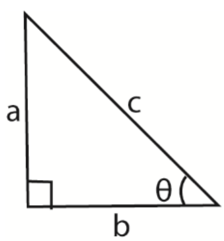
Trigonometry

The word 'trigonometry' suggests 'tri'-three, 'gono'-angle, 'metry'-measurement. As such, trigonometry is basically about triangles, most especially right-angled triangles.

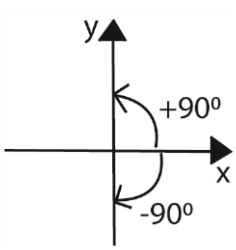
Important to note

(a) For a right angled triangle below

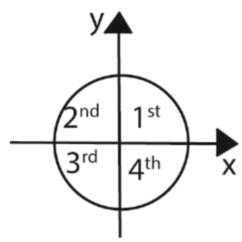


- $\sin\theta = \frac{a}{c}$ $\quad$ $\text{cosec}\theta = \frac{1}{\sin\theta} = \frac{c}{a}$
- $\cos\theta = \frac{b}{c}$ $\quad$ $\sec\theta = \frac{1}{\cos\theta} = \frac{c}{b}$
- $\tan\theta = \frac{\sin\theta}{\cos\theta}\frac{a}{b}$ $\quad$ $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{b}{a}$

(b) All positive angles are measured anticlockwise from positive x-axis

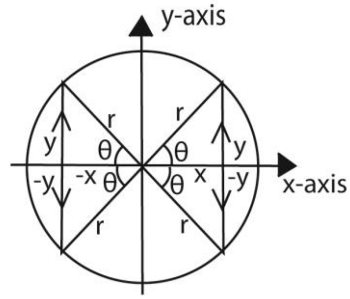


(c) A circle drawn with the centre O, divides the co-ordinate axis into four equal parts called quadrants



The quadrants are also labelled anti-clockwise from the positive x – axis.

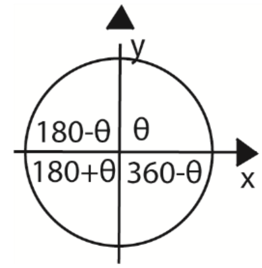
The signs the trigonometric ratios in the quadrants are given below



Ratio	Quadrant			
	1st	2nd	3rd	4th
cosθ	$\frac{+x}{r}$	$\frac{-x}{r}$	$\frac{-x}{r}$	$\frac{+x}{r}$
sinθ	$\frac{+y}{r}$	$\frac{+y}{r}$	$\frac{-y}{r}$	$\frac{-y}{r}$
tanθ	$\frac{+y}{x}$	$\frac{y}{-x}$	$\frac{y}{x}$	$\frac{-y}{x}$
secθ	$\frac{+r}{x}$	$\frac{-r}{x}$	$\frac{-r}{x}$	$\frac{+r}{x}$
cosecθ	$\frac{+r}{y}$	$\frac{+r}{y}$	$\frac{-r}{y}$	$\frac{-r}{y}$
cotθ	$\frac{+x}{y}$	$\frac{-x}{y}$	$\frac{+x}{y}$	$\frac{-x}{y}$

Note

- If θ is the angle in the 1st quadrat
- In the 2nd quadrat the angle is (180 – θ)
- In the 3rd quadrat the angle is (180 + θ)
- In the 4th quadrat the angle is (360 –θ)



Solving equations

We make use of the quadrants to find the ranges of values within which the angle follows

Example 1

Solve the following equations for $0^0 \leq \theta \leq 360^0$

(i) $3\cos\theta + 2 = 0$

Solution

$\cos\theta = -\frac{2}{3}$

But cos is negative in the 2^{nd} and 3^{rd} quadrants.

Ignoring the negative sign, the angle obtained is referred to as the key or principle angle, i.e. key angle = $\cos^{-1}\frac{2}{3} = 48.2^0$ (1d.p)

In the 2^{nd} quadrant, $\theta = 180 - 48.2 = 131.8^0$

In the 3^{rd} quadrant, $\theta = 180 + 48.2 = 228.2^0$

$\therefore\{\theta: \theta = 131.8^0, 228.2^0\}$

Note that: the key angle s not part of the solution but only a guide.

(ii) $4\cos^2\theta - 1 = 0$

Solution

$\cos\theta = \sqrt{\frac{1}{4}} = \pm\frac{1}{2}$

Key angle, $\theta = \cos^{-1}\frac{1}{2} = 60^0$

When $\cos\theta = ½$ (positive is 1^{st} and 4^{th} quadrants)

1^{st} quadrant $\theta = 60^0$

4^{th} quadrant $\theta = 360 - 60 = 300^0$

When $\cos\theta = -½$ (positive is 2^{nd} and 3^{rd} quadrants)

3^{rd} quadrant $\theta = 180 - 60 = 120^0$

4^{th} quadrant $\theta = 180 + 60 = 240^0$

$\therefore\{\theta: \theta = 60^0, 120^0, 240^0, 300^0\}$

(iii) $\text{cosec}\theta + 2 = 0$

Solution

$\text{cosec}\theta = -2 \Rightarrow \sin\theta = -½$ (taking reciprocal)

Key angle = $\sin^{-1}\frac{1}{2} = 30^0$

In the 3^{rd} quadrant $\theta = 180 + 30 = 210^0$

In the 4^{th} quadrant $\theta = 360 - 30 = 330^0$

$\therefore\{\theta: \theta = 210^0, 330^0\}$

(iv) $3\sec^2\theta - 4 = 0$

Solution

$\sec\theta = \pm\frac{2}{\sqrt{3}} \Rightarrow \cos\theta = \pm\frac{\sqrt{3}}{2}$

Key angle = $\cos^{-1}\frac{\sqrt{3}}{2} = 30^0$

For $\cos\theta = \frac{\sqrt{3}}{2}$; $\theta = 30^0, 330^0$

For $\cos\theta = -\frac{\sqrt{3}}{2}$; $\theta = 120^0, 210^0$

$\therefore\{\theta: \theta = 30^0, 120^0, 210^0, 330^0\}$

(d) Definitions of angle

(i) **Acute angle** is an angle between 0^0 and 90^0. It lies in the 1^{st} quadrant

(ii) **Right angle** is an angle = 90^0

(iii) **Obtuse angle** is an angle between 90^0 and 180^0. It lies in the 2^{nd} quadrant

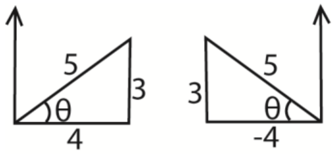
(iv) **Reflex angle** is an angle between 180^0 and 360^0. It lies in the 3^{rd} and 4^{th} quadrant

Example 2

(a) If $\sin\theta = \frac{3}{5}$ and $0^0 \leq \theta \leq 360^0$. Find the possible values of $3\tan\theta - \cot\theta$

Solution

If $\sin\theta = \frac{3}{5}$; θ lies in 1st or 2nd quadrants



In 1st quadrant

$3\tan\theta - \cot\theta = 3\left(\frac{3}{4}\right) - \left(\frac{4}{3}\right) = \frac{11}{12}$

In 2nd quadrant

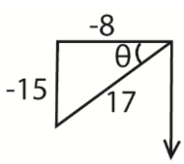
$3\tan\theta - \cot\theta = 3\left(-\frac{3}{4}\right) - \left(-\frac{4}{3}\right) = -\frac{11}{12}$

∴ the possible values are $\pm\frac{11}{12}$

(b) If $\cos\theta = -\frac{8}{17}$ and θ is reflex, find the value of $4\sec^2\theta + \tan\theta$

Solution

If $\cos\theta = -\frac{8}{17}$ and θ is reflex, θ lies in the 3rd quadrant



$4\sec^2\theta + \tan\theta = 4\left(-\frac{17}{8}\right)^2 + \frac{15}{8} = \frac{319}{16}$

Example 3

Solve for θ, where $\theta^0 \leq \theta \leq 360^0$

(i) $3\tan^2 3\theta = 1$

Solution

$\tan 3\theta = \pm\frac{1}{\sqrt{3}}$

taking $\tan 3\theta = \frac{1}{\sqrt{3}}$

⇨ $3\theta = 30^0, 210^0, 390^0, 570^0, 750^0, 930^0$
$\theta = 10^0, 70^0, 130^0, 190^0, 250^0, 310^0$

taking $\tan 3\theta = -\frac{1}{\sqrt{3}}$

⇨ $3\theta = 150^0, 330^0, 510^0, 690^0, 870^0, 1050^0$
$\theta = 50^0, 110^0, 170^0, 230^0, 290^0, 350^0$

$\therefore\{\theta: \theta = 10^0, 50^0, 70^0, 110^0, 130^0, 170^0, 190^0, 230^0, 250^0, 290^0, 310^0, 350^0\}$

Note

- If $\theta^0 \leq \theta \leq 360^0$ then $\theta^0 \leq 3\theta \leq 1080^0$ [multiply the interval through by 3]

(ii) $2\cos 2\theta + \sqrt{3} = 0$

Solution

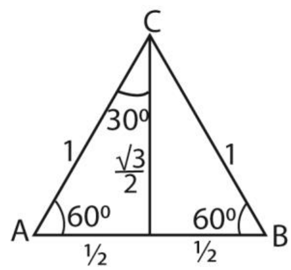
$\cos 2\theta = -\frac{\sqrt{3}}{2}$ and $\theta^0 \leq 2\theta \leq 720^0$

$2\theta = 150^0, 210^0, 510^0, 570^0$

$\therefore\{\theta: \theta = 75^0, 105^0, 255^0, 285^0\}$

Set square angles: 30^0, 45^0, and 60^0

(i) From equilateral triangle ABC with side equal to 1 unit



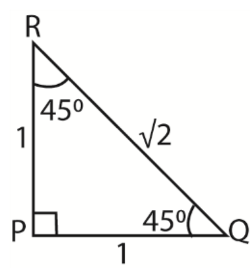
$\cos 60^0 = \sin 30^0 = \frac{1}{2}$

$\cos 30^0 = \sin 60^0 = \frac{\sqrt{3}}{2}$

$\tan 30^0 = \cot 60^0 = \frac{1}{\sqrt{3}}$

$\tan 60^0 = \cot 30^0 = \sqrt{3}$

(ii) From the right angled triangle PQR below



$\cos 45^0 = \sin 45^0 = \frac{1}{\sqrt{2}}$

$\tan 45^0 = 1$

Example 4

Without using tables or calculators find the value of

(i) $\cos 240^0$

Solution

$\cos 240^0 = -\cos(240 - 180)^0 = -\cos 60^0 = -\frac{1}{2}$

(ii) $\tan 3990^0$

Solution

$\tan 3990^0 = \tan[(360 \times 11) + 30]^0 = \tan 30^0 = \frac{1}{\sqrt{3}}$

(iii) $\sin 570^0$

Solution

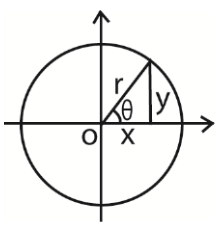
$\sin 570^0 = \sin\{(360 \times 1) + 210]^0 = -\sin 30 = -\frac{1}{2}$

(iv) $\sec 225^0$

Solution

$\sec 225^0 = \sec(225 - 180)0 = \sec 45^0 = -\sqrt{2}$

The Pythagoras theorem



For any acute angle θ

$x = r\cos\theta$ and $y = r\sin\theta$

By Pythagoras theorem

$x^2 + y^2 = r^2$

Substituting for x and y

$(r\cos\theta)^2 + (r\sin\theta)^2 = r^2$

$r^2(\cos^2\theta + \sin^2\theta) = r^2$

$\therefore \cos^2\theta + \sin^2\theta = 1$

Now $\tan\theta = \frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \frac{\sin\theta}{\cos\theta}$

$\therefore \frac{\sin\theta}{\cos\theta} = \tan\theta$

Identities

$\cos^2\theta + \sin^2\theta = 1$ …………………(i)

Identity (i) $\div \cos^2\theta$

$1 + \tan^2\theta = \sec^2\theta$ …………….. (ii)

Identiy (i) $\div \sin^2\theta$

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$ ………….(iii)

Example 5

Show that

(i) $\sin^2\theta + (1 + \cos\theta)^2 = 2(1 + \cos\theta)$

Solution

$\sin^2\theta + (1 + \cos\theta)^2$

$= \sin^2\theta + 1 + 2\cos\theta + \cos^2\theta$

$= \sin^2\theta + \cos^2\theta + 1 + 2\cos\theta$

$= 1 + 1 + 2\cos\theta$ (Recall that $\sin^2\theta + \cos^2\theta = 1$)
$= 2 + 2\cos\theta = 2(1 + \cos\theta)$

$\therefore \sin^2\theta + (1 + \cos\theta)^2 = 2(1 + \cos\theta)$

(ii) $\frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\sec\theta}{1+\csc\theta} = \tan\theta$

Solution

$$\frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\sec\theta}{1+\operatorname{cosec}\theta} = \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\frac{1}{\cos\theta}}{1+\frac{1}{\sin\theta}}$$

$$= \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{\frac{\cos\theta+1}{\cos\theta}}{\frac{\sin\theta+1}{\sin\theta}}$$

$$= \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{\cos\theta+1}{\cos\theta} \div \frac{\sin\theta+1}{\sin\theta}$$

$$= \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{\cos\theta+1}{\cos\theta} x \frac{\sin\theta}{\sin\theta+1}$$

$$= \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$\therefore \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\sec\theta}{1+\csc\theta} = \tan\theta$

(iii) $(\tan\theta + \sec\theta)^2 = \frac{1+\sin\theta}{1-\sin\theta}$

Solution

$(\tan\theta + \sec\theta)^2 = \left(\frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)^2 = \left(\frac{\sin\theta+1}{\cos\theta}\right)^2$

$$= \frac{(1+\sin\theta)^2}{\cos^2\theta} = \frac{(1+\sin\theta)^2}{1-\sin^2\theta}$$

$= \frac{(1+sin\theta)(1+sin\theta)}{(1+sin\theta)(1-sin\theta)} = \frac{1+sin\theta}{1-sin\theta}$

$\therefore(\tan\theta + \sec\theta)^2 = \frac{1+sin\theta}{1-sin\theta}$

Example 6

Solve the following equations for
$-180^0 \leq x \leq 180^0$

(i) $2\cos^2\theta + \sin\theta - 1 = 0$

Solution

$2(1 - \sin^2\theta) + \sin\theta - 1 = 0$

$2\sin2\theta - \sin\theta - 1 = 0$

$(\sin\theta - 1)(2\sin\theta + 1) = 0$

Either $\sin\theta = 1$ or $\sin\theta = -\frac{1}{2}$

When $\sin\theta = 1$; $\theta = 90^0$

When $\sin\theta = -\frac{1}{2}$; $\theta = -150^0, -30^0, 210^0, 330^0$

[θ: $\theta = -150^0, -30^0, 90^0$ for given range]

(ii) $\cos\theta + \sqrt{3}\sin\theta = 1$

Solution

1^st approach

$\sqrt{3}\sin\theta = 1 - \cos\theta$

Squaring both sides

$3\sin^2\theta = 1 - 2\cos\theta + \cos^2\theta$

$3(1 - \cos^2\theta) = 1 - 2\cos\theta + \cos^2\theta$

$4\cos^2\theta - 2\cos\theta - 1 = 0$

$(2\cos\theta + 1)(\cos\theta - 1) = 0$

$\cos\theta = -\frac{1}{2}$	$\cos\theta = 1$
$\theta = \pm 120^0$	$\theta = 0^0$

$\therefore[\theta: \theta = 0^0, \pm 120^0]$

2^nd approach

$\sqrt{3}\sin\theta = 1 - \cos\theta$

Dividing through by $\cos\theta$

$\sqrt{3}\tan\theta = \sec\theta - 1$

Squaring both sides

$3\tan^2\theta = \sec^2\theta - 2\sec\theta + 1$

$3\tan^2\theta = \sec^2\theta - 2\sec\theta + 1$

$3[\sec^2\theta - 1] = \sec^2\theta - 2\sec\theta + 1$

$2\sec^2\theta + 2\sec\theta - 4 = 0$

$\sec^2\theta + \sec\theta - 2 = 0$

$(\sec\theta + 2)(\sec\theta - 1) = 0$

$\sec\theta = -2$ or $\sec\theta = 1$

$\cos\theta = -\frac{1}{2}$ or $\cos\theta = 1$

$\therefore[\theta: \theta = 0^0, \pm 120^0]$

3^rd approach

$\sqrt{3}\sin\theta = 1 - \cos\theta$

Dividing through by $\sin\theta$

$\sqrt{3} = \csc\theta - \cot\theta$

Rearranging

$\sqrt{3} + \cot\theta = \csc\theta$

Squaring both sides

$3 + 2\sqrt{3}\cot\theta + \cot^2\theta = \csc^2\theta$

$3 + 2\sqrt{3}\cot\theta + \cot^2\theta = 1 + \cot^2\theta$

$\cot\theta = -\frac{1}{\sqrt{3}}$; => $\tan\theta = -\sqrt{3}$

$\therefore[\theta: \theta = -60^0, 120^0]$

Example 7

(a) Given that $7\tan\theta + \cot\theta = 5\sec\theta$, derive a quadratic equation for $\sin\theta$. Hence or otherwise, find all values of θ in the interval $0^0 \leq \theta \leq 180^0$ which satisfy the equation, giving your answer to the nearest 0.10 where necessary

Solution

$7\tan\theta + \cot\theta = 5\sec\theta$

$7\frac{sin\theta}{cos\theta} + \frac{cos\theta}{sin\theta} = \frac{5}{cos\theta}$

$7\frac{sin^2\theta + cos^2\theta}{cos\theta sin\theta} = \frac{5}{cos\theta}$

$7\sin^2\theta + \cos^2\theta = 5\sin\theta$

$7\sin^2\theta + (1 - \sin^2\theta) = 5\sin\theta$

$6\sin^2\theta - 5\sin\theta + 1 = 0$

$(3\sin\theta - 1)(2\sin\theta - 1) = 0$

$\sin\theta = \frac{1}{3}$	$\sin\theta = \frac{1}{2}$
$\theta = 19.5^0, 160.5^0$	$\theta = 30^0, 150^0$

$\therefore[\theta: \theta = 19.5^0, 30^0, 150^0, 160.5^0]$

Example 8

Find the solution of $3\cot\theta + \text{cosec}\theta = 2$ for $0^0 \leq \theta \leq 180^0$.

Solution

$3\cot\theta + \text{cosec}\theta = 2$

$3\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta} = 2$

$(3\cos\theta + 1)^2 = (2\sin\theta)^2$

$9\cos^2\theta + 6\cos\theta + 1 = 4\sin^2\theta$

$9\cos^2\theta + 6\cos\theta + 1 = 4(1 - \cos^2\theta)$

$13\cos^2\theta + 6\cos\theta - 3 = 0$

$\cos\theta = \frac{-6\pm\sqrt{6^2+4x3x13}}{2x13}$

$\cos\theta = 0.3021$	$\cos\theta = 0.7637$
$\theta = 72.40$	$\theta = 40.2$

$\therefore[\theta: \theta = 72.4^0, 40.2^0]$

Elimination of trigonometric parameter

This involves the use of identities to eliminate the trigonometric values in equation

Example 9

(a) If $x = \tan\theta + \sec\theta$ and $y = \tan\theta - \sec\theta$; show that $xy + 1 = 0$

Solution

$x + y = \tan\theta$

$x - y = 2\sec\theta$

$\sec\theta = \frac{1}{2}(x - y)$

Using identity: $1 + \tan^2\theta = \sec^2\theta$

$1 + (x + y)\ 2 = \left[\frac{1}{2}(x - y)\right]^2$

$4 + x^2 + 2xy + y^2 = x^2 - 2xy + y^2$

$4xy + 4 = 0$

$xy + 1 = 0$ as required

(b) $x = 2 + 3\sin\theta$ and $y = 3 + 2\cos\theta$ show that $4(x - 2)^2 + (y - 3)^2 = 36$

Solution

$x = 2 + 3\sin\theta \Rightarrow \sin\theta = \frac{x-2}{3}$

$y = 3 + 2\cos\theta \Rightarrow \cos\theta = \frac{y-3}{2}$

Using identity $\sin^2\theta + \cos^2\theta = 1$

$\left(\frac{x-2}{3}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1$

$4(x - 2)^2 + (y - 3)^2 = 36$ as reqyured

(c) $x = 2\sin\theta$ and $y = \tan\theta$, prove that $x = \pm\frac{2y}{\sqrt{(1+y^2)}}$

Solution

$x = 2\sin\theta; \Rightarrow \text{cosec}\theta = \frac{2}{x}$

$y = \tan\theta; \Rightarrow \cot\theta = \frac{1}{y}$

Using identity: $1 + \cot^2\theta = \text{cosec}^2\theta$

$1 + \left(\frac{1}{y}\right)^2 = \left(\frac{2}{x}\right)^2$

$x = \pm\frac{2y}{\sqrt{(1+y^2)}}$

Revision exercise 1

1. Solve for θ, where $\theta^0 \leq \theta \leq 360^0$

(a) $\sec\theta\,\text{cosec}\theta + 2\sec\theta - 2\text{cosec}\theta - 4 = 0$

[θ: $\theta = 60^0, 210^0, 300^0, 330^0$]

(b) $\tan^2\theta - (\sqrt{3} + 1)\tan\theta + \sqrt{3} = 0$

[θ: $\theta = 45^0, 60^0, 225^0, 240^0$]

2. Show that

(a) $\frac{1-\cos\theta+\sin\theta}{1-\cos\theta} = \frac{1+\cos\theta+\sin\theta}{\sin\theta}$

(b) $\tan\theta + \cot\theta = \sec\theta\,\text{cosec}\theta$

(c) $\cos^4\theta - \sin^4\theta + 1 = 2\cos^2\theta$

(d) $\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = 2\text{cosec}\theta$

(e) $\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \sec\theta + \tan\theta$

3. Solve the following equations for $-180^0 \leq x \leq 180^0$

(i) $2\cos^2\theta + \sin\theta - 1 = 0$

[θ: $\theta = -150^0, -30^0, 90^0$]

(ii) $\sin 2\theta + 5\cos 2\theta = 3$

[θ: $\theta = \pm45^0, \pm135^0$]

(iii) $4\cot^2\theta + 24\text{cosec}\theta + 39 = 0$

[θ: θ = 16.6^0, 23.6^0, 156.4^0, 163.4^0]

4. Solve each of the following equations in the stated range
(a) $4\cos^2\theta + 2\sin\theta = 4$ $0^0 \le \theta \le 360^0$
[θ: θ = 0^0, 48.6^0, 131.4^0, 180^0, 360^0]
(b) $2\sec^2\theta - 4\tan\theta - 2 =$ $-180^0 \le \theta \le 360^0$
[θ: θ = -135^0, -161.6^0, 18.4^0, 45^0]
(c) $5\cos^2 3\theta = 3(1 + \sin 3\theta)$, $0^0 \le \theta \le 360^0$
[θ: θ = 7.9^0, 52.1^0, 90^0, 127.9^0, 172.1^0]
5. Solve for θ; 00 ≤ θ ≤ 3600
(a) $\tan\theta + 3\cot\theta = 4$
[θ: θ = 45^0, 71.6^0, 225^0,251.6^0]
(b) $4\cos\theta - 3\sin\theta = 2$
[θ: θ = 29.50, 256.70]
6. Solve
(a) $\cos\theta + \sqrt{3}\sin\theta = 2$ $0 \le \theta \le \pi$
$\left[\theta = \frac{\pi}{3}\right]$
(b) $2\cos\theta - \operatorname{cosec}\theta = 0$ $0^0 \le \theta \le 270^0$
[θ: θ = 45^0, 225^0]
(c) $2\sin^2\theta + 3\cos\theta = 0$ $0^0 \le \theta \le 360^0$

[θ: θ = 240^0, 120^0]

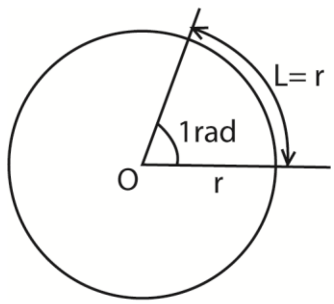
(d) $3\sin\theta + 4\cos\theta = 2$ $-180^0 \le \theta \le 180^0$
[θ: θ = -29.55^0, 103.29^0]
(e) $3\tan^2\theta + 2\sec^2\theta = 2(5 - 3\tan\theta)$ for $0^0 < \theta < 180^0$
[θ: θ = 38.66^0, 116.57^0]

7. Without using a tables or calculator , show that $\tan 15^0 = 2 - \sqrt{3}$
8. Solve equation
$8\cos^4\theta - 10\cos^2\theta + 3$ for $0^0 \le \theta \le 180^0$
[θ: θ = 30^0, 45^0, 135^0, 150^0]
9. Eliminate θ from the following equation
(a) $x = a\sec\theta$ and $y = b + c\cos\theta$
[ac = x(y – b)]
(b) $x = \sec\theta + \tan\theta$ and $y = \sec\theta - \tan\theta$
[xy = 1]
10. Solve the simultaneous equation
$\cos x + 4\sin y = 1$
$4\sec x - 3\operatorname{cosec} y = 5$ for values of x and y between 0^0 and 360^0
[x = 78.8^0, 281.5^0; y = 11.5^0, 168.5^0]

11. Prove each of the following identities
(a) Sinxtanx + cosx =secx
(b) Cosecx + tanxsecx = cosecx$\sec^2$x
(c) Cosecx – sinx = cotxcosx
(d) $(\sin x + \cos x)^2 - 1 = 2\sin x\cos x$
12. Eliminate θ from each of the following pairs of relationships
(a) x =3sinθ, y = cosecθ [xy = 3]
(b) 5x = sinθ, y = 2cosθ [$100x^2 + y^2 - 4 = 0$]
(c) x = 3 + sinθ, y = cosθ [$(x-3)^2 + y^2 = 1$]
(d) x = 2 + sinθ, cosθ = 1+y
[$(x-2)^2 + (y+1)^2 = 1$]

Measuring angles in radians

A radian is defined as an angle subtended at the centre of a circle by an arc that is equal to the radius of the circle. One radian is represented by π, where $\pi = \frac{22}{7}$



How to convert between degrees and radians

1 revolution = circumference of a circle

But circumference of a circle subtends an angle 2π at the centre.

⇨ 1 revolution = 2π = 360^0
π = 180^0
$1^0 = \frac{\pi}{180}$ radians
$x^0 = \frac{\pi}{180}x$ radians

Example 10

Convert the following angles to radians

(a) 330^0
(b) 90^0
(c) 30^0

Solution

(a) $330^0 = \frac{\pi}{180} x\ 330 = \frac{11\pi}{6}$ radians

(b) $90^0 = \frac{\pi}{180} x\ 90 = \frac{\pi}{2}$ radians

(c) $30^0 = \frac{\pi}{180} x\ 30 = \frac{\pi}{6}$ radians

Converting radians to degrees

2π radians = 360^0

1 radian = $\frac{180^0}{\pi}$

x radians $= \frac{180^0}{\pi} x$

Example 11

Convert each of the following radians to degrees

(i) $\frac{\pi}{3}$ radians

(ii) $\frac{2\pi}{5}$ radians

(iii) π radians

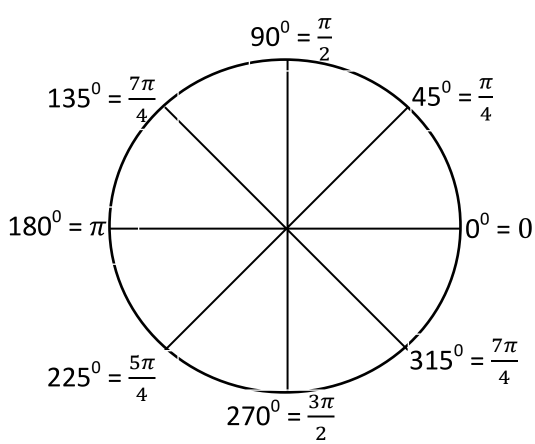
Solution

(i) $\frac{\pi}{3}$ radians $= \frac{180^0}{\pi}\ x \frac{\pi}{3} = 60^0$

(ii) $\frac{2\pi}{5}$ radians$= \frac{180^0}{\pi}\ x \frac{2\pi}{5} = 72^0$

(iii) π radians $= \frac{180^0}{\pi}\ x\ \pi = 180^0$

Some equivalent angles in degrees and radians



Example 12

Find each of the following values

(a) $\sin\left(\frac{2\pi}{3}\right)$

(b) $\cos\left(\frac{4\pi}{3}\right)$

(c) $\tan\left(\frac{7\pi}{4}\right)$

Solution

Convert the angles from radian to degrees

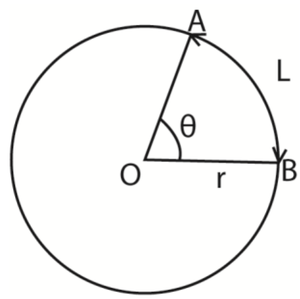
(a) $\sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{2\ x\ 180}{3}\right) = \sin 120^0 = \frac{\sqrt{3}}{2}$

(b) $\cos\left(\frac{4\pi}{3}\right) = \cos\left(\frac{4\ x\ 180}{3}\right) = \cos 240^0 = -\frac{1}{2}$

(d) $tan\left(\frac{7\pi}{4}\right) = \tan\left(\frac{7x\ 180}{4}\right) = \tan 60^0 = \sqrt{3}$

Length of an arc

Suppose that the angle subtended by the length L of an arc AB of a circle is θ as shown.



$\frac{L}{\theta} = \frac{2\pi r}{2\pi}$

L = rθ where θ must be in radians

Example 13

Find the length of an arc of a circle of radius 14 if it subtends an angle

(i) $\frac{\pi}{4}$

(ii) 150^0

Solution

(i) L = rθ = $14\ x\ \frac{\pi}{4} = 11$cm

(ii) Convert degrees to radians

$150^0 = \frac{\pi}{180} x 150 = \frac{5\pi}{6}$ radians

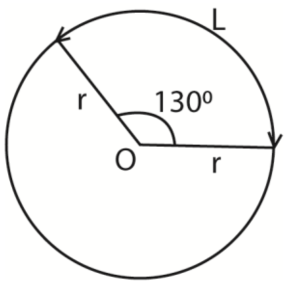
L = $14\ x\ \frac{5\pi}{6} = 36.67$cm

Example 14

A sector was drawn which had a perimeter of 80cm, and centre angle of 130^0. Calculate the radius

Solution

The sides of a sector are composed of an arc, and two more sides which are radii of a circle.



2r + L = 80

L = 80 – 2r

Converting 130^0 to radians

$130^0 = \frac{\pi}{180} x130 = \frac{13\pi}{18}$

But L = rθ

$80 - 2r = \frac{13\pi r}{18}$

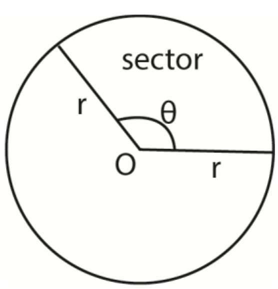
$2r + \frac{13\pi r}{18} = 80$

$\frac{(36+13\pi)r}{18} = 80$

r = 18.74cm

Area of a sector of a circle

A sector of a circle is a portion of the interior of a circle intercepted by a central angle.



The area of a sector of a circle of radius r and central angle θ is given by

$A = \left(\frac{\theta}{2\pi}\right)\pi r^2 = \left(\frac{\theta}{2}\right)r^2$

Where θ must be in radians

Example 15

Find the area of a sector with radius 14cm and angle (i) $\frac{\pi}{4}$ (ii) 1200

Solution

(i) $A = \left(\frac{\theta}{2}\right)r^2 = \left(\frac{\pi}{8}\right).14^2$=77cm^3

(ii) Converting 120^0 to radians

$120^0 = \frac{\pi}{180}\ x\ 120 = \frac{2\pi}{3}$

$A = \left(\frac{\theta}{2}\right)r^2 = \left(\frac{\pi}{3}\right).14^2$=205.25cm^3

Solving trigonometric functions whose range is in radians

When the range of the trigonometric function is in radians, the answer should be given in radians

Example 16

Solve the following equations for the ranges indicated

(i) cosθ + √3sinθ = 1 $\quad$ 0 ≤ θ ≤ π

Solution

√3sinθ = 1- cosθ

Squaring both sides

$3\sin^2\theta = 1 - 2\cos\theta + \cos^2\theta$

$3(1 - \cos^2\theta) = 1 - 2\cos\theta + \cos^2\theta$

$4\cos^2\theta - 2\cos\theta - 1 = 0$

$(2\cos\theta + 1)(\cos\theta - 1) = 0$

Cosθ = $-\frac{1}{2}$	cosθ = 1
θ = ± 120^0	θ = 0^0

$\pm 1200 = \pm\frac{\pi}{180} x\ 120 = \pm\frac{2\pi}{3}$ Radians

0^0 = 0 radians

$\therefore \left[\theta: \theta = 0, \pm\frac{2\pi}{3}\right]$

(ii) $2\cos^2\theta + \sin\theta - 1 = 0 \quad 0 \le \theta \le \pi$

Solution

$2(1 - \sin^2\theta) + \sin\theta - 1 = 0$

$2\sin2\theta - \sin\theta - 1 = 0$

$(\sin\theta - 1)(2\sin\theta + 1) = 0$

Either $\sin\theta = 1$ or $\sin\theta = -\frac{1}{2}$

When $\sin\theta = 1$; $\theta = 90^0$

When $\sin\theta = -\frac{1}{2}$; $\theta = -150^0, -30^0, 210^0, 330^0$

[θ: $\theta = \frac{\pi}{180} x\ 90 = \frac{\pi}{2}$ for given range]

Revision exercise 2

1. Express each of the following in radians
   (a) 30^0 $\left[\frac{\pi}{6}\right]$
   (b) 45^0 $\left[\frac{\pi}{4}\right]$
   (c) 120^0 $\left[\frac{2\pi}{3}\right]$
   (d) 300^0 $\left[\frac{5\pi}{3}\right]$

2. Express the following angle in degrees
   (a) $\frac{\pi}{3}$ rad [60^0]
   (b) $\frac{\pi}{8}$ rad [22.5^0]
   (c) 3π rad [540^0]
   (d) 5.2π rad[936^0]

3. A sector of the circle of radius 7 cm subtends an angle $\frac{\pi}{3}$ radians at the centre. Calculate the
   (a) Length of the arc $\left[6\frac{2}{3}cm\right]$
   (b) Perimeter of the sector$\left[20\frac{2}{3}cm\right]$
   (c) Area of the sector $\left[\frac{77}{3}cm^2\right]$

4. AOB is a sector of a circle, centre O, and is such that OA = OB = 7cm and angle AOB is 300. Calculate the
   (a) Perimeter of sector AOB$\left[17\frac{2}{3}cm\right]$
   (b) The area of AOB $\left[\frac{77}{6}cm^2\right]$

5. Find the value each of the following
   (a) Sinπ [0]
   (b) $\cos 3\pi$ [-1]
   (c) $\tan\frac{\pi}{3}$ [$\sqrt{3}$]

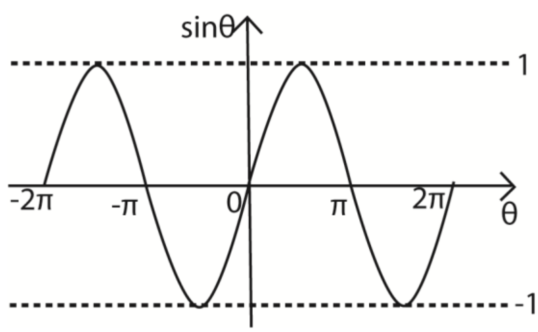
6. Solve the following equations for the ranges indicated
   (a) $2\sec^2\theta = 3 + \tan\theta$ for $0 \le \theta \le 2\pi$
   [θ: $\theta = 0.25\pi, 0.85\pi, 1.25\pi, 1.85\pi$]
   (b) $2\sin^2x\cos x + \cos x - 1$ for $0 \le \theta \le 2\pi$
   [θ: $\theta = 0.38\pi, 1.62\pi, 2\pi$]
   (c) $2\tan\theta + 4\cot\theta = \mathrm{cosec}\theta$ for $-\pi \le \theta \le \pi$
   $\left[\theta: \theta = \pm\frac{1}{3}\pi, \pm0.73\pi\right]$

Graphs of trigonometric functions

The following are the characteristic of the three major trigonometric functions

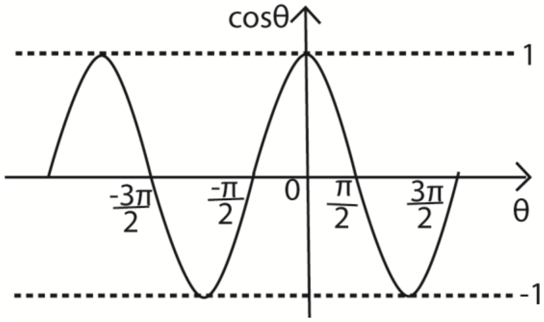
The sine function

- It is continuous (with no breaks)
- The range $-1 \le \sin\theta \le 1$
- The shape of the graph from $\theta = 0$ to $\theta = 2\pi$ is repeated every 2π radians
- This is called a periodic or cyclic function and the width of the repeating pattern that is measured on horizontal axis is called a **period**. The sine wave has a period of 2π, a maximum value of +1 and a minimum value of -1.
- The greatest value of sine wave is called the **amplitude**.



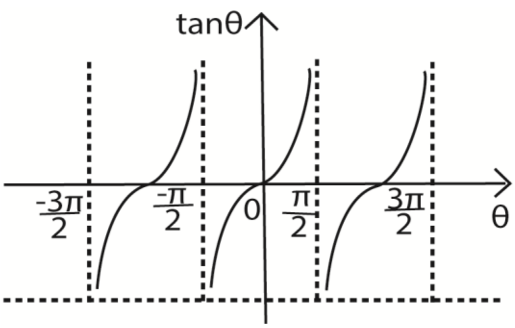
The coosine function

- It is continuous (with no breaks)
- The range $-1 \le \sin\theta \le 1$
- Has a period of 2π
- The shape is the same as the sine wave but displaced a distance $\frac{\pi}{2}$ to the left on the horizontal axis. This is called a **phase shift**



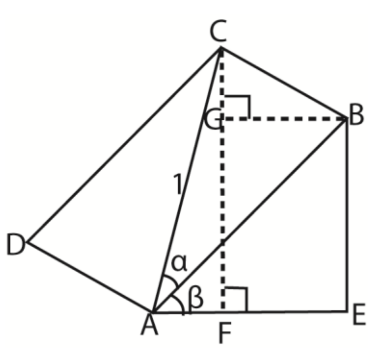
The tan function

- The tan function is found using; $tan\theta = \frac{sin\theta}{cos\theta}$. It follows that tanθ = 0 when sin θ = 0; and tanθ is undefined when cosθ = 0
- The graph is continuous, but undefined when θ= $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$
- The range of values for tanθ is unlimited
- It has a period π



Compound angles

Consider a cardboard ABCD of unit diagonal that stands on the edge A, making an angle β with the horizontal ground. Let the unit diagonal AC be inclined at an angle α to the side AB (see diagram)



Angles EAB= ABG(Alternative angles)

∴ Angle ABG = β

Angle [ABG + GBC] = 90^0

∴ Angle GBC = 90 – β

From triangle GBC,

Angle BCG = 180 – (90 + 90 – β)

∴ Angle BCG = β

From

(1) Triangle ABC:

cosα = $\frac{AB}{AC} = \frac{AB}{1}$; => AB = cosα

(2) Triangle ABE:

cosβ = $\frac{AE}{AB} = \frac{AE}{cos\alpha}$; => AE = cosβcosα

sinβ =$\frac{BE}{AB} = \frac{BE}{cos\alpha}$; => BE = cosαsinβ

(3) Triangle BCG:

cosβ = $\frac{CG}{BC} = \frac{CG}{sin\alpha}$; => CG = sinαcosβ

sinβ =$\frac{BG}{BC} = \frac{BG}{sin\alpha}$; => BG = sinαsinβ

(4) Triangle ACF:

cos(α + β) = $\frac{AF}{AC} = \frac{AF-BG}{1}$ = AE –BG

∴ **cos(α + β) = cosαcosβ – sinαsinβ**

sin(α + β) = $\frac{CF}{AC} = \frac{CG-GF}{1}$ = CG + GF

∴ **sin(α + β) = sinαcosβ + cosαsinβ**

It follows that

(i) cos(α + β) = cosαcosβ – sinαsinβ

(ii) cos(α - β) = cosαcosβ + sinαsinβ

[substituting –β for β)

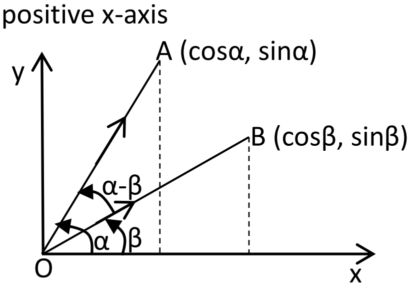
(iii) **sin(α + β) = sinαcosβ + cosαsinβ**

(iv) **sin(α - β) = sinαcosβ - cosαsinβ**

[substituting –β for β)

These can also be derived using vector approach.

Consider two unit vectors $\underline{OA}$ and $\underline{OB}$ each inclined at angles α and β, respectively to the positive x-axis



Using the definition of a vector product:

$\underline{OA}.\underline{OB} = |\underline{OA}|.|\underline{OB}|\cos(\alpha - \beta)$

Since $\underline{OA}$ and $\underline{OB}$ are unit vectors,

$|\underline{OA}| = |\underline{OB}| = 1$

$\therefore \underline{OA}.\underline{OB} = \cos(\alpha - \beta)$

$\Rightarrow$ $(\cos\alpha\underline{i} + \sin\alpha\underline{j}).(\cos\beta\underline{i} + \sin\beta\underline{j}) = \cos(\alpha - \beta)$

$\therefore \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$

Substituting 90 –α for α

$\cos(90 - \alpha)\cos\beta + \sin(90 - \alpha)\sin\beta$

$= \cos(90 - \alpha - \beta)$

$\therefore \sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha + \beta)$

Other expansions can be similar substitutions

i.e. $\tan(\alpha + \beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$

$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$

Dividing through by cosαcosβ

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

Similarly

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

The following is a summary of compound angles

1. $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
2. $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
3. $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$
4. $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$
5. $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$
6. $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

Example 17

Calculate the value of $\sin 15^0$ given that $\sin 45^0$ $= \cos 45 = \frac{1}{\sqrt{2}}$, $\sin 30^0 = \frac{1}{2}$ and $\cos 30^0 = \frac{\sqrt{3}}{2}$

$\sin 15^0 = \sin(45^0 - 30^0)$

$= \sin 45^0\cos 30^0 - \sin 30^0\cos 45^0$

$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{2}.\frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} = 0.2588$

Example 18

Prove that $\tan(45^0 + A) = \frac{1+\tan A}{1-\tan A}$

From $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$\tan(45^0 + A) = \frac{\tan 45^0 + \tan\beta}{1 - \tan 45^0\tan\beta}$

$= \frac{1+\tan A}{1-\tan A}$

Example 19

Acute angles A and B are such that: $\cos A = \frac{1}{2}$, $\sin B \frac{1}{3}$. Show without using tables or calculator that $\tan(A + B) = \frac{9\sqrt{3}+8\sqrt{2}}{5}$

Solution

Using $\cos^2\theta + \sin^2\theta = 1$

$\left(\frac{1}{2}\right)^2 + \sin^2 A = 1$

$\sin^2 A = \frac{3}{4} \Rightarrow \sin A = \frac{\sqrt{3}}{2}$

$\tan A = \frac{\sqrt{3}}{2} \div \frac{1}{2} = \sqrt{3}$

Similarly;

$\cos^2 B + \left(\frac{1}{3}\right)^2 = 1$

$\cos B = \frac{2\sqrt{2}}{3}$

$\tan B = \frac{2\sqrt{2}}{3} \div \frac{1}{3} = \frac{1}{2\sqrt{2}}$

But

From $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \frac{\sqrt{3} + \frac{1}{2\sqrt{2}}}{1 - \sqrt{3}.\frac{1}{2\sqrt{2}}}$

$= \frac{(2\sqrt{2}\sqrt{3}+1)(2\sqrt{2}+\sqrt{3})}{(2\sqrt{2}-\sqrt{3})(2\sqrt{2}+\sqrt{3})}$

$\tan(A + B) = \frac{9\sqrt{3}+8\sqrt{2}}{5}$

Example 20

Solve cos(θ + 35^0) = sin(θ + 25^0) for $0^0 \le θ \le 360^0$

Cosθcos35^0 – sinθsin35^0 =sinθcos25^0 + cosθsin25^0

Dividing through by cosθ

Cos35^0 – tanθsin35^0 = tanθcos25^0 + sin25^0

tanθ = $\frac{cos35^0 - sin25^0}{cos35^0 + sin25^0} = \frac{0.3965337825}{1.479884223}$

θ = 15^0, 195^0 for $0^0 \le θ \le 360^0$

Example 21

(a) Prove than $\frac{2tan\theta}{1+tan^2\theta} = sin2\theta$

Solution

$$\frac{2tan\theta}{1+tan^2\theta} = \frac{2sin\theta}{cos\theta} \div \left(1 + \frac{sin^2\theta}{cos^2\theta}\right)$$
$$= \frac{2sin\theta}{cos\theta} \div \left(\frac{cos^2\theta + sin^2\theta}{cos^2\theta}\right)$$
$$= \frac{2sin\theta}{cos\theta} \div \left(\frac{1}{cos^2\theta}\right)$$
$$= 2sin\theta cos\theta = sin2\theta$$

(b) Solve sin2θ = cosθ for $0^0 \le θ \le 90^0$

Solution

sin2θ = cosθ

2sinθcosθ = cosθ

sinθ = $\frac{1}{2}$

θ = 30^0 for $0^0 \le θ \le 90^0$

Example 22

Given that α, β and γ are angles of a triangle, show that tanα + tanβ + tanγ = tanαtanβtanγ

Hence find tanγ if tanα = 1 and tanγ = 2.

Solution

α + β + γ= 180^0 (angle sum of a triangle)

tan (α + β + γ) = tan180^0 = 0

tan[(α + β) + γ] = 0

$$\frac{tan(\alpha + \beta) + tan\gamma}{1 - tan(\alpha+\beta)\, tan\gamma} = 0$$

⇨ $tan(\alpha + \beta) + tan\gamma = 0$

$$\frac{tan\alpha + tan\beta}{1 - tan\alpha\, tan\beta} = -tan\gamma$$

$tan\alpha + tan\beta = -tan\gamma +$ tanαtanβtanγ

∴ tanα + tanβ + tanγ = tanαtanβtanγ

Example 23

In a triangle ABC, prove that $cot\frac{1}{2}A + cot\frac{1}{2}B + cot\frac{1}{2}C$

$= cot\frac{1}{2}Acot\frac{1}{2}Bcot\frac{1}{2}C$

Solution

$$\frac{1}{2}(A + B + C) = \frac{1}{2}(180^0) = 90^0$$

$$cot\left[\frac{1}{2}(A + B + C)\right] = cot90^0 = 0$$

$$⇨ \frac{1 - tan\left(\frac{1}{2}A + \frac{1}{2}B\right)tan\frac{1}{2}C}{tan\left(\frac{1}{2}A + \frac{1}{2}B\right) + tan\frac{1}{2}C} = 0$$

$$1 = tan\left(\frac{1}{2}A + \frac{1}{2}B\right) tan\frac{1}{2}C$$

$$1 = \left(\frac{tan\frac{1}{2}A + tan\frac{1}{2}B}{1 - tan\frac{1}{2}Atan\frac{1}{2}B}\right) tan\frac{1}{2}C$$

$$1 - tan\frac{1}{2}Atan\frac{1}{2}B$$
$$= tan\frac{1}{2}Atan\frac{1}{2}C + tan\frac{1}{2}Btan\frac{1}{2}C$$

$$1 = tan\frac{1}{2}Atan\frac{1}{2}B + tan\frac{1}{2}Atan\frac{1}{2}C + tan\frac{1}{2}Btan\frac{1}{2}C$$

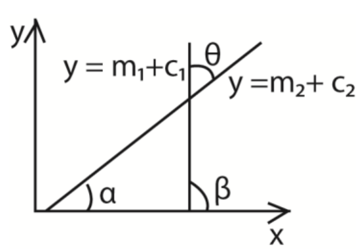
Dividing each side by $tan\frac{1}{2}Atan\frac{1}{2}Btan\frac{1}{2}C$

$$cot\frac{1}{2}Acot\frac{1}{2}Bcot\frac{1}{2}C = cot\frac{1}{2}A + cot\frac{1}{2}B + cot\frac{1}{2}C$$

Example 24

Prove that the angle θ, between the straight line y = m_1x + c_1 and the straight line y = m_2x + c_2 is given by tanθ = $\frac{m_2 - m_1}{1 + m_2 m_1}$

Let the lines be inclines at angles α and β with the x-axis respectively



From the diagram above

θ = β – α

⇨ tanθ = tan(β – α)

$= \frac{tan\beta - tan\alpha}{1+tan\beta tan\alpha}$

tanθ = $\frac{m_2 - m_1}{1+m_2 m_1}$

Revision exercise 3

1. (a) show that sin(α +β) - sin(α - β) = 2cosαsinβ
   (b) If sin(α +β) = 5cos(α - β) show that tanα = $\frac{5-tan\beta}{1+tan\beta}$
   (c) Without using tables or calculator, show that cos15^0 = sin75^0
   (d) If α +β = 45^0, show that tanα = $\frac{1-tan\beta}{1+tan\beta}$
2. Prove that:
   (i) $\frac{\sin(\alpha+\beta)}{\cos(\alpha-\beta)} + 1 = \frac{(1+tan\beta)(1+cot\alpha)}{cos\alpha + tan\beta}$
   (ii) $tan\alpha - tan\beta = \frac{\sin(\alpha-\beta)}{cos\alpha cos\beta}$
   (iii) $cot\alpha + cot\beta = \frac{\sin(\alpha+\beta)}{\sin\alpha \sin\beta}$
   (iv) $\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)} = \frac{tan\alpha - tan\beta}{tan\alpha + tan\beta}$
   (v) $\frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} = \frac{cot\alpha cot\beta + 1}{cot\alpha cot\beta - 1}$
   (vi) $\cot(A+B) = \frac{cotAcotB - 1}{cotA + cotB}$
3. (a) Determine solution of tan2x + 2sinx = 0 for 0^0 ≤ x ≤ 180^0 [x: x= 0^0, 60^0, 120^0, 180^0]
   (vii) Show that in triangle ABC,
   tanA + tanB + tanC = tanAtanBtanC
4. Find the values of tan α for each of the following
   (a) sin(α - 30^0) = cos α [$\sqrt{3}$]
   (b) sin(α + 45^0) = cos α [$\sqrt{2} - 1$]
   (c) cos(α + 60^0) = sin α [2 - √3]
   (d) sin(α + 60^0) = cos(α – 60^0) [1]
   (e) cos(α + 60^0) = 2cos(α + 30^0) [4 + 3√3]
   (f) sin(α + 60^0) = cos(45^0 – α)$\left[\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1}\right]$
5. Given that
   (a) tan(α –β) = ½ and tanα = 3 find the value of tanβ [1]
   (b) tan(α +β) = 5 and tanβ = 2 find the value of tanα$\left[\frac{3}{11}\right]$
6. Given that
   (a) tan(θ – 45^0) = 4, find the value of θ $\left[-\frac{5}{3}\right]$
   (b) tan (θ + 60^0) find the value of cotθ $[8 + 5\sqrt{3}]$

Double angles and half angles

(b) From cos (θ + θ) = cosθcosθ – sinθsinθ

⇨ cos2θ = cos^{2}θ – sin^{2}θ………… (i)

Either

cos2θ = cos^{2}θ -1 + cos^{2}θ (cos^{2}θ + sin^{2}θ = 1]

cos2θ = 2cos^{2}θ -1 ………………………… (ii)

Or

cos2θ = 1– sin^{2}θ – sin^{2}θ

⇨ cos2θ = 1 – 2sin^{2}θ …………….(iii)

It follows that

cos^{2}θ = $\frac{1}{2}(1 + cos2\theta)$ …………………..(iv)

sin^{2}θ = $\frac{1}{2}(1 - cos2\theta)$ …………………..(iv)

The identities imply

Cos6θ = cos^{2}3θ – sin^{2}3θ

=2cos^{2}3θ – 1 = 1 – 2 sin^{2}3θ

Cos θ = cos$^2\frac{\theta}{2}$ – sin$^2\frac{\theta}{2}$

=2cos$^2\frac{\theta}{2}$ – 1 = 1 – 2 sin$^2\frac{\theta}{2}$

(c) sin(θ + θ) = sinθcosθ + cosθsinθ

⇨ sin2θ = 2sinθcosθ

It follows that

sin6θ = 2sin3θcos3θ

sinθ = 2sin$\frac{\theta}{2}$cos$\frac{\theta}{2}$

(d) tan (θ + θ)=$\frac{tan\theta + tan\theta}{1 - tan\theta tan\theta}$

⇨ $tan2\theta = \frac{2tan\theta}{1 - tan^2\theta}$

It follows that

$$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$

$$\tan 6\beta = \frac{2\tan 3\beta}{1-\tan^2 3\beta}$$

Note that in all cases, the angles on the right hand side are half the angles on the left hand side [***half angle formulae]***

Example 25

Show that

(a) cosec2θ + cot2θ = cotθ

Solution

$$cosec2\theta + \cot 2\theta = \frac{1}{\sin 2\theta} + \frac{\cos 2\theta}{\sin 2\theta}$$

$$= \frac{1+\cos 2\theta}{\sin 2\theta}$$

$$= \frac{1+2\cos^2\theta - 1}{2\sin\theta\cos\theta}$$

$$= \frac{2\cos^2\theta}{2\sin\theta\cos\theta} = \cot\theta$$

(b) $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

Hence deduce that if 3θ + α = 45^0, then

$\tan\alpha = \frac{1-3\tan\theta-3\tan^2\theta+\tan^3\theta}{1+2\tan\theta-3\tan^2\theta-\tan^3\theta}$

Solution

$$\tan 3\theta = \tan(2\theta+\theta) = \frac{\tan 2\theta + \tan\theta}{1-\tan 2\theta\tan\theta}$$

$$=\left\{\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) + \tan\theta\right\} \div \left\{1 - \left(\frac{2\tan\theta}{1-\tan^2\theta}\right)\tan\theta\right\}$$

$$=\frac{2\tan\theta+\tan\theta-\tan^3\theta}{1-\tan^2\theta-2\tan^2\theta} = \frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$$

$$\therefore \tan 3\theta = \frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$$

Hence 3θ + α = 45^0 => α = 45^0 - 3θ

Tanα = tan(45^0 - 3θ)

$$= \frac{\tan 45^0 - \tan 3\theta}{1+\tan 45^0 \tan 3\theta} = \frac{1-\tan 3\theta}{1+\tan 3\theta}$$

$$= \frac{1-\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)}{1+\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)}$$

$$=\frac{1-3\tan\theta-3\tan^2\theta+\tan^3\theta}{1+2\tan\theta-3\tan^2\theta-\tan^3\theta}$$

$$\therefore \tan\alpha = \frac{1-3\tan\theta-3\tan^2\theta+\tan^3\theta}{1+2\tan\theta-3\tan^2\theta-\tan^3\theta}$$

Example 26

If tanα = $\frac{3}{4}$ and α is acute, without using tables or calculator work out the value of

(a) tan2α

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2x\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2} = \frac{\frac{2}{2}}{\frac{3}{4}} = \frac{24}{7}$$

(b) $\tan\frac{\alpha}{2}$

similarly $\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}} = \frac{3}{4}$

$$\Rightarrow 3\tan^2\frac{\alpha}{2} + 8\tan\frac{\alpha}{2} - 3 = 0$$

$$(3\tan\frac{\alpha}{2} - 1)(\tan\frac{\alpha}{2} + 3) = 0$$

$$\tan\frac{\alpha}{2} = \frac{1}{2} \text{ or } \tan\frac{\alpha}{2} = -3$$

Since α is acute, tanα cannot be negative

$$\therefore \tan\frac{\alpha}{2} = \frac{1}{3}$$

Example 27

(a) Show that cos3α = $4\cos^2$ α -3cos α. Hence solve the equation $4x^3 - 3x - \frac{\sqrt{3}}{3} = 0$ for $0^0 \leq \alpha \leq 180^0$

Solution

Cos3α = cos(2α +α)

= cos2αcosα – sin2αsinα

=($2\cos^2$α -1)cosα – $2\sin^2$αcosα

=($2\cos^2$α -1)cosα – 2(1-$\cos^2$α)cosα

= $2\cos^3$α – cosα -2cosα + $2\cos^3$α

=$4\cos^2$ α -3cos α

Hence $4x^3 - 3x = \frac{\sqrt{3}}{3}$

i.e. $4\cos^2$ α -3cos α =$\frac{\sqrt{3}}{3}$

$0^0 \leq \alpha \leq 180^0$; cos3α = $\frac{\sqrt{3}}{3}$

For the range $0^0 \leq \alpha \leq 180^0$

⇨ $0^0 \leq 3\alpha \leq 540^0$

$3\alpha = 54.7^0, 414.7^0$

$\alpha = 18.23^0, 138.23^0$ (2d.p)

[α: α= $18.23^0, 138.23^0$]

(b) Given that t = $\tan 22\frac{1}{2}$, show that $t^2 + 2t - 1 = 0$,

Hence show that $\tan 22\frac{1}{2}^0 = -1 + \sqrt{2}$

Solution

$\tan 45^0 = \frac{2\tan 22\frac{1}{2}^0}{1-\tan^2 22\frac{1}{2}^0}$

$1 = \frac{2t}{1-t^2}$

$1 - t^2 = 2t$

$t^2 + 2t - 1 = 0$ (as required)

solving

$t = \frac{-2\pm\sqrt{2^2-(4x1\ x-1)}}{2x1}$

$t = \frac{-2\pm2\sqrt{2}}{2} = -1 \pm \sqrt{2}$

Since $22\frac{1}{2}^0$ is an acute angle,

$\tan 22\frac{1}{2}^0 = -1 + \sqrt{2}$ is positive

$\therefore \tan 22\frac{1}{2}^0 = -1 + \sqrt{2}$

Example 28

(a) Show that 3sinθ = 3sinθ – 4sin³θ. Hence solve the equation sin3θ + sinθ = 0 for $0^0 \leq \theta \leq 360^0$

Solution

$\sin 3\theta = \sin(2\theta + \theta)$

$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$

$= 2\sin\theta\cos^2\theta + (1 - 2\sin^2\theta)\sin\theta$

$= 2\sin\theta(1 - \sin^2\theta) + (1 - 2\sin^2\theta)\sin\theta$

$= 3\sin\theta - 4\sin^3\theta$

Hence $\sin 3\theta + \sin\theta = 0$

$3\sin\theta - 4\sin^3\theta + \sin\theta = 0$

$4\sin\theta - 4\sin^3\theta = 0$

$4\sin\theta(1 - \sin^2\theta) = 0$

$4\sin\theta(1 - \sin\theta)(1 + \sin\theta) = 0$

$\sin\theta = 0$; $\theta = 0^0, 180^0, 360^0$

$\sin\theta = 1$; $\theta = 90^0$

sino = -1; $\theta = 270^0$

$\therefore$θ: θ= $0^0, 90^0, 180^0, 270^0, 360^0$

(b) Prove that $\cot 2\theta = \frac{\cot^2\theta-1}{2\cot\theta}$. Hence solve the equation cot2θ + 2cotθ = 2 for $0^0 \leq \theta \leq 360^0$

Solution

$\cot 2\theta = \frac{\cos 2\theta}{\sin 2\theta} = \frac{\cos^2\theta-\sin^2\theta}{2\sin\theta\cos\theta}$

dividing through by $\sin^2\theta$

$\cot 2\theta = \frac{\cot^2\theta-1}{2\cot\theta}$

Hence, cot2θ + 2cotθ =0

$\frac{\cot^2\theta-1}{2\cot\theta} + 2\cot\theta = 0$

$5\cot^2\theta - 4\cot\theta - 1 = 0$

$(5\cot\theta + 1)(\cot\theta - 1) = 0$

$\cot\theta = -\frac{1}{5}$ or $\cot\theta = 0$

⇨ tanθ = -5 ot tanθ = 1

When tanθ= -5; θ= $101.3^0, 281.3^0$

When tanθ = 1, θ = $45^0, 225^0$

$\therefore${θ: θ= $45^0, 101.3^0, 225^0, 281.3^0$}

Revison exercise 4

1. Prove that

(a) sinαcosecβ + cosαsecβ = 2sin(α + β)cosec2β

(b) $\cos^6\theta + \sin^6\theta = 1 - \frac{3}{4}\sin^2 2\theta$

(c) $\frac{\sin 3\alpha}{1+2\cos 2\alpha} = \sin\alpha$ and hence deduce that $\sin 15^0 = \frac{\sqrt{3}-1}{2\sqrt{2}}$

2. (a) Solve the equation for θ, $0^0 \leq \theta \leq 360^0$
$\sin^2\theta - 2\sin\theta\cos\theta - 3\cos^2\theta = 0$
[θ: θ = $71.6^0, 135^0, 251.6^0, 315^0$]

(b) show that $\frac{\cos\theta}{1+\sin\theta} = \cot(\frac{\theta}{2} + 45^0)$.
Hence or otherwise solve the equation $\frac{\cos\theta}{1+\sin\theta} = \frac{1}{2}$ for $0^0 \leq \theta \leq 360^0$ [θ= 36.8^0]

3. (a) solve the equation 4cos2θ – 2cosθ + 3 = 0 for $0^0 \leq \theta \leq 360^0$
[θ: θ= $60^0, 104.5^0, 255.5^0, 300^0$]

(c) Solve the equation $\sin\theta + \sin\frac{\theta}{2} = 0$ for $0^0 \leq \theta \leq 360^0$

[θ: θ= -360^0, -240^0, 0^0, 2405^0, 360^0]

4. (a) Prove that tan$\left(\frac{\pi}{4}+\theta\right)$ - tan$\left(\frac{\pi}{4}-\theta\right)$ = 2tan2θ
   (b) By expressing 2sinθsin(θ + α) as difference of cosines of two angles or otherwise, where α is constant, find its least value $\left[\frac{-\alpha}{2}\right]$
   (c) Solve for θ in the equation Cosθ – cos(θ + 60^0) = 0.4 for $0^0 \le \theta \le 360^0$ [θ = 126.4^0, 353.6^0]
5. (a) Show that cos3θ = $4\cos^3\theta - 3\cos\theta$.
   Hence solve the equation $4x^3 - 3x - \frac{\sqrt{3}}{3} = 0$
   [x: x = -0.746, -0.204, 0.959]
   (b) Find all solutions of the equation 5cosx – 4sinx = 6 in the range $-180^0 \le x \le 180^0$ [x: x = -59.1^0, -18.3^0]
6. (a) Express $\sqrt{\left(\frac{sin2\theta - cos2\theta - 1}{2 - 2sin2\theta}\right)}$ in terms of tanθ$\left[\frac{1}{\sqrt{(tan\theta - 1)}}\right]$
   (b) Find the solution of the equation $\sqrt{3sin\theta} - cos\theta + 1 = 0$ for 0≤θ≤2π
   $$\left[\theta: \theta = \frac{4}{3}\pi, 2\pi\right]$$
   (c) Factorize cosθ – cos3θ – cos7θ + cos9θ in form Acospθsinqθsinrθ where A, p, q and r are constants [A= -4, p =5, q = 5, r = 2]
7. (a) Given that sinα + sinβ = p and cosα + cosβ = q show that
   (i) tan$\left(\frac{\alpha+\beta}{2}\right) = \frac{p}{q}$
   (ii) cos(α + β)= $\frac{q^2-p^2}{q^2+p^2}$
   (b) Solve the simultaneous equation:
   cosα + 4sinβ = 1
   4secα – 3cosecβ = 5 [θ= 78.5^0, 281.5^0]
8. (a) Express sinθ + sin3θ in form pcosθsinqθ where p and q are constant [p =2, q = 2]
   (b) Find the solution of cos7θ + cos5θ = 2cosθ for $0^0 \le \theta \le 360^0$ [0^0, 60^0, 270^0, 360^0]
   (c) Prove that $\frac{sinA+sin4A+sin7A}{cosA+cos4A+cos7A} = tan4A$
9. Eliminate θ from each of the following pairs of expression
   (a) x+1 = cos2θ, y = sinθ [$x + 2y^2 = 0$]
   (b) x = cos2θ, y = cosθ -1 [$x = 2y^2 + 4y + 1$]
   (c) y – 3 = cos2θ, x = 2 – sinθ
   [$2x^2 - 8x + y + 4 = 0$]
10. Solve the following equations for $-180^0 \le \theta \le 180^0$
    (a) Sin2θ + sinθ = 0 [$\pm120^0$, $\pm180^0$]
    (b) Sin2θ – $2\cos^2\theta$ = 0 [-135^0, 45^0, $\pm90^0$]
    (c) 3cos2θ + 2+ cosθ = 0 [$\pm70.5^0$, $\pm120^0$]
    (d) sin2θ = tanθ [0^0, $\pm45^0$, $\pm135^0$, $\pm180^0$]
11. Solve the following equations for $-360^0 \le \theta \le 360^0$, giving your answer correct to 1 decimal place
    (a) sinθ = sin$\left(\frac{\theta}{2}\right)$[$0^0$, $\pm120^0$, $\pm360^0$]
    (b) 3cos$\left(\frac{\theta}{2}\right)$ = 2sinθ [$\pm180^0$, 97.2^0, 262.8^0]
    (c) 2sinθ = tan$\left(\frac{\theta}{2}\right)$
    [0^0, $\pm120^0$, $\pm240^0$, $\pm360^0$]
    (d) 2cosθ = 15cos$\left(\frac{\theta}{2}\right) + 2$ [$\pm209^0$]
12. Prove the following identities
    (a) $2\cos^2\theta$ – cos2θ = 1
    (b) 2cosec2θ = cosecθsecθ
    (c) $2\cos^3\theta$ + sin2θsinθ =2cosθ
    (d) tanθ + cotθ = 2cosec2θ
    (e) $\cos^4\theta - \sin^4\theta$ = cos2θ
    (f) $\frac{1-cos2\theta}{1+cos2\theta} = tan^2\theta$
    (g) Cotθ – tanθ = 2cot2θ
    (h) cot2θ + cosecθ = cotθ
    (i) $\frac{cos2\theta}{cos\theta+sin\theta} = cos\theta - sin\theta$
    (j) $\frac{sin2\theta}{1-cos2\theta} = cot\theta$
    (k) $cos2\theta = \frac{1-tan^2\theta}{1+tan^2\theta}$
    (l) $tan3\theta = \frac{3tan\theta - tan^3\theta}{1-3tan^2\theta}$
    (m) $tan\left(\frac{\theta}{2}\right) = \frac{sin\theta}{1+cos\theta}$
13. Express $\tan 22\frac{1}{2}^0$ in the form a + b√2 where a and b are integers [a = -1, b= ±1]
14. Solve the equation
    (i) 4cosθ – 2cos2θ = 3 for 0 ≤ θ ≤ π $\left[\frac{\pi}{3}\right]$
    (ii) Cos2θ + cos3θ + cosθ = 0 for $0^0 \le \theta \le 360^0$
    [θ = 45^0, 120^0, 135^0]
    (iii) cosθ + sin2θ = 0 for $0^0 \le \theta \le 360^0$
    [θ = 90^0, 210^0, 270^0, 330^0]
    (iv) 2sin2θ = 3cosθ for $-180^0 \le \theta \le 180^0$
    [θ = -90^0, 48.6^0, 90^0, 132.4^0]
    (v) Sinθ – 4sin4θ = sin2θ – sin3θ for $-\pi \le \theta \le \pi$ $\left[\frac{-\pi}{5}, \frac{-\pi}{2}, \frac{-3\pi}{5}, 0, \frac{\pi}{2}, \frac{\pi}{5}, \frac{3\pi}{5}\right]$

Harmonic form

These are trigonometric functions expressed in the form of **Rcos(x ± α) and Rsin(x ±α).** They are in two ways

(i) solving equations in the form
$a\cos\theta + b\sin\theta + c = 0$

(ii) determining the maximum and minimum values of the function
$a\cos\theta + b\sin\theta + c = 0$
where a, b and c are constants

A: Solving equations

Example 29

(a) Express $3\cos\theta - 4\sin\theta$ in the form $R\cos(\theta + \alpha)$, where R and α are constants

Solution

Let $3\cos\theta - 4\sin\theta = R\cos(\theta + \alpha)$
$= R(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$
$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

Comparing coefficient of cosθ and sinθ

$R\cos\alpha = 3$ .... (i)
$R\sin\alpha = 4$ .... (ii)
Eqn (ii) ÷ eqn (i)
$\tan\alpha = \frac{4}{3}$; $\alpha = 53.1^0$
Eqn. $(i)^2$ + eqn. $(ii)^2$
$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2 = 25$
$R^2[\cos^2\alpha + \sin^2\alpha] = 25$
$R^2 = 25$
$R = 5$
$\therefore 3\cos\theta - 4\sin\theta = 5\cos(\theta + 53.1^0)$

(b) Solve the equation $3\cos\theta - 4\sin\theta = 5$ for $0^0 \le \theta \le 360^0$.

Solution

$3\cos\theta - 4\sin\theta = 5\cos(\theta + 53.1^0)$

⇨ $5\cos(\theta + 53.1^0) = 5$
$\cos(\theta + 53.1^0) = 1$
$x + 53.1^0 = 0^0, 360^0$
$x = -53.1^0, 306.9^0$

Hence $x = 306.9^0$

Example 30

(a) Express $\sin\theta - \sqrt{3}\cos\theta$ in the form $R\sin(\theta - \alpha)$

Solution

Let $\sin\theta - \sqrt{3}\cos\theta = R\sin(\theta - \alpha)$
$= R(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$

Equating coefficients

$R\cos\alpha = 1$ .... (i)
$R\sin\alpha = \sqrt{3}$.................... (ii)
Eqn. (ii) ÷ eqn. (i)
$\tan\alpha = \sqrt{3}$; => $\alpha = 60^0$
$R^2[\cos^2\alpha + \sin^2\alpha] = 4$
$R^2 = 4$; $R = 2$
$\therefore \sin\theta - \sqrt{3}\cos\theta = 2\sin(\theta - 60^0)$

(b) Solve the equation
$\sin\theta - \sqrt{3}\cos\theta + 1 = 0$ for $0^0 \le \theta \le 360^0$
$\sin\theta - \sqrt{3}\cos\theta = 2\sin(\theta - 60^0)$
⇨ $2\sin(\theta - 60^0) + 1 = 0$

$\sin(\theta - 60^0) = -\frac{1}{2}$

$\theta - 60^0 = 210^0, 330^0$

$\theta = 270^0, 390^0$

Hence $\theta = 270^0$ for the given range

Example 31

(a) Express $4\cos\theta - 5\sin\theta$ in the form $A\cos(\theta + \beta)$, where A is constant and β is an acute angle

Let $4\cos\theta - 5\sin\theta = A\cos(\theta + \beta)$
$= A(\cos\theta\cos\beta - \sin\theta\sin\beta)$
$= A\cos\theta\cos\beta - R\sin\theta\sin\beta$

Comparing coefficient of cosθ and sinθ

$A\cos\beta = 4$ .... (i)
$A\sin\beta = 5$ .... (ii)
Eqn (ii) ÷ eqn (i)
$\tan\alpha = \frac{5}{4}$; $\alpha = 51.3^0$
Eqn. $(i)^2$ + eqn. $(ii)^2$
$A^2\cos^2\beta + A^2\sin^2\beta = 4^2 + 5^2 = 41$
$A^2[\cos^2\alpha + \sin^2\alpha] = 41$
$A^2 = 41$
$A = \sqrt{41}$
$\therefore 3\cos\theta - 4\sin\theta = \sqrt{41}\cos(\theta + 51.3^0)$

(b) Solve the equation $3\cos\theta - 4\sin\theta = 2.2$ for $0^0 \le \theta \le 360^0$

Solution

$3\cos\theta - 4\sin\theta = \sqrt{41}\cos(\theta + 51.3^0)$

⇨ $\sqrt{41}\cos(\theta + 51.3^0) = 2.2$

$\cos(\theta + 51.3^0) = \frac{2.2}{\sqrt{41}} = 0.3436$

$(\theta + 51.3^0) = 69.9^0,\ 290.1^0$

$\therefore \theta = 18.6^0, 238.3^0$

B: Maximum and minimum values

The maximum and minimum values of a circular function may be obtained using three methods

(i) Express the given function either in for Rcos (θ ± α) or Rsin(θ ± α) if possible, where R and α are constants.
(ii) Differentiating the given function with respect to the given function say θ
(iii) Sketching the graphs of the function given and noting their maximum and minimum points.

In this chapter approach I will be considered.

Example 32

Determine the maximum and minimum values of the following, stating the value of θ for which they occur

(a) $\sqrt{3}\sin\theta + \cos\theta + 7$

Let $\sqrt{3}\sin\theta + \cos\theta = R\sin(\theta + \alpha)$

$= R(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$

Equating coefficients

$R\sin\alpha = 1$ .... (i)

$R\cos\alpha = \sqrt{3}$.................. (ii)

Eqn. (i) ÷ eqn. (ii)

$\tan\alpha = \frac{1}{\sqrt{3}}$; => $\alpha = 30^0$

$R^2[\cos^2\alpha + \sin^2\alpha] = \left[1^2 + \left(\sqrt{3}\right)^2\right] = 2$

$R^2 = 4$; $R = 2$

$\therefore \sqrt{3}\sin\theta + \cos\theta = 2\sin(\theta + 30^0)$

⇨ $\sqrt{3}\sin\theta + \cos\theta + 7 = 2\sin(\theta + 30^0) + 7$

The minimum value occurs when

$\sin(\theta + 30^0) = -1$

⇨ Minimum value = 2(-1) + 7 = 5

Now for $\sin(\theta + 30^0) = -1$

$\theta + 30^0 = 270^0$

$\theta = 240^0 = \frac{4\pi}{3}$

The minimum value is $\left(\frac{4\pi}{3}, 5\right)$

And maximum value occurs when

$\sin(\theta + 30^0) = 1$

⇨ Minimum value = 2(1) + 7 = 9

Now for $\sin(\theta + 30^0) = 1$

$\theta + 30^0 = 90^0$

$\theta = 60^0 = \frac{\pi}{3}$

The maximum value is $\left(\frac{\pi}{3}, 9\right)$

(b) $5\cos\theta - 12\sin\theta - 13$

Solution

Let $5\cos\theta - 12\sin\theta = R\cos(\theta - \alpha)$

$= R(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$

$= R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

Comparing coefficient of cosθ and sinθ

$R\cos\alpha = 5$ .... (i)

$R\sin\alpha = 12$ .... (ii)

Eqn (ii) ÷ eqn (i)

$\tan\alpha = \frac{12}{5}$; $\alpha = 67.4^0$

Eqn. $(i)^2$ + eqn. $(ii)^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 5^2 + 12^2 = 169$

$R^2[\cos^2\alpha + \sin^2\alpha] = 169$

$R^2 = 169$

$R = 13$

$\therefore 2\cos\theta - 12\sin\theta = 13\cos(\theta - 67.4^0)$

⇨ $5\cos\theta - 12\sin\theta - 13 = 13\cos(\theta - 67.4^0) - 13$

The minimum value occurs when

$\cos(\theta - 67.4^0) = -1$

⇨ Minimum value = 13(-1) -13 = -26

Now for $\cos(\theta - 67.4^0) = -1$

$\theta - 67.4^0 = 180^0$

$\theta = 247.4^0$

The minimum value is $(247.4^0, -26)$

And maximum value occurs when

$\cos(\theta - 67.4^0) = 1$

⇨ Minimum value = 13(1) - 13 = 0

Now for $\cos(\theta - 67.4^0) = 1$

$\theta - 67.4^0 = 0^0$

$\theta = 67.4^0$

The maximum value is $(67.4^0, 0)$

Example 33

(a) Given that $p = 2\cos\theta + 3\cos 2\theta$ and $q = 2\sin\theta + 3\sin 2\theta$, show that $1 \le p^2 + q^2 \le 25$
If $p^2 + q^2 = 19$ and θ is acute, find θ and show that $pq = \frac{-5\sqrt{3}}{4}$

Solution

$p^2 = 4\cos^2\theta + 12\cos\theta\cos 2\theta + 9\cos^2 2\theta$ ....(i)

$q^2 = 4\sin^2\theta + 12\sin\theta\sin 2\theta + 9\sin^2 2\theta$ ....(ii)

Eqn. (i) + eqn. (ii)

$p^2 + q^2 = 4 + 12(\cos\theta\cos 2\theta + \sin\theta\sin 2\theta) + 9$

$p^2 + q^2 = 13 + 12\cos\theta$ $[\cos(-\theta) = \cos\theta]$

But $-1 \le \cos\theta \le 1$

Multiplying through by 12

$-12 \le 12\cos\theta \le 12$

Adding 13 throughout

$1 \le 12\cos\theta + 12 \le 25$

$\therefore 1 \le p^2 + q^2 \le 25$ as required

If $p^2 + q^2 = 19$, $\Rightarrow 13 + 12\cos\theta = 19$

$\cos\theta = \frac{1}{2}$; $\theta = 60^0$ [θ is acute]

⇨ $p = 2\cos 60^0 + 3\cos 120^0 = 1 - \frac{3}{2} = -\frac{1}{2}$

$q = 2\sin 60^0 + 3\sin 120^0 = \sqrt{3} + 3\frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2}$

$\therefore pq = \left(-\frac{1}{2}\right)\left(\frac{5\sqrt{3}}{2}\right) = \frac{5\sqrt{3}}{4}$

(b) Express $f(x) = 5\sin^2\theta - 3\sin\theta\cos\theta + \cos^2\theta$ in the form $p + q\cos(2\theta - \alpha)$
Hence show that $\frac{1}{2} \le f(x) \le 5\frac{1}{2}$

Solution

Using $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ and

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$f(x) = \frac{5}{2}(1 - \cos 2\theta) - 3\sin\theta\cos\theta + \frac{1}{2}(1 + \cos 2\theta)$

$= \frac{5}{2} - \frac{5}{2}\cos 2\theta - 3.\frac{2}{2}\sin\theta\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta$

$= 3 - 2\cos 2\theta - \frac{3}{2}\sin 2\theta$

$= 3 - [2\cos\theta + \frac{3}{2}\sin 2\theta]$

Now:

$3 - [2\cos\theta + \frac{3}{2}\sin 2\theta] \equiv p + q\cos(2\theta - \alpha)$

$= 3 + [q\cos 2\theta\cos\alpha + q\sin 2\theta\sin\alpha]$

By comparing: $p = 3$, $q\sin\alpha = \frac{3}{2}$ and $q\cos\alpha = 2$

⇨ $\tan\alpha = \frac{3}{4}$; $\alpha = 36.9^0$

And $q = \sqrt{\left\{\left(\frac{3}{2}\right)^2 + (2)^2\right\}} = \frac{5}{2}$

⇨ $3 - [2\cos\theta + \frac{3}{2}\sin 2\theta] = 3 - \frac{5}{2}\cos(2\theta - 36.9^0)$

But $-1 \le \cos(2\theta - 36.9^0) \le 1$

Multiplying through by $-\frac{5}{2}$

$\frac{5}{2} \ge -\frac{5}{2}\cos(2\theta - 36.9^0) \ge -\frac{5}{2}$

Adding 3 throughout

$3 - \frac{5}{2} \le 3 - \frac{5}{2}\cos(2\theta - 36.9^0) \le 3 + \frac{5}{2}$

$\therefore \frac{1}{2} \le f(x) \le 5\frac{1}{2}$

(c) Find the maximum and minimum points of the function; $f(x) = 3\cos\theta - 4\sin\theta + 20$ for $0^0 \le \theta \le 360^0$

Solution

Let $3\cos\theta - 4\sin\theta = R\cos(\theta + \alpha)$

$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

Comparing coefficient of cosθ and sinθ

$R\cos\alpha = 3$ .... (i)

$R\sin\alpha = 4$ .... (ii)

Eqn (ii) ÷ eqn (i)

$\tan\alpha = \frac{4}{3}$; $\alpha = 53.1^0$

Eqn. $(i)^2$ + eqn. $(ii)^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2 = 25$

$R^2[\cos^2\alpha + \sin^2\alpha] = 25$

$R^2 = 25$

$R = 5$

$\therefore 3\cos\theta - 4\sin\theta = 5\cos(\theta - 53.1^0)$

⇨ $3\cos\theta - 4\sin\theta + 20 = 5\cos(\theta - 53.1^0) + 20$

The minimum value occurs when

$\cos(\theta - 53.1^0) = -1$

⇨ Minimum value = 5(-1) +20 = 15

Now for $\cos(\theta - 53.1) = -1$

$\theta - 53.1^0 = 180^0$

$\theta = 126.8^0$

The minimum value is $(126.8^0\ , 15)$

And maximum value occurs when

$\cos(\theta - 53.1^0) = 1$

⇨ Minimum value = 5(1) 20 = 25

Now for $\cos(\theta - 53.1^0) = 1$

$\theta + 53.1^0 = 0^0, 360^0$

$\theta = -53.1^0, 306.8^0$

The maximum value is $(306.8^0, 25)$

Example 34

Find the maximum and minimum points of the following

(a) $f(\theta) = \frac{1}{3+sin\theta-2cos\theta}$

Solution

Let $\sin\theta - 2\cos\theta = R\sin(\theta - \alpha)$

$= R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

Comparing coefficient of cosθ and sinθ

$R\cos\alpha = 1$ .... (i)

$R\sin\alpha = 2$ .... (ii)

Eqn (ii) ÷ eqn (i)

$\tan\alpha = 2;\ \alpha = 63.4^0$

Eqn. $(i)^2$ + eqn. $(ii)^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 1^2 + 2^2 = 5$

$R^2[\cos^2\alpha + \sin^2\alpha] = 5$

$R^2 = 5$

$R = \sqrt{5}$

$\therefore \sin\theta - 2\cos\theta = \sqrt{5}\sin(\theta - 63.4^0)$

⇨ $3 + sin\theta - 2cos\theta = \sqrt{5}\sin(\theta - 63.4^0) + 3$

⇨ $f(\theta) = \frac{1}{3+\sqrt{5}\sin(\theta-63.4^0)}$

Note: for a fractional function, a maximum point is obtained when the denominator is minimum and the vice versa for the maximum point

The minimum value occurs when

$\sin(\theta - 63.4^0) = 1$

⇨ Minimum value $= \frac{1}{3+\sqrt{5}} = 0.31$

Now for $\sin(\theta - 63.4) = 1$

$\theta - 63.4^0 = 90^0$

$\theta = 153.4^0$

The minimum value is $(153.4^0.\ 0.31)$

And maximum value occurs when

$\sin(\theta - 63.4^0) = -1$

⇨ Maximum value $= \frac{1}{3+\sqrt{5(-1)}} = 1.31$

Now for $\sin(\theta - 63.4^0) = -1$

$\theta - 63.4^0 = 270^0$

$\theta = 333.4^0$

The maximum value is $(333.4^0, 1.31)$

(b) $f(\theta) = \frac{1}{4sin\theta-3cos\theta+6}$

Solution

Let $4\sin\theta - 3\cos\theta = R\sin(\theta - \alpha)$

$= R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

Comparing coefficient of cosθ and sinθ

$R\cos\alpha = 4$ .... (i)

$R\sin\alpha = 3$ .... (ii)

Eqn (ii) ÷ eqn (i)

$\tan\alpha = 0.75;\ \alpha = 36.9^0$

Eqn. $(i)^2$ + eqn. $(ii)^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2 = 25$

$R^2[\cos^2\alpha + \sin^2\alpha] = 25$

$R^2 = 25$

$R = 5$

$\therefore 4\sin\theta - 3\cos\theta = 5\sin(\theta - 36.9^0)$

⇨ $4\sin\theta - 3\cos\theta + 6 = 5\sin(\theta - 36.9^0) + 6$

⇨ $f(\theta) = \frac{1}{5\sin(\theta-36.9^0)+6}$

The minimum value occurs when

$\sin(\theta - 36.9^0) = 1$

⇨ Minimum value = $\frac{1}{5(1)+6} = \frac{1}{11}$

Now for sin(θ – 63.4) = 1

θ – 36.9^0= 90^0

θ = 126.9^0

The minimum value is (126.9^0. $\frac{1}{11}$)

And maximum value occurs when

sin(θ – 36.9^0) = -1

⇨ Maximum value = $\frac{1}{5(-1)+6} = 1$

Now for sin(θ – 36.9^0) = -1

θ – 36.9^0 = 270^0

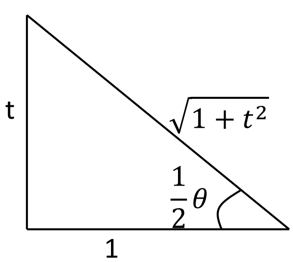
θ = 306.9^0

The maximum value is $(306.9^0, 1)$

The t-formula

Although this form has been tackled indirectly, it is formally stated here

Suppose that t = tan$\frac{\theta}{2}$, we have



From the triangle above

$cos\frac{1}{2}\theta = \frac{1}{\sqrt{1+t^2}}$ and $sin\frac{1}{2}\theta = \frac{t}{\sqrt{1+t^2}}$

But $cos\theta$ = $cos^2\frac{1}{2}\theta - sin^2\frac{1}{2}\theta$

= $\left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2$

$\therefore\ cos\theta = \frac{1-t^2}{1+t^2}$

And $sin\theta = 2sin\frac{1}{2}\theta cos\frac{1}{2}\theta$

$= 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right)$

$\therefore\ sin\theta = \frac{2t}{1+t^2}$

The t- formula is used widely in solving equations and proving trigonometric identities. These can be extended as follows

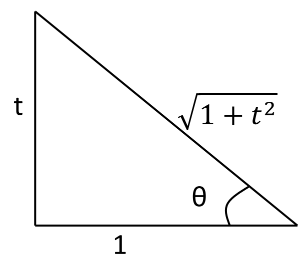
(i) For t= tanθ, sin2θ =$\frac{2t}{1+t^2}$ and $cos2\theta = \frac{1-t^2}{1+t^2}$

(ii) For t= tan$\left(\frac{5x}{4}\right)$, sin$\left(\frac{5x}{2}\right)$ =$\frac{2t}{1+t^2}$ and $cos\left(\frac{5x}{2}\right) = \frac{1-t^2}{1+t^2}$

Example 35

Show that if t = tanθ, then sin2θ =$\frac{2t}{1+t^2}$ and $2\theta = \frac{1-t^2}{1+t^2}$. Hence solve the equation √3cos2θ + sin 2θ = 1 for 0^0 ≤ θ ≤ 360^0.

Solution



From the triangle above

$cos\theta = \frac{1}{\sqrt{1+t^2}}$ and $sin\theta = \frac{t}{\sqrt{1+t^2}}$

But $cos2\theta$ = $cos^2\theta - sin^2\theta$

= $\left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2$

$\therefore\ cos2\theta = \frac{1-t^2}{1+t^2}$

And $sin2\theta = 2sin\theta cos\theta$

$= 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right)$

$\therefore\ sin2\theta = \frac{2t}{1+t^2}$

Hence √3cos2θ + sin 2θ = 1

⇨ $\sqrt{3\left(\frac{1-t^2}{1+t^2}\right)} + \left(\frac{2t}{1+t^2}\right) = 1$

$\sqrt{3} - \sqrt{3t^2} + 2t = 1 + t^2$

(1+ $\sqrt{3})t^1 - 2t + 1 - \sqrt{3} = 0$

$t = \frac{2\pm\sqrt{2^2-4(1+\sqrt{3})(1-\sqrt{3})}}{2(1+\sqrt{3})} = \frac{2\pm\sqrt{12}}{2(1+\sqrt{3})} = \frac{1\pm\sqrt{3}}{1+\sqrt{3}}$

t $=\frac{1+\sqrt{3}}{1+\sqrt{3}}$ = 1 or

t$=\left(\frac{1-\sqrt{3}}{1+\sqrt{3}}\right)\left(\frac{1-\sqrt{3}}{1-\sqrt{3}}\right)$ = -2 + √3

If tanθ = 1; θ = 450, 2250

If tanθ = -2 + √3; θ = 165^0, 345^0

∴ θ: θ = 45^0, 165^0, 225^0, 345^0

Example 36

Find all the solutions of the equation
5cosθ – 4sinθ = 6 for -180^0 ≤ θ ≤ 180^0

Solution

Let t = tan$\frac{\theta}{2}$ then

$cos\theta = \frac{1-t^2}{1+t^2}$

$sin\theta = \frac{2t}{1+t^2}$

⇨ $5\left(\frac{1-t^2}{1+t^2}\right) - 4\left(\frac{2t}{1+t^2}\right) = 6$

$5(1-t^2) - 8t = 6(1+t^2)$

5 – 5t^2 – 8t = 6 + 6t^2

11t^2 + 8t + 1 = 0

$t = \frac{-8\pm\sqrt{8^2-4\ x\ 11\ x\ 1}}{2\ x\ 11} = \frac{-8\pm4.4721}{22}$

t = $\frac{-8+4.4721}{22} = -0.1604$ or

t = $\frac{-8-4.4721}{22}$ = -0.5669

Taking t = -0.1604

$tan\frac{\theta}{2} = -0.1604;\ \theta = -18.2^0$

Taking t = -0.5669

$tan\frac{\theta}{2} = -0.5669;\ \theta = -59.1^0$

∴θ = -59.1^0, -18.2^0

Example 37

Solve the equation

3$\tan^2$θ + 2$\sec^2$ θ = 2(5 – 3tanθ) for
0^0 < θ <180^0

Let t = tan θ
3t^2- 2(1 + t^2) = 2(5 -3t)
5t^2 + 6t – 8 = 0
t = $\frac{-6\pm\sqrt{6^2-4(5)(-8)}}{2(5)} = \frac{-6\pm14}{10}$ = -2 or $\frac{4}{5}$
Taking t = -2; θ = $\tan^{-1}$(-2) = 116.57^0
Taking t = $\frac{4}{5}$; θ = $\tan^{-1}(\frac{4}{5})$ = 38.66^0
Hence θ = 38.66^0, 116.57^0

Example 38

Show that tan4θ = $\frac{4t(1-t^2)}{t^4-6t^2+1}$, where t = tanθ.

Solution

$$\tan 4\theta = \frac{2\tan 2\theta}{1-\tan^2 2\theta} \text{ and } \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$
$$= \frac{2\left(\frac{2t}{1-t^2}\right)}{1-\left(\frac{2t}{1-t^2}\right)^2}$$
$$= \frac{4t(1-t^2)}{t^4-6t^2+1}$$

Example 39

Solve the equation cosθ + sinθ + 1 = 0 for
0^0 ≤ θ ≤ 360^0

Solution

cosθ + sinθ + 1

Let t = tan$\frac{\theta}{2}$

$\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} = 1$

1 – t^2 + 2t = 1(1 + t^2)

2t + 2 = 0; t = -1

∴ tan$\frac{\theta}{2}$ = -1

$\frac{\theta}{2} = 135^0, 315^0$

θ = 270^0, 630^0

Hence θ = 270^0

Revision exercise 5

1. Solve equation 3cosθ + 4sinθ = 2 for
   0^0 ≤ ≤ 360^0 [119.6^0, 346.7^0]
2. (a) Show that cos4x = $\frac{tan^4x-6tan^2x+1}{tan^4x+2tan^2x+1}$
   (b) Show that if q = cos2x + sin2x, then
   (1+q)$\tan^2$x – 2tanx + q – 1 = 0.

Deduce that if the roots of the above equation are $\tan x_1$ and $\tan x_2$, the $\tan(x_1 + x_2) = 1$

3. Find the values of R and tanα in each of the following equations
   (a) 2cosθ + 5sinθ = Rsin(θ + α) $\left[\sqrt{29}, \frac{2}{5}\right]$
   (b) 2cosθ + 5sinθ = Rcos(θ - α) $\left[\sqrt{29}, \frac{5}{2}\right]$
   (c) √3cosθ + sinθ = Rcos(θ - α) $\left[2, \frac{1}{\sqrt{3}}\right]$
   (d) 5sinθ – 12cosθ = Rsin(θ - α) $\left[13, \frac{12}{5}\right]$
   (e) Cosθ – sinθ = Rcos(θ + α) $\left[\sqrt{2}, 1\right]$

4. Find the greatest and least values and state the smallest non-negative value of x for which each occurs
   (i) 12sinx + 5cosx [13, 67.4^0; -13, 247.4^0]
   (ii) 2cosx + sinx
   $[\sqrt{5}, 26.6^0;\ -\sqrt{5}, 206.6^0]$
   (iii) 7 + 3sinx – 4cosx
   [12, 143.1^0; 2, 323.1^0]
   (iv) 10 – 2sinx + cosx
   [10 + $\sqrt{5}, 296.6^0$; 10 - $\sqrt{5}$, 116.6^0]
   (v) $\frac{1}{2+sinx+cosx}$ $\left[\frac{2+\sqrt{2}}{2}, 225^0;\ \frac{2-\sqrt{2}}{2}, 45^0\right]$
   (vi) $\frac{1}{7-2cosx+\sqrt{5}sinx}$ $\left[\frac{1}{4}, 311.8^0;\ \frac{1}{10}, 131.8^0\right]$
   (vii) $\frac{3}{5cosx-12sinx+16}$ [1, 112.6^0; $\frac{3}{29}$, 292.6^0]

5. Solve each of the following equations for $0^0 \le x \le 360^0$
   (a) sinx + √3cosx = 1 [90^0, 330^0]
   (b) 4sinx – 3cosx = 2 [60.4^0, 193.3^0]
   (c) sinx + cosx = $\frac{1}{\sqrt{2}}$ [105^0, 345^0]
   (d) 5sinx + 12cosx = 7 [80.0^0, 325.2^0]
   (e) 7sinx – 4cosx = 3 [51.6^0, 187.9^0]
   (f) cosx – 3sinx = 2 [237.7^0, 339.2^0]
   (g) 5cosx + 2sinx = 4[63.8^0, 339.8^0]
   (h) 9cox2x – 4sin2x = 6 [14.2^0, 141.8^0, 194.2^0, 321.8^0]
   (i) 7cosx + 6sinx = 2 [118.1^0, 323.1^0]
   (j) 9cosx – 8sinx = 12 [313.6^0, 323.1^0]

The factor formulae

The following identities were developed from compound angles

cos (A + B) = cosAcosB – sinAsinB ……….(i)

cos(A – B) = coaAcosB + sinAsinB ……….. (ii)

sin(A + B) = sinAcosB + sinBcosA ………….(iii)

sin(A – B) = sinAcosB – sinBcosA ………….(iv)

eqn. (i) + eqn (ii)

cos(A + B) + cos(A – B) = 2cosAcosB

eqn. (i) – eqn (ii)

cos(A + B) - cos(A – B) = -2cosAcosB

eqn. (iii) + eqn (iv)

sin(A + B) + sin(A – B) = 2sinAcosB

eqn. (iii) - eqn (iv)

sin(A + B) - sin(A – B) = -2sinBcosA

For simplification, A + B =α and A – B = β

Add: 2A = α + β i.e. A = $\left(\frac{\alpha+\beta}{2}\right)$

Subtract 2B = α - β i.e. A = $\left(\frac{\alpha-\beta}{2}\right)$

Substituting for A and B in the above equation

cosα + cosβ = 2cos$\left(\frac{\alpha+\beta}{2}\right)cos\left(\frac{\alpha-\beta}{2}\right)$

cosα - cosβ = -2cos$\left(\frac{\alpha+\beta}{2}\right)cos\left(\frac{\alpha-\beta}{2}\right)$

sinα + sinβ = 2sin$\left(\frac{\alpha+\beta}{2}\right)cos\left(\frac{\alpha-\beta}{2}\right)$

sinα - sinβ = 2cos$\left(\frac{\alpha+\beta}{2}\right)sin\left(\frac{\alpha-\beta}{2}\right)$

Example 40

Show that if X, Y and Z are angles of a triangle, then

(a) cosX + cosY + cosZ – 1 = 4sin$\frac{X}{2}$sin$\frac{Y}{2}$sin$\frac{Z}{2}$

solution

LHS cosX + cosY + cosZ – 1

= 2cos $\frac{X+Y}{2}$ cos $\frac{X-Y}{2}$ + 1 – 2sin$^2\frac{Z}{2}$ – 1

(to eliminate -1)

= 2cos $\frac{180^0 - Z}{2}$ cos $\frac{X-Y}{2}$ – 2sin$^2\frac{Z}{2}$

(since X + Y = 180^0 – Z)

= 2sin $\frac{Z}{2}$ cos $\frac{X-Y}{2}$ −2 $\sin^2\frac{Z}{2}$

(Since cos$(90^0$ – A) = sinA)

= 2sin $\frac{Z}{2}\left[\cos\frac{X-Y}{2} - 2sin^2\left\{\frac{180^0-(X+Y)}{2}\right\}\right]$

= 2sin $\frac{Z}{2}\left[\cos\frac{X-Y}{2} - cos\left\{\frac{(X+Y)}{2}\right\}\right]$

(Since sin $(90^0$ – A) = cos A)

= 2sin$\frac{Z}{2}\left[-2sin\frac{X}{2}sin\frac{-y}{2}\right]$

=2sin$\frac{Z}{2}\left[2sin\frac{X}{2}sin\frac{Y}{2}\right]$

(Since sin(-A) = -sinA)

$4sin\frac{X}{2}sin\frac{Y}{2}sin\frac{Z}{2}$ as required

(b) Sin3X + sin3Y + sin3Z =

- 4cos$\frac{3X}{2}cos\frac{3Y}{2}cos\frac{3Z}{2}$

Solution

LHS: Sin3X + sin3Y + sin3Z

= 2sin$\frac{3(X+Y)}{2}$cos$\frac{3(X-Y)}{2}$ + 2sin$\frac{3Z}{2}$cos$\frac{3Z}{2}$

=2sin$\frac{3(180^0-Z)}{2}$ cos$\frac{3(X-Y)}{2}$ + 2sin$\frac{3Z}{2}$cos$\frac{3Z}{2}$

= -2cos$\frac{3Z}{2}$ cos$\frac{3(X-Y)}{2}$ + 2sin$\frac{3Z}{2}$cos$\frac{3Z}{2}$

Since sin$(270^0$ – A) = -cosA}

=-2cos$\frac{3Z}{2}\left[\cos\frac{3(X-Y)}{2} - sin\frac{3Z}{2}\right]$

=-2cos$\frac{3Z}{2}\left[\cos\frac{3(X-Y)}{2} - sin\frac{3\{180^0-(X+Y)\}}{2}\right]$

=-2cos$\frac{3Z}{2}\left[\cos\frac{3(X-Y)}{2} - cos\frac{3(X+Y)}{2}\right]$

=-2cos$\frac{3Z}{2}\left[2cos\frac{3X}{2} + cos\frac{-3Y}{2}\right]$

Since cos(-A) = cosA

=- 4cos$\frac{3X}{2}cos\frac{3Y}{2}cos\frac{3Z}{2}$

(c) cos4X + cos4Y + cos4Z + 1

= 4cos2Xcos2Ycos2Z

Solution

LHS: cos4X + cos4Y + cos4Z + 1

= 2cos2(X + Y)cos2(X – Y) + $2\cos^2 2Z$ -1 + 1

= 2cos2$(180^0$ – Z)cos2(X – Y) + $2\cos^2 2Z$

= 2cos2Z[cos2(X – Y) + cos2$\{180^0$ – (X + Y)}]

= 2cos2Z[cos2(X – Y) + cos2 (X + Y)]

= 2cos2Z[2cos2X cos-2Y]

Since cos(-A) = cosA

=4cos2Z2cos2X cos2Y

(d) $\sin^2$Y + $\sin^2$Z = 1 + cos(Y – Z)cosX

LHS: $\sin^2$Y + $\sin^2$Z

=$\frac{1}{2}(1-cos2Y)+\frac{1}{2}(1-cos2Z)$

=$\frac{1}{2}(2-cos2Y-cos2Z)$

=$1-\frac{1}{2}(cos2Y+cos2Z)$

= 1 – cos$(180^0$ –X)cos(Y – Z)

=1 + cos(Y – Z)cosX

Example 41

(a) Factorize cosθcos3θ – cos7θ + cos9θ and express it in the form Acospθsinqθsinrθ wher A, p, q and r are constants

Solution

f(θ)= cos9θ + cosθ – (cos7θ +cos3θ)

= 2cos5θcos4θ – 2cos5θcos2θ

=2cos5θ(cos4θ – cos2θ)

= -4cos5θ(-sin3θsinθ)

=-4cos5θsin3θsinθ

⇨ A = -4, p = 5, q = 3, r = 1

(b) Given that
p = sinα + sinβ
q = cosα + cosβ. Show that

$\frac{2pq}{p^2+q^2}$ = sin(α + β)

Solution

$\frac{2pq}{p^2+q^2}$

=$\frac{2(sin\alpha+sin\beta)(cos\alpha+cos\beta)}{sin^2\alpha+2sin\alpha sin\beta+sin^2\beta+cos^2\alpha+2cos\alpha cos\beta+cos^2\beta}$

$= \frac{2\left[2sin\frac{\alpha+\beta}{2}cos\frac{\alpha-\beta}{2}\right]\left[2cos\frac{\alpha+\beta}{2}cos\frac{\alpha-\beta}{2}\right]}{2+2(cos\alpha cos\beta+sin\alpha sin\beta}$

$= \frac{2\left[2sin\frac{\alpha+\beta}{2}cos\frac{\alpha+\beta}{2}\right]\left[2cos^2\frac{\alpha-\beta}{2}\right]}{2+2cos(\alpha-\beta)}$

$= \frac{2[\sin(\alpha+\beta)][1+\cos(\alpha-\beta)}{2[1+\cos(\alpha-\beta)]}$

= sin(α +β)

Example 42

Solve $5\cos^2 3\theta = 3(1 + \sin 3\theta)$ for $0^0 \leq \theta \leq 90^0$.

Solution

$5\cos^2 3\theta = 3(1 + \sin 3\theta)$

$5(1 - \sin^2 3\theta) = 3(1 + \sin 3\theta)$

$5 - 5\sin^2 3\theta = 3 + 3\sin 3\theta$

$5\sin^2 3\theta + 3\sin 3\theta - 0 = 0$

$(\sin 3\theta + 1)(5\sin 3\theta - 2) = 0$

$\sin 3\theta + 1 = 0$

$3\theta = \sin^{-1}(-1) = -90^0, 270^0$

Example 43

(a) solve the equation $\cos 2x = 4\cos^2 x - 2\sin^2 x$ for $0 \leq \theta \leq 180^0$

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$\cos^2 x - \sin^2 x = 4\cos^2 x - 2\sin^2 x$

$3\cos^2 x - \sin^2 x = 0$

$4\cos^2 x - 1 = 0$

$(2\cos x + 1)(2\cos x - 1) = 0$

Either

$2\cos x + 1 = 0$

$\cos x = -\frac{1}{2}$

$x = \cos^{-1}(-\frac{1}{2}) = 120^0$

Or

$2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$x = \cos^{-1}(\frac{1}{2}) = 60^0$

$\therefore x(60^0, 120^0)$

Alternatively

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$= \frac{4}{2}(1 + \cos 2x) - \frac{2}{2}(1 - cos2x)$

$= 2 + 2\cos 2x - 1 + \cos 2x$

$2\cos 2x + 1 = 0$

$\cos 2x = -\frac{1}{2}$

$2x = \cos_{-1}(-\frac{1}{2}) = 120^0, 240^0$

$x = 60^0, 120^0$

Alternatively

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$\cos^2 x - \sin^2 x = 4\cos^2 x - 2\sin^2 x$

$3\cos^2 x - \sin^2 x = 0$

$\sin^2 x = 3\cos^2 x$

$\tan^2 x = 3$

$\tan x = \pm\sqrt{3}$

Either

$\tan x = \sqrt{3}$

$x = \tan^{-1}\sqrt{3} = 60^0$

Or

$\tan x = -\sqrt{3}$

$x = \tan^{-1} -\sqrt{3} = 120^0$

Hence $x = 60^0, 120^0$

Alternatively

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$1 - 2\sin^2 x = 4(1 - \sin^2 x) - 2\sin^2 x$

$1 = 4 - 4\sin^2 x$

$4\sin^2 x = 3$

$\sin^2 x = \frac{3}{4}$

$\sin x = \pm\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

$x = 60^0, 120^0$

Alternatively

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$1 - 2\sin^2 x = 4\cos^2 x - 2\sin^2 x$

$4\cos^2 x = 1$

$\cos x = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$

$x = 60^0, 120^0$

(b) Show that if sin(x + α) = psin(x – α) then $\tan x = \left(\frac{p+1}{p-1}\right)\tan\alpha$.

Hence solve the equation sin(x + α) = psin(x – α) for p = 2 and α= 20^0.

sinxcosα + coxsinα = p(sinxcosα - coxsinα)

cosxsinα (p + 1) = sinxcosα (p - 1)

$\text{cosxsin}\alpha\left(\frac{p+1}{P-1}\right) = \text{sinxcos}\alpha$

$\frac{cosxsin\alpha}{\text{sinxcos}\alpha}\left(\frac{p+1}{p-1}\right) = \frac{sinxcos\alpha}{\text{sinxcos}\alpha}$

$tanx = \left(\frac{p+1}{P-1}\right)\tan\alpha$

For sin(x + 20^0) = 2sin(x – 20^0)

tanx = $\frac{2+1}{2-1}tan20^0$ = 3tan20^0

x = $\tan^{-1}(3tan20^0)$ = 47.52^0

Example 44

Prove that in any triangle ABC,

$\frac{\sin(A-B)}{\sin(A+B)} = \frac{a^2-b^2}{c^2}$

Solution

$\frac{a^2-b^2}{c^2} = \frac{(2Rsin\,A)^2-(2Rsin\,B)^2}{(2Rsin\,C)^2}$

$= \frac{4R^2(sin^2A-sin^2B}{4R^2sin^2C}$

$= \frac{(\sin A+\sin B)(\sin A-\sin B)}{sin^2[180^0-(A+B]}$

$= \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right).2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{sin^2(A+B)}$

$= \frac{\sin(A+B)\sin(A-B)}{sin^2(A+B)}$

$= \frac{\sin(A-B)}{sin(A+B)}$

Inverse trigonometric functions

Note that

(a) If θ = $\cos^{-1}\left(\frac{1}{2}\right)$ then cosθ = $\frac{1}{2}$

(b) $\tan^{-1}(tan\alpha)$= $\tan(\tan^{-1}\alpha)$ =α

(c) $\cos^{-1}[\cos(x+y)]$

= cos[$\cos^{-1}(x+y)$ = x+y

(d) $sin(\sin^{-1}\theta) = \sin^{-1}(\sin^{-1}\theta)$

To avoid errors test the values

Example 45

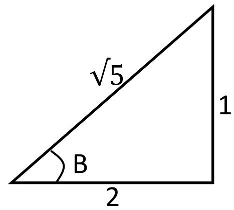
Show that

(a) $\tan^{-1}\frac{1}{3} + \sin^{-1}\frac{1}{\sqrt{5}} = \frac{\pi}{4}$

Solution

A = $\tan^{-1}\frac{1}{3}$ and B = $\sin^{-1}\frac{1}{\sqrt{5}}$

⇨ tanA = $\frac{1}{3}$ and B = $\frac{1}{\sqrt{5}}$



⇨ tan B = $\frac{1}{2}$

LHS = $\tan^{-1}\frac{1}{3} + \sin^{-1}\frac{1}{\sqrt{5}}$ = A + B

$=\tan^{-1}[\tan(A+B)]$

$=\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{2}}{1-\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)}\right)$

$=\tan^{-1}\frac{3+3}{6-1}$

$=\tan^{-1}\frac{5}{5} = \tan^{-1}1 = \frac{\pi}{4}$

(b) $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

Solution

Let A = $\tan^{-1}\frac{1}{3}$ and B = $\tan^{-1}\frac{1}{7}$

⇨ tanA = $\frac{1}{3}$ and tanB = $\frac{1}{7}$

LHS: $\tan^{-1}\tan(2A+B)$ but tan2A = $\frac{2\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2} = \frac{3}{4}$

$\therefore \tan^{-1}\tan(2A+B) = \frac{\frac{3}{4}+\frac{1}{7}}{1-\left(\frac{3}{4}\right)\left(\frac{1}{7}\right)}$

$= \tan^{-1}\left(\frac{21+4}{28-3}\right)$

$=\tan^{-1}1$

$=\frac{\pi}{4}$

(c) $\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

Solution

Let θ= $\cos^{-1}x$; => x = cosθ = $\sin\left(\frac{\pi}{2}-\theta\right)$

sinx = $\frac{\pi}{2} - \theta$

$\therefore \cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

Example 46

Solve the equations

(a) $\tan^{-1}(2\theta+1) + \tan^{-1}(2\theta-1) = \tan^{-1}(2)$

Solution

Let A = $\tan^{-1}(2\theta+1)$ and B =$\tan^{-1}(2\theta-1)$
⇨ tanA = $2\theta+1$ and tanB =$2\theta-1$

∴ A + B = $\tan^{-1}2$ or tan(A +B) =2

$\frac{2\theta+1+2\theta-1}{1-(2\theta+1)(2\theta-1)} = 2$

$4\theta = 2(1 - 4\theta^2 - 1)$

$2\theta^2 + \theta - 1 = 0$

$(2\theta - 1)(\theta + 1) = 0$

$\theta = \frac{1}{2}$ or $\theta = -1$

(b) $\tan^{-1}(1+\theta) + \tan^{-1}(1-\theta) = 32$
Let A = $\tan^{-1}(1+\theta)$ and B =$\tan^{-1}(1-\theta)$
⇨ tanA = $1+\theta$ and tanB =$1-\theta$

∴ A + B = 32 or tan(A +B) = tan32

Introducing tangents

$\frac{1+\theta+1-\theta}{1-(1+\theta)(1-\theta)} = tan32$

$\theta^2\tan 32 = 2$

$\theta = \sqrt{2cot32} = \pm 1.789$

Example 47

If x = $\tan^{-1}\alpha$ and y = $\tan^{-1}\beta$;

Show that x + y = $\tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$

Solution

$\tan x = \alpha$; $\tan y = \beta$

$(x + y) = \tan[\tan^{-1}(x + y)]$

$= \tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$

Example 48

Solve the equation

$\tan^{-1}\left(\frac{1}{x-1}\right) + \tan^{-1}(x + 1) = \tan(-2)$

Solution

Let A = $\tan^{-1}\left(\frac{1}{x-1}\right)$ and B = $\tan^{-1}(x + 1)$

⇨ A + B = $\tan^{-1}(-2)$

$\frac{\frac{1}{x-1}+(x+y)}{1-\left(\frac{1}{x-1}\right)(x+y)} = -2$

$\frac{1+x^2-1}{x-1-x-1} = -2$

$\therefore x^2 = 4; x = \pm 2$

Example 50

Without using tables or calculators determine the values of $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}$.

Solution

$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}$.

$=\frac{\frac{1}{2}+\frac{1}{5}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{5}\right)} + \tan^{-1}\frac{1}{8}$

$=\tan^{-1}\frac{7}{9} + \tan^{-1}\frac{1}{8}$

$=\frac{\frac{7}{9}+\frac{1}{8}}{1-\left(\frac{7}{9}\right)\left(\frac{1}{8}\right)} = \tan^{-1}\left(\frac{65}{65}\right) = \frac{\pi}{4}$

Example 51

Solve equations

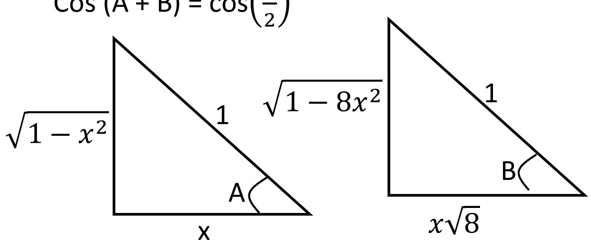
(a) $\cos^{-1}x + \cos^{-1}x\sqrt{8} = \frac{\pi}{2}$

Solution

Let A = $\cos^{-1}x$ and B =$\cos^{-1}x\sqrt{8}$

A + B = $\frac{\pi}{2}$

Cos (A + B) = $\cos\left(\frac{\pi}{2}\right)$



$$x(x\sqrt{8}) - \left(\sqrt{1-x^2}\right)\left(\sqrt{1-8x^2}\right) = 0$$

$$x(x\sqrt{8}) = \left(\sqrt{1-x^2}\right)\left(\sqrt{1-8x^2}\right)$$

$8x^4 = (1-x^2)(1-8x^2)$

$1 - 9x^2 = 0$

$(1 - 3x)(1 + 3x) = 0$

Either x = $\frac{1}{3}$ or x =$-\frac{1}{3}$

We discard the negative value, so the root is x = $\frac{1}{3}$

(b) $2\sin^{-1}\left(\frac{x}{2}\right) + \sin^{-1}(x\sqrt{2}) = \frac{\pi}{2}$

Solution

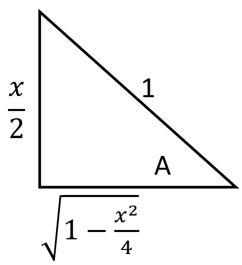
Let A = $\sin^{-1}\left(\frac{x}{2}\right)$ and B = $\sin^{-1}(x\sqrt{2})$

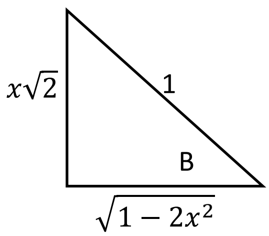
2A + B = $\frac{\pi}{2}$

2A = $\frac{\pi}{2} - B$

Sin(2A) = sin $\left(\frac{\pi}{2} - B\right)$

2SinAcosA = cosB





$2\left(\frac{x}{2}\right).\sqrt{1-\frac{x^2}{4}} = \sqrt{1-2x^2}$

$x.\sqrt{\frac{4-x^2}{4}} = \sqrt{1-2x^2}$

$\frac{x}{2}.\sqrt{4-x^2} = \sqrt{1-2x^2}$

$\frac{x^2}{4}.(4-x^2) = (1-2x^2)$

$X^4 - 12x^2 + 4 = 0$

$x^2 = \frac{12\pm\sqrt{144-4(4\,x\,1)}}{2\,x1}$

$x^2 = \frac{12\pm\sqrt{128}}{2} = 6 \pm 4\sqrt{2}$

$x = \sqrt{6 \pm 4\sqrt{2}}$

After testing for $x = \sqrt{6+4\sqrt{2}}$ and for $x = \sqrt{6-4\sqrt{2}}$, the value that satisfies the equation is $x = \sqrt{6-4\sqrt{2}}$ = 0.5858

Hence the value of x = 0.5858

Revision exercise 6

1. If p = sinα + sin β and q = cosα + cosβ show that $\frac{p}{q} = tan\frac{\alpha+\beta}{2}$
2. (a) Prove that:
   (i) (sin2x – sinx)(1+ 2cosx) = sin3x
   (ii) cos4θ =$\frac{tan^4\theta-6tan^2\theta+1}{tan^4\theta+2tan^2\theta+1}$
   (iii) $\frac{sinx+2sin2x+sin3x}{sinx+2sinx+sin3x} = tan^2\frac{x}{2}$
3. Solve the equation for $0^0 \le x \le 180^0$:
   (a) sinx + sn3x + sin5x + sin7x = 0
   [x: x = 0^0, 45^0, 90^0, 135^0, 180^0]
   (b) sin7x + sinx + sin5x + sin3x = 0
   [x: x = 60^0, 180^0]
   (c) sinx + sin4x = 0
   [x: x= 0^0, 60^0, 72^0, 144^0, 180^0]
   (d) cos (x + 10^0) – -cos(x + 30^0) = 0

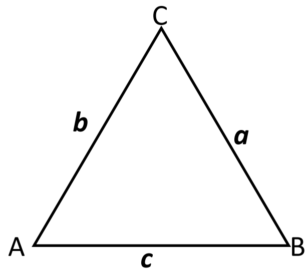
   [70^0]

   (e) cos5x – sin2x = cosx
   [x: x = 0^0, 70^0, 90^0, 110^0,180^0]
   (f) sin2x +sin10x + cos4x = 0
   [x: x = 22.5^0, 35^0, 55^0, 67.5^0, 95^0, 112.5^0, 115^0, 157.5^0, 175^0]

4. Show that
   (a) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$
   (b) $2\sin^{-1}\left(\frac{1}{2}\right)$ =$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$
   (c) the positive value that satisfies the equation $\tan^{-1}3x + \tan^{-1}2x = \frac{\pi}{4}$ is $\frac{1}{6}$
   (d) $\tan^{-1}(-x) = -\tan^{-1}x$
   (e) $\cos^{-1}\left(\frac{63}{65}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) = \sin^{-1}\left(\frac{3}{5}\right)$
5. Prove that
   (a) $\frac{sinA-sinB}{sinA+sinB} = tan\left(\frac{A-B}{2}\right)cot\left(\frac{A+B}{2}\right)$
   (b) Sin3x + sinx = 4sinx$\cos^2$x
   (c) $\frac{sinx+sin2x+sin3x}{cosx+cos2x+cos3x} = tan2xx$
   (d) sin(A + B) – sin(A – B) =2cosAsinB
   (e) $\frac{sin5x+sinx}{sin4x+sin2x} = 2cosx - secx$
   (f) Cos3x + cosx = 4$\cos^2$x – 2cosx

Solution to triangles

In a triangle ABC



(a) Six elements are considered: three angles and three sides

Capital letters denote angles and ***small bold and italics letters*** sides

(b) The opposite side of angle A is a, of angle B is b and of angle C is c.

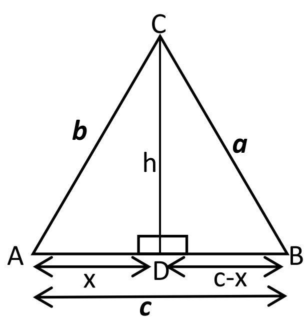
(c) The angle sum of a triangle is two right angles i.e. A + B + C = 180^0

(d) The sides are independent except that the sum of the two sides of the triangle should be equal to or greater than the third side

How to deal with triangles

1. The cosine rule

   (a) Given an acute angle A



From triangle

ACD: $x^2 + h^2 = \boldsymbol{b}^2$ ....(i)

BCD: $(\boldsymbol{c} - x)^2 + h^2 = \boldsymbol{a}^2$

$\boldsymbol{c}^2 - 2\boldsymbol{c}x + x^2 + h^2 = \boldsymbol{a}^2$ .... (ii)

Substituting eqn. (i) into eqn. (ii)

$\boldsymbol{c}^2 - 2\boldsymbol{c}x + \boldsymbol{b}^2 = \boldsymbol{a}^2$

But

$x = \boldsymbol{b}\cos A$

⇨ $\boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}\cos A = a^2$

$\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}\cos A$ (1)

Similarly;

$\boldsymbol{b}^2 = \boldsymbol{a}^2 + \boldsymbol{c}^2 - 2\boldsymbol{ac}\cos B$ (2)

$\boldsymbol{c}^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 - 2\boldsymbol{ab}\cos C$ (3)

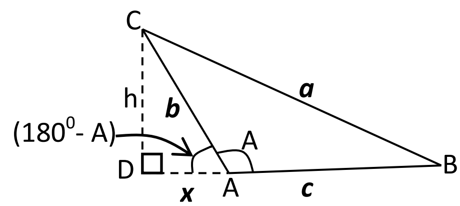
It follows that

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\cos B = \frac{a^2+c^2-b^2}{2ac}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

(b) Given an obtuse angle A



In triangle ABC, A is an obtuse angle and CD is the altitude.

From triangle

ACD: $x^2 + h^2 = \boldsymbol{b}^2$ ....(i)

BCD: $(\boldsymbol{c} - x)^2 + h^2 = \boldsymbol{a}^2$

$\boldsymbol{c}^2 - 2\boldsymbol{c}x + x^2 + h^2 = \boldsymbol{a}^2$ .... (ii)

Substituting eqn. (i) into eqn. (ii)

$\boldsymbol{c}^2 - 2\boldsymbol{c}x + \boldsymbol{b}^2 = \boldsymbol{a}^2$

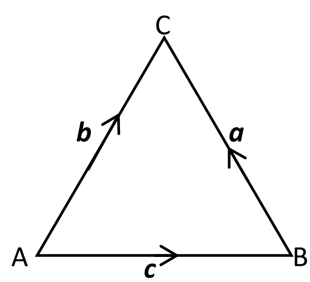
But

$x = \boldsymbol{b}\cos(180^0 - A) = -\boldsymbol{bc}\cos A$

From triangle ACD

$\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}\cos A$ as before

The cosine rule can be derived using the vector approach.



Given a triangle above with BC = ***a***, AC = ***c*** and AB = ***b***

BC = BA + AC = AC – AB

a = ***b*** – ***c***

⇨ $\boldsymbol{a}.\boldsymbol{a} = (\boldsymbol{b} - \boldsymbol{c})(\boldsymbol{b} - \boldsymbol{c})$

$= \boldsymbol{b}.\boldsymbol{b} - 2\boldsymbol{b}.\boldsymbol{c} + \boldsymbol{c}.\boldsymbol{c}$

$= \boldsymbol{b}.\boldsymbol{b} + \boldsymbol{c}.\boldsymbol{c} - 2\boldsymbol{b}.\boldsymbol{c}$

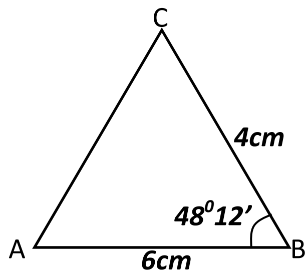
$\therefore \boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}\cos A$

since $\boldsymbol{b}.\boldsymbol{c} = |bc|\cos A$

Example 52

Solve the triangle in which AB = 6cm, BC = 4cm and angle ACB = $48^0 12'$

Solution



Using: $\boldsymbol{b}^2 = \boldsymbol{a}^2 + \boldsymbol{c}^2 - 2\boldsymbol{ac}\cos B$

$= 6^2 + 4^2 - 2(6)(4)\cos 48.2^0$

1^0 (degree) = 60' (minutes)

b = 4.47cm

Using: $\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{20.0+36-16}{2(4.47)(6)}$

A $=41.8^0$

But A + B + C = 180^0

$41.8^0 + 48.2^0 + C = 180^0$

C = 90^0

∴ AC = 4.47cm, angles BAC = 41.8^0 and ACB = 90^0

Example 53

In a triangle ABC, prove that

(a) $a^2 = (b - c)^2 + 4bc\sin^2\left(\frac{A}{2}\right)$ hence that

a = (b – c)secα where $\tan\alpha = \frac{\sqrt{bc}\sin\left(\frac{A}{2}\right)}{b-c}$

From $\cos A = 1 - 2\sin^2\left(\frac{A}{2}\right)$

Substituting for cosA into the cosine formula $\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}\cos A$

$\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}[1 - 2\sin^2\left(\frac{A}{2}\right)]$

$\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc} + 4\sin^2\left(\frac{A}{2}\right)$

$a^2 = (b - c)^2 + 4bc\sin^2\left(\frac{A}{2}\right)$

Hence, substituting for $\sin^2\left(\frac{A}{2}\right)$ into tanα expression we get

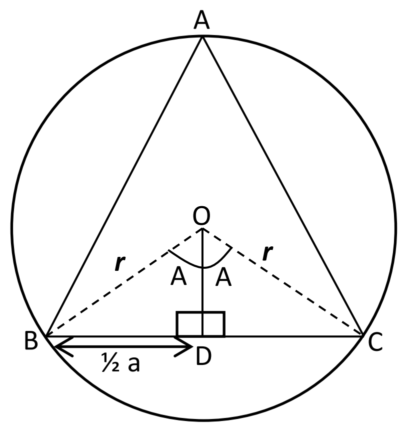
$a^2 = (b - c)^2 + (b - c)^2\tan^2\alpha$

$a^2 = (b - c)^2(1 + \tan^2\alpha)$

$a^2 = (b - c)^2\sec^2\alpha$

a = (b – c)secα

2. The Sine Rule



The figure shows a circle with centre O and radius r circumscribing triangle ABC

Angle BOC = 2A [angle subtended by the same arc at the centre of the circle is twice the angle formed at any point on the circumference]

Triangle BOC is isosceles

OD bisects angle BOC and side BC

∴BD = ½a

From triangle BOD

$\sin A = \frac{a}{2r}$ i.e. $\frac{a}{\sin A} = 2r$

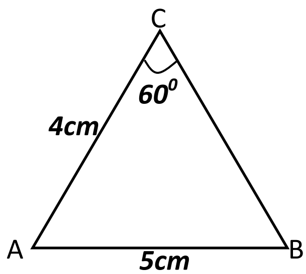
if instead we consider triangles AOC and AOB, we obtain $\frac{b}{\sin B} = 2r$ and $\frac{c}{\sin C} = 2r$

In general: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Example 54

Solve the triangle in which AB = 5cm, AC = 4cm and angle ACB = 600

Solution



Using sine rule

$\frac{b}{sinB} = \frac{c}{sinC}$ => B = $\sin^{-1}\left(\frac{bsinC}{c}\right)$

B = $\sin^{-1}\left(\frac{4}{5}sin60^0\right)$ = 43.9^0

From A + B + C+ = 180^0

A = $(180 – 60 – 43.9)^0$ = 76.1^0

Similarly a = $\frac{bsinA}{sinB} = \frac{4sin76.1^0}{sin43.9^0}$ = 5.6cm

$\therefore\ \overline{AB} = 5.6cm, B\hat{A}C = 76.1^0, A\hat{B}C = 43.9^0$

Example 55

Prove that in any triangle

$\frac{a^2-b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$

Solution

From sine rule formula;
a = 2rsinA, b = 2rsinB, c = 2rsinC

By substitution

$\frac{a^2-b^2}{c^2} = \frac{(2rsinA)^2-(2rsinB)^2}{(2rsinC)^2} = \frac{sin^2A-sin^2B}{sin^2C}$

But A + B + C = 180^0

C = 180^0 – (A + B)

sinC = sin[180^0 – (A + B)] = sin (A + B)

By substitution

$\frac{a^2-b^2}{c^2} = \frac{sin^2A-sin^2B}{\sin(A+B)} == \frac{(sinA+\sin B)(sinA-sinB}{\sin(A+B)}$

$=\frac{2sin\frac{1}{2}(A+B)cos\frac{1}{2}(A-B).2cos\frac{1}{2}(A+B)sin\frac{1}{2}(A-B)}{\sin(A+B)}$

$= \frac{2sin\frac{1}{2}(A-B)cos\frac{1}{2}(A-B)}{} = \frac{\sin(A-B)}{\sin(A+B)}$

Hence $\frac{a^2-b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$

Example 56

Prove that in any triangle ABC,

$sin\frac{1}{2}(B-C) = \frac{b-c}{a}cos\frac{1}{2}A$

Solution

From sine rule formula;
a = 2rsinA, b = 2rsinB, c = 2rsinC

By substitution

$\frac{b-c}{a} = \frac{2rsinB-2rsinC}{2rsinA} = \frac{SinB-sinC}{sinA}$

But A + B + C = 180^0

A = 180^0 – (B + C)

sinA = sin[180^0 – (B + C)] = sin(B + C)

By substitution

$\frac{b-c}{a} = \frac{SinB-sinC}{sinA} = \frac{SinB-sinC}{\sin(B+C)}$

$= \frac{2cos\frac{1}{2}(B+C)sin\frac{1}{2}(B+C)}{2cos\frac{1}{2}(B+C)cos\frac{1}{2}(B+C)}$

$= \frac{sin\frac{1}{2}(B+C)}{cos\frac{1}{2}(B+C)}$

From A + B + C = 180^0

B + C = 180^0 – A

$\frac{1}{2}(B+C) = \left(90^0 - \frac{1}{2}A\right)$

$sin\frac{1}{2}(B+C) = sin\left(90^0 - \frac{1}{2}A\right) = cos\frac{1}{2}A$

By substitution

$\frac{b-c}{a} = \frac{sin\frac{1}{2}(B+C)}{cos\frac{1}{2}A}$

$\therefore sin\frac{1}{2}(B-C) = \frac{b-c}{a}cos\frac{1}{2}A$

3. The Tangent Rule

It states that in a triangle ABC

$tan\frac{1}{2}(A-B) = \left(\frac{a-b}{a+b}\right)cot\frac{1}{2}C$

$tan\frac{1}{2}(C-A) = \left(\frac{c-a}{c+a}\right)cot\frac{1}{2}B$

$tan\frac{1}{2}(b-c) = \left(\frac{b-c}{b+c}\right)cot\frac{1}{2}A$

Proof

From $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$

a = 2rsinA, b = 2rsinB, c = 2rsinC

$\frac{a-b}{a+b} = \frac{2rsinA-2rsinB}{2rsinA+2rsinB} = \frac{sinA-sinB}{sinA+sinB}$

$= \frac{2cos\frac{1}{2}(A+B)sin\frac{1}{2}(A-B)}{2sin\frac{1}{2}(A+B)cos\frac{1}{2}(A-B)}$

$= \frac{2\cos(90-\frac{1}{2}C)sin\frac{1}{2}(A-B)}{2\sin(90-\frac{1}{2}C)cos\frac{1}{2}(A-B)}$

$= \frac{\cos(90-\frac{1}{2}C)tan\frac{1}{2}(A-B)}{\sin\frac{1}{2}(90-C)}$

$= \frac{\sin\frac{1}{2}Ctan\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}$

$\frac{a-b}{a+b} = tan\frac{1}{2}Ctan\frac{1}{2}(A-B)$

$\therefore tan\frac{1}{2}(A-B) = \left(\frac{a-b}{a+b}\right)cot\frac{1}{2}C$

Example 56

Show that in a triangle PQR

$tan\frac{1}{2}(Q-C) = \left(\frac{q-r}{q+r}\right)cot\frac{1}{2}P$

Hence solve the triangle in which q =15.32, r = 28.6 and P = $39^0 52'$

Solution

From $\frac{p}{sinP} = \frac{q}{sinQ} = \frac{r}{sinR}$

p = 2rsinP, q = 2rsinQ, r = 2rsinR

$\frac{q-r}{q+r} = \frac{2rsinQ-2rsinR}{2rsinQ+2rsinR} = \frac{sinQ-sinR}{sinQ+sinR}$

$= \frac{2cos\frac{1}{2}(Q+R)sin\frac{1}{2}(Q-R)}{2sin\frac{1}{2}(Q+R)cos\frac{1}{2}(Q-R)}$

$= \frac{2\cos(90-\frac{1}{2}P)sin\frac{1}{2}(Q-R)}{2\sin(90-\frac{1}{2}P)cos\frac{1}{2}(Q-R)}$

$= \frac{\cos(90-\frac{1}{2}P)tan\frac{1}{2}(Q-R)}{\sin\frac{1}{2}(90-P)}$

$= \frac{\sin\frac{1}{2}Ptan\frac{1}{2}(Q-R)}{\cos\frac{1}{2}P}$

$\frac{q-r}{q+r} = tan\frac{1}{2}Ptan\frac{1}{2}(Q-R)$

$\therefore tan\frac{1}{2}(Q-R) = \left(\frac{q-r}{q+r}\right)cot\frac{1}{2}P$

Hence

$$tan\frac{1}{2}(Q-R) = \frac{15.32-28.6}{15.32+29.6}cot39^0 52'$$

= -0.3621

$\frac{1}{2}(Q-R) = -19.9^0$ i.e. Q – R = -39.9^0

But P + Q + R = 180

Q + R = 180 - 39.9 = 140.1^0

Solving Q = 50.15^0 and R = 89.95^0

Now p = $\frac{qsinP}{sinQ} = \frac{15.32\sin\left[39+\frac{52}{60}\right]^0}{\sin 50.15}$= 12.79

$\therefore$p = 12.79, Q =50.15^0, R = 89.95^0

Example 57

Show that $\frac{a+b-c}{a+b+c} = tan\frac{1}{2}Atan\frac{1}{2}B$

Solution

LHS = $\frac{a+b-c}{a+b+c}$

$=\frac{2rsinA+2rsinB-2rsinC}{2rsinA+2rsinB+2rsinC}$

$= \frac{sinA+sinB-sinC}{sinA+sinB+sinC}$

$= \frac{2sin\frac{1}{2}(A+B)cos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}{2sin\frac{1}{2}(A+B)cos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}$

$= \frac{2sin\left(90-\frac{1}{2}C\right)cos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}{2sin\left(90-\frac{1}{2}C\right)cos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}$

$= \frac{2cos\frac{1}{2}Ccos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}{2os\frac{1}{2}Ccos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}$

$= \frac{cos\frac{1}{2}(A-B)-sin\frac{1}{2}C}{cos\frac{1}{2}(A-B)+sin\frac{1}{2}C}$

$=\frac{\cos\frac{1}{2}(A-B)-\sin\left(90-\frac{1}{2}(A+B)\right)}{\cos\frac{1}{2}(A-B)+\sin\left(90-\frac{1}{2}(A+B)\right)}$

$=\frac{\cos\frac{1}{2}(A-B)-\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)+\cos\frac{1}{2}(A+B)}$

$=\frac{-2\sin\frac{1}{2}A\sin\left(-\frac{1}{2}B\right)}{\cos\frac{1}{2}A+\cos\frac{1}{2}B}$

$=\tan\frac{1}{2}A\tan\frac{1}{2}B$

Expressions for sinA, sin$\frac{1}{2}A$ and cos$\frac{1}{2}A$ in terms of the sides of the triangle

(a) sinA

From the identity

$\sin^2A = 1 - \cos^2A = (1 - \cos A)(1 + \cos A)$

$=\left(1-\frac{b^2+c^2-a^2}{2bc}\right)\left(1+\frac{b^2+c^2-a^2}{2bc}\right)$

$=\left(\frac{2bc-b^2+c^2-a^2}{2bc}\right)\left(\frac{2bc+^2+c^2-a^2}{2bc}\right)$

$=\frac{[a^2-(b-c)^2][(b-c)^2-a^2]}{4b^2c^2}$

$\therefore \sin^2A =\frac{(a+c-b)(a+b-c)(b+c-a)(b+c+a)}{4b^2c^2}$

Let s $=\frac{1}{2}[perimeter\ of\ triangle]$

$=\frac{1}{2}[a+b+c]$

2s =[a + b + c]

a + b = 2s – c; i.e. a + b – c = 2s – c – c = 2(s – c)

a + c= 2s – b; i.e. a + c – b = 2s – b – b =2(s – b)

b + c = 2s – a; i.e. b + c – a = 2s- a – a = 2(s – a)

$\therefore \sin^2A =\frac{2(s-b).2(s-c).2(s-a).2s}{4b^2c^2}$

$\sin A =\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$

Similarly, $\sin B =\frac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)}$

$\sin C =\frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$

(b) sin$\frac{1}{2}A$ and cos$\frac{1}{2}A$

From $\sin^2\frac{1}{2}A =\frac{1}{2}(1-\cos A)$

$=\frac{1}{2}\left(1-\frac{b^2+c^2-a^2}{2bc}\right)$

$=\left(\frac{2bc-b^2+c^2-a^2}{4bc}\right)$

$=\left(\frac{a^2-(b-c)^2}{4bc}\right)$

$=\left(\frac{(a+c-b)(a+b-c)}{4bc}\right)$

$=\left(\frac{2(s-b).2(s-c)}{4bc}\right)$

$=\left(\frac{(s-b)(s-c)}{bc}\right)$

$\therefore \sin\frac{1}{2}A=\sqrt{\left(\frac{(s-b)(s-c)}{bc}\right)}$

Similarly;

$\sin\frac{1}{2}B=\sqrt{\left(\frac{(s-b)(s-c)}{ac}\right)}$

$\sin\frac{1}{2}C=\sqrt{\left(\frac{(s-b)(s-c)}{ab}\right)}$

Also;

$\cos^2\frac{1}{2}A=\frac{1}{2}(1+\cos A)$

$=\left(\frac{2bc+b^2+c^2-a^2}{4bc}\right)$

$=\left(\frac{(b+c)^2-a^2}{4bc}\right)$

$=\left(\frac{(b+c-a)(a+b+c)}{4bc}\right)$

$=\left(\frac{2(s-a).2s}{4bc}\right)$

$=\left(\frac{s(s-a)}{bc}\right)$

$\therefore \cos\frac{1}{2}A=\sqrt{\left(\frac{(s-a)}{bc}\right)}$

Similarly;

$\cos\frac{1}{2}B=\sqrt{\left(\frac{(s-b)}{ac}\right)}$

$\cos\frac{1}{2}C=\sqrt{\left(\frac{(s-c)}{ab}\right)}$

The expression for $\tan\frac{1}{2}A$ can be deduced as follows

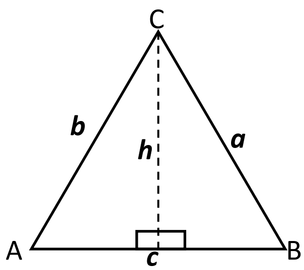
$\tan\frac{1}{2}A = \frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

Similarly;

$\tan\frac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

$\tan\frac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Area of a triangle



Area, $\Delta = \frac{1}{2}(base)(perpendicular\ height)$

$= \frac{1}{2}ch$

$= \frac{1}{2}cb\sin A$

Substituting for

$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$

$\Delta = \frac{1}{2}bc x \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

This a convenient form given the three sides of a triangle. The formula is called Hero's formula from the first mathematician who suggested it.

Example 58

The area of a triangle is $336m^2$. The sum of the three sides is 84m and one side is 28m. Calculate the length of the remaining two sides

Solution

Given $\Delta = 336$, a + b + c =84 and a = 28

$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(84) = 42$

28 + b + c = 84

b + c = 56, or c = 56 – b

But $\Delta^2 = s(s-a)(s-b)(s-c)$

$336^2 = 42(42–28)(42– b)(42 – 56 + b)$

$b^2 – 56b + 780 = 0$

$b = \frac{56 \pm \sqrt{56^2 - 4x\ 1\ x\ 780}}{2\ x\ 1}$

b = 30 or 26

substituting for c = 56 – b

c = 26 or 30

∴the remaining sides are 30m and 26m

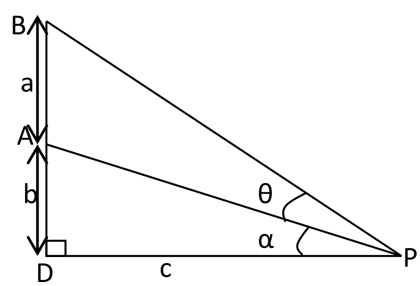
Applications of trigonometry in finding distances and bearings

Example 59

A vertical pole BAD stands with its base D on a horizontal plane where BA = a and AD = b. A point P is situated on the horizontal plane at a distance C from D and the angle APB = θ.

Prove that $\theta = \tan^{-1}\left(\frac{ac}{b^2+ab+c^2}\right)$

Solution



Let angle APD = α

For triangle APD: $\tan\alpha = \frac{b}{c}$

For triangle DPB: $\tan(\theta+\alpha) = \frac{a+b}{c}$

⇨ $\frac{\tan\theta + \tan\alpha}{1-\tan\theta\tan\alpha} = \frac{a+b}{c}$

Substituting for tanα

$\Rightarrow \frac{\tan\theta+\frac{b}{c}}{1-\left(\frac{b}{c}\right)\tan\theta} = \frac{a+b}{c}$

$c^2\tan\theta + bc = ac + bc - ab\tan\theta - b^2\tan\theta$

$(b^2 + ab + c^2)\tan\theta = ac$

$\tan\theta = \frac{ac}{b^2+ab+c^2}$

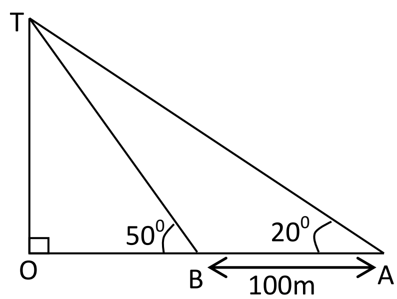
$\therefore \theta = \tan^{-1}\left(\frac{ac}{b^2+ab+c^2}\right)$

Example 60

The angle of the top of a vertical tower from a point A is 20^0 and from another point B is 50^0. Given that A and B lie on the same horizontal plane in the same direction where AB = 100m. Find the height of the tower

Solution

Let OT be the height of the tower



$A\hat{T}B = 50 - 30 = 30^0$

Using sine rule

$\frac{TB}{\sin 20^0} = \frac{100}{\sin 30^0}$

$TB = \frac{100\sin 20^0}{\sin 30^0}$

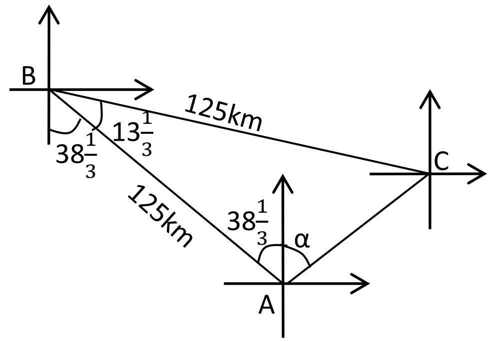
But $OT = TB\sin 50^0$

$OT = \frac{100\sin 20^0 \sin 50^0}{\sin 30^0} = 26.2\text{m}$

Example 61

From a point A, a pilot flies in the direction N$38^0$20'W to point B 125km from A. He then flies in the direction S$50^0$40'E for 125km. He wishes to return to A from this point. How far and in what direction must he fly.

Solution



From the digram

Let $B\hat{A}C = B\hat{C}A = \theta$

$\Rightarrow 2\theta + 13\frac{1}{3}^0 = 180^0$

$\theta = 83\frac{1}{3}^0$

But $38\frac{1}{3} + \alpha = \theta$

$38\frac{1}{3} + \alpha = 83\frac{1}{3}^0$

$\alpha = 45^0$

From the sine rule

$\frac{AC}{\sin 13\frac{1}{3}} = \frac{125}{\sin 83\frac{1}{3}}$

AC = 29km

$\therefore$ he has to fly 29km in the direction S45^0W

Example 62

(a) Prove that $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$

$\tan(A - B) = \frac{\sin(A-B)}{\cos(A-B)}$

$= \frac{\sin A\cos B - \cos A\sin B}{\cos A\cos B + \sin A\sin B}$

Diving numerator and denominator on the R.H.S by cosAcosB

$\tan(A - B) = \frac{\frac{\sin A\cos B}{\cos A\cos B} - \frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B} + \frac{\sin A\sin B}{\cos A\cos B}}$

$= \frac{\tan A - \tan B}{1 + \tan A\tan B}$

Hence show that $\frac{1-\tan 15^0}{1+\tan 15^0} = \frac{1}{\sqrt{3}}$

$\frac{1-\tan 15^0}{1+\tan 15^0} = \frac{\tan 45^0 - tan15^0}{1+\tan 45^0 tan15^0}$

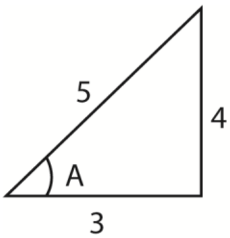
= tan $(45^0 - 15^0)$ tan30^0 = $\frac{1}{\sqrt{3}}$

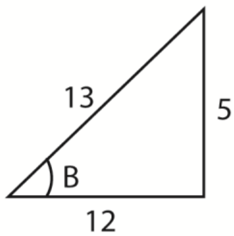
(b) Given that cos A = $\frac{3}{5}$ and cos B = $\frac{12}{13}$ where A and B are acute, find the values of

(i) tan (A + B)

(ii) cosec (A + B)

Solution





cos A = $\frac{3}{5}$ cos B = $\frac{12}{13}$

sin A = $\frac{4}{5}$ sin B = $\frac{5}{13}$

tan A = $\frac{4}{3}$ tanB = $\frac{5}{12}$

(i) tan (A + B)=$\frac{sinA\ cosB + cosA\ sinB}{cos\ AsinB - sinAsin\ B}$

$= \frac{\frac{4}{5}.\frac{12}{13}+\frac{3}{5}.\frac{5}{13}}{\frac{3}{5}.\frac{5}{13}-\frac{4}{5}.\frac{5}{13}}$ = 3.9375

(ii) cosec (A + B) = $\frac{1}{\sin(A+B}$

$= \frac{1}{sinAcosB + cosAsinB}$

$= \frac{1}{\frac{4}{5}.\frac{12}{13}+\frac{3}{5}.\frac{5}{13}}$

= 1.0317

Example 63

Express cos $(\theta + 30)^0$ – cos $(\theta + 48)^0$ in the form RsinPsinQ, where R is constant.

Hence solve th3 equation

cos $(\theta + 30)^0$ – cos $(\theta + 48)^0$ = 0.2

Solution

cos $(\theta + 30)^0$ – cos $(\theta + 48)^0$

= -2sin$\left(\frac{\theta + 30^0 + \theta + 48^0}{2}\right)$ sin $\left(\frac{\theta + 30^0 - \theta - 48^0}{2}\right)$

= -2sin $(\theta + 39^0)$sin(-9^0)

cos $(\theta + 30)^0$ – cos $(\theta + 48)^0$ = 0.

⇨ -2sin $(\theta + 39^0)$sin(-9^0) = 0.2

sin $(\theta + 39^0)$ = 0.63925

$\theta + 39^0 = 39.74^0$

$\theta = 0.74^0$

Example 64

Express 7cos 2θ + 6sin 2θ in form Rcos (2θ – α), where R is a constant and α is an acute angle.

7cos 2θ + 6sin 2θ ≡ Rcos (2θ – α)

7cos 2θ + 6sin 2θ ≡ Rcos 2θ cos α + Rsin2θsinα

Comparing both sides

Rcosα = 7 ………………… (i)

Rsinα = 6 ………………… (ii)

(i)2 + (ii)2 gives

R = $\sqrt{7^2 + 6^2} = \sqrt{85}$

From equation (i)

$\sqrt{85}$cosα = 7

$\alpha = cos^{-1}\left(\frac{7}{\sqrt{85}}\right)$ = 40.6^0

Hence solve 7cos 2θ + 6sin 2θ = 5 for $0^0 \leq \theta \leq 180^0$. (07marks)

∴ 7cos 2θ + 6sin 2θ = $\sqrt{85}$ cos$(2\theta - 40.6^0)$ =5

$2\theta - 40.6 = cos^{-1}\left(\frac{5}{\sqrt{85}}\right)$ = 57.16^0, 302.84^0

$\theta = 48.88^0, 171.72^0$

Revision exercise 7

1. Solve the triangles

(a) a = 17m, b = 21.42m, B = $51^034'$
[A =38.44^0, C = 90^0, c = 27.34m]

(b) b = 107.2m, c= 76.69m, B =$102^025'$
[A =33.26^0, C = 44.32^0, a = 60.21m]

(c) a = 7m, b = 3.59m, C= 47^0
[A= $103^02'$, B = $29^052'$, c= 5.25m]

(d) A = 60^0, b = 8m, C = 15
[a = 13, B= 32.2^0, C = 87.8^0]

2. Show that for all values of x

$cosx + cos\left(x + \frac{2\pi}{3}\right) + cos\left(x + \frac{3\pi}{3}\right) = 0$

3. (a) Simplify $\frac{sin3\theta}{sin\alpha} - \frac{cos3\theta}{cos\alpha}\left[\frac{2\sin(3\theta-\alpha)}{sin2\alpha}\right]$
   (b) Express 5sinθ + 12cosθ in the form rsin(θ + α) where r and α are constant. Hence determine the minimum value of 5sinθ + 12cosθ + 7.
   [r =13, α = 67.4^0, -6]
   (c) Given that tanθ = $\frac{3}{4}$, where θ is acute, find values of tan2θ and tan$\frac{\theta}{2}$
   [tan2θ =$\frac{24}{7}$ and tan$\frac{\theta}{2}$ = $\frac{1}{3}$]
4. (a) Show that $2\tan^{-1}\left(\frac{1}{3}\right) + tan\left(\frac{1}{7}\right) = \frac{\pi}{4}$
   (b) Find x given that
   $\tan^{-1}(1+x) + \tan^{-1}(1-x) = 32$
   [x = $\pm$1.789]
   (c) Given that sinα+ sinβ = p and cosα + cos β = q
   Show that sin(α + β) = $\frac{2pq}{p^2+q^2}$
5. (a) By expressing 2sinθsin(θ + α) as a difference of cosines of two angles or otherwise, where αis constant, find the least value [minimum value = cosα – 1. It occurs when θ = $\frac{-\alpha}{2}$]
   (b) Solve for x in the equation
   cosx – cos(x + 60^0) = 0.4 for
   $0^0 \leq x \leq 360^0$ [x: x = 126.4^0, 353.6^0]
6. (a) Prove that in any triangle ABC
   $\frac{b^2-c^2}{a^2} = \frac{\sin(B-C)}{\sin(B+C)}$
   (b) Show that for any isosceles triangle ABC with AB = c the base, is given by
   Δ =$\frac{1}{2}c\sqrt{s(s-c)}$ where s is the perimeter of the triangle
   Given that Δ= $\sqrt{3}$ and s = 4, determine the sides of the triangle [1, 3.5, 3.5]
7. Given that $\tan^{-1}\alpha = x$ and $\tan^{-1}\beta = y$, by expressing α and β as tangents ratio of x and y and manipulating the ratios show that x + y = $\tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$
   Hence or otherwise
   (i) Solve for x in

   $\tan^{-1}\left(\frac{1}{x-1}\right) + \tan(x+1) = \tan(-2)$
   [x = $\pm$2]

   (ii) Without using tables of calculators determine the value of
   $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}\left[\frac{\pi}{4}\right]$
8. (a) Prove that $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC} = 2R$ where ABC has all angles acute and R is the radius of the circumcircle.
   (b) From the top of a vertical cliff 10m high, the angle of depression of ship A is 10^0 and ship B is 15^0. The Bearings of A and B from the cliff are 162^0 and 202.5^0 respectively. Find the bearing of B from A [301.5^0]
9. (a) Prove that
   (sin2x –sinx)(1 + 2cosx) = sin3x
   (b) A vertical pole BAO stands with its base O on a horizontal plane, where BA = c and AO = b, a point P is situated on horizontal plane at a distance x from O and angle APB = θ
   Prove that tanθ =$\frac{cx}{x^2+b^2+bc}$
   As P takes different positions on the horizontal plane, find the value of x for which θ is greatest.
   [$18^0$26’, when x = b = c]
10. (a) Prove that sin3x = 3sinx – $4\sin^2$x.
    (b) Find all the solutions to $2\sin^2$x = 1 for 00≤ x≤ 360^0. [x =10^0, 50^0, 130^0, 170^0, 250^0, 290^0]
11. Solve cosx +$\sqrt{3}$sinx =2 for $0^0 \leq x \leq 360^0$
    [x = 60^0]
12. From the top of a tower 12.6m high, the angles of depression of ship A and B are 12^0 and 18^0 respectively. the bearing of ship A and ship B from the tower are 148^0 and 209.5^0 respectively
    Calculate
    (i) How far the ships are from each other [53.14m]
    (ii) The bearing of ship A from ship B[108.1^0]
13. (a) Solve sin3x + $\frac{1}{2}$ = $2\cos^2$x for
    $0^0 \leq x \leq 360^0$
    [x = 30^0, 60^0, 120^0, 150^0, 240^0, 300^0]
    (b) Given that in any triangle ABC,
    $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}cot\left(\frac{A}{a}\right)$ solve the triangle with two sides 5 and 7 and the included angle 45^0.
    [A = 45^0, B = 89.4^0, C= 45.6^0]

14. (a) Solve $\cot^2x = 5(\csc x + 1)$ for $0^0 \le x \le 360^0$ [9.6^0, 170.4^0, 270^0]
    (b) Use $\tan\frac{\theta}{2}$ =t to solve $5\sec\theta - 2\sin\theta = 2$ for $0^0 \le x \le 360^0$ [46.4^0, 270^0]
15. Given that sin2x = cos3x, fins the values of sinθ, $0 \le x \le \pi$ [0.309 3dp]
16. (a) Show that $\tan\left(\frac{A+B}{2}\right) - \tan\left(\frac{A-B}{2}\right) = \frac{2sinB}{cosA+cosB}$
    (b) Find in radians the solution of the equation cosθ + sin2θ = cos3θ for $0 \le \theta \le \pi$ $\left[0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right]$
17. (a) Show that cotA + tan2A = cotAsec2A
    (b) Show that $tan3x = \frac{3t-t^3}{1-3t^2}$, where t = tanx. Hence or otherwise show that $\tan^{-1}\left(\frac{\pi}{12}\right) = 2 - \sqrt{3}$
18. (a) Find all the values θ, 00 ≤ θ ≤ 3600, which satisfies the equation $\sin^2\theta - \sin2\theta - 3\cos^2\theta = 0$ [θ =135^0, 315^0]
    (b) Show that $\frac{cosx}{1+sinx} = cot\left(\frac{x}{2} + 45^0\right)$. Hence or otherwise solve $\frac{cosx}{1+sinx} = \frac{1}{2}$; $0^0 \le x \le 360^0$ [x = 36.8^0]
19. (a) Given that X, Y and Z are angles of a triangle XYZ. Prove that $tan\left(\frac{X-Y}{2}\right) = \frac{x-y}{x+y}cot\frac{Z}{2}$. Hence solve the triangle if x = 9cm, y = 5.7cm and Z = 57^0. [z = 7.6cm, X= 84.4^0]
    (b) Use the substitution t = $\tan\left(\frac{\theta}{2}\right)$ to solve the equation 3cosθ – 5sinθ = -1 for $0^0 \le \theta \le 360^0$ [40.84^0, 201.1^0]
20. Prove that $tan\left(\frac{\pi}{4} + \theta\right) - tan\left(\frac{\pi}{4} - \theta\right) = 2tan2\theta$
21. (a) Solve the equation 3cosx + 4sinx = 2 for $0^0 \le x \le 360^0$ [x = 119.5^0, 346.7^0]
    (b) If A, B, C are angles of a triangle. Show that

    cos2A + cos2B + cos2C = -1 –4cosAcosB

22. (a) Solve 2sin2θ = 3 for $-180^0 \le x \le 180^0$ [-90^0, 48.6^0,90^0, 131.4^0]
    (b) Solve sinx – sin4x = sin2x – sin3x for $-\pi \le x \le \pi$

    $\left[-\frac{\pi}{5}, -\frac{\pi}{2}, -\frac{3\pi}{5}, 0, \frac{\pi}{5}, \frac{\pi}{5}, \frac{3\pi}{5}\right]$
23. Without using tables or calculator, show that tan150 = 2-√3
24. (a) Solve the equation cosx + cos2x = 1 for $0^0 \le x \le 360^0$ [x = 38.67^0, 321.33^0]
    (b) (i) Prove that $\frac{cosA+cosB}{sinA+sinB} = cot\frac{A+B}{2}$
    (ii) $\frac{cosA+cosB}{sinA+sinB} = tan\frac{C}{2}$ where A, B and C are angles of a triangle
25. Given that $\sin(\theta - 45^0) = 3\cos(\theta + 45^0)$ show that tanθ = 1. Hence find θ if $0^0 \le \theta \le 360^0$ [45^0, 225^0]
26. (a) Use the factor formula to show that $\frac{\sin(A+2B)+sinA}{coa(A+2B)+cosA} = \tan(A+B)$
    (b) Express y = 8cosx + 6sinx in the form Rcos(x – α) where R is positive and α is acute
    Hence find the maximum and minimum values of $\frac{1}{8cosx+6sinx+15}$ [0.2, 0.04]
27. Express sinx + cosx in the form Rcos(x – α). Hence, find the greatest value of sinx + cos x – 1. [0.4142]
28. (a) Solve cosx + cos3x= cos2x, $0 \le x \le 360^0$ [x = 45^0, 60^0, 135^0, 225^0, 300^0, 315^0]
    (b) Show that $tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{1+sin\theta}{cos\theta}$
29. Show that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{7}{9}$
30. (a) Solve 3sinx + 4cosx = 2 for $-180^0 \le x \le 180^0$. [-29.55^0, 103.29^0]
    (b) Show that in any triangle ABC $\frac{a^2-b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$
31. (a) Prove that $\frac{2tanx}{1+tan^2x} = sin2x$
    (b) Solve sin2x = cosx; $0^0 \le x \le 90^0$ [x = 30^0, 90^0]
32. (a) Solve the equation $8\cos^4x - 10\cos^2x + 3 = 0$; $0^0 \le x \le 180^0$ [30^0, 45^0, 135^0, 150^0]
    (b) Prove that cos4A – cos4B – Cos4C = 4sin2Bsin2Ccos2A -1 given that A, B and C are angles of a triangle
33. Given that cos 2A - cos 2B = -p and sin2A – sin2B = q, prove that sec (A + B) = $\frac{1}{q}\sqrt{p^2 + q^2}$
34. Solve
(a) $4\sin^2\theta - 12\sin2\theta + 35\cos^2\theta = 0$; for $0^0 \le \theta \le 90^0$ [74.0^0]

(b) $3\cos\theta - 2\sin\theta = 2$, for $0^0 \leq \theta \leq 360^0$
$[\theta: \theta = 22.62^0, 270.00^0]$

35. Solve the equation $\sin 2\theta + \cos 2\theta\cos 4\theta = \cos 4\theta\cos 6\theta$ for $0 \leq \theta \leq \frac{\pi}{4}$. $\left[\theta = 0, \frac{3\pi}{16}\right]$

36. (a) solve the equation $\cos 2x = 4\cos^2 x - 2\sin^2 x$ for $0 \leq \theta \leq 180^0$ $[\theta = 60^0, 120^0]$

(b) Show that if $\sin(x + \alpha) = p\sin(x - \alpha)$ then $\tan x = \left(\frac{p+1}{p-1}\right)\tan\alpha$. Hence solve the equation $\sin(x + \alpha) = p\sin(x - \alpha)$ for $p = 2$ and $\alpha = 20^0$. $[x = 47.52^0]$

37. Solve the equation
$3\tan^2\theta + 2\sec^2\theta = 2(5 - 3\tan\theta)$
for $0^0 < \theta < 180^0$ $[\theta = 38.66^0, 116.57^0]$

38. (a) Show that $\tan 4\theta = \frac{4t(1-t^2)}{t^4-6t^2+1}$, where $t = \tan\theta$

(b) Solve the equation
$\sin x + \sin 5x = \sin 2x + \sin 4x$
for $0^0 < x < 90^0$. $[x = 60^0]$

39. Solve $2\cos 2\theta - 5\cos\theta = 4$
for $0^0 \leq \theta \leq 360^0$. $[\theta = 138.59^0, 221.41^0]$

Thank you

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