S6 PURE MATHS NOTES

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S.6 Notes in Pure Mathematics

H3 Proof by induction (mathematical induction)

Mathematical Induction is used to prove formulae that may not be directly proven.

The principle of induction:

i. Test the validity of the formulae for two lowest known values normally 1 or 2 or a given starting point.

ii. Assume the formula is true for some value of $n = k$

iii. Prove that it is also true for $n = k+1$. This is achieved when the result appears as if the n in the formula is replaced by $k+1$

iv. Draw the conclusion for the range of n for which the formula is true.

Eg 1 Prove the induction ~~that~~

$$1 + 2 + 3 + \cdots + n = \frac{n}{2}(n+1)$$

Soln:

Note that this is a proof not a derivation.

If $n = 1$; LHS $= 1$, RHS $= \frac{1}{2}(1+1) = 1$

$\therefore n = 1$ holds.

$n = 2$; LHS $= 1 + 2 = 3$, RHS $= \frac{2}{2}(2+1) = 3$

$\therefore n = 2$ holds

Assume $n = k$ holds

$\Rightarrow 1 + 2 + 3 + \cdots + k = \frac{k}{2}(k+1)$

For $n = k+1$;

$$\underbrace{1+2+3+\cdots+k}_{\text{Sum of previous } k \text{ terms}} + \overbrace{(k+1)}^{\text{Increase in previous sum}} = \frac{k(k+1)}{2} + (k+1)$$

$$= (k+1)\left[\frac{k}{2} + 1\right]$$

$$= \frac{(k+1)(k+2)}{2}$$

$\therefore n = k+1$ holds.

Thus the formula is true for all values of $n = 1, 2, 3,$

2. Prove by induction that

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{(n+1)}$$

If $n = 1$, $L.H.S = \frac{1}{1\times 2} = \frac{1}{2}$

$R.H.S = \frac{1}{1+1} = \frac{1}{2}$

$\therefore n = 1$ holds.

For $n = 2$; $L.H.S = \frac{1}{1\times 2} + \frac{1}{2\times 3} = \frac{2}{3}$

$R.H.S = \frac{2}{2+1} = \frac{2}{3}$

$\therefore n = 2$ holds

Assume $n = k$ holds

$$\Rightarrow \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

the R.H.S as a multiple of 8 or rearrange the R.H.S and create a multiple of 8 as shown below:

$$f(k+1) - f(k) = 9^{(k+1)} + 7 - (9^k + 7)$$

$$= 9^{(k+1)} - 9^k = 9.9^k - 9^k$$

$$= 8.9^k = 8B$$

$$\therefore f(k+1) = f(k) + 8B = 8A + 8B = 8C.$$

or $f(k+1) = 9^{(k+1)} + 7$

$$= \underbrace{9.9^k} + 7 = \underbrace{8.9^k + 9^k} + 7$$

$$= 8.9^k + 8A$$

$$= 8B + 8A = 8C$$

$\therefore n = k+1$ holds

Thus the formula is true for all values of $n = 1, 2, 3, \ldots$

4. Prove by induction that $\frac{d(x^n)}{dx} = n x^{(n-1)}$

For $=1$; $\frac{d x^1}{dx} = 1$

Let $n = k$ holds; $\Rightarrow \frac{d(x^k)}{dx} = k x^{(k-1)}$

For $n = k+1$; $\frac{d(x^{k+1})}{dx} = \frac{d(x^1 . x^k)}{dx}$

$$= x^k \frac{d(x^1)}{dx} + x^1 \frac{d(x^k)}{dx} = x^k + x^1 . k x^{k-1}$$

product rule

$$= x^k + kx^k = (1+k)x^k.$$

$\therefore n = k$ holds.
Thus the formula is true for all values of $n = 1, 2, 3, \cdots$
Note that $n = 2$ was not tested since it wasn't applied in the later ~~proof~~ stage of differentiation of a product.

Exercise: Prove by induction that
i. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

ii. $1\times2 + 2\times3 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$

(iii) $3^n + 5(4^n) - 1$ is a multiple of 11 for all values of $n = 1, 2, \cdots$

H4 <u>BINOMIAL EXPANSION</u>
A binomial expression has only two terms. The expansion of the terms has basically three methods, ie
- direct expansion (known)
- Pascal's triangle
- Binomial theorem

a Pascal's triangle:
This is a triangle that gives the number patterns of the coefficients of the terms of a binomial expansion
Consider the expansion of $(a+b)^n$ for $n = 0, 1, 2 \& 3.$

$(a+b)^0 = 1$

$(a+b)^1 = a + b$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Extracting the coefficients of $(a+b)^n$ we have:

$n = 0$	1
$n = 1$	1 1
$n = 2$	1 2 1
$n = 3$	1 3 3 1
$n = 4$	1 4 6 4 1
$n = 5$	1 5 10 10 5 1
$n = 6$	1 6 15 20 15 6 1
$n = 7$	1 7 21 35 35 21 7 1
$n = 8$	1 8 28 56 70 56 28 8 1

Generated ($n = 4$ to $n = 8$)

Note the following

- The first and the last coeff. is 1.
- The next coeff. are generated from the previous
- The highest power of a is n and decreases to zero and the power of b starts from 0 and increases to n.
- Every term is of the form ka^pb^q where $p+q = n$
- There are $n+1$ terms.

Eg 1. Expand $(a+b)^4$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

2. Find the 4^{th} term in the expansion of $(2x+3)^8$.

Coeff. of the 4^{th} is 56, $a = 2x$, $b = 3$

$$\therefore T = 56(2x)^5(3)^3 = 48{,}384x^5$$

3. Expand $(3x-y)^5$

Coeff. are $1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$, $a = 3x$, $b = (-y)$

$$\therefore (3x-y)^5 = (3x)^5 + 5(3x)^4(-y) + 10(3x)^3(-y)^2 + 10(3x)^2(-y)^3 + 5(3x)(-y)^4 + (-y)^5$$

$$= 243x^5 - 405x^4y + 270x^3y^2 - 90x^2y^3 + 15xy^4 - y^5.$$

4. Simplify $(1+\sqrt{2})^3 - (1-\sqrt{2})^3$

$$= 1 + 3(\sqrt{2}) + 3(\sqrt{2})^2 + (\sqrt{2})^3 - [1 + 3(-\sqrt{2}) + 3(-\sqrt{2})^2 + (-\sqrt{2})^3]$$

$$= 1 + 3\sqrt{2} + 6 + 2\sqrt{2} - 1 + 3\sqrt{2} - 6 + 2\sqrt{2}$$

$$= 6\sqrt{2} + 4\sqrt{2} = 10\sqrt{2}.$$

Exercise

1. Expand: i. $(x+2y)^4$, ii. $(2x+3y)^4$

iii. $(2x+1)^3$, iv. $(x-1)^5$

vi. Expand $\left(1+\frac{x}{4}\right)^4$. Taking the first three terms, put $x = 0.001$ and find the value of $(2.001)^5$ correct to 5 decimal places

b. Binomial Theorem:
When the power becomes large, Pascal's triangle becomes a tedious method to use. A careful look at at the coefficients indicate that they are actually combinations.
For instance, if $n = 4$,

$^4C_0 = 1$
$^4C_1 = 4$
$^4C_2 = 6$
$^4C_3 = 4$
$^4C_4 = 1$

These are the coefficients of $(a+b)^4$.

Thus from the binomial Theorem,

$$(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{(n-1)} b + {}^nC_2 a^{(n-2)} b^2 + \cdots + {}^nC_r a^{(n-r)} b^r + \cdots + {}^nC_n a^0 b^n$$

$$= a^n + {}^nC_1 a^{(n-1)} b + {}^nC_2 a^{(n-2)} b^2 + \cdots + {}^nC_r a^{(n-r)} b^r + \cdots + b^n$$

NB: * For the 1st term: $r = 0$, 2nd term $r = 1$
3rd term $r = 2$, etc for the m^{th} term
$r = m-1$.

- Any term can be written down at once without first obtaining all the expansions.
- If only the first few terms are required, we have

$$(a+b)^n = a^n + n a^{(n-1)} b + \frac{n(n-1)}{2!} a^{(n-2)} b^2 + \frac{n(n-1)(n-2)}{3!} a^{(n-3)} b^3 + \cdots$$

because $^nC_1 = \frac{n!}{(n-1)!1!} = n$

$$^nC_2 = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2!}$$

$$^nC_3 = \frac{n!}{(n-3)!3!} = \frac{n(n-1)(n-2)}{3!}$$

$$\vdots \qquad\qquad \vdots$$

Eg 1 Write down the term in x^5 if $(x+2)^8$ is expanded.

Soln:

$(x+2)^8 \Rightarrow n=8,\ a=x,\ b=2$

Since power of x is 5 $\Rightarrow$ power of 2 is 3 $\therefore r=3$

$$\therefore T = {}^8C_3 \cdot x^5 \cdot (2)^3 = 448x^5$$

ii. Find the coefficient of the term in u^3 in the expansion of $(3u-2)^5$

Power of u is 3 $\Rightarrow$ power of (-2) is ~~2~~ 2

$\therefore r = 2$.

$\therefore T = {}^5C_2 (3u)^3(-2)^2 = 10 \times 27 \times u^3 \times 4$

$= 1080u^3$

$\therefore$ Coeff. $= 1080$.

(iii) Find the 4^{th} term in the expansion of $(2x-3)^7$.

Soln:

Recall that m^{th} term $\Rightarrow r = m-1$

$\therefore$ for the 4^{th} term, $r = 3$

$\therefore T = {}^7C_3 (2x)^4(-3)^3$

$= 35 \times 16 \times (-27) x^4$

$= -15120x^4$.

iv. Write down the term involving $x^4\left(\frac{1}{x}\right)^2$ in the expansion of $\left(x + \frac{1}{x}\right)^6$

Soln: $r = 2 \Rightarrow T = {}^6C_2 (x)^4\left(\frac{1}{x}\right)^2$

$= \ldots^2$

From the above expansion determine the constant term.

Soln:

This is a term that doesnt contain x. $\Rightarrow$ the power of x and that of $\frac{1}{x}$ ~~one~~ is the same

$$\therefore T = {}^6C_r\, x^{(6-r)} \cdot \left(\frac{1}{x}\right)^r$$

$$\therefore (6-r) = r \quad \therefore 2r = 6 \quad \therefore r = 3$$

$$\therefore T = {}^6C_3\, x^3 \cdot \left(\frac{1}{x}\right)^3$$

$$= 20.$$

Exercise: From Backhouse 1

1: c, d, 2: b, 3: b, 4: d, 6: b, 7: b, 9,
11: c, e, 12: c, 13: a, d, g

c. Binomial theorem for any index

From

$$(a+b)^n = a^n + na^{(n-1)}b + \frac{n(n-1)}{2!}a^{(n-2)}b^2 + \cdots + b^n$$

If $a = 1$ and $b = x$, we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + x^n.$$

If n is a positive integer, the above expansion is true for any value of x and it is finite.

For any other value of n, the expansion is an infinite power series of x and it is true only provided $|x| < 1$ i.e $-1 < x < 1$

Thus $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$

provided $-1 < x < 1$

This means you cannot directly use to evaluate $(1+4)^{-2}$ but you can evaluate $(1+0.4)^{-2}$ using the expansion.

Eg Find the first four terms in the expansion of $(1+x)^{\frac{1}{2}}$ and state the range for which the expansion is valid.

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{-1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{3!}x^3$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \cdots$$

Valid for $-1 < x < 1$

ii. Expand $(1-3x)^{3/2}$ upto the term in x^3. State the range for which the expansion is valid.

$$(1-3x)^{3/2} = 1 + \frac{3}{2}(-3x) + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)(-3x)^2}{2!} +$$

$$\frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{3!}(-3x)^2$$

$$= 1 - \frac{9}{2}x + \frac{27}{8}x^2 - \frac{9x^3}{16}$$

Valid for $-1 < 3x < 1$, $\frac{-1}{3} < x < \frac{1}{3}$.

(iii) Expand $\frac{1}{(1-2x)^2}$ in ascending powers of x upto the fourth term

~~powers of x upto the 4th term.~~

Soln:

$$\frac{1}{(1-2x)^2} = (1-2x)^{-2}$$

$$= 1 + (-2)(-2x) + \frac{(-2)(-2-1)}{2!}(-2x)^2 +$$

$$\frac{(-2)(-2-1)(-2-2)}{3!}(-2x)^3 + \cdots$$

$$= 1 + 4x + 12x^2 + 32x^3.$$

Valid for $-1 < 2x < 1 \Rightarrow -\frac{1}{2} < x < \frac{1}{2}$.

NB. Showing the ~~range for~~ validity is part of the solution.

iv Expand $(3+x)^{\frac{1}{3}}$ upto the term in x^2 and state the range of validity ~~for~~ of x in the expansion

The binomial term must be in the form $(1+x)$. Thus we shall factor out 3 before the expansion.

$$(3+x)^{\frac{1}{3}} = 3^{\frac{1}{3}}\left(1+\frac{x}{3}\right)^{\frac{1}{3}}$$

$$= 3^{\frac{1}{3}}\left[1 + \frac{1}{3}\left(\frac{x}{3}\right) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{x}{3}\right)^2\right]$$

$$= 3^{\frac{1}{3}}\left[1 + \frac{x}{9} - \frac{x^2}{81}\right]$$

$\therefore (3+x)^{\frac{1}{3}} = \sqrt[3]{3} + \frac{\sqrt[3]{3}\,x}{9} - \frac{\sqrt[3]{3}\,x^2}{81}$

Valid for $-1 < \frac{x}{3} < 1$

$\therefore -3 < x < 3.$

v. Find the value of $\frac{1}{(1\cdot 02)^2}$ correct to 4 decimal places:

$\frac{1}{(1\cdot 02)^2} = (1+0\cdot 02)^{-2}$

$= 1 + (-2)(0\cdot 02) + \frac{(-2)(-3)}{2!}(0\cdot 02)^2$

$+ \frac{(-2)(-3)(-4)}{3!}(0\cdot 02)^3$

$= 1 - 0\cdot 04 + 0\cdot 0012 - \underline{0\cdot 000032}$

$= 0\cdot 9612$

We could work to only the 3rd term. Check with the exact value from CALC.

vi. Find the first ~~four~~ 3 terms of the expansion $\frac{(1+2x)^2}{(2-x)^2}$ and state the range for validity

Soln

$(1+2x)^2 = 1 + 4x + 4x^2$

$(2-x)^2 = 2^2\left(1 - \frac{x}{2}\right)^2$

$$\therefore \frac{(1+2x)^2}{(2-x)^2} = \frac{(1+4x+4x^2)}{4}\left(1-\frac{x}{2}\right)^{-2}$$

$$= \frac{(1+4x+4x^2)}{4}\left(1+(-2)\left(\frac{-x}{2}\right)+\frac{(-2)(-3)}{2!}\left(\frac{-x}{2}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{-x}{2}\right)^3\right]$$

$$= \frac{(1+4x+4x^2)}{4}\left\{1+x+\frac{3}{4}x^2+\frac{x^3}{4}\right\}$$

$$= \frac{1}{4}\left\{1+5x+\frac{19}{4}x^2+5x^3+3x^3\right\}$$

$$= \frac{1}{4}+\frac{5}{4}x+\frac{19}{16}x^2+2x^3$$

Validity: The numerator is valid for any value of x. The denominator is valid for $-1 < \frac{x}{2} < 1 \Rightarrow -2 < x < 2$

The power series must satisfy both ranges, the common range is therefore $-2 < x < 2$.

VII. Expand $(1-8x)^{\frac{1}{2}}$ in ascending powers of x upto the fourth term. By putting $x = \frac{1}{100}$, find $\sqrt{23}$ correct to five significant figures

$$(1-8x)^{\frac{1}{2}} = 1+\frac{1}{2}(-8x)+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)}{2!}(-8x)^2+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)(-8x)^3}{3!}$$

$$= 1 - 4x - 8x^2 - 32x^3.$$

Valid for $-1 < 8x < 1 \Rightarrow -\frac{1}{8} < x < \frac{1}{8}$

$$x = \frac{1}{100}$$

$$\left(1 - 8\left(\frac{1}{100}\right)\right)^{\frac{1}{2}} = \left(1 - \frac{2}{25}\right)^{\frac{1}{2}}$$

$$= \left(\frac{23}{25}\right)^{\frac{1}{2}} = 1 - 4\left(\frac{1}{100}\right) - 8\left(\frac{1}{100}\right)^2 - 32\left(\frac{1}{100}\right)^3$$

$$\therefore \sqrt{23} = (0.959168) \times 5$$

$$= 4.79584$$

$$\underline{\underline{\simeq 4.7958}}$$

VIII. Expand $\frac{1}{1+x+2x^2}$ upto the term in x^3

$$\frac{1}{1+x+2x^2} = \left[1 + (x+2x^2)\right]^{-1}$$

$$= 1 + (-1)(x+2x^2) + \frac{(-1)(-2)}{2!}(x+2x^2)^2$$

$$= 1 - x - 2x^2 + x^2 + 4x^3 + 4x^4$$

$$= 1 - x - x^2 + 4x^3.$$ (Validity later)

Exercise

1. Expand the following in ascending powers of x upto the term in x^3 Stating the values of x for which the expansion is valid.

i. $(1+x)^{1/3}$, ii. $\frac{1}{(1+3x)}$, (iii). $\sqrt{1-x^2}$

iv. $\frac{1}{(3-x)^2}$, ii $\frac{(x+3)}{\sqrt[3]{(1-3x)}}$

2. Obtain the first four terms of $(1-16x)^{\frac{1}{4}}$. Substituting $x = \frac{1}{10,000}$ and use the first two terms to find $\sqrt[4]{39}$

3. Expand $\frac{1}{(1+2x+3x^2)^2}$ upto the term in x^3

d Binomial expansion for $|x| > 1$ ie $x < -1$ or $x > 1$

We factor out x from $(1+x)^n$ and obtain $x^n(1+\frac{1}{x})^n$.

Thus $\frac{1}{x}$ now lies between -1 and 1

So the expansion is obtained in ascending powers of $\frac{1}{x}$ or descending powers. From the question it should be easy to deduce whether $|x| < 1$ or $|x| > 1$.

Eg 1. Expand $(2+x)^{-2}$ in ascending powers of $\frac{1}{x}$ stating the range of values of x for which the expansion is valid (1st 3 terms)

Soln

Since the expansion is required in ascending powers of $\left(\frac{1}{x}\right)$ this implies that the x in $(2+x)^{-2}$ is much greater than 2, ∴ we factor out x instead of 2.

$\therefore (2+x)^{-2} = x^{-2}\left(1+\frac{2}{x}\right)^{-2}$. Terms within the brackets have been rearranged so that it is in the form $(1+x)^n$

$$\therefore (2+x)^{-2} = x^{-2}\left(1+\frac{2}{x}\right)^{-2}$$

$$= \frac{1}{x^2}\left\{1 + (-2)\left(\frac{2}{x}\right) + \frac{(-2)(-3)}{2!}\left(\frac{2}{x}\right)^2\right\}$$

$$= \frac{1}{x^2}\left[1 - \frac{4}{x} + \frac{12}{x^2}\right]$$

$$= \frac{1}{x^2} - \frac{4}{x^3} + \frac{12}{x^4}.$$

Valid for $\left|\frac{1}{x}\right| < 1 \Rightarrow |x| > 1$

$\therefore x < -1$ or $x > 1$.

ii. Expand $(3-x)^{-3}$ in ascending powers of $\frac{1}{x}$ upto the 3^{rd} term.

$$(3-x)^{-3} = (-x)^{-3}\left(1-\frac{3}{x}\right)^{-3}$$

$$= \frac{-1}{x^3}\left\{1 + (-3)\left(\frac{-3}{x}\right) + \frac{(-3)(-4)}{2!}\left(\frac{-3}{x}\right)^2\right\}$$

$$= -\frac{1}{x^3}\left\{1 - \frac{9}{x} + \frac{54}{x^2}\right\}$$

$$= -\frac{1}{x^3} + \frac{9}{x^4} - \frac{54}{x^5}$$

Valid for $\left|\frac{3}{x}\right| < 1$

$\Rightarrow |3| < |x| \therefore |x| > 3$

$\Rightarrow x < -3$ or $x > 3$.

Exercise 1. Expand the following in ascending powers of $\frac{1}{x}$ upto the third term; State the validity range.

i. $(1+3x)^{-2}$, ii. $(1-2x)^{-2}$,

iii. $\left(\frac{x+2}{x+1}\right)$

2. Expand $(x-2)^{1/2}$ in descending powers of x upto the 3^{rd} term. By substituting $x = 100$, evaluate $\sqrt{2}$ correct to 5 sf.

Classic

3. Use binomial theorem to find:
i. $\sqrt{9.09}$ correct to six decimal places

ii $\frac{1}{\sqrt{17}}$ correct to four decimal places.

H5. PARTIAL FRACTIONS:

These are fractions that add up to give a sing rational function.

Eg $$\underbrace{\frac{1}{(x+1)} + \frac{3}{(2-x)}}_{\text{Partial fractions}} = \frac{2-x+3x+3}{(x+1)(2-x)}$$
$$= \frac{2x+5}{2+x-x^2}$$

rational function

Our task here is to break down a rational function to partial fractions.

The numerators ~~powers~~ of the partial fractions depend the nature of the denominators. If the denominator is linear $(ax+b)$, the numerator is a constant (A).
If the denominator is a quadratic which is not factorisable (ax^2+bx+c), the numerator is linear $(Ax+B)$ and so on.
If the ~~numerator~~ denominator is ~~a multiple (the~~ factor has a power greater than 1, it is with all its powers starting from one.

Eg. $\frac{2}{(2+x)(x+1)} = \frac{A}{2+x} + \frac{B}{x+1}$

Both are linear and have a power of 1

ii $\frac{2}{(2+x)^2(x+1)} = \frac{A}{(2+x)} + \frac{B}{(2+x)^2} + \frac{C}{(x+1)}$

iii $\frac{2}{(2+x)^3(x+1)} = \frac{A}{(2+x)} + \frac{B}{(2+x)^2} + \frac{C}{(2+x)^3} + \frac{D}{(x+1)}$

NB

- The same applies for a repeated quadratic factor.
- ~~If~~ Only rational functions which are proper are broken down to partial fractions. If they are improper, we first divide using long division and express the proper fraction as partials
- A rational function $\frac{f(x)}{g(x)}$ is proper if the degree of $f(x)$ < degree of $g(x)$

otherwise it is improper

Eg $\frac{x^2+3x}{(x+2)(x-3)}$, $\frac{x^3}{(x+2)(x-3)}$

The constants in the numerators are obtained by comparing the coeff. of the L.H.S with that of the simplified R.H.S (Principle of undetermined coefficients) or substitution of some values of x which will ~~eliminate~~ eliminate some terms (Heaviside method)

Eg: Find constants below which makes the terms identical

$$(5x+3) \equiv Ax(x-3) + Bx(x-1) + C(x-1)(x+3)$$

Method 1: Expand the RHS and compare the coeff.

$$5x+3 = Ax^2 - 3Ax + Bx^2 - Bx + Cx^2 + 2Cx - 3C$$

Coeff. of x^2: $0 = A + B + C$

Coeff. of x: $5 = -3A - B + 2C$

Const: $3 = -3C$

$\Rightarrow C = -1$

Subst. $A + B = 1$

$-3A - B = 7$

Add: $-2A = +8 \quad \therefore A = -4$

$\Rightarrow B = +4 + 1 = 5$

$\therefore$ ~~A = 3, B = -2, C = -1~~ $\therefore A = -4,\ B = 5,\ C = -1$

Method 2: Since the R.H.S contains factors $(x-3)$, $(x+3)$ and $(x-1)$ which go to zero if we substitute x as $3, -3$ and 1 respectively, we have.

$x = 3$, $5(3)+3 = 0 + 6B + 12C$

$\therefore 3 = B + 2C$

$x = -3$, $-12 = +18A + 12B$

$\therefore -2 = 3A + 2B$

$x = 1$, $8 = -2A$ $\therefore A = -4$

Subst. $-2 = 3(-4) + 2B$

$\Rightarrow B = 5$

$\therefore 3 = 5 + 2C \Rightarrow C = -1.$

The two methods can be used together for quick results:

Eg 1. Express $\frac{x-11}{(x+3)(x-4)}$ as partial fractions.

Soln:

All the denominators are linear and the fraction is proper,

$$\therefore \frac{x-11}{(x+3)(x-4)} = \frac{A}{(x+3)} + \frac{B}{(x-4)}$$

$$\Rightarrow \quad = \frac{A(x-4) + B(x+3)}{(x+3)(x-4)}$$

Comparing the numerators, we have $x - 11 \equiv A(x-4) + B(x+3)$

Let $x = 4$, $\Rightarrow 4 - 11 = 0 + 7B$

$7B = -7$ $\therefore B = -1$

Classic

Let $x = -3$ $\quad \therefore -7A = -14$
$\Rightarrow -3-11 = -7A$

$\therefore A = 2$

$\therefore \dfrac{x-11}{(x+3)(x-4)} = \dfrac{2}{x+3} - \dfrac{1}{x-4}$

2. $\dfrac{7}{(2x-3)(x+2)} = \dfrac{A}{2x-3} + \dfrac{B}{(x+2)}$

$$= \frac{A(x+2) + B(2x-3)}{(2x-3)(x+2)}$$

$\therefore 7 = A(x+2) + B(2x-3)$

$x=-2;\ 7 = -7B \quad \therefore B = -1$

Comparing coeff. of x,

$0 = A + 2(-1) \Rightarrow A = 2$

$\therefore \dfrac{7}{(2x-3)(x+2)} = \dfrac{2}{(2x-3)} - \dfrac{1}{(x+2)}$

3. $\dfrac{5x^2+2}{(3x+1)(x+1)^2}$ This is ~~impro~~ proper but with a repeated factor.

$\therefore \dfrac{5x^2+2}{(3x+1)(x+1)^2} = \dfrac{A}{(3x+1)} + \dfrac{B}{(x+1)} + \dfrac{C}{(x+1)^2}$

$\therefore 5x^2+2 = A(x+1)^2 + B(x+1)(3x+1) + C(3x+1)$

$x=0;\ 2 = A + B + C$

$x=-1;\ 7 = -2C \Rightarrow C = -\frac{7}{2}$

Coeff of x: $0 = 2A + 4B + 3C$

$\Rightarrow 2 = A + B - \frac{7}{2} \quad \therefore A+B = \frac{11}{2}$

$2A + \cancel{4}4B + 3\left(\frac{-7}{2}\right) = 0$

$\Rightarrow 2A + 4B = \frac{21}{2}$

$2 \times \left(A + B = \frac{11}{2}\right)$

$\rightarrow 2A + 2B = 11$

Subt: $0 + 2B = -\frac{1}{2} \Rightarrow B = -\frac{1}{4}$

$\Rightarrow A = \frac{11}{2} - \left(\frac{-1}{4}\right) = \frac{23}{4}$

$\therefore \frac{5x^2+2}{(3x+1)(x+1)^2} = \frac{23}{4(3x+1)} - \frac{1}{4(x+1)} - \frac{7}{2(x+1)^2}$

4. $\frac{x^3+2x^2-2x+2}{(x-1)(x+3)}$

The above expression is improper so we must first divide.

$$\begin{array}{r|l} & x \\ x^2+2x-3 & x^3+2x^2-2x+2 \\ & \underline{x^3+2x^2-3x} \\ & \quad\quad\quad\quad x+2 \end{array}$$

$$\therefore \frac{x^3+2x^2-2x+2}{(x-1)(x+3)} = x + \frac{x+2}{(x-1)(x+3)}$$

Thus, $\dfrac{x+2}{(x-1)(x+3)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x+3)}$

$$\therefore x+2 \equiv A(x+3) + B(x-1)$$

When $x=1$, $3 = 4A \Rightarrow A = \frac{3}{4}$.

$x=-3$, $-1 = -4B \Rightarrow B = -\frac{1}{4}$.

$$\therefore \frac{x^3+2x^2-2x+2}{(x-1)(x+3)} = x + \frac{3}{4(x-1)} - \frac{1}{4(x+3)}.$$

5. $\dfrac{x^2-7}{x^2-x-2}$. This is improper.

$$\begin{array}{r|l} & 1 \\ \hline x^2-x-2 & x^2-7 \\ & x^2-x-2 \\ \hline & x-5 \end{array}$$

$$\therefore \frac{x^2-7}{x^2-x-2} = 1 + \frac{x-5}{(x^2-x-2)}$$

But $x^2-x-2 \equiv (x-2)(x+1)$

$$\therefore \frac{x-5}{(x-2)(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x+1)}$$

$$\therefore x-5 \equiv A(x+1) + B(x-2)$$

$\therefore x=-1$, $-6 = -3B$ $\therefore B = 2$

$x=2$, $-3 = 3A$, $A = -1$

$$\therefore \frac{x^2-7}{x^2-x-2} \equiv 1 - \frac{1}{x-2} + \frac{2}{(x+1)}$$

6. $\frac{8x-1}{(x-2)(x^2+1)}$ (x^2+1) is a quadratic factor, numerator will be $(Bx+C)$

$$\therefore \frac{8x-1}{(x-2)(x^2+1)} = \frac{A}{(x-2)} + \frac{(Bx+C)}{(x^2+1)}$$

$$\therefore (8x-1) \equiv A(x^2+1) + (Bx+C)(x-2)$$

$x=0$; $-1 = A - 2C$

$x=2$; $15 = 5A \Rightarrow A = 3$.

$\Rightarrow -1 = 3-2C \therefore C = 2$

Coeff. of x: $8 = -2B + C$

$\Rightarrow -2B = 8-2 = -6$

$\therefore B = 3$

$$\therefore \frac{(8x-1)}{(x-2)(x^2+1)} = \frac{3}{(x-2)} + \frac{(3x+2)}{(x^2+1)}$$

Exercise:

Express the following as partial fractions:

i. $\frac{x}{25-x^2}$, (ii). $\frac{(3x+7)}{x(x+2)(x-1)}$ iii. $\frac{2x^4+7x-1}{(x-2)(x^2+5)}$

iv. $\dfrac{x^2+2x+18}{x(x^2+3)^2}$ v. $\dfrac{2x+1}{x^3-1}$

vi. $\dfrac{10+6x-3x^2}{(2x-1)(x+3)^2}$

More problems in UPM ~~Ch~~ Ex 18A and Backhouse 2 Ex 3A and 3B.

Applications of Partial fractions:

i. To Binomial expansions:

- Expand $\dfrac{4}{(x+3)(1+x)}$ in ascending powers of x upto the third term.

$$\frac{4}{(x+3)(1+x)} = \frac{A}{(x+3)} + \frac{B}{1+x}$$

$$\Rightarrow 4 \equiv A(1+x) + B(x+3)$$

$\Rightarrow$ If $x=-1$; $4 = 2B$; $B = 2$

$x=-3$; $4 = -2A$ $\therefore A = -2$

$$\therefore \frac{4}{(x+3)(1+x)} = -\frac{2}{x+3} + \frac{2}{1+x}$$

$$= 2(1+x)^{-1} - \frac{2}{3}\left(1+\frac{x}{3}\right)^{-1}$$

$$= 2\left(1+(-1)x+\frac{(-1)(-2)x^2}{2!}\right) - \frac{2}{3}\left\{1+(-1)\left(\frac{x}{3}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{3}\right)^2\right\}$$

$= 2 - 2x + 2x^2 - \frac{2}{3} + \frac{2x}{9} - \frac{2x^2}{27}$

$= \frac{4}{3} - \frac{16}{9}x + \frac{52}{27}x^2$

$(1+x)$ is valid for $-1 < x < 1$

$\left(1 + \frac{x}{3}\right)$ is valid for $-3 < x < 3$

$\therefore$ common range is $-1 < x < 1$.

ii. Expand $\frac{1}{x^2 - 5x + 6}$ in ascending powers of $\frac{1}{x}$ and state ~~the~~ any restrictions upto, to the 3rd Term.

$\frac{1}{x^2 - 5x + 6} = \frac{1}{(x-3)(x-2)}$

$= \frac{A}{(x-3)} + \frac{B}{(x-2)}$

$\therefore 1 \equiv A(x-2) + B(x-3)$

$x = 2, \; 1 = -B \;\therefore\; B = -1$

$x = 3, \; 1 = A \Rightarrow B = 1$

$\frac{1}{x^2 - 5x + 6} \equiv \frac{1}{x-3} - \frac{1}{x-2}$

ascending powers of $\frac{1}{x}$ means x

is a big value in both cause so we factor out x.

$$\frac{1}{x-3} = \frac{1}{x\left(1-\frac{3}{x}\right)} = \frac{1}{x}\left(1-\frac{3}{x}\right)^{-1}$$

$$= \frac{1}{x}\left\{1 + (-1)\left(\frac{-3}{x}\right) + \frac{(-1)(-2)}{2!}\left(\frac{-3}{x}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(\frac{-3}{x}\right)^3\right\}$$

$$= \frac{1}{x} + \frac{3}{x^2} + \frac{9}{x^3} + \frac{27}{x^4}$$

$$\frac{1}{x-2} = \frac{1}{x\left(1-\frac{2}{x}\right)} = \frac{1}{x}\left(1-\frac{2}{x}\right)^{-1}$$

$$= \frac{1}{x}\left\{1 + (-1)\left(\frac{-2}{x}\right) + \frac{(-1)(-2)}{2!}\left(\frac{-2}{x}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(\frac{2}{x}\right)^3\right\}$$

$$= \frac{1}{x} + \frac{2}{x^2} + \frac{4}{x^3} + \frac{8}{x^4}$$

$$\therefore \frac{1}{x^2-5x+6} = \left(\frac{1}{x} + \frac{3}{x^2} + \frac{9}{x^3} + \frac{27}{x^4}\right) - \left(\frac{1}{x} + \frac{2}{x^2} + \frac{4}{x^3} + \frac{8}{x^4}\right)$$

$$= \frac{1}{x^2} + \frac{5}{x^3} + \frac{19}{x^4}$$

$\left(1 - \frac{3}{x}\right)$ is valid for $\left|\frac{3}{x}\right| < 1$

$\Rightarrow |x| > 3$: $x < -3$ and $x > 3$

$\left(1 - \frac{2}{x}\right)$ is valid for $\left|\frac{2}{x}\right| < 1 \Rightarrow 2 < |x|$

$\therefore$ $x < -2$ or $x > 2$.

$\therefore$ common range is $x < -3$ or $x > 3$.

Exercise: 1. Expand $\frac{7x+1}{(1+x)(1+3x)}$ in ascending powers of x upto the term in x^3 and state the range of validity for the expansion.

2. Expand $\frac{2x+4}{(x-1)(x+3)}$ in ascending powers of $\frac{1}{x}$ upto the 3^{rd} term and state the range of values of x for which the expansion is valid.

ii. Application to sum of series:
Partial fractions enable us to obtain the sum of series whose general term can be decomposed to partial fractions. If each term is decomposed to ~~its~~ its partial fractions and added, the terms in the middle cross of leaving ~~a~~ few terms at the beginning and at the end. These can then be simplified to give the required sum.

Eg: 1. Show that

$$\frac{1}{1\times3}+\frac{1}{2\times4}+\cdots+\frac{1}{n(n+2)}=\frac{3}{4}-\frac{(2n+3)}{2(n+1)(n+2)}$$

Hence deduce that the series converges to $\frac{3}{4}$ as $n\to\infty$.

Gen. term $\frac{1}{n(n+2)} \equiv \frac{A}{n}+\frac{B}{(n+2)}$

$$\Rightarrow 1 = A(n+2)+Bn$$

$n=0$, $1 = 2A \Rightarrow A=\frac{1}{2}$

$n=-2$, $\Rightarrow -2B=1 \therefore B=-\frac{1}{2}$

$$\therefore \frac{1}{n(n+2)}=\frac{1}{2n}-\frac{1}{2(n+2)}.$$

Rewriting each term as partial fractions:

$$\frac{1}{1\times3}=\frac{1}{2(1)}-\frac{1}{2(3)}$$

$$\frac{1}{2\times4}=\frac{1}{2(2)}-\frac{1}{2(4)}$$

$$\frac{1}{3\times5}=\frac{1}{2(3)}-\frac{1}{2(5)}$$

$$\frac{1}{(n-2)n}=\frac{1}{2(n-2)}-\frac{1}{2(n)}$$

$$\frac{1}{(n-1)(n+1)}=\frac{1}{2(n-1)}-\frac{1}{2(n+1)}$$

$$\frac{1}{n(n+2)} = \frac{1}{2(n)} - \frac{1}{2(n+2)}$$

All the values between the dotted lines cross off leaving;

$$\frac{1}{1\times3} + \frac{1}{2\times4} + \ldots + \frac{1}{n(n+2)} = \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$$

$$= \frac{3}{4} - \frac{1}{2}\left[\frac{(n+2)+(n+1)}{(n+1)(n+2)}\right]$$

$$= \frac{3}{4} - \frac{1}{2}\frac{(2n+3)}{(n+1)(n+2)}$$

As $n \to \infty$, $\frac{1}{2}\frac{(2n+3)}{(n+1)(n+2)} \to 0$

$\therefore S_\infty = \frac{3}{4}$.

Exercise: Find the sum of the first n terms of the series below and deduce its sum to infinity.

i. $\frac{1}{1\times4} + \frac{1}{2\times5} + \ldots + \frac{1}{n(n+3)}$

ii. $\frac{1}{3\times6} + \frac{1}{6\times9} + \ldots + \frac{1}{3n(3n+9)}$