PHY 1201: MATHEMATICAL PHYSICS II

LAPLACE AND FOURIER TRANSFORMS

Laplace Transform
The Laplace transformation method is generally useful for obtaining solutions of linear differential equations (both ordinary and partial). It enables us to reduce a differential equation to an algebraic equation, thus avoiding going to the trouble of finding the general solution and then evaluating the arbitrary constants.

The Laplace transform $F(s)$ of a function $f(\cancel{x}t)$ is an integral defined by

$$F(s) = \int_0^\infty f(t)e^{-st}dt \qquad (3.1a)$$

The function $F(s)$ is called the Laplace transform of $f(t)$. It may also be written using the Laplace transform operator L which transforms $f(t)$ in to $F(s)$ as

$$L\{f(t)\} = F(s) \qquad (3.1b)$$

Illustration
Find $L\{e^{ax}\}$, where a is a constant

The transform is obtained by replacing $f(t)$ by e^{ax} as

$$L\{e^{ax}\} = \int_0^\infty e^{ax}e^{-sx}dx = \int_0^\infty e^{-(s-a)x}dx$$

For $s \le a$, the exponent is either positive or zero and the integral diverges. For $s > a$, the integral converges:

$$L\{e^{ax}\} = \int_0^\infty e^{-(s-a)x}dx = \frac{1}{-(s-a)}\left[e^{-(s-a)x}\right]_0^\infty = \frac{1}{(s-a)}$$

This example enables us to investigate the existence of equation (3.1) for a general function $f(t)$ or $f(x)$.

Properties of Laplace transform

(i) Linearity property: $L\{c_1f(t) + c_2g(t)\} = \int_0^\infty \{c_1f(t) + c_2g(t)\}e^{-st}dt = c_1L\{f(t)\} + c_2L\{g(t)\}$

Where c_1 and c_2 are constants.

(ii) Shifting (or first translation) property: $L\{e^{-at}f(t)\} = F(s+a)$

(iii) Shifting (or second translation) property: $L\{f(t-t_0)\} = e^{-at_0}F(s)$

(iv) Scaling property: $L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$

(v) Differentiation property: $L\{f^{(n)}(t)\} = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$

(vi) Integration property: $L\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s)$

(vii) Convolution property: $L\{f_1(t) + f_2(t)\} = F_1(s)F_2(s)$

Definition: $f_1(t) * f_2(t) = \int_0^t f_1(\tau)f_2(t-\tau)d\tau$ (convolution of $f_1(t)$ and $f_2(t)$)

It is commutative, i.e. $f_1(t) * f_2(t) = f_2(t) * f_1(t)$

(viii) Limits property: $lim_{t\to 0} f(t) = lim_{s\to\infty} sF(s)$ and $lim_{t\to\infty} f(t) = lim_{s\to 0} sF(s)$

Inverse Laplace Transform

The inverse Laplace transform of $F(s)$ is a function $f(t)$ such that $L\{f(t)\} = F(s)$. We denote the operation of taking an inverse Laplace transform by L^{-1}:

$$L^{-1}\{F(s)\} = f(t) \qquad (3.2)$$

That is, we operate algebraically with the operators L and L^{-1}, bringing them from one side of an equation to the other side just as we would in writing $ax = b$ implies $x = a^{-1}b$. The following example illustrates the calculation of Laplace transform and the inverse Laplace transforms which is equally important in solving differential equations.

Laplace transforms of some elementary functions

Using the definition (3.1) we now obtain the transforms of polynomials, exponential and trigonometric functions.

1. $f(x) = 1$ for $x > 0$

   By definition, we have

   $L\{1\} = \int_0^\infty e^{-sx}dx = \frac{1}{s} \quad s > 0$

2. $f(x) = x^n$ where n is a positive integer

$$L\{x^n\} = \int_0^\infty x^n e^{-sx} dx$$

Using integration by parts with $u = x^n$ and $\frac{dv}{dx} = e^{-sx} \Rightarrow v = \frac{-1}{s}e^{-sx}$, we get

$$L\{x^n\} = \left[\frac{-x^n e^{-sx}}{s}\right]_0^\infty + \frac{n}{s}\int_0^\infty x^{n-1}e^{-sx}dx$$

For $s > 0$ and $n > 0$, the first term on the right hand side of the above equation is zero, and so we have

$\int_0^\infty x^n e^{-sx} dx = \frac{n}{s}\int_0^\infty x^{n-1}e^{-sx} dx$ or $L\{x^n\} = \frac{n}{s}L\{x^{n-1}\}$

from which we may obtain for $n > 1$

$L\{x^{n-1}\} = \frac{n-1}{s}L\{x^{n-2}\}$

Iteration of this process yields

$L\{x^n\} = \frac{n(n-1)(n-2)\ldots 2.1}{s^n}L\{x^0\}$

$L\{x^0\} = L\{1\} = \frac{1}{s}$

Hence we finally have

$L\{x^n\} = \frac{n!}{s^{n+1}}$, $\quad s > 0$

3. $f(x) = sinax$ where a is a real constant

$$L\{sinax\} = \int_0^\infty sinaxe^{-sx} dx$$

Using integration by parts with $u = e^{-sx}$ and $\frac{dv}{dx} = sinax\left(v = \frac{-1}{a}cosax\right)$
and $\int e^{mx}sinnx\, dx = \frac{e^{mx}(msinnx-ncosnx)}{m^2+n^2}$ we get

$$L\{sinax\} = \int_0^\infty sinaxe^{-sx} dx = \left[\frac{e^{-sx}(-ssinax - acosax)}{s^2 + a^2}\right]_0^\infty$$

Since s is positive, $e^{-sx} \to 0$ as $x \to \infty$, but $sinax$ and $cosax$ are bounded as $x \to \infty$, so we obtain
$L\{sinax\} = \int_0^\infty sinaxe^{-sx} dx = 0 - \frac{1(0-a)}{s^2+a^2} = \frac{a}{s^2+a^2} \quad s > 0$

4. $f(x) = cosax$ where a is a real constant

Using a similar approach in 3 above and

$\int e^{mx}sinnx\, dx = \frac{e^{mx}(mcosnx+nsinmx)}{m^2+n^2}$

We obtain

$$L\{cosax\} = \int_0^\infty cosaxe^{-sx} dx = \left[\frac{e^{-sx}(-scosax + asinsx)}{s^2 + a^2}\right]_0^\infty = \frac{a}{s^2 + a^2}$$

5. $f(x) = sinhax$ where a is a real constant

Using the linearity property of the Laplace transform operator L, we obtain

$$L\{coshax\} = L\left\{\frac{e^{ax} + e^{-ax}}{2}\right\} = \frac{1}{2}L(e^{ax}) + \frac{1}{2}L(e^{-ax}) = \frac{a}{s^2 + a^2}$$

$$L\{coshax\} = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2 - a^2}$$

6. Using the definition (3.2), for $s > 0$, obtain the following inverse Laplace transforms

(i) $L^{-1}\left\{\frac{5}{s+2}\right\}$

Recall $L\{e^{ax}\} = \frac{1}{s-a}$, hence $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{ax}$. It follows that

$L^{-1}\left\{\frac{5}{s+2}\right\} = 5L^{-1}\left\{\frac{1}{s+2}\right\} = 5e^{-2x}$

(ii) $L^{-1}\left\{\frac{1}{p^s}\right\}$

Recall $L\{x^k\} = \int_0^\infty x^k e^{sx}\,dx = \frac{1}{p^{k+1}}\int_0^\infty u^k e^{-u}\,du = \frac{\Gamma_{(k+1)}}{p^{k+1}}$
From this we have

$L\left\{\frac{x^k}{\Gamma_{(k+1)}}\right\} = \frac{1}{p^{k+1}},$

hence
$L^{-1}\left\{\frac{1}{p^{k+1}}\right\} = \frac{x^k}{\Gamma_{(k+1)}}$

If we now let $k + 1 = s$, then

$L^{-1}\left\{\frac{1}{p^s}\right\} = \frac{x^{s-1}}{\Gamma_{(s)}}$

(iii) Find $L^{-1}\left\{\frac{4s+10}{s^2+6s+8}\right\}$

Expressing in partial fraction,

$F(s) = \frac{4s+10}{s^2+6s+8} \equiv \frac{4s+10}{(s+2)(s+4)}$

$\frac{4s+10}{(s+2)(s+4)} \equiv \frac{A}{s+2} + \frac{B}{s+4}$

$\frac{4s+10}{(s+2)(s+4)} \equiv \frac{A(s+4)+B(s+2)}{(s+2)(s+4)}$

Hence

$4s + 10 = A(s+4) + B(s+2)$

When $s = -4$,

$-6 = -2B \quad \mapsto \quad B = 3$

When $s = -2$,

$2 = 2A \quad \mapsto \quad A = 1$

$F(s) = \frac{1}{s+2} + \frac{3}{s+4}$

Now

$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+2} + \frac{3}{s+4}\right\} = L^{-1}\left\{\frac{1}{s+2}\right\} + L^{-1}\left\{\frac{3}{s+4}\right\}$

$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+2}\right\} + 3L^{-1}\left\{\frac{1}{s+4}\right\}$

From tables

$f(t) = e^{-2t} + 3e^{-4t}$

Exercise

1. Find $L\{f(x)\}$ given that
   (i) $f(x) = cosax$ where a is a real constant

   Hint: $\int e^{mx} sinnx\, dx = \frac{e^{mx}(mcosnx+nsinmx)}{m^2+n^2}$. **Answer:** $L\{cosax\} = \frac{s}{s^2+a^2}$

   (ii) $f(x) = coshax$ where a is a real constant

   Hint: Use linearity property of the Laplace transform operator L. **Answer:** $L\{coshax\} = \frac{s}{s^2-a^2}$

Shifting (or translation) theorems
In practical applications, we often meet functions multiplied by exponential factors. If we know the Laplace transform of a function, then multiplying it by an exponential factor does not require a new computation as shown by the following theorem.

The first shifting theorem

If $L\{f(t)\} = F(s),\ s > b$; then $L\{e^{at}f(t)\} = F(s-a), \quad s > a+b$

The following examples illustrate the use of this theorem.

1. Show that

(a) $L\{e^{-ax}x^n\} = \frac{s!}{(s+a)^{n+1}}$, $s > -a$

Recall

$L\{x^n\} = \frac{n!}{s^{n+1}}$, $s > 0$

The shifting theorem then gives

$L\{e^{-ax}x^n\} = \frac{s!}{(s+a)^{n+1}}$, $s > -a$

(b) $L\{e^{-ax}sinbx\} = \frac{s!}{(s+a)^{n+1}}$, $s > -a$

Since $L\{sinax\} = \frac{a}{s^2+a^2}$ $s > 0$

it follows from the shifting theorem that

$L\{e^{-ax}sinbx\} = \frac{b}{(s+a)^2+b^2}$, $s > -a$

The second shifting theorem

This second shifting theorem involves the shifting t variable and states that:
Given that $L\{f(t)\} = F(s)$, where $f(t) = 0$ for $x < 0$; and if $g(t) = f(t-a)$, then

$L\{g(t)\} = e^{-as}L\{f(t)\}$

Example

Given that $f(x) = \begin{cases} x,\wedge x \geq 0 \\ 0,\wedge x < 0 \end{cases}$ and $g(x) = \begin{cases} 0,\wedge x < 5 \\ x-5,\wedge x \geq 5 \end{cases}$.

Show that $L\{g(x)\} = \frac{e^{-5s}}{s^2}$

We first notice that

$$g(x) = f(x-5)$$

Then the second shifting theorem gives

$$L\{g(x)\} = e^{-5s}L\{x\} = \frac{e^{-5s}}{s^2}$$

Laplace Transforms of Derivatives

Laplace transform can be used to solve ordinary linear DE's with constant coefficients. The Laplace transform of the first derivative of $f(t)$ is

$$L\left[\frac{df}{dt}\right] = \int_0^\infty \frac{df}{dt}e^{-st}dt$$

Integrating by parts

$$L\left[\frac{df}{dt}\right] = [f(t)e^{-st}]_0^\infty + s\int_0^\infty f(t)e^{-st}dt$$

$$= -f(0) + sF(s)$$

The Laplace transform for the second derivative is

$$L\left[\frac{d^2f(t)}{dt^2}\right] = \int_0^\infty \frac{d^2f(t)}{dt^2}e^{-st}$$

$$= \left[\frac{df(t)}{dt}e^{-st}dt\right]_0^\infty + s\int_0^\infty \frac{df(t)}{dt}e^{-st}dt$$

$$= -\frac{df(0)}{dt} + s\big(sF(s) - f(0)\big)$$

In general,

$$L\left[\frac{d^nf(t)}{dt^n}\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}\frac{df(0)}{dt} - \cdots - \frac{d^{n-1}f(0)}{dt^{n-1}}$$

Similarly you can get the Laplace transform of integrals

$$L\left[\int_0^t f(u)du\right] = \int_0^\infty dte^{-st}\int_0^t f(u)du$$

Integrating by parts

Let $dv = e^{-st}dt$ and $u = \int_0^t f(u)du$

$$L\left[\int_0^t f(u)du\right] = \left[-\frac{1}{s}e^{-st}\int_0^t f(u)du\right]_0^\infty + \int_0^\infty \frac{1}{s}e^{-st}f(u)dt$$

The first term on the RHS vanishes. Thus

$$L\left[\int_0^t f(u)du\right] = \frac{1}{s}L[f(u)]$$

Properties and use of tables.

f(t)	F(s)	f(t)	F(s)
1. 1	$\frac{1}{s}$	19. $\frac{sinat}{t}$	$tan^{-1}\left(\frac{a}{s}\right)$
2. e^{-at}	$\frac{1}{s+a}$	20. $\frac{1}{t}sinatcosbt$	$\frac{1}{2}\left[tan^{-1}\left(\frac{a+b}{s}\right) + tan^{-1}\left(\frac{a-b}{s}\right)\right]$
3. $sinat$	$\frac{a}{s^2+a^2}$	21. $\frac{e^{-at}-e^{-bt}}{t}$	$ln\left(\frac{s+b}{s+a}\right)$
4. $cosat$	$\frac{s}{s^2+a^2}$	22. $1-erf\left(\frac{a}{2\sqrt{t}}\right), a > 0$	$\frac{1}{s}e^{-a\sqrt{s}}$
5. $t^k, k > -1$	$\frac{k!}{s^{k+1}}$	23. $J_0(at)$	$\frac{1}{s}(s^2+a^2)^{\frac{-1}{2}}$
6. $t^k e^{-at}, k > -1$	$\frac{k!}{(s+a)^{k+1}}$	24. $u(t-a)$	$\frac{1}{s}e^{-sa}$
7. $\frac{e^{-at}-e^{-bt}}{b-a}$	$\frac{1}{(s+a)(s+b)}$	25. $\delta(t-a)$	e^{-sa}
8. $\frac{ae^{-at}-be^{-bt}}{a-b}$	$\frac{s}{(s+a)(s+b)}$	**General properties**	
9. $sinhat$	$\frac{a}{s^2-a^2}$	26. $e^{-at}f(t)$	$F(s+a)$
10. $coshat$	$\frac{s}{s^2-a^2}$	27. $f(t-a), t > 0$	$e^{-sa}F(s)$
11. $tsinat$	$\frac{2as}{(s^2+a^2)^2}$	28. $f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
12. $tcosat$	$\frac{s^2-a^2}{(s^2+a^2)^2}$	29. $f''(t)$	$s^2F - sf(0) - f'(0)$
13. $e^{-at}sinbt$	$\frac{b}{(s+a)^2+b^2}$	30. $f'(t)$	$sF - f(0)$
14. $e^{-at}cosbt$	$\frac{s+a}{(s+a)^2+b^2}$	31. $\int_0^t f(\tau)\,d\tau$	$\frac{1}{s}F(s)$
15. $1-cosat$	$\frac{a^2}{s(s^2+a^2)}$	32. $f_1(t) * f_2(t)$	$F_1(s)F_2(s)$
16. $at - sinat$	$\frac{a^3}{s^2(s^2+a^2)}$	33. $\frac{f(t)}{t}$	$\int_s^\infty f(u)\,du$
17. $sinat - atcosat$	$\frac{2a^3}{(s^2+a^2)^2}$	34. $f(t) = f(t+T)$	$\frac{1}{1-e^{-st}}\int_0^T e^{-su}F(u)du$
18. $e^{-at}(1-at)$	$\frac{s}{(s+a)^2}$		

Examples

1. Consider a simple pendulum released from rest with an initial displacement A. If the pendulum performs a simple harmonic motion (SHM) described by the differential equation $\ddot{x} + \omega^2 x = 0$, where ω is the angular frequency, find the displacement and velocity of the pendulum at any time.

Taking Laplace transform of the differential equation

$$L\{\ddot{x}\} + L\{\omega^2 x\} = 0$$

Expanding using $L\{y''\} = s^2Y(s) - sy_0 - y_0'$

$s^2X - sx(0) - \dot{x}(0) + \omega^2 X = 0$

$X(s^2 + \omega^2) = sx(0) - \dot{x}(0)$

$X = \frac{sx(0)-\dot{x}(0)}{(s^2+\omega^2)} = A\frac{s}{s^2+\omega^2}$

From tables (i.e. 19), we find the inverse Laplace transform of X, which gives the displacement as

$$x(t) = AL^{-1}\left\{\frac{s}{s^2 + \omega^2}\right\}$$

$x(t) = Acos\omega t$

The velocity of the pendulum

$v(t) = \dot{x} = -\omega Asin\omega t$

2. Solve the DE $y' - y = 2e^t$, given $y_0 = 3$

$L\{y'\} - L\{y\} = L\{2e^t\}$

Expanding using $L\{y'\} = sY(s) - y_0$,

$sY - y_0 - Y = 2\frac{1}{s-1}$

$sY - 3 - Y = \frac{2}{s-1}$

$Y(s-1) = \frac{2}{s-1} - 3 = \frac{3s-1}{s-1}$

$Y = \frac{3s-1}{(s-1)^2} = \frac{3s}{(s-1)^2} - \frac{1}{(s-1)^2}$

Taking the inverse Laplace transform by using $f(t) = e^{-at}(1 - at)$ (i.e. 18), with $a = -1$

$y = 3L^{-1}\left\{\frac{s}{(s-1)^2}\right\} - L^{-1}\left\{\frac{1}{(s-1)^2}\right\}$

$y = 3e^t(1+t) - te^t = (3+2t)e^t$

3. Solve the DE $y'' + 2y' + 5y = 10cost$, given $y_0 = 0, y' = 3$

Rewriting using $L\{y''\} = s^2Y(s) - sy_0 - y_0'$ and $L\{y'\} = sY(s) - y_0$ gives

$$s^2Y - sy_0 - y_0' + 2(sY - y_0) + 5Y = 10\frac{s}{s^2+1}$$

$$Y(s^2+2s+5) = 10\frac{s}{s^2+1} + 3$$

$$Y(s^2+2s+5) = \frac{10s + 3(s^2+1)}{s^2+1} = \frac{3s^2+10s+3}{s^2+1}$$

$$Y = \frac{3s^2+10s+3}{(s^2+1)(s^2+2s+5)}$$

This Laplace transform is not listed in the table (appendix). We shall use partial fraction to reduce the RHS of Y.

$$\frac{3s^2+10s+3}{(s^2+1)(s^2+2s+5)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+5}$$

$$3s^2+10s+3 = (As+B)(s^2+2s+5) + (Cs+D)(s^2+1)$$

When $s = 0$, $3 = 5B + D$
When $s = 1$, $16 = 8A + 8B + 2C + 2D$
When $s = -1$, $-4 = 4A + 4B - 2C + 2D$
When $s = 2$, $35 = 26A + 13B + 10C + 5D$

Solving the above simultaneous equation gives

$A = 1$, $B = 2$, $C = -2$ and $B = -2$

Therefore,

$$Y = \frac{s+2}{s^2+1} - 2\frac{s+1}{s^2+2s+5} = \frac{s+2}{s^2+1} - 2\frac{s+1}{(s+1)^2+2^2} = \frac{s}{s^2+1} + \frac{2}{s^2+1} - 2\frac{S+1}{(s+1)^2+2^2}$$

The inverse Laplace transform gives

$$y = L^{-1}\left\{\frac{s+2}{s^2+1}\right\} + 2L^{-1}\left\{\frac{1}{s^2+1}\right\} - 2L^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\}$$

$$y = sint + 2cost - 2e^{-t}cos2t$$

4. Solve the initial value problem:
   $y'' + y = 0$, $y'(0) = y(0) = 0$ and $f(t) = 0, t < 0$ but $f(t) = 1, t \geq 0$

   Hint $y' = \frac{dy}{dt}$
   Answer: $y = 1 - cost$ for $t \geq 0$, $y = 0, t < 0$

FOURIER TRANSFORM

The Fourier transform of a function $f(t)$ is given by

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

The inverse Fourier transform is

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

Example:

Find the Fourier transform of the exponential decay function $f(t) = 0$ for $t < 0$ and $f(t) = Ae^{-\gamma t}$ for $t \geq 0, \gamma > 0$

Soln:

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} 0 * e^{-i\omega t} dt + \frac{A}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\gamma t} e^{-i\omega t} dt$$

$$= \frac{A}{\sqrt{2\pi}} * \frac{1}{(\gamma + i\omega)}$$

Fourier transform for odd and even functions

For even or odd functions, alternative forms of the Fourier transform may be obtained.

For odd function i.e $f(t) = -f(-t)$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)(\cos \omega t - i \sin \omega t) dt$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} f(t) \cos \omega t - \frac{2i}{\sqrt{2\pi}} \int_{0}^{\infty} f(t) \sin \omega t$$

Since $f(t)$ and $\sin \omega t$ are odd, no cosine terms exist

Thus $F(\omega) = \frac{-2i}{\sqrt{2\pi}} \int_{0}^{\infty} f(t) \sin \omega t dt$

So

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

$$= \frac{2i}{\sqrt{2\pi}} \int_0^\infty F(\omega) \sin \omega t d\omega$$

$$= \frac{2}{\pi} \int_0^\infty d\omega \sin \omega t \left\{ \int_0^\infty f(u) \sin \omega u \, du \right\}$$

Therefore, for an odd function, the Fourier sine transform is

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \omega t dt$$

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\omega) \sin \omega t d\omega$$

Note: For the even functions, the Fourier cosine transform is

$$F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \omega t dt$$

Reading assignment:

Applications: spectral analysis of signals in electronic circuits

SPECIAL AND GENERALIZED FUNCTIONS

The functions discussed in this chapter arise as solutions of second-order differential equations which appear in special, rather than in general, physical problems. So these functions are usually known as the special functions of mathematical physics.

A generalized function is an ideal function which is a generalization of the classical concept of a function. The concept makes it possible to express mathematically idealized concepts.

Bessel functions (1st kind);

The Bessel differential equation is of the form

$$x^2 y'' + xy' + (x^2 - s^2) y = 0 \qquad (3.1)$$

The solutions to the Bessel equation are called **Bessel functions**. They are like damped sines and cosines. The importance of this equation and its solutions lies in the fact that they occur frequently in the boundary-value problems of mathematical physics and engineering. Therefore, Bessel functions are sometimes called cylindrical functions because of their application to problems involving cylindrical symmetry. The constant s is a real and positive constant called the order of the Bessel function y.

Bessel function y is usually denoted by $J_s(x)$, called the Bessel function of the 1st kind and order s.

A series solution of the Bessel equation for integer values of s gives $J_s(x)$ as:

$$J_s(x) = x^s \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+s} n!(n+s)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma^{(n+1)}\Gamma^{(n+s)}} \left(\frac{x}{2}\right)^{2n+s} \qquad (3.2a)$$

Where $\Gamma^{(n+1)} = n!$ is the Gamma function

If s is not an integer, this leads to an independent second solution that can be written

$$J_{-s}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(-s+n+1)!} \left(\frac{x}{2}\right)^{2n-s} \qquad (3.2b)$$

The complete solution of Bessel's equation is then given by

$$y(x) = AJ_s(x) + BJ_{-s}(x) \qquad (3.3)$$

Where A and B are arbitrary constants. When s is a positive integer, it can be shown that $J_{-s}(x) = (-1)^s J_s(x)$.

Bessel functions $J_s(x)$ are tabulated for positive values of the order s. The values for $J_{-s}(x)$ are obtained from the relation $J_{-s}(x) = (-1)^s J_s(x)$. Also tabulated are the zeros of the Bessel function, since they do not occur at regular intervals.

Orthogonality of Bessel functions

If

$$\int_0^1 xJ_s(a_m x)J_s(a_n x)dx = \begin{cases} \frac{1}{2}J_{s-1}^2(a) = \frac{1}{2}J_s^2(a) = \frac{1}{2}J_{s+1}^2(a), \wedge\, a_m = a_n = a \\ 0, \wedge\, m \neq n \end{cases} \qquad (3.4)$$

Where $a_n, n = 1,2, \ldots$ are the zeros of $J_s(x)$, then the functions $J_s(a_n x)$ are orthogonal on the interval $(0,1)$ w.r.t the weight x. Alternatively we say that the function $\sqrt{x}J_s(a_n x)$ are orthogonal on the interval $(0,1)$.

The fact that the Bessel functions $J_s(a_n x)$ obey the orthogonality condition (3.4) makes it possible to expand a given function in a series of Bessel functions.

Bessel functions of the second kind $Y_n(x)$

For integer $s = n$, $J_n(x)$ and $J_{-n}(x)$ are linearly dependent and do not form a fundamental system. We shall now obtain a second independent solution, starting with the case $n = 0$. In this case Bessel's equation may be written

$$xy'' + y' + xy = 0 \qquad (3.5)$$

The desired solution of equation (3.5) must be of the form

$$y_2(x) = J_0(x)lnx + \sum_{m=1}^{\infty} A_m x^m \tag{3.6}$$

Solving equation (3.5) by substituting y_2 and its derivatives gives

$$\begin{cases} y_2(x) = J_0(x)lnx + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}h_m}{2^{2m}(m!)^2} x^{2m} \\ y_2(x) = J_0(x)lnx + \frac{1}{4}x^2 - \frac{3}{128}x^4 + \cdots \end{cases} \tag{3.7}$$

Where $h_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$. Since J_0 and y_2 are linearly independent functions, they form a fundamental system of equation (3.5)

Another fundamental system is obtained by replacing y_2 by an independent particular solution of the form $a(y_2 + bJ_0)$ where $a \neq 0$ and b are constants.

It is customary to choose $a = \frac{\pi}{2}$ and $b = \gamma - ln2$, where $\gamma = 0.57721566490 \ldots$ is the so-called Euler constant, which is defined as the limit of $1 + \frac{1}{2} + \cdots + \frac{1}{s} - lns$ as s approaches infinity.

The standard particular solution thus obtained is known as the Bessel function of the second kind of order zero or Neumann's function of order zero and is denoted by $Y_n(x)$:

$$Y_0(x) = \frac{\pi}{2}\left[J_0(x)\left(ln\frac{x}{2} + \gamma\right) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}h_m}{2^{2m}(m!)^2} x^{2m}\right] \tag{3.8}$$

Note that the second solution is defined differently, depending on whether the order s is integral or not. To provide uniformity of formalism and numerical tabulation, it is desirable to adopt a form of the second solution that is valid for all values of the order. The common choice for the standard second solution defined for all is given by the formula

$$Y_s(x) = \frac{J_s(x)coss\pi - J_{-s}(x)}{sins\pi}, Y_n(x) = lim_{s \to n} Y_s(x) \tag{3.9}$$

This function is known as the Bessel function of the second kind of order s. It is also known as Neumann's function of order and is denoted by $N_s(x)$. It can also be shown that $Y_{-n}(x) = (-1)^n Y_n(x)$. A more general solution of Bessel's equation for all values of s can now be written as:

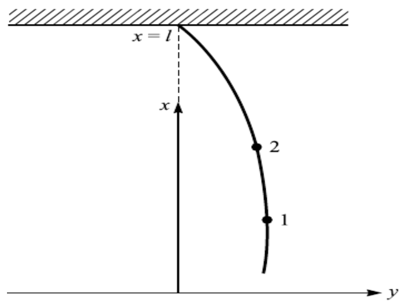
$$y(x) = c_1 J_{(s)} + c_2 Y_s(x) \tag{3.10}$$

In some applications it is convenient to use solutions of Bessel's equation that are complex for all values of x. Therefore, the following solutions were introduced

$$\begin{cases} H_s^{(1)}(x) = J_{(s)} + iY_s(x) \\ H_s^{(2)}(x) = J_{(s)} - iY_s(x) \end{cases} \tag{3.11}$$

These linearly independent functions are known as Bessel functions of the third kind of order s or first and second Hankel functions of order s.

To illustrate how Bessel functions, enter into the analysis of physical problems, we consider the problem of small oscillations in the vertical x-y plane caused by small displacements from the stable equilibrium position. In this system, a uniform heavy flexible chain of length l hanging vertically under its own weight. The x-axis is the position of stable equilibrium of the chain and its lowest end is at $x = 0$.



Here the tension T at a given point of the chain is equal to the weight of the chain below that point, and now one end of the chain is free, whereas before both ends were fixed. To derive an equation for y, consider an element dx, then Newton's second law gives

$$\left(T\frac{\partial y}{\partial x}\right)_2 - \left(T\frac{\partial y}{\partial x}\right)_1 = \rho dx\frac{\partial^2 y}{\partial t^2} \quad \text{or} \qquad \rho dx\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right)dx =$$

(3.12)

from which we obtain

$$\rho\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right) \tag{3.13}$$

Substituting $T = \rho g x$ into the equation (3.13), we obtain

$$\rho\frac{\partial^2 y}{\partial t^2} = g\frac{\partial y}{\partial x} + gx\frac{\partial^2 y}{\partial x^2} \tag{3.14}$$

Since y is a function of two variables x and t, separating the variables, we attempt a solution of the form

$$y(x,t) = u(x)f(t) \tag{3.15}$$

Substitution of this into the partial differential equation yields two equations:

$$\begin{cases} f''(t) + \omega^2 f(t) = 0 \\ xu''(x) + u'(x) + \left(\frac{\omega^2}{g}\right)u(x) = 0 \end{cases} \tag{3.16}$$

Where ω^2 is the separation constant. The differential equation for $f(t)$ is ready for integration and the result is

$$f(t) = cos(\omega t - \delta) \qquad (3.17)$$

with a phase constant. The differential equation for $u(x)$ is not in a recognizable form yet. To solve it, first change variables by putting

$$x = \frac{gz^2}{4}, w(z) = u(x) \qquad (3.18)$$

Then the differential equation for $u(x)$ becomes Bessel's equation of order zero:

$$zw''(x) + w'(x) + \omega^2 zw(x) = 0$$

Its general solution is

$$\begin{cases} w(z) = AJ_0(\omega z) + BY_0(\omega z) \\ u(x) = AJ_0\left(2\omega\sqrt{\left(\frac{g}{x}\right)}\right) + BY_0\left(2\omega\sqrt{\left(\frac{g}{x}\right)}\right) \end{cases} \qquad (3.19)$$

Since $Y_0\left(2\omega\sqrt{\left(\frac{g}{x}\right)}\right) \to -\infty$ as $x \to 0$, we are forced by to choose $B = 0$, so that the solution is then

$$y(x,t) = AJ_0\left(2\omega\sqrt{\left(\frac{g}{x}\right)}\right) cos(\omega t - \delta) \qquad (3.20)$$

The upper end of the chain at $x = l$ is fixed, requiring that

$$J_0\left(2\omega\sqrt{\left(\frac{g}{x}\right)}\right) = 0 \qquad (3.21a)$$

The frequencies of the normal vibrations of the chain are given by

$$2\omega_n\sqrt{\left(\frac{l}{g}\right)} = s_n \qquad (3.21b)$$

where s_n are the roots of J_0. Some values of $J_0(x)$ and $J_1(x)$ are tabulated at the end of this chapter.

Generating function for $J_n(x)$

The generating function for Bessel functions of the first kind of integral order n is given by

$$\Phi(x,t) = exp\left[\frac{1}{2}x(t - t^{-1})\right] = \sum_{n=-\infty}^{\infty} t^n J_n(x) \qquad (3.22)$$

The generating function is very useful in obtaining properties of $J_n(x)$ for integral values of n which can then often be proved for all values of n.

Bessel's integral representation

With the help of the generating function, we can express $J_n(x)$ in terms of a definite integral with a parameter. To do this, let $t = e^{i\theta}$ in the generating function, then

$$exp\left[\frac{1}{2}x(t-t^{-1})\right] = exp\left[\frac{1}{2}x\left(e^{i\theta}-e^{-i\theta}\right) = e^{ixsin\theta} = cos(xsin\theta) + isin(xsin\theta)\right]$$

Substituting this into equation (3.22) we obtain

$$cos(xsin\theta) + isin(xsin\theta) = \sum_{n=-\infty}^{\infty} J_n(x)(cos\theta + isin\theta)^n = \sum_{n=-\infty}^{\infty} J_n(x)cosn\theta + i\sum_{n=-\infty}^{\infty} J_n(x)sinn\theta$$

Since $J_{-n}(x) = (-1)^n J_n(x)$, $cos(n\theta) = cos(-n\theta)$ and $sin(n\theta) = -sin(-n\theta)$, we have, upon equating the real and imaginary parts of the above equation.

$$\begin{cases} cos(xsin\theta) = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)cos2n\theta \\ sin(xsin\theta) = 2\sum_{n=1}^{\infty} J_{2n-1}(x)sin(2n-1)\theta \end{cases} \tag{3.23}$$

It is interesting to note that these are the Fourier cosine and sine series of $cos(xsin\theta)$ and $sin(xsin\theta)$. Multiplying the first equation in (3.23) by $cos(k\theta)$ and integrating from 0 to π, we obtain

$$\frac{1}{\pi}\int_0^{\pi} cosk\theta\, cos(xsin\theta)d\theta = \begin{cases} J_k(x), \wedge\ k = 0,2,4, \ldots \\ 0, \wedge\ k = 1,3,5, \ldots \end{cases} \tag{3.24}$$

Now multiplying the second equation in (3.23) by $sink\theta$ and integrating from 0 to π, we obtain

$$\frac{1}{\pi}\int_0^{\pi} sink\theta\, sin(xsin\theta)d\theta = \begin{cases} J_k(x), \wedge\ k = 1,3,5, \ldots \\ 0, \wedge\ k = 0,2,4, \ldots . \end{cases} \tag{3.25}$$

Adding equations (3.24) and (3.25), we obtain Bessel's integral representation

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} cos(k\theta - xsin\theta)d\theta, \quad k = positiveinteger. \tag{3.26}$$

Recurrence formulas for $J_n(x)$

Bessel functions of the first kind, $J_n(x)$, are the most useful, because they are bounded near the origin. And there exist some useful recurrence formulas between Bessel functions of different orders and their derivatives.

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x) \tag{3.27a}$$

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x) \tag{3.27b}$$

$$xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x) \tag{3.27c}$$

$$J_n'(x) = \frac{J_{n-1}(x) - J_{n+1}(x)}{2} \tag{3.27d}$$

Note:
Adding equations (3.27b) and (3.27c) and dividing by 2x, we obtain the required result in (3.27d). If we subtract (3.27b) from (3.27c), $J_n'(x)$ is eliminated and we obtain the result in equation (3.27a). These recurrence formulas (or important identities) are very useful.

Examples.

1. Show that $J_0'(x) = J_{-1}(x) = -J_1(x)$
   Solution
   Putting $n = 0$ in equation (3.27d), we obtain

   $$J_0'(x) = \frac{J_{-1}(x) - J_1(x)}{2}$$

   Then using the fact that $J_{-n}(x) = (-1)^n J_n(x)$ on the RHS (i.e. $n = 1$), we obtain the required result.
2. Show that $J_3(x) = \left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x)$
   Solution
   Putting $n = 2$ in equation (3.27a) gives

   $$J_3(x) = \frac{4}{x}J_2(x) - J_1(x)$$

   Similarly, putting $n = 1$ in equation (3.27a) gives

   $$J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$$

   Substituting $J_2(x)$ into the expression for $J_3(x)$, we obtain

   $$J_3(x) = \frac{4}{x}\left(\frac{2}{x}J_1(x) - J_0(x)\right) - J_1(x) = \left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x)$$

Approximations to the Bessel functions
For very large or very small values of x we might be able to make some approximations to the Bessel functions of the first kind $J_n(x)$. By a rough argument, we can see that the Bessel functions behave

something like a damped cosine function when the value of x is very large. Bessel's equation in equation (3.1) gives

$$x^2y'' + xy' + (x^2 - s^2)y = 0$$

This can be re-written as

$$y'' + \frac{1}{x}y' + \left(1 - \frac{s^2}{x^2}\right)y = 0 \tag{3.28}$$

If x is very large, let us drop the term $\frac{s^2}{x^2}$ and then the differential equation reduces to

$$y'' + \frac{1}{x}y' + y = 0 \tag{3.29}$$

Let $u = yx^{\frac{1}{2}}$, then $u' = y'x^{\frac{1}{2}} + \frac{1}{2}x^{\frac{-1}{2}}y$, and $u'' = y''x^{\frac{1}{2}} + x^{\frac{-1}{2}}y' - \frac{1}{4}x^{\frac{-3}{2}}y$

From u'' we have

$$y'' + \frac{1}{x}y' = x^{\frac{-1}{2}}u'' + \frac{1}{4x^2}y$$

Adding y on both sides, we obtain

$$y'' + \frac{1}{x}y' + y = x^{\frac{-1}{2}}u'' + \frac{1}{4x^2}y + y = 0$$

Now

$$x^{\frac{-1}{2}}u'' + \frac{1}{4x^2}y + y = 0$$

$$u'' + \left(\frac{1}{4x^2} + 1\right)yx^{\frac{1}{2}} = 0$$

$$u'' + \left(\frac{1}{4x^2} + 1\right)u = 0$$

The solution to this equation is of the form

$$u = Acosx + Bsinx$$

Thus the approximate solution to Bessel's equation for very large values of x is given by

$$y = ux^{\frac{-1}{2}} = x^{\frac{-1}{2}}(Acosx + Bsinx)$$

This can be written as

$y = Cx^{\frac{-1}{2}} cos(x + \beta)$

A more rigorous argument leads to the following asymptotic formula

$J_n(x) \approx Cx^{\frac{-1}{2}} cos(x + \beta)$

Where $C = \frac{\pi}{2}$ and $\beta = -\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)$

For very small values of x (that is, near 0), by examining the solution itself and dropping all terms after the first, we find

$J_n(x) \approx \frac{x^n}{2^n \Gamma(n+1)}$

Vibrations of a circular membrane: Read about it

Legendre's equation
Legendre's differential equation (DE) is of the form

$$(1 - x^2)y'' - 2xy' + l(l + 1)y = 0, l = constant \qquad (4.1)$$

Where l is a positive constant.

This DE is of great importance in classical and quantum physics. In general, Legendre's equation appears in problems in classical mechanics, electromagnetic theory, heat, and quantum mechanics, with spherical symmetry.

The solution of the Legendre DE is a set of polynomials called the Legendre polynomials $P_n(x)$, where $l = 0,1,2, \ldots$, is the degree of the polynomial. The Legendre polynomials are also called Legendre functions of the first kind.

Dividing through by $(1 - x^2)$ in equation 4.1, we obtain the standard form

$$y'' - \frac{2x}{(1-x^2)} y' + \frac{l(l+1)}{(1-x^2)} y = 0, l = constant \qquad (4.2)$$

The resulting solution of Legendre's equation is called the Legendre polynomial $P_n(x)$, where $n = 0,1,2, \ldots$, is the degree of the polynomial.

$$\begin{cases} P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m} \\ P_n(x) = \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n 1!(n-1)!(n-2)!} x^{n-2} + \cdots \end{cases} \qquad (4.3)$$

Where $M = \frac{n}{2}$ or $\frac{(n-1)}{2}$ whichever is an integer.

Rodrigues' formula for $P_n(x)$
The Legendre polynomials $P_n(x)$ are given by the formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n] \tag{4.4}$$

Examples
$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2-1), \quad P_3(x) = \frac{1}{2}(5x^3-3x),$
$P_4(x) = \frac{1}{8}(35x^4-30x^2+3), \quad P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$

Note: Generally $P_n(1) = 1$ (i.e. $P_0(1) = 1, \quad P_1(1) = 1, \quad P_2(1) = 1, \quad P_3(1) = 1$) etc.)

The generating function for $P_n(x)$
The generating function for Legendre polynomials $P_n(x)$ is given by
$$\phi(x,z) = (1-2xz+z^2)^{\frac{-1}{2}} = \sum_{n=0}^{\infty} P_n(x)z^n, |z| < 1 \tag{4.5}$$

We shall be concerned only with the case in which

$x = cos\theta \quad (-\pi < \theta \leq \pi)$ and then $z^2 - 2xz + 1 \equiv \left(z - e^{i\theta}\right)\left(z - e^{i\theta}\right)$

We can use equation (4.5) to find successive polynomials explicitly. Thus, differentiating equation (4.5) with respect to z so that

$$(x-z)(1-2xz+z^2)^{\frac{-3}{2}} = \sum_{n=1}^{\infty} nz^{n-1}P_n(x) \tag{4.6}$$

Using equation (4.5) again gives

$$(x-z)[P_0(x) + \sum_{n=1}^{\infty} P_n(x)z^n] = (1-2xz+z^2)\sum_{n=1}^{\infty} nz^{n-1}P_n(x) \tag{4.7}$$

Then expanding coefficients of z^n in equation (4.6) leads to the recurrence relation

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \tag{4.8}$$

This gives P_4; P_5; P_6, etc. very quickly in terms of P_0; P_1, and P_3. Recurrence relations are very useful in simplifying work, helping in proofs or derivations. We list four more recurrence relations below without proofs or derivations:

$$xP_n'(x) - P_{n-1}'(x) = nP_n(x) \tag{4.9a}$$

$$P_n'(x) - xP_{n-1}'(x) = nP_{n-1}(x) \tag{4.9b}$$

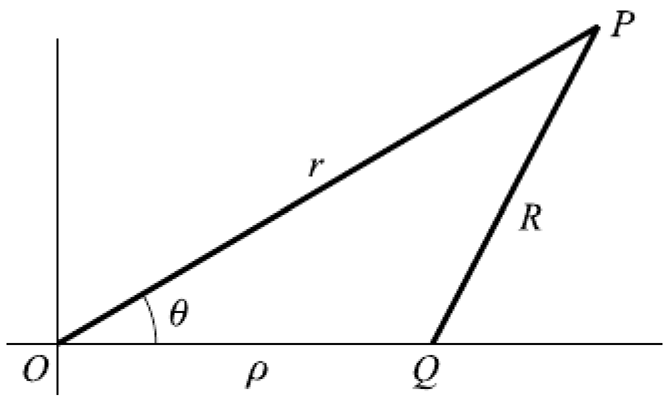
$$(1-x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x) \tag{4.9c}$$

$$(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x) \tag{4.9d}$$

With the help of the recurrence formulas (4.8) and (4.9b), we can straight away establish the other three. Full details are left for the reader.

Example

The physical significance of expansion of the generating function is apparent in this simple example: find the potential V of a point charge at point P due to a charge $+q$ at Q



$$V_P = \frac{q}{R} = q(\rho^2 - 2r\rho cos\theta + r^2)^{\frac{-1}{2}}$$

$V_P = \frac{q}{\rho}(\rho^2 - 2zcos\theta + z^2)^{\frac{-1}{2}}$, where $z = \frac{r}{\rho}$

This gives

$$V_P = \begin{cases} \frac{q}{\rho}\sum_{n=0}^{\infty}\left(\frac{r}{\rho}\right)^n P_n(cos\theta) ,\wedge r < \rho \\ \frac{q}{\rho}\sum_{n=0}^{\infty}\left(\frac{r}{\rho}\right)^{n+1} P_n(cos\theta) ,\wedge r > \rho \end{cases} \tag{4.10}$$

There are many problems in which it is essential that the Legendre polynomials be expressed in terms of θ, the colatitude angle of the spherical coordinate system.

Note:

$P_0(cos\theta) = 1$, $P_1(cos\theta) = cos\theta$, $P_2(cos\theta) = \frac{(3cos2\theta+1)}{4}$, $P_3(cos\theta) = \frac{(5cos3\theta+3cos\theta)}{8}$

$P_4(cos\theta) = \frac{(35cos4\theta+20cos2\theta+9)}{64}$, $P_5(cos\theta) = \frac{(63cos5\theta+35cos3\theta+30cos\theta+9)}{128}$, $P_6(cos\theta) = \frac{(231cos6\theta+126cos4\theta+105cos2\theta+50)}{512}$

Orthogonality of Legendre polynomials

The set of Legendre polynomials $\{P_n(x)\}$ is orthogonal for $-1 < x < +1$. In particular, we can show that

$$\int_{-1}^{+1} P_n(x)P_m(x)\,dx = \begin{cases} \frac{2}{(2n+1)}, \wedge\, m = n \\ 0, \wedge\, m \neq n \end{cases} \quad (4.11)$$