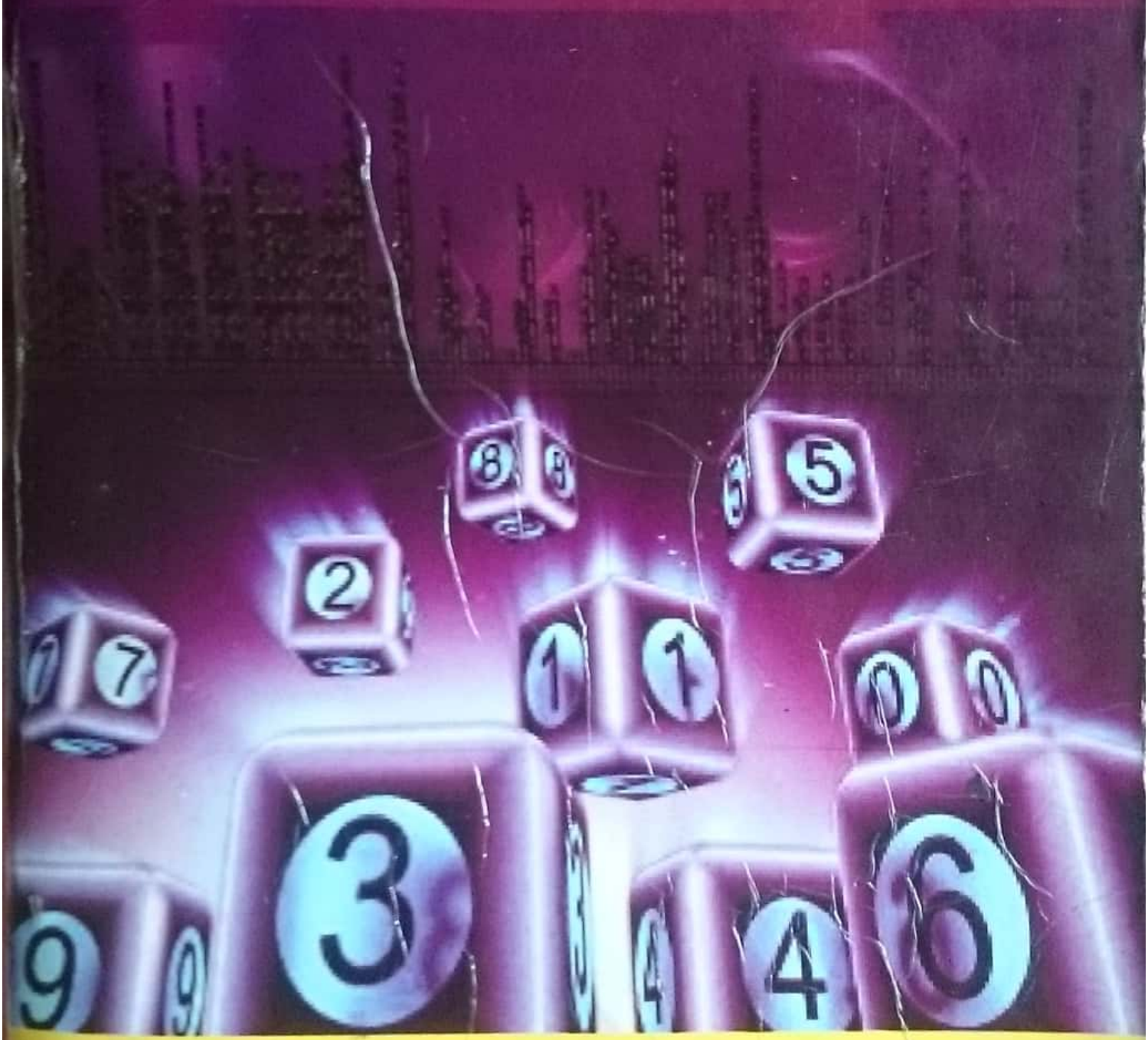
# A COMPREHENSIVE APPROACH TO ADVANCED LEVEL STATISTICS AND NUMERICAL METHODS



MUKOSE MUHAMMED

THIRD EDITION

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2022

# A COMPREHENSIVE APPROACH TO ADVANCED LEVEL STATISTICS AND NUMERICAL METHODS

2022

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THIRD EDITION 2008

M.MUKOSE BSC/ED (HONS), M.A DEMOGRAPHY (MUK)
MATHEMATICS DEPARTMENT
NABISUNSA GIRLS' SCHOOL

**CONTACTS:**
TEL. NO. 0772429998 OR 0701429988
EMAIL mukosm@yahoo.com

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**MUKOSE MUHAMMED**
**MATHEMATICS DEPARTMENT**
**NABISUNSA GIRLS' SCHOOL**
**P.O. BOX 3027 KAMPALA**



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# INTRODUCTION

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I have written this book purposely to meet the needs for students offering Mathematics at advanced level. This book is based on the UACE syllabus for paper two which include Statistics, Mechanics and Numerical methods. However, in this book Statistics and Numerical Methods have been catered for. The topics there in have been comprehensively covered to cater for the increasing demand by most schools to cover the different papers concurrently. Accordingly, topical issues such as combinations and permutations have been looked at to help in the student's understanding for probability and binomial distribution.

Bearing in mind that students will find it useful as study aid, I designed exercises at the end of each topic for discussion, to stimulate the student's understanding of the subject matter. I hope students offering the following subject – combinations MEG, PEM, PCM, BCM, PCB, PCB/M will find this book extremely useful. So will those undergrates in higher institutions of learning.

As for the teachers, this book is designed to serve as a key teaching – aid.

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# CHAPTER ONE

# DESCRIPTIVE STATISTICS

## 1.1 INTRODUCTION

This is the branch of mathematics dealing with collection, interpretation, presentation and analysis of data. Data refers to facts and figures collected for a purpose.

## STATISTICAL DATA

Data can be categorized into two, namely Qualitative and Quantitative.

Qualitative data measures attributes such as sex, colour and so on. While quantitative data can be represented by numerical quantity. For instance, height, mass, time and so on. In this part of statistics it is only the quantitative data that will be considered. Quantitative data is mainly of two forms either continuous or discrete.

### (a) Discrete and Continuous data

Statistical data can be of two forms; Discrete or Continuous. Information collected by counting is discrete and usually takes integral values e.g. numbers of students in a class, school, etc.. Continuous data can take on any value, for instance height, weight, time, distance, mass, temperature.

The quantity, which is counted or measured, is called the variable. Other terms used include



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### (a) Crude and Classified data

Crude data are individual values of variable in no particular order of magnitude, written down as they occurred or were measured. When the numbers have been arranged in order and grouped in a small number of classes, they are called classified or grouped data

### (b) Population and Sample

A population is the total set of items under consideration and is defined by some characteristics of these items. A sample is a finite subset of a population.

### Frequency

This refers to the number of times an item occurs. In most cases many items occur more than once, therefore a frequency distribution table is used to remove repetitions.

Example

The raw data gives the marks of 10 students in class.

35, 40, 45, 50, 60, 35, 40, 35, 45, 70

Represent the above data using frequency table



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A frequency table can be used to represent the data

| marks | 35 | 40 | 45 | 50 | 60 | 70 |
|---|---|---|---|---|---|---|
| frequency | 3 | 2 | 2 | 1 | 1 | 1 |

This kind of representation is commonly used for ungrouped data. However if the data is large then grouping is necessary.

**Grouped data**

| Weight (kg) | Number of students |
|---|---|
| 40 - 44 | 3 |
| 45 - 49 | 10 |
| 50 - 54 | 16 |
| 55 - 59 | 10 |
| 60 - 64 | 4 |

The first column represents the classes. So 40 – 44 is a class. The second column represents the class frequencies. For any given class ( 50 – 54), the lower value is called lower class limit and upper value is upper class limit.

**Class boundaries** are the true class limits. For the class 60 -64, the lower class boundary is 59.5 and the upper is 64.5.

**Note** that the class boundaries depend on degree of accuracy.



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For example (7.0 – 7.4). Lower class boundary is 6.95 and upper class boundary is 7.45.

**Class width** is difference between upper class boundary and lower class boundary.

For 50 – 54. The class width = 54.5 – 49.5 = 5

For 7.0 – 7.4. The class width = 7.45 – 6.95 = 0.5

**Note:** that the class width , size and length are the same.

**Class mark** is the average of lower and upper limit

Example (40 – 44)

Class mark = ( 40 + 44) /2 = 42

Note: Tallies are obtained when forming the classes.

## 1.2 PRESENTATION OF DATA

Ways of presenting data include

i. Bar graphs

ii. Histogram

iii. Frequency polygon

iv. The Ogive

v. Pie chart



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## BAR GRAPH

A bar graph or bar chart is a graph where the class frequencies are plotted against class limits.

## HISTOGRAM

A histogram is a graph where the class frequencies are plotted against class boundaries.

Example

The table below shows the weights of some freshers in 2000/2001 academic year who under went medical examination at the sick bay.

Use the data given to plot a bar graph and histogram.

| Weight (kg) | Number of students |
|---|---|
| 40 – 44 | 3 |
| 45 – 49 | 10 |
| 50 – 54 | 16 |
| 55 – 59 | 10 |
| 60 – 64 | 4 |
| 65 – 69 | 5 |
| 70 – 74 | 4 |
| 75 – 79 | 6 |
| 80 - 84 | 1 |



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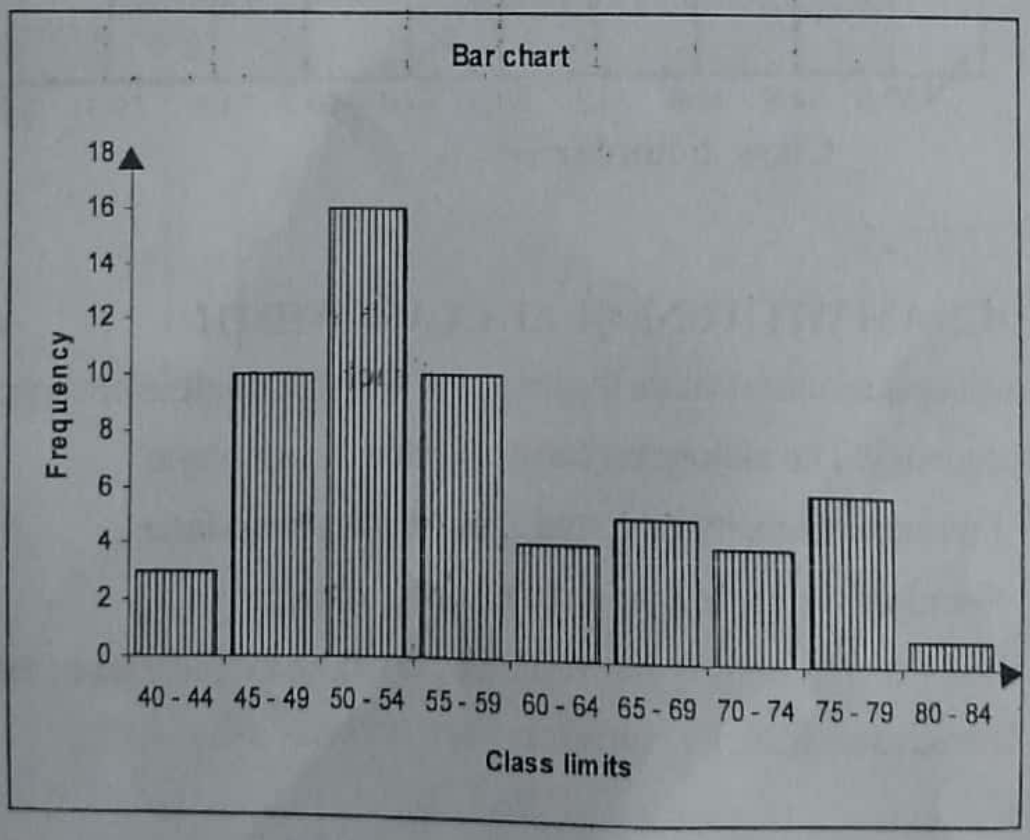
**Solution**

| Class limits | Class boundaries | frequency |
|---|---|---|
| 40 - 44 | 39.5 – 44.5 | 3 |
| 45 - 49 | 44.5 – 49.5 | 10 |
| 50 - 54 | 49.5 – 54.5 | 16 |
| 55 - 59 | 54.5 – 59.5 | 10 |
| 60 - 64 | 59.5 – 64.5 | 4 |
| 65 - 69 | 64.5 – 69.5 | 5 |
| 70 - 74 | 69.5 – 74.5 | 4 |
| 75 - 79 | 74.5 – 79.5 | 6 |
| 80 - 84 | 79.5 – 84.5 | 1 |



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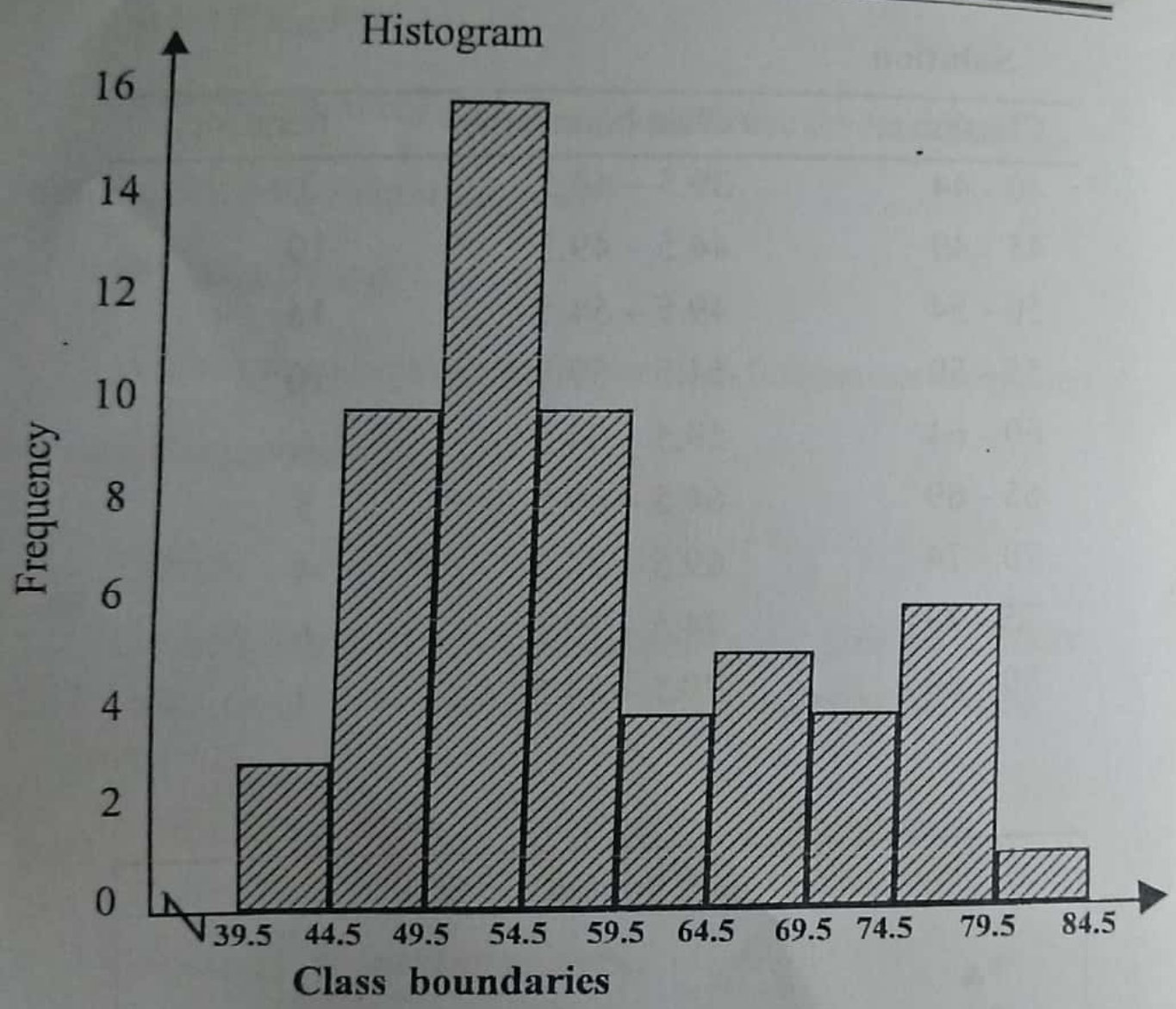


## HISTOGRAM WITH UNEQUAL CLASS WIDTH

In case of unequal class width, the height of each rectangle is not equal to the frequency. The histogram can be plotted in two ways:

- Frequency density is plotted against class boundaries.
- Standard frequency against class boundaries.

Where the frequency density is obtained by dividing frequencies of that class – by – class width.



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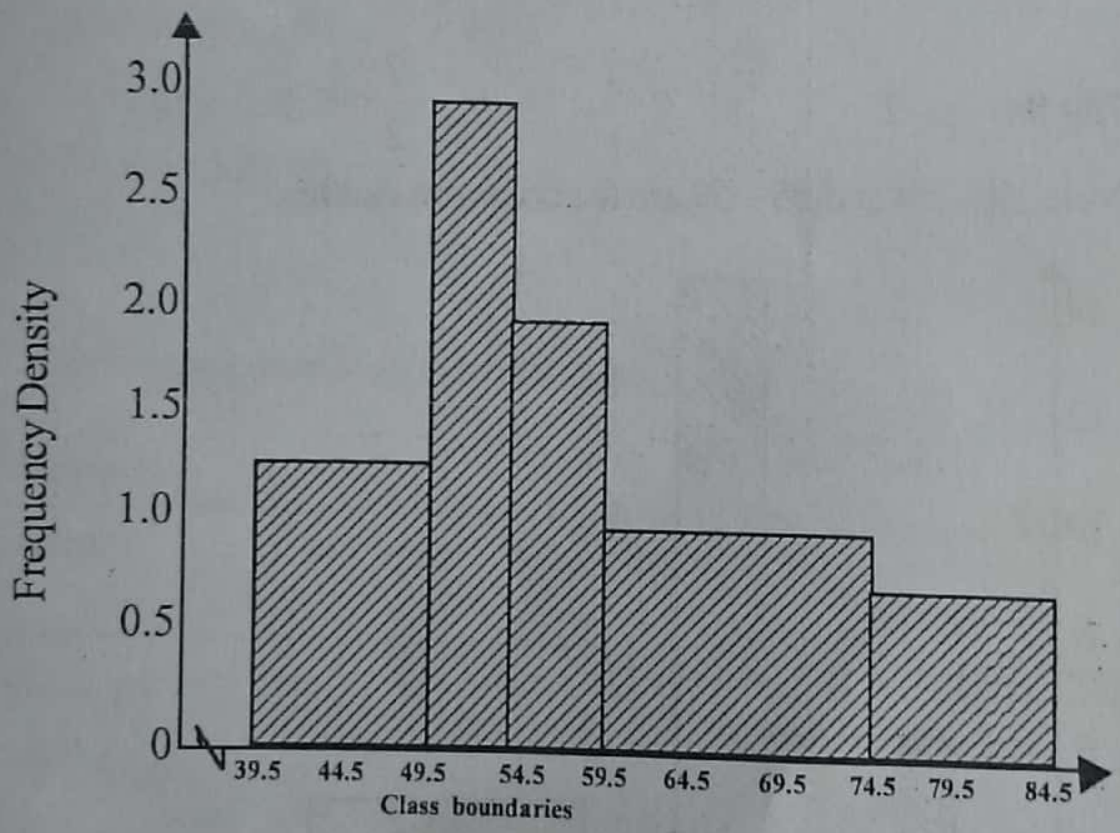
**Example**

| Class boundaries | Frequency | Frequency density |
|---|---|---|
| 39.5 – 49.5 | 13 | 1.3 |
| 49.5 – 54.5 | 15 | 3.0 |
| 54.5 – 59.5 | 10 | 2.0 |
| 59.5 – 74.5 | 15 | 1.0 |
| 74.5 – 84.5 | 7 | 0.7 |




On the other hand standard frequency is obtained by taking the common class width as the standard one and then divide the frequency values by the number of times the class is the standard one.



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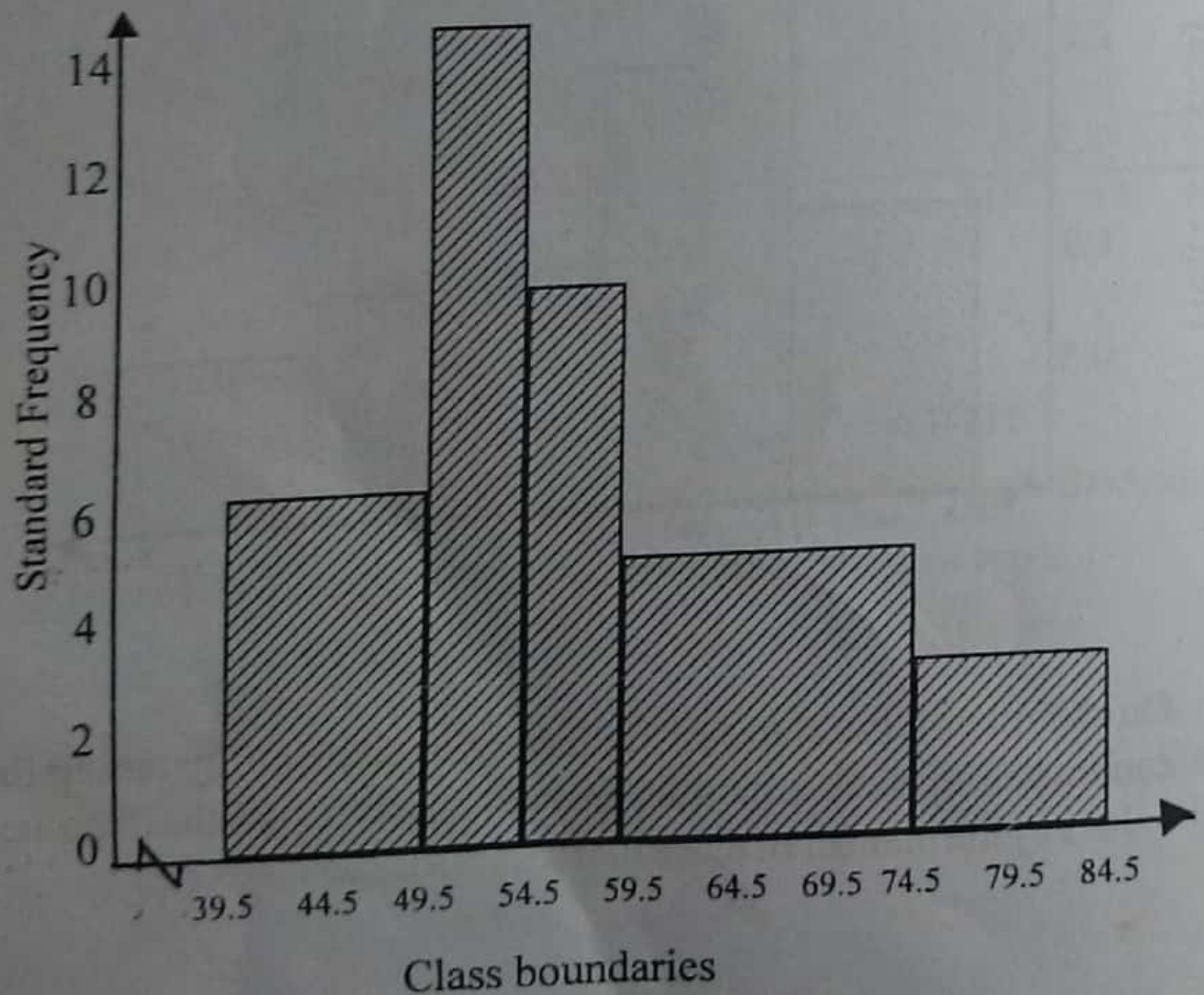
Example

| Class limits | No. of times | Frequency | *Standard frequency* |
|---|---|---|---|
| 40 - 49 | 2 | 13 | 6.5 |
| 50 - 54 | 1 | 15 | $\frac{15}{1}=15$ |
| 55 - 59 | 1 | 10 | 10 |
| 60 - 74 | 3 | 15 | $\frac{15}{3}=5$ |
| **75 - 84** | 2 | 7 | $\frac{7}{2}=3.5$ |

Note: 50 – 54 and 55 – 59 are the common classes.



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**Example**

The table below shows the population of Cairo in millions for the different age groups.

| Age group | Population in millions |
|---|---|
| Below 10 | 2 |
| 10 and under 20 | 8 |
| 20 and under 30 | 10 |
| 30 and under 40 | 14 |
| 40 and under 50 | 10 |
| 50 and under 70 | 5 |
| 70 and under 90 | 1 |

Draw a histogram to represent the data above

**Solution**

| Class | Class width | Frequency | *Frequency density* |
|---|---|---|---|
| 0 - < 10 | 10 | 2 | 0.2 |
| 10 - < 20 | 10 | 8 | 0.8 |
| 20 - < 30 | 10 | 10 | 1.0 |
| 30 - < 40 | 10 | 14 | 1.4 |
| 40 - < 50 | 10 | 10 | 1.0 |
| 50 - < 70 | 20 | 5 | 0.25 |
| **70 - < 90** | 20 | 1 | 0. 05 |



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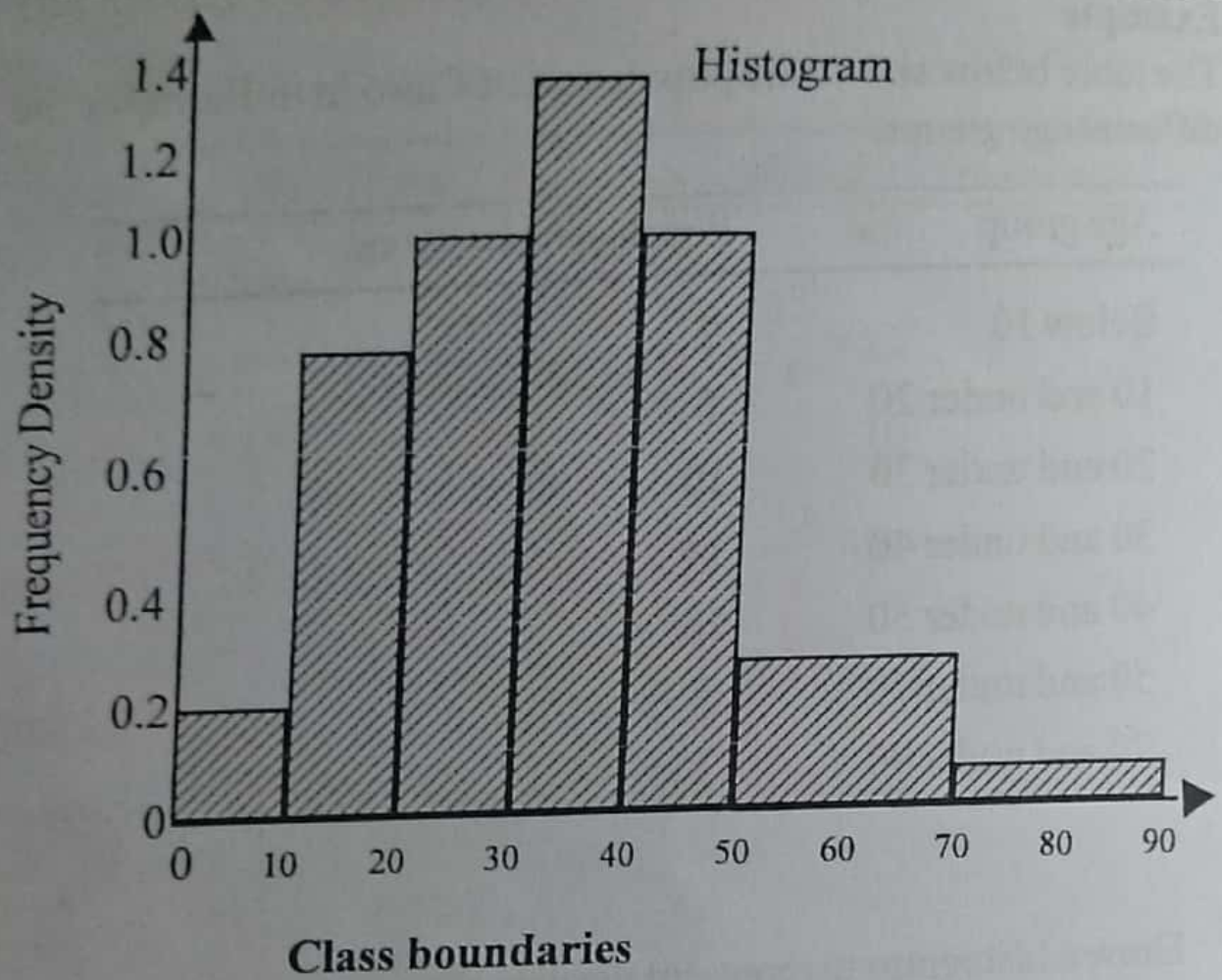


**Assignment**: Repeat the above example using standard frequency instead of frequency density.

**Example 1**

On a particular day the length of stay of each taxi at the New taxi park was recorded as shown in the table below

| Length of stay ( min) | Frequency |
|---|---|
| $0 \le t < 4$ | 2 |
| $4 \le t < 8$ | 6 |
| $8 \le t < 12$ | 8 |
| $12 \le t < 16$ | 8 |
| $16 \le t < 18$ | 6 |
| $18 \le t < 20$ | 4 |
| $20 \le t < 28$ | 8 |



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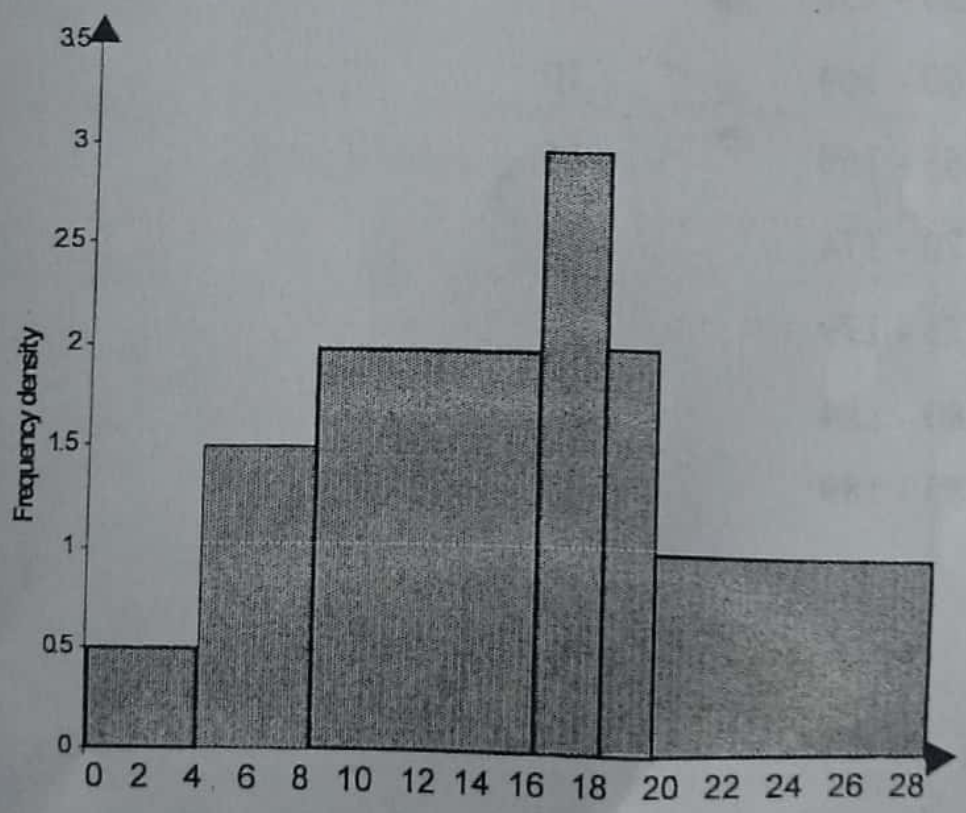
Draw a histogram for the data above

**Solution**

| Class | Class width | Frequency | *Frequency density* |
|---|---|---|---|
| 0 - < 4 | 4 | 2 | 0.5 |
| 4 - < 8 | 4 | 6 | 1.5 |
| 8 - < 12 | 4 | 8 | 2.0 |
| 12 - < 16 | 4 | 8 | 2.0 |
| 16 - < 18 | 2 | 6 | 3.0 |
| 18 - < 20 | 2 | 4 | 2.0 |
| **20 - < 28** | 8 | 8 | 1.0 |



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## Frequency Polygon

A frequency polygon is obtained by plotting class frequencies versus class marks. Then consecutive points are joined using a straight line.
*Note*: Additional classes may be added to either end with zero frequency to make the polygon a closed figure.

### Example

The frequency distribution below shows the heights to the nearest cm, of 80 patients at a Hospital.

| Height in cm | frequency |
|---|---|
| 150 - 154 | 3 |
| 155 - 159 | 7 |
| 160 - 164 | 10 |
| 165 - 169 | 15 |
| 170 - 174 | 25 |
| 175 - 179 | 12 |
| 180 - 184 | 6 |
| 185 - 189 | 2 |



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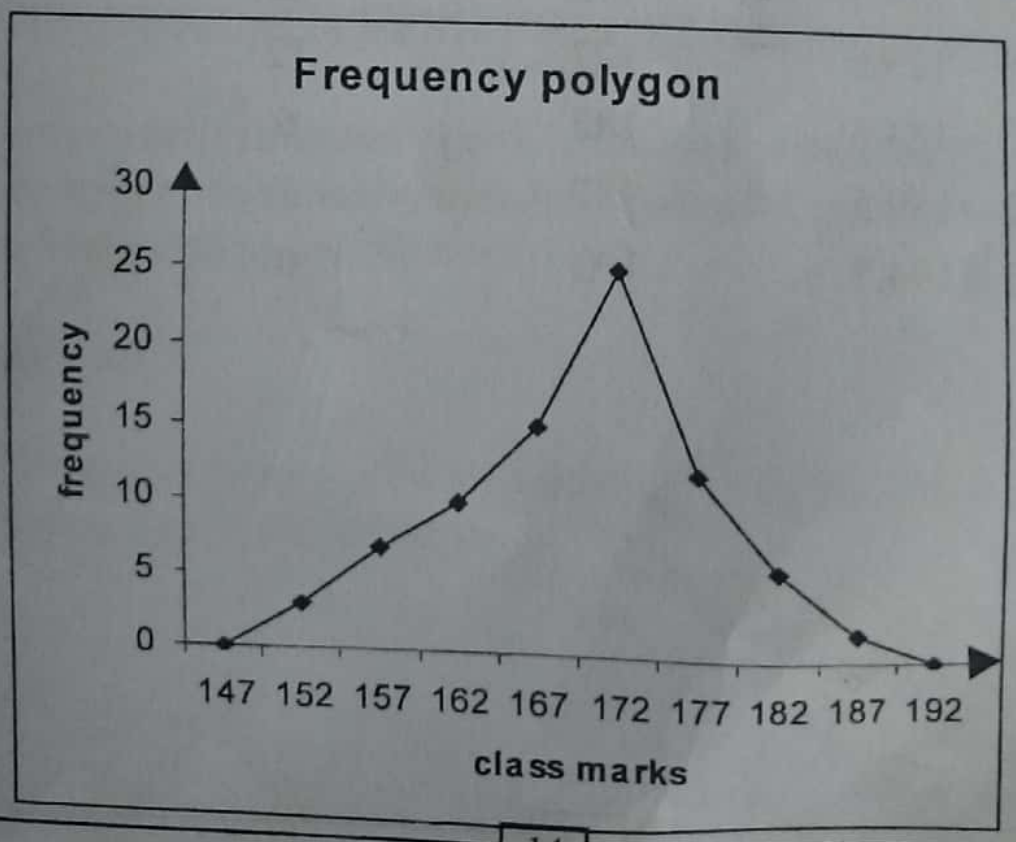
Construct a frequency polygon for the above data?

***Solution***

| Class limits | Class mark | frequency |
|---|---|---|
| Additional class | 147 | 0 |
| 150 - 154 | 152 | 3 |
| 155 - 159 | 157 | 7 |
| 160 – 164 | 162 | 10 |
| 165 – 169 | 167 | 15 |
| 170 – 174 | 172 | 25 |
| 175 – 179 | 177 | 12 |
| 180 – 184 | 182 | 6 |
| 185 – 189 | 187 | 2 |
| 190 – 194 | 192 | 0 |



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**Assignment: Using example 1**

Draw a frequency polygon for the data

***Note*:** A frequency polygon can be put on the same graph with the histogram. This is known as superimposing a frequency polygon.

**Example**

Using example below, draw a histogram and superimpose a frequency polygon.

| *Class boundaries* | *Class mark* | *frequency* |
|---|---|---|
| 144.5 – 149.5 | 147 | 0 |
| 149.5 – 154.5 | 152 | 3 |
| 154.5 – 159.5 | 157 | 7 |
| 159.5 – 164.5 | 162 | 10 |
| 164.5 – 169.5 | 167 | 15 |
| 169.5 – 174.5 | 172 | 25 |
| 174.5 – 179.5 | 177 | 12 |
| 179.5 – 184.5 | 182 | 6 |
| 184.5 – 189.5 | 187 | 2 |
| 189.5 – 194.5 | 192 | 0 |



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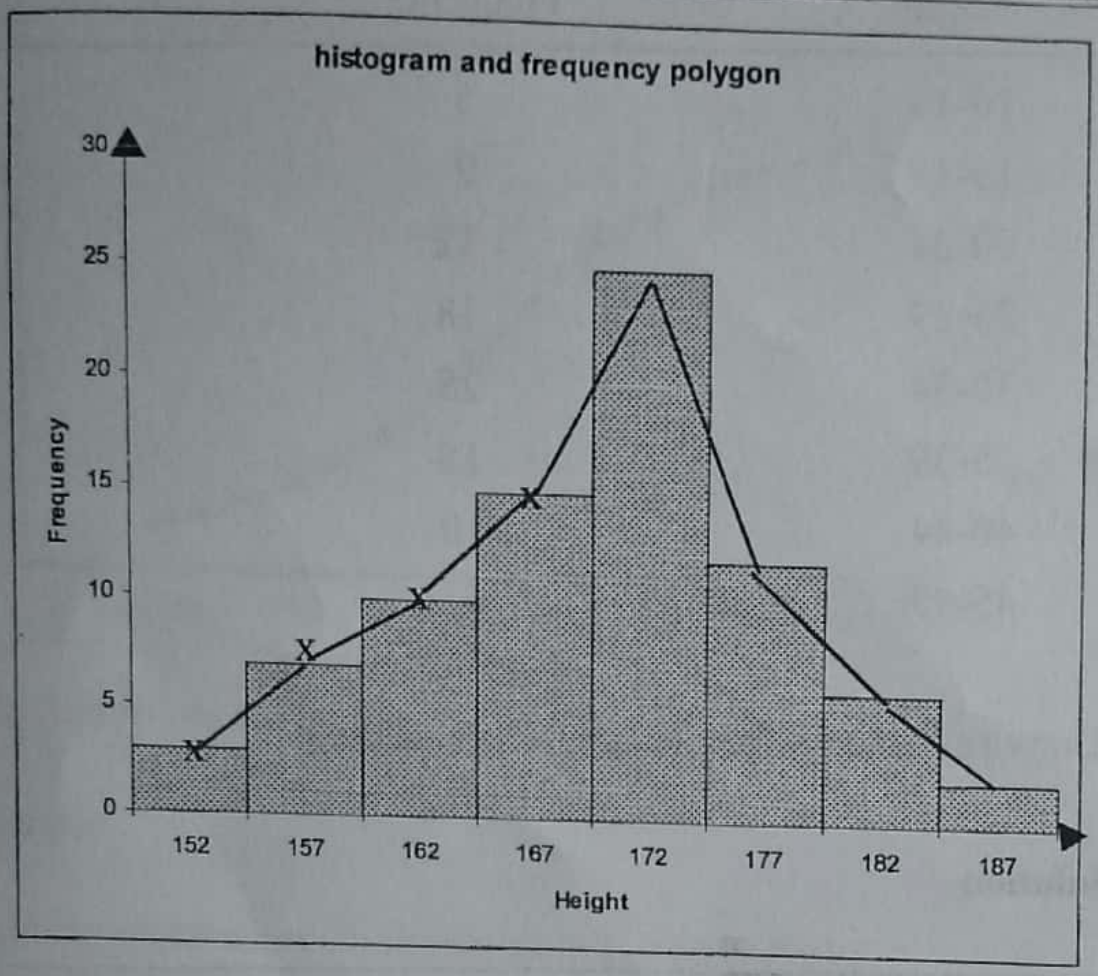


## THE CUMULATIVE FREQUENCY CURVE (OGIVE)

The cumulative frequency curve is obtained by plotting cumulative frequency versus class boundaries and then the consecutive points are joined using a smooth curve.

### Example 2

The frequency distribution table shows the weights of 100 children measured to the nearest kg.



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| Weight | Frequency |
|---|---|
| 10-14 | 5 |
| 15-19 | 9 |
| 20-24 | 12 |
| 25-29 | 18 |
| 30-34 | 25 |
| 35-39 | 15 |
| 40-44 | 10 |
| 45-49 | 6 |

Draw a cumulative frequency curve for the data

**Solution**

| Class boundaries | frequency | Cumulative frequency |
|---|---|---|
| 9.5 | 0 | 0 |
| 14.5 | 5 | 5 |
| 19.5 | 9 | 14 |
| 24.5 | 12 | 26 |
| 29.5 | 18 | 44 |
| 34.5 | 25 | 69 |
| 39.5 | 15 | 84 |
| 44.5 | 10 | 94 |
| 49.5 | 6 | 100 |



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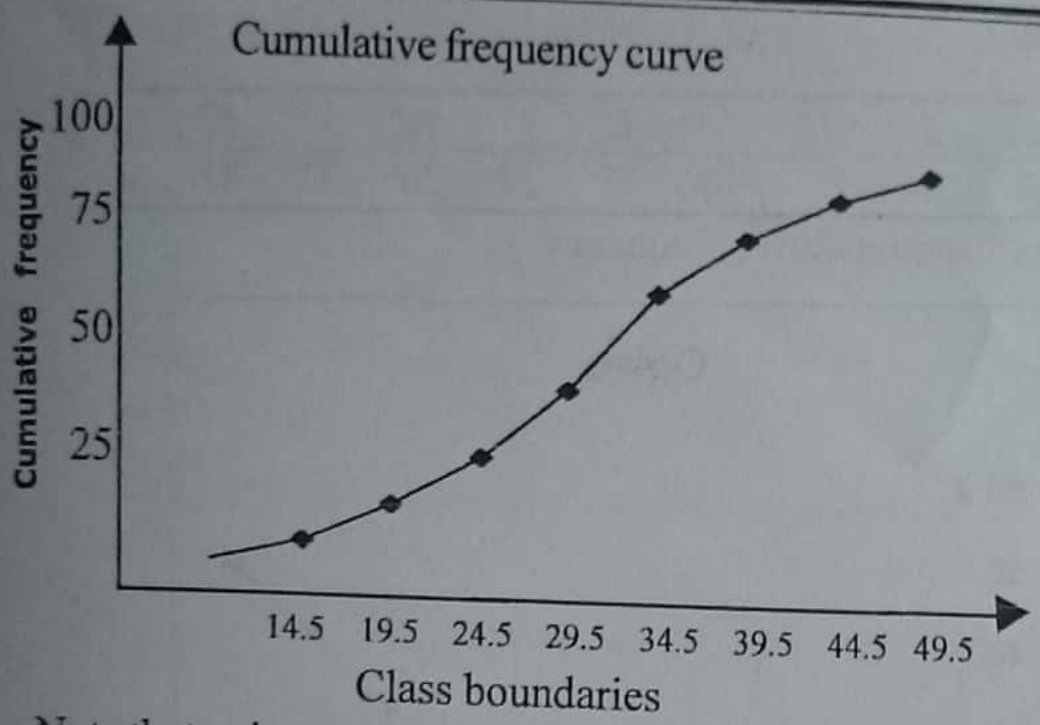


Note that points correspond i.e (9.5, 0 ) , (14.5, 5) e.t.c

**Example**

The table below shows the earnings to nearest dollar of Employees of a certain farm

| *Earnings to nearest dollar* | *Frequency* |
|---|---|
| 18 - < 22 | 2 |
| 22 - < 27 | 2 |
| 27 - < 32 | 2 |
| 32 - < 37 | 4 |
| 37 - < 42 | 5 |
| 42 - < 47 | 7 |
| 47 - < 52 | 4 |
| 52 - < 57 | 7 |
| 57 - < 62 | 9 |
| 62 - < 67 | 8 |

Draw an Ogive for the data



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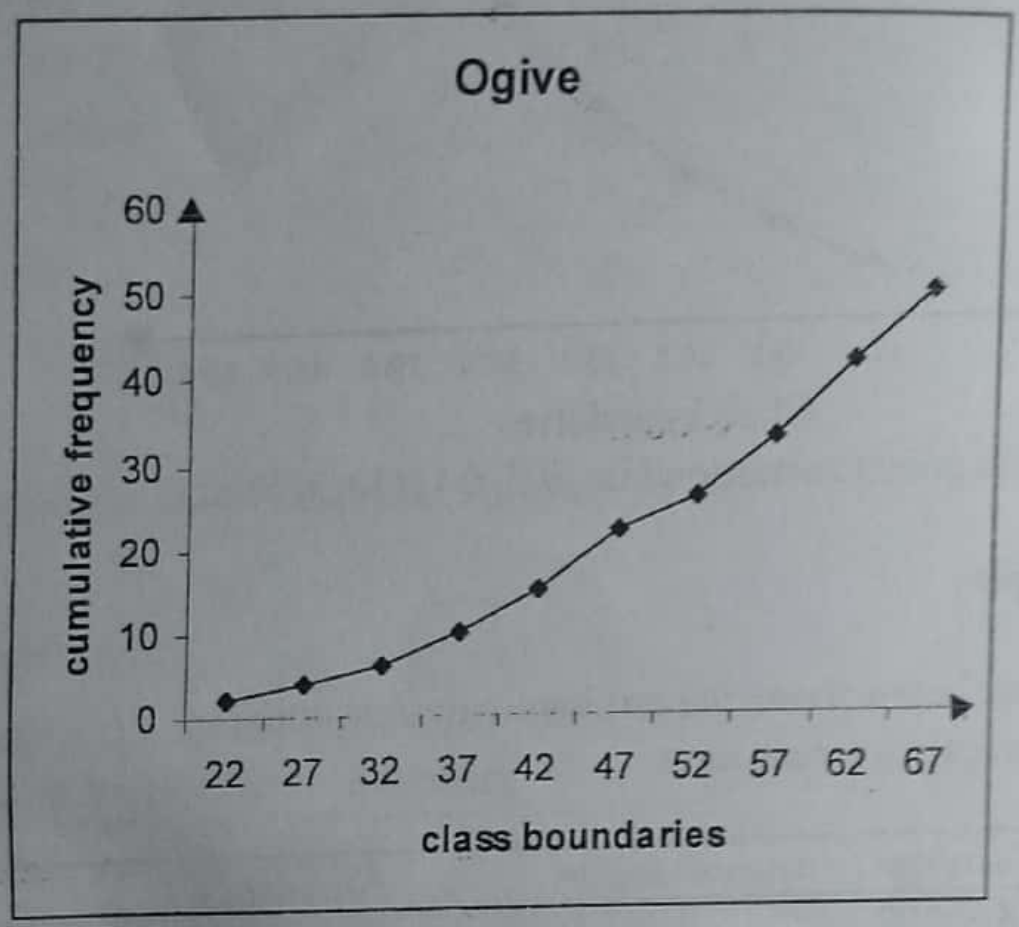
***Solution***

| | 22 | 27 | 32 | 37 | 42 | 47 | 52 | 57 | 62 | 67 |
|---|---|---|---|---|---|---|---|---|---|---|
| Cf | 2 | 4 | 6 | 10 | 15 | 22 | 26 | 33 | 42 | 50 |

Note: Cf is cumulative frequency




**Example**

The table below shows the weights to the nearest kg of 150 patients who visited a certain health unit during a certain week

| *Weight (kg)* | *Number of patients (f)* |
|---|---|
| 0 - 19 | 30 |
| 20 - 29 | 16 |
| 30 - 39 | 24 |
| 40 - 49 | 32 |
| 50 - 59 | 28 |
| 60 - 79 | 20 |



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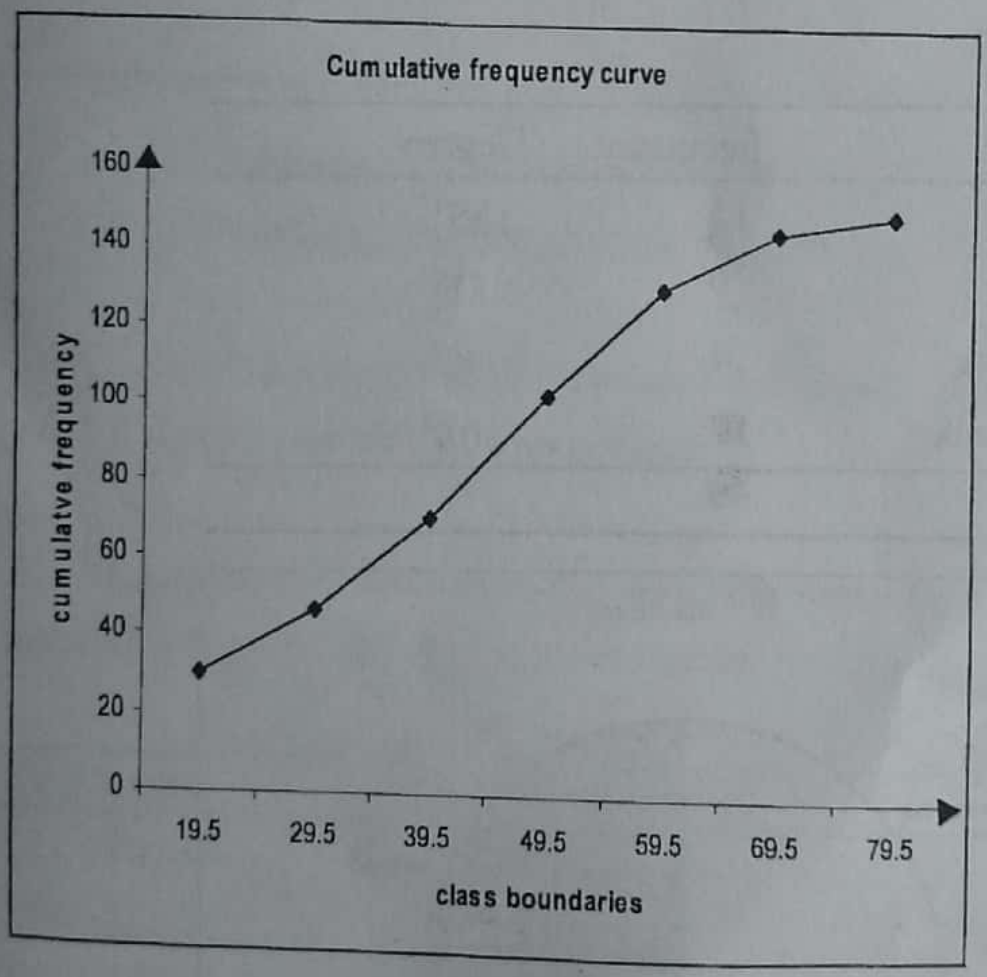
Draw a cumulative frequency curve for the data

**Solution**

| *Class boundaries* | *frequency* | *Cumulative frequency* |
|---|---|---|
| 19.5 | 30 | 30 |
| 29.5 | 16 | 46 |
| 39.5 | 24 | 70 |
| 49.5 | 32 | 102 |
| 59.5 | 28 | 130 |
| 79.5 | 20 | 150 |



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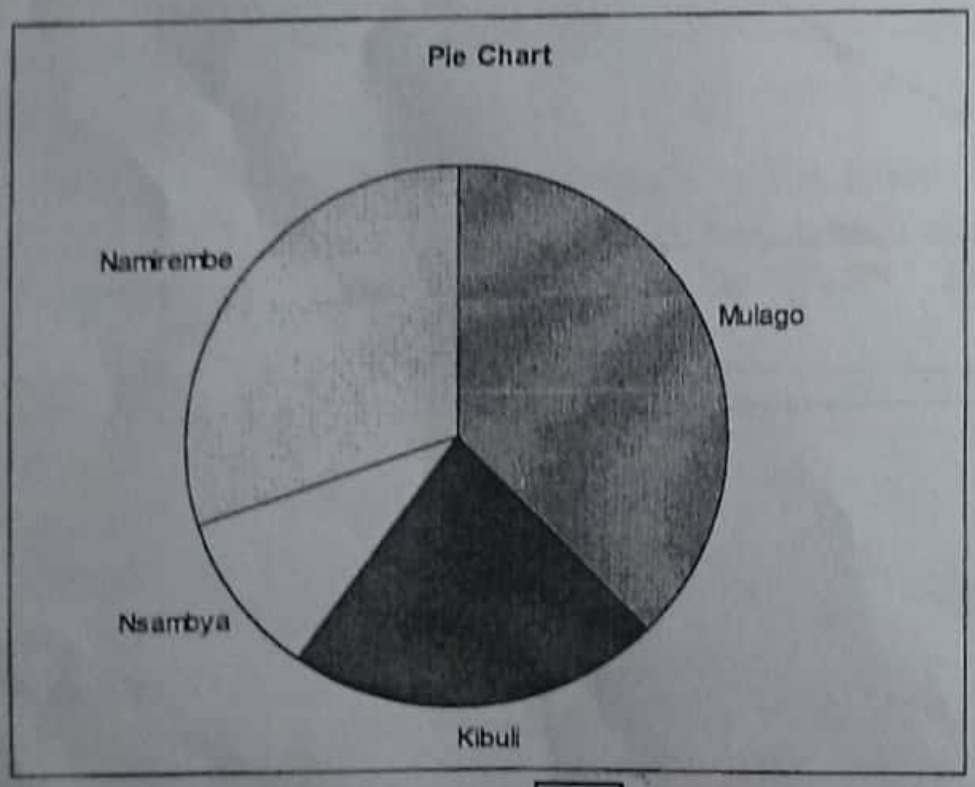
## Pie Chart

Draw a pie chart for the information given below

| Hospital | Number of children |
|---|---|
| Mulago | 15 |
| Kibuli | 9 |
| Namirembe | 12 |
| Nsambya | 4 |

**Solution**

| Hospital | frequency | Degree |
|---|---|---|
| Mulago | 15 | $135^0$ |
| Kibuli | 9 | $81^0$ |
| Nsambya | 4 | $36^0$ |
| Namirembe | 12 | $108^0$ |
| Total | 40 | |



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**Exercise**

1.The table below shows the expenditure of examiners in thousands of shillings during the year 1990 ordinary level marking exercise.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 10 | 11 | 10 | 12 | 14 | 16 | 20 | 25 |
| 21 | 22 | 13 | 17 | 18 | 24 | 30 | 32 |
| 27 | 35 | 40 | 44 | 39 | 50 | 54 | 53 |
| 44 | 37 | 36 | 39 | 52 | 51 | 57 | 15 |
| 16 | 19 | 34 | 43 | 26 | 38 | 53 | 40 |

i. Form a frequency distribution table with class interval of 5,000 shillings, the lowest class limit being 10,000.

ii. Draw a histogram to represent the above data and superimpose a frequency polygon.

2. The heights of students in S.5 Mathematics class were according to the following frequency table

| *Height cm* | *Frequency (f)* |
|---|---|
| 151 - 153 | 2 |
| 154 - 156 | 14 |
| 157 - 159 | 13 |
| 160 - 162 | 13 |
| 163 – 165 | 2 |
| 166 - 168 | 1 |



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Plot the cumulative frequency curve for the students' heights. Hence estimate the median, lower and upper quartiles for the heights of the students.

**Answer** median = 158.3 cm, lower quartile = 155.8 cm,
upper quartile = 160.7 cm

3.The information shows the amount of money in millions (A) given to some districts in Uganda for 'Entandikwa" scheme

| A | 25-<30 | 30-<40 | 40-<50 | 50-<60 | 60-<80 |
|---|---|---|---|---|---|
| f | 4 | 10 | 4 | 3 | 5 |

i. Use the above information to draw a histogram

ii. Use the histogram to estimate the mode of the distribution.

4. The table below shows the weights in kilograms of 250 boys. Each weight was recorded to nearest 100 grams

| *weight kgs* | *Frequency (f)* |
|---|---|
| 44.0 – 47.9 | 3 |
| 48.0 – 51.9 | 17 |
| 52.0 – 55.9 | 50 |
| 56.0 – 59.9 | 45 |
| 60.0 – 63.9 | 46 |
| 64.0 – 67.9 | 57 |
| 68.0 – 71.9 | 23 |
| 72.0 – 75.9 | 9 |

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Draw an Ogive and use it to estimate

i. The semi-interquartile range

ii. The second decile

iii. The percentage of boys weighing over 59 kilograms

**Answer** (i) 5 (ii) 54.55 (iii) 59%

5. At 7.30 am daily a bus leaves Kampala for Jinja. The times (minutes) taken to cover the journey were recorded over a certain period of time and were grouped as shown in the table below.

| *Time minutes* | *Frequency (f)* |
|---|---|
| 80 -84 | 10 |
| 85 - 89 | 15 |
| 90 - 94 | 35 |
| 95 - 99 | 40 |
| 100 - 104 | 28 |
| 105 – 109 | 15 |
| 110 - 114 | 4 |
| 115 - 119 | 2 |
| 120 - 124 | 1 |

a) Calculate the mean time of travel from Kampala to Jinja by bus.

b) Draw a cumulative frequency curve for the data. Use your curve to estimate the:

i. Median time for the journey,



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i. Number of times the bus arrived in Jinja between 9.00 – 9.25 am.

ii. Semi-interquartile range of time of travel from Kampala to Jinja.

**Answer**

(a) mean time = 96.6 minutes

(b) (i) median (96 – 97) minutes

(ii) (118 - 122) times

(iii) (4.5 – 5.5) minutes.

6. The table below is the distribution of weights of a group of animals.

| *Mass (kg)* | *Frequency* |
|---|---|
| 21- 25 | 10 |
| 26 – 30 | 20 |
| 31 - 35 | 15 |
| 36 - 40 | 10 |
| 41 -50 | 30 |
| 51 - 65 | 45 |
| 66 - 75 | 5 |

(a) Draw a cumulative frequency curve and use it to estimate the semi-interquartile range.

(b) Find the mode.



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## 1.3 MEASURES OF CENTRAL LOCATION

These are average values that locate values of a variable in a particular part of the number line.
They include: Mean, Median and Mode

### Mean or Average

The mean is the sum of the observations divided by the number of items

$$\bar{X} = \frac{\sum x}{n}$$

Where $\sum$ sigma, is used for sum.

**Example**
Determine the average height of the class using the data below
171, 180, 154, 160, 155 and 164

**Solution**

$$\text{Mean} = \frac{171+180+154+160+155+164}{6} = 164$$

If the class had many students then, some might have the same height. So in this case frequency table is needed

**Example**

| Height | 154 | 155 | 160 | 164 | 171 | 180 |
|---|---|---|---|---|---|---|
| Frequency | 4 | 6 | 8 | 5 | 4 | 3 |



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**Solution**

$$\text{Mean} = \bar{X} = \frac{\sum fx}{\sum f}$$

| *Height (x)* | *Frequency (f)* | *fx* |
|---|---|---|
| 154 | 4 | 616 |
| 155 | 6 | 930 |
| 160 | 8 | 1280 |
| 164 | 5 | 820 |
| 171 | 4 | 684 |
| 180 | 3 | 540 |
| **Total** | **30** | **4870** |

$$\text{Mean} = \frac{4870}{30} = 162.3333$$

Note: depending on the nature of data, a suitable formula should be used.

**Example**

The table below shows the weight of some exotic cows in kilograms.

**Example**

The table below shows the weight of some exotic cows in kilograms.

| weight | 990 | 1000 | 1010 | 1020 | 970 | 980 |
|---|---|---|---|---|---|---|
| frequency | 5 | 6 | 3 | 2 | 20 | 10 |



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Calculate the average weight for the cows

**Solution**

$$\bar{X} = A + \frac{\sum fd}{\sum f}$$

Let A the working mean be 1000 and d = x - A

| *Weight (x)* | *frequency* | *d* | *fd* |
|---|---|---|---|
| 970 | 20 | 970 -1000 = -30 | -600 |
| 980 | 10 | -20 | -200 |
| 990 | 5 | -10 | -50 |
| 1000 | 6 | 0 | 0 |
| 1010 | 3 | 10 | 30 |
| 1020 | 2 | 20 | 40 |
| **Total** | **46** | | **-780** |

$$\bar{X} = 1000 + \frac{-780}{46} = 983.04 kgs$$

Mean = 983.04 kgs.

**Mean for grouped data**

**Example**

The table shows the frequency distribution of the rate of unemployment in Kampala in 1990.



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| *Unemployment rate* | *Frequency* |
|---|---|
| 7.0 – 7.4 | 2 |
| 7.5 – 7.9 | 4 |
| 8.0 – 8.4 | 5 |
| 8.5 – 8.9 | 4 |
| 9.0 – 9.4 | 3 |
| **9.5 – 9.9** | 2 |

Calculate the mean of unemployment rate

**Solution**

| *Unemployment (x)* | *Frequency (f)* | *fx* |
|---|---|---|
| 7.2 | 2 | 14.4 |
| 7.7 | 4 | 30.8 |
| 8.2 | 5 | 41.0 |
| 8.7 | 4 | 34.8 |
| 9.2 | 3 | 27.6 |
| 9.7 | 2 | 19.4 |
| **Total** | **20** | **168.0** |

$$\bar{X} = \frac{\sum fx}{\sum f}$$

$$\text{Mean} = \frac{168}{20} = 8.4$$

So mean unemployment rate is 8.4

**Use of working mean**



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**Example 3**

The table below show the weight of 250 patients at kibuli Hospital

| Weight (kgs) | Frequency |
|---|---|
| 44.0 – 47.9 | 3 |
| 48.0 – 51.9 | 17 |
| 52.0 – 55.9 | 50 |
| 56.0 – 59.9 | 45 |
| 60.0 – 63.9 | 46 |
| 64.0 – 67.9 | 57 |
| 68.0 – 71.9 | 23 |
| 72.0 – 75. 9 | 9 |

Find the average weight of Patients

**Solution**

A suitable working mean is selected from any of the middle values of the class marks.

Let A = 57.95

| *Weight* | *x* | *f* | *d* | *fd* |
|---|---|---|---|---|
| 44.0 – 47.9 | 45.95 | 3 | -12 | -36 |
| 48.0 – 51.9 | 49.95 | 17 | -8 | -136 |
| 52.0 – 55.9 | 53.95 | 50 | -4 | -200 |
| 56.0 – 59.9 | 57.95 | 45 | 0 | 0 |
| 60.0 – 63.9 | 61.95 | 46 | 4 | 184 |
| 64.0 – 67.9 | 65.95 | 57 | 8 | 456 |
| 68.0 – 71. 9 | 69.95 | 23 | 12 | 276 |
| 72.0 – 75. 9 | 73.95 | 9 | 16 | 144 |
| **Total** | | **250** | | **688** |



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$$\overline{X} = 57.95 + \frac{688}{250} = 60.702kgs$$

**Advantages of using the mean include**

1) All values are used in the calculation

2) It is easy to understand and calculate

**Disadvantages of using mean**

One or two very low or high values can distort the information

**(II) MEDIAN**

The median of a set of numbers is the middle one when they are arranged in order of magnitude. If the number of observations is even, then the median is the average of the two middle values.

**Example**

Determine the median for given set of data?

(i) 5, 10, 10, 6, 6, 9, 10, 7, 7

(ii) 5,5,5, 9,9,8,9,6,6,8

**Solution**

Arrange in either ascending or descending order

(i) 5, 6, 6, 7, 7, 9, 10, 10, 10.

Therefore middle value is 7

Median = 7

(ii) 5, 5, 5, 6, 6, 8, 8, 9, 9, 9



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The middle values are 6 and 8

$$\text{Median} = \frac{6+8}{2} = 7$$

Therefore the median is 7.

**For the grouped data the formula below is used**

$$\text{Median} = L_1 + \frac{\left(\frac{N}{2} - F_b\right)C}{f_m}$$

Where

$L_1$ = Lower class boundary of median class

N = Total number of observations

Fb = Cumulative frequency before the median class

$f_m$ = Frequency of the median class

C = Class width.

**Example**

**Using example 3**

Find the median weight of Patients

**Solution**

$$\text{Median} = L_1 + \frac{(\frac{N}{2} - F_b)C}{f_m}$$



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| *Weight (kgs)* | *f* | *Cumulative frequency* |
|---|---|---|
| 44.0 – 47.9 | 3 | 3 |
| 48.0 – 51.9 | 17 | 20 |
| 52.0 – 55.9 | 50 | 70 |
| 56.0 – 59.9 | 45 | 115 |
| 60.0 – 63.9 | 46 | 161 |
| 64.0 – 67.9 | 57 | 218 |
| 68.0 – 71.9 | 23 | 241 |
| 72.0 – 75. 9 | 9 | 250 |

Median class is obtained from cumulative frequency just above or equal to $\frac{250}{2} = 125$. Note: $\frac{N}{2}$

Median class = 60.0 – 63.9

$$\text{Median} = 59.95 + \frac{(125-115)}{46} x4$$

$$= 60.82 \text{ kgs}$$

**Advantages of median include**

1. It is not affected by extreme values
2. It is easy to understand and calculate

**Disadvantages of using median**

Is that it is only one or two values that decide the median

**Note:** that you can use the Ogive to determine the median



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## (III) Mode

The mode is the value that occurs most frequently in given set of data

Find the mode
3, 3, 3, 4, 4, 6, 6, 7, 7, 7, 7
Since 7 appears four times
Therefore the mode = 7

**For grouped data the mode is given by**

$$\text{Mode} = L_1 + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right)C$$

Where

$L_1$ = Lower class boundary of modal class

$\Delta_1$ = Difference between highest frequency and value before it

$\Delta_2$ = Difference between highest frequency and value after it

C = Class width.

**Example**

**Using example 3**

Calculate the mode for data

**Solution**

Modal class = 64.0 – 67.9 since 57 is the highest frequency.



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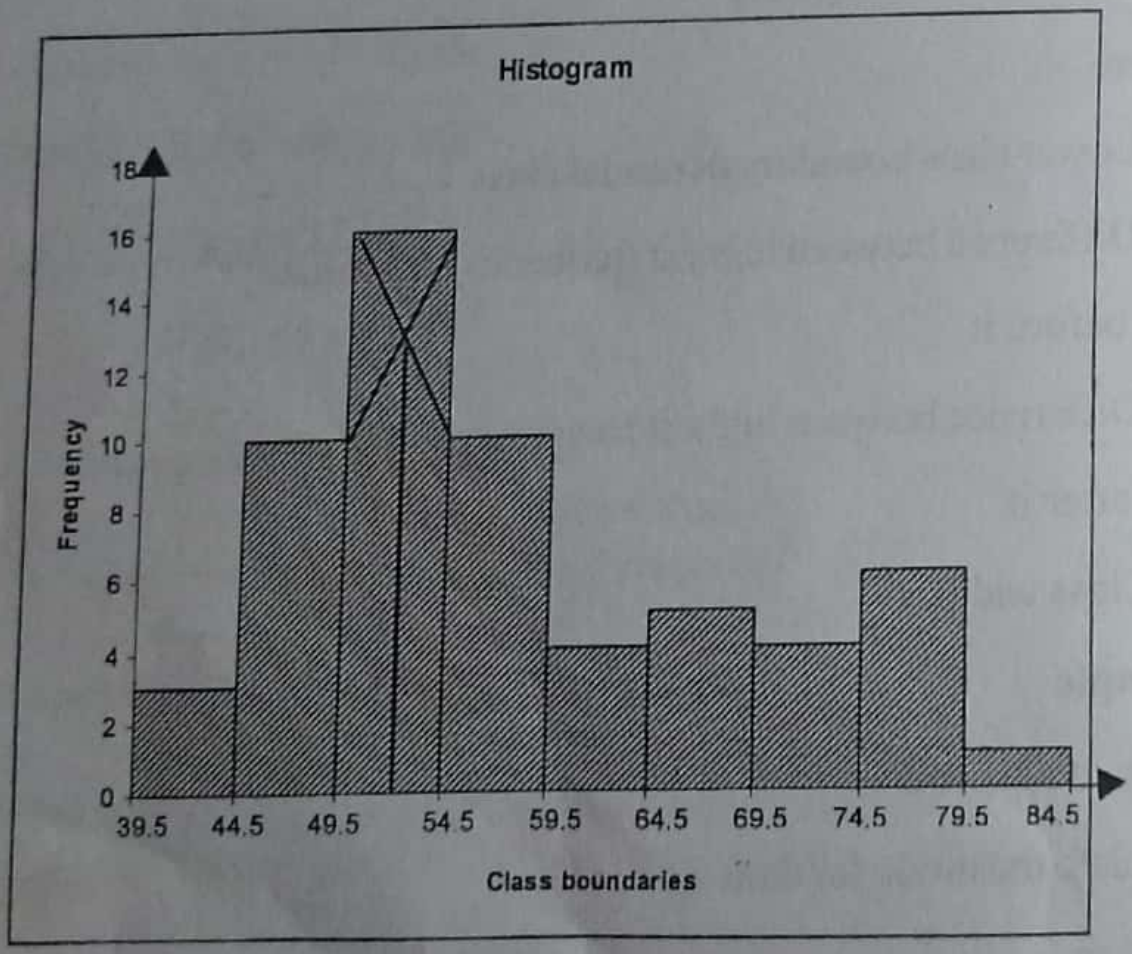
$$\text{Mode} = L_1 + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right)C$$

$$\text{Mode} = 63.95 + \frac{11x4}{11+34}$$

Mode = 64.93 kgs

**Note:** The mode can be obtained from the histogram. The modal class is identified, the class with highest frequency or frequency density. The position of the mode within the modal class can be estimated from the histogram as shown in the figure below.




Advantage of mode

Very easy to understand and calculate



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**Example**

The age of people in town were as follows

| Age | Number in thousands |
|---|---|
| 0 - < 5 | 4.4 |
| 5 - < 15 | 8.1 |
| 15 - < 30 | 10.5 |
| 30 - < 50 | 14.6 |
| 50 - < 70 | 9.8 |
| 70 - < 90 | 4.7 |

Determine the median and mean

**Solution**

| *age* | *x* | *f* | *fx* | *Cf* |
|---|---|---|---|---|
| 0 - < 5 | 2.5 | 4.4 | 11 | 4.4 |
| 5 - < 15 | 10 | 8.1 | 81 | 12.5 |
| 15 - < 30 | 22.5 | 10.5 | 236.25 | 23 |
| 30 - < 50 | 40 | 14.6 | 584 | 37.6 |
| 50 - < 70 | 60 | 9.8 | 588 | 47.4 |
| 70 - < 90 | 80 | 4.7 | 376 | 52.1 |
| **Total** | | **52.1** | **1876.25** | |

$$\text{Median} = 30 + \frac{(26.05 - 23)}{14.6} x20$$

$$= 34.178 \text{ years}$$

$$\text{Mean} = \frac{\sum fx}{\sum f} = \frac{1876.25}{52.1}$$

Mean = 36. 012 years



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## 1.4 MEASURES OF SPREAD ( DISPERSION)

These are measures used to find out how the observations are spread out from the average. These include; range, the mean deviation,quartile range, semi-interquartile range, variance and standard deviation.

### (a) RANGE

The range of a set of numbers is the difference between the largest and the smallest value of the set.
For example S = { 45, 54, 64, 76, 86}
The range of S is = 86 – 45 = 41

### (b) THE MEAN DEVIATION

The mean deviation of a set of numbers is given by

$$\sum_{1}^{n} \frac{|x_i - M|}{n} \quad \text{Where M is the Mean}$$

**Example**

Find the mean deviation of the set of numbers
45, 54, 64, 76, 86

**Solution**

$$M = \frac{\sum x}{n}$$

$$M = \frac{45 + 54 + 64 + 76 + 86}{5}$$

$$= 65$$

**Mean deviation**



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$$= \frac{|45-65|+|54-65|+|64-65|+|76-65|+|86-65|}{5}$$

$$= \frac{20+11+1+11+21}{5}$$

Mean deviation = 12.8

## (c) VARIANCE FOR POPULATION

This is sum of the mean deviations squared divided by the number of observations.

$$\text{Variance} = \frac{\sum_{1}^{n}(x_i - M)^2}{n}$$ Note:

Where M is the mean

For given set of numbers, $x_1, x_2, x_3, \ldots, x_n$

**Example**

Find the variance of 45, 54, 64, 76, 86

A simplified form of the formula normally used for computations is

$$\text{Variance} = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2$$



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**Solution**

| x | $x^2$ |
|---|---|
| 45 | 2025 |
| 54 | 2916 |
| 64 | 4096 |
| 76 | 5776 |
| 86 | 7396 |
| **325** | **22209** |

$$\text{Variance} = \frac{\sum x^2}{n} - (\frac{\sum x}{n})^2$$

$$= \frac{22209}{5} - (65)^2$$

Variance = 216.8

$\text{Variance} = \frac{\sum fx^2}{\sum f} - (\frac{\sum fx}{\sum f})^2$ is used for repeated values or grouped data

**Note**: Standard deviation is the positive square root of the variance

**Example**

The frequency distribution table shows the marks of some students from a certain school.



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| x | 55 | 63 | 65 | 66 | 70 | 72 | 75 | 80 | 90 |
|---|---|---|---|---|---|---|---|---|---|
| f | 2 | 2 | 3 | 1 | 2 | 2 | 4 | 3 | 1 |

Calculate the standard deviation of the data

**Solution**

| x | f | $x^2$ | fx | $fx^2$ |
|---|---|---|---|---|
| 55 | 2 | 3025 | 110 | 6050 |
| 63 | 2 | 3969 | 126 | 7938 |
| 65 | 3 | 4225 | 195 | 12675 |
| 66 | 1 | 4356 | 66 | 4356 |
| 70 | 2 | 4900 | 140 | 9800 |
| 72 | 2 | 5184 | 144 | 10368 |
| 75 | 4 | 5625 | 300 | 22500 |
| 80 | 3 | 6400 | 240 | 19200 |
| 90 | 1 | 8100 | 90 | 8100 |
| **Total** | **20** | | **1411** | **100987** |

$\sum fx = 1411$ *and* $\sum fx^2 = 100987$

$$\text{Variance} = \frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2$$

$$= \frac{100987}{20} - \left(\frac{1411}{20}\right)^2$$

$$= 72.05$$

Therefore Standard deviation = $\sqrt{72.05}$

Standard deviation = 8.488 marks.



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**Example**

The frequency distribution table shows the heights of some students from a certain school

| Height | 154 | 155 | 160 | 164 | 171 | 180 |
|---|---|---|---|---|---|---|
| Frequency | 4 | 6 | 8 | 5 | 4 | 3 |

Determine the variance of the above data

**Solution**

$$\text{Variance} = \frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2$$

D = x – A, let A be 160

| x | f | d | fd | $fd^2$ |
|---|---|---|---|---|
| 154 | 4 | -6 | -24 | 144 |
| 155 | 6 | -5 | -30 | 150 |
| 160 | 8 | 0 | 0 | 0 |
| 164 | 5 | 4 | 20 | 80 |
| 171 | 4 | 11 | 44 | 484 |
| 180 | 3 | 20 | 60 | 1200 |
| Total | **30** | | **70** | **2058** |

$$\text{Variance} = \frac{2058}{30} - \left(\frac{70}{30}\right)^2$$

$$= 68.6 - 5.4444$$

$$\text{Variance} = 63.1556$$



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**Example**

**Using example 2**

Determine the variance and standard deviation

**Solution**

| x | f | $x^2$ | fx | $f x^2$ |
|---|---|---|---|---|
| 12 | 5 | 144 | 60 | 720 |
| 17 | 9 | 289 | 153 | 2601 |
| 22 | 12 | 484 | 264 | 5808 |
| 27 | 18 | 729 | 486 | 13122 |
| 32 | 25 | 1024 | 800 | 25600 |
| 37 | 15 | 1369 | 555 | 20535 |
| 42 | 10 | 1764 | 420 | 17640 |
| 47 | 6 | 2209 | 282 | 13254 |
| **Total** | **100** | | **3020** | **99280** |

$$\text{Variance} = \frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2$$

$$= \frac{99280}{100} - \left(\frac{3020}{100}\right)^2$$

$$= 992.8 - 912.04$$

$$= 80.76$$

Therefore variance = 80.76

Standard deviation = $\sqrt{80.76}$

= 8.987 kgs.



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**Working mean can be used to obtain variance.**

The formula used is

$$\text{Variance} = \frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2$$

Where d = x – A and A is working mean

**Example**

Using **example 3** and working mean of 57.95 find the variance for given data.

**Solution**

| *Weight* | *x* | *f* | *d* | *fd* | *fd²* |
|---|---|---|---|---|---|
| 44.0 – 47.9 | 45.95 | 3 | -12 | -36 | 432 |
| 48.0 – 51.9 | 49.95 | 17 | -8 | -136 | 1088 |
| 52.0 – 55.9 | 53.95 | 50 | -4 | -200 | 800 |
| 56.0 – 59.9 | 57.95 | 45 | 0 | 0 | 0 |
| 60.0 – 63.9 | 61.95 | 46 | 4 | 184 | 736 |
| 64.0 – 67.9 | 65.95 | 57 | 8 | 456 | 3648 |
| 68.0 – 71. 9 | 69.95 | 23 | 12 | 276 | 3312 |
| 72.0 – 75. 9 | 73.95 | 9 | 16 | 144 | 2304 |
| **Total** | | **250** | | **688** | **12320** |



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Variance = $\frac{\sum fd^2}{\sum f} - (\frac{\sum fd}{\sum f})^2$

$$= \frac{12320}{250} - \left(\frac{688}{250}\right)^2$$

$= 49.28 - 7.573504$

Variance = 41.7065

**Example**

The frequency table shows the height (H) of 80 students to nearest cm.

| *Height (cm)* | ***Frequency*** |
|---|---|
| 150 - 154 | 3 |
| 155 - 159 | 7 |
| 160 – 164 | 10 |
| 165 – 169 | 15 |
| 170 – 174 | 25 |
| 175 – 179 | 12 |
| 180 – 184 | 6 |
| **185 - 189** | 2 |

Calculate the mean and standard deviation

**Solution**

Let A = 167, $t = \frac{d}{c}$ *where* $d = x - A$ and c = 5



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| $x$ | $f$ | $d$ | $t$ | $ft$ | $ft^2$ |
|---|---|---|---|---|---|
| 152 | 3 | -15 | -3 | -9 | 27 |
| 157 | 7 | -10 | -2 | -14 | 28 |
| 162 | 10 | -5 | -1 | -10 | 10 |
| 167 | 15 | 0 | 0 | 0 | 0 |
| 172 | 25 | 5 | 1 | 25 | 25 |
| 177 | 12 | 10 | 2 | 24 | 48 |
| 182 | 6 | 15 | 3 | 18 | 54 |
| 187 | 2 | 20 | 4 | 8 | 32 |
| **Total** | **80** | | | **42** | **224** |

$$Mean = A + \frac{\sum ft}{\sum f} c$$

$$Mean = 167 + \frac{42 x 5}{80}$$

$$= 169.625\, cm$$

$$\text{Standard deviation} = c\sqrt{\left(\frac{\sum ft^2}{\sum f} - (\frac{\sum ft}{\sum f})^2\right)}$$

$$= 5\sqrt{\left(\frac{224}{80} - (\frac{42}{80})^2\right)}$$

$$= 7.944 \text{ cm}$$

**Note**: The formula above is used only when the classes are of the same class width.

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**Example**

The information below shows the amount of money in millions (S) given to some districts in Uganda for "Entandikwa" scheme.

| S | 25 - < 30 | 30 - < 40 | 40-<50 | 50-< 60 | 60-<80 |
|---|---|---|---|---|---|
| f | 4 | 10 | 4 | 3 | 5 |

Find the mean and standard deviation of the distribution

**Solution**

Note: The classes above are of different class width.

| *class* | *x* | *f* | *fx* | $x^2$ | $fx^2$ |
|---|---|---|---|---|---|
| 25 - < 30 | 27.5 | 4 | 110 | 756.25 | 3025 |
| 30 - < 40 | 35 | 10 | 350 | 1225 | 12250 |
| 40 - < 50 | 45 | 4 | 180 | 2025 | 8100 |
| 50 - < 60 | 55 | 3 | 165 | 3025 | 9075 |
| 60 - < 80 | 70 | 5 | 350 | 4900 | 24500 |
| **Total** | | **26** | **1155** | | **56950** |

$$Mean = \frac{\sum fx}{\sum f}$$

$$= \frac{1155}{26}$$

$$Mean = 44.423\ millions$$



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$$\sqrt{\left(\frac{\sum fx^2}{\sum f}-(\frac{\sum fx}{\sum f})^2\right)}$$

Standarddeviation= $\sqrt{\left(\frac{56950}{26}-(\frac{1155}{26})^2\right)}$

$\sqrt{(2190.384615-1973.409763)}$

$\sqrt{216.974852}$

$=14.73$ millions

**Variance for Sample**

**Note:** Variance for a population is donated by

Variance of sample ($s^2$)

$$= \left(\frac{n}{n-1}\right) \quad \frac{\sum x^2}{n}-(\frac{\sum x}{n})^2$$

**Example**

A sample of 8 students gave the following as the estimates for the length of certain plot in meters as, 44, 48, 57, 53, 50, 52, 49 and 51. Calculate the standard deviation of the plot?



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**Solution**

Sample variance = $\left(\frac{n}{n-1}\right) \quad \frac{\sum x^2}{n} - (\frac{\sum x}{n})^2$

| $x$ | $x^2$ |
|---|---|
| 44 | 1936 |
| 48 | 2304 |
| 49 | 2401 |
| 50 | 2500 |
| 51 | 2601 |
| 52 | 2704 |
| 53 | 2809 |
| 57 | 3249 |
| 404 | 20504 |

Sample variance = $\frac{8}{7}\left(\frac{20504}{8} - (\frac{404}{8})^2\right)$

$= 14.5714$

Standard deviation = $\sqrt{14.5614}$

$= 3.817$

**Example**

A certain factory produces ball bearings. A sample of the bearing from the factory produced the following results



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| Diameter of bearing in mm | frequency |
|---|---|
| 91 – 93 | 4 |
| 94 – 96 | 6 |
| 97 – 99 | 34 |
| 100 - 102 | 40 |
| 103 – 105 | 13 |
| 106 - 108 | 3 |

Determine the mean and variance of the diameter of the sample bearings.

**Solution**

$$\text{Mean} = \frac{\sum fx}{\sum f}$$

$$\text{Sample variance} = \left(\frac{\sum f}{\sum f - 1}\right)\left(\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right)$$

| *class* | *f* | *x* | $x^2$ | *fx* | $fx^2$ |
|---|---|---|---|---|---|
| 91-93 | 4 | 92 | 8464 | 368 | 33856 |
| 94 – 96 | 6 | 95 | 9025 | 570 | 54150 |
| 97 – 99 | 34 | 98 | 9604 | 3332 | 326536 |
| 100 - 102 | 40 | 101 | 10201 | 4040 | 408040 |
| 103 – 105 | 13 | 104 | 10816 | 1352 | 140608 |
| 106 - 108 | 3 | 107 | 11449 | 321 | 34347 |
| **Total** | **100** | | | **9983** | **997537** |



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$$Mean = \frac{9983}{100}$$
$$= 99.83\,mm$$

$$\text{Sample Variance} = \left(\frac{100}{99}\right)\left(\frac{997537}{100} - \left(\frac{9983}{100}\right)^2\right)$$
$$= 9.4355 \text{ mm}$$

## 1.5 PERCENTILES, DECILES, AND QUARTILES

### a) PERCENTILES

These are values that divide a set of observations into 100 equal parts. These values, denoted by $P_1, P_2, P_3. - . , P_{99}$, are such that 1% of the data falls below $P_1$, 20% of the data below $P_{20}$, etc.

**Example**

Suppose the number of observation n is 200, then

$$P_{10} = \left(\frac{10}{100} x\, 200\right) = 20$$

This implies that 20 observations are below it.

### b) DECILES

The values divide a set of observation in 10 equal parts. These are denoted by $D_1, D_2, D_3 \ldots D_9$. Such that 20% of the total frequency fall below $D_2$



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Suppose n = 500, determine $D_6$

$$D_6 = \frac{6}{10} x 500 = 300$$

## c) QUARTILES

Quartiles are values that divide a set of observation into 4 equal parts. These values are denoted by $Q_1$, $Q_2$, and $Q_3$, are such that 25% of the data falls below $Q_1$, 50% below $Q_2$ and 75% below $Q_3$.

**NOTE:** Percentiles, deciles and quartiles can be obtained from the Ogive. $Q_3$ is known as the upper quartile while $Q_1$ is known as the lower quartile.

## THE INTERQUARTILE RANGE ( QUARTILE RANGE)

= Upper quartile – lower quartile = $Q_3 - Q_1$

$$Upper\ quartile = L_3 + \left(\frac{\frac{3N}{4} - F_b}{f_q}\right) x C$$

Where:

$L_3$ = lower class boundary of upper quartile class

N = Total number of observations

$F_b$ = Cumulative frequency before the upper quartile class

$f_q$ = frequency of the upper quartile class.

C = class width



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$$Lower\ quartile = L_1 + \left(\frac{\frac{N}{4} - F_b}{f_q}\right) x C$$

Where:

$L_1$ = lower class boundary of lower quartile class
N = Total number of observations
$F_b$ = Cumulative frequency before the lower quartile class
$f_q$ = frequency of the lower quartile class.
C = class width

## THE SEMI-INTERQUARTILE RANGE

$$= \frac{1}{2}(Q_3 - Q_1)$$

**Example**

The data below gives the weight of some students from a certain school.

| *Mass* | *Frequency* |
|---|---|
| 40 – 44 | 3 |
| 45 – 49 | 30 |
| 50 – 54 | 29 |
| 55 – 59 | 33 |
| 60 – 64 | 13 |
| 65 – 69 | 1 |
| 70 - 74 | 1 |

Draw an Ogive and use it to estimate

i. The median mass
ii. 10 -90 percentile range
iii. Interquartile range

(iv) The number of students who weigh above 47 kgs.



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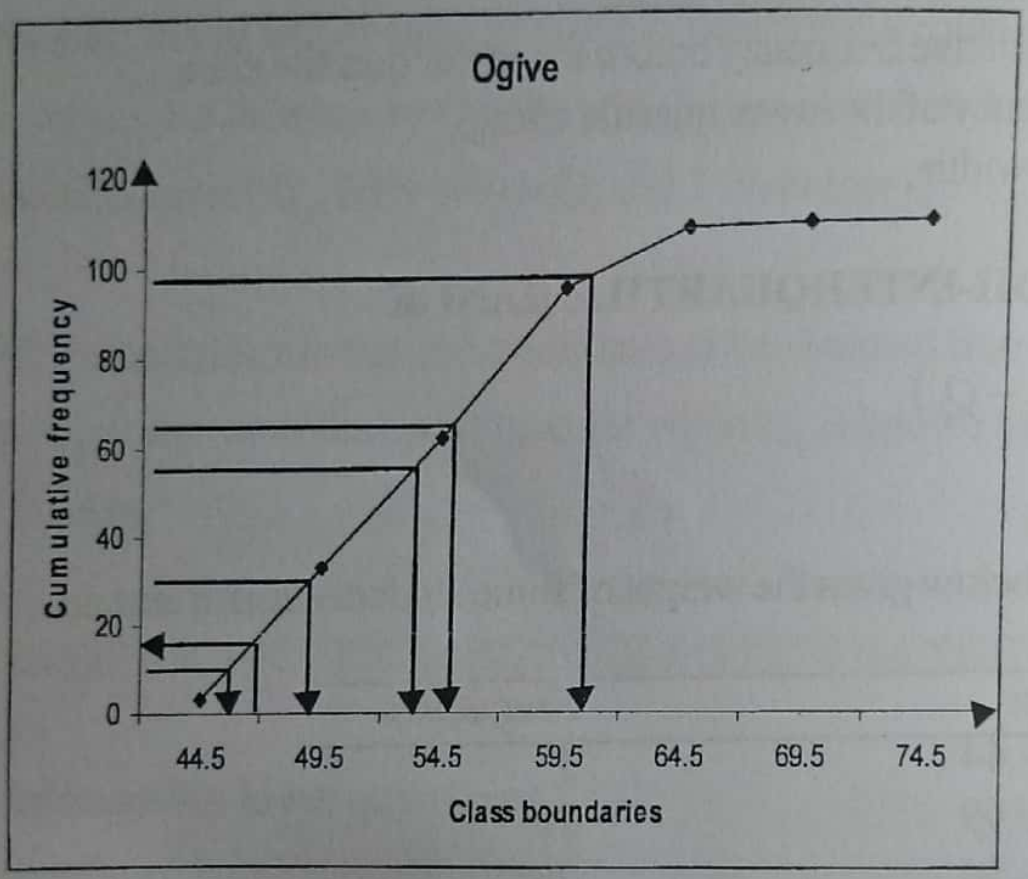
**Solution**

| | 44.5 | 49.5 | 54.5 | 59.5 | 64.5 | 69.5 | 74.5 |
|---|---|---|---|---|---|---|---|
| Cf | 3 | 33 | 62 | 95 | 108 | 109 | 110 |




i. Median = 53.5 kgs from the graph

ii. $P_{90} - P_{10} = 60.5 - 46.5 = 14$ kgs ( graph)

iii. $Q_3 - Q_1 = 57.5 - 48.5 = 9$ kgs ( Graph)

iv. Number of people weighing below 47 kgs = 17

Therefore the number above 47 kgs =

$$110 - 17 = 93 \text{ students}$$



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## Exercise

1. The table below shows the marks obtained by students of mathematics in a certain school

| *Marks* | *Number of students* |
|---|---|
| 30 - < 40 | 2 |
| 40 - < 50 | 15 |
| 50 - < 55 | 10 |
| 55 - < 60 | 11 |
| 60 - < 70 | 30 |
| 70 - < 90 | 29 |
| 90 - < 100 | 3 |

a) Draw a histogram for the data and hence determine the modal mark
b) Draw an ogive for the data
c) Calculate the mean, median and standard deviation

**Answer** (c) mean = 64.575, median = 64
Standard deviation = 13.852

2. The data below shows the amount of cotton (in 100's of bales) produced by Growers' Unions over a certain period of time.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 70 | 41 | 34 | 55 | 45 | 66 | 73 | 77 | 80 | 30 |
| 50 | 45 | 72 | 50 | 27 | 70 | 55 | 70 | 85 | 70 |
| 30 | 50 | 60 | 53 | 40 | 45 | 35 | 55 | 20 | 81 |
| 25 | 51 | 35 | 62 | 60 | 30 | 45 | 35 | 50 | 89 |
| 53 | 23 | 28 | 65 | 68 | 50 | 65 | 34 | 35 | 76 |



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i. Beginning with the 20 – 29 class and using intervals of equal width, construct a frequency table for the data. Using the frequency table

ii. Draw a cumulative frequency curve for the data and hence estimate the median production

iii. Calculate the mean and standard deviation of production.

Answer (ii) median = 5450 bales (iii) mean = 5370 bales, standard deviation = 1798 bales

3. The amount of money collected by children in a charity walk in thousands of shillings is recorded in the table below:

| Amount('000 Ushs) | 1 | 2- 4 | 5 -1 | 11 - 15 | 16 -25 |
|---|---|---|---|---|---|
| Frequency | 20 | 93 | 90 | 58 | 39 |

i. Calculate the mean and standard deviation of the children's collection

ii. Draw a histogram and use it to estimate the modal collection

iii. Construct an ogive and use it to estimate the number of children who collected between 3,000/= and 7,000/=.

**Answer** (i) mean = 8425 /=

standard deviation = 6004.7 /=

From the graph modal collection = 2900 (iii) 68



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4. The table below shows the weights to the nearest kg of 150 patients who visited a certain health unit during acertain week.

| *weight (kg)* | *Number of patients (f)* |
|---|---|
| 0 - 19 | 30 |
| 20 -29 | 16 |
| 30 - 39 | 24 |
| 40 - 49 | 32 |
| 50 - 59 | 28 |
| 60 – 69 | 12 |
| 70 - 79 | 8 |

a) Calculate the approximate mean and modal weights of the patients.

b) Plot an ogive for the data. Use the ogive to estimate the

i. Median and semi- interquartile range for the weights of patients,

**Answer** (a) mean = 38.8333 kgs,

Modal weight = 46. 167 kgs, Median = ( 41.0 – 42.0) kgs

Semi-interquartile range = ( 14.5 – 15.5) kgs.

5. The ages of people in Kampala were as follows



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| *Age (years)* | *Number in thousands* |
|---|---|
| 0 - < 5 | 4.4 |
| 5 - <15 | 8.1 |
| 15 - < 30 | 10.5 |
| 30 - < 50 | 14.6 |
| 50 - < 70 | 9.8 |
| 70 - < 90 | 4.7 |

a) Draw a histogram for the data

b) State the modal age interval

c) Estimate

i. Average age of Kampala

ii. Number of people under 18 years

iii. Median age

**Answer** (b) modal age interal = ( 0 - < 5)

(c ) mean = 36.01 yrs (ii) 14600 (iii) 34. 178yrs

6. Find the mean and standard deviation of the first n integers

**Answer**: mean = $\frac{1}{2}(n+1)$

Standard deviation = $\sqrt{\frac{(n-1)(n+1)}{12}}$



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7. The table below gives the weekly earnings of a random sample of workers in Mukwano soap factory

| Weekly earnings (Shs) | Number of workers |
|---|---|
| Under 1500 | 1 |
| 1500 and under 2000 | 4 |
| 2000 and under 2500 | 28 |
| 2500 and under 3000 | 42 |
| 3000 and under 3500 | 33 |
| 3500 and under 4000 | 18 |
| 4000 and under 4500 | 13 |
| 4500 and under 5000 | 9 |
| 5000 and over | 2 |

Calculate

i. The percentage number of workers earning Shs 3050 and above.

ii. The mean weekly earning and standard deviation of the distribution

iii. If the wages are increased by 10%, determine the new mean weekly wage and new standard deviation



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**Answer** (i) 45% (ii) mean = 3133.333,

standard deviation = 799.364 (iii) mean = 3446.67,

standard deviation = 879.30

8. The heights of a senior five class in a certain school were recorded as in the frequency table below

| *Height (cm)* | *Frequency* |
|---|---|
| 170 - 175 | 19 |
| 175 - 180 | 36 |
| 180 - 185 | 70 |
| 185 - 190 | 64 |
| 190 - 195 | 39 |
| 195 - 200 | 22 |

(a) estimate the mean and standard deviation of the students' height

(b) Plot an Ogive

(c) Use your Ogive to estimate the:

(i) median and inter- quartile range for the data



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## 1.6 INDEX NUMBERS

An index number is a statistical measure, which represents the change in a variable or group of variables with respect to time, environment or other characteristics

**Base year**

This is the year or period against which all other years or periods are compared.

**Current year**

This the year (period) *for* which the index is to be calculated

**SIMPLE INDEX NUMBER**

Price index, quantity index, wage index e.t.c.. are examples of the simple index numbers. Simple price index is often known as a price relative and it is given by $\frac{p_1}{p_0} \times 100$ or $\frac{p_1}{p_0}$

Where $p_1$ is the price of unit of one commodity in the current period while $p_0$ is the price of unit of one commodity in the base period.

**Example**

One litre of petrol costs shs. 1200 in 1998 and shs. 1800 in 2004.Taking 1998 as the base year, find the price relative in 2004

**Solution**

$$\text{Price relative} = \frac{1800}{1200} x100 = 150$$

*This indicates that the price of petrol has gone up by 50%*



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NOTE: The quantity index can be obtained by using

$$Quantity\ index = \frac{q_1}{q_0} x100$$

**Example**

The wage of Nurses in Uganda in 1994 was Shs 30,000.The wage of the same Nurse in 1996 was increased by 15,000/=. Using 1994 as the base year calculate the nurses wage index for 1996.

**Solution**

$W_1$=30,000+15,000=45,000

$W_o$=30,000

$$\text{Wage index} = \frac{45000}{30000} x100 = 150$$

Therefore the nurses wage increased by 50 % in 1994.

**Note:** The percentage sign is always omitted in the answer

**Example**

In 1990 the price index of a commodity, using 1986 as base year was 112. In 1996, the index using 1990 as base year was 85. What would have been the index in 1996, using 1986 as base year?

**Solution**

$$\frac{p_{1990}}{p_{1986}} = 1.12$$

$$\frac{p_{1996}}{p_{1990}} = 0.85$$



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$$\frac{p_{1996}}{p_{1986}} = 1.12 x 0.85$$

$$= 0.952$$

$$= 95.2$$

Sometimes you can calculate the total price of group of items as ratio of the total price of the same group of items in the base year This index is known as the simple aggregate price index

$$\text{Simple aggregate price index} = \frac{\sum p_1}{\sum p_0}$$

**Example**

The table shows the price in dollars of sugar and cassava in 1990 and 1995

| Item | 1990 | 1995 |
|---|---|---|
| Sugar (bag) | 80 | 120 |
| Cassava (bag) | 10 | 15 |

Determine simple aggregate price index for the total cost of one bag of cassava and sugar.

**Solution**

| Item | 1990 ( $P_0$) | 1995 ( $P_1$) |
|---|---|---|
| Sugar | 80 | 120 |
| Cassava | 10 | 15 |
| Total | 90 | 135 |



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Simple aggregate price index = $\frac{135}{90} = 1.5$

**Example**

An average family in Kampala spent the following amounts per month on the items shown, in the years 1999 and 2000

| *ITEM* | *1999 Amount in Uganda sh.* | *2000 Amount in Uganda Sh.* |
|---|---|---|
| Housing | 80,000 | 100,000 |
| Clothing | 20,000 | 20,000 |
| Electricity | 40,000 | 50,000 |
| Water | 10,000 | 12,000 |
| Food | 140,000 | 160,000 |
| Transport | 50,000 | 60,000 |
| Medical | 30,000 | 35,000 |
| Miscellaneous | 30,000 | 40,000 |

( They save Shs 50,000 per month)



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Using 1999 as the base year, calculate the simple aggregate expenditure index for 2000.

**Solution**

| *ITEM* | *1999 Amount in Uganda sh.* | *2000 Amount in Uganda Sh.* |
|---|---|---|
| Housing | 80,000 | 100,000 |
| Clothing | 20,000 | 20,000 |
| Electricity | 40,000 | 50,000 |
| Water | 10,000 | 12,000 |
| Food | 140,000 | 160,000 |
| Transport | 50,000 | 60,000 |
| Medical | 30,000 | 35,000 |
| Miscellaneous | 30,000 | 40,000 |
| Total | 400,000 | 477,000 |

Simple aggregate expenditure index =

$$= \frac{477,000}{400,00} = 1.1925$$



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Therefore the expenditure went by 19.25 % in 2000.

## COMPOSITE INDEX NUMBER

Index numbers are at times needed where there is more than just one item, for example an index number that compares the cost of living depends on food, clothing, housing,entertainment,etc...

**Example**

Find the cost of living based on the following data.

| *Item* | *Price index* | *Weight* |
|---|---|---|
| Housing | 125 | 170 |
| Clothing | 124 | 160 |
| Electricity | 150 | 120 |
| Water | 110 | 180 |
| Food | 120 | 172 |
| Transport | 135 | 210 |
| Others | 104 | 140 |

**Solution**

| *ITEM* | *Price Index (A)* | *Weight (W)* | *WA* |
|---|---|---|---|
| Housing | 125 | 170 | 21,250 |
| Clothing | 124 | 160 | 19,840 |
| Electricity | 150 | 120 | 18,000 |
| Water | 110 | 180 | 19,800 |
| Food | 120 | 172 | 20,640 |
| Transport | 135 | 210 | 28,350 |
| Others | 104 | 140 | 14,560 |
| **Total** | | **1152** | **142,440** |



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$$\text{Cost of living index} = \frac{142,440}{1152} = 123.65$$

**Example**

The following items are used in the assembly of a TV set. 8 transistors,22 resistors, 9 capacitors, 2 diodes and a circuit board. Due to inflation the price of each component has increased as shown in the table:

| *Year* | *Transistor* | *Resistor* | *Capacitor* | *Diodes* | *Circuit* |
|---|---|---|---|---|---|
| 2000 | 1200/= | 1650/= | 1500/= | 1600/= | 2000/= |
| 2002 | 1800/= | 2100/= | 1700/= | 1800/= | 2500/= |

Calculate the composite index number of the assembled TV set in 2002 using 2000 as the base year

**Solution**

$P_0$ (2000)

| *Item* | *Weight (W)* | *Price ($P_0$)* | *$WP_0$* | *Price ($P_1$)* | *$WP_1$* |
|---|---|---|---|---|---|
| Transistor | 8 | 1200 | 9600 | 1800 | 14400 |
| Resistor | 22 | 1650 | 36300 | 2100 | 46200 |
| Capacitor | 9 | 1500 | 13500 | 1700 | 15300 |
| Diode | 2 | 1600 | 3200 | 1800 | 3600 |
| Circuit | 1 | 2000 | 2000 | 2500 | 2500 |
| **Total** | | | **64600** | | **82000** |

$$\text{Compositeindex} = \frac{82000}{64600} = 1.27$$



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**Note**: Composite index is the same as weighted aggregate index.

$$\text{Composite index} = \frac{\sum p_1 w}{\sum p_0 w} x100 = \frac{\sum q_1 w}{\sum q_0 w} x100$$

This is weighted aggregate method

Sometimes the prices of the commodities can be quoted in different units. In such situations the price relatives are used.

**Example**

Wavamuno imported the following commodities A, B, C from Kenya, U.S.A and England respectively.

| *Commodity* | *Price* | |
|---|---|---|
| | 1992 = 100 | 1996 |
| A | K sh 20,000 | K sh 25,000 |
| B | $ 4,000 | $ 4,800 |
| C | £8,000 | £10,,000 |

Calculate the simple relative price index for 1996
Note: 1992 = 100 means that 1992 is taken as the base year.

**Solution**

| *Commodity* | *Price* | | $\frac{p_1}{p_0}$ |
|---|---|---|---|
| | 1992 = 100 | 1996 | |
| A | K sh 20,000 | K sh 25,000 | 1.25 |
| B | $ 4,000 | $ 4,800 | 1.2 |
| C | £8,000 | £10,000 | 1.25 |
| **Total** | | | **3.7** |



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Index for 1996 = $\frac{3.7}{3} x100=123.3$

Therefore the price increased by 23 % in 1996 as compared to 1992.

**Note**: Price relatives can be used for commodities in the same units too.

**Example**

Nabisunsa Girls' School bought three types of chicken feed in 1998 and 2000. The weights and the corresponding prices are given in the table. Using 1998 as the base year calculate the weighted average relative price index.

| *Commodity* | *Weight* | *Price 1998* | *Price 2000* |
|---|---|---|---|
| A | 120 | 500 | 600 |
| B | 60 | 300 | 360 |
| C | 50 | 250 | 400 |

**Solution**

| *Commodity* | *weight* | *Price 1998* | *Price 2000* | $\frac{p_1}{p_0}$ | $\frac{p_1}{p_0} x w$ |
|---|---|---|---|---|---|
| A | 120 | 500 | 600 | 1.2 | 144 |
| B | 60 | 300 | 360 | 1.2 | 72 |
| C | 50 | 250 | 400 | 1.6 | 80 |
| **Total** | **230** | | | | **296** |

Weighted average relative price index

$$=\frac{296}{230} x100=128.7$$

The commodities increased by 28.7%



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**Example**

The wages and wage bills for a group of workers in a factory for two years are shown in the table.

Let S- skilled, U- unskilled and M- semi-skilled

| | *Year 1986 Wage (shs)* | *Total) wage Bill (shs)* | *Year 1986 Wage (shs)* | *Total wage Bill (shs)* |
|---|---|---|---|---|
| S | 7500 | 225,000 | 10,000 | 320,000 |
| M | 4000 | 160,000 | 4800 | 240,000 |
| U | 3500 | 157,000 | 4200 | 193,200 |

Determine

i. The weighted average wage for each year
ii. The wage index in 1989 taking 1986 as the base year.

**Solution**

Let N be the number of workers, W be wage in shillings.

| $W_0$ | $N$ | *Total Wage* | $W_1$ | $N$ | *Total Wage* |
|---|---|---|---|---|---|
| 7500 | 30 | 225,000 | 10,000 | 32 | 320,000 |
| 4000 | 40 | 160,000 | 4800 | 50 | 240,000 |
| 3500 | 45 | 157,500 | 4200 | 46 | 193,200 |
| Total | 115 | 542,500 | | 128 | 753,200 |

Weighted average wage for 1986 is

$$\frac{542,500}{115} = 4,717.40/=$$

Weighted average for 1989 is

$$\frac{753,200}{128} = 5,884.40/=$$



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The wage index in 1989 taking 1986 as the base year

$$= \frac{5,884.4}{4,717.4} = 124.7$$

Wage index = 124.7

**Example**

Musa reckons that in 1988, his farm produced 60% of his income, and that farm production in the years 1988, 1990, 1992 were in the ratio 60:55:60. His other income, a salary, in millions shillings was:

| 1988 | 1990 | 1992 |
|---|---|---|
| 8 | 20 | 32 |

However, in real terms this salary must be related to cost of living index, whch was

| 1988 | 1990 | 1992 |
|---|---|---|
| 100 | 160 | 240 |

i) Find his salary in real terms of each of the years given and express it as an index with 1988 as 100.

ii) still keeping 1988 as the base year, find an index of the total real income for 1990 and 1992.

iii) Amend (ii) if the farm production for 1988 formed 75% of his income (the ratio 60:55:50 remaining valid)



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**Solution.**

| | *Salary* Millions | *Farm* Millions | *Cost* of Living | *Total* income (m) |
|---|---|---|---|---|
| 1988 | 8 | 12 | 100 | 20 |
| 1990 | 20 | 24.4 | 160 | 44.4 |
| 1992 | 32 | 32 | 240 | 64 |
| **Total** | **60** | **68.4** | | **128.4** |

| *Year* | *Real salary* | *Salary index (1988=100)* |
|---|---|---|
| 1988 | $8x\frac{100}{100}=8$ | $\frac{8}{8}x100=100$ |
| 1990 | $20x\frac{100}{160}=12.5$ | $\frac{12.5}{8}x100=156.3$ |
| 1992 | $32x\frac{100}{240}=13.33$ | $\frac{13.33}{8}x100=166.6$ |

(ii) Index for total real income

| Year | Real income | index for total real income |
|---|---|---|
| 1988 | 20 | $\frac{20}{20}x100=100$ |
| 1990 | 44.4 | $\frac{44.4}{20}x100=222$ |
| 1992 | 64 | $\frac{64}{20}x100=320$ |

Salary was 25% of his total real income

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Therefore $\frac{25}{100} x(total\ real\ income) = 8)$

Total income = 32 millions

If 60 corresponds to 75%
Then 55 will correspond to y

Therefore y = $\frac{75 x 55}{60} = 68.75$ %

Salary was 31.25 % of total real income in 1990

| *Year* | *Salary* | *Real salary* | *Salary index (1988=100)* |
|---|---|---|---|
| 1988 | 8 | $8 x \frac{100}{25} = 32$ | $\frac{32}{32} x100 = 100$ |
| 1990 | 20 | $20 x \frac{100}{31.25} = 64$ | $\frac{64}{32} x100 = 200$ |
| 1992 | 32 | $32 x \frac{100}{37.5} = 85.3$ | $\frac{85.3}{32} x100 = 266.6$ |

**Example**

The cost of making a cake is calculated from the cost of baking flour, sugar, milk and eggs. The table gives the cost of these items in 1990 and 1996

| *Item* | *1990* | *1996* | *Weight* |
|---|---|---|---|
| Flour per kg | 600 | 780 | 12 |
| Sugar per kg | 500 | 400 | 5 |
| Milk per litre | 250 | 300 | 2 |
| Eggs per egg | 100 | 150 | 1 |



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Using 1990 as the base year

i. Calculate the price relatives for each item. Hence find simple price index for the cost of making a cake

ii. Find the weighted aggregate price index for the cost of making a cake

If the cost of making a cake in 1996 was Sh 300.

Find the cost in 1990 using the two indices in (i) and (ii)

**Solution**

| *Item* | *1990* | *1996* | *W* | $\frac{p_1}{p_0}$ | $p_0W$ | $p_1W$ |
|---|---|---|---|---|---|---|
| Flour / kg | 600 | 780 | 12 | 1.3 | 7200 | 9360 |
| Sugar / kg | 500 | 400 | 5 | 0.8 | 2500 | 2000 |
| Milk/l | 250 | 300 | 2 | 1.2 | 500 | 600 |
| Eggs/ egg | 100 | 150 | 1 | 1.5 | 100 | 150 |
| Total | | 20 | 4.8 | 10300 | | 12110 |

The fifth column has the price relatives

Simple price index = $\frac{4.8}{4} x100 = 120$

Weighted aggregate index = $\frac{12110}{10300} x100 = 117$

Cost in 1990

Using simple price index = 120

(i) From $\frac{p_1}{p_0} x100 = 120$



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$$\frac{300}{P_0} x100 = 120$$

$$P_0 = \frac{300}{120} x100 = 250$$

Cost in 1990 was 250/=

(ii) From $\frac{P_1}{P_0} x100 = 117$

$$\frac{300}{P_0} x100 = 117$$

$$P_0 = \frac{300}{117} x100 = 256.4/=$$

Cost in 1990 was 256.4/=

Sometimes different quantities are quoted for the base year and current year.

In such situations the methods that can be used include

i. Paasche aggregate price index
ii. Laspeyre aggregate price index.

**Paasche aggregate price index**

The quantities of the current year are used. This implies that you allow the changes in the prices.

**Laspeyre aggregate price index**

The quantities of the base year are used



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**Example**

Using the figures given below, calculate:

i. Paasche aggregate price index

ii. Laspeyre aggregate price index

| | 1980 | | 1986 | |
|---|---|---|---|---|
| | Quantity | Price | Quantity | Price |
| Maize | 20 | 650 | 25 | 700 |
| Wheat | 10 | 1500 | 8 | 1600 |
| Beans | 5 | 150 | 8 | 200 |

Let Q denote quantity, P denote price

**Solution**

| | 1980 | | 1986 | | | |
|---|---|---|---|---|---|---|
| | $Q_0$ | $P_0$ | $Q_1$ | $P_1$ | $P_0Q_1$ | $P_1Q_1$ |
| Maize | 20 | 650 | 25 | 700 | 16250 | 17500 |
| Wheat | 10 | 1500 | 8 | 1600 | 12000 | 12800 |
| Beans | 5 | 150 | 8 | 200 | 1200 | 1600 |
| **Total** | | | | | **29450** | **31900** |

$$\text{Paasche aggregate price index} = \frac{\sum P_1Q_1}{\sum P_0Q_1} x100$$

$$= \frac{31900}{29450} x100 = 108.3$$

The prices of 1986 are higher than those of 1980 by 8%



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(ii)

| | 1980 | | 1986 | | | |
|---|---|---|---|---|---|---|
| | $Q_0$ | $P_0$ | $Q_1$ | $P_1$ | $P_0Q_0$ | $P_1Q_0$ |
| Maize | 20 | 650 | 25 | 700 | 13000 | 14000 |
| Wheat | 10 | 1500 | 8 | 1600 | 15000 | 16000 |
| Beans | 5 | 150 | 8 | 200 | 750 | 1000 |
| Total | | | | | 28750 | 31000 |

Laspeyre aggregate price index = $\frac{\sum P_1Q_0}{\sum P_0Q_0} x100$

$$= \frac{31000}{28750} x100 = 107.8$$

The prices of 1986 are higher than those of 1980 by 8%.

**Note**: In most cases the two methods give different answers.

**Value index number**

This is obtained from = $\frac{\sum P_1Q_1}{\sum P_0Q_0} x100$

Using the above example

The value index = $\frac{31900}{28750} x100 = 111$

This shows an increase of 11% in 1986 compared to 1980



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**Sometimes the value and quantity is given without the price of the commodity.**

**Example**

| Commodity | 1990 | | 1992 | |
|---|---|---|---|---|
| | Quantity $Q_0$ (kgs) | Value $P_0Q_0$ | Quantity $P_1$ (kgs) | Value $P_1Q_1$ |
| A | 25 | 7500 | 40 | 16000 |
| B | 30 | 12000 | 50 | 15000 |
| C | 10 | 8000 | 25 | 10000 |
| D | 20 | 12000 | 15 | 12000 |

Determine the index for the change in the price and weighted average of price relatives.

**Solution**

| Item | 1990 (100) | | 1992 | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $Q_0$ | $P_0Q_0$ | $Q_1$ | $P_1Q_1$ | $P_0$ | $P_1$ | $P_1Q_0$ | $\frac{P_1}{P_0} x Q_1$ |
| A | 25 | 7500 | 40 | 16000 | 300 | 400 | 10000 | 53.33 |
| B | 30 | 12000 | 50 | 15000 | 400 | 300 | 9000 | 37.5 |
| C | 10 | 8000 | 25 | 10000 | 800 | 400 | 4000 | 12.5 |
| D | 20 | 12000 | 15 | 12000 | 600 | 800 | 16000 | 20.0 |
| | | 39500 | 130 | | | | 39000 | 123.33 |

Index for change in price = $\frac{39000}{39500} x 100$



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$$= 98.7$$

Weighted average of price relatives

$$= \frac{123.33}{130} x100 = 94.9$$

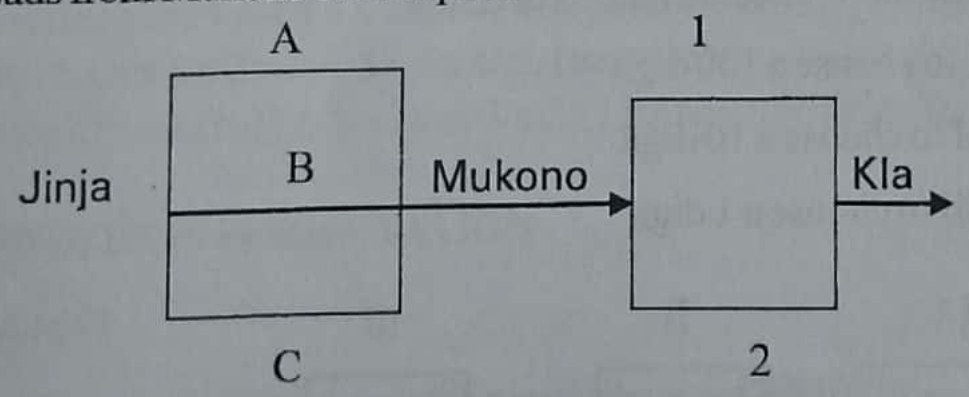
**Changing the base year**

If you wish to change the base year, multiply the current indices

by $\dfrac{100}{\text{Price index of new base year}}$

## 1.7 PERMUTATION AND COMBINATION

Consider the fundamental counting principle. A bus driver has to drive from Jinja to Kampala via Mukono. There are three different roads that he can take from Jinja to Mukono and two different roads from Mukono to Kampala.




One of the possible route is first take road A and then take 1 or simply A.1 Other possible routes are A2, B1, B2 C1 and C2. Therefore there are six possible routes



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**NOTE**: The number of possible routes can be found without listing any of the routes as follows

Make two boxes for two stages

| Jinja I | Mukono II |
|---|---|
| 3 | 2 |

Stage I for Jinja to Mukono there are three choices A, B, C.

Stage II for Mukono to Kampala there are two choices 1 and 2

Total number of ways 3 x 2 = 6 ways.

Therefore, permutation is an arrangement of objects in a definite order.

**Example**

How many three digit numbers can be made from the digits 3, 4, 5, 6, 7 and 8 if no digit is repeated.

There are three stages to make a decision

Stage I to choose a 100 digit

Stage II to choose a 10 digit

Stage III to choose a 1 digit

| I | II | III |
|---|---|---|
| 6 | 5 | 4 |

There are 6 choices for the first position, then 5 choices for the second position and 4 choices for the last position. The total numbers of ways are 6 x 5 x 4 = 120



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**Example**

How many three digits can be formed from the digits 1, 2, 3, 4 if repetitions are allowed?

| I | II | III |
|---|---|---|
| 4 | 4 | 4 |
| 100s digits | 10s digits | 1s digits |

The total number of ways are 4x4x4 = 64 ways.

**Example**

Find the number of possible arrangements for five people to sit together o a bench

| 5 | 4 | 3 | 2 | 1 |
|---|---|---|---|---|
| Seat 1 | Seat 2 | Seat 3 | Seat 4 | Seat 5 |

Thus there are 5 x 4 x 3 x 2 x 1 = 120 possible arrangements.
Note: 5 x 4 x 3 x 2 x 1 = *5!* Factorial notation.
Hence n!= n x (n-1) x (n - 2)...x 3 x 2 x 1 for positive integers.

## CONDITION PERMUTATION

**Example**

How many permutations of all letters in the word BRIDGE end with the letter I or B

| 1st | 2nd | 3rd | 4th | 5th | 6th |
|---|---|---|---|---|---|
| 5 | 4 | 3 | 2 | 1 | 2 |



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Therefore the possibilities are: 5x 4 x 3 x 2 x 1 x 2
= 120 x 2 = 240 ways

## THE OR CONDITION

### Example

How many even numbers containing one or more digits can be formed from the digits 2, 3, 4, 5, 6 if no digit may be repeated. Since the numbers required are even, last unit digits must be one of the three digits 2, 4, 6. Note there are five digits so we can form one digit, two digit, three digit, four digit or five digit numbers as follows

| | |
|---|---|
| One digit numbers is either 2 0r 4 or 6 | 3 = 3 |
| Two digit numbers | 4x3 = 12 |
| Three digit numbers | 4x3x3 = 36 |
| Four digit numbers | 4x3x2x3 = 72 |
| Five digit numbers | 4x3x2x1x3 = 72 |
| **Total** | **195** |

There are 195 such even numbers

## THE AND CONDITION

### Example

Four different English books and three different mathematics books are to be arranged on a shelf. In how many ways can these be arranged if the English books must be put together on the left.

| English books on left | | Math books on the right |
|---|---|---|
| 4 x 3 x 2 x1 | x | 3 x 2 x1 |
| 4! | x | 3! = 144ways |



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**The symbol** $^{n}p_{r}$

The number of r objects chosen from n unlike objects is

$$^{n}p_{r}$$

**Example**

How many permutations are there of 3 letters chosen from eight unlike letters of the word RELATION

$$^{8}p_{3} = \frac{8!}{(8-3)!} = \frac{8!}{5!}$$

$$= 8 \times 7 \times 6 = 336 \text{ ways}$$

**DISTINGUISHABLE PERMUTATION**

**Example**

How many distinguishable six digit numbers can be formed fom the digits of 5,4,8,4,5,4.

The two 5's can be arranged in 2! Ways and the 4's can be arranged in 3! Ways. Hence the number of the six digits are

$$\frac{6!}{3!2!} = \frac{720}{6x2} = 60$$

**Example**

How many signals can be formed by displaying seven flags if three of them are red, two are green and the other two are blue?

**Example** $\frac{7!}{3!2!2!} = \frac{5040}{6x2x2} = 210$



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How many arrangements can be made of the letters in word TERRITORY?

$$\frac{9!}{2!3!}=\frac{362880}{12}=30240$$

**Example**

Find the number of ways in which the letters of ISOSCELES can be arranged if the two E's are separated.

**Solution**

Arrangement without restriction

$$\frac{9!}{3!2!}=\frac{362880}{12}=30240\ ways$$

Let the two E's be together. Then they can be treated as one letter. Total number of letters is 8

$$\frac{8!}{3!}=\frac{40320}{6}=6720$$

The number of ways when the E's are separated

$= 30240 - 6720 = 23520$ ways

$= 23520$ ways.

## CIRCULAR PERMUTATION

The number of circular permutation of n distinct objects is (n-1)!

**Example**

In how many ways can 5 people sit at a round table?

One person is considered fixed and the other 4 can be arranged in 4! Ways

$n = 5 \Rightarrow (5-1)! = 4! = 24$ ways



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## COMBINATIONS

A combination is a selection of an object in which no regard is paid to the order.

**Example**

List all the three combinations that can be formed from the five students A, B, C, D, E and hence state the number of combinations.

ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE and CDE

Thus 10 three-student combinations can be formed.

The number of permutations of three letters from A, B, C, D, E are

$$^5p_3 = \frac{5!}{(5-3)!} = \frac{120}{2} = 60$$

**NOTE**: The number of permutations

= 3! x (the number of combinations)

$$\therefore \text{The number of combinations} = \frac{5!}{3!2!} = 10$$

In general, the number of combinations of r objects chosen unlike objects is written as $^nC_r$ or

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} \qquad r \le n$$

**Example**

A committee of four people is formed from nine people. In how many ways can this be done

$$^9C_4 = \frac{9!}{(9-4)!4!} = \frac{9!}{5!4!}$$

$$= \frac{9x8x7x6x5!}{4x3x2x1x5!} = \frac{9x8x7}{4} = 126 \text{ committee's}$$



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**Example**

A committee of three men and one woman is obtained from five men and three women. In how many ways can the members be chosen?

Three men can be selected from five men in ${}^5C_3$ ways and 1 woman can be selected from three women in ${}^3C_1$ ways. Now for each of the ${}^5C_3$ ways of selecting men, there are ${}^3C_1$ ways of selecting the woman. Hence there are ${}^5C_3 \times {}^3C_1$ ways of selecting the committee.

$${}^5C_3 \times {}^3C_1 =$$

$$= \frac{5!}{2!3!} x \frac{3!}{2!1!} = \frac{5!}{2!2!}$$

Therefore 30 such committees can be formed

$$\frac{120}{4} = 30$$

**Example**

There are 3 boys and 4 girls at a birthday party. In how many ways can a team of 3 students be formed so as to include at least one boy.

Possible number of boys are 1,2, or all 3. The total number of students should be 3.

Hence number of teams with I boy and 2 girls

$${}^3C_1 \times {}^4C_2 = 3 \times 6 = 18$$

Number of teams with 2 boys and 1 girl

${}^3C_2 \times {}^4C_1 = 3 \times 4 = 12$

Number of teams with 3 boys and no girl

${}^3C_3 \times {}^4C_0 = 1 \times 1 = 1$

Total is $18 + 12 = 1 = 31$

$\therefore$ 31 such teams can be formed .



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**Exercise**

1. In how many ways can 4 people be seated on a bench?

   **Answer** 24 ways

2. In how many ways can 9 different books be arranged on a shelf such that

   (a) Three of the books are always together

   (b) Three of the books are all never together.

   **Answer** (a) 30240 (b) 332640

3. Seven men and two women are to sit on a bench. In how many ways can they arrange themselves so that the women do not sit next to each other?

   **Answer** 282240

4. How many arrangements can be made of the letters in THIRTIETH?

   **Answer** 15120

5. Solve for n in ${}^{n}C_4 = {}^{n}C_2$

   **Answer** $n = 6$

6. A committee of five students to comprise the school council is to be selected from eight male students and five female students. Find how many possible committees can be obtained.

   **Answer** 1287 committees

7. From 7 capitals, 3 vowels and 5 consonants, how many words of 4 letters each can be formed if each word begins with a capital, contains at least one vowel, all the letters of each word being different.

   **Answer** 1932 words.



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# CHAPTER TWO

## PROBABILITY THEORY

### 2.1 INTRODUCTION

The term probability arose from the games of chance. For example tossing a coin, rolling a die, playing cards etc…

**Sample Space and generation of the sample space.**

Sample space (S) is the set of all possible outcomes of an experiment. Each possible outcome is called a sample point.

**Example:** rolling a die: S = (1, 2, 3, 4, 5, 6)

Note: 1, 2, 3, 4, 5 and 6 are sample points

**Generation of sample space**

Ways of generating a sample space include

a) Table of outcomes

b) Permutations and combinations

c) Tree diagram



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## Review of terms used in set theory

Subset of a sample space S is called an Event.

**Example of an Event**

(a) Toss a die, then S = {1,2,3,4,5,6}. If the

desired event is that an 'Odd' number shows up.

Odd is the event E and thus E ={1,3,5}.

**Intersection of *events*:**

For any two events A and B, we define a new event called the intersection of A and B and consists of outcomes that are in both A and B or $(A \cap B)$

**Union of Events.**

Denoted by $A \cup B$ is the set of all sample points in either A or B or both.

**Compliment of events**

If A is an event of a sample space S, the compliment of A is given by the set containing all sample points in S that are not in A. Denoted by $\overline{A}$

**Mutually Exclusive events**

If two events A and B have no sample points in common i.e if $A \cap B = \{\ \}$, we say that A and B are mutually exclusive. These are disjoint as shown in Venn diagram.



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Therefore $P(A)=\frac{n(A)}{n(S)}$

**Example**

Find the probability of choosing a defective computer in a lot of 12 out of which 4 are defective if a single draw is made.

**Solution**

The number of ways the event can happen is 4. Total number of possibilities is 12

Hence probability $= \frac{4}{12}=\frac{1}{3}$

**Example**

What is the probability of throwing a number greater than 4 for a die whose faces are numbered from 1 to 6?

$\{1, 2, 3, 4, 5, 6\}$ $\quad$ n(S) = 6

A = $\{5, 6\}$ $\quad \Rightarrow n(A)=2$

$$P(A)=\frac{2}{6}=\frac{1}{3}$$

$$\therefore P(A)=\frac{1}{3}$$

## 2.2 INTERACTION WITH SET THEORY

The following results can be deducted from set theory.

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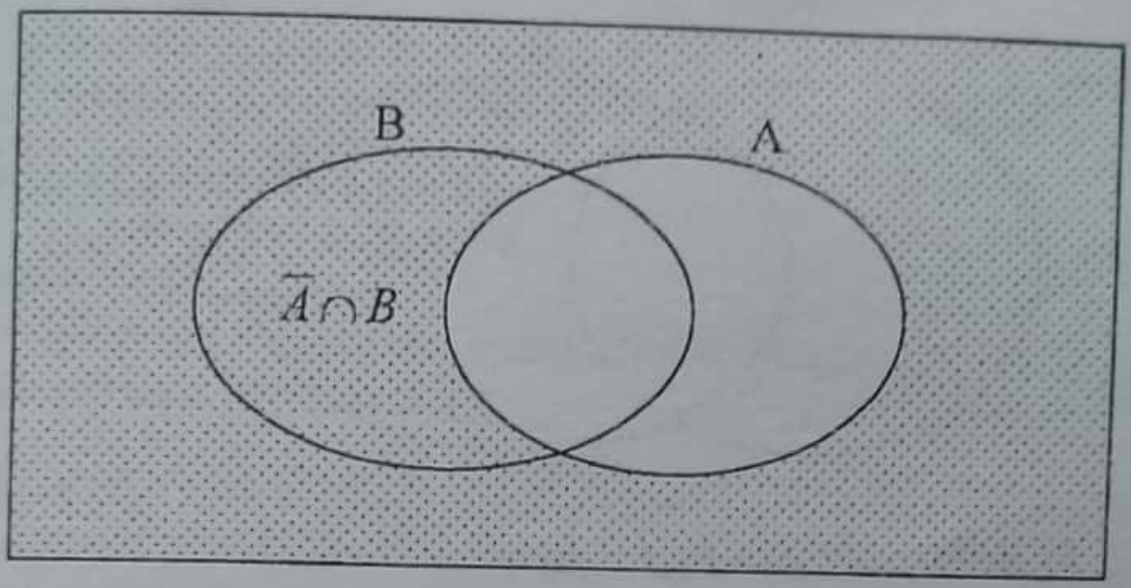
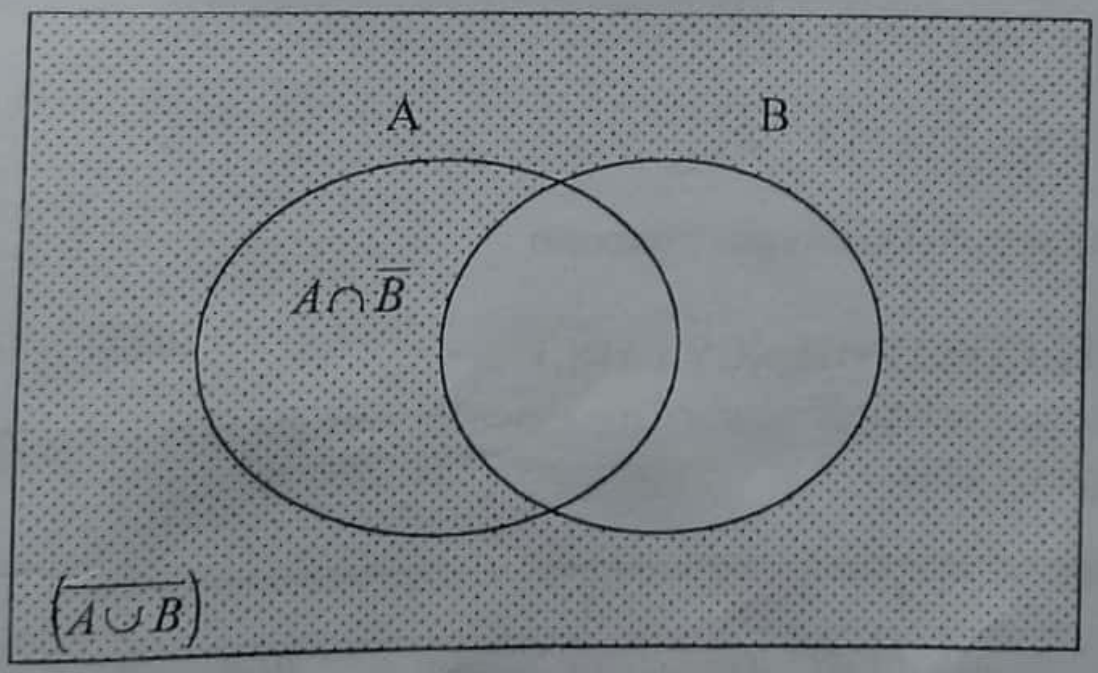



Note that $\left(\overline{A \cup B}\right) = \overline{A} \cap \overline{B}$

**Result 4**

$$P(\overline{B}) = P(A \cap \overline{B}) + P(\overline{A} \cap \overline{B})$$




**Result 5**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



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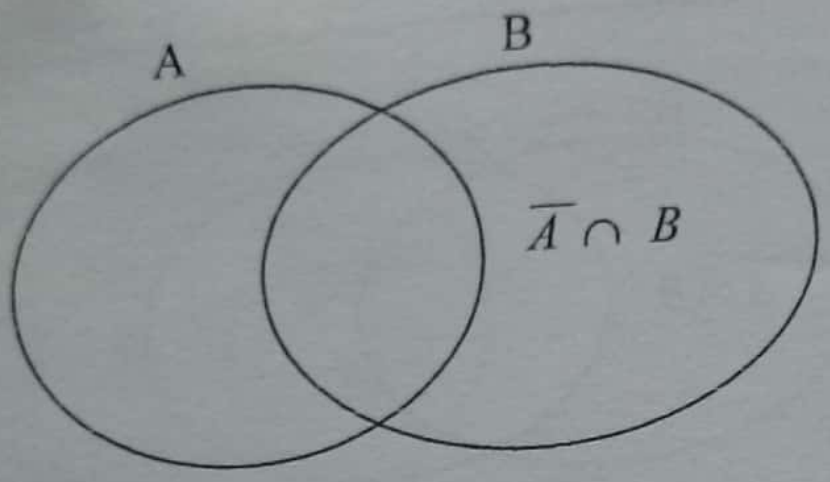


$$P(A \cup B) = P(A) + P(\overline{A} \cap B)$$

$$But\ P(\overline{A} \cap B) = P(B) - P(A \cap B)\ from\ result\ 2)$$

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

**Result 6**

For any two events A and B

i. $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B})$

ii. $P(\overline{A} \cup \overline{B}) = P(\overline{A \cap B})$

This is from De-morgan's theorem

## 2.3 THE CONTINGENCY TABLE

The alternative way of recalling the above four results, is by using the contingency table.

| | $A$ | $\overline{A}$ | |
|---|---|---|---|
| B | $P(A \cap B)$ | $P(\overline{A} \cap B)$ | $P(B)$ |
| $\overline{B}$ | $P(A \cap \overline{B})$ | $P(\overline{A} \cap \overline{B})$ | $P(\overline{B})$ |
| | $P(A)$ | $P(\overline{A})$ | 1 |



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**Example**

Using the second row implies that

$$P(B)=P(A\cap B)+P(\overline{A}\cap B)$$

**Example**

Given that A and B are two events such that $P(A)=0.5$, $P(B)=0.7$ and $P(A\cup B)=0.8$

Find (i) $P(A\cap B)$ (ii) $P(A\cap \overline{B})$

**Solution**

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$0.8 = 0.5+0.7-P(A\cap B)$$

$$\therefore\ P(A\cap B)=1.2-0.8=0.4$$

(ii)

$$P(A\cap \overline{B})=P(A)-P(A\cap B)$$

$$=0.5-0.4$$

$$\therefore\ P(A\cap \overline{B})=0.1$$

**Example**

The probability that a student passes Mathematics is $\frac{2}{3}$ probability that he passes Chemistry is $\frac{4}{9}$. If the probability that he passes at least one of them is $\frac{4}{5}$, find the probability that he passes both subjects.

**Solution**

Let M represent mathematics and C chemistry

That implies $P(M)=\frac{2}{3},\ P(C)=\frac{4}{9},\ P(M\cup C)=\frac{4}{5}$



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$$P(M \cup C) = P(M) + P(C) - P(M \cap C)$$

$$\Rightarrow \frac{4}{5} = \frac{2}{3} + \frac{4}{9} - P(M \cap C)$$

$$\therefore P(M \cap C) = \frac{2}{3} + \frac{4}{9} - \frac{4}{5} = \frac{14}{45}$$

**Example**

Given that A and B are mutually exclusive events such that $P(A) = 0.5$, $P(A \cup B) = 0.9$, *find*;

i. $P(\overline{A} \cup B)$

ii. $P(\overline{A} \cap \overline{B})$

**Solution**

$$P(\overline{A} \cup B) = P(\overline{A}) + P(B) - P(\overline{A} \cap B)$$

$P(A) = 0.5 \quad \Rightarrow P(\overline{A}) = 1 - 0.5 = 0.5$

$P(B) = 0.9 - 0.5 = 0.4$ since $A$ and $B$ are mutually exclusive

$$P(\overline{A} \cap B) = P(B) - P(A \cap B)$$

$$= 0.4 - 0 = 0.4$$

$$P(\overline{A} \cup B) = P(\overline{A}) + P(B) - P(\overline{A} \cap B)$$

$$= 0.5 + 0.4 - 0.4$$

$$= 0.5$$

$$\therefore P(\overline{A} \cup B) = 0.5$$

(ii) $$P(\overline{A} \cap \overline{B}) = 1 - P(A \cup \mathbf{B})$$

$$= 1 - 0.9 = 0.1$$

$$\therefore P(\overline{A} \cap \overline{B}) = 0.1$$



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**Example**

In a race, the probability that Grace wins is 0.4, the probability that Umar wins is 0.2 and the probability that Denis wins is 0.3. Find the probability that

a) Grace or Denis wins

b) Neither Denis nor Umar wins

Assume that there are no dead heats.

**Solution**

Let G, U, D represent Grace, Umar and Denis respectively

a) $P(G \cup D) = 0.4 + 0.3 = 0.7$

$\therefore P(Grace\ \ or\ \ Denis\ wins) = 0.7$

b) $P\left(\overline{D \cup U)}\right) = 1 - P(D \cup U)$

$= 1 - (0.3 + 0.2)$

$= 1 - 0.5 = 0.5$

$\therefore P(Neither\ \ Denis\ \ nor\ \ Umar) = 0.5$

## 2.4 PROBABILITY SITUATIONS

### THE OR SITUATION

If A and B are two events, the probability that either event A or B or even both occur is denoted by $P(A \cup B)$,

Where $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

**Example**

Two dice are thrown. What is the probability of scoring either a double or a sum greater than 8?



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**Solution**

Table of outcome can be used to generate a sample space

| | *First die* | | | | | |
|---|---|---|---|---|---|---|
| Second die | 1, 1 | 1, 2 | 1, 3 | 1, 4 | 1, 5 | 1, 6 |
| | 2, 1 | 2, 2 | 2, 3 | 2, 4 | 2, 5 | 2, 6 |
| | 3, 1 | 3, 2 | 3, 3 | 3, 4 | 3, 5 | 3, 6 |
| | 4, 1 | 4, 2 | 4, 3 | 4, 4 | 4, 5 | 4, 6 |
| | 5, 1 | 5, 2 | 5, 3 | 5, 4 | 5, 5 | 5, 6 |
| | 6, 1 | 6, 2 | 6, 3 | 6, 4 | 6, 5 | 6, 6 |

Table of sums

| | | | | | |
|---|---|---|---|---|---|
| 2 | 3 | 4 | 5 | 6 | 7 |
| 3 | 4 | 5 | 6 | 7 | 8 |
| 4 | 5 | 6 | 7 | 8 | 9 |
| 5 | 6 | 7 | 8 | 9 | 10 |
| 6 | 7 | 8 | 9 | 10 | 11 |
| 7 | 8 | 9 | 10 | 11 | 12 |

Let A represent a double and B a sum greater than 8

$A = \{(1,1)(2,2)(3,3)(4,4)(5,5)(6,6)\}$

$B = \{(3,6)\ (4,5)\ (6,3)\ (5,4)\ (5,5)\ (5,6)\ (6,5)\ (6,4)\ (4,6)\ (6,6)\}$

$n(A) = 6,\quad n(B) = 10$ and $n(A \cap B) = 2$

Using $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{6}{36} + \frac{10}{36} - \frac{2}{36} = \frac{14}{36}$$

$$\therefore P(\textit{Double or sum greater than } 8) = \frac{7}{18}$$



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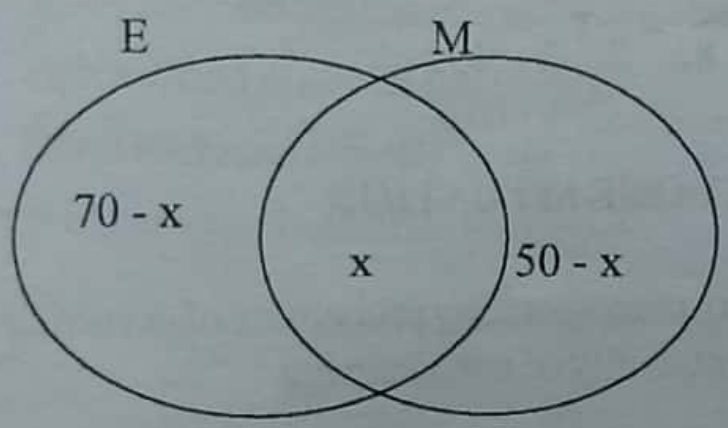
## THE AND SITUATION

a) When there is joint occurrence of events
b) Considering a sequence of events.

### Example (a)

In a class of 100 students, 70 offer Economics while 50 students offer Mathematics. Each student offers at least one of the subjects. Determine the probability for the number of students who offer both subjects?




From the venn diagram

$70 - x + x + 50 - x = 100$

$120 - x = 100$

$x = 20$ students

$\therefore P(E \cap M) = 0.2$

### Example (b)

Three balls are drawn at random one after the other without replacement from a bag containing 21 white, 9 blue, 40 red and 12 orange balls. Determine the probability that the first ball is blue, the second white and the third red.



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**Solution**

Let $B_1$ indicate the first ball is blue
$W_2$ indicate the second ball is white
$R_3$ indicate the third ball is red

| *RED* | *BLUE* | *WHITE* | *ORANGE* | *TOTAL* |
|---|---|---|---|---|
| 40 | 9 | 21 | 12 | 82 |

$$P(B_1 \cap W_2 \cap R_3) = \frac{9}{82} x \frac{21}{81} x \frac{40}{80}$$
$$= \frac{1}{82} x \frac{7}{3} x \frac{1}{2} = \frac{7}{492}$$

## THE ONE AND ONLY ONE SITUATION

If A, B, C, are events with corresponding probability of occurring P(A), P(B), P(C) and probability of not occurring

$P(\overline{A}),\ P(\overline{B})\ and\ P(\overline{C})$.

Let a = event A occurs, B and C don't occur
b = event B occurs, A and C don't occur
c = event C occurs, A and B don't occur

$$\Rightarrow P(a\, or\, b\, or\, c) = P(a) + P(b) + P(c)$$
$$P(a) = P(A)P(\overline{B})P(\overline{C})$$
$$P(b) = P(\overline{A})P(B)P(\overline{C})$$
$$P(c) = P(\overline{A})P(\overline{B})P(C)$$
$$\therefore P(a \cup b \cup c) = P(A)P(\overline{B})P(\overline{C}) + P(\overline{A})P(B)P(\overline{C}) + P(\overline{A})P(\overline{B})P(C)$$



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**Example**

When three marksmen take part in a shooting contest,

their chances of hitting a target are $\frac{1}{2}, \frac{1}{3}, and \frac{1}{4}$

a) Calculate the chance that one and only one bullet will hit the target if all men shoot simultaneously.

b) Determine the probability that the target will be hit (Assume independence)

**Solution**

Let $E_1$ = First man hits the target

$E_2$ = Second man hits the target

$E_3$ = Third man hits the target

Let N be only one bullet hits the target

$$P(E_1)=\frac{1}{2} \qquad P(\overline{E}_1)=\frac{1}{2}$$

$$P(E_2)=\frac{1}{3} \qquad P(\overline{E}_2)=\frac{2}{3}$$

$$P(E_3)=0.25 \qquad P(\overline{E}_3)=0.75$$

$$P(N)=P(E_1)P(\overline{E}_2)P(\overline{E}_3)+P(\overline{E}_1)P(E_2)P(\overline{E}_3)$$
$$+P(\overline{E}_1)P(\overline{E}_2)P(E_3)$$

$$P(N)=\frac{1}{2}x\frac{2}{3}x\frac{3}{4}+\frac{1}{2}x\frac{1}{3}x\frac{3}{4}+\frac{1}{2}x\frac{2}{3}x\frac{1}{4}$$

$$=\frac{1}{4}+\frac{1}{8}+\frac{1}{12}$$

$$=\frac{11}{24}$$



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(b) $P(H) = P(E_1 \cup E_2 \cup E_3)$

The target can be hit by any three or two of them or all the three.

$$P(E_1 \cup E_2 \cup E_3) =$$

$$P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2)$$
$$-P(E_1 \cap E_3) - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$
$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} x \frac{1}{3} - \frac{1}{2} x \frac{1}{4} - \frac{1}{3} x \frac{1}{4} + \frac{1}{2} x \frac{1}{3} x \frac{1}{4}$$

$$= \frac{3}{4}$$

## 2.5 INDEPENDENT EVENTS

Events are said to be independent if and only if the occurrence of one event does not influence the occurrence of the other event. Or the non occurrence of one event does not influence the non-occurrence of the other event.

Therefore $P(A \text{ and } B) = P(A \cap B) = P(A)\,P(B)$

**Example**

The probability that a man and his wife pass their driving test at first attempt is $\frac{1}{3}$ and $\frac{2}{5}$ respectively. Assuming that these events are independent, determine the probability that both pass the test at first attempt



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**Solution**

Let M man passes and W woman passes

$$P(M \cap W) = P(M)\,P(W)$$

$$= \frac{1}{3} x \frac{2}{5}$$

$$\textit{The probability that both pass} = \frac{2}{15}$$

**Example**

The probability that a student S can solve a certain problem is 0.4 and that of student L can solve it is 0.5. Find the probability that the problem will be solved if both S and L try to solve it independently.

**Solution**

$$P(S \cup L) = P(S) + P(L) - P(S \cap L)$$

$$\textit{But } P(S \cap L) = P(S)P(L) \textit{ for independent events}$$

$$P(S \cup L) = 0.4 + 0.5 - 0.4 x 0.5$$

$$= 0.9 - 0.2$$

$$= 0.7$$

$$\therefore \textit{the probability that the probability will be solved} = 0$$

**Example**

A and B are two independent events, show that A and $\overline{B}$ are independent.

**Solution**

$$P(A \cap B) = P(A)P(B) \textit{ for independent events}$$



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$$P(A)=P(A\cap B)+P(A\cap \overline{B})$$
$$\Rightarrow P(A\cap \overline{B})=P(A)-P(A\cap B)$$
$$P(A\cap \overline{B})=P(A)-P(A)P(B)$$
$$=P(A)(1-P(B))$$
$$=P(A)P(\overline{B})$$

*Since* $P(\overline{B})=1-P(B)$

*Hence* $P(A\cap \overline{B})=P(A)P(\overline{B})$

Therefore A and $\overline{B}$ are indpendent events

**Example**

Events A and C are independent. Probabilities relating to events A, B and C are as follows

$$P(A)=\frac{1}{5}, P(B)=\frac{1}{6}, P(A\cap C)=\frac{1}{20}, P(B\cup C)=\frac{3}{8}$$

Calculate P(C) and show that B and C are independent

**Solution**

A and C are independent events then

$P(A\cap C) = P(A)P(C)$

$$\frac{1}{20}=\frac{1}{5}P(C) \Rightarrow P(C)=\frac{1}{4}$$

*But* $P(B\cap C)=P(B)+P(C)-P(B\cup C)$

$$\frac{1}{6}+\frac{1}{4}-\frac{3}{8}=\frac{4+6-9}{24}=\frac{1}{24}$$



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$$P(B)P(C)=\frac{1}{6}x\frac{1}{4}=\frac{1}{24}$$

$$P(B\cap C)=P(B)P(C)=\frac{1}{24}$$

Hence B and C are independent events

## 2.6 CONDITIONAL PROBABILITY

Suppose we have the information about 20 students given in table below

| | Female | Male |
|---|---|---|
| Below 18 years old | 5 | 6 |
| Above 18 years old | 1 | 8 |
| **Total** | **6** | **14** |

If a person is selected at random what is probability that is above 18 years

$$=\frac{9}{20}$$

Note if above 18 years is denoted by A and below by B.

Then M to denote male while F to represent female.

The expression P( B| M) is conditional probability that means

The conditional probability of below 18 years given a male.



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**Definition**

If A and B are events, then the conditional probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{provided } P(B) \neq 0$$

**The example above**

$$P(B|M) = \frac{P(B \cap M)}{P(M)}$$

$$P(B \cap M) = = 0.3$$

$$P(M) = \frac{14}{20}$$

$$P(B|M) = \frac{0.3}{0.7} = \frac{3}{7}$$

**Example**

If A and B are events and $P(B) = \frac{1}{6}$, $P(A \text{ and } B) = , \frac{1}{12}$

$$P(B|A) = \frac{1}{3}$$

Calculate P(A), P(A|B) and $P(A|\overline{B})$



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$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

$$\frac{1}{3}P(A) = \frac{1}{12}$$

$$\therefore\ P(A) = \frac{1}{4}$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$= \left( \frac{1/12}{\frac{1}{6}} \right)$$

$$P(A|B) = \frac{1}{2}$$

$$P(A \mid \overline{B}) = \frac{P(A \cap \overline{B})}{P(\overline{B})}$$

$$= \frac{P(A) - P(A \cap B)}{1 - P(B)}$$

$$= \frac{\frac{1}{4} - \frac{1}{12}}{1 - \frac{1}{6}} = \frac{1}{6} x \frac{6}{5}$$

$$\therefore\ P\left(A \mid \overline{B}\right) = 0.2$$



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**Solution**

**Example**

In a certain University

75% of the students are full-time students, 45% of the students are female, 40% of the students are male full-time students. Find the probability that

a) A student chosen at random from the students in the University is a part-time student,

b) A student chosen at random from all students in the University is female and a part-time student,

c) A student chosen at random from all the female students in the University is a part-time student.

**Solution**

$P(Female) = P(F) = 0.45,\ P(M) = 0.55$

$(a)\ P(S) = 0.75, P(\bar{S}) = 0.25$

$(b)\ P(F \cap S) = 0.75 - 0.4 = 0.35$

$But\ P(F \cap \bar{S}) = P(F) - P(F \cap S)$

$= 0.45 - 0.35$

$\therefore\ P(F \cap \bar{S}) = 0.10$

$$P(\bar{S} \mid F) = \frac{0.1}{0.45}$$

$$\Rightarrow P(\bar{S} \mid F) = \frac{2}{9}$$



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## Exercise

1. When a die is thrown the score was an even number. What is the probability that it was a prime number.

Answer: P( Prime, given Even) $= \frac{1}{3}$

2. A and B are two events such that P (A|B) = 0.4, P( B) = 0.25 and P( A) = 0.2. Find
   (a) P (B|A) (b) P( A ∩ B) (c ) P( A ∪ B)
   Answer (a) 0.5 (b) 0.1 (c) 0.35
3. Events A and B are such that

$$P(A) = \frac{19}{30},\ P(B) = \frac{2}{5}\ and\ P(A \cup B) = \frac{4}{5}$$

Find P(A ∩ B)

**Answer** $P(A \cap B) = \frac{7}{30}$

4. The probability that a regular scheduled flight departs on time is 0.83 and the probability that it arrives on time is 0.92. The probability that it departs on time and arrives on time is 0.78. Find the probability that the plane;
   a) Arrives on time given that it departs on time
   b) Departs on time given that it arrives on time
   **Answer** (a) 0.94 (b) 0.85

5. Prove that for any two events A and B,

$$P(A \mid B) + P(\overline{A} \mid B) = 1$$

6. Three events A, B and C are such that A and B are independent, A and C are mutually exclusive.

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Given that

$$P(A) = 0.4,\ P(B) = 0.2,\ P(C) = 0.3\ and\ P(B \cap C) = 0.1$$

Calculate

(i) $P(A \cup B)$ (ii) $P(C|B)$ (iii) $P(B|A \cup C)$

(iv) the probability that one and only one of the events B, C will occur.

**Answer** (i) 0.52 (ii) 0.5 (iii) 0.26 (iv) 0.3

7. A box contains 3 red, 2 green and 5 blue crayons. Two crayons are randomly selected from the box without replacement. Find the probability that:

i. The crayons are of the same colour.

ii. At least one red crayon is selected

**Answer**

i. $P(\text{same colour}) = \frac{14}{45}$

ii. $P(\text{at least one red crayon}) = \frac{8}{15}$

8. A bag contains 3 black and 5 white balls. 2 balls are drawn at random one at a time without replacement. Find

i. The probability that the second ball is white

ii. The probability that the first ball is white given that the second is white

**Answer**

$$P(W_2) = \frac{5}{8},\ (ii)\ P(W_1|W_2) = \frac{4}{7}$$



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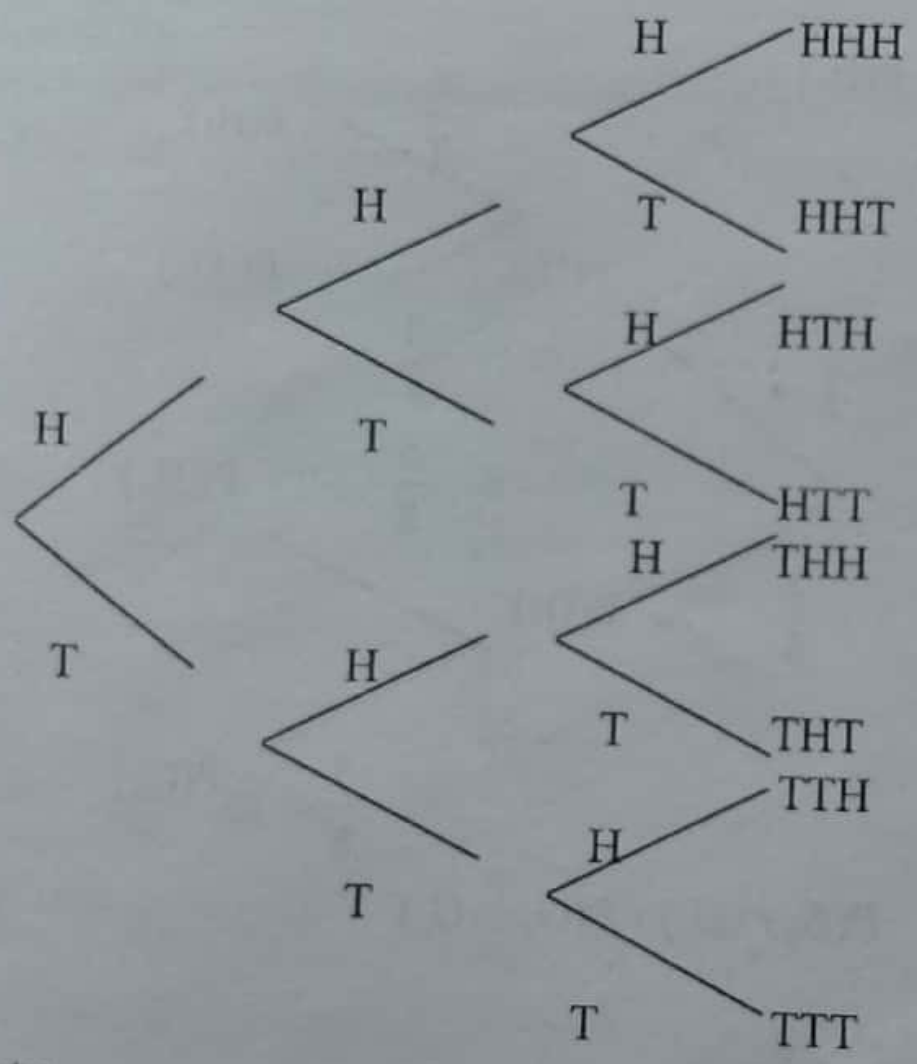
## 2.7 PROBABILITY TREE DIAGRAMS

Tree diagrams can be used to obtain the possible outcomes of an experiment.

**Example**

Find the possible outcomes when three coins are tossed.

The possible outcomes are




**Note:**

The total probability for any one set of branches = 1

The sum of final probabilities is = 1

i. The tree shows conditional probabilities



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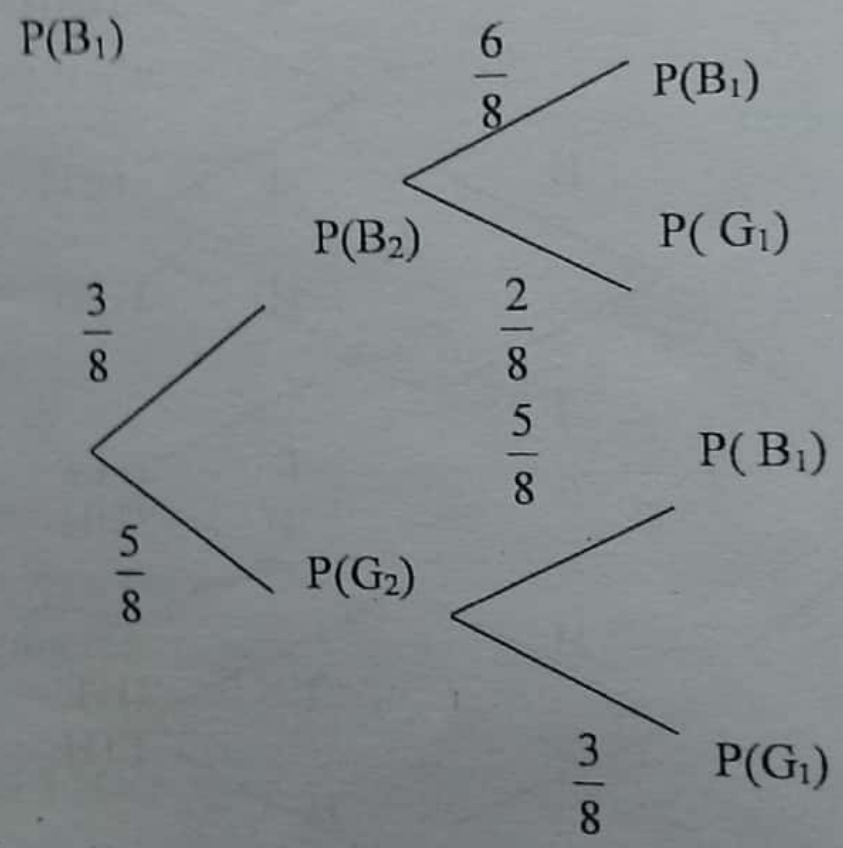
**Example**

In an experiment, a box contains 2 green and 5 blue balls. A second box contains 5 green and 3 blue balls. One ball is drawn at random from the second box and placed into the first box. What is the probability that a ball drawn now from the first box is green?

**Solution**

Let $B_2$ indicate blue ball from second box,

$B_1$ indicates blue ball from first box.




$$P(G) = P(B_2 \cap G_1) + P(G_2 \cap G_1)$$

$$= \frac{3}{8} x \frac{2}{8} + \frac{5}{8} x \frac{3}{8}$$

$$= \frac{6}{64} + \frac{15}{64}$$

$$\therefore P(G) = \frac{21}{64}$$



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**Example**

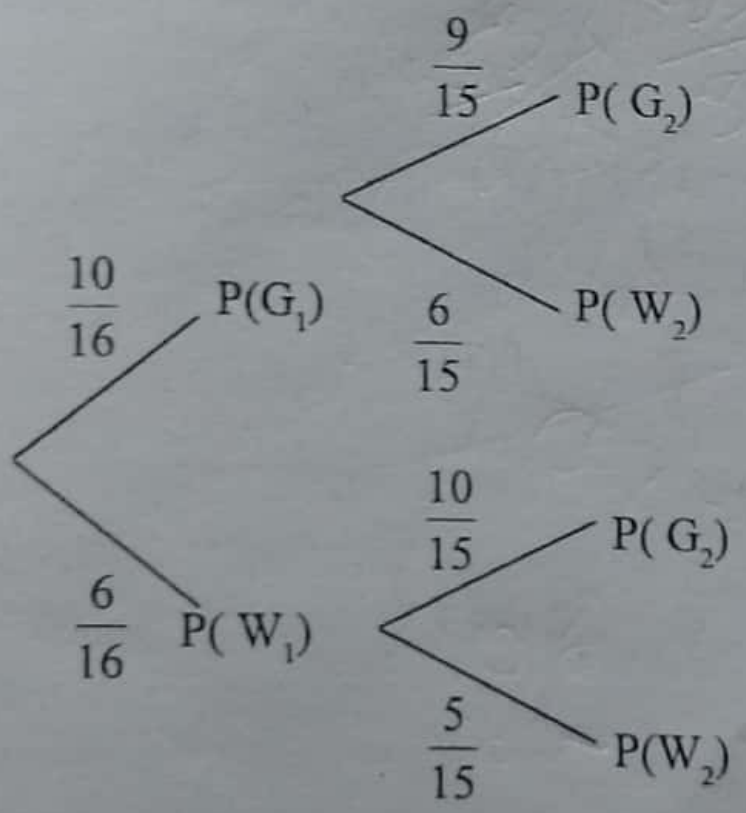
A box contains ten green and six white marbles. A marble is chosen at random, its colour noted and it is not replaced. This is repeated once more.

What is the probability that the marble chosen are of the same colour?

**Solution**

Let $W_1$ and $W_2$ indicate white marble is picked first and second respectively.

Let $G_1$ and $G_2$ indicate green marble is picked first and second respectively.




P( marble of the same colour) $= P(W_1 \cap W_2) + P(G_1 \cap G_2)$

$$= \frac{6}{16} x \frac{5}{15} + \frac{10}{16} x \frac{9}{15}$$

$$= \frac{1}{2}$$



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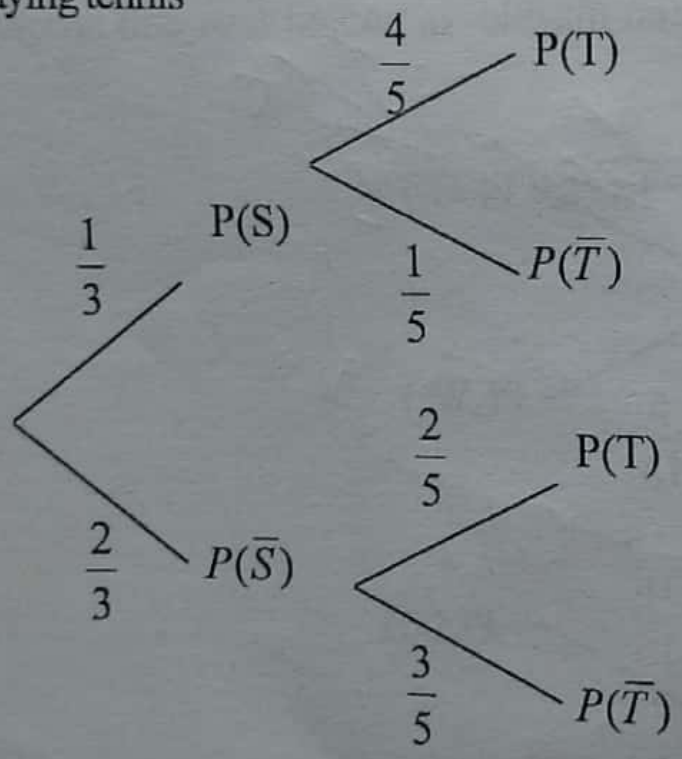
**Example**

The probability that it will be sunny tomorrow is $\frac{1}{3}$. If it is sunny the probability that Shamim plays tennis tomorrow is $\frac{4}{5}$. If it is not sunny, the probability that she plays tennis is $\frac{2}{5}$. Find the probability that Shamim plays tennis tomorrow?

**Solution**

Let S represent sunny, denote not sunny,
T playing tennis




$$P(T) = P(S \cap T) + P(\bar{S} \cap T)$$

$$= \frac{1}{3} x \frac{4}{5} + \frac{2}{3} x \frac{2}{5}$$

$$\therefore P(T) = \frac{8}{15}$$



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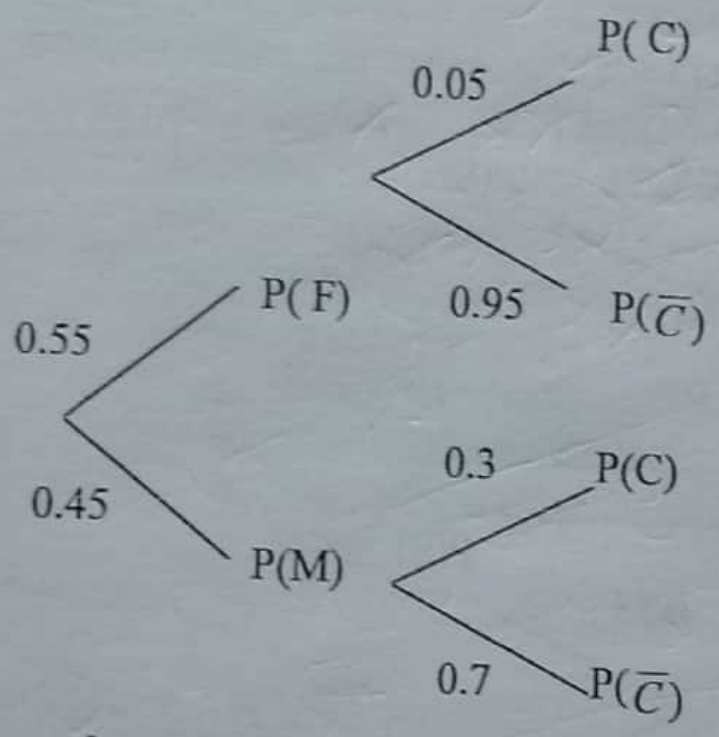
**Example**

The proportion of female students at Kyambogo college is 55%. If 30% of the male students and 5% of female students study Chemistry. What is the probability that a chemistry student chosen at random is a girl?

**Solution**

Let F be for female, M for male while C is for chemistry




**Example**

$$P(F \mid C) = \frac{P(F \cap C)}{P(C)}$$

$$But\ \ P(C) = P(F \cap C) + P(M \cap C)$$
$$= 0.55x0.05 + 0.45x0.3$$

$$P(F \mid C) = \frac{0.55x0.05}{0.55x0.05 + 0.45x0.3}$$
$$= 0.1692$$



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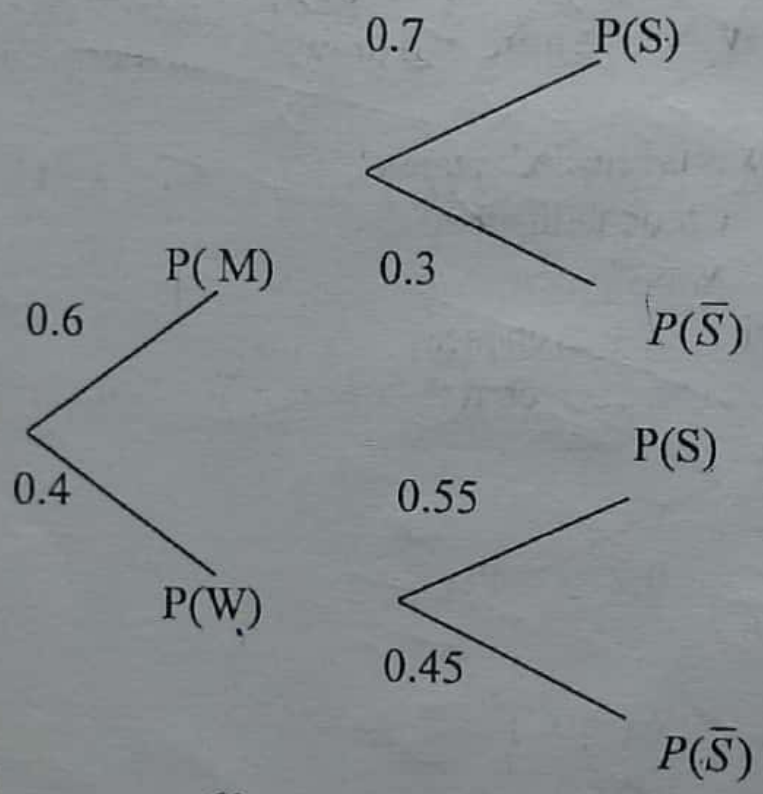
In a survey, 70% of men and 55% of women said that they smoked. If the proportion of men to women is 60 : 40 and a person from the survey was chosen at random and found to be a smoker, what is the probability that this person is a woman?

**Solution**

Let M and W denote man and woman respectively

Let S indicate a person is a smoker

The required probability is P(W|S)




**Note:**

$$P(M)=\frac{60}{100},\ P(W)=\frac{40}{100}$$

$$P(W\mid S)=\frac{P(W\cap S)}{P(S)}$$

$$\begin{aligned} But\ \ P(S) &= P(W\cap S)+P(M\cap S) \\ &= 0.4x0.55+0.6x0.7 \\ &= 0.64 \end{aligned}$$



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$$P(W \mid S) = \frac{0.22}{0.64}$$

$$\therefore P(W \mid S) = \frac{11}{32}$$

**Example**

Three boxes X, Y and Z contain coloured balls. X contains 5 black balls and 4 white balls, Y contains 7 black and 5 white balls and Z contains 3 black and 5 white balls.

a) If the balls are withdrawn from box Z, with replacement, find the probability that the third ball drawn is the second white ball.

b) One of the boxes is selected at random and a ball is withdrawn from it. Find the probability that:

   i. Box X was chosen and the ball was black

   ii. A white ball was chosen

   iii. The ball was selected from box Z, given that it was black.

**Solution**

For Box Z, $P(\text{black}) = P(B) = \frac{3}{8}$

$P(\text{white}) = P(W) = \frac{5}{8}$

P( the third ball is the second white one)

$$\Rightarrow P(BWW) + P(WBW)$$

$$P(BWW) + P(WBW) = \frac{3}{8} x \frac{5}{8} x \frac{5}{8} + \frac{5}{8} x \frac{3}{8} x \frac{5}{8}$$

$$\frac{75}{512} + \frac{75}{512} = \frac{75}{256}$$



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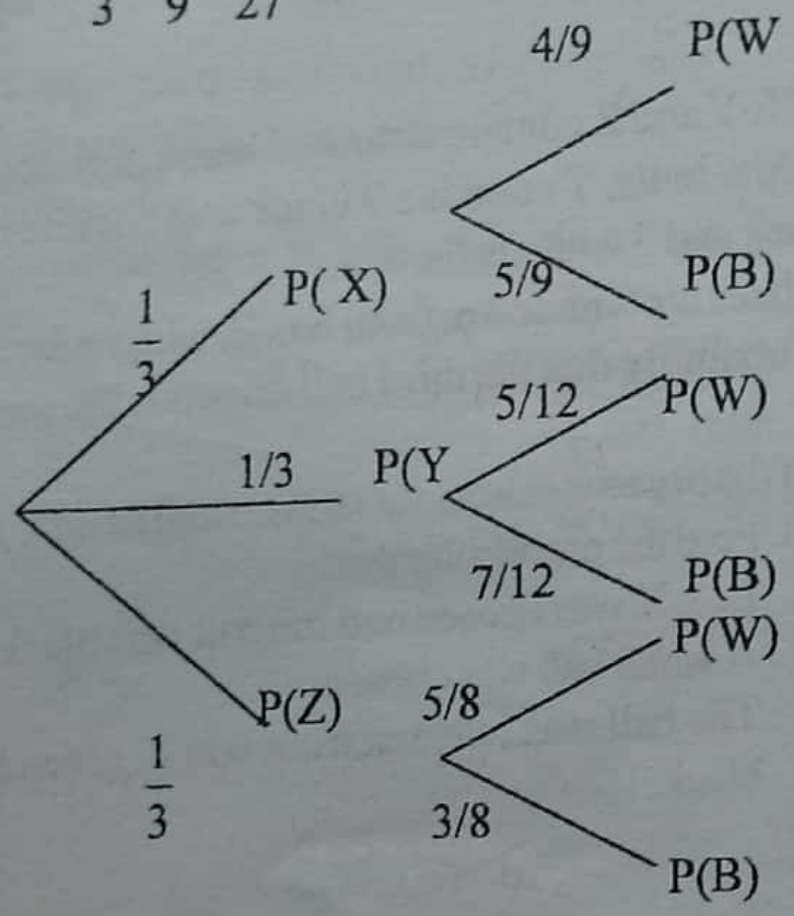
b(i) $P(X \cap B) = P(X)\,P(B|X)$

$$= \frac{1}{3} x \frac{5}{9} = \frac{5}{27}$$




b(ii)

$$P(W) = P(X \cap W) + P(Y \cap W) + P(Z \cap W)$$

$$P(W) = \frac{1}{3} x \frac{4}{9} + \frac{1}{3} x \frac{5}{12} + \frac{1}{3} x \frac{5}{8}$$

$$P(W) = \frac{107}{216}$$



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(iii) $P(Z|B) = \dfrac{P(Z \cap B)}{P(B)}$

But $P(W) + P(B) = 1$

$$P(B) = 1 - \frac{107}{216} = \frac{109}{216}$$

$$P(Z \cap B) = \frac{1}{3} x \frac{3}{8} = \frac{1}{8}$$

$$P(Z \mid B) = \frac{1}{8} x \frac{216}{109} = \frac{27}{109}$$

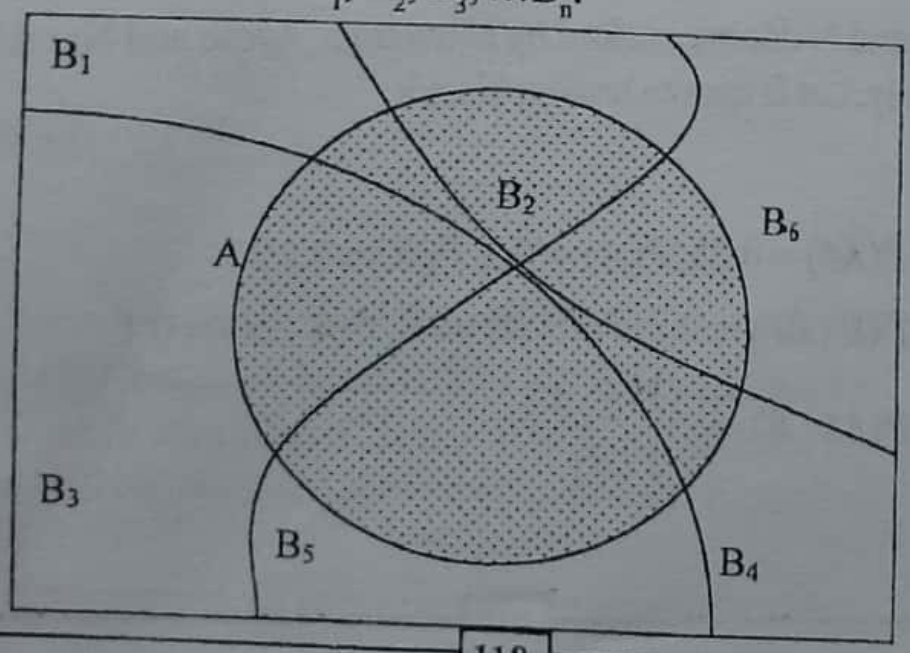
$$\therefore\ P(Z \mid B) = \frac{27}{109}$$

## 2.8 BAYES' THEOREM

This theorem is obtained from the extension of the conditional probability.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B \mid A)}{P(B)}$$

Suppose S contains $B_1, B_2, B_3, \ldots B_n$.



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Event A can be given by:

$$P(A)=P(B_1 \cap A)+P(B_2 \cap A)+....+P(B_n \cap A)$$

$$\therefore P(B_i \mid A)=\frac{P(B_i \cap A)}{P(A)}=$$

$$\frac{P(B_i)P(A \mid B_i)}{P(B_1 \cap A)+P(B_2 \cap A)+...+P(B_n \cap A)}$$

$$\therefore P(B_i \mid A)=\frac{P(B_i)P(A \mid B_i)}{\sum_{k=1}^{n} P(B_k\ P(A \mid B_k)}$$

Where i = 1, 2, 3, …n.

**Example**

Three girls, Muwanga, Ayolo and Nambi pack biscuits in a factory, from the batch allotted to them, Muwanga packs 55%, Ayolo 30% and Nambi 15%. The probability that Muwanga breaks some biscuits in a packet is 0.7, and the respective probabilities of Ayolo and Nambi are 0.2 and 0.1. What is the probability that a packet with broken biscuits found by the checker was packed by Muwanga

**Solution**

Let M, A and N denote packed by Muwanga, Ayolo and Nambi respectively. Let B denote broken biscuit.

$$P(M)=0.55, P(A)=0.3, P(N)=0.15$$

$$P(B \mid M)=0.7, P(B \mid A)=0.2, P(B \mid N)=0.1$$

$$P(M \mid B)=\frac{P(M \cap B)}{P(B)}$$



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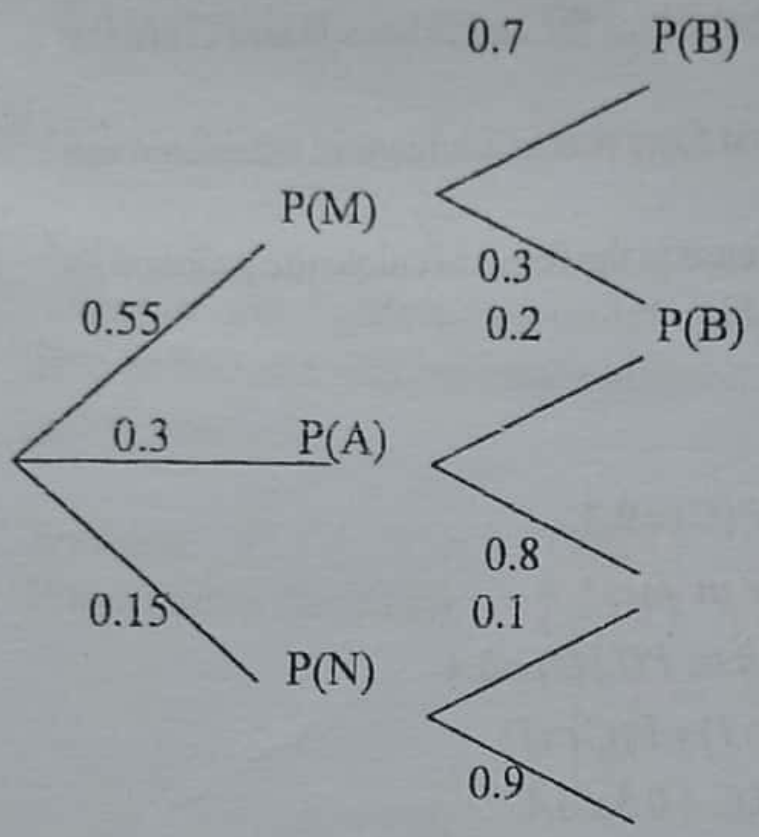


$$P(B)=P(M\cap B)+P(A\cap B)+P(N\cap B)$$
$$=0.55x0.7+0.3x0.2+0.15x0.1=0.46$$
$$P(M\cap B)=0.55x0.7=0.385$$
$$\Rightarrow P(M\mid B)=\frac{P(M\cap B)}{P(B)}$$
$$=\frac{0.385}{0.46}$$
$$\therefore\ P(M\mid B)=0.84$$

**xample**

hree candidates have been nominated for post of welfare prefect in hool. The probability that candidate A will be elected is 0.1, that for ndidate B is 0.2 while candidate C is 0.3. It is expected that, the tertainment fee will be increased if any one of these is elected as



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prefect. The probability of an increase in the fee when A is elected is 0.5, the corresponding probabilities for candidates B and C are 0.6 and 0.4 respectively.

Determine the probability that there was no increase in entertainment fees?

Given that there was an increase in the fees, calculate the probability that candidate A was elected?

**Solution**

$P(A)=0.1, P(B)=0.2, P(C)=0.3,$

*Let I represent increase in fees*

$P(I|A)=0.5, P(I|B)=0.6,\ P(I|C)=0.4$

$P(I)=P(A\cap I)+P(B\cap I)+P(C\cap I)$

$=0.1x0.5+0.2x0.6+0.3x0.4$

$=0.29$

*Probability of no increase in fees* $=1-0.29=0.71$

$$P(A|I)=\frac{P(A\cap I)}{P(I)}$$

$$=\frac{0.1x0.5}{0.29}$$

$$=0.1724$$

## 2.9 Calculations involving combinations

**Example**

In class with 5 boys and 3 girls. Two students are selected at random from the class. What is the probability that they are all girls.

**Solution**

The two girls can be selected in ${}^3C_2$ ways from the girls.



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But two students can be selected in ${}^{8}C_2$ ways from the class

$$P(\text{All girls}) = \frac{{}^{3}C_2}{{}^{8}C_2} = \frac{3}{28}$$

**Example**

There are only 3 girls in a group of 8 students. A group of 5 students is to be selected. Determine the probability that two girls are in the group selected.

**Solution**

$$P(\text{two girls in the group}) = \frac{\binom{3}{2}\binom{5}{3}}{\binom{8}{5}} = \frac{3x10}{56}$$

$$= \frac{15}{28}$$

**Example**

A bag contains 5 green, 4 yellow and 3 blue balls, from which 4 balls are picked at random. Determine the probability that the 4 balls selected will contain.

i. Exactly 3 yellow balls
ii. At least one green ball

**Solution**

$$P(Three\ yellow\ balls) = \frac{{}^{4}C_3 x\, {}^{8}C_1}{{}^{12}C_4}$$

$$= \frac{4x8}{495} = \frac{32}{495}$$



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P(At least one green)

$$= \frac{{}^5C_1 x\, {}^7C_3}{{}^{12}C_4} + \frac{{}^5C_2 x\, {}^7C_2}{{}^{12}C_4} + \frac{{}^5C_3 x\, {}^7C_1}{{}^{12}C_4} + \frac{{}^5C_4}{{}^{12}C_4}$$

$$= \frac{175}{495} + \frac{210}{495} + \frac{70}{495} + \frac{5}{495}$$

$$= \frac{92}{99}$$

## 2.10 OTHER EXAMPLES

When it is fine day, the probability that Aisha plays basket ball is 0.9 and the probability that Zakia plays is 0.75. If it is not fine, Aisha's probability is 0.5 and Zakia's is 0.25. Their chances are independent. In general it is twice likely to be fine as not.

i. Determine the probability that they both go to play
ii. If they both go to play, what is the probability that it is a fine day?

**Solution**

Let F, A and Z represent fine, Aisha and Zakia respectively

$$P(A|F) = \frac{9}{10}, \quad P(Z|F) = \frac{3}{4},$$

$$P(A|\bar{F}) = \frac{1}{2}, \quad P(Z|\bar{F}) = \frac{1}{4}$$

$$P(F) = \frac{2}{3}, \quad P(\bar{F}) = \frac{1}{3}$$



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$$P(F)=\frac{2}{3},\ P(\overline{F})=\frac{1}{3}$$

P(Both Aisha and Zakia play)

$$= P(A\cap Z\cap F)+\ P(A\cap Z\cap \overline{F})$$

$$=\frac{2}{3}x\frac{9}{10}x\frac{3}{4}+\frac{1}{3}x\frac{1}{4}x\frac{1}{2}$$

$$=\frac{59}{120}$$

$$P(F\mid A\cap B)=\frac{P(A\cap Z\cap F)}{P(A\cap B)}$$

$$=\frac{54}{120}x\frac{120}{59}$$

$$=\frac{54}{59}$$

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**Example**

The table below shows the likely hood where Linda and Sheila spend their Satarday evening

| | *Linda* | *Sheila* |
|---|---|---|
| Go dancing | $\frac{1}{2}$ | $\frac{2}{3}$ |
| Visit neighbour | $\frac{1}{3}$ | $\frac{1}{6}$ |
| Stays at home | $\frac{1}{6}$ | $\frac{1}{6}$ |



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i. Find the probability that they both go out?
ii. If we know that they both go out, what is the probability that they went dancing?

**Solution**

$$P(L_{out})=\frac{1}{2}+\frac{1}{3}=\frac{5}{6},\quad P(S_{out})=\frac{2}{3}+\frac{1}{6}=\frac{5}{6}$$

$$P(L_{out}\cap S_{out})=\frac{5}{6}x\frac{5}{6}=\frac{25}{36}$$

P(Going to dance | Both go out)

$$=\frac{P(Going\ to\ dance)}{P(Both\ Going\ out)}$$

$$=\frac{1}{2}x\frac{2}{3}x\frac{36}{25}=\frac{12}{25}$$

**Example**

The probability that two events occur is $\frac{2}{15}$ and the probability that either or both events occur is $\frac{3}{5}$. Determine the individual probabilities of both events.

**Solution**

$$P(A\cap B)=P(A)P(B)=\frac{2}{15}$$

$$\Rightarrow P(B)=\frac{2}{15P(A)}\quad \text{if let P(A) =Y}$$



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$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{3}{8} = Y + \frac{2}{15Y} - \frac{2}{15}$$

$$15Y^2 - 11Y + 2 = 0$$

$$Y = \frac{11 \pm 1}{30} = \frac{2}{5} or \frac{1}{3}$$

$$\therefore P(A) = \frac{2}{5} \text{ and } P(B) = \frac{1}{3}$$

$$Or\ P(A) = \frac{1}{3} \text{ and } P(B) = \frac{2}{5}$$

**Example**

Bag A contains 2 green and 2 blue balls, while bag B contains 2 green and 3 lue balls. A bag is selected at random and two balls drawn from it without replacement. Find the probability that the balls are of different colours.

**Solution**

$$P(B_A \cap G_A \cap B_A) + P(B_A \cap B_A \cap G_A) +$$
$$P(B_B \cap G_B \cap B_B) + P(B_B \cap B_B \cap G_B)$$

$$= \frac{1}{2} x \frac{2}{4} x \frac{2}{3} + \frac{1}{2} x \frac{2}{4} x \frac{2}{3} + \frac{1}{2} x \frac{2}{5} x \frac{3}{4} + \frac{1}{2} x \frac{3}{5} x \frac{2}{4}$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{3}{20} + \frac{3}{20}$$

$$P(\text{balls are of different colours}) = \frac{19}{30}$$



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## Exercise

1. (a) Define the independence of two events A and B. Given that A and B are independent events in a sample space such that

i. $P(A)=\frac{2}{5}$, *and* $P(A\cup B)=0.8$, *find*

ii. $P(B)$ $P(\overline{A}\cup\overline{B})$

(b) In a certain town the probability that a person owns a car is 0.25. Given that the probability that a person who owns a car is a university graduate is 0.2. Find the probability that a person selected at random owns a car and is a university graduate.

**Answer** (a) (i) P(B) = $\frac{2}{3}$, $(ii)\ P(\overline{A}\cup\overline{B})=\frac{11}{15}$

(b) $P(C\cap G)=0.05$

2. In a certain country, 60% of the cars are privately owned, of the privately owned cars 70% are small, whereas of the cars, which are not privately owned 40% are small. Independent of ownership 20% of the small cars and 30% of the large cars are less than two years old. If a car is choosen at random, calculate the probability that:

i. The car is small
ii. The car is privately owned given that it is large
iii. The car is privately owned, large and more than two years old.
iv. The car is large, given that it is privately owned and more than to years old.

**Answer** $(i)\ P(S)=0.58$, $(ii)\ P(P\mid L)=0.43$
$(iii)\ P(P\cap L\cap O)=0.126$ $(iii)\ P(L\mid P\cap O)=0.273$



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3. Tom is to travel from Lira to Kampala for an interview. The probabilities that he will be in time for the interview when he travels by bus and taxi are 0.1 and 0.2 respectively. The probabilities that he will travel by bus and taxi are 0.6 and 0.4 repectively.

   i. Find the probability that he will be on time.
   ii. Given that he is not on time what is the probability that he traveled by a taxi?

**Answer**

(i) $P(T) = 0.14$ (ii) $P(t \mid \overline{T}) = 0.372$

4. When visiting a friend Grace may go by road, air or rail. The proability of using road, air and rail are 0.3, 0.8 and 0.6 respectively. The corresponding probabilities of arriving on agreed time are 0.2, 0.8 and 0.1 respectively. Find the probability of having used the road given that he arrived on time.

**Answer** $P(R|T) = 0.08$

5. The events A and B are such that $P(A) = 0.5$,

$P(\text{A or B but not both A and B}) = \frac{1}{3} P(B) = 0.25.$

Calculate $P(A \cap B),\ P(\overline{A} \cap B),\ P(A \mid B)$

*where $\overline{A}$ is A doesnot occur.*

State with reasons whether A and B are:

   i. Independent
   ii. Mutually exclusive



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**Answer**

$$P(A \cap B) = \frac{5}{24}, \quad P(\bar{A} \cap B) = \frac{1}{24}, \quad P(A \mid B) = \frac{5}{6}$$

6. George Wear was a striker for AC Milan. In any match he may score, 0, 1, 2, 3 and 4 goals for his side with probabilities 0.2, 0.4, 0.2, 0.1 and 0.1 respectively. Assuming that his performance in successive matches are independent, what is the probability that in three consecutive matches:

   i. He scores a total of 10 goals or more

   ii. He does not score exactly three goals in any match

**Answer** (i) 0.013 (ii) 0.729

7. 95% of car drivers wears seat belts, 64% of cars drivers involved in serious accidents die if not wearing seat belts, where as 12% of those that do wear seat belt die. Calculate to three significant figures, the percentage of drivers involved in serious accidents who died and were not wearing seat belts.

**Answer** 21.9%

8. A court in Kampala may return any one of these verdicts namely Guilty(G), Not guilty(N) or Not proven(V). Of all the cases tried by this court, 70% of the verdicts were guilty, 20% were not guilty and 10% not proven. Suppose that when the courts verdict is guilty, not guilty and not proven, the respective probabilities of the accused person being innocent are 0.05, 0.95 and 0.25. Find the

i. probability that the person is innocent



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ii. an innocent person being tried by this court is found guilty

**Answer** (i) 0.25 (ii) 0.14

9. Two cards are drawn without replacement from ten cards, which are numbered fom 1 to 10. Find the probability that:

i. the numbers on both cards are even

ii. the number on one card is odd and the number on the other is even.

iii. The sum of the numbers on the two cards exceeds 4.

**Answer** (i) $\frac{2}{9}$ (ii) $\frac{5}{9}$ (iii) $\frac{43}{45}$

10. Two independent events are such that the probability of both is $\frac{1}{6}$ and that of neither is $\frac{1}{3}$. Determine the probability of each of the events occurring?

**Answer** $P(A)=\frac{1}{3}$, $P(B)=\frac{1}{2}$, *Or* $P(A)=\frac{1}{2}$, $P(B)=\frac{1}{3}$

11. Three balls are chosen at random without replacement from a bag containing 3 red, 8 blue and 7 white balls. Determine the probability that the balls chosen will be

(i) all blue (ii) all red (iii) one of each colour

**Answer** (i) $\frac{7}{102}$, (ii) $\frac{1}{816}$, (iii) $\frac{7}{34}$

12. A bag contains 5 white , 3 red and green counters. 3 counters are drawn without replacement. What is the probability there

*(i) is no green counter*

*(ii) are 2 white counters and a green counter*

*Answer* (i) $\frac{7}{15}$ (ii) $\frac{1}{6}$



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13. A football star has injury problems. When he is playing the probability that his team will win is $\frac{3}{4}$ but otherwise it is only $\frac{1}{2}$. The probability that the player will be fit this weekend is $\frac{1}{3}$.

Determine the probability that his team will win the match.

**Answer** $\frac{7}{12}$

14. Three dice are to be rolled. Determine the probability of scoring a double but not a triple

**Answer** $\frac{5}{12}$

15. Mariam's chances of passing physics are 0.6, of economics 0.75 and of mathematics 0.80.

i. Determine the chance that she passes one subject only

ii. If it is known that she passed at least two subjects, what is the probability that she failed economics?

**Answer** (i) 0.17 (ii) $\frac{4}{27}$

16. There are 3 black and 2 white balls in each of the two bags. A ball is taken from the first bag and placed in the second, and then a ball is taken from the second into the first, what is the probability that there is now the same number of black and white balls in each bag as there to begin with?

**Answer** 0.6



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17. A bag contains 5 black beads and 3 white beads. A second bag contains 3 black beads and 5 white beads. A bead is drawn at random from the first bag and placed in the second bag. A bead is now drawn from the second bag and placed in the first. Find the probability that each bag now contains

a) 4 black and 4 white beads

b) The same numbers of each colour as it did initially

**Answer** (a) $\frac{25}{72}$ (b) $\frac{19}{36}$

18. A box A contains 3 red balls and 4 black balls. A box B contains 3 red balls and 2 black balls. One box is selected at random and then from it, one ball is selected at random.

(a) Find the probability that the ball is red,

(b) the probability that the ball came from box A, given that it is red.

**Answer** (a) $\frac{18}{35}$, (b) $\frac{5}{12}$

19. A school is divided into two sections: A-level, 400 boys and 200 girls, O-level, 400 girls and 300 boys. A student is selected at random from the school. If this student comes from O-level, a second student is selected from the A- level; if the first students comes from the A- level, the second student is chosen from the O-level. Find the probability that

a) The second student will be a girl

b) If the second student is a boy, he is a member of A-level.

**Answer** (a) $\frac{121}{273}$ (b) $\frac{49}{76}$

20. A boat hiring company at the local boating lake has two types of crafts 20 blue and 35 herons. The customer has to take the next boat available when hiring. The boats are all distinguishable

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by their numbers. A regular customer Amina carefully notes that the numbers of the boats, which she uses, and finds that she has used 15 different blue birds and 20 different herons. Each boat is equally likely to be the next in line. Amina hires two boats at the same time. (one is for a friend). If event X is: Amina has not hired either boat before, and event Y is: both are herons, determine:

(a) P(X) (b) P(Y) (c) P(X|Y)

(d) whether the events are mutually exclusive

**Answer**

(a) $P(X)=\frac{38}{297}$, $P(Y)=\frac{119}{297}$, $P(X|Y)=\frac{3}{17}$

(d) $P(X \cap Y) \neq 0$, hence not mutually exclusive

21. Two girls Faridah and Monica play a game in which each throws a tetrahedron in turn. The first to get a 3 wins, Faridah tries first. What is the probability that

i. Monica wins on his second trial

ii. Faridah wins

(b) A player can play on either of the two gambling machines A and B. He chooses one of the machines at random, and plays two games. The probability of winning a game on A is $\frac{1}{3}$ and the probability of winning on B is $\frac{1}{4}$. If he loses both of these two games, he plays a third game on the other machine; otherwise he plays the third game on the same machine. Find the probability that he



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i. Wins the first game

ii. Changes the machine after the second game

iii. Plays the third game on A

iv. Wins the third game

**Answer** $(a)\ \frac{27}{256},\ \frac{4}{7}$ $(b)\ (i)\ \frac{7}{24},\ (ii)\ \frac{145}{288}$

$(iii)\ \frac{161}{288}$ $(iv)\ 0.3242$

22. A bag contains 4 white balls and 1 black ball. A second bag contains 1 white ball and 4 black balls. Aball is drawn at random from the first bag and put into the second bag, then a ball is taken from the second bag and put into the first bag. Find the probability that a white ball will be picked when a ball is selected from the first bag.

**Answer** 0.7



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# CHAPTER THREE

## DISCRETE PROBABILITY DISTRIBUTION

### 3.1 INTRODUCTION

A probability density function P(X=x) is discrete if it has countable domain. A random variable following such a probability density function (P.D.F) is called a discrete random variable.

**Example**

Write the probability distribution of the score when an ordinary die is thrown

**Solution**

(i) S = (1, 2, 3, 4, 5, 6)

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| P( X = x) | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

**Properties of discrete probability distribution**

(i) $\sum_{\text{all } x} P(X = x) = 1$

Total probability is 1

(ii) $P(X = x) \geq 0$ for all values of X.

**Note:** $P(X = x) = f(x)$. f(x) can be used to represent a discrete probability density function.



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**Example**

A random variable X has the following probability distribution

| x | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| P(X = x) | $\frac{1}{5}$ | $\frac{3}{10}$ | $\frac{1}{10}$ | $\frac{2}{5}$ |

Determine

(i) $P(X>2)$, (ii) $P(<2)$ (iii) $P(X\leq 2)$

**Solution**

(i) $P(X>2) = P(X=3) + P(X=4)$

$$= \frac{1}{10}+\frac{2}{5}$$

$$P(X>2) = 0.5$$

(ii) $P(X<2) = P(X=1)$

$$P(X<2) = 0.2$$

(iii) $P(X\leq 2) = P(X=1) + P(X=2)$

$$= \frac{1}{5}+\frac{3}{10}$$

$$P(X\leq 2) = 0.5$$

**Example**

A discrete random variable X has P.D.F as shown below

| *x* | *1* | *2* | *3* | *4* | *5* |
|---|---|---|---|---|---|
| P(X = x) | 0.2 | 0.25 | 0.4 | a | 0.05 |

Determine

i. Value of a

ii. $P(1<X<3)$



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iii. $P(2 \le X \le 4)$

**Solution**

(i) $\sum P(X = x) = 1$

$$0.2 + 0.25 + 0.4 + a + 0.05 = 1$$

$$0.9 + a = 1$$

Therefore $a = 0.1$

(ii) $P(1 < X < 3) = P(X = 2)$

$P(1 < X < 3) = 0.25$

(iii) $P(2 \le X \le 4) = P(X = 2) + P(X = 3) + P(X = 4)$

$$= 0.25 + 0.4 + 0.1$$

$P(2 \le X \le 4) = 0.75$

**Example**

Two biased tetrahedrons have each of their faces numbered 1 to 4. The chances of getting anyone face showing uppermost is inversely proportional to the number on it. If the two tetrahedrons are drawn and the number on the uppermost face noted, determine the probability that the faces show the same number.

**Solution**

$P(X = x) \ \alpha \ \frac{1}{x} \Rightarrow P(X = x) = k\frac{1}{x}$

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| P(X=x) | $k$ | $\frac{k}{2}$ | $\frac{k}{3}$ | $\frac{k}{4}$ |

$$k + \frac{k}{2} + \frac{k}{3} + \frac{k}{4} = 1$$



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$$12k + 6k + 4k + 3k = 12$$

$$25k = 12$$

$$k = \frac{12}{25}$$

P(Same number) = P(1 and 1) + P(2 and 2)
+ P( 3 and 3) + P( 4 and 4)

$$= k^2 + (\frac{k}{2})^2 + (\frac{k}{3})^2 + (\frac{k}{4})^2$$

$$= k^2(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16})$$

$$= (\frac{12}{25})^2(\frac{205}{144}) = \frac{205}{625}$$

P( same number) = $\frac{41}{125}$ =0.328

**Example**

The probability distribution of a random variable X is given by the function

$$f(x) = \frac{1}{84}\binom{5}{x}\binom{4}{3-x} \quad x = 0,1,2,3$$

calculate the numerical probabilities?

**Solution** $\binom{n}{r}$ *mean combinations*

Note:

$$f(0) = \frac{1}{84}\binom{5}{0}\binom{4}{3-0}$$



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$$= \frac{1}{84} x 1 x 4 = \frac{4}{84}$$

$$f(1) = \frac{1}{84}\binom{5}{1}\binom{4}{3-1}$$

$$= \frac{1}{84} x 5 x 6 = \frac{30}{84}$$

$$f(2) = \frac{1}{84}\binom{5}{2}\binom{4}{3-2}$$

$$= \frac{1}{84} x 10 x 4 = \frac{40}{84}$$

$$f(3) = \frac{1}{84}\binom{5}{3}\binom{4}{3-3}$$

$$= \frac{1}{84} x 10 x 1 = \frac{10}{84}$$

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| P( X= x) | $\frac{4}{84}$ | $\frac{30}{84}$ | $\frac{40}{84}$ | $\frac{10}{84}$ |

## 3.2 MEAN OR EXPECTED VALUE. E(X)

$$E(x) = \sum xP(X = x)$$

The symbol for mean is E(X) or μ



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## 3.3 VARIANCE ( VAR(X))

$$Var(X) = EX^2 - (E(X))^2$$

Where $EX^2 = \sum x^2 P(X = x)$

and $E(x) = \sum xP(X = x)$

**Example**

A fair coin is tossed four times and X represents the number of tails that appear. Determine the probability distribution of X and hence expected number of tails.

**Solution**

$$S = \left\{ \begin{array}{c} TTTT, TTTH, TTHT, THTT, HTTT, \\ TTHH, THTH, HTHT, HHTT, THHT, \\ HTTH, HHHT, HHTH, HTHH, \\ THHH, HHHH \end{array} \right\}$$

| $X$ | $P(X=x)$ | $xP(X=x)$ |
|---|---|---|
| 0 | $\frac{1}{16}$ | 0 |
| 1 | $\frac{4}{16}$ | $\frac{4}{16}$ |
| 2 | $\frac{6}{16}$ | $\frac{12}{16}$ |
| 3 | $\frac{4}{16}$ | $\frac{12}{16}$ |
| 4 | $\frac{1}{16}$ | $\frac{4}{16}$ |
| **Total** | 1 | 2 |



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Therefore expected value = 2

**Example**

The random variable X takes on integral values only and had P.D.F given by $P(X=x)=kx \qquad x=1,2,3,4,5$

$P(X=x)=k(10-x) \quad x=6,7,8,9$

Find

a) The value of the constant k.
b) Expected value of X
c) Var(X)

**Solution**

| $x$ | $x^2$ | $P(X=x)$ | $x\,P(X=x)$ | $x^2P(X=x)$ |
|---|---|---|---|---|
| 1 | 1 | k | k | k |
| 2 | 4 | 2k | 4k | 8k |
| 3 | 9 | 3k | 9k | 27k |
| 4 | 16 | 4k | 16k | 64k |
| 5 | 25 | 5k | 25k | 125k |
| 6 | 36 | 4k | 24k | 144k |
| 7 | 49 | 3k | 21k | 147k |
| 8 | 64 | 2k | 16k | 128k |
| 9 | 81 | K | 9k | 81k |
| Total | | 25k | 125k | 725k |

(a) $25k=1$

$$\Rightarrow k=\frac{1}{25}$$

(b) $E(X)=\sum x\,P(X=x)$

E( X) = 125k



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$$= 125x\frac{1}{25} = 5$$

$E(X) = 5$

(c) $Var(X) = EX^2 - (E(X))^2$

$$= (725\ x\frac{1}{25}) - 5^2$$

$$29 - 25$$

$$= 4$$

Variance (x) = 4

**Example**

At school, projector shows follow a discrete random variable. 3 students, Aisha, Brenda and Cathy will watch the show with probability of $\frac{1}{4}$, $\frac{1}{3}$, *and* $\frac{2}{5}$ respectively.

The decision of a student makes to watch the show is independent of the other students.

Find

(i) the probability function of the number of students who watch the projector show

(ii) Hence the expected number of students who watch the Show

**Solution**

Let X be discrete random variable representing the number of students who watch the show and A, B and C represent the respective events that a student watches the show



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$$P(X=0)=P(\overline{A}\cap\overline{B}\cap\overline{C})=P(\overline{A})P(\overline{B})P(\overline{C})$$

$$\frac{3}{4}x\frac{2}{5}x\frac{3}{5}=\frac{3}{10}$$

$$P(X=1)=P(A\cap\overline{B}\cap\overline{C})+$$
$$P(\overline{A}\cap B\cap\overline{C})+P(\overline{A}\cap\overline{B}\cap C)$$
$$=\frac{1}{4}x\frac{2}{3}x\frac{3}{4}+\frac{1}{4}x\frac{2}{3}x\frac{3}{4}+\frac{3}{4}x\frac{2}{3}x\frac{2}{4}$$
$$\frac{1}{10}+\frac{3}{20}+\frac{1}{5}=\frac{9}{20}$$

$$P(X=2)=P(A\cap B\cap\overline{C})+P(A\cap\overline{B}\cap C)+$$
$$P(\overline{A}\cap B\cap C)$$
$$=\frac{1}{4}x\frac{1}{3}x\frac{3}{5}+\frac{1}{4}x\frac{2}{3}x\frac{2}{5}+\frac{3}{4}x\frac{1}{3}x\frac{2}{5}$$
$$\frac{1}{20}+\frac{1}{15}+\frac{2}{20}=\frac{13}{60}$$

$$P(X=3)=P(A\cap B\cap C)$$
$$=\frac{1}{4}+\frac{1}{3}+\frac{2}{3}=\frac{1}{30}$$

Probability function

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| P(X=x) | $\frac{3}{10}$ | $\frac{9}{20}$ | $\frac{13}{60}$ | $\frac{1}{30}$ |



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Expected numbers

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| P(X=x) | $\frac{3}{10}$ | $\frac{9}{20}$ | $\frac{13}{60}$ | $\frac{1}{30}$ |
| xP(X=x) | 0 | $\frac{9}{20}$ | $\frac{26}{60}$ | $\frac{1}{10}$ |

Means $\frac{59}{60} \cong 1$

**Example**

A box contains 4 red balls and 2 blue balls. A ball is drawn without replacement until a blue is drawn. If X represents the number of draws required to draw a blue ball, find the probability distribution of X and hence mean and variance.

$$P(X=1) = P(B_1) = \frac{1}{3}$$

$$P(X=2) = P(R_1B_2) = \frac{2}{3}x\frac{2}{5}$$

$$P(X=3) = P(R_1R_2B_3) = \frac{2}{3}x\frac{3}{5}x\frac{2}{4}$$

$$P(X=4) = P(R_1R_2R_3B_4) = \frac{2}{3}x\frac{3}{5}x\frac{2}{4}x\frac{2}{3}$$

$$P(R_1R_2R_3R_4B_5) = \frac{2}{3}x\frac{3}{5}x\frac{2}{4}x\frac{1}{3}x1$$

| x | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| P(X=x) | $\frac{1}{3}$ | $\frac{4}{15}$ | $\frac{1}{5}$ | $\frac{2}{15}$ | $\frac{1}{15}$ |



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| x | P (X = x) | xP (X = x) | $x^2$P (X = x) |
|---|---|---|---|
| 1 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| 2 | $\frac{4}{15}$ | $\frac{8}{15}$ | $\frac{16}{15}$ |
| 3 | $\frac{1}{5}$ | $\frac{3}{5}$ | $\frac{9}{5}$ |
| 4 | $\frac{2}{15}$ | $\frac{8}{15}$ | $\frac{32}{15}$ |
| 5 | $\frac{1}{15}$ | $\frac{5}{15}$ | $\frac{25}{15}$ |
| Total | 1 | $\frac{7}{3}$ | 7 |

$$E(X) = \frac{7}{3} = 2.3333$$

$$Var(X) = EX^2 - (E(X))^2$$

$$Var(X) = 7 - \left(\frac{7}{3}\right)^2 = 1.556$$

## 3.4 PROPERTIES OF MEAN

1. $E(a) = a$ where a is constant
2. $E(aX) = a\,E(X)$
3. $E(aX + b) = aE(X) + b$ where a and b consants.



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**Example**

The discrete random variable X has p.d.f given by

| x | -1 | 0 | 1 | 2 |
|---|---|---|---|---|
| P(X=x) | 0.25 | 0.10 | 0.45 | 0.20 |

If Y is random variable defined by Y = 0.5X + 3. Determine the E(X) and E(Y)

**Solution**

| x | -1 | 0 | 1 | 2 |
|---|---|---|---|---|
| P(X=x) | 0.25 | 0.10 | 0.45 | 0.20 |
| xP(X=x) | -0.25 | 0 | 0.45 | 0.40 |

$\sum xP(X = x) = 0.6$

Therefore E(X) = 0.6

$$
\begin{aligned}
E(Y) &= E(0.5X + 3) \\
&= 0.5E(X) + 3 \\
&= 0.5 \times 0.6 + 3 \\
&= 3.3
\end{aligned}
$$

E(Y) = 3.3

## 3.5 PROPERTIES OF VARIANCE

1. Var(a) = 0 where a is constant
2. $Var(aX) = a^2 Var(X)$
3. $Var(aX + b) = a^2 Var(X)$



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**xample**

ne discrete random variable X has p.d.f given by

| : | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| P(X= x) | 0.2 | 0.25 | 0.4 | 0.1 | 0.05 |

etermine

Var (X) (ii) Var(3X-2)

**lution**

| : | x | P(X=x) | xP(X=x) | $x^2P(x=x)$ |
|---|---|---|---|---|
| | 1 | 0.2 | 0.2 | 0.2 |
| 2 | 4 | 0.25 | 0.5 | 1 |
| 3 | 9 | 0.4 | 1.2 | 3.6 |
| 4 | 16 | 0.1 | 0.4 | 1.6 |
| 5 | 25 | 0.05 | 0.25 | 1.25 |
| tal | | 1 | 2.55 | 7.65 |

$\text{Var}(X) = 7.65 - 2.55^2$

$\text{Var}(X) = 1.1475$

i) $\text{Var}(Y) = \text{Var}(3X - 2)$

$= 9\ \text{Var}(X) - 0$

$= 9 \times 1.1475$

$= 10.3275$

herefore Var( 3X-2) = 10. 3275.



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**Example**

The discrete random variable X is defined by p.d.f given by

| x | 0 | 1 | 2 |
|---|---|---|---|
| P(X=x) | 0.4 | 0.4 | 0.2 |

If Y is a random variable defined by Y= 2X-1. Determine the mean and variance of X and Y.

**Solution**

| x | y | P(X) | P(Y) | xP(X) | yP(Y) | $x^2P(X)$ | $y^2P(Y)$ |
|---|---|---|---|---|---|---|---|
| 0 | -1 | 0.4 | 0.4 | 0 | -0.4 | 0 | 0.4 |
| 1 | 1 | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 |
| 2 | 3 | 0.2 | 0.2 | 0.4 | 0.6 | 0.8 | 1.8 |
| Total | | 1 | 1 | 0.8 | 0.6 | 1.2 | 2.6 |

$E(X) = 0.8,$

$Var(X) = 1.2 - 0.8^2$

$Var(X) = 0.56$

$E(Y) = 0.6$

$Var(Y) = 2.6 - 0.6^2$

$Var(Y) = 2.24$

**Example**

The packets of Omo sold in a shop are of four categories namely, small, medium, large and giant. On a particular day, the stock is such that the ratio of small: medium: large: giant is equal to 4: 2: 1: 1. The costs of the packets are in the ratio: small: medium: large: giant are 350: 500: 800: 1400 respectively.



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a) 30 packets are sold randomly on that particular day, the total cost of the sales being S shillings. Calculate
   i. The expected value of S
   ii. The standard deviation of S.

**Solution**

Let s denote small, m medium, l large and g giant.

$P(s)=\frac{4}{8}=\frac{1}{2}$, $P(m)=\frac{2}{8}=\frac{1}{4}$

$P(l)=\frac{1}{8}$, $P(g)=\frac{1}{8}$

Let X be the cost of a packet

| $x$ | $P(X=x)$ | $xP(X=x)$ | $x^2$ | $x^2P(X=x)$ |
|---|---|---|---|---|
| 350 | $\frac{1}{2}$ | 175 | 122,500 | 61,250 |
| 500 | $\frac{1}{4}$ | 125 | 250,000 | 62,500 |
| 800 | $\frac{1}{8}$ | 100 | 640,000 | 80,000 |
| 1400 | $\frac{1}{8}$ | 175 | 1,960,000 | 245,000 |
| Total | 1 | 575 | | 448,750 |



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$E(X) = 575$

$E(S) = E(30X)$

$= 30\ E(X)$

$E(S) = 30 \times 575 = 17{,}250/=$

$Var(X) = EX^2 - (E(X))^2$

$= 448{,}750 - 575^2$

$Var(X) = 118,\ 125$

$Var(S) = 30^2 Var(X)$

$= 900 \times 118,\ 125$

Standard deviation $= \sqrt{900 x 118{,}125}$

Standard deviation $= 10{,}310.8/=$

## 3.6 MODE

The mode of a discrete probability density function is the value of X with the highest probability.

**Example**

A random variable has the following probability distribution

| x | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| P(X=x) | 0.1 | 0.4 | 0.3 | 0.2 |

Determine the mode

**Solution**

$P(X=2) = 0.4$ is the highest probability therefore the mode is 2.

$X = 2$ is the mode



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**Example**

A random variable has the following distribution

| x | 1 | 2 | 3 |
|---|---|---|---|
| P(X=x) | 0.35 | 0.35 | 0.3 |

Find the mode

P(X=1)=P(X=2)=0.35. This is highest probability. Therefore the mode is 1 and 2.

**Example**

A die is loaded such that the chance of throwing a one is $\frac{x}{4}$, the chance of a two is $\frac{1}{4}$, and the chance of a six is $\frac{1-x}{4}$. The chance of a three, four and five is $\frac{1}{6}$. The die is thrown twice. Prove that the chance of throwing total of 7 is $\frac{9x-9x^2+10}{72}$,

Find the value of x, which makes this chance maximum, and find this maximum probability.

**Solution**

Let r be the number that appears when a die is thrown

| *r* | *1* | *2* | *3* | *4* | *5* | *6* |
|---|---|---|---|---|---|---|
| P(R=r) | $\frac{x}{4}$ | $\frac{1}{4}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1-x}{4}$ |

Possibilities for sum of seven



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$$P(\text{sum of } 7) = P(6 and 1) + P(1 and 6) + P(5 and 2) + P(2 and 5) + P(4 and 3) + P(3 and 4)$$

$$\left(\frac{1-x}{4}\right)\left(\frac{x}{4}\right)+\left(\frac{x}{4}\right)\left(\frac{1-x}{4}\right)+\frac{1}{6}x\frac{1}{4}+\frac{1}{4}x\frac{1}{6}+\frac{1}{6}x\frac{1}{6}+\frac{1}{6}x\frac{1}{6}$$

$$=\left(\frac{x-x^2}{16}\right)+\left(\frac{x-x^2}{16}\right)+\frac{1}{24}+\frac{1}{24}+\frac{1}{18}$$

$$=\left(\frac{x-x^2}{8}\right)+\frac{5}{36}$$

$$=\frac{9x-9x^2+10}{72} \text{ as required}$$

For maximum differentiate the expression = 0

$$\frac{d}{dx}\left(\frac{9x-9x^2+10}{72}\right)=0$$

$$9-18x=0$$

Therefore $x=\frac{1}{2}$

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$$\text{Maximum probability} = \frac{9x0.5-9x0.5^2+10}{72}$$

$$=\frac{4.5-2.25+10}{72}=\frac{12.25}{72}$$

$$=\frac{49}{288}$$



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## 3.7 CUMULATIVE MASS FUNCTION (DISTRIBUTION FUNCTION).

The cumulative distribution function of a discrete p.d.f is given by:
$F(X) = P(X \leq x)$
Note: Related to how you obtain cumulative frequencies.

**Example**
Consider the following probability distribution

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $P(X=x)$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

Determine the cumulative mass function

**Solution**

$$F(1) = P(X \leq 1) = \frac{1}{6}$$

$$F(2) = P(X \leq 2) = P(X = 1) + P(X = 2) = \frac{1}{3}$$

$$F(3) = P(X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3) = \frac{1}{2}$$

$$F(4) = P(X \leq 4) = \frac{4}{6}$$

$$F(5) = P(X \leq 5) = \frac{5}{6}$$

$$F(6) = P(X \leq 6) = 1$$

Normally the last value Of $F(X) = 1$.
Note: $F(X)$ can be given as an formula.



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**Example**

The discrete random variable X has cumulative mass function

$$F(X) = \frac{x}{5} \; for \; x = 1, 2, 3, 4, 5.$$

Find the probability distribution of X.

**Solution**

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| F(X) | $\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{3}{5}$ | $\frac{4}{5}$ | 1 |

$P(X=1) = F(1) = 0.2$

$P(X=2) = F(2) - F(1)$

$= 0.4 - 0.2$

$P(X=2) = 0.2$

$P(X=3) = F(3) - F(2)$

$= 0.6 - 0.4$

$P(X=3) = 0.2$

$P(X=4) = F(4) - F(3)$

$P(X=4) = 0.8 - 0.6 = 0.2$

$P(X=5) = 1 - 0.8 = 0.2$

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| P(X= x) | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ |



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## 3.8 MEDIAN

The median of the probability distribution of random variable X is the smallest value for which F(x) is at least 0.5. If m is the median then

(i) $F(m) \geq 0.5$

(ii) $1- \ F(m-1) \geq 0.5$

## 3.9 GRAPH OF f(x)

This is composed of vertical lines parallel to the y-axis drawn at every specified value of x.

## 3.10 GRAPH OF F(x)

This is composed of horizontal straight lines drawn parallel to x – axis.

**Example**

A random variable has the following distribution,
P(X= 1)=0.1, P(x=2) = 0.4, P(X=3)=0.3, P(X=4) = 0.2
Find

i. The cumulative distribution function

ii. The median

iii. The graph of f(x) and F(x)



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**Solution**

| $x$ | $P(X=x)$ | $P(X\leq x) = F(X)$ |
|---|---|---|
| 1 | 0.1 | 0.1 |
| 2 | 0.4 | 0.5 |
| 3 | 0.3 | 0.8 |
| 4 | 0.2 | 1.0 |

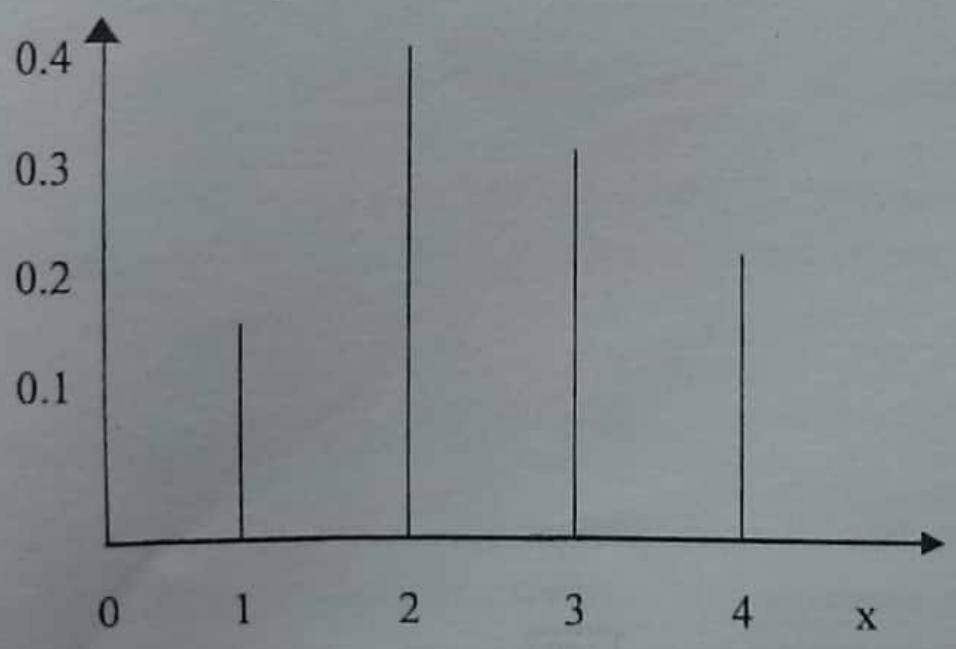
(i) $F(X) = P(X \leq x)$

(ii) $F(m) \geq 0.5$ where m is the median

$F(2) = 0.5$

Therefore the median is 2

graph of f(x)



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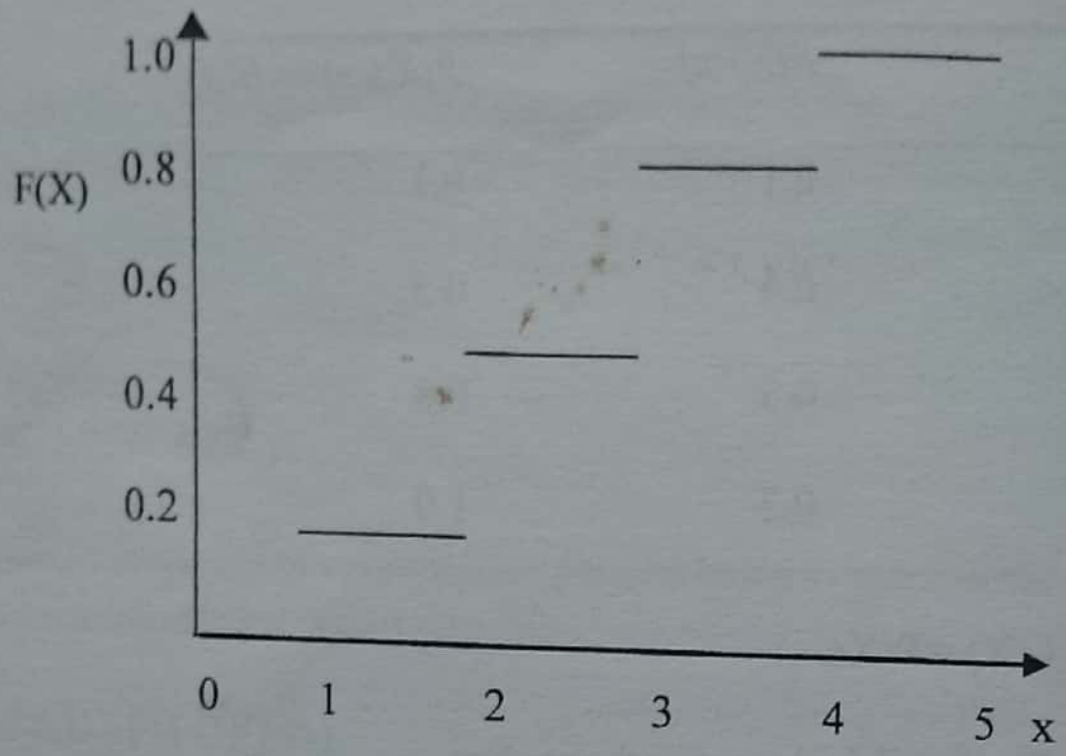
Graph F(X)



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## Exercise

1. A random variable T has the following probability distribution. $P(T=1) = 0.1$, $P(T=2) = 0.2$, $P(T=3) = 0.3$, $P(T=4) = 0.4$, show that the distribution above is a probability distribution

Determine:

i. The expected value and variance of T
ii. $P(T=2 \mid T \geq 2)$

**Answer** $E(T) = 3$, $Var(T) = 1$ $P(T=2 \mid T \geq 2) = \frac{2}{9}$

2. For each of the following random vaiable write out the proability distributions

a) The number of heads obtained when two fair coins are tossed
b) The sum of the scores when two ordinary dice are thrown.
c) The number of threes obtained when two tetrahedral are thrown
d) The numerical value of a digit chosen from a set of random number table
e) The number of tails obtained when three fair coins are tossed.
f) The difference between the number when two ordinary dice are thrown

3. A computer is made to produce randomly the number 0, 1, 2, … 9, what is the expected value and variance of the numbers so produced?

**Answer** $E(X) = 4.5$, $Var(X) = 8.25$

4. A discrete random variable X takes integer values between 0 and 5 inclusive with probabilities given by



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$$P(X=r)=\begin{cases}\dfrac{2r+1}{20} & r=0,1,2,3\\[2ex] \dfrac{11-2r}{20} & r=4,5\end{cases}$$

*Find the expectation and variance of X*

*Answer* : $E(X)=2.55,\quad Var(X)=1.45$

5. A random variable X can assume values 10 and 20 only. If E(X) = 16, write the P.D.F of X in table form

**Answer**

| x | P(X=x) |
|---|---|
| 10 | 0.4 |
| 20 | 0.6 |

6. Two tetrahedral dice are thrown and score is the product of the number on which the dice fall. What is the expected score for a throw?

**Answer**: 6.25

7. A random variable X has the probability function

$$f(x)=\begin{cases}k\,2^x & x=0,1,2,\ldots 6\\ 0 & \text{otherwise}\end{cases}$$

*Determine* :

(i) *the value of* $k$

(ii) $E(X)$

(iii) $P(X<4\,|\,X>1)$

*Answer* (i) $=\dfrac{1}{127}$ (ii) $E(X)=5$ (iii) $\dfrac{3}{31}$



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8. A discrete random variable X has probability function

$$P(X=x)=\begin{cases}\dfrac{x}{k} & x=1,2,3...n\\ 0 & otherwise\end{cases}$$

*where k is a constant*

*Given that the expectation of X is 3, find :*

*(i) the value of n and the constant k*

*(ii) the median and variance of X*

*(iii)* $P(X=2 \mid X \geq 2)$

Answer : (i) $n=4,\ k=10$

(ii) median $=3$, $\mathrm{Var}(x)=1$

(iv) $P(X=2 \mid X \geq 2)=\dfrac{2}{9}$

9. The number of times a machine breaks down every month is a discrete random variable X with the probability distribution

$$P(X=x)=\begin{cases}k\ 0.25^{x} & x=0,1,2,3,...\\ 0 & otherwise\end{cases}$$

*where k is a constant*

*Determine the probability that the machine will break down not more than two times a month.*

*Answer :* 0.984

10. A 5- sided die with sides numbered 1,2,3,4 and 5 is constructed so that 1 and 5 occur twice as often as 2 and 4, which occur three times as often as 3. What is the probability that a prefect square occurs when this die is tossed once?

**Answer:** $P(4\ Or\ 1)=\dfrac{9}{19}$



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11. A bag contains two red and eight black marbles. A sample of four marbles is to be drawn at random from the bag without replacement:

(a) Show that the probability of obtaining exactly two red marbles in the sample is 2/ 15

(b) Show that the probability of obtaining exactly one red marbles in the sample is 8/ 15

(c) Calculate the expected number of red marbles that will be drawn.

**Answer:** E(X) = 0.8

12. The probability of a number appearing on the top face when an unbalanced die is tossed is proportional to the number. Find the probability that an odd or prme number will appear. If the die is tossed twice, find the probability distribution function for the sum sum of the two numbers that appear on the top face. What is the most likely sum?

| $x$ | $P(X=x)$ |
|---|---|
| 2 | $1/441$ |
| 3 | $4/441$ |
| 4 | $10/441$ |
| 5 | $20/441$ |
| 6 | $35/441$ |
| 7 | $56/441$ |
| 8 | $70/441$ |
| 9 | $76/441$ |
| 10 | $73/441$ |
| 11 | $60/441$ |
| 12 | $36/441$ |



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Answer: $P(O \cup P) = \frac{11}{21}$

The likely sum is 9

13.A bag contains one 200 Shs note, three 100 Shs notes and n 50 Shs notes. A note is selected at random from the bag, its value noted and then replaced. The process is repeated many times. If the average of the values of the notes after many trials is 110 Shs, determine:

i. The value of n
ii. The expected value of the sum of the two notes selected at random without replacement

**Answer:** (i) n = 1 (ii) E(X) = 220 /=

14. A basket contains 7 ripe mangoes and 8 raw mangoes. Two mangoes are picked in succession with replacement. Find the most likely number of ripe mangoes picked.

**Answer** : 1

15. A bag Z contains 3 red balls and 6 white balls and the second bag X contains 5 red balls and 4 white balls. A ball is chosen at random from bag Z and placed in bagX. A ball is then chosen at random from bag X and placed in bag Z. If R is the number of red balls in bag Z after these operations, find the probability distribution of R. Hence the expected value of R.

**Answer:**

$$P(R=2) = \frac{2}{15},\ P(R=3) = \frac{8}{15}, P(R=4) = \frac{1}{3}, E(R) = 3.2$$

16. The discrete random variable X has distribution function F(X) where



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$F(x)=1-(1-x/4)^x$

For $x=1,2,3,4$

(a) Show that $F(3)=63/64$ and $F(2)=3/4$

(b) Obtain the probability distribution of $X$

(c) Find the $E(X)$ and $Var(X)$

(d) Find $P(X>E(X))$

Answer:

(b)

| x | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| P(X=x) | 1/4 | 1/2 | 15/64 | 1/64 |

(c) $2\frac{1}{64}$, 0.547 (d) 0.25

17. (a) An examination question has two parts A and B. The probability of a student getting part A correct is $2/3$. If she gets part A correct, the probability that she gets part B correct is $3/4$, otherwise it is $1/6$. There are three marks for a correct solution of part A, two marks for part B and a bonus mark if both parts are correct. Calculate the expecte value and variance of the student's total mark for the question?

(b) In a game where a gambler rolls a dice on a \$ 1 stake, the Casino pays out \$ 10 for a double six and \$ 3 for a score of sum seven ( the stake money returned as well) and for other score the gambler loses his stake. Whatis the Casino's expected profit on a \$ 100 stake?

**Answer**

(a) E(X) = 3.61, Var(X) = 6.683, (b) E(P) = \$ 2.78



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# CHAPTER FOUR

# THE BINOMIAL DISTRIBUTION

## 4.1 INTRODUCTION

Binomial distribution is an example of discrete probability distribution with a countable domain. A binomial experiment is characterized with repeated trials, and two possible outcomes, one is termed as "success" and other "failure"

**Example**

A student attempts three questions in a commerce objective section, where each question has four possible alternatives with only one correct answer. Determine the probability distribution of the number of correct answers.

**Solution**

$P(C)=\frac{1}{4},\ P(\overline{C})=\frac{3}{4}$, *where C represents correct answer.*

Using the theory of Binomial expansion

$(a+b)^3=a^3+3a^2b+3ab^2+b^3$ Note: $n=3$

If let $a=\frac{3}{4}, b=\frac{1}{4}$

$$\Rightarrow\left(\frac{3}{4}+\frac{1}{4}\right)^3=\left(\frac{3}{4}\right)^3+3\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)+3\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3$$

**Note:** First term has zero correct answers. The second term has one correct answer. Third term has two correct answers. The fourth term has all correct answers.



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| $x$ | | $P(X=x)$ |
|---|---|---|
| 0 | $\left(\frac{3}{4}\right)^3$ | $\frac{27}{64}$ |
| 1 | $3\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)$ | $\frac{27}{64}$ |
| 2 | $3\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2$ | $\frac{9}{64}$ |
| 3 | $\left(\frac{1}{4}\right)^3$ | $\frac{1}{64}$ |

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| P(X=x) | $\frac{27}{64}$ | $\frac{27}{64}$ | $\frac{9}{64}$ | $\frac{1}{64}$ |

## 4.2 PROPERTIES OF BINOMIAL DISTRIBUTION

1. The experiment has a number of repeated trials (n)
2. Each trial results in only two possible outcomes "Success" and "failure"
3. The trials are independent
4. The probability of success p is constant.

## 4.3 FORMULA FOR BINOMIAL EXPERIMENT

A random variable X is said to have a binomial distribution B(n, p),

If $P(X = r) = \binom{n}{r} q^{n-r} p^r, r = 0, 1, 2 \ldots n$

where $0 \le p \le 1$ and $q = 1 - p$



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Where p is probability of success

where $\binom{n}{r} = {}^{n}C_{r}$

**EXAMPLE**

Find the probability of obtaining exactly three 4's if an ordinary die is tossed 5 times.

**Solution:**

n = 5, p = 1/6, r = 3

$$P(X = r) = \binom{n}{r} q^{n-r} p^{r}$$

$$P(X = 3) = \binom{5}{3}\left(\frac{5}{6}\right)^{2}\left(\frac{1}{6}\right)^{3}$$

$$= 10 \times \frac{5^2}{6^5} = 0.03215$$

$P(X=3) = 0.03215$

**Example**

The probability that a pen drawn at random from a box of pens is defective is 0.1. If a sample of 6 pens is taken, find the probability that it will contain

(a) No defective pens
(b) 5 or 6 defective pens.
(c) Less than 3 defective pens.



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**Solution**

n= 6, p = 0.1, let X be number of defective pens.

(a) $P(X=0)=\binom{6}{0}(0.9)^6(0.1)^0$

$P(X=0) = (0.9)^6 = 0.5314$

$P(X=0) = 0.5314$

(b) P (X=5) + (X=6).

$P(X=r)=\binom{n}{r}q^{n-r}P^r$ ).

$P(X=5)=\binom{6}{5}(0.9)^1(0.1)^5$

$= 6(0.9) \times (0.1)^5$

$= 0.000054$

$P(X=6)=\binom{6}{6}(0.9)^0(0.1)^6$

$= (0.1)^6 = 0.000001$

P(5 or 6) = 0.000054 + 0.000001

= 0.000055.

(c) $P(X<3) = P(X \leq 2)$

**Note:** this is a discrete function.

$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$P(X = 0) = 0.5314$ from part (a)

$P(X = 1)=\binom{6}{1}(0.9)^5(0.1)^1$

$0.6 \times (0.9)^5$

$P(X = 1) = 0.3543$



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$$P(X=2) = \binom{6}{2}(0.9)^4(0.1)^2$$

$$15 \times 0.01 \times (0.9)^4$$

$P(X = 2) = 0.0984$

$\therefore P(X \leq 2) = 0.5314 + 0.3543 + 0.0984$

$P(X \leq 2) = 0.9841$

**Note** $\binom{n}{r}$ can be obtained from tables.

Some of the above probabilities can be obtained from the table. The value of n is located from the table and its corresponding p (probability of success noted). The probability can be read off from the table for r = 0, 1, ….n.

**Example**

Using the example above obtain P(X = 0)

**Solution:**

n = 6, p = 0.1

Table

B(n, p), Individual terms.

| n | r | Probability of success(p) 0.01 | 0.05 | 0.1 | 0.15 | 0.2 | 0.25… |
|---|---|---|---|---|---|---|---|
| 6 | 0 | | | 0.5314 | | | |
| | 1 | | | 0.3543 | | | |
| | 2 | | | 0.0984 | | | |
| | 3 | | | 0.0146 | | | |
| | 4 | | | 0.0012 | | | |
| | 5 | | | 0.0001 | | | |
| | 6 | | | - | | | |



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From the table $P(X=0)=0.5314$. Obtained by reading $n=6, r=0$ and $p=0.1$.

Then $P(X=1)=0.3543$ (Tab)
And $P(X=2)=0.0984$ (Tab).
Check part c. for $P(X=1)$ and $P(X=2)$.

**Example**

A multiple –choice quiz has 15 questions, each with 4 – possible answers of which only one is the correct answer. Determine the probability that sheer guess work yields.

(a) Exactly five correct answers
(b) Five incorrect answers

**Solution**

$n=15, p=¼=0.25$
Let X be number of correct answers.

(a) $P(X=5)$

Use $P(X=r)=\binom{n}{r}q^{n-r}p^{r}$

From the table when $n=15, r=5, p=0.25$

$$\Rightarrow P(X=5)=\binom{15}{5}(0.25)^5(0.75)^{10}=0.1651 \text{ (Tab)}$$

(b) P(five incorrect answers) = P( ten correct answers)

$P(X=10)$
$n=15, r=10, P=0.25$

$$P(X=10)=\binom{15}{10}(0.25)^{10}(0.75)^5=0.0007 \text{ (Tab)}$$



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P(five incorrect answers) = 0.0007.

**Example**

In certain clan the probability of a family having a girl is 0.6. If there are 5 children in a family, determine the probability that they are all girls.

**Solution**

$n=5, r=5 \; and \;\; p=0.6$

$But \;\; P(all\ girls) = P(zero\ boys)$

$Therefore \;\; using \;\; the\ boys$

$\Rightarrow n=5, r=0, p=0.4$

$$P(X=0)=\binom{5}{0}(0.4)^0(0.6)^5$$

$$=0.0778\,(Tab)$$

The table above gives individual probabilities. Sometimes sums of binomial probability are needed. These are obtained from the cumulative binomial probability ie $\Sigma P$. It gives the sum of probabilities from that value and above ( $i \geq r$).

Probabilities below a certain value are obtained by using
$P(X \leq r) = 1 - P(X \geq r+1)$

**EXAMPLE**

The probability that a patient recovers from a disease is 0.4. If 15 people are known to have contracted the disease, what is the probability that

(a) at least 10 will survive
(b) between 3 and 8 inclusive will survive



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(c) atmost 5 will survive
(d) between 6 to 10 will survive

**Solution:**
n = 15, p = 0.4, and q = 0.6
Let X be the number of patients who recover

(a) $P(X \geq 10) = \sum_{10}^{15} \binom{15}{x} (0.4)^x (0.6)^{15-x} = 0.0338 (Tab)$.

Read n = 15, p = 0.4 and x = r = 10

(b) $P(3 \leq X \leq 8)$

$$\sum_{3}^{15} \binom{15}{x} (0.4)^x (0.6)^{15-x} - \sum_{9}^{15} \binom{15}{x} (0.4)^x (0.6)^{15-x}$$

$= 0.9729 - 0.0950 = 0.8777$ (Tab)

(c) $P(X \leq 5) = 1 - P(X \geq 6)$.

$$P(X \geq 6) = \sum_{6}^{15} \binom{15}{x} (0.4)^x (0.6)^{15-x} = 0.5968 (Tab)$$

$P(X \leq 5) = 1 - 0.5968 = 0.4032$

(d) $P(6 < X \leq 10) = P(7 \leq X \leq 10)$

$$\sum_{7}^{15} \binom{15}{x} (0.4)^x (0.6)^{15-x} - \sum_{11}^{15} \binom{15}{x} (0.4)^x (0.6)^{15-x}$$



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$= 0.3902 - 0.0093 = 0.3809$ (Tab)

## 4.4 FORMAT FOR TABLE FOR SUMS

| | | *Probability of success (p)* | | | | | | | | $P(x \geq r)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| n | r | 0.01 | 0.05 | 0.1 | 0.15 | 0.2 | 0.25 | 0.3 | .35 | 0.40 |
| 15 | 1 | | | | | | | | | 0.9995 |
| | 2 | | | | | | | | | 0.9948 |
| | 3 | | | | | | | | | 0.9729 |
| | 4 | | | | | | | | | 0.9095 |
| | 5 | | | | | | | | | 0.7827 |
| | 6 | | | | | | | | | 0.5968 |
| | 7 | | | | | | | | | 0.3902 |
| | 8 | | | | | | | | | 0.2131 |
| | 9 | | | | | | | | | 0.0950 |
| | 10 | | | | | | | | | 0.0338 |
| | 11 | | | | | | | | | 0.0093 |
| | 12 | | | | | | | | | 0.0019 |
| | 13 | | | | | | | | | 0.0003 |
| | 14 | | | | | | | | | |

All the above sums of probability were obtained from the table.

**Example**

In a certain manufacturing process, it is know that approximately 10% of the items produced are defective. A quality control scheme is set up by selecting twenty items out of a large batch, and rejecting the whole batch if three or more are defective. Find the probability that the batch is rejected.

**Solution:**

$n = 20, p = 0.1, P(X \geq 3)$

$X \sim B(n, p)$



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$$P(X \geq 3) = \sum_{3}^{20} \binom{20}{x} (0.1)^x (0.9)^{20-x} = 0.3231 (Tab)$$

**Example**

The probability of winning a game is $\frac{4}{5}$. Ten games are played. What is the probability of at least 8 successes in the ten games?

**Solution**

$n = 10, p = 0.8, \ q = 0.2$

$$P(X \geq 8) = \sum_{8}^{10} \binom{10}{x} (0.8)^x (0.2)^{10-x}$$

$$= \sum_{0}^{2} \binom{10}{x} (0.2)^x (0.8)^{10-x}$$

$$P(X \geq 8) = 1 - \sum_{3}^{10} \binom{10}{x} (0.2)^x (0.8)^{10-x}$$

$$= 1 - 0.3222 = 0.6778 (Tab)$$

Sometimes the probability of success in not contained in the table.

**Example**

A coin is tossed such that is twice as likely to show heads as tails. Find the probability that in five tosses of the coin

(a) exactly three heads are obtained

(b) more than three heads are obtained

**Solution:**

(a) Let H denote head and T denote tail

$P(H) = 2(T)$

$P(H) + P(T) = 1$



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$$2P(T) + P(T) = 1$$

$P(T) = 1/3$

$P(H) = 2\,P\,(T) = 2/3$

Probability of success $p = 2/3$

Let X be the number of heads that appear.

$n = 5$

$$P(X = r) = \binom{n}{r} P^r q^{n-r}$$

$$P(X = 3) = \binom{5}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2$$

$$= \frac{10 \times 8}{243} = \frac{80}{243} = 0.3292$$

$P(X = 3) = 0.3292$

(b) $P(X > 3) = P(X = 4) + P\,(X = 5)$

$$P(X = 4) = \binom{5}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right) = \frac{5x16}{243} = \frac{80}{243}$$

$P(X = 4) = 0.32922$

$$P(X = 5) = \binom{5}{5}\left(\frac{2}{3}\right)^5 = \frac{2^5}{3^5} = \frac{32}{243}$$

$P(X = 5) = 0.131687$



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$P(X > 3) = 0.131687 + 0.32922 = 0.460907$
$P(X > 3) = 0.4609$

## 4.5 EXPECTATION AND VARIANCE

### MEAN (EXPECTATION)

If the random variable X is such that
$X \sim B(n, p)$ then

$$E(X) = \sum_{allx} x\,P(X = x) = np$$

$\therefore E(X) = \mu = np$

VARIANCE
$Var(X) = npq$

$$\sigma = \sqrt{Vax(x)} = \sqrt{(npq)}$$

Example
In Kampala city, it is known that 1/3 of the voters support the U.P.C party. In a sample of 12 voters, what is the expected value and standard deviation of the number of U.P.C's.

**Solution**

$X \square B(n,p)$, Let X be the number of U.P.C voters.
$n = 12,\ p = 1/3,\ q = 1 - p = 2/3$

$$E(X) = np = 12 \times \frac{1}{3} = 4$$

$$\sigma = \sqrt{Vax(x)} = \sqrt{(npq)}$$

$$= \sqrt{12x\frac{1}{3}x\frac{2}{3}}$$

$$= \sqrt{\frac{8}{3}} \cong 1.63$$



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**Example**

Given that the binomial distribution B (n, p) has mean 9.6 and standard deviation 2.4, find n and p.

**Solution:**

$E(x) = np$

$\sqrt{Var(x)} = \sqrt{npq}$

$\sigma = \sqrt{npq}$

$9.6 = np$ .....(1)

$2.4 = \sqrt{npq}$..... (1)

From (2) $npq = (2.4)^2$

$npq = 5.76$

but $np = 9.6$

$\Rightarrow 9.6q = 5.76$

$q = \frac{5.76}{9.6}$

$q = 0.6$

$\Rightarrow p = 1 - q = 0.4 \Rightarrow p = 0.4$

From (1) $np = 9.6$

$0.4n = 9.6$

$n = \frac{9.6}{0.4} = 24$

$n = 24, p = 0.4$

Hence $n = 24, p = 0.4$

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**Example**

The probability of a student arriving at the school late on any given day is 1/10. What is the probability of his being punctual for a whole week (i.e 5 school days). Calculate the mean and variance of the number of days he will be late in school term consisting of 14 weeks (i.e 70 days). Also calculate the expected number of completely punctual weeks in the term.

**Solution:**

Let X be number of days he is late.

$n = 5, p = 0.1, P(X = 0)$.

P(punctual for a whole week)

$$P(X = 0) = \binom{5}{0}(0.1)^0(0.9)^5 = 0.5905 \text{ (Tab)}.$$

$E(X) = np$

14 weeks = 14 x 5 = 70 days

$E(X) = 70 \times 0.1 = 7$ days.

$Var(X) = npq$

$= 70 \times 0.1 \times 0.9$

$7 \times 0.9 = 6.3$

$Var(X) = 6.3$

P( punctual for all whole week) = 0.5905.

n = 14 weeks

$E(X) = 14 \times 0.5905$

$= 8.267$

$E(X) = 8.267$



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**Example**

If a random variable X is binomially distributed with B(10,p) with p < ½ an variance 15/8. Determine p, the mean and probability of at least one.

**Solution:**

np = ? n = 10

$$npq = 15/8$$

$$pq = \frac{15/8}{10} = 3/16$$

$$p(1-p) = 3/16$$

$$p - p^2 = 3/16$$

$$16p - 16p^2 - 3 = 0$$

$$16p^2 - 16p + 3 = 0$$

p = ¼ or ¾ (Cal)

since p < ½ then p = ¼

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mean = np

= 10 x ¼

= 2.5

$$P(x \geq 1) = \sum_{1}^{10} \binom{10}{x} (0.25)^x (0.75)^{10-x}$$

= 0.9437 (Tab)

**Example**

A box contains a large number of red and yellow bulbs in the ratio 1:3. Bulbs are picked at random from the box. How many bulbs must be picked so that the probability that there is at least one red bulb among them is greater than 0.95?



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**Solution**

$P = \frac{1}{4}, q = \frac{3}{4}$

$X \sim B(n, p)$

$P(X \geq 1) > 0.95$

$= 1 - P(X = 0)$

$$1 - \binom{n}{0}(0.25)^0(0.75)^n > 0.95$$

$1 - 0.75^n > 0.95$

$0.05 > 0.75^n$

$\log 0.05 > n \log 0.75$

$-1.301 > -0.1249n$

$n > 10.42$

$n \approx 11$ *therefore the least value of* $n$ *is* 11

## 4.6 THE MOST LIKELY VALUE OF X TO OCCUR

The value of X that is most likely to occur is the one with the highest probability. However the working could be tedious to calculate all probabilities.

Therefore the mean is obtained and normally the values close to mean tend to have the highest probabilities.

**Example**

Of the inhabitants of Katwe village, 80% are known to have Malaria. If 12 people are waiting to see the nurse, what is the most likely number of them to have malaria?



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**Solution**

Mean = np
= 12 x 0.8
= 9.6

Therefore the most likely number is either 9 or 10

P(X = 9) = P( three people without Malaria)
n = 12, p = 0.2, q = 0.8, r = 3

$$P(X=9)=\binom{12}{9}(0.8)^9(0.2)^3$$

$$Or\ P(X=3)=\binom{12}{3}(0.2)^3(0.8)^9$$

$$=0.2362\,(Tab)$$

$$P(X=10)=\binom{12}{10}(0.8)^{10}(0.2)^2$$

$$Or\ P(X=2)=\binom{12}{2}(0.2)^2(0.8)^{10}$$

$$=0.2835\,(Tab)$$

Therefore the most likely number of people is 10 since it has the highest probability.



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**Exercise.**

1. In a certain clan the probability of a family having a boy 0.6. If there are 5 children in a family, determine.
   (i) The expected number of girls.
   (ii) Probability that there are at least three girls.

   **Answer** (i) $E(X) = 2$. (ii) $P(X \geq 3) = 0.3174$

2. Three people play a game in which each person tosses a coin. The game is a success if one of the players gets an outcome different from the others. Determine the probability that.
   (i) a success will occur at the first trial.

   (ii) in two trials at least one success will occur.

   **Answer** (i) P (a successes at first trial) = ¾

   (ii) P( at least one success in two trials) = $15/16$

3. Muwanga makes 5 practice runs in the 100m sprint. A run is successful if he runs it in less than 11 seconds. There are 8 chances out of 10 that he is successful. Find the probability that
   (a) (i) he records no success at all

   (ii) he records at least 2 successes.

   (b) If he is successful in the 5 practice runs he makes two additional runs, the probability of success in either of the additional runs is 0.7. Determine the probability that Muwanga will make 7 successful runs in total.

   **Answer**: (a) (i) 0.0003 (ii) 0.9933 (b) 0.16



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4. The packets of Omo sold in a shop are of four categories namely, small, medium, large and giant. On a particular day, the stock is such that the ratio of small: medium: large: gaint is equal to 4:2:1:1 respectively. Ten packets are picked at random. Determine the probability that six are medium size packets.

   **Answer:** $P(X = 6) = 0.0162$

5. Assuming that a couple is equal likely to produce a girl or a boy. Find the probability that in a family of 5 children there will be more boys than girls.

**Answer:** 0.5

6. Of the students in a school, 30% travel to school by bus. From a sample of 10 students chosen at random, find the probability that:

   (a) Only 3 travel by bus
   (b) more than 8 travel by bus
   (c) at most 8 travel by bus

   **Answer:** (a) 0.2668 (b) 0.0001 (c) 0.9999

7. In a group of people the expected number who wear glasses is 2 and the variance is 1.6. Find the probability that:-

   (a) A person, chosen at random from the group wears glasses
   (b) 6 people in the group wear glasses

   **Answer** a) 0.2, b) 0.0055

8. Seedlings are planted in 10 rows of six each. The probability of a seedling dying before it flowers is 1/8. Calculate the mean and variance of the number of rows in which all the seedlings flower.

   **Answer**: 4.488, 2. 4738

9. A certain electronic system contains ten components. Suppose that the probability that any individual component will fail is 0.2, and that the components fail independently of each other. Given that at least one of the components has failed, what is the probability that at least two of the components have failed.



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**Answer**: 0.6993

10. A cross world puzzle is published in the times each day of the week, except Sunday. A man is able to complete on average 8 out of 10 of the cross word puzzles.

(a) Find the excepted value and the standard deviation of the number of completed cross words in a given week.

(b) Show that the probability that he will complete at least 5 in a given week is 0.655 (to 3 significant figure)

**Answer**: (a) 4.8, 0.98

11. At a certain hatchery, records taken over several months have shown that the ratio of male to female chicks hatched has been 4 : 6. Basing on this experience, what is the probability that there will be more female chicks in a random collection of a dozen chicks from this hatchery?

**Answer** 0.6652

12. A machine producing electrical components occasionally makes faulty ones. In a batch of 20 such components, 5 are found to be faulty. A sample of 3 of this batch is taken and examined. What is the probability that:-

(i) it contains faulty components

(ii) it contains exactly one faulty component?

**Answer** (i) 0.5781 (ii) 0.4219

13. A fair die is rolled 6 times. Calculate the probability that

(i) a 2 or 4 appears on the first throw,

(ii) four 5s will appear in the six throws

**Answer** (i) $\frac{1}{3}$ (ii) 0.0080



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# CHAPTER FIVE

## CONTINUOUS PROBABILITY DENSITY FUNCTION

### 5.1 INTRODUCTION

A continuous random variable is theoretical representation of continuous variable such as weight, temperature , time, distance, mass, height. A probability density function f(x) of random variable X is said to be continuous it has a continuous domain.

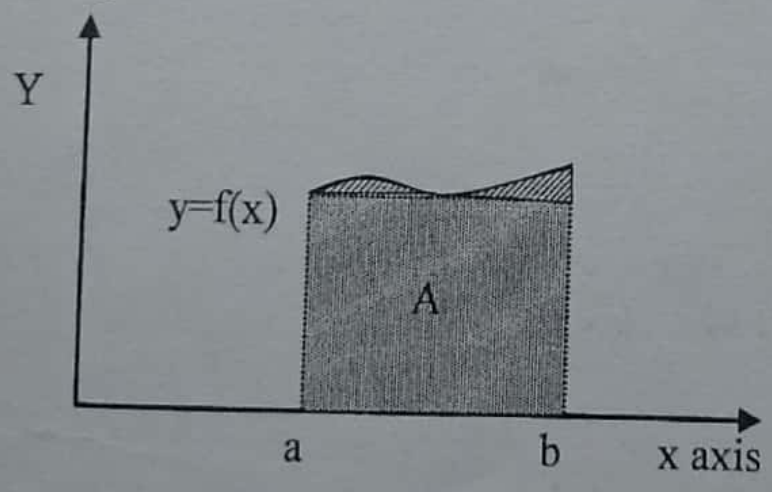
**Properties of f (x)**

(i) $f(x) > 0$ for all values of x.

(ii) $\int_{\text{all}x} f(x)\,dx = 1$

i.e the total area under a curve is 1.

$$A = \int_a^b f(x)dx = 1$$




### 5.2 HOW TO OBTAIN PROBABILITIES

The probability that a random variable attains values between $x_1$ and $x_2$ given by $P(x_1 < X < x_2)$ is obtained from the area under the curve between $x_1$ and $x_2$.



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$$\therefore P(x_1 < X < x_2) = \int_{x_1}^{x_2} f(x)dx$$

**Example**

A continuous random variable has p.d.f given by

$$f(x) = \begin{cases} kx & 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

Determine (a) The value of the constant k.

(b) $P(1 \leq X \leq 2)$

(c) Sketch the graph of f(x)

**Solution:**

(a) $\int f(x)\,dx = 1 \Rightarrow \int_0^4 kx\,dx = 1$

$$k\left[\frac{x^2}{2}\right]_0^4 = 1$$

$8k = 1 \Rightarrow \quad k = 1/8$

$$\therefore f(x) = \begin{cases} \frac{x}{8} & 0 \leq x \leq 4 \\ 0 & \textit{otherwise} \end{cases}$$



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(b) $P(1 \le X \le 2) = \int_1^2 f(x)dx =$

$$\int_1^2 kx\,dx =$$

$$k\left[\frac{x^2}{2}\right]_1^2 = \frac{3}{2}k$$

$$P(1 \le X \le 2) = \frac{3}{2} x \frac{1}{8}$$

$$= \frac{3}{16}$$

## 5.3 GRAPH OF f(x)

Look at the function and note whether it is a straight line or simple curve in the interval given. i.e for the above example $f(x) = \frac{x}{8}$ which is a straight line between 0 and 4. With a gradient of $\frac{1}{8}$ and zero intercept

When $x = 0$, $f(0) = 0$.
When $x = 4$, $f(4) = 0.5$



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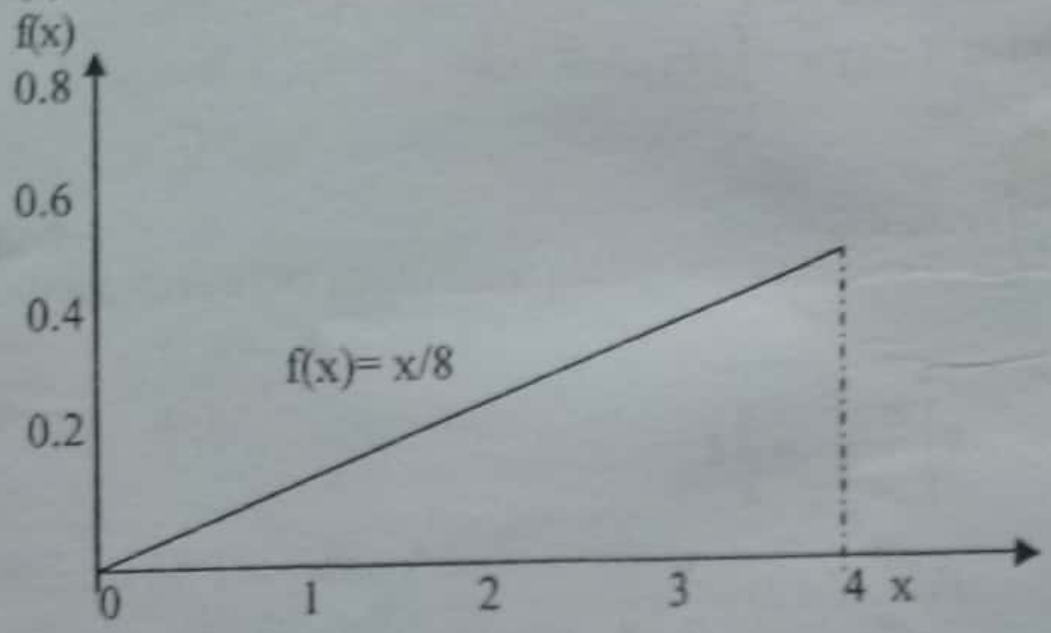
(c) GRAPH OF f(x)




**Example**

A continuous random variable has a p.d.f where

$$f(x) = \begin{cases} kx & 0 \le x \le 2 \\ k(4-x) & 2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Determine:

(a) Value of the constant k.

(b) $P(1 \le X \le 3)$

(c) Sketch the graph of f(x)

**Solution:**

(a) $\int_{\text{all}x} f(x)\,dx = 1$



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$$\int_0^2 kx\,dx + \int_2^4 k(4-x)\,dx = 1$$

$$k\left[\frac{x^2}{2}\right]_0^2 + k\left[4x - \frac{x^2}{2}\right]_2^4 = 1$$

$$2k + 2k = 4k = 1$$

$$\therefore k = 1/4$$

(b) $P(1 \le X \le 3) = \int_1^2 kx\,dx + \int_2^3 k(4-x)\,dx$

i.e $P(1 \le X \le 3) = P(1 \le X \le 2) + P(2 \le X \le 3)$

$$k\left[\frac{x^2}{2}\right]_1^2 + k\left[4x - \frac{x^2}{2}\right]_2^3$$

$$= \frac{1}{4}\left(2 - \frac{1}{2}\right) + \frac{1}{4}(7.5 - 6)$$

$$= \frac{1}{4} x \frac{3}{2} + \frac{1}{4} x \frac{3}{2} = \frac{3}{4}$$

$$P(1 \le X \le 3) = 0.75$$

(c) GRAPH f(x)

$$f(x) = \begin{cases} x/4 & 0 \le x \le 2 \\ 1 - x/4 & 2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$



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for $0 < x < 2$, $y = f(x)$ $y = x/4$

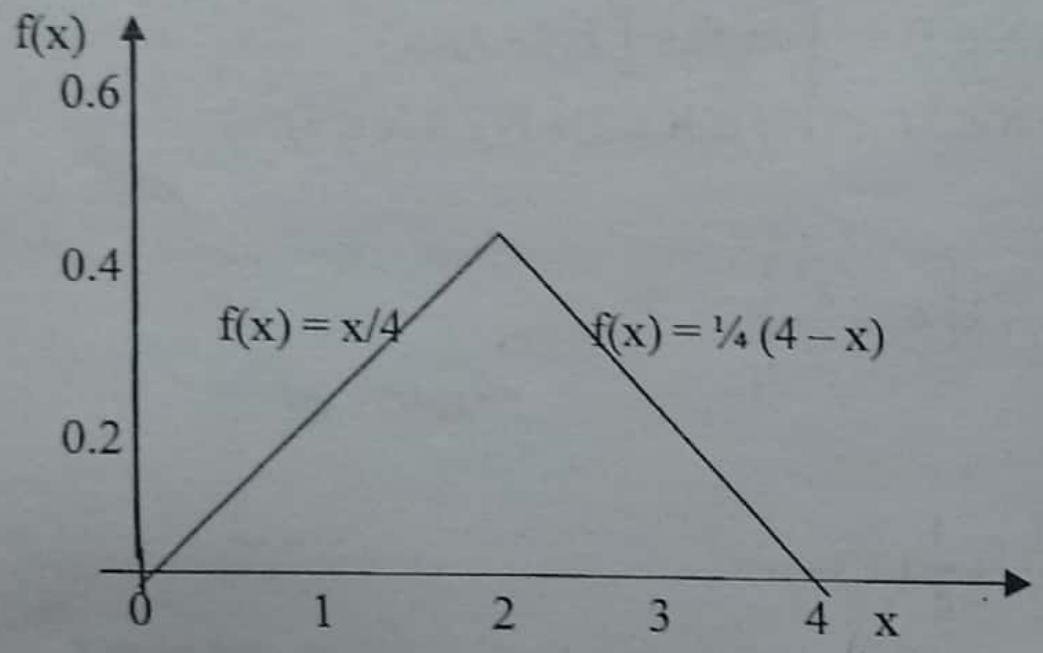
Which is a straight line

With $m = ¼$ and $c = 0$, consider $y = mx + c$

When $x = 0, y = 0, x = 2, y = \frac{1}{2}$

For $2 \leq x \leq 4$, $y = 1 - \frac{x}{4}$

When $x = 2, y = 0.5, x = 4, y = 0$.




**Example**

A continuous random variable X has p.d.f f(x) given by

$$f(x) = \begin{cases} k & 0 \leq x \leq 2 \\ k(2x-3) & 2 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Determine:

(a) value of constant k.

(b) sketch the graph of f(x)

(c) (i) $P(X < 1)$ (ii) $P(X = 1)$

(iii) $P(X > 2.5)$ (iv) $P(0 < X < 2 \mid X > 1)$



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**Solution:**

(a) from $\int_{\text{all}x} f(x)dx = 1$

$$k[x]_0^2 + k\left[x^2 - 3x\right]_2^3 = 1$$

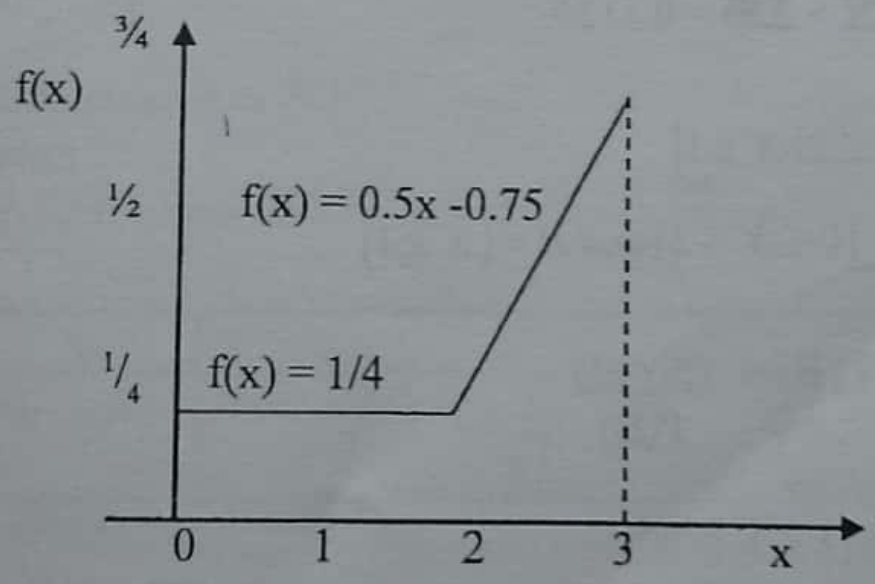
$$2k + 2k = 1$$
$$k = \tfrac{1}{4}$$

(b) Graph of f (x)

$$f(x) = \begin{cases} 1/4 & 0 \le x \le 2 \\ 0.5x - \tfrac{3}{4} & 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

for $0 \le X \le 2$ $\quad y = \tfrac{1}{2}x - \tfrac{3}{4}$

when $x = 3$, $y = 3/2 - \tfrac{3}{4} = \tfrac{3}{4}$, $\quad x = 2$, $y = \tfrac{1}{4}$




(c) (i) $P(X < 1) =$

$$\int_0^1 f(x)dx \text{ Or } Area = \left(\tfrac{1}{4}\right)x1 = \tfrac{1}{4}$$



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$$\int_0^1 \frac{1}{4}dx = \left[\frac{x}{4}\right]_0^1 = \frac{1}{4}$$

$\therefore P(X < 1) = ¼$

(ii) $P(X=1) = 0$ Since X is continuous random variable.

(iii) $P(X > 2.5) = P(2.5 < X < 3) = \int_{2.5}^{3} f(x)dx$

$$= \int_{2.5}^{3} k(2x-3)dx = k\left[x^2 - 3x\right]_{2.5}^{3}$$

$$= ¼\left[x^2 - 3x\right]_{2.5}^{3}$$

$$= 0.3125$$

$\Rightarrow$ $P(X > 2.5) = 0.3125.$

(iv) $P\left[0 \le X \le 2 \mid X \le 1\right]$

Let $A = (0 \le X \le 2)$ and $B = (X \ge 1)$

$$P(A \mid B) = \frac{(A \cap B)}{P(B)}$$

$A \cap B = (1 \le X \le 2)$

$$P\left[0 \le X \le 2 \mid X \ge 1\right] = \frac{P(1 \le X \le 2)}{P(1 \le X \le 3)}$$



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$$\Rightarrow \quad \frac{\int_1^2 1/4\,dx}{\int_1^2 1/4\,dx + \int_2^3 1/4(2x-3)\,dx} = \frac{1/4}{1/4+1/2}$$

$= \frac{1}{4} \div \frac{3}{4} = \frac{1}{3}$

$$P(0 < X < 2 \mid x \geq 1) = \frac{1}{3}$$

**Example**

A continuous random variable X has p.d.f given by

$$f(x) = \begin{cases} \frac{3}{32}(x^2) & 0 \leq x \leq 2 \\ \frac{3}{32}(6-x) & 2 \leq x \leq 6 \\ 0 & otherwise \end{cases}$$

Sketch the graph of f(x)

**Solution**

**Note:** for the interval $0 \leq x \leq 2$ the function is a curve

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| f(x) | 0 | $\frac{3}{32}$ | $\frac{12}{32}$ |

For the interval $2 \leq x \leq 6$. This is a straight line

| $x$ | 2 | 6 |
|---|---|---|
| f(x) | $\frac{12}{32}$ | 0 |



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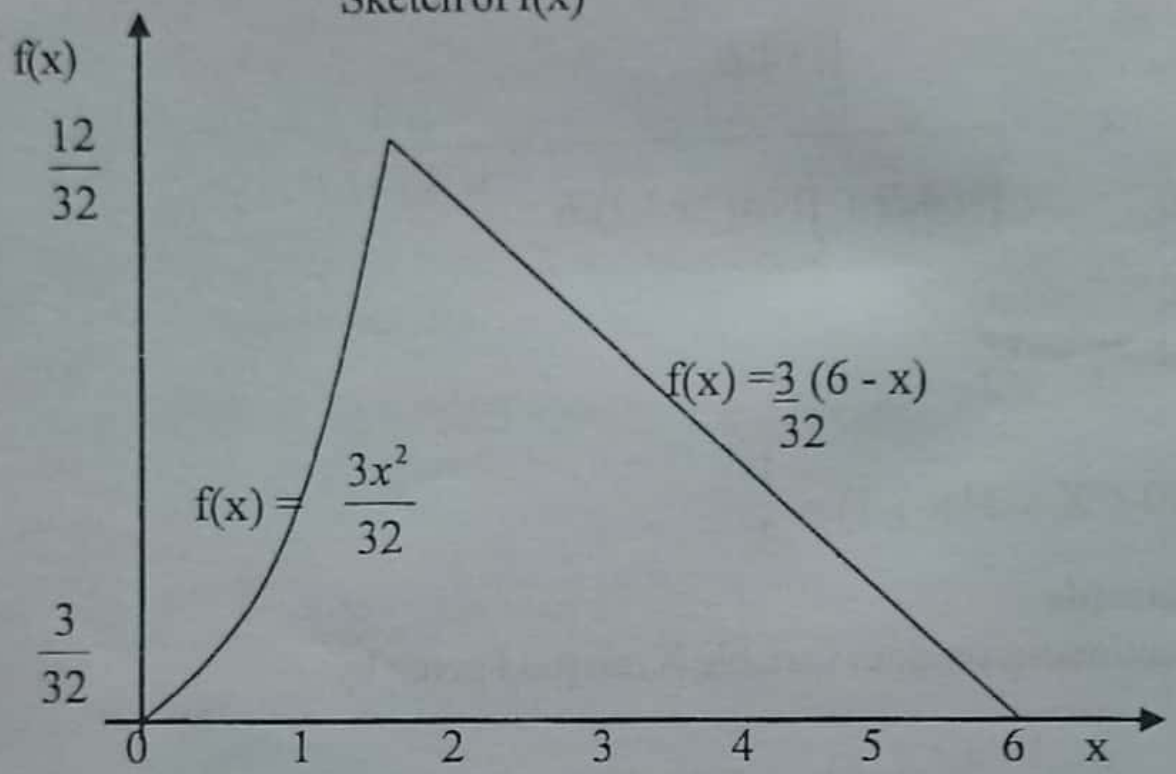
Therefore you can use three points.

Sketch of f(x)




**Example**

A continuous random variable X has a probability density function given by:

$$f(x) = \begin{cases} \alpha, & 2<x<3 \\ \alpha(x-2), & 3<x<4 \\ 0 & otherwise \end{cases}$$

(i) Sketch f(x)

(ii) Find the value of α, hence f(x)



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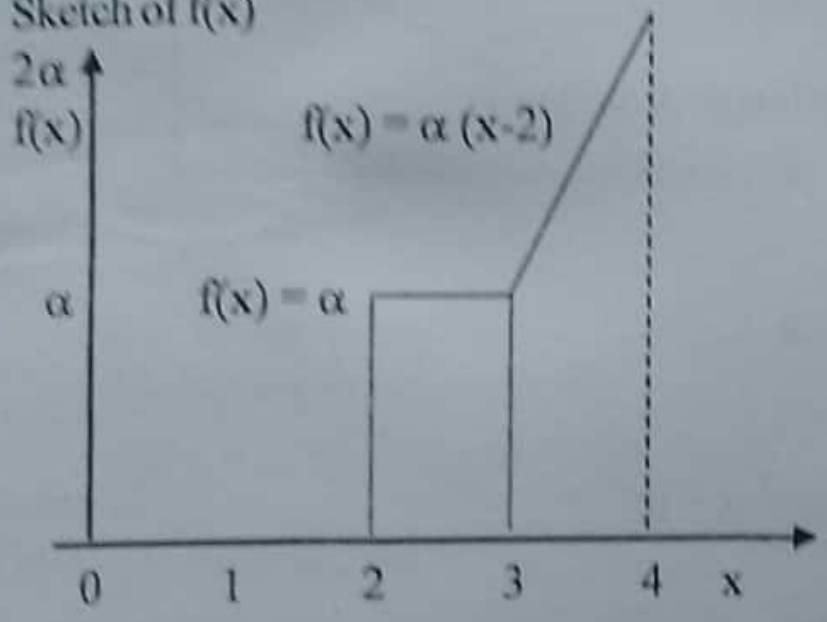
(iii) $P(|x-2.5|<0.5)$

**Solution**

Sketch of f(x)




(ii) From the sketch of f(x)

$\Rightarrow \alpha x 1 + 0.5 x 1(\alpha + 2\alpha) = 1$

$So\ 5\alpha = 2$

$\therefore \alpha = \frac{2}{5}$

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$$f(x) = \begin{cases} \frac{2}{5}, & 2 < x < 3 \\ \frac{2}{5}(x-2), & 3 < x < 4 \\ 0 & elsewhere \end{cases}$$

(iii) $P(|x-2.5|<0.5)$



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$P(-0.5<x-2.5<0.5)$

$P(2.5-0.5<x<2.5+0.5)$

$P(2<x<3)$

$=Area\ of\ the\ rectangle\ from\ 2\ to\ 3$

$1x\alpha=\frac{2}{5}$

$P(2<x<3)=0.4$

**Note:**

$P(|x|<a)=P(-a<x<a)$

$P(|x|>a)=1-P(-a<x<a)$

## 5.4 How to obtain probability density function from the sketch.

**Example**

A continuous random variable X has a probability density function f(x) given by $f(x)=0$ for $x<0$ and $x>3$ and between $x=0$ and $x=3$ its form is as shown in the graph.



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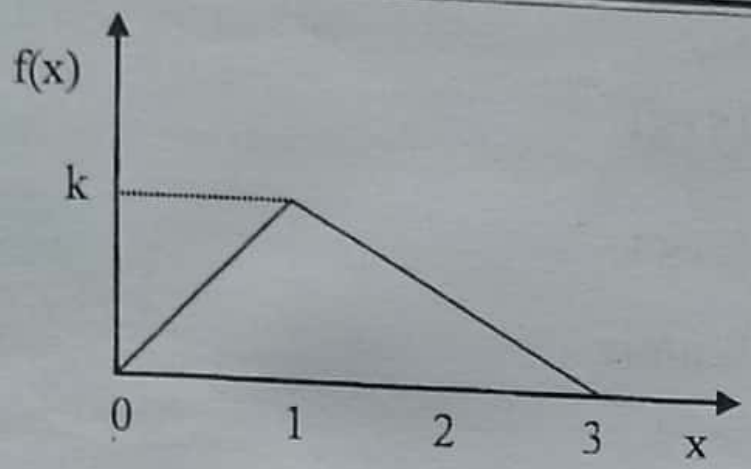


(a) Find the value of k
(b) Express f(x) algebraically

**Solution**

Total area under the curve = 1

(a)

$$\frac{1}{2} x 3 x k = 1$$

$$3k = 2$$

$$Therefore \quad k = \frac{2}{3}$$

$$0 \le x \le 1, \; f(x) = \frac{2}{3}x + 0 = \frac{2}{3}x$$

$$1 \le x \le 3, \; (1, \frac{2}{3}) \; and \, (3, 0)$$

(b)

$$Gradient = \frac{0 - \frac{2}{3}}{3 - 1} = \frac{-1}{3}$$

$$Equation = \frac{y - 0}{x - 3} = \frac{-1}{3}$$

$$y = 1 - \frac{x}{3}$$

$$Otherwise\, the\, expression\, of\, f(x) = 0$$



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## 5.5 EXPECTATION (MEAN) AND VARIANCE

The mean for continuous p.d.f is given by

$$E(X) = \int_{allx} xf(x)dx$$

$$Var(X) = E(X^2) - E(X)^2$$

$$= \int x^2 f(x)dx - [E(X)]^2$$

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Not that $\mu = E(X)$

$$\Rightarrow Var(X) = \int_{allx} x^2 f(x)dx - \mu^2$$

and $Var(X) = \sigma^2$

**Example**

The continuous random variable X has p.d.f f(x) where

$$f(x) = \begin{cases} \frac{x}{8} & 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

Determine (i)E(X) (ii) Var (X)
(iii) standard deviation of X.

(i) $E(X) = \int_{allx} xf(x)dx$



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$$\int_{all\,x} xf(x)dx = \int_0^4 x\frac{1}{8}x\,dx$$

$$= \int_0^4 x\frac{x}{8}dx = \int_0^4 \frac{x^2}{8}dx$$

$$\left[\frac{x^2}{24}\right]_0^4 = \frac{64}{24} = \frac{8}{3}$$

Mean = $8/3$

Variance = $E(X^2) - [E(X)]^2$

$$E(X^2) = \int_0^4 \frac{x^3}{8}dx$$

$$= \left[\frac{x^4}{32}\right]_0^4 = 256/32 = 8$$

$$Var(X) = 8 - \left(\frac{8}{3}\right)^2 = 8 - \frac{64}{9} = \frac{8}{9}$$

$$Var(X) = \frac{8}{9}$$

(iii) Standard deviation ($\sigma$) = $\sqrt{8/9}$

$$= \frac{2\sqrt{2}}{3}$$

**Properties of E(X)**

1. $E(a) = a$
2. $E(ax) = a\,E(x)$
3. $E(aX) + b) = E(X) + b$.



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**Properties of Var(X)**

1. $Var(a) = 0$
2. $Var(aX) = a^2 Var(X)$
3. $Var(aX + b) = a^2 Var(X)$ where a and b are constants.

**Example**

The outputs of 9 machines in a factory are independent random variables each with p.d.f given by:

$$f(x) = \begin{cases} ax & 0 \le x \le 10 \\ a(20-x) & 10 \le x \le 20 \\ 0 & \text{otherwise} \end{cases}$$

Find

(i) the value of a.

(ii) the expected value and variance of the output of each machine. Hence or otherwise find the expected value and variance of the total output from all machines.

Solution: (i)

$$\int_{all\,x} f(x)dx = 1$$

$$\int_0^{10} axdx + \int_{10}^{20} a(20-x)dx = 1$$

$$a\left[\frac{x^2}{2}\right]_0^{10} + a\left[20x - \frac{x^2}{2}\right]_{10}^{20} = 1$$

$$50a + a[200 - 150] = 1$$

$$100 = 1 \Rightarrow a = 0.01$$

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(ii) $E(X) = \int_{all} xf(x)dx$

$$= \int_0^{10} xf(x)dx + \int_{10}^{20} xf(x)dx$$

$$= \int_0^{10} \frac{1}{100}x^2dx + \int_{10}^{20} \frac{1}{100}x(20-x)dx$$

$$= \frac{1}{100}\left[\frac{x^3}{3}\right]_0^{10} + \left[10x^2 - \frac{x^3}{3}\right]_{10}^{20}$$

$$= \frac{1000}{300} + \frac{20}{3} = \frac{10}{3} + \frac{20}{3} = \frac{30}{3} = 10$$

For 9 machines $E(9X) = 9E(X)$
$9 \times 10 = 90$, Expected value is 90.

$Var(X) = E(X^2) - [E(X)]^2$
but $E(X^2) = \int_{allx} x^2 f(x)dx$

$$\frac{1}{100}\int_0^{10} x^3dx + \int_{10}^{20}(20x^2 - x^3)dx$$

$$E(X^2) = \frac{1}{100}\left[\frac{x^4}{4}\right]_0^{10} + \frac{1}{100}\left[\frac{20x^3}{3} - \frac{x4}{4}\right]_{10}^{20}$$

$$E(X^2) = \frac{1}{100}\left[\left(\frac{160,000}{3} - 40,000\right) - \left(\frac{20,000}{3} - \frac{10,000}{4}\right)\right]$$



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$$E(X^2) = 25 + \frac{1,400}{3} + 25 - 400$$

$$E(X^2) = 116.667$$

$$Var(X) = 116.6667 - (10)^2 = 16.6667$$

$$Var(9X) = 9^2 Var(X) = 81 Var(X) =$$

$$81 \times 16.6667 = 1,350.$$

**Assignment**

The mass X kg of loaves of bread produced per hour in certain bakery is modeled by a continuous random variable with a probability function given by:

$$f(x) = \begin{cases} kx^2 & 0 \le x \le 2 \\ k(6-x) & 2 \le x \le 6 \\ 0 & otherwise \end{cases}$$

(a) find the value of k

(b) Given that a 1 kg loaf is sold at Ushs. 1500 and the running costs of baking is 1200/= per hour. Taking Y/= as the profit made in each express Y in terms of X, hence find E(Y).



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## 5.6 THE MEDIAN

The median divides the area under the curve into two halves, consider m to be the median then,

$$\int_{-\infty}^{m} f(x)dx = 0.5 \quad or \quad \int_{m}^{\infty} f(x)dx = 0.5$$

**Example**

If X is a continuous random variable with p.d.f given by

$$f(x) = \begin{cases} \frac{x}{8} & 0 \le x \le 4 \\ 0 & otherwise \end{cases}$$

Obtain the median m.

**Solution**

$$\int_{0}^{m} f(x)dx = 0.5 \qquad \Rightarrow \int_{0}^{m} \frac{x}{8} dx = 0.5$$

$$\left[\frac{x^2}{16}\right]_0^m = 0.5$$

$$m^2 = 8 \therefore m = \pm 2.828$$

Note $0 < m < 4$, Median = 2.828 since -2.828 is outside the interval.

**Example**

A continuous random variable X has p.d.f f(x)

$$f(x) = \begin{cases} 0.4x + 0.8 & -1 \le x \le 0 \\ 0.8 - 0.8x & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Obtain the median

We first integrate the first interval to establish if it is at least 0.5. If not then median will be in the second interval.



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$$\int_{-1}^{0} f(x)dx = \int_{-1}^{0}(0.4x+0.8)dx$$

$$\left[0.2x^2+0.8x\right]_{-1}^{0} = 0-(0.2-0.8) = 0.6$$

Therefore the median lies in the first interval $-1 \leq x\ 0$

$$\left[0.2x^2+0.8x\right]_{-1}^{m} = .05$$

$0.2m^2 + 0.8m + 0.6 = 0.5$
either m = -0.129 or -3.871 (Cal)
m = -0.129 since -3871 is outside the interval

**Example**
A random variable X has probability density function given by

$$f(x) = \begin{cases} \frac{2}{3}(x+1); & -1 < x \leq 0 \\ \frac{1}{3}(2-x); & 0 < x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

where a is constant, determine the median of X

**Solution:**
Let m be the median
We integrate the first interval to establish whether it is at least 0.5



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$$\int_{-1}^{0} f(x)dx = \int_{-1}^{0} \frac{2}{3}(x+1)dx$$

$$= \frac{2}{3}\left[\frac{(x^2+x)}{2}\right]_{-1}^{0}$$

$$\frac{2}{3}x\left(0+1-\frac{1}{2}\right)=\frac{1}{3}$$

$$\Rightarrow \frac{1}{3}<\frac{1}{2}$$

m lies between 0 and 2.

$$\Rightarrow \int_{-1}^{0} f(x)dx + \int_{0}^{m} \tfrac{1}{3}(2-x)dx = 0.5$$

$$\frac{1}{3}+\frac{1}{3}\left(2x-\frac{x^2}{2}\right)_0^m = 0.5$$

$$\frac{1}{3}+\frac{1}{3}\left(2m-\frac{m^2}{2}\right)=\frac{1}{2}$$

$$2m-\frac{m^2}{2}=\frac{1}{2}$$

$$4m-m^2=1 \Rightarrow m^2-4m+1=0$$

$$m=\frac{-b+\sqrt{b^2-4ac}}{2a}$$

$$m=\frac{4+\sqrt{16-4}}{2}$$



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$$m = 2 + \frac{\sqrt{12}}{2}$$

$$m = 2 \pm \frac{3.464}{2} \quad or \quad 2 \pm 1.732$$

0.268 lies between 0 and 2 while 3.732 does not.
Hence the median is 0.268.

## 5.7 MODE

This is the value of x for which $f^1(x) = \frac{d}{dx} f(x) = 0$

**Example**

The continuous random variable x has p.d.f f(x) where

$$f(x) = \frac{3}{80}(2+x)(4-x) \quad 0 \leq x \leq 4.$$

Find the mode.

**Solution**

For mode use $f^1(x) = 0$

$$f(x) = \frac{3}{80}\left(-x^2 + 2x + 8\right) \Rightarrow f'(x) = \frac{3}{80}(-2x+2) = 0$$

$$\frac{3}{80}(2-2x) = 0 \text{ , when } 2 - 2x = 0 \Rightarrow x = 1$$

**Note** if you have more than one value check for one which gives a maximum i.e $f^{11}(x) < 0$.



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$$f(x)=\begin{cases}\frac{x}{108}(6-x)^2 & 0<x\leq 6\\ 0 & \text{otherwise}\end{cases}$$

obtain the mode

$$f(x)=\frac{x}{108}(6-x)^2$$

$$f(x)=\left(\frac{36x-12x^2+x^3}{108}\right)$$

$$f'(x)=\frac{36-24x+3x^2}{108}=0$$

$$\Rightarrow\frac{3}{108}\left(x^2-8x+12\right)=0$$

$$\frac{3}{108}(x-6)(x-2)=0 \; Either\; x=6 \;\; 0r \;\; x=2$$

$$f''(x)<0, f'(x)=\frac{3}{108}\left(x^2-8x+12\right)$$

$$f''(x)=\frac{3}{108}(2x-8), when\, x=6\Rightarrow f''(6)>0$$

$$f''(2)=\frac{3}{108}(4-8)=\frac{-12}{108}\Rightarrow f''(2)<0$$

Hence the mode is 2.

## 5.8 THE CUMULATIVE DISTRIBUTION FUNCTION

The cumulative probability function for any variable X is defined by

$$F(x)=P(X\leq x)=\int_{-\infty}^{x} f(t)dt$$



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**Properties of F(x)**

(i) $F(-\infty) = 0$ (ii) $F(\infty) = 1$

(iii) None decreasing function.

**Example**

Given the following p.d.f f(x) determine F(x)

$$f(x) = \begin{cases} \frac{x}{8} & 0 \le x \le 4 \\ 0 & otherwise \end{cases}$$

**Solution**

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt$$

For $x < 0$, $f(t) = 0 \Rightarrow F(x) = 0$ and $F(0) = 0$

For $0 < x < 4$, $f(t) = \frac{t}{8}$

$$F(x) = F(0) + \int_{0}^{x} \frac{t}{8} dt$$

$$= 0 + \left[\frac{t^2}{16}\right]_0^x$$

$$\Rightarrow F(x) = \frac{x^2}{16} \qquad \Rightarrow F(4) = \frac{16}{16} = 1$$

for $x > 4$ $f(t) = 0$

$$F(x) = F(4) + \int_{0}^{x} f(t)dt$$

$$F(x) = F(4) + 0 = 1$$



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$$F(x) = \begin{cases} 0 & x<0 \\ \dfrac{x^2}{16} & 0 \le x \le 4 \\ 1 & x \ge 4 \end{cases}$$

**Example**

Given the following p.d.f f(x) determine F (x)

$$f(x) = \begin{cases} \dfrac{x}{3} & 0 \le x \le 2 \\ \dfrac{-2x}{3}+2 & 2 \le x \le 3 \\ 0 & otherwise \end{cases}$$

**Solution**

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt$$

For the interval $x < 0, f(t) = 0, F(0) = 0$

for $0 \le x \le 2$ $f(t) = \dfrac{t}{3}$

$$\Rightarrow F(x) = F(0) + \int_0^x \frac{t}{3} dt,$$

$$F(x) = \frac{x^2}{6}, \Rightarrow F(2) = \frac{4}{6} = \frac{2}{3}$$



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For $2 < x < 3 \Rightarrow f(t) = \frac{-2t}{3} + 2$

$$\Rightarrow F(x) = F(2) + \int_2^x \left(\frac{-2t}{3} + 2\right) dt$$

$$F(x) = \frac{2}{3} + \left[\frac{-t^2}{3} + 2t\right]_2^x$$

$$= \frac{2}{3} + \left(\frac{-x^2}{3} + 2x\right) - \left(\frac{-4}{3} + 4\right)$$

$$= \frac{2}{3} - \frac{x^2}{3} + 2x + \frac{4}{3} - 4$$

$$F(x) = -\frac{x^2}{3} + 2x - 2$$

$$\Rightarrow F(3) = \frac{-9}{3} + 6 - 2 = 1$$

for $x > 3$, $f(t) = 0$, $F(x) = F(3) + 0 = 1$

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{6} & 0 \le x \le 2 \\ \frac{-x^2}{3} + 2x - 2 & 2 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$$



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**Note**: The cumulative probability function F(x) can be used to obtain.

## 5.9 Probaliites, median, p.d.f and constants

**Example**

Given the following distribution function F(x) determine:

(i)p($1 \le x \le 4$) ii) Median iii) p.d.f f(x)

$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x^2}{12} & 0 \le x \le 3 \\ 2x - \dfrac{x^2}{4} - 3 & 3 \le x \le 4 \\ 1 & x \ge 4 \end{cases}$$

i) $P(x_1 \le X \le x_2) = P(X \le x_2) - P(X \le x_1)$

$= F(x_2) - F(x_1)$

$P(1 \le X \le 4) = P(X \le 4) - P(X \le 1)$

$= F(4) - F(1)$

$F(4) = 1$

$F(1) = 1/12$

$\therefore P(1 \le X \le 4) = 1 - \dfrac{1}{12} = \dfrac{11}{12}$

ii) $F(m) = 0.5$ ie $\int_{-\infty}^{m} f(x)dx = 0.5$



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For $0 < x < 3$, $F(3) = \frac{9}{12} = \frac{3}{4}$

$F(3) > 0.5$

$\Rightarrow$ m lies between $0 < x < 3$

$F(m) = 0.5$

$\Rightarrow \quad F(x) = \frac{x^2}{12}, \qquad \frac{1}{2} = \frac{m^2}{12}$

$6 = m^2 \Rightarrow m = \pm\sqrt{6}$

$m = 2.449$ the median is 2.449 since -2.449 is outside the range

iii) p.d.f(x) $f(x) = F^1(x)$

$$f(x) = \frac{d}{dx}\left(\frac{x^2}{12}\right) = \frac{x}{6} \quad \text{for } 0 < x < 3$$

$$\text{for } 3 < x < 4,\ f(x) = F^1(x) \Rightarrow \frac{d}{dx}\left(2x - \frac{x^2}{4} + 3\right)$$

$= 2 - \frac{1}{2}x$

otherwise $f(x) = 0$

$$\therefore f(x) = \begin{cases} \frac{x}{6} & 0 \le x \le 3 \\ 2 - \frac{1}{2}x & 3 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$



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**Example**

A continuous random variable X has a probability density function:

$f(x) = kx(3 - x)$ for $0 \le x \le 2$

$f(x) = k(4 - x)$ for $2 \le x \le 4$

$f(x) = 0$ otherwise

Determine

i) The value of k

ii) Mean

iii) F(x) the cumulative distribution function

iv) $P(1 \le X \le 3)$

**Solution**

(i) $$\int_{all} f(x)dx = 1$$

$$k\int_0^2 (3x - x^2)dx + K\int_2^4 (4 - x)dx = 1$$

$$k\left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^2 + k\left[4x - \frac{x^4}{2}\right]_2^4 = 1$$

$$k\left[6 - \frac{8}{3}\right] + k[8 - 6] = 1$$

$$\frac{16}{3}k = 1 \quad k = \frac{3}{16}$$



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(ii) The mean $E(X) = \int_{allx} xf(x)dx$

$$E(X) = \frac{3}{16}\left[\int_0^2 (3x^2 - x^3)dx + \int_2^4 (4x - x^2)dx\right]$$

$$\frac{3}{16}\left(\left[x^3 - \frac{x^4}{4}\right]_0^2 + \left[2x^2 - \frac{x^3}{3}\right]_2^4\right)$$

$$E(X) = \frac{3}{16}\left[(8-4) + (32 - \frac{64}{3}) - (8 - \frac{8}{3})\right]$$

$$= \frac{7}{4} = 1\frac{3}{4} \quad \Rightarrow E(X) = 1\frac{3}{4}$$

(iii) $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt$

For $x < 0$, $F(x) =$ since $f(t) = 0$
For $0 < x < 2$, $f(t) = kt(3 - t)$

$$F(x) = F(0) + \frac{3}{16}\int_0^x (3t - t^2)dt$$

$$= \frac{3}{16}\left[\frac{3t^2}{2} - \frac{t^3}{3}\right]_0^x \Rightarrow F(x) = \frac{3}{16}\left(\frac{3x^2}{2} - \frac{x^3}{3}\right)$$



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$$F(2) = \frac{3}{16}\left(6 - \frac{8}{3}\right) = \frac{10}{16}$$

$$2 < x < 4$$

$$F(x) = F(2) + \frac{3}{16}\int_2^x (4-t)\,dt$$

$$= \frac{10}{16} + \frac{3}{16}\int_2^x (4-t)\,dt$$

$$= \frac{10}{16} + \frac{3}{16}\left[4t - \frac{t^2}{2}\right]_2^x$$

$$= \frac{10}{16} + \frac{3}{16}\left[(4x - \frac{x^2}{2}) - (8-2)\right]$$

$$= \frac{10}{16} + \frac{3}{16}\left[4x - \frac{x^2}{2} - 6\right]$$

$$F(x) = \frac{3}{16}(4x - \frac{x^2}{2}) - \frac{1}{2}$$

$F(4) = 1$

For $x > 4$ $\qquad F(x) = 1$



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$$F(x) = \begin{cases} 0 & x \le 0 \\ \frac{3}{16}\left(\frac{3x^2}{2} - \frac{x^3}{3}\right) & 0 \le x \le 2 \\ \frac{3}{16}\left(4x - \frac{x^2}{2}\right) - \frac{1}{2} & 2 \le x \le 4 \\ 1 & x \ge 4 \end{cases}$$

(iv) $P(1 \le X \le 3) = P(X \le 3) - P(X \le 1)$

$= F(3) - F(1)$

$$F(3) = \frac{3}{16}(12 - \frac{9}{2})\frac{1}{2} = \frac{3}{16}\left(\frac{15}{2}\right) - \frac{1}{2}$$

$$= \frac{45}{32} - \frac{1}{2} = \frac{29}{32}$$

$$F(1) = \frac{3}{16}\left(\frac{3}{2} - \frac{1}{3}\right) = \frac{3}{16} x \frac{7}{6} = \frac{7}{32}$$

$$P(1 \le X \le 3) = \frac{29}{32} - \frac{7}{32} = \frac{22}{32} = \frac{11}{16}$$

$$\therefore P(1 \le X \le 3) = \frac{11}{16}$$

**Assignment**

A random variable X has F(x) given by



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$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x^2}{4} & 0 \le x \le 1 \\ \alpha x + k & 1 \le x \le 2 \\ \dfrac{1}{4}(5-x)(x-1) & 2 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$$

**Determine**

(a) The value of the constants $\alpha$ and k

(b) $P(3 < 2X < 5)$

(c) $P(|X-2| < 1)$

**Note**: Graph F(x) is plotted the same way as that of f(x) however its sketch should be given from $-\infty$ to $+\infty$

**Example**

Sketch the graph of F(x) for the function given below

$$F(x) = \begin{cases} 0 & x \le 0 \\ \dfrac{x^2}{6} & 0 \le x < 2 \\ -\dfrac{x^2}{3} + 2x - 2 & 2 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$



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**Solution**

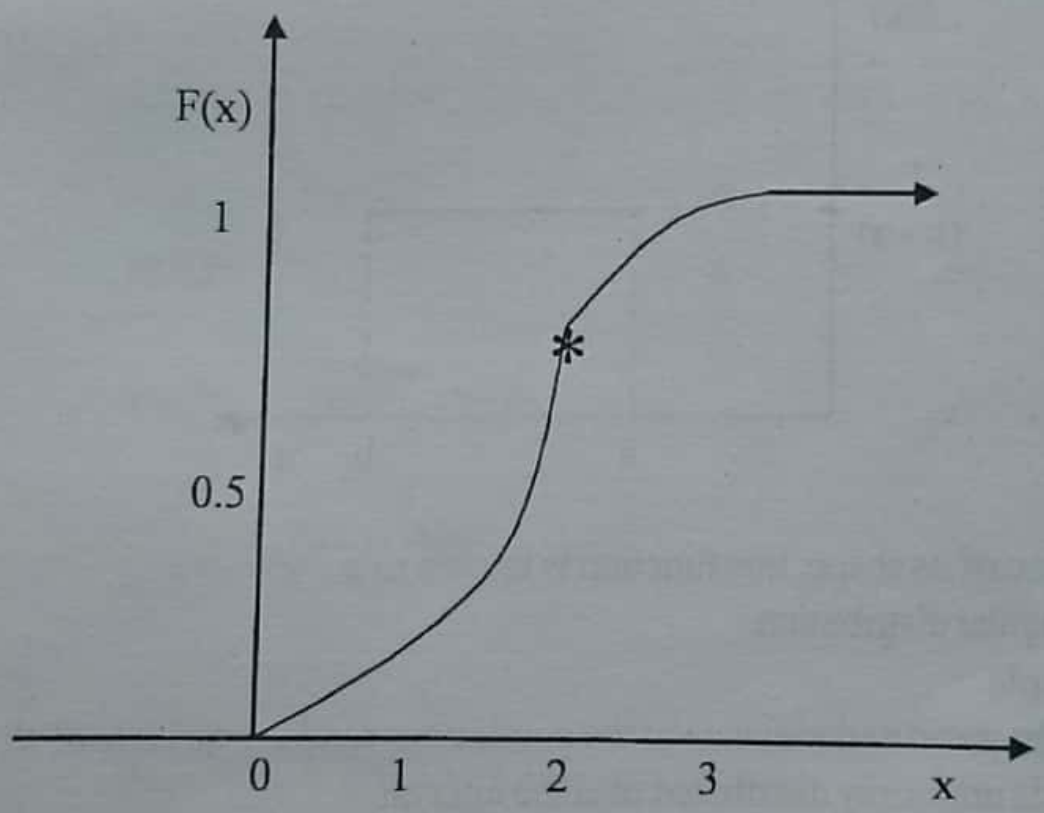
When $x < 0$, $F(0) = 0$

For $0 \leq x \leq 2$, $F(x) = x^2/6$ and $F(2) = 2/3$

For $2 \leq x \leq 3$, $F(x) = \frac{-x^2}{3} + 2x - 2$ and

$$F(3) = \frac{-3^2}{3} + 2x3 - 2 = 1$$

For $x > 3$, $F(x) = 1$



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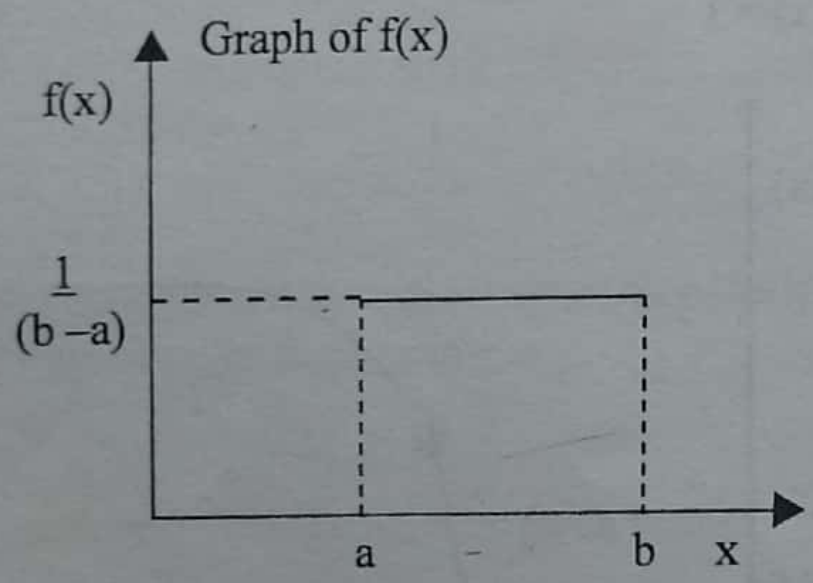
## 5.10 UNIFORM DISTRIBUTION

This is a continuous random variable whose density function is defined by

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

Where $a \le b$




Because of its shape, this function is known as a rectangular distribution.

**Example**

Find the mean and variance of the continuous random variable of X, which is uniformly distributed over the interval.

(a) 0 to 1

(b) 2 to k



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$$\text{(a) } f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & otherwise \end{cases}$$

$$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

where a = 0 and b = 1

$$E(X) = \int_{all} x f(x) dx \qquad = \int_0^1 x dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

$$E(X) = \frac{1}{2}$$

$$Var(X) = EX^2 - [E(X)]^2$$

$$E(X^2) = \int_{all} x^2 f(x) dx$$

$$\int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$Var\,(X) = \frac{1}{3} - \left(\frac{1}{2}\right)^2$$



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$$\text{Var}(X) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\text{(b) } E(X) = \int_2^k \frac{x}{k-2}\,dx$$

$$= \frac{1}{k-2}\left[\frac{x^2}{2}\right]_2^k$$

$$= \frac{1}{k-2}\left(k^2 - 2^2\right) = \frac{k+2}{2}$$

$$E(X) = \frac{k+2}{2}$$

$$E(X^2) = \int_2^k \frac{x^2}{k-2}\,dx = \frac{1}{k-2}\left[\frac{x^3}{3}\right]_2^k$$

$$= \frac{1}{k-2}\left(\frac{k^3 - 2^3}{3}\right)$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

But $\Rightarrow k^3 - 2^3 = (k-2)^3 + 6k(k-2)$

$$E(X^2) = = \frac{k-2}{k-2}\left(\frac{(k-2)^2 + 6k}{3}\right)$$

$$= \left(\frac{k^2 + 2k + 4}{3}\right)$$



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$$\text{Var(X)} = = \left(\frac{k^2+2k+4}{3}\right) - \left(\frac{k+2}{2}\right)^2$$

$$= \left(\frac{4k^2+8k+16}{12} - \left(\frac{3k^2+12k+12}{12}\right)\right)$$

$$= \left(\frac{k^2-4k+4}{12}\right) = \frac{(k-2)^2}{12}$$

**Note for uniform distribution,**

$$\text{E(X)} = \frac{a+b}{2} \text{ and } \text{Var(X)} = \frac{(b-a)^2}{12}$$

$$\therefore \text{Var(X)} = \frac{(k-2)^2}{12}$$

where k = b and a = 2

**Assignment**

During rush hours, it was observed that the number of vehicles departing for Entebbe from Kampala old taxi park is a random variable X with a uniform distribution over the interval $[x_1, x_2]$
If in one hour, the expected number of vechiles leaving the stage is 12, with variance of 3, calculate the:

i. Values of $x_1$ and $x_2$,
ii. Probability that at least 11 vechiles leave the stage



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**Exercise**

1.A continuous random variable X has the distribution function

$$F(x) = \begin{cases} 3kx(1-\frac{x^2}{3}) & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

Determine

i) the value of k

ii) the probability density function of X

iii) the mean of X

iv) $P(X > 0.5 \mid 0.25 < X < 1)$

**Answer:** i) $k = ½$

(ii) $f(x) = \begin{cases} \frac{3}{2}(1-x^2) & 0 \le x \le 1 \\ 0 & \textit{otherwise} \end{cases}$

(iii) $E(x) = \frac{3}{8}$ iv) 0.494

2. The probability density function of a random variable X is

$$f(x) = \begin{cases} k\sin x \text{ for } & 0 \le x \le \pi \\ 0 & \text{otherwise} \end{cases}$$

Determine

i) the value of k

ii) $P(X > \frac{\pi}{3})$ (iii) the median of X



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**Answer**

(i) $k = \frac{1}{2}$, (ii) 0.75 (iii) $\frac{\pi}{2}$

3. A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} \frac{1}{4} & 0 < x < 1 \\ \frac{1}{8} & 2 < x < 8 \\ 0 & \text{elsewhere} \end{cases}$$

Find

(i) the distribution function and expectation of X

(ii) $P[0.5 < X < 3]$

**Answer**

(i) $$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{4} & 0 \leq x \leq 1 \\ \frac{1}{4} & 1 \leq x \leq 2 \\ \frac{x}{8} & 2 \leq x \leq 8 \\ 1 & x \geq 8 \end{cases}$$

$E(X) = 3.875$ ii) 0.25

4. A random variable X has the probability density function given by



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$$f(x) = \begin{cases} k(1-x^2) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant, find

i) the value of the constant k

ii) the mean of X

iii) the variance of X

**Answer**

i) $k = 1.5$ ii) $E(X) = \frac{3}{8}$ (iii) $\frac{19}{320}$

5. The number X of cars crossing Owen falls Dam daily is uniformly distributed between 1,026 to 3,025 cars.

(i) Find the probability that at least 1,625 cars cross the bridge.

(ii) What is the expected number of cars that will cross the bridge on any given day?

**Answer**

i) 0.7 ii) E(X) = 2,026 cars.

6. A random variable X has the probability density function

$$f(x) = \begin{cases} \frac{2}{3a}(x+a) & -a \le x \le 0 \\ \frac{1}{3a}(2a-x) & 0 \le x \le 2a \\ 0 & \textit{otherwise} \end{cases}$$

where a is constant

Determine

i. the value of a

ii. the median of X



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(iii) $P(X \leq 1.5 \mid X > 0)$

(iv) The cumulative distribution function F(x)

(v) Sketch the graph of F(x)

**Answer**

i) a = 1 ii) median = 0.268,

iii) $P[(x \leq 1.5) \mid (x \leq )] = 0.9375$

$$F(x) = \begin{cases} 0 & x \leq -1 \\ \frac{2}{3}\left(\frac{x^2}{2} + x + \frac{1}{2}\right) & -1 \leq x \leq 0 \\ \frac{1}{3} + \frac{2x}{3} - \frac{-x^2}{6} & 0 \leq x \leq 2 \\ 1 & x \geq 2 \end{cases}$$

7. A.p.d.f is given by

$$f(x) = \begin{cases} \frac{1}{6}x & for \quad 0 \leq x \leq 3 \\ \frac{1}{2}(4-x) & for \quad 3 \leq x \leq 4 \\ 0 & for \quad x < 0 \ and \ x > 4 \end{cases}$$

sketch the graph of f(x)

calculate

(a) the probability that X occurs in the interval (1,2)

(b) the probability that x > 2. Obtain the cumulative probability function and hence or otherwise, find the median of the distribution.



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**Answer** (a) 0.25 (b) $\frac{2}{3}$

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{12} & 0 \le x \le 3 \\ 2x - \frac{x^2}{4} - 3 & 3 \le x \le 4 \\ 1 & x > 4 \end{cases}$$

$m = \sqrt{6} = 2.449$

8. The probability that a transistor in a radio lasts less than t hours is $1 - e^{-t/2000}$. Find the p.d.f for the lifetime of a transistor?

a) What is the probability that the transistor lasts more than 4,000hours?

b) What is the probability that a transistor ceases to function after 2,000 hours, of use but before 3,000 hours?

c) If a radio contains 8 transistors, what is the probability that none of them fails before 1,000 hours of use?

Answer a)0.135 b)0.145 c)0.18

$$f(t) = \frac{1}{2000} e^{\frac{-t}{2000}}$$

9. A random variable X has cumulative probability function.



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$$F(x) = \begin{cases} 0 & x \le a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & x \ge b \end{cases}$$

Find the p.d.f f(x) and sketch the graph of f(x). Obtain the mean and variance of X.

**Answer**

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$E(X) = \frac{a+b}{2} \qquad Var(x) = \frac{(b-a)^2}{12}$$

10. The continuous random variable X has p.d.f f(x) given by

$$f(x) = \begin{cases} k & 0 \le x \le 2 \\ k(3-x) & 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of k

(b) Hence evaluate E(X) and the median of X

(c) Prove that $\sigma$ the standard deviation of X is 0.75, correct to 2 decimal places.

(d) Denoting E(X) by $\mu$, find the $P[(X < \mu - \sigma)]$



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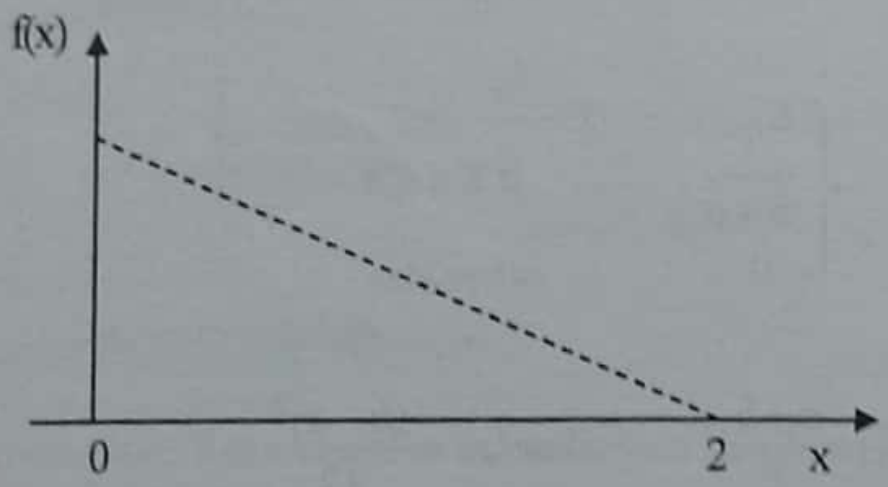
**Answer**

a) $k = 0.4$ b) $E(X) = \frac{19}{15}, \frac{5}{4}$ d) 0.207

11. The probability density function f(x) of a random variable X takes on the form shown in the diagram below.




Determine the expression for f(x). Hence obtain the expression for the cumulative probability density function of X.

**Answer** $f(x) = \begin{cases} 1 - x/2 & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$

$$F(x) = \begin{cases} 0 & x < 0 \\ x - \frac{x^2}{4} & 0 \le x \le 2 \\ 1 & x > 2 \end{cases}$$

12. The quality of an animal feedstuff depends both upon the raw materials used and the production process. One measure of this quality is the nutritional index, which varies between 0 and 1. A particular mill produces an animal feedstuff in batches of constant size, and the nutritional index of any batch may be considered to be having probability density function.



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$$f(x) = \begin{cases} kx(1-x)^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(i) show that $k = 12$ and sketch $f(x)$

ii. calculate $P(x < 0.25)$

iii. batches of this mill's feedstuff may be sold for 500/= each if the index is 0.8 or more, and 350/= otherwise. The cost of producing a batch is 300/=. What is the expected profit per batch?

**Answer**

(ii)0.262 (iii) $E(x) = 0.4$ (iv) 54.08/=

13. The continuous random variable X has cumulative distribution function F(x) given by

$$F(x) = \begin{cases} 0 & x < -1 \\ \dfrac{1+x}{8} & -1 \le x \le 0 \\ \dfrac{1+3x}{8} & 0 \le x \le 2 \\ \dfrac{5+x}{8} & 2 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$$

i) Sketch the graph of the probability density function f(x)

ii) Determine the expectation of X and the variance of X

iii) Determine $P(3 \le 2X \le 5)$



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**Answer**

ii) $E(X) = 1$ $Var(X) = \frac{5}{6}$ iii) $P(3 \le 2X \le 5) = 0.25$

14. Petrol is delivered to a garage every Monday morning. At this garage the weekly demand of petrol, in thousands of units is a continuous random variable X distributed with a probability function of the form.

$$f(x) = \begin{cases} ax^2(b-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

I. Given that the mean weekly demand is 600 units, determine the value of a and b.

II. If the storage tanks at this garage are filled to their capacity of 900 units every Monday morning, what is the probability that in any given week the garage will be unable to meet the demand of petrol.

III. What is the mode of X.

**Answer**

I) $a = 12$ $b = 1$ (II) 0.0523 (III) $\frac{2}{3}$

15. A random variable X has F(x) given by

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{4} & 0 \le x \le 1 \\ ax + k & 1 \le x \le 2 \\ \frac{1}{4}(5-x)(x-1) & 2 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$$



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Determine

(a) The value of the constants a and k

(b) $P(3 \leq 2X \leq 5)$

(c) The probability density function f(x) sketch graph of f(x),and hence or otherwise, deduce the mean of X.

**Answer**

(a) $a = 0.5, k = -0.25$ b) $\frac{3}{2}$, Var (X) $= \frac{5}{12}$

$$f(x) = \begin{cases} \frac{1}{2}x & 0 \leq x \leq 1 \\ \frac{1}{2} & 1 \leq x \leq 2 \\ -\frac{1}{2}x + \frac{3}{2} & 2 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

16. A continuous random variable has probability function f(x)

$f(x) = 0 \quad x < 0$

$f(x) = x \quad 0 \leq x \leq 1$

$f(x) = ½ \quad 1 \leq x \leq 2$

$f(x) = 0 \quad x \geq 2$

(a) Sketch f(x) and show that it satisfies the conditions to be a probability density function.

(b) Find the cumulative distribution function F(x).

(c) Calculate the mean and variance of X.



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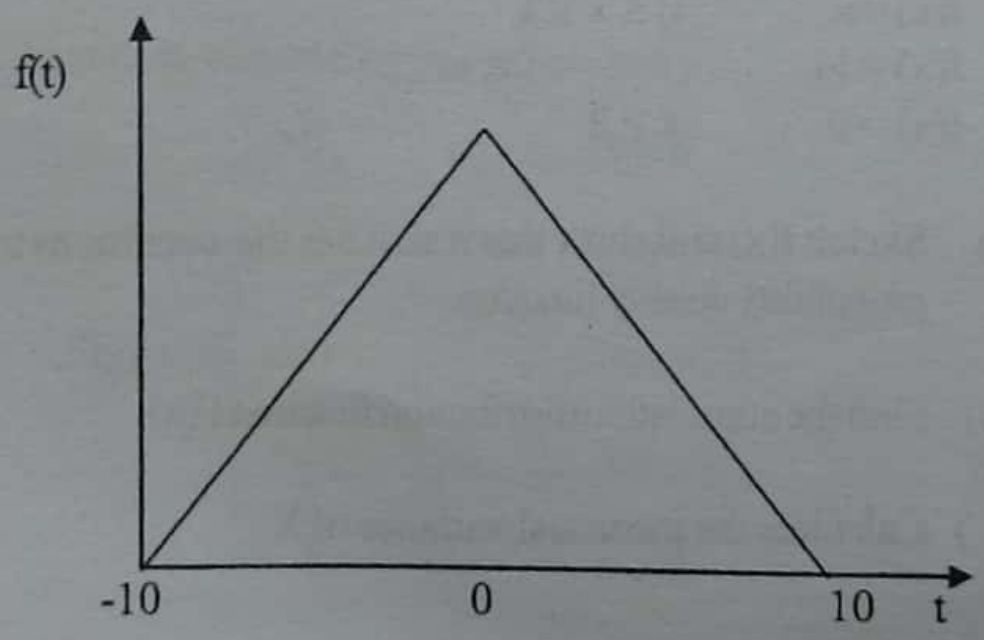
**Answer**

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}x^2 & 0 \le x \le 1 \\ \frac{1}{2}x & 1 \le x \le 2 \\ 1 & x > 2 \end{cases}$$

(c) $E(X) = \frac{13}{12}$ $\quad Var(X) = 0.243$

17. A student cycles to school each day. He is supposed to be at school by 8:50 am, but aims at getting there by 8:45. Over a full year his arrival times are noted and recorded. The student never arrives earlier than 8.35am and never later than 8.55am. An examination of this data suggests that an appropriate model of the distribution of his arrival times is given by the probability density function f(t) shown in the figure below, where the students arrives t minutes after 8.45am.



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State the value of f(t) when $t = 0$

Find for a day chosen at random

a) Probability that he arrives by 8.42am

b) The probability in terms of t that he arrives before the time given by t where $-10 < t < 0$.

c) The probability in terms of t that he arrives before the time given by t where $0 < t < 10$, hence specify fully the distribution function F(t), sketch the graph of F(t).

**Answer** 0.1 a)0.245 b) $\frac{1}{200}(10+t)^2$

c) 1- $\frac{1}{200}(10+t)^2$

18. A continuous random variable X has p.d.f given by

$f(x) = ax - bx^2$ for $0 \le x \le 2$

$f(x) = 0$ elsewhere

observation on X indicates that the mean is 1

a) Obtain two simultaneous equations for a and b, show that $a = 1.5$ and find the value of b.

b) Find the variance of X.

c) If F(x) is the probability that $X \le x$, find F(x) and verify that $F(2) = 1$

d) If two independent observations are made on X what is the probability that at least one of them is less than 0.5?

**Answer** a) $b = 0.75$ b) $Var(x) = 0.2$ d) 0.288

19. The time taken to perform a particular task, t hours, has the p.d.f given by

$$f(t) = \begin{cases} 10Ct^2 & 0 \quad t \quad 0.6 \\ 9C(1-t) & 0.6 \quad t \quad 1.0 \\ 0 & \text{otherwise} \end{cases}$$



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Where C is a constant.

a) Find the value of C and sketch the graph of this distribution.
b) Write down the most likely time.
c) Find the expected time.
d) Determine the probability that the time will be-
   i. More than 48 minutes
   (ii) Between 24 and 48 minutes

**Answer**

a) $C = \frac{25}{36}$ b) most likely time is 36 minutes

c) $E(T) = 35.5$ minutes d) (i) 0.125 (ii) 0.727

20. The continuous random variable X has cumulative distribution function.

$$F(x) = \begin{cases} 0; & x \le 0 \\ k_1 x; & 0 \le x \le 1 \\ x/3 + k_2 & 1 \le x \le 2 \\ 1; & x \ge 2 \end{cases}$$

Find:

(i) the values of $k_1$ and $k_2$
(ii) p.d.f f(x) and sketch it
(iii) mean and variance of X
(iv) $P(X < 1.5 \mid X > 1)$

**Answer** (i) $k_1 = 2/3$, $k_2 = 1/3$

$$\text{(ii) } f(x) = \begin{cases} 2/3; & 0 \le x \le 1 \\ 1/3 & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$



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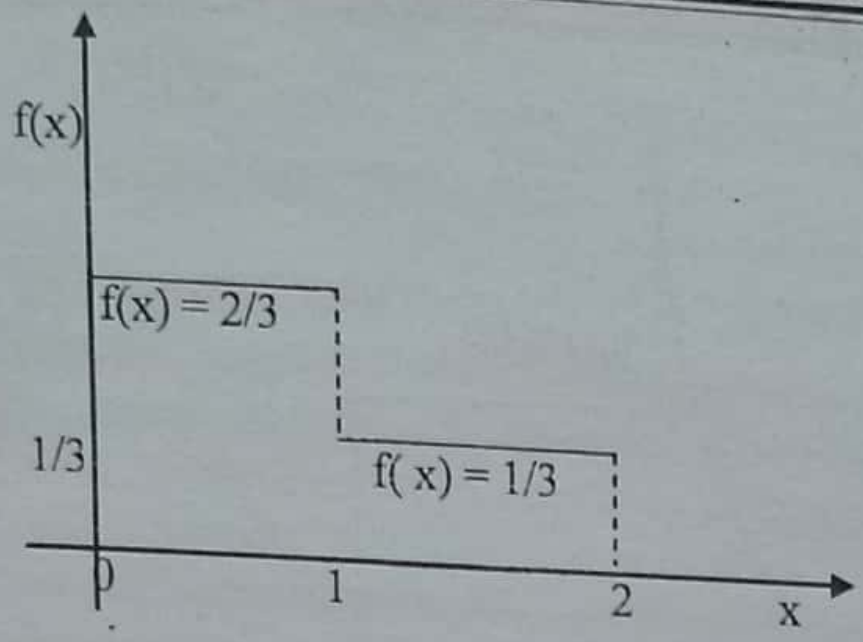


Mean = 5/6 variance = 19/36 (iv) 0.4998

21. A probability density function of a random variable X is defined by

$$f(x) = \begin{cases} 0 & x < 0 \\ x^2; & 0 \le x \le 1 \\ ½ & 1 < x < k \\ 0 & x > k \end{cases}$$

Where k is constant

i. sketch the graph of f(x)
ii. find the value of k and hence the mean of the distribution.
iii. Calculate the median of the distribution

Answer k = 7/3, mean = 49/36, median = 4/3



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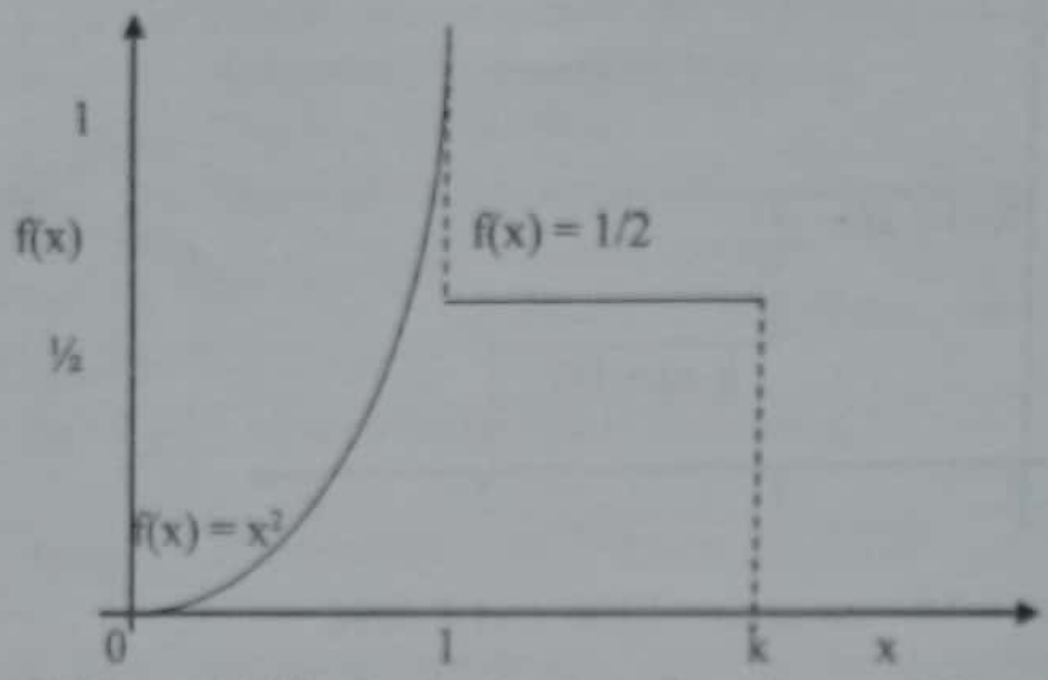
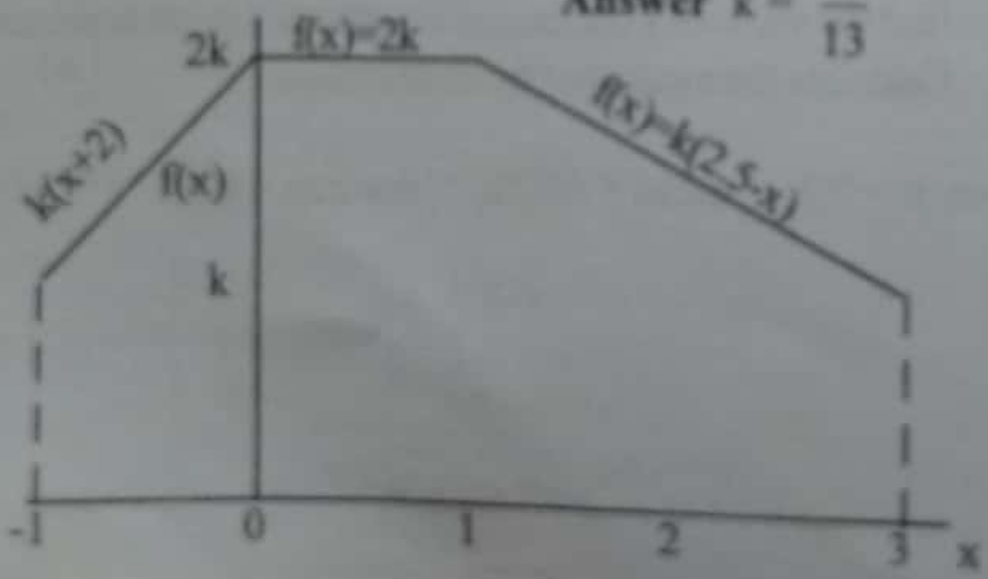



22. The probability density function of a random variable X is given by

$$f(x) = \begin{cases} k(x+2); & -1 < x < 0, \\ 2k & 0 \le x \le 1, \\ k(2.5-x); & 1 < x \le 3 \\ 0 & \textit{elsewhere} \end{cases}$$

Sketch the graph of f(x) and find the value of k

**Answer** $k = \dfrac{2}{13}$



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# CHAPTER SIX

# THE NORMAL DISTRIBUTION

## 6.1 INTRODUCTION

The normal distribution is one of the most important distributions in statistics. Many quantities in natural science follow a normal distribution and under certain circumstances it is also a useful approximation to binomial distribution.

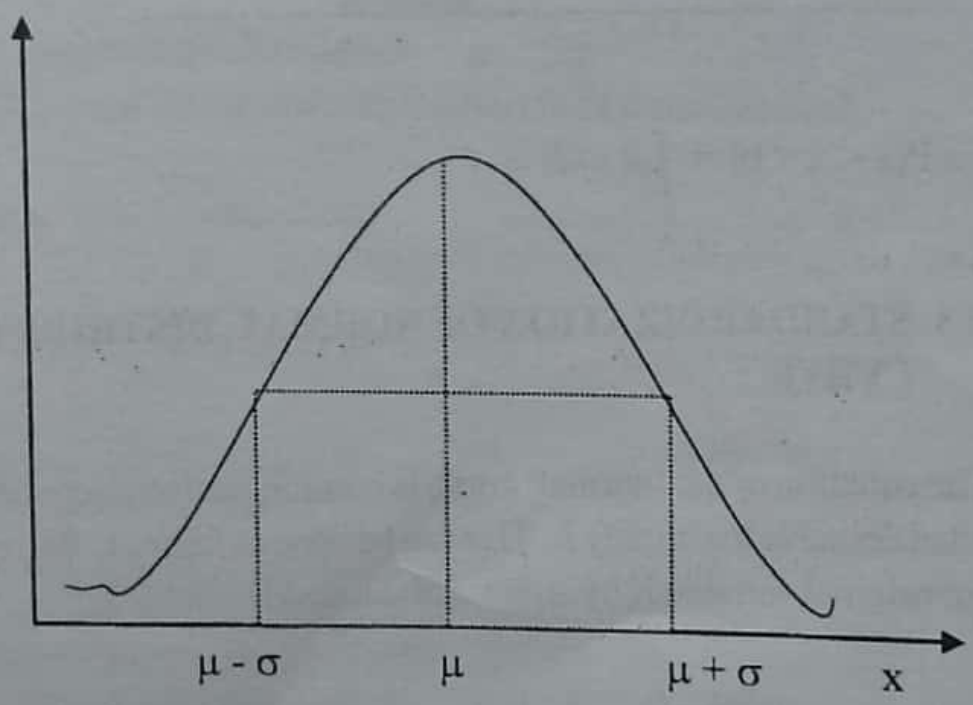
If X follows a normal distribution, then we can represent it by

$X \sim N(\mu, \delta^2)$

Where $\mu$ is the mean and $\delta^2$ is the variance

The sketch of $\phi(x)$ is given below

$$\phi(x) = \frac{1}{\sqrt{2\pi}\,\delta} e - (x-\mu)^2 / 2\sigma^2 \quad -\infty < x < \infty$$




The curve for $\phi(x)$ known as normal curve



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## 6.2 PROPERTIES OF THE NORMAL CURVE

(i) The curves is symmetrical about the mean $\mu$.

(ii) The curve never touches the x-axis i.e it is asymptotic to the x-axis.

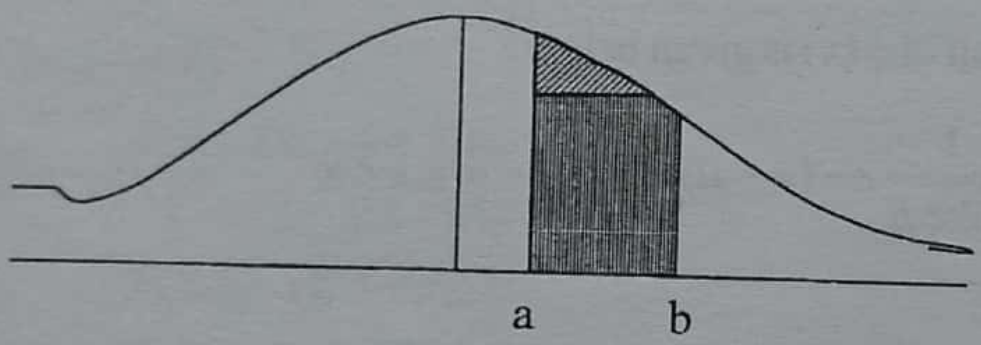
(iii) The total area under the curve and above the x-axis is 1.

(iv) The mean, median and mode coincide at the maximum value of the function i.e the mode occurs at a point on the horizontal axis where $x = \mu$.

**Area under the normal curve.**

The area under the normal curve is used to obtain probabilities

$$P(a < x < b)$$




$$\Rightarrow P(a < x < b) = \int_a^b \phi(x)dx = A$$

## 6.3 STANDARDIZATION OF NORMAL DISTRIBUTION CURVE

The equation of the 'normal' curve is complicated to integrate. It is often desirable to simplify it. This can be done as follows. We replace our original variable X by a new standardized variable Z,

$$\text{where } Z = \frac{x - \mu}{\sigma}$$



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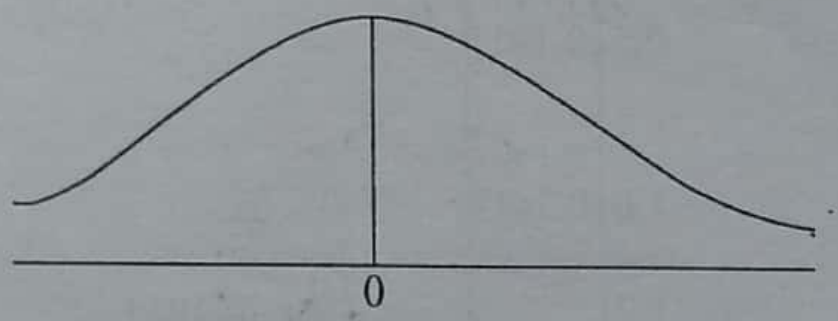
The equation $\phi(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$

Then reduces to $y = \dfrac{1}{\sqrt{2\pi}} e^{-z^2/2}$

This curve still has the same basic shape but is now symmetrical about the y-axis, so that its mean is 0 and its standard deviation is 1.

Standard normal curve




**How to obtain probabilities.**

The values of the normal distribution are first standardized

$$U\sin g \;\; Z = \frac{X-\mu}{\sigma}$$

$$\Rightarrow Z_1 = \frac{X_1-\mu}{\sigma} \; and \; Z_2 = \frac{X_2-\mu}{\delta}$$

$$P(X_1 < X < X_2) = P(Z_1 < Z < Z_2)$$

## 6.4 HOW TO READ TABLES

Probabilities are given in the table for all positive values of Z.
The $P(0 < Z < Z_1)$ can easily be obtained from the table.
Using the property of symmetry $P(Z < 0)$ can be obtained.



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**Example**

(i) $P(0 < Z < 2) = 0.4772$ (Tab)

From table $P(0 < Z < 2) = 0.4772$. The first column gives Z values to one decimal place. The other nine columns give Z to two decimal places. Therefore $Z = 2.00$ implies read $Z = 2.0$ and its probability is obtained by reading the second column with zero in the first row.

| Z | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | | | 0.0080 | | | | | | | |
| 0.5 | 0.1915 | | | | | | | | | |
| 1.0 | 0.3413 | | | | | | | | | |
| 1.2 | | | | | 0.3944 | | | | | |
| 1.8 | | | | | | | | | | |
| 1.9 | | | | | | | | | | |
| 2.0 | 0.4772 | | | | | | | | | |
| 2.1 | | | | | | | | | | |
| 2.2 | | | | | 0.4878 | | | | | |

Probabilities of $P(0 < Z < z_p)$ can easily be obtained from the table

$P(0 < Z < 1) = 0.3413$ (Tab)

$P(0 < Z < 2.25) = 0.4878$ (Tab)

$P(0 < Z\ 0.024) = 0.0080 + 0.0016 = 0.0096$

**Note** that the table above gives $Z = 0.02$.



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$P(0 < Z < 0.02) = 0.0080$. For Z values with more than two decimal places, the third decimal place is obtained from additional columns not shown.

(i) $P(0 < z < 1.252) = 0.3944 + 0.0004 = 0.3948$

(ii)$(P(-0.5 < Z < 0) = P(0 < Z < 0.5)$ by symmetry.

$P(0 < Z < 0.5) = 0.1915$

$\Rightarrow p(-0.5 < Z < 0) = 0.1915$

(iii) $P(Z < -2) = P(Z > 2)$ by symmetry.

$P(Z > 2) = 0.5 - P(0 < Z < 2)$

$P(Z > 2) = 0.5 - 0.4772$

$P(Z > 2) = 0.0228$ (Tab)

$\therefore P(Z < -2) = 0.0228$

(iv) $P(0 < Z < 0.42) = 0.1628$ (Tab)

(v) $P(-1.25 < Z < 0) = 0.3944$ (Tab)

(vi) Note: $P(-3.5 < Z < 0) = 0.5000$ (Tab)

(vii) $P(Z > 1.36) = 0.0869$ (Tab)

(viii) $P(Z > 0.25) = 0.4013$ (Tab)

(ix) $P(Z < -0.25) = 0.4013$ (Tab)

(x) $P(Z < 1) = 0.5 + 0.3413 = 0.8413$

(xi) $P(Z > -2) = 0.5 + 0.4772 = 0.9772$

(xii) $P(-1 < Z < -2) = 0.4772 - 0.3413$ (Tab)

$= 0.1359$

(xiii) $P(-1 < Z < 2) = 0.3413 + 0.4772$ (Tab)

$= 0.8185$

(xiv) $P(0.5 < Z < 2.5) = 0.4938 - 0.1915$ (Tab)

$= 0.3023$

(xv) $P(|Z| < 1.78) = P(-1.78 < Z < 1.78)$

$= 2 \times 0.4625$ (Tab)

$= 0.9250$

(xvi) $P(|Z| > 2.326) = 1 - P(-2.326 < Z < 2.326)$

$= 1 - 2 \times 0.4900$ (Tab)

$= 0.0200$



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**Note.** All above probabilities can be verified using a calculator. PRESS MODE, MODE, 1 SHIFT, 3 then area needed. Either P Or , Q Or R.

**Example**

The heights of students at a particular University follow a normal distribution with mean 150.3 cm and standard deviation 5 cm. Find the probability that a student picked at random from the university has height

a) Less than 153 cm,
b) More than 158 cm
c) Between 150 cm and 158 cm

**solution**

$X \sim N(150.3, 25)$

$$P(X < 153) = P\left(Z < \frac{153 - 150.3}{5}\right)$$

$P(Z < 0.54) = 0.5 + 0.2054(Tab)$

$\therefore P(X < 153) = 0.7054$

(b) $$P(X > 158) = P\left(Z > \frac{158 - 150.3}{5}\right)$$

$P(Z > 1.54) = 0.5 - 0.4382(Tab)$

$\therefore P(X > 158) = 0.0618$

(c) $P(150 < X < 158) =$

$$P\left(\frac{150 - 150.3}{5} < Z < \frac{158 - 150.3}{5}\right)$$

$P(-0.06 < Z < 1.54) = 0.4382 + 0.0239(Tab)$

$\therefore P(150 < X < 158) = 0.4621$



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**Example**

A random variable X is normally distributed with mean 50 and standard deviation 10. Find the probability that X assumes a value between 45 and 62.

Solution: $\sigma = 10, \mu = 50, \quad P(45 < X < 62)$

$$Z = \frac{X - \mu}{\sigma}$$

$\Rightarrow P(45<X<62) =$

$$P\left(\frac{45-50}{10} < Z < \frac{62-50}{10}\right)$$

$$P(-0.5 < Z < 1.2) = 0.1915 + 0.3849\,(Tab)$$

$$P(45 < X < 62) = 0.5764$$

**Example**

In an external examination, the scores are normally distributed with mean 120 and standard deviation 15. If a score of 100 is required to pass the examination, what is the probability that a candidate chosen at random has failed the examination?

**Solution:**

$\mu = 120, \sigma = 15$, from $Z = \dfrac{x - \mu}{\sigma}$

$$Z = \frac{100-120}{15} = -1.333$$

P(He scores less than 100) = P(Z < -1.333)

P(Z < -1.333) = (Z > 1.333) by symmetry

P(Z > 1.333) = 0.5 – P (0 < Z < 1.333)

= 0.5– 0.4087 (Tab)

P (Z < -1333) = 0.0913.

P(A candidate failed) = 0.0913.



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**Example**

A certain firm sells flour bags of mean weight 40 kg and standard deviation 2kg. Given that the weight is normally distributed find:

(i) the probability that the weight of any bag taken at random will lie between 41.0 and 42.5kg.

(ii) the percentage of bags whose weight exceeds 43kg.

(iii) the number of bags rejected out of a 500 bag purchase by a retailer whose consumers cannot accept a bag whose weight is below 38.5kg.

**Solution:**

(i) $\mu = 40, \sigma = 2$

$P(41.0 < X < 42.5) = P(Z_1 < Z < Z_2)$.

$\Rightarrow Z_1 = \frac{41-40}{2} = 0.5$ and

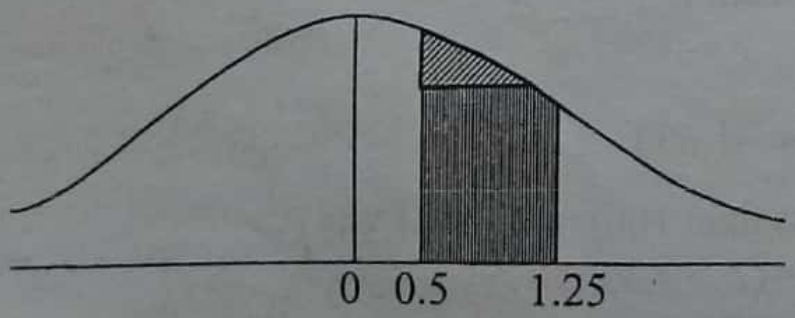
$Z_2 = \frac{42.5-40}{2} = 1.25$

$\Rightarrow \quad P(41.0 < X < 42.5) = P(0.5 < Z < 1.25)$

$P(0.5 < Z < 1.25) = P(0 < Z < 1.25) - P(0 < Z < 0.5)$

$= 0.3944 - 0.1915$ (Tab) $= 0.2029$

$P(41.0 < X < 42.5) = 0.2029$




(ii) $P(X > 43) = P(Z > \frac{43-40}{2} = 1.5)$

$P(X > 43) = P(Z > 1.5)$



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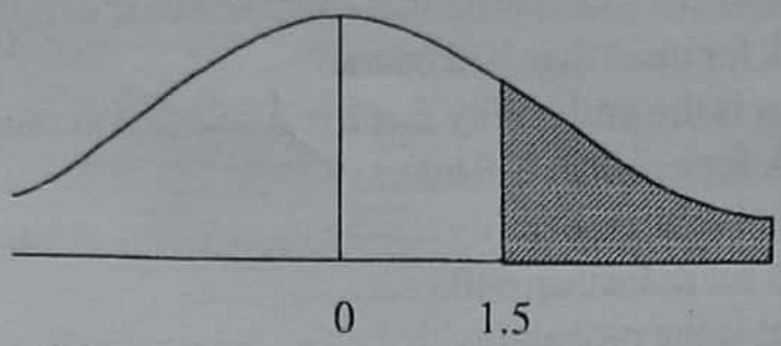


$P(Z > 1.5) = 0.5 - P(0 < Z < 1.5)$

$0.5 - 0.4332 = 0.0668$

Percentage $= 0.0668 \times 100 = 6.68\%$

(iii) Total number of bags is 500

$P(X < 38.5) = P(Z <) \quad \frac{38.5 - 40}{2}$

$Z_1 = -0.75$

$P(Z < -0.75) = P(Z > 0.75)$ by symmetry.

$P(Z > 0.75) = 0.5 - P(0 < Z < 0.75)$

$= 0.5 - 0.2734(\text{Tab}) = 02266$

$P(Z < -0.75) = 0.2266$

Bags rejected are $0.2266 \times 500 = 113.3$ bags.

**Assignement**

1.Cartons of milk from a particular supermarket are advertised as containing 1 litre, but in fact the volume of the contents is normally distributed with a mean of 1012 ml and a standard deviaion of 5 ml.

a) Find the probability that a randomly chosen carton contains more than 1010 mls
b) In a batch of 1000 cartons, estimate the number of cartons that contain less than the advertised volume of milk.

2. The life time of pens produced by a certain factory is normally distributed with mean of 800 hours and standard deviation 80 hours.



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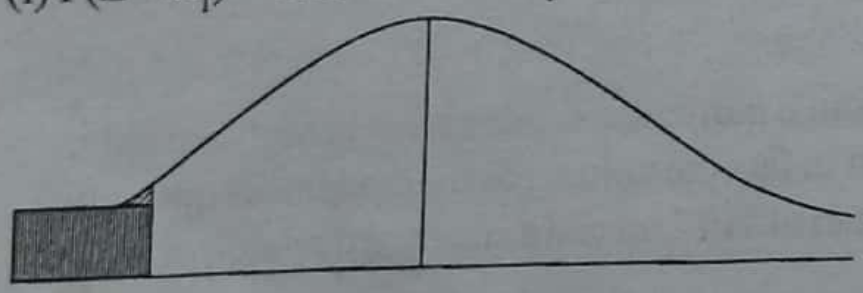
i. What is the probability that a pen selected at random will work for more than 920 hours.
ii. What is the probability that a pen selected at random will work for less than 720 hours.
iii. What is the probability that a pen selected at random will work for more than 640 hours.
iv. What is the probability that a pen selected at random will work for less than 880 hours.
v. What is the probability that a pen selected at random will work for between 840 hours and 920 hours.

## 6.5 HOW TO OBTAIN Z-VALUES FROM A GIVEN PROBABILITY

Sometimes, you are given information leading to the probability of a certain value of Z. This probability can be used to determine the value of Z using the tables. e.g

(i) $P(Z < Z_1) = 0.0968$. Find $Z_1$



$P(Z < Z_1) < 0.5$

Means $Z_1$ is less than mean $\mu = 0$.

$P(Z < Z_1) = P(Z > -Z_1) = 0.0968$

$P(Z > -Z_1) = 0.0968$.

$P(0 < Z < -Z_1) = 0.5 - 0.0968 = 0.4032$

$P(0 < Z < -Z_1) = -1.3$, (Tab)

$\therefore Z_1 = -1.3$



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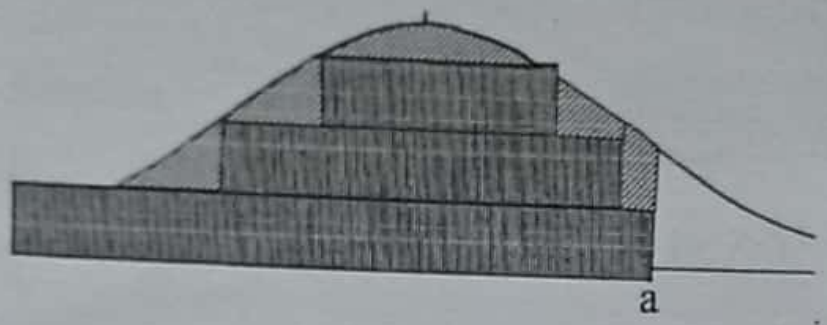
(ii) $P(Z < a) = 0.787$




$P(Z < a) = P(Z < 0) + P(0 < Z < a)$
$P(Z < 0) = 0.5$
$P(0 < Z < a) = 0.787 - 0.5$
$P(0 < Z < a) = 0.287$
From the table $a = 0.796$

Note: From the table the value indicated is 0.2852; the difference is obtained from the last part where 6 corresponds to 18, to make the value 0.2870.

(iii) $P(Z_1 > b) = 0.0100$
$P(0 < Z < b) = 0.5 - 0.0100$
$P(0 < Z < b) = 0.49$
$b = 2.326$

**Note:** the value of b can be obtained by using the critical points of the normal distribution. 0.01 corresponds to Q and this gives the value of b as 2.326 (Tab)

(iv) $P(Z > a) = 0.812.$
$P(Z > a) = P(a < Z < 0) + P(Z > 0)$
$P(a < Z < 0) = 0.812 - 0.5$
$P(a < Z < 0) = 0.312$



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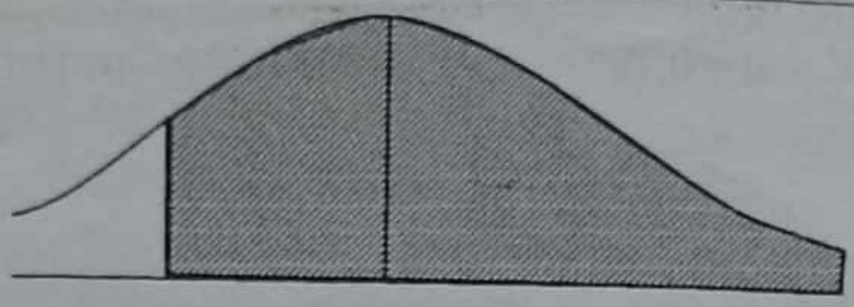

$P(a < Z < 0) = P(0 < Z < -a) = 0.312$

$-a = 0.885$

$a = -0885$

(v) $P(Z > b) = 0.030$

| P | Q | z |
|---|---|---|
| 0.470 | 0.030 | 1.881 |

Therefore $z = 1.881$.

(vi) $P(Z < b) = 0.040$

| P | Q | z |
|---|---|---|
| 0.460 | 0.040 | 1.751 |

Therefore $z = -1.751$. Since z is on the left of zero.

(vii) $P(Z < b) = 0.650$

| P | Q | z |
|---|---|---|
| 0.150 | 0.350 | 0.385 |

Therefore $z = 0.385$.

**Example**

The period of a certain machine approximately follows a normal distribution with mean of 5 years and standard deviation of 1 year. Given that the manufacturer of this machine replaces the machines that fail under guarantee, determine the length of the guarantee required so that not more than 2% of the machines that fail are replaced.



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Determine the proportion of the machines that would be replaced if the guarantee period was 4 years.

**Solution:**

$\mu = 5, \sigma = 1$

let $x_0$ be guarantee period

$X \sim N(\mu, \sigma^2)$

$P(X < x_0) = 0.02$

$-Z_0$ corresponds to 2.054 (Tab) using the critical points

$Z_0 = -2.054$

but $Z = \dfrac{X - \mu}{\sigma} \Rightarrow X = Z\sigma + \mu$

$Xo = -2.054 \times 1 + 5$

$Xo = 2.946$ years,

the guarantee period is 2.946 years.

When $X_0 = 4$, $P(X < Xo) = P(X < 4)$

$$Z = \frac{X - \mu}{\sigma} = \frac{4 - 5}{1} = -1$$

$P(X < 4) = P(Z < -1)$

$P(Z < -1) = P(Z > 1)$ by symmetry

but $P(0 < Z < 1) = 0.5 - P(Z > 1)$

$P(Z > 1) = 0.5 - P(0 < Z < 1)$

$P(Z - 1) = 0.5 - 0.3413$

$P(Z > 1) = P(Z < -1) = 0.1587$

$P(X < 4) = 0.1587$

The proportion is $0.1587 \times 100\%$

15.87%



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**Example**

A total population of 700 students sat an examination for which the pass mark was 50. Their marks were normally distributed. 28 students scored below 40 marks while 35 scored above 60.

(a) Find the mean mark and standard deviation of the students.

(b) What is the probability that a student chosen at random passed the examination?

(c) Suppose the pass mark is lowered by 2 marks, how many more students will pass?

**Solution:**

(a) Let X be the marks obtained by a student.

$X \sim N(\mu, \sigma^2)$.

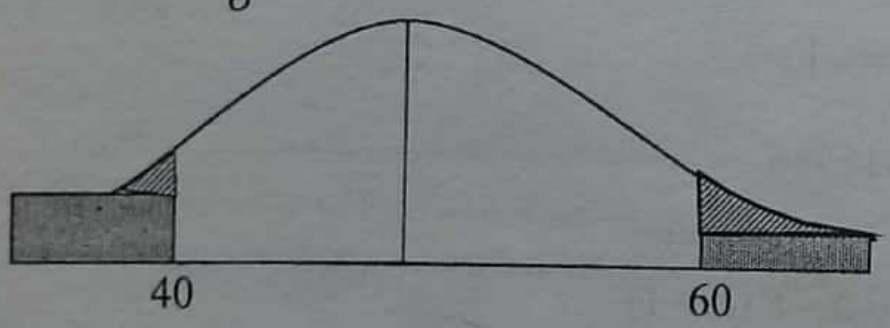
Let the mean be $\mu$ and standard deviation $\sigma$.

$P(X < 40) = {}^{28}/_{700} = 0.04$

$P(X < 40) = 0.04$

$P(X > 60) = {}^{35}/_{700} = 0.05$.

Note $Z = \dfrac{X - \mu}{\sigma}$




$P(X < 40) = P(Z < Z_1) = 0.04$

$P(Z < Z_1) = 0.04$

$-Z_1 = 1.751$ from the table of critical values

$\therefore Z1 = -1.751$

$$\Rightarrow -1.751 = \frac{40 - \mu}{\sigma}$$



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$\Rightarrow \mu - 1.75\sigma = 40 \ldots(1)$

$P(X > 60) = 0.05$

$P(X < 60) = P(Z > Z_2) = 0.05$

$Z_2 = 1.645$ from table of critical values

$$\Rightarrow 1.645 = \frac{60 - \mu}{\sigma}$$

$\Rightarrow \mu - 1.645\sigma = 60 \ldots(2)$

$\Rightarrow \mu - 1.75\sigma = 40 \ldots..(1)$

Equation (2) – (1)

$1.645\sigma + 1.751\sigma = 20 \ldots(1)$

$3.396\sigma = 20$

$\sigma = 5.889$

From (1) $\mu = 1.751\sigma + 40$

$\mu = 10.3 + 40$

$\mu = 50.3$

Standard deviation $\sigma = 5.889, \mu = 50.3$

(b) The pass mark is 50.

$P(X < 50)$

$$Z_1 = \frac{X - \mu}{\sigma} = \frac{50 - 50.3}{5.889}$$

$Z_1 = -0.051$

$P(Z > -0.051) = 0.5 + P(-0.0051 < Z < 0)$

$P(-0.051 < Z < 0) = P(0 < Z < 0.051)$ By symmetry

$P(0 < Z\ 0.051) = 0.203$ from the table

$P(0 < Z < 0.051) = 0.5203$

$P(Z > -0051) = 0.5203$



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(c) The pass mark

$P(X > 48) = P(Z > Z_1)$

$$Z_1 = \frac{48 - 50.3}{5.889}$$

$Z_1 = -0.391$

$P(Z > -0.391) = P(-0.391 < Z < 0) + P(Z > 0)$

$P(Z > 0) = 0.5$

$P(-0.391 < Z < 0) = P(0 < Z\ 0.391)$ By symmetry

$P(0 < Z < 0.391) = 0.1521$

$P(Z > -0.391) = 0.5 + 0.1521$

$P(X > 48) = 0.6521$

The probability of more students is

$0.6521 - 0.5203 = 0.1318$

number of students

$0.1318 \times 700$

$= 92$ students

## 6.6 DISTRIBUTION OF SAMPLE MEAN FROM A NORMAL POPULATION

If $x_1, x_2 \ldots x_n$ is a random sample of size n taken from a normal distribution with mean $\mu$ and variance $\sigma^2$ such that $X \sim N(\mu, \sigma^2)$ then the distribution of $\bar{x}$ is also normally distributed with mean $\mu$ and variance $\sigma^2/n$.

**Example**

A random sample of size 16 is taken from a normal population with mean 60 and standard deviation 4. Find the probability that the mean of the sample is less than 58.



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**Solution**

$$P(\overline{X} < 58) = P(Z < \frac{58-60}{4/4})$$

$$P(Z < -2) = 0.5 - 0.4772$$

$$P(\overline{X} < 58) = 0.0228$$

**Example**

The height of students are normally distributed with mean 164cm and standard deviation 7.2cm. Calculate the probability that the mean height of a sample of 36 students will be between 162 and 166cm

**Solution**

$$P(162 < x < 166) = (P(\frac{162-164}{(7.2/6)} < Z < \frac{166-164)}{(7.2/6)}$$

$$= P(-1.667 < Z < 1.667)$$

$$= 2P(0 < Z < 1.667)$$

$$= 2 \times 0.4522$$

$$P(162 < x < 166) = 0.9044$$

**Assignment**

The days production of milk sold in packets of certain firm is normally distributed with mean of 0.5 litres and standard deviation of 0.08 litres. If a sample of 16 packets is drawn from days production, find the probability that the mean is between 0.46 and 0.47 litres.



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**Exercise**

1. The length of a type A rod is normally distributed with mean of 15cm and a standard deviation of 0.1cm. The length of another type B rod is also normally distributed with a mean of 20cm and standard deviation 0.16cm. For a type A rod to be acceptable, its length must be between 14.8cm and 15.2cm and for a type B rod the length must be between 19.8cm and 20.2cm.

   i. What proportion of type A rod is of acceptable length?
   ii. What is the probability that one of them is of acceptable length?

**Answer**

i) Acceptable length for type A is 95.44%.
Acceptable length for type B is 78.88% (ii) 0.7528 (iii) 0.2375

2. The lifetime of a bulb is normally distributed with a mean of 800 hours and standard deviation of 80 hours. The manufacturer guarantees to replace bulbs, which blow before 660 hours.

   i. What percentage of the bulbs will he have to replace under the guarantee?
   ii. The manufacturer is only willing to replace a maximum of 1% of the bulbs. What should be the guaranteed lifetime of the bulbs?
   iii. Instead of reducing the guaranteed life time as in (ii). the mean lifetime was increased by superior technology. What should be the new mean so that only 1% are replaced if the guaranteed life time remains at 660 hours but the standard deviation is reduced to 70 hours.

**Answer:** i) 4.01% ii) 613.92 hours
iii) $\mu = 822.82$ hours



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3.In an examination 30% of the candidates fail.and 10% achieve distinctions. Last year the pass mark (out of 200) was 84 and the minimum mark required for a distinction was 154. Assuming that the marks of the candidates were normally distributed, estimate the mean mark and standard deviation.

**Answer:** $\mu = 104.31$ $\sigma = 38.76$

4. The marks of 500 candidates in an examination are normally distributed with a mean of 45 marks and standard deviation of 20 marks.

a) Given that the pass mark is 41, estimate the number of candidates who passed the examination.
b) If 5% of the candidates obtained a distinction by scoring x marks, or more, estimate the value of x.
c) Estimate the interquartile range of the distribution.

**Answer** a) 290 b) 78 c) 27

5. Given a normal distribution with mean 40 and standard deviation 6, find

a) The area below $x = 32$.
b) The area above $x = 34$
c) The area between $x = 42$ and $x = 51$
d) The x – value that has 45% of the area below it
e) The x-value that has 13% of the area above it

**Answer:** a) 0.0913 b) 0.8413, c) 0.3362 d) 39.244 e) 46.756

6.The marks obtained by 1000 candidates in an examination were normally distributed with a mean of 55 and standard deviation 8,

a) If a mark of 71 or more is required for A – pass, estimate the number of A passes awarded.
b) If 15% of the candidates failed the examination, estimate the minimum mark required for a pass.



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c) Calculate the probability that two candidates chosen at random both passed the examination.

**Answer:** a) 23, b) 47 c) 0.7225

7. A certain type of electricity light bulb has a burning life of H hours, where it has a normal distribution with mean 1300 hours and standard deviation 125 hours.

a) What is the probability that a bulb selected at random will burn for more than 1500 hours?

b) If a manufacturer guarantees to replace any bulb which burns for less than 1050 hours, what percentage of the bulbs will have to be replaced?

c) If two bulbs are installed at the same time, what is the probability that both will burn less than 1400 hours but more than 1200 hours?

**Answer:** a) 0.0548 b) 2.28 c) 0.3320

8. The heights of boys in a certain school are normally distributed. 10% are over 1.8 meters and 20% are below 1.6 meters. Determine the mean height $\mu$ meters and standard deviation $\sigma$ meters. Hence, find the interquartile range

**Answer:** $\mu = 1.68$ $\sigma = 0.09, 0.13$

9. Tests made on two types of electric bulbs show the following. Type A, lifetime was distributed normally with an average of 1150 hours and standard deviation of 30 hours. Type B, long life bulbs, average lifetime of 1900 hours, with standard deviation of 50 hours.

a) What percentage of bulbs of type A could be expected to have a life time more than 1200 hours.

b) What percentage of type B would you expect to last longer than 1800 hours?

c) What lifetime limit would you estimate would contain the central 80% of production of type A.?



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**Answer:** a) 4.78% b) 97.72%
c) 1111.54 hours, 1188.46 hours

10. A factory produces two types of bars of soap A and B. Their lengths are normally distributed with type A having mean 115cm and standard deviation 3 cm. Type B having mean 190 cm and standard deviation 5 cm

Determine the percentage of type.

(i) A bars that have a length of more than 120cm

(ii) B bars that have a length of more than 180 cm

**Answer** (i) 4.78% (ii) 97.72%

11. The weight of the contents of a "2Kg" packet of frozen food is normally distributed with a mean of 2.05 kg and a standard deviation of 0.015kg. Estimate 99% confidence limits for the weight of contents in a single 2kg packet.

**Answer** 2.011 kg, 2.089kg

12. The r. v X is such that $X \sim N(\mu, 4)$. A random sample size n, is taken from the population. Find the least value of n such that

$$P(|X - \mu| < 0.5) > 0.95.$$

**Answer 62**

13. Boxes made in a factory have weights which are normally distributed with a mean of 4.5 kg and a standard deviation of 2.0 kg. Find the probability of there being a box with a weight of more than 5.4 kg when a box is chosen at random. If a sample of 16 boxes is drawn, find the probability that the mean is between

(i) 4.6 and 4.7 kg

(ii) 4.3 and 4.7 kg

**Answer:** 0.3264 (i) 0.0761 (ii) 0.3108



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14. Biscuits are produced with Weight (Wg) where W is N(10, 4) and are packed at random into boxes consisting of 25 biscuits. Find the probability that;

(a) a biscuit chosen at random weighs between 9.25g and 10.75g
(b) the contents of a box weighs between 245g and 255g.
(c) the average weight of the biscuits in the box lies between 9.7g and 10.3g.

**Answer** (a) 0.2924 (b) 0.0796 (c) 0.5468

15. Observation of a very large number of cars at a certain point on a motorway establishes that the speeds are normally distributed. 90% of cars have speeds less than 77.7 km/h and only 5% of cars have speeds less than 63.1 km/h.
Determine the mean speed $\mu$ and standard deviation $\sigma$

**Answer** $\mu = 71.305$, $\sigma = 4.988$.

16. The marks in an examination were normally distributed with mean $\mu$ and standard deviation $\sigma$. 10% of the candidates scored more than 75 marks and 20% scored less than 40 marks.

i. 25 candidates were chosen at random from those who sat for the examination.
Find the probability that their average mark exceeds 60.

ii. If a sample of 8 candidates were chosen, find the probability that not more than 3 scored between 45 and 65 marks

**Answer** $\mu = 53.87$, $\sigma = 16.473$,
(i)0.0313 (ii) 0.5419

17. The life time of batteries produced by a certain factory is normally distributed. Out of 10,000 batteries selected at random, 668 have life time less than 130 hours and 228 have life time more than 200 hours.

i. find the mean and standard deviation of the battery life time.



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ii. find the percentage of the batteries with life time between 150 and 180 hours.

iii. If the sample of 25 batteries is selected at random, find the probability that the mean of the life time exceeds 165 hours.

**Answer** (i) $\mu = 160$, $\sigma = 20$
(ii) 53.28% (iii) 0.1056

18. The volume of a soft drink bottled by a certain company is approximately normally distributed with mean 300 mls and standard deviation 2 mls. Determine the probability that in a sample of 10 bottles at least two contain less than 297.4 mls.

**Answer** 0.2515

## 6.7 THE NORMAL APPROXIMATION TO THE BINOMIAL DISTRIBUTION

Under certain circumstances, the normal distribution can be used as an approximation to the binomial distribution. One practical advantage is that calculations are much less tedious to perform.

**Condition**

For a large value of $n > 20$ and the probability of success not too small or too large, i.e p is very close to 0.5

If $X \sim B(n, p)$, then

$E(X) = np$

$Var(X) = npq$ where $q = 1 - p$.

Then $X \sim N(np, npq)$. Approximately

The value of Z can be obtained from



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$$Z = \frac{X \pm 0.5 - np}{\sqrt{npq}}$$

i.e substituting for $\mu = E(X) = np$ and

$$\sigma = \sqrt{npq} \text{ in } \frac{x-\mu}{\sigma}$$

The $\pm$ 0.5 is used to make the binomial distribution continuous.

**Note**: $P(X \le x_1) = P(Z \le Z_1)$ and corresponding value of $Z_1$ is

$$Z_1 = \frac{X + 0.5 - np}{\sqrt{npq}}$$

$$P(X \ge X_2) = P(Z \ge Z_2) \Rightarrow Z_2 = \frac{X - 0.5 - np}{\sqrt{npq}}$$

**Example**

If an unbiased coin is tossed 100 times, what is the probability that

(a) There will be more than 60 heads?

(b) There will be at least 45 and at most 55 heads?

(c) There will be fewer than 43 heads?

**Solution**

a) $n = 100, p = ½$

$X \sim B(n, p)$

$X \sim N(np, npq)$ since n is large

$\mu = np = 100 \times 0.5 = 50$

$Var(X) = npq = 50 \times ½ = 25$

$P(X > 60) = P(X \ge 61)$

$P(X \ge 61) = P(Z > Z_1) =$

$$\Rightarrow Z_1 = \frac{61 - 0.5 - 50}{\sqrt{25}} = \frac{60.5 - 50}{5}$$

$Z1 = 2.1$

$P(Z > 2.1) = 0.5 - P(0 < Z < 2.1)$



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$= 0.5 - 0.4821$

$P(Z > 2.1) = 0.0179$

$P(X > 60) = 0.0179$

b) $P(45 \leq X \leq 55)$

$\mu = np, \sigma = \sqrt{npq} = 5$

$P(45 \leq X \leq 55) = P(44.5 \leq X \leq 55.5)$

$$\Rightarrow P\left(\frac{44.5-50}{5} < Z < \frac{55.5-50}{5}\right)$$

$P(-1.1 < Z < 1.1) =$

$2\ x P(0 < Z < 1.1) = 2x0.3643(Tab)$

$P(45 \leq X \leq 55) = 0.7286$

P(at least 45 and at most 55 heads)

$= 0.7286$

c) $P(X < 43)$

$P(X \leq 42) = P(Z \leq Z_1)$

$$P(Z < \frac{42.5-50}{5}) =$$

$P(Z < -1.5) = 0.0668(Tab)$

P(will be fewer than 43 heads) = 0.0668

**Example**

Among the spectators watching a football match 80% were the home team's supporters while 20% were the visiting team's supporters. If 2,500 of the spectators are selected randomly, what is the probability that there are more than 540 visitors in the sample?



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**Solution**

$n = 2500,\ p = 0.2$ and $q = 0.8$

n is large, hence $X \sim N(np, npq)$

$\mu = np = 2500 \times 0.2 = 500$

$$\delta = \sqrt{2500x0.8x0.2} = 20$$

$P(X > 540) = P(X \geq 541) =$

$$P(Z > \frac{540.5 - 500}{20})$$

$$P(Z > 2.025) = 0.5 - 0.4785(Tab)$$

$$P(X > 540) = 0.0215$$

**Example**

It is known that 72% of UBC viewers watch a particular rogramme known as Cuando Seas Mia. What is the probability that in a sample of 500 viewers chosen at random

(a) More than 350 watch the programme.

(b) Fewer than 340 watch the programme.

(c) Exactly 350 watch the program

**Solution**

$n = 500,\ p = 0.72$

$\mu = np = 500 \times 0.72 = 360$

$\sigma^2 = npq = 360 \times 0.28 = 100.8$

$\sigma = 10.04$

$P(X > 350) = P(x \geq 351)$



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$$P(Z > \frac{350.5-360}{10.04}) =$$

$$P(Z > -0.946) = 0.5 + 0.3280(Tab)$$

$$P(X > 350) = 0.8280$$

b) $P(X < 340) = P(X \leq 339)$

$$P(Z < \frac{339.5-360}{10.04})$$

$$P(Z < -2.042) = 0.5 - 0.4794(Tab)$$

$$P(X < 340) = 0.0206$$

(c)

$$P(X = 350) = P(349.5 < X < 350.5)$$

$$= P(\frac{349.5-360}{10.04} < Z < \frac{350.5-360}{10.04})$$

$$= P(-1.046 < Z < -0.946)$$

$$= 0.3522 - 0.3280\,(Tab)$$

$$= 0.0242$$

**Assignment**

1. Twenty percent of the eggs supplied by a poultry farm have racks on them. Determine the probability that a sample of 900 eggs supplied by the farm will have more than 200 eggs with cracks.

2. A question paper contained 100 objective questions with 4 alternative answers but only one correct answer. A candidate who sat for the paper attempted 80 questions purely by guessing and left out the remaining 20. Determine the probability that;

   (i) he got 35%

   (ii) he scored between 20% and 30%



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## Exercise

1. Statistics records from Uganda Police traffic department shows that on weekend nights, one out of every ten drivers on the road is drunk. A random sample of four hundred drivers is checked on a weekend.
Find the probability that the number of drunk drivers is at least 35 but less than 47.

**Answer**: 0.6807

2. A pair of dice is tossed 144 times and the sum of the out comes recorded. Find the probability that a sum of 7 occurs at least 26 times.

**Answer:** 0.3688

3. In an examination which consists of 100 questions, a student has a probability of 0.6 of getting each question correct. The student fails the examination if he obtains a mark less than 55, and obtains a distinction for a mark of 68, or more. Calculate:

(a) the probability that he fails the examination.

(b) The probability that he obtains a distinction.

**Answer**: a) 0.1308 b) 0.0629

4. If a fair die is thrown 300 times, what is the probability that:

(a) there will be more than 60 sixes

(b) There will be fewer than 45 sixes.

**Answer:** a) 0.0519 b) 0.1971

5. A coin is biased such that head is twice as likely to occur as a tail. The coin is tossed 120 times. Find the probability that there will be:

a) between 42 and 51 tails inclusive

b) 48 tails or less

c) less than 34 tails

d) between 72 and 90 heads inclusive.



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**Answer:** a) 0.3729 b) 0.9501
c) 0.1039 d) 0.9290

6. Four hundred students sat a test which consists of 80 true / false questions. None of the candidates knows any of the answers and so guesses.

(a) If the pass mark is 38, how many of the candidates would be expected to pass?

(b) What should the new pass mark, if it is decided that only 115 candidates pass?

**Answer:** a) 285 b) 43

7. A lorry of potatoes has on average one rotten potato in 6. A green grocer tests a random sample of 100 potatoes and decides to turn away the lorry if he finds more than 18 rotten potatoes in the sample. Find the probability that he accepts the consignment.

**Answer:** 0.6886



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# CHAPTER SEVEN

## ESTIMATION

### 7.1 INTRODUCTION

Statistical estimation is a statistical procedure used to describe the unknown characteristics of the population by using sample characteristics.A sample is a representation of a population. Parameters are population constants, e.g population mean $\mu$, population variance $\sigma^2$

**There are two types of estimation**

a) Point estimation
b) Interval estimation.

### 7.2 Point Estimation

Point estimation means single value estimation of a parameter. The value got is the *estimate* while the method used is called the *estimator*. An estimator is a statistical technique used to obtain a sample characteristic. It is a function of the sample observations,

**Example** If $x = 3, 4, 6, 7$

$$\bar{x} = \frac{\sum x}{n} = \frac{3+4+6+7}{4} = 5$$

$\bar{x}$

$\bar{x} = 5$ which is an estimate.

and the formula $\frac{\sum x}{n}$ *is* the estimator

### 7.3 Interval Estimation

Interval estimation deals with a range between which the parameter lies e.g $5 < \mu < 10$.



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## 7.4 Confidence Interval

If a and b are limits of a parameter, i.e $a \leq \mu \leq b$, a and b are called confidence limits, a being the lower confidence limit while b is the upper confidence limit.

## 7.5 Confidence Coefficient or Degree of Confidence

The probability that the true population parameter will lie within the stated interval is denoted by $(1 - \alpha)$ and this is referred to as confidence coefficient.

## 7.6 Estimation of the Mean

Consider the population to be normally distributed. The point estimator for $\mu$ is $\bar{x}$ (the sample mean)

**Note**: The mean of a random sample drawn from a population which is normally distributed with mean $\mu$ and variance $\sigma^2$ is also normally distributed with mean $\mu$ and variance $\sigma^2/n$

$\text{Var}(\bar{x}) = \dfrac{\sigma^2}{n}$ Remember $Z = \dfrac{x-\mu}{\sigma}$ .Implies $Z = \dfrac{\bar{x} - E(\bar{x})}{\sqrt{\text{var}(\bar{x})}}$

**Note**

If $\bar{x}$ is the mean of a random sample of size n from a normal population with unknown mean $\mu$ and known variance $\sigma^2$, then

$(1 - \alpha)\%$ confidence limit for $\mu$ is

**Note:** For a population the (confidence limit for

$\mu$ is $\bar{x} \pm Z_{\alpha/2}\,\delta$



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**Example**

A factory produces two types of bars of soap A and B. Their lengths are normally distributed with type A having mean 115cm and standard deviation 3 cm. Find the 95% confidence limits for the mean of lengths of type A bars of soap.

**Solution**

$95\% C.I \Rightarrow Z_{0.025} = 1.96 (Tab)$

$Upper \lim it = 115 + 1.96 x 3 = 120.88$

$Lower \lim it = 115 - 1.96 x 3 = 109.12$

**Example**

It is known that an examination paper is marked in such a way that the standard deviation of the marks is 15.1. In a certain school, 80 candidates take the examination and they have an average mark of 57.4. Find

(a) 95% and

(b) 99% confidence limits for the mean mark in the examination.

**Solution**

$\sigma = 15.1n, n = 80,$

$(1 - \alpha)\%$ confidence interval is given by $\bar{x} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$

(a) $95\% \Rightarrow 1 - \alpha = 0.95$

$0.05 = \alpha$, implies that $\alpha/2 = 0.025$

$P(-Z_{0.025} \leq Z \leq Z_{0.025}) = 1 - 0.05 = 0.95$

But $P(0 < Z < Z_{0.025}) = 0.475$



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$Z_{0.025} = 1.96(Tab)$

$$Upper\ \lim it = 57.4 + 1.96x\frac{15.1}{\sqrt{80}} = 60.709$$

$$Lower\ \lim it = 57.4 - 1.96x\frac{15.1}{\sqrt{80}}$$

$$= 54.091$$

$Z_{0.005} = 2.575(Tab)\ from\ \ critical\ \ points$

$$Upper\ \lim it = 57.4 + 2.575x\frac{15.1}{\sqrt{80}}$$

$$= 61.746$$

$$Lower\ \lim it = 57.4 - 2.575x\frac{15.1}{\sqrt{80}}$$

$$= 53.053$$

**Example**

The mean and standard deviation of a random sample of size 100 is 900 and 60 respectively. Given that the population is normally distributed, find a 96% confidence interval of the population mean.

**Solution**

$n = 100,\ \bar{x} = 900,\ \delta = 60\ \ and\ \ Z_{0.02} = 2.054(Tab)$

$$Upper\ \lim it = 900 + 2.054\,x\frac{60}{\sqrt{100}} = 912.324$$

$$Lower\ \lim it = 900 - 2.054\,x\frac{60}{\sqrt{100}} = 887.676$$



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**Example**

The weights of a sample of 36 chicken from a poultry farm were recorded as follows

| *Weight (kg)* | *3.30* | *3.60* | *3.90* | *4.20* | *4.50* | 4.80 |
|---|---|---|---|---|---|---|
| Frequency | 3 | 6 | 9 | 11 | 5 | 2 |

Calculate (i) the mean sample weight and standard deviation of the chicken

(ii) 97.5% confidence interval of the mean weight of the poultry on the farm

**Solution**

| $x$ | $f$ | $fx$ | $fx^2$ |
|---|---|---|---|
| 3.30 | 3 | 9.90 | 32.67 |
| 3.60 | 6 | 21.60 | 77.76 |
| 3.90 | 9 | 35.10 | 136.89 |
| 4.20 | 11 | 46.20 | 194.04 |
| 4.50 | 5 | 22.50 | 101.25 |
| 4.80 | 2 | 9.60 | 46.08 |
| **Total** | **36** | **144.9** | **588.69** |

$$Mean = \frac{\sum fx}{\sum f} = \frac{144.9}{36}$$

$$\therefore Sample\ mean = 4.025$$

$$Sample\,Variance = \frac{n}{n-1}\left(\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right)$$



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$$Sample\,Variance = \frac{36}{35}\left(\frac{588.69}{36} - (\frac{144.9}{36})^2\right)$$

$$= 0.156$$

$$\therefore s\tan dard\ deviaton = \sqrt{0.156} = 0.395$$

$$97.5\% \Rightarrow 1 - 0.975 = 0.025$$

$$\Rightarrow Z_{0.0125} = 2.24\,(Tab)\ Used\ P = 0.4875(Tab)$$

$$Upper\ \lim it = 4.025 + 2.24\,x\frac{0.395}{6}$$

$$= 4.1725$$

$$Lower\ \lim it = 4.025 - 2.24\,x\frac{0.395}{6}$$

$$= 3.8775$$

$$\therefore Therefore\ confidence\ \text{int}erval\ is\,[3.8775, 4.1725]$$

**Note:**

$$\text{Standard error} = \frac{\delta}{\sqrt{n}}$$

$$\text{Maximum error} = Z_{\alpha/2}\frac{\delta}{\sqrt{n}}$$

Class width = Upper limit – lower limit

**Confidence interval for μ (when n < 30) and $\sigma^2$ is unknown. Here the t- distribution is used**



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$$T = \frac{x-\mu}{s/\sqrt{n}} \text{ from which the } (1-\alpha)\%$$

$$C.I \text{ for } \mu \text{ is given by}$$

Where $\bar{x} \pm t_{\alpha/2}, (n-1)\frac{s}{\sqrt{n}}$, where

$n-1$ indicates the degree of freedom

Assignment

The table below shows the distribution of weights of a random sample of 16 tins taken from a large consignment.

| Weight (gm) | 97 | 98 | 99 | 100 | 101 | 102 |
|---|---|---|---|---|---|---|
| Frequency | 2 | 1 | 2 | 3 | 6 | 2 |

Assuming the weghts are normally distributed, determine a 95% confidence interval for the mean weight of all the tins.



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# CHAPTER EIGHT

## SCATTER GRAPHS AND CORRELATION

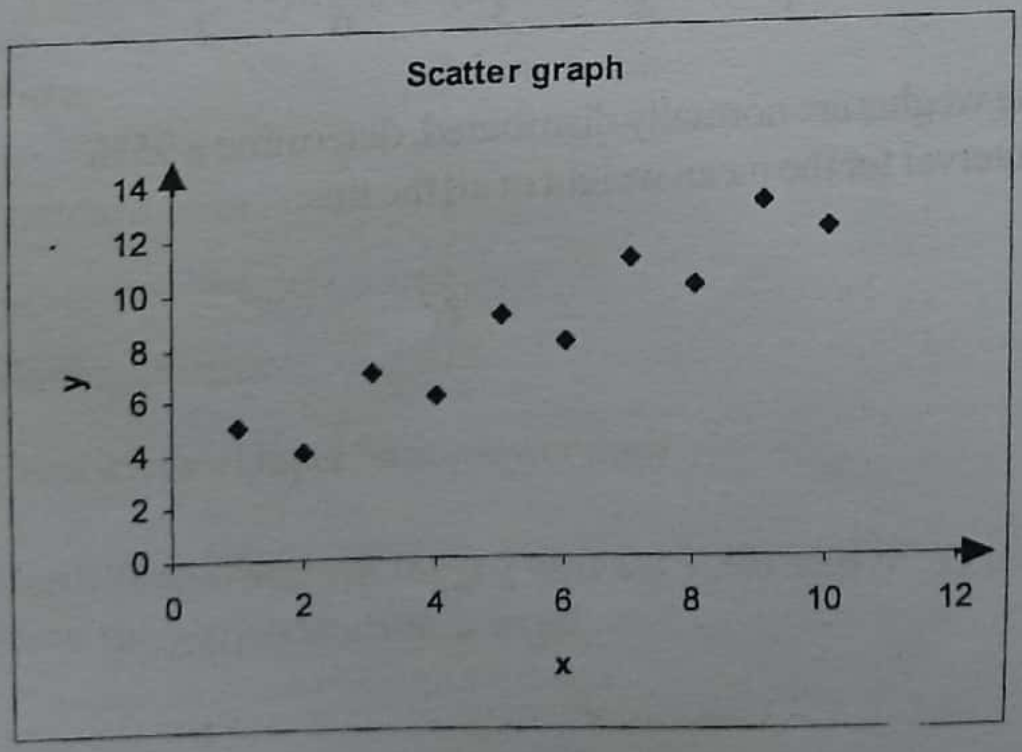
### 8.1 INTRODUCTION – SCATTER GRAPHS

Relation between two variables can be shown on diagrams or graphs known as scatter diagrams or graphs. A scatter graph is obtained by representing scores of one variable on the vertical axis and the other scores on the horizontal axis.

**Example**

Draw a scatter diagram for the following data.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| x | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| y | 5 | 4 | 7 | 6 | 9 | 8 | 11 | 10 | 13 | 12 |



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## 8.2 REGRESSION LINE

When a scatter graph is plotted, a line of best fit can be drawn through the points. This line is called the regression line.

**Note:** The regression line should pass through $(\bar{x}\ \bar{y},)$

where

$$\bar{x} = \frac{\sum x}{n} \text{ and } \bar{y} = \frac{\sum y}{n}$$

**Drawing a regression line "by eye"**

If there is very little scatter, a line of best – fit can drawn by eye.

**Example**

The table below shows the percentage preference of nine most popular holiday destinations as sampled by a tour company for two years 2000 and 2002.

| *Holiday Place* | *2000 (x)* | *2002 (y)* |
|---|---|---|
| F | 90 | 79 |
| G | 80 | 90 |
| S | 78 | 80 |
| I | 78 | 60 |
| A | 50 | 60 |
| Y | 40 | 35 |
| C | 30 | 30 |
| H | 20 | 60 |
| B | 10 | 22 |

i. Plot a scatter diagram for the data, and include the line of best fit.



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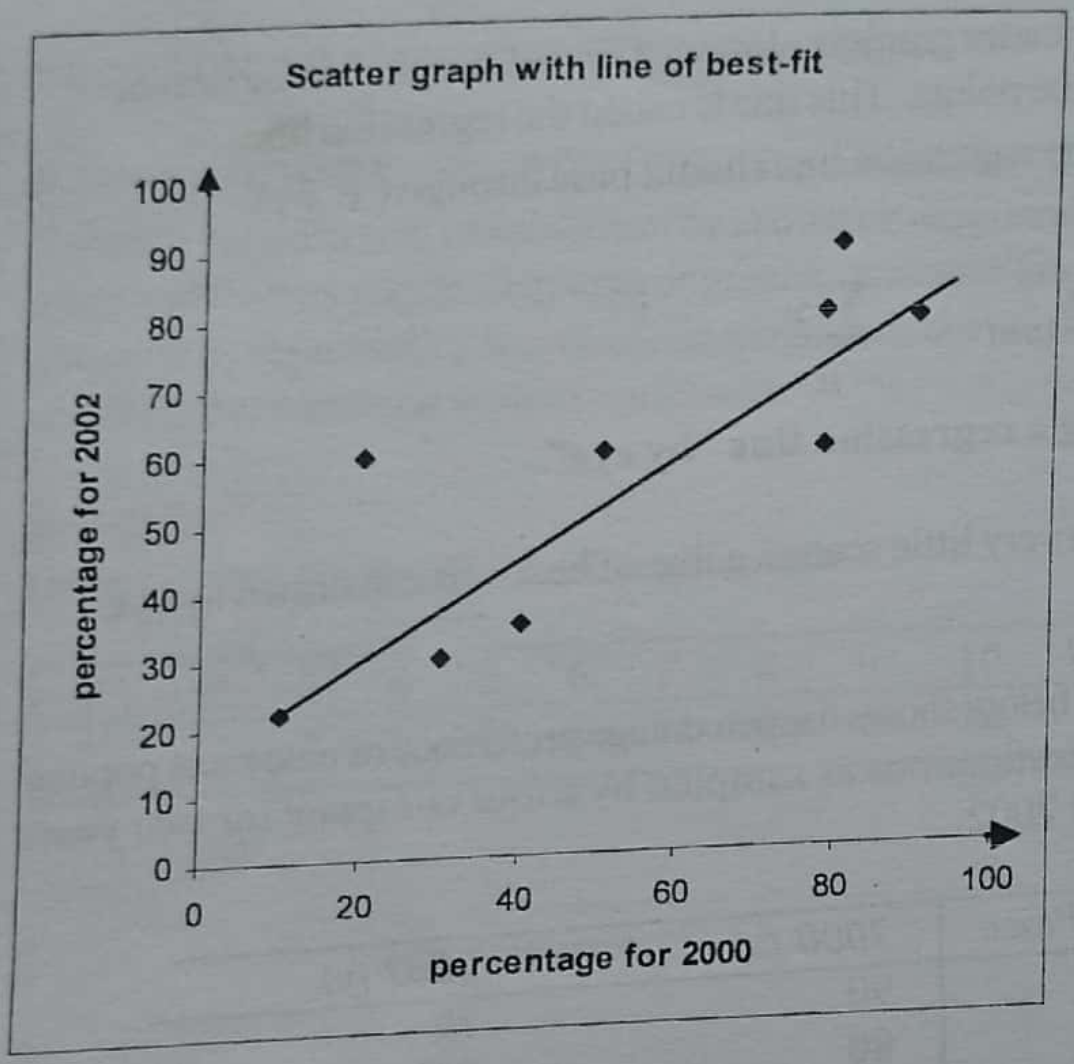


The line above is used to predict y from x and hence known as y on x.

**If the points are widely scattered two different lines can be drawn. One is y on x and the other x on y.**

**Steps for drawing the line y on x by eye.**

Assume the values of x are accurate.

(i) Find the mean M ( $\overline{x}, \overline{y}$ ) of the distribution.

(ii) Through M draw a line parallel to y – axis.

(iii) Find the (mean) $M_L$ of points on the left

(iv) Find the (mean) $M_R$ of the points on the right.



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(v) Then draw the line of best fit through M, $M_L$ and $M_R$

**Steps for drawing the line x on y by eye.**

Assume the values of y are accurate.

i) Find the mean M $(\bar{x}, \bar{y})$ of the distribution.
ii) Through M draw a line parallel to x-axis.
iii) Find the $M_A$ of the points above
iv) Find the $M_B$ of the points below.
v) Then draw the line of best fit through M, $M_A$ and $M_B$

**Example:**

| X | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 10 |
|---|---|---|---|---|---|---|---|---|
| Y | 13 | 7 | 11 | 16 | 8 | 15 | 12 | 16 |

Draw on a scatter diagram and the line of best fit for predicting y from x.

**Solution**

$$M(\bar{x}, \bar{y}) = \left(\frac{\sum x}{n}, \frac{\sum y}{n}\right) = (5.6, 12.3)$$

$$M_L = \left(\frac{2+3+4+5}{4}\right), \left(\frac{13+7+11+16}{4}\right) = (3.5, 11.8)$$

$$M_R = \left(\frac{6+7+8+10}{4}\right), \left(\frac{8+15+12+16}{4}\right) = (7.8, 12.8)$$

The dotted line is x = 5.6 which is parallel to y – axis

**Note**: In order to draw x on y two other points are needed and the line y = 12.3. Therefore line x on y has been left as an exercise.



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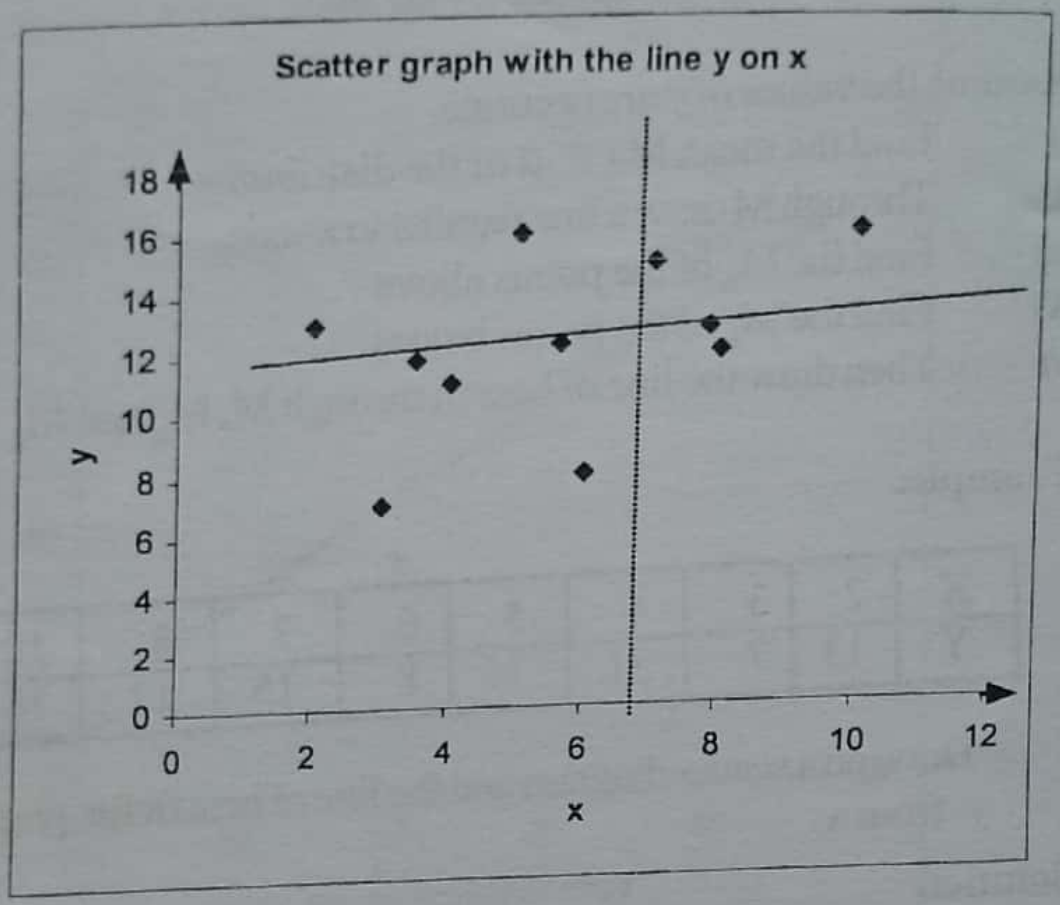
**Note:** The slope of a regression line is also known as the regression coefficient.




## 8.3 CORRELATION

Correlation is method used to determine relationships between two or more variables. Correlation coefficient is the index used to measure the degree of correlation

### Types of Correlation

**Positive correlation**

If y tends to increase as x increases, then there is positive correlation. The correlation coefficient is between 0 and 1



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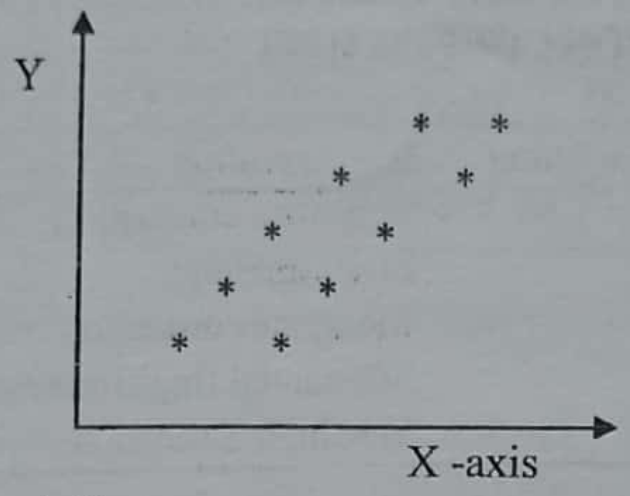
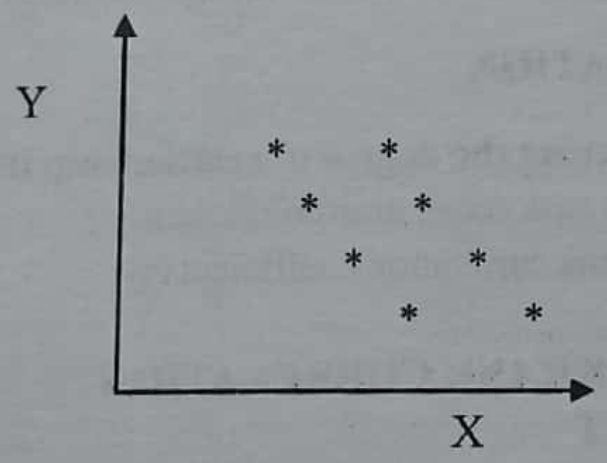



**Negative correlation**

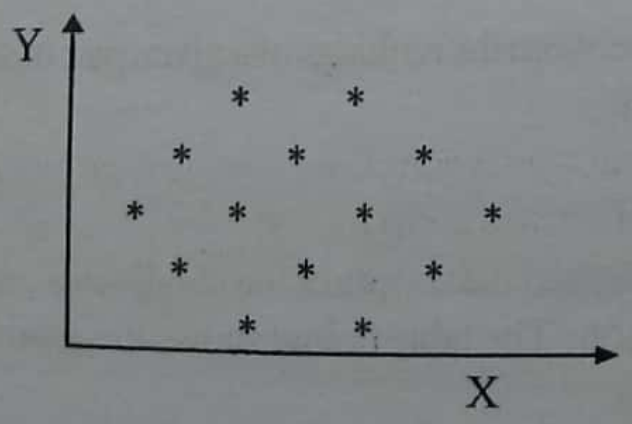
If y tends to decrease as x increases, then there is negative correlation.

The correlation coefficient is between -1 and 0.




**Zero or No correlation**

If there is no relationship between x and y, then there is no correlation.



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## 8.4 INTERPRETATION OF THE MAGNITUDE OF CORRELATION COEFFICIENT

| *Correlation coefficient* | *interpretation* |
|---|---|
| 0 – 0.19 | Very low correlation |
| 0.2 – 0.39 | Low correlation |
| 0.4 – 0.59 | Moderate correlation |
| 0.6 – 0.79 | Substantial (high) correlation |
| 0.8 – 1.0 | Very high correlation |

**Note:**
The sign associated with correlation coefficient will be the one responsible for the type of correlation.
Example –0.5 would indicate a moderate negative linear correlation.

## RANK CORRELATION

**Method for measuring the degree of relationship include:-**

1. Spearman's rank correlation coefficient (ρ)
2. Kendall's rank correlation coefficient (τ)

## 8.5 SPEARMAN'S RANK CORRELATION COEFFICIENT

$$\rho = 1 - \frac{6\sum d^2}{n\left(n^2 - 1\right)}$$

where d is the difference between the rankings of a given pair of scores and n is number of pairs.

**Example**
Two examiners x and y marked the scripts of ten candidates who sat a mathematics examination. The table below shows the examiners rankings of the candidates.



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| Examiner | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| x | 8 | 5 | 9 | 2 | 10 | 1 | 7 | 6 | 3 | 4 |
| y | 5 | 3 | 6 | 1 | 4 | 7 | 2 | 10 | 8 | 9 |

Calculate spearman's correlation coefficient (ρ)

**Solution:**

| | $Rx$ | $Ry$ | $D=Rx-Ry$ | $D^2$ |
|---|---|---|---|---|
| A | 8 | 5 | 3 | 9 |
| B | 5 | 3 | 2 | 4 |
| C | 9 | 6 | 3 | 9 |
| D | 2 | 1 | 1 | 1 |
| E | 10 | 4 | 6 | 36 |
| F | 1 | 7 | -6 | 36 |
| G | 7 | 2 | 5 | 25 |
| H | 6 | 10 | -4 | 16 |
| I | 3 | 8 | -5 | 25 |
| J | 4 | 9 | -5 | 25 |
| | | | | 186 |

But $\rho = 1-\frac{6\sum D^2}{n(n^2-1)}$

$\rho = 1- \frac{6x186}{990} = 1-\frac{1116}{990}$

$\rho = -0.13$

**Example**

Two examiners Y and Z each marked the scripts of eight candidates who sat a mathematics contest. The table below shows the examiner's marks of the candidates.

Calculate spearman's correlation coefficient and comment about your results.



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| Examiners | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| y | 72 | 60 | 56 | 76 | 68 | 52 | 80 | 64 |
| z | 56 | 44 | 60 | 74 | 66 | 38 | 68 | 52 |

**Solution:**

| | $Rx$ | $Ry$ | $D=Rx-Ry$ | $D^2$ |
|---|---|---|---|---|
| A | 3 | 5 | -2 | 4 |
| B | 6 | 7 | -1 | 1 |
| C | 7 | 4 | 3 | 9 |
| D | 2 | 1 | 1 | 1 |
| E | 4 | 3 | 1 | 1 |
| F | 8 | 8 | 0 | 0 |
| G | 1 | 2 | -1 | 1 |
| H | 5 | 6 | -1 | 1 |

$$\sum D^2 = 18$$

But $\rho = 1 - \frac{6\sum D^2}{n(n^2-1)}$

$$\rho = 1 - \frac{6x18}{8(8^2-1)} = \frac{11}{14}$$

$\therefore \rho = 0.7857$

Since $\rho = 0.7857$ shows that there is a high positive linear correlation between the two examiners x and y.

**Example**

The following table shows the marks of eight students in biology and chemistry. Rank the results and find the value of spearman's coefficient of rank correlation. Test for the significance of this at 5% level.



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| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Biology x | 65 | 65 | 70 | 75 | 75 | 80 | 85 | 85 |
| Chemistry y | 50 | 55 | 58 | 55 | 65 | 58 | 61 | 65 |

**Solution:**

| $x$ | $y$ | $Rx$ | $Ry$ | $D=Rx-Ry$ | $D^2$ |
|---|---|---|---|---|---|
| 65 | 50 | 7.5 | 8 | -0.5 | 0.25 |
| 65 | 55 | 7.5 | 6.5 | 1 | 1 |
| 70 | 58 | 6 | 4.5 | -1.5 | 2.25 |
| 75 | 55 | 4.5 | 6.5 | -2 | 4 |
| 75 | 65 | 4.5 | 1.5 | 3 | 9 |
| 80 | 58 | 3 | 4.5 | -1.5 | 2.25 |
| 85 | 61 | 1.5 | 3 | -1.5 | 2.25 |
| 85 | 65 | 1.5 | 1.5 | 0 | 0 |
| | | | | | $\sum D^2 = 21$ |

But $\rho = 1 - \dfrac{6\sum D^2}{n(n^2-1)}$

$$\rho = 1 - \frac{6x21}{8(8^2-1)}$$

$$\rho = 1 - \frac{126}{504}$$

$$\therefore \rho = 0.75$$

From the table, for eight pairs at 5% level of significance $\rho = 0.71$ since from above $\rho = 0.75$ which is greater than 0.71
$\Rightarrow$ it is significant at 5% level.



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## 8.7 KENDALL'S RANK CORRELATION COEFFICIENT ($\tau$)

Kendall's rank correlation coefficient is given by

$$\tau = \frac{agreements - disagreements}{total\ number\ of\ pairs} = \frac{S}{\frac{1}{2}n(n-1)}$$

Where S is the total sum of all scores. For set of n objects the total number of pairs is $\frac{1}{2}n(n-1)$

**Obtaining scores.**

1. The data is arranged in two rows, the first row should be in descending order.
2. Arrange the data in the second row in accordance with that of the first row.
3. Obtain ranks for each of the rows.
4. Compare each of the score in the second row with the rest of the members starting from the first member. A score of + 1 is allotted to a pair of objects in the right order and – 1 for a pair not in the right order.

**The total sum of scores given (S).**

**Example**

Two examiners Y and Z each marked the scripts of ten candidates who sat a mathematics examination. The table below shows the examiners ranking of the candidates.

| Examiner | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| Y | 5 | 3 | 6 | 1 | 4 | 7 | 2 | 10 | 8 | 9 |
| Z | 6 | 3 | 7 | 2 | 5 | 4 | 1 | 10 | 9 | 8 |

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Calculate Kendell's correlation coefficient for the two examiners.

**Solution**

| Examiner | D | G | B | E | A | C | F | I | J | H |
|---|---|---|---|---|---|---|---|---|---|---|
| Ry | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| RZ | 2 | 1 | 3 | 5 | 6 | 7 | 4 | 9 | 8 | 10 |

| DG | DB | DE | DA | DC | DF | DI | DJ | DH | Score |
|---|---|---|---|---|---|---|---|---|---|
| -1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 7 |
| | GB | GE | GA | GC | GF | GI | GJ | GH | |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 8 |
| | | BE | BA | BC | BF | B1 | BJ | BH | |
| | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 7 |
| | | | EA | EC | EF | EI | EJ | EH | |
| | | | 1 | 1 | -1 | 1 | 1 | 1 | 4 |
| | | | | AC | AF | A1 | AJ | AH | |
| | | | | 1 | -1 | 1 | 1 | 1 | 3 |
| | | | | | CF | C1 | CJ | CH | |
| | | | | | -1 | 1 | 1 | 1 | 2 |
| | | | | | | F1 | FJ | FH | |
| | | | | | | 1 | 1 | 1 | 3 |
| | | | | | | | IJ | IH | |
| | | | | | | | -1 | 1 | 10 |
| | | | | | | | | JH | |
| | | | | | | | | 1 | 1 |

Total score is 35 $\therefore S = 35$

$$\tau = \frac{2S}{n(n-1)} = \frac{35 \times 2}{10 \times 9} = \frac{35}{45}$$

$$\tau = 0.78$$

$$1 - \frac{\Sigma d^2}{n(n^2-1)}$$



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**Example**

In many Government Institutions, officers complain about typing errors. A test was designed to investigate the relationship between typing speed and errors made.

Twelve typists A, B, C, D ...L. were picked at random to type the same text. The table below shows the rankings of the typists according to speed and errors made. [N.B lowest ranking in errors implies least errors made]

| Typist | A | B | C | D | E | F | G | H | I | J | K | L |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Speed | 3 | 4 | 2 | 1 | 8 | 11 | 10 | 6 | 7 | 12 | 5 | 9 |
| Error | 2 | 6 | 5 | 1 | 10 | 9 | 8 | 3 | 4 | 12 | 7 | 11 |

Calculate the rank correlation coefficient (Kendall's). Test the assertion made by the officers and comment on your result.

| | D | C | A | B | K | H | I | E | L | G | F | J |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Rx | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| Ry | 1 | 5 | 2 | 6 | 7 | 3 | 4 | 10 | 11 | 8 | 9 | 12 |
| P | 11 | 7 | 9 | 6 | 5 | 6 | 5 | 2 | 1 | 2 | 1 | =55 |
| Q | 0 | 3 | 0 | 2 | 2 | 0 | 0 | 2 | 2 | 0 | 0 | =11 |

S = Agreements (P) – Disagreements (Q)
= 55 – 11 = 44

$$\tau = \frac{2S}{n(n-1)} = \frac{88}{12\times 11} = \frac{88}{132}$$

$\tau = 0.67$

From the table for 12 pairs of observation at 1% level of significance, $\tau = 0.58$



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But $0.67 > 0.58$

$\Rightarrow$ Typing speed causes errors at 1% level of significance based on twelve observations.

**Example**

The following are the final examination scores, which the 12 students obtained in Mathematics (X) and Economics (Y)

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| X | 35 | 56 | 65 | 78 | 49 | 82 | 22 | 90 | 77 | 53 | 52 | 93 |
| Y | 57 | 72 | 63 | 76 | 53 | 100 | 38 | 82 | 82 | 19 | 43 | 79 |

**Solution**

| *Rx* | *Ry* | *P* | *Q* |
|---|---|---|---|
| 1 | 4 | 8 | 3 |
| 2 | 2.5 | 8 | 1 |
| 3 | 1 | 9 | 0 |
| 4 | 5 | 7 | 1 |
| 5 | 2.5 | 7 | 0 |
| 6 | 7 | 5 | 1 |
| 7 | 6 | 5 | 0 |
| 8 | 12 | 0 | 4 |
| 9 | 10 | 1 | 2 |
| 10 | 9 | 1 | 1 |
| 11 | 8 | 1 | 0 |
| 12 | 11 | | |
| Total | | 52 | 13 |

$S = 52 - 13 = 39$



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$$\tau = \frac{39x2}{12x11}$$

$$\tau = 0.591$$

**Note:** In case of ties in the first row, the best alternative would be to use Spearman's method since the questions are normally open.

Otherwise if you have the ties in first row, the student with a better mark in the second paper is ranked first. Also when you are comparing the second row make sure you put into consideration the firs row because of the ties.

**Exercise**

1.Three persons P, Q and R were asked in order of importance nine features of a house (A, B, C ...I). Calculate spearman's rank correlation coefficients between the pairs of preferences as shown in the following table.

| | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| P | 1 | 2 | 4 | 8 | 9 | 7 | 6 | 3 | 5 |
| Q | 1 | 4 | 5 | 8 | 7 | 9 | 2 | 3 | 6 |
| R | 1 | 9 | 6 | 8 | 7 | 4 | 2 | 3 | 5 |

How far does this help to decide which pair from the three would be most likely to be able to compromise on a suitable house.

**Answer**: P, Q: 0.75 Q, R: 0.567 P, R: 0.317

2. The price (p) of matooke is found to depend on the distance (d) the market is away from the nearest town. The table below gives the average price of matooke for markets around Kampala City.

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| d(km) | 40 | 8 | 17 | 20 | 24 | 30 | 10 | 28 | 16 | 28 |
|---|---|---|---|---|---|---|---|---|---|---|
| p(sh) | 120 | 160 | 140 | 130 | 135 | 125 | 150 | 130 | 145 | 125 |

(i) Plot this data on a scatter diagram.

(i) Draw the line of best fit on your diagram.

(iii) Find the equation of your line in form of

$$P = \alpha + \beta d$$

where $\alpha$ and $\beta$ are constants.

Hence estimate the price of matooke when $d = 5$

**Answer:** (iii) $p = 161.4 - 1.111d$,

When $d = 5$ Implies $p = 156$ shs.

3. The following table gives the order in which six candidates were ranked in two tests x and y

| x: | E | C | B | F | D | A |
|---|---|---|---|---|---|---|
| y: | F | A | D | E | C | C |

Calculate the coefficient of rank correlation(Kendall's) and comment on your results.

**Answer:** $\tau = 0.4$

4. In a certain commercial institution, a speed and error typing examination was administered to 12 randomly selected candidates A, B, C, … L of the institution. The table below shows their speeds (y) in seconds and the number of errors in their typed script (x).

| | A | B | C | D | E | F | G | H | I | J | K | L |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| x | 12 | 24 | 20 | 10 | 32 | 30 | 28 | 15 | 18 | 40 | 27 | 35 |
| y | 130 | 136 | 124 | 120 | 153 | 160 | 155 | 142 | 145 | 172 | 140 | 157 |



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(i) Plot the data on a scatter diagram.

(ii) Draw the line of best fit on your diagram and comment on the likely association between speed and the errors made.

(iii) determine the equation of your line in the form $y = xk + b$ where k and b are constants.

(iv) By giving rank 1 to the fastest student and the student with the fewest errors, rank the above data and use it to calculate the rank correlation coefficient. Comment on your results.

**Answer:** (iii) $y = 1.42x + 110.1$

(iv) $\rho = 0.84$, $\tau_0 = 0.7$

5. The following table shows the marks of 8 students in two different subjects

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Maths | 65 | 60 | 70 | 75 | 75 | 80 | 85 | 50 |
| Geography | 50 | 55 | 54 | 60 | 65 | 58 | 65 | 61 |

Calculate the rank correlation coefficient

**Answer** $\tau = \frac{2}{7}$, $\rho = 0.435$

6. 8 students took an exam in Math and Physics an their grades were as follows:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Math | A | B | A | D | O | D | B | F |
| Physics | C | B | C | E | E | C | A | F |

Calculate Spearman's rank correlation coefficient

**Answer**: $\rho = 0.7083$



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# CHAPTER NINE

## ERRORS

### 9.1 INTRODUCTION

Errors arise from the many calculations and approximations used when working out numbers.
Suppose a calculation involved the fraction $\frac{1}{3}$, then using 0.33 ( to two decimal places) causes a round-off error.

### 9.2 TYPES OF ERRORS

**Random errors**
These type of errors that occur due machine or human failure and therefore they can not be treated numerically.
Suppose a student is given 54 and uses 45 instead.

**Rounding errors**
Some numbers are normally corrected to a number of decimal places or significant figures. An error usually arises because most values used in the computations are normally approximations of the true values and such are referred to as rounding errors.

**Example**
Round off 3.896234 to four decimal places

$3.896234 = 3.8962$ ( to four decimal places)

Round off $\frac{2}{3}$ to three decimal places

$\frac{2}{3} = 0.66666666666666 = 0.667$ ( 3 decimal places)



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Round off 0.00652673 to three significant figures

0.00652673 = 0.00653 ( 3 significant figures)

Round off 7.00214 to four significant figures

7.00214 = 7. 002 (to 4 significant figures)

Round 5,415,000 to 3 significant figures

5,415,000 = 5,420,000 ( 3 significant figures)

**Note that rounding is done in a single step.**

**Truncation errors**

Occurs when an infinite process is terminated at a some point.

**Example**

Truncate $\frac{2}{3}$ to three significant figures

$\frac{2}{3} = 0.666666666666 = 0.666$ ( 3 s.f's)

Sometimes infinite series are terminated at some point such that higher terms are neglected since these values that are likely to be too small to affect the result.

**Example**

Find the approximate value of $e^{0.6}$ to 2 decimal places

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$e^{0.6} = 1 + 0.6 + \frac{0.6^2}{2!} + \frac{0.6^3}{3!} + . \frac{0.6^4}{4!} + \ldots \quad = 1.822$$

$= 1.82$(2.d.p's)



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## COMMON TREMS USED

Include error, absolute error, relative error and percentage error.

**Error**. If x represents an approximation to the value of X and $\partial x$ is the error in this approximation.

Then $\partial x$ = X- x = True value - Approimate value

**Absolute error** is the actual size of error disregarding the sign.

Therefore absolute error $|\partial x| = |X - x| = e$ which is at times used. So absolute error only takes positive values only.

Therefore

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0 \end{cases}$$

Example

Round off 32.52632 to 2 decimal places and determine the bsolute error.

**Solution**

32.52632 = 32.53 ( to 2 decimal places)

So X = 32.52632 and x = 32.53

Error = X- x = 32.52632 – 32.53 = -0.00368

Therefore the absolute error

$= |-0.00368| = 0.00368$

**Relative error** = $\dfrac{\text{Absolute error}}{\text{Exact value or Approximate value}}$

The relative error gives an indication of whether the error is relatively large or small.



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For example if in measuring a distance of 50 metres an error of 1 cm is introduced, then relative error is

$$\frac{1}{5000} = 0.0002$$

However if an error of 1 cm is made in 1 metre

Then $\frac{1}{100} = 0.01$

Relative error = $\left|\frac{\partial x}{X}\right|$ or $\left|\frac{\partial x}{x}\right|$

**Percentage error or percentage relative error**

The relative error is expressed as a percentage i.e multiplied by 100.

So percentage error = Relative error x 100

$$\left|\frac{\partial x}{X}\right| \text{ x } 100$$

**Example**

Using your calculator, state as accurately as possible the value of $\sqrt{3}$. Find the percentage error involved in approximating this value to two decimal places.

**Solution**

$\sqrt{3} = 1.732050808$ which is 1.73 to two decimal places

Absolute error = $|1.732050808 - 1.73|$

$= 0.002050808$

Percentage error = $\left|\frac{0.002050808}{1.732050808}\right| \times 100\% = 0.118$

Note that the percentage (%) sign can be ignored.



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## Maximum possible error(s)

This depend on the number of decimal places or significant figures used in the computation. If a number is corrected to n decimal places, then the maximum possible error in it becomes $0.5 \times 10^{-n}$
If the maximum and minimum possible errors are known then the

$$\text{absolute error} = \frac{\text{max } value - \text{min } value}{2}$$

## 9.3 Limits of accuracy

We can state the upper and lower limit of given figures
**Example** If a student weighs 70 kgs.
Then this value can range from 69.5 kg to 70.5 kg

**Example**
If X is given to a stated level of accuracy. State the lower and upper bounds of X.
(a) 7.47
(b) 0.184
(c ) 7.0

**Solution**
(a) error = $0.5 \times 10^{-2} = 0.005$
Lower bound = 7.47 – 0.005 = 7.465
Upper bound = 7.47 + 0.005 = 7. 475

(b) error = $0.5 \times 10^{-3} = 0.0005$
Lower bound = 0.184 – 0.0005 = 0.1835
Upper bound = 0.184 + 0.0005 = 0.1845

(c ) error = $0.5 \times 10^{-1} = 0.05$



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Lower bound = 7.0 – 0.05 = 6.95
Upper bound = 7.0 + 0.05 = 7.05

**Example**

The sides of a rectangle are measured as 5.24 m and 6.38 m, to the nearest centimeter.
(a) Calculate minimum value of the perimeter.
(b) Calculate the maximum value of the area
(c) Calculate the minimum value of the area hence determine the absolute error

**Solution**

W = 5.24, then it ranges from 5.235 to 5.245
L = 6.38, then it ranges from 6.375 to 6.385

Perimeter = 2( L + W)

Minimum perimeter = 2 ( 5.235 + 6.375)
= 23.22 m

Maximum area = 5.245 x 6.385
= 33.489325 $m^2$

(c) Minimum area = 5.235 x 6.375
= 33.373125

$$\text{absolute error} = \frac{\max value - \min value}{2}$$

$$= \frac{33.489325 - 33.373125}{2}$$

= 0.0581



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**Example**

If $x = 5.356$ and $y = 6.81$, both numbers are rounded. Find the maximum and minimum value of

(a) $y - x$

(b) $\frac{y}{x}$

**Solution**

y ranges from 6.805 to 6.815

Then x ranges from 5.3555 to 5.3565

Maximum $(y - x) = 6.815 - 5.3555$
$= 1.4595$

Minimum $(y - x) = 6.805 - 5.3565$
$= 1.4485$

(b) Maximum $\frac{y}{x} = \frac{\max y}{\min x} = \frac{6.815}{5.3555}$

$= 1.272524$

Minimum $\frac{y}{x} = \frac{\min y}{\max x} = \frac{6.805}{5.3565}$

$= 1.270419$

**Example**

Given that $N = \frac{12.4}{4.20} - \frac{10.80}{6.124}$

(a) Write the possible error in each of the values given

(b) Estimate the range of values within which N Lies .Hence determine the absolute error in N

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**Solution**

Error in 12.4 is 0.05

Error in 4.20 is 0.005

Error in 10.80 is 0.005

Error in 6.124 is 0.0005

(b) Maximum value of N = $\frac{12.45}{4.195} - \frac{10.795}{6.1245} = 1.20523$

Minimum value of N = $\frac{12.35}{4.205} - \frac{10.805}{6.1235} = 1.17247$

Range of value is [1.17247, 1.20523]

$$\text{absolute error} = \frac{\max value - \min value}{2}$$

$$= \frac{1.20523 - 1.17247}{2}$$

$$= 0.01638$$

**Example**

Obtain the range of values within which the exact value of $2.7654 + \frac{3.8006x15.178}{0.9876}$ lies

**Solution**

Max value = $2.76545 + \frac{3.80065}{0.98755} x 15.1785 = 61.18089$

Min value = $2.7635 + \frac{3.80055}{0.98765} x 15.1775 = 61.16949$

Range of values is [61.16949, 61.18089]



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**Example**

A trader in tea and coffee makes an annual profit in tea of Shs 1080 million with a margin error of $\pm 10\%$ and an annual loss in coffee of Shs 560 million with a margin of error of $\pm 5\%$.

(i) Find the range of values corresponding to his gross income.

(ii) Given that his annual income tax is Shs 75 million, express this as a percentage of his gross income giving your answer as a range of values.

**Solution**

Max profit from tea = 1080 x 110% = 1188 m

Min profit from tea = 1080 x 90% = 972 m

Max loss from coffee = 560 x 105% = 588 m

Min loss from coffee = 560 x 95% = 532 m

Range of values for the gross income

Max income = max profit – min loss

= 1188 – 532 = 656 millions

Min income = min profit – max loss

= 972 – 588 = 384 millions

Range [384, 656]

(ii) lower limit $= \dfrac{75x100}{656} = 11.43\%$

Upper limit $= \dfrac{75x100}{384} = 19.53\%$

**Example**

A value of L = 200.27m was obtained when measuring the length of football pitch. Given that the relative error in this value was 0.08%, find the limits within which the value of L lie.



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**Solution**

L = 200.27

$$\frac{\partial L x 100}{L} = 0.08 \quad \text{so} \quad |\partial L| = \frac{200.27 x 0.08}{100} = 0.160$$

Upper limit = 200.27 + 0.160 = 200.43
Lower limit = 200.27 – 0.160 = 200.11

**Triangular inequality**

It states that $|a \pm b| \leq |a| + |b|$ . It is important when deducing maximum possible error.

**Example**

If a = 3 and b = 7 then

$$|3+7| \leq |3| + |7|$$

If a =7 and b = -4

$$|7-4| \leq |7| + |-4|$$

$$3 \quad \leq \quad 11$$

**9.4 Deriving formular for error propagation**

**ADDITION**

If x and y are approximation to X and Y,

and $\partial x$ and $\partial y$ be error respectively.

then true value = X + Y

Working value x + y = z

So X + Y = $(x + \partial x) + (y + \partial y)$

$$z + \partial z = (x + \partial x) + (y + \partial y)$$

$$\partial z = \partial x + \partial y$$



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Maximum possible absolute error in addition

$$|\partial x + \partial y| \leq |\partial x| + |\partial y|$$

from triangular inequality

Maximum possible error in addition $= |\partial x| + |\partial y|$

Maximum relative error in addition

$$\frac{|\partial x| + |\partial y|}{x + y}$$

**Percentage error**

$$\left(\frac{|\partial x| + |\partial y|}{x + y}\right) \text{x } 100$$

**Example**

If x and y are numbers rounded to 2 and 3 decimal places respectively, Calculate the max possible error in x + y if x = 2.84 and y = 6.364

**Solution**

$\partial x = 0.005$

$\partial y = 0.0005$

$e_{x+y} = 0.005 + 0.0005 = 0.0055$

Maximum possible error = 0.0055

**SUBTRACTION**

If x and y are approximation to X and Y,

And $\partial x$ and $\partial y$ be error respectively.

Then true value = X - Y

Working value x - y = z

So X - Y = $(x + \partial x) - (y + \partial y) = \partial x - \partial y$



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Max possible error $|\partial x - \partial y| \leq |\partial x| + |-\partial y| = |\partial x| + |\partial y|$

**Example**

If x = 6.364 and y = 2.84

Determine lower and upper limit in (x-y)

Max possible error

$$\partial x = 0.005$$

$$\partial y = 0.0005$$

$$e_{x-y} = 0.005 + 0.0005 = 0.0055$$

Lower limit = (6.364-2.84)-0.0055 = 3.5185

Upper limit = (6.364-2.84) +0.0055 = 3.5295

**MULTIPLICATION**

If x and y are approximation to X and Y,

And $\partial x$ and $\partial y$ be errors respectively.

True value = XY

z = xy

$$z + \partial z = (x + \partial x)(y + \partial y)$$

$$z + \partial z = xy + x\partial y + y\partial x + \partial x\partial y$$

*dx is too small and dy is too small*

Implies $\partial x\partial y \cong 0$

$$z + \partial z = xy + x\partial y + y\partial x$$

$$dz = x\partial y + y\partial x$$

$$|dz| = |x\partial y + y\partial x| \leq |x\partial y| + |y\partial x|$$

From the triangular inquality

Therefore maximum absolute error in the product is $|x\partial y| + |y\partial x|$



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Relative error

$$\frac{\partial z}{z} = \frac{x\partial y + y\partial x}{xy}$$

$$= \frac{\partial x}{x} + \frac{\partial y}{y}$$

Max possible relative error

$$\left|\frac{\partial x}{x} + \frac{\partial y}{y}\right| \leq \left|\frac{\partial x}{x}\right| + \left|\frac{\partial y}{y}\right|$$

Percentage error

$$\left( \left|\frac{\partial x}{x}\right| + \left|\frac{\partial y}{y}\right| \right) \text{x } 100$$

**Example**

The sides of a rectangle are measured as 5.24 m and 6.38 m, to the nearest centimeter.

(i) Determine the maximum possible area of the rectangle

(ii) Calculate the percentage error in measuring of the area

**Solution**

$$\partial w = 0.005$$

$$\partial l = 0.005$$

$$\partial lw = l|\partial w| + w|\partial l|$$

$= 6.38 \text{ x } 0.005 + 5.24 \text{ x } 0.005$

$= 0.0581$

Maximum possible area $= 5.24 \text{ x } 6.38 + 0.0581$

$= 33.4893$



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$$\text{Percentage error} = \left( \left|\frac{\partial w}{w}\right| + \left|\frac{\partial l}{l}\right| \right) \text{x } 100$$

$$\left( \left|\frac{0.005}{5.24}\right| + \left|\frac{0.005}{6.38}\right| \right) \text{x } 100 = 0.174$$

**Example**

Two positive numbers $y_1$ and $y_2$ are each rounded off to three decimal places to give $x_1$ and $x_2$ respectively. Find in terms of $x_1$ and $x_2$ the maximum relative error in using $x_1x_2$ as an approximation for $y_1y_2$.

**Solution**

$$\partial x_1 = 0.0005 \qquad y_1 = x_1 + 0.0005$$

$$\partial x_2 = 0.0005 \qquad y_2 = x_2 + 0.0005$$

$$y_1y_2 = x_1x_2 + e$$

$$= (x_1 + 0.0005)(x_2 + 0.0005)$$

$$x_1x_2 + 0.0005x_2 + 0.0005x_1 + 0.0005x0.0005$$

$0.0005$ *is too small*

$$0.0005x0.0005 \cong 0$$

$$e = 0.0005x_1 + 0.0005x_2$$

$$\text{Relative error} = \left| \frac{0.0005x_1 + 0.0005x_2}{x_1x_2} \right|$$



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Max relative error $\leq \left|\frac{0.0005}{x_1}\right| + \left|\frac{0.0005}{x_2}\right|$

Max relative error $= \left|\frac{0.0005}{x_1}\right| + \left|\frac{0.0005}{x_2}\right|$

Note if m = xyz

Then the absolute error in the product

$$|\partial m| = yz|\partial x| + xz|\partial y| + yx|\partial z|$$

The max relative error

$$\left|\frac{\partial m}{m}\right| \leq \left|\frac{\partial x}{x}\right| + \left|\frac{\partial y}{y}\right| + \left|\frac{\partial z}{z}\right|$$

$$= \left|\frac{\partial x}{x}\right| + \left|\frac{\partial y}{y}\right| + \left|\frac{\partial z}{z}\right|$$

**DIVISION**

If x and y are approximation to X and Y,

And $\partial x$ and $\partial y$ be errors respectively

True value $= \frac{X}{Y}$ but when z $= \frac{x}{y}$ is used

$$z + \partial z = \frac{x + \partial x}{y + \partial y}$$

$$= \frac{x + \partial x}{y + \partial y} \quad \frac{(y - \partial y)}{(y - \partial y)}$$

$$z + \partial z = \frac{xy - x\partial y + y\partial x - \partial x\partial y}{y^2 - y\partial y + y\partial y - (\partial y)^2}$$



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$$z+\partial z=\frac{xy-x\partial y+y\partial x-\partial x\partial y}{y^2-(\partial y)^2}$$

*dx is too small and dy is too small*

Implies $\partial x\partial y \cong 0, (\partial y)^2 \cong 0$

$$z+\partial z=\frac{xy-x\partial y+y\partial x}{y^2}$$

$$\partial z=\frac{\partial x}{y}-\frac{x\partial y}{y^2}$$

Max absolute error $= |\partial z| = \dfrac{y|\partial x|+x|\partial y|}{y^2}$

**Relative error in division**

$$\frac{\partial z}{z}=\frac{\partial x}{x}-\frac{\partial y}{y}$$

**Max possible relative error in division**

$$\left|\frac{\partial x}{x}-\frac{\partial y}{y}\right| \le \left|\frac{\partial x}{x}\right|+\left|\frac{-\partial y}{y}\right|$$

$$=\left|\frac{\partial x}{x}\right|+\left|\frac{\partial y}{y}\right|$$

**Example**

Given the numbers a = 23.037 and b = 8.4658,
measured to their nearest number of decimal places indicated,

(i) State the maximum possible errors in a and b.
(ii) determine the absolute error in $\dfrac{a}{b}$



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(iii) find the limits within which $\frac{a}{b}$ lies, correct to 4 decimal places

**Solution**

(i) $\partial a = 0.0005$

$\partial b = 0.00005$

(ii) $\partial(\frac{a}{b}) = \frac{b|\partial a| + a|\partial b|}{b^2}$

$= \frac{8.4658|0.0005| + 23.037|0.00005|}{8.4658^2}$

$= 0.0000751$

(iii) lower limit $= \frac{23.037}{8.4658} - 0.0000751 = 2.7211$

Upper limit $= \frac{23.037}{8.4658} + 0.0000751 = 2.7213$

**Example**

Given that $Y_1$ and $Y_2$ are approximations to $X_1$ and $X_2$ with errors $E_1$ and $E_2$ respectively, show that the maximum possible relative error in $\frac{X_1}{X_2}$ is $\left|\frac{E_1}{Y_1}\right| + \left|\frac{E_2}{Y_2}\right|$



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**Solution**

$$X_1 = Y_1 + E_1, \quad X_2 = Y_2 + E_2$$

$$= \frac{X_1}{X_2} = \frac{Y_1 + E_1}{Y_2 + E_2}$$

$$= \frac{(Y_1 + E_1)(Y_2 - E_2)}{(Y_2 + E_2)(Y_2 - E_2)}$$

$$= \frac{Y_1Y_2 + Y_2E_1 - Y_1E_2 - E_1E_2}{Y_2^{\,2} - E_2^{\,2}}$$

Assumptions

$E_1$ is too small, $E_2$ is too small

$$E_1E_2 \cong 0, \ (E_2)^2 \cong 0$$

$$\frac{Y_1}{Y_2} + E = \frac{Y_1}{Y_2} + \frac{E_1}{Y_2} - \frac{Y_1E_2}{Y_2^{\,2}}$$

$$\mathrm{E} = \frac{E_1}{Y_2} - \frac{Y_1E_2}{Y_2^{\,2}}$$

$$\text{Relative error} = \left(\frac{E_1}{Y_2} - \frac{Y_1E_2}{Y_2^{\,2}}\right) x \frac{Y_2}{Y_1}$$

$$= \frac{E_1}{Y_1} - \frac{E_2}{Y_2}$$

$$= \left|\frac{E_1}{Y_1} - \frac{E_2}{Y_2}\right| \le \left|\frac{E_1}{Y_1}\right| + \left|\frac{-E_2}{Y_2}\right|$$



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$$\text{Max relative error} = \left|\frac{E_1}{Y_1}\right| + \left|\frac{E_2}{Y_2}\right|$$

**NOTE**: The maximum possible relative error in multiplication and division are the same

**ERRORS IN FUNCTIONS** maclaurin's theorem

Let z = f(x)

$$z + \partial z = f(x + \partial x)$$

$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots$

$$f(x + \partial x) = f(x) + f'(x)\partial x + \frac{f''(x)(\partial x)^2}{2!} + \ldots$$

*dx is too small*

$$(\partial x)^2 \cong 0$$

$$\partial z = f'(x)\partial x$$

$$|\partial z| = |f'(x)||\partial x|$$

$$\text{Relative error} = \frac{|\partial z|}{z} = \left|\frac{f'(x)\partial x}{f(x)}\right|$$

**Example.**

If $z = x^5$. Find expressions for the absolute error and maximum relative error in z.

**Solution**

$z = f(x) = x^5$

$$f'(x) = 5x^4$$

$$\partial z = 5x^4 \partial x$$

$$|\partial z| = 5x^4 |\partial x|$$



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$$R.E = \left|\frac{\partial z}{z}\right| = \left|\frac{5x^4 \partial x}{x^5}\right|$$

$$= 5\left|\frac{\partial x}{x}\right|$$

**Example**

If z = sinx. Determine the expression for the absolute error and maximum relative error.

**Solution**

f(x) = sinx

$$f'(x) = \cos x$$

$$|\partial z| = |\cos x||\partial x|$$

$$R.E = \left|\frac{\cos x}{\sin x}\right||\partial x|$$

$$= |\cot x||\partial x|$$

**Example**

Given that the error in measuring an angle is $0.5^0$. Find the maximum possible error in $\frac{\sin x}{\cos x}$ If $x = 30^0$.

**Solution**

$$\text{Max relative error} = \left|\frac{E_1}{Y_1}\right| + \left|\frac{E_2}{Y_2}\right|$$



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Let $Y_1 = \sin x$ and $Y_2 = \cos x$

$$E_1 = |\cos x \partial x|$$

$$E_2 = |-\sin x \partial x| = |\sin x \partial x|$$

Note: $E = f'(x)\partial x$

$$R.E = \left|\frac{\cos x \partial x}{\sin x}\right| + \left|\frac{\sin x \partial x}{\cos x}\right|$$

$$= |\cot x|\partial x|| + |\tan x||\partial x|$$

$$\partial x = 0.5^0 = \frac{\pi}{360}$$

$$R.E = |\cot 30|\left|\frac{\pi}{360}\right| + |\tan 30|\left|\frac{\pi}{360}\right|$$

$$= 0.015115 + 0.00504$$
$$= 0.02015$$

Percentage error $= 0.02015 \times 100$
$= 2.015$

**Example**

If m, y and z are 3 numbers which have been rounded.
Evaluate $\frac{m^2 + y}{y + z}$ with its error bounds
if $m = 2.12$, $y = 3.8$ and $z = 0.31$



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**Solution**

$$\frac{m^2+y}{y+z}=\frac{2.12^2+3.8}{3.8+0.31}$$

Error in numerator =

$$|f'(m)\partial m|+|\partial y|=2m\partial m+|\partial y|$$

$$=2 \text{ x } 2.12 \text{ x } 0.005+0.05=0.0712$$

Error in denominator = $\frac{p}{n}=\frac{m^2+y}{y+z}=\frac{8.2944}{4.11}=2.0181$

$$|\partial y|+|\partial z| = 0.05+0.005=0.055$$

If we let u= $\partial u=\left|\frac{\partial p}{p}\right|+\left|\frac{\partial n}{n}\right|=(\left|\frac{0.0712}{8.2944}\right|+\left|\frac{0.055}{4.11}\right|)x2.0181$

$$=0.0443$$

Range $2.0181\pm0.0443$

[1.9738, 2.0624]



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**Exercise**

1. The numbers A and B are approximated by the numbers X and Y respectively such that A = X –a, B = Y-b, where a, b are small numbers compared to A and B. Given that Y= f(X) and B = f(A). Show that $|b| = |a||f'(A)|$. If $f(A) = A^p$ where p is a constant deduce that $|b| = |a|pA^{p-1}$ and find the expression for the relative error.

**Answer** $\dfrac{|ap|}{|A|}$

2. The numbers 2.6754, 4.8006, 15.175 and 0.92 have been rounded off correct to the given number of decimal places. Find the range of values within which the exact value of

$2.6754(4.8006 - \dfrac{15.175}{0.92})$ can be expected to lie.

**Answer** [-31.529, -31.045]

3. Given the numbers; x = 2.678 and y = 0.8765 measured to the nearest number of decimal places indicated,

(a) state the maximum possible errors in x and y

(b) determine the absolute error in xy

(c) find the limits within which the product xy lies, correct to 4 decimal places

**Answer** (a) $\partial x = 0.0005$ $\partial y = 0.00005$

(b) $|\partial xy| = 0.000572$

(c) Lower limit = 2.3467, Upper limit = 2.3478



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4. The radius of a circle is measured as 5.34 m to the nearest cm. Calculate the lower bound of its area, correct to three significant figures.

**Answer**: 89.4 $m^2$

5. Two decimal number x and y are rounded to give X and Y with errors $E_1$ and $E_2$ respectively. Show that the maximum relative error made in approximating $x^2y$ by $X^2Y$ is given by

$$2\left|\frac{E_1}{X}\right| + \left|\frac{E_2}{Y}\right|$$

6. Determine the maximum absolute error in $\frac{\sqrt{z}}{x^2y^3}$,

given that x = 2.4, y = 5.4 and z = 1.8 all numbers rounded

**Answer** 0.000123

7. If the error in each of the values of $e^x$ and $e^{-x}$ is $\pm$ 0.0005,

find the maximum and minimum value of the quotient, $\frac{e^x}{e^{-x}}$

when x = 0.004, giving your nswer correct to 3 decimal places.

**Answer**: Max = 1.084 and Min = 1.082

8. The variable v, r and h are related by the formula

$$v = \frac{r^2}{3h} + 5$$

In an experiment, the value of r and h were found to be 4.1 and 40.8 respectively. Calculate the lower and upper limits of v, correct 3 decimal places.



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**Answer**

lower limit = 5.134, upper limit = 5.141

9. Given that Y and Z are measured with possible Errors $\Delta Y$ and $\Delta Z$ respectively, show that the relative error in the product YZ is

$$\frac{Z|\Delta Y| + Y|\Delta Z|}{YZ}$$

10. A company had a capital of sh. 500 million. The profit in a certain year was sh. 25.8 million in section A of the company and sh. 14.56 million in section B of the company. There was possible error of 5% in section A and an 8% error in section B. Find the maximum and minimum values of the total profit of the sections as a percentage of the capital.

**Answer**: Max = 8.56%, Min = 7.58%

11. Given that $A = |x||y| \sin\theta$

(a) Deduce that the maximum possible relative error in A is given by $\left|\frac{\partial x}{x}\right| + \left|\frac{\partial y}{y}\right| + \cot\theta|\partial\theta|$

Where $\partial x$, $\partial y$ and $\partial\theta$ are small numbers compared to x, y and $\theta$ respectively. Find the percentage error made in the area if |x| and |y| are measured with errors of $\pm$ 0.05 and angle with an error $\pm 0.5^0$, given that |x| = 2.5 cm, |y| = 3.4 cm and $\theta = 30^0$

**Answer** 4.98



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12. Two numbers A and B are approximated by a and b with errors x and y respectively. Show that maximum percentage error in approximating $\frac{A}{B}$ is given by $\left(\left|\frac{x}{a}\right| + \left|\frac{y}{b}\right|\right)x100\%$

Hence find the maximum percentage error in $\frac{3.25^2}{4.562}$ to 2 decimal places.

**Answer** 0.32

13. Given that a = 42.326, b = 27.26 and c = -12.93 are rounded off to the given decimal places, find the range within which the exact value of the expression, $\frac{A}{B+C}$ lies

Note, the numbers A and B are rounded off to a and b.

**Answer**

14. A cylindrical pipe has a radius of 2.5 cm measured to the nearest unit. If the relative absolute error made in calculating its volume is 0.125, find the absolute relative error made in measuring its height.

**Answer** 0.085



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# CHAPTER TEN

## LINEAR INTERPOLATION

### 10.1 Interpolation

Linear interpolation is a process where by the non-tabulated values of the function are estimated on the assumption that the function behaves sufficiently smooth between the tabular points.

**Example**

| x | 1 | 2 | 3 |
|---|---|---|---|
| f(x) | 2 | 8 | 11 |

Using interpolation, find f(x) when x = 1.15

**Solution**

| x | 1 | 1.15 | 2 |
|---|---|---|---|
| f(x) | 2 | f | 8 |

$$\frac{Change\ in\ Y}{Change\ in\ X} = \frac{8-2}{2-1} = \frac{f-2}{1.15-1}$$

$$f = 2 + 6x0.15 = 2.9$$

$$Therefore\ f(1.15) = 2.9$$

**Example**

| $X^0$ | $50.0^0$ | $50.2^0$ | $50.4^0$ | $50.6^0$ |
|---|---|---|---|---|
| Cos $X^0$ | 0.6428 | 0.6401 | 0.6374 | 0.6347 |

Find the cos $50.3^0$



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**Solution**

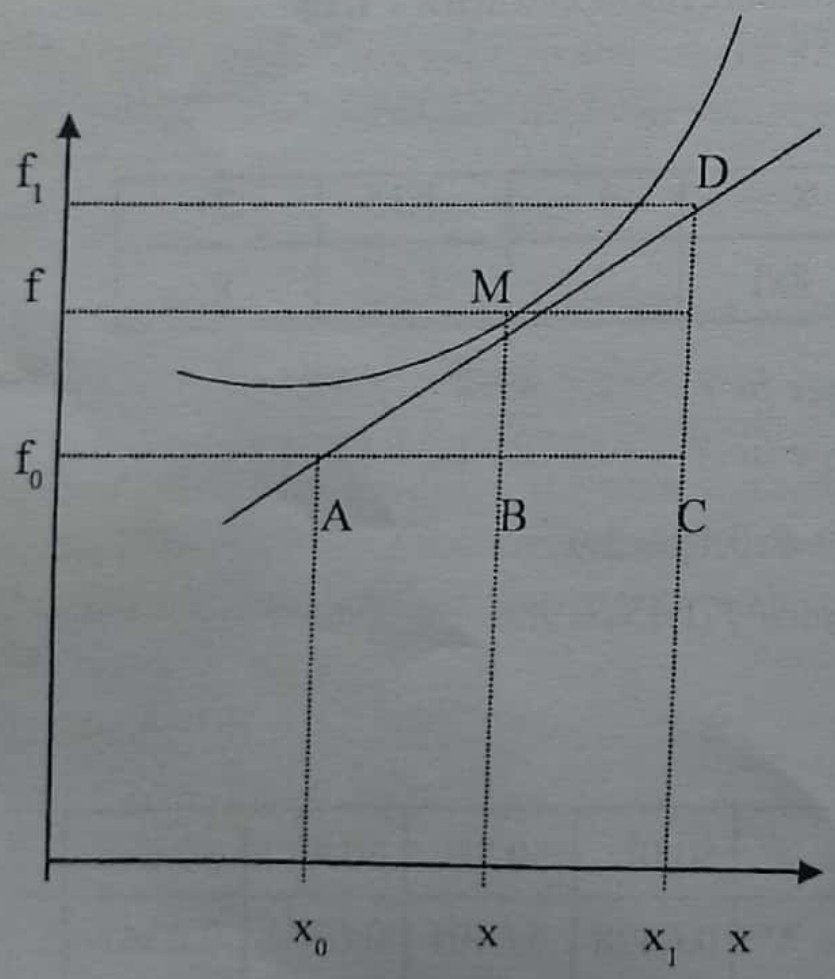
| $X^0$ | $50.2^0$ | $50.3^0$ | $50.4^0$ |
|---|---|---|---|
| $\cos X^0$ | 0.6401 | f | 0.6374 |

$$\frac{0.6374-0.6401}{50.4^0-50.2^0}=\frac{f-0.6401}{50.3^0-50.2^0}$$

$$f=0.6401-0.00135=0.63875$$

$$\therefore \cos 50.3^0=0.6388$$

## 10.2 Expression for linear interpolation




$f(x)=f$ $\quad f(x_0)=f_0$ $\quad f(x_1)=f_1$



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i) Identify the curve.

ii) Draw a straight line for the curve.

iii) Find the gradient of the straight line $= \frac{CD}{AC}$

$$= \frac{f_1 - f_0}{x_1 - x_0}$$

$$\text{Gradient AM} = \frac{BM}{AB} = \frac{f - f_0}{x - x_0}$$

$$f - f_0 = \frac{(x - x_0}{(x_1 - x_0)}(f_1 - f_0)$$

let $\frac{x - x_0}{x_1 - x_0} = \delta, let(f - f_0) = \Delta f_0$

$$\Rightarrow f = f_0 + \delta \Delta f_0$$

**Example**

Certain curve y = f(x) passes through the points (4, 1.88) and (5, 1.84). Find the value of f (4.2)

**Solution**

$$\frac{f_1 - f_0}{x_1 - x_0} = \frac{f - f_0}{x - x_0}$$

| X | 4 | 4.2 | 5 |
|---|---|---|---|
| f(x) | 1.88 | f | 1.84 |

$$\frac{1.84 - 1.88}{5 - 4} = \frac{f - 1.88}{4.2 - 4}$$

$-0.04 \times 0.2 = f - 1.88$

$1.88 - 0.008 = f$



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$f(4.2) = 1.872$

**Example**

Given the following values

| $X^0$ | $0.85^0$ | $0.86^0$ | $1.85^0$ | $1.86^0$ |
|---|---|---|---|---|
| $\sin x^0$ | 0.7513 | 0.7578 | 0.9612 | 0.9585 |

Use linear interpolation to estimate;

i) $\sin 0.857^0$

ii) $\sin 1.857^0$

**Solution**

$x = 0.857^0$

$$\frac{f_1 - f_0}{x_1 - x_0} = \frac{f - f_0}{x - x_0}$$

$$\frac{0.7578 - 0.7513}{0.86 - 0.85} = \frac{f - 0.7513}{0.857 - 0.85}$$

$$\frac{6.5x10^{-3}}{0.01} = \frac{f - 07513}{7 \times 10^{-3}}$$

$\Rightarrow 4.55 \times 10^{-3} + 7.513 \times 10^{-3} = 0.01f$

$f = 0.75585$

$\Rightarrow \sin 0.857 = 0.7559$

If $x = 1.857^0$

$$\frac{f_1 - f_0}{x_1 - x_0} = \frac{f - f_0}{x - x_0}$$



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$$\frac{0.9585-0.9612}{1.86-1.85}=\frac{f-0.9612}{1.857-1.85}$$

$$\frac{0.0027}{0.01}=\frac{f-0.9612}{7\times10^{-3}}$$

$$f=0.95931$$

$$\Rightarrow \sin 1.857=0.9593$$

**Example**

Use 4 figure tables and linear interpolation to evaluate

(a) $\sqrt{60.535}$

(b) sec $78.38^0$

**Solution**

(a)

$\sqrt{60.535}$

$\sqrt{60.535}=7.78, \quad \sqrt{60.54}=7.781$

| x | 60.53 | 60.535 | 60.54 |
|---|---|---|---|
| f(x) | 7.780 | f | 7.781 |

$$f=f_0+\frac{(x-x_0)}{(x_1-x_0)}(f_1-f_0)$$

$$=7.78+\left(\frac{60.535-60.53}{60.54-60.53}\right)(7.781-7.78)$$



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$$= 7.78 + \left(\frac{60.535 - 60.53}{60.54 - 60.53}\right)(7.781 - 7.78)$$

$$= \quad 7.78 + \frac{5\times10^{-3}}{0.01} 1\times10^{-3}$$

$$= 7.7805$$

(b) Sec $78.38^0$

| x | 78.3 | 78.38 | 78.4 |
|---|---|---|---|
| f(x) | 4.9313 | f | 4.9732 |

$$f = f_0 + \frac{(x - x_0)}{x_1 - x_0}(f_1 - f_0)$$

$$\Rightarrow 4.9313 + ) \left(\frac{78.38 - 78.3}{78.4 - 78.3}\right)(4.9732 - 4.9313$$

$$4.9313 + \left(\frac{0.08}{0.01}\right)(0.0419)$$

$$4.9313 + 0.03352 = 4.96482$$

sec $78.38^0 = 4.9648$

**10.3 Inverse interpolation**

From $(f - f_0) = \frac{(f_1 - f_0}{(x_1 - x_0}(x - x_0)$

$$(x - x_0) = ) \frac{f - f_0}{f_1 - f_0}(x_1 - x_0$$

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$$x = x_0 + \frac{f - f_0}{f_1 - f_0}(x_1 - x_0)$$

$$\left(\frac{x_1 - x_0}{f_1 - f_0}\right) = \left(\frac{x - x_0}{f - f_0}\right)$$

**Example**

In an experiment the following obscrvations were noted:

| T | 0 | 12 | 20 | 30 |
|---|---|---|---|---|
| σ | 6.6 | 2.9 | -0.1 | -2.9 |

Use interpolation to find T when σ = -1

$$\frac{x_1 - x_0}{f_1 - f_0} = \frac{x - x_0}{f - f_0}$$

$$\frac{30 - 20}{-2.8 - -0.1} = \frac{x - 20}{-1 - -0.1}$$

$$\frac{10}{-2.8} = \frac{x - 20}{-0.9}$$

-9 = -2.8x + 56

-9 - 56 = -2.8x

$23.21^0 = x = T$

**Example**

Give the table below



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| x | -0.11 | -0.10 | -0.09 | -0.08 |
|---|---|---|---|---|
| f(x) | 1.1821 | 1.1800 | 1.1781 | 1.1764 |

Find the value of x such that f(x) = 1.1785

**Solution**

$$\frac{x_1 - x_0}{f_1 - f_0} = \frac{x - x_0}{f - f_0}$$

$$\frac{-0.10 - 0.09}{1.1800 - 1.1781} = \frac{x + 0.09}{1.1785 - 1.1781}$$

$-4 \times 10^{-6} = 1.09 \times 10^{-3}x + 1.71 \times 10^{-4}$

$$\frac{-0.01}{1.09 \times 10^{-3}} = \frac{x + 0.09}{4 \times 10^{-4}}$$

$-4 \times 10^{-6} = 1.09 \times 10^{-3}x + 1.71 \times 10^{-4}$

x = -0.092

## 10.4 Extrapolation

Involves approximating the value of a function f(x) for given values of x outside the given tabulated values.

**Example**

| x | $x^2$ |
|---|---|
| 3.35 | 11.223 |
| 3.36 | 11.280 |

Find $(3.363)^2$



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Using: $\frac{f_0 - f_0}{x_1 - x_0} = \frac{f - f_0}{x - x_0}$

$$\frac{11.290 - 11223}{3.36 - 3.35} = \frac{f - 11.223}{3.363 - 3.35}$$

$$\frac{0.067}{0.01} = \frac{f - 11.223}{0.013}$$

$0.0871 + 11.223 = f$

$11.3101 = f$

$3.363^2 = 11.3101$

**ASSIGNMENT**

1. In an experiment to measure the rate of cooling of an object, the following temperatures, ($\theta^0$ C) against times, T were recorded:

| Temperature, $\theta^0$ C | 80 | 70.2 | 65.8 | 61.9 | 54.2 |
|---|---|---|---|---|---|
| Time, T (s) | 0 | 10 | 15 | 20 | 30 |

Use linear interpolation to find

(i) the value of $\theta$ when T = 18 s

(ii) T when $\theta = 60^0$.

2. In the table below is an extract of part of log x to base 10, $\log_{10} x$:

| $x$ | 80.00 | 80.20 | 80.50 | 80.80 |
|---|---|---|---|---|
| $\text{Log}_{10} x$ | 1.9031 | 1.9042 | 1.9058 | 1.9074 |



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Use linear interpolation to estimate:

(a) $\text{Log}_{10}\ 80.759$,
(b) The number whose logarithm is 1.90388.

3. The bus stages a long Jinja- Kampala road are 10 kms apart. An express bus traveling between the two towns only stops at these stages except in case of emergency when it is permtted to stop at a point in between the two stages. The fares up to the firs, second, third and fourth stages from Jinja are Sh 110, Sh 150, Shs 185 and Sh 200 respectively. On certain day a passenger paid to travel from Jinja in the bus up to the fourth stage but fell sick and had to be left at a health center 33 kms away from Jinja. Given that he was refunded money for part of journey he had not traveled, find the approximate amountt he recieved.
Another person had only Sh 165 was allowed t board the bus but be left at a point worth his money. How far from Jinja was he to be left?
**Answer** balance Sh 10.5, 24.29 km

4. The table shows the values of function f(x) at a set of points

| x | 0.9 | 1.0 | 1.1 | 1.2 |
|---|---|---|---|---|
| f(x) | 0.266 | 0.242 | 0.218 | 0.192 |

Using linear interpolation to find
(i) the value of f (1.04)
(ii) the value of x corresponding to f(x) = 0.25

**Answer** (i) 0.2324 (ii) 0.967

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# CHAPTER ELEVEN

## APPROXIMATE NUMERICAL METHODS

### 11.1 LOCATION OF ROOTS $f(x) = 0$

**Two methods can be used.**

i – Graphical
ii – Analytical

**Analytical method.**

**Example**

Locate the ranges of root of the equation
$x^3 - x - 1 = 0$

Let $f(x) = x^3 - x - 1 = 0$

| x | -3 | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|---|
| f(x) | -25 | -7 | -1 | -1 | -1 | 5 |

The root f(x) of the equation lies between 1 and 2 because
$f(1) < 0$ and $f(2) > 0$

**Example**

Show that the equation $x^4 - 12x^3 + 12 = 0$ has a root between 1 and 2

$f(x) = x^4 - 12x^3 + 12 = 0$
$f(1) = 1 - 12 \times 1 + 12 = 1$
$f(2) = 16 - 12 \times 8 + 12 = -68$



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Since $f(1) > 0$ and $f(2) < 0$, implies the root lies between 1 and 2.

**Example**

Show that the equation $x = \ln(8 - x)$ has a root between 1 and 2

**Solution**

$f(x) = x - \ln(8 - x)$
$f(1) = 1 - \ln(7) = -0.946$
$f(2) = 2 - \ln(6) = 0.208$

Since $f(1) x f(2) < 0$, implies $1 < x_r < 2$

**GRAPHICAL METHOD**

Using the graphical method, single graph could be plotted or two or more graphs could be plotted.

**Example**

Show graphically that there is **one** positive real root of the equation $x^3 + 2x - 2 = 0$

**Solution**



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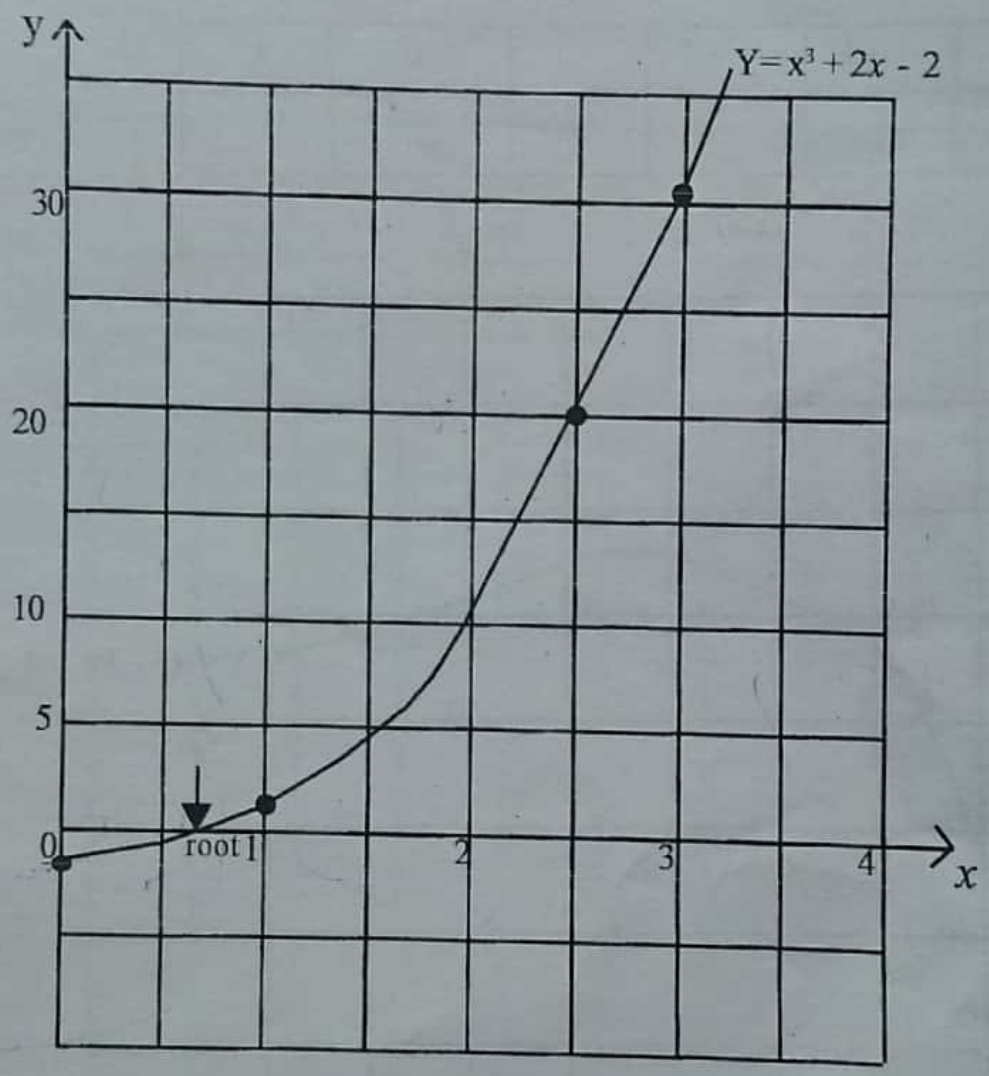
| x | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| f(x) | -2 | 1 | 10 | 31 |




From the graph the curve cuts the x –axis between 1 and 2 therefore there is positive root.



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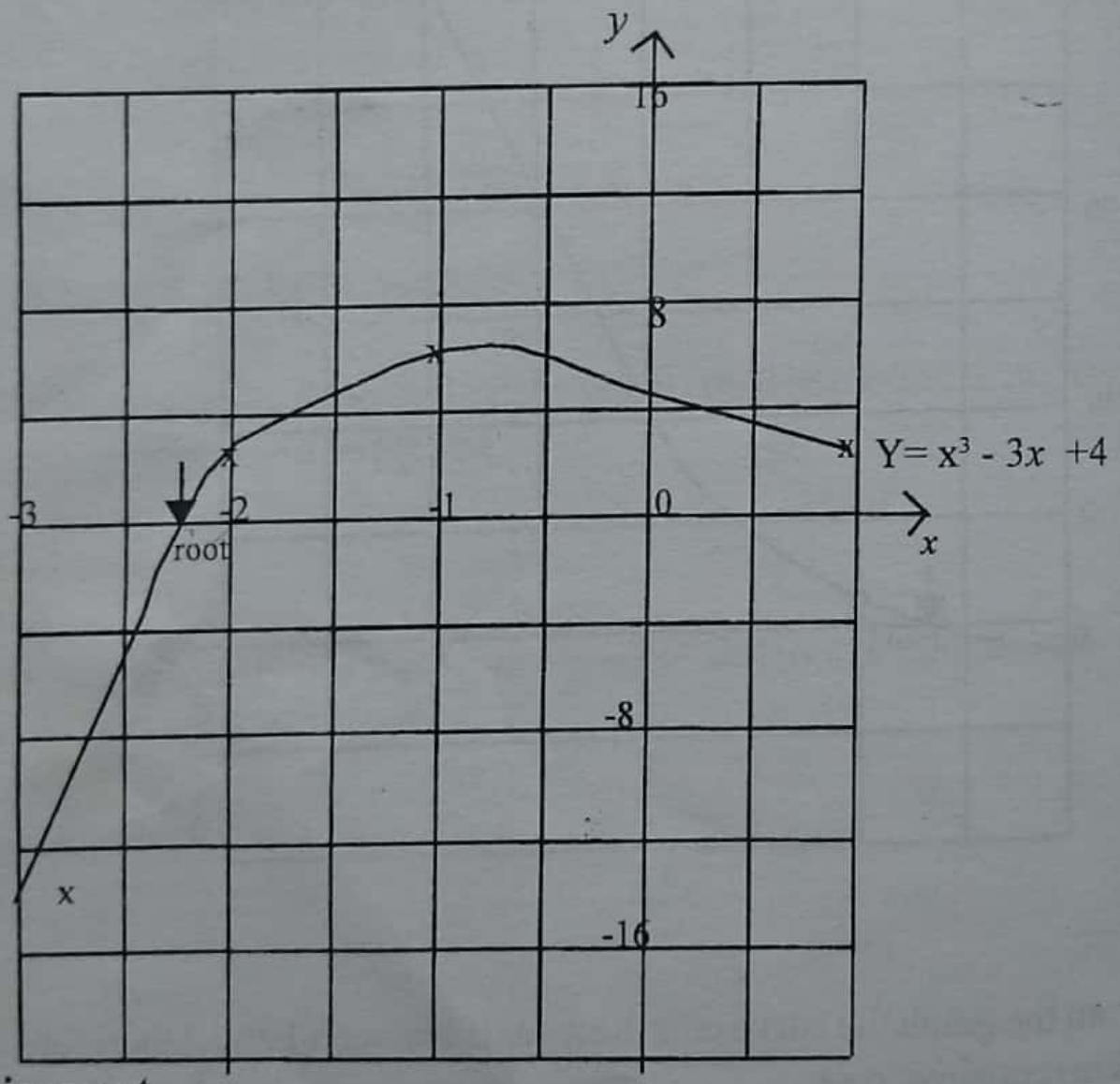
**Example**

Use a graphical method to find a first approximation to the real root of $x^3 - 3x + 4 = 0$

**Solution**

| x | -3 | -2 | -1 | 0 | 1 |
|---|---|---|---|---|---|
| f(x) | -14 | 2 | 6 | 4 | 2 |




**Assignment**

Using $f(x) = x^3 - 3x - 3$, show that it's root lies between 2 and 3.



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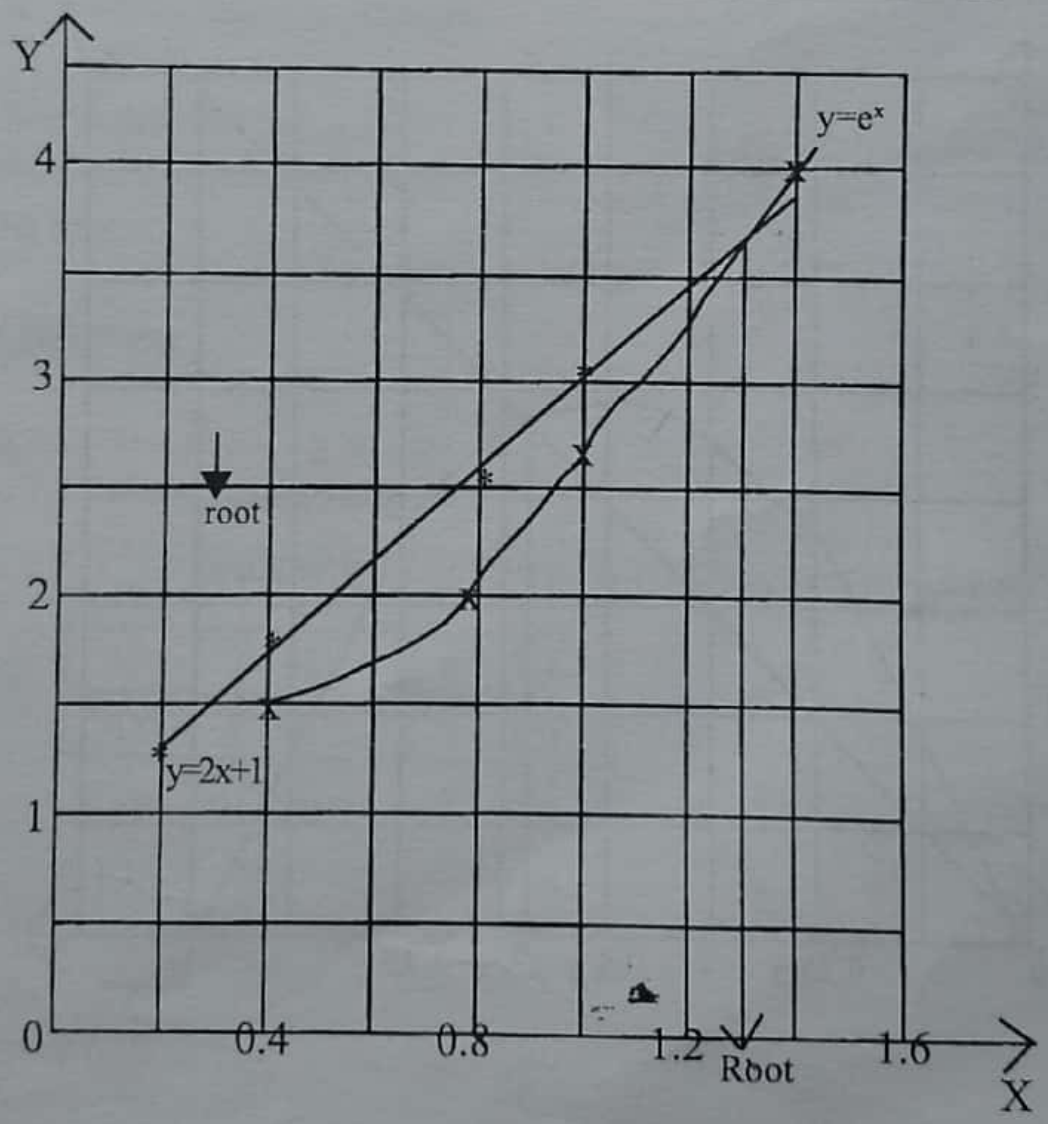
**Example**

Use a graphical method to show that the equation $e^x + x - 4 = 0$ has only one real root.

$f(x) = e^x - 2x - 1$ $\qquad$ $e^x = 2x + 1$

Let $y_1 = e^x$, $y_2 = 2x + 1$

| x | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2 | 1.4 | |
|---|---|---|---|---|---|---|---|---|
| $y_1$ | 1.2 | 1.5 | 1.8 | 2.2 | 2.7 | 3.3 | 4.1 | |
| $y_2$ | 1.4 | 1.8 | 2.2 | 2.6 | 3.0 | 3.4 | 3.8 | |



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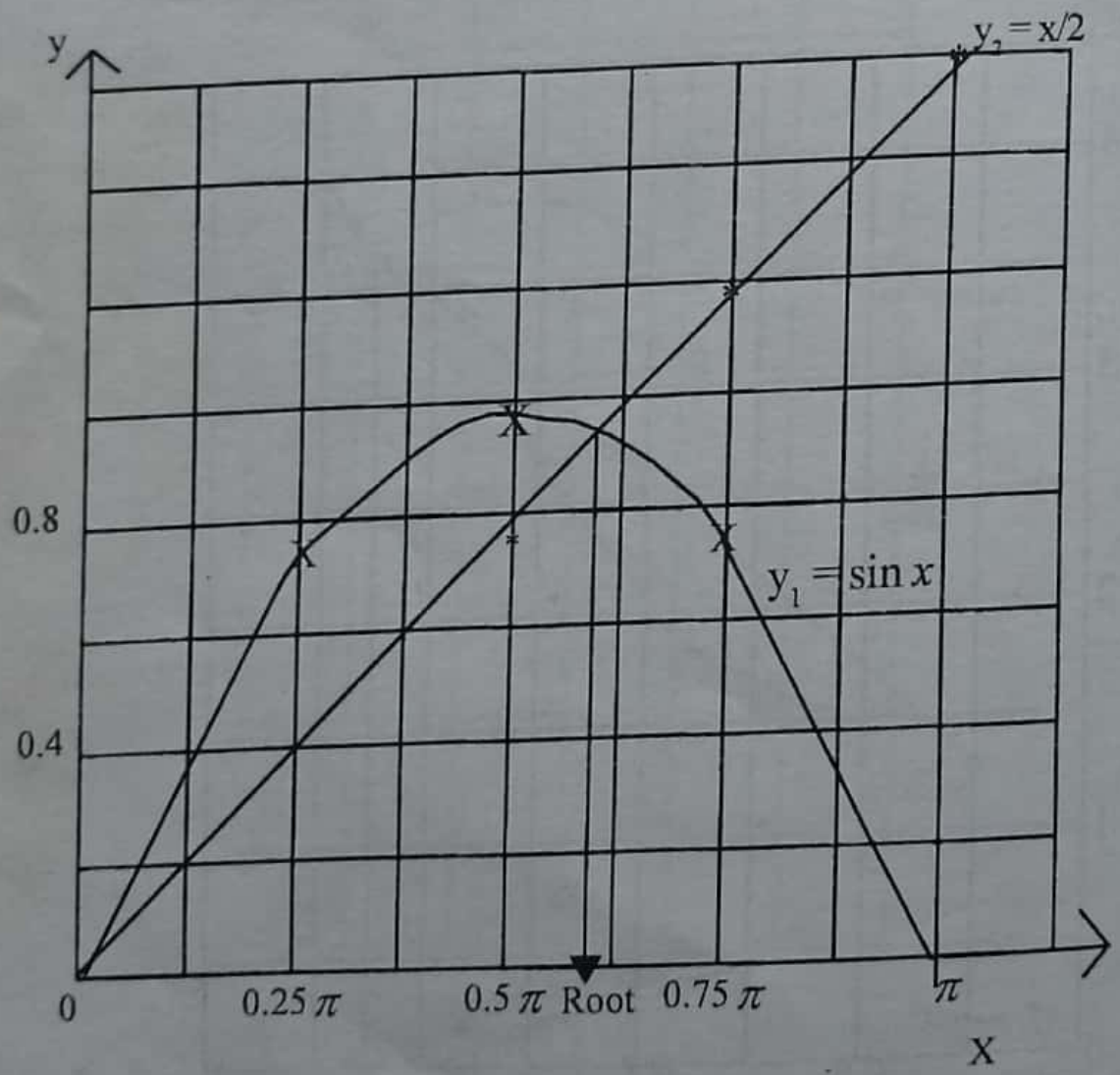
**Example**

Given the equation $y = \sin x - x/2$, show by plotting suitable graphs on the same axes that the root lies between $\pi/2$ and $3\pi/4$.

$y_1 = \sin x, y_2 = x/2$

| x | 0 | $\pi/4$ | $\pi/2$ | $3\pi/4$ | $\pi$ |
|---|---|---|---|---|---|
| $y_1$ | 0 | 0.71 | 1.00 | 0.71 | 0 |
| $y_2$ | 0 | 0.39 | 0.76 | 1.18 | 1.57 |



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Therefore the root lies between $\pi/2$ and $3\pi/4$

## 11.2 Approximations / Iterations.

The process of finding successive approximations to a quantity is known as an iterative process. Each use of particular formulae is an iteration.

**The methods of finding real roots of f(x) = 0 include:-**

- Interpolation
- General iteration method
- Newton Raphson method

### 11. 2.1 INTERPOLATION

Show that the equation $x^3 - 3x - 12$ has a root between 2 and 3. Hence use linear interpolation once to get the first approximation to the root.

**Solution**

$f(2) = 2^3 - 3x2 - 12 = -10$
$f(3) = 3^3 - 3x3 - 12 = 6$
Since $f(2) < 0$ and $f(3) > 0$ implies that root lies between 2 and 3.

| X | 2 | x | 3 |
|---|---|---|---|
| f(x) | -10 | 0 | 6 |

$$\frac{3-2}{6+10} = \frac{x-2}{10}$$

$16x - 32 = 10$
$16x = 42$
$x_0 = 2.625 \Rightarrow$ first approximation



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**Example**

Show that the equation $3x^2 + x - 5 = 0$ has a real root between $x = 1$ and $x = 2$. Hence use linear interpolation to calculate the root to 2 decimal places.

**Solution**

$f(1) = 3x\ 1^2 + 1 - 5 = -1$

$f(2) = 3x2^2 + 2 - 5 = 9$

Since $f(1)\,x\,f(2) < 0$, implies $1 < x_r < 2$

Since $x = 2$ gives 9 which is too far from zero, then other values can be used, $x = 1.5$

$f(1.5) = 3x\ 1.5^2 + 1.5 - 5 = 3.25$

| x | 1 | x | 1.5 |
|---|---|---|---|
| f(x) | -1 | 0 | 3.25 |

$$\frac{1.5-1}{3.25+1} = \frac{x-1}{1}$$

$x = 1.1176$

$f(1.1176) = 3x\ 1.1176^2 + 1.1176 - 5 = -0.1353$

We can use any other x value between 1.1 and 1.5

Using $f(1.2) = f(1.2) = 3x\ 1.2^2 + 1.2 - 5 = 0.52$

| x | 1.1176 | x | 1.2 |
|---|---|---|---|
| f(x) | -0.1353 | 0 | 0.52 |



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$$\frac{1.2-1.1176}{0.52+0.1353}=\frac{x-1.1176}{0.1353}$$

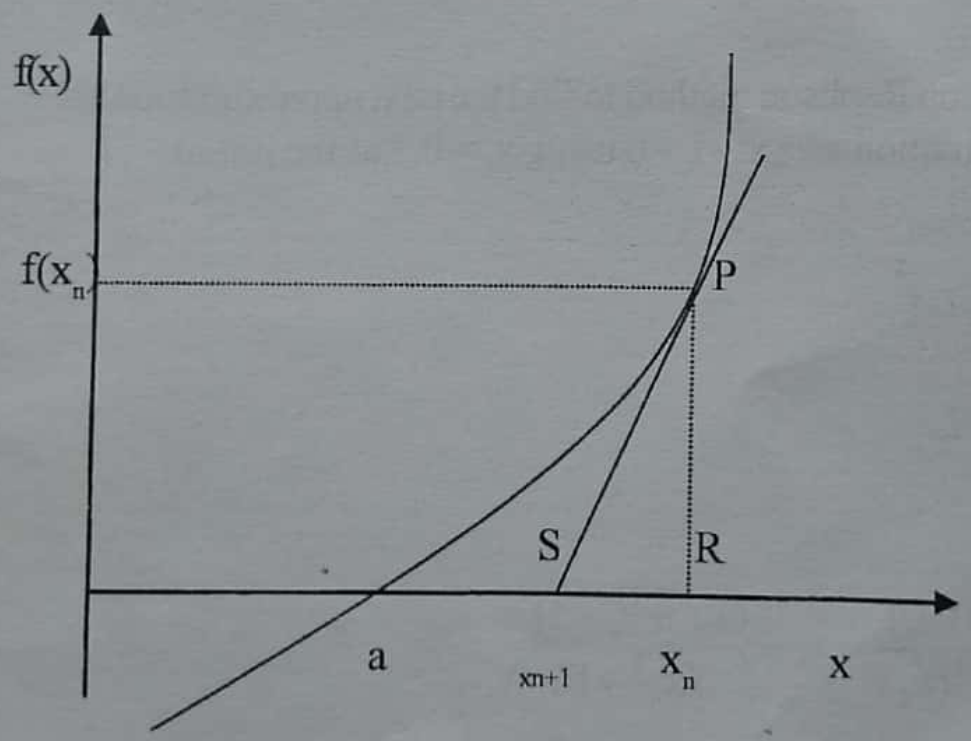
$x = 1.1176 + 0.0170 = 1.1346$

$f(1.1346) = 3x\ 1.1346^2 - 5 = -0.0034$

$x = 1.13$ root correct to 2 d.p

## 11.2.2 NEWTON RAPHSON METHOD

Expression for Newton Raphson method




From the graph, $f(a) = 0$ if we let $x_n$ be the first approximation

The gradient of point $P = \frac{PR}{SR}$



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$$= \frac{f(x_n)}{x_n - (x_{n+1})}$$

From the graph, Grad $P = f^1(x_n)$

$$\Rightarrow \frac{f(x_n)}{x_n - x_{n+1}} = f'(x_n)$$

$$x_n - x_{n+1} = \frac{f(x_n)}{f'(x_n)}$$

$$\therefore x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

**Example**

Use the Newton Raphson method to find the next approximation to root of the equation $x^3 + x - 1 = 0$ using $x_0 = 0.5$ as the initial approximation

$f(x) = x^3 + x - 1$

$f^1(x) = 3x^2 + 1$

$x_0 = 0.5$

$$x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)} = x_n - \frac{(x_n^3 + x_n - 1)}{3x_n^2 + 1}$$

$$x_{n+1} = \frac{2x_n^2 + 1}{3x_n^2 + 1}$$



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note $x_o = 0.5$

$$\frac{2x^2_0+1}{3x^2_o+1} = \frac{2x(0.5)^2+1}{3x(0.5)^2+1}$$

$x_1 = \underline{0.7143}$

**Example**

Use the N.R.M to show the cube root of a number N is given as

$$\frac{1}{3}(2x_n + \frac{N}{x_n^{\ 2}})$$

**Solution**

$x = \sqrt[3]{N} \Rightarrow x^3 = N$

$x^3 - N = 0 \Rightarrow f(x) = x^3 - N = 0$

using N.R.M

$$x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)}$$

$f(x) = x^3 - N \Rightarrow f^1(x) = 3x^2$

$$x_{n+1} = x_n - \frac{(x^3_n - N)}{3x^2_n}$$

$$= \frac{3x_n^{\ 3} - x^3_n + N}{3x_n^{\ 2}} = \frac{2x^3_n + N}{3x_n^{\ 2}}$$

$$\frac{2x^2_n + N}{3x^2_n} = \frac{1}{3}\left(2x_n + \frac{N}{x^2_n}\right)$$



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**Example**

Using the interactive formula, show that the fourth root of the number N is $\frac{3}{4}x_n + \frac{N}{4x_n^3}$

(b) and hence show that

$(45.7)^{1/4} = 2.600$ (correct to 3 decimal places)

**Solution**

$x = \sqrt[4]{N} \Rightarrow x^4 = N$

$x^4 - N = 0 \Rightarrow f(x) = x^4 - N = 0$

Using NRM; $x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)}$

$f(x) = x^4 - N \Rightarrow f^1(x) = 4x^3$

$$x_{n+1} = x_n - \frac{(x_n^4 - N)}{4x_n^3}$$

$$= \frac{4x_n^4 - x_n^4 + N}{4x_n^3}$$

$$= \frac{3x_n^4 + N}{4x_n^3}$$

$$= \frac{3x_n^4 + N}{4x_n^3} = \frac{3}{4}x_n + \frac{N}{4x_n^3}$$



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b) $x = \sqrt[4]{45.7}$

let $x_o = 2.5$ , $N = 45.7$

$$x_{n+1} = \frac{3}{4}x_n + \frac{N}{4x_n^{\ 3}}$$

Substituting

$$\Rightarrow \frac{3}{4} \times 2.5 + \frac{45.7}{4(2.5)^3} = 1.875 + 0.7312$$

$$x_1 = 2.6062$$

$$x_2 = 2.6062 \text{ x } \frac{3}{4} + \frac{45.7}{4(2.6062)^3} = 1.95465 + 0.64540$$

$$= 2.60006$$

$$x_2 \cong 2.6001$$

$$\Rightarrow \sqrt[4]{45.7} \cong 2.600 \text{ (3 decimal places)}$$

**Assignment**

Show that the Newton – Raphson formula for for approximating the $K^{th}$ root of a number N is given by:

$$x_{n+1} = \frac{1}{k}\left[(K-1)x_n + \frac{N}{x_n^{\ K-1}}\right]$$

(ii) Use your formula to find the positive square root of 67 correct to **four** significant figures

**Example**



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Derive the simplest interactive formula for the NRM for the root of the equation $e^{3x} - 3 = 0$. Using your formula with $x_o = \frac{1}{3}$, find the root correct to 4 decimal places and hence find and log $e^3$ correct to 4 decimal places.

**Solution.**

$f(x) = e^{3x} - 3 = 0$

$f^1(x) = 3e^{3x} \Rightarrow$

$$\text{Using } x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)}$$

$$= x_n - \frac{(e^{3x_n} - 3)}{3e^{x_n}}$$

$$x_{n+1} = \frac{(3x_n e^{3x_n} - e^{3x_n} + 3)}{3e^{3x_n}}$$

$$= \frac{e^{3x_n}(3x_n - 1) + 3}{3e^{3x_n}}$$

$$x_o = \frac{1}{3} = x_n$$

$$x_1 = \frac{e^{3x\frac{1}{3}}(3 \times \frac{1}{3} - 1) + 3}{3e^{3\times\frac{1}{3}}}$$



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$$x_1 = \frac{e(0)+3}{3e} = \frac{3}{3e} = \frac{1}{e} = 0.36787944$$

$$x_2 = \frac{e^{3\times 0.36787944}(3\times 0.36787944-1)+3}{3e^{3\times 0.36787944}}$$

$$= \frac{3.0151(0.103638+3}{9.045345} = 0.36621$$

$$x_3 = \frac{e^{3\times 0.36621}(3\times 0.36621-)+3}{3e^{3\times 0.36621}}$$

$$= \frac{3.0000(0.0986)+3}{9} = \frac{3.2958}{9}$$

$$= 0.36620$$

$\Rightarrow$ Root is 0.3662

Hence find log $e^3$

Let $\log e^3 = x \Rightarrow e^x = 3$

$x \ln e = \ln 3 \Rightarrow x = \dfrac{\ln 3}{\ln e}$

$e^{3x} - 3 = 0 \Rightarrow 3x \ln e = \ln 3$

$\Rightarrow \ln 3 = 3x$, but $x = 0.3662$

$\Rightarrow \ln 3 = 3 \times 0.3662 = 1.0986$



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## 11.2.3 THE GENERAL ITERATIVE METHOD

If $f(x) = 0$, then let $x = g(x)$
Now x can be expressed as the root to f(x)

**Example**

Given $f(x) = x^3 - 3x - 12 = 0$, generate equations in the form of $x_{n+1} = g(x_n)$ that can be used to solve the equation $f(x) = 0$.

**Solution**

$$f(x) = x^3 - 3x - 12 = 0$$

$$3x = x^3 - 12$$

$$\Rightarrow x = \frac{x^3 - 12}{3} = g(x)$$

$$\therefore x_{n+1} = \frac{x_n^{\ 3} - 12}{3} \quad .....(1)$$

$$x^3 = 3x + 12$$

$$x = \sqrt[3]{(3x+12)}$$

$$\therefore x_{n+1} = \sqrt[3]{(3x_n + 12)} .....(2)$$

$$x^3 = 3x + 12$$

$$x^3 - 3x = 12$$

$$x\left(x^2 - 3\right) = 12$$

$$x = \frac{12}{x^2 - 3}$$

$$\therefore x_{n+1} = \frac{12}{x_n^{\ 2} - 3} .....(3)$$



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$$x^3 = 3x + 12$$

$$x\left(x^2 - 3\right) = 12$$

$$\left(x^2 - 3\right) = \frac{12}{x}$$

$$x = \sqrt{\left(3 + \frac{12}{x}\right)}$$

$$\therefore x_{n+1} = \sqrt{\left(3 + \frac{12}{x_n}\right)} \text{.....}(4)$$

$$x^3 = 3x + 12$$

$$x = \frac{3}{x} + \frac{12}{x^2}$$

$$\therefore x_{n+1} = \frac{3x_n + 12}{x_n^{\ 2}} \text{.....}(5)$$

**Example**

Show that the iteratve formula for solving the equation

$$x^3 - x - 1 = 0 \ \ is \ \ x_{n+1} = \sqrt{\left(1 + \frac{1}{x_n}\right)}$$

Hence find the root of the equation correct to 3 s.fs.



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**Solution**

$$f(x) = x^3 - x - 1 = 0$$

$$divide\ by\ x \Rightarrow \frac{x^3}{x} = \frac{1+x}{x}$$

$$x = \sqrt{\left(1 + \frac{1}{x}\right)}$$

$$\therefore x_{n+1} = \sqrt{\left(1 + \frac{1}{x_n}\right)}$$

Using $x_0 = 1.2$

$$x_1 = \sqrt{\left(1 + \frac{1}{1.2}\right)} = 1.3540$$

$$|x_1 - x_o| = |1.354 - 1.2| = 0.154$$

$$x_2 = \sqrt{\left(1 + \frac{1}{1.354}\right)} = 1.3185$$

$$|x_2 - x_1| = |1.3185 - 1.354| = 0.0355$$

$$x_3 = \sqrt{\left(1 + \frac{1}{1.3185}\right)} = 1.3261$$

$$|x_3 - x_2| = |1.3261 - 1.3185| = 0.0076$$



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$$x_4 = \sqrt{\left(1+\frac{1}{1.3261}\right)} = 1.3244$$

$$|x_4 - x_3| = |1.3244 - 1.3261| = 0.0017$$

$$|x_4 - x_3| = 0.0017 < 0.005$$

$$\therefore root = 1.32$$

## 11.3 Test for convergence

Using the general iterative method a suitable formula can be rearranged in the form $x_{n+1} = g(x_n)$

In most cases at least one equation can be obtained.

The best equation is the one whose $|g'(x_n)| < 1$, otherwise it tends to diverge.

**Example**

Given the two iterative formulae

(i) $x_{n+1} = \frac{x_n^{\ 3} - 1}{5}$ $\quad (ii)\ x_{n+1} = \sqrt{\left(5+\frac{1}{x_n}\right)}$

Using $x_0 = 2$, deduce a more suitable formula for solving the equation.

Hence, find the root correct to 2 decimal places.

**Solution**



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(i) $g(x) = \frac{x^3 - 1}{5}, g^1(x) = 3x^2/5$

$$g'(2) = \frac{3x2^2}{5} = 2.4$$

since $|g'(2)| > 1$ then it will not form a convergent sequence

(ii) $g(x) = \sqrt{\left(5 + \frac{1}{x}\right)} \Rightarrow g'(x) = -\frac{1}{2}x^{-2}\left(5 + \frac{1}{x}\right)^{-0.5}$

$$g'(2) = -\frac{1}{2}x2^{-2}\left(5 + \frac{1}{2}\right)^{-0.5} = -0.0533$$

since $|g^1(2)| < 1$, then the second formula will produce a convergent sequence.

Hence using equation (ii)

$$x_{n+1} = \sqrt{\left(5 + \frac{1}{x_n}\right)} \; but \; x_0 = 2$$

$$x_1 = \sqrt{\left(5 + \frac{1}{2}\right)} = 2.3452, |x_1 - x_0| = 0.3452$$

$$x_2 = \sqrt{\left(5 + \frac{1}{2.3452}\right)} = 2.3295, |x_2 - x_1| = 0.0157$$



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$$x_3 = \sqrt{\left(5 + \frac{1}{2.3295}\right)} = 2.3301, \; |x_2 - x_1| = 0.0006$$

$Since\ 0.0006 < 0.005$

$Therefore\ the\ \ root = 2.33$

## 11.4 Trapezium rule

$$\text{Area} = \frac{1}{2}d(y_1 + y_2) + \frac{1}{2}d(y_2 + y_3) + \ldots + \frac{1}{2}d(y_{n-1} + y_n)$$

$$\text{Area} = \frac{1}{2}d\left[y_1 + 2(y_2 + y_3 + \ldots + y_{n-1}) + y_n\right]$$

$$\int_{x_0}^{x_n} f(x)dx \cong \int_{x_o}^{x_n} \frac{1}{2}h[y_o + 2(y_1 + y_2 + \ldots + y_{n-1}) + y_n]$$

**Example**

Use the trapezium rule to estimate the area under the curve $\frac{1}{x}$ from $x = 1$ to $x = 2$

| x | 1.0 | 1.2 | 1.4 | 1.6 | 1.8 | 2.0 |
|---|---|---|---|---|---|---|
| y | 1 | 0.833 | 0.7143 | 0.625 | 0.556 | 0.50 |
| $y_o$ | $y_1$ $y_0$ | $y_2$ $y_1$ | $y_3$ $y_2$ | $y_5$ $y_3$ | $y_5$ $y_4$ | $y_5$ |

$\int f(x).dx = ½\, h[y_o + 2(y_1 + y_2 + y_3 + y_4) + y_5]$

$$\int_{1.0}^{2.0} \frac{1}{x}dx \cong \frac{1}{2} \times 0.2[1 + 2(0.833 + 0.7143 + 0.625 + 0.556) + 0.5]$$

$= 0.69572 \cong 0.696$



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**Example**

Use the trapezium rule with 7 ordinates to estimate

$$\int_0^3 \frac{1}{1+x}dx, \ correct\ to\ 3\ decimal\ places$$

**Solution**

| $x$ | $y_0, y_6$ | $y_1, \cdots y_5$ |
|---|---|---|
| 0 | 1.0000 | |
| 0.5 | | 0.6667 |
| 1.0 | | 0.5 |
| 1.5 | | 0.4 |
| 2.0 | | 0.3333 |
| 2.5 | | 0.2857 |
| 3.0 | 0.2500 | |
| **Sum** | 1.25 | 2.1857 |

$$\int_0^3 \frac{1}{1+x}dx = \frac{1}{2}x0.5(1.25+2x2.1857)$$

$$=1.40535 \cong 1.405$$

**Note:** $d = \dfrac{x_n - x_0}{number\ of\ sub-intervals} = \dfrac{3-0}{6} = 0.5$

**Sub-intervals, sub-divisions and strips are same**
**However ordinates = (sub-intervals + 1)**

**Example**

i. Use the trapezium rule to estimate the area of $y = 3^x$ between the x-axis, x = 1 and x = 2 using five strips. Give your answer correct to 4 significant figures.



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ii. Find the exact value of $\int_1^2 3^x\,dx$.

iii. Find the percentage error in calculations (i) and (ii) above

**Solution**

| $x$ | $y_0, y_5$ | $y_1, \cdots y_4$ |
|---|---|---|
| 1.0 | 3.0000 | |
| 1.2 | | 3.7372 |
| 1.4 | | 4.6555 |
| 1.6 | | 5.7995 |
| 1.8 | | 7.2247 |
| 2.0 | 9.0000 | |
| **Sum** | 12.0000 | 21.4169 |

$$\int_1^2 3^x\,dx = \frac{1}{2}x0.2(12+2x21.4169)$$

$$=5.48338 \cong 5.483$$

(ii) $\int_1^2 3^x\,dx = \left[\frac{3^x}{\ln 3}\right]_1^2 = 5.461$

(iii) percentage error =

$$\frac{|5.483-5.461|}{5.461}x100 = 0.401$$

**Example**

Use the trapezium rule with six sub-intervals to estimate

$\int_0^{\pi} x\sin x\,dx$ correct to 2 decimal places. Determine the error in your estimation and suggest how this error may be reduced .



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**Solution**

| x | 0 | π/6 | 2π/6 | 3π/6 | 4π/6 | 5π/6 | π |
|---|---|---|---|---|---|---|---|
| x sinx | 0 | 0.262 | 0.907 | 1.571 | 1.814 | 1.309 | 0 |

$$=\frac{1}{2}x\frac{\pi}{6}\left(0+2(0.262+0.907+1.571+1.814+1.309)\right)$$

$$=0.5236 \text{ x } 5.862$$

$$=3.069$$

$$=3.07$$

$$\int_0^{\pi} x\sin x\,dx=[-x\cos x]_0^{\pi}+\int_0^{\pi}\cos x\,dx$$

$$=\pi+\sin x=\cong 3.14$$

error = 3.14 – 3.07 = 0.07

**Increasing the number of sub intervals can reduce the error.**



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**Exercise**

1. An iterative formula for solving an equation is given by:

$$x_{n+1} = \sqrt[3]{(3x_n + 3)} \quad \text{for } n = 0, 1, 2, ..$$

i) Find the equation whose root is being sought

ii) Show that the equation has one root and using an appropriate starting value, find the root correct to 2 decimal places.

**Answer** 2.10

2. By drawing a graph, show that the root of the equation 2tanx = 3x lies between $\frac{\pi}{6}$ and $\frac{\pi}{3}$. Hence by using Newton Raphson's method find the root of the equation to 2 decimal places.

**Answer** 0.97

3. Draw using the same axes, the graphs of $y = x^2$ and $y = \sin 2x$ for $0 \le x \le \frac{\pi}{2}$. From your graph obtain to one decimal place an approximation of the non-zero root of the equation $x^2 - \sin 2x = 0$. Using Newton's method, calculate to 2 decimal places a more suitable approximation.

**Answer** 0.97

4. Show that the equation $3x^3 + x - 5$ has a real root between $x = 1$ and $x = 2$.

(i) Using linear interpolation find the first approximation for this root to two decimal places.



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(ii) Using Newton Raphson formula twice, find the value of this root correct to 2 decimal places.

**Answer** (i) 1.13 (ii) 1.10

5. By sketching the graphs of 2x and tanx show that the equation 2x = tanx has only one root between x = 1.1 and 1.2. Use linear interpolation to find the value of the root correct to 2 decimal places.

**Answer** 1.17

6. Show graphically that there is only one positive real root of the equation; $xe^{-x} - 2x + 5 = 0$. Using the Newton Raphson method, find this root correct to 1 decimal place.

**Answer** 2.6

7. (a) Show that the Newton Raphson's formular for finding the smallest positive root of the equation

3tan + x = 0 is $\dfrac{6x_n - 3\sin 2x_n}{6 + 2\cos^2 x_n}$

(b) By sketching the graphs of $y = \tan x$, $y = \dfrac{-x}{3}$ or otherwise, find the first approximation to the required root and use it to find the actual root correct to 3 decimal places.

(hint work in radians)

**Answer** 2.456

8. Find an approximation value of $\int_0^{0.5} 1 - x^2 dx$ by using the trapezoidal rule with intervals of 0.1. Show by integration that the magnitude of the error in the approximation is less than 0.001.

**Answer** 0.4778; 0.4783



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9. Show that the equation $x = \cos x - 3$ has a root between -4 and -3. Use an iterative method to calculate this root correct to 3 decimal places.

**Answer** -3.794

10. Show that the equation $f(x) = x^3 + 3x - 9$ has a root between x=1 and x=2. Using the Newton Raphson formula once, estimate the root of the equation, rounded off to **two** significant figures.

**Answer** 1.6

11. Use the trapezium rule with 7 ordinates to find the value of

$\int_0^{\pi} \sqrt{(1+\sin x)}\, dx$, *correct to* **two** decimal places

**Answer** 3.98

12. (a) Use the trapezium rule to estimate the area of $y = 5^{2x}$ between the x-axis, x = 0 and x = 1 using **five** sub-intervals. Give your answer correct to 3 decimal places.

(b) Find the exact value of $\int_0^1 5^{2x}\, dx$.

(c) Find the percentage error in the two calculations in (a) and (b) above

**Answer** (a) 7.712 (b) 7.4560 (c) 3.43

13 Show that the equation $e^x + x - 4 = 0$ has a real root between 1 and 1.2. Use the Newton-Raphson method to find the root of the equation correct to 3 significant figures.



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**Answer 1.07 ( 3 sf)**

14. Show graphical that equation $e^x + x - 8 = 0$ has only one real root. Use Newton's Raaphson method to find the approximation of $x = \ln(x-8)$ correct to 3 decimal places

**Answer** 1.821

15. ) Use the trapezium rule to estimate the area of $y = e^{-2x}$ between the x – axis, x = 1 and x =2, using six ordinates. Give your answer correct to 3 significant fiures.

**Answer** 0.0593



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# CHAPTER TWELVE

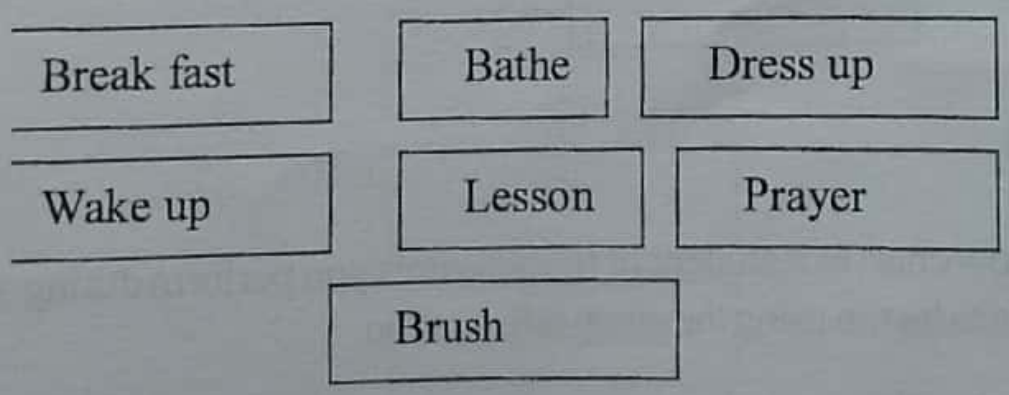
## FLOW DIAGRAM/ CHARTS

### 12.1 INTRODUCTION

If one has a problem to solve, you can write down steps that he/she needs to solve it. These steps required are called an algorithm. So an algorithm is a step by step procedure required to solve a problem. These steps can be represented in a diagrammatic form using a flow chart. A flow chart or diagram is one that shows how the logical sequence of steps that must be followed in order to solve a problem. Or it is a visual picture that gives the steps of an algorithm and the flow of control between the various steps.

**Example**

1. Draw a flow chart as a student of events you perform before the 8.00 am lesson using the given activities.



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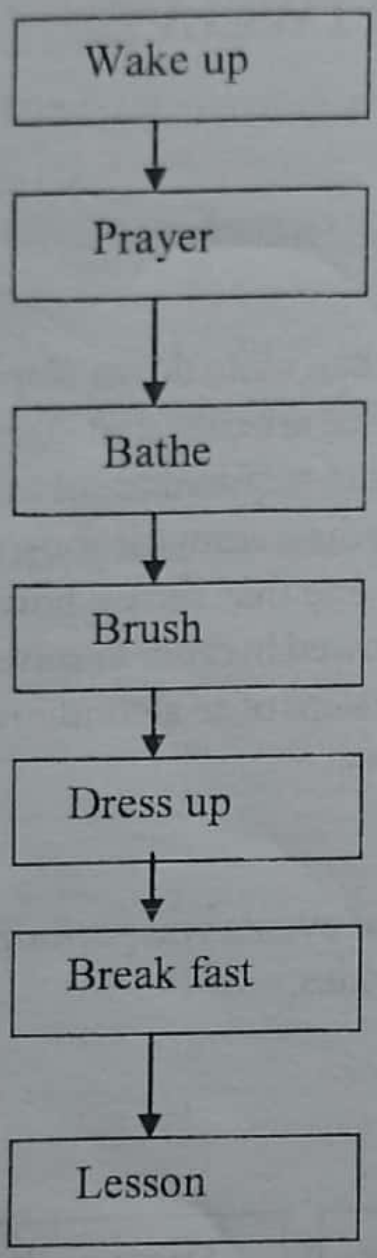
**Solution**




**Example**

Draw a flow chart as a student of the activities you perform during a Mathematics lesson using the given information.



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Enter class

Books

Teacher

Review of work

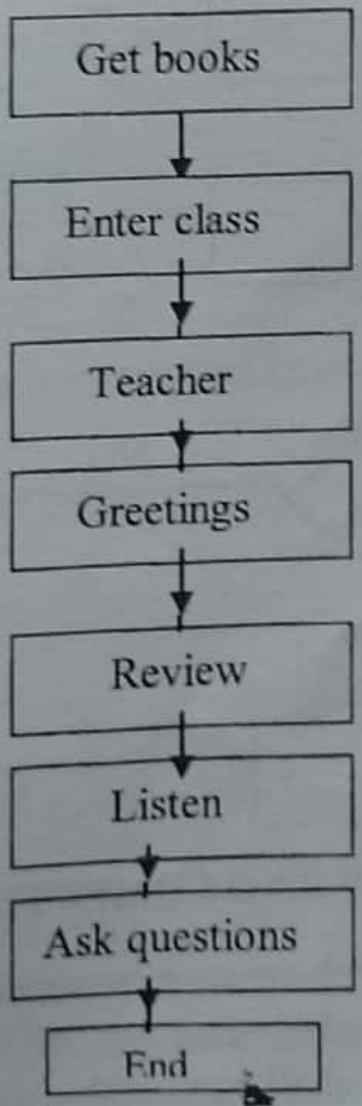
Greetings

Listen

Lesson ends

Ask questions

**Solution**



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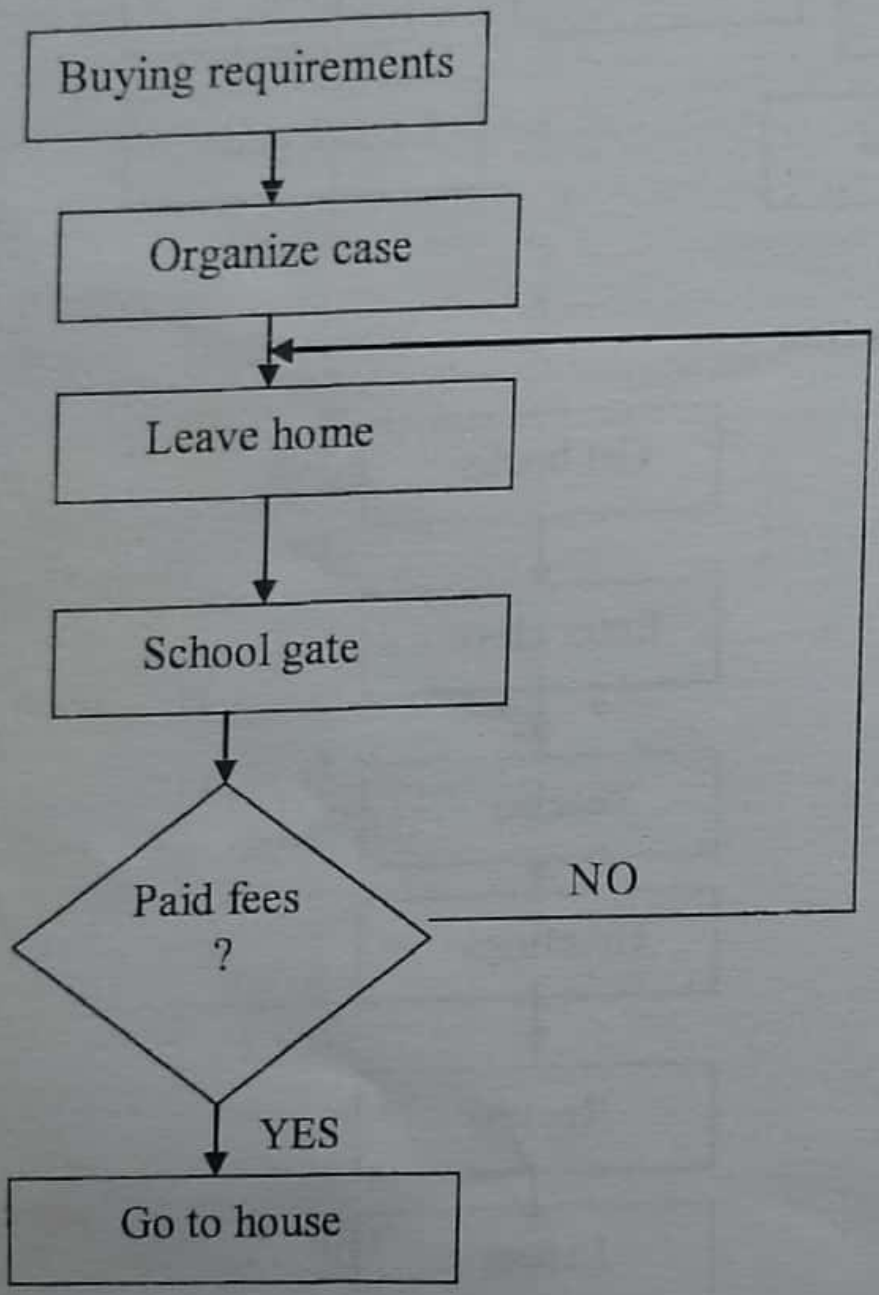
**Example**

Draw a flow chart for the events you perform when reporting to school




## 12.2 FLOW CHARTS AND COMPUTER PROGRAMS

Computer based flow charts are more modified for every day life type. They are constructed in such a way that a mathematical problem is broken down into small computable steps. These figures are connected to each other by arrows such that the information flow is followed. In such steps different figures are used.



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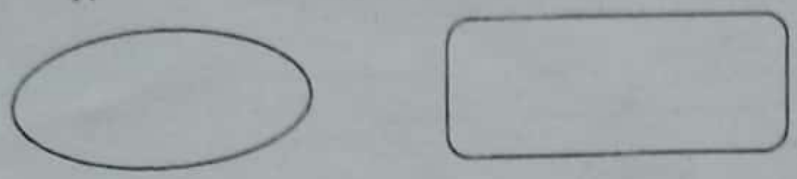
**They include the following**

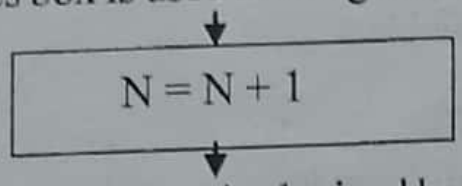
1.



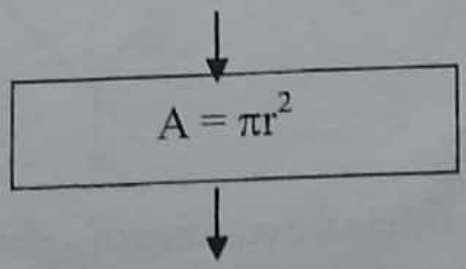
Starts with word **Start/Begin** in first box
Ends with word **Stop/End** in the final box.
Any of the above boxes can be used.

In between the start and end boxes, there are a number of other boxes with different shapes that can be used depending on the nature of the instructions.

2. Rectangles box is used as assignment box




This indicates that the new N is obtained by adding one to the previous N.
So it means let N become N + 1.




**Example**

So the box calculates the area of circle. Therefore output will be the area.



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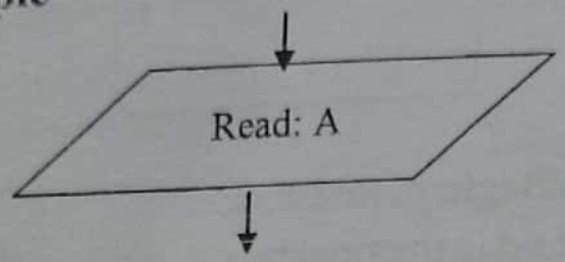
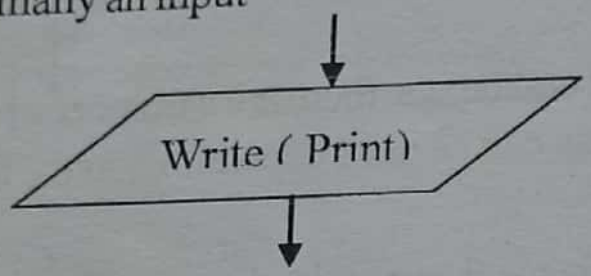
3. Other shapes include rhombus, parallelogram
These can be used for information input or output

**Example**

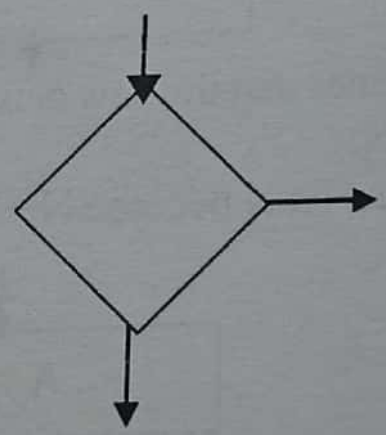



Read is normally an input




Print or Write is an output box

Diamond box is a conditional box



## 12.3 Dry run

This is the method for predicting the outcome of a given flow chart that produces a table of contents that can be continually updated

**Example**

Perform a dry run and state the purpose of the flow chart.



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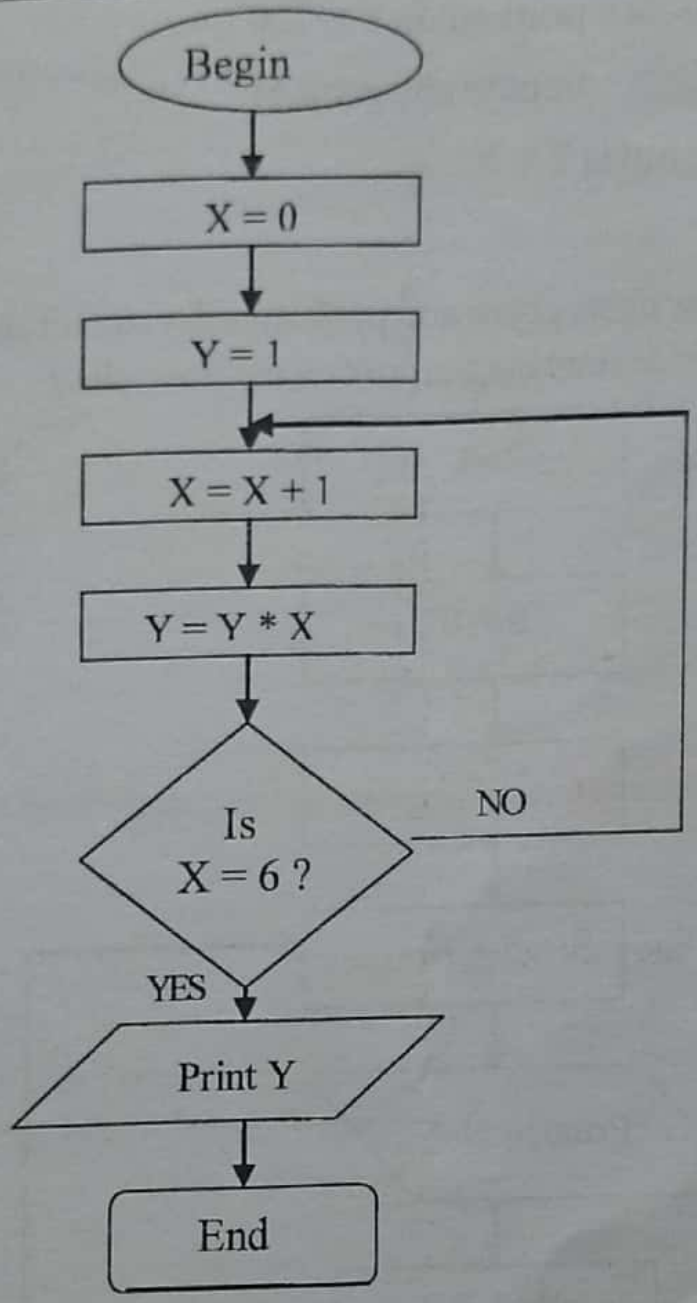


**Solution**

| $X$ | Y |
|---|---|
| 0 | 1 |
| 1 | 1 |
| 2 | 2 |
| 3 | 6 |
| 4 | 24 |
| 5 | 120 |
| **6** | **720** |



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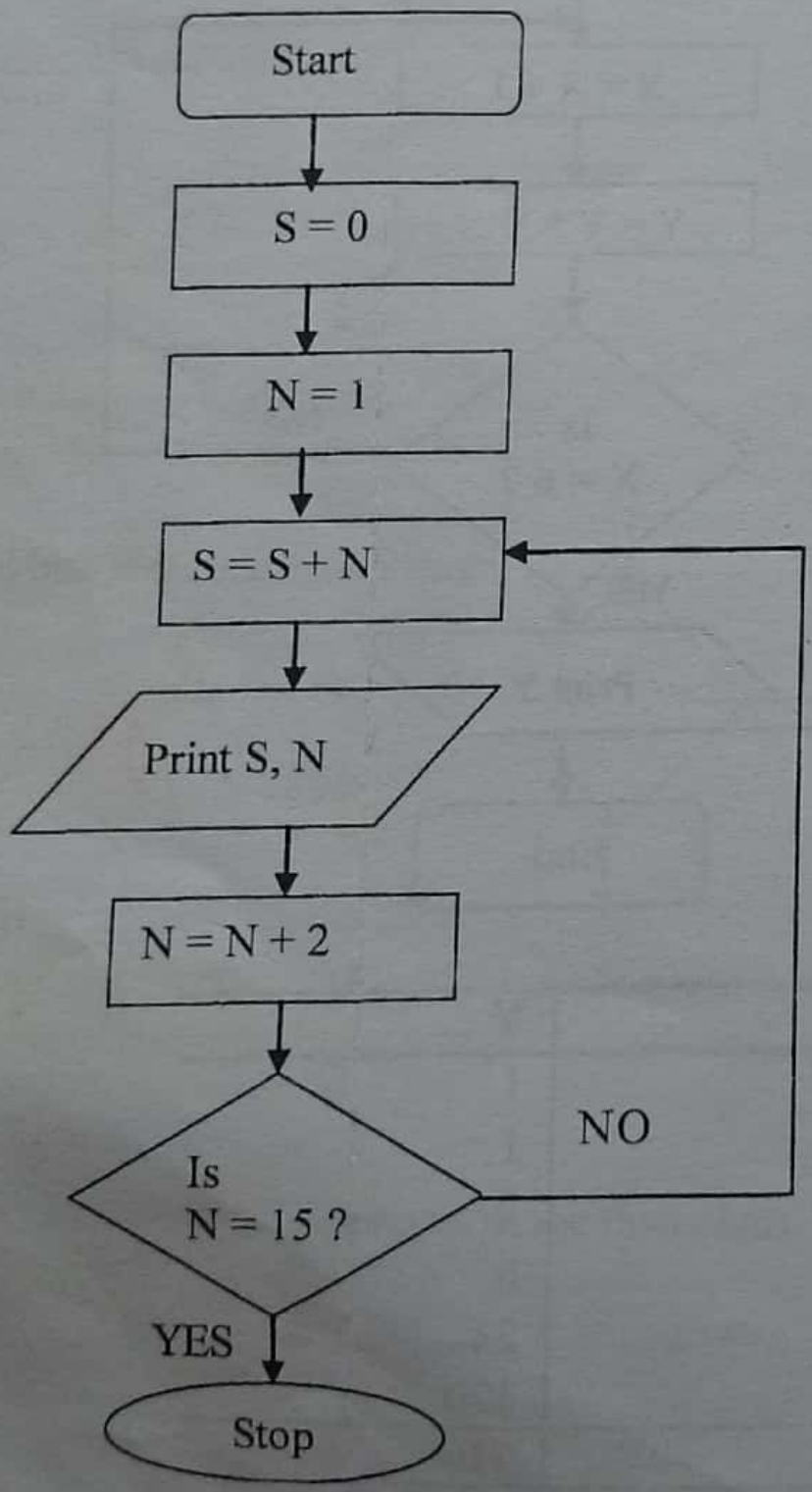
Therefore the print out is Y = 720

Purpose is to compute and print 6!

Relationship is Y = X!

**Example**

Study the flow chart given and perform a dry run for the flow chart. Hence state the purpose of the flow chart



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**Solution**

| S | N |
|---|---|
| 0 | 1 |
| 1 | 1 |
| 4 | 3 |
| 9 | 5 |
| 16 | 7 |
| 25 | 9 |
| 36 | 11 |
| 49 | 13 |
| - | 15 |

Prints the $n^{th}$ odd number and sum of odd numbers up to $7^{th}$

.

**Example**

Study the flow chart given. Perform a dry run of the flow chart and state the relationship between N and M.



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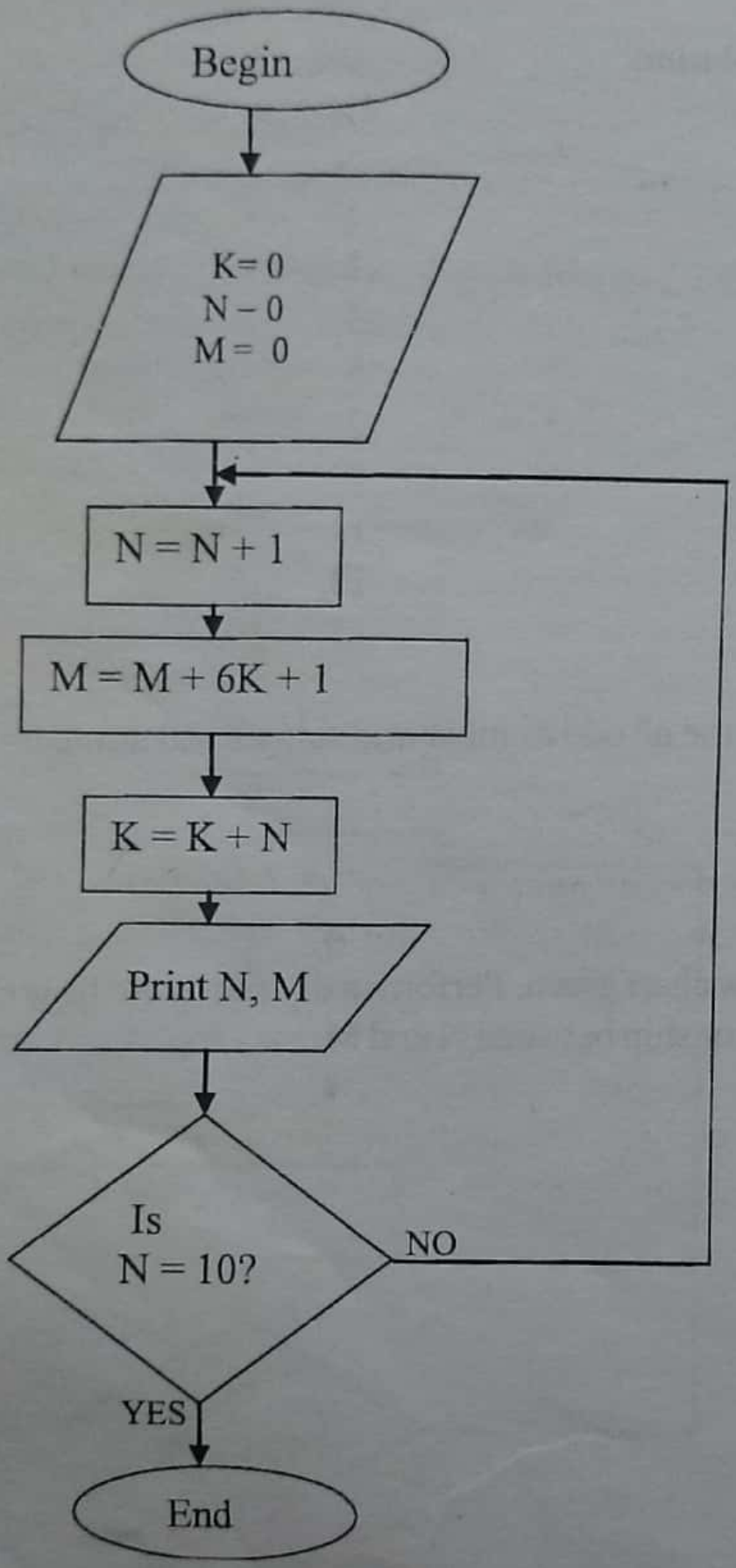




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**Solution**

| N | M | K |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |
| 2 | 8 | 3 |
| 3 | 27 | 6 |
| 4 | 64 | 10 |
| 5 | 125 | 15 |
| 6 | 216 | 21 |
| 7 | 343 | 28 |
| 8 | 572 | 36 |
| 9 | 729 | 45 |
| 10 | | |

M is the cube of N
Therefore the purpose is to compute and print the number and its cube. Hence $M = N^3$

**Example 4**

In the flow diagram C = counter, SSN = Social security number, W = monthly wage of a factory employee in Shillings.



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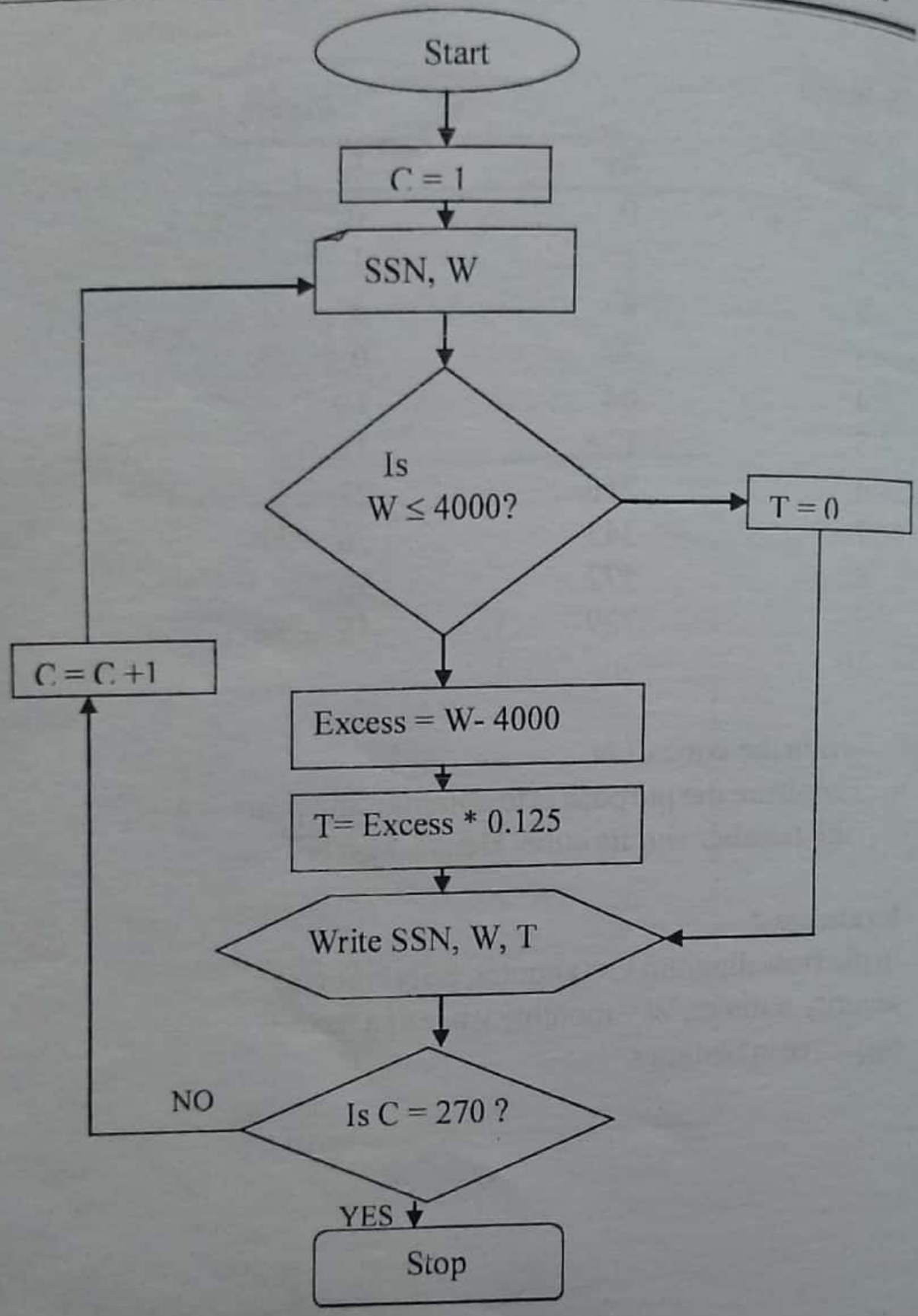




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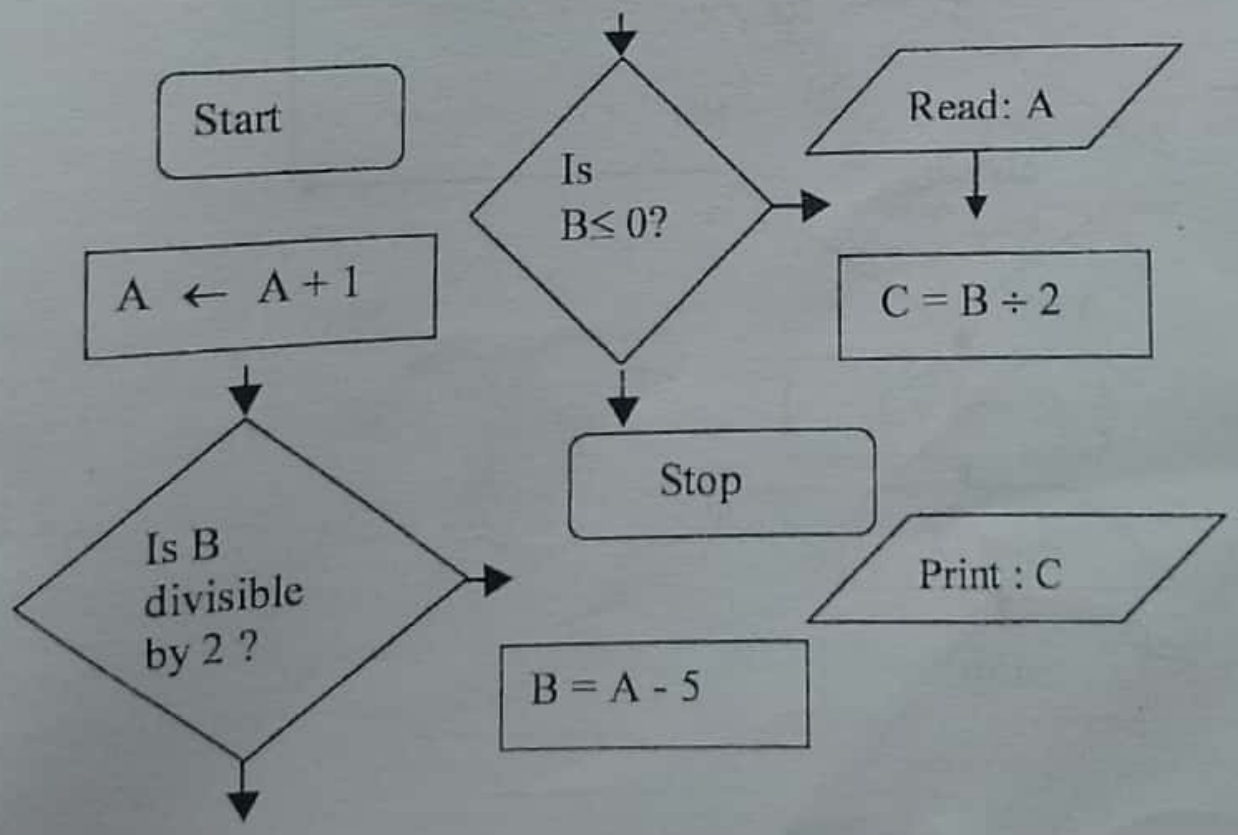
Copy and complete the following table.
For how many employees is this program designed?

| SSN | W | T |
|---|---|---|
| 01-86-003 | 8400 | |
| 03- 86- 095 | 8200 | |
| 04-86-064 | 7500 | |
| 02-86-035 | 8000 | |
| 04- 86 - 066 | 6400 | |
| 01-87-098 | 4800 | |
| 02-87-105 | 6300 | |
| 03-87-135 | 5500 | |
| 01-88-215 | 3800 | |
| 01-89-217 | 3500 | |

**Example**

Given below are parts of a flow chart not arranged in order.



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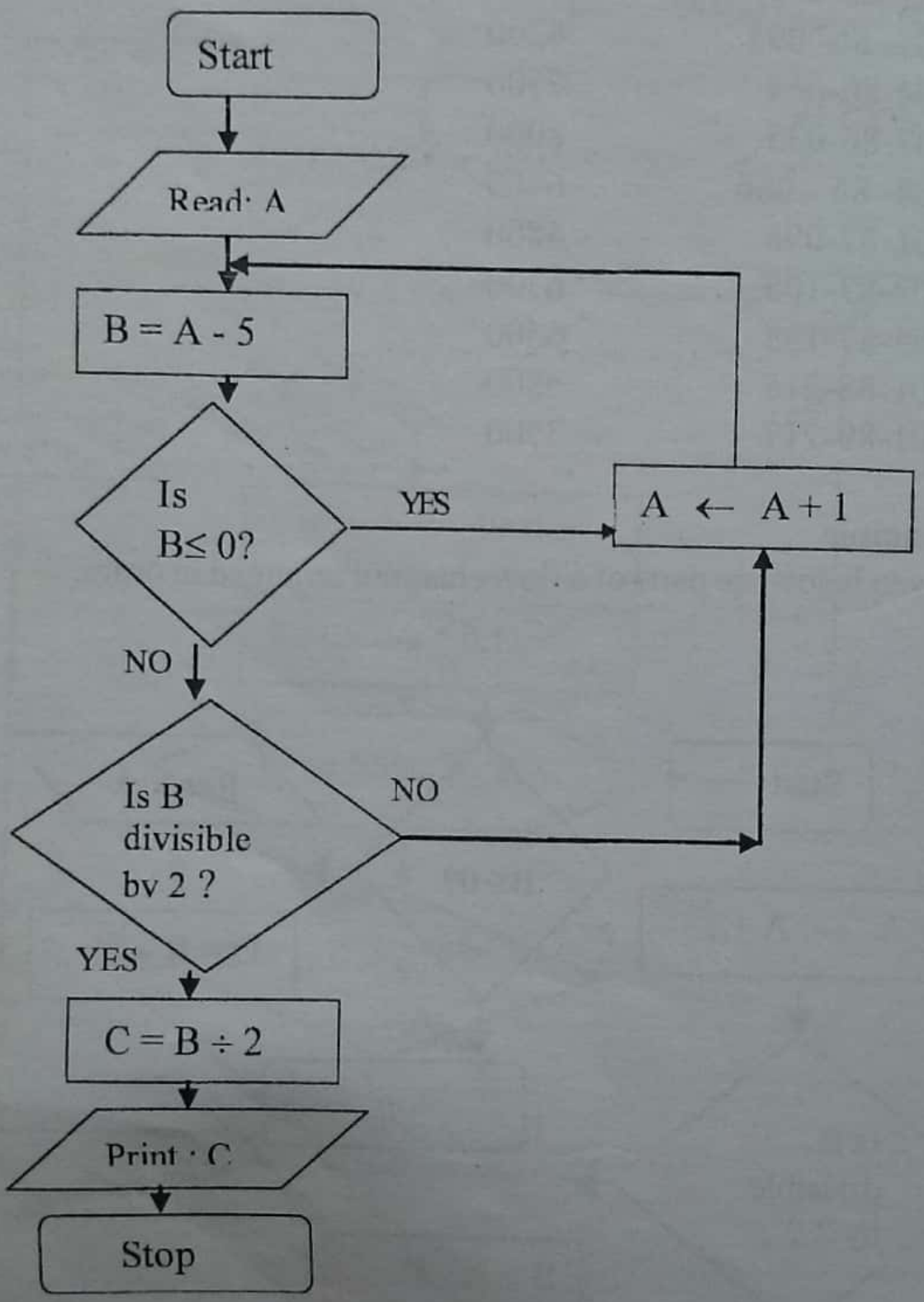
Rearrange them and draw a complete logical flow chart

**Solution**



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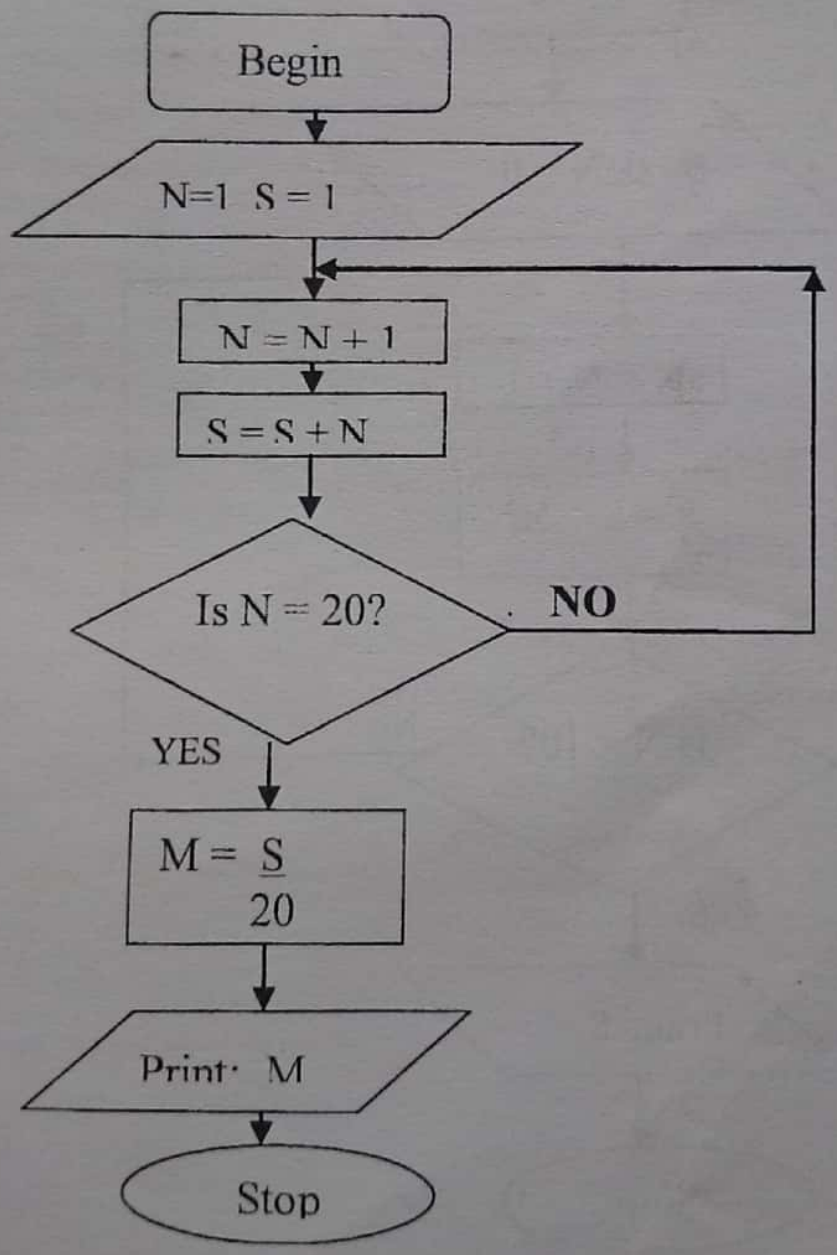
## 12.4 Constructing flow charts

**Example**

Draw a flow chart that reads and prints the mean of the first twenty counting numbers.

**Solution**

Let S represent sum and M the mean



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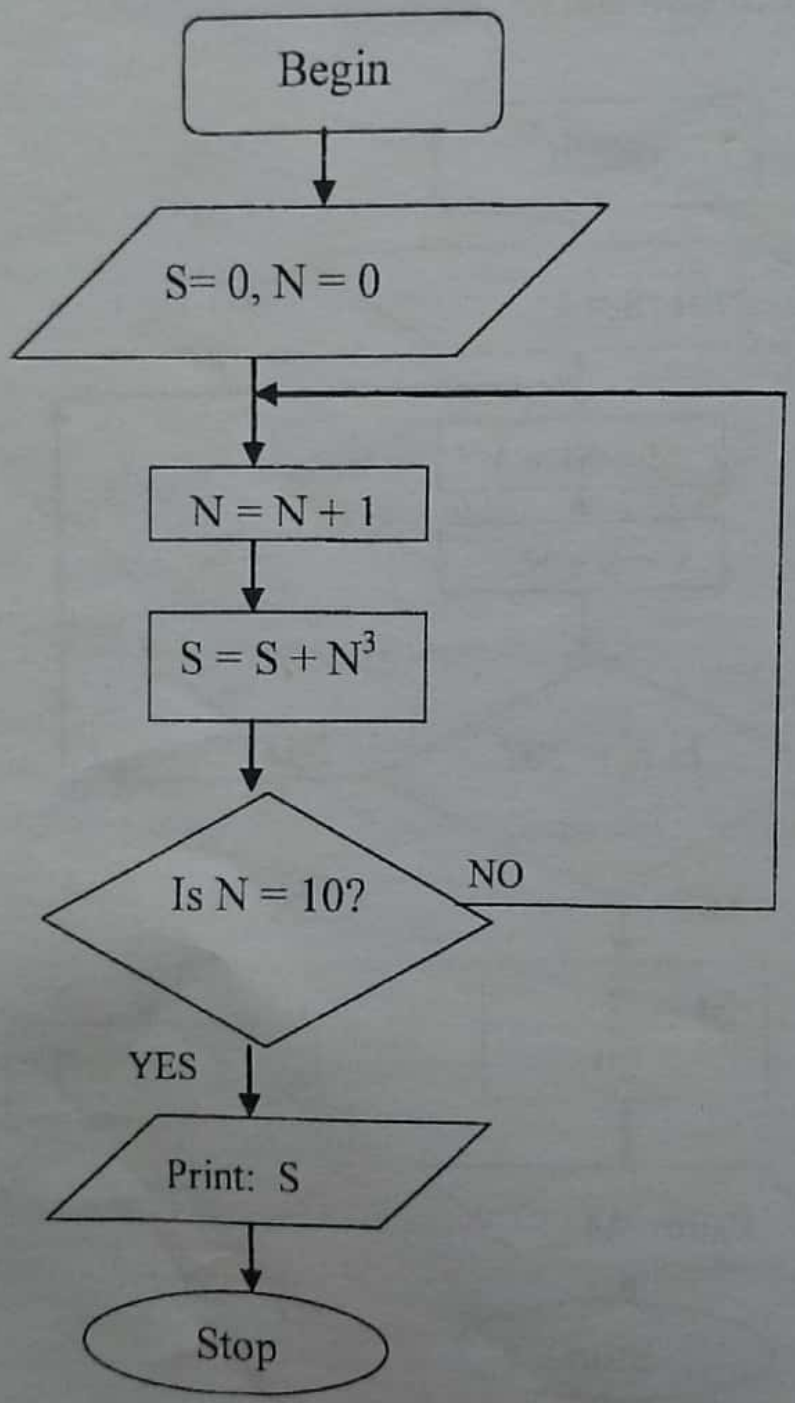
**Example**

Draw a flow chart that computes and prints the sum of the cubes of the first ten natural numbers.

**Solution**

Let S represent sum



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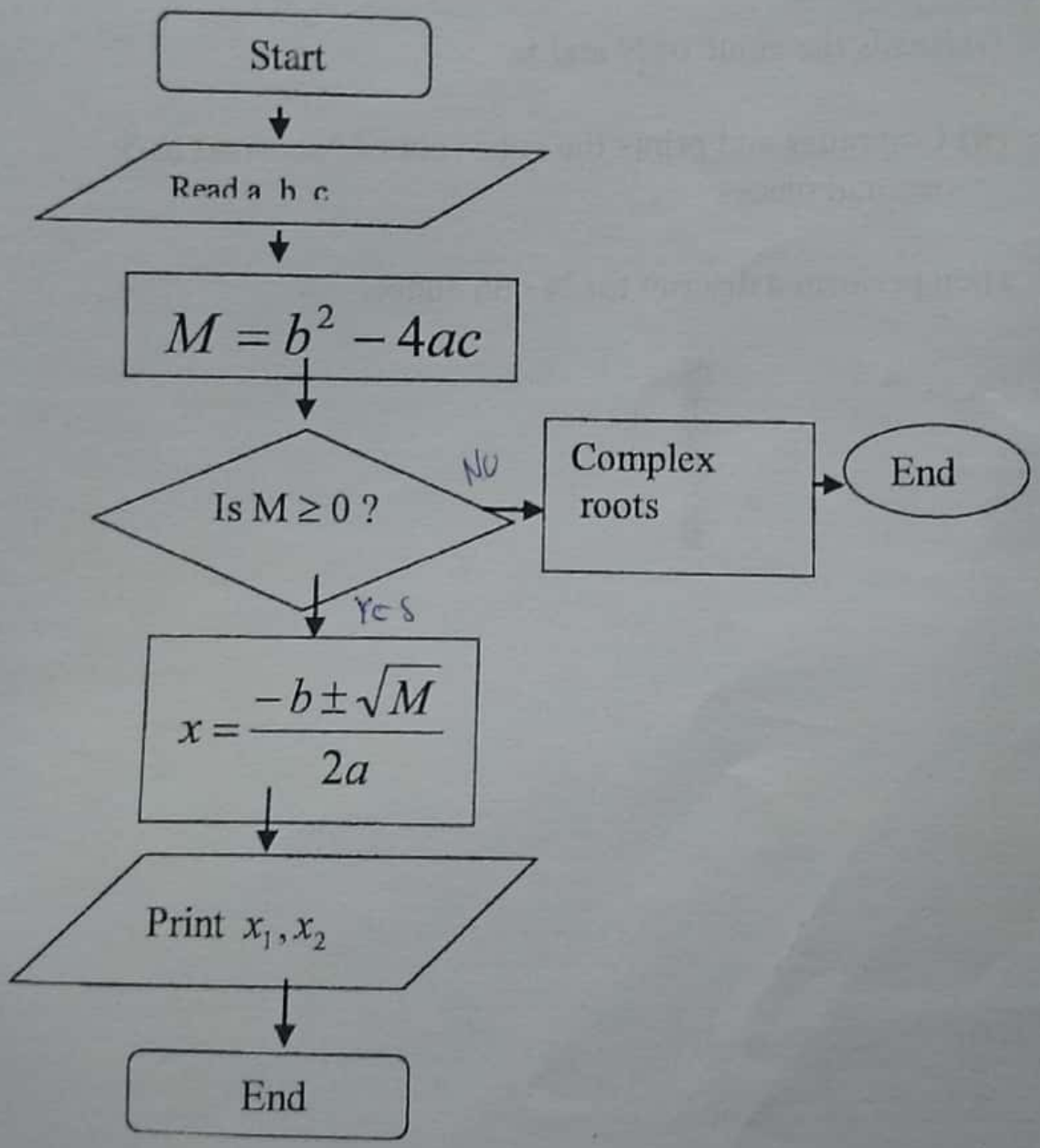
**Example**

Draw a flow chart that computes the real root of the equation $ax^2 + bx + c$, where a $\neq 0$

**Solution**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{for real roots, } b^2 - 4ac \geq 0$$

$$M = b^2 - 4ac \quad \Rightarrow M \geq 0$$



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**Example 9**

An iterative method of finding the cube root of a number N , is given by

$$x_{n+1} = \frac{1}{3}(2x_n + \frac{N}{x_n^{\ 2}}); \ n = 0, 1, 2 \ldots$$

Draw a flow chart that

(i) Reads the value of N and $x_0$

(ii) Computes and prints the cube root of N correct to 3 decimal places

Then perform a dry run for N= 66 and $x_0 = 4$



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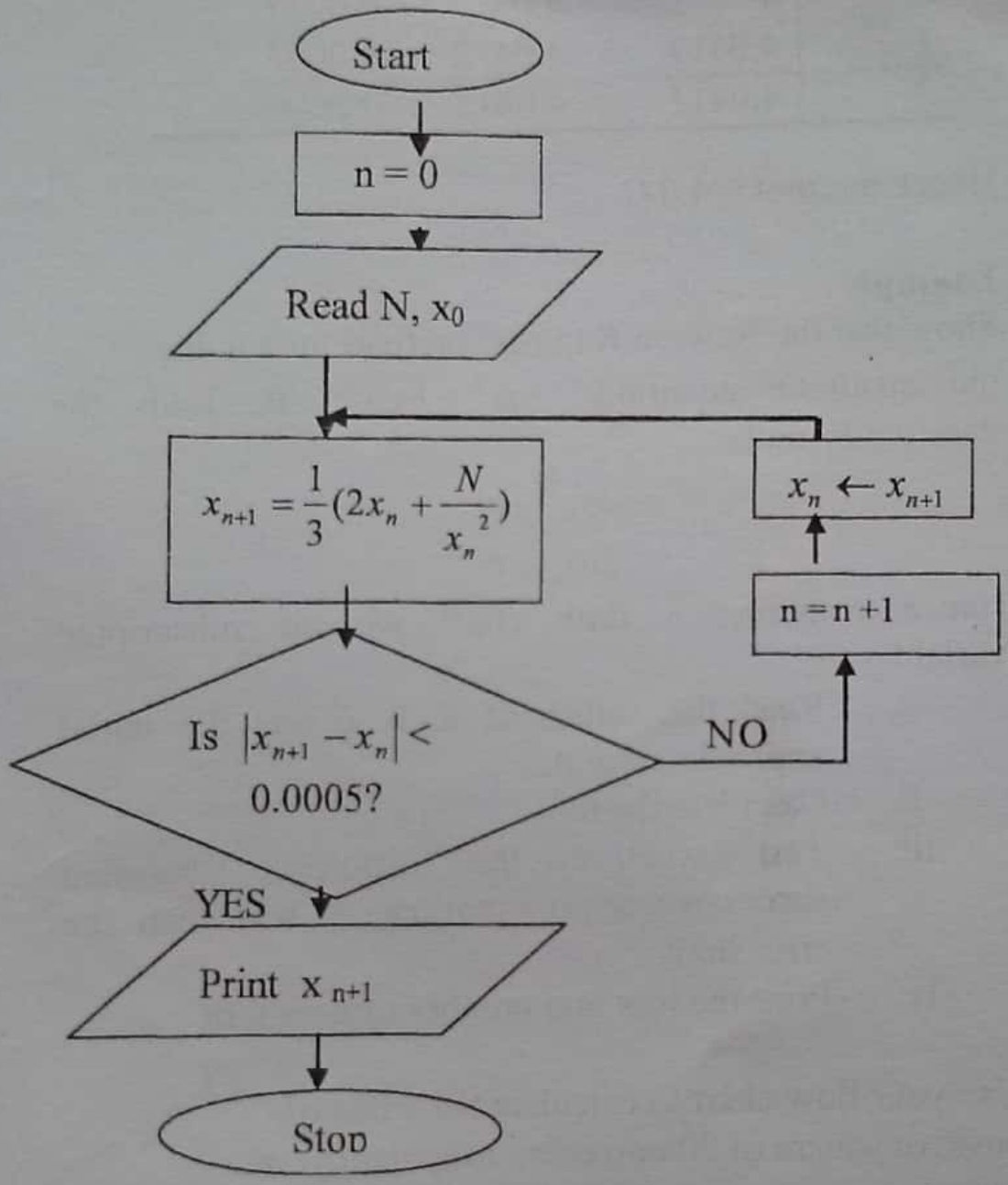
**Solution**



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| $n$ | $x_n$ | $x_{n+1}$ | $\lvert x_{n+1} - x_n \rvert$ |
|---|---|---|---|
| 0 | 4 | 4.0417 | 0.0417 |
| 1 | 4.0417 | 4.0412 | 0.0005 |
| 2 | 4.0412 | 4.0412 | 0.0000 |

Hence the root = 4.041

**Example**

Show that the Newton-Raphson method for solving the quadratic equation $ax^2 + bx + c = 0$, leads the iterative formula

$$x_{n+1} = \frac{ax_n^{\ 2} - c}{2ax_n + b}$$

Hence construct a flow chart without subscripted variables to;

i. Read the values of a, b, c and the initial approximation A.
ii. Calculate the root
iii. Test whether the difference between successive approximations is less than the error limit $\varepsilon$ ,
iv. Print the root and number of iterations

Use your flow chart to calculate the value of positive square of 20 correct to four significant figures.



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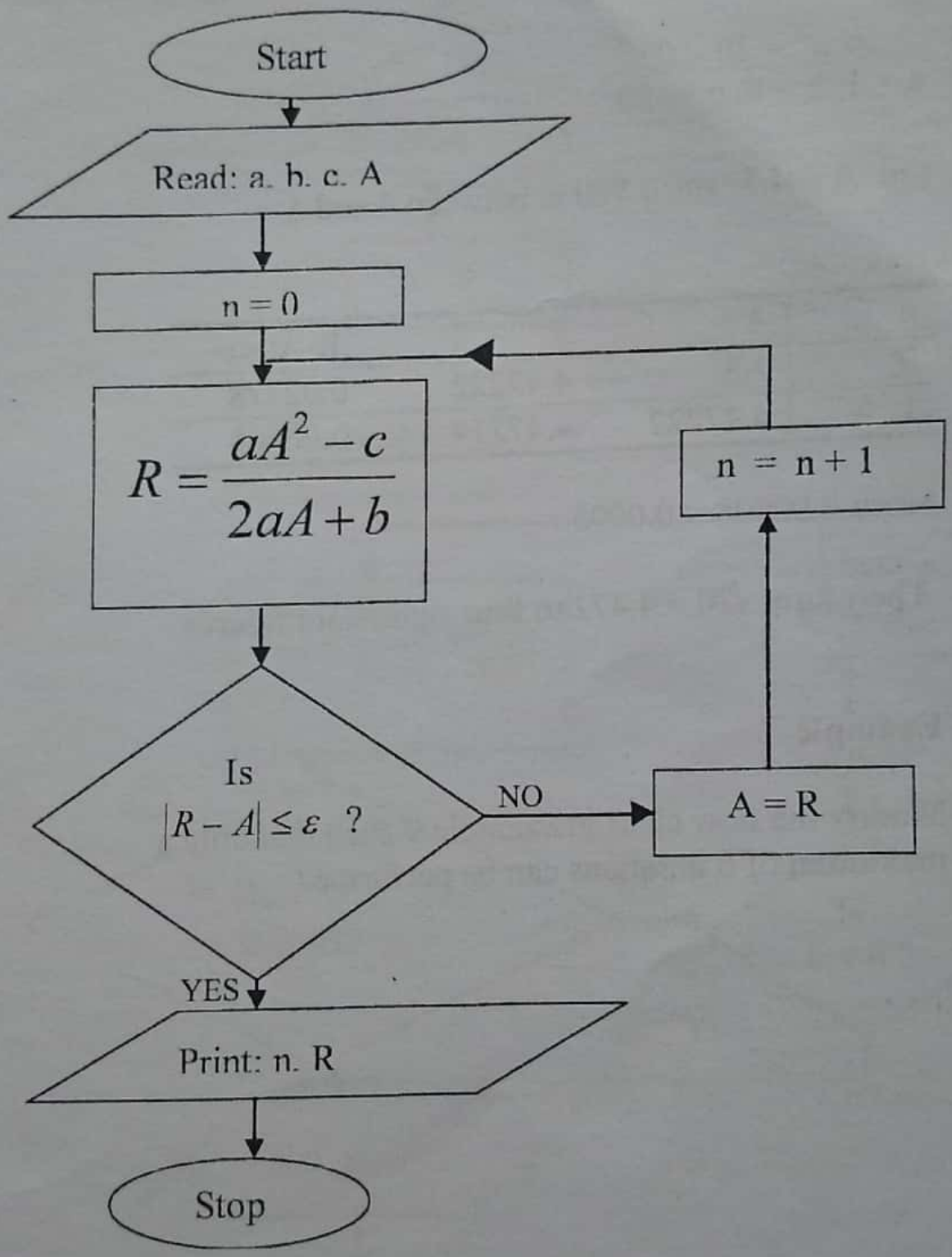
**Solution**



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Dry run

$$x = \sqrt{20}$$

$$\Rightarrow x^2 - 20 = 0$$

a = 1, b = 0, c = -20

and A = 4.5 since √20 is between 4 and 5

| $n$ | $A$ | $R$ | $\lvert R-A \rvert < \epsilon$ |
|---|---|---|---|
| 0 | 4.5 | 4.47222 | 0.02778 |
| 1 | 4.47222 | 4.47214 | 0.00008 |

Since 0.00008 < 0.0005

Therefore √20 = 4.472 to four significant figures

**Example**

Modify the flow chart in example 9 such that only a maximum of 6 iterations can be performed



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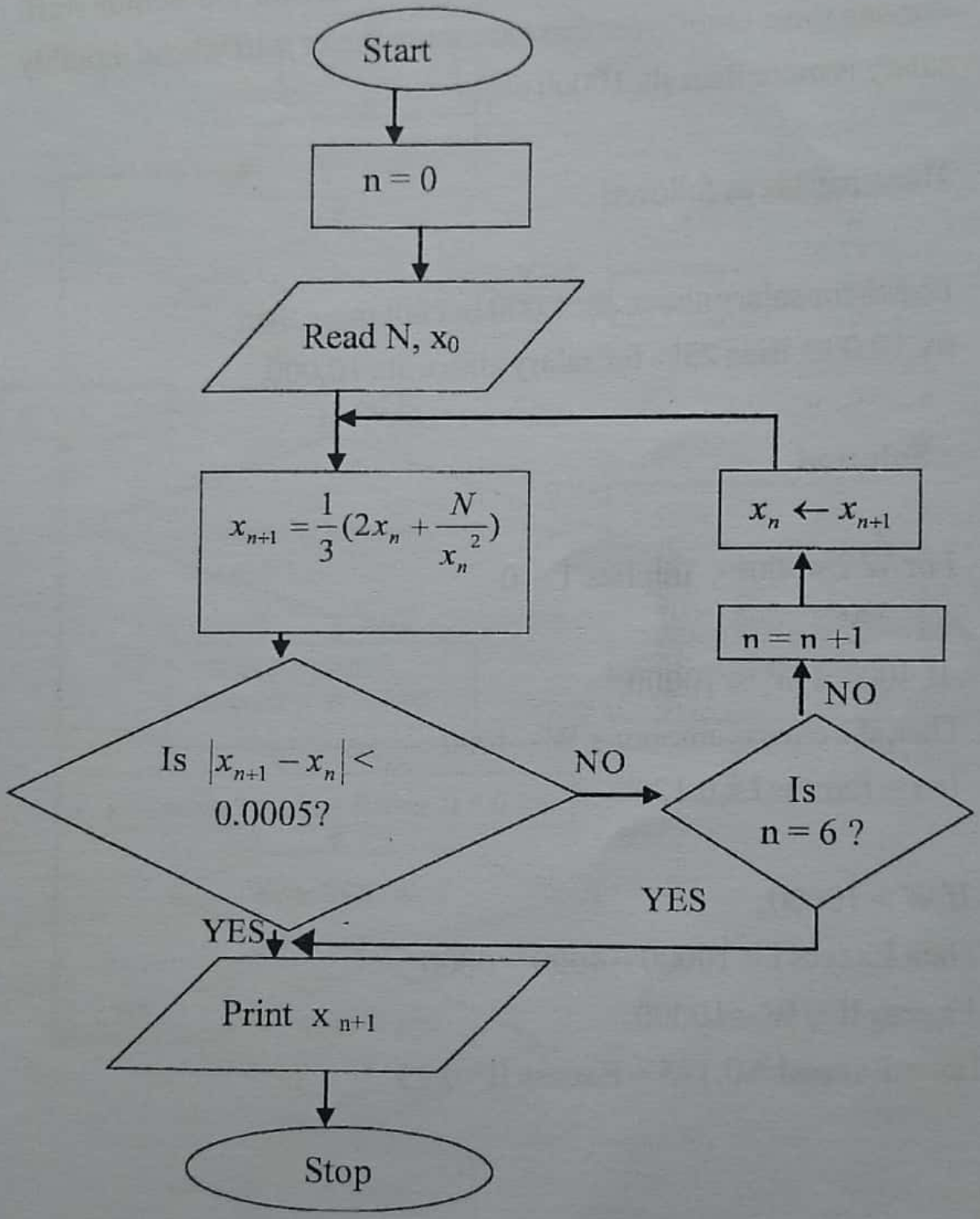
**Modification**



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**Example**

Modify the flow chart in **example 4** to cater for the Senior staff. Among these employees there are some senior staff whose monthly salary is more than sh. 10000 each.

These pay tax as follows:

12.5% for salary above sh. 4,000 but not more than sh. 10,000; then 25% for salary above sh. 10,000.

**Solution**

For $W \leq 4000/=$, Implies $T = 0$

If $4000 \leq W \leq 10000$,
Then the excess amount $= W - 4000$
Tax = Excess I x 0.125

If $W > 10000$,
Then Excess I $= 10000 - 4000 = 6000$
Excess II $= W - 10000$
Tax = ExcessI * 0.125 + Excess II* 0.25



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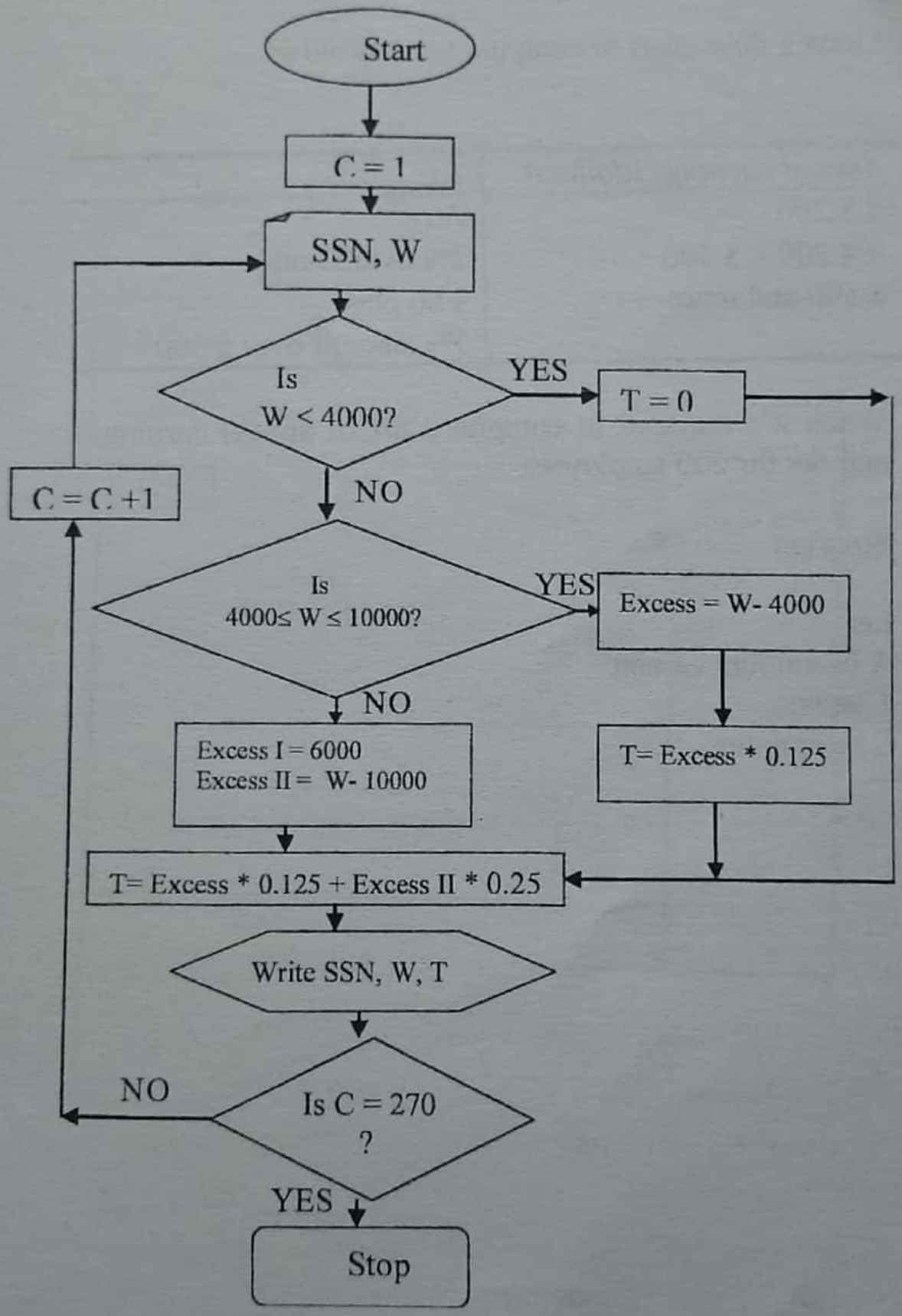




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Example

Draw a flow chart to compute tax as follows

| *Annual earnings (dollars)* | *Tax* |
|---|---|
| < \$ 200 | Zero |
| ≥ \$ 200 < \$ 400 | 2% of amount |
| \$ 500 and more | \$ 60 plus |
| | 5% amount over \$ 500 |

When it's required to compile a list of annual earnings and tax for 200 employees.

Solution

Let
A be amount earned
T be tax
C be counter



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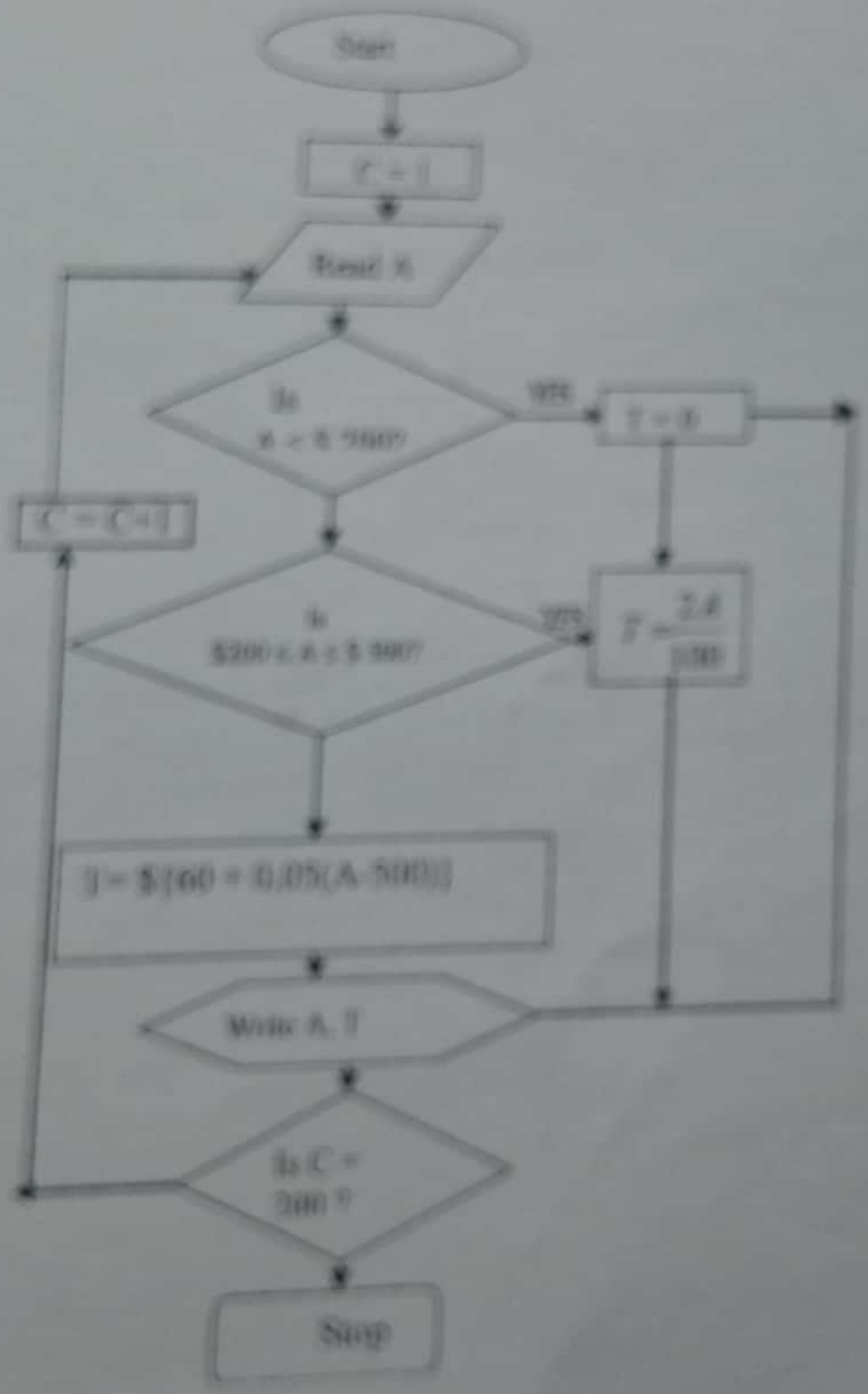




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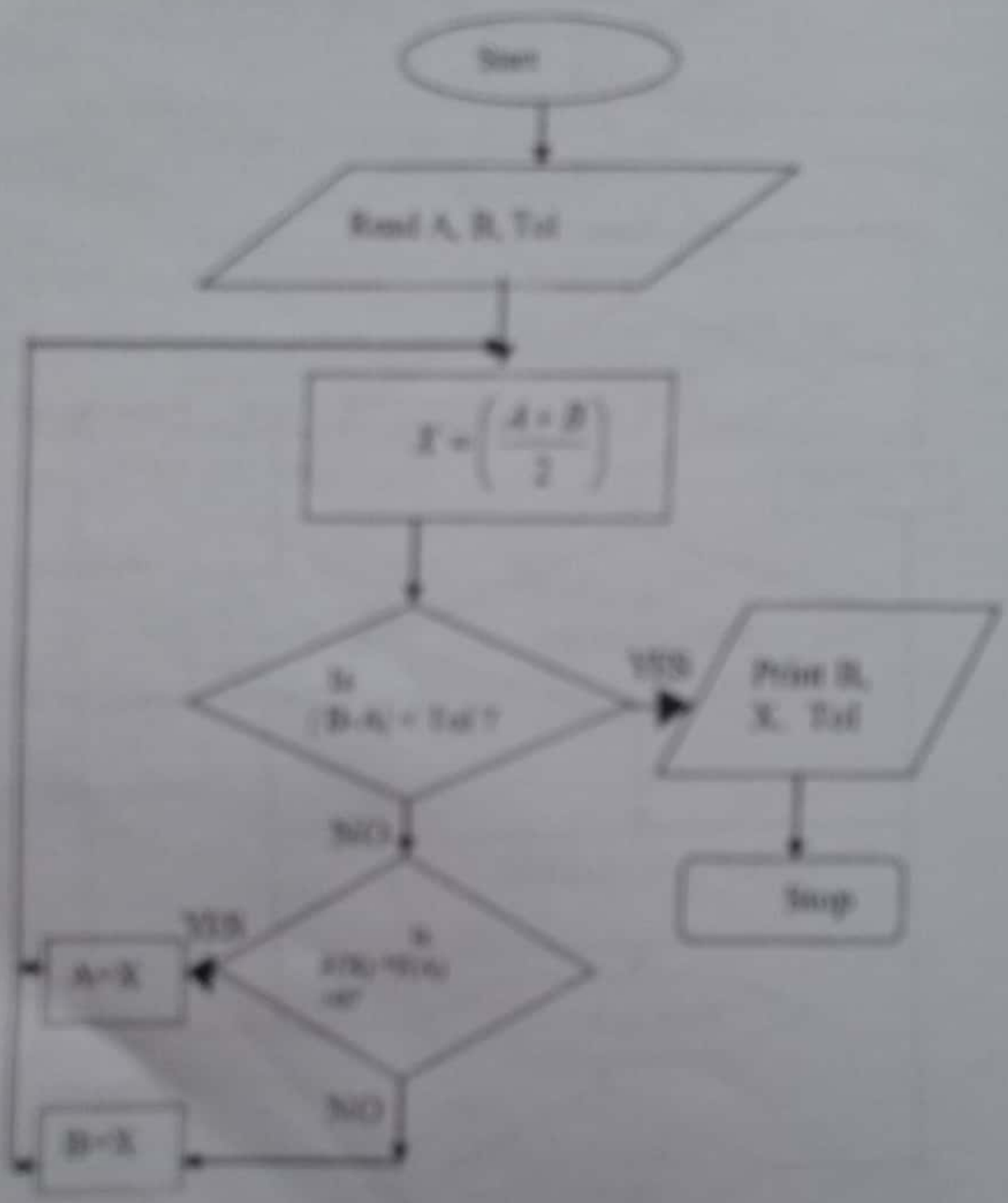
An iteration method for approximating a root of the equation $f(x)=0$ is described in the follow chart below.



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Given that A = 1.6875, B = 1.8750, TOL = 0.01 perform a dry run for the flow chart to determine $\sqrt[3]{5}$ tabulating the values of A,B and X at each stage.

Solution

$$f(x)=0 \Rightarrow \sqrt[3]{N}=x$$
$$f(x)=x^3-N=0$$
$$f(x)=x^3-N$$
$$When\ N=5$$
$$\Rightarrow f(1.6875)=1.6875^3-5=-0.1946$$
$$Then\ f(1.78125)=1.78125^3-5=0.6516$$

| A | B | X | \|B-A\| | F(X)*F(A) |
|---|---|---|---|---|
| 1.6875 | 1.8750 | 1.78125 | 0.1875 | < 0 |
| 1.6875 | 1.78125 | 1.734375 | 0.099375 | <0 |
| 1.6875 | 1.734375 | 1.7109375 | 0.046875 | <0 |
| 1.6875 | 1.7109375 | 1.6992188 | 0.0224 | <0 |
| 1.69922 | 1.7109375 | 1.7050788 | 0.0118 | <0 |
| 1.705079 | 1.7109375 | 1.708007 | 0.0059 | <0 |
| 1.708007 | 1.7109375 | 1.70947 | 0.002937 | |

$$\therefore \sqrt[3]{5}=1.70947=1.71$$



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## COMPOUND INTEREST

Sometimes people invest their money in the bank. It is important that the person get know how much he will earn after some time. If we let the amount be A and interest be 2% per year.

Then at end of the year he earns = $A + \frac{2A}{100} = 1.02A$

The following year he earns =

$$1.02A + \frac{2}{100} x 1.02A = 1.02^2 A$$

So an expression can be obtained for any given number of years and flow chart can be constructed

$$P = 1.02^n A$$

**Example**

Construct a flow chart for public servant who earns A millions and he/she invest in bank at rate of 8% compound interest for n years. Perform a dry run for
$n = 4, A_0 = 2$ million



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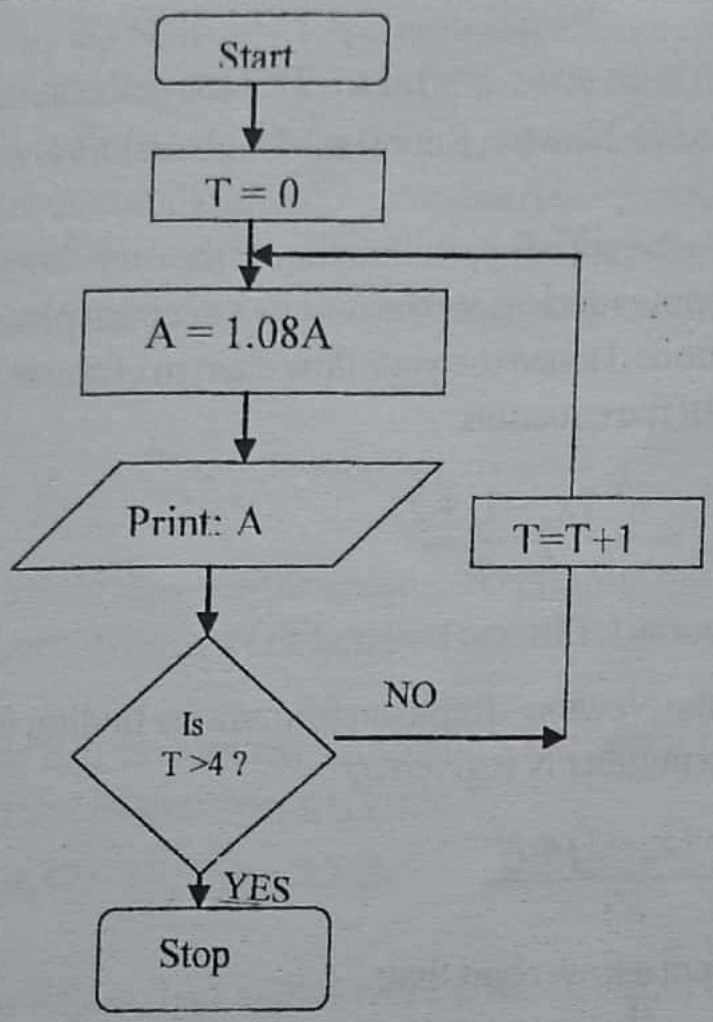


Dry run

| *T* | *A(millions)* |
|---|---|
| 0 | 2 |
| 1 | 2.16 |
| 2 | 2.3328 |
| 3 | 2.5194 |
| 4 | 2.939 |

Therefore A = 2.939 millions after four years



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## Exercise

1. An equation is given by $x = \ln(x+2)$. Derive the iterative formulae based on Newton Raphson's Method. Draw a flow chart that:

   i. Reads the initial approximation of the root
   ii. Computes and prints the root to 3 decimal places or after 4 iterations. Hence use your flow chart to evaluate the positive root of the equation.

**Answer**

$$x_{n+1} = \frac{e^{x_n}(x_n - 1) + 2}{e^{x_n} - 1}$$

$$Root = 1.146$$

2. Show that the Newton–Raphson formula for finding the natural logarithm of a number N is given by

$$x_{n+1} = \frac{e^{x_n}(x_n - 1) + N}{e^{x_n}}$$

Hence construct a flow chart that:

   i. Reads N and the initial approximation $x_0$
   ii. Computes the logarithm to 2 decimal places
   iii. Prints out the natural logarithm $\alpha$

Perform a dry run for N=1.7, $x_0 = 0.5$

**Answer**

| n | $x_n$ | $x_{n+1}$ | $\lvert x_{n+1} - x_n \rvert$ |
|---|---|---|---|
| 0 | 0.5 | 0.5675 | 0.0675 |
| 1 | 0.5675 | 0.5653 | 0.0022 |



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3.Derive by the Newton – Raphson's formula, the iterative that can be used for finding the fifth root of a number N. Write a simple algorithm that can be used for finding the root. Construct a simple flow chart that reads and computes the root. Taking N= 37 and initial approximation $x_0 = 2$, perform a dry for your flow chart to read the root correct to three decimal places.

**Answer** $x_{n+1} = \dfrac{4x_n^5 + 5}{5x_n^4}$, $root = 2.059(3d.p)$

4. Draw a flow chart for computing and printing the mean of the square roots of the first one hundred natural numbers

5. (i)Show that the Newton – Raphson's formula for approximating the reciprocal of a number n is given by

$$x_{n+1} = x_n(2 - Nx_n);\ n = 0,1,2...$$

(ii) Draw a flow chart to illustrate the use of the algorithm for computing andprinting of a close approximation to α. Take the initial approximation to be a and ensure that the computation ceases as soon as the absolute value of the difference between consecutive iterates is less than 0.005 or twenty iterations have been performed.

(iii) Perform a dry run for your flow chart for

N = 55.7; a = 0.02

**Answer** 0.02

6. Show that the Newton-Rahson formula for finding the root of the equation $2x^3 + 5x - 8 = 0$ is

$$\frac{4x_n^3 + 8}{6x_n^2 + 5}$$



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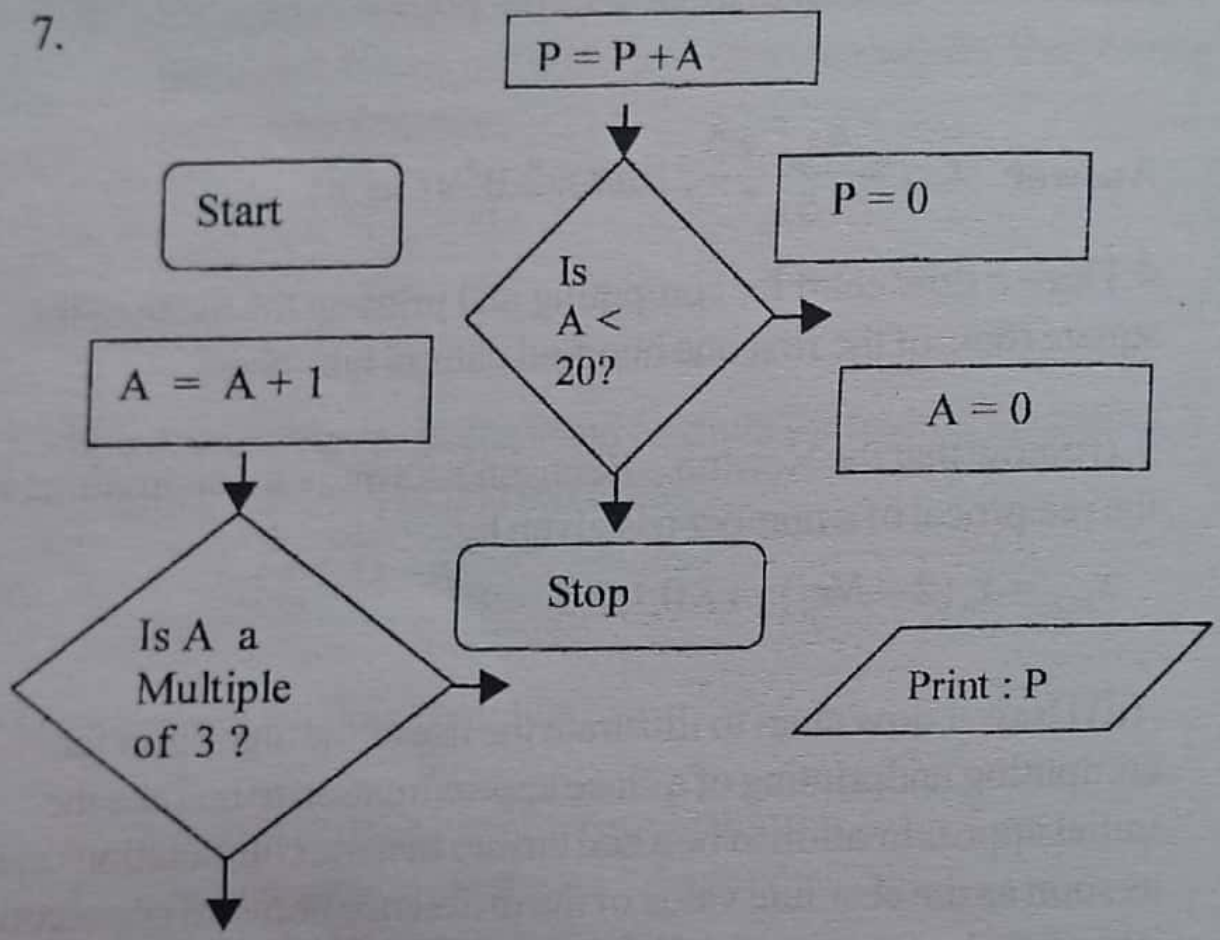
Taking the first approximation to the root of the above equation as 1.2. Draw a flow chart that reads and prints the number of iterations and root. Carry out a dry run of the flow chart and obtain the root with an error of less than 0.001

**Answer** 1.087

7.




Rearrange them and draw a complete logical flow chart Perform a dry run for the flow chart



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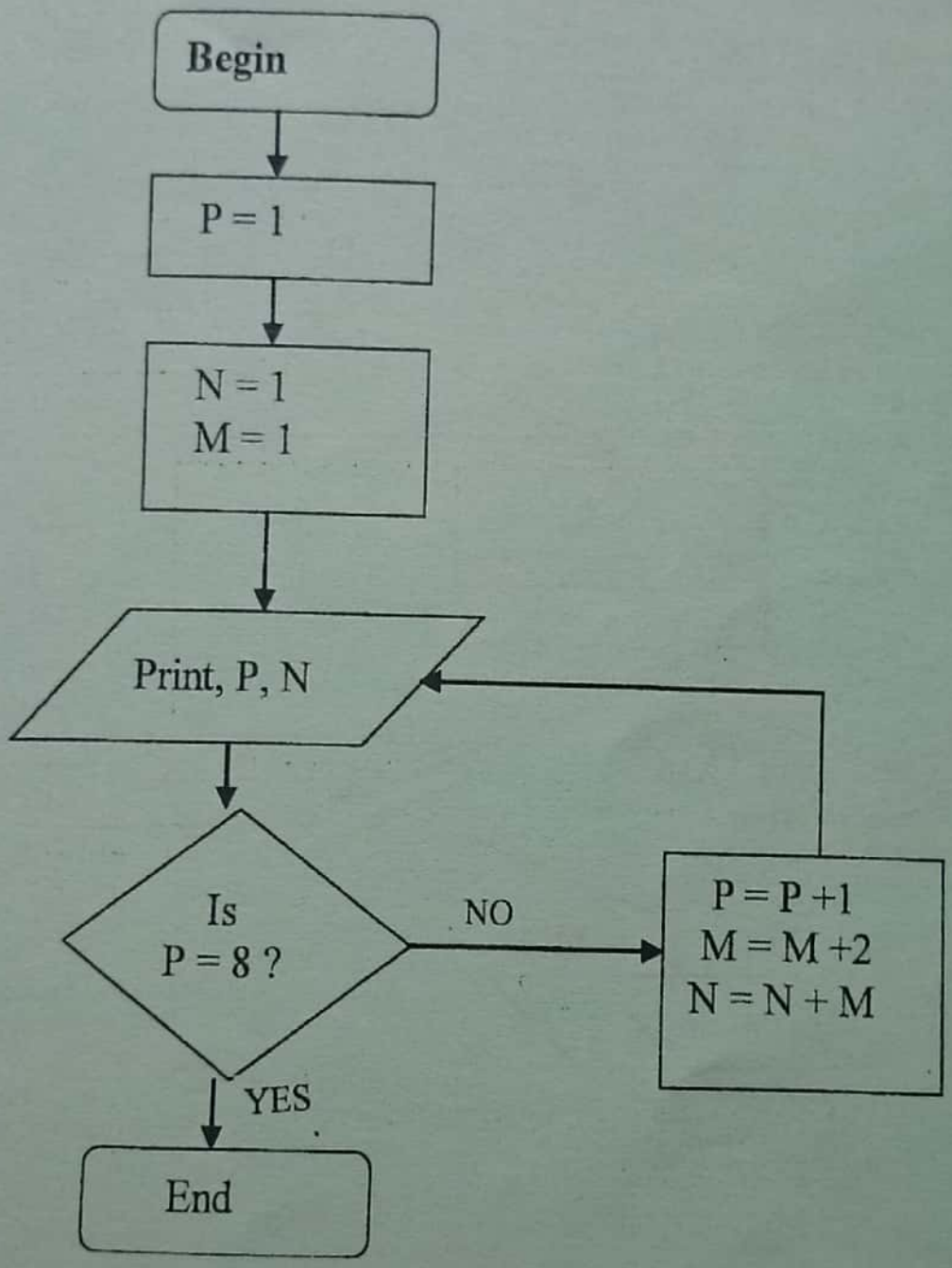
8. Study the flow chart below:-




(i.) Perform a dry run of the flow chart

(ii) State the relationship between P and N



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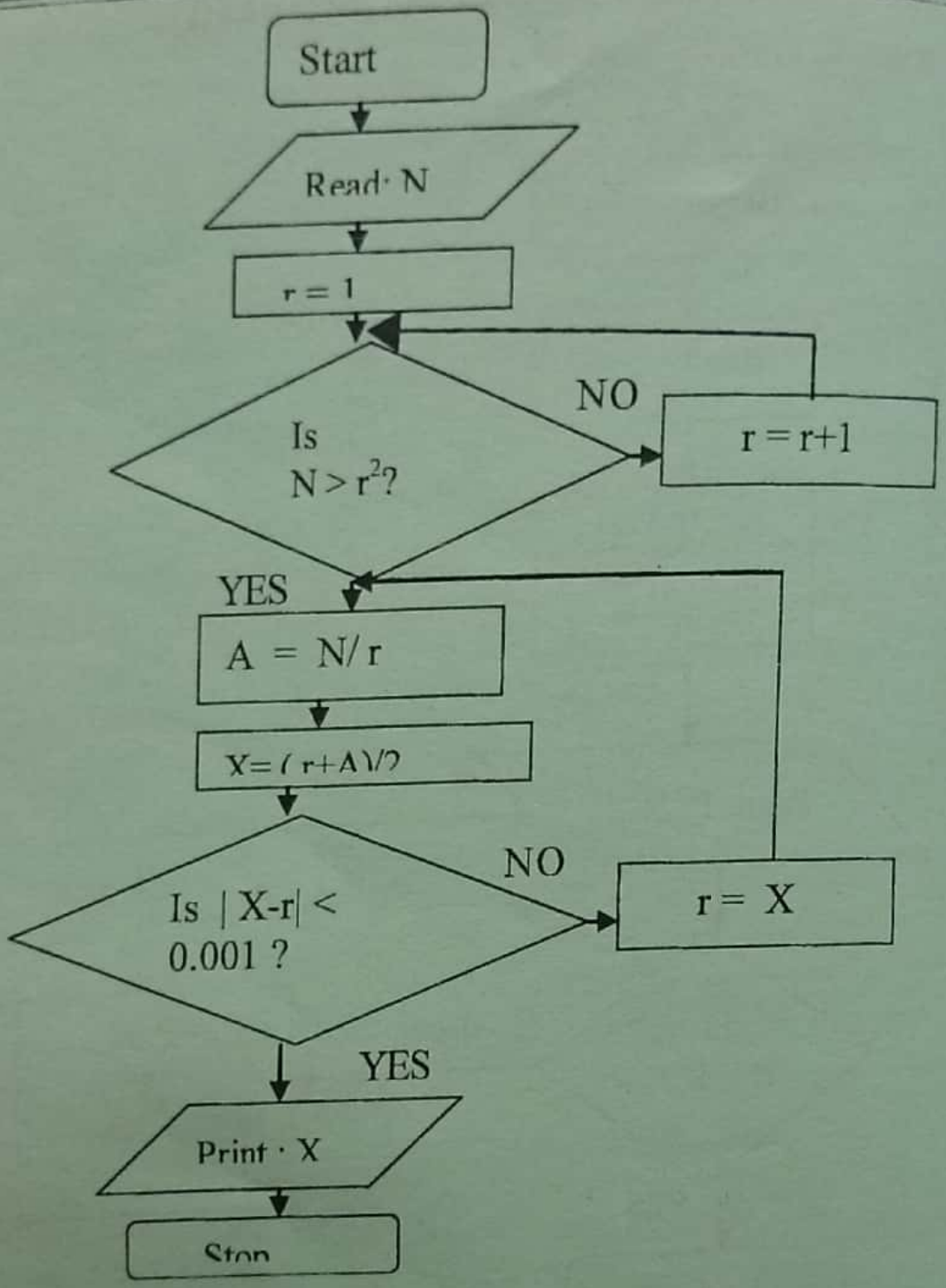


i. Perform a dry run for N= 20, showing clearly the contents of each store and values of X printed out.

ii. What process does the flow chart represent?



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# FORMULAE

## DESCRIPTIVE STATISTICS

Mean $\bar{x} = \dfrac{\sum x}{n}$ or $\bar{x} = \dfrac{\sum fx}{\sum f}$

Or $\bar{x} = A + \left[\dfrac{\sum ft}{\sum f}\right]c$ Let $t = \dfrac{x - A}{c}$

$$\text{Variance } (\sigma^2) = \left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]$$

$$\left[\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2\right]$$ Note: d = x -A

Note A is working mean

$$\text{Variance } (\sigma^2) = c^2\left[\frac{\sum ft^2}{\sum f} - \left(\frac{\sum ft}{\sum f}\right)^2\right]$$

$$\text{Standard Deviation (s)} = \left[\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2\right]^{0.5}$$



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$$\text{Median} = L_1 + \left(\frac{N/_2 - fb}{fm}\right)c$$

$$\text{Mode} = L_1 + \left(\frac{\Delta_1}{\Delta_1 + \Delta_1}\right)c$$

**PROBABILITY THEORY**

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (additional rule)

$P(A \cap B) = P(A) + P(B)$ (mutually exclusive)

$P(A \cap B) = P(A)\,P(B \mid A)$ (multiplication rule)

$P(A \cap B) = P(A)\ P(B)$ (for independent events)

Conditional probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Bayes' theorem $P(B_1|A) = \dfrac{P(B_1 \cap B)}{P(A)}$

$\Rightarrow P(B_1|A) =$

$$\frac{P(B_1)P(A \mid B_1)}{P(B_1)P(A \mid B_2) + P(B_2)P(A \mid B_2) + \ldots + P(B_k)P(A \mid B_k)}$$



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## DISCRETE PROBABILITY DISTRIBUTION

$\sum(X = x) = 1$

Mean $E(x) = \sum xP(X = x)$

$Var(X) = E(X^2) - [E(X)^2]$

$= \sum x^2 \quad P(X = x) - [\sum xP(X = x)]^2$

Note for series

$1 + 2 + 3 + 4 + 5 \ldots\ldots + n = \frac{1}{2}\, n(n + 1)$

$1^2 + 2^2 \quad + 3^2 + \ldots + n^2 = 1/6\, n(n + 1)(2n + 1)$

If $Sn = a + ar + ar^2 + \ldots ar^{n-1}$

$$S_n = \frac{a(r^2 - 1)}{r - 1} = \frac{a(1 - r^n}{1 - r}$$

If $-1 < r > 1$ then $S_n = \dfrac{a}{1 - r}$



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BINOMIAL DISTRIBUTION n cr or $\binom{n}{n-r} = \binom{n}{r}$

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | | | | | | | | | | |
| 2 | 1 | 2 | 1 | | | | | | | | | |
| 3 | 1 | 3 | 3 | 1 | | | | | | | | |
| 4 | 1 | 4 | 6 | 4 | 1 | | | | | | | |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | | | | | | |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 | | | | | |
| 7 | 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 | | | | |
| 8 | 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 | | | |
| 9 | 1 | 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 | | |
| 10 | 1 | 10 | 45 | 120 | 210 | 252 | 210 | 120 | 45 | 10 | 1 | |
| 11 | 1 | 11 | 55 | 165 | 330 | 462 | 462 | 330 | 165 | 55 | 11 | 1 |
| 12 | 1 | 12 | 66 | 220 | 495 | 792 | 924 | 792 | 495 | 220 | 66 | 12 |
| 13 | 1 | 13 | 78 | 286 | 715 | 1287 | 1716 | 1716 | 1287 | 715 | 286 | 78 |
| 14 | 1 | 14 | 91 | 364 | 1001 | 2002 | 3003 | 3432 | 3003 | 2002 | 1001 | 364 |
| 15 | 1 | 15 | 105 | 455 | 1365 | 3003 | 5005 | 6435 | 6435 | 5005 | 3003 | 1365 |
| 16 | 1 | 16 | 120 | 560 | 1820 | 4368 | 8008 | 11440 | 12870 | 11440 | 8008 | 4368 |
| 17 | 1 | 17 | 136 | 680 | 2380 | 6188 | 12376 | 19448 | 24310 | 24310 | 19448 | 123% |
| 18 | 1 | 18 | 153 | 816 | 3060 | 8568 | 18564 | 31824 | 43758 | 48620 | 43758 | 31824 |
| 19 | 1 | 19 | 171 | 969 | 3876 | 11628 | 27132 | 50388 | 75582 | 92378 | 92378 | 75582 |
| 20 | 1 | 20 | 190 | 1140 | 4845 | 15504 | 38760 | 77520 | 125970 | 167960 | 184756 | 167960 |
| 21 | 1 | 210 | 1330 | 5985 | 20349 | 54264 | 116280 | 203490 | 293930 | 352716 | 352716 | 352716 |
| 22 | 1 | 22 | 231 | 1540 | 7315 | 26334 | 74613 | 170544 | 319770 | 497420 | 646646 | 705432 |
| 23 | 1 | 23 | 253 | 1771 | 8855 | 33649 | 100947 | 245157 | 490314 | 817190 | 1144066 | 1352078 |



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# BINOMIAL DISTRIBUTION B (n,p) INDIVIDUAL TERMS

| n | r | 0.01 | 0.050 | 0.10 | 0.15 | 0.20 | 0.25 | 0.30 | 6.35 | 040 | 045 | 050 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 0 | 0.9801 | 9025 | 8100 | 7225 | 6400 | 5625 | 4900 | 4225 | 3600 | 3025 | 2500 |
| | 1 | 0.0198 | 0950 | 1800 | 2550 | 3200 | 3750 | 4200 | 4550 | 4800 | 4950 | 5000 |
| | 2 | 0.0001 | 0025 | 0100 | 0225 | 0400 | 0625 | 0900 | 1225 | 1600 | 2025 | 2500 |
| 3 | 0 | 0.9703 | 8574 | 7290 | 6141 | 5120 | 4219 | 3430 | 2746 | 2160 | 1664 | 1250 |
| | 1 | 0.0294 | 1354 | 2430 | 3251 | 3840 | 4219 | 4410 | 4436 | 4320 | 4084 | 3750 |
| | 2 | 0.0003 | 0071 | 0270 | 0574 | 0960 | 1406 | 1890 | 2389 | 2880 | 3341 | 3750 |
| | 3 | | 0001 | 0010 | 0034 | 0080 | 0156 | 0270 | 0429 | 0640 | 0911 | 1250 |
| 4 | 0 | 0.9606 | 8145 | 6561 | 5220 | 4096 | 3164 | 2401 | 1785 | 1296 | 0915 | 0625 |
| | 1 | 0.0388 | 1715 | 2916 | 3685 | 4096 | 4219 | 4116 | 3845 | 3456 | 2995 | 2500 |
| | 2 | 0.0006 | 0135 | 0486 | 0975 | 1536 | 2109 | 2646 | 3105 | 3456 | 3675 | 3750 |
| | 3 | | 0005 | 0036 | 0115 | 0256 | 0469 | 0756 | 1115 | 1536 | 2005 | 2500 |
| | 4 | | | 0001 | 0005 | 0016 | 0039 | 0081 | 0150 | 0256 | 0410 | 0625 |
| 5 | 0 | 0.9510 | 7738 | 5905 | 4437 | 3277 | 2373 | 1681 | 1160 | 0778 | 0503 | 0312 |
| | 1 | 0.0480 | 2036 | 3280 | 3915 | 4096 | 3955 | 3602 | 3124 | 2592 | 2059 | 1562 |
| | 2 | 0.0010 | 0214 | 0729 | 1382 | 2048 | 2637 | 3087 | 3364 | 3456 | 3369 | 3125 |
| | 3 | | 0011 | 0081 | 0244 | 0512 | 0879 | 1323 | 1811 | 2304 | 2757 | 3125 |
| | 4 | | | 0004 | 0022 | 0064 | 0146 | 0284 | 0488 | 0768 | 1128 | 1562 |
| | 5 | | | | 0001 | 0003 | 0010 | 0024 | 0053 | 0102 | 0185 | 0312 |
| 6 | 0 | 0.9415 | 7351 | 5314 | 3771 | 2621 | 1780 | 1176 | 0754 | 0467 | 0277 | 0156 |
| | 1 | 0.0571 | 2321 | 3543 | 3993 | 3932 | 3560 | 3025 | 2437 | 1866 | 1359 | 0938 |
| | 2 | 0.0014 | 0305 | 0984 | 1762 | 2458 | 2966 | 3241 | 3280 | 3110 | 2780 | 2344 |
| | 3 | | 0021 | 0146 | 0415 | 0819 | 1318 | 1852 | 2355 | 2765 | 3032 | 3125 |
| | 4 | | 0001 | 0012 | 0055 | 0154 | 0330 | 0595 | 0953 | 1382 | 1861 | 2344 |
| | 5 | | | 0001 | 0004 | 0015 | 0044 | 0102 | 0205 | 0369 | 0609 | 0938 |
| | 6 | | | | | 0001 | 0002 | 0007 | 0018 | 0041 | 0083 | 0156 |
| 7 | 0 | 0.9321 | 6983 | 4783 | 3206 | 2097 | 1335 | 0824 | 0490 | 0280 | 0152 | 0078 |
| | 1 | 0.0659 | 2573 | 3720 | 3960 | 3670 | 3115 | 2471 | 1848 | 1306 | 0872 | 0547 |
| | 2 | 0.0020 | 0406 | 1240 | 2097 | 2753 | 3115 | 3177 | 2985 | 2613 | 2140 | 1641 |
| | 3 | | 0036 | 0230 | 0617 | 1147 | 1730 | 2269 | 2679 | 2903 | 2918 | 2734 |
| | 4 | | 0002 | 0026 | 0109 | 0287 | 0577 | 0972 | 1442 | 1935 | 2388 | 2734 |
| | 5 | | | 0002 | 0012 | 0043 | 0115 | 0250 | 0466 | 0774 | 1172 | 1641 |
| | 6 | | | | 0001 | 0004 | 0013 | 0036 | 0084 | 0172 | 0320 | 0547 |
| | 7 | | | | | | 0001 | 0002 | 0006 | 0016 | 0037 | 0078 |
| 8 | 0 | 0.9227 | 6634 | 4305 | 2725 | 1678 | 1001 | 0576 | 0319 | 0168 | 00&4 | 0039 |
| | 1 | 0.0746 | 2793 | 3826 | 3847 | 3355 | 2670 | 1977 | 1373 | 0896 | 0548 | 0312 |
| | 2 | 0.0026 | 0515 | 1488 | 2376 | 2936 | 3115 | 2965 | 2587 | 2090 | 1569 | 1094 |
| | 3. | 0001 | 0054 | 0331 | 0839 | 1468 | 2076 | 2541 | 2786 | 2787 | 2568 | 2188 |
| | 4. | | 0004 | 0046 | 0185 | 0459 | 0865 | 1361 | 1875 | 2322 | 2627 | 2734 |
| | 5 | | | 0004 | 0026 | 0092 | 0231 | 0467 | 0808 | 1239 | 1719 | 2188 |
| | 6 | | | | 0002 | 0011 | 0038 | 0109 | 0217 | 0413 | 0703 | 1094 |
| | 7 | | | | | 0001 | 0004 | 0012 | 0033 | 0079 | 0164 | 0312 |
| | 8. | | | | | | | 0001 | 0002 | 0007 | 0017 | 0039 |



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| n | r | 0.01 | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 | 0.30 | 0.35 | 0.40 | 0.45 | 0.50 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 9 | 0 | 0.9135 | 6302 | 3874 | 2316 | 1342 | 0751 | 0404 | 0207 | 0101 | 0046 | 0020 |
| | 1 | 0.0830 | 2985 | 3874 | 3679 | 3020 | 2253 | 1556 | 1004 | 0605 | 0339 | 0176 |
| | 2 | 0.0034 | 0629 | 1722 | 2597 | 3020 | 3003 | 2668 | 2162 | 1612 | 1110 | 0703 |
| | 3 | 0.0001 | 0077 | 0446 | 1069 | 1762 | 2336 | 2668 | 2716 | 2508 | 2119 | 1641 |
| | 4 | | 0006 | 0074 | 0283 | 0661 | 1168 | 1715 | 2194 | 2508 | 2600 | 2461 |
| | 5 | | | 0008 | 0050 | 0165 | 0389 | 0735 | 1181 | 1672 | 2128 | 2461 |
| | 6 | | | 0001 | 0006 | 0028 | 0087 | 0210 | 0424 | 0743 | 1160 | 1641 |
| | 7 | | | | | 0003 | 0012 | 0039 | 0098 | 0212 | 0407 | 0703 |
| | 8 | | | | | | 0001 | 0004 | 0013 | 0035 | 0083 | 0176 |
| | 9 | | | | | | | | 0001 | 0003 | 0008 | 0020 |
| 10 | 0 | 0.9044 | 5987 | 3487 | 1969 | 1074 | 0563 | 0282 | 0135 | 0060 | 0025 | 0010 |
| | 1 | 0.0914 | 3151 | 3874 | 3474 | 2684 | 1877 | 1211 | 0725 | 0403 | 0207 | 0098 |
| | 2 | 0.0042 | 0746 | 1937 | 2759 | 3020 | 2816 | 2335 | 1757 | 1209 | 0763 | 0439 |
| | 3 | 0.0001 | 0105 | 0574 | 1298 | 2013 | 2503 | 2668 | 1522 | 2150 | 1665 | 1172 |
| | 4 | | 0010 | 0112 | 0401 | 0881 | 1460 | 2001 | 2377 | 2508 | 2384 | 2051 |
| | 5 | | 0001 | 0015 | 0085 | 0264 | 0584 | 1029 | 1536 | 2007 | 2340 | 2461 |
| | 6 | | | 0001 | 0012 | 0055 | 0162 | 0368 | 0689 | 1115 | 1596 | 2051 |
| | 7 | | | | 0001 | 0008 | 0031 | 0090 | 0212 | 0425 | 0746 | 1172 |
| | 8 | | | | | 0001 | 0004 | 0014 | 0043 | 0106 | 0229 | 0439 |
| | 9 | | | | | | | 0001 | 0005 | 0016 | 0042 | 0098 |
| | 10 | | | | | | | | | 0001 | 0003 | 0010 |
| 11 | 0 | 0.8953 | 5688 | 3138 | 1673 | 0859 | 0422 | 0198 | 0088 | 0036 | 0014 | 0005 |
| | 1. | 0.0995 | 3293 | 3835 | 3248 | 2362 | 1549 | 0932 | 0518 | 0266 | 0125 | 0054 |
| | 2 | 0.0050 | 0867 | 2131 | 2866 | 2953 | 2581 | 1998 | 3395 | 0887 | 0513 | 0269 |
| | 3 | 0.0002 | 0137 | 0710 | 1517 | 2215 | 2581 | 2568 | 2254 | 1774 | 1259 | 0806 |
| | 4 | | 0014 | 0158 | 0536 | 1107 | 1721 | 2201 | 2428 | 2365 | 2060 | 1611 |
| | 5 | | 0001 | 0025 | 0132 | 0388 | 0803 | 1321 | 1830 | 2207 | 2360 | 2256 |
| | 6 | | | 0003 | 0023 | 0097 | 0268 | 0566 | 0985 | 1471 | 1931 | 2256 |
| | 7 | | | | 0003 | 0017 | 0064 | 0173 | 0379 | 0701 | 1128 | 1611 |
| | 8 | | | | | 0002 | 0011 | 0037 | 0102 | 0234 | 0462 | 0806 |
| | 9 | | | | | | 0001 | 0005 | 0018 | 0052 | 0126 | 0269 |
| | 10 | | | | | | | | 0002 | 0007 | 0021 | 0054 |
| | 11 | | | | | | | | | | 0002 | 0005 |
| 12 | 0 | 0.8864 | 5404 | 2824 | 1422 | 0687 | 0317 | 0138 | 0057 | 0022 | 0008 | 0002 |
| | 1 | 0.1074 | 3413 | 3766 | 3012 | 2062 | 1267 | 0712 | 0368 | 0174 | 0075 | 0029 |
| | 2 | 0.0060 | 0988 | 2301 | 2924 | 2835 | 2323 | 1678 | 1088 | 0639 | 0339 | 0161 |
| | 3 | 0.0002 | 0173 | 0852 | 1720 | 2362 | 2581 | 2397 | 1954 | 1419 | 0923 | 0537 |
| | 4 | | 0021 | 0213 | 0683 | 1329 | 1936 | 2311 | 2367 | 2128 | 1700 | 1208 |
| | 5 | | 0002 | 0038 | 0193 | 0532 | 1032 | 1585 | 2039 | 2270 | 2225 | 1934 |
| | 6. | | | 0005 | 0040 | 0155 | 0401 | 0792 | 1281 | 1766 | 2124 | 2256 |
| | 7 | | | | 0006 | 0033 | 0115 | 0291 | 0591 | 1009 | 1489 | 1934 |
| | 8 | | | | 0001 | 0005 | 0024 | 0078 | 0199 | 0420 | 0762 | 1208 |



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| n | r | 0.01 | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 | 0.30 | 0.35 | 040 | 0.45 | 0.50 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 9 | | | | | 0001 | 0004 | 0015 | 0048 | 0125 | 0277 | 0537 |
| | 10 | | | | | | | 0002 | 0008 | 0025 | 0068 | 0161 |
| | 11 | | | | | | | | 0001 | 0003 | 0010 | 0029 |
| | 12 | | | | | | | | | | 0001 | 0002 |
| 15 | 0 | 0.8601 | 4633 | 2059 | 0874 | 0352 | 0134 | 0047 | 0016 | 0005 | 0001 | |
| | 1 | 0.1301 | 3658 | 3432 | 2312 | 1319 | 0668 | 0305 | 0126 | 0047 | 0016 | 0005 |
| | 2 | 0.0092 | 1348 | 2669 | 2856 | 2309 | 1559 | 0916 | 0476 | 0219 | 0090 | 0032 |
| | 3 | 0.0004 | 0307 | 1285 | 2184 | 2501 | 2252 | 1700 | 1110 | 0634 | 0318 | 0139 |
| | 4 | | 0049 | 0428 | 1156 | 1876 | 2252 | 2186 | 1792 | 1268 | 0780 | 0417 |
| | 5 | | 0006 | 0105 | 0449 | 1032 | 1651 | 2061 | 2123 | 1859 | 1404 | 0916 |
| | 6. | | | 0019 | 0132 | 0430 | 0197 | 1472 | 1906 | 2066 | 1914 | 1527 |
| | 7. | | | 0003 | 0030 | 0138 | 0393 | 0811 | 1319 | 1771 | 2013 | 1964 |
| | 8. | | | | 0005 | 0035 | 0131 | 0348 | 0710 | 1181 | 1647 | 1964 |
| | 9. | | | | 0001 | 0007 | 0034 | 0116 | 0298 | 0612 | 1048 | 1527 |
| | 10. | | | | | 0001 | 0007 | 0030 | 0096 | 0245 | 0515 | 0916 |
| | 11. | | | | | | 0001 | 0006 | 0024 | 0074 | 0191 | 0417 |
| | 12. | | | | | | | 0001 | 0004 | 0016 | 0052 | 0139 |
| | 13. | | | | | | | | 0001 | 0003 | 0010 | 0032 |
| | 14. | | | | | | | | | | 0001 | 0005 |
| 20. | 0 | 0.8179 | 3585 | 1216 | 0388 | 0115 | 0032 | 0008 | 0002 | | | |
| | 1 | 0.1652 | 3774 | 2702 | 1368 | 0576 | 0211 | 0068 | 0020 | 0005 | 0001 | |
| | 2. | 0.0159 | 1887 | 2852 | 2293 | 1369 | 0669 | 0278 | 0100 | 0031 | 0008 | 0002 |
| | 3. | 0.0010 | 0596 | 1901 | 2428 | 2054 | 1339 | 0716 | 0323 | 0123 | 0040 | 0011 |
| | 4. | | 0133 | 0898 | 1821 | 2182 | 1897 | 1304 | 0738 | 0350 | 0139 | 0046 |
| | 5. | | 0022 | 0319 | 1028 | 1746 | 2023 | 1789 | 1272 | 0746 | 0365 | 0148 |
| | 6. | | 0003 | 0089 | 0454 | 1091 | 1686 | 1916 | 1712 | 1244 | 0746 | 0370 |
| | 7. | | | 0020 | 0160 | 0545 | 1124 | 1643 | 1844 | 1659 | 1221 | 0739 |
| | 8. | | | 0004 | 0046 | 0222 | 0609 | 1144 | 1614 | 1797 | 1623 | 1201 |
| | 9. | | | 0001 | 0011 | 0074 | 0271 | 0654 | 1158 | 1597 | 1771 | 1602 |
| | 10. | | | | 0002 | 0020 | 0099 | 0308 | 0686 | 1171 | 1593 | 1762 |
| | 11. | | | | | 0005 | 0030 | 0120 | 0336 | 0710 | 1185 | 1605 |
| | 12. | | | | | 0001 | 0008 | 0039 | 0136 | 0355 | 0727 | 1201 |
| | 13. | | | | | | 0002 | 0010 | 0045 | 0146 | 0366 | 0739 |
| | 14. | | | | | | | 0002 | 0012 | 0049 | 0150 | 0370 |
| | 15. | | | | | | | | 0003 | 0013 | 0049 | 0148 |
| | 16. | | | | | | | | | 0003 | 0013 | 0046 |
| | 17. | | | | | | | | | | 0002 | 0011 |
| | 18. | | | | | | | | | | | 0002 |



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# CUMULATIVE BINOMIAL DISTRIBUTION $\sum b(x, n, p)$

| n | r | 0.01 | 0.05 | 0.10 | 0.15 | 0.20 P | 0.25 | 0.30 | 0.35 | 040 | 0.45 | 0.50 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 0.0199 | 0975 | 1900 | 2775 | 3600 | 4375 | 5100 | 5775 | 6400 | 6975 | 7500 |
| | 2 | 0.0001 | 0025 | 0100 | 0225 | 0400 | 0625 | 0900 | 1225 | 1600 | 2025 | 2500 |
| 3 | 1 | 00297 | 1426 | 2710 | 3859 | 4880 | 5781 | 6570 | 7254 | 7840 | 8336 | 8750 |
| | 2 | 0.0803 | 0072 | 0280 | 0608 | 1040 | 1562 | 2160 | 2818 | 3520 | 4252 | 5000 |
| | 3 | | 0001 | 0010 | 0034 | 0080 | 0156 | 0270 | 0429 | 0640 | 0911 | 1250 |
| 4 | 1 | 0.0394 | 1855 | 3439 | 4780 | 5904 | 6836 | 7599 | 8215 | 8704 | 9085 | 9375 |
| | 2 | 0.0006 | 0140 | 0523 | 1095 | 1808 | 2617 | 3483 | 4370 | 5248 | 6090 | 6875 |
| | 3 | | 0005 | 0037 | 0120 | 0272 | 0508 | 0837 | 1265 | 1792 | 2415 | 3125 |
| | 4 | | | 0001 | 0005 | 0016 | 0039 | 0081 | 0150 | 0256 | 0410 | 0625 |
| 5 | 1 | 0.0490 | 2262 | 4095 | 5563 | 6723 | 7627 | 8319 | 8840 | 9222 | 9497 | 9688 |
| | 2 | 0.0010 | 0226 | 0815 | 1648 | 2627 | 3672 | 4718 | 5716 | 6630 | 7438 | 8125 |
| | 3 | | 0012 | 0086 | 0266 | 0579 | 1035 | 1631 | 2352 | 3174 | 4069 | 5000 |
| | 4 | | | 0005 | 0022 | 0067 | 0156 | 0308 | 0540 | 0870 | 1312 | 1875 |
| | 5 | | | | 0001 | 0003 | 0010 | 0024 | 0053 | 0102 | 0185 | 0312 |
| 6 | 1 | 0.0585 | 2649 | 4686 | 6229 | 7379 | 8230 | 8824 | 9246 | 9533 | 9723 | 9844 |
| | 2 | 0.0015 | 0328 | 1143 | 2235 | 3446 | 4661 | 5798 | 6809 | 7667 | 8364 | 8906 |
| | 3 | | 0022 | 0158 | 0473 | 0989 | 1694 | 2557 | 3529 | 4557 | 5585 | 6562 |
| | 4 | | 0001 | 0013 | 0059 | 0170 | 0376 | 0705 | 1174 | 1792 | 2553 | 3438 |
| | 5 | | | 0001 | 0004 | 0016 | 0046 | 0109 | 0223 | 0410 | 0692 | 1094 |
| | 6 | | | | | 0001 | 0002 | 0007 | 0018 | 0041 | 0083 | 0156 |
| 7 | 1 | 0.0679 | 3017 | 5217 | 6794 | 7903 | 805 | 9176 | 9510 | 9720 | 9848 | 9922 |
| | 2 | 0.0020 | 0444 | 1497 | 2834 | 4233 | 5551 | 6706 | 7662 | 8414 | 8976 | 9375 |
| | 3 | | 0038 | 0257 | 0738 | 1480 | 2436 | 3529 | 4677 | 5801 | 6836 | 7734 |
| | 4 | | 0002 | 0027 | 0121 | 0333 | 0706 | 1260 | 1998 | 2898 | 3917 | 5000 |
| | 5 | | | 0002 | 0012 | 0047 | 0129 | 0288 | 0556 | 0963 | 1529 | 2266 |
| | 6 | | | | 0001 | 0004 | 0013 | 0038 | 0090 | 0188 | 1357 | 0625 |
| | 7 | | | | | | 0001 | 0002 | 0006 | 0016 | 0037 | 0078 |
| 8 | 1 | 0.0773 | 3366 | 5695 | 7275 | 8322 | 8999 | 9424 | 9681 | 9832 | 9916 | 9961 |
| | 2 | 0.0027 | 0572 | 1869 | 3428 | 4967 | 6329 | 7447 | 8309 | 8936 | 9368 | 9648 |
| | 3 | 0.0001 | 0058 | 0381 | 1052 | 2031 | 3215 | 4482 | 5722 | 6846 | 7799 | 8555 |
| | 4 | | 0004 | 0050 | 0214 | 0563 | 1138 | 1941 | 2936 | 4059 | 5230 | 6367 |
| | 5 | | | 0004 | 0029 | 0104 | 0273 | 0580 | 1061 | 1737 | 2604 | 3633 |
| | 6 | | | | 0002 | 0012 | 0042 | 0113 | 0253 | 0498 | 0885 | 1445 |



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# CUMULATIVE BINOMIAL DISTRIBUTION $\sum b(x, n, p)$

| n | r | 0.01 | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 | 0.30 | 0.35 | 040 | 0.45 | 0.50 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 7 | | | | | 0001 | 0004 | 0013 | 0036 | 0085 | 0181 | 0352 |
| | 8 | | | | | | | 0001 | 0002 | 0007 | 0017 | 0039 |
| 9 | 1 | 0.0865 | 3698 | 6126 | 7684 | 8658 | 9249 | 9596 | 9793 | 9899 | 9954 | 9980 |
| | 2 | 0.0034 | 0712 | 2252 | 4005 | 5638 | 6997 | 8040 | 8789 | 9295 | 9615 | 9805 |
| | 3. | 0.0001 | 0084 | 0530 | 1409 | 2618 | 3993 | 5372 | 6627 | 7682 | 8505 | 9102 |
| | 4 | | 0006 | 0083 | 0339 | 0856 | 1657 | 2703 | 3911 | 5174 | 6386 | 7461 |
| | 5 | | | 0009 | 0056 | 0196 | 0489 | 0988 | 1717 | 2666 | 3786 | 5000 |
| | 6 | | | 0001 | 0006 | 0031 | 0100 | 0253 | 0536 | 0994 | 1658 | 2539 |
| | 7 | | | | | 0003 | 0013 | 0043 | 0112 | 0250 | 0498 | 0898 |
| | 8 | | | | | | 0001 | 0004 | 0014 | 0038 | 0091 | 0195 |
| | 9 | | | | | | | | 0001 | 0003 | 0008 | 0020 |
| 10 | 1 | 0.0956 | 4013 | 6513 | 8031 | 8926 | 9437 | 9718 | 9865 | 9940 | 9975 | 9990 |
| | 2 | 0.0043 | 0961 | 2639 | 4557 | 6242 | 7560 | 8507 | 9140 | 9536 | 9787 | 9893 |
| | 3 | 00001 | 0115 | 0702 | 1798 | 3222 | 4744 | 6172 | 7384 | 8327 | 9004 | 9453 |
| | 4 | | 0010 | 0128 | 0500 | 1209 | 2241 | 3504 | 4862 | 6177 | 7340 | 8281 |
| | 5. | | 0001 | 0016 | 0099 | 0328 | 0781 | 1503 | 2485 | 3669 | 4956 | 6230 |
| | 6. | | | 0001 | 0014 | 0064 | 0197 | 0473 | 0949 | 1662 | 2616 | 3770 |
| | 7. | | | | 0001 | 0009 | 0035 | 0106 | 0260 | 0548 | 1020 | 1719 |
| | 8 | | | | | 0001 | 0004 | 0016 | 0048 | 0123 | 0274 | 0547 |
| | 9. | | | | | | | 0001 | 0005 | 0017 | 0045 | 0107 |
| | 10 | | | | | | | | | 0001 | 0003 | 0010 |
| 11 | 1 | 0.1047 | 4312 | 6862 | 8327 | 9141 | 9578 | 9802 | 9912 | 9964 | 9986 | 9995 |
| | 2 | 00052 | 1019 | 3006 | 5078 | 6779 | 8029 | 8870 | 9394 | 9698 | 9861 | 9941 |
| | 3 | 0.0002 | 0152 | 0896 | 2212 | 3826 | 5448 | 6873 | 7999 | 8811 | 9348 | 9673 |
| | 4 | | 0016 | 0185 | 0694 | 1611 | 2867 | 4304 | 5744 | 7037 | 8089 | 8867 |
| | 5 | | 0001 | 0028 | 0159 | 0504 | 1146 | 2103 | 3317 | 4672 | 6029 | 7256 |
| | 6 | | | 0003 | 0027 | 0117 | 0343 | 0782 | 1487 | 2465 | 3669 | 5000 |
| | 7 | | | | 0003 | 0020 | 0076 | 0216 | 0501 | 0994 | 1738 | 2744 |
| | 8 | | | | | 0002 | 0012 | 0043 | 0122 | 0293 | 0610 | 1133 |
| | 9 | | | | | | 0001 | 0006 | 0020 | 0059 | 0148 | 0327 |
| | 10 | | | | | | | | 0002 | 0007 | 0022 | 0059 |
| | 11 | | | | | | | | | | 0002 | 0005 |
| 12 | 1 | 0.1136 | 4596 | 7176 | 8578 | 9313 | 9683 | 9862 | 9943 | 9978 | 9992 | 9998 |
| | 2 | 0.0062 | 1184 | 3410 | 5565 | 7251 | 8416 | 9150 | 9576 | 9804 | 9917 | 9968 |
| | 3 | 0.0002 | 0196 | 1109 | 2642 | 4417 | 6093 | 7472 | 8487 | 9166 | 9579 | 9807 |
| | 4 | | 0022 | 0256 | 0922 | 2054 | 3512 | 5075 | 6533 | 7747 | 8655 | 9270 |
| | 5 | | 0002 | 0043 | 0239 | 0726 | 1576 | 2763 | 4167 | 5618 | 6956 | 8062 |



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# CUMULATIVE BINOMIAL DISTRIBUTION $\sum b(x, n, p)$

| n | r | 0.01 | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 | 0.30 | 0.35 | 040 | 0.45 | 0.50 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 6 | | | 0005 | 0046 | 0194 | 0544 | 1178 | 2127 | 3348 | 4731 | 6128 |
| | 7 | | | 0001 | 0007 | 0039 | 0143 | 0386 | 0846 | 1582 | 2607 | 3872 |
| | 8 | | | | 0001 | 0006 | 0028 | 0095 | 0255 | 0573 | 1117 | 1938 |
| | 9 | | | | | 0001 | 0004 | 0017 | 0056 | 0153 | 0956 | 0730 |
| | 10 | | | | | | | 0002 | 0008 | 0028 | 0079 | 0193 |
| | 11 | | | | | | | | 0001 | 0003 | 0011 | 0032 |
| | 12 | | | | | | | | | | 0001 | 0002 |
| 15 | 1 | 0.1399 | 5367 | 7941 | 9126 | 9648 | 9866 | 9953 | 9984 | 9995 | 9999 | 1.0000 |
| | 2 | 0.0096 | 1710 | 4510 | 6814 | 8329 | 9198 | 9647 | 9858 | 9948 | 9983 | 9995 |
| | 3 | 0.0004 | 0362 | 1841 | 3958 | 6020 | 7639 | 8732 | 9383 | 9729 | 9893 | 9963 |
| | 4 | | 0055 | 0556 | 1773 | 3518 | 5387 | 7031 | 8273 | 9095 | 9576 | 9824 |
| | 5 | | 0006 | 0127 | 0617 | 1642 | 3135 | 4845 | 6481 | 7827 | 8796 | 9408 |
| | 6 | | 0001 | 0022 | 0168 | 0611 | 1484 | 2784 | 4357 | 5968 | 7392 | 8491 |
| | 7 | | | 0003 | 0036 | 0181 | 0566 | 1311 | 2452 | 3902 | 5478 | 6964 |
| | 8 | | | | 0006 | 0042 | 0173 | 0500 | 1132 | 2131 | 3465 | 5000 |
| | 9 | | | | 0001 | 0008 | 0042 | 0152 | 0422 | 0950 | 1818 | 3036 |
| | 10 | | | | | 0001 | 0008 | 0037 | 0124 | 0338 | 0769 | 1509 |
| | 11 | | | | | | 0001 | 0007 | 0028 | 0093 | 0255 | 0592 |
| | 12 | | | | | | | 0001 | 0005 | 0019 | 0063 | 0176 |
| | 13 | | | | | | | | 0001 | 0003 | 0011 | 0037 |
| | 14 | | | | | | | | | | 0001 | 0005 |
| 20 | 1 | 0.1821 | 6415 | 8784 | 9612 | 9885 | 9968 | 9992 | 9998 | 1.0000 | 1.0000 | 1.0000 |
| | 2 | 0.0169 | 2642 | 6083 | 8244 | 9308 | 9757 | 9934 | 9979 | 9995 | 9999 | 1.0000 |
| | 3 | 0.0010 | 0755 | 3231 | 5951 | 7939 | 9087 | 9645 | 9879 | 9964 | 9991 | 9998 |
| | 4 | | 0159 | 1330 | 3523 | 5886 | 7748 | 8929 | 9556 | 9840 | 9951 | 9987 |
| | 5 | | 0026 | 0432 | 1702 | 3704 | 5852 | 7625 | 8818 | 9490 | 9811 | 9941 |
| | 6 | | 0003 | 0113 | 0673 | 1958 | 3828 | 5836 | 7546 | 8744 | 9447 | 9793 |
| | 7 | | | 0024 | 0219 | 0867 | 2142 | 3920 | 5834 | 7500 | 8701 | 9423 |
| | 8 | | | 0004 | 0059 | 0321 | 1018 | 2277 | 3990 | 5841 | 7480 | 8684 |
| | 9 | | | 0001 | 0013 | 0100 | 0409 | 1133 | 2376 | 4044 | 5857 | 7483 |
| | 10 | | | | 0002 | 0026 | 0139 | 0480 | 1218 | 2447 | 4086 | 5881 |
| | 11 | | | | | 0006 | 0039 | 0171 | 0532 | 1275 | 2483 | 4119 |
| | 12 | | | | | 0001 | 0009 | 0051 | 0196 | 0565 | 1308 | 2517 |
| | 13 | | | | | | 0002 | 0013 | 0060 | 0210 | 0580 | 1316 |
| | 14 | | | | | | | 0003 | 0015 | 0065 | 0214 | 0577 |
| | 15 | | | | | | | | 0003 | 0016 | 0064 | 0207 |
| | 16 | | | | | | | | | 0003 | 0015 | 0069 |
| | 17 | | | | | | | | | | 0003 | 0013 |
| | 18 | | | | | | | | | | | 0002 |



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NOBMAL DISTRIBUTION NN(0,1) (p(Z)

| Z | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | SUBTRACT 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.3989 | 3989 | 3989 | 3988 | 3986 | | | | | | | | | | | | | | |
| | | | | | | 3984 | 3982 | 3980 | 3977 | 3973 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| 0.1 | 0.3970 | 3965 | 3961 | 3956 | 3951 | | | | | | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| | | | | | | 3945 | 3939 | 3932 | 3925 | 3918 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 6 |
| 02 | 0.3910 | 3902 | 3894 | 3885 | 3876 | | | | | | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 5 |
| | | | | | | 3867 | 3857 | 3847 | 3836 | 3825 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 |
| 0.3 | 0.3814 | 3802 | 3790 | 3778 | 3765 | | | | | | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| | | | | | | 3752 | 3739 | 3725 | 3712 | 3697 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 13 |
| 0.4 | O.3683 | 3668 | 3653 | 3637 | 3621 | 3605 | 3589 | 3572 | 3555 | 3538 | 2 | 3 | 5 | 6 | 8 | 10 | 11 | 13 | 14 |
| 0.5 | 0.3521 | 3503 | 3485 | 3467 | 3448 | 3429 | 3410 | 3391 | 3372 | 3352 | 2 | 4 | 6 | 8 | 10 | 11 | 13 | 15 | 17 |
| 0.6 | 0.3332 | 3312 | 3292 | 3271 | 3251 | 3230 | 3209 | 3187 | 3166 | 3144 | 2 | 4 | 6 | 8 | 10 | 13 | 15 | 17 | 19 |
| 0.7 | 0.3123 | 3101 | 3079 | 3056 | 3034 | 3011 | 2989 | 2966 | 2943 | 2920 | 2 | 5 | 7 | 9 | 11 | 14 | 16 | 18 | 21 |
| 0.8 | 0.2897 | 2874 | 2850 | 2827 | 2803 | 2780 | 2756 | 2732 | 2709 | 2685 | 2 | 5 | 7 | 10 | 12 | 14 | 17 | 19 | 22 |
| 0.9 | 0.2661 | 2637 | 2613 | 2589 | 2565 | 2541 | 2516 | 2492 | 2468 | 2444 | 2 | 5 | 7 | 10 | 12 | 14 | 17 | 19 | 22 |
| 1.0 | 0.2420 | 2396 | 2371 | 2347 | 2323 | 2299 | 2275 | 2251 | 2227 | 2203 | 2 | 5 | 7 | 10 | 12 | 14 | 17 | 19 | 22 |
| 1.1 | 0.2179 | 2155 | 2131 | 2107 | 2083 | 2059 | 2036 | 2012 | 1989 | 1965 | 2 | 5 | 7 | 10 | 12 | 14 | 17 | 19 | 22 |
| 1.2 | 0.1942 | 1919 | 1895 | 1872 | 1849 | 1826 | 1804 | 1781 | 1758 | 1736 | 2 | 5 | 7 | 9 | 12 | 14 | 16 | 18 | 21 |
| 1.3 | 0.1714 | 1691 | 1669 | 1647 | 1626 | 1604 | 1582 | 1561 | 1539 | 1518 | 2 | 4 | 7 | 9 | 11 | 13 | 15 | 18 | 30 |
| 1.4 | 0.1497 | 1476 | 1456 | 1435 | 1415 | 1394 | 1374 | 1354 | 1334 | 1315 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| 1.5 | 0.1295 | 1276 | 1257 | 1238 | 1219 | 1200 | 1182 | 1163 | 1145 | 1127 | 2 | 4 | 6 | 8 | 9 | 11 | 13 | 15 | 17 |
| 1.6 | 0.1109 | 1092 | 1074 | 1057 | 1040 | 1023 | 1006 | 0989 | 0973 | 0957 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 14 | 15 |
| 1.7 | 0.0940 | 0925 | 0909 | 0893 | 0878 | 0863 | 0848 | 0833 | 0818 | 0804 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| 1.8 | 0.0790 | 0775 | 0761 | 0748 | 0734 | | | | | | I | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 11 |
| | | | | | | 0721 | 0707 | 0694 | 0681 | 0669 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 10 | 12 |
| 1.9 | 0.0656 | 0644 | 0632 | 0620 | 0608 | 0596 | 0584 | 0.773 | 0561 | 0551 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| 2.0 | 0.0540 | 0529 | 0519 | 0508 | 0498 | 0488 | 0478 | 0468 | 0459 | 0449 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2.1 | 0.0440 | 0431 | 0422 | 0413 | 0404 | 0396 | 0389 | 0379 | 0371 | 0363 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| 2.2 | 0.0355 | 0347 | 0339 | 0332 | 0325 | 0317 | 0310 | 0303 | 0297 | 0290 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 2.3 | 0.0283 | 0277 | 0270 | 0264 | 0258 | 0252 | 0246 | 0241 | 0235 | 0229 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 2.4 | 0.0244 | 0219 | 0213 | 0208 | 0203 | 0198 | 0194 | 0189 | 0184 | 0180 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 15 | 0.0175 | 0171 | 0167 | 0163 | 0158 | 0154 | 0151 | 0147 | 0143 | 0139 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| 2.6 | 0.0136 | 0132 | 0129 | 0126 | 0122 | 0119 | 0116 | 0113 | 0110 | 0107 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 |
| 2.7 | 0.0104 | 0101 | 0099 | 0096 | 0093 | 0091 | 0088 | 0086 | 0084 | 0081 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| 2.8 | 0.0079 | 0077 | 0075 | 0073 | 0071 | 0069 | 0067 | 0065 | 0063 | 0061 | | | | | | | | | |
| 2.9 | 0.0060 | 0058 | 0056 | 0055 | 0053 | 0051 | 0050 | 0048 | 0047 | 0046 | | | | | | | | | |
| 3. | 0.0044 | 0033 | | | | | | | | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| | | | 0024 | 0017 | | | | | | | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| | | | | | 0012 | 0009 | 0006 | 0004 | 0003 | 0002 | | | | | | | | | |



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CUMULATIVE NORMAL DISTRIBUTION P(z)

| Z | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.0000 | 0040 | 0080 | 0120 | 0160 | 0199 | 0239 | 0279 | 0319 | 0859 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 |
| 0.1 | 0.398 | 0438 | 0478 | 0517 | 0557 | 0596 | 0636 | 0675 | 0714 | 0753 | 4 | 8 | 12 | 15 | 19 | 22 | 27 | 31 | 35 |
| 0.2 | 0.0793 | 0832 | 0671 | 0910 | 0948 | 0987 | 1026 | 1064 | 1103 | 1141 | 4 | 8 | 12 | 15 | 19 | 22 | 27 | 31 | 35 |
| 0.3 | 0.1179 | 1217 | 1255 | 1293 | 1331 | 1366 | 1406 | 1443 | 1480 | 1517 | 4 | 8 | 11 | 15 | 19 | 22 | 26 | 30 | 34 |
| 0.4 | 0.1554 | 1591 | 1628 | 1664 | 1700 | 1736 | 1772 | 1808 | 1844 | 1879 | 4 | 7 | 11 | 14 | 18 | 22 | 25 | 29 | 32 |
| 0.5 | 0.1915 | 1950 | 1985 | 2019 | 2054 | 2088 | 2123 | 2157 | 2190 | 2224 | 3 | 7 | 10 | 14 | 17 | 21 | 24 | 27 | 31 |
| 0.6 | 0.2257 | 2291 | 2324 | 2357 | 2389 | 2422 | 2454 | 2486 | 2517 | 2549 | 3 | 6 | 10 | 13 | 16 | 19 | 23 | 36 | 29 |
| 0.7 | 0.2580 | 2611 | 2642 | 2673 | | | | | | | 3 | 6 | 9 | 12 | 15 | 19 | 22 | 25 | 28 |
| | | | | | 2704 | 2734 | 2764 | 2794 | 2823 | 2852 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |
| 0.8 | 0.2881 | 2910 | 2939 | 2967 | 2995 | 3023 | | | | | 3 | 6 | 8 | 11 | 14 | 17 | 20 | 22 | 25 |
| | | | | | | | 3051 | 3078 | 3106 | 3133 | 3 | 5 | 8 | 11 | 13 | 16 | 19 | 22 | 24 |
| 0.9 | 0.3159 | 3186 | 3212 | 3238 | 3264 | 3289 | | | | | 3 | 5 | 8 | 10 | 13 | 16 | 18 | 21 | 23 |
| | | | | | | | 3315 | 3340 | 3365 | 3389 | 2 | 5 | 7 | 10 | 12 | 15 | 17 | 20 | 22 |
| 1.0 | 0.3413 | 3438 | 3461 | 3485 | 3508 | | | | | | 2 | 5 | 7 | 10 | 12 | 14 | 17 | 19 | 22 |
| | | | | | | 3531 | 3554 | 3577 | 3399 | 3621 | 2 | 4 | 7 | 9 | 11 | 13 | 15 | 18 | 20 |
| 1.1 | 0.366 | 3665 | 3686 | 3708 | | | | | | | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 |
| | | | | | 3729 | 3749 | 3770 | 3790 | 3810 | 3830 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| 1.2 | 0.3849 | 3869 | 3888 | 3907 | 3925 | | | | | | 2 | 4 | 6 | 8 | 10 | 11 | 13 | 15 | 17 |
| | | | | | | 3944 | 3962 | 3980 | 3997 | 4015 | 2 | 4 | 5 | 7 | 9 | 11 | 13 | 14 | 16 |
| 1.3 | 0.432 | 4049 | 4066 | 4082 | 4099 | 4115 | 4131 | 4147 | 4162 | 4177 | 2 | 3 | 5 | 6 | 8 | 10 | 11 | 13 | 14 |
| 1.4 | 0.4192 | 4207 | 4222 | 4236 | 4251 | 4265 | 4279 | 4292 | 4306 | 4319 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 13 |
| 1.5 | 0.4332 | 4345 | 4357 | 4370 | 4382 | 4394 | 4406 | 4418 | 4429 | 4441 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| 1.6 | 0.4452 | 4463 | 4474 | 4484 | 4495 | 4905 | 4515 | 4535 | 4535 | 4545 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1.7 | 0.4554 | 4564 | 4573 | 4382 | 4591 | 4599 | 4608 | 4616 | 4625 | 4633 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1.8 | 0.4641 | 4619 | 4656 | 4661 | 4671 | 4678 | 4686 | 4693 | 4899 | 4706 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 1.9 | 0.4713 | 4719 | 4726 | 4732 | 4738 | 4744 | 4750 | 4756 | 4761 | 4767 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 2.0 | 0.4772 | 4778 | 4783 | 4788 | 4793 | 4798 | 4803 | 4808 | 4812 | 4817 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 2.1 | 0.4821 | 4836 | 4830 | 4834 | 4838 | 4842 | 4846 | 4850 | 4854 | 4857 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| 2.2 | 0.4861 | 4864 | 4868 | 4871 | 4875 | 4878 | 4881 | 4884 | 4887 | 4890 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| 2.3 | 0.4893 | 4896 | 4898 | 4901 | 4904 | 4906 | 4909 | 4911 | 4913 | 4916 | 0 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| 2.4 | 0.4918 | 4920 | 4922 | 4925 | 4927 | 4929 | 4931 | 4932 | 4934 | 4936 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| 2.5 | 0.4938 | 4940 | 4941 | 4943 | 4945 | 4946 | 4948 | 4949 | 4951 | 4952 | | | | | | | | | |
| 2.6 | 0.4953 | 4955 | 4956 | 4957 | 4959 | 4960 | 4961 | 4982 | 4963 | 4964 | | | | | | | | | |
| 2.7 | 0.4965 | 4966 | 4967 | 4968 | 4969 | 4970 | 4971 | 4972 | 4973 | 4974 | | | | | | | | | |
| 2.8 | 0.4974 | 4975 | 4976 | 4977 | 4977 | 4978 | 4979 | 4979 | 4980 | 4981 | | | | | | | | | |
| 2.9 | 0.4961 | 4982 | 4982 | 4983 | 4984 | 4984 | 4985 | 4985 | 4986 | 4986 | | | | | | | | | |
| 3.0 | 0.4987 | 4990 | 4993 | 4995 | 4997 | 4998 | 4998 | 4999 | 4999 | 5000 | | | | | | | | | |



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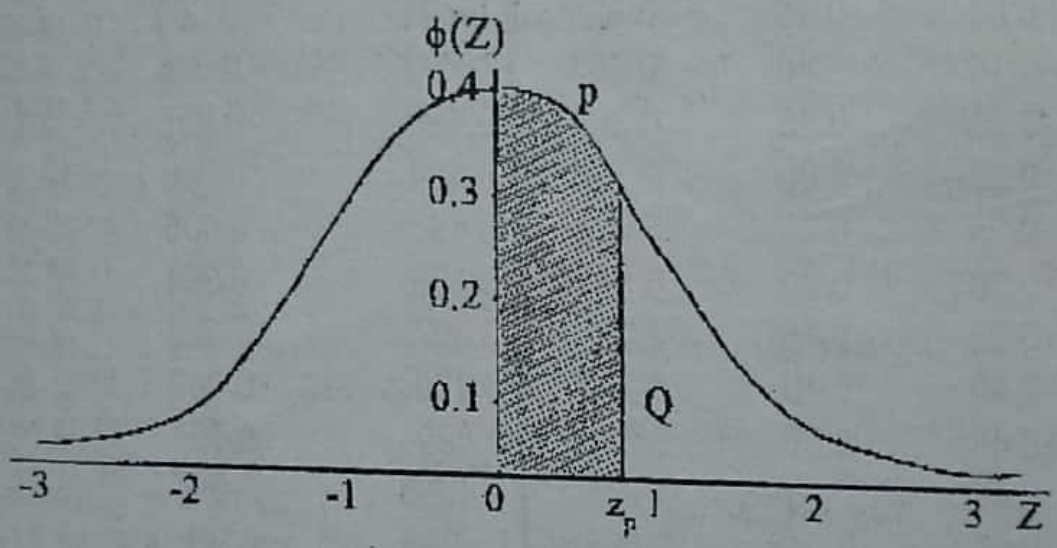
Normal Curve




If the random variable Z is distributed as the standard normal distribution N(0, 1) then

1. $P(0 < Z, < z_p) = P$ (shaded area)
2. $P(Z > z_p) = Q = 0.5 - P$
3. $P(Z > |Z_p|) = 1 - 2P = 2Q$



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## CRITICAL POINTS OF THE NORMAL DISTRIBUTION $z_p$

| P | Q | z | P | Q | |
|---|---|---|---|---|---|
| 0.00 | 0.50 | 0.000 | 0.490 | 0.010 | 2.326 |
| 0.05 | 0.45 | 0.126 | 0.491 | 0.009 | 2.366 |
| 0.10 | 0.40 | 0.253 | 0.492 | 0.008 | 2.409 |
| 0.15 | 0.35 | 0.385 | 0.493 | 0.007 | 2.457 |
| 0.20 | 0.30 | 0.524 | 0.494 | 0.006 | 2.512 |
| 0.25 | 0.25 | 0.674 | 0.495 | 0.005 | 2.575 |
| 0.30 | 0.20 | 0.842 | 0.496 | 0.004 | 2.652 |
| 0.35 | 0.15 | 1.036 | 0.497 | 0.003 | 2.748 |
| 0.40 | 0.10 | 1.282 | 0.498 | 0.002 | 2-878 |
| 0.45 | 0.05 | 1.645 | 0.499 | 0.001 | 3.090 |
| 0.450 | 0.050 | 1.645 | 0.4995 | $0.0^{3}5$ | 3.291 |
| 0.452 | 0.048 | 1.665 | 0.4999 | $0.0^{3}1$ | 3.719 |
| 0.454 | 0.046 | 1.685 | 0.49995 | $0.0^{4}5$ | 3.891 |
| 0.456 | 0.044 | 1.706 | 0.49999 | $0.0^{4}1$ | 4.265 |
| 0-458 | 0.042 | 1.728 | 0.499995 | $0.0^{5}5$ | 4.417 |
| 0.460 | 0.040 | 1.751 | | $0.0^{5}1$ | 4.753 |
| 0.462 | 0.038 | 1.774 | | $0.0^{6}5$ | 4.892 |
| 0.464 | 0.036 | 1.799 | | $0.0^{6}1$ | 5.199 |
| 0.466 | 0.034 | 1.825 | | $0.0^{7}5$ | 5.327 |
| 0.468 | 0.032 | 1.852 | | $0.0^{7}1$ | 5.612 |
| 0.470 | 0.030 | 1.881 | | $0.0^{8}5$ | 5.731 |
| 0.472 | 0.028 | 1.911 | | $0.0^{8}1$ | 5.998 |
| 0,474 | 0.026 | 1.943 | | $0.0^{9}5$ | 6.109 |
| 0.476 | 0.024 | 1.977 | | 25% | 0.674 |
| 0.478 | 0.022 | 2.014 | | 10% | 1.282 |
| 0.480 | 0.020 | 2.054 | | 5% | 1.645 |
| 0.482 | 0.018 | 2.097 | | 1% | 2.326 |
| 0.484 | 0.016 | 2.144 | | 0.5% | 2.575 |
| 0.486 | 0.014 | 2.197 | | 0.1% | 3.090 |
| 0.488 | 0.012 | 2.257 | | 0.05% | 3.291 |

$1^\circ = \frac{x}{180}$ $0.5^\circ = x$



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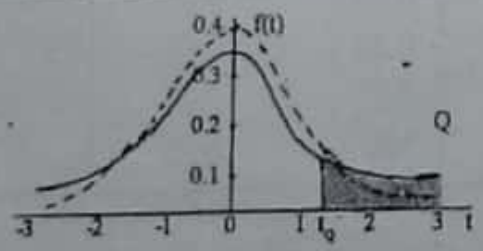
PERCENTAGE POINTS OF STUDENTS t- DISTRIBUTION t,
Probability*

| v | 0.25<br>0.50 | 0.10<br>0.20 | 0.05<br>0.10 | 0.025<br>0.050 | 0.01<br>0.02 | 0.005<br>0.010 | 0.0025<br>0.0050 | 0.001<br>0.002 | 0.0005<br>0.0010 | Q<br>2Q |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.000 | 3.078 | 6.314 | 12.71 | 31.82 | 63.66 | 127.3 | 318.3 | 636.6 | |
| 2 | 0.816 | 1.886 | 2.920 | 4.303 | 6.965 | 9.925 | 14.09 | 22.33 | 31.60 | |
| 3 | 0.765 | 1.638 | 2.353 | 3.182 | 4.541 | 5.841 | 7,453 | 10.21 | 12.92 | |
| 4 | 0.741 | 1.533 | 2.132 | 2.776 | 3.747 | 4.604 | 5,598 | 7.173 | 8.610 | |
| 5 | 0.727 | 1.476 | 2.015 | 2.571 | 3.365 | 4.032 | 4.773 | 5.893 | 6.869 | |
| 6 | 0.718 | 1.440 | 1.943 | 2.447 | 3.143 | 3.707 | 4.317 | 5.208 | 5.959 | |
| 7 | 0.711 | 1.415 | 1.895 | 2.365 | 2.998 | 3.499 | 4.029 | 4.785 | 5.408 | |
| 8 | 0.706 | 1.397 | 1.860 | 2.306 | 2.896 | 3.355 | 3.833 | 4.501 | 5.041 | |
| 9 | 0.703 | 1.383 | 1.833 | 2.262 | 2.821 | 3.250 | 3.690 | 4297 | 4.781 | |
| 10 | 0.700 | 1.372 | 1.812 | 2.228 | 2.764 | 3.169 | 3.581 | 4.144 | 4.587 | |
| 11 | 0.697 | 1.363 | 1.796 | 2.201 | 2.718 | 3.106 | 3.497 | 4.025 | 4.437 | |
| 12 | 0.695 | 1.356 | 1.771 | 2.160 | 2.650 | 3.012 | 3.372 | 3.852 | 4.221 | |
| 14 | 0.692 | 1.345 | 1.761 | 2.145 | 2.624 | 2.977 | 3.326 | 3.787 | 4.140 | |
| 15 | 0.691 | 1.341 | 1.753 | 2.131 | 2.602 | 2.947 | 3.286 | 3.733 | 4.073 | |
| 16 | 0.690 | 1.337 | 1.746 | 2.120 | 2.583 | 2.921 | 3.252 | 3.686 | 4.015 | |
| 17 | 0.689 | 1.333 | 1.740 | 2.110 | 2.567 | 2.898 | 3.222 | 3.646 | 3.965 | |
| 18 | 0.688 | 1.330 | 1.734 | 2.101 | 2.552 | 2.878 | 3.197 | 3.610 | 3.922 | |
| 19 | 0.688 | 1.328 | 1.729 | 2.093 | 2.539 | 2.861 | 3.174 | 3.579 | 3.883 | |
| 20 | 0.687 | 1.325 | 1.725 | 2.086 | 2.528 | 2.845 | 3.153 | 3.552 | 3.850 | |
| 21 | 0.686 | 1.323 | 1.721 | 2.080 | 2.518 | 2.831 | 3.135 | 3.527 | 3.819 | |
| 22 | 0.686 | 1.321 | 1.717 | 2.074 | 2.508 | 2.819 | 3.119 | 3.505 | 3.792 | |
| 23 | 0.685 | 1.319 | 1.714 | 2.069 | 2.500 | 2.807 | 3.104 | 3.485 | 3.767 | |
| 24 | 0.685 | 1.318 | 1.711 | 2.064 | 2.492 | 2.797 | 3.091 | 3.467 | 3.745 | |
| 25 | 0.684 | 1.316 | 1.708 | 2.060 | 2.485 | 2.787 | 3.078 | 3.450 | 3.725 | |
| 26 | 0.684 | 1.315 | 1.706 | 2.056 | 2.479 | 2.779 | 3.067 | 3.435 | 3.707 | |
| 27 | 0.684 | 1.314 | 1.703 | 2.052 | 2.473 | 2.771 | 3.057 | 3.421 | 3.690 | 120 |
| 28 | 0.683 | 1.313 | 1.701 | 2.048 | 2.467 | 2.763 | 3.047 | 3.408 | 3.674 | V |
| 29 | 0.683 | 1.311 | 1.699 | 2.045 | 2.462 | 2.756 | 3.038 | 3.396 | 3.659 | |
| 30 | 0.683 | 1.310 | 1.697 | 2.042 | 2.457 | 2.750 | 3.030 | 3.385 | 3.646 | 4 |
| 40 | 0.681 | 1.303 | 1.684 | 2.021 | 2.423 | 2.704 | 2.971 | 3.307 | 3.551 | 3 |
| 60 | 0.679 | 1.296 | 1.671 | 2.000 | 2.390 | 2.660 | 2.915 | 3.232 | 3.460 | 2 |
| 120 | 0.677 | 1.289 | 1.658 | 1.980 | 2.358 | 2.617 | 2.860 | 3.160 | 3.373 | 1 |
| α | 0.674 | 1.282 | 1.645 | 1.960 | 2.326 | 2.576 | 2.807 | 3.090 | 3.291 | 0 |



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## SIGNIFICANCE LEVELS FOR CORRELATION COEFFICINTS

| | Product-moment coefficient of correlation($r_{xy}$) | | Spearman's rank correlation coefficient ($\rho$) | | Kendall's rarik correlation coefficient ($\tau$) | |
|---|---|---|---|---|---|---|
| No of Pairs | Signifant If $\lvert r_{xy} \rvert$ exceeds at 5% | at 1% | Significance if $\lvert \rho \rvert$ exceeds at 5% | at 1% | Significance if $\lvert \tau \rvert$ exceeds 5% | 1% |
| 3 | 1.00 | 1.00 | | | | |
| 4 | 0.95 | 0.99 | | | | |
| 5 | 0.88 | 0.96 | 1.00 | | | |
| 6 | 0.81 | 0.92 | 0.89 | 1.00 | 0.87 | 1.00 |
| 7 | 0.75 | 0.88 | 0.75 | 0.89 | 0.71 | 0.81 |
| 8 | 0.71 | 0.83 | 0.71 | 0.86 | 0.64 | 0.79 |
| 9 | 0.67 | 0.80 | 0.68 | 0.83 | 0.56 | 0.72 |
| 10 | 0.63 | 0.77 | 0.65 | 0.79 | 0.51 | 0.64 |
| 11 | 0.60 | 0.74 | 0.60 | 0.74 | 0.49 | 060 |
| 12 | 0.58 | 0.71 | 0.58 | 0.71 | 0.45 | 0.58 |
| 13 | 0.55 | 0.68 | 0.55 | 0.68 | | |
| 14 | 0.53 | 0.66 | 0.53 | 0.66 | | |
| 15 | 0.51 | 0.64 | 0.51 | 0.64 | | |
| 16 | 0.50 | 0.62 | 0.50 | 0.62 | | |
| 17 | 0.48 | 0.61 | 0.48 | 0.61 | | |
| 18 | 0.47 | 0.59 | 047 | 0.59 | | |
| 19 | 0.46 | 0.58 | 0.46 | 0.58 | | |
| 20 | 0.44 | 0.56 | 0.44 | 0.56. | 0.33 | |
| 30 | 0.35 | 0.45 | 0.35 | 0.45 | | |
| 40 | 0.31 | 0.39 | 0.31 | 0.39 | | |
| 50 | 0.27 | 0.35 | 0.27 | 0.35 | | |
| 60 | 0.25 | 0.33 | 0.25 | 0.33 | | |
| 70 | 0.23 | 0.31 | 0.23 | 0.31 | | |
| 80 | 0.22 | 0.29 | 0.22 | 0.29 | | |
| 90 | 0.21 | 0.27 | 0.21 | 0.27 | | |
| 100 | 0.20 | 0.25 | 020 | 0.25 | | |



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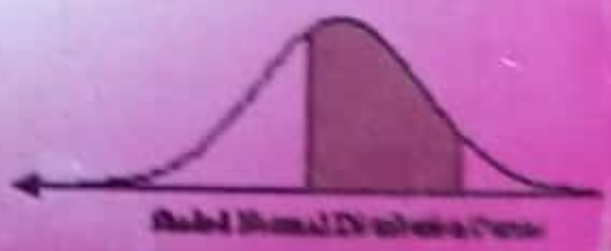
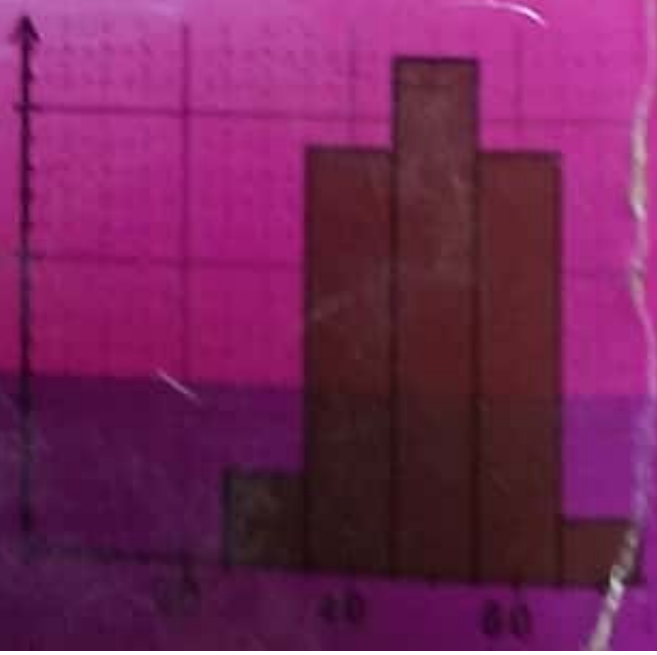
This book cover Statistics, Probability and Numerical methods. It provides about Eighty five percent of Advanced Level Mathematics Paper II Course. The scope of Third Edition has been expanded to include Errors, Interpolation, Approximation methods and Flow charts.

The topics are presented in a logic sequence and are arranged in an orderly format to stimulate student's understanding. There are a variety of worked examples and numerous exercises, providing both routine practice and opportunities to develop the understanding.

Getting Things Done



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