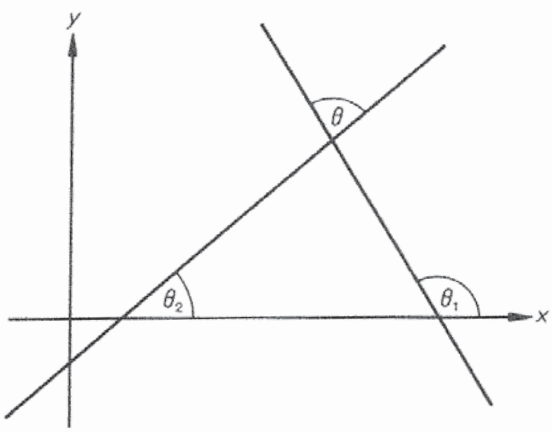
13 Coordinate geometry

13.1 Angles, distances and areas



The diagram shows two lines which make angles θ_1 and θ_2 with the positive direction of the x-axis. If the gradients of these lines are m_1 and m_2 respectively, then

$$m_1 = \tan\theta_1, \qquad m_2 = \tan\theta_2.$$

One angle between the lines is θ, where $\theta = \theta_1 - \theta_2$,

$$\therefore \qquad \tan\theta = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}.$$

Hence the angle θ between two lines with gradients m_1 and m_2 is given by

$$\boxed{\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}}$$

Example 1 Find the acute angle between the straight lines $3x - y + 2 = 0$ and $x - 2y - 1 = 0$.

The equations of the lines may be re-written as:

$y = 3x + 2$ and $y = \frac{1}{2}x - \frac{1}{2}$.

Hence the gradients of the lines are 3 and $\frac{1}{2}$.

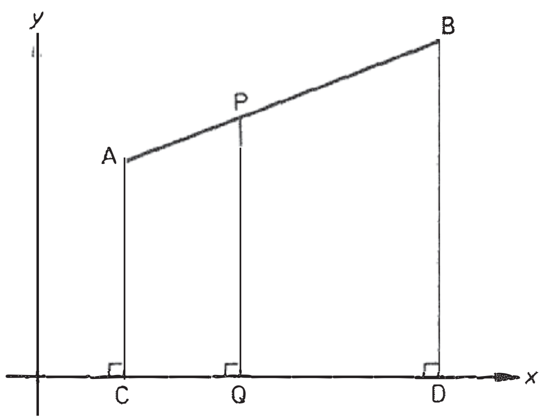
If θ is one angle between the lines, then

$$\tan\theta = \frac{3-\frac{1}{2}}{1+3\times\frac{1}{2}} = \frac{2\frac{1}{2}}{2\frac{1}{2}} = 1$$

$\therefore$ the acute angle between the lines is $45°$.

[Note that the angle between two curves at any point of intersection is defined as the angle between the tangents to the curves at the point.]

Consider now the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$. Let P be the point which divides AB in the ratio $\mu:\lambda$. Then if C, D and Q are the feet of the perpendiculars from A, B and P to the x-axis, Q must divide CD in the ratio $\mu:\lambda$. It follows that the x-coordinate of P



$$= x_1 + \frac{\mu}{\lambda+\mu}(x_2 - x_1) = \frac{\lambda x_1 + \mu x_2}{\lambda + \mu}$$

A similar expression is obtained for the y-coordinate of P.

i.e. The point dividing the line AB in the ratio $\mu:\lambda$ has coordinates

$$\boxed{\left(\frac{\lambda x_1 + \mu x_2}{\lambda+\mu}, \frac{\lambda y_1 + \mu y_2}{\lambda+\mu}\right).}$$

This formula can be used when P divides AB externally in a given ratio if μ and λ are taken to have opposite signs.
[We recall that the corresponding formula for the position vector of a point which divides a line in the ratio $\mu:\lambda$ was obtained in §7.1.]

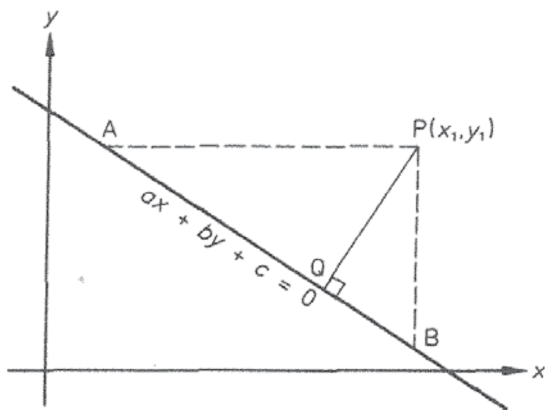
Example 2 If A and B are the points $(2, 1)$ and $(-4, 4)$ respectively, find the coordinates of the point C which divides AB internally in the ratio $2:1$ and of the point D which divides AB externally in the ratio $3:2$.

The coordinates of C are

$$\left(\frac{1\times 2 + 2(-4)}{1+2}, \frac{1\times 1 + 2\times 4}{1+2}\right) \quad \text{i.e.} \quad (-2, 3)$$

Regarding D as the point which divides AB in the ratio $3:-2$, the coordinates of D are

$$\left(\frac{(-2)2 + 3(-4)}{-2+3}, \frac{(-2)1 + 3\times 4}{-2+3}\right) \quad \text{i.e.} \quad (-16, 10).$$



Next we derive a formula for the perpendicular distance from a point $P(x_1, y_1)$ to a straight line $ax + by + c = 0$. In the diagram A and B are points on the line $ax + by + c = 0$, such that AP is parallel to the x-axis and BP is parallel to the y-axis. The point Q is the foot of the perpendicular from P to the line and so PQ is the required perpendicular distance.

Substituting $y = y_1$ in the equation $ax + by + c = 0$, we find that the x-coordinate of A is $-(by_1 + c)/a$.

$$\therefore \qquad AP = \left| x_1 - \left\{ -\frac{(by_1 + c)}{a} \right\} \right| = \left| \frac{ax_1 + by_1 + c}{a} \right|$$

Similarly $BP = \left| \dfrac{ax_1 + by_1 + c}{b} \right|$

$$\therefore \qquad AB = \sqrt{\{AP^2 + BP^2\}}$$

$$= \sqrt{\left\{ \frac{(ax_1 + by_1 + c)^2}{a^2} + \frac{(ax_1 + by_1 + c)^2}{b^2} \right\}}$$

$$= \sqrt{\left\{ \frac{(ax_1 + by_1 + c)^2(a^2 + b^2)}{a^2b^2} \right\}}$$

$$= \left| \frac{(ax_1 + by_1 + c)\sqrt{(a^2 + b^2)}}{ab} \right|$$

However, the area of $\triangle APB = \frac{1}{2} \times PQ \times AB = \frac{1}{2} \times AP \times BP$

$$\therefore \qquad PQ = \frac{AP \times BP}{AB}$$

$$= \left| \frac{(ax_1 + by_1 + c)^2}{ab} \times \frac{ab}{(ax_1 + by_1 + c)\sqrt{(a^2 + b^2)}} \right|$$

$$= \left| \frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}} \right|$$

i.e.

> The perpendicular distance from the point (x_1, y_1) to the line
>
> $ax + by + c = 0$ is $\left| \dfrac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}} \right|$

Example 3 Find the distance of the point $(3, -5)$ from the line $2x - y = 1$.

The distance of the point $(3, -5)$ from the line $2x - y - 1 = 0$ is

$$\left|\frac{2 \times 3 - (-5) - 1}{\sqrt{\{2^2 + (-1)^2\}}}\right| = \frac{10}{\sqrt{5}} = 2\sqrt{5}.$$

Example 4 Find the locus of points equidistant from the lines $y = 2x$ and $2x + 4y - 3 = 0$.

Let $P(x_1, y_1)$ be a point equidistant from the lines $2x - y = 0$ and $2x + 4y - 3 = 0$, then

$$\left|\frac{2x_1 - y_1}{\sqrt{\{2^2 + (-1)^2\}}}\right| = \left|\frac{2x_1 + 4y_1 - 3}{\sqrt{\{2^2 + 4^2)}}\right|$$

$$\therefore \quad \frac{1}{\sqrt{5}}\left|2x_1 - y_1\right| = \frac{1}{\sqrt{20}}\left|2x_1 + 4y_1 - 3\right|$$

$$\therefore \quad 2|2x_1 - y_1| = |2x_1 + 4y_1 - 3|$$

Thus either

$$\begin{aligned} 2(2x_1 - y_1) &= 2x_1 + 4y_1 - 3 \\ 4x_1 - 2y_1 &= 2x_1 + 4y_1 - 3 \\ 2x_1 - 6y_1 + 3 &= 0 \end{aligned}$$

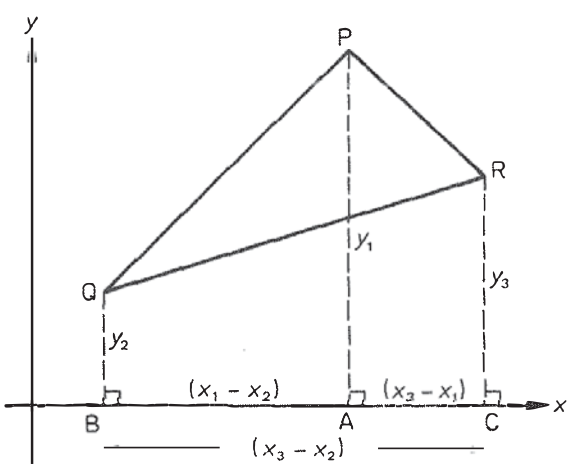
or

$$\begin{aligned} 2(2x_1 - y_1) &= -(2x_1 + 4y_1 - 3) \\ 4x_1 - 2y_1 &= -2x_1 - 4y_1 + 3 \\ 6x_1 + 2y_1 - 3 &= 0 \end{aligned}$$

Hence the locus of all points P equidistant from the given lines is the pair of straight lines

$$2x - 6y + 3 = 0, \qquad 6x + 2y - 3 = 0.$$

It follows that these lines are the bisectors of the angles between the given lines.



Finally we consider the area of the triangle whose vertices are $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$.

As shown in the diagram, the area of ΔPQR = area $PQBA$ + area $PACR$ − area $QBCR$.

Since each of these areas is a trapezium,

$$\begin{aligned} \text{area of } \Delta PQR &= \tfrac{1}{2}(x_1 - x_2)(y_1 + y_2) + \tfrac{1}{2}(x_3 - x_1)(y_1 + y_3) - \tfrac{1}{2}(x_3 - x_2)(y_2 + y_3) \\ &= \tfrac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\} \end{aligned}$$

It is found that, although this result holds for any points P, Q and R, it sometimes gives a negative value for the area. We must therefore include modulus signs in the general formula.

$$\text{Area of triangle} = \tfrac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

[Note that this formula is rarely needed in elementary work. Problems on area usually involve right-angled triangles, isosceles triangles or other triangles whose heights and bases are easily calculated.]

Exercise 13.1

1. Find the tangent of the acute angle between the following pairs of lines
(a) $y = 2x - 3$, $y = 3x + 1$
(b) $3x - 2y + 1 = 0$, $2x - y - 7 = 0$
(c) $x + 3y - 2 = 0$, $5x - 6y + 2 = 0$
(d) $7x + 4y + 5 = 0$, $y = 0$
(e) $y = 4x$, $2x - 8y - 5 = 0$
(f) $4x + 3y + 2 = 0$, $x = 0$.

2. Find the coordinates of the points which divide the lines joining the given pairs of points internally in the stated ratio.
(a) $(7, 3)$, $(1, -6)$; $1{:}2$
(b) $(2, -4)$, $(-8, 1)$; $2{:}3$
(c) $(0, 6)$, $(2\frac{1}{2}, -4)$; $4{:}1$
(d) $(-3, 5)$, $(1, -1)$; $5{:}3$

3. Find the coordinates of the points which divide the lines joining the given pairs of points externally in the stated ratio.
(a) $(5, -1)$, $(-2, 3)$; $1{:}2$
(b) $(2, -3)$, $(0, 5)$; $2{:}3$
(c) $(4, 3)$, $(-2, 0)$; $4{:}1$
(d) $(-3, 4)$, $(1, -1)$; $5{:}3$

4. Find the distances between the given points and the given lines.
(a) $(1, 2)$, $4x - 3y - 3 = 0$
(b) $(-3, -1)$, $5x + 12y + 1 = 0$
(c) $(2, 5)$, $x - y + 5 = 0$
(d) $(0, 0)$, $2x + 3y = 13$
(e) $(-6, 10)$, $x = 3$
(f) $(7, -4)$, $3y - x - 1 = 0$

5. Find the areas of the triangles with the following vertices:
(a) $(2, 3)$, $(5, -1)$, $(2, -1)$
(b) $(-3, 4)$, $(5, 1)$, $(-3, -2)$
(c) $(-1, 3)$, $(2, -3)$, $(4, 5)$
(d) $(3, -1)$, $(-2, 0)$, $(1, 4)$
(e) $(5, 7)$, $(-4, -11)$, $(0, -3)$
(f) $(-2, 6)$, $(3, -7)$, $(4, 5)$

6. Find the equations of the bisectors of the angles between the given pairs of lines
(a) $3x - y - 3 = 0$
$x + 3y + 1 = 0$
(b) $2x + 3y = 2$
$3x + 2y = 3$
(c) $3x + 6y - 1 = 0$
$x - 2y + 1 = 0$
(d) $y = 7x - 1$
$y = x + 5$

7. Find the angles of the triangle PQR with vertices $P(1, 4)$, $Q(3, -2)$ and $R(5, 2)$.

8. Find the angles of the triangle ABC with vertices $A(1, -3)$, $B(4, 6)$ and $C(-1, 1)$.

9. A triangle has vertices $A(-4, 1)$, $B(3, 0)$ and $C(1, 2)$. If the internal bisector of $\angle B$ meets AC at P, use the fact that $AP:PC = AB:BC$ to find the coordinates of P.

10. A triangle has vertices $P(1, -2)$, $Q(5, 1)$ and $R(6, 10)$. Find the coordinates of the point on QR which is equidistant from the sides PQ and PR.

11. Find the tangent of the acute angle between the parabola $y^2 = 4x$ and the circle $x^2 + y^2 = 5$ at their points of intersection.

12. The curves $y = x^2$ and $y^2 = 8x$ intersect at the origin and at a point A. Find the angle between the curves at A.

13. Find the incentre of the triangle with vertices $(-1, 4)$, $(2, -2)$ and $(7, 8)$.

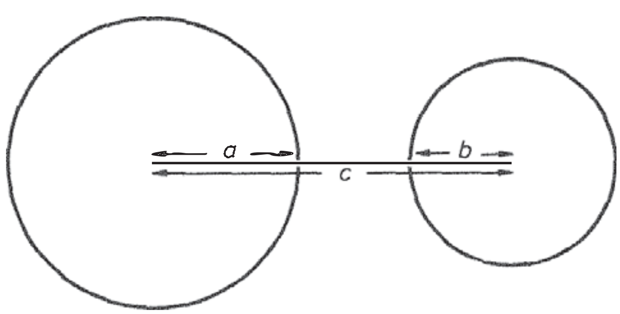
14. Find the incentre of the triangle whose sides have equations $x - y + 1 = 0$, $x + 7y + 9 = 0$ and $7x - y - 2 = 0$.

15. The points $A(-8, 9)$ and $C(1, 2)$ are opposite vertices of a parallelogram $ABCD$. The sides BC, CD of the parallelogram lie along the lines $x + 7y - 15 = 0$, $x - y + 1 = 0$, respectively. Calculate (i) the coordinates of D, (ii) the tangent of the acute angle between the diagonals of the parallelogram, (iii) the length of the perpendicular from A to the side CD, (iv) the area of the parallelogram. (JMB)

13.2 Further work on circles

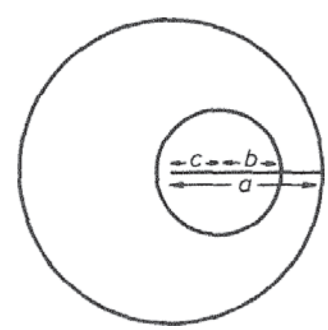
When a problem involves two circles, of radii a and b, the relative positions of the circles can be determined by finding the distance c between their centres. Assuming that $a > b$, there are various different possibilities.

(1)



$c > a + b$

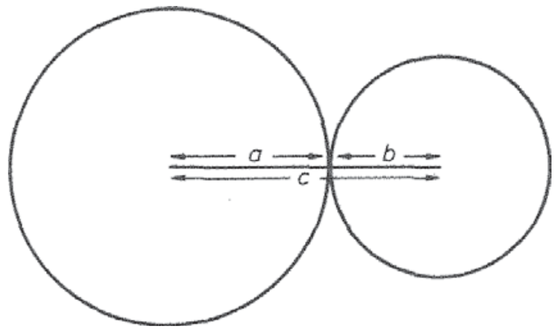
(2)



$c < a - b$

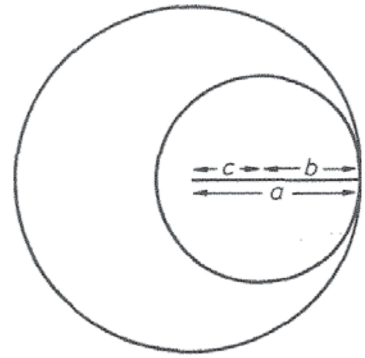
In diagrams (1) and (2) the circles do not intersect.

(3)



$c = a + b$

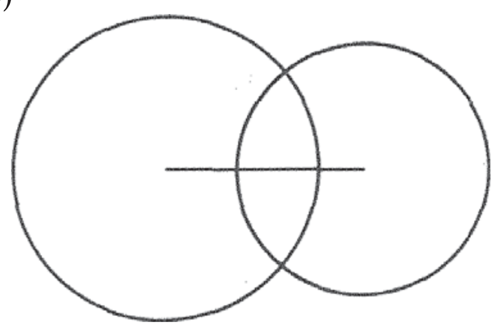
(4)



$c = a - b$

In diagrams (3) and (4) the circles touch either externally or internally.

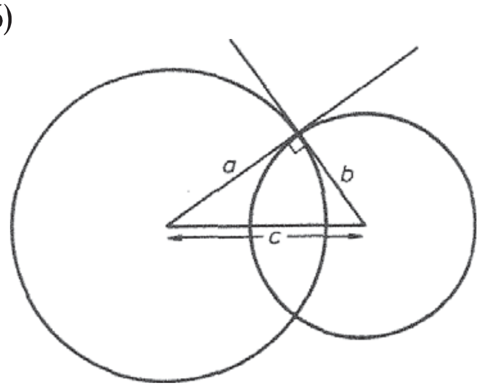
(5)



$a - b < c < a + b$

In diagram (5) the circles have two distinct points of intersection.

(6)



$c^2 = a^2 + b^2$

Two circles which cut at right-angles are called *orthogonal* circles. Diagram (6) shows two such circles.

The *common chord* of two circles is the line joining their points of intersection. The equation of this chord can be obtained without finding the coordinates of the points of intersection.

Example 1 The circles $x^2 + y^2 + 4x - 3y + 1 = 0$ and $x^2 + y^2 + x - y - 2 = 0$ intersect at the points A and B. Find the equation of the common chord AB.

$$x^2 + y^2 + 4x - 3y + 1 = 0 \qquad (1)$$
$$x^2 + y^2 + x - y - 2 = 0 \qquad (2)$$

Subtracting (2) from (1) we obtain:

$$3x - 2y + 3 = 0$$

Since the coordinates of A and B satisfy equations (1) and (2), they must also satisfy this new equation, which represents a straight line.
Hence $3x - 2y + 3 = 0$ is the equation of AB.

We now extend the method of Example 1 by considering two intersecting circles with equations:

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{(i)}$$
$$x^2 + y^2 + 2Gx + 2Fy + C = 0 \qquad \text{(ii)}$$

From these equations we can form a new equation:

$$x^2 + y^2 + 2gx + 2fy + c + k(x^2 + y^2 + 2Gx + 2Fy + C) = 0 \qquad \text{(iii)}$$

i.e. $$(1 + k)x^2 + (1 + k)y^2 + 2(g + kG)x + 2(f + kF)y + c + kC = 0$$

This is the equation of a circle for all values of k, except $k = -1$. Moreover, any pair of values of x and y which satisfy equations (i) and (ii) must also satisfy equation (iii). Hence, in general, equation (iii) represents a circle passing through the points of intersection of circles (i) and (ii). When $k = -1$ equation (iii) represents a straight line through the points of intersection i.e. the common chord of the circles.

Example 2 The circles $x^2 + y^2 + 3x - y - 5 = 0$ and $x^2 + y^2 - 2x + y - 1 = 0$ intersect at points A and B. Find the equation of the circle which passes through the origin and the points A and B.

Any circle which passes through A and B has an equation of the form:

$$x^2 + y^2 + 3x - y - 5 + k(x^2 + y^2 - 2x + y - 1) = 0$$

If this circle also passes through the origin,

$$-5 + k(-1) = 0 \quad \text{i.e.} \quad k = -5$$

Hence the equation of the circle which passes through the origin and the points A and B is

$$x^2 + y^2 + 3x - y - 5 - 5(x^2 + y^2 - 2x + y - 1) = 0$$

i.e. $$4x^2 + 4y^2 - 13x + 6y = 0.$$

Exercise 13.2

1. Two circles have equations $x^2 + y^2 - 2x - 6y - 54 = 0$ and $x^2 + y^2 - 8x + 2y + 13 = 0$.

Show that one circle lies entirely inside the other.

2. Prove that the circles whose equations are $x^2 + y^2 - 2x - 2y - 2 = 0$, $x^2 + y^2 - 6x - 10y + 33 = 0$ lie entirely outside one another.

3. A circle C_1 has equation $x^2 + y^2 - 32x - 24y + 300 = 0$ and a second circle C_2 has as diameter the line joining the points (8, 0) and (0, 6). Show that the circles C_1 and C_2 touch externally.

4. A circle C has equation $x^2 + y^2 - 4x + 2y = 40$. Find the equations of the circles with centre $(3, 1)$ which touch circle C. Find also the coordinates of the points of contact.

5. Show that the circles $x^2 + y^2 - 2x - 2y - 2 = 0$ and
$$x^2 - y^2 - 8x - 10y + 32 = 0$$
touch externally and find the coordinates of the point of contact.

6. Find the equations of the two circles with centres $A(1, 1)$ and $B(9, 7)$ which have equal radii and touch each other externally. Find also the equations of the common tangents to these circles.

7. Two circles with centres $(3, -5)$ and $(-6, 7)$ pass through the point $(4, 2)$. Find the equations of these circles. Find also the equation and the length of their common chord.

8. Prove that the circles whose equations are $x^2 + y^2 - 4y - 5 = 0$, $x^2 + y^2 - 8x + 2y + 1 = 0$ cut orthogonally and find the equation of the common chord.

9. Find the equation of the circle with centre $(3, -2)$ and radius 5 units. Find also the equation of the circle with centre $(-7, 3)$ which intersects the original circle at right angles.

10. Show that any circle which passes through the points $(1, 0)$ and $(-1, 0)$ has an equation of the form $x^2 + y^2 - 2\lambda y - 1 = 0$. Prove also that if the circles given by $\lambda = \lambda_1$ and $\lambda = \lambda_2$ cut orthogonally then $\lambda_1 \lambda_2 = -1$.

11. Show that if the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and
$$x^2 + y^2 + 2Gx + 2Fy + C = 0$$
cut orthogonally, then $2gG + 2fF = c + C$.

12. The circles $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 4x - 6y - 3 = 0$ intersect at the points A and B. Find (a) the equation of AB, (b) the equation of the circle which passes through A, B and the point $(1, 2)$.

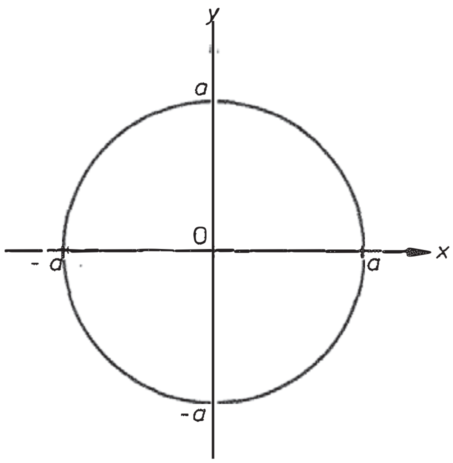
13. Given that the circles $x^2 + y^2 - 3y = 0$ and $x^2 + y^2 + 5x - 8y + 5 = 0$ intersect at P and Q, find the equation of the circle which passes through P, Q and (a) the point $(1, 1)$, (b) the origin, (c) touches the x-axis.

14. The circle $x^2 + y^2 + 3x - 5y - 4 = 0$ and the straight line $y = 2x + 5$ intersect at the points A and B. Find the equation of the circle which passes through A, B and (a) the point $(3, 1)$, (b) the origin, (c) has its centre on the y-axis.

15. Show that the circles $x^2 + y^2 + 8x + 2y + 8 = 0$, $x^2 + y^2 - 2x + 2y - 2 = 0$ have three common tangents, and find their equations. (C)

13.3 Loci using parametric forms

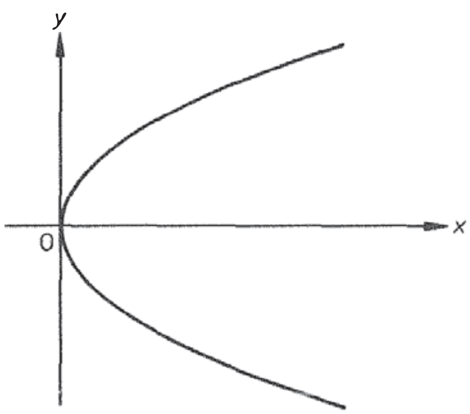
Many problems in coordinate geometry are best solved using the parametric equations of curves. The basic techniques involved were introduced in Book 1 and used to investigate the properties of various curves including the parabola $y^2 = 4ax$. We now consider further applications of these methods. As clear diagrams are often useful in this type of work, we begin by giving sketches of several important curves together with the most commonly used parametric forms of their equations.



The circle $x^2 + y^2 = a^2$.
Parametric equations:

$$x = a\cos\theta,$$
$$y = a\sin\theta.$$

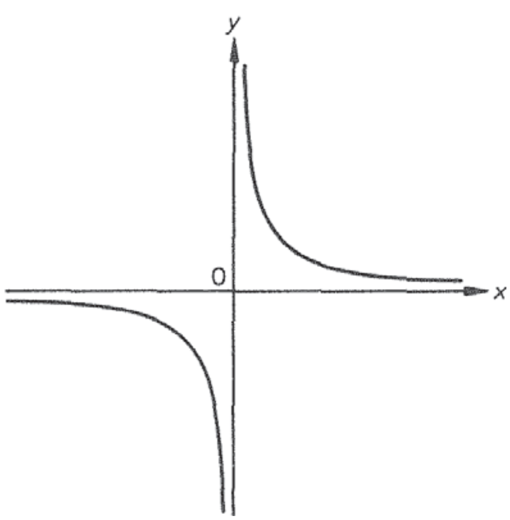
This is the circle with centre O and radius a.



The parabola $y^2 = 4ax$.
Parametric equations:

$$x = at^2,$$
$$y = 2at.$$

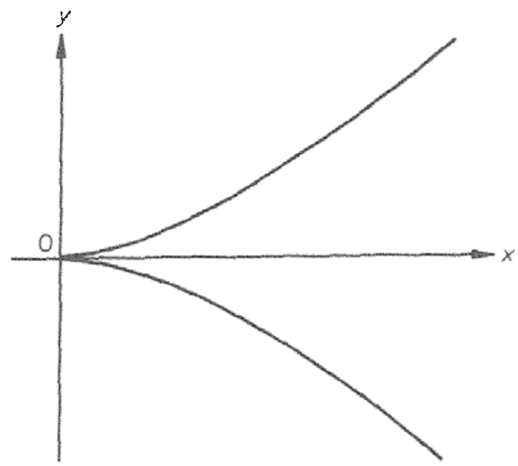
This parabola has vertex O and is symmetrical about the x-axis.



The rectangular hyperbola $xy = c^2$.
Parametric equations:

$$x = ct,$$
$$y = c/t.$$

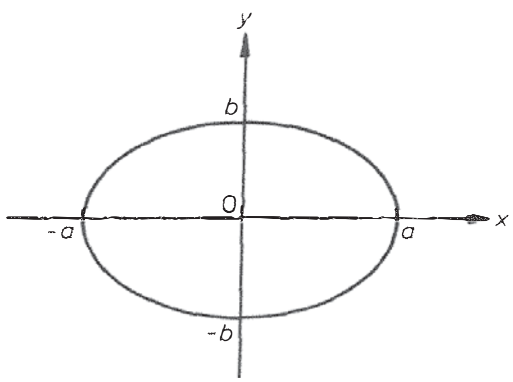
The x- and y-axes are asymptotes to the curve and the curve is symmetrical about the lines $y = \pm x$.



The semi-cubical parabola $ay^2 = x^3$.
Parametric equations:

$$x = at^2,$$
$$y = at^3.$$

The curve has a *cusp* at O and is symmetrical about the x-axis.



The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.
Parametric equations:

$$x = a\cos\theta,$$
$$y = b\sin\theta.$$

The curve is symmetrical about the x- and y-axes.

Example 1 Find the equation of the chord of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ joining the points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$.

$$\text{Gradient of } PQ = \frac{b\sin\theta - b\sin\phi}{a\cos\theta - a\cos\phi}$$

$$= \frac{2b\cos\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\theta-\phi)}{-2a\sin\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\theta-\phi)} = -\frac{b\cos\frac{1}{2}(\theta+\phi)}{a\sin\frac{1}{2}(\theta+\phi)}$$

Thus the equation of the chord PQ is

$$y - b\sin\theta = -\frac{b\cos\frac{1}{2}(\theta+\phi)}{a\sin\frac{1}{2}(\theta+\phi)}(x - a\cos\theta)$$

i.e. $ay\sin\frac{1}{2}(\theta+\phi) - ab\sin\theta\sin\frac{1}{2}(\theta+\phi)$

$$= -bx\cos\tfrac{1}{2}(\theta+\phi) + ab\cos\theta\cos\tfrac{1}{2}(\theta+\phi)$$

$\therefore \quad bx\cos\frac{1}{2}(\theta+\phi) + ay\sin\frac{1}{2}(\theta+\phi)$

$$= ab\{\cos\theta\cos\tfrac{1}{2}(\theta+\phi) + \sin\theta\sin\tfrac{1}{2}(\theta+\phi)\}$$
$$= ab\cos\{\theta - \tfrac{1}{2}(\theta+\phi)\} = ab\cos\tfrac{1}{2}(\theta-\phi).$$

Hence the equation of the chord PQ may be written:

$$\frac{x\cos\frac{1}{2}(\theta+\phi)}{a} + \frac{y\sin\frac{1}{2}(\theta+\phi)}{b} = \cos\tfrac{1}{2}(\theta-\phi).$$

Example 2 Find the equations of the tangent and the normal to the curve $xy = c^2$ at the point $P(ct, c/t)$. Given that the normal at P meets the curve again at Q, find the coordinates of Q. If the tangent at P meets the y-axis at R, find the equation of the locus of the mid-point M of PR.

The parametric equations of the curve are

$$x = ct, \qquad y = \frac{c}{t}$$

Differentiating with respect to t we have

$$\frac{dx}{dt} = c, \qquad \frac{dy}{dt} = -\frac{c}{t^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt}\bigg/\frac{dx}{dt} = \left(-\frac{c}{t^2}\right)\bigg/c = -\frac{1}{t^2}$$

Hence the gradients of the tangent and the normal to the curve at P are $-1/t^2$ and t^2 respectively.

$\therefore$ the equation of the tangent at P is

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

i.e. $\qquad t^2 y - ct = -x + ct$

i.e. $\qquad x + t^2 y = 2ct \qquad (1)$

The equation of the normal at P is

$$y - \frac{c}{t} = t^2(x - ct)$$

i.e. $\qquad ty - c = t^3 x - ct^4$

i.e. $\qquad t^3 x - ty = c(t^4 - 1) \qquad (2)$

Let $(cu, c/u)$ be a point which lies on the curve $xy = c^2$ and on the normal at P, then from equation (2)

$$t^3 \times cu - t \times \frac{c}{u} = c(t^4 - 1)$$

$\therefore \quad t^3u^2 + (1 - t^4)u - t = 0$

$\therefore \quad (u - t)(t^3 u + 1) = 0$

$\therefore$ either $u = t$ or $u = -1/t^3$

Since the solution $u = t$ gives the point P, the remaining solution $u = -1/t^3$ must give the point Q.

Hence the coordinates of Q are $(-c/t^3, -ct^3)$.

Substituting $x = 0$ in equation (1) we have,

$$t^2 y = 2ct \quad \text{i.e.} \quad y = 2c/t$$

$\therefore$ the coordinates of the point R are $(0, 2c/t)$.

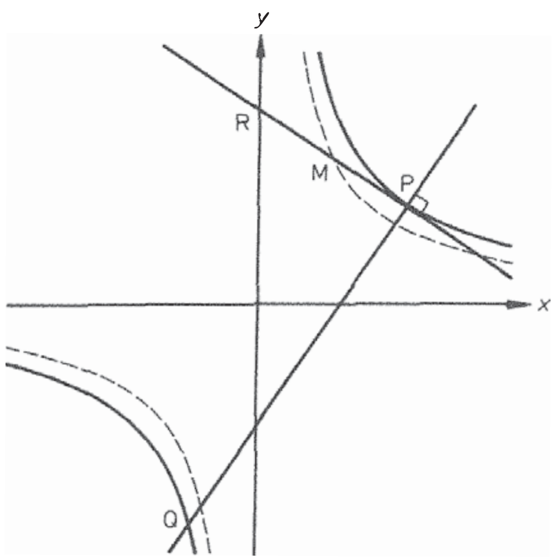
Thus the coordinates of the mid-point M of PR are

$$\left(\frac{1}{2}\{ct+0\}, \frac{1}{2}\left\{\frac{c}{t}+\frac{2c}{t}\right\}\right) \quad \text{i.e.} \quad \left(\frac{1}{2}ct, \frac{3c}{2t}\right)$$

Hence as t varies the parametric equations of the locus of M are

$$x = \frac{1}{2}ct, \qquad y = \frac{3c}{2t}$$

Eliminating t from these equations we obtain the Cartesian equation of the locus, namely $xy = \frac{3}{4}c^2$.



As shown in the diagram the locus of M is another rectangular hyperbola.

Exercise 13.3

1. Write down parametric coordinates for a point on each of the following curves and sketch the curves.
(a) $y^2 = 12x$, (b) $xy = 9$, (c) $y^2 = x^3$,
(d) $4xy = 25$, (e) $4y^2 = x^3$, (f) $y^2 = 10x$.

2. Write down parametric coordinates for a point on each of the following curves and sketch the curves.
(a) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$, (b) $x^2 + y^2 = 9$, (c) $\dfrac{x^2}{5} + y^2 = 1$,
(d) $4x^2 + 9y^2 = 4$, (e) $9x^2 + y^2 = 9$, (f) $x^2 + (y-1)^2 = 25$.

3. Find the equations of the tangent and normal to the curve $xy = 16$ at the point $(4t, 4/t)$.

4. Find the equations of the tangent and normal to the curve $x = t$, $y = 1/t$ at the point where $t = 2$.

5. Find the equations of the tangent and normal to the curve $x = 2\cos\theta$, $y = \sin\theta$ at the point where $\theta = \frac{1}{3}\pi$.

6. P is the point $(5\cos\theta, 4\sin\theta)$ on the curve $x^2/25 + y^2/16 = 1$. Given that S and S' are the points with coordinates $(-3,0)$ and $(3,0)$, show that the value of $PS + PS'$ is independent of θ.

7. Find the equations of the tangents to the curve $xy = c^2$ which have gradient -4 and write down the coordinates of their points of contact.

8. The points $P(cp, c/p)$, $Q(cq, c/q)$ and $R(cr, c/r)$ lie on the curve $xy = c^2$. Show that if $\angle RPQ$ is a right angle, then $p^2qr = -1$. Prove further that QR is perpendicular to the tangent at P to the curve.

9. The parametric equations of a curve are $x = 2t$, $y = 2/t$. The tangent to the curve at $P(2t, 2/t)$ meets the x-axis at A; the normal to the curve at P meets the y-axis at B. The point Q divides AB in the ratio 3:1. Find the parametric equations of the locus of Q as P moves on the given curve. (JMB)

10. Let P and Q be points on the rectangular hyperbola $xy = a^2$, with coordinates $(ap, a/p)$ and $(aq, a/q)$. Obtain the equation of the line PQ, and deduce, or obtain otherwise, the equation of the tangent at P to the hyperbola. Show that, if the line PQ meets the coordinate axes at A and B, then the mid-point M of AB is the mid-point of PQ also. Obtain the coordinates of the point T where the tangents at P and Q meet, and show that the line MT passes through the origin. (W)

11. Prove that the normal to the hyperbola $xy = c^2$ at the point $P(ct, c/t)$ has equation $y = t^2x + \dfrac{c}{t} - ct^3$. If the normal at P meets the line $y = x$ at N, and O is the origin, show that $OP = PN$ provided that $t^2 \neq 1$. The tangent to the hyperbola at P meets the line $y = x$ at T. Prove that $OT \,.\, ON = 4c^2$. (O&C)

12. The normal at the point $P(ct, c/t)$ on the rectangular hyperbola $xy = c^2$ meets the curve again at Q. If the normal meets the x-axis at A and the y-axis at B, show that the mid-point of AB is also the mid-point of PQ.

13. Show that the tangent at $P(cp, c/p)$ to the rectangular hyperbola $xy = c^2$ has the equation $p^2y + x = 2cp$. The perpendicular from the origin to this tangent meets it at N, and meets the hyperbola again at Q and R. Prove that (i) the angle QPR is a right angle, (ii) as p varies, the point N lies on the curve whose equation is $(x^2 + y^2)^2 = 4c^2xy$. (C)

14. The tangents at the points $P(cp, c/p)$ and $Q(cq, c/q)$ on the rectangular hyperbolar $xy = c^2$ intersect at the point R. Given that R lies on the rectangular hyperbola $xy = \frac{1}{2}c^2$, find the equation of the locus of the mid-point M of PQ as p and q vary.

15. Find the equation of the tangent to the curve $ay^2 = x^3$ at the point (at^2, at^3) and prove that, apart from one exceptional case, the tangent meets the curve again. Find the coordinates of the point of intersection. What is the exceptional case? (O&C)

16. The tangents to the curve $9y^2 = x^3$ at the points $P(9p^2, 9p^3)$ and $Q(9q^2, 9q^3)$ intersect at the point R. Find the coordinates of R. If p and q vary in such a way that $\angle PRQ$ is always a right angle, find the equation of the locus of R.

17. Show that the equation of the tangent to the curve $x = a\cos t$, $y = b\sin t$ at the point $P(a\cos p, b\sin p)$ is $\frac{x}{a}\cos p + \frac{y}{b}\sin p = 1$. This tangent meets the curve $x = 2a\cos\theta$, $y = 2b\sin\theta$ at the points Q and R, which are given by $\theta = q$ and $\theta = r$ respectively. Show that p differs from each of q and r by $\pi/3$. (JMB)

18. Show that if the tangents to the ellipse $x^2/a + y^2/b = 1$ at the points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ intersect at the point R, then the coordinates of R are $\left(\frac{a\cos\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}, \frac{b\sin\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}\right)$. If P and Q move on the ellipse in such a way that $\phi = \theta + \frac{1}{2}\pi$, find the equation of the locus of R.

19. Find the equation of the normal to the curve $x^2 - y^2 = 1$ at the point $P(\sec\theta, \tan\theta)$, where $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$. If this normal cuts the x-axis at A and the y-axis at B, show that P is the centre of the circle which passes through A, B and the origin O. If the line OP cuts this circle again at the point Q, find the equation of the locus of Q as θ varies.

20. (a) The points $A(d,0)$, $B(-d,0)$ and $P(d\cos\theta, d\sin\theta)$ lie on the circle $x^2 + y^2 = d^2$. Show that the equation of the tangent to the circle at P is $x\cos\theta + y\sin\theta = d$. Show that the sum of the perpendicular distances from A and B to this tangent is independent of θ, and calculate the product of these distances.
(b) The point $P(p, p^3)$ lies on the curve $y = x^3$. Show that the equation of the tangent to the curve at P is $y - 3p^2x + 2p^3 = 0$. Show that when $p = 3$ this tangent passes through the point $A(7/3, 9)$, and find the other two values of p for which the tangent passes through A. (W)

21. A curve has the parametric equations $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, where $0 \leqslant \theta \leqslant \pi$. A point P of the curve has parameter ϕ, where $\phi \neq 0$.

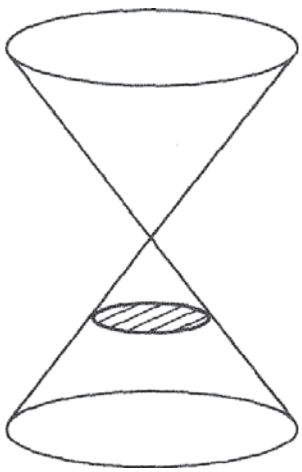
(i) Show that, at P, $\frac{dy}{dx} = \frac{\sin\phi}{1 - \cos\phi}$.

(ii) The normal to the curve at P meets the x-axis at G, and O is the origin. Show that $OG = a\phi$.

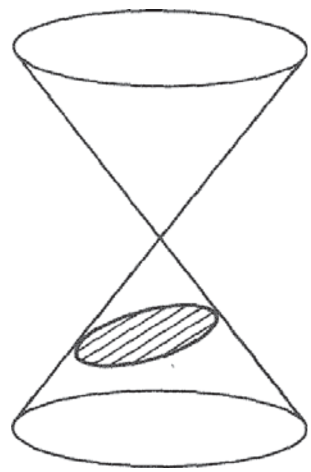
(iii) The tangent to the curve at P meets at K the line through G parallel to the y-axis. Show that $GK = 2a$. (C)

13.4 Conic sections

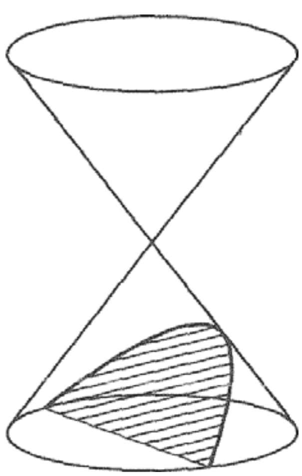
Some of the curves we have been studying arise naturally as cross-sections of a circular cone. The following diagrams show the sections produced when a double cone of semi-vertical angle θ is cut by a plane which does not pass through the vertex. The nature of the curve obtained in each case is determined by the value of the angle α between the plane and the axis of the cone, where $0 \leqslant \alpha \leqslant \frac{1}{2}\pi$.



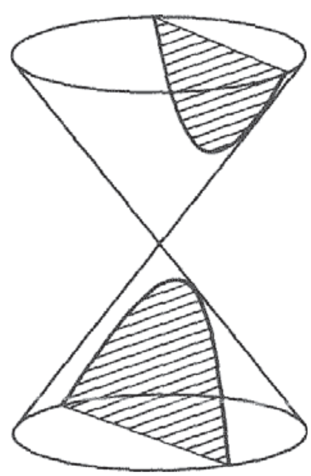
Circle, $\alpha = \frac{1}{2}\pi$



Ellipse, $\alpha > \theta$



Parabola, $\alpha = \theta$

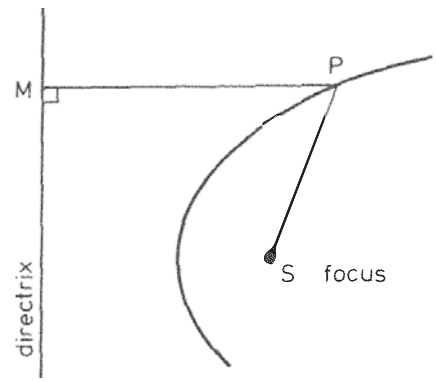


Hyperbola, $\alpha < \theta$

The complete set of conics includes the sections produced by planes through the vertex of the double cone, namely:

for $\alpha < \theta$, a pair of straight lines;
for $\alpha = \theta$, a single straight line;
for $\alpha > \theta$, a single point.

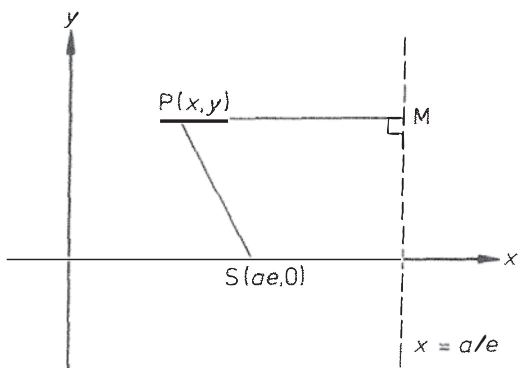
It can be shown that the definition of a parabola as a conic section is consistent with the locus definition used in Book 1. We now consider the corresponding locus definitions of the ellipse and hyperbola.



Suppose that P is a point which moves so that its distance from a fixed point S, the *focus*, is a constant multiple of its distance from a fixed line called the *directrix*. If M is the foot of the perpendicular from P to the directrix then

$$PS = ePM$$

where e is a positive constant called the *eccentricity*. The nature of the locus of P depends on the value taken by e, as follows:



for an ellipse, $0 < e < 1$;
for a parabola, $e = 1$;
for a hyperbola, $e > 1$.

The standard form of the equation of the ellipse is obtained by taking the point $(ae, 0)$ as focus and the line $x = a/e$ as directrix.

$$PS^2 = e^2PM^2$$

$$\therefore \quad (x - ae)^2 + y^2 = e^2\left(\frac{a}{e} - x\right)^2$$

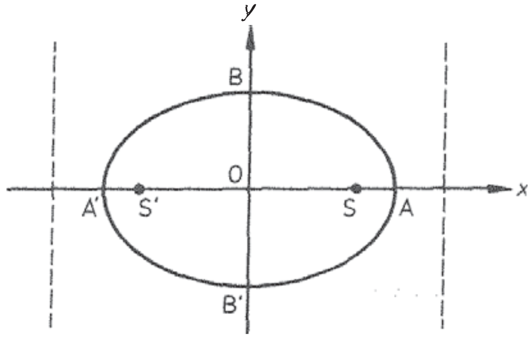
$$x^2 - 2aex + a^2e^2 + y^2 = a^2 - 2aex + e^2x^2$$

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

i.e.
$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

Hence the equation of the ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{where} \quad b^2 = a^2(1 - e^2).$$



From this equation we see that the ellipse is symmetrical about both x- and y-axes. It cuts the x-axis at the points $A(a, 0)$ and $A'(-a, 0)$. It cuts the y-axis at the points $B(b, 0)$ and $B'(-b, 0)$. AA' is called the *major axis* and BB' the *minor axis*. The origin O is the *centre* of the ellipse and any chord through O is a *diameter*. From symmetry of the curve it is clear that the same locus would have been produced using the point $S'(-ae, 0)$ as focus and the line $x = -a/e$ as directrix. Thus the ellipse is said to have foci $(\pm ae, 0)$ and directrices $x = \pm ae$.

[Note that in terms of a and b the coordinates of the foci are $(\pm\sqrt{\{a^2 - b^2\}}, 0)$ and the equations of the directrices are $x = \pm a^2/\sqrt{(a^2 - b^2)}$.]

Example 1 Find the eccentricity, the foci and the directrices of the ellipse $\dfrac{x^2}{9} + y^2 = 1$.

For the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, we have $b^2 = a^2(1 - e^2)$.

Substituting $a = 3$, $b = 1$ we find that for the given ellipse,

$$
\begin{aligned}
1 &= 9(1 - e^2) \\
\therefore \quad 1 &= 9 - 9e^2 \\
\therefore \quad e^2 &= 8/9
\end{aligned}
$$

Hence the eccentricity of the ellipse is $\frac{2}{3}\sqrt{2}$. Thus the foci are the points $(\pm 2\sqrt{2}, 0)$ and the directrices are the lines $x = \pm\frac{9}{4}\sqrt{2}$.

The standard form of the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \text{where} \quad b^2 = a^2(e^2 - 1).$$

Using the methods applied to the ellipse it can be shown that this hyperbola has foci $(\pm ae, 0)$ and directrices $x = \pm a/e$.

From the equation we see that the hyperbola is also symmetrical about both x- and y-axes. It cuts the x-axis at the points $A(a, 0)$ and $A'(-a, 0)$, but does not cut the y-axis.

Rearranging the equation we have:

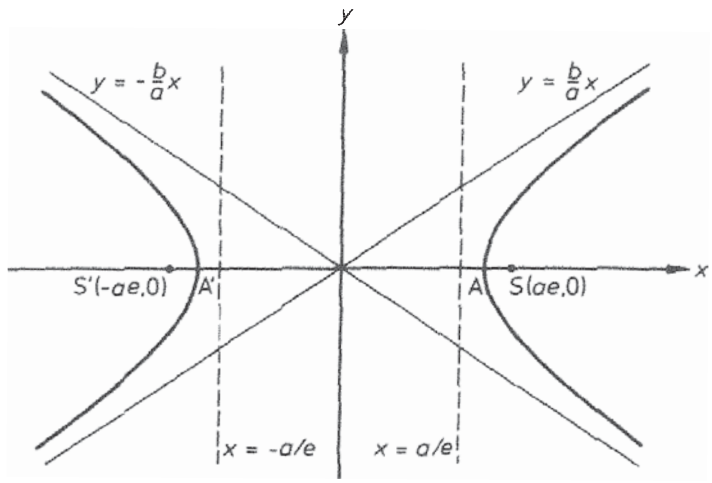
$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

$$\therefore \quad \frac{y^2}{x^2} = \frac{b^2}{a^2} - \frac{b^2}{x^2}$$

$$\therefore \quad \text{as } |x| \to \infty, \quad \frac{y^2}{x^2} \to \frac{b^2}{a^2} \quad \text{and} \quad \frac{y}{x} \to \pm\frac{b}{a}$$

i.e. as $|x|$ increases the curve approaches the lines $y = \pm\dfrac{b}{a}x$.

Hence the lines $y = \pm\dfrac{b}{a}x$ are asymptotes to the hyperbola.



A hyperbola with perpendicular asymptotes is called a *rectangular hyperbola.* Since the lines $y = \pm\frac{b}{a}x$ are perpendicular when $a = b$, the equation of a rectangular hyperbola is of the form

$$x^2 - y^2 = a^2.$$

When the asymptotes are used as x- and y-axes the equation of the rectangular hyperbola takes the more familiar form $xy = c^2$.

Exercise 13.4

1. Find the eccentricities, the foci and the directrices of the following ellipses.

(a) $\frac{x^2}{25} + \frac{y^2}{9} = 1$, (b) $16x^2 + 25y^2 = 100$, (c) $\frac{x^2}{4} + y^2 = 1$,

(d) $\frac{x^2}{9} + \frac{y^2}{5} = 1$, (e) $\frac{x^2}{4} + \frac{y^2}{9} = 1$, (f) $4x^2 + y^2 = 1$.

2. Find the eccentricities, the foci, the directrices and the asymptotes of the following hyperbolas.

(a) $\frac{x^2}{16} - \frac{y^2}{9} = 1$, (b) $x^2 - y^2 = 4$, (c) $x^2 - 4y^2 = 4$

3. Find the eccentricities of the following ellipses

(a) $x = 2\cos\theta$, $y = \sin\theta$, (b) $x = 5\cos\theta$, $y = 3\sin\theta$,
(c) $x = 2\cos\theta$, $y = 3\sin\theta$, (d) $x = 5\sin\theta$, $y = 3\cos\theta$.

4. Find in the form $x^2/a^2 + y^2/b^2 = 1$ the equations of the ellipses with (a) eccentricity 1/2, foci $(\pm 2, 0)$, (b) eccentricity 3/5, foci $(\pm 9, 0)$.

5. Use the locus definition of the ellipse to find the equation of the ellipse with eccentricity $\frac{2}{3}$, focus (2, 1) and directrix $x = -\frac{1}{2}$.

6. Find the equations of the tangent and the normal to each of the following ellipses at the given point:

(a) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$; (3, 0), (b) $\dfrac{x^2}{8} + \dfrac{y^2}{2} = 1$; (−2, 1),

(c) $9x^2 + 16y^2 = 40$; $(2, \frac{1}{2})$, (d) $4x^2 + 5y^2 = 120$; (−5, −2).

7. Find the equations of the tangents at the point (x_1, y_1) to the following curves:

(a) $4x^2 + 9y^2 = 36$, (b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, (c) $\dfrac{x^2}{3} - \dfrac{y^2}{2} = 1$.

8. Use the locus definition of the hyperbola to find the equation of the hyperbola with eccentricity $\sqrt{2}$, focus $(2k, 2k)$ and directrix $x + y = k$.

9. The line $y = mx + c$ is a tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$, $(a > b > 0)$. Show that $c^2 = a^2m^2 + b^2$. The perpendicular distances from the points $(\sqrt{(a^2 - b^2)}, 0)$ and $(-\sqrt{(a^2 - b^2)}, 0)$ to any tangent to the ellipse are p_1, p_2. Show that $p_1p_2 = b^2$. (C)

10. A line with gradient m is drawn through the fixed point $C(h, 0)$, where $0 < h < a$, to meet the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the points P and Q. Prove that the mid-point R of PQ has the coordinates $\left(\dfrac{a^2hm^2}{a^2m^2 + b^2}, \dfrac{-b^2hm}{a^2m^2 + b^2}\right)$. Show that, as m varies, R always lies on the curve whose equation is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{hx}{a^2}$. (C)

11. The ellipse $x^2/a^2 + y^2/b^2 = 1$ intersects the positive x-axis at A and the positive y-axis at B. Determine the equation of the perpendicular bisector of AB. (i) Given that this line intersects the x-axis at P and that M is the mid-point of AB, prove that the area of triangle PMA is $b(a^2 + b^2)/8a$. (ii) If $a^2 = 3b^2$, find, in terms of b, the coordinates of the points where the perpendicular bisector of AB intersects the ellipse. (JMB)

12. Prove that the hyperbola H_1, with equation $x^2 - y^2 = a^2$, cuts the hyperbola H_2, with equation $xy = c^2$, at right angles at two points P and Q. If the distance between the tangents to H_1 at P and Q is equal to the distance between the tangents to H_2 at P and Q, find the relation between a and c. (O)

Exercise 13.5 (miscellaneous)

1. A straight line parallel to the line $2x + y = 0$ intersects the x-axis at A and the y-axis at B. The perpendicular bisector of AB cuts the y-axis at C. Prove that the gradient of the line AC is $-\frac{3}{4}$. Find also the tangent of the acute angle between the line AC and the bisector of the angle AOB, where O is the origin. (JMB)

2. The triangle ABC has vertices $A(0, 12)$, $B(-9, 0)$, $C(16, 0)$. Find the equations of the internal bisectors of the angles ABC and ACB. Hence, or otherwise, find the equation of the inscribed circle of the triangle ABC. Find also the equation of the circle passing through A, B and C. (C)

3. Write down the perpendicular distance from the point (a, a) to the line $4x - 3y + 4 = 0$. The circle, with centre (a, a) and radius a, touches the line $4x - 3y + 4 = 0$ at the point P. Find a, and the equation of the normal to the circle at P. Show that P is the point $(1/5, 8/5)$. Show that the equation of the circle which has centre P and which passes through the origin is $5(x^2 + y^2) - 2x - 16y = 0$. (L)

4. Find the centre and the radius of the circle C which passes through the points $(4, 2)$, $(2, 4)$ and $(2, 6)$. If the line $y = mx$ is a tangent to C, obtain the quadratic equation satisfied by m. Hence or otherwise find the equations of the tangents to C which pass through the origin O. Find also (i) the angle between the two tangents, (ii) the equation of the circle which is the reflection of C in the line $y = 3x$. (AEB 1977)

5. Show that the equation of the tangent to the curve $y^2 = 4ax$ at the point $(at^2, 2at)$ is $ty = x + at^2$ and the equation of the tangent to the curve $x^2 = 4by$ at the point $(2bp, bp^2)$ is $y = px - bp^2$. The curves $y^2 = 32x$ and $x^2 = 4y$ intersect at the origin and at A. Find the equation of the common tangent to these curves and the coordinates of the points of contact B and C between the tangent and the curves. Calculate the area of the triangle ABC. (AEB 1976)

6. On the same diagram sketch the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 10x = 0$. The line $ax + by + 1 = 0$ is a tangent to both these circles. State the distances of the centres of the circles from this tangent. Hence, or otherwise, find the possible values of a and b and show that, if 2ϕ is the angle between the common tangents, then $\tan\phi = \frac{3}{4}$. (L)

7. Find the centre and radius of each of the circles C_1 and C_2 whose equations are $x^2 + y^2 - 16y + 32 = 0$ and $x^2 + y^2 - 18x + 2y + 32 = 0$ respectively and show that the circles touch externally. Find the coordinates of their point of contact and show that the common tangent at that point passes through the origin. The other tangents from the origin, one to each circle, are drawn. Find, correct to the nearest degree, the angle between these tangents. (SU)

8. Two circles, C_1 and C_2, have equations $x^2 + y^2 - 4x - 8y - 5 = 0$ and $x^2 + y^2 - 6x - 10y + 9 = 0$, respectively. Find the x-coordinates of the points P and Q at which the line $y = 0$ cuts C_1, and show that this line touches C_2. Find the tangent of the acute angle made by the line $y = 0$ with the tangents to C_1 at P and Q. Show that, for all values of the constant λ, the circle C_3 whose equation is $\lambda(x^2 + y^2 - 4x - 8y - 5) + x^2 + y^2 - 6x - 10y + 9 = 0$ passes through the points of intersection of C_1 and C_2. Find the two possible values of λ for which the line $y = 0$ is a tangent to C_3. (JMB)

9. The circles whose equations are
$$x^2 + y^2 - x + 6y + 7 = 0$$
$$\text{and} \quad x^2 + y^2 + 2x + 2y - 2 = 0$$
intersect at the points A and B. Find (i) the equation of the line AB, (ii) the coordinates of A and B. Show that the two given circles intersect at right angles and obtain the equation of the circle which passes through A and B and which also passes through the centres of the two circles. (AEB 1978)

10. A curve C_1 has equation $2y^2 = x$; a curve C_2 is given by the parametric equations $x = 4t$, $y = 4/t$. Sketch C_1 and C_2 on the same diagram and calculate the coordinates of their point of intersection, P. The tangent to C_1 at P crosses the x-axis at T and meets C_2 again at Q. (a) Show that T is a point of trisection of PQ. (b) Find the area of the finite region bounded by the x-axis, the line PT and the curve C_1. (L)

11. Show that the tangent at the point P, with parameter t, on the curve $x = ct$, $y = c/t$ has equation $x + t^2y = 2ct$. This tangent meets the x-axis in a point Q and the line through P parallel to the x-axis cuts the y-axis in a point R. Show that, for any position of P on the curve, QR is a tangent to the curve with parametric equations $x = ct$, $y = c/(2t)$. (L)

12. Prove that the equation of the normal to the rectangular hyperbola $xy = c^2$ at the point $P(ct, c/t)$ is $ty - t^3x = c(1 - t^4)$. The normal at P and the normal at the point $Q(c/t, ct)$, where $t > 1$, intersect at the point N. Show that $OPNQ$ is a rhombus, where O is the origin. Hence, or otherwise, find the coordinates of N. If the tangents to the hyperbola at P and Q intersect at T, prove that the product of the lengths of OT and ON is independent of t. (JMB)

13. The line of gradient $m(\neq 0)$ through the point $A(a, O)$ is a tangent to the rectangular hyperbola $xy = c^2$ at the point P. Find m in terms of a and c, and show that the coordinates of P are $(\frac{1}{2}a, 2c^2/a)$. The line through A parallel to the y-axis meets the hyperbola at Q, and the line joining Q to the origin O intersects AP at R. Given that OQ and AP are perpendicular to each other, find the numerical value of c^2/a^2 and the numerical value of the ratio $AR:RP$. (JMB)

14. The point P in the first quadrant lies on the curve with parametric equations $x = t^2$, $y = t^3$. The tangent to the curve at P meets the curve again at Q and is normal to the curve at Q. Find the coordinates of P and of Q. (L)

15. Sketch the parabola whose parametric equations are $x = t^2$, $y = 2t$ and on the same diagram sketch the curve with parametric equations $x = 10(1 + \cos\theta)$, $y = 10\sin\theta$. These curves touch at the origin and meet again at two other points A and B. The normals at A and B to the parabola meet at P and the tangents to the other curve at A and B meet at Q. Calculate the length of PQ. (L)

16. Obtain an equation of the tangent, at the point with parameter t, to the curve $\mathscr{C}$ whose parametric equations are given by

$$x = 2\sin^3 t, \quad y = 2\cos^3 t, \quad 0 \leqslant t \leqslant \pi/2.$$

Show that, if the tangent meets the coordinate axes in points R and S, then RS is of constant length. Sketch the curve $\mathscr{C}$. Find the area of the finite region enclosed by the curve $\mathscr{C}$ and the coordinate axes. (L)

17. Sketch the curve whose parametric equations are $x = a\cos\phi$, $y = b\sin\phi$, $0 \leqslant \phi \leqslant 2\pi$, where a and b are positive constants. The point P is given by $\phi = \pi/4$. Find (a) the equation of the tangent to the curve at P, (b) the equation of the normal to the curve at P. By evaluating a suitable integral, calculate the area of the region in the first quadrant between the curve and the coordinate axes. Hence deduce the area of the region enclosed by the curve. (L)

18. Prove that the equation of the tangent at the point (x_1, y_1) on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$. The tangent at the point $(2\cos\theta, \sqrt{3}\sin\theta)$ on the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$ passes through the point $P(2, 1)$. Show that $\sqrt{3}\cos\theta + \sin\theta = \sqrt{3}$. Without using tables, calculator or slide rule, find all the solutions of this equation which are in the range $0° \leqslant \theta < 360°$. Hence obtain the coordinates of the points of contact, Q and R, of the tangents to the ellipse from P. Verify that the line through the origin and the point P passes through the mid-point of the line QR. (JMB)

19. Prove that the equation of the normal at $(\alpha\cos\phi, \beta\sin\phi)$ to the ellipse $\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{\beta^2} = 1$ is $\alpha x \sec\phi - \beta y \operatorname{cosec}\phi = \alpha^2 - \beta^2$. P is the point $(a\cos\theta, b\sin\theta)$ on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. M and N are the feet of the perpendiculars from P to the axes. Find the equation of MN. Prove that, for variable θ, MN is always normal to a fixed concentric ellipse and find the equation of this ellipse. (O&C)

20. Prove that the equation of the tangent to the curve $xy^2 = c^3$ $(c > 0)$ at the point $P(ct^2, c/t)$ is $2t^3y + x = 3ct^2$. Prove that the parameters of the points of contact of the tangents which pass through the point $Q(h, k)$ $(k \neq 0)$ satisfy a cubic equation and, by considering the turning values of this cubic, or otherwise, prove that there are three distinct tangents to the curve which pass through Q if $hk^2 < c^3$

and $h > 0$. State a necessary and sufficient condition on the parameters t_1, t_2, t_3 for the tangents at the correspon~~ding points to~~ be concurrent.

When Q lies on the curve, the tangent at Q cuts the curve again at R; if the parameter of Q is t, determine the parameter of R. Prove that, if the tangents Q_1, Q_2, Q_3 are concurrent and cut the curve again at R_1, R_2, R_3 then the tangents at R_1, R_2, R_3 are concurrent. (O&C)