10

Permutations and combinations

syllabus reference

Core topic:
Structures and patterns

In this chapter

Introduction

Combinatorics deals with determining the number of ways in which activities or events may occur.

The study of combinatorics provides ways of answering questions such as:

1. How many doubles teams can be selected from a group of 6 volleyball players?
2. From a group of 4 candidates, in how many ways can a class captain and deputy class captain be selected?
3. How many different outfits can be chosen from 3 skirts and 5 tops?
4. If a Lotto ticket consists of a choice of 6 numbers from 45, how many different tickets are there?
5. How many different car number plates of 3 digits and 3 letters can be made using the digits 0 to 9 and the letters A, B and C?

The addition and multiplication principles

To count the number of ways in which an activity can occur, first make a list. Let each outcome be represented by a letter and then systematically list all the possibilities.

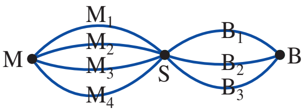
Consider the following question:

In driving from Melbourne to Sydney I can take any one of 4 different roads *and* in driving from Sydney to Brisbane there are 3 different roads I can take. How many different routes can I take in driving from Melbourne to Brisbane?

To answer this, let M_1, M_2, M_3, M_4 stand for the 4 roads from Melbourne to Sydney and B_1, B_2, B_3 stand for the 3 roads from Sydney to Brisbane.

Use the figure to systematically list the roads:

M_1B_1, M_1B_2, M_1B_3
M_2B_1, M_2B_2, M_2B_3
M_3B_1, M_3B_2, M_3B_3
M_4B_1, M_4B_2, M_4B_3



Hence, there are 12 different ways I can drive from Melbourne to Brisbane.

In the above example it can be argued logically that if there are 4 ways of getting from Melbourne to Sydney and 3 ways of getting from Sydney to Brisbane then there are 4 × 3 ways of getting from Melbourne to Brisbane.

This idea is formalised in the *multiplication principle*.

The multiplication principle should be used when there are two operations or events (say, A and B), where one event is followed by the other.

It states:

If there are n ways of performing operation A and m ways of performing operation B, then there are $n \times m$ ways of performing A *and* B.

Note: In this case '*and*' means to multiply.

A useful technique for solving problems based on the multiplication principle is to use boxes. In the example above we would write

1st	2nd
4	3

The value in the ‘1st’ column represents the number of ways the first operation — the trip from Melbourne to Sydney — can be performed.

The value in the ‘2nd’ column stands for the number of ways the second operation — the trip from Sydney to Brisbane — can be performed.

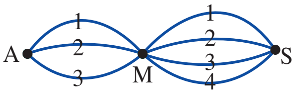
To apply the multiplication principle you multiply the numbers in the lower row of boxes.

Now consider a different situation, one in which the two operations do not occur one after the other.

I am going to travel from Melbourne to either Sydney *or* Adelaide. There are 4 ways of travelling from Melbourne to Sydney and 3 ways of travelling from Melbourne to Adelaide.

How many different ways can I travel to my destination?

It can be seen from the figure that there are $4 + 3 = 7$ ways of completing the journey. This idea is summarised in the *addition principle*.



The addition principle should be used when two distinct operations or events occur in which one event is not followed by another.

It states:

If there are *n* ways of performing operation A and *m* ways of performing operation B then there are *n* + *m* ways of performing A *or* B.
***Note*: In this case ‘*or*’ means to add.**

WORKED Example 1

Two letters are to be chosen from A, B, C, D and E, where order is important.

a **List all the different ways that this may be done.**

b **State the number of ways that this may be done.**

THINK	WRITE
a 1 Begin with A in first place and make a list of each of the possible pairs.	**a** AB AC AD AE
2 Make a list of each of the possible pairs with B in the first position.	BA BC BD BE
3 Make a list of each of the possible pairs with C in the first position.	CA CB CD CE
4 Make a list of each of the possible pairs with D in the first position.	DA DB DC DE
5 Make a list of each of the possible pairs with E in the first position. *Note*: AB and BA need to be listed separately as order is important.	EA EB EC ED

Continued over page →

THINK

b Method 1
Count the number of ordered pairs and answer the question.

Alternatively, the multiplication principle could have been used to determine the number of ordered pairs.

b Method 2

1. Rule up two boxes which represent the pair.
2. Write down the number of letters which may be selected for the first box. That is, in first place any of the 5 letters may be used.
3. Write down the number of letters which may be selected for the second box. That is, in second place, any of the 4 letters may be used.
   Note: One less letter is used to avoid repetition.
4. Evaluate.
5. Answer the question.

WRITE

b There are 20 ordered pairs.

b

5	4

5 × 4 = 20 ways

There are 20 ways in which 2 letters may be selected from a group of 5 where order is important.

A selection where order is important is called an *arrangement*.

WORKED Example 2

How many ways could an arrangement of 5 letters be chosen from A, B, C, D, E and F?

THINK

1. Instead of listing all possibilities, draw 5 boxes to represent the 5 letters chosen.
   Label each box on the top row as 1st, 2nd, 3rd, 4th and 5th.
   Note: The word arrangement implies order is important.
2. Fill in each of the boxes showing the number of ways a letter may be chosen.
   (a) In the 1st box there are 6 choices for the first letter.
   (b) In the 2nd box there are 5 choices for the second letter as 1 letter has already been used.
   (c) In the 3rd box there are 4 choices for the third letter as 2 letters have already been used.
   (d) Continue this process until each of the 5 boxes is filled.
3. Use the multiplication principle as this is an ‘*and*’ situation.
4. Answer the question.

WRITE

1st	2nd	3rd	4th	5th
6	5	4	3	2

$$\text{No. of ways} = 6 \times 5 \times 4 \times 3 \times 2$$
$$= 720$$

An arrangement of 5 letters may be chosen 720 ways.

WORKED Example 3

One or two letters are to be chosen from 6 letters A, B, C, D, E and F. In how many ways can this be done?

THINK	WRITE
① Determine the number of ways of choosing 1 letter.	No. of ways of choosing 1 letter = 6.
② Rule up two boxes for the first and second letters.	
③ Determine the number of ways of choosing 2 letters from 6. (a) In the 1st box there are 6 choices for the first letter. (b) In the 2nd box there are 5 choices for the second letter as 1 letter has already been used.	1st: 6 2nd: 5
④ Use the multiplication principle (as this is an '*and*' situation) to evaluate the number of ways of choosing 2 letters from 6.	No. of ways of choosing 2 letters $= 6 \times 5$ $= 30$
⑤ Determine the number of ways of choosing 1 or 2 letters from 6 letters. Use the addition principle as this is an '*or*' situation.	The number of ways of choosing 1 or 2 letters is $6 + 30 = 36$.
⑥ Answer the question.	There are 36 ways of choosing 1 or 2 letters from 6.

WORKED Example 4

Jeannine's restaurant offers its patrons a choice of 3 entrees, 9 main courses and 4 desserts.

a How many choices of 3-course meals (entree, main, dessert) are available?

b How many choices of entree and main course are offered?

c How many choices of main course and dessert are offered?

d How many choices of 2- or 3-course meals are available (assuming that a main course is always ordered)?

THINK	WRITE
a ① Rule up 3 boxes to represent each course — entree, main, dessert. Label each box on the top row as E, M and D.	**a** E: 3, M: 9, D: 4
② Determine the number of ways of choosing each meal: entree = 3, main = 9, dessert = 4.	
③ Use the multiplication principle (as this is an '*and*' situation) to evaluate the number of choices of 3-course meals.	No. of choices $= 3 \times 9 \times 4$ $= 108$
④ Answer the question.	There are 108 choices of 3-course meals.

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<table>
<tr><th>THINK</th><th>WRITE</th></tr>
<tr><td>b 1 Rule up 2 boxes to represent each course — entree, main. Label each box on the top row as E and M.</td><td>b
<table><tr><td>E</td><td>M</td></tr><tr><td>3</td><td>9</td></tr></table></td></tr>
<tr><td>2 Determine the number of ways of choosing each meal: entree = 3, main = 9.</td><td></td></tr>
<tr><td>3 Use the multiplication principle (as this is an ‘<i>and</i>’ situation) to evaluate the number of choices of entree and main courses.</td><td>No. of choices = 3 × 9
= 27</td></tr>
<tr><td>4 Answer the question.</td><td>There are 27 choices of entree and main course.</td></tr>
<tr><td>c 1 Rule up 2 boxes to represent each course — main and dessert. Label each box on the top row as M and D.</td><td>c
<table><tr><td>M</td><td>D</td></tr><tr><td>9</td><td>4</td></tr></table></td></tr>
<tr><td>2 Determine the number of ways of choosing each meal: main = 9, dessert = 4.</td><td></td></tr>
<tr><td>3 Use the multiplication principle (as this is an ‘<i>and</i>’ situation) to evaluate the number of choices of main course and dessert.</td><td>No. of choices = 9 × 4
= 36</td></tr>
<tr><td>4 Answer the question.</td><td>There are 36 choices of main course and dessert.</td></tr>
<tr><td>d 1 Determine the number of ways of choosing 2- or 3-course meals, assuming that a main course is always ordered.
Use the addition principle as this is an ‘<i>or</i>’ situation.</td><td>d The number of ways of choosing 2- or 3-course meals, assuming that a main course is always ordered, is:
108 + 27 + 36 = 171</td></tr>
<tr><td>2 Answer the question.</td><td>There are 171 ways of choosing 2- or 3-course meals, assuming that a main course is always ordered.</td></tr>
</table>

remember

1. The multiplication principle should be used when there are two operations or events (say, A and B) where one event is followed by the other. It states:
   If there are n ways of performing operation A and m ways of performing operation B, then there are $n \times m$ ways of performing A *and* B.
2. The addition principle should be used when two distinct operations or events occur in which one event is not followed by another. It states:
   If there are n ways of performing operation A and m ways of performing operation B, then there are $n + m$ ways of performing A *or* B.
3. A selection where order is important is called an *arrangement*.

EXERCISE 10A The addition and multiplication principles

WORKED Example 1

1 Two letters are to be chosen from A, B and C, where order is important.
  - **a** List all the different ways that this may be done.
  - **b** State the number of ways that this may be done.

2 List all the different arrangements possible for a group of 2 colours to be chosen from B (blue), G (green), Y (yellow) and R (red).

3 List all the different arrangements possible for a group of 3 letters to be chosen from A, B and C.

WORKED Example 2

4
  - **a** In how many ways can an arrangement of 2 letters be chosen from A, B, C, D, E, F and G?
  - **b** In how many ways can an arrangement of 3 letters be chosen from 7 different letters?
  - **c** In how many ways can an arrangement of 4 letters be chosen from 7 different letters?
  - **d** How many different arrangements of 5 letters can be made from 7 letters?

5
  - **a** A teddy bear's wardrobe consists of 3 different hats, 4 different shirts and 2 different trousers. How many different outfits can the teddy bear wear?
  - **b** A surfboard is to have 1 colour on its top and a different colour on its bottom. The 3 possible colours are red, blue and green. In how many different ways can the surfboard be coloured?
  - **c** A new computer system comes with a choice of 3 keyboards, 2 different monitors and 2 different mouse attachments. With these choices, how many different arrangements are possible?
  - **d** Messages can be sent by placing 3 different coloured flags in order on a pole. If the flags come in 4 colours, how many different messages can be sent?
  - **e** A yacht race has a field of 12 competitors. In how many different ways can first, second and third place be filled by these 12 yachts?

WORKED Example 3

6
  - **a** One or 2 letters are to be chosen from the letters A, B, C, D, E, F and G. In how many ways can this be done?
  - **b** Two or 3 letters are to be chosen from the letters A, B, C, D, E, F and G. In how many ways can this be done?
  - **c** How many 1- or 2-digit numbers can be made using the digits 1, 3, 5 and 7 if no digit can be used more than once?

7 Nadia is in a race with 10 other girls.
  - **a** If we are only concerned with the first, second and third placings, in how many ways can:
    - **i** Nadia finish first?
    - **ii** Nadia finish second?
  - **b** In how many ways can Nadia finish first or second?

8 White Wolf is a horse in a race with 7 other runners. If we are concerned only with the first, second and third placings, in how many ways can White Wolf finish first or second or third?

9 **multiple choice**

There are 12 people on the committee at the local softball club. In how many ways can a president and a secretary be chosen from this committee?

A 2 **B** 23 **C** 132 **D** 144 **E** 66

10 **multiple choice**

Phone numbers consist of 8 digits. The first must be a 9. The second digit can be a 3, 4, 5 or 8. There are no restrictions on the remaining digits. How many different telephone numbers are possible?

A 4320 **B** 499 999

C 4 000 000 **D** 4 999 999

E 10 000 000

11 **multiple choice**

A TV station runs a cricket competition called *Classic Catches*. Six catches, A to F, are chosen and viewers are asked to rank them in the same order as the judges. The number of ways in which this can be done is:

A 1 **B** 6 **C** 30

D 720 **E** 128

12 The local soccer team sells 'doubles' at each of their games to raise money. A 'double' is a card with 2 digits on it representing the score at full time. The card with the actual full time score on it wins a prize. If the digits on the cards run from 00 to 99, how many different tickets are there?

13 Marcus has a briefcase that has a 4-digit security code. He remembers that the first number in the code was 9 and that the others were 3, 4 and 7 but forgets the order of the last 3 digits. How many different trials must he make to be sure of unlocking the briefcase?

14 Julia has a briefcase that has two 4-digit locks. She remembers that she used the digits 1, 3, 5 and 7 on the left lock and 2, 4, 6 and 8 on the right lock, but can not remember the order. What is the maximum number of trials she would need to make before she has opened both the left lock and the right lock?

15 How many different 4-digit numbers can be made from the numbers 1, 3, 5 and 7 if the numbers can be repeated (that is 3355 and 7777 are valid)?

16 How many 4-digit numbers can be made from the numbers 1, 3, 5, 7, 9 and 2 if the numbers can be repeated?

17 How many 4-digit numbers can be made from the numbers 1, 3, 5, 7, 9 and 0 if the numbers can be repeated? (Remember — a number cannot start with 0.)

18 How many numbers less than 5000 can be made using the digits 2, 3, 5, 7 and 9 if repetition is not permitted?

19 A combination lock has 3 digits each from 0 to 9.

a How many combinations are possible?

The lock mechanism becomes loose and will open if the digits are within one either side of the correct digit. For example if the true combination is 382 then the lock will open on 271, 272, 371, 493 and so on.

b What is the minimum number of guesses required before being sure of opening the lock?

c List the possible combinations that would open the lock if the true combination is 382.

WORKED Example 4

20 Hani and Mary's restaurant offers its patrons a choice of 4 entrees, 10 main courses and 5 desserts.

a How many choices of 3-course meals (entree, main, dessert) are available?

b How many choices of entree and main course are offered?

c How many choices of main course and dessert are offered?

d How many choices of 2- or 3-course meals are available (assuming that a main course is always ordered)?

21 Jake is able to choose his work outfits from the following items of clothing: 3 jackets, 7 shirts, 6 ties, 5 pairs of trousers, 7 pairs of socks and 3 pairs of shoes.

a How many different outfits are possible if he wears one of each of the above items? (He wears matching socks and shoes.)

b If Jake has the option of wearing a jacket and each of the above items, how many different outfits are possible? Explain your answer.

Factorials and permutations

Factorials

The Physical Education department is to display 5 new trophies along a shelf in the school foyer and wishes to know how many ways this can be done.

Using the multiplication principle from the previous section, the display may be done in the following way:

Position 1	Position 2	Position 3	Position 4	Position 5
5	4	3	2	1

That is, there are $5 \times 4 \times 3 \times 2 \times 1 = 120$ ways.

Depending on the number of items we have, this method could become quite time consuming.

In general when we need to multiply each of the integers from a particular number, n, down to 1, we write $n!$, which is read as n factorial.

Hence:
$$\begin{aligned} 6! &= 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 720 \\ 8! &= 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 40\,320 \\ n! &= n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1 \end{aligned}$$

Factorials may be evaluated on the TI-83 graphics calculator.

To evaluate 12! we enter 12, then press MATH and select **PRB** and **4:!** and press ENTER.

Check to see that you can use this function by evaluating 12! = 479 001 600.

1. **The number of ways n distinct objects may be arranged is $n!$ (n factorial) where:**
$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$$
That is, $n!$ is the product of each of the integers from n down to 1.
2. **A special case of the factorial function is: $0! = 1$.**

WORKED Example 5

Evaluate the following factorials.

a $7!$ **b** $13!$ **c** $\frac{18!}{5!}$ **d** $\frac{9!}{3!}$ **e** $\frac{(n-1)!}{(n-3)!}$

THINK	**WRITE**
a ① Write 7! in its expanded form and evaluate. ② Verify the answer obtained using the factorial function on the calculator.	**a** $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ $= 5040$
b ① Write 13! in its expanded form and evaluate. ② Verify the answer obtained using the factorial function on the calculator.	**b** $13! = 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ $= 6\,227\,020\,800$
c ① Write each factorial term in its expanded form. ② Cancel down like terms. ③ Evaluate. ④ Verify the answer obtained using the factorial function on the calculator.	**c** $\frac{8!}{5!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1}$ $= 8 \times 7 \times 6$ $= 336$
d ① Write each factorial term in its expanded form. ② Cancel down like terms. ③ Evaluate. ④ Verify the answer obtained using the factorial function on the calculator.	**d** $\frac{9!}{3!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1}$ $= 9 \times 8 \times 7 \times 6 \times 5 \times 4$ $= 60\,480$
e ① Write each factorial term in its expanded form. ② Cancel down like terms.	**e** $\frac{(n-1)!}{(n-3)!}$ $= \frac{(n-1)(n-2)(n-3)(n-4) \times \ldots \times 3 \times 2 \times 1}{(n-3)(n-4) \times \ldots \times 3 \times 2 \times 1}$ $= (n-1)(n-2)$

In parts **c**, **d** and **e** of worked example 5, there was no need to fully expand each factorial term.

The factorial $\frac{8!}{5!}$ could have first been simplified to $\frac{8 \times 7 \times 6 \times 5!}{5!}$ and then the 5! terms cancelled.

The factorial $\frac{9!}{3!}$ could have first been simplified to $\frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3!}{3!}$ and then the 3! terms cancelled.

The factorial $\frac{(n-1)!}{(n-3)!}$ could have first been simplified to $\frac{(n-1)(n-2)(n-3)!}{(n-3)!}$ and then the $(n-3)!$ terms cancelled.

Permutations

The term *permutation* is often used instead of the term *arrangement* and in this section we begin by giving a formal definition of permutation.

Previously, we learned that if you select 3 letters from 7 where order is important, the number of possible arrangements is:

1st	2nd	3rd
7	6	5

$$\begin{aligned}\text{The number of arrangements} &= 7 \times 6 \times 5 \\ &= 210\end{aligned}$$

This value may also be expressed in factorial form: $7 \times 6 \times 5 = \frac{7 \times 6 \times 5 \times 4!}{4!} = \frac{7!}{4!}$

Using more formal terminology we say that in choosing 3 things from 7 things where order is important the number of permutations is ${}^7P_3 = 7 \times 6 \times 5$. The letter P is used to remind us that we are finding permutations.

The number of ways of choosing r things from n distinct things is given by the rule:

$$\begin{aligned}{}^nP_r &= n \times (n-1) \times \ldots \times (n-r+1) \\ &= \frac{n \times (n-1) \times \ldots \times (n-r+1)(n-r)!}{(n-r)!} \\ &= \frac{n!}{(n-r)!}\end{aligned}$$

The definition of ${}^nP_r$ may be extended to the cases of ${}^nP_n$ and ${}^nP_0$.

${}^nP_n$ represents the number of ways of choosing n objects from n distinct things.

$$\begin{aligned}{}^nP_n &= n \times (n-1) \times (n-2) \times \ldots \times (n-n+1) \\ &= n \times (n-1) \times (n-2) \times \ldots \times 1 \\ &= n!\end{aligned}$$

From the definition:

$$\begin{aligned}{}^nP_n &= \frac{n!}{(n-n)!} \\ &= \frac{n!}{0!}\end{aligned}$$

Therefore, equating both sides, we obtain: $n! = \frac{n!}{0!}$.

This can occur only if $0! = 1$.

$$\begin{aligned} {}^{n}P_{0} &= \frac{n!}{(n-0)!} \\ &= \frac{n!}{n!} \\ &= 1 \end{aligned}$$

In summary, the two special cases are:

1. $^{n}P_{n} = n!$
2. $^{n}P_{0} = 1$

WORKED Example 6

a **Calculate the number of permutations for $^{6}P_{4}$ by expressing it in expanded form.**
b **Write $^{8}P_{3}$ as a quotient of factorials and hence evaluate.**

THINK	WRITE
a ① Write down the first 4 terms beginning with 6.	**a** $^{6}P_{4} = 6 \times 5 \times 4 \times 3$
② Evaluate.	$= 360$
b ① Write down the rule for permutations.	**b** $^{n}P_{r} = \frac{n!}{(n-r)!}$
② Substitute the given values of n and r into the permutation formula.	$^{8}P_{3} = \frac{8!}{(8-3)!}$ $= \frac{8!}{5!}$
③ Use a calculator to evaluate 8! and 5!	$= \frac{40\,320}{120}$
④ Evaluate.	$= 336$

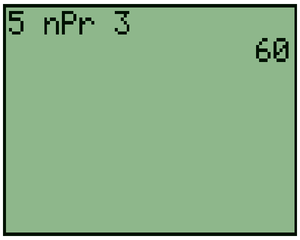
WORKED Example 7

The netball club needs to appoint a president, secretary and treasurer. From the committee 7 people have volunteered for these positions. Each of the 7 nominees is happy to fill any one of the 3 positions. In how many different ways can these positions be filled?

THINK	WRITE
① Write down the rule for permutations. *Note*: Order is important, so use permutations.	$^{n}P_{r} = \frac{n!}{(n-r)!}$
② Substitute the given values of n and r into the permutation formula.	$^{7}P_{3} = \frac{7!}{(7-3)!}$ $= \frac{7!}{4!}$
③ Use a calculator to evaluate 7! and 4!	$= \frac{5040}{24}$
④ Evaluate.	$= 210$
⑤ Answer the question.	There are 210 different ways of filling the positions of president, secretary and treasurer.

Graphics Calculator tip!

Calculating permutations


```
5 nPr 3
             60
```

To find the number of permutations of n objects taken r at a time, enter the number of objects, then press MATH, select **PRB**, and choose **2:nPr** and enter the value of r.

Arrangements in a circle

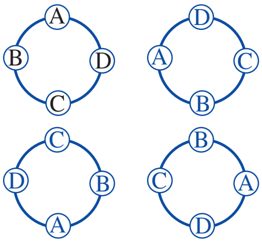
Consider this problem: In how many different ways can 7 people be seated, 4 at a time, on a bench?

By now you should quickly see the answer: $^7P_4 = 840$.

Let us change the problem slightly: In how many different ways can 7 people be seated, 4 at a time, at a circular table?

The solution must recognise that when people are seated on a bench, each of the following represents a different arrangement:

ABCD BCDA CDAB DABC



However, when sitting in a circle each of the following represents the *same* arrangement.

In each case B has A on the left and C on the right.

We conclude that the number $^7P_4$ gives 4 times the number of arrangements of 7 people in a circle 4 at a time. Therefore, the number of arrangements is $\dfrac{^7P_4}{4} = 210$.

In general, the number of different ways n people can be seated, r at a time, in a circle is: $\dfrac{^nP_r}{r}$

WORKED Example 8

How many different arrangements are possible if, from a group of 8 people, 5 are to be seated at a round table?

THINK	WRITE
1 Write down the rule for the number of arrangements in a circle.	$\dfrac{^nP_r}{r}$
2 Substitute the given values of n and r into the formula.	$= \dfrac{^8P_5}{5}$
3 Use a calculator to evaluate $^8P_5$.	$= \dfrac{6720}{5}$
4 Evaluate.	$= 1344$
5 Answer the question.	The number of ways of seating 5 from a group of 8 people at a round table is 1344.

remember

1. (a) The number of ways n distinct objects may be arranged is $n!$ (n factorial) where:
$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$$
   (b) $0! = 1$
   (c) $1! = 1$
2. (a) The number of different arrangements (permutations) when r things are chosen from n things and order is important is given by the rule ${}^nP_r$, where:
$${}^nP_r = \frac{n!}{(n-r)!}$$
   (b) ${}^nP_n = n!$
   (c) ${}^nP_0 = 1$
3. The number of different ways n people can be seated, r at a time, in a circle is:
$$\frac{{}^nP_r}{r}$$

EXERCISE 10B Factorials and permutations

Mathcad Factorials

1 Write each of the following in expanded form.

a $4!$ **b** $5!$ **c** $6!$ **d** $7!$

WORKED Example 5a, b

2 Evaluate the following factorials.

a $4!$ **b** $5!$ **c** $6!$ **d** $10!$
e $14!$ **f** $9!$ **g** $7!$ **h** $3!$

WORKED Example 5c, d

3 Evaluate the following factorials.

a $\frac{9!}{5!}$ **b** $\frac{10!}{4!}$ **c** $\frac{7!}{3!}$ **d** $\frac{6!}{0!}$

WORKED Example 5e

4 Evaluate the following factorials.

a $\frac{n!}{(n-5)!}$ **b** $\frac{(n+3)!}{(n+1)!}$ **c** $\frac{(n-3)!}{n!}$ **d** $\frac{(n-2)!}{(n+2)!}$

SkillSHEET 10.1

WORKED Example 6a

5 Calculate each of the following by expressing it in expanded form.

a ${}^8P_2$ **b** ${}^7P_5$ **c** ${}^8P_7$

WORKED Example 6b

Mathcad Permutations

6 Write each of the following as a quotient of factorials and hence evaluate.

a ${}^9P_6$ **b** ${}^5P_2$ **c** ${}^{18}P_5$

7 Use your calculator to find the value of:

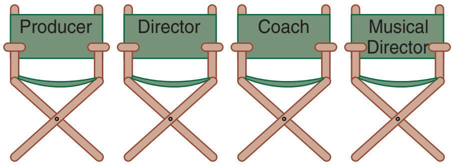
a ${}^{20}P_6$ **b** ${}^{800}P_2$ **c** ${}^{18}P_5$

WORKED Example 7

8 A soccer club will appoint a president and a vice-president. Eight people have volunteered for either of the two positions. In how many different ways can these positions be filled?

EXCEL Spreadsheet Permutations

9 The school musical needs a producer, director, musical director and script coach. Nine people have volunteered for any of these positions. In how many different ways can the positions be filled? (*Note*: One person cannot take on more than 1 position.)



10 There are 14 horses in a race. In how many different ways can the 1st, 2nd and 3rd positions be filled?

11 There are 26 horses in a race. How many different results for 1st, 2nd, 3rd and 4th can occur?

12 A rowing crew consists of 4 rowers who sit in a definite order. How many different crews are possible if 5 people try out for selection?

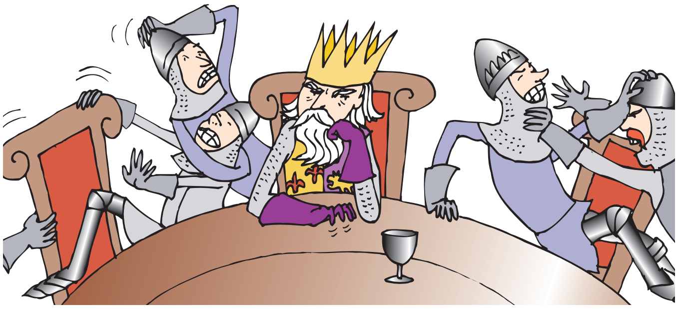
WORKED Example 8

13 How many different arrangements are possible if, from a group of 15 people, 4 are to be seated in a circle?

14 A round table seats 6 people. From a group of 8 people, in how many ways can 6 people be seated at the table?

15 At a dinner party for 10 people all the guests were seated at a circular table. How many different arrangements were possible?

16 At one stage in the court of Camelot, King Arthur and 12 knights would sit at the round table. If each person could sit anywhere how many different arrangements were possible?



17 **multiple choice**

Which one of the following permutations cannot be calculated?

A ${}^{1000}P_{100}$ **B** ${}^{1}P_{0}$ **C** ${}^{8}P_{8}$ **D** ${}^{4}P_{8}$ **E** ${}^{5}P_{4}$

18 **multiple choice**

The result of 100! is greater than 94!.
Which of the following gives the best comparison between these two numbers?

A 100! is 6 more than 94!
B 100! is 6 times bigger than 94!
C 100! is about 6! times bigger than 94!
D 100! is about 10 000 more than 94!
E 100! is ${}^{100}P_6$ times bigger than 94!

For questions **19** to **21** show your answers in the form ${}^{n}P_{r}$ and then evaluate.

19 In how many ways can the letters of the word TODAY be arranged if they are used once only and taken:

a 3 at a time? **b** 4 at a time? **c** 5 at a time?

20 In how many ways can the letters of the word TUESDAY be arranged if they are used once only and taken:

a 3 at a time? **b** 4 at a time? **c** 7 at a time?

21 In how many ways can the letters of the word NEWTON be arranged if they are used once only and taken 6 at a time, assuming:

a the first N is distinct from the second N?

b there is no distinction between the two Ns?

Arrangements involving restrictions and like objects

A 5-letter word is to be made from 3 As and 2 Bs. How many different arrangements can be made?

If the 5 letters were all different, it would be easy to calculate the number of arrangements. It would be 5! = 120. Perhaps you can see that when letters are repeated, the number of different arrangements will be less than 120. To analyse the situation let us imagine that we can distinguish one A from another. We will write A_1, A_2, A_3, B_1 and B_2 to represent the 5 letters.

As we list some of the possible arrangements we notice that some are actually the same, as shown in the table.

$A_1A_2B_1A_3B_2$ $A_1A_3B_1A_2B_2$ $A_2A_1B_1A_3B_2$ $A_2A_3B_1A_1B_2$ $A_3A_1B_1A_2B_2$ $A_3A_2B_1A_1B_2$	$A_1A_2B_2A_3B_1$ $A_1A_3B_2A_2B_1$ $A_2A_1B_2A_3B_1$ $A_2A_3B_2A_1B_1$ $A_3A_1B_2A_2B_1$ $A_3A_2B_2A_1B_1$	Each of these 12 arrangements is the same — AABAB.
$B_2A_1A_2B_1A_3$ $B_2A_1A_3B_1A_2$ $B_2A_2A_1B_1A_3$ $B_2A_2A_3B_1A_1$ $B_2A_3A_1B_1A_2$ $B_2A_3A_2B_1A_1$	$B_1A_1A_2B_2A_3$ $B_1A_1A_3B_2A_2$ $B_1A_2A_1B_2A_3$ $B_1A_2A_3B_2A_1$ $B_1A_3A_1B_2A_2$ $B_1A_3A_2B_2A_1$	Each of these 12 arrangements is the same — BAABA.

The number of repetitions is 3! for the As and 2! for the Bs. Thus, the number of different arrangements is $\frac{5!}{3! \times 2!}$.

The number of different ways of arranging n things made up of groups of indistinguishable things, n_1 in the first group, n_2 in the second group and so on is:

$$\frac{n!}{n_1!n_2!n_3!\ldots n_r!}.$$

Note: If there are elements of the group which are not duplicated, then they can be considered as a group of 1. It is not usual to divide by 1!; it is more common to show only those groups which have duplications.

WORKED Example 9

How many different arrangements of 8 letters can be made from the word PARALLEL?

THINK	WRITE
① Write down the number of letters in the given word.	The word PARALLEL contains 8 letters; therefore $n = 8$.
② Write down the number of times any of the letters are repeated.	The letter A is repeated twice; therefore $n_1 = 2$. The letter L is repeated 3 times; therefore $n_2 = 3$.
③ Write down the rule for arranging groups of like things.	$\frac{n!}{n_1!n_2!n_3!\ldots n_r!}$
④ Substitute the values of n, n_1 and n_2 into the rule.	$= \frac{8!}{2! \times 3!}$
⑤ Evaluate each of the factorials.	$= \frac{40\,320}{2 \times 6}$
⑥ Simplify the fraction.	$= \frac{40\,320}{12}$
⑦ Evaluate.	$= 3360$
⑧ Answer the question.	3360 arrangements of 8 letters can be made from the word PARALLEL.

WORKED Example 10

How many different arrangements of 7 counters can be made from 4 black and 3 white counters?

THINK	WRITE
① Write down the total number of counters.	There are 7 counters in all; therefore $n = 7$.
② Write down the number of times any of the coloured counters are repeated.	There are 3 white counters; therefore $n_1 = 3$. There are 4 black counters; therefore $n_2 = 4$.
③ Write down the rule for arranging groups of like things.	$\frac{n!}{n_1!n_2!n_3!\ldots n_r!}$
④ Substitute the values of n, n_1 and n_2 into the rule.	$= \frac{7!}{3! \times 4!}$
⑤ Evaluate each of the factorials.	$= \frac{5040}{6 \times 24}$
⑥ Simplify the fraction.	$= \frac{5040}{144}$
⑦ Evaluate.	$= 35$
⑧ Answer the question.	Thirty-five different arrangements can be made from 7 counters, of which 3 are white and 4 are black.

WORKED Example 11

A rowing crew of 4 rowers is to be selected, in order from the first seat to the fourth seat, from 8 candidates. How many different arrangements are possible if:

a **there are no restrictions?**

b **Jason or Kris must row in the first seat?**

c **Jason must be in the crew, but he can row anywhere in the boat?**

THINK	WRITE
a 1 Write down the permutation formula. *Note*: 4 rowers are to be selected from 8 and the order is important.	**a** $^nP_r = \frac{n!}{(n-r)!}$
2 Substitute the given values of *n* and *r* into the permutation formula.	$^8P_4 = \frac{8!}{(8-4)!}$ $= \frac{8!}{4!}$
3 Use a calculator to evaluate 8! and 4!.	$= \frac{40\,320}{24}$
4 Evaluate.	$= 1680$
5 Answer the question.	There are 1680 ways of arranging 4 rowers from a group of 8.
b 1 Apply the multiplication principle since two events will follow each other; that is, Jason will fill the first seat and the remaining 3 seats will be filled in 7 × 6 × 5 ways or Kris will fill the first seat and the remaining 3 seats will be filled in 7 × 6 × 5 ways. $\boxed{J}\boxed{7}\boxed{6}\boxed{5}$ or $\boxed{K}\boxed{7}\boxed{6}\boxed{5}$	**b** No. of arrangements = no. of ways of filling the first seat × no. of ways of filling the remaining 3 seats. No. of arrangements $= 2 \times {}^nP_r$
2 Substitute the values of *n* and *r* into the formula.	No. of arrangements $= 2 \times {}^7P_3$
3 Evaluate.	$= 2 \times 210$ $= 420$
4 Answer the question.	There are 420 ways of arranging the 4 rowers if Jason or Kris must row in the first seat.
c 1 Apply the addition principle, since Jason must be in either the first, second, third or fourth seat. The remaining 3 seats will be filled in 7 × 6 × 5 ways each time. $\boxed{J}\boxed{7}\boxed{6}\boxed{5} + \boxed{7}\boxed{J}\boxed{6}\boxed{5} +$ $\boxed{7}\boxed{6}\boxed{J}\boxed{5} + \boxed{7}\boxed{6}\boxed{5}\boxed{J}$	**c** No. of arrangements = No. of arrangements with Jason in seat 1 + No. of arrangements with Jason in seat 2 + No. of arrangements with Jason in seat 3 + No. of arrangements with Jason in seat 4.
2 Substitute the values of *n* and *r* into the formula.	No. of arrangements $= 1 \times {}^7P_3 + 1 \times {}^7P_3$ $+ 1 \times {}^7P_3 + 1 \times {}^7P_3$ $= 4 \times {}^7P_3$ $= 4 \times 210$
3 Evaluate.	$= 840$
4 Answer the question.	There are 840 ways of arranging the 4 rowers if Jason must be in the crew of 4.

WORKED Example 12

a **How many permutations of the letters in the word COUNTER are there?**

b **In how many of these do the letters C and N appear side by side?**

c **In how many permutations do the letters C and N appear apart?**

THINK	WRITE
a ① Count the number of letters in the given word.	**a** There are 7 letters in the word COUNTER.
② Determine the number of ways the 7 letters may be arranged.	The 7 letters may be arranged 7! = 5040 ways.
③ Answer the question.	There are 5040 permutations of letters in the word COUNTER.
b ① Imagine the C and N are ‘tied’ together and are therefore considered as 1 unit. Determine the number of ways C and N may be arranged: CN and NC.	**b** Let C and N represent 1 unit. They may be arranged 2! = 2 ways.
② Determine the number of ways 6 things can be arranged. *Note*: There are now 6 letters: the ‘CN’ unit along with O, U, T, E and R.	Six things may be arranged 6! = 720 ways.
③ Determine the number of permutations in which the letters C and N appear together.	The number of permutations $= 2 \times 6!$
④ Evaluate.	$= 2 \times 720$ $= 1440$
⑤ Answer the question.	There are 1440 permutations in which the letters C and N appear together.
c ① Determine the total number of arrangements of the 7 letters.	**c** Total number of arrangements = 7! = 5040
② Write down the number of arrangements in which the letters C and N appear together, as obtained in **a**.	Arrangements with C and N together = 1440
③ Determine the difference between the values obtained in steps 1 and 2. *Note*: The number of arrangements in which C and N are apart is the total number of arrangements less the number of times they are together.	The number of arrangements $= 5040 - 1440$ $= 3600$
④ Answer the question.	The letters C and N appear apart 3600 times.

remember

1. The number of different ways of arranging n things made up of groups of indistinguishable things, n_1 in the first group, n_2 in the second group and so on is: $\dfrac{n!}{n_1!n_2!n_3!\ldots n_r!}$.
2. When restrictions apply to arrangements, use the multiplication and addition principles as well as ${}^nP_r$.

EXERCISE 10C Arrangements involving restrictions and like objects

WORKED Example 9

Mathcad Permutations

1 How many different arrangements can be made using the 6 letters of the word NEWTON?

2 How many different arrangements can be made using the 11 letters of the word ABRACADABRA?

WORKED Example 10

EXCEL Spreadsheet Permutations

3 How many different arrangements of 5 counters can be made using 3 red and 2 blue counters?

4 How many different arrangements of 9 counters can be made using 4 black, 3 red and 2 blue counters?

Mathcad Factorials

5 A collection of 12 books is to be arranged on a shelf. The books consist of 3 copies of *Great Expectations*, 5 copies of *Catcher in the Rye* and 4 copies of *Huntin', Fishin' and Shootin'*. How many different arrangements of these books are possible?

6 A shelf holding 24 cans of dog food is to be stacked using 9 cans of *Yummy* and 15 cans of *Ruff for Dogs*. In how many different ways can the shelf be stocked?

WORKED Example 11

7 A cricket team of 11 players is to be selected, in batting order, from 15. How many different arrangements are possible if:
- a there are no restrictions?
- b Mark must be in the team at number 1?
- c Mark must be in the team but he can be anywhere from 1 to 11?

8 The Student Council needs to fill the positions of president, secretary and treasurer from 6 candidates. Each candidate can fill only one of the positions. In how many ways can this be done if:
- a there are no restrictions?
- b Jocelyn must be secretary?
- c Jocelyn must have one of the 3 positions?

9 The starting 5 in a basketball team is to be picked, in order, from the 10 players in the squad. In how many ways can this be done if:
- a there are no restrictions?
- b Jamahl needs to play at number 5?
- c Jamahl and Anfernee must be in the starting 5?

WORKED Example 12

10
- a How many permutations of the letters in the word MATHS are there?
- b In how many of these do the letters M and A appear together?
- c In how many permutations do the letters M and A appear apart?

11 A rowing team of 4 rowers is to be selected in order from 8 rowers.

a In how many different ways can this be done?

b In how many of these ways do 2 rowers, Jane and Lee, sit together in the boat?

c In how many ways can the crew be formed without using Jane or Lee?

12 A horse race has 12 runners.

a In how many ways can 1st, 2nd and 3rd be filled?

b In how many ways can 1st, 2nd and 3rd be filled if Najim finishes first?

13 **multiple choice**

If the answer is 10, which of the following statements best matches this answer?

A The number of ways 1st and 2nd can occur in a race with 5 entrants.

B The number of distinct arrangements of the letters in NANNA.

C The number of permutations of the letters in POCKET where P and O are together.

D The number of permutations of the letters in POCKET where P and O are apart.

E ${}^{10}P_2 \div {}^{4}P_2$

14 **multiple choice**

If the answer is 480, which of the following statements best matches this answer?

A The number of ways 1st and 2nd can occur in a race with 5 entrants.

B The number of distinct arrangements of the letters in NANNA.

C The number of permutations of the letters in POCKET where P and O are together.

D The number of permutations of the letters in POCKET where P and O are apart.

E ${}^{10}P_2 \div {}^{4}P_2$

15 The clue in a crossword puzzle says that a particular answer is an anagram of STOREY. An anagram is another word that can be obtained by rearranging the letters of the given word.

a How many possible arrangements of the letters of STOREY are there?

b The other words in the crossword puzzle indicate that the correct answer is O--T-- How many arrangements are now possible?

c Can you see the answer?

16 There are 30 students in a class. The students are arranged in order and asked to give the month and date of their birthday.

a How many different arrangements of these dates are possible?

b How many arrangements of these dates are possible if no 2 students have the same birthday?

Combinations

A group of things chosen from a larger group where order is not important is called a *combination*. In previous sections we performed calculations of the number of ways a task could be done where order is important — permutations or arrangements. We now examine situations where order does not matter.

Suppose 5 people have nominated for a committee consisting of 3 members. It does not matter in what order the candidates are placed on the committee, it matters only whether they are there or not. If order was important we know there would be ${}^5P_3$, or 60, ways in which this could be done. Here are the possibilities:

ABC	ABD	ABE	ACD	ACE	ADE
BDE	BCD	BCE	CDE	CAB	DAB
EAB	DAC	EAC	EAD	EBD	DBC
EBC	ECD	BCA	BDA	BEA	CDA
CEA	DEA	DEB	CDB	CEB	DEC
CBA	DBA	EBA	DCA	ECA	EDA
EDB	DCB	ECB	EDC	BAC	BAD
BAE	CAD	CAE	DAE	DBE	CBD
CBE	DCE	ACB	ADB	AEB	ADC
AEC	AED	BED	CDB	BEC	CED

The 60 arrangements are different only if we take order into account; that is, ABC is different from CAB and so on. You will notice in this table that there are 10 distinct committees corresponding to the 10 distinct rows. Each column merely repeats, in a different order, the committee in the first row. This result (10 distinct committees) can be arrived at logically:

1. There are ${}^5P_3$ ways of choosing or selecting 3 from 5 in order.
2. Each choice of 3 is repeated 3! times.
3. The number of distinct selections or combinations is ${}^5P_3 \div 3! = 10$.

This leads to the general rule of selecting r things from n things:

1. **The number of ways of choosing or selecting r things from n distinct things where order is not important is given by the rule ${}^nC_r$, where:**

$$ {}^nC_r = \frac{{}^nP_r}{r!} $$

2. **The letter C is used to represent combinations.**

WORKED Example 13

Write these combinations as statements involving permutations, then calculate them.

a $^{7}C_{2}$ **b** $^{20}C_{3}$

THINK	WRITE
a 1 Write down the rule for ^nC_r.	**a** $^nC_r = \frac{^nP_r}{r!}$
2 Substitute the given values of n and r into the combination formula.	$^7C_2 = \frac{^7P_2}{2!}$
3 Simplify the fraction.	$= \frac{\left(\frac{7!}{5!}\right)}{2!}$ $= \frac{7!}{5!} \div 2!$ $= \frac{7!}{5!} \times \frac{1}{2!}$ $= \frac{7!}{5!2!}$ $= \frac{7 \times 6 \times 5!}{5! \times 2 \times 1}$
4 Evaluate.	$= \frac{7 \times 6}{2 \times 1}$ $= \frac{42}{2}$ $= 21$
b 1 Write down the rule for ^nC_r.	**b** $^nC_r = \frac{^nP_r}{r!}$
2 Substitute the values of n and r into the formula.	$^{20}C_3 = \frac{^{20}P_3}{3!}$
3 Simplify the fraction.	$= \frac{\left(\frac{20!}{17!}\right)}{3!}$ $= \frac{20!}{17!} \div 3!$ $= \frac{20!}{17!} \times \frac{1}{3!}$ $= \frac{20!}{17!3!}$

Continued over page →

THINK **WRITE**

$$= \frac{20 \times 19 \times 18 \times 17!}{17! \times 3 \times 2 \times 1}$$

④ Evaluate.

$$= \frac{20 \times 19 \times 18}{3 \times 2 \times 1}$$

$$= \frac{6840}{6}$$

$$= 1140$$

WORKED Example 14

In how many ways can a basketball team of 5 players be selected from a squad of 9 if the order in which they are selected does not matter?

THINK **WRITE**

① Write down the rule for ${}^nC_r$.
Note: Since order does not matter use the ${}^nC_r$ rule.

$${}^nC_r = \frac{{}^nP_r}{r!}$$

② Substitute the values of n and r into the formula.

$${}^9C_5 = \frac{{}^9P_5}{5!}$$

③ Simplify the fraction.

$$= \frac{\left(\frac{9!}{4!}\right)}{5!}$$

$$= \frac{9!}{4!} \div 5!$$

$$= \frac{9!}{4!} \times \frac{1}{5!}$$

$$= \frac{9!}{4!5!}$$

$$= \frac{9 \times 8 \times 7 \times 6 \times 5!}{4 \times 3 \times 2 \times 1 \times 5!}$$

④ Evaluate.

$$= \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1}$$

$$= \frac{3024}{24}$$

$$= 126$$

The formula we use to determine the number of ways of selecting r things from n distinct things, where order is not important, is useful but needs to be simplified.

$${}^nC_r = \frac{{}^nP_r}{r!}$$

$$= \frac{\dfrac{n!}{(n-r)!}}{r!}$$

$$= \frac{n!}{(n-r)!r!}$$

$$\boldsymbol{^nC_r = \frac{n!}{(n-r)!r!}}$$

Note: ^nC_r may also be written as $\binom{n}{r}$.

WORKED Example 15

Determine the value of the following. **a** $^{12}C_5$ **b** $\binom{10}{2}$

THINK	WRITE
a 1 Write down the rule for ^nC_r.	**a** $^nC_r = \frac{n!}{(n-r)!r!}$
2 Substitute the given values of n and r into the combination formula.	$^{12}C_5 = \frac{12!}{(12-5)!5!}$ $= \frac{12!}{7!5!}$
3 Simplify the fraction.	$= \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7!}{7! \times 5 \times 4 \times 3 \times 2 \times 1}$ $= \frac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2 \times 1}$
4 Evaluate.	$= \frac{95\,040}{120}$ $= 792$
b 1 Write down the rule for $\binom{n}{r}$.	**b** $\binom{n}{r} = {}^nC_r$ $= \frac{n!}{(n-r)!r!}$
2 Substitute the given values of n and r into the combination formula.	$\binom{10}{2} = \frac{10!}{(10-2)!2!}$ $= \frac{10!}{8!2!}$
3 Simplify the fraction.	$= \frac{10 \times 9 \times 8!}{8! \times 2 \times 1}$ $= \frac{10 \times 9}{2 \times 1}$
4 Evaluate.	$= \frac{90}{2}$ $= 45$

WORKED Example 16

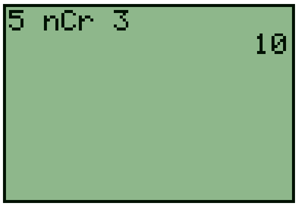
A committee consisting of 3 men and 4 women is to be chosen from 7 men and 9 women. In how many ways can this be done?

THINK	WRITE
① Write down the rule for ${}^nC_r$. *Note*: Since order does not matter, use the ${}^nC_r$ rule.	${}^nC_r = \frac{n!}{(n-r)!r!}$
② Write down the number of ways of choosing 3 men from 7.	Number of ways of choosing 3 men = ${}^7C_3$.
③ Write down the number of ways of choosing 4 women from 7.	Number of ways of choosing 4 women = ${}^9C_4$.
④ Evaluate each of the combinations obtained in steps 2 and 3.	$\begin{aligned} {}^7C_3 &= \frac{7!}{(7-3)!3!} \\ &= \frac{7!}{4!3!} \\ &= \frac{7 \times 6 \times 5 \times 4!}{4!3 \times 2 \times 1} \\ &= \frac{7 \times 6 \times 5}{3 \times 2 \times 1} \\ &= \frac{210}{6} \\ &= 35 \end{aligned}$ $\quad$ $\begin{aligned} {}^9C_4 &= \frac{9!}{(9-4)!4!} \\ &= \frac{9!}{5!4!} \\ &= \frac{9 \times 8 \times 7 \times 6 \times 5!}{5! \times 4 \times 3 \times 2 \times 1} \\ &= \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} \\ &= \frac{3024}{24} \\ &= 126 \end{aligned}$
⑤ Use the multiplication principle to find the number of ways of choosing men *and* women.	The number of ways of choosing 3 men and 4 women $= {}^7C_3 \times {}^9C_4$ $= 35 \times 126$ $= 4410$
⑥ Answer the question.	There are 4410 ways of choosing 3 men and 4 women.

Graphics Calculator tip!

Calculating combinations

To find the number of combinations of n objects taken r at a time, enter the number of objects, then press **MATH**, select **PRB**, choose **3:nCr** and enter the value of r.



```
5 nCr 3
                10
```

WORKED Example 17

Evaluate the following using your calculator and comment on your results.

a ${}^9C_3$ **b** ${}^9C_6$ **c** ${}^{15}C_5$ **d** ${}^{15}C_{10}$ **e** ${}^{12}C_7$ **f** ${}^{12}C_5$

THINK	WRITE
a-f Use a graphics calculator (see above for steps) to evaluate the listed combinations.	**a** ${}^9C_3 = 84$ **b** ${}^9C_6 = 84$ **c** ${}^{15}C_5 = 3003$ **d** ${}^{15}C_{10} = 3003$ **e** ${}^{12}C_7 = 792$ **f** ${}^{12}C_5 = 792$
Comment on your results.	So ${}^9C_3 = {}^9C_6$, ${}^{15}C_5 = {}^{15}C_{10}$ and ${}^{12}C_7 = {}^{12}C_5$.

For each of the preceding examples, it can be seen that ${}^nC_r = {}^nC_{n-r}$. This may be derived algebraically:

$$\begin{aligned}
{}^nC_{n-r} &= \frac{{}^nP_{n-r}}{(n-r)!} \\
&= \frac{\left(\dfrac{n!}{[n-(n-r)]!}\right)}{(n-r)!} \\
&= \frac{\left(\dfrac{n!}{r!}\right)}{(n-r)!} \\
&= \frac{n!}{r!} \div (n-r)! \\
&= \frac{n!}{r!} \times \frac{1}{(n-r)!} \\
&= \frac{n!}{r!(n-r)!} \\
&= \frac{n!}{(n-r)!r!} \\
&= {}^nC_r
\end{aligned}$$

remember

1. The number of ways of selecting r things from n things when order is important is ${}^nP_r$.
2. The number of ways of selecting r things from n things when order is not important is ${}^nC_r$.
3. $${}^nC_r = \frac{{}^nP_r}{r!}$$
$$= \frac{n!}{(n-r!)r!}$$
4. ${}^nC_r$ may also be written as $\binom{n}{r}$.
5. ${}^nC_r = {}^nC_{n-r}$

EXERCISE 10D Combinations

WORKED Example 13

1 Write each of the following as statements in terms of permutations.

a $^{8}C_{3}$ **b** $^{19}C_{2}$ **c** $^{1}C_{1}$ **d** $^{5}C_{0}$

2 Write each of the following using the notation $^{n}C_{r}$.

a $\frac{^{8}P_{2}}{2!}$ **b** $\frac{^{9}P_{3}}{3!}$ **c** $\frac{^{8}P_{0}}{0!}$ **d** $\frac{^{10}P_{4}}{4!}$

WORKED Example 14

3 A committee of 3 is to be chosen from a group of people. In how many ways can this be done if the group contains:

a 3 people? **b** 6 people? **c** 10 people? **d** 12 people?

Mathcad Combinations

4 A cricket team of 11 players is to be chosen from a squad of 15 players. In how many ways can this be done?

5 A basketball team of 5 players is to be chosen from a squad of 10 players. In how many ways can this be done?

WORKED Example 15

EXCEL Spreadsheet Combinations

6 Determine the value of the following:

a $^{12}C_{4}$ **b** $^{11}C_{1}$ **c** $^{15}C_{0}$ **d** $^{12}C_{12}$

e $\binom{21}{15}$ **f** $\binom{10}{7}$ **g** $\binom{100}{1}$ **h** $\binom{17}{14}$

SkillSHEET 10.2

7 From a pack of 52 cards, a hand of 5 cards is dealt.

a How many different hands are there?

b How many of these hands contain only red cards?

c How many of these hands contain only black cards?

d How many of these hands contain at least one red and one black card?

WORKED Example 16

8 A committee of 3 men and 3 women is to be chosen from 5 men and 8 women. In how many ways can this be done?

9 A mixed netball team must have 3 women and 4 men in the side. If the squad has 6 women and 5 men wanting to play, how many different teams are possible?

10 A rugby union squad has 12 forwards and 10 backs in training. A team consists of 8 forwards and 7 backs. How many different teams can be chosen from the squad?

11 A *quinella* is a bet made on a horse race which pays a win if the punter selects the first 2 horses in any order. How many different quinellas are possible in a race that has:

a 8 horses?

b 16 horses?

12 A CD collection contains 32 CDs. In how many ways can 5 CDs be chosen from the collection?

13 **multiple choice**

At a party there are 40 guests and they decide to have a toast. Each guest ‘clinks’ glasses with every other guest. How many clinks are there in all?

A 39 **B** 40 **C** 40! **D** 780 **E** 1560

14 **multiple choice**

On a bookshelf there are 15 books — 7 geography books and 8 law books. Jane selects 5 books from the shelf — 2 geography books and 3 law books. How many different ways can she make this selection?

A 3003 **B** 360 360 **C** 1176 **D** 366 **E** 15

Questions **15**, **16** and **17** refer to the following information. The Maryborough Tennis Championships involve 16 players. The organisers plan to use 3 courts and assume that each match will last on average 2 hours and that no more than 4 matches will be played on any court per day.

15 In a 'round robin' each player plays every other player once. If the organisers use a round robin format:
- **a** How many games will be played in all?
- **b** For how many days does the tournament last?

16 The organisers split the 16 players into two pools of 8 players each. After a 'round robin' within each pool, a final is played between the winners of each pool.
- **a** How many matches are played in the tournament?
- **b** How long does the tournament last?

17 A 'knock out' format is one in which the loser of every match drops out and the winners proceed to the next round until there is only one winner left.
- **a** If the game starts with 16 players how many matches are needed before a winner is obtained?
- **b** How long would the tournament last?

18 Lotto is a gambling game played by choosing 6 numbers from 45. Gamblers try to match their choice with those numbers chosen at the official draw. No number can be drawn more than once and the order in which the numbers are selected does not matter.
- **a** How many different selections of 6 numbers can be made from 45?
- **b** Suppose the first numbers drawn at the official draw are 42, 3 and 18. How many selections of 6 numbers will contain these 3 numbers?
- **c** Suppose the first numbers drawn at the official draw are 42, 3, 18 and 41. How many selections of 6 numbers will contain these 4 numbers?

Note: This question ignores supplementary numbers. Lotto is discussed further in the next section.

WORKED Example 17

19 Calculate the value of:
- **a** $^{12}C_3$ and $^{12}C_9$
- **b** $^{15}C_8$ and $^{15}C_7$
- **c** $^{10}C_1$ and $^{10}C_9$
- **d** $^{8}C_3$ and $^{8}C_5$
- **e** $^{10}C_0$ and $^{10}C_{10}$
- **f** What do you notice?

Give your answer as a general statement such as 'The value of $^{n}C_r$ is . . .'.

Applications of permutations and combinations

Counting techniques, particularly those involving permutations and combinations, can be applied in gambling, logistics and various forms of market research. In this section we investigate when to use permutations and when to use combinations as well as examining problems associated with these techniques.

Permutations are used to count when order is important. Some examples are:

1. the number of ways the positions of president, secretary and treasurer can be filled
2. the number of ways a team can be chosen from a squad *in distinctly different positions*
3. the number of ways the first three positions of a horse race can be filled.

Combinations are used to count when order is not important. Some examples are:

1. the number of ways a committee can be chosen
2. the number of ways a team can be chosen from a squad
3. the number of ways a hand of 5 cards can be dealt from a deck.

An interesting application of combinations as a technique of counting is a game that Australians spend many millions of dollars on each week — Lotto. To play, a player selects 6 numbers from 45 numbers. The official draw chooses 6 numbers and 2 supplementary numbers. Depending on how the player's choice of 6 numbers matches the official draw, prizes are awarded in different divisions.

Division 1: 6 winning numbers
Division 2: 5 winning numbers and one or both supplementary
Division 3: 5 winning numbers
Division 4: 4 winning numbers
Division 5: 3 winning numbers and one or both supplementary
If the official draw was:

Winning numbers						**Supplementaries**	
13	42	6	8	20	12	2	34

A player who chose:

8 34 13 12 20 45

would win a Division 4 prize and a player who chose:

8 34 13 12 22 45

would win a Division 5 prize.

Lotto systems

A player may have 7 lucky numbers 4, 7, 12, 21, 30, 38 and 45, and may wish to include all possible combinations of these 7 numbers in a Lotto entry.
This can be done as follows:

4 7 12 21 30 38
4 7 12 21 30 45
4 7 12 21 38 45
4 7 12 30 38 45
4 7 21 30 38 45
4 12 21 30 38 45
7 12 21 30 38 45

The player does not have to fill out 7 separate entries to enter all combinations of these 7 numbers 6 at a time but rather can complete a ‘System 7’ entry by marking 7 numbers on the entry form.

A System 9 consists of all entries of 6 numbers from the chosen 9 numbers.

WORKED Example 18

a **Ten points are marked on a page and no three of these points are in a straight line. How many triangles can be drawn joining these points?**

b **How many different 3-digit numbers can be made using the digits 1, 3, 5, 7 and 9 without repetition?**

THINK	WRITE
a ① Write down the rule for ^nC_r. *Note*: A triangle is made by choosing 3 points. It does not matter in what order the points are chosen, so ^nC_r is used.	**a** $^nC_r = \dfrac{n!}{(n-r)!r!}$
② Substitute the given values of n and r into the combination formula.	$^{10}C_3 = \dfrac{10!}{(10-3)!3!}$ $= \dfrac{10!}{7!3!}$
③ Simplify the fraction.	$= \dfrac{10 \times 9 \times 8 \times 7!}{7! \times 3 \times 2 \times 1}$ $= \dfrac{10 \times 9 \times 8}{3 \times 2 \times 1}$
④ Evaluate.	$= \dfrac{720}{6}$ $= 120$
⑤ Answer the question.	120 triangles may be drawn by joining 3 points.
⑥ Verify the answer obtained by using the combination function on the calculator.	
b ① Write down the rule for ^nP_r. *Note*: Order is important here.	**b** $^nP_r = \dfrac{n!}{(n-r)!}$
② Substitute the given values of n and r into the permutation formula.	$^5P_3 = \dfrac{5!}{(5-3)!}$ $= \dfrac{5!}{2!}$
③ Evaluate.	$= \dfrac{5 \times 4 \times 3 \times 2!}{2!}$ $= 5 \times 4 \times 3$ $= 60$
④ Answer the question.	Sixty 3-digit numbers can be made without repetition from a group of 5 numbers.
⑤ Verify the answer obtained by using the permutation function on the calculator.	

WORKED Example 19

Jade and Kelly are 2 of the 10 members of a basketball squad. In how many ways can a team of 5 be chosen if:

a **both Jade and Kelly are in the 5?**
b **neither Jade nor Kelly is in the 5?**
c **Jade is in the 5 but Kelly is not?**

THINK	WRITE
a ① Write down the rule for ^nC_r. *Note*: Order is not important, so ^nC_r is used.	**a** $^nC_r = \frac{n!}{(n-r)!r!}$
② Substitute the given values of n and r into the combination formula. *Note*: If Jade and Kelly are included then there are 3 positions to be filled from the remaining 8 players.	$^8C_3 = \frac{8!}{(8-3)!3!}$ $= \frac{8!}{5!3!}$
③ Simplify the fraction.	$= \frac{8 \times 7 \times 6 \times 5!}{5! \times 3 \times 2 \times 1}$ $= \frac{8 \times 7 \times 6}{3 \times 2 \times 1}$
④ Evaluate.	$= \frac{336}{6}$ $= 56$
⑤ Answer the question.	If Jade and Kelly are included, then there are 56 ways to fill the remaining 3 positions.
b ① Write down the rule for ^nC_r. *Note*: Order is not important, so ^nC_r is used.	**b** $^nC_r = \frac{n!}{(n-r)!r!}$
② Substitute the given values of n and r into the combination formula. *Note*: If Jade and Kelly are not included then there are 5 positions to be filled from 8 players.	$^8C_5 = \frac{8!}{(8-5)!5!}$ $= \frac{8!}{3!5!}$
③ Simplify the fraction.	$= \frac{8 \times 7 \times 6 \times 5!}{3 \times 2 \times 1 \times 5!}$ $= \frac{8 \times 7 \times 6}{3 \times 2 \times 1}$
④ Evaluate.	$= \frac{336}{6}$ $= 56$
⑤ Answer the question.	If Jade and Kelly are not included, then there are 56 ways to fill the 5 positions.

THINK **WRITE**

c ① Write down the rule for ${}^nC_r$.
Note: Order is not important, so ${}^nC_r$ is used.

c $${}^nC_r = \frac{n!}{(n-r)!r!}$$

② Substitute the given values of n and r into the combination formula.
Note: If Jade is included and Kelly is not then there are 4 positions to be filled from 8 players.

$${}^8C_4 = \frac{8!}{(8-4)!4!}$$
$$= \frac{8!}{4!4!}$$

③ Simplify the fraction.

$$= \frac{8 \times 7 \times 6 \times 5 \times 4!}{4 \times 3 \times 2 \times 1 \times 4!}$$
$$= \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1}$$

④ Evaluate.

$$= \frac{1680}{24}$$
$$= 70$$

⑤ Answer the question.

If Jade is included and Kelly is not, then there are 70 ways to fill the 4 positions.

⑥ Verify each of the answers obtained by using the combination function on the calculator.

WORKED Example 20

Use the information on Lotto systems given on page 454.

A player uses a System 8 entry with the numbers 4, 7, 9, 12, 22, 29, 32 and 36.

The official draw for this game was 4, 8, 12, 15, 22, 36 with supplementaries 20 and 29.

a To how many single entries is a System 8 equivalent?

b List 3 of the player's entries that would have won Division 4.

c How many of the player's entries would have won Division 4?

THINK **WRITE**

a ① Write down the rule for ${}^nC_r$.
Note: Order is not important, so ${}^nC_r$ is used.

a $${}^nC_r = \frac{n!}{(n-r)!r!}$$

② Substitute the given values of n and r into the combination formula.
Note: A System 8 consists of all entries consisting of 6 numbers chosen from 8.

$${}^8C_6 = \frac{8!}{(8-6)!6!}$$
$$= \frac{8!}{2!6!}$$

③ Simplify the fraction.

$$= \frac{8 \times 7 \times 6!}{2 \times 1 \times 6!}$$
$$= \frac{8 \times 7}{2 \times 1}$$

Continued over page →

THINK	**WRITE**
4 Evaluate.	$= \frac{56}{2}$ $= 28$
5 Answer the question.	A System 8 is equivalent to 28 single entries.
6 Verify each of the answers obtained by using the combination function on the calculator.	
b List 3 of the player's entries that would have won Division 4. *Note*: Division 4 requires 4 winning numbers. The player's winning numbers are 4, 12, 22 and 36. Any of the other 4 numbers can fill the remaining 2 places.	**b** Some of the possibilities are: 4 12 22 36 7 9 4 12 22 36 7 29 4 12 22 36 7 32
c 1 Write down the rule for ^nC_r. *Note*: Order is not important, so ^nC_r is used.	**c** $^nC_r = \frac{n!}{(n-r)!r!}$
2 Substitute the given values of n and r into the combination formula. *Note*: To win Division 4 the numbers 4, 12, 22 and 36 must be included in the entry. The other 2 spaces can be filled with any of the other 4 numbers in any order.	$^4C_2 = \frac{4!}{(4-2)!2!}$ $= \frac{4!}{2!2!}$
3 Simplify the fraction.	$= \frac{4 \times 3 \times 2!}{2 \times 1 \times 2!}$ $= \frac{4 \times 3}{2 \times 1}$
4 Evaluate.	$= \frac{12}{2}$ $= 6$
5 Answer the question.	Six of the player's entries would have won Division 4.
6 Verify each of the answers obtained by using the combination function on the calculator.	

remember

1. Permutations are used to count when order is important.
2. Combinations are used to count when order is not important.

EXERCISE 10E Applications of permutations and combinations

WORKED Example 18

1 How many ways are there:
- **a** to draw a line segment between 2 points on a page with 10 points on it?
- **b** to make a 4-digit number using the digits 2, 4, 6, 8 and 1 without repetition?
- **c** to allocate 5 numbered singlets to 5 players?
- **d** to choose a committee of 4 people from 10 people?
- **e** for a party of 15 people to shake hands with one another?

2 How many ways are there:
- **a** for 10 horses to fill 1st, 2nd and 3rd positions?
- **b** to give only 5 players in a group of 10 an unnumbered singlet?
- **c** to choose a team of 3 cyclists from a squad of 5?
- **d** to choose 1st, 2nd and 3rd speakers for a debating team from 6 candidates?
- **e** for 20 students to seat themselves at 20 desks arranged in rows?

3 The French flag is known as a tricolour flag because it is composed of the 3 bands of colour. How many different tricolour flags can be made from the colours red, white, blue and green, if each colour can be used only once in one of the 3 bands?

4 In a taste test a market research company has asked people to taste 4 samples of coffee and try to identify each as one of four brands. Subjects are told that no 2 samples are the same brand. How many different ways can the samples be matched to the brands?

5 In the gambling game roulette, if a gambler puts $1 on the winning number he will win $35. Suppose a gambler wishes to place five $1 bets on 5 different numbers in one spin of the roulette wheel. If there are 36 numbers in all, in how many ways can the five bets be placed?

WORKED Example 19

6 A volleyball team of 6 players is to be chosen from a squad of 10 players. In how many ways can this be done if:
- **a** there are no restrictions?
- **b** Stephanie is to be in the team?
- **c** Stephanie is not in the team?
- **d** two players, Stephanie and Alison, are not both in the team together?

7 A cross-country team of 4 runners is to be chosen from a squad of 9 runners. In how many ways can this be done if:
- **a** there are no restrictions?
- **b** Tony is to be one of the 4?
- **c** Tony and Michael are in the team?
- **d** either Tony or Michael but not both are in the team?

8 A soccer team of 11 players is to be chosen from a squad of 17. If one of the squad is selected as goalkeeper and any of the remainder can be selected in any of the positions, in how many ways can this be done if:
- **a** there are no restrictions?
- **b** Karl is to be chosen?
- **c** Karl and Andrew refuse to play in the same team?
- **d** Karl and Andrew are either both in or both out?

9 **multiple choice**

A netball team consists of 7 different positions: goal defence, goal keeper, wing defence, centre, wing attack, goal attack and goal shooter. The number of ways a squad of 10 players can be allocated to these positions is:

A 10! **B** 7! **C** $\frac{10!}{7!}$ **D** ${}^{10}P_7$ **E** ${}^{10}C_7$

10 **multiple choice**

A secret chemical formula requires the mixing of 3 chemicals. A researcher does not remember the 3 chemicals but has a shortlist of 10 from which to choose. Each time she mixes 3 chemicals and tests the result, she takes 15 minutes. How long does the researcher need, to be absolutely sure of getting the right combination?

A 1 hour **B** 7.5 hours **C** 15 hours **D** 30 hours **E** 120 hours

WORKED Example 20

11 Use the information on Lotto given on page 454.
A player uses a System 8 entry with the numbers 9, 12, 14, 17, 27, 34, 37 and 41. The official draw for this game was 9, 13, 17, 20, 27, 41 with supplementaries 25 and 34.
- **a** To how many single entries is a System 8 equivalent?
- **b** List 3 of the player's entries that would have won Division 4.
- **c** How many of the player's entries would have won Division 4?

12 Use the information on Lotto given on page 454.

A player uses a System 9 entry with the numbers 7, 10, 12, 15, 25, 32, 35, 37 and 41. The official draw for this game was 7, 11, 15, 18, 25, 39 with supplementaries 23 and 32.

a To how many single entries is a System 9 equivalent?

b List 3 of the player's entries that would have won Division 5.

c How many of the player's entries would have won Division 5?

Questions **13** and **14** refer to the following information: Keno is a popular game in clubs and pubs around Australia. In each round a machine randomly generates 15 numbers from 1 to 50. In one entry a player can select up to 15 numbers.

13 Suppose a player selects an entry of 6 numbers.

The payout for a $1 bet on 6 numbers is:

Match 6	$1500	Match 5	$85
Match 4	$5	Match 3	$1

a In how many ways can an entry of 6 numbers contain 6 winning numbers?

Suppose an entry of 6 numbers has exactly 3 winning numbers in it.

b In how many ways can the 3 winning numbers be chosen?

c In how many ways can the 3 losing numbers be chosen?

d How many entries of 6 numbers contain 3 winning numbers and 3 losing numbers?

14 Suppose a player selects an entry of 12 numbers.

The payout for a $1 bet on 12 numbers is:

Match 12	$160 000
Match 11	$56 000
Match 10	$7600
Match 9	$600
Match 8	$80
Match 7	$15
Match 6	$4
Match 5	$1
Match 1	$1
Match 0	$4

a In how many ways can an entry of 12 numbers contain 12 winning numbers?

b Suppose an entry of 12 numbers has exactly 6 winning numbers in it.

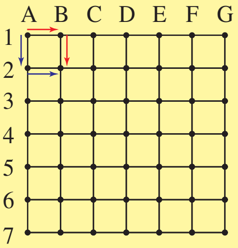
- **i** In how many ways can the 6 winning numbers be chosen?
- **ii** In how many ways can the 6 losing numbers be chosen?
- **iii** How many entries of 12 numbers contain 6 winning numbers and 6 losing numbers?
- **iv** How many entries of 12 numbers contain no winning numbers?

WorkSHEET 10.2

Pascal's triangle, the binomial theorem and the pigeonhole principle

Combinations are useful in other areas of mathematics, such as probability and binomial expansions. If we analyse the ${}^nC_r$ values closely, we notice that they produce the elements of any row in Pascal's triangle or each of the coefficients of a particular binomial expansion.

Counting paths



Dorothy and Toto enter a maze, and they have a compass. To prevent themselves from going round in circles they decide that they will only travel south or east and never north or west. The maze is shown at left and each intersection is labelled.

Dorothy and Toto enter the maze at A1.

There are 2 ways to get to B2:

A1 → A2 → B2 or A1 → B1 → B2

Are you able to list the 6 ways of going from A1 to C3? (Remember that they can only travel south or east.)

If you answered:

A1 → A2 → A3 → B3 → C3
A1 → A2 → B2 → B3 → C3
A1 → A2 → B2 → C2 → C3
A1 → B1 → B2 → B3 → C3
A1 → B1 → B2 → C2 → C3
A1 → B1 → C1 → C2 → C3

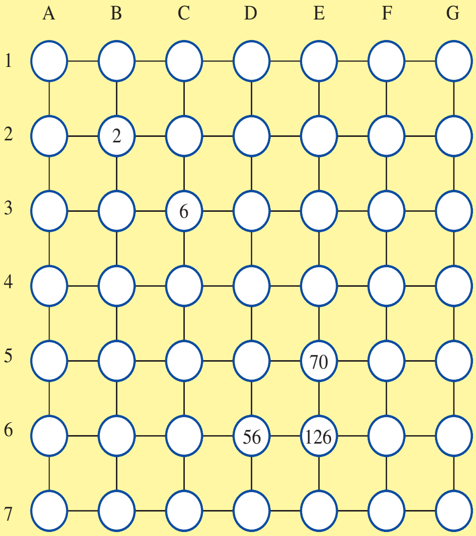
then you are correct.

How many different ways can they travel to get to D6, E5 and E6?

If you answered 56, 70 and 126 respectively, then you are correct.

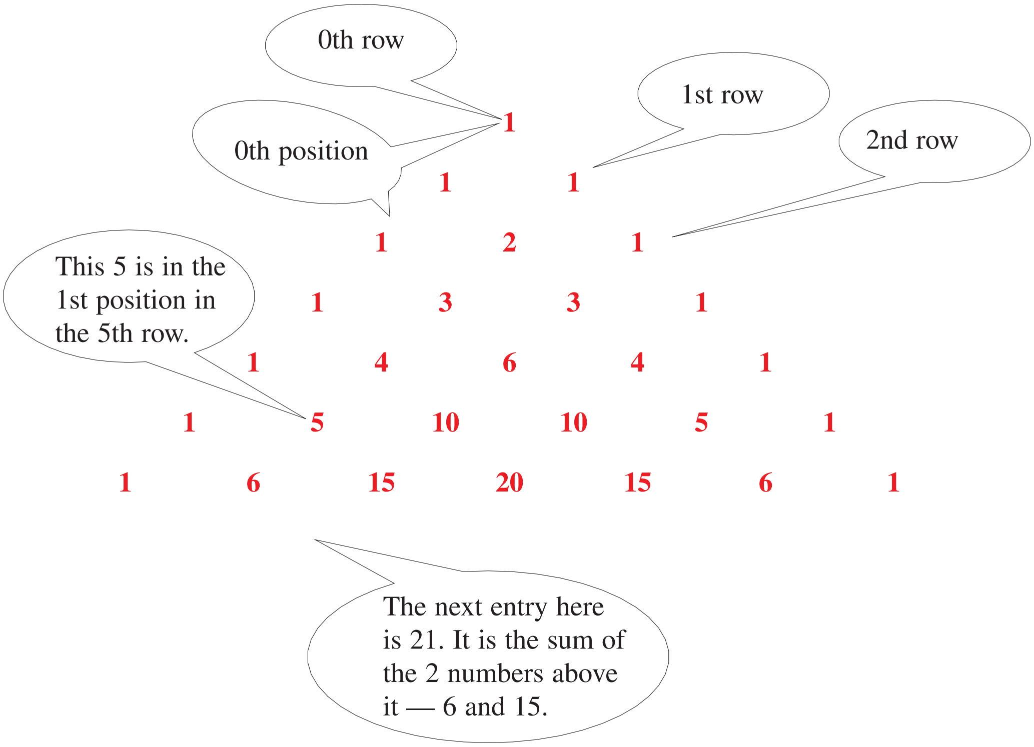
Can you devise a method of counting the number of ways of getting to each intersection and so show that there are 924 ways of getting to G7? Use the grid at right to help record your results.

This activity provides a link between two apparently separate ideas — combinations and Pascal's triangle. Let us consider Pascal's triangle and then look at its connection with combinations.



Pascal's triangle

The triangle below was named after the French mathematician Blaise Pascal. He was honoured for his application of the triangle to his studies in the area of probability.



Each element in Pascal's triangle can be calculated using combinations. For example, 10 is the 2nd element in the 5th row of Pascal's triangle; that is, ${}^5C_2 = 10$ (this assumes 1 is the zeroth (0th) element).

Compare the values from Pascal's triangle with the number of ways of getting to each intersection in the investigation 'Counting paths'. What do you notice?

Pascal's triangle shows that the *r*th element of the *n*th row of Pascal's triangle is given by ${}^nC_r$. It is assumed that the 1 at the beginning of each row is the 0th element.

Another application of combinations is the binomial theorem. This theorem gives a rule for expanding an expression such as $(a + b)^n$. Expanding expressions such as this may become quite difficult and time consuming using the usual methods of algebra.

$$(a + b)^1 = a + b$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

To use the binomial theorem you will need to recall the following conventions and terminology of algebra:

1. In the term $3a^2b$, '3' is called the coefficient of the term.
2. In the term a^2b, the coefficient of the term is 1 even though it is not written.
3. In the term $3a^2b$, the power of 'a' is 2 and the power of 'b' is 1.

The binomial theorem is defined by the rule:

$$\begin{aligned}(a + b)^n &= {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_r\,a^{n-r}b^r + \ldots + {}^nC_nb^n \\ &= a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots\ {}^nC_ra^{n-r}b^r + \ldots + b^n\end{aligned}$$

When expanding brackets which are in the form $(a + b)^n$ using the binomial theorem, recall:

1. The power of a in the first term of the expansion corresponds to the power of n and in each successive term decreases by 1 until it corresponds to the power of 0.
2. The power of b starts at 0 and in each successive term increases by 1 until it corresponds to the power of n.
3. The coefficient of the rth term is ${}^nC_r$.
4. The rth term is obtained by using ${}^nC_r a^{n-r} b^r$.

Again, this assumes that the initial term of the expansion is the 0th element.

The binomial theorem is particularly useful in probability calculations.

WORKED Example 21

Refer to Pascal's triangle on page 463 and answer the following questions.

a What number is in the 4th position in the 6th row?

b Complete the 7th row in Pascal's triangle.

c The numbers 7 and 21 occur side by side in the 7th row. What element in the 8th row occurs below and in between these numbers?

THINK	WRITE
a 1 Locate the 6th row and the 4th position. *Note*: Remember the 0th row is 1 and the first row is 1 1. In the 6th row the 1 on the left is in the 0th position.	**a** 6th row ⇒ 1 6 15 20 **15** 6 1
2 Answer the question.	The number in the 4th position in the 6th row is 15.
b 1 Write down the elements of the 6th row.	**b** 6th row ⇒ 1 6 15 20 15 6 1
2 Obtain the 7th row. (a) Place the number 1 at the beginning of the row. (b) Add the first 2 adjacent numbers from the 6th row (1 and 6). (c) Place this value next to the 1 on the new row and align the value so that it is in the middle of the 2 numbers (directly above) which created it. (d) Repeat this process with the next 2 adjacent numbers from the 6th row (6 and 15). (e) Once the sums of all adjacent pairs from the sixth row have been added, place a 1 at the end of the row.	7th row ⇒ 1 7 21 35 35 21 7 1
3 Answer the question.	The 7th row is 1 7 21 35 35 21 7 1.
c 1 Add the numbers 7 and 21 in order to obtain the element in the 8th row which occurs below and in between these numbers.	**c** 7 21 28
2 Answer the question.	The element in the 8th row which occurs below and in between 7 and 21 is 28.

WORKED Example 22

Use combinations to calculate the number in the 5th position in the 9th row of Pascal's triangle.

THINK	WRITE
1 Write down the combination rule.	${}^nC_r$
2 Substitute the values for n and r into the rule. *Note*: The row is represented by $n = 9$. The position is represented by $r = 5$.	${}^9C_5 = 126$
3 Evaluate using a calculator.	
4 Answer the question.	The value of the number in the 5th position in the 9th row is 126.

WORKED Example 23

Use the binomial theorem to expand $(a + 2)^4$.

THINK	WRITE
1 Write down the rule for the binomial theorem.	$(a + b)^n = a^n + {}^nC_1a^{n-1}b^1 + \ldots {}^nC_ra^{n-r}b^r + \ldots b^n$
2 Substitute the values for a, b and n into the rule: $a = a$, $b = 2$ and $n = 4$.	$(a + 2)^4 = a^4 + {}^4C_1a^32^1 + {}^4C_2a^22^2 + {}^4C_3a^12^3 + 2^4$
3 Simplify.	$= a^4 + 4 \times a^3 \times 2 + 6 \times a^2 \times 4 + 4 \times a \times 8 + 16$ $= a^4 + 8a^3 + 24a^2 + 32a + 16$

WORKED Example 24

What is the 4th term in the expansion of $(x + y)^7$?

THINK	WRITE
1 Write down the rule for the rth term. *Note*: The rule for the 4th term is obtained from the binomial theorem: $(a + b)^n = a^n + {}^nC_1a^{n-1}b^1 + \ldots {}^nC_ra^{n-r}b^r + \ldots b^n$	rth term $= {}^nC_ra^{n-r}b^r$
2 Substitute the values for a, b, n and r into the rule: $a = x$, $b = y$, $n = 7$ and $r = 4$.	${}^nC_ra^{n-r}b^r = {}^7C_4x^{7-4}y^4$
3 Simplify. *Note*: The 0th term corresponds to the first element of the expansion.	$= 35x^3y^4$
4 Answer the question.	The 4th term is equal to $35x^3y^4$.

Pigeonhole principle

Henri Poincaré, a famous mathematician, once described mathematics as 'the art of giving the same name to different things'. Consider three phenomena, which on the surface appear different — population growth, the value of investments and radioactive decay. Each can be described by one mathematical concept: exponential change. The mathematician gives three seemingly different things the same name.

The *pigeonhole principle* is a good example of how mathematics gives the same name to different things.

The pigeonhole principle states that:
If there are $n + 1$ pigeons to be placed in n pigeonholes then there is at least one pigeonhole with at least two pigeons in it.

In this statement:

1. Note the precise use of language; in particular the importance of the word 'least'.
2. Some may view the pigeonhole principle as an obvious statement, but used cleverly it is a powerful problem-solving tool.

WORKED Example 25

In a group of 13 people show that there are at least 2 whose birthday falls on the same month.

THINK	WRITE
① Think of each person as a pigeon and each month as a pigeonhole.	There are 12 months and 13 people.
② If there are 13 pigeons to be placed in 12 holes at least one hole must contain at least two pigeons.	Using the pigeonhole principle: 13 people to be assigned to 12 months. At least one month must contain two people. That is, at least two people have birthdays falling on the same month.

Generalised pigeonhole principle:
If there are $nk + 1$ pigeons to be placed in n pigeonholes then there is at least one pigeonhole with at least $k + 1$ pigeons in it.

WORKED Example 26

In a group of 37 people show that there are at least 4 whose birthdays lie in the same month.

THINK	WRITE
① Think of each person as a pigeon and each month as a pigeonhole.	There are 12 months and 37 people.
② Use the generalised pigeonhole principle.	Using the generalised pigeonhole principle: 37 people to be assigned to 12 months.
③ $nk + 1$ pigeons to be allocated to n holes; $n = 12 \rightarrow k = 3$	The value of n is 12 and k is 3. So at least one month has 4 people in it. That is, at least 4 people have birthdays falling on the same month.

WORKED Example 27

On resuming school after the Christmas vacation, many of the 22 teachers of Eastern High School exchanged handshakes. Mr Yisit, the social science teacher said 'Isn't that unusual — with all the handshaking, no two people shook hands the same number of times'.

Not wanting to spoil the fun, the mathematics teacher, Mrs Pigeon said respectfully, 'I am afraid you must have counted incorrectly. What you say is not possible.'

How can Mrs Pigeon make this statement?

THINK	WRITE
1 Think of the possible number of handshakes by a person as a pigeonhole.	For each person there are 22 possible numbers of handshakes; that is 0 to 21.
2 If two or more people have 0 handshakes, the problem is solved. Consider the cases where there is 1 person with 0 handshakes or 0 persons with 0 handshakes.	1 person with 0 handshakes: If there is 1 person with 0 handshakes there can be no person with 21 handshakes. Thus there are 21 people to be assigned to 20 pigeonholes. Therefore there must be at least one pigeonhole with two people in it. 0 people with 0 handshakes: If there is no person with 0 handshakes there are 22 people to be assigned to 21 pigeonholes. Therefore there must be at least one pigeonhole with two people in it (at least two people have made the same number of handshakes).
3 Conclude using a sentence.	Thus, there are at least two people who have made the same number of handshakes.

remember

1. Pascal's triangle shows that the rth element of the nth row of Pascal's triangle is given by ${}^nC_r$.
2. Each new row in Pascal's triangle is obtained by first placing a 1 at the beginning and end of the row and then adding adjacent entries from the previous row.
3. Row 1 is the row containing the elements '1 and 1'.
4. The '1' on the left-hand side of each row is in the 0th position of that row.
5. The binomial theorem is defined by the rule:

$$(a+b)^n = \underbrace{a^n}_{\text{0th term}} + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + \underbrace{{}^nC_ra^{n-r}b^r}_{r\text{th term}} + \ldots + b^n$$

6. When expanding brackets which are in the form $(a+b)^n$ using the binomial theorem, recall:
   (a) The power of a in the first term of the expansion corresponds to the power of n and in each successive term decreases by 1 until it corresponds to the power of 0.
   (b) The power of b starts at 0 and in each successive term increases by 1 until it corresponds to the power of n.
   (c) The coefficient of the rth term is ${}^nC_r$.
   (d) The rth term is obtained by using ${}^nC_ra^{n-r}b^r$.
   Again, this assumes that the initial term of the expansion is the 0th element.
7. The pigeonhole principle: If there are $n+1$ pigeons to be placed in n pigeonholes then there is at least one pigeonhole with at least two pigeons in it.

EXERCISE 10F Pascal's triangle, the binomial theorem and the pigeonhole principle

1 Write the first 8 rows in Pascal's triangle.

EXCEL Spreadsheet Pascal's triangle

WORKED Example 21

2 Refer to Pascal's triangle on page 463 and answer the following questions.
- **a** What number is in the 4th position in the 8th row?
- **b** Complete the 9th row in Pascal's triangle.
- **c** If 9 and 36 occur side by side in the 9th row, what element in the 10th row occurs below and in between these numbers?

WORKED Example 22

3 Use combinations to:
- **a** calculate the number in the 7th position of the 8th row of Pascal's triangle
- **b** calculate the number in the 9th position of the 12th row of Pascal's triangle
- **c** generate the 10th row of Pascal's triangle.

Mathcad Binomial theorem

WORKED Example 23

4 Use the binomial theorem to expand:
a $(x+y)^2$ **b** $(n+m)^3$ **c** $(a+3)^4$

WORKED Example 24

5 **a** What is the 4th term in the expansion of $(x + 2)^5$?
b What is the 3rd term in the expansion of $(p + q)^8$?
c What is the 7th term in the expansion of $(x + 2)^9$?

6 **multiple choice**

x

1 9 36 84 126 126 84 36 9 1

A row of Pascal's triangle is given above. What number is located at position x?

A 8 **B** 28 **C** 45 **D** 120 **E** 136

7 **multiple choice**

$16x^3$ is definitely a term in the binomial expansion of:

A $(x + 2)^3$ **B** $(x + 4)^3$ **C** $(x + 2)^4$ **D** $(x + 4)^4$ **E** $(x + 2)^5$

8 **a** In Pascal's triangle, calculate the sum of all elements in the:
i 0th row **ii** 1st row **iii** 2nd row **iv** 3rd row
v 4th row **vi** 5th row

b **i** What do you notice?
ii Complete the statement: 'The sum of the elements in the *n*th row of Pascal's triangle is . . .'

9 Use the result from question **8** to deduce a simple way of calculating:

$$^6C_0 + {^6C_1} + {^6C_2} + {^6C_3} + {^6C_4} + {^6C_5} + {^6C_6}$$

WORKED Example 25

10 In a cricket team consisting of 11 players, show that there are at least 2 whose phone numbers have the same last digit.

11 Two whole numbers add to give 21. Show that at least one of the numbers is greater than 10.

WORKED Example 26

12 A squad of 10 netballers is asked to nominate when they can attend training. They can choose Tuesday only, Thursday only or Tuesday and Thursday. Show that there is at least one group of at least 3 players who agree with one of these options.

13 S&M lollies come in five great colours — green, red, brown, yellow and blue. How many S&Ms do I need to select to be sure I have 6 of the same colour?

14 The new model WBM roadster comes in burgundy, blue or yellow with white or black trim. That is, the vehicle can be burgundy with white or burgundy with black and so on. How many vehicles need to be chosen to ensure at least 3 have the same colour combination?

15 Is it possible to show that in a group of 13 people, there are at least 2 whose birthdays fall in February?

WORKED Example 27

16 Nineteen netball teams entered the annual state championships. However, it rained frequently and not all games were completed. No team played the same team more than once. Mrs Organisit complained that the carnival was ruined and that no two teams had played the same number of games. Show that she is incorrect in at least part of her statement and that at least two teams played the same number of games.

History of mathematics

BLAISE PASCAL — (1623–1662)

During his life . . .

Construction of the Taj Mahal is started.

Rembrandt completes many of his famous paintings.

Oliver Cromwell governs England.

Blaise Pascal was a French mathematician and physicist who studied combinatorics and developed the theory of probability.

He was born in the town of Clermont in France. His father was a taxation officer. His mother died when he was only 4. Pascal was a sickly child and so was not sent to school initially but was educated at home by his father. Because he was not healthy his father forbad him from studying mathematics. It took about 5 years before Blaise could convince his father to let him try.

When Blaise was 16, his father was in trouble with the courts because he would not set any more taxes. He had to leave Paris, and the family moved to Rouen.

Blaise Pascal discovered and proved a major theorem of geometry when he was only 16 years old. This theorem was about the intersections of points on a conic plane.

When he was 18 he became very ill. He eventually recovered, after being temporarily paralysed and close to death. After this scare he became very religious and started to study philosophy and religion. His research into mathematics and science often conflicted with his religious beliefs.

At age 19, Pascal invented a calculating machine that could do simple addition and subtraction. He sold many of these machines and they were so well made that some still exist today.

He demonstrated that air pressure decreases with height by taking accurate measurements at various levels on the side of the Puy de Dôme mountain. He persuaded his brother to climb the mountain and take measurements using a heavy barometer.

Like many mathematicians, Blaise Pascal had arguments with other mathematicians, including René Descartes, who came to visit him. Descartes did not believe that Pascal was capable of such difficult mathematics and claimed that Pascal had stolen some of his ideas from Descartes himself. Blaise Pascal developed the pattern of numbers now known as *Pascal's triangle* that is used in probability, permutations and combinations.

When Blaise Pascal's father died, his sister went into a monastery and he was left to live free of family and spiritual conflicts. His health improved and he took up an active social life including gambling and driving a fast, horse-drawn carriage!

In late 1654 he was involved in an accident. His horses went over the edge of a bridge and were killed, but he survived. Pascal was shaken up by this and again saw the event as a message from God. In 1655 he moved in with his married sister. Later that year, Pascal became ill and eventually died from the effects of a brain tumour and stomach ulcer in 1662.

The computer language 'Pascal' is named after him.

Questions

1. How old was Pascal when he proved his theorem on conics?
2. What did he develop at age 19 that earned him a lot of money?
3. Upon which mountain was his work on air pressure done and who did the real work?
4. What is 'Pascal's triangle' used for?
5. What did he die from?

summary

The addition and multiplication principles

- Combinatorics is often called ‘counting’ and deals with counting the number of ways in which activities or events can happen.
- The multiplication principle should be used when there are two operations or events (say, A and B) where one event is followed by the other. It states that: If there are n ways of performing operation A and m ways of performing operation B, then there are $n \times m$ ways of performing A *and* B.
- The addition principle should be used when two distinct operations or events occur in which one event is not followed by another. It states that: If there are n ways of performing operation A and m ways of performing operation B then there are $n + m$ ways of performing A *or* B.
- A selection where order is important is called an *arrangement*.

Factorials and permutations

- The number of ways in which n distinct objects may be arranged is $n!$ (n factorial) where:
  $n! = n \times (n - 1) \times (n - 2) \times (n - 3) \times \ldots \times 3 \times 2 \times 1$
- $0! = 1$
- $1! = 1$
- The number of different arrangements or permutations when r things are chosen from n things and order is important is given by the rule ${}^nP_r$, where:
- ${}^nP_r = \dfrac{n!}{(n-r)!}$
- ${}^nP_n = n!$
- ${}^nP_0 = 1$
- The number of different ways in which n people can be seated, r at a time, in a circle is:

$$\frac{{}^nP_r}{r}.$$

Arrangements involving restrictions and like objects

- The number of different ways of arranging n things made up of groups of indistinguishable things, n_1 in the first group, n_2 in the second group and so on is:

$$\frac{n!}{n_1!n_2!n_3!\ldots n_r!}.$$

- When restrictions apply to arrangements use the multiplication and addition principles as well as ${}^nP_r$.

Combinations

- The number of ways of selecting r things from n things when order is not important is ${}^nC_r$.
- $${}^nC_r = \frac{{}^nP_r}{r!}$$
  $$= \frac{n!}{(n-r)!r!}$$
- ${}^nC_r$ may also be written as $\binom{n}{r}$.
- ${}^nC_r = {}^nC_{n-r}$

Applications of permutations and combinations

- Permutations are used to count when order is important.
- Combinations are used to count when order is not important.

Pascal's triangle and the binomial theorem

- Pascal's triangle shows that the rth element of the nth row of Pascal's triangle is given by ${}^nC_r$.
- Each new row in Pascal's triangle is obtained by first placing a 1 at the beginning and end of the row and then adding adjacent entries from the previous row.
- The top row is row 0.
- Row 1 is the row containing the elements '1 1'.
- The '1' on the left-hand side of each row is in the 0th position of that row.
- The binomial theorem is defined by the rule:
- $$(a+b)^n = \underbrace{a^n}_{\text{0th term}} + {}^nC_1a^{n-1}b + {}^nC_2\,a^{n-2}b^2 + \ldots + \underbrace{{}^nC_ra^{n-r}b^r}_{r\text{th term}} + \ldots + b^n$$
- When expanding brackets which are in the form $(a+b)^n$ using the binomial theorem, recall:
  1. The power of a in the first term of the expansion corresponds to the power of n and in each successive term decreases by 1 until it corresponds to the power of 0.
  2. The power of b starts at 0 and in each successive term increases by 1 until it corresponds to the power of n.
  3. The coefficient of the rth term is ${}^nC_r$.
  4. The rth term is obtained by using ${}^nC_ra^{n-r}b^r$.
- Points 3 and 4 both assume that the initial term of the expansion is the 0th element.

The pigeonhole principle

- If there are $n + 1$ pigeons to be placed in n pigeonholes then there is at least one pigeonhole with at least two pigeons in it.
- The generalised pigeonhole principle:
  If there are $nk + 1$ pigeons to be placed in n pigeonholes then there is at least one pigeonhole with at least $k + 1$ pigeons in it.

CHAPTER review

1 **multiple choice** 10A

Barbie's wardrobe consists of 5 different tops, 4 different skirts and 3 different pairs of shoes. The number of different outfits Barbie can wear is:

A 5 **B** 12 **C** 60 **D** 80 **E** 120

2 **multiple choice** 10A

How many different 3-digit numbers can be made from the numbers 1, 3, 5, 7 and 9 if the numbers can be repeated?

A 60 **B** 125 **C** 243 **D** 729 **E** 999

3 **multiple choice** 10A

There are 7 candidates seeking election to the positions of either president or secretary of the Soccer Club Committee. If one of these candidates, George, is to be either president or secretary, in how many ways can positions be filled?

A 12 **B** 21 **C** 42 **D** 49 **E** 56

4 How many numbers less than 4000 can be made using the digits 1, 2, 3, 5, 7 and 9 if: 10A

a repetition is not permitted?

b repetition is permitted?

5 **multiple choice** 10B

The permutation ${}^{9}P_{6}$ is equal to:

A $9 \times 8 \times 7$ **B** $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3$ **C** $\frac{9!}{6!}$ **D** $\frac{9!}{3!}$ **E** $\frac{9!}{4!}$

6 **multiple choice** 10B

There are 12 horses in a race. In how many different ways can the 1st, 2nd and 3rd positions be filled?

A ${}^{12}P_{3}$ **B** 12^{3} **C** 3^{12} **D** ${}^{12}C_{3}$ **E** ${}^{12}C_{12}$

10B

7 **multiple choice**

A round table seats 5 people. From a group of 8 people, in how many ways can 5 people be seated at the table:

A $\frac{9!}{6!}$ **B** ${}^{8}P_{5}$ **C** $\frac{{}^{8}P_{5}}{5}$ **D** $\frac{{}^{8}P_{5}}{5!}$ **E** ${}^{8}C_{5}$

10B

8 Use your calculator to place these in ascending order: ${}^{19}P_{6}$, ${}^{12}P_{9}$, ${}^{2000}P_{2}$.

10C

9 **multiple choice**

How many different arrangements can be made using the 8 letters of the word NONSENSE?

A 1680 **B** 2520 **C** 3360 **D** 5040 **E** 40 320

10C

10 How many different arrangements of 4 letters can be made from the letters of the word PILL?

10D

11 **multiple choice**

Which of the following is equivalent to ${}^{8}C_{2}$?

A $\frac{{}^{6}P_{2}}{2!}$ **B** $\frac{{}^{8}P_{6}}{2!}$ **C** $\frac{{}^{8}P_{2}}{6!2!}$ **D** $\frac{{}^{8}P_{2}}{6!}$ **E** $\frac{{}^{8}P_{2}}{2!}$

10D

12 **multiple choice**

A committee of 4 men and 3 women is to be formed from 5 men and 8 women. In how many ways can this be done?

A 61 **B** 280 **C** 1320 **D** 20 160 **E** 40 320

10D

13 Use your calculator to place these in ascending order: ${}^{19}C_{6}$, ${}^{22}C_{15}$, ${}^{2000}C_{2}$.

10D

14 A committee of 3 men and 4 women is to be formed from 7 men and 5 women. In how many ways can this be done?

10D

15 Two cards are dealt from a pack of 52. What is the number of ways that:

a both are black?
b both are aces?
c the cards are of different colours?

10E

16 **multiple choice**

A cycling team of 3 riders is to be chosen from a squad of 8 riders. In how many ways can this be done if one particular rider, Jorge, must be in the team?

A 56 **B** 336 **C** 21 **D** 210 **E** 420

17 A ward in a city hospital has 15 nurses due to work on Friday. There are 3 shifts that need to be staffed by 5 nurses on each shift. How many different arrangements for staffing these 3 shifts are possible, assuming that each nurse only works 1 shift? **10E**

18 **multiple choice** **10F**

What is the 4th term in the expansion of $(p + 1)^7$?

A p^4 **B** $35p^3$ **C** $35p^4$ **D** $21p^3$ **E** $21p^4$

19 **multiple choice** **10F**

A row of Pascal's triangle is given below. What number is located at position x?

1 9 36 84 126 126 84 36 9 1

x

A 48 **B** 120 **C** 56 **D** 210 **E** 252

20 **a** In the 10th row of Pascal's triangle, what is the 6th entry? **10F**

b Write the 10th row of Pascal's triangle using combinations.

c What is the sum of the elements of the 10th row?

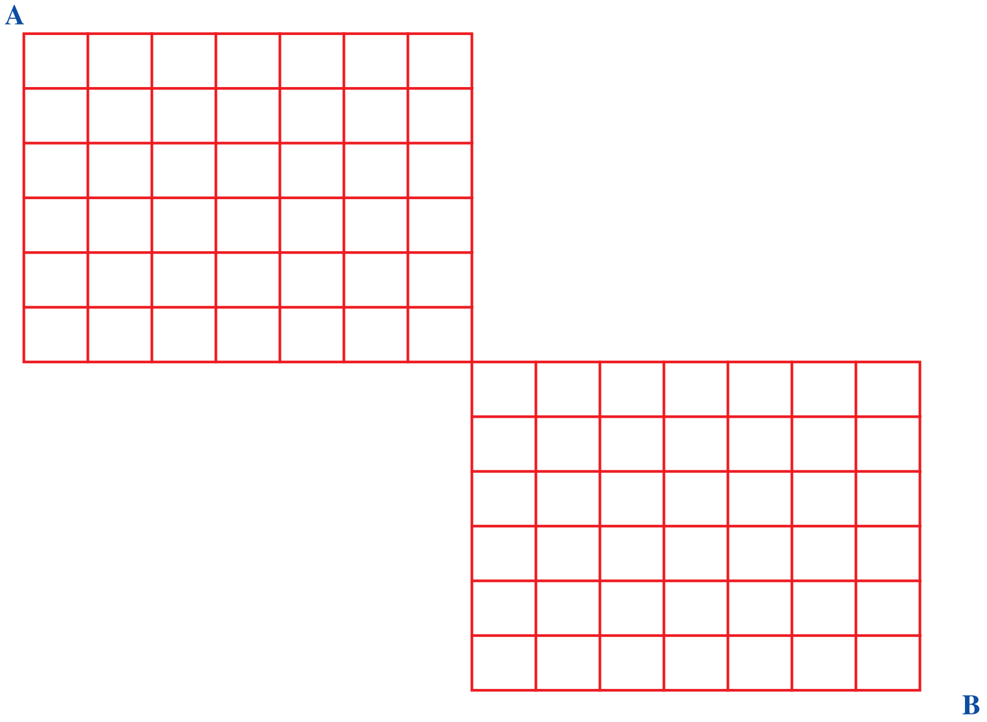
21 In the expansion of $(x + 3)^{10}$, what is the: **10F**

a 2nd term? **b** 3rd term? **c** 9th term?

Modelling and application

1 Assume that car number plates are sequenced as follows: DLV334 → DLV335 → ... DLV339 → DLV340 → ... DLV999 → DLW000 and so on. Using this sequence, how many number plates are there between DLV334 and DNU211 inclusive?

2 How many paths are there from A to B if you are only allowed to move either down or to the right on the lines of the grid?



test yourself CHAPTER 10

3 Poker is a card game in which initially each person is dealt 5 cards.

a How many different hands are possible? (Order is not important.)

b How many hands contain only diamonds?

c How many hands contain only red cards?

d When Wild Bill Hickock died at Deadwood, Dakota he was holding in his hand 2 pairs — aces and eights. This is called the *dead man's hand*. In how many ways can you be dealt the dead man's hand?