7 & 3 \\ 4 & 8 & 10 \\ 4 & 7 & 8 \end{bmatrix} \begin{bmatrix} 600 \\ 400 \\ 500 \end{bmatrix}$$

$$= \begin{bmatrix} (10 \times 600) + (7 \times 400) + (3 \times 500) \\ (4 \times 600) + (8 \times 400) + (10 \times 500) \\ (4 \times 600) + (7 \times 400) + (8 \times 500) \end{bmatrix}$$

$$= \begin{bmatrix} 6000 + 2800 + 1500 \\ 2400 + 3200 + 5000 \\ 2400 + 2800 + 4000 \end{bmatrix}$$

$$= \begin{bmatrix} 10,300 \\ 10,600 \\ 9,200 \end{bmatrix} \begin{matrix} \text{Nelly} \\ \text{Marvin} \\ \text{Becky} \end{matrix}$$

Example (4)

To

Ms Jane went to a market to purchase 3kg of Sugar, 5 kg of wheat and 1kg of salt. In a shop near Ms Jane's residence, these commodities are priced at 3400, 6000 and 1000 shillings respectively whereas in a local market, these commodities are priced at 3000, 5500 and 700 shillings respectively. If the cost of travelling to the local market is shs 1000, find the net savings of Ms Jane using matrix multiplication method.

Solution:

Let  $J$  denote the matrix of items to be purchased by Jane  
 $P$  denote the price of these items at nearby shop  
 $\Omega$  denote the price of these items in the market.

$$J \times P = \begin{bmatrix} S & W & Sa \\ 3 & 10 & 1 \end{bmatrix} \times \begin{bmatrix} 3400 \\ 6000 \\ 1000 \end{bmatrix}$$

$$= 3 \times 3400 + 10 \times 6000 + 1 \times 1000$$

$$= 10,200 + 60,000 + 1000$$

$$= 71,200 \text{ at nearby shop. : Prices}$$

$$J \times \Omega = \begin{bmatrix} S & W & Sa \\ 3 & 10 & 1 \end{bmatrix} \times \begin{bmatrix} 3000 \\ 5500 \\ 700 \end{bmatrix}$$

$$= 9000 + 55000 + 700$$

$$= 64,700 \text{ at market}$$

Add Transport of 1000

$$= 65,700 \text{ at market.}$$

$$\begin{bmatrix} 71,200 & 64,700 \end{bmatrix}$$

$$= 71,200 - 65,700$$

$$= 5500 = \text{net savings.}$$

$$\begin{array}{r} \text{Item} \\ \hline S & \begin{bmatrix} 3400 & 6000 & 1000 \end{bmatrix} \\ W & \begin{bmatrix} 3000 & 5500 & 700 \end{bmatrix} \\ Sa & \begin{bmatrix} 3 \\ 10 \\ 1 \end{bmatrix} \end{array}$$

$$3400 \times 3 + 6000 \times 10 + 1000 \times 1$$

$$3000 \times 3 + 5500 \times 10 + 700 \times 1$$

=

coefficient matrix  
 $A$   
 matrix for unknowns  
 $X = A^{-1}B$

## Solving Systems of linear Equations

For equations with more than one unknown, we use the matrix algebra to solve for the unknowns using the matrix inverse.

Given  $AX = B$

where  $A$  = matrix for the coefficients

$X$  = matrix for the unknowns

$B$  = matrix for the constants.

multiply by  $A^{-1}$  all through

$$A \cdot A^{-1}X = A^{-1}B$$

but  $A \cdot A^{-1} = I_n$ .

$$\Rightarrow IX = A^{-1}B$$

$$\therefore X = A^{-1}B$$

$$3x + 2y = 8$$

$$4x - 3y = 5$$

$$X = \frac{Dx}{D}, Y = \frac{Dy}{D}$$

Cramer's rule

$D$  = Determinant of original matrix

$D_x$  = Determinant of original matrix but with solution on RHS substituted in for  $X$

$D_y$  = Determinant of original matrix but with solution on RHS substituted in for  $Y$ .

### Example ①

Given  $3x + 2y = 8$ , find the value of  $X$  and  $y$ .

$$4x - 3y = 5$$

Solution:

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} B \\ 8 \\ 5 \end{bmatrix}$$

$$\det A = -3 \times 3 - (4 \times 2) \\ = -9 - 8 \\ = -17$$

$$\text{Adjoint } A = \begin{bmatrix} -3 & -2 \\ -4 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \cdot \text{Adjoint } A$$

$$= \frac{1}{-17} \begin{bmatrix} -3 & -2 \\ -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{17} & \frac{2}{17} \\ \frac{4}{17} & -\frac{3}{17} \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{17} & \frac{2}{17} \\ \frac{4}{17} & -\frac{3}{17} \end{bmatrix} \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

$$x = \left( \frac{3}{17} \times 8 \right) + \left( \frac{2}{17} \times 5 \right) = \frac{24 + 10}{17}$$

$$x = \frac{24 + 10}{17} \\ = \frac{34}{17} \\ = 2$$

$$X = \frac{Dx}{D} \\ = \frac{-34}{-17} \\ = 2$$

$$x = 2,$$

$$Y = \frac{Dy}{D}$$

$$y = \left( \frac{4}{17} \times 8 \right) + \left( \frac{-3}{17} \times 5 \right)$$

$$Dy = \begin{bmatrix} 3 & 8 \\ 4 & 5 \end{bmatrix}$$

$$= \frac{32 - 15}{17} \\ = \frac{17}{17} \\ = 1$$

$$= (8 \times 3) - (4 \times 8)$$

$$= 15 - 32$$

$$= -17$$

$$\therefore Y = \frac{Dy}{D}$$

$$= \frac{-17}{-17}$$

$$y = 1$$

$$(Using Cramer's rule)$$

$$x = \frac{Dx}{D}$$

$$D = -17$$

$$Dx = \begin{bmatrix} 8 & 2 \\ 5 & -3 \end{bmatrix}$$

$$= (-3 \times 8) - (5 \times 2)$$

$$= -24 - 10$$

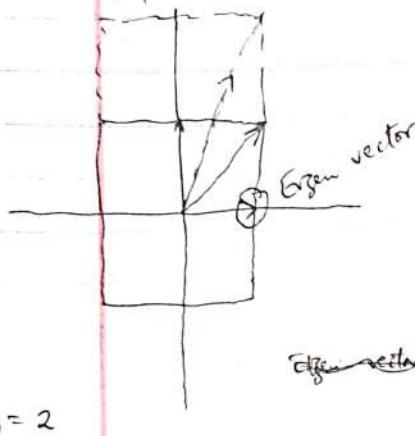
$$= -34$$

# Eigen Vectors

02/12/2021  
Thursday.

An Eigen vector is a non zero vector whose direction doesn't change when linear transformation is applied to it.

Eigen values or scalars are a special type of scalars that are associated with the linear system of equations or matrix equations that are known as the x-tic roots, values, proper scalar or latent variables



$$\lambda = 2$$

Mathematical expression:

$$\text{LHS } AX = \lambda X$$

$A \rightarrow$  matrix of transformation

$X \rightarrow$  Eigen vector

RHS

$\lambda$  = Eigen value.

Proof

$$\text{Given } AX = \lambda X$$

$$\Rightarrow AX - \lambda X = 0$$

$$(A - \lambda I)X = 0$$

$$\Rightarrow (A - \lambda I)X = 0 \quad \begin{matrix} \text{produces Eigen} \\ \text{vector} \end{matrix}$$

$$|A - \lambda I| = 0 \quad \begin{matrix} \text{produces Eigen} \\ \text{values.} \end{matrix}$$

## Example ①

Show that  $X = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$  is the Eigen vector for  $A = \begin{pmatrix} 3 & 2 \\ 2 & +2 \end{pmatrix}$  corresponding to  $\lambda = 4$ .

Solution

$$AX = \lambda X$$

$$\text{LHS} = AX = \begin{pmatrix} 3 & 2 \\ 2 & +2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \times 2 + 2 \times 1 \\ 2 \times 2 + (+2 \times 1) \end{pmatrix}$$

$$AX = \begin{pmatrix} 8 \\ 6 \end{pmatrix} .$$

$$\text{RHS} = \lambda X = 4 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

$\therefore$  Since LHS  $\neq$  RHS i.e.  $AX \neq \lambda X$ , vector  $X = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$  is not an eigen vector for  $A = \begin{pmatrix} 3 & 2 \\ 2 & +2 \end{pmatrix}$

## Example ②

Given  $A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$ , find all the Eigen vectors and their Eigen values.

Solution:

$$|A - \lambda I| = 0$$

$$\left| \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

Eigen vector for matrix  $A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$  corresponding to Eigen value  $\lambda = 2$  is  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\left| \begin{pmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{pmatrix} \right| = 0$$

$$(4-\lambda)(1-\lambda) + 2 = 0$$

$$4 - 4\lambda - \lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$\lambda^2 - 2\lambda - 3\lambda + 6 = 0$$

$$\lambda(\lambda-2) - 3(\lambda-2) = 0$$

$$(\lambda-3)(\lambda-2) = 0$$

$$\lambda-3 = 0$$

$$\Rightarrow \lambda_1 = 3$$

$$\text{or } \lambda-2 = 0$$

$$\lambda_2 = 2$$

Eigen values are  $\lambda_1 = 3$  and  $\lambda_2 = 2$ .

$$\text{using } \lambda_1 = 3$$

$$(A - \lambda_1 I)X = 0$$

$$\left[ \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] X = 0$$

$$\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{c|c|c} R_1 & -2 & 1 & | 0 \\ R_2 & -2 & 1 & | 0 \end{array}$$

$$C_{ij} = -1^{(i+j)} \cdot M_{ij}$$

$$R_1 \rightarrow R_1 - R_2$$

$$\begin{matrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{matrix}$$

$$\Rightarrow x_1 - \frac{1}{2}x_2 = 0$$

$$\Rightarrow x_1 = \frac{1}{2}x_2$$

$$\Rightarrow 2x_1 = x_2$$

let  $x_2 = r$

~~let  $x_1 = s$~~

~~2s = r~~

~~2x\_1 = r~~

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =$$

using  $\lambda_2 = 2$ ,

$$(A - \lambda_2 I)x = 0$$

$$\left[ \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] x = 0$$

$$\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$R_1 \left[ \begin{array}{cc|c} -1 & 1 & 0 \end{array} \right]$$

$$R_2 \left[ \begin{array}{cc|c} -2 & 2 & 0 \end{array} \right]$$

$$\Rightarrow R_1 \rightarrow R_1 + R_2$$

$$-2R_1 + R_2 \left[ \begin{array}{cc|c} 0 & 0 & 0 \end{array} \right]$$

$$\Rightarrow x_1 - x_2 = 0$$

$$x_1 = x_2$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}$$

let  $x_1 = v$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} v$$

i.e.  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  is the eigenvector corresponding

to eigen value  $\lambda = 3$

and  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  is the eigenvector corresponding

to eigen value  $\lambda = 2$ .

Example ③ Given  $B = \begin{pmatrix} 7 & 3 \\ -3 & 1 \end{pmatrix}$ , find all the Eigen vectors & their corresponding Eigen values

Solutions:

$$|A - \lambda I| = 0$$

$$B = \begin{pmatrix} 7 & 3 \\ 3 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 7 & 3 \\ -3 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = 0$$

$$\begin{pmatrix} 7-\lambda & 3 \\ -3 & 1-\lambda \end{pmatrix} = 0$$

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{ E.V.} \quad \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$(7-\lambda)(1-\lambda) + 9 = 0$$

$$7 - 7\lambda - \lambda + \lambda^2 + 9 = 0$$

$$\lambda^2 - 8\lambda + 16 = 0$$

$$\lambda^2 - 4\lambda - 4\lambda + 16 = 0$$

$$\lambda(\lambda-4) - 4(\lambda-4) = 0$$

$$(\lambda-4)(\lambda-4) = 0$$

$\lambda = 4$ .  
Eigen value = 4.

$$(A - \lambda I)x = 0$$

$$\begin{pmatrix} 7 & 3 \\ -3 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} x = 0$$

$$\begin{pmatrix} 3 & 3 \\ -3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$R_1 \left[ \begin{array}{cc|c} 3 & 3 & 0 \\ -3 & -3 & 0 \end{array} \right]$$

$$\frac{1}{3}R_1 \rightarrow R_1 \left[ \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$R_1 + R_2 \rightarrow R_2 \left[ \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$\Rightarrow x_1 = -x_2 \Rightarrow x_2 = -x_1$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix}$$

let  $x_1 = r$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} r \\ -r \end{pmatrix} = r \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Example (4)

Given  $A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$  find all the Eigen vectors and the corresponding Eigen values.

Solution:

$$|A - \lambda I| = 0$$

$$\left| \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{pmatrix} \right| = 0$$

$$M_{11} = \begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = \frac{(2-\lambda)(5-\lambda)-4}{\cancel{10}-2\lambda-\cancel{5}\lambda+\cancel{\lambda^2}-4} = \underline{\underline{\lambda^2-7\lambda+6}}$$

$$M_{12} = \begin{vmatrix} 4 & 2 \\ 2 & 2-\lambda \end{vmatrix} = \frac{4(2-\lambda)}{\cancel{8}-4\lambda-4} = \underline{\underline{-4\lambda+4}}$$

$$M_{13} = \begin{vmatrix} 4 & 5-\lambda \\ 2 & 2 \end{vmatrix} = \frac{8-2(5-\lambda)}{8-10+2\lambda} = \underline{\underline{2\lambda-2}}$$

$$M_{21} = \begin{vmatrix} 4 & 2 \\ 2 & 2-\lambda \end{vmatrix} = \frac{4(2-\lambda)}{8-4\lambda-4} = \underline{\underline{-4\lambda+4}}$$

$$M_{22} = \begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = \frac{(2-\lambda)(5-\lambda)}{8-10+2\lambda} = \underline{\underline{\lambda^2-7\lambda+6}}$$

$$M_{23} = \begin{vmatrix} 5-\lambda & 4 \\ 2 & 2 \end{vmatrix} = \frac{2(5-\lambda)}{10-2\lambda-8} = \underline{\underline{-2\lambda+2}}$$

$$M_{31} = \begin{vmatrix} 4 & 2 \\ 5-\lambda & 2 \end{vmatrix} = \frac{8-2(5-\lambda)}{8-10+2\lambda} = \underline{\underline{2\lambda-2}}$$

$$M_{32} = \begin{vmatrix} 5-\lambda & 2 \\ 4 & 2 \end{vmatrix} = \frac{2(5-\lambda)}{10-2\lambda-8} = \underline{\underline{-2\lambda+2}}$$

$$M_{33} = \begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = \frac{(5-\lambda)(5-\lambda)}{25-5\lambda-5\lambda+\lambda^2-16} = \underline{\underline{\lambda^2-10\lambda+9}}$$

$$C_{11} = -1^{(11)} \cdot M_{11} = \lambda^2 - 7\lambda + 6$$

$$= \lambda^2 - 7\lambda + 6$$

$$C_{12} = -1^{(12)} \cdot M_{12} = 4\lambda - 4$$

$$C_{13} = -1^{(13)} \cdot M_{13} = 2\lambda - 2$$

$$C_{21} = -1^{(21)} \cdot M_{21} = 4\lambda - 4$$

$$C_{22} = -1^{(22)} \cdot M_{22} = \lambda^2 - 7\lambda + 6$$

$$C_{23} = -1^{(23)} \cdot M_{23} = 2\lambda - 2$$

$$C_{31} = -1^{(31)} \cdot M_{31} = 2\lambda - 2$$

$$C_{32} = -1^{(32)} \cdot M_{32} = 2\lambda - 2$$

$$C_{33} = -1^{(33)} \cdot M_{33} = \lambda^2 - 10\lambda + 9$$

Now

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= (5-\lambda)(\lambda^2 - 7\lambda + 6) + 4(4\lambda - 4) + 2(2\lambda - 2)$$

$$= 5\lambda^2 - \lambda^3 - 35\lambda + 7\lambda^2 + 30 - 6\lambda + 16\lambda - 16 + 4\lambda - 4$$

$$= -\lambda^3 + 12\lambda^2 - 21\lambda + 10 = 0$$

$\Rightarrow$  solve for  $\lambda$  to get  $\lambda_1 = 10$  &  $\lambda_2 = 1$ .

for  $\lambda_1 = 10$ ,

$$(A - \lambda I)x = 0$$

$$\left( \begin{matrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{matrix} - \begin{matrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{matrix} \right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$R_1 \left[ \begin{array}{ccc|c} -5 & 4 & 2 & 0 \end{array} \right]$$

$$R_2 \left[ \begin{array}{ccc|c} 4 & -5 & 2 & 0 \end{array} \right]$$

$$R_3 \left[ \begin{array}{ccc|c} 2 & 2 & -8 & 0 \end{array} \right]$$

$$\rightarrow R_1 \left[ \begin{array}{ccc|c} -5 & 4 & 2 & 0 \end{array} \right]$$

$$4R_1 + 5R_2 \rightarrow R_2 \left[ \begin{array}{ccc|c} 0 & -9 & 18 & 0 \end{array} \right]$$

$$2R_1 + 5R_3 \rightarrow R_3 \left[ \begin{array}{ccc|c} 0 & 18 & -36 & 0 \end{array} \right]$$

$$R_1 \left[ \begin{array}{ccc|c} -5 & 4 & 2 & 0 \end{array} \right]$$

$$R_2 \left[ \begin{array}{ccc|c} 0 & -9 & 18 & 0 \end{array} \right]$$

$$2R_2 + R_3 \rightarrow R_3 \left[ \begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array} \right]$$

$$\Rightarrow -9X_2 + 18X_3 = 0$$

$$\frac{-9X_2}{-9} = \frac{18X_3}{-9}$$

$$36 \quad \text{or} \quad 38$$

back  $X_2 = 2X_3$

Substitute for  $X_2$

$$\Rightarrow -5X_1 + 4X_2 + 2X_3 = 0$$

$$\Rightarrow -5X_1 + 4(2X_3) + 2X_3 = 0$$

$$-5X_1 + 8X_3 + 2X_3 = 0$$

$$-5X_1 + 10X_3 = 0$$

$$\frac{-5X_1}{-5} = \frac{-10X_3}{-5}$$

$$X_1 = 2X_3$$

$$\begin{array}{c} \xrightarrow{\text{R}_1 - 5\text{R}_2} \\ \xrightarrow{\text{R}_3 - \text{R}_2} \\ \xrightarrow{\text{R}_1 - 5\text{R}_3} \\ \xrightarrow{\text{R}_2 - 2\text{R}_3} \\ \xrightarrow{\text{R}_3 - \text{R}_2} \\ \xrightarrow{\text{R}_1 - 5\text{R}_3} \\ \xrightarrow{\text{R}_2 - 2\text{R}_3} \\ \xrightarrow{\text{R}_3 - \text{R}_2} \\ \xrightarrow{\text{R}_1 - 5\text{R}_3} \\ \xrightarrow{\text{R}_2 - 2\text{R}_3} \\ \xrightarrow{\text{R}_3 - \text{R}_2} \end{array}$$

$$\Rightarrow \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 2X_3 \\ 2X_3 \\ X_3 \end{pmatrix} = X_3 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$\therefore \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$  is the eigen vector for the matrix

$$\begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix} \text{ corresponding to the eigen vector } \lambda_1 = 10.$$

for  $\lambda_2 = 1$ ,

$$(A - \lambda_2 I)X = 0$$

$$\begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} X = 0$$

$$\begin{pmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{r} R_1 \\ R_2 \\ R_3 \end{array} \left| \begin{array}{ccc|c} 4 & 4 & 2 & 0 \\ 4 & 4 & 2 & 0 \\ 2 & 2 & 1 & 0 \end{array} \right.$$

$$\begin{array}{r} R_1 \\ R_2 \\ -R_1 + 2R_3 \end{array} \left| \begin{array}{ccc|c} 4 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right.$$

$$\Rightarrow 4X_1 + 4X_2 + 2X_3 = 0$$

$$\Rightarrow 2X_3 = -4X_1 - 4X_2$$

$$\Rightarrow X_3 = -2X_1 - 2X_2$$

$$\begin{cases} X_1 \\ X_2 \\ X_3 \end{cases} = \begin{pmatrix} X_1 \\ X_2 \\ -2X_1 - 2X_2 \end{pmatrix}$$

$$= X_1 \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + X_2 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}.$$

$\therefore \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$  are the eigen vectors for the matrix  $\begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$  corresponding to the eigen value  $\lambda_2 = 1$ .

Algebraic multiplicity & Geometric

Diagonalization: produces a diagonal matrix if it is diagonalizable. The application of Eigen vectors. we use the formula

$$D = P^{-1}AP,$$

where  $P$  is a matrix of Eigen vectors  $(X_1, X_2, X_3, \dots, X_n)$

and  $P^{-1}$  is the inverse of  $P$ .

Example ①

Diagonalize matrix  $A = \begin{pmatrix} -3 & 4 \\ 5 & 6 \end{pmatrix}$ ,

Solution:

$$\begin{aligned} |A - \lambda I| &= 0 & \lambda_1 = 2, \lambda_2 = 1 \\ (-3 - \lambda) &- (4 - \lambda) = 0 & \begin{pmatrix} -4 & 1 \\ 1 & 1 \end{pmatrix} \\ (5 - \lambda) &- (6 - \lambda) = 0 & P = \begin{pmatrix} -4 & -1 \\ 1 & 1 \end{pmatrix} \\ (-3 - \lambda) &- (4 - \lambda) = 0 & P^{-1} = \begin{pmatrix} 5 & 5 \\ -5 & -4 \end{pmatrix} \\ (6 - \lambda)(-3 - \lambda) - 20 &= 0 & \\ -18 - 6\lambda + 3\lambda + \lambda^2 - 20 &= 0 & \\ \lambda^2 - 3\lambda - 38 &= 0 & \end{aligned}$$

Conditions for diagonalization:

- Have same number of eigen vectors as eigen values
- $n$  eigen vectors that are linearly independent.

Example ④

Given  $A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$  find all the Eigen vectors and the corresponding Eigen values.

Solution:

$$(A - \lambda I) = 0$$

$$\left| \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{pmatrix} \right| = 0$$

$$M_{11} = \begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = \frac{(2-\lambda)(5-\lambda)-4}{\cancel{\lambda^2-7\lambda+6}} = 10-2\lambda-5\lambda+\cancel{\lambda^2-4}$$

$$M_{12} = \begin{vmatrix} 4 & 2 \\ 2 & 2-\lambda \end{vmatrix} = \frac{4(2-\lambda)-4}{\cancel{8-4\lambda-4}} = \underline{-4\lambda+4}$$

$$M_{13} = \begin{vmatrix} 4 & 5-\lambda \\ 2 & 2 \end{vmatrix} = \frac{8-2(5-\lambda)}{\cancel{8-10+2\lambda}} = \underline{2\lambda-2}$$

$$M_{21} = \begin{vmatrix} 4 & 2 \\ 2 & 2-\lambda \end{vmatrix} = \frac{4(2-\lambda)-4}{\cancel{8-4\lambda-4}} = \underline{-4\lambda+4}$$

$$M_{22} = \begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = \frac{(2-\lambda)(5-\lambda)-4}{\cancel{\lambda^2-7\lambda+6}} = \underline{\lambda^2-7\lambda+6}$$

$$M_{23} = \begin{vmatrix} 5-\lambda & 4 \\ 2 & 2 \end{vmatrix} = \frac{2(5-\lambda)-8}{\cancel{10-2\lambda-8}} = \underline{-2\lambda+2}$$

$$M_{31} = \begin{vmatrix} 4 & 2 \\ 5-\lambda & 2 \end{vmatrix} = \frac{8-2(5-\lambda)}{\cancel{8-10+2\lambda}} = \underline{2\lambda-2}$$

$$M_{32} = \begin{vmatrix} 5-\lambda & 2 \\ 4 & 2 \end{vmatrix} = \frac{2(5-\lambda)-8}{\cancel{10-2\lambda-8}} = \underline{-2\lambda+2}$$

$$M_{33} = \begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = \frac{(5-\lambda)(5-\lambda)-16}{\cancel{25-5\lambda-5\lambda+\lambda^2-16}} = \underline{\lambda^2-10\lambda+9}$$

$$C_{11} = -1^{(1+1)} \cdot M_{11} = \lambda^2 - 7\lambda + 6 \quad \text{if } j = \text{even} +$$

$$C_{11} = -1^{(1+1)} \cdot M_{11} = \lambda^2 - 7\lambda + 6$$

$$C_{12} = -1^{(1+2)} \cdot M_{12} = 4\lambda - 4$$

$$C_{13} = -1^{(1+3)} \cdot M_{13} = 2\lambda - 2$$

$$C_{21} = -1^{(2+1)} \cdot M_{21} = 4\lambda - 4$$

$$C_{22} = -1^{(2+2)} \cdot M_{22} = \lambda^2 - 7\lambda + 6$$

$$C_{23} = -1^{(2+3)} \cdot M_{23} = 2\lambda - 2$$

$$C_{31} = -1^{(3+1)} \cdot M_{31} = 2\lambda - 2$$

$$C_{32} = -1^{(3+2)} \cdot M_{32} = 2\lambda - 2$$

$$C_{33} = -1^{(3+3)} \cdot M_{33} = \lambda^2 - 10\lambda + 9$$

Ans

$$\begin{aligned} |A| &= a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} \\ &= (5-\lambda)(\lambda^2 - 7\lambda + 6) + 4(4\lambda - 4) + 2(2\lambda - 2) \\ &= 5\lambda^2 - \cancel{3\lambda^3} - 35\lambda + 7\lambda^2 + 30 - 6\lambda + 16\lambda - 16 + 4\lambda - 4 \\ &= -\lambda^3 + 12\lambda^2 - 21\lambda + 10 = 0 \\ \Rightarrow \text{solve for } \lambda \text{ to get } \lambda_1 = 10 \text{ & } \lambda_2 = 1. \end{aligned}$$

for  $\lambda_1 = 10$ ,

$$(A - \lambda I)X = 0$$

$$\left( \begin{matrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{matrix} \right) - \left( \begin{matrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{matrix} \right) \left( \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \right) = \left( \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right)$$

$$\left( \begin{matrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{matrix} \right) \left( \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \right) = \left( \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right)$$

$$R_1 \left[ \begin{array}{ccc|c} -5 & 4 & 2 & 0 \end{array} \right]$$

$$R_2 \left[ \begin{array}{ccc|c} 4 & -5 & 2 & 0 \end{array} \right]$$

$$R_3 \left[ \begin{array}{ccc|c} 2 & 2 & -8 & 0 \end{array} \right]$$

$$\rightarrow R_1 \left[ \begin{array}{ccc|c} -5 & 4 & 2 & 0 \end{array} \right]$$

$$4R_1 + SR_2 \rightarrow R_2 \left[ \begin{array}{ccc|c} 0 & -9 & 18 & 0 \end{array} \right]$$

$$2R_1 + SR_3 \rightarrow R_3 \left[ \begin{array}{ccc|c} 0 & 18 & -36 & 0 \end{array} \right]$$

$$R_1 \left[ \begin{array}{ccc|c} -5 & 4 & 2 & 0 \end{array} \right]$$

$$R_2 \left[ \begin{array}{ccc|c} 0 & -9 & 18 & 0 \end{array} \right]$$

$$2R_2 + R_3 \rightarrow R_3 \left[ \begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{aligned} \Rightarrow -9x_2 + 18x_3 &= 0 & 36 \\ -9x_2 &= -18x_3 & 38 \\ -9 & \end{aligned}$$

back  $x_2 = 2x_3$

Substitute for  $x_2$

$$\begin{aligned} \Rightarrow -5x_1 + 4x_2 + 2x_3 &= 0 \\ \Rightarrow -5x_1 + 4(2x_3) + 2x_3 &= 0 \\ -5x_1 + 8x_3 + 2x_3 &= 0 \\ -5x_1 + 10x_3 &= 0 \\ -5x_1 &= -10x_3 \\ -5 & \end{aligned}$$

$x_1 = 2x_3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ 2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$\therefore \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$  is the eigen vector for the matrix

$$\begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix} \text{ corresponding to the eigen vector } \lambda_1 = 10.$$

for  $\lambda_2 = 1$ ,

$$(A - \lambda_2 I)x = 0$$

$$\left[ \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] x = 0$$

$$\begin{pmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{r} R_1 \\ R_2 \\ R_3 \end{array} \left[ \begin{array}{ccc|c} 4 & 4 & 2 & 0 \\ 4 & 4 & 2 & 0 \\ 2 & 2 & 1 & 0 \end{array} \right]$$

$$R_1 \left[ \begin{array}{ccc|c} 4 & 4 & 2 & 0 \end{array} \right]$$

$$-R_1 + R_2 \rightarrow R_2 \left[ \begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array} \right]$$

$$-R_1 + 2R_3 \rightarrow R_3 \left[ \begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array} \right]$$

$$\Rightarrow 4x_1 + 4x_2 + 2x_3 = 0$$

$$\Rightarrow 2x_3 = -4x_1 - 4x_2$$

$$\Rightarrow x_3 = -2x_1 - 2x_2$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ -2x_1 - 2x_2 \end{pmatrix}$$

$$= x_1 \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}.$$

$\therefore \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$  are the Eigen vectors for the matrix  $\begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$  corresponding to the eigen value  $\lambda_2 = 1$ .

Algebraic multiplicity & Geometric

Diagonalization:  $\Rightarrow$  produces a diagonal matrix if at all the matrix is diagonalizable.

The application of Eigen vectors.

We use the formula

$$D = P^{-1}AP,$$

where  $P$  is a matrix of Eigen vectors  $(x_1, x_2, x_3, \dots, x_n)$

and  $P^{-1}$  is the inverse of  $P$ .

Example ①

Diagonalize matrix  $A = \begin{pmatrix} -3 & 4 \\ 5 & 6 \end{pmatrix}$ .

Solution:

$$\begin{array}{l} |A - \lambda I| = 0 \\ \begin{pmatrix} -3-\lambda & 4 \\ 5 & 6-\lambda \end{pmatrix} = 0 \\ (-3-\lambda)(6-\lambda) = 0 \end{array} \quad \begin{array}{l} \lambda_1 = 2, \lambda_2 = 1 \\ \begin{pmatrix} -4 & 1 \\ 1 & 1 \end{pmatrix} \\ P = \begin{pmatrix} -4 & 1 \\ 1 & 1 \end{pmatrix} \\ P^{-1} = \begin{pmatrix} 5 & 5 \\ -5 & -4 \end{pmatrix} \end{array}$$

$$(6-\lambda)(-3-\lambda) - 20 = 0$$

$$-18 - 6\lambda + 3\lambda^2 - 20 = 0$$

$$\lambda^2 - 3\lambda - 38 = 0$$

Conditions for diagonalization

- Same number of eigen vectors as eigen values
- n eigen vectors that are linearly independent.

Example ①  $\lambda_1 = \lambda_2 = \lambda_3 = 2$

Diagonalize  $B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2^3 & 0 & 0 \\ 0 & 2^3 & 0 \\ 0 & 0 & 1^3 \end{pmatrix} \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

Ans:  $\lambda_1 = 2 \quad \lambda_2 = 1 \quad \lambda_3 = -1$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$D = P^{-1} B P$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

Q: Show that  $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  is not diagonalizable.

### Application of diagonalization

Used in getting higher powers of a given matrix  $\Rightarrow A^K = P^{-1} D^K P$ .

#### Example ②

Given  $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$ , find  $A^{13}$ .

Ans:

$$\lambda_1 = 2 \quad \lambda_2 = 1$$

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^{13} = P^{-1} D^{13} P$$

$$A^{13} = \begin{pmatrix} -8190 & 0 & -16382 \\ 8191 & 8192 & 8191 \\ 8191 & 0 & 16383 \end{pmatrix}$$

Solution:

$$|A - \lambda I| = 0$$

$$(A - \lambda I)X = 0$$

$$|A - \lambda I| = \begin{vmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix}$$

$$M_{11} = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix}, \quad M_{23} = \begin{vmatrix} -\lambda & 0 \\ 1 & 0 \end{vmatrix}$$

$$= (2-\lambda)(3-\lambda) - 0 = 0 - 0$$

$$M_{11} = \lambda^2 - 5\lambda + 6$$

$$M_{22} = 0$$

$$M_{12} = \begin{vmatrix} 1 & 1 \\ 1 & 3-\lambda \end{vmatrix}$$

$$M_{31} = \begin{vmatrix} 0 & -2 \\ 2-\lambda & 1 \end{vmatrix}$$

$$= (3-\lambda) - 1$$

$$= 0 - [-2(2-\lambda)]$$

$$M_{12} = 2-\lambda$$

$$= 0 - (2\lambda - 4)$$

$$M_{13} = \begin{vmatrix} 1 & 2-\lambda \\ 1 & 0 \end{vmatrix}$$

$$= 0 - (2\lambda - 4)$$

$$= 0 - (2-\lambda)$$

$$M_{33} = 4-2\lambda$$

$$M_{13} = \lambda - 2$$

$$M_{32} = \begin{vmatrix} -\lambda & -2 \\ 1 & 1 \end{vmatrix}$$

$$= 0 - (2-\lambda)$$

$$= -\lambda - -2$$

$$M_{21} = 0$$

$$M_{32} = 2-\lambda$$

$$= 0 - 0$$

$$M_{33} = \begin{vmatrix} -\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= -\lambda(2-\lambda) - 0$$

$$M_{21} = 0$$

$$M_{33} = \lambda^2 - 2\lambda$$

$$M_{22} = \begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= -\lambda(3-\lambda) - (-2)$$

$$M_{22} = \lambda^2 - 3\lambda + 2$$

$$\begin{aligned}
 C_{11} &= \lambda^2 - 5\lambda + 6 & a_{11} &= 4 - \lambda \\
 C_{12} &= \lambda - 2 & a_{12} &= 0 \quad 0 \\
 C_{13} &= 2\cancel{\lambda} \cancel{\lambda} - 2 & a_{13} &= \cancel{2} \cancel{\lambda} - 2
 \end{aligned}$$

$$\begin{aligned}
 C_{21} &= 0 & a_{21} &= 1 \quad 1 \\
 C_{22} &= \lambda^2 - 3\lambda + 2 & a_{22} &= 4 - 2\lambda \\
 C_{23} &= 0 & a_{23} &= 1 \quad 1 \\
 C_{31} &= 4 - 2\lambda & a_{31} &= 1 \quad 1 \\
 C_{32} &= \lambda - 2 & a_{32} &= 0 \quad 0 \\
 C_{33} &= \lambda^2 - 2\lambda & a_{33} &= 3 - \lambda
 \end{aligned}$$

$$\begin{aligned}
 |A - \lambda I| &= a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} \\
 &= -\lambda(\lambda^2 - 5\lambda + 6) + 0 + -2(\lambda - 2) \\
 &= -\lambda^3 + 5\lambda^2 - 6\lambda - 2\lambda + 4
 \end{aligned}$$

$$\begin{aligned}
 |A - \lambda I| &= -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0 \\
 \lambda_1 &= 1, \quad \lambda_2 = 2
 \end{aligned}$$

$$(A - \lambda_1 I)X = 0$$

when  $\lambda = 1$ ,

$$\left[ \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[ \begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array} \right]$$

$$\begin{aligned}
 -R_1 \rightarrow R_1 &\left( \begin{array}{ccc|c} 1 & 0 & 2 & 0 \end{array} \right) \\
 R_2 &\left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \end{array} \right) \\
 R_3 &\left( \begin{array}{ccc|c} 1 & 0 & 2 & 0 \end{array} \right)
 \end{aligned}$$

$$\begin{aligned}
 R_1 &\left( \begin{array}{ccc|c} 1 & 0 & 2 & 0 \end{array} \right) \\
 -R_1 + R_2 \rightarrow R_2 &\left( \begin{array}{ccc|c} 0 & 1 & -1 & 0 \end{array} \right) \\
 -R_1 + R_3 \rightarrow R_3 &\left( \begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array} \right)
 \end{aligned}$$

$$\Rightarrow X_2 - X_3 = 0$$

$$X_2 = X_3$$

$$X_1 + 2X_3 = 0$$

$$X_1 = -2X_3$$

$$\therefore \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} -2X_3 \\ X_3 \\ X_3 \end{pmatrix} = X_3 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \text{ eigen vector corresponding to eigen value } \lambda_1 = 1$$

when  $\lambda = 2$ ,

$$\left[ \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right] \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[ \begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array} \right]$$

$$\begin{aligned}
 -\frac{1}{2}R_1 \rightarrow R_1 &\left( \begin{array}{ccc|c} 1 & 0 & 1 & 0 \end{array} \right) \\
 \rightarrow R_2 &\left( \begin{array}{ccc|c} 1 & 0 & 1 & 0 \end{array} \right) \\
 R_3 &\left( \begin{array}{ccc|c} 1 & 0 & 1 & 0 \end{array} \right)
 \end{aligned}$$

$$\begin{aligned}
 R_1 &\left( \begin{array}{ccc|c} 1 & 0 & 1 & 0 \end{array} \right) \\
 -R_1 + R_2 \rightarrow R_2 &\left( \begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array} \right) \\
 -R_1 + R_3 \rightarrow R_3 &\left( \begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array} \right)
 \end{aligned}$$

$$X_1 + X_3 = 0$$

$$\Rightarrow X_1 = -X_3$$

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} -X_3 \\ 0 \\ X_3 \end{pmatrix} + X_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
 = X_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + X_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\therefore \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$  are the eigen vectors corresponding to eigen value  $\lambda_2 = 2$ .

P = matrix of eigen vectors

$$\therefore P = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$M_{11} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$$

$$M_{12} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$$

$$M_{13} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 - 0 = 1$$

$$M_{21} = \begin{vmatrix} -1 & 0 \\ 1 & 0 \end{vmatrix} = 0 - 0 = 0$$

$$M_{22} = \begin{vmatrix} -2 & 0 \\ 1 & 0 \end{vmatrix} = 0 - 0 = 0$$

$$M_{23} = \begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} = -2 - 1 = -1$$

$$M_{31} = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1 - 0 = -1$$

$$M_{32} = \begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix} = -2 - 0 = -2$$

$$M_{33} = \begin{vmatrix} -2 & -1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = 1$$

$$C = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ -1 & 2 & 1 \end{pmatrix}$$

$$\begin{aligned} |P| &= -2(-1) + -1(1) + 0 = 2 - 1 = 1 \\ &= 0 + 0 + 1(1) = 1 \\ &= 1(-1) + 1(2) + 0 = -1 + 2 = 1 \end{aligned}$$

$$C^T = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$P^{-1} = \frac{1}{|P|} C^T$$

$$= \frac{1}{1} \begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$D = P^{-1} A P$$

$$= \begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & -1 \\ 2 & 0 & 4 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{aligned} A^{13} &= \bar{P}^{-1} D^{13} P \\ &= \begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 8192 & 0 \\ 0 & 0 & 8192 \end{pmatrix} \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & -8192 \\ 1 & 0 & 16384 \\ 1 & -8192 & 8192 \end{pmatrix} \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \\ A^{13} &= \begin{pmatrix} -8190 & -8191 & 0 \\ 16382 & 16383 & 0 \\ 16382 & 8191 & 8192 \end{pmatrix} \end{aligned}$$

change arrangement of P:

$$P = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$M_{11} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$M_{12} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

$$M_{13} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

06/12/2021 - Monday

## LEONTIF INPUT-OUTPUT ANALYSIS

The main feature of the input-output model is that it incorporates the interactions between industries or sectors which make up an economy.

The aim of the Leontif model is to allow economists to forecast the future production levels of each industry in order to meet the future demands for the various products.

Such forecasting is complicated by linkages between the different industries because the change in the demand of one industry can induce a change in the production levels of another industry e.g. an increase in the demand of automobiles leads not only to the increase in production of automobiles but also an increase in the levels of a variety of other industries within the economy such as steel, rubber, leather, etc.

Consider an economy with  $n$  industries or sectors, each industry/sector producing a homogeneous product. Part of the industry or sector is required / used by households as final goods.

In a modern economy, production of one good requires the input of many other goods in the production process. (intermediate goods)

Therefore, the demands of an individual product will need to find the output required to meet the final demand from within and outside the industry.

i.e.  $X = \text{summation of all intermediate goods} + \text{External demand.}$

Mathematically, the total demand,  $X$  is given by the summation of all intermediate products  $\alpha_{ij}$  and the final demand.

i.e. for example if an economy with 3 sectors, 1, 2 & 3,

$$X_1 = \alpha_{11}X_1 + \alpha_{12}X_2 + \alpha_{13}X_3 + D_1$$

$$X_2 = \alpha_{21}X_1 + \alpha_{22}X_2 + \alpha_{23}X_3 + D_2$$

$$X_3 = \alpha_{31}X_1 + \alpha_{32}X_2 + \alpha_{33}X_3 + D_3$$

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} + \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix}$$

$$X = AX + D$$

$X$  = Total output matrix

$A$  = Matrix of technical coefficients

$D$  = External demand.

$$X - AX = D$$

$$X(1-A) = D \quad \text{Introduce Identity matrix}$$

$$X(I-A) = D$$

$$(I-A)^{-1}(I-A)X = (I-A)^{-1}D$$

$$X = (I-A)^{-1}D \quad \equiv \text{Leontif matrix.}$$

Example ①:

Consider a hypothetical two industry economy with the matrix of technical coefficients given as  $C$  and

$$C \begin{pmatrix} 0.2 & 0.3 \\ 0.4 & 0.1 \end{pmatrix}$$

external demand as  $D = C \begin{pmatrix} 24 \\ 15 \end{pmatrix}$ ,

determine the total output required to meet the internal and external demands.

Solution:

$$X = (I - A)^{-1} D$$

$$(I - A) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.4 & 0.3 \\ 0.5 & 0.7 \end{pmatrix}$$

$$= \begin{pmatrix} 0.6 & -0.3 \\ -0.5 & 0.3 \end{pmatrix}$$

$$|(I - A)| = (0.6 \times 0.3) - (-0.5 \times 0.3)$$

$$= 0.72 + 0.15$$

$$|(I - A)| = 0.60$$

$$(I - A)^T = \begin{pmatrix} 0.9 & +0.3 \\ +0.4 & 0.8 \end{pmatrix}$$

$$(I - A)^{-1} = \frac{1}{0.6} \begin{bmatrix} 0.9 & +0.3 \\ +0.4 & 0.8 \end{bmatrix}$$

$$= \begin{pmatrix} 1.5 & 0.5 \\ 0.67 & 1.33 \end{pmatrix}$$

$$X = (I - A)^{-1} D$$

$$= \begin{pmatrix} 1.5 & 0.5 \\ 0.67 & 1.33 \end{pmatrix} \begin{pmatrix} 24 \\ 15 \end{pmatrix}$$

$$= \begin{pmatrix} (1.5 \times 24) + (0.5 \times 15) \\ (0.67 \times 24) + (1.33 \times 15) \end{pmatrix}$$

$$= \begin{pmatrix} 36 + 7.5 \\ 16.08 + 19.95 \end{pmatrix}$$

$$X = \begin{pmatrix} 43.5 \\ 36.03 \end{pmatrix}$$

⇒ Industry C needs to produce 43.5 units in order to meet the internal & external d.d.

⇒ Industry I needs to produce 36.03 units in order to meet the internal & external d.d.

Example ②

Suppose an economy is divided into 2 sectors, A & B with a matrix of technical coefficients given as

$$\begin{array}{c} A \& B \\ \begin{pmatrix} 0.4 & 0.3 \\ 0.5 & 0.7 \end{pmatrix} & \xrightarrow{\text{output}} \\ & \downarrow \begin{pmatrix} 0.6 & -0.3 \\ -0.5 & 0.3 \end{pmatrix} \xleftarrow{\text{inputs}} \end{array}$$

Interpret the coefficient of technical matrix and discuss the economic feasibility of the two industries.

b) If the input-output model is set up as

$X = AX + D$ , derive the final solution of the model using the Leontif approach.

c) Given the final demand as \$2000 and \$1500 for industry A and B respectively, determine the final output matrix.

Solution:

a)  $\frac{0.4}{0.6} \text{ or } 40\%$  of the output of industry A is used as its input

$\frac{0.3}{0.3} \text{ or } 30\% \rightarrow$  industry A uses 30% of the output of B as its input.

$\frac{0.5}{0.6} \text{ or } 50\%$  of the output of A is used as its input.

$\frac{0.7}{0.7} \text{ or } 70\%$  of its output as input in production in the very industry.

There is a high level of interdependence between industries A and B just laying off 0.1 units to the external demand.

b) → covered.

c)  $X = (I - A)^{-1} D$

$$X = \frac{1}{0.6} \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.4 & 0.3 \\ 0.5 & 0.7 \end{pmatrix} \right]^{-1} \begin{pmatrix} 2000 \\ 1500 \end{pmatrix}$$

$$\left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) - \left( \begin{array}{cc} 0.4 & 0.3 \\ 0.5 & 0.7 \end{array} \right)$$

$$(I - A) = \begin{pmatrix} 0.6 & -0.3 \\ -0.5 & 0.3 \end{pmatrix}$$

$$(I - A) = (0.3 \times 0.6) - (-0.3 \times -0.5)$$

$$= 0.18 - 0.15$$

$$= 0.03.$$

$$(I - A)^T = \begin{pmatrix} 0.3 & 0.3 \\ 0.5 & 0.6 \end{pmatrix}$$

$$(I - A)^{-1} = \frac{1}{0.03} \begin{pmatrix} 0.3 & 0.3 \\ 0.5 & 0.6 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 10 & 10 \\ 16.67 & 20 \end{pmatrix}$$

$$\therefore X = (I - A)^{-1} D$$

$$= \begin{pmatrix} 10 & 10 \\ 16.67 & 20 \end{pmatrix} \begin{pmatrix} 2000 \\ 1500 \end{pmatrix}$$

$$= \begin{pmatrix} 10 \times 2000 + 10 \times 1500 \\ 16.67 \times 2000 + 20 \times 1500 \end{pmatrix}$$

$$= \begin{pmatrix} 20,000 + 15,000 \\ 33,340 + 30,000 \end{pmatrix}$$

$$X = \begin{pmatrix} 35,000 \\ 63,340 \end{pmatrix}$$

$$\text{Final output matrix} = X = \begin{pmatrix} 35,000 \\ 63,340 \end{pmatrix}.$$

### Example ③

	Industry I	Industry II	Final Demand	Gross Output
Industry I	240	750	210	1200
Industry II	720	450	330	1500
Primary Inputs	240	300		

- a) Determine the input-output matrix A  
 b) Determine the output matrix if the final demand changes to 312 units for industry I and 299 units for industry II respectively.

### Example ④

Q) What will be the new primary inputs for the two industries?

Solution

$$A = \begin{matrix} & I & II \\ I & \begin{pmatrix} 240 & 750 \\ 1200 & 1500 \end{pmatrix} \\ II & \begin{pmatrix} 720 & 450 \\ 1200 & 1500 \end{pmatrix} \end{matrix}$$

$$\underline{A = \begin{pmatrix} 0.2 & 0.5 \\ 0.6 & 0.3 \end{pmatrix}}$$

$$b) D = \begin{matrix} I \\ II \end{matrix} \begin{pmatrix} 312 \\ 299 \end{pmatrix}$$

$$X = (I - A)^{-1} D$$

$$(I - A) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.2 & 0.5 \\ 0.6 & 0.3 \end{pmatrix} = \begin{pmatrix} 0.8 & -0.5 \\ -0.6 & 0.7 \end{pmatrix}$$

$$(I - A)^{-1} = (0.7 \times 0.8) - (-0.6 \times -0.5)$$

$$= 0.56 - 0.30$$

$$= 0.26.$$

$$(I - A)^T = \begin{pmatrix} 0.7 & 0.5 \\ 0.6 & 0.8 \end{pmatrix}$$

$$(I - A)^{-1} = \frac{1}{(I - A)} \cdot (I - A)^T$$

$$= \frac{1}{0.26} \begin{pmatrix} 0.7 & 0.5 \\ 0.6 & 0.8 \end{pmatrix}$$

$$= \begin{pmatrix} 2.692 & 1.923 \\ 2.308 & 3.077 \end{pmatrix}$$

$$X = (I - A)^{-1} D$$

$$= \begin{pmatrix} 2.692 & 1.923 \\ 2.308 & 3.077 \end{pmatrix} \begin{pmatrix} 312 \\ 299 \end{pmatrix}$$

$$= ((2.692 \times 312) + (1.923 \times 299)) = 705744 + 574.9$$

$$= (2.308 \times 312) + (3.077 \times 299) = 720.096 + 920.0$$

$$= 1280.721 \quad \# \quad X = \begin{pmatrix} 1415 \\ 1640.119 \end{pmatrix}$$

c) for industry I, initial fraction of inputs

$$= \frac{240}{1200} = 0.2 \text{ of the output}$$

New primary input =  $0.2 \times 1415$

$$= 283$$

For industry II, initial fraction of inputs

$$= \frac{300}{1500} = 0.2 \text{ of the output}$$

New primary input =  $0.2 \times 1640.119$

$$= 328$$

Example ④ = Exercise

	Agriculture	Manufacturing	Final demand	Gross output
Agriculture	240	270	90	600
Manufacturing	300	90	60	450
Labour	60	90		

a) Determine the input-output matrix A

b) Suppose that in 3 years, the demand for agricultural products decreases to 63 units and increases to 105 units for manufactured products, determine the new output vector to meet the new demand.

c) What will be the new requirement for each sector?

Example ⑤ = Exercise

Given a 3 sector economy with the input-output matrix A given as

$$A = \begin{matrix} C & I & A \\ \end{matrix}$$

$$C \begin{pmatrix} 0.5 & 0.1 & 0.2 \end{pmatrix}$$

$$I \begin{pmatrix} 0.2 & 0.8 & 0.1 \end{pmatrix}$$

$$A \begin{pmatrix} 0.1 & 0 & 0.4 \end{pmatrix}$$

$$\text{with the external demand as } D = \begin{pmatrix} 100 \\ 150 \\ 200 \end{pmatrix},$$

determine the total output to meet the new demand.

# Accounting Principles.

Fundamental Accounting Principle  
This is a way to figure out the number of outcomes in a probability problem.

or

A way to figure out a given situation.

Addition principle

If an operation can be performed  $M$  ways and also if another operation can be performed  $N$  ways and only one operation can be done at a time then either of the two operations can be performed in  $M+N$  ways.

Example.

A student is shopping for a new computer he is decide between three desk top computer and four laptop computer what is the total number of computer options.

Solution.

$$M+N$$

$$3+4$$

7 computer options.

Multiplication.

If an operation can be performed in  $M$  ways and if an operation can be performed in  $N$  ways independent of the 1st then the total number of ways of performing the operations sumanl is  $M \times N$  ways.

## Examples

Fabby has 2 t-shirts and 3 skirts. How many outfits can be put together.

$$2 \times 3 \\ 6 \text{ outfits}$$

Example 8:  
You all take a survey with 3 (Yes, No)ans. How many diff't ways would you complete the survey.

$$2 \times 2 \times 2 \times 2 \\ 32 \text{ ways.}$$

You toss 3 coins. How many possible outcomes are there?

$$2 \times 2 \times 2 \\ 8 \text{ outcomes.}$$

Note:  
This works when all choices are independent of each other. If one choice depends on another then simple multiplication is not right.

## Factorial notation.

This is a product of positive integers from 1 to n, it is always denoted by (!) written as  $n!$

$$\text{i.e } n! = n \times (n-1) \times (n-2) \dots 1$$

$$\therefore n! = 1 \cdot 2 \cdot 3 \cdots (n-2)(n-1) \times n$$

$$0! = 1$$

$$2^0 = 1$$

5!

$$5 \times 4 \times 3 \times 2 \times 1$$

$$= 120$$

$$\frac{10!}{8!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$$= 90$$

Evaluat

$$\frac{10!}{2! 3! 5!}$$

$$\frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 3 \times 2 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$$\frac{30240}{12}$$

$$2520$$

$$\frac{6! 2!}{8!}$$

$$\frac{6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 2!}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2!}$$

$$\frac{12}{336}$$

$$\frac{1}{28}$$

Exerc

$$\frac{(n+1)!}{2(n!)}$$

$$\frac{(n+1)(n+1-1)!}{2 n!}$$

$$\frac{1}{2} (n+1)$$

## PERMUTATIONS

Arrangement of a number of objects in a particular order.

OR.  
Diff<sup>n</sup> arrangements that can be made out of the diff<sup>n</sup> set of objects or items by taking same or all of them at a time.

When given a set of  $N$  objects and we are required to arrange them in a particular order the total number of arrangements is denoted as  ${}^n P_r$ . It means that the number of possible arrangements of  $r$  items taken from  $n$  items.

$${}^n P_r = \frac{n!}{(n-r)!}$$

Diff<sup>n</sup> types of permutation.

① Permutations in which objects are replaced.

② Permutation of objects not all diff<sup>n</sup>

$$\frac{n!}{p! q! r!}$$

③ C  
There we fix.

From a class of 20 students we need to select  
3 for a com

1 to be president, 1 as VP and the others as  
how many diff<sup>t</sup> Committee can be selected performed.

$$n = 20$$

$$r = 3$$

$${}^n P_r = \frac{n!}{(n-r)!}$$

$$\frac{20!}{(20-3)!}$$

$$\frac{20!}{17!}$$

$$\frac{20 \times 19 \times 18 \times 17!}{17!}$$

$$= 6840 \text{ ways.}$$

Example 2.

How many unique ways are there to arrange in  
the word THAT.

$$\frac{n!}{P!}$$
$$\frac{4!}{2!}$$

$$\frac{4 \times 3 \times 2 \times 1}{2!}$$

$$= 12 \text{ ways}$$

How many different two digit numbers can be formed from the 1st natural numbers if repetition is allowed.

$$n^r$$

$$5^2$$

A Father mother two boys and 3 girls are asked to line up for a pic. determine the number of ways

if (a) There are no restrictions.

(b) If the parents stand together.

(c) If the parents don't stand together.

(d) all females stand together.

Solution:

a)  $7! = 5040$  ways.

b)  $6! \times 2! = 1440$  ways.

c)  $7! - (6! \times 2!)$

$$5040 - 1440$$

$$= 3600$$

d)  $4! \times 4!$   
= 576 ways.

In how many ways can 5 people be seated on a round table.

$$(n-1)!$$

$$(5-1)!$$

$$= 4!$$

$$= 24 \text{ ways.}$$

# MATHEMATICS OF FINANCE

The study of financial mathematics is concerned with financial decisions we help in allocating, paying and receiving money at intervals. It helps us to take efficient financial decisions including how best to invest money.

Example of financial Calculations.

- Compounding
- discounting
- annuities
- Amortization.
- Net present Value (NPV)
- Internal Rate of Return.

READ: Principal, Rate of interest & Accrued Amount.

Interest rate / Cost of Capital.

Simple Interest:  
The interest earned under this approach can be calculated as,  $I = P \times R \times T$  where R is rate of interest in %.

Example:  
Suppose that 200000 was invested at 10% interest per annum. Compute the simple interest earned after 3 years and the cumulative amount accrued.

$$I = P \times R \times T$$

$$I = \frac{200,000 \times 10}{100} \times 3$$

$$I = 60,000$$

$$\text{Amount accrued} = P + I$$

$$200,000 + 60,000$$

$$260,000$$

Compound interest:

In this technique, the amount accrued can be calculated as:  $S = P(1+r)^n$  where S is the sum value or amount accrued, r is rate of interest in %, n is the time period.

Example 2.

$$S = P(1+r)^n$$

$$260,000 = 200,000 \left(1 + \frac{10}{100}\right)^3$$

$$260,000 = 200$$

Compounding and Discounting: Future Value from  
Compounding is the process of obtaining future value from  
present value of money. For example determining the money  
that will be realized after depositing money in a bank  
for a specific period of time into the future.

$$A = P(1+r)^n$$

Discounting:

Is the process of obtaining present value of money  
from the future value of money for example obtain amount  
to deposit today in order to achieve a certain amount  
in the future.

other definitions

a) An annuity

Is a sequence of equal payments (or receipts) over  
uniform time intervals. Annuities can be ordinary  
(paid at end of the period) or due (paid at the  
beginning of the period). They normally begin  
and end on fixed dates (ordinary), but occasionally  
carry on indefinitely (contingent).

b) An amortized debt.

Is one that is repaid with an annuity that  
takes account of both interest and repayment  
of principal amount borrowed.

Investment Appraisal.

Since investment involves risks, there is need to  
evaluate the investments.

Traditional techniques that can be used include  
Accounting Rate of Return (ARR)  
payback Period.

Modern techniques (discounting techniques) *(consider time)*  
Net Present Value (NPV)  
Internal Rate of Returns (IRR)

Capital investments.  
Are projects that involve an initial outlay of capital and a set of estimated cash inflows and outflows over the life of a project.

Present Value and Investment appraisal /  
This is a technique that enables a future cash flow to be represented in equivalent of today's money terms.  
The present Value P of amount A payable in n-years time subject to a discount rate of r% is given by:

$$P = \frac{A}{(1+r)^n}$$

Recall: Formula for Compounding,  $S = P(1+r)^n$

$P = A$   
 $\frac{1}{(1+r)^n}$  can be rearranged as  $P = A \times \frac{1}{(1+r)^n}$

where  $\frac{1}{(1+r)^n}$  is known as the discount factor or Present Value factor.

Example  
Find the present value of \$ 1,500 "expans subject to a discount rate of 19%

$$PV = \frac{A}{(1+i)^n} = \frac{1,500}{(1+0.19)^6} = \$ 528.15$$

Net Present Value (NPV)  
It involves calculating the sum of the present values of all cash flows associated with the project. The sum is known as the net present value. The cash flows are normally tabulated per year.

$$\text{It can be computed as } NPV = \sum_{n=0}^t PV_n \text{ or } NPV = \sum_{n=1}^t PV_n - I_0$$

where  $I_0$  is the initial outlay.

Interpretation of NPV:

$NPV > 0$ ; The project earns more than the discount rate hence it breaks the initial outlay in profit (worthwhile)

$NPV = 0$  The project earns same as the discount rate hence it is ~~+ a loss~~ (not breaks even) (Revenue = Cost)

$NPV < 0$ ; The project earns less than the discount rate hence it is in loss (not worthwhile)

Example:  
A business project is considered to have \$ 12000 initial costs and the associated inflows (Revenues) and outflows (Costs) over the following 4 years are given below-

Number Year	Cash inflow(\$)	Cash outflows(\$)
1	8600	8500
2	12000	3000
3	10,000	1500
4	6500	1500

Evaluate the Project's NPV at the discount rate of 18.5% and comment on your results.

Solution:

Note since this involves many years, we can use a table.  
DF (discount factor)

$$DF = \frac{1}{(1+r)^n} \text{, and } PV = A \times DF$$

Year	Inflow \$	Outflows \$	Net cashflow \$	DF	PV
0	0	12,000	-(12,000)	1	0.84388
1	8000	8500	500	0.712137	
2	12000	3,000	9000	0.600959	
3	10,000	1,500	8500	0.507139	
4	6500	1,500	5000		