# VECTORS

A vector is a quantity with both magnitude and direction unlike a scalar which has got only magnitude.

Examples of vector quantities: Displacement, Velocity, Acceleration, Momentum, Force

Examples of scalar quantities: Distance, speed, mass, time, temperature etc.

## BASIC CONCEPTS USED IN VECTORS

1. Position vector.

If a point A in a two dimensional geometry has Cartesian coordinates $(x, y)$, the position vector of A is given by $\underset{\sim}{OA} = \underset{\sim}{a} = \begin{pmatrix} x \\ y \end{pmatrix}$

or $\underset{\sim}{OA} = a = x\underset{\sim}{i} + y\underset{\sim}{j}$

If A has coordinates $(x, y, z)$ in a three dimensional geometry, its position vector is given by

$\underset{\sim}{OA} = \underset{\sim}{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ OR $\underset{\sim}{OA} = a = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

2. Displacement Vector

If points A and B have coordinates $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ respectively, the displacement vector denoted by $\overrightarrow{AB}$ is defined as

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$
$$= \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$$

3 Modulus of a vector

Modulus of a vector is the same as the magnitude of the vector.

If $\vec{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$, where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit

Vectors in the direction of x, y-, and z-axes respectively, then the magnitude of $\vec{r}$ is given by

$$|\vec{r}| = \sqrt{x^2+y^2+z^2}$$

Example:

Given $\underset{\sim}{r} = 3\underset{\sim}{i} - 4\underset{\sim}{j} + 12\underset{\sim}{k}$, find $|\underset{\sim}{r}|$

Soln

$$|\underset{\sim}{r}| = \sqrt{(3)^2 + (-4)^2 + (12)^2}$$
$$= \sqrt{169}$$
$$= 13.$$

4 The unit vector parallel to a given vector

Given $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$, the unit vector in the direction of $\vec{r}$ is defined by

$$\hat{\underset{\sim}{r}} = \frac{\vec{r}}{|\underset{\sim}{r}|}$$

$$\hat{\underset{\sim}{r}} = \frac{x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}}{\sqrt{x^2+y^2+z^2}}$$

Example:

If $\underset{\sim}{r} = 4\underset{\sim}{i} + 3\underset{\sim}{j} + 12k$, find the unit vector parallel to $\vec{r}$.

Soln.

$$|\underset{\sim}{r}| = \sqrt{(4)^2 + (3)^2 + 12^2} = 13$$

$$\hat{\underset{\sim}{r}} = \frac{\underset{\sim}{r}}{|\underset{\sim}{r}|}$$

$$= \frac{1}{13}(4\underset{\sim}{i} + 3\underset{\sim}{j} + 12k)$$

$$\hat{\underset{\sim}{r}} = \frac{4}{13}\underset{\sim}{i} + \frac{3}{13}\underset{\sim}{j} + \frac{12}{13}\underset{\sim}{k}$$

5 Multiplication of a vector by a scalar

for any scalar $\lambda$, the vector $\lambda\underset{\sim}{a}$ is parallel to $\underset{\sim}{a}$ and has a magnitude $|\lambda|$ times the magnitude of $\underset{\sim}{a}$

Illustration:

Let $\underset{\sim}{a} = 3\underset{\sim}{i} + 4\underset{\sim}{j} - 7\underset{\sim}{k}$ and $\lambda = 2$

$$\lambda a = 2(3\underset{\sim}{i} + 4\underset{\sim}{j} - 7\underset{\sim}{k}) = 6\underset{\sim}{i} + 8\underset{\sim}{j} - 14\underset{\sim}{k}$$

$|\underset{\sim}{a}| = \sqrt{3^2+4^2+(-7)^2} = \sqrt{74}$

$|\lambda \underset{\sim}{a}| = \sqrt{6^2+8^2+(-14)^2} = \sqrt{4\times74} = 2\sqrt{74} = |\lambda||\underset{\sim}{a}|$

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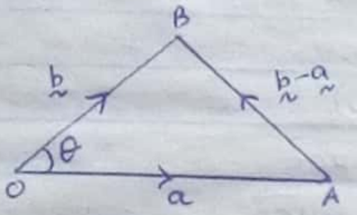
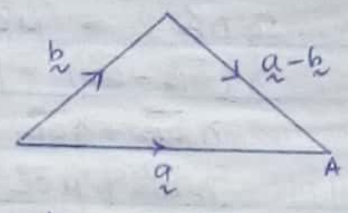



If the position vectors of points A and B with respect to the origin O are $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively, then $\overrightarrow{OA} = a$, $\overrightarrow{OB} = \underset{\sim}{b}$ and

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \underset{\sim}{b} - \underset{\sim}{a}$

$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \underset{\sim}{a} - \underset{\sim}{b}$

(i) the distance between A and B is given by
$|\overrightarrow{AB}| = |\underset{\sim}{b} - \underset{\sim}{a}|$ or $|\overrightarrow{BA}| = |\underset{\sim}{a} - \underset{\sim}{b}|$

(ii) If θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ then by using cosine rule

$|\overrightarrow{AB}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 - 2|\overrightarrow{OA}||\overrightarrow{OB}|\cos\theta$

$|\underset{\sim}{b} - \underset{\sim}{a}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$

(iii) A point C with position vector c lies on the straight line passing through A and B if and only if there exists a scalar λ such that $\overrightarrow{AC} = \lambda\overrightarrow{AB}$

ie $\underset{\sim}{c} - \underset{\sim}{a} = \lambda(\underset{\sim}{b} - \underset{\sim}{a})$

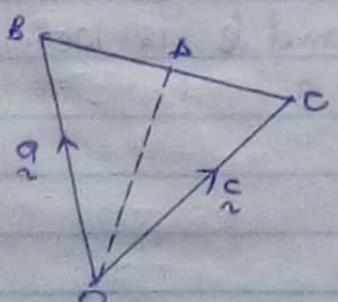
## THE RATIO THEOREM.

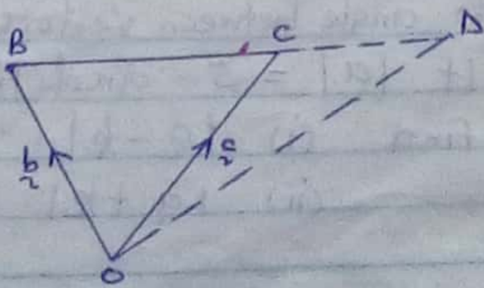
It states that if λ and μ are scalar quantities then;

$\lambda\overrightarrow{OB} + \mu\overrightarrow{OC} = (\lambda+\mu)\overrightarrow{OD}$

or $\overrightarrow{OD} = \frac{\lambda}{\lambda+\mu}\overrightarrow{OB} + \frac{\mu}{\lambda+\mu}\overrightarrow{OC}$

where $\overrightarrow{CD} : \overrightarrow{DB}$ is in the ratio λ:μ



$$\overrightarrow{CD} : \overrightarrow{DB} = \lambda : \mu$$

$$\frac{\overrightarrow{CD}}{\overrightarrow{DB}} = \frac{\lambda}{\mu}$$

$$\lambda \overrightarrow{DB} = \mu \overrightarrow{CD}$$

$$\lambda(\overrightarrow{OB} - \overrightarrow{OD}) = \mu(\overrightarrow{OD} - \overrightarrow{OC})$$

$$\lambda \overrightarrow{OB} - \lambda \overrightarrow{OD} = \mu \overrightarrow{OD} - \mu \overrightarrow{OC}$$

$$\lambda \overrightarrow{OB} + \mu \overrightarrow{OC} = \lambda \overrightarrow{OD} + \mu \overrightarrow{OD}$$

$$\lambda \overrightarrow{OB} + \mu \overrightarrow{OC} = (\lambda + \mu) \overrightarrow{OD}$$

$$\overrightarrow{OD} = \frac{\lambda}{\lambda+\mu} \overrightarrow{OB} + \frac{\mu}{\lambda+\mu} \overrightarrow{OC}.$$

Convention

(i) If $\lambda$ and $\mu$ have the same sign, then D lies between B and C (D divides line BC internally)

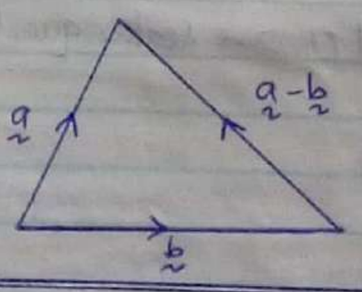
(ii) If $\lambda$ and $\mu$ have opposite signs, then D does not lie between B and C (D divides the line BC externally)

EXAMPLES

1. Find the angle between the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ given that $|\underset{\sim}{a}| = 3$, $|\underset{\sim}{b}| = 8$ and $|\underset{\sim}{a} - \underset{\sim}{b}| = 7$

Soln

Let $\theta$ be the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$




using cosine rule

$$|\underset{\sim}{a} - \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2|\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$$

$$7^2 = 3^2 + 8^2 - 2(3)(8)\cos\theta$$

$$48\cos\theta = 24$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = 60^\circ$$

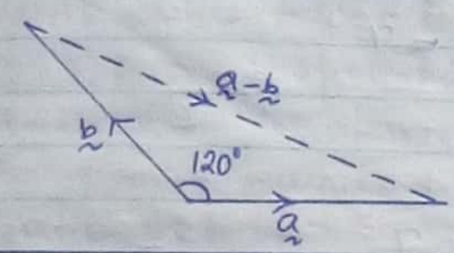
2. The angle between vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is $120^\circ$. If $|\underset{\sim}{a}| = 5$ and $|\underset{\sim}{b}| = 3$

Find (i) $|\underset{\sim}{a} - \underset{\sim}{b}|$

(ii) $|\underset{\sim}{a} + \underset{\sim}{b}|$

Sol.

(i)




using cosine rule

$|\underset{\sim}{a}-\underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2+|\underset{\sim}{b}| - 2|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$

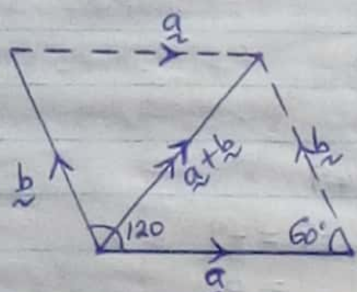
$|\underset{\sim}{a}-\underset{\sim}{b}|^2 = 5^2+3^2-2(5)(3)(-\frac{1}{2})$

$|\underset{\sim}{a}-\underset{\sim}{b}|^2 = 25+9+15$

$|\underset{\sim}{a}-\underset{\sim}{b}|^2 = 49$

$|a-b| = 7$ units

(ii)




$|\underset{\sim}{a}+\underset{\sim}{b}|^2 = |\underset{\sim}{a}^2| + |\underset{\sim}{b}|^2 - 2|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$

$= 5^2+3^2-2(5)(3)\cos 60$

$= 25+9-15$

$|\underset{\sim}{a}+\underset{\sim}{b}|^2 = 19$

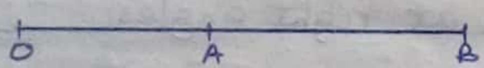
$|\underset{\sim}{a}+\underset{\sim}{b}| = \sqrt{19}$

3. The points A and B have position vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively relative to the origin. $\underset{\sim}{a} = 2\underset{\sim}{i}+\underset{\sim}{j}-3\underset{\sim}{k}$ and $\underset{\sim}{b} = -4\underset{\sim}{i}+p\underset{\sim}{j}+t\underset{\sim}{k}$. find the possible values of $p$ and $t$ if

a) the points O, A and B are collinear

(b) $|AB| = 7$, $p = 2t$

Soln.




a) If O, A and B are collinear

$\overrightarrow{OB} = \lambda\overrightarrow{OA}$

$\underset{\sim}{b} = \lambda\underset{\sim}{a}$

$\begin{pmatrix}-4\\p\\t\end{pmatrix} = \lambda\begin{pmatrix}2\\1\\-3\end{pmatrix}$

Equating corresponding Components

$2\lambda = -4$, $\lambda = -2$

$\lambda = p$, $p = -2$

$-3\lambda = t$, $t = (-3)(-2) = 6$

b) $|\underset{\sim}{b}-\underset{\sim}{a}| = 7$

$$\left|\begin{pmatrix}-4\\p\\t\end{pmatrix}-\begin{pmatrix}2\\1\\-3\end{pmatrix}\right| = 7$$

$$\left|\begin{matrix}-6\\p-1\\t+3\end{matrix}\right| = 7$$

$$\sqrt{(-6)^2+(p-1)^2+(t+3)^2} = 7$$

But $p = 2t$

$$36 + (2t-1)^2 + (t+3)^2 = 49$$
$$36 + 4t^2-4t+1+t^2+6t+9 = 49$$
$$5t^2+2t-3 = 0$$
$$(5t-3)(t+1) = 0$$

Either $t = 3/5$ or $t = -1$

when $t = 3/5$, $s = 6/5$

when $t = -1$, $s = -2$

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4 Given that $\underset{\sim}{a} = 4\underset{\sim}{i}+6\underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i}+t\underset{\sim}{j}$. find the value of $t$ given that

a) $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel

b) $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are at right angles

soln

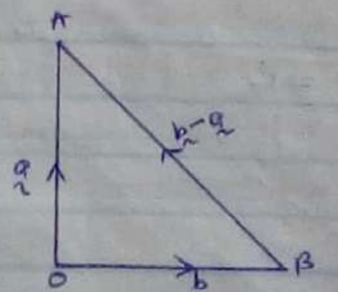
a) If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel, then

$$\underset{\sim}{b} = \lambda\underset{\sim}{a}$$
$$\begin{pmatrix}3\\t\end{pmatrix} = \lambda\begin{pmatrix}4\\6\end{pmatrix}$$

Equating Corresponding Components

$$4\lambda = 3, \quad \lambda = 3/4$$
$$t = 6\lambda = 6(3/4) = 9/2$$

(b)




$$|\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2 = |\underset{\sim}{b}-\underset{\sim}{a}|^2$$
$$4^2+6^2+3^2+t^2 = (3-4)^2+(6-t)^2$$
$$52+9+t^2 = 1+36-12t+t^2$$
$$12t = -24, \quad t = -2$$

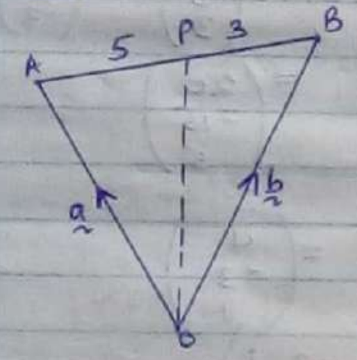
5 If the points A and B have position vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively. Find in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ the position vector of;

a) the point P which divides AB internally in the ratio 5:3

b) The point Q which divides line AB externally in the ratio 3:1

Soln

a)



$$\overrightarrow{AP} : \overrightarrow{PB} = 5:3$$

$$\frac{\overrightarrow{AP}}{\overrightarrow{PB}} = \frac{5}{3}$$

$$3\overrightarrow{AP} = 5\overrightarrow{PB}$$

$$3(\overrightarrow{OP} - \overrightarrow{OA}) = 5(\overrightarrow{OB} - \overrightarrow{OP})$$

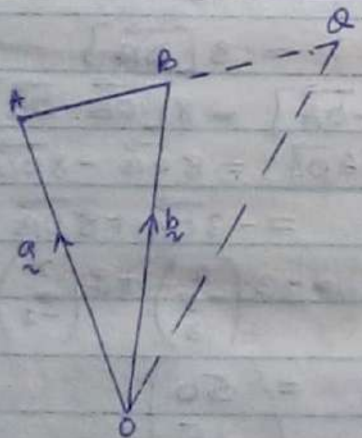
$$3\overrightarrow{OP} - 3\overrightarrow{OA} = 5\overrightarrow{OB} - 5\overrightarrow{OP}$$

$$8\overrightarrow{OP} = 3\overrightarrow{OA} + 5\overrightarrow{OB}$$

$$\overrightarrow{OP} = \frac{3}{8}\overrightarrow{OA} + \frac{5}{8}\overrightarrow{OB}$$

$$\overrightarrow{OP} = \frac{3}{8}\underset{\sim}{a} + \frac{5}{8}\underset{\sim}{b}$$

b)



$$\overrightarrow{AQ} : \overrightarrow{QB} = 3:-1$$

$$\frac{\overrightarrow{AQ}}{\overrightarrow{QB}} = \frac{3}{-1}$$

$$-\overrightarrow{AQ} = 3\overrightarrow{QB}$$

$$-[\overrightarrow{OQ} - \overrightarrow{OA}] = 3[\overrightarrow{OB} - \overrightarrow{OQ}]$$

$$-\overrightarrow{OQ} + \overrightarrow{OA} = 3\overrightarrow{OB} - 3\overrightarrow{OQ}$$

$$2\overrightarrow{OQ} = -\overrightarrow{OA} + 3\overrightarrow{OB}$$

$$\overrightarrow{OQ} = -\tfrac{1}{2}\overrightarrow{OA} + \tfrac{3}{2}\overrightarrow{OB}$$

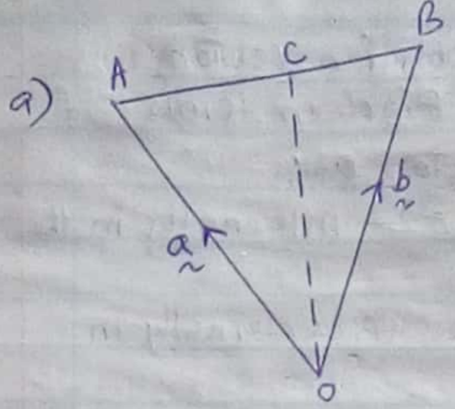
$$\overrightarrow{OQ} = -\tfrac{1}{2}\underset{\sim}{a} + \tfrac{3}{2}\underset{\sim}{b}$$

6. Points A and B have coordinates (2,-5,3) and (7,0,-2) respectively.

a) find the coordinates of point C which divides AB internally in the ratio 2:3

(b) the point D divides AB externally in the ratio 8:3

a)




$$\vec{AC} : \vec{CB} = 2 : 3$$

$$\frac{\vec{AC}}{\vec{CB}} = \frac{2}{3}$$

$$3\vec{AC} = 2\vec{CB}$$

$$3[\vec{OC} - \vec{OA}] = 2[\vec{OB} - \vec{OC}]$$

$$3\vec{OC} - 3\vec{OA} = 2\vec{OB} - 2\vec{OC}$$

$$5\vec{OC} = 3\vec{OA} + 2\vec{OB}$$

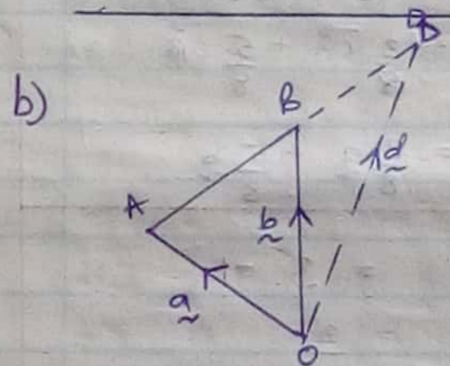
$$5\vec{OC} = 3\begin{pmatrix}2\\-5\\3\end{pmatrix} + 2\begin{pmatrix}7\\0\\-2\end{pmatrix}$$

$$5\vec{OC} = \begin{pmatrix}20\\-15\\5\end{pmatrix}$$

$$\vec{OC} = \begin{pmatrix}4\\-3\\1\end{pmatrix}$$

$C(4, -3, 1)$

b)




$$\vec{AD} : \vec{DB} = 8 : -3$$

$$\frac{\vec{AD}}{\vec{DB}} = \frac{8}{-3}$$

$$-3\vec{AD} = 8[\vec{DB}]$$

$$-3[\vec{OD} - \vec{OA}] - 8[\vec{OB} - \vec{OD}]$$

$$-3\vec{OD} + 3\vec{OA} = 8\vec{OB} - 8\vec{OD}$$

$$5\vec{OD} = -3\vec{OA} + 8\vec{OB}$$

$$5\vec{OD} = -3\begin{pmatrix}2\\-5\\3\end{pmatrix} + 8\begin{pmatrix}7\\0\\-2\end{pmatrix}$$

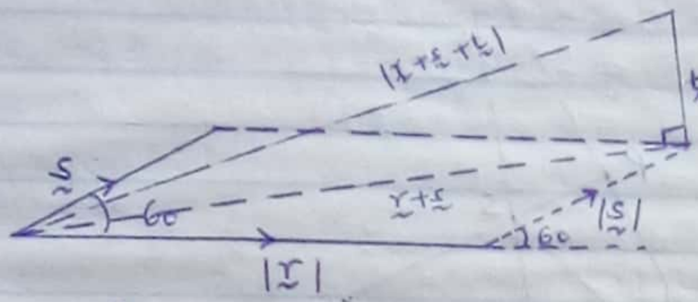
$$5\vec{OD} = \begin{pmatrix}50\\15\\-25\end{pmatrix}$$

$$\vec{OD} = \begin{pmatrix}10\\3\\-5\end{pmatrix}$$

$D(10, 3, -5)$

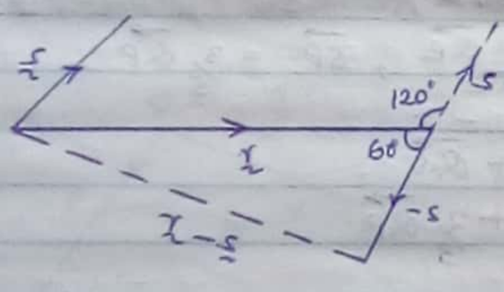
7. Given that $\underset{\sim}{r}$ and $\underset{\sim}{s}$ are inclined at $60°$ and $\underset{\sim}{t}$ perpendicular to $\underset{\sim}{r}+\underset{\sim}{s}$ and $|\underset{\sim}{r}| = 8$, $|\underset{\sim}{s}| = 5$, $|\underset{\sim}{t}| = 10$. Find $|\underset{\sim}{r}+\underset{\sim}{s}+\underset{\sim}{t}|$ and $|\underset{\sim}{r}-\underset{\sim}{s}|$

Soln.




$$|\underset{\sim}{r}+\underset{\sim}{s}|^2 = |\underset{\sim}{r}|^2 + |\underset{\sim}{s}|^2 - 2|\underset{\sim}{r}||\underset{\sim}{s}|\cos 120$$
$$= 8^2 + 5^2 - 2(8)(5)(-\tfrac{1}{2})$$
$$|\underset{\sim}{r}+\underset{\sim}{s}|^2 = 129$$

$$|\underset{\sim}{r}+\underset{\sim}{s}+\underset{\sim}{t}|^2 = |\underset{\sim}{r}+\underset{\sim}{s}|^2 + |\underset{\sim}{t}|^2$$
$$= 129 + 100$$
$$= 229$$
$$|\underset{\sim}{r}+\underset{\sim}{s}+\underset{\sim}{t}| = \sqrt{229}$$




$$|\underset{\sim}{r}-\underset{\sim}{s}|^2 = |\underset{\sim}{r}|^2 + |\underset{\sim}{s}|^2 - 2|\underset{\sim}{r}||\underset{\sim}{s}|\cos 60$$
$$= 8^2 + 5^2 - 2(8)(5)(\tfrac{1}{2})$$
$$|\underset{\sim}{r}-\underset{\sim}{s}|^2 = 49$$
$$|\underset{\sim}{r}-\underset{\sim}{s}| = 7$$

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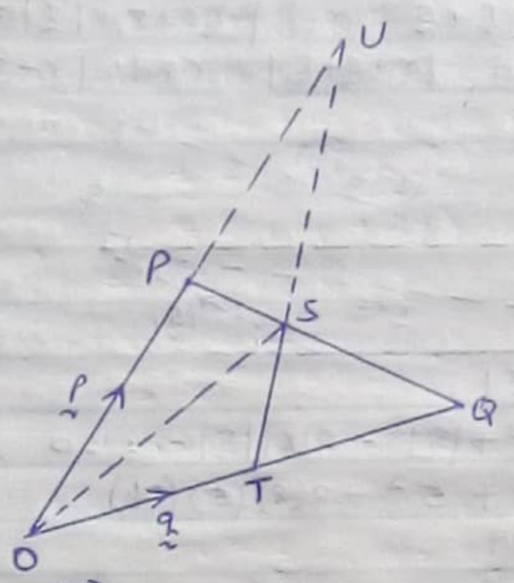
8. Given that $OP = \underset{\sim}{p}$ and $OQ = \underset{\sim}{q}$. Point T is on $\overrightarrow{OQ}$ such that $\overrightarrow{OT} : \overrightarrow{TQ} = 4:1$, Point S is on $\overrightarrow{QP}$ Such that $\overrightarrow{QS} : \overrightarrow{SP} = 2:3$ and $\overrightarrow{TS}$ and $\overrightarrow{OP}$ are both produced to meet at point U

Find (i) $\overrightarrow{OT}$ and $\overrightarrow{OS}$ in terms of $\underset{\sim}{p}$ and $\underset{\sim}{q}$

(ii) OU in terms of $\underset{\sim}{p}$




(i) $\overrightarrow{OT} : \overrightarrow{TQ} = 4:1$

$$\overrightarrow{OT} = \frac{4}{5}\overrightarrow{OQ}$$

$$= \frac{4}{5}\underset{\sim}{q}$$

$\overrightarrow{QS} : \overrightarrow{SP} = 2:3$

$$\overrightarrow{QS} = \frac{2}{5}\overrightarrow{QP}, \quad \overrightarrow{SP} = \frac{3}{5}\overrightarrow{QP}$$

$$\overrightarrow{OS} = \overrightarrow{OQ} + \overrightarrow{QS}$$

$$= \overrightarrow{OQ} + \frac{2}{3}\overrightarrow{QP}$$

$$= \overrightarrow{OQ} + \frac{2}{3}(\overrightarrow{OP} - \overrightarrow{OQ})$$

$$= \underset{\sim}{q} + \frac{2}{3}(\underset{\sim}{p} - \underset{\sim}{q})$$

$$= \underset{\sim}{q} + \frac{2}{3}\underset{\sim}{p} - \frac{2}{3}\underset{\sim}{q}$$

$$= \frac{1}{5}(2\underset{\sim}{p} + 3\underset{\sim}{q})$$

(ii) $\overrightarrow{OU} = \lambda\overrightarrow{OP}$

$$= \lambda\underset{\sim}{p}$$

$$\overrightarrow{OU} = \overrightarrow{OT} + \overrightarrow{TU}$$

$$= \overrightarrow{OT} + \mu\overrightarrow{TS}$$

$$= \frac{4}{5}q + \mu(\overrightarrow{OS} - \overrightarrow{OT})$$

$$\vec{OU} = \frac{4}{5}\underset{\sim}{q} + \mu\left[\frac{1}{5}(2\underset{\sim}{p}+3\underset{\sim}{q}) - \frac{4}{5}\underset{\sim}{q}\right]$$

$$= \frac{4}{5}\underset{\sim}{q} + \mu\left[\frac{2}{5}\underset{\sim}{p} - \frac{1}{5}\underset{\sim}{q}\right]$$

$$\vec{OU} = \left(\frac{4}{5} - \frac{\mu}{5}\right)\underset{\sim}{q} + \frac{2\mu}{5}\underset{\sim}{p}$$

On Comparison.

$$\frac{4}{5} - \frac{\mu}{5} = 0$$

$$\mu = 4$$

$$\lambda = \frac{2}{5}\mu = \left(\frac{2}{5}\right)(4) = \frac{8}{5}$$

$$\therefore \vec{OU} = \frac{8}{5}\underset{\sim}{p}$$

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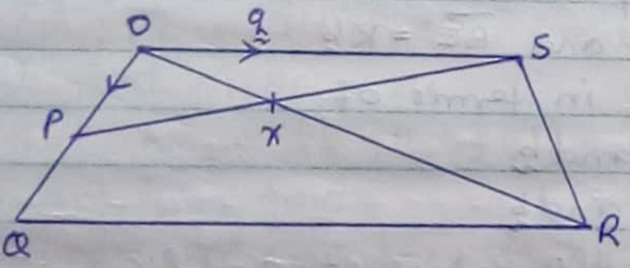
9. The diagram below shows a quadrilateral OSRQ $\vec{OS} = \underset{\sim}{q}$, $\vec{OP} = \underset{\sim}{p}$ and $\vec{SX} = k\vec{SP}$




(i) Express Vectors $\vec{SP}$ and $\vec{OX}$ in terms of $\underset{\sim}{p}$, $\underset{\sim}{q}$ and $k$

(ii) If $\vec{OQ} = 3\underset{\sim}{p}$ and $\vec{QR} = 2\vec{OS}$, and $\vec{OX} = l\,\vec{OR}$, find the values of $k$ and $l$

Soln.

$$\vec{SP} = \vec{SO} + \vec{OP}$$

$$= -\underset{\sim}{q} + \underset{\sim}{p}$$

$$= \underset{\sim}{p} - \underset{\sim}{q}$$

$$\vec{OX} = \vec{OS} + \vec{SX}$$

$$= \underset{\sim}{q} + k\vec{SP}$$

$$\overrightarrow{OX} = q + k(p - q)$$
$$= kp + (1-k)q$$

(ii) Also $\overrightarrow{OX} = l\,\overrightarrow{OR}$

$$= l(\overrightarrow{OQ} + \overrightarrow{QR})$$
$$= l(3p + 2q)$$
$$= 3lp + 2lq$$

By equating corresponding unit vectors;

for $p$; $\quad k = 3l$ ---- (1)

for $q$; $\quad 1 - k = 2l$ ---- (2)

(1) + (2); $\quad 1 = 5l$

$l = \frac{1}{5}$, $k = 3/5$

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10. Triangle OAB has $\overrightarrow{OA} = a$ and $\overrightarrow{OB} = b$. C is a point on $\overrightarrow{OA}$ such that $\overrightarrow{OC} = \frac{2a}{3}$. D is the mid-point of $\overline{AB}$. When $\overline{CD}$ is produced it meets $\overline{OB}$ produced at E, such that $\overrightarrow{DE} = n\overrightarrow{CD}$ and $\overrightarrow{BE} = kb$

Express $\overrightarrow{DE}$ in terms of

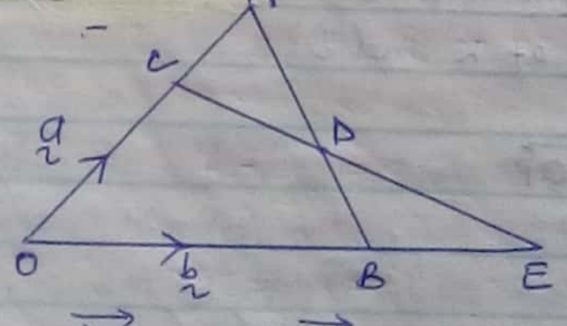
a) $n$, $a$ and $b$

b) $k$, $a$ and $b$

Hence find the values of $n$ and $k$

Soln.

a)




$$\overrightarrow{DE} = n\overrightarrow{CD}$$
$$= n(\overrightarrow{CA} + \overrightarrow{AD})$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = b - a$$

$$\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(\underset{\sim}{b}-\underset{\sim}{a})$$

$$\overrightarrow{DE} = n\left[\frac{1}{3}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}\right]$$

$$= n\left[-\frac{1}{6}a + \frac{1}{2}\underset{\sim}{b}\right]$$

$$= -\frac{n}{6}\underset{\sim}{a} + \frac{n}{2}\underset{\sim}{b}$$

b) $\overrightarrow{DE} = \overrightarrow{DB} + \overrightarrow{BE}$

$$= \frac{1}{2}\overrightarrow{AB} + k\underset{\sim}{b}$$

$$= \frac{1}{2}[\underset{\sim}{b} - \underset{\sim}{a}] + k\underset{\sim}{b}$$

$$= \left(\frac{1}{2}+k\right)\underset{\sim}{b} - \frac{\underset{\sim}{a}}{2}$$

Equating $\overrightarrow{DE}$ in (a) to $\overrightarrow{DE}$ in (b)

$$-\frac{n}{6}\underset{\sim}{a} + \frac{n}{2}\underset{\sim}{b} = \left(\frac{1}{2}+k\right)b - \frac{1}{2}\underset{\sim}{a}$$

$$-\frac{n}{6} = -\frac{1}{2}, \quad n = 3$$

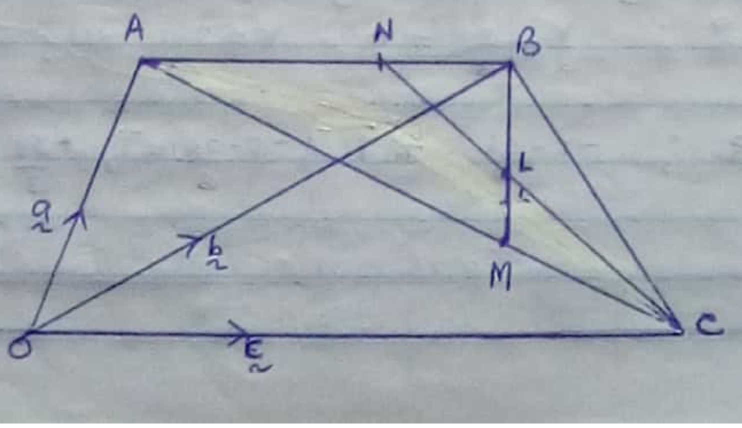
11. Three non-collinear points A, B and C have position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively with respect to an origin O. The point M on AC is such that $\overrightarrow{AM} : \overrightarrow{MC} = 2:1$ and the point N on AB is such that $AN : NB = 2:1$

i) Show that $\overrightarrow{BM} = \frac{1}{3}\underset{\sim}{a} - \underset{\sim}{b} + \frac{2}{3}\underset{\sim}{c}$, and find a similar expression for CN

ii) The lines BM and CN intersect at L. Given that $\overrightarrow{BL} = r\overrightarrow{BM}$ and $CL = s\overrightarrow{CN}$, where r and s are scalars, express BL and CL in terms of r, s, $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$

iii) Hence by using triangle BLC, or otherwise, find r and s.

Soln



i) $\overrightarrow{BM} = \overrightarrow{BA} + \overrightarrow{AM}$

$= \overrightarrow{OA} - \overrightarrow{OB} + \frac{2}{3}\overrightarrow{AC}$

$= \underset{\sim}{a} - \underset{\sim}{b} + \frac{2}{3}(\overrightarrow{OC} - \overrightarrow{OA})$

$= \underset{\sim}{a} - \underset{\sim}{b} + \frac{2}{3}(\underset{\sim}{c} - \underset{\sim}{a})$

$= \frac{1}{3}\underset{\sim}{a} - \underset{\sim}{b} + \frac{2}{3}\underset{\sim}{c}$

$\overrightarrow{CN} = \overrightarrow{CA} + \overrightarrow{AN}$

$= -\overrightarrow{AC} + \frac{2}{3}\overrightarrow{AB}$

$= -[\underset{\sim}{c} - \underset{\sim}{a}] + \frac{2}{3}(\underset{\sim}{b} - \underset{\sim}{a})$

$= \underset{\sim}{a} - \underset{\sim}{c} + \frac{2}{3}\underset{\sim}{b} - \frac{2}{3}\underset{\sim}{a}$

$= \frac{1}{3}\underset{\sim}{a} + \frac{2}{3}\underset{\sim}{b} - \underset{\sim}{c}$

(ii) $\overrightarrow{BL} = r\overrightarrow{BM}$

$= r\left(\frac{1}{3}\underset{\sim}{a} - \underset{\sim}{b} + \frac{2}{3}\underset{\sim}{c}\right)$

$= \frac{r}{3}\underset{\sim}{a} - r\underset{\sim}{b} + \frac{2r}{3}\underset{\sim}{c}$

$CL = s\overrightarrow{CN}$

$= s\left(\frac{1}{3}\underset{\sim}{a} + \frac{2}{3}\underset{\sim}{b} - \underset{\sim}{c}\right)$

$= \frac{s}{3}\underset{\sim}{a} + \frac{2s}{3}\underset{\sim}{b} - s\underset{\sim}{c}$

$\overrightarrow{BC} = \overrightarrow{BL} + \overrightarrow{LC}$

$= \frac{r}{3}\underset{\sim}{a} - r\underset{\sim}{b} + \frac{2r}{3}\underset{\sim}{c} - \frac{s}{3}\underset{\sim}{a} - \frac{2s}{3}\underset{\sim}{b} + s\underset{\sim}{c}$

$= \left(\frac{r}{3} - \frac{s}{3}\right)\underset{\sim}{a} - \left(r + \frac{2s}{3}\right)\underset{\sim}{b} + \left(\frac{2r}{3} + s\right)\underset{\sim}{c}$

Also $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= \underset{\sim}{c} - \underset{\sim}{b}$

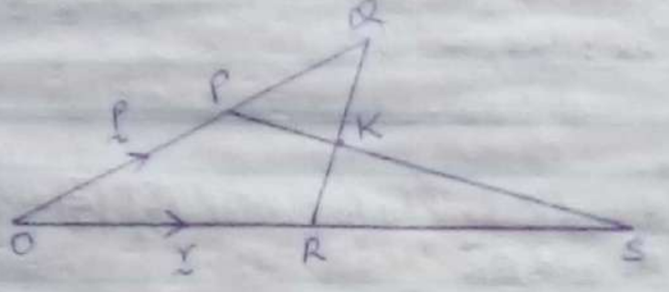
Equating Corresponding unit vectors

for $\underset{\sim}{a}$ ; $\frac{r}{3} - \frac{s}{3} = 0$, $r = s$

for $\underset{\sim}{b}$ ; $-r - \frac{2s}{3} = 1$ ; $-s - \frac{2s}{3} = -1$, $+\frac{5s}{3} = 1$, $s = +\frac{3}{5}$

$\therefore r = 3/5$

12 In the figure below $\overrightarrow{OS} = 2\underset{\sim}{r}$ and $\overrightarrow{OQ} = \frac{3}{2}\underset{\sim}{p}$
Given that $\overrightarrow{QK} = m\,\overrightarrow{QR}$ and $\overrightarrow{PK} = n\overrightarrow{PS}$, find two distinct expressions in terms of $\underset{\sim}{p}$, $\underset{\sim}{r}$, $m$ and $n$ for $\overrightarrow{OK}$. By equating these expressions, find the values of $m$ and $n$ and hence calculate the ratios $\overrightarrow{QK} : \overrightarrow{KR}$ and $\overrightarrow{PK} : \overrightarrow{KS}$




Soln

$\overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR}$

$= -\frac{3}{2}\underset{\sim}{p} + \underset{\sim}{r}$

Consider OK

$\overrightarrow{OK} = \overrightarrow{OQ} + \overrightarrow{QK}$

$= \overrightarrow{OQ} + m\overrightarrow{QR}$

$= \frac{3}{2}\underset{\sim}{p} + m\left(r - \frac{3}{2}\underset{\sim}{p}\right)$ - - - - - - (1)

$\overrightarrow{OK} = \overrightarrow{OP} + \overrightarrow{PK}$

$= \overrightarrow{OP} + n\overrightarrow{PS}$ ; but $\overrightarrow{PS} = \overrightarrow{OS} - \overrightarrow{OP} = 2r - p$

$= \underset{\sim}{p} + n(2r - p)$ - - - - - (2)

Equating equations (1) and (2)

$\frac{3}{2}\underset{\sim}{p} + m\left(\underset{\sim}{r} - \frac{3}{2}\underset{\sim}{p}\right) = \underset{\sim}{p} + n(2\underset{\sim}{r} - p)$

for $p$; $\frac{3}{2} - \frac{3m}{2} = 1 - n$ ; $3 - 3m = 2 - 2n$

$2n - 3m = -1$ - - - (3)

for $r$; $m = 2n$ - - - (4)

Substituting (4) in (3); we shall have

$m - 3m = -1$

$-2m = -1$

$m = \frac{1}{2}$

Substituting in (4); $\frac{1}{2} = 2n$

$n = \frac{1}{4}$

$\overrightarrow{QK} = m\overrightarrow{QR} = \frac{1}{2}\overrightarrow{QR}$

$\overrightarrow{QK} = \overrightarrow{KR}$

there fore $\overrightarrow{QK} : \overrightarrow{KR} = 1:1$

Also $\overrightarrow{PK} = n\overrightarrow{PS}$

$\overrightarrow{PK} = \frac{1}{4}\overrightarrow{PS}$

$4PK = PS$

$\therefore \quad PK : KS = 1:3$

---

Exercise:

1. Points P, Q and R have position vectors $\begin{pmatrix}5\\4\\1\end{pmatrix}$, $\begin{pmatrix}7\\5\\4\end{pmatrix}$ and $\begin{pmatrix}11\\7\\10\end{pmatrix}$ respectively
   a) Find $\overrightarrow{PQ}$ and $\overrightarrow{QR}$
   b) Deduce that P, Q and R are collinear and find the ratio $PQ : QR$
2. Given that $A(2,13,-5)$, $B(3,\beta,-3)$ and $C(6,-7,\alpha)$ are collinear, find the values of the constants $\alpha$ and $\beta$
3. OABC is a parallelogram with $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$. S is the point of AB such that $\overrightarrow{AS} : \overrightarrow{SB} = 3:1$ and T is a point on BC such that $BT : TC = 1:3$
   a) Express each of the following in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$
   (i) $\overrightarrow{AC}$ (ii) $\overrightarrow{SB}$ (iii) $\overrightarrow{BT}$ (iv) $\overrightarrow{ST}$
   b) state the value of the ratio $ST : AC$
4. In the triangle OAB, $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$. M is the Mid-point of AB and N is the point on OB such that $ON : NB = 1:4$. OM meets AN at P
   a) find an expression for $\overrightarrow{OP}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$
   b) Deduce that $\overrightarrow{AP} : \overrightarrow{PN} = 5:1$
5. In the trapezium OABC, $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OC} = \underset{\sim}{c}$ and $\overrightarrow{CB} = 3\underset{\sim}{a}$. T is the point on $\overrightarrow{BC}$ such that $\overrightarrow{BC} : \overrightarrow{TC} = 1:2$. $\overrightarrow{OT}$ meets $\overrightarrow{AC}$ at P
   a) Find an expression for $\overrightarrow{OP}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$
   b) Deduce that P is the point of trisection of both AC & OT
6. In the triangle OAB, M is the Mid-point of AB and N is the point on OB such that $ON : NB = \lambda : \mu$. OM meets AN at P. Find the ratio of the following
   a) $\overrightarrow{AP} : \overrightarrow{PN}$ (b) $\overrightarrow{OP} : \overrightarrow{PM}$

## THE SCALAR PRODUCT (DOT PRODUCT)

Definition: The dot or scalar product of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ denoted by $a \cdot b$ is defined as

$\boxed{a \cdot b = |\underset{\sim}{a}| \times |\underset{\sim}{b}| \cos\theta}$ where $0 \le \theta \le \pi$

$|\underset{\sim}{a}|, |\underset{\sim}{b}|$ are magnitudes of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively, $\theta$ is the angle between the two Vectors

## PROPERTIES OF SCALAR PRODUCT

1 (i) $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{a}$

(ii) $\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$

(iii) $m(\underset{\sim}{a} \cdot b) = (m\underset{\sim}{a}) \cdot b = a \cdot (m\underset{\sim}{b})$

(iv) $- i \cdot i = \underset{\sim}{j} \cdot \underset{\sim}{j} = \underset{\sim}{k} \cdot \underset{\sim}{k} = 1$

$- \underset{\sim}{i} \cdot \underset{\sim}{j} = \underset{\sim}{j} \cdot \underset{\sim}{k} = \underset{\sim}{k} \cdot \underset{\sim}{i} = 0$

2. If $\underset{\sim}{A} = A_1\underset{\sim}{i} + A_2\underset{\sim}{j} + A_3\underset{\sim}{k}$ and $B = B_1\underset{\sim}{i} + B_2\underset{\sim}{j} + B_3\underset{\sim}{k}$ then $A \cdot B = A_1B_1 + A_2B_2 + A_3B_3$

3 If $A \cdot B = 0$ where $A$ and $B$ are not null vectors then $A$ and $B$ are perpendicular

Proof

$\underset{\sim}{i} \cdot \underset{\sim}{i} = |\underset{\sim}{i}||\underset{\sim}{i}| \cos 0$; using the definition

$= 1 \cdot 1 \cdot 1$

$i \cdot i = 1$

$\underset{\sim}{j} \cdot \underset{\sim}{j} = |\underset{\sim}{j}||\underset{\sim}{j}| \cos 0$

$= 1 \cdot 1 \cdot 1$

$\underset{\sim}{j} \cdot \underset{\sim}{j} = 1$

$\underset{\sim}{k} \cdot \underset{\sim}{k} = |\underset{\sim}{k}||\underset{\sim}{k}| \cos 0$

$= 1 \cdot 1 \cdot 1$

$\underset{\sim}{k} \cdot \underset{\sim}{k} = 1$

$\underset{\sim}{i} \cdot \underset{\sim}{j} = |\underset{\sim}{i}||\underset{\sim}{j}| \cos 90$; Since the two Vectors are mutually

$= 1 \cdot 1 \cdot 0$ at right angles

$\underset{\sim}{i} \cdot \underset{\sim}{j} = 0$

$\underset{\sim}{j} \cdot \underset{\sim}{k} = |\underset{\sim}{j}||\underset{\sim}{k}| \cos 90$

$= 1 \cdot 1 \cdot 0$

$\underset{\sim}{j} \cdot \underset{\sim}{k} = 0$

$\underset{\sim}{k} \cdot \underset{\sim}{i} = |\underset{\sim}{k}||\underset{\sim}{i}| \cos 90$

$= 1 \cdot 1 \cdot 0$

$k \cdot i = 0$

3 from $A \cdot B = |A||B| \cos\theta$

If $|A| \neq 0$, $|B| = 0$

$\Rightarrow \cos\theta = 0$

$\theta = \cos^{-1}(0)$

$\theta = 90^\circ$

Examples

1. Given that $\underset{\sim}{p} = \underset{\sim}{i} - 2\underset{\sim}{k}$ and $\underset{\sim}{q} = 3\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$, find the angle between $\underset{\sim}{p}$ and $\underset{\sim}{q}$ correct to the nearest degree.

Soln.

Definition of dot product; $\underset{\sim}{p} \cdot \underset{\sim}{q} = |p||q| \cos\theta$; where $\theta$ is the angle between $\underset{\sim}{p}$ and $\underset{\sim}{q}$

$$\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} = \sqrt{(1)^2 + 0^2 + (-2)^2}\ \sqrt{(3)^2 + (-3)^2 + 1^2}\ \cos\theta$$

$$3 + 0 - 2 = \sqrt{5}\sqrt{19} \cos\theta$$

$$\cos\theta = \frac{1}{\sqrt{95}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{95}}\right)$$

$$\theta = 84.11$$

$$\theta = 84 \text{ (to the nearest degree)}$$

2. Show that the following vectors are perpendicular

(i) $a = 2\underset{\sim}{i} + 6\underset{\sim}{j} + 4\underset{\sim}{k}$ and $\underset{\sim}{b} = -2\underset{\sim}{i} - 2\underset{\sim}{j} + 4k$

(ii) $r_1 = 3\underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{k}$ and $r_2 = 2\underset{\sim}{i} + 3\underset{\sim}{j} + 6\underset{\sim}{k}$

soln

(i) $a \cdot b = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix} = -4 - 12 + 16 = 0$

Since $a \cdot b = 0$, the two vectors are perpendicular

$r_1 \cdot r_2 = \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ +6 \end{pmatrix} = 6 - 12 + 6 = 0$

Since $r_1 \cdot r_2 = 0$, the two Vectors are perpendicular

3. (a) Find the value(s) of the scalar $\lambda$ if the two vectors $\underset{\sim}{r_1} = 2\lambda\underset{\sim}{i} + 7\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{r_2} = 3\lambda\underset{\sim}{i} + \lambda\underset{\sim}{j} + 3\underset{\sim}{k}$ are perpendicular

b) If the angle between the vectors $\underset{\sim}{a} = \beta\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + \underset{\sim}{j}$ is $60°$, find the possible values of $\beta$

Soln

a) $\underset{\sim}{r_1} \cdot \underset{\sim}{r_2} = 0$, for Perpendicular vectors

$$\begin{pmatrix} 2\lambda \\ 7 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3\lambda \\ \lambda \\ 3 \end{pmatrix} = 0$$

$$6\lambda^2 + 7\lambda - 3 = 0$$

$$6\lambda^2 - 2\lambda + 9\lambda - 3 = 0$$

$$2\lambda(3\lambda - 1) + 3(3\lambda - 1) = 0$$

$$(2\lambda + 3)(3\lambda - 1) = 0$$

$$\lambda = -3/2 \,,\; \lambda = 1/3$$

b) $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

$$\begin{pmatrix} \beta \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \sqrt{\beta^2+4} \cdot \sqrt{9+1} \cos 60$$

$$3\beta + 2 = \sqrt{(\beta^2+4)10} \left(\tfrac{1}{2}\right)$$

$$6\beta + 4 = \sqrt{10(\beta^2+4)}$$

$$36\beta^2 + 48\beta + 16 = 10\beta^2 + 40$$

$$28\beta^2 + 48\beta - 24 = 0$$

$$\beta = \frac{-48 \pm \sqrt{48^2 - (4)(28)(-24)}}{(2)(28)}$$

~~either~~ $\beta = \dfrac{-48 \pm 70.65}{56}$

either $\beta = 0.405$ or $\beta = -2.12$

# THE VECTOR (CROSS) PRODUCT

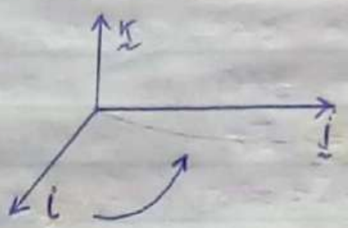
Given two non-zero vectors $a$ and $b$, their vector (cross) product denoted by $a \times b$ is defined as

$$\boxed{a \times b = |a|\,|b| \sin\theta \cdot \hat{n}} \quad \text{where } 0 \leq \theta \leq \pi$$

Where $\theta$ is the angle between the vectors $a$ and $b$ and $\hat{n}$ is a unit vector perpendicular to both $a$ and $b$

## PROPERTIES OF CROSS PRODUCT

1




a) $i \times j = |i|\,|j| \sin 90^\circ \cdot k = k$

$j \times k = |j|\,|k| \sin 90 \cdot i = i$

$k \times i = |j|\,|k| \sin 90 \cdot i = j$

b) $i \times k = |i|\,|k| \sin(-90) \cdot j = -j$

$k \times j = |k|\,|j| \sin(-90) \cdot i = -i$

$j \times i = |j|\,|i| \sin(-90) \cdot k = -k$

c) $i \times i = j \times j = k \times k = 0$

Hence the cross product of two vectors is a vector quantity

Note: We use the cross product to show that the two vectors are parallel

2. $b \times a = -(a \times b)$

3 For any three vectors $a$, $b$ and $c$

$$a \times (b + c) = a \times b + a \times c$$

4 The cross product gives a vector that is perpendicular to either of the vectors crossed.

5 Suppose we have vectors $a = a_1 i + a_2 j + a_3 k$ and $b = b_1 i + b_2 j + b_3 k$, the cross product of $a$ and $b$ is

$$\mathbf{a}\times\mathbf{b} = (a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$$
$$= a_1\mathbf{i}\times(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$$
$$+ a_2\mathbf{j}\times(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$$
$$+ a_3\mathbf{k}\times(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$$
$$= a_1b_1(\mathbf{i}\times\mathbf{i}) + a_1b_2(\mathbf{i}\times\mathbf{j}) + a_1b_3(\mathbf{i}\times\mathbf{k})$$
$$+ a_2b_1(\mathbf{j}\times\mathbf{i}) + a_2b_2(\mathbf{j}\times\mathbf{j}) + a_2b_3(\mathbf{j}\times\mathbf{k})$$
$$+ a_3b_1(\mathbf{k}\times\mathbf{i}) + a_3b_2(\mathbf{k}\times\mathbf{j}) + a_3b_3(\mathbf{k}\times\mathbf{k})$$
$$= a_1b_2\mathbf{k} + a_1b_3\mathbf{j} - a_2b_1\mathbf{k} + a_2b_3\mathbf{i} + a_3b_1\mathbf{j} - a_3b_2\mathbf{i}$$
$$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Alternative

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} i & -j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= i\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$
$$= (a_2b_3 - a_3b_2)i - j(a_1b_3 - a_3b_1) + (a_1b_2 - a_2b_1)\mathbf{k}$$
$$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Example:

Given $\mathbf{a} = 4\mathbf{i} - \mathbf{k}$ , $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$. Find $\mathbf{a}\times\mathbf{b}$

Soln

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} i & -j & k \\ 4 & 0 & -1 \\ 2 & 1 & 3 \end{vmatrix}$$
$$= i\begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} - j\begin{vmatrix} 4 & -1 \\ 2 & 3 \end{vmatrix} + k\begin{vmatrix} 4 & 0 \\ 2 & 1 \end{vmatrix}$$
$$= \mathbf{i} - 14\mathbf{j} + 4\mathbf{k}.$$

## APPLICATION OF CROSS PRODUCT

1. Determining the angle between two vectors

Example:

Find the angle between the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

Soln

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} i & -j & k \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= i \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - j \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} + k \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$$

$$= \underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}.$$

$|\underset{\sim}{a} \times \underset{\sim}{b}| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$

$|\underset{\sim}{a}| = \sqrt{(1)^2 + 2^2 + 3^2} = \sqrt{14}$

$|\underset{\sim}{b}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$

from the definition; $\underset{\sim}{a} \times \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta \, \hat{n}$

Taking magnitude both sides; $|\underset{\sim}{a} \times \underset{\sim}{b}| = |\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta \, |\hat{n}|$; but $|\hat{n}| = 1$

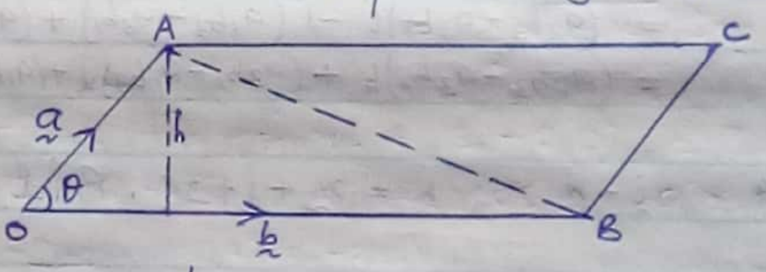
$$|\underset{\sim}{a} \times \underset{\sim}{b}| = |\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta$$

$$\sqrt{3} = \sqrt{14}\sqrt{6} \sin\theta$$

$$\sin\theta = \frac{\sqrt{3}}{\sqrt{14}\sqrt{6}}$$

$$\theta = 10.9$$

2. find the area of a parallelogram/triangle




Area of triangle $OAB = \frac{1}{2} \times |OB| \times h$

But $\sin\theta = \frac{h}{|OA|}$, $h = |OA| \sin\theta$.

Area of triangle $= \frac{1}{2} \times |OA| \times |OB| \sin\theta$

$= \frac{1}{2} |\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta$

from the definition of cross product -

$$|\underset{\sim}{a} \times \underset{\sim}{b}| = |\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta$$

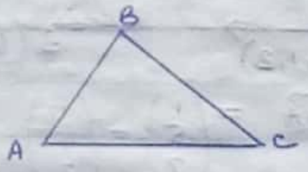
Area of triangle $= \frac{1}{2} |\underset{\sim}{a} \times \underset{\sim}{b}|$

Area of a parallelogram $= |\underset{\sim}{a} \times \underset{\sim}{b}|$

where $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are the vectors representing the adjacent sides.

Examples:

1. Find the area of the triangle that has the following vertices, A(0,0), B(0,1), C(1,1)



$$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\vec{AC} = \vec{OC} - \vec{OA} \quad \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & -j & k \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= -k$$

$$|\vec{AB} \times \vec{AC}| = \sqrt{(-1)^2} = 1$$

But Area of a triangle $= \frac{1}{2}|\vec{AB} \times \vec{AC}|$

$= \frac{1}{2} \cdot 1$

$= \frac{1}{2}$ sq. units

2. Find the area of a parallelogram of which the given vectors are adjacent sides

$\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{j} + \underset{\sim}{k}$

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} i & -j & k \\ 1 & 2 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 3\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$$

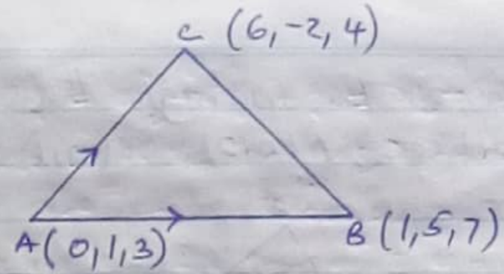
$$|\underset{\sim}{a} \times \underset{\sim}{b}| = \sqrt{3^2 + (-1)^2 + 1^2} = \sqrt{11}$$

But Area of a parallelogram $= |\underset{\sim}{a} \times \underset{\sim}{b}|$

$= \sqrt{11}$

3 Find the area of triangle ABC with vertices A(0,1,3), B(1,5,7) and C(6,−2,4)

<u>Soln</u>




$$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix}1\\5\\7\end{pmatrix} - \begin{pmatrix}0\\1\\3\end{pmatrix} = \begin{pmatrix}1\\4\\4\end{pmatrix}$$

$$\vec{AC} = \vec{OC} - \vec{OA} = \begin{pmatrix}6\\-2\\4\end{pmatrix} - \begin{pmatrix}0\\1\\3\end{pmatrix} = \begin{pmatrix}6\\-3\\1\end{pmatrix}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & -j & k \\ 1 & 4 & 4 \\ 6 & -3 & 1 \end{vmatrix}$$

$$= i \begin{vmatrix} 4 & 4 \\ -3 & 1 \end{vmatrix} - j \begin{vmatrix} 1 & 4 \\ 6 & 1 \end{vmatrix} + k \begin{vmatrix} 1 & 4 \\ 6 & -3 \end{vmatrix}$$

$$= 16i + 23j - 27k$$

$$|\vec{AB} \times \vec{AC}| = \sqrt{16^2 + 23^2 + (-27)^2}$$

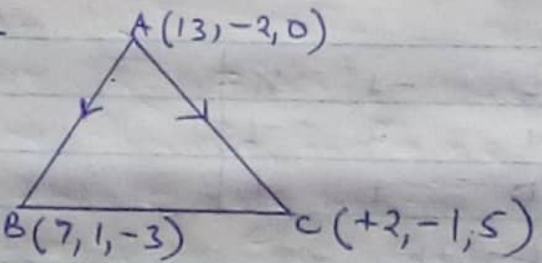
$$= \sqrt{1514}$$

Area of a triangle $= \frac{1}{2} |\vec{AB} \times \vec{AC}|$

$= \frac{1}{2}\sqrt{1514}$ sq-units

4 Show that the points $A(13,-2,0)$, $B(7,1,-3)$ and $C(+2,-1,5)$ are vertices of a triangle

Soln




$$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix}7\\1\\-3\end{pmatrix} - \begin{pmatrix}13\\-2\\0\end{pmatrix} = \begin{pmatrix}-6\\3\\-3\end{pmatrix}$$

$$\vec{AC} = \vec{OC} - \vec{OA} = \begin{pmatrix}+2\\-1\\5\end{pmatrix} - \begin{pmatrix}13\\-2\\0\end{pmatrix} = \begin{pmatrix}-15\\1\\5\end{pmatrix}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & -j & k \\ -6 & 3 & -3 \\ -11 & 1 & 5 \end{vmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = i\begin{vmatrix} 3 & -3 \\ 1 & 5 \end{vmatrix} - j\begin{vmatrix} -6 & -3 \\ -11 & 5 \end{vmatrix} + k\begin{vmatrix} -6 & 3 \\ -11 & 1 \end{vmatrix}$$

$$= 18\underset{\sim}{i} + 63\underset{\sim}{j} + 27\underset{\sim}{k}$$

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(18)^2 + (63)^2 + (27)^2}$$

$$= \sqrt{5022}$$

$$|AB| = \sqrt{(-6)^2 + 3^2 + (-3)^2} = \sqrt{54}$$

$$|AC| = \sqrt{(-11)^2 + 1^2 + 5^2} = \sqrt{147}$$

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = |\overrightarrow{AB}||\overrightarrow{AC}| \sin A$$

$$\sqrt{5022} = \sqrt{54}\sqrt{147} \sin A$$

$$\sin A = \sqrt{\frac{5022}{7938}}$$

$$A = 39.2^\circ$$

ABC is atriangle since $39.2^\circ \neq 180^\circ$ and is not equal to $0^\circ$

ALT:

$$\overrightarrow{AB} = \begin{pmatrix} -6 \\ 3 \\ -3 \end{pmatrix}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$

$$\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} - \begin{pmatrix} 7 \\ 1 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ -2 \\ 8 \end{pmatrix}$$

Since $\overrightarrow{AB} \neq \lambda \overrightarrow{BC}$; then ABC is atriangle

Exercise

1 Given that $\underset{\sim}{a} = 4\underset{\sim}{i}+5\underset{\sim}{j}$ , $\underset{\sim}{b} = \lambda\underset{\sim}{i}-8\underset{\sim}{j}$ and $\underset{\sim}{c} = \underset{\sim}{i}+\mu\underset{\sim}{j}$

a) find the value of the constant, $\lambda$ given that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular

b) find the value of the constant, $\mu$ given that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel

2. Given that $\begin{pmatrix} \lambda \\ 2+\lambda \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \\ 4-\lambda \end{pmatrix}$ are perpendicular vectors, find the value of the constant $\lambda$

3. Given that the vectors $\begin{pmatrix} t \\ 4 \\ 2t+1 \end{pmatrix}$ and $\begin{pmatrix} t+2 \\ 1-t \\ -1 \end{pmatrix}$ are perpendicular, find the possible values of the constant $t$

4 Three points P, Q and R have position vectors $\underset{\sim}{p}$, $\underset{\sim}{q}$ and $\underset{\sim}{r}$ respectively, where $\underset{\sim}{p} = 7\underset{\sim}{i}+10\underset{\sim}{j}$, $\underset{\sim}{q} = 3\underset{\sim}{i}+12\underset{\sim}{j}$ and $\underset{\sim}{r} = -\underset{\sim}{i}+4\underset{\sim}{j}$

(i) Write down the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RQ}$, and show that they are perpendicular

(ii) Using scalar product, or otherwise, find angle PRQ

(iii) find the position vector of S, the mid-point of PR

(iv) show that $|\overrightarrow{QS}| = |\overrightarrow{RS}|$, Using your previous results or otherwise, find angle PSQ

5. The points A and B have position vectors $\underset{\sim}{i}+2\underset{\sim}{j}+2\underset{\sim}{k}$ and $4\underset{\sim}{j}+3\underset{\sim}{k}$ respectively, relative to an origin O.

a) find the lengths of $\overrightarrow{OA}$ and $\overrightarrow{OB}$

b) find the scalar product of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ and hence find angle OAB

c) find the area of triangle OAB, correct to 2d.p

d) The point C divides AB in the ratio $\lambda : 1-\lambda$

(i) find an expression for $\overrightarrow{OC}$

(ii) Show that $OC^2 = 14\lambda^2+2\lambda+9$

(iii) find the position vectors of the two points on AB which distance from O is $\sqrt{21}$

# THE EQUATION OF A STRAIGHT LINE

The equation of a straight line can be obtained if either

a) It has a known direction and passes through a known point

b) OR It passes through two fixed points

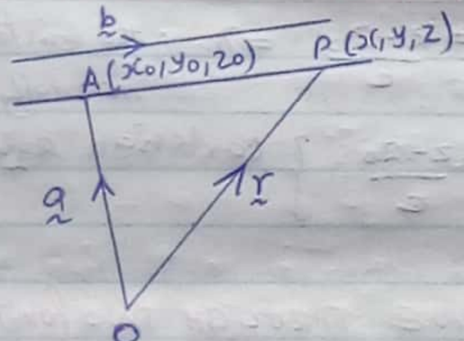
The equation of a line can be written in three forms.

(i) vector equation

(ii) Parametric equation

(iii) Cartesian/Symmetric equation

## 1 EQUATION OF A LINE THROUGH SOME POINT, A AND PARALLEL TO SOME VECTOR $\underset{\sim}{b}$




Suppose O is the origin and $\underset{\sim}{a}$ is the position vector of A. Let P, with position vector $\underset{\sim}{r}$ be any point on the line.

Then $\overrightarrow{AP}$ is parallel to $\underset{\sim}{b}$

So
$$\overrightarrow{AP} = \lambda \underset{\sim}{b}$$
$$\overrightarrow{OP} - \overrightarrow{OA} = \lambda \underset{\sim}{b}$$
$$\underset{\sim}{r} - \underset{\sim}{a} = \lambda \underset{\sim}{b}$$
$$\boxed{r = \underset{\sim}{a} + \lambda \underset{\sim}{b}}$$ - Vector equation of a line

### PARAMETRIC FORM

The equation above can be written in a parametric form. When we assume that P has cartesian coordinates $(x, y, z)$; b is a vector $a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$ and A has coordinates $(x_0, y_0, z_0)$

It follows that $r = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$, $\underset{\sim}{a} = x_0\underset{\sim}{i} + y_0\underset{\sim}{j} + z_0\underset{\sim}{k}$

and using the above equation

$$x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} = x_0\underset{\sim}{i} + y_0\underset{\sim}{j} + z_0\underset{\sim}{k} + \lambda(a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k})$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\left.\begin{aligned} x &= x_0 + \lambda a \\ y &= y_0 + \lambda b \\ c &= z_0 + \lambda \underset{\sim}{c} \end{aligned}\right\} \text{Parametric equation of a line}$$

CARTESIAN / SYMMETRIC FORM

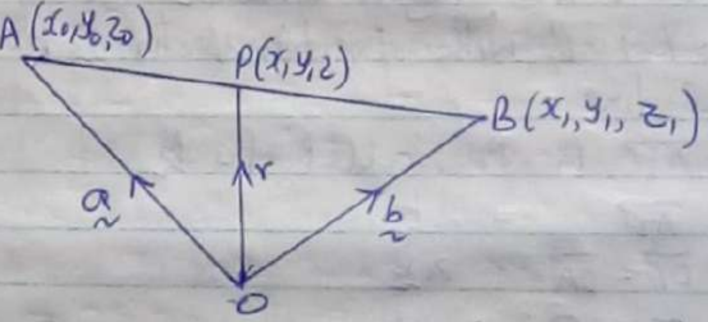
Make $\lambda$ the subject in the parametric form

$$\frac{x - x_0}{a} = \lambda, \quad \frac{y - y_0}{b} = \lambda \quad \frac{z - z_0}{c} = \lambda$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = \lambda$$ ; the constant $\lambda$ is normally dropped

$$\left.\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}\right\} \text{cartesian equation}$$

2 EQUATION OF A LINE THROUGH TWO POINTS A AND B




Suppose $\underset{\sim}{a}$ is the position vector of A, $b$ is the position vector of B. Let P( with position vector, r) be any point on the line

Then $\overrightarrow{AP}$ is parallel to $\overrightarrow{AB}$

$$\begin{aligned} \overrightarrow{AP} &= \lambda \overrightarrow{AB} \\ \overrightarrow{OP} - \overrightarrow{OA} &= \lambda(OB - \overrightarrow{OA}) \\ \underset{\sim}{r} - \underset{\sim}{a} &= \lambda(\underset{\sim}{b} - \underset{\sim}{a}) \\ \underset{\sim}{r} &= \underset{\sim}{a} + \lambda(\underset{\sim}{b} - \underset{\sim}{a}) \end{aligned}$$ — vector equation

## PARAMETRIC EQUATION

$P$ is a point $(x, y, z)$, $A$ is a point $(x_0, y_0, z_0)$ and $B$ is a point $(x_1, y_1, z_1)$; then

$$x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} = x_0\underset{\sim}{i} + y_0\underset{\sim}{j} + z_0\underset{\sim}{k} + \lambda\left[(x_1-x_0)\underset{\sim}{i} + (y_1-y_0)\underset{\sim}{j} + (z_1-z_0)\underset{\sim}{k}\right]$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} x_1-x_0 \\ y_1-y_0 \\ z_1-z_0 \end{pmatrix}$$

$$\left.\begin{aligned} x &= x_0 + \lambda(x_1 - x_0) \\ y &= y_0 + \lambda(y_1 - y_0) \\ z &= z_0 + \lambda(z_1 - z_0) \end{aligned}\right\} \text{Parametric equation}$$

Cartesian equation

$$\frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{z-z_0}{z_1-z_0}.$$

## EXAMPLES

1. Find the cartesian equation of the line passing through the point $(1, 2, 4)$ and parallel to $\underset{\sim}{b} = \underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$

Soln

$$\underset{\sim}{r} = a + \lambda \underset{\sim}{b}$$

$$\underset{\sim}{r} = \underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k} + \lambda(\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}) \quad \text{- vector equation}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{- vector equation}$$

$$\left.\begin{aligned} x &= 1 + \lambda \\ y &= 2 + \lambda \\ z &= 4 + 2\lambda \end{aligned}\right\} \text{Parametric equation}$$

$$\left. x - 1 = y - 2 = \frac{z-4}{2} \right\} \text{Cartesian equation}$$

2. Find the equation of the line passing through points A and B whose position vectors are

$\underset{\sim}{i}+2\underset{\sim}{j}-5\underset{\sim}{k}$ and $2\underset{\sim}{i}-5\underset{\sim}{j}+8\underset{\sim}{k}$

Soln.

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}2\\-5\\8\end{pmatrix}-\begin{pmatrix}1\\2\\-5\end{pmatrix}=\begin{pmatrix}1\\-7\\13\end{pmatrix}$$

Using $\underset{\sim}{r}=\underset{\sim}{a}+\lambda\overrightarrow{AB}$

$$\underset{\sim}{r}=\begin{pmatrix}1\\2\\-5\end{pmatrix}+\lambda\begin{pmatrix}1\\-7\\13\end{pmatrix}$$ – vector equation

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\2\\-5\end{pmatrix}+\lambda\begin{pmatrix}1\\-7\\13\end{pmatrix}$$

$$\left.\begin{aligned}x&=1+\lambda\\y&=2-7\lambda\\z&=-5+13\lambda\end{aligned}\right\}$$ Parametric equation

$$x-1=\frac{y-2}{-7}=\frac{z+5}{13}\Bigg\}$$ Cartesian equation

3 Find the cartesian equation of the line passing through the point $A(1,2,3)$ and is parallel to $2\underset{\sim}{i}-\underset{\sim}{j}+\underset{\sim}{k}$

4 Find the equation of a line through $A(0,1,0)$ and $B(1,0,1)$ in parametric form and cartesian form.

5 Find the vector equation of a straight line which passes through the points $A(1,0,-2)$ and $B(2,3,-1)$

To show that a given point lies on a line

Examples

1. Show that the point with coordinates (4, -1, 12) lies on the line $r = 2i + 3j + 4k + \lambda(i - 2j + 4k)$

Soln

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

Substituting for $(x, y, z)$

$$\begin{pmatrix} 4 \\ -1 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

$4 = 2 + \lambda, \quad \lambda = 2$

$-1 = 3 - 2\lambda, \quad \lambda = 2$

$12 = 4 + 4\lambda; \quad \lambda = 2$

Since the value of $\lambda$ is constant, therefore the point lies on the line

2. Show that the point with position vector $i - 9j + k$ lies on the line $r = 3i + 3j - k + \lambda(i + 6j - k)$

## INTERSECTION OF TWO LINES

To find the point of intersection of two lines, express the equation of the lines in parametric form, then determine unique values of parameters for which the two equations are simultaneously satisfied.

Examples

1. Show that the lines whose equations are:

$L_1: \dfrac{x+5}{2} = \dfrac{y+4}{2} = \dfrac{z+9}{4}$ and

$L_2: \dfrac{x+1}{2} = \dfrac{y-1}{1} = \dfrac{z-2}{1}$

intersect and find their point of intersection.

Soln

Let $\dfrac{x+5}{2} = \dfrac{y+4}{2} = \dfrac{z+9}{4} = \lambda$

$$\left.\begin{aligned} x &= -5+2\lambda \\ y &= -4+2\lambda \\ z &= -9+4\lambda \end{aligned}\right\} \text{ - - - - (1)}$$

Let $\frac{x+1}{2} = \frac{y-1}{1} = \frac{z-2}{1} = \mu$

$$\left.\begin{aligned} x &= -1+2\mu \\ y &= 1+\mu \\ z &= 2+\mu \end{aligned}\right\} \text{ - - - - (2)}$$

Equating equations (1) and (2)

$-5+2\lambda = -1+2\mu$

$2\lambda - 2\mu = 4$

$\lambda - \mu = 2$ - - - (3)

$-4+2\lambda = 1+\mu$

$2\lambda - \mu = 5$ - - - (4)

(4) - (3) ; $\lambda = 3$ and $\mu = 1$

for z ; $-9+4\lambda = 2+\mu$

$4\lambda - \mu = 11$

Substituting for $\lambda$ and $\mu$

$4(3) - 1 = 11$

Since the third equation is satisfied, the two lines do intersect.

Point of intersection is $(x, y, z)$

$x = -5+2(3) = 1$

$y = -4+2(3) = 2$

$z = -9+4(3) = 3$

Therefore the point of intersection is $(1, 2, 3)$

2. Show that the lines $l_1$ and $l_2$ with equations $\vec{r} = 2\underset{\sim}{i} + \underset{\sim}{j} + \mu(3\underset{\sim}{j} + 5\underset{\sim}{k})$ and $\vec{r} = \underset{\sim}{k} + \lambda(\underset{\sim}{i} - \underset{\sim}{j} - 3\underset{\sim}{k})$ respectively intersect and find the position vector of the point of intersection of $l_1$ and $l_2$

Soln

$$\vec{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix}, \quad \vec{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$$

At the point of intersection $l_1 = l_2$

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$$

$$2 = \lambda$$
$$1 + 3\mu = -\lambda$$
$$1 + 3\mu = -2$$
$$3\mu = -3$$
$$\mu = -1$$
$$5\mu = 1 - 3\lambda$$
$$5(-1) = 1 - 3(2)$$
$$-5 = -5$$

Since the third equation is satisfied the lines intersect;

$$r = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix}$$

$$r = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + -1 \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix}$$

$$r = \begin{pmatrix} 2 \\ -2 \\ -5 \end{pmatrix}$$ Is the position vector for the point of intersection

3. Find the point of intersection of the following lines

$L_1$: $\frac{x+1}{4} = y - 3, \quad z = 1$ and

$L_2$: $\frac{x+13}{12} = \frac{y-1}{6} = \frac{z-2}{3}$

4 Show that the lines

a) $r_1 = \begin{pmatrix} -2 \\ 8 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ and $r_2 = \begin{pmatrix} 8 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$

b) $\frac{x}{1} = \frac{y+2}{2} = \frac{z-5}{7}$ and $\frac{x-1}{-1} = \frac{y+3}{-3} = \frac{z-4}{1}$

intersect and find their points of intersection

5. show that the lines

a) $\frac{x-1}{2} = \frac{y}{1} = \frac{z-4}{4}$ and $\frac{x}{3} = \frac{y+2}{-2} = \frac{z}{-1}$

are perpendicular

b) $\underset{\sim}{r_1} = \underset{\sim}{i} - \underset{\sim}{j} + t_1(\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k})$ and $\underset{\sim}{r_2} = 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k} + t_2(-2\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k})$

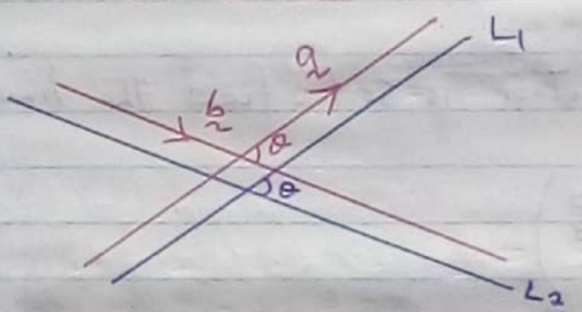
are parallel.

6. show that the lines with vector equations

$\vec{\underset{\sim}{r}} = 2\lambda i - 3\underset{\sim}{j} + (\lambda - 2)\underset{\sim}{k}$ and $\vec{r} = (\mu+1)\underset{\sim}{i} + (2-\mu)\underset{\sim}{j} + (2\mu - 5)\underset{\sim}{k}$

do not intersect

## ANGLE BETWEEN TWO LINES




The angle between two lines, is the angle between the two vectors which are parallel to the lines.

Examples

1. find the angle between the lines

$L_1: \frac{x-1}{2} = y = \frac{z+1}{3}$ and

$L_2: x = \frac{y-1}{4} = \frac{z+5}{3}$

Soln

$L_1$: Parallel vector; $\underset{\sim}{b_1} = 2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$

$L_2$: Parallel vector; $\underset{\sim}{b_2} = \underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}$

$\underset{\sim}{b_1} \cdot \underset{\sim}{b_2} = |\underset{\sim}{b_1}||\underset{\sim}{b_2}| \cos\theta$

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} = \sqrt{4+1+9}\,\sqrt{1+16+9}\,\cos\theta$$

$2+4+9 = \sqrt{14}\sqrt{26}\cos\theta$

$\cos\theta = \dfrac{15}{\sqrt{14}\sqrt{26}}$

$\theta = 38.27°$

2. Determine the acute angle between the lines

a) $\underset{\sim}{r_1} = 3\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k} + \lambda(2\underset{\sim}{i}+3\underset{\sim}{j}+6\underset{\sim}{k})$ and

$\underset{\sim}{r_2} = \underset{\sim}{i}+2\underset{\sim}{j}-3\underset{\sim}{k} + \mu(2\underset{\sim}{i}-2\underset{\sim}{j}+\underset{\sim}{k})$

(b) $\dfrac{x-2}{-4} = \dfrac{y-3}{3} = \dfrac{z+1}{-1}$ and $\dfrac{x-3}{2} = \dfrac{y-1}{6} = \dfrac{z+1}{-5}$

3. Given that the equations of two lines are $y = m_1x + c_1$ and $y = m_2x + c_2$, show that

(i) their vector equations are respectively

$\begin{pmatrix}0\\c_1\end{pmatrix} + \mu\begin{pmatrix}1\\m_1\end{pmatrix}$ and $\begin{pmatrix}0\\c_2\end{pmatrix} + \lambda\begin{pmatrix}1\\m_2\end{pmatrix}$

where $\mu$ and $\lambda$ are constants

(ii) the angle, $\alpha$ between them is $\tan^{-1}\left(\dfrac{m_1-m_2}{1+m_1m_2}\right)$

Soln

(i) Given $y = m_1x + c_1$ and $y = m_2x + c_2$

$y - c_1 = m_1x$ $\qquad$ $y - c_2 = m_2x$

$\dfrac{y-c_1}{m_1} = \dfrac{x}{1}$ $\qquad$ $\dfrac{y-c_2}{m_2} = \dfrac{x}{1}$

$\dfrac{x-0}{1} = \dfrac{y-c_1}{m_1}$ $\qquad$ $\dfrac{x-0}{1} = \dfrac{y-c_2}{m_2}$

$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$

$\underset{\sim}{r} = \begin{pmatrix}0\\c_1\end{pmatrix} + \mu\begin{pmatrix}1\\m_1\end{pmatrix}$ $\qquad$ $r = \begin{pmatrix}0\\c_2\end{pmatrix} + \lambda\begin{pmatrix}1\\m_2\end{pmatrix}$

(ii) $\begin{pmatrix}1\\m_1\end{pmatrix}\cdot\begin{pmatrix}1\\m_2\end{pmatrix} = \sqrt{1+m_1^2}\sqrt{1+m_2^2}\cos\alpha$

$1 + m_1m_2 = \sqrt{(1+m_1^2)(1+m_2^2)}\cos\alpha$

Square both sides

$(1+m_1m_2)^2 = (1+m_2^2+m_1^2+m_1^2m_2^2)\cos^2\alpha$

Divide @ term by $\cos^2\alpha$

$$(1+m_1m_2)^2\sec^2\alpha = 1+m_1^2+m_2^2+m_1^2m_2^2$$

$$(1+m_1m_2)^2[1+\tan^2\alpha] = 1+m_1^2+m_2^2+m_1^2m_2^2$$

$$(1+m_1m_2)^2+(1+m_1m_2)^2\tan^2\alpha = 1+m_1^2+m_2^2+m_1^2m_2^2$$

$$(1+m_1m_2)^2\tan^2\alpha = 1+m_1^2+m_2^2+m_1^2m_2^2-(1+m_1m_2)^2$$

$$(1+m_1m_2)^2\tan^2\alpha = 1+m_1^2+m_2^2+m_1^2m_2^2-(1+2m_1m_2+m_1^2m_2^2)$$

$$(1+m_1m_2)^2\tan^2\alpha = m_1^2+m_2^2-2m_1m_2$$

$$(1+m_1m_2)^2\tan^2\alpha = m_1^2-2m_1m_2+m_2^2$$

$$(1+m_1m_2)^2\tan^2\alpha = (m_1-m_2)^2$$

$$\tan^2\alpha = \frac{(m_1-m_2)^2}{(1+m_1m_2)^2}$$

$$\alpha = \frac{m_1-m_2}{1+m_1m_2}$$

$$\alpha = \tan^{-1}\left(\frac{m_1-m_2}{1+m_1m_2}\right)$$

4 Given the vectors $\underset{\sim}{a} = 3\underset{\sim}{i}-2\underset{\sim}{j}+\underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i}-2\underset{\sim}{j}+2\underset{\sim}{k}$, find

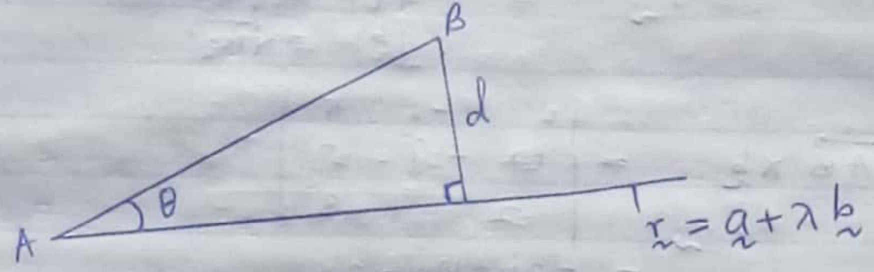
(i) the acute angle between the vectors

(ii) Vector, n such that it is perpendicular to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$

5

## PERPENDICULAR DISTANCE OF A POINT FROM A LINE

Consider the line $\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$, the perpendicular (shortest) distance of a point B from $\underset{\sim}{r}$ is d as shown in the diagram below




Now $\sin\theta = \dfrac{d}{|AB|}$

$$d = |\overrightarrow{AB}| \sin\theta \quad --- (1)$$

By using the cross product of two vectors

$$|\overrightarrow{AB} \times \hat{b}| = |\overrightarrow{AB}||\hat{b}| \sin\theta \quad ; \text{ But } |\hat{b}| = 1$$

$$|\overrightarrow{AB} \times \hat{b}| = |\overrightarrow{AB}| \sin\theta \quad ------ (2)$$

Equating equations (1) and (2)

$$d = |\overrightarrow{AB} \times \hat{b}|$$

But $\hat{b} = \dfrac{\underset{\sim}{b}}{|\underset{\sim}{b}|}$

$$d = \left| \overrightarrow{AB} \times \frac{b}{|\underset{\sim}{b}|} \right|$$

$$d = \frac{1}{|\underset{\sim}{b}|} \left| \overrightarrow{AB} \times \underset{\sim}{b} \right|$$

Examples:

1 Find the perpendicular distance of the point (1,1,4) from the line $\dfrac{x-1}{2} = y = \dfrac{z+1}{3}$

Soln

Method 1

Equation of the line : $\dfrac{x-1}{2} = y = \dfrac{z+1}{3}$

A(1, 0, −1) , B(1, 1, 4)

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}1\\1\\4\end{pmatrix} - \begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}0\\1\\5\end{pmatrix}$$

Parallel vector to the line, $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$

$$|\underset{\sim}{b}| = \sqrt{4+1+9}$$

$$|\underset{\sim}{b}| = \sqrt{14}$$

$$\overrightarrow{AB} \times \underset{\sim}{b} = \begin{vmatrix} i & j & k \\ 0 & 1 & 5 \\ 2 & 1 & 3 \end{vmatrix}$$

$$= -2\underset{\sim}{i} + 10\underset{\sim}{j} - 2\underset{\sim}{k}$$

$$|\overrightarrow{AB} \times \underset{\sim}{b}| = \sqrt{4+100+4}$$

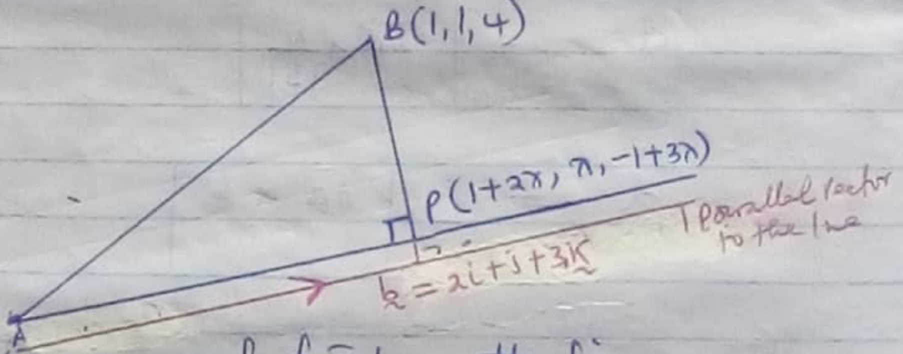
$$= \sqrt{108}$$

$$d = \frac{1}{|b|} \left| \overrightarrow{AB} \times \underset{\sim}{b} \right|$$

$$= \frac{1}{\sqrt{14}} \sqrt{108}$$

$$= \sqrt{\frac{54}{7}}$$

Method II




Let P be a general point on the line

Equation of the line: $\frac{x-1}{2} = \frac{y}{1} = \frac{z+1}{3} = \lambda$

$$x = 1+2\lambda, \; y = \lambda, \; z = -1+3\lambda$$

$$BP \cdot \underset{\sim}{b} = 0$$

$$(OP - OB) \cdot b = 0$$

$$\left[\begin{pmatrix}1+2\lambda\\ \lambda\\ -1+3\lambda\end{pmatrix} - \begin{pmatrix}1\\1\\4\end{pmatrix}\right] \cdot \begin{pmatrix}2\\1\\3\end{pmatrix} = 0$$

$$\begin{pmatrix} 2\lambda \\ \lambda-1 \\ 3\lambda-5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 0$$

$$4\lambda + \lambda - 1 + 9\lambda - 15 = 0$$

$$14\lambda = 16$$

$$\lambda = \frac{8}{7}$$

$$\overrightarrow{BP} = \begin{pmatrix} 2\lambda \\ \lambda-1 \\ 3\lambda-5 \end{pmatrix} = \begin{pmatrix} 2(8/7) \\ 8/7 - 1 \\ 3(8/7) - 5 \end{pmatrix} = \begin{pmatrix} 16/7 \\ 1/7 \\ -11/7 \end{pmatrix}$$

$$|\overrightarrow{BP}| = \sqrt{\left(\frac{16}{7}\right)^2 + \left(\frac{1}{7}\right)^2 + \left(\frac{-11}{7}\right)^2}$$

$$= \sqrt{\frac{54}{7}}$$

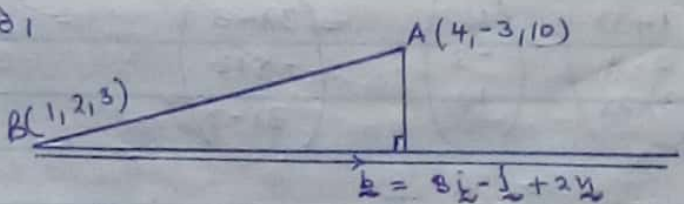
2(a) Find the perpendicular distance from the point $A(4,-3,10)$ to the line L with vector equation

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

(b) The points A, B and C have position vectors $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{j} - 4\underset{\sim}{k}$, $\underset{\sim}{c} = 6\underset{\sim}{i} + 4\underset{\sim}{j} + 5\underset{\sim}{k}$ respectively. Find the position vector of the point R on BC such that $\overrightarrow{AR}$ is perpendicular to $\overrightarrow{BC}$. Hence find the perpendicular distance of A from the line BC

<u>Soln</u>

a) Method 1




$$\overrightarrow{BA} = \begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 7 \end{pmatrix}$$

$$\underset{\sim}{b} = 3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$$

$$|\underset{\sim}{b}| = \sqrt{9+1+4} = \sqrt{14}$$

$$\overrightarrow{BA} \cdot \underset{\sim}{b} = \begin{vmatrix} i & -j & k \\ 3 & -5 & 7 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= -3\underset{\sim}{i} + 15\underset{\sim}{j} + 12\underset{\sim}{k}$$

$$|\overrightarrow{BA} \cdot \underset{\sim}{b}| = \sqrt{9+225+144}$$

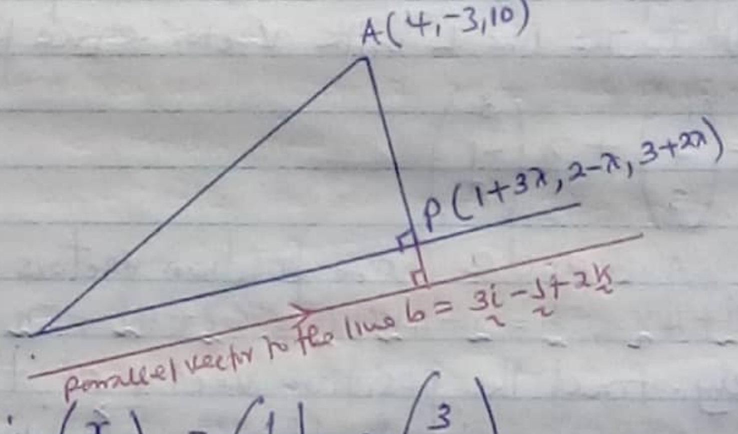
$$= \sqrt{378}$$

Perpendicular distance; $d = \frac{1}{|\underset{\sim}{b}|} |\overrightarrow{BA} \cdot \underset{\sim}{b}|$

$$= \frac{1}{\sqrt{14}} \sqrt{378}$$

$$= \sqrt{27}$$

Method II




line: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

$x = 1+3\lambda, \ y = 2-\lambda, \ z = 3+2\lambda$

$$\overrightarrow{AP} = \begin{pmatrix} 1+3\lambda \\ 2-\lambda \\ 3+2\lambda \end{pmatrix} - \begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix} = \begin{pmatrix} 3\lambda-3 \\ -\lambda+5 \\ 2\lambda-7 \end{pmatrix}$$

$$\overrightarrow{AP} \cdot b = 0$$

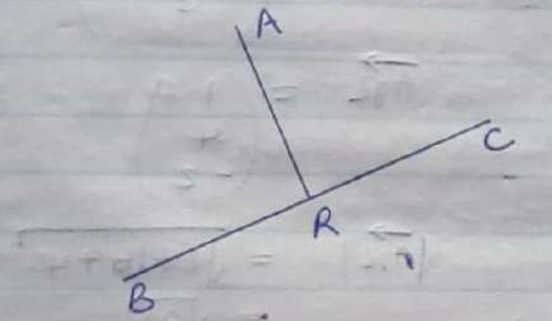
$$\begin{pmatrix} 3\lambda-3 \\ -\lambda+5 \\ 2\lambda-7 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 0$$

$$9\lambda - 9 + \lambda - 5 + 4\lambda - 14 = 0$$
$$= 14\lambda = 28$$
$$= \lambda = 2$$

$$\overrightarrow{AP} = \begin{pmatrix} 3(2)-3 \\ -2+5 \\ 2(2)-7 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$$

Perpendicular distance, $|\overrightarrow{AP}| = \sqrt{(3)^2+(3)^2+(-3)^2}$
$= \sqrt{27}$

(b)




$$\overrightarrow{BR} = \lambda\overrightarrow{BC}$$
$$\overrightarrow{OR} - \overrightarrow{OB} = \lambda(\overrightarrow{OC} - \overrightarrow{OB})$$
$$\overrightarrow{OR} = \overrightarrow{OB} + \lambda(\overrightarrow{OC} - \overrightarrow{OB})$$
$$x = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + \lambda\left[\begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}\right]$$

$$\underset{\sim}{r} = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + \lambda\begin{pmatrix} 6 \\ 3 \\ 9 \end{pmatrix}$$
$$\underset{\sim}{r} = 6\lambda\underset{\sim}{i} + (3\lambda+1)\underset{\sim}{j} + (9\lambda-4)\underset{\sim}{k}$$

Perpendicular distance $= |\overrightarrow{AR}|$
Since $\overrightarrow{AR}$ is perpendicular to $\overrightarrow{BC}$,

$$\overrightarrow{AR} \cdot \overrightarrow{BC} = 0$$
$$(OR - OA) \cdot BC = 0$$
$$\left[\begin{pmatrix} 6\lambda \\ 3\lambda+1 \\ 9\lambda-4 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}\right] \cdot \begin{pmatrix} 6 \\ 3 \\ 9 \end{pmatrix} = 0$$

$$\begin{pmatrix} 6\lambda-3 \\ 3\lambda+2 \\ 9\lambda-8 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 3 \\ 9 \end{pmatrix} = 0$$

$$36\lambda - 18 + 9\lambda + 6 + 81\lambda - 72 = 0$$
$$126\lambda = 84$$
$$\lambda = \tfrac{2}{3}$$

Substituting for $\lambda$ into $\overrightarrow{AR} = \begin{pmatrix} 6\lambda - 3 \\ 3\lambda + 2 \\ 9\lambda - 8 \end{pmatrix}$

$$\overrightarrow{AR} = \begin{pmatrix} 6 \times \tfrac{2}{3} - 3 \\ 3 \times \tfrac{2}{3} + 2 \\ 9 \times \tfrac{2}{3} - 8 \end{pmatrix}$$

$$\overrightarrow{AR} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$$

$$|\overrightarrow{AR}| = \sqrt{1 + 16 + 4}$$
$$= \sqrt{21}$$

3 For each of the following parts find the perpendicular distance from the given point to the given line

a) the point with position vector $\begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$ and the line

$$\underset{\sim}{x} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

b) the point with position vector $3\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ and the line $\underset{\sim}{x} = \underset{\sim}{i} - 6\underset{\sim}{j} - 2\underset{\sim}{k} + \lambda(\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k})$

c) the point $(1, 1, 3)$ and the line $\frac{x+4}{2} = \frac{y+1}{3} = \frac{z-1}{3}$

d) the point $(-6, -4, -5)$ and the line $x - 5 = \frac{y-6}{2} = \frac{z-3}{4}$

## Distance between Parallel lines

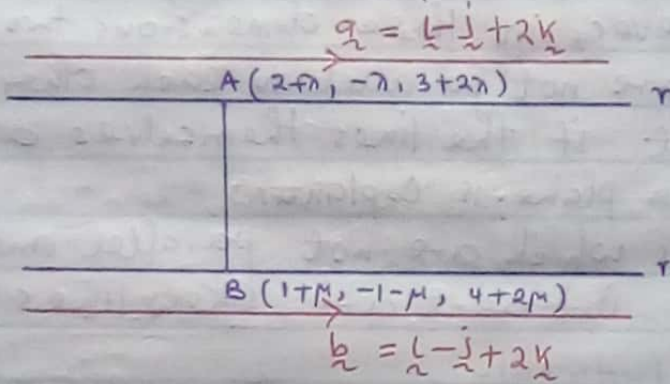
Examples:

Determine the distance between the following pairs of parallel lines

a) $\underset{\sim}{r_1} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\underset{\sim}{r_2} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$ ~~and~~ $+ \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

b) $\frac{x-2}{1} = \frac{y-1}{-1} = \frac{z-3}{2}$ and $\frac{x+1}{1} = \frac{y-3}{-1} = \frac{z-1}{2}$

Soln




$$\overrightarrow{AB} = \begin{pmatrix} 1+\mu \\ -1-\mu \\ 4+2\mu \end{pmatrix} - \begin{pmatrix} 2+\lambda \\ -\lambda \\ 3+2\lambda \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} \mu-\lambda-1 \\ -\mu+\lambda-1 \\ 2\mu-2\lambda+1 \end{pmatrix}$$

$$AB \cdot \underset{\sim}{a} = 0$$

$$\begin{pmatrix} \mu-\lambda-1 \\ -\mu+\lambda-1 \\ 2\mu-2\lambda+1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 0$$

$$\mu-\lambda-1+\mu-\lambda+1+4\mu-4\lambda+2 = 0$$

$$6\mu-6\lambda = -2$$

$$6(\mu-\lambda) = -2$$

$$\mu-\lambda = -\tfrac{1}{3}$$

On substitution in $\overrightarrow{AB}$

$$\overrightarrow{AB} = \begin{pmatrix} \mu-\lambda-1 \\ -\mu+\lambda-1 \\ 2\mu-2\lambda+1 \end{pmatrix} = \begin{matrix} -\tfrac{1}{3}-1 \\ \tfrac{1}{3}-1 \\ 2(-\tfrac{1}{3})+1 \end{matrix} = \begin{matrix} -\tfrac{4}{3} \\ -\tfrac{2}{3} \\ \tfrac{1}{3} \end{matrix}$$

$$|\overrightarrow{AB}| = \sqrt{\left(-\tfrac{4}{3}\right)^2 + \left(-\tfrac{2}{3}\right)^2 + \left(\tfrac{1}{3}\right)^2}$$

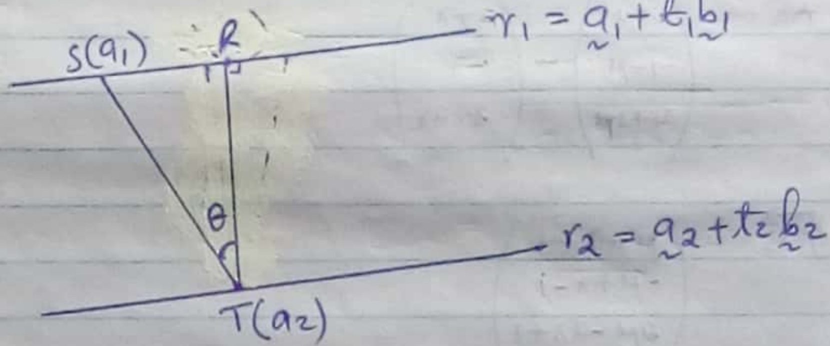
$$|\overrightarrow{AB}| = \sqrt{\tfrac{7}{3}} = \frac{\sqrt{7}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{21}}{3}$$

## SKEW LINES

When considering a two-dimensional situation, two lines that are not parallel to each other will intersect.

However, in three dimensions two lines which are not parallel to each other will intersect if the lines themselves are in the same plane.

Two lines which are not parallel and yet do not intersect are skew lines




If two lines are skew, the shortest distance between them is the perpendicular distance RT to each of these lines

$$\cos\theta = \frac{|RT|}{|ST|}$$

$$RT = |ST| \cos\theta \quad ---- (1)$$

$\theta$ is the angle between RT and ST

Using the dot product

$$\overrightarrow{ST} \cdot \hat{b} = |\overrightarrow{ST}||\hat{b}| \cos\theta ; \text{ but } |\hat{b}| = 1$$

$$\overrightarrow{ST} \cdot \hat{b} = |\overrightarrow{ST}| \cos\theta \quad ---- (2)$$

$\underset{\sim}{b}$ is a unit vector in the direction of RT

from (1) and (2)

$RT = \vec{ST} \cdot \hat{b}$ - - - - - - - (3)

We need to determine $\hat{b}$

Since $\underset{\sim}{b}$ is the vector in the direction of $\vec{RT}$, and because $\vec{RT}$ is perpendicular to both $r_1$ and $r_2$

then $\underset{\sim}{b} = \underset{\sim}{b_1} \times \underset{\sim}{b_2}$

Hence $\hat{b} = \dfrac{\underset{\sim}{b_1} \times \underset{\sim}{b_2}}{|\underset{\sim}{b_1} \times \underset{\sim}{b_2}|}$ - - - - - - (4)

Substituting (4) in (3)

$$RT = \vec{ST} \cdot \frac{\underset{\sim}{b_1} \times \underset{\sim}{b_2}}{|\underset{\sim}{b_1} \times \underset{\sim}{b_2}|}$$

Now $\vec{ST} = \vec{OT} - \vec{OS} = a_2 - a_1$

Hence $RT = (a_2 - a_1) \cdot \dfrac{\underset{\sim}{b_1} \times \underset{\sim}{b_2}}{|\underset{\sim}{b_1} \times \underset{\sim}{b_2}|}$

$$RT = \frac{(a_2 - a_1) \cdot (\underset{\sim}{b_1} \times \underset{\sim}{b_2})}{|\underset{\sim}{b_1} \times \underset{\sim}{b_2}|} \quad - - - - (5)$$

Condition for two lines to intersect

from equation (5), we deduce that if two lines do intersect then $RT = 0$, hence

$(\underset{\sim}{a_2} - \underset{\sim}{a_1}) \cdot (\underset{\sim}{b_1} \times \underset{\sim}{b_2}) = 0$

N.B This is just a condition for the lines to intersect, and does not give the position vector for the point of intersection

Examples

1. Show that the following lines don't intersect and find the shortest distance between them

$L_1: \dfrac{x-1}{2} = \dfrac{y+1}{3} = z, \quad L_2: \dfrac{x+1}{5} = \dfrac{y-2}{1}, \quad z = 2$

Soln.

for $L_1$; $A(1, -1, 0)$, $a_1 = \underset{\sim}{i} - \underset{\sim}{j}$

for $L_2$; $B(-1, 2, 2)$, $\underset{\sim}{a_2} = -\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$

for $L_1$; $\underset{\sim}{b_1} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$

for $L_2$; $\underset{\sim}{b_2} = 5\underset{\sim}{i} + \underset{\sim}{j}$

$$\underset{\sim}{a_2} - \underset{\sim}{a} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$$

$$\underset{\sim}{b_1} \times \underset{\sim}{b_2} = \begin{vmatrix} i & -j & k \\ 2 & 3 & 1 \\ 5 & 1 & 0 \end{vmatrix}$$

$$= -\underset{\sim}{i} + 5\underset{\sim}{j} - 13\underset{\sim}{k}$$

$$(\underset{\sim}{a_2} - \underset{\sim}{a_1}) \cdot (\underset{\sim}{b_1} \times \underset{\sim}{b_2}) = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ -13 \end{pmatrix}$$

$$= -9$$

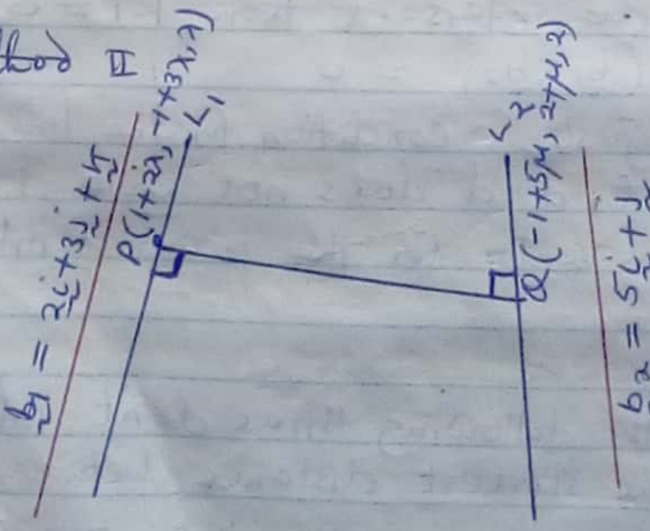
Since $(\underset{\sim}{a_2} - \underset{\sim}{a_1}) \cdot (\underset{\sim}{b_1} \times \underset{\sim}{b_2}) \neq 0$, then the lines dont intersect.

Shortest distance, $d = \left| \dfrac{(\underset{\sim}{a_2} - \underset{\sim}{a_1}) \cdot (\underset{\sim}{b_1} \times \underset{\sim}{b_2})}{|\underset{\sim}{b_1} \times \underset{\sim}{b_2}|} \right|$

$$= \frac{|-9|}{\sqrt{(-1)^2 + (5)^2 + (-13)^2}}$$

$$= \frac{9}{\sqrt{195}}$$

Method II




$L_1$: $\dfrac{x-1}{2} = \dfrac{y+1}{3} = z = \lambda$;

General point on the line $x = 1 + 2\lambda$, $y = -1 + 3\lambda$, $z = \lambda$

$L_2$: $\frac{x+1}{5} = \frac{y-2}{1}$ , $z = 2$

$$\frac{x+1}{5} = \frac{y-2}{1} = \frac{z-2}{0} = \mu$$

General point on the line; $x = -1+5\mu$, $y = 2+\mu$, $z = 2$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= \begin{pmatrix} -1+5\mu \\ 2+\mu \\ 2 \end{pmatrix} - \begin{pmatrix} 1+2\lambda \\ -1+3\lambda \\ \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 5\mu - 2\lambda - 2 \\ \mu - 3\lambda + 3 \\ 2-\lambda \end{pmatrix}$$

$$\overrightarrow{PQ} \cdot \underset{\sim}{b_1} = 0$$

$$\begin{pmatrix} 5\mu - 2\lambda - 2 \\ \mu - 3\lambda + 3 \\ 2-\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 0$$

$$10\mu - 4\lambda - 4 + 3\mu - 9\lambda + 9 + 2 - \lambda = 0$$

$$13\mu - 14\lambda = -7 \quad ---- (1)$$

$$\overrightarrow{PQ} \cdot b_2 = 0$$

$$\begin{pmatrix} 5\mu - 2\lambda - 2 \\ \mu - 3\lambda + 3 \\ 2-\lambda \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$25\mu - 10\lambda - 10 + \mu - 3\lambda + 3 = 0$$

$$26\mu - 13\lambda = +7 \quad ---- (2)$$

Eqn (1) × 2; $26\mu - 28\lambda = -14$

$$15\lambda = +21$$

$$\lambda = \frac{+7}{15}$$

Substituting in (1) $13\mu - 14\left(\frac{+7}{15}\right) = -7$

$$13\mu = \frac{-63}{15}$$

$$\mu = \frac{-63}{165}$$

$$\vec{PQ} = \begin{pmatrix} 5\mu - 2\lambda - 2 \\ \mu - 3\lambda + 3 \\ 2 - \lambda \end{pmatrix}$$ but $\lambda =$

$$= \begin{pmatrix} 3/65 \\ -3/13 \\ 3/5 \end{pmatrix}$$

$$|\vec{PQ}| = \sqrt{\left(\frac{3}{65}\right)^2 + \left(\frac{-3}{13}\right)^2 + \left(\frac{3}{5}\right)^2}$$

$$= \sqrt{\frac{27}{65}}$$

$$= 0.6445.$$

2. The equations $\frac{x-3}{2} = \frac{y-5}{1} = \frac{z-7}{-4}$, $\frac{x+1}{3} = \frac{y+4}{1} = \frac{z-2}{-2}$ represent pipes A and B in a chemical plant, where length is measured in metres. A by-pass is to be installed connecting A and B. Find the length of the shortest pipe that may be fitted, and the location of its end points

Soln.

Method I:

$L_1$: $\frac{x-3}{2} = \frac{y-5}{1} = \frac{z-7}{-4}$ ; $a_1 = (3,5,7)$, $\underline{b}_1 = 2\underline{i} + \underline{j} - 4\underline{k}$

$L_2$: $\frac{x+1}{3} = \frac{y+4}{1} = \frac{z-2}{-2}$ ; $a_2 = (-1,-4,2)$, $b_2 = 3\underline{i} + \underline{j} - 2\underline{k}$

$$\underline{a}_2 - \underline{a}_1 = \begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} -4 \\ -9 \\ -5 \end{pmatrix}$$

$$\underline{b}_1 \times \underline{b}_2 = \begin{vmatrix} i & -j & k \\ 2 & 1 & -4 \\ 3 & 1 & -2 \end{vmatrix}$$

$$= 2i + 8\underline{j} - \underline{k}$$

$$|\underset{\sim}{b}_1 \times b_2| = \sqrt{4+64+1} = \sqrt{69}$$

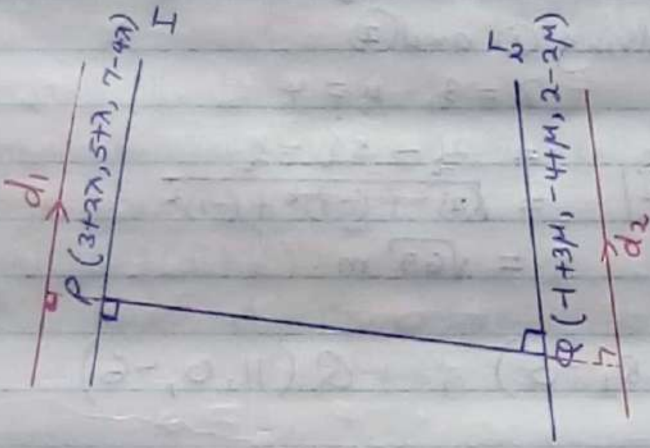
$$(\underset{\sim}{a}_2 - \underset{\sim}{a}_1) \cdot (b_1 \times b_2) = \begin{pmatrix} -4 \\ -9 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -8 \\ -1 \end{pmatrix} = 69$$

Shortest distance; $d = \dfrac{|(a_2 - \underset{\sim}{a}_1) \cdot (\underset{\sim}{b}_1 \times \underset{\sim}{b}_2)|}{|\underset{\sim}{b}_1 \times \underset{\sim}{b}_2|}$

$$= \frac{69}{\sqrt{69}}$$

$$= \sqrt{69}$$

Method II




$L_1: \dfrac{x-3}{2} = \dfrac{y-5}{1} = \dfrac{z-7}{-4} = \lambda$

General point on the line; $x = 3+2\lambda,\ y = 5+\lambda,\ z = 7-4\lambda$

$L_2: \dfrac{x+1}{3} = \dfrac{y+4}{1} = \dfrac{z-2}{-2} = \mu$

General point on the line; $x = -1+3\mu,\ y = -4+\mu,\ z = 2-2\mu$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= \begin{pmatrix} -1+3\mu \\ -4+\mu \\ 2-2\mu \end{pmatrix} - \begin{pmatrix} 3+2\lambda \\ 5+\lambda \\ 7-4\lambda \end{pmatrix}$$

$$= \begin{pmatrix} -4+3\mu-2\lambda \\ -9+\mu-\lambda \\ -5-2\mu+4\lambda \end{pmatrix}$$

But $\vec{PQ} \cdot d_1 = 0$ and $\vec{PQ} \cdot d_2 = 0$

$$\begin{pmatrix} -4+3\mu-\lambda \\ -9+\mu-\lambda \\ -5-2\mu+4\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} = 0$$

$$5\mu - 7\lambda = -1 \quad --- (1)$$

Similarly

$$\begin{pmatrix} -4+3\mu-2\lambda \\ -9+\mu-\lambda \\ -5-2\mu+4\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 0$$

$$14\mu - 15\lambda = 11 \quad --- (2)$$

On solving (1) and (2)

$$\lambda = 3, \quad \mu = 4$$

$$\vec{PQ} = 2\underset{\sim}{i} - 8\underset{\sim}{j} - \underset{\sim}{k}$$

$$|\vec{PQ}| = \sqrt{(2)^2 + (-8)^2 + (-1)^2}$$

$$= \sqrt{69} \text{ m}$$

$$P(9, 8, -5), \quad Q(11, 0, -6)$$

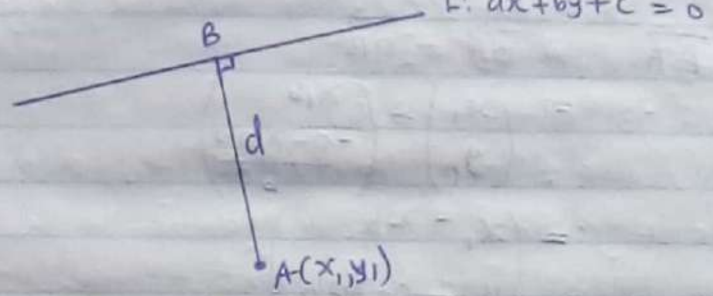
3. Find the shortest distance between the two skew lines $L_1$ and $L_2$ given that $L_1$ has vector equation $\underset{\sim}{r} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $L_2$ has vector equation $\underset{\sim}{r} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

4 Find the shortest distance between the two skew lines given in each of the following parts

a) $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\underset{\sim}{r} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

b) $\dfrac{x-2}{0} = \dfrac{y+1}{1} = \dfrac{z}{2}$ and $\dfrac{x+1}{1} = \dfrac{y-1}{-3} = \dfrac{z-1}{-2}$

Qn. The diagram shows the line L and the point A both lying in the x-y plane. L has Cartesian equation $ax+by+c=0$ and A is the point $(x_1, y_1)$. A is a perpendicular distance d from L




Show that the vector equation $\underset{\sim}{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -c/b \end{pmatrix} + \lambda \begin{pmatrix} b \\ -a \end{pmatrix}$

also represents the line

If $\lambda$ takes the value $\lambda_1$ at point B show that

$$\lambda_1 = \frac{b^2x_1 - aby_1 - ac}{b(a^2+b^2)}$$

Hence show that $d = \dfrac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$

Soln.

$$ax+by+c=0$$

$$\frac{ax}{ab} = \frac{-by-c}{ab}$$

$$\frac{x}{b} = \frac{-by-c}{ab} \quad ; \text{ divide each term on the RHS by } -b$$

$$\frac{x}{b} = \frac{y+c/b}{-a}$$

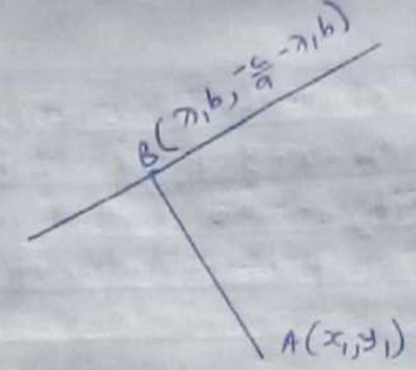
Equation of the line is given by $\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$

$$\underset{\sim}{r} = \begin{pmatrix} 0 \\ -c/b \end{pmatrix} + \lambda \begin{pmatrix} b \\ -a \end{pmatrix}$$

If $\lambda$ takes the value $\lambda_1$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -c/b \end{pmatrix} + \lambda_1 \begin{pmatrix} b \\ -a \end{pmatrix}$$

$$x = \lambda_1 b$$
$$y = -c/b - \lambda_1 \underset{\sim}{a}$$




$$\vec{BA} = \vec{OA} - \vec{OB}$$

$$= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} \lambda_1 b \\ \frac{-c-\lambda_1 ab}{b} \end{pmatrix}$$

$$= \begin{pmatrix} x_1 - \lambda_1 b \\ y_1 + \frac{c+\lambda ab}{b} \end{pmatrix}$$

$$= \begin{pmatrix} x_1 - \lambda_1 b \\ \frac{by_1 + c + \lambda_1 ab}{b} \end{pmatrix}$$

$$\vec{BA} \cdot d = 0$$

$$\begin{pmatrix} x_1 - \lambda_1 b \\ \frac{by_1 + c + \lambda_1 ab}{b} \end{pmatrix} \cdot \begin{pmatrix} b \\ -a \end{pmatrix} = 0$$

$$bx_1 - \lambda_1 b^2 - \frac{aby_1 - ac - \lambda a^2 b}{b} = 0$$

$$b^2 x_1 - \lambda_1 b^3 - aby_1 - ac - \lambda_1 a^2 b = 0$$

$$b^2 x_1 - aby_1 - ac = \lambda_1 b^3 + \lambda_1 a^2 b$$

$$b^2 x_1 - aby_1 - ac = \lambda_1 (b^3 + a^2 b)$$

$$\lambda_1 = \frac{b^2 x_1 - aby_1 - ac}{b^3 + a^2 b}$$

$$\lambda_1 = \frac{b^2 x_1 - aby_1 - ac}{b(a^2+b^2)}$$

Exercise

1. Find the vector equation for the line passing through the point
   a) $(4,3)$ and parallel to the vector $\underset{\sim}{i}-2\underset{\sim}{j}$
   b) $(5,-2,3)$ and parallel to the vector $4\underset{\sim}{i}-3\underset{\sim}{j}+\underset{\sim}{k}$
   c) $(5,-1)$ and perpendicular to the vector $\underset{\sim}{i}+\underset{\sim}{j}$
2. Find a cartesian equation for the line joining the points
   a) $(2,6)$ and $(5,-2)$
   b) $(-1,2,-3)$ and $(6,3,0)$
3(a) Points A and B have coordinates $(4,1)$ and $(2,-5)$ respectively. Find the vector equation for the line which passes through the point A and perpendicular to the line AB.
(b) Points P and Q have coordinates $(3,5)$ and $(-3,-7)$ respectivectively. Find a vector equation for the line which passes through point P and which is perpendicular to the line PQ
4 Two lines $L_1$ and $L_2$, have equations

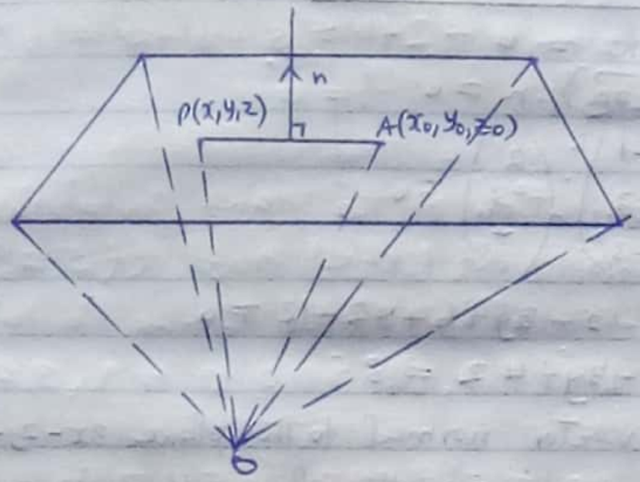
$$L_1: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$$

and $L_2$: $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

   a) Show that $L_1$ and $L_2$ are concurrent (meet at a common point) and find the position vector of their point of intersection
   b) Find the angle between $L_1$ and $L_2$
5. Two lines $L_1$ and $L_2$ have vector equations
   $\underset{\sim}{r_1} = (2-3\lambda)\underset{\sim}{i} + (1+\lambda)\underset{\sim}{j} + 4\lambda\underset{\sim}{k}$ and
   $\underset{\sim}{r_2} = (-1+3\mu)\underset{\sim}{i} + 3\underset{\sim}{j} + (4-\mu)\underset{\sim}{k}$, respectively
   Find
   a) the position vector of their common point of intersection
   b) the angle between the lines

# PLANES

1 Equation of the plane given the point on the plane and a normal vector, $\underline{n}$




The point on the plane is $A(x_0, y_0, z_0)$; $P(x, y, z)$ is a general point on the plane.
Normal vector is $\underline{n} = a\underline{i} + b\underline{j} + c\underline{k}$
Using dot product

$$\overrightarrow{AP} \cdot \underline{n} = 0$$

$$(\overrightarrow{OP} - \overrightarrow{OA}) \cdot \underline{n} = 0$$

$$\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}\right] \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$ax - ax_0 + by - by_0 + cz - cz_0 = 0$$

$$ax + by + cz = ax_0 + by_0 + cz_0 \text{; let } ax_0 + by_0 + cz_0 = d$$

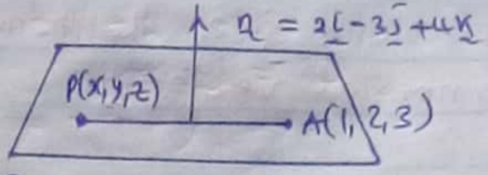
$ax + by + cz = d$ is the equation of a plane.

Examples:

1. Find the equation of a plane through the point $(1, 2, 3)$ parallel to the plane $2x - 3y + 4z = 1$

Soln

If two planes are parallel, they have the same normal vector to the plane; $\underline{n} = 2\underline{i} - 3\underline{j} + 4\underline{k}$
(ie to coefficients of $x$, $y$ and $z$ give the normal vector to the plane)




$$\overrightarrow{AP} \cdot n = 0$$

$$(\overrightarrow{OP} - \overrightarrow{OA}) \cdot n = 0$$

$$\begin{pmatrix} x-1 \\ y-2 \\ z-3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = 0$$

$$2x - 2 - 3y + 6 + 4z - 12 = 0$$

$$2x - 3y + 4z = 8$$

2(a) Find the vector normal to the plane $3x - 2y - z = 10$

(b) Find the equation of the plane that is normal to $5\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ and passes through the point $A(4, 1, -3)$

c) Find the equation of the plane that is normal to $4\underset{\sim}{i} + 6\underset{\sim}{j} + 5\underset{\sim}{k}$ and passes through the point with position vector $\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$

## 2 EQUATION OF THE PLANE GIVEN THREE NON-COLLINEAR POINTS

Two coterminant vectors are determined in the plane using three points.
The cross product of the two vectors is the normal vector to the plane. Using a general point $(x, y, z)$ and any of the three given points and the normal vector, the equation of the plane is obtained in the same way as in the previous section

Examples:

1. Find the equation of the plane through $A(3, 2, 1)$, $B(2, 1, -1)$ and $C(-1, 3, 3)$

Soln

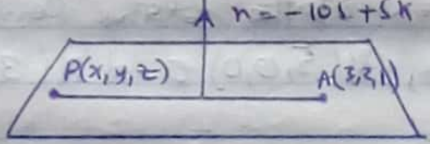
$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix}$$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix}-1\\3\\3\end{pmatrix} - \begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}-4\\1\\2\end{pmatrix}$

Normal vector, $\underset{\sim}{n} = \overrightarrow{AB} \times \overrightarrow{AC}$

$$\begin{vmatrix} i & -j & k \\ -1 & -1 & -2 \\ -4 & 1 & 2 \end{vmatrix}$$

$= -10\underset{\sim}{j} + 5k$




$\overrightarrow{AP} \cdot n = 0$

$\begin{pmatrix}x-3\\y-2\\z-1\end{pmatrix} \cdot \begin{pmatrix}0\\-10\\5\end{pmatrix} = 0$

$-10y + 20 + 5z - 5 = 0$

$-10y + 5z - 15 = 0.$

2. Find the equation of the plane through the points $A(3,2,1)$, $B(1,1,3)$ and $C(2,2,3)$

Soln

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}1\\1\\3\end{pmatrix} - \begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}-2\\-1\\2\end{pmatrix}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix}2\\2\\3\end{pmatrix} - \begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}-1\\0\\2\end{pmatrix}$

$AB \times AC$

Normal vector, $n = \begin{pmatrix} i & -j & k \\ -2 & -1 & 2 \\ -1 & 0 & 2 \end{pmatrix}$

$= -2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$

Using A and a general point $P(x,y,z)$ on the Plane

$\overrightarrow{AP} \cdot n = 0$

$$\begin{pmatrix} x-3 \\ y-2 \\ z-1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} = 0$$

$$-2x+6+2y-4-z+1=0$$

$$-2x+2y-z+1 = 0$$

3 Find the equation of the plane containing the points $P(1,2,5)$, $Q(1,0,4)$ and $R(5,2,1)$

4 Find the equation of the plane containing the points $L(1,1,1)$, $M(5,0,0)$ and $N(3,2,1)$

3 EQUATION OF THE PLANE GIVEN A POINT AND A LINE IN THE PLANE

Two points on the line are chosen arbitrary. These points would be lying in the plane. These points and the given one are three points in the plane and the procedure would be in the (2) above

Example

Find the equation of the plane through $(2,0,1)$ Containing the line $\frac{x}{1} = \frac{y}{-1} = \frac{z}{2}$

Soln:

Let $\frac{x}{1} = \frac{y}{-1} = \frac{z}{2} = \lambda$

Put $\lambda = 1$ ; $x=1, y=-1, z=2$

Put $\lambda = 0$, $x=0, y=0, z=0$

$A(2,0,1)$ , $B(1,-1,2)$ , $C(0,0,0)$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$$

$$n = \overrightarrow{AB} \times \overrightarrow{AC}$$

$$n = \begin{pmatrix} i & -j & k \\ -1 & -1 & 1 \\ -2 & 0 & -1 \end{pmatrix}$$

$$= \underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k}$$

Let $P(x,y,z)$ be a general point.

$$\overrightarrow{AP} \cdot n = 0$$

$$\begin{pmatrix} x-2 \\ y-0 \\ z-1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} = 0$$

$$x-2-3y-2z+2=0$$

$$x-3y-2z=0$$

4 EQUATION OF THE PLANE GIVEN TWO INTERSECTING LINES IN THE PLANE

The parallel vector to each of the lines is written down. The cross product of the two vectors is the vector perpendicular to the plane. A point on either of the lines is chosen arbitrary and the process that follows is one above.

Example:

Verify that the lines

$$L_1: \frac{x-2}{1} = \frac{y-1}{2} = \frac{z-0}{3}$$

$$L_2: \frac{x-1}{4} = \frac{y-2}{1} = \frac{z-3}{-2}$$

intersect and find the equation of the plane containing them.

Soln.

for lines to intersect $(a_2-a_1)\cdot(b_1 \times b_2) = 0$

$a_1 = 2i + j + 0k$ , $a_2 = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$b_1 = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$, $b_2 = 4\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$.

$$\underset{\sim}{a_2} - \underset{\sim}{a_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

$$\underset{\sim}{b_1} \times \underset{\sim}{b_2} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 4 & 1 & -2 \end{vmatrix}$$

$$= -7\underset{\sim}{i} + 14\underset{\sim}{j} - 7\underset{\sim}{k}$$

$$(a_2 - a_1) \cdot b_1 \times b_2 = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -7 \\ 14 \\ -7 \end{pmatrix} = 7 + 14 - 21 = 0$$

Since $(a_2 - a_1)\cdot(b_1 \times b_2) = 0$, the lines do intersect

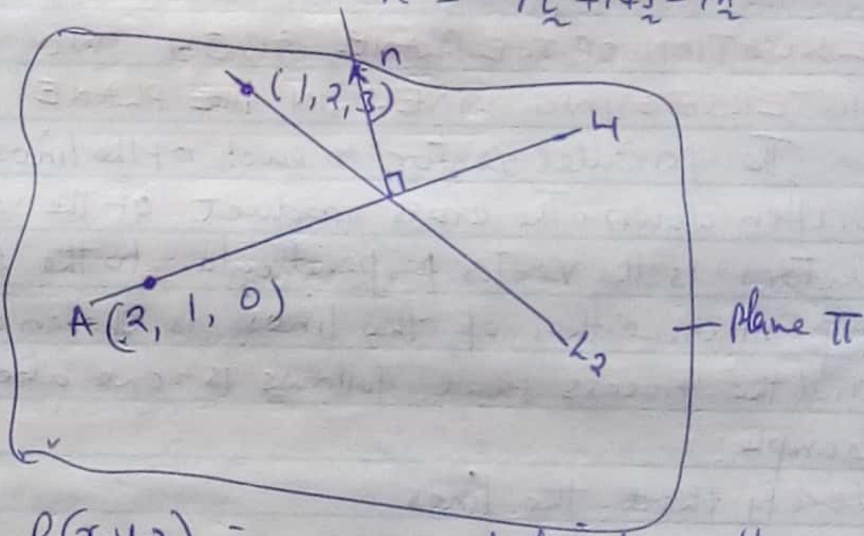
A vector parallel to $L_1$; $b_1 = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$

A vector parallel to $L_2$: $b_2 = 4\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$

normal vector to the plane, $n = \underset{\sim}{b_1} \times b_2$

$$n = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 4 & 1 & -2 \end{vmatrix}$$

$$n = -7\underset{\sim}{i} + 14\underset{\sim}{j} - 7\underset{\sim}{k}$$




If $P(x, y, z)$ is a general point on the plane

$$\overrightarrow{AP} \cdot n = 0$$

$$\begin{pmatrix} x-2 \\ y-1 \\ z-0 \end{pmatrix} \cdot \begin{pmatrix} -7 \\ 14 \\ -7 \end{pmatrix} = 0$$

$$-7x + 14 + 14y - 14 - 7z = 0$$

$$x - 2y + z = 0$$

2. Verify that the lines

$L_1$: $\frac{x-3}{5} = \frac{y+1}{2} = \frac{z-3}{1}$ and

$L_2: \frac{x-3}{2} = \frac{y+1}{4} = \frac{z-3}{3}$ intersect and find the equation of the plane containing these lines

5 EQUATION OF THE PLANE GIVEN TWO PARALLEL LINES IN THE PLANE:

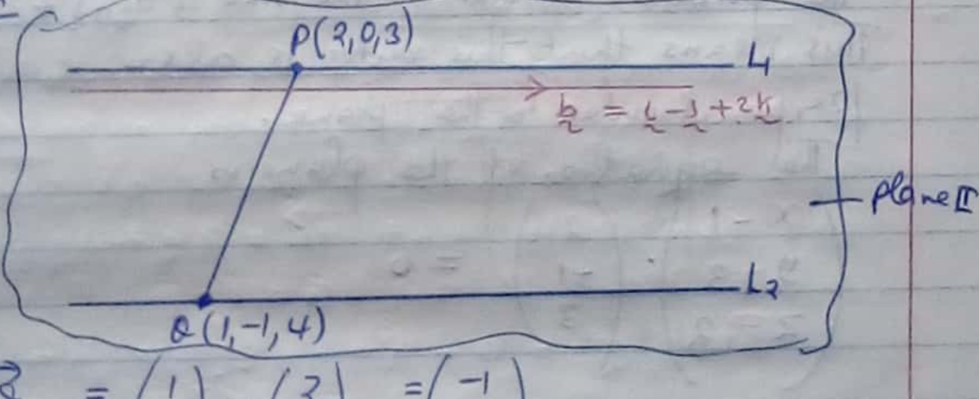
Two points $P$ and $Q$ are chosen on each of the lines. The cross product of $PQ$ and the vector parallel to the two lines gives the normal vector to the plane. Once the normal vector is obtained with any point chosen on the lines, the equation of the plane can be obtained

Example:

Find the equation of the plane containing the following parallel lines

$L_1: r_1 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $r_2 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

Soln




$$\overrightarrow{PQ} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

normal vector to the plane, $n = \overrightarrow{PQ} \times b$.

$$n = \begin{vmatrix} i & -j & k \\ -1 & -1 & 1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$n = -i + 3j + 2k$$

$$\begin{pmatrix} x-2 \\ y-0 \\ z-3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = 0$$

$$-x+2+3y+2z-6=0$$
$$-x+3y+2z-4=0$$

2. Find the equation of the plane containing the following parallel lines

$L_1: \frac{x-2}{1} = \frac{y-1}{7} = \frac{z-3}{2}$ and

$L_2: \frac{x+1}{1} = \frac{y-3}{7} = \frac{z-1}{2}$

6 EQUATION OF THE PLANE GIVEN ONE POINT IN THE PLANE AND A PERPENDICULAR LINE

Example.

Find the equation of the plane perpendicular to the line $\frac{x-1}{2} = \frac{y}{-1} = \frac{z}{3}$ and passing through the point $B(1,-3,2)$

Soln

The parallel vector to the line is $2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$

This means that this vector is also perpendicular to the plane.

The equation of the plane is

$$\begin{pmatrix} x-1 \\ y+3 \\ z-2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 0$$

$$2x-2-y-3+3z-6=0$$
$$2x-y+3z=11$$

7 EQUATION OF THE PLANE GIVEN TWO POINTS IN THE PLANE

Example;

Find the equation of the plane containing the points $A(1,2,-1)$ and $B(4,-3,2)$

Soln

$\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{b} = 4\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$

Normal vector to the plane, $n = \underset{\sim}{a} \times \underset{\sim}{b}$

$$= \begin{vmatrix} i & -j & k \\ 1 & 2 & -1 \\ 4 & -3 & 2 \end{vmatrix}$$

$$= \underset{\sim}{i} - 6\underset{\sim}{j} - 11\underset{\sim}{k}$$

Equation of the plane

$$\begin{pmatrix} x-1 \\ y-2 \\ z+1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -6 \\ -11 \end{pmatrix} = 0$$

$$x - 1 - 6y + 12 - 11z - 11 = 0$$

$$x - 6y - 11z = 0$$

8 EQUATION OF A PLANE GIVEN A LINE IN THE PLANE AND A PARALLEL VECTOR

Find the equation of the plane containing

$$x = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \text{ and parallel to } \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Soln

Parallel vector, $b_1 = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$, Parallel vector, $\underset{\sim}{b}_2 = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$

Normal vector to the plane, $n = \underset{\sim}{b_1} \times \underset{\sim}{b_2}$

$$= \begin{vmatrix} i & -j & k \\ 1 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$n = -5\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$$

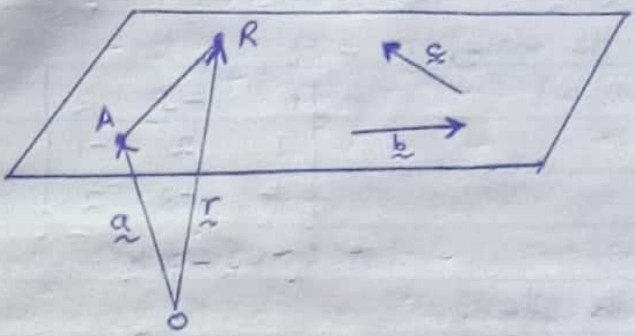
Equation of the plane

$$\begin{pmatrix} x-1 \\ y-2 \\ z-1 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ -2 \\ 3 \end{pmatrix} = 0$$

$$-5x + 5 - 2y + 4 + 3z - 3 = 0$$

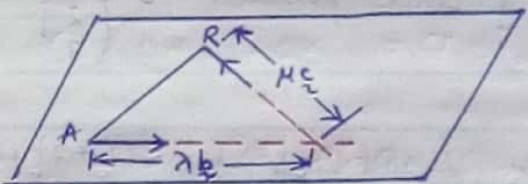
$$-5x - 2y + 3z + 6 = 0.$$

# VECTOR EQUATION OF A PLANE



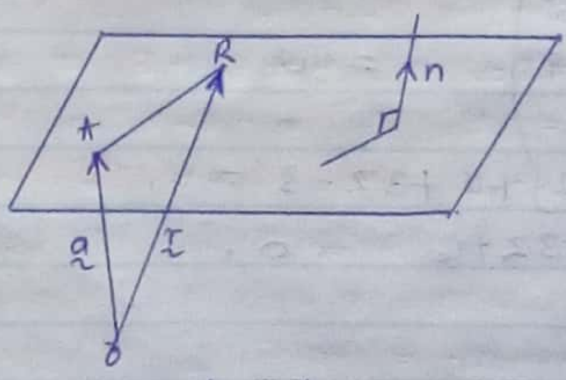

If we are given two non-parallel vectors $\underset{\sim}{b}$ and $\underset{\sim}{c}$ that are parallel to a particular plane and the position vector $\underset{\sim}{a}$ on a point in the plane then the plane is uniquely defined. To obtain a vector equation of the plane we consider some general point R in the plane having position vector $\underset{\sim}{r}$. From the diagram

$$\underset{\sim}{r} = \underset{\sim}{a} + \overrightarrow{AR}$$




Now it is possible to move from A to R by combining a suitable number of vector b's with a suitable number of vector c's. Thus we can write that $\overrightarrow{AR} = \lambda\underset{\sim}{b} + \mu\underset{\sim}{c}$ where $\lambda$ and $\mu$ are scalars

Thus $\underset{\sim}{r} = a + \lambda\underset{\sim}{b} + \mu\underset{\sim}{c}$ [vector equation / parametric vector equation]




A plane can also be uniquely defined by stating the vector that is perpendicular

to the plane and the position vector of a point on the plane.
Suppose that $\underset{\sim}{n}$ is a vector perpendicular to the plane and A is a point in the plane having position vector, $\underset{\sim}{a}$
Consider some general point R in the plane having position vector $\underset{\sim}{r}$.
Thus $\overrightarrow{AR}$ will lie in the plane and will therefore be perpendicular to $\underset{\sim}{n}$
ie $\overrightarrow{AR} \cdot n = 0$.

$$(\overrightarrow{OR} - \overrightarrow{OA}) \cdot \underset{\sim}{n} = 0$$
$$(\underset{\sim}{r} - \underset{\sim}{a}) \cdot \underset{\sim}{n} = 0$$
$$\underset{\sim}{r} \cdot \underset{\sim}{n} - \underset{\sim}{a} \cdot \underset{\sim}{n} = 0$$
$$\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n}$$

The equation $\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n}$ is the vector equation of the plane.
if $\underset{\sim}{a} \cdot n = P$

$\underset{\sim}{r} \cdot \underset{\sim}{n} = P$ [referred to scalar product form of the vector equation of a plane]

Examples:
1. Find a vector equation for the plane containing the three points A, B and C whose position vectors are $2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$, $3\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ and $5\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$ respectively

Soln
As A, B and C all lie in the plane then the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ will lie in the plane

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b} + \mu \underset{\sim}{c}$$

$$\underset{\sim}{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$$ — Vector equation of the plane

2. A plane contains a point A, position vector $3\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k}$ and is perpendicular to vector $\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$,
find the vector equation of the plane

Soln

$$\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n}$$

$$r \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

$$r \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = 3 + 8 - 4$$

$$r \cdot (\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}) = 7$$ — scalar product vector equation of a plane

3. Find the Cartesian equation of the plane with parametric vector equation

$$r = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Soln

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$x = 3 + 2\lambda + \mu$ ; $x - 3 = 2\lambda + \mu$ --- (1)

$y = 0 - \lambda + \mu$ ; $y = -\lambda + \mu$ --- (2)

$z = 1 + \mu$ ; $z - 1 = \mu$ --- (3)

Eliminate $\lambda$ from (1) and (2)

Eqn (1) × (1) ; $x - 3 = 2\lambda + \mu$

Eqn (2) × 2 + $2y = -2\lambda + 2\mu$

$x - 3 + 2y = 3\mu$ ---- (4)

Substitute (3) in (4)

$$x-3+2y = 3(z-1)$$
$$x-3+2y = 3z-3$$
$$x+2y-3z = 0 \quad \text{- Cartesian equation of a plane}$$

4 Find the Cartesian equation of the plane containing the point with position vector $\begin{pmatrix}1\\3\\1\end{pmatrix}$ and parallel to vectors $\begin{pmatrix}1\\-1\\3\end{pmatrix}$ and $\begin{pmatrix}2\\1\\-3\end{pmatrix}$

Soln

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b} + \mu \underset{\sim}{c}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\3\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\3\end{pmatrix} + \mu\begin{pmatrix}2\\1\\-3\end{pmatrix}$$

$x = 1+\lambda+2\mu$; $\quad x-1 = \lambda+2\mu$ ----(1)

$y = 3-\lambda+\mu \quad y-3 = -\lambda+\mu$ ---(2)

$z = 1+3\lambda-3\mu \quad z-1 = 3\lambda-3\mu$ ---(3)

Eliminate $\lambda$ from ① and ②; $\quad x-1 = \lambda+2\mu$

$$+ \quad \underline{y+3 = -\lambda+\mu}$$
$$x+y-4 = 3\mu \quad ----$$
$$\mu = \frac{1}{3}(x+y-z) \quad ---(4)$$

Eliminate $\lambda$ from ② and ③;

Eqn ② $\times 3$ ; $3y-9 = -3\lambda+3\mu$

Eqn 3 $\times 1$ $\quad + \underline{\; z-1 = 3\lambda-3\mu \;}$

$$3y+z-10 = 0$$
$$3y+z = 10 \quad \text{- Cartesian equation of a plane.}$$

5 Find the Cartesian equation of the plane containing the points with position vectors $\begin{pmatrix}1\\2\\-1\end{pmatrix}$, $\begin{pmatrix}2\\1\\-2\end{pmatrix}$ and $\begin{pmatrix}3\\-3\\3\end{pmatrix}$

Soln

$A(1,2,-1)$, $B(2,1,-2)$ and $C(3,-3,3)$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}2\\1\\-2\end{pmatrix} - \begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}1\\-1\\-1\end{pmatrix}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix}3\\-3\\3\end{pmatrix} - \begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}2\\-5\\4\end{pmatrix}$$

Equation of the plane is given by

$$r = \underset{\sim}{a} + \lambda \underset{\sim}{b} + \mu \underset{\sim}{c}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\-1\end{pmatrix} + \mu\begin{pmatrix}2\\-5\\4\end{pmatrix}$$

$x = 1+\lambda+2\mu$ ; $x-1 = \lambda+2\mu$ --- (1)

$y = 2-\lambda-5\mu$ ; $y-2 = -\lambda-5\mu$ --- (2)

$z = -1-\lambda+4\mu$ ; $z+1 = -\lambda+4\mu$ --- (3)

Eliminate $\lambda$ from (1) and (2)

$$\begin{aligned} x-1 &= \lambda+2\mu \\ + \quad y-2 &= -\lambda-5\mu \\ \hline x+y-3 &= -3\mu \quad \text{--- (4)} \end{aligned}$$

Eliminate $\lambda$ from (2) and (3)

$$\begin{aligned} y-2 &= -\lambda-5\mu \\ - \quad z+1 &= -\lambda+4\mu \\ \hline y-z-3 &= -9\mu \quad \text{--- (5)} \end{aligned}$$

Make $\mu$ the subject in (4) and (5) and then equate the two equations

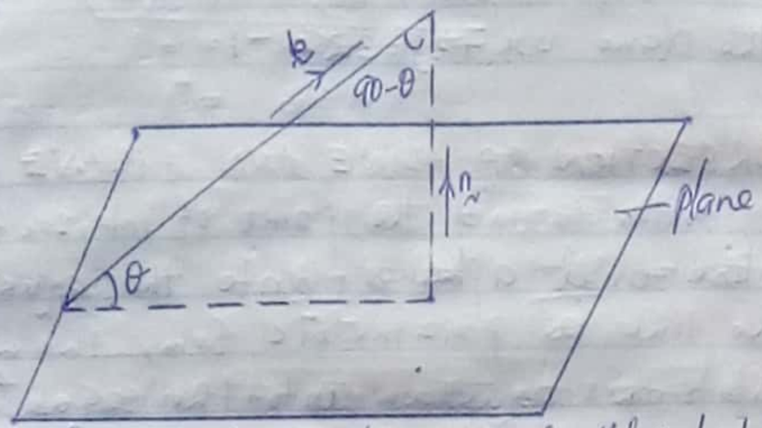
$$-\frac{1}{3}(x+y-3) = -\frac{1}{9}(y-z-3)$$

$$3(x+y-3) = y-z-3$$

$$3x+3y-9 = y-z-3$$

$$3x+2y+z = 6.$$

# ANGLE BETWEEN A LINE AND A PLANE




n - normal vector to the plane, b - parallel vector to the line

$$\underset{\sim}{n}\cdot\underset{\sim}{b} = |\underset{\sim}{n}||\underset{\sim}{b}|\cos(90-\theta)$$

$$\underset{\sim}{n}\cdot\underset{\sim}{b} = |\underset{\sim}{n}||\underset{\sim}{b}|\sin\theta$$

$$\sin\theta = \frac{\underset{\sim}{n}\cdot\underset{\sim}{b}}{|\underset{\sim}{n}||\underset{\sim}{b}|}$$

Examples

1. Find the angle between the line $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and the plane $x+y+z=0$

Soln

parallel vector to the ~~plane~~ line; $\underset{\sim}{b} = \underset{\sim}{i}+2\underset{\sim}{j}+3\underset{\sim}{k}$

normal vector to the plane; $\underset{\sim}{n} = \underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}$

$$\underset{\sim}{n}\cdot\underset{\sim}{b} = |\underset{\sim}{n}||\underset{\sim}{b}|\sin\theta$$

$$\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\end{pmatrix} = \sqrt{1+1+1}\cdot\sqrt{1+4+9}\sin\theta$$

$$1+2+3 = \sqrt{3}\cdot\sqrt{14}\sin\theta$$

$$\sin\theta = \frac{6}{\sqrt{3}\sqrt{14}}$$

$\theta = 67.89$; where $\theta$ is the angle btn the line & Plane

2. Determine the angle between the line ~~r~~ $\vec{r} = \underset{\sim}{i}+2\underset{\sim}{j}-\underset{\sim}{k}+\lambda(\underset{\sim}{i}-\underset{\sim}{j}+\underset{\sim}{k})$ and the plane $2x-y+z=4$

3. Find the angle between the line $\frac{x+1}{4} = \frac{y-2}{1} = \frac{z-3}{-1}$ and the plane $3x-5y+4z=5$

4 Find the angle between the line $r = \begin{pmatrix} -4 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ 2 \\ -4 \end{pmatrix}$
and the plane $4x + 3y - 3z = -1$

INTERSECTION OF A LINE AND A PLANE

In order to find the point of intersection of a line and a plane, write the equation of the line in parametric form, then substitute this equation into the equation of the plane to determine the unique value of the parameter for which the two equations are simultaneously satisfied

Examples

1. Find the point of intersection of the line $\frac{x-4}{5} = y+2 = z-4$ and the plane $3x - y + 7z + 8 = 0$

Soln

Let $\frac{x-4}{5} = \frac{y+2}{1} = \frac{z-4}{1} = \lambda$

$$\left.\begin{array}{l} x = 4 + 5\lambda \\ y = -2 + \lambda \\ z = 4 + \lambda \end{array}\right\} \text{------ (1)}$$

Equation of the plane

$$3x - y + 7z + 8 = 0 \quad \text{-- (2)}$$

Substitute (1) in (2)

$$3(4+5\lambda) - (-2+\lambda) + 7(4+\lambda) + 8 = 0$$

$$12 + 15\lambda + 2 - \lambda + 28 + 7\lambda + 8 = 0$$

$$\lambda = -\frac{50}{21}$$

Substituting $\lambda$ in equation (1), the point of intersection is $M\left(-\frac{166}{21}, -\frac{92}{21}, \frac{34}{21}\right)$

2 Find the point where the line $\frac{x-3}{-1} = \frac{y-1}{2} = \frac{z+3}{4}$ meets the plane $3x - y + 2z = 8$.

3 Find the position vector of the point where the line $r = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$ meets the plane $r. \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 15$

4 find the position vector of the point where the line $r = -i - 3j + 4k + \lambda(2i + j - 3k)$ Cuts the plane $r \cdot (i - j + 2k) = -5$

5 Show that lines $r = 2i - 3j + 4k + \lambda(3i - 2j + k)$ and $r = i + 3j + k + \mu(-i - 2j + k)$ intersect and that their point of intersection lies on the plane $r \cdot (2i + j + 2k) = 3$

6. Find the point where the line $\frac{x+2}{-1} = \frac{y-1}{2} = z-4$ cuts the plane $2x - y + 3z = 10$

7 Find the point where the line $x+1 = \frac{y-2}{4} = z-3$ cuts the plane $r.(2i - j + 3k) = 8$

## INTERSECTION OF A PLANE AND A PLANE

Two planes meet in a line. To find the line of intersection, express one of the variables x, y or z in terms of each of the remaining others.

Examples:

1. find the line of intersection of the planes $2x+3y+4z = 1$ and $x+y+3z = 0$

Soln.

~~Eliminate x~~

Eq $2x + 3y + 4z = 1$ - - - (1)

$x + y + 3z = 0$ - - - - (2)

Eliminate x;

Eqn ① x 1 $2x + 3y + 4z = 1$

Eqn ② x 2; $2x + 2y + 6z = 0$

$y - 2z = 1$ - - - (3)

Eliminate y;

Eqn ① × 1 ; $2x+3y+4z=-1$

Eqn ② × 3 ; $3x+3y+9z=-6$

$-x-5z=+1$

$x+5z=-1$ —— ④

Make the subject, the variable that is common to eqns ③ and ④

$$z=\frac{y-1}{2}=\frac{x+1}{-5}$$

Equation for the line of intersection

$$\frac{x+1}{-5}=\frac{y-1}{2}=z$$

2 find the vector form the equation of the line of intersection of the two planes $2x+3y-z=4$ and $x-y+2z=5$

Soln

The planes are: $2x+3y-z=4$ ---(1)

$x-y+2z=5$ ---(2)

Eliminate x; Eqn ① × 1 ; $2x+3y-z=4$.

Eqn ② × 2 ; $-2x-2y+4z=10$

$5y-5z=-6$ ③

Eliminate y ; Eqn ① × ① ; $2x+3y-z=4$

Eqn ② × 3 $3x-3y+6z=15$

$5x+5z=19$ ④

from ③ $z=\frac{5y+6}{5}$

from ④ $z=\frac{5x-19}{-5}$

$$\therefore\ \frac{5x-19}{-5}=\frac{5y+6}{5}=z$$

$$\frac{x-\frac{19}{5}}{-1}=\frac{y+6/5}{1}=\frac{z}{1}$$

The Vector equation of the line of intersection is

$$\underset{\sim}{r}=\begin{pmatrix}19/5\\-6/5\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

3 Find the Cartesian equations of the lines of intersection of the following Planes

a) $3x-5y+z=8$ and $2x-3y+z=3$

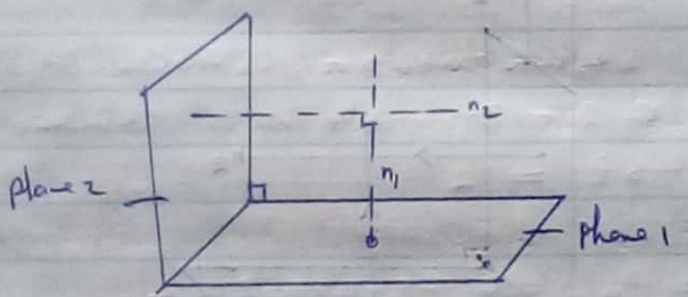
b) $3x+4y+2z=3$ and $2x-3y-z=1$

## ANGLE BETWEEN A PLANE AND A PLANE

The angle $\theta$ between two planes is the same as the angle between their normal vectors, which can be obtained by using the dot-product

Examples

1. Find the angle between the planes $3x+4y+z=11$ and $x-y+2z=0$




$\underset{\sim}{n}_1 = 3\underset{\sim}{i}+4\underset{\sim}{j}+\underset{\sim}{k}$, $\quad \underset{\sim}{n}_2 = \underset{\sim}{i}-\underset{\sim}{j}+2\underset{\sim}{k}$

$$\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_3 = |\underset{\sim}{n}_1||\underset{\sim}{n}_2|\cos\theta$$

$$\begin{pmatrix}3\\4\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix} = \sqrt{9+16+1}\sqrt{1+1+4}\cos\theta$$

$$3-4+2 = \sqrt{26}\sqrt{6}\cos\theta$$

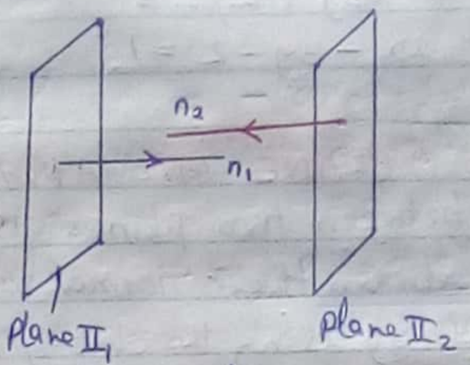
$$\cos\theta = \frac{1}{\sqrt{26}\sqrt{6}}$$

$$\theta = 85.4^\circ$$

2 Determine the angle between the planes $4x+3y+12z=10$ and $4x-3y=7$

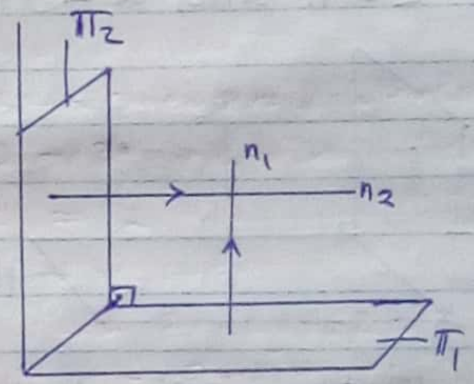
## PARALLEL AND PERPENDICULAR PLANES

If two planes are parallel, their normal vectors must be parallel




$n_1$ is parallel to $n_2$

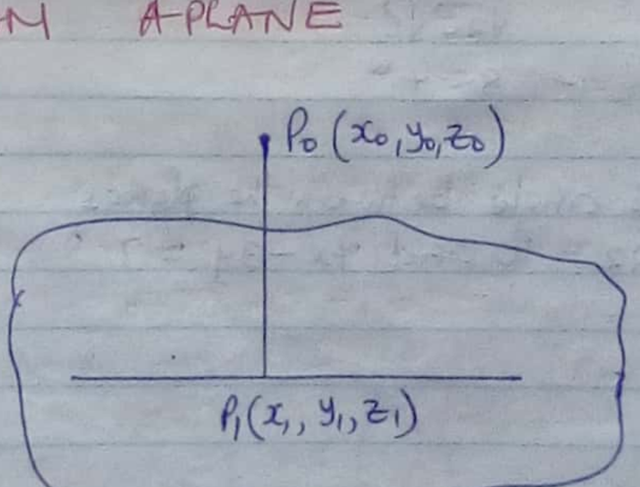
If two planes are perpendicular, then their normal vectors are perpendicular




$n_1$ is perpendicular to $n_2$

To show whether two planes are parallel or perpendicular, find out whether their normal vectors are parallel or perpendicular to one another.

## PERPENDICULAR DISTANCE OF A POINT FROM A PLANE



Consider a plane whose equation is $ax+by+cz+d=0$ --(1)
Suppose $P_0$ is a point whose perpendicular distance from the plane is required, and that $P_1$ is the point where the perpendicular from $P_0$ meets the plane.
We require $|P_0P_1|$
Now since $P_0P_1$ is perpendicular to the plane, then its parallel vector is normal vector to the plane.
But normal vector is

$$x = ai + bj + ck$$

Hence equation of $P_0P_1$ is

$$\left.\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}\right\} \quad (2)$$

If $P_1(x_1, y_1, z_1)$ is the point of intersection of the plane and the line; Equations (1) and (2) become

$$ax_1 + by_1 + cz_1 + d = 0 \quad (3)$$

$$\left.\begin{aligned} x_1 &= x_0 + at \\ y_1 &= y_0 + bt \\ z_1 &= z_0 + ct \end{aligned}\right\} \quad (4)$$

Substituting (4) in (3), we shall have

$$a(x_0+at) + b(y_0+bt) + c(z_0+ct) + d = 0$$

$$ax_0 + a^2t + by_0 + b^2t + cz_0 + c^2t + d = 0$$

$$a^2t + b^2t + c^2t = -(ax_0+by_0+cz_0+d)$$

$$t = \frac{-(ax_0+by_0+cz_0+d)}{a^2+b^2+c^2}$$

from (4)

$$x_1 - x_0 = at$$

$$y_1 - y_0 = bt$$

$$z_1 - z_0 = ct$$

But perpendicular distance $|P_0P_1| = \sqrt{(x_1-x_0)^2 + (y_1-y_0)^2 + (z_1-z_0)^2}$

$$|P_0P_1| = \sqrt{(at)^2 + (bt)^2 + (ct)^2}$$

$$= t\sqrt{a^2+b^2+c^2}$$

$$|P_0P_1| = t\sqrt{a^2+b^2+c^2}$$

~~Subst~~

Substitution $-$ ? for $t$

$$|P_0P_1| = -\frac{(ax_0+by_0+cz_0+d)}{a^2+b^2+c^2} \cdot \sqrt{a^2+b^2+c^2}$$

$$|P_0P_1| = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

> take modulus in order to avoid getting a negative value for the distance

Examples

1 Find the perpendicular distance of the point $P(5,12,-13)$ from the plane $3x+4y+5z=12$

Soln

Equation of the plane $3x+4y+5z=12$

$a=3,\ b=4,\ c=5$

Point $P_0(5,12,-13)$

$x_0=5,\ y_0=12,\ z_0=-13$

Perpendicular distance, $d = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$

$$= \frac{|(3)(5)+(4)(12)+(5)(-13)+-12|}{\sqrt{3^2+4^2+5^2}}$$

$$= \frac{14}{\sqrt{50}}$$

2. Calculate the perpendicular distance of the point $(1,2,3)$ from the plane $x+2y+2z+7=0$

Soln

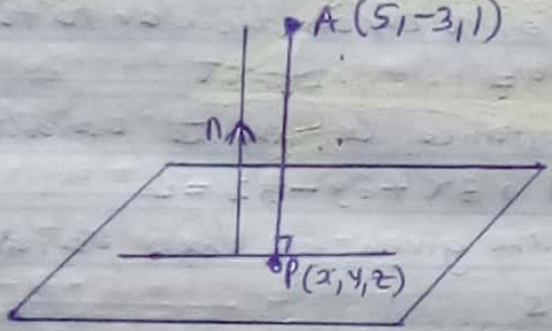
Perpendicular distance, $d = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$

$= \frac{(1)(1) + (2)(2) + (2)(3) + 7}{\sqrt{1^2+2^2+2^2}}$

$= \frac{18}{3}$

$= 6$ units

3. Determine the distance from the plane $12x - 3y - 4z = 39$
1. to the point $(5, -3, 1)$




$\overrightarrow{AP} = \lambda n$

$p - a = \lambda n$

$p = a + \lambda n$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 12 \\ -3 \\ -4 \end{pmatrix}$

$x = 5 + 12\lambda, \quad y = -3 - 3\lambda, \quad z = 1 - 4\lambda$ ----- **

Substituting * in the equation of the plane

$12x - 3y - 4z = 39$

$12(5 + 12\lambda) - 3(-3 - 3\lambda) - 4(1 - 4\lambda) = 39$

$60 + 144\lambda + 9 + 9\lambda - 4 + 16\lambda = 39$

$169\lambda = -26$

$\lambda = -\frac{26}{169}$

$\lambda = -\frac{2}{13}$

$\overrightarrow{AP} = -\frac{2}{13}\begin{pmatrix} 12 \\ -3 \\ -4 \end{pmatrix}$

$|\overrightarrow{AP}| = \frac{2}{13}\sqrt{(12)^2 + (-3)^2 + (-4)^2} = \frac{2}{13} \times 13 = 2$ Units

Exercise

1. Find the equation of the plane containing points $P(1,1,1)$, $Q(1,2,0)$ and $R(-1,2,1)$
2. Find the equation of the plane containing point $(4,-2,3)$ and parallel to the plane $3x-7z=12$
3. Show that the point with position vector $7\underset{\sim}{i}-5\underset{\sim}{j}-4\underset{\sim}{k}$ lies in the plane $x = 4\underset{\sim}{i}+3\underset{\sim}{j}+2\underset{\sim}{k}+\lambda(\underset{\sim}{i}-\underset{\sim}{j}-\underset{\sim}{k})+\mu(2\underset{\sim}{i}+3\underset{\sim}{j}+\underset{\sim}{k})$.
   Find the point at which the line $x = y-1 = 2z$ meets the plane $4x-y+3z=8$
4. Find the parametric equations for the line through the point $(0,1,2)$ that is parallel to the plane $x+y+z=2$ and perpendicular to the line $x=1+t$, $y=1-t$, $z=2t$
5. Find the distance between the parallel planes $z=x+2y+1$, $3x+6y-3z=4$
6. Two places are given by their parametric equations

   $x = r+s$ and $x = 1+r+s$
   $y = 3s$ $\quad$ $y = 2+r$
   $z = 2r$ $\quad$ $z = -3+s$

   Find the Cartesian equation of the intersection point
7. The equation of the plane $P$ is given by $r\cdot\begin{pmatrix}2\\6\\9\end{pmatrix} = 33$ where $r$ is the position Vector of $P$

   Find the perpendicular distance from the Origin to the plane.
8. The line through the point $P(1,-2,3)$ and Parallel to the line $\frac{x}{3} = \frac{y+1}{-1} = z+1$ Meets the plane $x+2y+2z=8$ at $Q$. Find the Coordinates of $Q$
9. (a) Find the angle between the plane $x+4y-z=72$ and the line $\underset{\sim}{i}-2\underset{\sim}{j}+3\underset{\sim}{k}+\lambda(9\underset{\sim}{i}+6\underset{\sim}{j}+8\underset{\sim}{k})$

   (b) Obtain the equation of the plane that passes through $(1,-2,2)$ and perpendicular to the line $\frac{x-9}{4} = \frac{y-6}{-1} = \frac{z-8}{-1}$

(c) find the parametric equations of the line of intersection of the plane $x+y+z=4$ and $x-y+2z+2=0$

10 Find the point of intersection of the three planes $2x-y+3z=4$, $3x-2y+6z=3$ and $7x-4y+5z=11$

11 Find the perpendicular distance from the plane $r\cdot(2\underset{\sim}{i}-14\underset{\sim}{j}+5\underset{\sim}{k})=10$ to the origin

12 Two lines have vector equations

$$r=\begin{pmatrix}3\\-1\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-1\end{pmatrix} \text{ and } r=\begin{pmatrix}4\\4\\1\end{pmatrix}+\mu\begin{pmatrix}-1\\1\\2\end{pmatrix}$$

Find the position vector of the point of intersection of the two lines and the cartesian equation of the plane containing the two lines.

# UNEB 2019

No4. Given the plane $4x+3y-3z-4=0$;

a) Show that the point $A(1,1,1)$ lies on the plane (02mks)

b) Find the perpendicular distance from the plane to the point $B(1,5,1)$ (03mks)

Soln

(a) $4x+3y-3z-4=0$

$LHS = 4x+3y-3z-4$

at $A(1,1,1)$; $LHS = 4(1)+3(1)-3(1)-4$ M1

$= 0$ A1

Since LHS = RHS, the point $A(1,1,1)$ lies on the plane

(b) Perpendicular distance, $d = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$

$= \frac{|4(1)+3(5)+(-3)(1)+-4|}{\sqrt{4^2+3^2+(-3)^2}}$ M1

$= \frac{|4+15-3-4|}{\sqrt{16+9+9}}$ M1

$= \frac{12}{\sqrt{34}}$ A1

05

No9 (a) Determine the perpendicular distance of the point $(4,6)$ from the line $2x+4y-3=0$ (03mks)

(b) Show that the angle, $\theta$ between two lines with gradients $\lambda_1$ and $\lambda_2$ is given by

$$\theta = \tan^{-1}\left(\frac{\lambda_1-\lambda_2}{1+\lambda_1\lambda_2}\right)$$

Hence find the acute angle between the lines $x+y+7=0$ and $\sqrt{3}x-y+5=0$ (09mks)

Soln

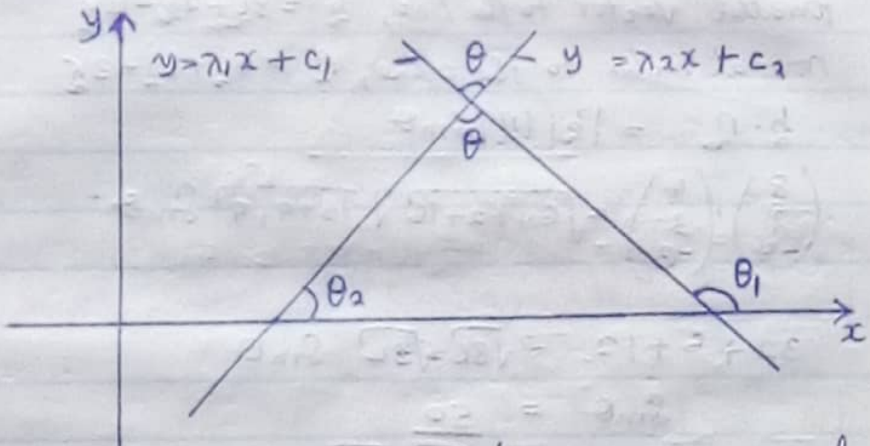
$d = \frac{|ax+by+c|}{\sqrt{a^2+b^2}}$

$= \frac{|(2)(4)+(4)(4)+-3|}{\sqrt{(2)^2+(4)^2}}$ M1

$$d = \frac{|8 + 16 + -3|}{\sqrt{4+16}}$$ M1

$$= \frac{21}{\sqrt{20}}$$ A1

(b)




Consider two lines $y = \lambda_1 x + c_1$ and $y = \lambda_2 x + c_2$ and suppose that they make angles $\theta_1$ and $\theta_2$ respectively with the positive x-axis. Let $\theta$ be the acute angle between the two lines, then

$$\theta_2 + \theta = \theta_1$$ B1

$$\theta = \theta_1 - \theta_2$$ B1

$$\tan\theta = \tan(\theta_1 - \theta_2)$$

$$= \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$ M1

$$\tan\theta = \frac{\lambda_1 - \lambda_2}{1 + \lambda_1 \lambda_2}$$ A1

$$\theta = \tan^{-1}\left(\frac{\lambda_1 - \lambda_2}{1 + \lambda_1 \lambda_2}\right)$$ B1

Acute angle btn the lines: $x + y + 7 = 0$, $y = -x - 7$, $\lambda_2 = -1$ B1

$\sqrt{3}x - y + 5 = 0$, $y = \sqrt{3}x + 5$, $\lambda = \sqrt{3}$ B1

$$\theta = \tan^{-1}\left(\frac{\lambda_1 - \lambda_2}{1 + \lambda_1 \lambda_2}\right)$$

$$= \tan^{-1}\left[\frac{\sqrt{3} - -1}{1 + \sqrt{3}(-1)}\right]$$ M1

$$= -75$$

$\therefore$ Acute angle $= 75°$ A1

12

# UNEB 2018



No2. Determine the angle between the line $\frac{x+4}{8} = \frac{y-2}{2} = \frac{z+1}{-4}$ and the plane $4x+3y-3z+1=0$ (05mks)

Soln

Parallel vector to the line, $\underset{\sim}{b} = 8\underset{\sim}{i}+2\underset{\sim}{j}-4\underset{\sim}{k}$

normal vector to the plane, $\underset{\sim}{n} = 4\underset{\sim}{i}+3\underset{\sim}{j}-3\underset{\sim}{k}$ B1

$\underset{\sim}{b}\cdot\underset{\sim}{n} = |\underset{\sim}{b}||\underset{\sim}{n}|\sin\theta$

$\begin{pmatrix}8\\2\\-4\end{pmatrix}\cdot\begin{pmatrix}4\\3\\-3\end{pmatrix} = \sqrt{64+2+16}\sqrt{16+9+9}\sin\theta$ M1

$32+6+12 = \sqrt{86}\sqrt{34}\sin\theta$ M1

$\sin\theta = \frac{50}{\sqrt{86}\sqrt{34}}$

$\theta = \sin^{-1}\left[\frac{50}{\sqrt{86}\sqrt{34}}\right]$ M1

$\theta = 67.62$ A1

05

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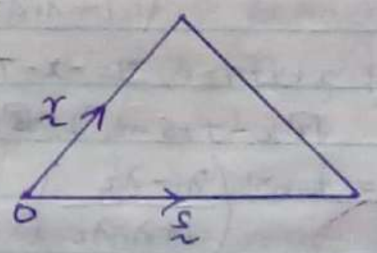
No9. The position vectors of the vertices of a triangle are O, r and s, where O is the origin. Show that $4A^2 = |\underset{\sim}{r}|^2|\underset{\sim}{s}|^2 - (\underset{\sim}{r}\cdot\underset{\sim}{s})^2$ (06mks)

Hence, find the area of a triangle when $\underset{\sim}{r} = \begin{pmatrix}2\\3\end{pmatrix}$ and $\underset{\sim}{s} = \begin{pmatrix}1\\4\end{pmatrix}$ (06mks)

Soln




Area of a triangle, $A = \frac{1}{2}|\underset{\sim}{r}||\underset{\sim}{s}|\sin\theta$, where $\theta$ is the angle between vectors r and s M1

$A^2 = \frac{1}{4}|\underset{\sim}{r}|^2|\underset{\sim}{s}|^2\sin^2\theta$ - - - (1) M1

Using definition of dot product

$r\cdot s = |\underset{\sim}{r}||\underset{\sim}{s}|\cos\theta$

$(r\cdot s)^2 = |\underset{\sim}{r}|^2|\underset{\sim}{s}|^2\cos^2\theta$ M1

$$(\underset{\sim}{r}\cdot\underset{\sim}{s})^2 = |r|^2|\underset{\sim}{s}|(1-\sin^2\theta) \quad m1$$
$$= |r|^2|\underset{\sim}{s}|^2 - |\underset{\sim}{r}|^2|\underset{\sim}{s}|^2\sin^2\theta$$
$$|\underset{\sim}{r}|^2|\underset{\sim}{s}|\sin^2\theta = |r|^2|\underset{\sim}{s}|^2 - (r.s)^2 \quad ------ (2) \quad m1$$

from (1); $4A^2 = |\underset{\sim}{r}|^2|\underset{\sim}{s}|^2\sin^2\theta$ ---- (3)

Substituting (3) in (2) we shall have

$$4A^2 = |r|^2|\underset{\sim}{s}|^2 - (\underset{\sim}{r}.\underset{\sim}{s})^2 \quad A1$$

Given $\underset{\sim}{r} = \begin{pmatrix}2\\3\end{pmatrix}$ ; $s = \begin{pmatrix}1\\4\end{pmatrix}$

$$4A^2 = \left|\begin{matrix}2\\3\end{matrix}\right|^2\left|\begin{matrix}1\\4\end{matrix}\right| - \left[\begin{pmatrix}2\\3\end{pmatrix}\cdot\begin{pmatrix}1\\4\end{pmatrix}\right]^2 \quad m1$$

$$= \left(\sqrt{2^2+3^2}\right)^2\cdot\left(\sqrt{1^2+4^2}\right)^2 - [2+12]^2 \quad m1$$

$$= 13 * 17 - (14)^2 \quad m1$$

$$4A^2 = 221 - 196 \quad m1$$

$$4A^2 = 25$$

$$A^2 = \frac{25}{4} \quad m1$$

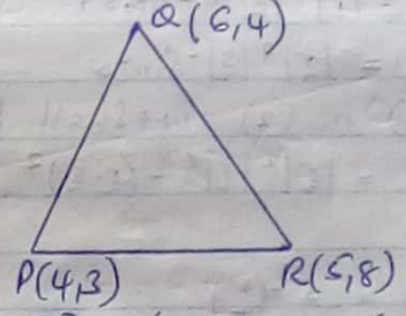
$$A = 5/2 \quad A1$$

---

12

# UNEB 2017

No5. The vertices of a triangle are $P(4,3)$, $Q(6,4)$ and $R(5,8)$; find angle RPQ using vectors (05marks)




$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix}6\\4\end{pmatrix} - \begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$ B1

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \begin{pmatrix}5\\8\end{pmatrix} - \begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}1\\5\end{pmatrix}$ B1

$\overrightarrow{PQ} \cdot \overrightarrow{PR} = |\overrightarrow{PQ}||\overrightarrow{PR}|\cos\theta$; where $\theta$ is the angle btn $\overrightarrow{PQ}$ and $\overrightarrow{PR}$

$\begin{pmatrix}2\\1\end{pmatrix} \cdot \begin{pmatrix}1\\5\end{pmatrix} = \sqrt{4+1}\sqrt{1+26}\cos\theta$ M1

$7 = \sqrt{5}\sqrt{26}\cos\theta$ M1

$\cos\theta = \dfrac{7}{\sqrt{5}\sqrt{26}}$

$\theta = 52.1$ A1

05

---

14(a) The points A and B have position vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$. A point C with a position vector $\underset{\sim}{c}$ lies on AB such that $\dfrac{AC}{AB} = \lambda$

Show that $\underset{\sim}{c} = (1-\lambda)\underset{\sim}{a} + \lambda\underset{\sim}{b}$ (04marks)

(b) The vector equations of two lines are

$r_1 = 2\underset{\sim}{i} + \underset{\sim}{j} + \lambda(\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k})$ and

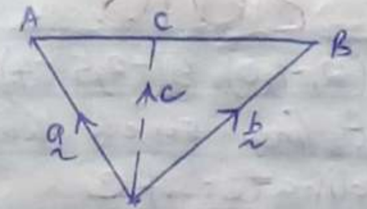
$\underset{\sim}{r_2} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + t\underset{\sim}{k} + \mu(\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k})$

where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors and $\lambda, \mu$ & $t$ are constants.

Given that the two vectors intersect, find

(i) the value of $t$

(ii) the coordinates of the point of intersection (07marks)




$\frac{\overrightarrow{AC}}{\overrightarrow{AB}} = \lambda$

$\overrightarrow{AC} = \lambda \overrightarrow{AB}$

$\overrightarrow{OC} - \overrightarrow{OA} = \lambda(\overrightarrow{OB} - \overrightarrow{OA})$ M1

$\overrightarrow{OC} = \overrightarrow{OA} + \lambda(\overrightarrow{OB} - \overrightarrow{OA})$ M1

$\underset{\sim}{c} = \underset{\sim}{a} + \lambda(\underset{\sim}{b} - \underset{\sim}{a})$ M1

$\underset{\sim}{c} = \underset{\sim}{a} + \lambda \underset{\sim}{b} - \lambda \underset{\sim}{a}$

$\underset{\sim}{c} = \underset{\sim}{a} - \lambda \underset{\sim}{a} + \lambda \underset{\sim}{b}$ M1

$\underset{\sim}{c} = (1-\lambda)\underset{\sim}{a} + \lambda \underset{\sim}{b}$ B1

(b) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$; $x = 2+\lambda$, $y = 1+\lambda$, $z = 2\lambda$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ t \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$; $x = 2+\mu$, $y = 2+2\mu$, $z = t+\mu$ B1 – Parametric form

If $(x, y, z)$ is the point of intersection

for $x$; $2+\lambda = 2+\mu$, $\lambda = \mu$ M1

for $y$; $1+\lambda = 2+2\mu$.

Substituting for $\lambda$; $1+\mu = 2+2\mu$

$-\mu = 1$, $\mu = -1$ A1

$\Rightarrow \lambda = -1$ A1

for $z$; $2\lambda = t+\mu$

$2(-1) = t + -1$

$t = -1$ A1

Point of intersection is $(x, y, z)$

$x = 2+\mu = 2+-1, = 1$

$y = 2+2\mu = 2+2(-1) = 0$

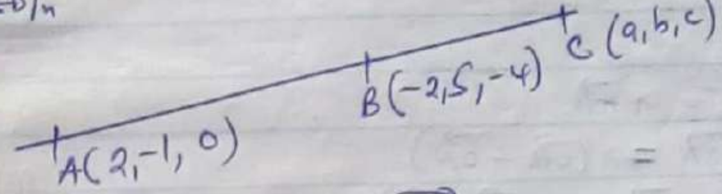
$z = (-1) + -1 = -2$ M1

$P(1, 0, -2)$ A1

12

No8 Three points $A(2,-1,0)$, $B(-2,5,-4)$ and C are on a straight line such that $3\overrightarrow{AB} = 2\overrightarrow{AC}$. Find the coordinates of C (05)

Soln




$$3\overrightarrow{AB} = 2\overrightarrow{AC}$$

$$3\left[\overrightarrow{OB}-\overrightarrow{OA}\right] = 2\left[\overrightarrow{OC}-\overrightarrow{OA}\right]$$

$$3\left[\begin{pmatrix}-2\\5\\-4\end{pmatrix}-\begin{pmatrix}2\\-1\\0\end{pmatrix}\right] = 2\left[\begin{pmatrix}a\\b\\c\end{pmatrix}-\begin{pmatrix}2\\-1\\0\end{pmatrix}\right]$$

$$3\begin{pmatrix}-4\\6\\-4\end{pmatrix} = 2\begin{pmatrix}a-2\\b+1\\c\end{pmatrix}$$ B1 B1 M1 — vector $\overrightarrow{AB}$ — vector $\overrightarrow{AC}$ — Equating the vectors

$-12 = 2a-4$; $a=-4$

$18 = 2b+2$; $b = 8$

$-12 = 2c$; $c=-6$ A1 for values of a,b,c

$C(-4, 8, -6)$ B1 — Coordinate form

05

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No12(a) Line A is the intersection of two planes whose equations are $3x-y+z=2$ and $x+5y+2z=6$. Find the Cartesian equation of the line (05)

(b) Given that line B is perpendicular to the plane $3x-y+z=2$ and passes through point $C(1,1,0)$, find the

(i) Cartesian equation of B

(ii) angle between line B and line A in (a) above (07)

Soln

a) $3x-y+z=2$, $x+5y+2z=6$

Eliminate x; 

$$\begin{aligned} 3x-y+z &= 2\\ -\;\;3x+15y+6z &= 18\\ \hline -16y-5z &= -16 \quad ----(1) \quad M1\end{aligned}$$

8) Eliminate, $y$; $15x - 5y + 6z = 10$

$+ \; x + 5y + 2z = 6$

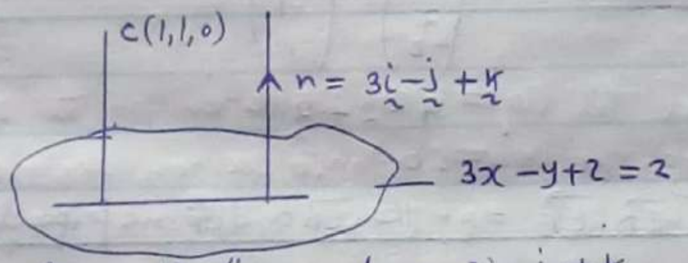
$16x + 7z = 16 \quad \text{-----} (2)$ M1

From ①; Make z the subject; $z = \frac{16y-16}{-5}$ A1

From ②; make z the subject; $z = \frac{16x-16}{-7}$ A1

$$\frac{16x-16}{-7} = \frac{16y-16}{-5} = z$$

$$\frac{x-1}{-7/16} = \frac{y-1}{-5/16} = \frac{z}{1}$$ B1




Parallel vector to the line; $\underset{\sim}{b} = 3\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ B1

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$$

$$\underset{\sim}{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\1\end{pmatrix}$$ M1

Cartesian $\frac{x-1}{3} = \frac{y-1}{-1} = \frac{z}{1}$ A1

Parallel vector to line A, $\underset{\sim}{a} = -\frac{7}{16}\underset{\sim}{i} - \frac{5}{16}\underset{\sim}{j} + \underset{\sim}{k}$ ✓

Parallel vector to line B, $\underset{\sim}{b} = 3\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ ✓ B1

Using dot product

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$; where $\theta$ is the angle btn $\underset{\sim}{a}$ and $\underset{\sim}{b}$

$$\begin{pmatrix}-7/16\\-5/16\\1\end{pmatrix}\begin{pmatrix}3\\-1\\1\end{pmatrix} = \sqrt{\frac{49}{256} + \frac{25}{256} + \frac{1}{1}}\,\sqrt{9+1+1}\,\cos\theta$$ M1

$$0 = \sqrt{\frac{3630}{256}}\cos\theta$$

$\cos\theta = 0$ M1

$\theta = \cos^{-1}(0)$

$\theta = 90°$ A1

12

# UNEB 2015



No4. Given that $D(7,1,2)$, $E(3,^-1,4)$ and $F(4,^-2,5)$ are points on a plane, show that ED is perpendicular to EF.

Soln

$\overrightarrow{ED} = \overrightarrow{OD} - \overrightarrow{OE} = \begin{pmatrix}7\\1\\2\end{pmatrix} - \begin{pmatrix}3\\-1\\4\end{pmatrix} = \begin{pmatrix}4\\2\\-2\end{pmatrix}$ B1

$\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE} = \begin{pmatrix}4\\-2\\5\end{pmatrix} - \begin{pmatrix}3\\-1\\4\end{pmatrix} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$ B1

$\overrightarrow{ED} \cdot \overrightarrow{EF} = \begin{pmatrix}4\\2\\-2\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\1\end{pmatrix}$ M1

$= 4 - 2 - 2$

$= 0$ A1

Since $\overrightarrow{ED} \cdot \overrightarrow{EF} = 0$, then $\overrightarrow{ED}$ and $\overrightarrow{EF}$ are perpendicular B1

05

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No14. Show that the line $\underset{\sim}{a} = \begin{pmatrix}3\\-4\\2\end{pmatrix} + \alpha\begin{pmatrix}-1\\1\\2\end{pmatrix}$ and

$\underset{\sim}{b} = \begin{pmatrix}5\\0\\-2\end{pmatrix} + \beta\begin{pmatrix}-1\\-1\\2\end{pmatrix}$ intersect. (06mks)

(b) find the

(i) point of intersection, P of the two lines in (a)

(ii) Cartesian equation of the plane which contains $\underset{\sim}{a}$ and $\underset{\sim}{b}$ (06mks)

Soln

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\-4\\2\end{pmatrix} + \alpha\begin{pmatrix}-1\\1\\2\end{pmatrix}$$

$x = 3 - \alpha$, $y = -4 + \alpha$, $z = 2 + 2\alpha$ B1

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}5\\0\\-2\end{pmatrix} + \beta\begin{pmatrix}-1\\-1\\2\end{pmatrix}$$

$x = 5 - \beta$, $y = -\beta$, $z = -2 + 2\beta$ B1

If $(x, y, z)$ is a point of intersection

$3-\alpha = 5-\beta \; ; \quad \alpha-\beta = -2 \quad ---- (1)$

$-4+\alpha = -\beta \; ; \quad \alpha+\beta = 4 \quad --- (2)$

$2+2\alpha = -2+2\beta \; ; \quad 2\alpha-2\beta = -4 \quad --- (3)$ M1

$(1)+(2) ; \; 2\alpha = 2, \; \alpha = 1$ A1

$(2)-(1) ; \; 2\beta = 6, \; \beta = 3$ A1

Substituting the values of $\alpha$ and $\beta$ in equation 3

$LHS = 2\alpha - 2\beta = 2(1) - 2(3) = 2-6 = -4 = RHS$

Since equation (3) is satisfied, then the two lines intersect B1

---

(b) Point of intersection ; Substitute for $\alpha$

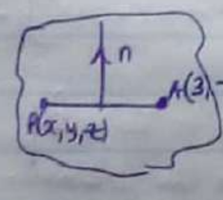
$x = 3-1 = 2, \; y = -4+1 = -3, \; z = 2+2(1) = 4$ M1

$P(2, -3, 4)$ A1

Normal vector to the plane

$$n = \begin{vmatrix} i & -j & k \\ -1 & 1 & 2 \\ -1 & -1 & 2 \end{vmatrix}$$ M1

$= 4i + 2k$ B1




$\vec{AP} \cdot n = 0$

$$\begin{pmatrix} x-3 \\ y+4 \\ z-2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} = 0$$ M1

$= 2x + z - 8 = 0$ A1

OR $\quad r = a + \lambda b + \mu c$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

$x-3 = -\lambda-\mu \;\text{---}(1), \quad y+4 = \lambda-\mu \;\text{---}(2), \quad z-2 = 2\lambda+2\mu \;\text{---}(3)$

$(1)+(2) ; \; x+y+1 = -2\mu \; ---4$

Eqn (2) $\times 2$ ; $\; 2y+8 = 2\lambda - 2\mu$

Eqn 3 $\times 1$ ; $\; \underline{z-2 = 2\lambda + 2\mu}$

$2y - z + 10 = -4\mu \; -- (5)$

Make $\mu$ the subject in (4) and (5) and equate them

$-\frac{1}{2}(x+y+1) = -\frac{1}{4}(2y - z + 10)$

$2(x+y+1) = 2y - z + 10$

$2x + 2y + 2 = 2y - z + 10$

$2x + z = 8$.

12

# UNEB 2014



No5. Find the equation of a line through $S(1,0,2)$ and $T(3,2,1)$ in the form $\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$

Soln

$$\overrightarrow{ST} = \overrightarrow{OT} - \overrightarrow{OS} = \begin{pmatrix}3\\2\\1\end{pmatrix} - \begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}2\\2\\-1\end{pmatrix}$$ M1B1

$$\underset{\sim}{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\2\\-1\end{pmatrix}$$ M1M1A1

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05

No12(a) Find the cartesian equation of the plane through the points whose position vectors are $2\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$, $3\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$ and $-2\underset{\sim}{j} + 4\underset{\sim}{k}$

(b) Determine the angle between the plane in (a) and the line $\frac{x-2}{2} = \frac{y}{-4} = z-5$

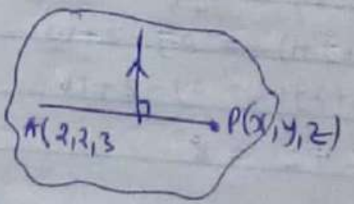
Soln

(a) $$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}3\\1\\2\end{pmatrix} - \begin{pmatrix}2\\2\\3\end{pmatrix} = \begin{pmatrix}1\\-1\\-1\end{pmatrix}$$ B1

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix}0\\-2\\4\end{pmatrix} - \begin{pmatrix}2\\2\\3\end{pmatrix} = \begin{pmatrix}-2\\-4\\1\end{pmatrix}$$ B1

$$n = \begin{vmatrix} i & -j & k \\ 1 & -1 & -1 \\ -2 & -4 & 1 \end{vmatrix}$$ M1

$$= -5\underset{\sim}{i} + \underset{\sim}{j} - 6\underset{\sim}{k}$$ B1




$$\overrightarrow{AP} \cdot n = 0$$

$$\begin{pmatrix} x-2 \\ y-2 \\ z-3 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ 1 \\ -6 \end{pmatrix} = 0$$ M1

$-5x+10+y-2-6z+18=0$

$-5x+y-6z=26$ A1

$5x-y+6z+26=0$

Parallel vector to the line, $\underset{\sim}{b} = 2\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}$ B1

normal vector to the plane, $\underset{\sim}{n} = 5\underset{\sim}{i} - \underset{\sim}{j} + 6\underset{\sim}{k}$ B1

$\underset{\sim}{n} \cdot \underset{\sim}{b} = |\underset{\sim}{n}||\underset{\sim}{b}| \sin\theta$

$$\begin{pmatrix} 5 \\ -1 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = \sqrt{25+1+36}\,\sqrt{4+16+1}\,\sin\theta$$ M1 M1

$10+4+6 = \sqrt{62}\,\sqrt{21}\,\sin\theta$

$\sin\theta = \dfrac{20}{\sqrt{62}\sqrt{21}}$ M1

$\sin\theta = 0.5543$

$\theta = \sin^{-1}(0.5543)$

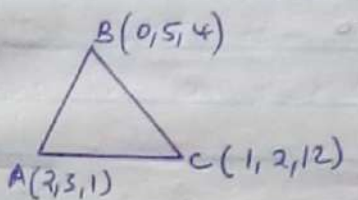
$\theta = 36.66°$ A1

12

# UNEB 2013



No4. The position vector of point A is $2\underset{\sim}{i}+3\underset{\sim}{j}+\underset{\sim}{k}$, of B is $5\underset{\sim}{j}+4k$ and of C is $\underset{\sim}{i}+2\underset{\sim}{j}+12\underset{\sim}{k}$. Show that ABC is a triangle




$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}0\\5\\4\end{pmatrix}-\begin{pmatrix}2\\3\\1\end{pmatrix}=\begin{pmatrix}-2\\-2\\3\end{pmatrix}$ B1

$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\begin{pmatrix}1\\2\\12\end{pmatrix}-\begin{pmatrix}2\\3\\1\end{pmatrix}=\begin{pmatrix}-1\\-1\\11\end{pmatrix}$ B1

$\overrightarrow{AB}\cdot\overrightarrow{AC}=|\overrightarrow{AB}||\overrightarrow{AC}|\cos A$

$\begin{pmatrix}-2\\-2\\3\end{pmatrix}\begin{pmatrix}-1\\-1\\11\end{pmatrix}=\sqrt{17}\sqrt{123}\cos A$ M1

$\cos A = \dfrac{37}{\sqrt{17}\sqrt{123}}$

$A = 35.99°$

Since $\angle A = 35.99 \neq 180$ and A1 B1

$\angle A = 35.99 \neq 0$; then ABC is a triangle

ALT $\overrightarrow{AB}=\begin{pmatrix}-2\\-2\\3\end{pmatrix}$ B1

$\overrightarrow{AC}=\begin{pmatrix}-1\\-1\\11\end{pmatrix}$ B1

Since $\overrightarrow{AB} = \lambda\,\overrightarrow{AC}$ M1 A1

then ABC is a triangle B1

05

No11(a) find the point of intersection of the lines

$\dfrac{x-5}{4}=\dfrac{y-7}{4}=\dfrac{z+3}{-5}$ and $\dfrac{x-8}{7}=\dfrac{y-4}{1}=\dfrac{z-5}{3}$

(b) The equations of a line and a plane are

$\dfrac{x-2}{1}=\dfrac{y-2}{2}=\dfrac{z-3}{2}$ and $2x+y+4z=9$ respectively.

P is a point on the line where $x=3$. N is the foot of the perpendicular from point P to the plane. Find the coordinates of N.

Soln

$\dfrac{x-5}{4}=\dfrac{y-7}{4}=\dfrac{z+3}{-5}=\lambda$

$x=5+4\lambda,\ y=7+4\lambda,\ z=-3-5\lambda$ ------(1) B1

$\dfrac{x-8}{7}=\dfrac{y-4}{1}=\dfrac{z-5}{3}=\mu$

$x=8+7\mu,\ y=4+\mu,\ z=5+3\mu$ ----(2) B1

$P(x,y,z)$ is the point of intersection; Equating equations

$5+4\lambda=8+7\mu;\quad 4\lambda-7\mu=3$ ---(3)

$7+4\lambda=4+\mu;\quad 4\lambda-\mu=-3$ ---(4)

(4) − (3); $6\mu = -6$, $\mu = -1$ m1

Put $\mu$ in (4); $4\lambda - -1 = -3$, $4\lambda = -4$, $\lambda = -1$ m1

Substitute for $\lambda$ in equation(1)

$x = 5 + 4(-1)$; $x = 1$, $y = 7 + 4(-1) = 3$ $z = -3 - 5(-1) = 2$ M1

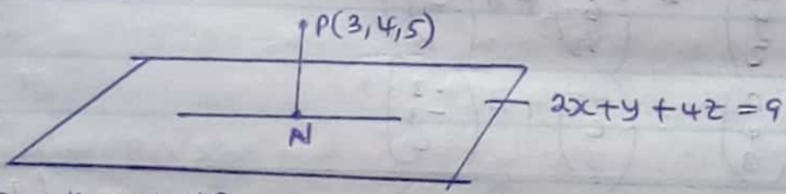
Point of intersection $P(1,3,2)$ A1

Let $\frac{x-2}{1} = \frac{y-2}{2} = \frac{z-3}{2} = \lambda$

$\frac{x-2}{1} = \lambda$; but $x = 3$; $\frac{3-2}{1} = \lambda$, $\lambda = 1$ M1

$\frac{y-2}{2} = 1$, $y = 4$; $\frac{z-3}{2} = 1$, $z = 5$

$P(3,4,5)$ B1




Equation of NQ

$\frac{x-3}{2} = \frac{y-4}{1} = \frac{z-5}{4} = \mu$

$x = 3 + 2\mu$, $y = 4 + \mu$, $z = 5 + 4\mu$ ---- (1) B1

Substituting equation (1) in the equation of the plane

$2(3+2\mu) + 4 + \mu + 4(5+4\mu) = 9$ M1

$6 + 4\mu + 4 + \mu + 20 + 16\mu = 9$

$21\mu = -21$

$\mu = -1$

$x = 3 + 2(-1) = 1$, $y = 4 + -1 = 3$, $z = 5 + 4(-1) = 1$ M1

$N(1,3,1)$ A1

12

No4. A line passes through the points $A(4,6,3)$ and $B(1,3,3)$

a) Find the vector equation of the line

b) Show that the point $C(2,4,3)$ lies on the line in (a) above

<u>Soln</u>

Parallel vector, $\underset{\sim}{b} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}1\\3\\3\end{pmatrix} - \begin{pmatrix}4\\6\\3\end{pmatrix} = \begin{pmatrix}-3\\-3\\0\end{pmatrix}$ M1

Vector equation of the line, $\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$

$$\underset{\sim}{r} = \begin{pmatrix}4\\6\\3\end{pmatrix} + \lambda\begin{pmatrix}-3\\-3\\0\end{pmatrix}$$ A1

To show that $C(2,4,3)$ lies on the line

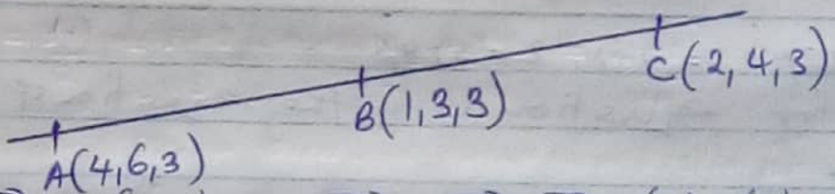
$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}4\\6\\3\end{pmatrix} + \lambda\begin{pmatrix}-3\\-3\\0\end{pmatrix}$$

$$\begin{pmatrix}2\\4\\3\end{pmatrix} = \begin{pmatrix}4\\6\\3\end{pmatrix} + \lambda\begin{pmatrix}-3\\-3\\0\end{pmatrix}$$

$2 = 4 - 3\lambda$; $\lambda = 2/3$, $4 = 6 - 3\lambda$, $\lambda = 2/3$ M1 A1

Since it gives the same value of $\lambda = 2/3$, the point $C(2,4,3)$ lies on a line B1

OR




$\overrightarrow{AB} = \begin{pmatrix}-3\\-3\\0\end{pmatrix}$, $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix}2\\4\\3\end{pmatrix} - \begin{pmatrix}1\\3\\3\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}$

$$\overrightarrow{AB} = \lambda\overrightarrow{BC}$$

$$\begin{pmatrix}-3\\-3\\0\end{pmatrix} = \lambda\begin{pmatrix}1\\1\\0\end{pmatrix}; \lambda = -3$$ M1 A1

Since $\overrightarrow{BC}$ is a scalar multiple of $\overrightarrow{AB}$ and share a point B then $C(2,4,3)$ lies on the line B1

05

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No12. Triangle OAB has $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$. C is a point on $\overrightarrow{OA}$ such that $\overrightarrow{OC} = \frac{2}{3}\underset{\sim}{a}$. D is a mid-point of AB. When $\overrightarrow{CB}$ is produced it meets $\overline{OB}$ produced at E., such that $\overrightarrow{DE} = n\overrightarrow{CB}$ and $\overrightarrow{BE} = K\underset{\sim}{b}$

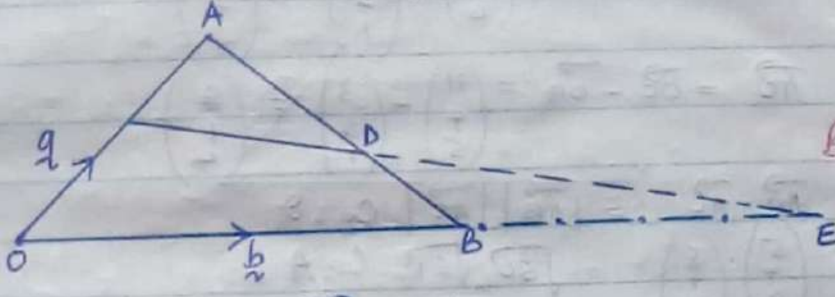
Express $\overrightarrow{DE}$ in terms of

a) $n$, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ (06mks)

b) $k$, $\underset{\sim}{a}$ and $\underset{\sim}{b}$

Hence find the values of $n$ and $k$

Soln




B1

a) $\overrightarrow{OC} = \frac{2}{3}\underset{\sim}{a}$ , $\overrightarrow{DE} = n\overrightarrow{CD}$ , $\overrightarrow{BE} = k\underset{\sim}{b}$

$\overrightarrow{DE} = n\overrightarrow{CD}$

But $\overrightarrow{CD} = \overrightarrow{CO} + \overrightarrow{OB} + \overrightarrow{BD}$

$= -\overrightarrow{OC} + \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BA}$ M1

$= -\frac{2}{3}a + \underset{\sim}{b} + \frac{1}{2}(\underset{\sim}{a} - \underset{\sim}{b})$ M1M1

$= -\frac{1}{6}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$ A1

$\overrightarrow{DE} = n[-\frac{1}{6}a + \frac{1}{2}\underset{\sim}{b}]$

$= -\frac{n}{6}\underset{\sim}{a} + \frac{n}{2}\underset{\sim}{b}$ B1

b) Also $\overrightarrow{DE} = \overrightarrow{DB} + \overrightarrow{BE}$

$= \frac{1}{2}\overrightarrow{AB} + k\underset{\sim}{b}$ M1

$= \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a}) + k\underset{\sim}{b}$ M1

$= -\frac{1}{2}\underset{\sim}{a} + (\frac{1}{2} + k)\underset{\sim}{b}$ A1

Equating the two expressions

$-\frac{n}{6}\underset{\sim}{a} + \frac{n}{2}\underset{\sim}{b} = -\frac{1}{2}\underset{\sim}{a} + (\frac{1}{2} + k)\underset{\sim}{b}$

Comparing the same vectors

$-\frac{n}{6} = -\frac{1}{2}$ , M1

$n = 3$ A1

$\frac{n}{2} = \frac{1}{2} + k$

$\frac{3}{2} = \frac{1}{2} + k$

$k = 1$ A1

12

# UNEB 2011



No 7. Show that the points A, B and C with position vectors $3\underset{\sim}{i}+3\underset{\sim}{j}+\underset{\sim}{k}$, $8\underset{\sim}{i}+7\underset{\sim}{j}+\underset{\sim}{k}$ and $11\underset{\sim}{i}+4\underset{\sim}{j}+5\underset{\sim}{k}$ respectively are vertices of a triangle (0

Soln

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}8\\7\\4\end{pmatrix} - \begin{pmatrix}3\\3\\1\end{pmatrix} = \begin{pmatrix}5\\4\\3\end{pmatrix}$$ B1

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix}11\\4\\5\end{pmatrix} - \begin{pmatrix}3\\3\\1\end{pmatrix} = \begin{pmatrix}8\\1\\4\end{pmatrix}$$ B1

$$\overrightarrow{AB}\cdot\overrightarrow{AC} = |\overrightarrow{AB}||\overrightarrow{AC}|\cos\theta$$

$$\begin{pmatrix}5\\4\\3\end{pmatrix}\cdot\begin{pmatrix}8\\1\\4\end{pmatrix} = \sqrt{50}\sqrt{81}\cos A$$ M1

$$56 = 9\sqrt{50}\cos A$$

$$\cos A = \frac{56}{9\sqrt{50}}$$

$$A = 28.36^\circ$$ A1

Since $\angle A \neq 180$ and $\angle A \neq 0^\circ$, then ABC is a triangle. B1

---

No 12 (a) Find the angle between the lines

$$x = \frac{y-1}{2} = \frac{z-2}{3} \text{ and } \frac{x}{2} = \frac{y+1}{3} = \frac{z+2}{4}$$ (06

(b) Find in vector form the equation of the line of intersection of the two planes $2x+3y-z=4$ and $x-y+2z=5$ (06m

Soln

Parallel vector to $L_1$: $b_1 = \underset{\sim}{i}+2\underset{\sim}{j}+3\underset{\sim}{k}$ B1

Parallel vector to $L_2$: $b_2 = 2\underset{\sim}{i}+3\underset{\sim}{j}+4\underset{\sim}{k}$ B1

$$\underset{\sim}{b_1}\cdot\underset{\sim}{b_2} = |\underset{\sim}{b_1}||\underset{\sim}{b_2}|\cos\theta$$

$$\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}2\\3\\4\end{pmatrix} = \sqrt{1+4+9}\sqrt{4+9+16}\cos\theta$$ M1 M1

$$2+6+12 = \sqrt{14}\sqrt{29}\cos\theta$$

$$\cos\theta = \frac{20}{\sqrt{14}\sqrt{29}}$$ M1

$$\theta = 6.98^\circ$$ A1

The Planes are; $2x + 3y - z = 4$ ---- (1)

$x - y + 2z = 5$ ---- (2)

Eliminate x

Equation (1) × 1 ; $2x + 3y - z = 4$

Equation (2) × 2 $-\ \underline{2x - 2y + 4z = 10}$

$5y - 5z = -6.$ M1 – Eliminating x

$y - z = -6/5$

$z = y + 6/5$ A1

Eliminate y.

Equation (1) × 1 ; $2x + 3y - z = 4$

Equation (2) × 3 ; $+\ \underline{3x - 3y + 6z = 15}$

$5x + 5z = 19$ M1 – Eliminating y

$x + z = 19/5$

$z = -x + 19/5$ A1

$$\frac{-x + \frac{19}{5}}{1} = \frac{y + 6/5}{1} = z$$

$$\frac{x - 19/5}{-1} = \frac{y + 6/5}{1} = \frac{z}{1}$$ B1

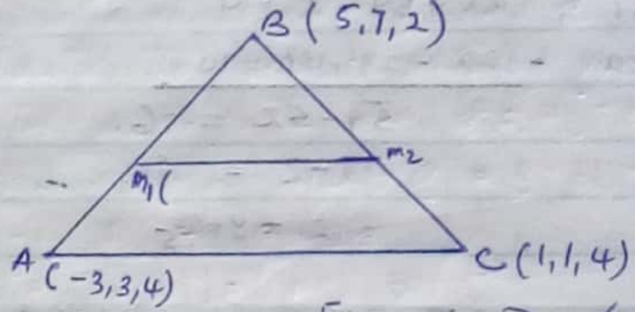
$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$

$$\underset{\sim}{r} = \begin{pmatrix} 19/5 \\ -6/5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$ B1

12

No5. Given the points $A(-3,3,4)$, $B(5,7,2)$ and $C(1,1,4)$. Find the vector equation of a line which joins the mid-points of AB and BC

Soln




Mid-point of AB $= \frac{1}{2}\left[\begin{pmatrix}-3\\3\\4\end{pmatrix}+\begin{pmatrix}5\\7\\2\end{pmatrix}\right] = \begin{pmatrix}1\\5\\3\end{pmatrix}$ B1

Mid-point of BC $= \frac{1}{2}\left[\begin{pmatrix}5\\7\\2\end{pmatrix}+\begin{pmatrix}1\\1\\4\end{pmatrix}\right] = \begin{pmatrix}3\\4\\3\end{pmatrix}$ B1

Parallel vector to $M_1M_2$, $\underset{\sim}{b} = \begin{pmatrix}3\\4\\3\end{pmatrix} - \begin{pmatrix}1\\5\\3\end{pmatrix} = \begin{pmatrix}2\\-1\\0\end{pmatrix}$ B1

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$$

$$\underset{\sim}{r} = \begin{pmatrix}1\\5\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\0\end{pmatrix}$$ M1 A1

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No11(a) The equation of the plane R is $x \cdot \begin{pmatrix}4\\3\\-2\end{pmatrix} = 16$, where $x$ is the position vector of R. Find the perpendicular distance of the plane from the origin (04

(b) Find the Cartesian equation of the plane through the points $P(1,0,-2)$ and $Q(3,-1,1)$ parallel to the line with a vector equation.

$\underset{\sim}{x} = 3i + (2\alpha-1)\underset{\sim}{j} + (5-\alpha)\underset{\sim}{k}$ (08

Soln

$r \cdot \begin{pmatrix}4\\3\\-2\end{pmatrix} = 16$, $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}4\\3\\-2\end{pmatrix} = 16$

$$4x + 3y - 2z - 16 = 0$$ B1

$$d = \frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2+b^2+c^2}}$$

$$d = \frac{|0+0+0+-16|}{\sqrt{4^2+3^2+(-2)^2}}$$ M1

$$= \frac{16}{\sqrt{29}}$$ M1

$$d = 2.97$$ A1

(b) Method I

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= \begin{pmatrix}3\\-1\\1\end{pmatrix} - \begin{pmatrix}1\\0\\-2\end{pmatrix} = \begin{pmatrix}2\\-1\\3\end{pmatrix}$ B1

$\underset{\sim}{r} = 3i + (2\alpha - 1)j + (5-\alpha)k$

$\underset{\sim}{r} = 3\underset{\sim}{i} - \underset{\sim}{j} + 5\underset{\sim}{k} + \alpha(2\underset{\sim}{j} - \underset{\sim}{k})$ B1

$n = \begin{vmatrix} i & -j & k \\ 2 & -1 & 3 \\ 0 & 2 & -1 \end{vmatrix}$ M1

$n = -5\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$ M1 M1

$A(x, y, z)$, $P(1, 0, -2)$

$\overrightarrow{PA} \cdot n = 0$

$\begin{pmatrix}x-1\\y\\z+2\end{pmatrix} \cdot \begin{pmatrix}-5\\2\\4\end{pmatrix} = 0$ M1

$-5x + 5 + 2y + 4z + 8 = 0$ M1 A1

$-5x + 2y + 4z + 13 = 0$

Method II

Parallel vector, $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= \begin{pmatrix}2\\-1\\3\end{pmatrix}$ B1

$\underset{\sim}{r} = \begin{pmatrix}3\\-1\\5\end{pmatrix} + \lambda\begin{pmatrix}0\\2\\-1\end{pmatrix}$ B1

The vector equation of a plane

$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b} + \mu\underset{\sim}{c}$

$\underset{\sim}{r} = \begin{pmatrix}1\\0\\-2\end{pmatrix} + \lambda\begin{pmatrix}0\\2\\-1\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\3\end{pmatrix}$ M1

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\0\\-2\end{pmatrix} + \lambda\begin{pmatrix}0\\2\\-1\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\3\end{pmatrix}$

$x - 1 = 2\mu$ ---- (1) B1

$y = 2\lambda - \mu$ ---- (2)

$z + 2 = -\lambda + 3\mu$ ---- (3)

Eqn (2) x1 ; $y = 2\lambda - \mu$ M1

Eqn (3) x2 ; $2z + 4 = -2\lambda + 6\mu$

$y + 2z + 4 = 5\mu$ ---- (5) M1

Make $\mu$ the subject in (1) and (5) and equate the two equations

$\frac{1}{2}(x-1) = \frac{1}{5}(y+2z+4)$ M1

$5(x-1) = 2(y+2z+4)$

$5x - 5 = 2y + 4z + 8$

$5x - 2y - 4z - 13 = 0$ A1

12

No.7: Find the equation of a line through the point $(1,3,-2)$ and perpendicular to the plane whose equation is $4x+3y-2z-16=0$

Soln

Normal vector to the plane is parallel vector to the line

parallel vector to the line, $\underset{\sim}{b} = 4\underset{\sim}{i}+3\underset{\sim}{j}-2\underset{\sim}{k}$ B1

A point where the line passes, $\underset{\sim}{a} = \underset{\sim}{i}+3\underset{\sim}{j}-2\underset{\sim}{k}$ B1

$$\underset{\sim}{r} = \underset{\sim}{a}+\lambda\underset{\sim}{b}$$

$$\underset{\sim}{r} = \begin{pmatrix}1\\3\\-2\end{pmatrix}+\lambda\begin{pmatrix}4\\3\\-2\end{pmatrix}$$ M1 M1 A1

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NO13(a) Find the angle between the planes $x-2y+z=0$ and $x-y=1$

(b) Two lines are given by the parametric equations; $-\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k}+t(\underset{\sim}{i}-2\underset{\sim}{j}+3\underset{\sim}{k})$ and $-3\underset{\sim}{i}+p\underset{\sim}{j}+7\underset{\sim}{k}+s(\underset{\sim}{i}-\underset{\sim}{j}+2\underset{\sim}{k})$

If the lines intersect, find the

(i) Values of $t$, $s$ and $p$

(ii) Coordinates of the point of intersection

Soln

a) Perpendicular vectors to the plane

$\underset{\sim}{n_1} = \underset{\sim}{i}-\underset{\sim}{j}$ , $\underset{\sim}{n_2} = \underset{\sim}{i}-2\underset{\sim}{j}+\underset{\sim}{k}$ B1

$$\underset{\sim}{n_1}\cdot\underset{\sim}{n_2} = |\underset{\sim}{n_1}||\underset{\sim}{n_2}|\cos\theta$$

$$\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix} = \sqrt{1+1+0}\sqrt{1+4+1}\cos\theta$$ M1

$$1+2+0 = \sqrt{2}\sqrt{6}\cos\theta$$

$$\cos\theta = \frac{3}{\sqrt{12}}$$

$$\theta = 30^\circ$$ M1 A1

(b) $$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-1\\2\\1\end{pmatrix}+t\begin{pmatrix}1\\-2\\3\end{pmatrix}$$

$$\left.\begin{aligned} x &= -1+t \\ y &= 2-2t \\ z &= 1+3t \end{aligned}\right\} \text{-----} (1) \quad M1$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ p \\ 7 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\left.\begin{aligned} x &= -3+s \\ y &= p-s \\ z &= 7+2s \end{aligned}\right\} \text{-----} (2) \quad M1$$

Equating equations (1) and (2)

for $x$ $\quad -1+t = -3+s \; ; \; t-s = -2$ ---(3)

for $z$ $\quad 1+3t = 7+2s \; ; \; 3t-2s = 6$ ---(4)

from (3); $t = s-2$

$3(s-2)-2s = 6$ M1

$3s-6-2s = 6$

$s = 12.$ A1

$t = 10$ A1

for $y$; $\quad 2-2t = p-s$

$2-2(10) = p-12$

$p = -6$ A1

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + 10\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \quad M1$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -18 \\ 31 \end{pmatrix}$$

Coordinates for the point of intersection.
$(9, -18, 31)$ A1

12

# UNEB 2008 

No 4. Given the vectors $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{b} = -\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$

Find the

(i) acute angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$

(ii) Equation of the plane containing $\underset{\sim}{a}$ and $\underset{\sim}{b}$ (0

Soln

(i) $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

$$\begin{pmatrix}1\\3\\3\end{pmatrix} \cdot \begin{pmatrix}-1\\-3\\2\end{pmatrix} = \sqrt{1+9+9}\ \sqrt{1+9+4}\ \cos\theta$$ m1

$$-1 + -9 + 6 = \sqrt{19}\sqrt{14} \cos\theta$$

$$\cos\theta = \frac{-4}{\sqrt{19}\sqrt{14}}$$

$$\theta = 104.2$$

Acute angle $= 180 - 104.2 = 75.8$ A1

(ii) normal vector, $n = \begin{vmatrix} i & -j & k \\ 1 & 3 & 3 \\ -1 & -3 & 2 \end{vmatrix}$ m1

$$= 15\underset{\sim}{i} - 5\underset{\sim}{j}$$

$$\begin{pmatrix}x-1\\y-3\\z-3\end{pmatrix} \cdot \begin{pmatrix}15\\-5\\ \end{pmatrix} = 0$$ m1

$$15x - 15 - 5y + 15 = 0$$

$$15x - 5y = 0$$

$$3x - y = 0$$ A1

05

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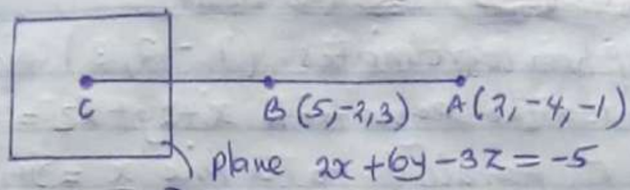
No. 12 The position vectors of points A and B are $\overrightarrow{OA} = 2\underset{\sim}{i} - 4\underset{\sim}{j} - \underset{\sim}{k}$ and $\overrightarrow{OB} = 5\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$ respectively. The line $\overrightarrow{AB}$ is produced to meet the plane $2x + 6y - 3z = -5$ at a point C.

Find the

a) Co-ordinates of C (07

b) the angle between $\overrightarrow{AB}$ and the plane (05 soln:




Equation of the line $\overrightarrow{AB}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}5\\-2\\3\end{pmatrix} - \begin{pmatrix}2\\-4\\-1\end{pmatrix} = \begin{pmatrix}3\\2\\4\end{pmatrix}$$ B1

$$\underset{\sim}{r} = \begin{pmatrix}2\\-4\\-1\end{pmatrix} + \lambda\begin{pmatrix}3\\2\\4\end{pmatrix}$$ B1 — Eqn of the line

Intersection of a line and a plane

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\-4\\-1\end{pmatrix} + \lambda\begin{pmatrix}3\\2\\4\end{pmatrix}$$

$x = 2+3\lambda,\ y = -4+2\lambda,\ z = -1+4\lambda$ ——— (1) M1

Substitute in the equation of the plane

$$2x+6y-3z=-5$$

$$2(2+3\lambda)+6(-4+2\lambda)-3(-1+4\lambda) = -5$$ M1 — substitution

$$6\lambda = 12$$

$$\lambda = 2.$$ B1

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\-4\\-1\end{pmatrix} + 2\begin{pmatrix}3\\2\\4\end{pmatrix}$$ M1

Coordinates of C $(8,0,7)$ A1

(b) Parallel vector to $\overrightarrow{AB}$; $\underset{\sim}{b} = 3\underset{\sim}{i}+2\underset{\sim}{j}+4\underset{\sim}{k}$ B1

normal vector to the plane, $\underset{\sim}{n} = 2\underset{\sim}{i}+6\underset{\sim}{j}-3\underset{\sim}{k}$. B1

$$\underset{\sim}{b}\cdot\underset{\sim}{n} = |\underset{\sim}{b}||\underset{\sim}{n}|\sin\theta$$

$$\begin{pmatrix}3\\2\\4\end{pmatrix}\cdot\begin{pmatrix}2\\6\\-3\end{pmatrix} = \sqrt{9+4+16}\sqrt{4+36+9}\sin\theta$$ M1

$$6+12-12 = \sqrt{29}\sqrt{49}\sin\theta$$

$$\sin\theta = \frac{6}{7\sqrt{29}}$$ M1

$$\theta = \sin^{-1}\left(\frac{6}{7\sqrt{29}}\right)$$

$$\theta = 9.16^\circ$$ A1

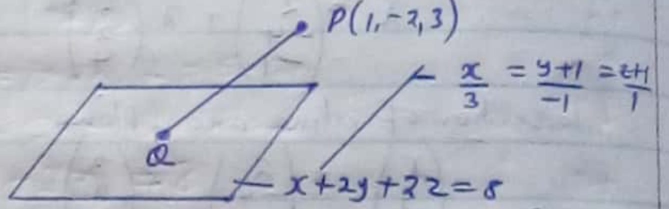
12

# UNEB 2007



No5. A point P has coordinates (1, −2, 3) and a certain plane has the equation $x+2y+2z=8$. The line through P parallel to the line $\frac{x}{3}=\frac{y+1}{-1}=z+1$ meets the plane at a point Q. Find the coordinates of Q.

Soln




Parallel lines have the same ~~grad~~ parallel vector

Parallel vector of PQ, $\underline{b} = 3\underline{i}-\underline{j}+\underline{k}$ B1

Equation of the line PQ; $r=\begin{pmatrix}1\\-2\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\-1\\1\end{pmatrix}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\-2\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\-1\\1\end{pmatrix}$$

$x=1+3\lambda,\ y=-2-\lambda,\ z=3+\lambda$ B1

Equation of ~~the~~ plane; $x+2y+2z=8$

$1+3\lambda+2(-2-\lambda)+2(3+\lambda)=8$ M1

$3\lambda=5$

$\lambda=5/3$ B1

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\-2\\3\end{pmatrix}+\frac{5}{3}\begin{pmatrix}3\\-1\\-1\end{pmatrix}$$

Point of intersection, $Q\left(6, -\frac{11}{3}, \frac{1}{3}\right)$ A1

05

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No15 Given that the position vectors of A, B and C are $\overrightarrow{OA}=\begin{pmatrix}1\\-2\\2\end{pmatrix}$, $\overrightarrow{OB}=\begin{pmatrix}3\\2\\-1\end{pmatrix}$ and $\overrightarrow{OC}=\begin{pmatrix}7\\10\\-7\end{pmatrix}$

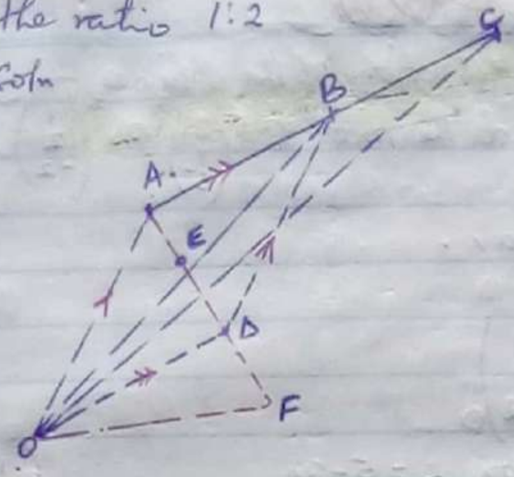
(i) Prove that A, B and C are collinear

(ii) Find the acute angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$

(iii) If OABD is a parallelogram, find the position vectors of E and F such that E divides DA in the ratio 1:2 and F divides it externally in the ratio 1:2 (12mks)

Soln

(i)




$\vec{AB} = \vec{OB} - \vec{OA}$

$= \begin{pmatrix}3\\2\\-1\end{pmatrix} - \begin{pmatrix}1\\-2\\2\end{pmatrix}$

$= \begin{pmatrix}2\\4\\-3\end{pmatrix}$ B1

$\vec{BC} = \vec{OC} - \vec{OB}$

$= \begin{pmatrix}7\\10\\-7\end{pmatrix} - \begin{pmatrix}3\\2\\-1\end{pmatrix}$

$= \begin{pmatrix}4\\8\\-6\end{pmatrix}$ B1

$\vec{AB} = \lambda \vec{BC}$

$\begin{pmatrix}3\\4\\-3\end{pmatrix} = \lambda \begin{pmatrix}4\\8\\-6\end{pmatrix}$ M1

and $\lambda = \frac{1}{2}$

Since $\vec{AB} = \lambda \vec{BC}$; where $\lambda = \frac{1}{2}$, the A, B, C are collinear B1

(ii) $\vec{OA} \cdot \vec{OB} = |\vec{OA}||\vec{OB}| \cos\theta$

$\begin{pmatrix}1\\-2\\2\end{pmatrix} \cdot \begin{pmatrix}3\\2\\-1\end{pmatrix} = \sqrt{1+4+4}\,\sqrt{9+4+1}\,\cos\theta$ M1 M1

$3-4-2 = 3\sqrt{14}\,\cos\theta$

$\cos\theta = \frac{-3}{3\sqrt{14}}$ M1

$\theta = 105.5$

Acute angle $= 180 - 105.5 = 74.5°$ A1

(iii) $\vec{AB} = \vec{OD}$

$\vec{DA} = \vec{DO} + \vec{OA} = -\vec{AB} + \vec{OA} = \begin{pmatrix}-2\\-4\\3\end{pmatrix} + \begin{pmatrix}1\\-2\\2\end{pmatrix} = \begin{pmatrix}-1\\-6\\5\end{pmatrix}$

$DE : EA = 1:2$

$\frac{DE}{EA} = \frac{1}{2}$

$2\vec{DE} = \vec{EA}$

$2[\vec{OE} - \vec{OD}] = \vec{OA} - \vec{OE}$

$2\vec{OE} - 2\vec{OD} = \vec{OA} - \vec{OE}$

$3\vec{OE} = \vec{OA} + 2\vec{OD}$ M1 A1

$OE \quad \frac{1}{3}\left[\begin{pmatrix}1\\-2\\2\end{pmatrix} + 2\begin{pmatrix}2\\4\\-3\end{pmatrix}\right] = \begin{pmatrix}5/3\\2\\-4/3\end{pmatrix}$

$\vec{OE} = \frac{1}{3}\begin{pmatrix}5\\6\\-4\end{pmatrix} = \begin{pmatrix}5/3\\2\\-4/3\end{pmatrix}$

$\vec{DF} : \vec{FA} = 1 : -2$

$\frac{DF}{FA} = \frac{1}{-2}$

$\vec{FA} = -2\vec{DF}$

$\vec{OF} = -\vec{OA} + 2\vec{OD}$

$= +\begin{pmatrix}-1\\2\\-2\end{pmatrix} + 2\begin{pmatrix}2\\4\\-3\end{pmatrix}$ M1

$\vec{OF} = \begin{pmatrix}3\\10\\-8\end{pmatrix}$ A1

12

# UNEB 2006



No 5. Find the point of intersection of the plane $11x - 3y + 7z = 8$ and the line $r = \begin{pmatrix}3\\1\\5\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\-2\end{pmatrix}$

where $\lambda$ is a scalar.

Soln

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\1\\5\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\-2\end{pmatrix}$$

$x = 3+\lambda,\quad y = 1+2\lambda,\quad z = 5-2\lambda$ B1

Substitute in the equation of the plane

$$11x - 3y + 7z = 8$$

$$11(3+\lambda) - 3(1+2\lambda) + 7(5-2\lambda) = 8$$ M1

$$33 + 11\lambda - 3 - 6\lambda + 35 - 14\lambda = 8$$

$$-9\lambda = -57$$

$$\lambda = \frac{57}{9}$$ B1

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\1\\5\end{pmatrix} + \frac{57}{9}\begin{pmatrix}1\\2\\-2\end{pmatrix} = \begin{pmatrix}28/3\\44/3\\-23/3\end{pmatrix}$$ M1

Coordinates for the point of intersection $\left(\frac{28}{3}, \frac{44}{3}, -\frac{23}{3}\right)$ A1

05

---

11(a) Given the vectors $a = 3i - 2j + k$ and $b = i - 2j + 2k$

Find (i) the acute angle between the vectors

(ii) Vector c such that it is perpendicular to both vectors $a$ and $b$ (06

(b) Given that $\overrightarrow{OA} = a$ and $\overrightarrow{OB} = b$, point R is on $\overline{OB}$ such that $\overrightarrow{OR} : \overrightarrow{RB} = 4:1$. Point P is on $\overline{BA}$ such that $\overrightarrow{BP} : \overrightarrow{PA} = 2:3$ and when $\overline{RP}$ and $\overline{OA}$ are both produced they meet at Q

Find (i) $\overrightarrow{OR}$ and $\overrightarrow{OP}$ in terms of $a$ and $b$

(ii) $\overrightarrow{OQ}$ in terms of a (06

Soln

(a) $a \cdot b \quad |a||b|\cos\theta$

$$\begin{pmatrix}3\\-2\\1\end{pmatrix} \cdot \begin{pmatrix}1\\-2\\2\end{pmatrix} = \sqrt{9+4+1}\,\sqrt{1+4+4}\cos\theta$$ M1 M1

$$3+4+2 = 3\sqrt{14}\cos\theta$$ M1

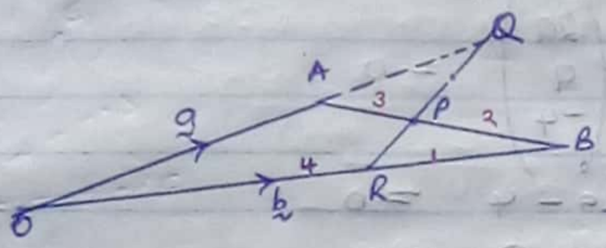
$\cos\theta = \frac{9}{3\sqrt{14}}$

$= 36.7°$ A1

a(ii) $\underset{\sim}{c} = \begin{vmatrix} i & j & k \\ 3 & -2 & 1 \\ 1 & -? & 2 \end{vmatrix}$ M1

$= 2i - 7j - 4k$ A1

(b)



$\overrightarrow{OR} = \frac{4}{5}\overrightarrow{OB} = \frac{4}{5}\underset{\sim}{b}$ B1

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \underset{\sim}{b} - \underset{\sim}{a}$.

$\overrightarrow{AP} = \frac{3}{5}\overrightarrow{AB} = \frac{3}{5}(b - \underset{\sim}{a})$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$= \underset{\sim}{a} + \frac{3}{5}(b - \underset{\sim}{a})$ M1

$= \frac{5a + 3b - 3a}{5}$

$= \frac{1}{5}(2\underset{\sim}{a} + 3\underset{\sim}{b})$ A1

$\overrightarrow{OQ} = \lambda\overrightarrow{OA} = \lambda\underset{\sim}{a}$.

$\overrightarrow{RP} = \overrightarrow{RO} + \overrightarrow{OA} + \overrightarrow{AP}$

$= -\frac{4}{5}\underset{\sim}{b} + \underset{\sim}{a} + \frac{3}{5}(b - a)$

$= \frac{1}{5}(-4\underset{\sim}{b} + 5\underset{\sim}{a} + 3\underset{\sim}{b} - 3\underset{\sim}{a})$

$= \frac{1}{5}(2\underset{\sim}{a} - \underset{\sim}{b})$ B1

$\overrightarrow{OQ} = \overrightarrow{OR} + \mu\overrightarrow{RP}$

$= \frac{4}{5}\underset{\sim}{b} + \mu \cdot \frac{1}{5}(2\underset{\sim}{a} - \underset{\sim}{b})$

$= \left(\frac{4}{5} - \frac{\mu}{5}\right)\underset{\sim}{b} + \frac{2\mu}{5}a$. B1

Equating the two vectors; $\frac{4}{5} - \frac{\mu}{5} = 0$, $\mu = 4$

$\frac{2\mu}{5} = \lambda$, $\lambda = \left(\frac{2}{5}\right)(4) = \frac{8}{5}$

$\therefore OQ = \frac{8}{5}\underset{\sim}{a}$ A1

12

# UNEB 2005

No 4. Given the vectors $a\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and $2a\underset{\sim}{i} + a\underset{\sim}{j} - 4\underset{\sim}{k}$ are perpendicular, find the values of a

Soln:

$$\begin{pmatrix} a \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2a \\ a \\ -4 \end{pmatrix} = 0$$ M1

$2a^2 - 2a - 4 = 0$ B1

$a^2 - a - 2 = 0$

$a^2 - 2a + a - 2 = 0$

$a(a-2) + 1(a-2) = 0$ M1

$(a+1)(a-2) = 0$

$a = -1, \quad a = 2$ A1 A1

05

10 (i) Determine the coordinates of the point of intersection of the line $\frac{x+1}{2} = \frac{y-3}{5} = \frac{z+2}{-1}$

and the plane $x + y + z = 12$

(ii) Find the angle between the line $\frac{x+1}{2} = \frac{y-3}{5} = \frac{z+1}{-1}$ and the plane $x+y+z = 12$

Soln

(i) Let $\frac{x+1}{2} = \frac{y-3}{5} = \frac{z+2}{-1} = \lambda$

$$\left.\begin{aligned} x &= -1 + 2\lambda \\ y &= 3 + 5\lambda \\ z &= -2 - \lambda \end{aligned}\right\} \text{---- (1)}$$ M1

Equation of the plane

$x + y + z = 12$ ---(2)

If $(x, y, z)$ is the point of intersection; substitute (1) in (2)

$-1 + 2\lambda + 3 + 5\lambda + 2 - \lambda = 12$ M1

$6\lambda = 12$ M1

$\lambda = 2$ A1

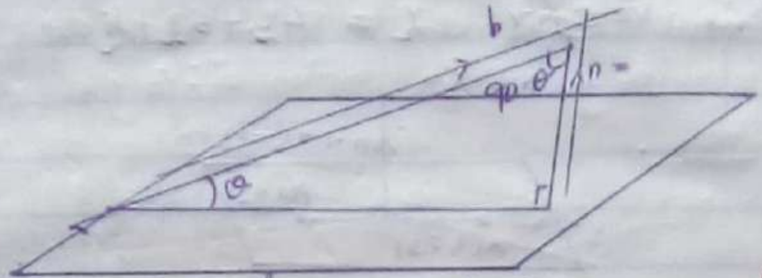
$x = -1 + 2(2) = 3$

$y = 3 + 5(2) = 13$ M1

$z = -2 - 2 = -4$

Coordinates for the point of intersection $P(3, 13, -4)$ A1

(ii)




normal vector to the plane, $\underset{\sim}{n} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ B1

Parallel vector to the line, $\underset{\sim}{b} = 2\underset{\sim}{i} + 5\underset{\sim}{j} - \underset{\sim}{k}$ B1

$$\underset{\sim}{n} \cdot \underset{\sim}{b} = |\underset{\sim}{n}||\underset{\sim}{b}| \cos(90-\theta)$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} = \sqrt{1+1+1}\sqrt{4+25+1} \sin\theta$$ M1 M1

$$2+5-1 = \sqrt{3}\sqrt{30} \sin\theta$$ M1

$$\sin\theta = \frac{6}{\sqrt{3}\sqrt{30}}$$

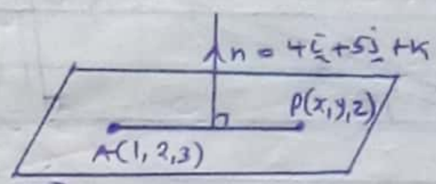
$$\theta = \sin^{-1}\left(\frac{6}{\sqrt{90}}\right)$$

$$\theta = 39.2^\circ$$ A1

12

# UNEB 2004

No6. Find the equation of a plane through the point $(1,2,3)$ and perpendicular to the vector $\underset{\sim}{r} = 4\underset{\sim}{i} + 5\underset{\sim}{j} + \underset{\sim}{k}$




$\overrightarrow{AP} \cdot n = 0$

$$\begin{pmatrix} x-1 \\ y-2 \\ z-3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} = 0 \qquad M1M1$$

$$4x - 4 + 5y - 10 + z - 3 = 0 \qquad M1M1$$

$$4x + 5y + z - 17 = 0 \qquad A1$$

05

---

11 (a) Find the equation of the line through $A(2,2,5)$ and $B(1,2,3)$

(b) If the line in (a) above meets the line $\frac{x-1}{1} = \frac{y-2}{0} = \frac{z-1}{3}$ at $P$, find the

(i) coordinates of $P$,

(ii) angle between the two lines

(12)

Soln

$A(2,2,5)$ , $B(1,2,3)$

(a) Parallel vector to line AB; $\underset{\sim}{b} = \overrightarrow{AB}$

$$= \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} \qquad M1$$

$$= \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} \qquad A1$$

Equation of the line is given by

$$r = \underset{\sim}{a} + \lambda \underset{\sim}{b}$$

$$\underset{\sim}{r} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} \qquad M1A1$$

-1M-

(b) Write equations in parametric form.

$L_1$: $\frac{x-1}{1} = \frac{y-2}{0} = \frac{z-1}{3} = \lambda$

$$\left.\begin{array}{l} x = 1+\lambda \\ y = 2 \\ z = 1+3\lambda \end{array}\right\} \text{-----} (1)$$

M1

$L_2$: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$

$$\left.\begin{array}{l} x = 2-\mu \\ y = 2 \\ z = 5-2\mu \end{array}\right\} \text{-----} (2)$$

M1

If $(x,y,z)$ is the point of intersection; equate the two equations

$1+\lambda = 2-\mu$

$\lambda+\mu = 1$ --- (3)

$1+3\lambda = 5-2\mu$

$3\lambda+2\mu = 4$ --- (4)

Eqn (3) x 2; $-\underline{2\lambda+2\mu = 2}$

$\lambda = 2$, $-\cancel{\mu}$

$\mu = -1$

M1 A1

$x = 1+2 = 3$, $y = 2$, $z = 1+3(2) = 7$

Point of intersection $P(3,2,7)$ A1

Parallel vector to $L_1$: $b_1 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ ~~~~

Parallel vector to $L_2$; $b_2 = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$

$\underset{\sim}{b_1} \cdot \underset{\sim}{b_2} = |\underset{\sim}{b_1}||b_2| \cos\theta$

$\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} = \sqrt{4+0+9}\sqrt{1+0+4} \cos\theta$ M1

$-2+0-6 = \sqrt{13}\sqrt{5} \cos\theta$ M1

$\cos\theta = \frac{-8}{\sqrt{65}}$

$= 172.9°$

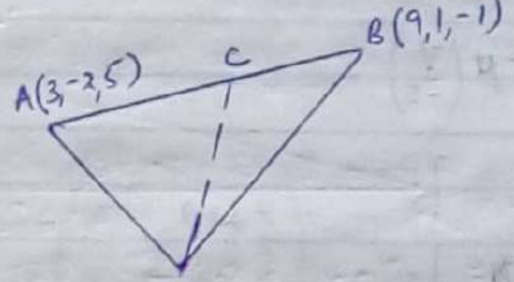
Acute angle, $\theta = 180 - 172.9$

$= 7.1°$ A1

12

No2. Given the position vectors $\overrightarrow{OA} = (3, -2, 5)$ and $\overrightarrow{OB} = (9, 1, -1)$, find the position vector of point C, such that C divides $\overline{AB}$ internally in the ratio 5 : 3

Soln




$\overrightarrow{AC} : \overrightarrow{CB} = 5 : 3$

$\dfrac{\overrightarrow{AC}}{\overrightarrow{CB}} = 5/3$ M1

$3\overrightarrow{AC} = 5\overrightarrow{CB}$

$3[\overrightarrow{OC} - \overrightarrow{OA}] = 5[\overrightarrow{OB} - \overrightarrow{OC}]$

$8\overrightarrow{OC} = 3\overrightarrow{OA} + 5\overrightarrow{OB}$ M1

$= 3\begin{pmatrix}3\\-2\\5\end{pmatrix} + 5\begin{pmatrix}9\\1\\-1\end{pmatrix}$ M1M1

$\overrightarrow{OC} = \begin{pmatrix}54/8\\-1/8\\10/8\end{pmatrix}$ A1

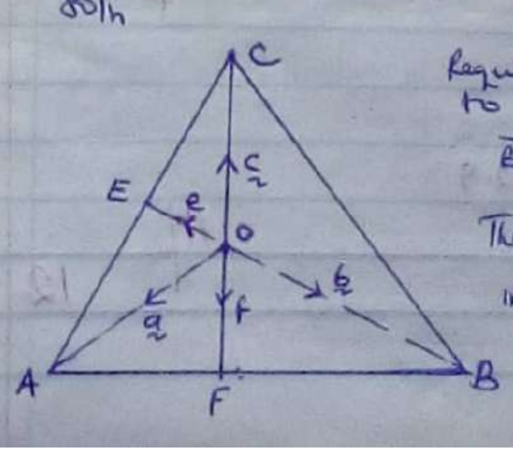
05

---

No 1(a) In a triangle ABC, the altitudes from B and C meet the opposite sides at E and F respectively. BE and CF intersect at O. Taking O as the origin, use the dot product to prove that AO is perpendicular to BC (6)

(b) Prove that $\angle ABC = 90°$ given that A is $(0, 5, -3)$, $B(2, 3, -4)$ and $C(1, -1, 2)$. Find the coordinates of D, if ABCD is a rectangle.

Soln




Required is to show that AO is perpendicular to BC

$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = -\underset{\sim}{b} + \underset{\sim}{c}$

The triangle intersects at its centre in the ratio 2 : 1

Then BO + OE = 2+1 = 3 = BE

$\overrightarrow{BO} = \frac{2}{3}\overrightarrow{BE} = \frac{2}{3}h$. M1

(Letting the altitudes of the triangle from every point to be h. ie $CF = BE = h$)

$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$

$= -\frac{2}{3}\overrightarrow{BE} + \overrightarrow{OC}$

$= -\frac{2}{3}h + \frac{2}{3}h = 0$. B1

$\overrightarrow{OA} = \frac{2}{3}h$.

$\overrightarrow{AO} \cdot \overrightarrow{BE} = \left(\frac{2}{3}h\right) \cdot 0 = 0$ M1 A1

Since $AO \cdot BC = 0$, then AO is perpendicular to BC B1

ALT.

$\overrightarrow{OF}$ is perpendicular to $\overrightarrow{AB}$

$\overrightarrow{OF} \cdot \overrightarrow{AB} = 0$; but $\overrightarrow{OF} = -t\overrightarrow{OC}$

$-t\overrightarrow{OC} \cdot \overrightarrow{AB} = 0$

$-t\underset{\sim}{c} \cdot (\underset{\sim}{b} - \underset{\sim}{a}) = 0$. M1

$\underset{\sim}{c} \cdot (b - a) = 0$ ---(1)

OE is perpendicular to AC

$\overrightarrow{OE} \cdot \overrightarrow{AC} = 0$; but $\overrightarrow{OE} = -\lambda\overrightarrow{OB}$

$-\lambda\overrightarrow{OB} \cdot \overrightarrow{AC} = 0$

$-\lambda\underset{\sim}{b} \cdot (\underset{\sim}{c} - \underset{\sim}{a}) = 0$ M1

$\underset{\sim}{b} \cdot (\underset{\sim}{c} - \underset{\sim}{a}) = 0$ -- (2)

(2)-(1); $b \cdot (c - a) - c \cdot (b - a) = 0$ M1

$b \cdot c - b \cdot a - c \cdot b + c \cdot a = 0$

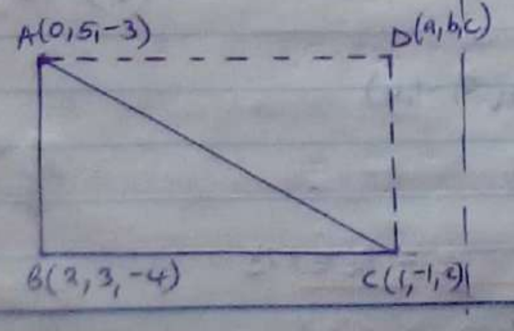
$\underset{\sim}{c} \cdot \underset{\sim}{a} - \underset{\sim}{b} \cdot \underset{\sim}{a} = 0$

$\underset{\sim}{a} \cdot (\underset{\sim}{c} - \underset{\sim}{b}) = 0$ A1

$\overrightarrow{OA} \cdot \overrightarrow{BC} = 0$ B1

Hence $\overrightarrow{OA}$ and $\overrightarrow{BC}$ are perpendicular

(b)




$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix}0\\5\\-3\end{pmatrix} - \begin{pmatrix}2\\3\\-4\end{pmatrix} = \begin{pmatrix}-2\\2\\1\end{pmatrix}$ B1

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix}1\\-1\\2\end{pmatrix} - \begin{pmatrix}2\\3\\-4\end{pmatrix} = \begin{pmatrix}-1\\-4\\6\end{pmatrix}$ B1

$\overrightarrow{BA} \cdot \overrightarrow{BC} = \begin{pmatrix}-2\\2\\1\end{pmatrix} \cdot \begin{pmatrix}-1\\-4\\6\end{pmatrix}$ M1

$= 2 - 8 + 6$

$= 0$ A1

Since $\overrightarrow{BA} \cdot \overrightarrow{BC} = 0$; then $\overrightarrow{BA}$ is perpendicular to BC and $\angle ABC = 90°$. B1

If ABCD is a rectangle, then

$\overrightarrow{AD} = \overrightarrow{BC}$

$\begin{pmatrix}a\\b-5\\c+3\end{pmatrix} = \begin{pmatrix}-1\\-4\\6\end{pmatrix}$ M1

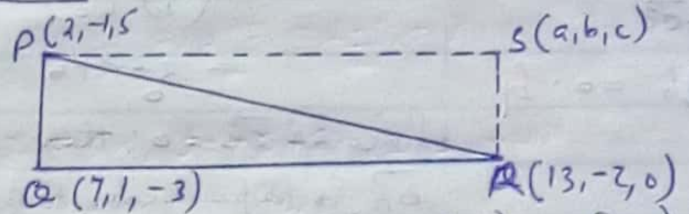
$a = -1$, $b - 5 = -4$, $b = 1$

$c + 3 = 6$, $c = 3$ M1

$D(-1, 1, 3)$ A1

No2. The vertices of a triangle are $P(2,-1,5)$; $Q(7,1,-3)$ and $R(13,-2,0)$.
Show that $\angle PQR = 90°$. Find the coordinates of S if PQRS is a rectangle.

Soln.




$$\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = \begin{pmatrix}2\\-1\\5\end{pmatrix} - \begin{pmatrix}7\\1\\-3\end{pmatrix} = \begin{pmatrix}-5\\-2\\8\end{pmatrix}$$

$$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = \begin{pmatrix}13\\-2\\0\end{pmatrix} - \begin{pmatrix}7\\1\\-3\end{pmatrix} = \begin{pmatrix}6\\-3\\3\end{pmatrix} \quad \text{B1}$$

$$\overrightarrow{QP} \cdot \overrightarrow{QR} = \begin{pmatrix}-5\\-2\\8\end{pmatrix} \cdot \begin{pmatrix}6\\-3\\3\end{pmatrix} = -30+6+24 = 0 \quad \text{A1}$$

Since $\overrightarrow{QP} \cdot \overrightarrow{QR} = 0$; $\overrightarrow{QP}$ is perpendicular to QR and so $\angle PQR = 90°$ B1

$$\overrightarrow{PS} = \overrightarrow{QR}$$

$$\begin{pmatrix}a-2\\b+1\\c-5\end{pmatrix} = \begin{pmatrix}6\\-3\\3\end{pmatrix}; \quad \begin{matrix}a-2=6, & a=8\\ b+1=-3, & b=-4\\ c-5=3, & c=8\end{matrix} \quad \text{M1}$$

$$S(8,-4,8) \quad \text{A1}$$

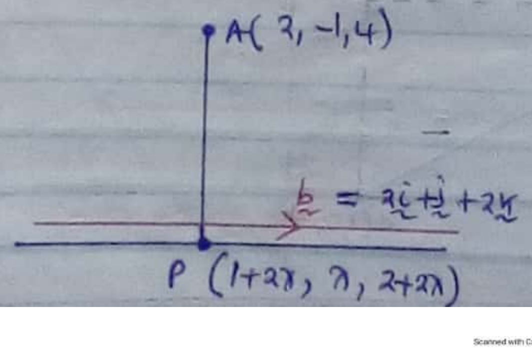
05

No11(a) Find the equation of the perpendicular line from point $A = \begin{pmatrix}2\\-1\\4\end{pmatrix}$ onto the line $r = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\2\end{pmatrix}$.

What is the distance of A from r.

(b) Find the angle contained between line OR and the x-y plane, where $OR = \begin{pmatrix}2\\1\\2\end{pmatrix}$

Soln



Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$x = 1 + 2\lambda, \; y = \lambda, \; z = 2 + 2\lambda$ M1

$\overrightarrow{AP} \cdot \underline{b} = 0$

$\left[ \begin{pmatrix} 1+2\lambda \\ \lambda \\ 2+2\lambda \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 0$

$\begin{pmatrix} 2\lambda - 1 \\ \lambda + 1 \\ 2\lambda - 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 0$ M1

$4\lambda - 2 + \lambda + 1 + 4\lambda - 4 = 0$ M1

$9\lambda = 5$

$\lambda = 5/9$ A1

$\overrightarrow{AP} = \begin{pmatrix} 2\lambda - 1 \\ \lambda + 1 \\ 2\lambda - 2 \end{pmatrix} = \begin{pmatrix} 2(5/9) - 1 \\ 5/9 + 1 \\ 2(5/9) - 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{9} \\ 14/9 \\ -8/9 \end{pmatrix}$ B1

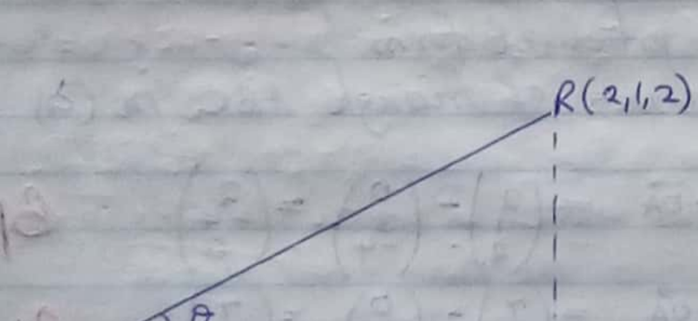
Equation $\underline{r} = \underline{a} + \lambda \underline{b}$

$= \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{9} \\ 14/9 \\ -8/9 \end{pmatrix}$ A1

$|\overrightarrow{AP}| = \sqrt{\left(\frac{1}{9}\right)^2 + \left(\frac{14}{9}\right)^2 + \left(-\frac{8}{9}\right)^2}$ M1

$= 1.795$ A1

(b)




$\overrightarrow{OS} \cdot \overrightarrow{OR} = |\overrightarrow{OS}| \cdot |\overrightarrow{OR}| \cos\theta$

$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \sqrt{4+1+0} \cdot \sqrt{4+1+4} \cos\theta$ M1 M1

$4 + 1 + 0 = \sqrt{5}\sqrt{9} \cos\theta, \quad \cos\theta = \frac{5}{\sqrt{5}\sqrt{9}}$ M1

$\theta = 41.8$ A1

12

# UNEB 2001

No 4. Find the point of intersection of the line $\frac{x}{5} = \frac{y+2}{2} = \frac{z-1}{4}$ with the plane $3x+4y+2z-25=0$ (05)

Soln

– write the equation of the line in parametric form.

Let $\frac{x}{5} = \frac{y+2}{2} = \frac{z-1}{4} = \lambda$

$x = 5\lambda,\ y = -2+2\lambda,\ z = 1+4\lambda$ ---- (1) M1

Equation of the Plane:

$3x+4y+2z-25=0$ ---- (2)

Substitute (1) in (2)

$3(5\lambda)+4(-2+2\lambda)+2(1+4\lambda)-25=0$ M1

$15\lambda - 8 + 8\lambda + 2 + 8\lambda - 25 = 0$

$31\lambda = 31$

$\lambda = 1$ A1

$x = 5(1) = 5,\ y = -2+2(1) = 0,\ z = 1+4(1) = 5$ M1

Point of intersection $(5, 0, 5)$ A1

05

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14(a) Find the Cartesian equation of the plane through $A(0,3,-4)$, $B(2,-1,2)$ and $C(7,4,-1)$. Show that $Q(10,13,-10)$ lies in the plane

(b) Express the equation of the plane in (a) in the scalar product form

(c) Find the area of triangle ABC in (a) (12)

Soln

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}2\\-1\\2\end{pmatrix} - \begin{pmatrix}0\\3\\-4\end{pmatrix} = \begin{pmatrix}2\\-4\\6\end{pmatrix}$ B1

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix}7\\4\\-1\end{pmatrix} - \begin{pmatrix}0\\3\\-4\end{pmatrix} = \begin{pmatrix}7\\1\\3\end{pmatrix}$ B1

$n = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & -j & k \\ 2 & -4 & 6 \\ 7 & 1 & 3 \end{vmatrix}$ M1

$= -18i + 36j + 30k$ A1

If $P(x,y,z)$ is a general point on the plane

$\overrightarrow{AP} \cdot n = 0$

$$\begin{pmatrix} x \\ y-3 \\ z+4 \end{pmatrix} \cdot \begin{pmatrix} -18 \\ 36 \\ 30 \end{pmatrix} = 0$$ M1

$-18x + 36y - 108 + 30z + 120 = 0$

$-18x + 36y + 30z + 12 = 0$

$-3x + 6y + 5z + 2 = 0$ A1

for $Q(10, 13, -10)$, LHS $= -3(10) + 6(13) + 5(-10) + 2 = 0$ M1B1

(b) $r \cdot n = a \cdot n$

$$r \cdot \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix}$$

$r \cdot (-3i + 6j + 5k) = -2$

$-3x + 6y + 5z = -2$

$r \cdot n = P$

$r \cdot (-3\underset{\sim}{i} + 6\underset{\sim}{j} + 5\underset{\sim}{k}) = -2$ M1A1

(c) Area $= \frac{1}{2}|\overrightarrow{AB} \times AC|$

$= \frac{1}{2}|-18\underset{\sim}{i} + 36\underset{\sim}{j} + 30k|$

$= \frac{1}{2}\sqrt{(-18)^2 + (36)^2 + (30)^2}$ M1

$= 25.1$ sq units A1

12

# UNEB 2000

No8: Show that the equation of the line through the points (1, 2, 1) and (4, −2, 2) is given as (05m)

$$\frac{x-1}{3} = \frac{y-2}{-4} = z-1$$

Soln

Parallel vector to the line; $\underset{\sim}{b} = \begin{pmatrix}4\\-2\\2\end{pmatrix} - \begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}3\\-4\\1\end{pmatrix}$ M1B1

Equation of the line

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$$

$$\underset{\sim}{r} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}3\\-4\\1\end{pmatrix} \quad A1$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}3\\-4\\1\end{pmatrix}$$

$$x = 1+3\lambda, \quad y = 2-4\lambda, \quad z = 1+\lambda$$

$$\lambda = \frac{x-1}{3}, \quad \lambda = \frac{y-2}{-4}, \quad \lambda = z-1 \quad M1$$

$$\frac{x-1}{3} = \frac{y-2}{-4} = z-1 \quad \# \quad B1$$

05

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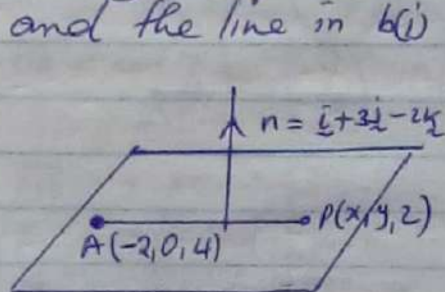
12(a) Show that the equation of the plane through points A with the position vector $-2\underset{\sim}{i} + 4\underset{\sim}{k}$ perpendicular to the vector $\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}$ is $x + 3y - 2z + 10 = 0$ (04marks)

b(i) Show that the vector $2\underset{\sim}{i} - 5\underset{\sim}{j} + 3.5\underset{\sim}{k}$ is perpendicular to the line $\underset{\sim}{r} = 2\underset{\sim}{i} - \underset{\sim}{j} + \lambda(4\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k})$ (03mks)

(ii) Calculate the angle between the vector $3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and the line in b(i) above (05mks)

Soln



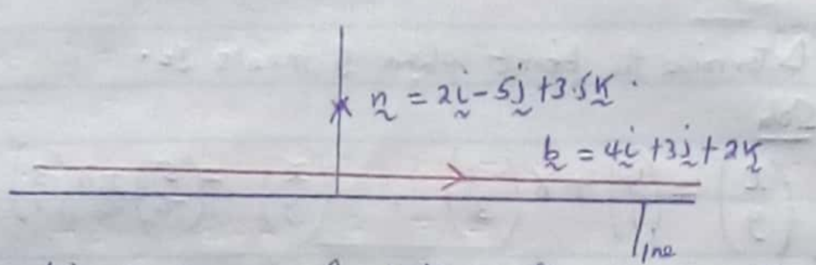

$$\overrightarrow{AP} \cdot n = 0$$

$$\begin{pmatrix}x+2\\y\\z-4\end{pmatrix} \cdot \begin{pmatrix}1\\3\\-2\end{pmatrix} = 0 \quad M1M1$$

$x+2+3y-2z+8=0$ M1

$x+3y-2z+10=0$ $\boxed{\cdot}$ A1

b(i)




If the line is perpendicular to the vector, the parallel vector should be perpendicular to the vector and so.

$n \cdot b = 0$

$n \cdot b = \begin{pmatrix} 2 \\ -5 \\ 3.5 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = 8 - 15 + 7 = 0$ M1A1

Since $n \cdot b = 0$, the two are perpendicular. B1

$\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \sqrt{9+4+1}\,\sqrt{16+9+4}\cos\theta$ M1 M1

$12 - 6 + 2 = \sqrt{14}\,\sqrt{29}\cos\theta$ M1

$\cos\theta = \dfrac{8}{\sqrt{14}\sqrt{29}}$ M1

$\theta = 66.6°$ A1

# UNEB 1999



No2. The vector equations of the two lines are
$r_1 = \begin{pmatrix}4\\1\end{pmatrix} + r\begin{pmatrix}-2\\3\end{pmatrix}$ and $r_2 = \begin{pmatrix}5\\-6\end{pmatrix} + \lambda\begin{pmatrix}3\\1\end{pmatrix}$

Determine the point where $r_1$ meets $r_2$.

Soln

$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}4\\1\end{pmatrix} + \gamma\begin{pmatrix}-2\\3\end{pmatrix}$ , $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}5\\-6\end{pmatrix} + \lambda\begin{pmatrix}3\\1\end{pmatrix}$

$x = 4-2\gamma$ $\quad$ $x = 5+3\lambda$

$y = 1+3\gamma$ $\quad$ $y = -6+\lambda$

If $(x,y)$ is the point of intersection;

$4-2\gamma = 5+3\lambda$

$2\gamma+3\lambda = -1$ —— (1)

$1+3\gamma = -6+\lambda$

$3\gamma-\lambda = -7$ ---(2)

Eqn (2) × 3; $\quad 9\gamma-3\lambda = -21$ ---(3)

(1)+(3); $11\gamma = -22$

$\gamma = -2$

$\lambda = 1$

Point of intersection $(8,-5)$

No16(a) Find in cartesian form, the equation of the that passes through the points $A(1,2,5)$, $B(1,0,4)$ and $C(5,2,1)$

(b) Find the angle between the line $\frac{x+4}{8} = \frac{y-2}{2} = \frac{z+1}{-4}$ and the plane $4x+3y-3z+1$

Soln

(a) $\overrightarrow{AB} = \begin{pmatrix}1\\0\\4\end{pmatrix} - \begin{pmatrix}5\\2\\1\end{pmatrix} = \begin{pmatrix}-4\\-2\\3\end{pmatrix}$ B1

$\overrightarrow{AC} = \begin{pmatrix}5\\2\\1\end{pmatrix} - \begin{pmatrix}1\\2\\5\end{pmatrix} = \begin{pmatrix}4\\0\\-4\end{pmatrix}$ B1

$r = a + \lambda b + \mu c$

$$\therefore r = \begin{pmatrix}1\\2\\5\end{pmatrix} + \lambda\begin{pmatrix}-4\\-2\\3\end{pmatrix} + \mu\begin{pmatrix}4\\0\\-4\end{pmatrix}$$ A1

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\5\end{pmatrix} + \lambda\begin{pmatrix}-4\\-2\\3\end{pmatrix} + \mu\begin{pmatrix}4\\0\\-4\end{pmatrix}$$

$x-1 = -4\lambda + 4\mu$ ---(1)

$y-2 = -2\lambda$ -----(2)

$z-5 = 3\lambda - 4\mu$ ---(3) M1

(1) + (3); $x-1+z-5 = -\lambda$

$x+z-6 = -\lambda$ -----(4)

from (2); $\lambda = -\frac{1}{2}(y-2)$, from (4); $\lambda = -(x+z-6)$ M1

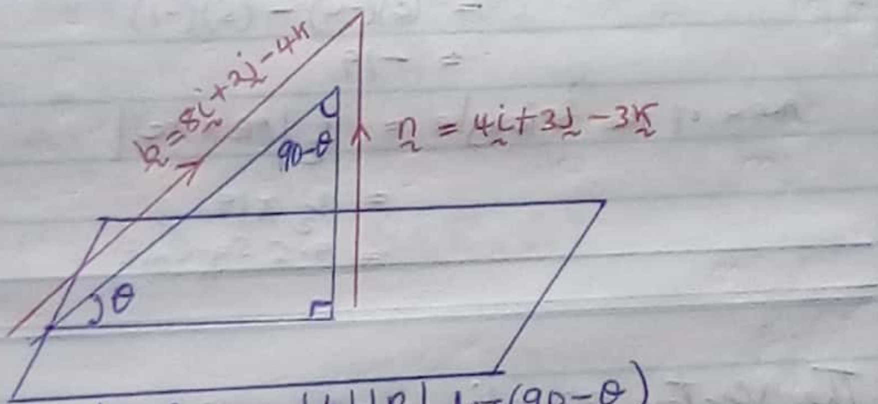
Equating the two; $-\frac{1}{2}(y-2) = -(x+z-6)$

$y-2 = 2(x+z-6)$

$y-2 = 2x+2z-12$

$2x - y + 2z - 10 = 0$ A1

(b)




$b \cdot n = |b||n| \cos(90-\theta)$

$\begin{pmatrix}8\\2\\-4\end{pmatrix} \cdot \begin{pmatrix}4\\3\\-3\end{pmatrix} = \sqrt{64+4+16}\,\sqrt{16+9+9}\,\sin\theta$ M1M1

$32+6+12 = \sqrt{84}\,\sqrt{34}\,\sin\theta$ M1

$50 = \sqrt{2856}\,\sin\theta$ M1

$\sin\theta = \frac{50}{\sqrt{2856}}$ M1

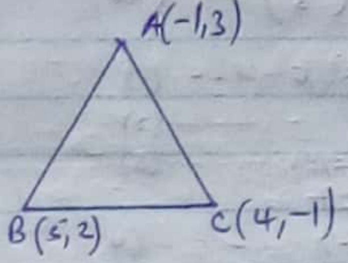
$\theta = 69.3°$ A1

12

# UNEB 1998

No7. Calculate the area of a triangle with vertices $(-1,3)$, $(5,2)$ and $(4,-1)$

Soln



$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}5\\2\end{pmatrix} - \begin{pmatrix}-1\\3\end{pmatrix} = \begin{pmatrix}6\\-1\end{pmatrix}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix}4\\-1\end{pmatrix} - \begin{pmatrix}-1\\3\end{pmatrix} = \begin{pmatrix}5\\-4\end{pmatrix}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & -j & k \\ 6 & -1 & 0 \\ 5 & -4 & 0 \end{vmatrix}$$

$= (6)(-4) - (5)(-1)$

$= -19$

Area of the triangle $= \frac{1}{2}|AB \times AC|$

$= \frac{1}{2} \times 19$

$= 9.5$ sq-units

12. The vector equations of the lines P and Q are given as $r_p = r(4\underset{\sim}{i}+3\underset{\sim}{j})$ and $r_q = 2\underset{\sim}{i}+12\underset{\sim}{j}+s(\underset{\sim}{i}-\underset{\sim}{j})$

(a) Use the dot product to find the angle between lines P and Q

(b) If the lines P and Q meet at M, find the Coordinates of M.
Find also the equation of the line through M perpendicular to the line Q.

Soln

Parallel vector the first line, $\underset{\sim}{b}_1 = 4\underset{\sim}{i}+3\underset{\sim}{j}$

Parallel vector to the second line; $\underset{\sim}{b}_2 = i - \underset{\sim}{j}$

$\underset{\sim}{b}_1 \cdot \underset{\sim}{b}_2 = |\underset{\sim}{b}_1||\underset{\sim}{b}_2| \cos\theta$

$$\begin{pmatrix}4\\3\end{pmatrix}\cdot\begin{pmatrix}1\\-1\end{pmatrix} = \sqrt{16+9}\cdot\sqrt{1+1}\cos\theta$$

$$4+-3 = 5\sqrt{2}\cos\theta$$

$$\cos\theta = \frac{1}{5\sqrt{2}}$$

$$\theta = 81.9^{\circ}$$

(b) $\begin{pmatrix}x\\y\end{pmatrix} = r\begin{pmatrix}4\\3\end{pmatrix}$ , $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2\\12\end{pmatrix} + s\begin{pmatrix}1\\-1\end{pmatrix}$

$x = 4r$ $\quad x = 2+s$

$y = 3r$ $\quad y = 12-s$

If $(x,y)$ is the point of intersection.

$4r = 2+s$

$4r-s = 2$ --- (1)

$3r = 12-s$

$3r+s = 12$ --- (2)

(1) + (2) ; $7r = 14$ , $r = 2$

from (2) ; $6+s = 12$

$s = 6$.

Point of intersection $P(8,6)$

Vector perpendicular to $G$ ; $n = \underset{\sim}{b_1} \times \underset{\sim}{b_2}$

$$n = \begin{vmatrix} i & j & k \\ 4 & 3 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

~~$n = (4)(-1) - (1)(3)$~~

~~$n = -7$~~ $0\underset{\sim}{i} + 0\underset{\sim}{j} - 7\underset{\sim}{k}$.